

# On large gaps between consecutive zeros, on the critical line, of some zeta-functions



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## Notation

When writing  $f(x) = O(g(x))$  or  $f(x) \ll g(x)$  we mean that  $g(x)$  is a positive function and that there exists a positive constant  $C$  such that  $|f(x)| \leq C \cdot g(x)$ . Also,  $f(x) \asymp g(x)$  will mean  $f(x) \ll g(x) \ll f(x)$ . Furthermore,  $f(x) \sim g(x)$  stands for that  $\lim \frac{f(x)}{g(x)} = 1$ .

With  $p$  being a prime,  $p^k || n$  means that  $p^k | n$  but  $p^{k+1} \nmid n$ . By  $\int_{(c)}$  we mean integration along the line  $\operatorname{Re}(s) = c$ .

Euler's constant is denoted by  $\gamma$  ( $\approx 0.577$ ). The divisor function  $d(n)$ , the Möbius function  $\mu(n)$  and Euler's totient function  $\varphi(n)$  are defined as usual (see for example Chapter 2 in Apostol [1]).

Notice that when we let  $A$  stand for a positive absolute number, it need not always be the same at each occurrence. This practice is also used with  $\epsilon$ , which denotes a small positive number.

# Chapter 1

## Overview of the thesis

In Chapter 2 we go through general background theory of the Riemann zeta-function  $\zeta(s)$  and Dirichlet  $L$ -functions, with particular focus on their zeros.

Hall [30] has shown the existence of large gaps between consecutive zeros, on the critical line<sup>1</sup>, of  $\zeta(s)$ . In Chapter 3 we illustrate his method in the simplest case.

The aim of Chapter 4 is basically to show that analogous results to the ones of the above type for the Riemann zeta-function  $\zeta(s)$ , can be shown for any fixed Dirichlet  $L$ -function.

A generalisation of Hall's article [34] is done in Chapter 5 by introducing an amplifier. To find asymptotics for some integrals, we use the article [44] by Hughes and Young. Our deduction is that for any sufficiently large  $T$ , there exists a subinterval of  $[T, 2T]$  of length at least  $2.766 \times \frac{2\pi}{\log T}$ , in which the function  $t \mapsto \zeta(\frac{1}{2} + it)$  has no zeros<sup>2</sup>.

The aim of Chapter 6 is to show the existence of large gaps between consecutive zeros, on the critical line, of some Dirichlet  $L$ -functions  $L(s, \chi)$ , with  $\chi$  being an even primitive Dirichlet character, of length at least 3.54 times the average<sup>3</sup>. The article [14] by Conrey, Iwaniec and Soundararajan enables us to handle our integral-calculations.

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<sup>1</sup>On the assumption of the Riemann Hypothesis, all the non-trivial zeros of  $\zeta(s)$  lie on the critical line, i.e. have real part equal to  $1/2$ .

<sup>2</sup>Notice that on the Riemann Hypothesis, the average gap between two consecutive zeros at height  $T$ , on the critical line, of  $\zeta(s)$  has length  $\frac{2\pi}{\log T}$ .

<sup>3</sup>On assuming the Generalised Riemann Hypothesis, all the non-trivial zeros of  $L(s, \chi)$  lie exactly on the critical line — see Theorem 6.1 and Remark 6.1.

# Chapter 2

## Background

### 2.1 Introducing the Riemann zeta-function $\zeta(s)$

In 1737, Euler [20] studied for real  $s > 1$  the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots \quad (2.1)$$

Since by the Fundamental Theorem of Arithmetic each integer greater than 1 can be factorised into a product of primes in a unique way, we see formally that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left( \frac{1}{1^s} + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right) = \prod_p (1 - p^{-s})^{-1}, \quad (2.2)$$

where the product is over all primes  $p$ . Euler used this together with the well-known fact that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots \quad (2.3)$$

diverges to deduce that

$$\sum_p \frac{1}{p} \quad (2.4)$$

diverges. The latter obviously implies that there are infinitely many primes.

Riemann considered the sum in (2.1) more generally, seeing it as a function of the complex variable  $s$ . Writing as usual  $s = \sigma + it$ , we define for  $\sigma > 1$  the Riemann zeta-function to be

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (2.5)$$

Clearly the sum in (2.5) is absolutely convergent whenever  $\operatorname{Re}(s) = \sigma > 1$ . Indeed, it is uniformly convergent on any closed disc contained in the half-plane  $\sigma > 1$ , which

tells us that  $\zeta(s)$  is analytic in said half-plane with

$$\zeta'(s) = \sum_{n=1}^{\infty} -\frac{\log n}{n^s}. \quad (2.6)$$

The Euler product identity (2.2) for the Riemann zeta-function is indeed valid when  $\sigma > 1$ . By using the Dirichlet convolution relation  $u * \mu = I$ , which says that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (2.7)$$

one may deduce for  $\sigma > 1$  that

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1, \quad (2.8)$$

which in particular tells us that  $\zeta(s)$  cannot be zero when  $\operatorname{Re}(s) > 1$ .

In his famous article [57] from 1859, Riemann showed that  $\zeta(s)$  can be extended as a meromorphic function in the whole complex plane, with a simple pole at  $s = 1$  with residue 1. Moreover, he established that

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \xi(1-s), \quad (2.9)$$

which is one way to write the so called functional equation for  $\zeta(s)$ . What follows is a brief sketch of his proof. Suppose first that  $\sigma > 1$ . Having the definition of the gamma function  $\Gamma(s)$  in one's mind, it is easily seen that

$$\int_0^{\infty} x^{\frac{1}{2}s-1} e^{-n^2\pi x} dx = \frac{\Gamma\left(\frac{1}{2}s\right)}{n^s \pi^{\frac{1}{2}s}}. \quad (2.10)$$

Summing this up over  $n \in \mathbb{N}$  and interchanging the order of summation and integration<sup>1</sup>, we obtain

$$\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s) = \int_0^{\infty} \psi(x)x^{\frac{1}{2}s-1} dx, \quad (2.11)$$

where

$$\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}. \quad (2.12)$$

For  $x > 0$  we have

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left\{ 2\psi\left(\frac{1}{x}\right) + 1 \right\}, \quad (2.13)$$

---

<sup>1</sup>This is allowed due to absolute convergence.

this essentially being an identity of Jacobi and it may be proved using Poisson's summation formula (see Chapter 10 of [19]). Using this in (2.11) yields

$$\begin{aligned}\Gamma\left(\frac{1}{2}s\right)\pi^{-\frac{1}{2}s}\zeta(s) &= \int_1^\infty \psi(x)x^{\frac{1}{2}s-1} dx + \int_0^1 \left\{ \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \right\} x^{\frac{1}{2}s-1} dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty \frac{\psi(x)}{x} \left[ x^{\frac{1}{2}s} + x^{\frac{(1-s)}{2}} \right] dx.\end{aligned}\quad (2.14)$$

Since  $\psi(x)$  decreases faster than any power of  $x$ , the last integral in (2.14) converges for all  $s$ . Thus  $\zeta(s)$  can be analytically continued as stated. Also, (2.9) follows from noting that (2.14) remains unchanged when  $s$  is replaced by  $1-s$ .

Using (2.9) we may conclude that  $\zeta(s)$  is zero when  $s = -2, -4, -6, \dots$  (these are the so called trivial zeros). We have already mentioned that  $\zeta(s)$  is never zero when  $\sigma > 1$ . This fact together with (2.9) yields that all other zeros of  $\zeta(s)$  (the so called non-trivial zeros) coincide with the zeros of the entire function  $\xi(s)$  and must lie in the so called critical strip, i.e. satisfy  $0 \leq \sigma \leq 1$ .

Let us next briefly discuss the order of  $\zeta(s)$ . It is standard to denote by  $\mu(\sigma)$  the infimum of all numbers  $c$  such that<sup>2</sup>  $\zeta(\sigma + it) \ll t^c$ . It is known that  $\mu(\sigma)$  is a non-increasing and continuous function. Now whenever  $\text{Re}(s) \geq \sigma_0 > 1$  say, then trivially

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} n^{-\sigma_0} < \infty. \quad (2.15)$$

Hence  $\mu(\sigma) = 0$  for  $\sigma \geq 1$ . The functional equation (2.9) can also be formulated as

$$\zeta(s) = \chi(s)\zeta(1-s), \quad (2.16)$$

where

$$\chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{1}{2}s\right)}. \quad (2.17)$$

By using Stirling's formula for the gamma function, one obtains in any fixed strip  $\alpha \leq \sigma \leq \beta$ , as  $t \rightarrow \infty$  that

$$|\chi(s)| \sim \left( \frac{t}{2\pi} \right)^{\frac{1}{2}-\sigma}. \quad (2.18)$$

We see that  $\mu(\sigma) = 1/2 - \sigma$  for  $\sigma \leq 0$ . By using the Phragmén–Lindelöf principle, one may show that  $\mu(\sigma)$  is a convex function. Hence in the critical strip  $\mu(\sigma) \leq 1/2 - \sigma/2$ . In particular we find  $\mu(1/2) \leq 1/4$ . This result has been improved upon many times, in 2005 Huxley [45] showed that  $\mu(1/2) \leq 32/205$ . The Lindelöf Hypothesis<sup>3</sup> says that  $\mu(1/2) = 0$ .

<sup>2</sup>Notice that there is no loss in generality in supposing here that  $t > 0$  since  $\zeta(\sigma \pm it)$  are complex conjugates.

<sup>3</sup>This conjecture is expected to be true, indeed it is known to be true upon assuming the famous Riemann Hypothesis (see Section 2.2.2).

## 2.2 The non-trivial zeros of $\zeta(s)$

### 2.2.1 Initial observations

$\xi(s)$  is an integral function of order 1, indeed we have

$$|\xi(s)| < \exp(A|s| \log |s|), \quad (2.19)$$

as  $|s| \rightarrow \infty$  (see Chapter 12 in [15]). By using Hadamard's factorisation theorem we may write

$$\xi(s) = e^{A_0 + A_1 s} \prod_{\rho} (1 - s/\rho) e^{s/\rho}, \quad (2.20)$$

where the sum runs over the zeros  $\rho$  of  $\xi(s)$  (including multiplicity) and where  $A_0, A_1$  are computable constants. Furthermore we have that  $\sum_{\rho} |\rho|^{-1-\epsilon}$  converges for any  $\epsilon > 0$ . However,  $\sum_{\rho} |\rho|^{-1}$  diverges, which in particular shows that  $\xi(s)$  has infinitely many zeros. On the other hand, defining

$$N(T) := \#\{\rho = \beta + i\gamma : 0 \leq \beta \leq 1, 0 \leq \gamma \leq T\}, \quad (2.21)$$

one can easily show that  $N(T)$  can not grow too fast. Let  $n(r)$  denote the number of zeros of  $\zeta(s)$  in the circle with radius  $r$ , centred at  $2 + iT$ . Clearly

$$N(T+1) - N(T) \leq n(\sqrt{5}) \ll \int_{\sqrt{5}}^3 \frac{n(\sqrt{5})}{r} dr \leq \int_{\sqrt{5}}^3 \frac{n(r)}{r} dr \leq \int_0^3 \frac{n(r)}{r} dr. \quad (2.22)$$

But by means of Jensen's formula we obtain

$$\int_0^3 \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |\zeta(2 + iT + 3e^{i\theta})| d\theta - \log |\zeta(2 + iT)| \ll \log T, \quad (2.23)$$

the last step following from the fact that  $|\zeta(s)| \leq t^A$  in the relevant region.

The next piece in our study of the vertical distribution of the zeros of  $\zeta(s)$  will be the Riemann–von Mangoldt formula, which says that

$$N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + O(\log T). \quad (2.24)$$

When proving (2.24) we may without loss of generality assume that  $T$  is not the ordinate of a zero. Applying the Principle of Argument to  $\xi(s)$  and using Stirling's formula, one can show that

$$N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right), \quad (2.25)$$

with

$$S(T) := \pi^{-1} \operatorname{Im} \log(\zeta(\frac{1}{2} + iT)), \quad (2.26)$$

where the value of the right-hand side (RHS) of (2.26) is obtained by continuous variation along the two lines joining  $2$ ,  $2 + iT$  and  $\frac{1}{2} + iT$ , starting with the value  $0$ . Combining logarithmic differentiation of (2.20) and (2.9) yields

$$\frac{\zeta'}{\zeta}(s) = A_1 - \frac{1}{s-1} + \frac{\log \pi}{2} - \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} + 1 \right) + \sum_{\rho} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right). \quad (2.27)$$

Careful use of (2.27) yields  $S(T) \ll \log T$ , which allows us to conclude (2.24).

## 2.2.2 Horizontal distribution

Let us begin by showing that  $\zeta(s) \neq 0$  on the lines  $\sigma = 1$  and  $\sigma = 0$ . For  $\sigma > 1$  we have

$$\operatorname{Re} \log(\zeta(\sigma + it)) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} \cos(t \log(p^m)). \quad (2.28)$$

Combining this with the inequality

$$3 + 4 \cos \theta + \cos(2\theta) = 2(1 + \cos \theta)^2 \geq 0, \quad \forall \theta \in \mathbb{R}, \quad (2.29)$$

we obtain

$$3 \log(\zeta(\sigma)) + 4 \operatorname{Re} \log(\zeta(\sigma + it)) + \operatorname{Re} \log(\zeta(\sigma + 2it)) \geq 0. \quad (2.30)$$

Hence for all  $\sigma > 1$  we have

$$\zeta^3(\sigma) |\zeta^4(\sigma + it) \zeta(\sigma + 2it)| \geq 1. \quad (2.31)$$

Suppose now that  $\zeta(1 + it_0) = 0$  for some  $t_0 \neq 0$ . Let us study the left-hand side (LHS) of (2.31) as  $\sigma \rightarrow 1$ . From our assumption we find  $\zeta^4(\sigma + it_0) \ll (\sigma - 1)^4$ . By continuity  $\zeta(\sigma + 2it_0) \ll 1$ . And we know that  $\zeta(\sigma) \ll (\sigma - 1)^{-1}$ . All-in-all we would have

$$\zeta^3(\sigma) |\zeta^4(\sigma + it) \zeta(\sigma + 2it)| \ll (\sigma - 1). \quad (2.32)$$

Since clearly  $(\sigma - 1) \rightarrow 0$  as  $\sigma \rightarrow 1$ , we get a contradiction to (2.31). Thus if  $\sigma = 1$ , then  $\zeta(s)$  is not equal to zero<sup>4</sup>. This result was shown independently by Hadamard and de la Vallée Poussin in 1896. It enabled them to prove the Prime Number Theorem which says that

$$\pi(x) := \sum_{p \leq x} 1 \sim \frac{x}{\log x}. \quad (2.33)$$

---

<sup>4</sup>By the functional equation, there are no zeros with  $\sigma = 0$  either.

Let us also mention that it is quite easy to improve the above argument to deduce that there exists  $c > 0$  such that  $\zeta(s)$  has no zero in the region<sup>5</sup>

$$\sigma \geq 1 - \frac{c}{\log t}, \quad t \geq 2. \quad (2.34)$$

The non-trivial zeros of  $\zeta(s)$  thus satisfy  $0 < \sigma < 1$ . They are symmetrically distributed with respect to the real axis (since  $\overline{\zeta(s)} = \zeta(\bar{s})$ ) and the critical line<sup>6</sup> (use the functional equation). The famous Riemann Hypothesis (RH) is a conjecture which says that in fact all of them lie exactly on the critical line. This has not yet been proved (or disproved), but it is believed to be true. There are many known consequences of the truth of the RH, an example is von Koch's observation from 1901 (see [50]) that the RH implies that

$$\pi(x) = \text{li}(x) + O(\sqrt{x} \log x), \quad (2.35)$$

where

$$\text{li}(x) := \lim_{\epsilon \rightarrow 0^+} \left\{ \int_0^{1-\epsilon} \frac{1}{\log t} dt + \int_{1+\epsilon}^x \frac{1}{\log t} dt \right\}. \quad (2.36)$$

It follows trivially from (2.36) that

$$p_{n+1} - p_n \ll p_n^{1/2} \log^2 p_n, \quad (2.37)$$

where  $p_n$  denotes the  $n$ -th prime number. Numerical calculations will of course never yield a proof of the RH, however, they are interesting and (at least so far!) comforting for people who hope that the RH holds. Gourdon [28] has verified that the first  $10^{13}$  non-trivial zeros of  $\zeta(s)$  satisfy  $\text{Re}(s) = 1/2$ . There are also some theoretical results worthy of mention in this context. Bohr and Landau [3] proved in 1914 that  $\forall \delta > 0$ , the number of zeros of  $\zeta(s)$  with  $\text{Re}(s) \geq 1/2 + \delta$  and  $0 \leq \text{Im}(s) \leq T$  is  $\ll_\delta T$ . Since  $N(T) \sim \frac{T \log T}{2\pi}$ , their result thus shows that for any  $\delta > 0$ , all but an infinitesimal proportion of the non-trivial zeros of  $\zeta(s)$  lie within a distance of  $\delta$  from the critical line. In the same year (1914), Hardy [36] showed the existence of infinitely many zeros on the critical line. In 1942, Selberg [58] showed that a positive proportion of the zeros are on the critical line. Levinson [51] showed in 1974 that at least a third of the zeros must lie on the critical. This was shown to hold for  $2/5$  of the zeros by Conrey [9] in 1989 and for 41.7 % of the zeros by Feng [21] in 2010.

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<sup>5</sup>Due to work of Korobov and Vinogradov in 1958 (see e.g. Chapter 6 in [47]) we know that  $\exists C > 0$  such that  $\zeta(s) \neq 0$  for  $\sigma \geq 1 - C(\log t)^{-2/3}(\log \log t)^{-1/3}$  and  $t \geq A$ .

<sup>6</sup>The critical line is the line  $\text{Re}(s) = 1/2$ .

### 2.2.3 Vertical distribution

Now (2.24) tells us that the average difference of the ordinates of two consecutive zeros of  $\zeta(s)$  at height  $T$  is approximately  $2\pi/\log T$ . One might wonder in more detail what the vertical distribution of the zeros looks like. Many have worked on problems in this area. We will briefly go through the history of such results next.

Denote by  $\{\gamma_n\}$  the sequence of ordinates of all zeros of  $\zeta(s)$  in the upper halfplane, ordered in non-decreasing order. As the distribution of the  $\gamma_n$  becomes denser as we move up in the critical strip, it makes more sense to consider the normalised vertical distance between consecutive zeros, i.e. to ask what we can say about

$$\mu := \liminf_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{(2\pi/\log \gamma_n)} \quad \text{and} \quad \lambda := \limsup_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{(2\pi/\log \gamma_n)}. \quad (2.38)$$

In 1946, Selberg [59] remarked that

$$\mu < 1 < \lambda. \quad (2.39)$$

Although this is still the only known unconditional result, it is believed to be far from the whole truth.

Indeed, in 1973, Montgomery [52] assumed the RH and made his famous pair correlation conjecture for the non-trivial zeros of  $\zeta(s)$ . If we denote such a zero by  $1/2 + i\gamma$ , then it says that for fixed  $0 < \alpha < \beta$ ,

$$\sum_{\substack{\gamma, \gamma' \in [0, T] \\ 2\pi\alpha/\log T \leq \gamma - \gamma' \leq 2\pi\beta/\log T}} 1 \sim \frac{T \log T}{2\pi} \int_{\alpha}^{\beta} 1 - \left( \frac{\sin(\pi u)}{\pi u} \right)^2 du \quad (2.40)$$

as  $T \rightarrow \infty$ . This conjecture is easily seen to imply that  $\mu = 0$ . Montgomery also made the prediction that  $\lambda = \infty$ . It is believed that the statistical distribution of the zeros high up on the critical line is basically similar to that of the eigenvalues of large random unitary matrices. The physicist Dyson had studied the latter phenomena (see [18]). When he met Montgomery, the connection between their pair correlation functions was noticed. For a more detailed account of the connection between the Riemann zeta-function and random matrix theory, see for example [49]. There is also numerical “evidence” of this connection, most noticeably due to Odlyzko [55], supporting that the pair correlation and neighbour spacing for the non-trivial zeros of  $\zeta(s)$  are close to those for the Gaussian Unitary Ensemble.

Going back to discussing actual results, Montgomery also showed in his article [52] that the RH<sup>7</sup> implies that  $\mu < 0.68$ . Then in 1980, Mueller [54] managed to prove,

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<sup>7</sup>If the RH is false infinitely many times, then trivially there are infinitely many pairs of non-trivial zeros of  $\zeta(s)$  with the same ordinate, and hence we would have  $\mu = 0$ .

albeit on the RH, the existence of what one may actually begin to properly call large gaps. The underlying idea in her proof is nice and easy to explain, so we give a brief sketch here. Let us assume the RH and that

$$\limsup_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{(2\pi / \log \gamma_n)} < 1.9. \quad (2.41)$$

We may therefore pick a (sufficiently) large value of  $T$  such that all gaps between  $T$  and  $2T$  are at most  $1.9 \cdot 2\pi / \log T$ . Our goal is to show that the latter leads to a contradiction. We get that

$$\int_T^{2T} |\zeta(1/2 + it)|^2 dt \leq \sum_{T < \gamma \leq 2T} \int_{\gamma - 0.95 \cdot 2\pi / \log T}^{\gamma + 0.95 \cdot 2\pi / \log T} |\zeta(1/2 + it)|^2 dt. \quad (2.42)$$

Already in 1918, it was shown by Hardy and Littlewood [37] that the LHS of (2.42) is asymptotic to  $T \log T$ , as  $T \rightarrow \infty$ . In order to estimate the RHS of (2.42), Mueller used a result of Gonek (see [25] or [26]), namely that on assuming the RH we have

$$\sum_{0 < \gamma \leq T} \left| \zeta \left( \frac{1}{2} + i(\gamma + 2\pi c / \log T) \right) \right|^2 = \frac{T}{2\pi} \log^2 T \left\{ 1 - \left( \frac{\sin(\pi c)}{\pi c} \right)^2 \right\} + O(T \log T) \quad (2.43)$$

as  $T \rightarrow \infty$ , uniformly for  $|c| \leq (4\pi)^{-1} \log(\frac{T}{2\pi})$ . Clearly we may apply (2.43) twice and take the difference in order to obtain

$$\sum_{T < \gamma \leq 2T} \left| \zeta \left( \frac{1}{2} + i(\gamma + 2\pi c / \log T) \right) \right|^2 = \frac{T}{2\pi} \log^2 T \left\{ 1 - \left( \frac{\sin(\pi c)}{\pi c} \right)^2 \right\} + O(T \log T). \quad (2.44)$$

Now we simply integrate (2.44) with respect to  $c$  over the interval  $[-0.95, 0.95]$  and obtain that the RHS of (2.42) is asymptotic (as  $T \rightarrow \infty$ ) to  $aT \log T$ , where

$$a = \int_{-0.95}^{0.95} \left\{ 1 - \left( \frac{\sin(\pi c)}{\pi c} \right)^2 \right\} dc \approx 0.9973. \quad (2.45)$$

Since  $0.9973 < 1$  we have a contradiction to (2.42), as desired. Montgomery and Odlyzko [53] improved the value of 1.9 to 1.9799 and also obtained  $\mu < 0.5179$ . Their method turned out to be the other side of Mueller's coin, as is explained by Conrey, Ghosh and Gonek [12]. The latter article also improved the numerical value of the constants, to 2.337 and 0.5172 respectively.

Hall (see [30–35]) only deals with large gaps and he also only looks at the zeros of the Riemann zeta-function which lie on the critical line (regardless of whether these are all of the non-trivial zeros or not). He then analogously defines lim sup of the gaps between such zeros (this naturally coincides with the usual definition if we assume

the RH). For this he obtained the value 2.26 in [30]. This was done by using (for details on Hall’s method, the reader is recommended to read Chapter 3) the simplest fourth power version of Wirtinger’s inequality, namely that if  $y(t)$  is a continuously differentiable function satisfying  $y(0) = y(\pi) = 0$  then

$$\int_0^\pi y(t)^4 dt \leq (4/3) \int_0^\pi y'(t)^4 dt. \quad (2.46)$$

Hall later [31] chose to instead work with the Wirtinger-inequality that says that if  $y(t)$  is a continuously differentiable function satisfying  $y(0) = y(\pi) = 0$  and  $v \geq 0$  then

$$\int_0^\pi y'(t)^4 + 6vy(t)^2y'(t)^2 dt \geq 3\lambda_0(v) \int_0^\pi y(t)^4 dt, \quad (2.47)$$

where

$$\lambda_0(v) := \frac{1}{8} \{1 + 4v + \sqrt{1 + 8v}\}. \quad (2.48)$$

By applying this inequality to Hardy’s function  $Z(t)$  (see (3.1)) with  $v = 22/49$ , he found the value  $\sqrt{11/2} \approx 2.34$  for his lim sup. Finally, in 2005, Hall [34] applied the simplest version of Wirtinger’s inequality, which says that if  $y(t)$  is continuously differentiable and is zero at  $t = 0$  and  $t = \pi$  then

$$\int_0^\pi y(t)^2 dt \leq \int_0^\pi y'(t)^2 dt, \quad (2.49)$$

to the function  $Z(t)Z(t + 1.315 \times \frac{2\pi}{\log T})$  and obtained the value 2.63 for his lim sup.

On the assumption of the RH<sup>8</sup>,  $\mu \leq 0.5154$  and  $\lambda \geq 2.7327$  were recently (2010) obtained by Feng and Wu [22]<sup>9</sup>.

## 2.3 Dirichlet $L$ -functions

### 2.3.1 Preliminary definitions

There are generalisations of the Riemann zeta-function  $\zeta(s)$ . For a short survey of such so called “zeta-functions”, see for example [47]. In this thesis we will focus on  $\zeta(s)$  and Dirichlet  $L$ -functions. Before we introduce the latter, we will now go through some background theory.

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<sup>8</sup>On the assumption of the Generalised Riemann Hypothesis (see Section 2.3.3), Bui [5] has established  $\lambda \geq 3.033$ .

<sup>9</sup>Note that by working similarly to Hall, we will show in Theorem 5.1 that the RH implies that  $\lambda \geq 2.766$ .

**Definition 2.1.** Let  $G$  be the group of reduced residue classes modulo  $q$ . Corresponding to each character<sup>10</sup>  $f$  of  $G$ , we define an arithmetical function  $\chi_f$  as follows:

$$\chi_f(n) := \begin{cases} f(\bar{n}) & \text{if } (n, q) = 1, \\ 0 & \text{if } (n, q) > 1. \end{cases} \quad (2.50)$$

Such a function  $\chi = \chi_f$  is called a Dirichlet character modulo  $q$ .

Since there are  $\varphi(q)$  characters mod  $q$ , there are  $\varphi(q)$  Dirichlet characters mod  $q$ . The latter are all clearly completely multiplicative and periodic with period  $q$ . If  $(n, q) = 1$  then  $\chi(n)$  is a  $\varphi(q)$ -th root of unity, so that in particular  $|\chi(n)| = 1$ . Also, from the orthogonality property of characters, we have that

$$\sum_{n=1}^q \chi(n) = \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ \varphi(q) & \text{if } \chi = \chi_0, \end{cases} \quad (2.51)$$

where the so called principal character  $\chi_0$  is defined as

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n, q) = 1, \\ 0 & \text{if } (n, q) > 1. \end{cases} \quad (2.52)$$

**Definition 2.2.** Given a Dirichlet character  $\chi$  mod  $q$  and an integer  $n$ , we have the associated so called Gauss sum given by

$$G(n, \chi) = \sum_{m=1}^q \chi(m) e^{2\pi i mn/q}. \quad (2.53)$$

**Definition 2.3.** The Gauss sum  $G(n, \chi)$  is said to be separable if

$$G(n, \chi) = \overline{\chi(n)} G(1, \chi). \quad (2.54)$$

**Definition 2.4.** Let  $\chi$  be a Dirichlet character mod  $q$ . A positive divisor  $d$  of  $q$  is called an induced modulus for  $\chi$  if  $\chi(a) = 1$  is satisfied whenever we have  $(a, q) = 1$  and  $a \equiv 1 \pmod{d}$ .

**Definition 2.5.** A Dirichlet character  $\chi$  mod  $q$  is said to be primitive if it has no induced modulus  $d < q$ .

**Remark 2.1.** It is well-known (see e.g. Chapter 8 in Apostol [1]) that every Dirichlet character  $\chi$  mod  $q$  can be expressed as a product  $\chi(n) \equiv \psi(n)\chi_0(n)$ , where  $\psi$  is a primitive Dirichlet character modulo the conductor<sup>11</sup> of  $\chi$ .

**Remark 2.2.** It is also well-known that  $\chi$  is primitive if and only if  $G(n, \chi)$  is separable for every  $n$ .

<sup>10</sup> $f : G \rightarrow \mathbb{C}$  is said to be a character of  $G$  if  $f(ab) = f(a)f(b)$  for all  $a, b \in G$  and if  $f$  is not identically zero.

<sup>11</sup>The conductor of a Dirichlet character  $\chi$  mod  $q$  is the smallest induced modulus for  $\chi$ .

### 2.3.2 Introducing Dirichlet $L$ -functions

**Definition 2.6.** Given a Dirichlet character  $\chi \bmod q$  we define for  $\sigma > 1$  the associated Dirichlet  $L$ -function by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \quad (2.55)$$

The sum in (2.55) is trivially absolutely convergent for  $\sigma > 1$ . It has therefore (recall that  $\chi(n)$  is completely multiplicative) in this region an Euler product:

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}. \quad (2.56)$$

Provided that  $\chi \neq \chi_0$  one can use Dirichlet's test for convergence (recall (2.51)) and the concept of the abscissa of convergence of a Dirichlet series, to show that the sum in (2.55) is convergent when  $\sigma > 0$ . Moreover, this convergence is naturally uniform on any closed disc in said region, implying that  $L(s, \chi)$  is then analytic for  $\sigma > 0$ .

Let us for a moment assume that  $q > 1$  and restrict  $\chi$  to be primitive<sup>12</sup>. Then, just as in the case of the Riemann zeta-function,  $L(s, \chi)$  satisfies a functional equation. If we put

$$\xi(s, \chi) = (q/\pi)^{\frac{1}{2}(s+\mathbf{a})} \Gamma[\frac{1}{2}(s+\mathbf{a})] L(s, \chi), \quad (2.57)$$

where

$$\mathbf{a} := \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases} \quad (2.58)$$

then we have

$$\xi(1-s, \bar{\chi}) = \frac{i^{\mathbf{a}} q^{\frac{1}{2}}}{G(1, \chi)} \xi(s, \chi), \quad (2.59)$$

where of course  $G(1, \chi)$  is as in (2.53)<sup>13</sup>.

### 2.3.3 On the zeros of Dirichlet $L$ -functions

With  $q > 1$ , suppose first that  $\chi$  is primitive. Using the Euler product in (2.56) we see that  $L(s, \chi) \neq 0$  for  $\sigma > 1$ . Next we note (using (2.59)) that  $L(s, \chi)$  has trivial zeros at  $s = 0, -2, -4, \dots$  if  $\mathbf{a} = 0$  and at  $s = -1, -3, -5, \dots$  if  $\mathbf{a} = 1$ . The other zeros lie in the critical strip, but unless  $\chi$  is real, these non-trivial zeros do not necessarily come in pairs of complex conjugates. However, they are symmetrically distributed with respect to the critical line<sup>14</sup>.

<sup>12</sup>Then in particular  $\chi$  cannot be principal (since 1 would then have been an induced modulus).

<sup>13</sup>By Remark 2.2,  $G(n, \chi)$  is separable for every  $n$ . This can be used to show that  $|G(1, \chi)| = \sqrt{q}$ .

<sup>14</sup>As we shall see soon, there is no zero of  $L(s, \chi)$  at  $s = 1$ .

The Generalised Riemann Hypothesis (GRH) is a conjecture saying that no Dirichlet  $L$ -function  $L(s, \chi)$  has a zero with real part greater than  $1/2$ . In [42] Hilano extends to Dirichlet  $L$ -functions the ideas used by Levinson, when the latter showed that at least a third of the non-trivial zeros of the Riemann zeta-function lie on the critical line.

Working as with  $\zeta(s)$ , one can establish (see Chapter 14 in [15]) that there exists an absolute  $A > 0$  such that if  $\chi$  is any<sup>15</sup> complex Dirichlet character mod  $q$ , then  $L(s, \chi)$  has no zero in the region given by

$$\sigma \geq \begin{cases} 1 - \frac{A}{\log(q|t|)} & \text{if } |t| \geq 1, \\ 1 - \frac{A}{\log q} & \text{if } |t| \leq 1. \end{cases} \quad (2.60)$$

Furthermore, if  $\chi$  is a real non-principal Dirichlet character, the only possible zero of  $L(s, \chi)$  in said region is a single (simple) real zero. Such exceptional zeros are called Landau–Siegel zeros<sup>16</sup>. Siegel [61] has shown that  $\forall \epsilon > 0$  there exists  $C(\epsilon) > 0$  such that if  $\chi$  is any real non-principal Dirichlet character mod  $q$ , then  $L(s, \chi) \neq 0$  for  $s > 1 - C(\epsilon)q^{-\epsilon}$ . As a historical side-note, we mention here that the key step in Dirichlet’s [17] proof from 1837 that there are infinitely many primes in any arithmetical sequence

$$qn + a, \quad n = 0, 1, 2, \dots, \quad (2.61)$$

provided that  $(a, q) = 1$ , is to show that  $L(1, \chi) \neq 0$  for all non-principal  $\chi$ .

Let us now consider the vertical distribution of the zeros of Dirichlet  $L$ -functions  $L(s, \chi)$ , where  $q > 1$  and  $\chi$  is primitive. As the zeros of a Dirichlet  $L$ -function are in general not symmetrically distributed with respect to the real axis, it is natural to define

$$N(T, \chi) := \#\left\{s = \sigma + it : L(s, \chi) = 0, 0 < \sigma < 1 \text{ and } |t| < T\right\} \quad (2.62)$$

and study how  $N(T, \chi)$  behaves as a function of  $T$  and of the modulus  $q$ . By generalising the proof of the corresponding result for the Riemann zeta-function, one may show (see Chapter 16 in [15]) that for  $T \geq 2$ ,

$$\frac{1}{2}N(T, \chi) = \frac{T}{2\pi} \log(Tq/(2\pi)) - \frac{T}{2\pi} + O(\log(Tq)). \quad (2.63)$$

<sup>15</sup>In this paragraph,  $\chi$  does not have to be primitive.

<sup>16</sup>Notice that it may be the case that no Landau–Siegel zeros exist.

One may analogously directly obtain an expression for the number of zeros of  $L(\sigma + it, \chi)$  with  $T \leq t < 2T$  and unsurprisingly the answer is

$$\begin{aligned} & \left\{ \frac{2T}{2\pi} \log(2Tq/(2\pi)) - \frac{2T}{2\pi} \right\} - \left\{ \frac{T}{2\pi} \log(Tq/(2\pi)) - \frac{T}{2\pi} \right\} + O(\log(Tq)) \\ &= \frac{T}{2\pi} \log(Tq) + O(T) + O(\log q). \end{aligned} \quad (2.64)$$

This tells us that if we treat  $q$  as fixed, then just as with the Riemann zeta-function, the number of zeros in said region is asymptotic to  $\frac{T \log T}{2\pi}$ . Fujii remarks in his article [23] that a similar result to (2.39) holds for Dirichlet  $L$ -functions. However, (2.64) also tells us that if we let  $T$  be large but fixed and then study  $N(T, \chi)$  as a function of  $q$ , the average difference of the ordinates of two consecutive zeros at height  $T$  is approximately  $2\pi/\log q$  for large  $q$ . Hence roughly speaking  $q$  overtakes the role of  $T$ , an example of a so called  $q$ -analogue result<sup>17</sup>. Assuming the GRH, Selberg [60] proved a result which holds uniformly for all  $T$  and all primes  $q$ . Denoting by  $N_2(T, \chi)$  the number of zeros of  $L(s, \chi)$  with  $0 < \sigma < 1$  and  $0 \leq t \leq T$ , possible zeros with  $t = 0$  or  $t = T$  counting one-half only, his result is

$$N_2(T, \chi) = \frac{T}{2\pi} \log(Tq/(2\pi)) - \frac{T}{2\pi} + O\left(\frac{\log(q(1+T))}{\log \log(q(3+T))}\right). \quad (2.65)$$

Finally, we now comment on the case when  $\chi$  is imprimitive (i.e. not primitive). Remembering Remark 2.1, let us write  $\chi = \psi\chi_0$ , where  $\psi$  is a primitive Dirichlet character modulo  $q_1$  say. By the Euler product formula we have for  $\sigma > 1$  that

$$L(s, \chi) = \prod_{p \nmid q} [1 - \chi(p)p^{-s}]^{-1} = \prod_{p \nmid q} [1 - \psi(p)p^{-s}]^{-1} = L(s, \psi) \prod_{p|q} [1 - \psi(p)p^{-s}]. \quad (2.66)$$

However, by analytic continuation (2.66) remains valid for all  $s$ . The zeros of  $L(s, \chi)$  are thus the union of the zeros of  $L(s, \psi)$  and the zeros of  $\prod_{p|q} [1 - \psi(p)p^{-s}]$ . Clearly all of the latter have real part equal to 0. To see how many zeros  $L(s, \chi)$  has in the critical strip with  $|t| < T$ , we once again consider (2.66). The number of zeros of  $L(s, \psi)$  can be found via (2.63), provided that  $q$  is now replaced by  $q_1$ . Next, for each prime  $p$  not dividing  $q_1$ , the zeros of  $\prod_{p|q} [1 - \psi(p)p^{-s}]$  are spaced at equal distances  $2\pi/\log p$  apart. Therefore the number of them (with  $|t| < T$ ) is

$$\ll \sum_{p|q} (T \log p + 1) \ll T \log q. \quad (2.67)$$

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<sup>17</sup>For an elaboration of this, see for example Paley [56].

## 2.4 Mean value results

### 2.4.1 $2k^{\text{th}}$ moments of $\zeta(s)$

Let us consider

$$M_k(T) := \int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} dt \quad (2.68)$$

with  $k$  being a positive integer<sup>18</sup>. Hardy and Littlewood [37] showed in 1918 that

$$M_1(T) \sim T \log T. \quad (2.69)$$

In his article [46] from 1928, Ingham improved the error term for the second moment of  $\zeta(s)$ , obtaining

$$M_1(T) = T \log T + (2\gamma - 1 - \log(2\pi))T + O(T^{\frac{1}{2}+\epsilon}). \quad (2.70)$$

In the same article, Ingham also established

$$M_2(T) \sim \frac{T \log^4 T}{2\pi^2}. \quad (2.71)$$

In 1979, Heath-Brown [40] showed that there are constants  $a_4, a_3, a_2, a_1, a_0$  such that

$$M_2(T) = a_4 T \log^4 T + a_3 T \log^3 T + a_2 T \log^2 T + a_1 T \log T + a_0 T + O(T^{7/8+\epsilon}). \quad (2.72)$$

For  $k > 2$  there are only conjectures. Conrey and Ghosh [11] predicted in 1998 that

$$M_3(T) \sim \frac{42}{9!} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right\} T \log^9 T \quad (2.73)$$

and Conrey and Gonek [13] predicted in 2001 that

$$M_4(T) \sim \frac{24024}{16!} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^9 \left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) \right\} T \log^{16} T. \quad (2.74)$$

Roughly speaking they were led to (2.73) and (2.74) by predicting how certain mean values involving “long” Dirichlet polynomials should behave.

In 2000, Keating and Snaith [48] conjectured an asymptotic result for  $M_k(T)$  for more general  $k$ . To do this, they used random matrix theory, arguing that the value-distribution of  $\zeta(s)$  near height  $T$  can be modelled by the characteristic polynomial

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<sup>18</sup>There are some results concerning the integral in (2.68) for  $k \notin \mathbb{N}$ , but we will here focus on the case  $k \in \mathbb{N}$ .

of large random unitary matrices. Explicitly, their conjecture says that for any fixed  $k$  satisfying  $\operatorname{Re}(k) > -1/2$ ,

$$M_k(T) \sim a_k \frac{G^2(k+1)}{G(2k+1)} T(\log T)^{k^2}, \quad (2.75)$$

where  $G(z)$  is Barnes'  $G$ -function and

$$a_k = \prod_p \left( \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left( \frac{\Gamma(m+k)}{m! \Gamma(k)} \right)^2 p^{-m} \right). \quad (2.76)$$

When  $n$  is a positive integer, we have the relation

$$G(n) = \prod_{j=1}^{n-1} \Gamma(j) = \prod_{j=1}^{n-1} (j-1)!, \quad (2.77)$$

and this may be used to conclude that (2.75) is indeed consistent with (2.69), (2.71), (2.73) and (2.74).

Let us also mention another path, due to Gonek, Hughes and Keating [27], which again leads to the conjecture (2.75). Some of their results are unconditional. However, let us assume the RH here, as this simplifies things when we briefly discuss the ideas involved in their article. Then the first ingredient is loosely speaking that

$$\zeta(1/2 + it) \approx P_X(1/2 + it) Z_X(1/2 + it), \quad (2.78)$$

with

$$P_X(1/2 + it) \approx \prod_{p \leq X} (1 - p^{-1/2-it})^{-1} \quad (2.79)$$

and

$$Z_X(1/2 + it) \approx \prod_{\substack{\gamma_n \\ |\gamma_n - t| < 1/\log X}} (i(t - \gamma_n) e^{\gamma} \log X), \quad (2.80)$$

where  $\gamma_n$  refers to the ordinates of the non-trivial zeros of  $\zeta(s)$  and  $\gamma \approx 0.577$  is Euler's constant. Fittingly, the name of their article from 2007 is "A hybrid Euler-Hadamard product for the Riemann zeta function". The second ingredient is the following conjecture:

**Conjecture 2.1** (Splitting conjecture). *Let  $X, T \rightarrow \infty$  with  $X = O((\log T)^{2-\epsilon})$ . Then for any fixed  $k > -1/2$ , we have*

$$\frac{1}{T} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \sim \left( \frac{1}{T} \int_T^{2T} \left| P_X\left(\frac{1}{2} + it\right) \right|^{2k} dt \right) \times \left( \frac{1}{T} \int_T^{2T} \left| Z_X\left(\frac{1}{2} + it\right) \right|^{2k} dt \right). \quad (2.81)$$

With conditions as in Conjecture 2.1, they show that for any fixed  $k > 0$  we have

$$\frac{1}{T} \int_T^{2T} \left| P_X \left( \frac{1}{2} + it \right) \right|^{2k} dt \sim a_k (e^\gamma \log X)^{k^2}, \quad (2.82)$$

and conjecture, using random matrix theory, that

$$\frac{1}{T} \int_T^{2T} \left| Z_X \left( \frac{1}{2} + it \right) \right|^{2k} dt \sim \frac{G^2(k+1)}{G(2k+1)} \left( \frac{\log T}{e^\gamma \log X} \right)^{k^2}. \quad (2.83)$$

The result coming from putting together (2.82) and (2.83) is consistent with (2.75).

## 2.4.2 Mollifiers and amplifiers

Sometimes it is useful to consider mean values looking like

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} |A_y(\frac{1}{2} + it)|^2 dt, \quad (2.84)$$

where

$$A_y(s) = \sum_{n \leq y} \frac{a(n)}{n^s}. \quad (2.85)$$

The Dirichlet polynomial  $A_y(s)$  can act as a mollifier or as an amplifier.

Balasubramanian, Conrey and Heath-Brown [2] have shown that if  $y = T^{1/2-\epsilon}$  and  $a(n) \ll_\epsilon n^\epsilon$ , then

$$\int_0^T |\zeta(\frac{1}{2} + it)|^2 |A_y(\frac{1}{2} + it)|^2 dt \sim \sum_{h,k \leq y} \frac{a(h) \overline{a(k)}(h,k)}{hk} \left( \log \left( \frac{T(h,k)^2}{2\pi hk} \right) + 2\gamma - 1 \right). \quad (2.86)$$

Using coefficients  $a(n)$  similar to  $\mu(n)$ , Conrey [9] was able to let the length of the Dirichlet polynomial be  $4/7 - \epsilon$ . It was via this route that he could show that at least  $2/5$  of the non-trivial zeros of  $\zeta(s)$  are simple and lie on the critical line.

An asymptotic evaluation of (2.84) for the case  $k = 2$  was done in 1997 by Gaggero Jara [24] provided that the length of the Dirichlet polynomial is  $T^{4/589-\epsilon}$  (we naturally still require that  $a(n) \ll_\epsilon n^\epsilon$ ). Hughes and Young [44] have shown that the length may be increased to  $T^{1/11-\epsilon}$ . We will make use of the latter result in Chapter 5.

## 2.4.3 $q$ -analogue results

It was shown in 1931 by Paley [56] that

$$\frac{1}{\varphi(q)} \sum_{\chi} |L(1/2, \chi)|^2 \sim \frac{\varphi(q)}{q} \log q, \quad \text{as } q \rightarrow \infty, \quad (2.87)$$

where the sum runs over all characters mod  $q$ , except the principal one.

In 1981, Heath-Brown [41] showed that for those  $q$  which do not have too many prime factors, we have

$$\frac{1}{\phi^*(q)} \sum_{\chi(\bmod q)}^* |L(1/2, \chi)|^4 \sim \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} \log^4 q, \quad (2.88)$$

where the star indicates that the sum runs over all primitive characters mod  $q$  and where  $\phi^*(q)$  denotes the number of primitive characters mod  $q$ . As Heath-Brown mentions, his result is to be compared with the fact that

$$\frac{1}{T} \int_0^T |L(\frac{1}{2} + it, \chi)|^4 dt \sim \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} \log^4 T. \quad (2.89)$$

Soundararajan [62] showed in 2005 that (2.88) is valid for all  $q$ .

Conrey, Iwaniec and Soundararajan [14] have found a nice expression for the sixth power moment of Dirichlet  $L$ -functions, *but* only when one averages over all even (or odd) primitive Dirichlet characters mod  $q$  over a range of values of  $q$  and they also need to average over an interval in  $t$ . We will return to this in Chapter 6.

The result corresponding to (2.75) (see also [10]) is that for any fixed  $k$  with  $\operatorname{Re}(k) \geq 0$ , we have that

$$\frac{1}{\phi^*(q)} \sum_{\chi(\bmod q)}^* |L(1/2, \chi)|^{2k} \sim a_k \frac{G^2(k+1)}{G(2k+1)} \prod_{p|q} \left[ \sum_{m=0}^{\infty} \left( \frac{\Gamma(m+k)}{m! \Gamma(k)} \right)^2 p^{-m} \right]^{-1} (\log q)^{k^2}. \quad (2.90)$$

Finally, we mention that one can let both  $T$  and  $q$  become large. The following result due to Bui and Heath-Brown [6] is of that type. They show that for  $q, T \geq 2$ , we have

$$\begin{aligned} \sum_{\chi(\bmod q)}^* \int_0^T |L(\frac{1}{2} + it, \chi)|^4 dt &= \left( 1 + O\left( \frac{\omega(q)}{\log q} \sqrt{\frac{q}{\varphi(q)}} \right) \right) \frac{\phi^*(q)T}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} \log^4(qT) \\ &\quad + O(qT(\log(qT))^{7/2}), \end{aligned} \quad (2.91)$$

where  $\omega(n)$  denotes the number of distinct prime factors of  $n$ .

# Chapter 3

## Hall's method in the simplest case

The overall strategy that we will use throughout this thesis for finding large gaps between consecutive zeros on the critical line was introduced by Hall in his article [30] from 1999. For the reader's convenience we will now present a sketch of his method in the simplest case.

With  $T$  being large, begin by labelling the zeros of the function  $t \mapsto \zeta(\frac{1}{2} + it)$  lying in the interval  $[T, 2T]$  as  $t_1, \dots, t_N$ , ordered in non-decreasing order. Suppose that  $\kappa \in \mathbb{R}$  is such that all the gaps between consecutive zeros (in  $[T, 2T]$  of course) are at most  $\kappa \times \frac{2\pi}{\log T}$ .

Hardy's function  $Z(t)$  is defined as (see e.g. Chapter 6 in Edwards [19])

$$Z(t) := \exp(i\theta(t))\zeta(1/2 + it), \quad (3.1)$$

where

$$\theta(t) := \operatorname{Im}\left(\log\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right)\right) - \frac{t}{2}\log\pi. \quad (3.2)$$

$Z(t)$  is an even real function for real  $t$  and is clearly zero exactly when  $\zeta(1/2 + it) = 0$ . The latter observation allows us to work with  $Z(t)$  rather than  $t \mapsto \zeta(\frac{1}{2} + it)$ .

The simplest version of Wirtinger's inequality says that if  $y(t)$  is continuously differentiable and is zero at  $t = 0$  and  $t = \pi$  then<sup>1</sup>

$$\int_0^\pi y(t)^2 dt \leq \int_0^\pi y'(t)^2 dt. \quad (3.3)$$

One may make a linear substitution in (3.3), so that the role of the points 0 and  $\pi$  is overtaken by two general points, say  $a$  and  $b$ . Applying the outcoming result to  $Z(t)$

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<sup>1</sup>See Theorem 256 in [38] for a proof.

between two consecutive zeros  $t_n$  and  $t_{n+1}$  yields

$$\int_{t_n}^{t_{n+1}} Z(t)^2 dt \leq \left( \frac{t_{n+1} - t_n}{\pi} \right)^2 \int_{t_n}^{t_{n+1}} Z'(t)^2 dt \leq \left( \frac{2\kappa}{\log T} \right)^2 \int_{t_n}^{t_{n+1}} Z'(t)^2 dt. \quad (3.4)$$

We proceed by summing up the inequality (3.4) for all pairs of consecutive zeros of  $Z(t)$  in  $[T, 2T]$ , thereby getting

$$\int_{t_1}^{t_N} Z(t)^2 dt \leq \left( \frac{2\kappa}{\log T} \right)^2 \int_{t_1}^{t_N} Z'(t)^2 dt. \quad (3.5)$$

If the zeros near the endpoints were not very close to the endpoints, that would a priori give us a large gap. Thus we may assume that  $t_1$  is approximately  $T$  and that  $t_N$  is approximately  $2T$ . We basically obtain that

$$\int_T^{2T} Z(t)^2 dt \leq \left( \frac{2\kappa}{\log T} \right)^2 \int_T^{2T} Z'(t)^2 dt. \quad (3.6)$$

It is easy to show that

$$\int_T^{2T} Z(t)^2 dt \sim T \log T \quad (3.7)$$

and that

$$\int_T^{2T} Z'(t)^2 dt \sim T(\log T)^3/12. \quad (3.8)$$

Putting in these two well-known estimates into our inequality (3.6), we basically see that  $\kappa \geq \sqrt{3}$  is necessary. Thus we have established the existence of large gaps between consecutive zeros, *on the critical line*, of  $\zeta(s)$ . However, on assuming the RH, we immediately obtain that  $\lambda$ , as defined in (2.38), is at least  $\sqrt{3}$ .

# Chapter 4

## Hall's method and Dirichlet $L$ -functions

### 4.1 Statement of our goal

This chapter has two purposes. Firstly, to show that large gaps between consecutive zeros, on the critical line, of any fixed Dirichlet  $L$ -function can be shown to exist, by working as one does in the case of  $\zeta(s)$ . In fact our result holds uniformly for the modulus  $q$  in a certain range. Secondly, we become more familiar with Hall's method, here working with a fourth power version of it, rather than the simple second power version explained in Chapter 3. Explicitly, our goal here is to show the following:

**Theorem 4.1.** *Let  $\chi$  be a primitive Dirichlet character mod  $q$ . Suppose that  $\varepsilon(T) \rightarrow 0$  as  $T \rightarrow \infty$  in such a way that  $\varepsilon(T) \log^{1/4} T \rightarrow \infty$ . Then, for sufficiently large  $T$ , there exists an interval  $I$  say, contained in  $[T, T(1 + \varepsilon(T))]$ , such that  $L(\frac{1}{2} + it, \chi) \neq 0$  for  $t \in I$  and such that*

$$|I| \geq \sqrt{\frac{11}{2}} \left(1 + O\left(\frac{1}{\varepsilon(T) \log^{1/4} T}\right)\right) \left(\frac{2\pi}{\log T}\right). \quad (4.1)$$

*This is true uniformly for all  $q \ll \sqrt{\log T}$ , i.e. the implied big-oh-constant in (4.1) can then be chosen to depend on  $T$  only.*

**Remark 4.1.** Our result and proof will deal with the case  $t > 0$  only. However, since once  $q$  is chosen our choice of primitive character  $\chi$  is irrelevant and since  $L(\frac{1}{2} + it, \chi) = 0$  if and only if  $L(\frac{1}{2} - it, \bar{\chi}) = 0$  we could thus formulate an analogous result for  $t < 0$ .

**Remark 4.2.** Take any (fixed) Dirichlet  $L$ -function  $L(s, \chi)$  and let  $\{t_n\}$  denote the sequence of positive zeros of  $t \mapsto L(\frac{1}{2} + it, \chi)$ , ordered in non-decreasing order. Then

we thus have that

$$\limsup_{n \rightarrow \infty} \frac{t_{n+1} - t_n}{(2\pi/\log t_n)} \geq \sqrt{\frac{11}{2}} \approx 2.34. \quad (4.2)$$

## 4.2 The path to our goal

Firstly, we introduce a function  $Y(t, \chi)$  (see Section 4.3). It has two key properties, namely that it is real for real  $t$  and that  $Y(t, \chi) = 0$  *exactly* when  $L(\frac{1}{2} + it, \chi) = 0$ . Denote by  $\{t_n\}$  the sequence of positive zeros of  $Y(t, \chi) = 0$ , ordered in non-decreasing order.

Secondly, we let

$$\kappa = \kappa(T, \chi) := \max_{\substack{I \subseteq [T, T(1+\varepsilon(T))] \\ Y(t, \chi) \neq 0 \text{ for } t \in I}} \frac{|I|}{(2\pi/\log T)}. \quad (4.3)$$

Then

$$t_{n+1} - t_n \leq \frac{2\pi\kappa}{\log T} \quad (4.4)$$

for all pairs of zeros  $(t_n, t_{n+1})$  lying in the “short” interval  $[T, T(1 + \varepsilon(T))]$ . Say that  $t_l$  is the first zero in our sequence not less than  $T$  and that  $t_m$  is the last one not exceeding  $T(1 + \varepsilon(T))$ .

Thirdly, we use an inequality coming from the calculus of variations, namely Theorem 4.3. Taking the function to be  $Y(t, \chi)$  and the endpoints to be pairs  $(t_n, t_{n+1})$  of its distinct consecutive zeros in  $[T, T(1 + \varepsilon(T))]$ , we find that (remembering (4.4))

$$\int_{t_n}^{t_{n+1}} \left(\frac{2\kappa}{\log T}\right)^4 Y'(t, \chi)^4 + 6v \left(\frac{2\kappa}{\log T}\right)^2 Y(t, \chi)^2 Y'(t, \chi)^2 dt \geq \int_{t_n}^{t_{n+1}} 3\lambda_0(v) Y(t, \chi)^4 dt. \quad (4.5)$$

Summing up (4.5) for  $l \leq n < m$ , we obtain

$$\int_{t_l}^{t_m} \left(\frac{2\kappa}{\log T}\right)^4 Y'(t, \chi)^4 + 6v \left(\frac{2\kappa}{\log T}\right)^2 Y(t, \chi)^2 Y'(t, \chi)^2 dt \geq \int_{t_l}^{t_m} 3\lambda_0(v) Y(t, \chi)^4 dt. \quad (4.6)$$

At this point we notice that since certainly  $\kappa \geq \sqrt{\frac{11}{2}}$  would imply Theorem 4.1, the end-points  $t_l$  and  $t_m$  may be assumed to lie extremely close to  $T$  and  $T(1 + \varepsilon(T))$  respectively. Clearly one is allowed to use the crude estimates  $t_l = T + O(\sqrt{T})$  and

$t_m = T(1 + \varepsilon(T)) + O(\sqrt{T})$ . Hence

$$\begin{aligned} & \int_{T+O(\sqrt{T})}^{T(1+\varepsilon(T))+O(\sqrt{T})} \left( \frac{2\kappa}{\log T} \right)^4 Y'(t, \chi)^4 + 6v \left( \frac{2\kappa}{\log T} \right)^2 Y(t, \chi)^2 Y'(t, \chi)^2 dt \\ & \geq \int_{T+O(\sqrt{T})}^{T(1+\varepsilon(T))+O(\sqrt{T})} 3\lambda_0(v) Y(t, \chi)^4 dt. \end{aligned} \quad (4.7)$$

Employing the integral estimates in Lemmas 4.13 and 4.14, we obtain

$$D_q \left\{ \frac{\kappa^4}{70\pi^2} + \frac{v\kappa^2}{5\pi^2} - \frac{3\lambda_0(v)}{2\pi^2} \right\} \varepsilon(T) T \log^4 T + O(Tq \log^3 T) \geq 0, \quad (4.8)$$

where

$$D_q := \prod_{p|q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})}. \quad (4.9)$$

Hence

$$\kappa^4 + 14v\kappa^2 - 105\lambda_0(v) \geq -\frac{Aq}{D_q \varepsilon(T) \log T}. \quad (4.10)$$

From (4.10) we deduce

$$\kappa^2 \geq -7v + \sqrt{49v^2 + 105\lambda_0(v)} + O\left(\frac{q}{D_q \varepsilon(T) \log T}\right) \quad (4.11)$$

and we now pick  $v = 22/49$  and hence  $\lambda_0(v) = 121/196$ , since that is the value of  $v$  that maximizes the main term in the RHS of (4.11). We insert said values into the inequality (4.11), which is then seen to imply that

$$\kappa \geq \sqrt{\frac{11}{2}} + O\left(\frac{q}{D_q \varepsilon(T) \log T}\right). \quad (4.12)$$

Merten's theorem (see Theorem 429 in [39]) gives that certainly

$$\frac{1}{D_q} = \prod_{p|q} \frac{(1 + p^{-1})}{(1 - p^{-1})^3} \leq \prod_{p|q} \frac{1}{(1 - p^{-1})^4} \ll \log^4(q + 2) \ll \sqrt{q}. \quad (4.13)$$

Substituting this upper bound into the big-oh-term in (4.12) concludes the proof of Theorem 4.1, since  $q \ll \log^{1/2} T$ .

### 4.3 Introducing the function $Y(t, \chi)$

Recall from Section 2.3.2 that the well-known function

$$\xi(s, \chi) := (q/\pi)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi) \quad (4.14)$$

satisfies the functional equation

$$\xi(1-s, \bar{\chi}) = K(\chi)\xi(s, \chi), \quad (4.15)$$

where

$$K(\chi) = i^a q^{1/2} / G(1, \chi). \quad (4.16)$$

We recall that

$$G(1, \chi) = \sum_{l=1}^q \chi(l) e^{2\pi i l / q} \quad (4.17)$$

and notice that

$$G(1, \chi)G(1, \bar{\chi}) = q(-1)^a. \quad (4.18)$$

Now define

$$H(s, \chi) := \sqrt{K(\chi)}\xi(s, \chi), \quad (4.19)$$

where<sup>1</sup> we choose to define  $\log z = \log |z| + i \arg z$ , with  $-\pi < \arg z < \pi$ .

Let us first show that  $H(1/2 + it, \chi)$  is real (for  $t \in \mathbb{R}$ ). To do this, we first notice that both sides of

$$H(s, \chi) = \overline{H(\bar{s}, \bar{\chi})}. \quad (4.20)$$

represent analytic (or meromorphic in the case  $q = 1$ ) functions, so that the identity theorem tells us that we only need to establish the result for  $s = \sigma > 1$ . Then

$$\begin{aligned} H(\sigma, \chi) &= \sqrt{K(\chi)}\xi(\sigma, \chi) = \sqrt{K(\chi)} \overline{\xi(\bar{\sigma}, \bar{\chi})} \\ &= \left( \sqrt{K(\chi)} / \sqrt{K(\bar{\chi})} \right) \overline{\sqrt{K(\bar{\chi})}\xi(\bar{\sigma}, \bar{\chi})} \\ &= \left( \frac{\sqrt{K(\chi)}\sqrt{K(\bar{\chi})}}{|K(\bar{\chi})|} \right) \overline{H(\bar{\sigma}, \bar{\chi})} = \frac{1}{1} \cdot \overline{H(\bar{\sigma}, \bar{\chi})} = \overline{H(\bar{\sigma}, \bar{\chi})}. \end{aligned} \quad (4.21)$$

This property of  $H(s, \chi)$  tells us that

$$H\left(\frac{1}{2} + it, \chi\right) = \overline{H\left(\frac{1}{2} - it, \bar{\chi}\right)}. \quad (4.22)$$

On the other hand, from the functional equation (4.15) we have

$$\begin{aligned} \overline{H(1/2 - it, \bar{\chi})} &= \overline{\sqrt{K(\bar{\chi})}\xi(1/2 - it, \bar{\chi})} = \overline{\sqrt{K(\bar{\chi})}K(\chi)\xi(1/2 + it, \chi)} \\ &= \overline{\sqrt{K(\chi)}\xi(1/2 + it, \chi)} = \overline{H(1/2 + it, \chi)}. \end{aligned} \quad (4.23)$$

---

<sup>1</sup>If  $K(\chi) = -1$  then  $\chi$  cannot be real and in this case we define  $\sqrt{K(\chi)}$  to be  $i$  or  $-i$  corresponding to whether or not the imaginary part of the first non-real number in the sequence  $\chi(1), \chi(2), \dots$  is positive.

Thus  $H(1/2 + it, \chi)$  is real (for  $t \in \mathbb{R}$ ). Moreover, from the definition we have

$$H(1/2 + it, \chi) = \left[ \left( \frac{q}{\pi} \right)^{1/4+a/2} e^{\operatorname{Re}\{\log\{\Gamma(\frac{1}{4} + \frac{a}{2} + \frac{it}{2})\}\}} \right] e^{i\theta(t, \chi)} L(1/2 + it, \chi), \quad (4.24)$$

where

$$\theta(t, \chi) := \operatorname{Im}(\log(\sqrt{K(\chi)})) + \frac{t}{2} \log(q/\pi) + \operatorname{Im}(\log(\Gamma(\frac{1}{4} + \frac{a}{2} + \frac{it}{2}))). \quad (4.25)$$

Clearly the expression in the first bracket in (4.24) is just a positive real number. Thus if we define

$$Y(t, \chi) := e^{i\theta(t, \chi)} L(1/2 + it, \chi), \quad (4.26)$$

then we must have that  $Y(t, \chi) \in \mathbb{R}$  for  $t \in \mathbb{R}$ . Also, it follows immediately from (4.26) that  $Y(t, \chi) = 0$  if and only if  $L(\frac{1}{2} + it, \chi) = 0$ .

For future need, we find an estimate of the derivative of the function  $\theta(t, \chi)$  which was defined in (4.25). The digamma function is defined by  $\psi(s) := \Gamma'(s)/\Gamma(s)$ . Now

$$\begin{aligned} \theta'(t, \chi) &= \frac{1}{2} \log(q/\pi) + \operatorname{Im}\left(\frac{i}{2} \left(\psi\left(\frac{1}{4} + \frac{a}{2} + \frac{it}{2}\right)\right)\right) \\ &= \frac{1}{2} \log(q/\pi) + \frac{1}{2} \operatorname{Re}\left(\psi\left(\frac{1}{4} + \frac{a}{2} + \frac{it}{2}\right)\right) = \frac{1}{2} \log(tq/2\pi) + O\left(\frac{1}{t}\right), \end{aligned} \quad (4.27)$$

where we in the last step used an estimate for the digamma function. Basically we used Stirling's formula together with Cauchy's integral formula, for more details see [26] (or Chapter 6 in [19]). Thus

$$\theta'(t, \chi) = \frac{1}{2} \log\left(\frac{tq}{2\pi}\right) + O\left(\frac{1}{t}\right). \quad (4.28)$$

## 4.4 Some integral results

### 4.4.1 An approximate functional equation for $L(s, \chi)$

The functional equation given in (2.59) for  $L(s, \chi)$  may be written as

$$L(s, \chi) = B(s, \chi) L(1 - s, \bar{\chi}), \quad (4.29)$$

where

$$B(s, \chi) := iG(1, \chi) q^{-s} (2\pi)^{s-1} \Gamma(1 - s) [\chi(-1) e^{-\pi is/2} - e^{\pi is/2}]. \quad (4.30)$$

In the following lemma we obtain an approximate functional equation for  $L(s, \chi)$ .

**Lemma 4.1.** *Let  $\chi$  be a primitive Dirichlet character modulo  $q$ . Then for  $\sigma \in [0, 1]$ ,  $x, y \geq 1$ ,  $t \geq A > 0$ ,  $q \ll t$  and  $2\pi xy = qt$  we have uniformly that*

$$\begin{aligned} L(s, \chi) &= \sum_{n \leq x} \chi(n) n^{-s} + B(s, \chi) \sum_{n \leq y} \overline{\chi(n)} n^{s-1} \\ &\quad + O\left((q^{1/2} x^{-\sigma} + q^{1-\sigma} t^{1/2-\sigma} y^{\sigma-1}) \log(q+2)\right), \end{aligned} \quad (4.31)$$

where as usual  $s = \sigma + it$  and  $B(s, \chi)$  of course is as defined in (4.30).

*Proof.*<sup>2</sup> We will first assume that  $x \leq y$ . Then we may without loss of generality assume that  $y \equiv \frac{1}{2} \pmod{1}$  since by changing  $y$ , and hence  $x$ , by no more than  $1/2$  will not change the size of the error term, and any individual term in the sums is clearly absorbable into the error term. Let  $M = [x]$  and  $N = [y]$ .

For  $\sigma > 0$  we have

$$\int_0^\infty x^{s-1} e^{-nx} dx = n^{-s} \int_0^\infty y^{s-1} e^{-y} dy = n^{-s} \Gamma(s). \quad (4.32)$$

Hence for  $\sigma > 1$  we get

$$\begin{aligned} L(s, \chi) &= \sum_{n \leq M} \chi(n) n^{-s} + \sum_{l=1}^q \chi(l) \sum_{\substack{n > M \\ n \equiv l \pmod{q}}} n^{-s} \\ &= \sum_{n \leq M} \chi(n) n^{-s} + \frac{1}{\Gamma(s)} \sum_{l=1}^q \chi(l) \sum_{\substack{n > M \\ n \equiv l \pmod{q}}} \int_0^\infty x^{s-1} e^{-nx} dx \\ &= \sum_{n \leq M} \chi(n) n^{-s} + \frac{1}{\Gamma(s)} \sum_{l=1}^q \chi(l) \int_0^\infty x^{s-1} \sum_{\substack{n > M \\ n \equiv l \pmod{q}}} e^{-nx} dx \\ &= \sum_{n \leq x} \chi(n) n^{-s} + \sum_{l=1}^q \frac{\chi(l)}{\Gamma(s)} \int_0^\infty x^{s-1} \frac{e^{-([\frac{M-l}{q}]q+l)x}}{e^{qx} - 1} dx, \end{aligned} \quad (4.33)$$

where the inversion of order of summation and integration is justified by absolute convergence. Now let  $P = [\frac{M-l}{q}]q + l$  and  $C$  be the contour that starts at infinity on the positive real axis, encircles the origin once in the positive (anti-clockwise) direction (suitably small radius) and then returns to infinity. Then

$$\begin{aligned} L(s, \chi) &= \sum_{n \leq x} \chi(n) n^{-s} + \sum_{l=1}^q \frac{\chi(l)}{\Gamma(s)} \frac{1}{(e^{2\pi is} - 1)} \int_C w^{s-1} \frac{e^{-Pw}}{e^{qw} - 1} dw \\ &= \sum_{n \leq x} \chi(n) n^{-s} + \sum_{l=1}^q \frac{\chi(l) e^{-\pi is} \Gamma(1-s)}{2\pi i} \int_C w^{s-1} \frac{e^{-Pw}}{e^{qw} - 1} dw. \end{aligned} \quad (4.34)$$

---

<sup>2</sup>The general structure of this proof comes from the proof of Theorem 4.1 in [47].

Now let  $\eta = 2\pi y/q$  so that  $x = t/\eta$ . We “replace”  $C$  by  $C_1, C_2, C_3, C_4$  which are the lines joining  $\infty, c\eta + i\eta(1+c), -c\eta + i\eta(1-c), -c\eta - \frac{(2N+1)\pi i}{q}, \infty$ , where  $c$  is a small absolute positive constant. Applying Cauchy’s Residue Theorem, we obtain

$$\begin{aligned} L(s, \chi) &= \sum_{n \leq x} \chi(n)n^{-s} + \sum_{l=1}^q \left[ \frac{\chi(l)e^{-\pi is}\Gamma(1-s)}{2\pi i} \sum_{j=1}^4 \int_{C_j} w^{s-1} \frac{e^{-Pw}}{e^{qw} - 1} dw \right] \\ &- 2\pi i \sum_{l=1}^q \frac{\chi(l)}{2\pi i} \sum_{\text{Residues}} \frac{e^{-\pi is}\Gamma(1-s)w^{s-1}e^{-Pw}}{e^{qw} - 1}, \end{aligned} \quad (4.35)$$

where the residues are at  $\pm \frac{2\pi in}{q}$ , for  $n = 1, \dots, N$ . The contribution to the Residue-sum in (4.35) coming from the pair of residues at  $\pm \frac{2\pi in}{q}$  is

$$\begin{aligned} &\left( \frac{e^{-\pi is}}{q} \right) \Gamma(1-s) \left( \pm \frac{2\pi in}{q} \right)^{s-1} e^{\mp 2\pi in/q} \\ &= iq^{-s}\Gamma(1-s)(2\pi n)^{s-1} [-e^{-\pi is/2}e^{-2\pi in/q} + e^{\pi is/2}e^{2\pi in/q}]. \end{aligned} \quad (4.36)$$

Thus recalling the  $-\chi(l)$  and summing over  $l$  and  $n$  yields that the third part of (4.35) equals

$$\begin{aligned} &\left( iq^{-s}\Gamma(1-s)(2\pi)^{s-1} \right) \\ &\times \sum_{n \leq N} \left[ n^{s-1} \left\{ \chi(-1)e^{-\pi is/2} \sum_{l=1}^q \chi(-l)e^{-2\pi iln/q} - e^{\pi is/2} \sum_{l=1}^q \chi(l)e^{2\pi iln/q} \right\} \right] \\ &= B(s, \chi) \sum_{n \leq N} \overline{\chi(n)} n^{s-1}. \end{aligned} \quad (4.37)$$

Hence we have

$$\begin{aligned} L(s, \chi) &= \sum_{n \leq x} \chi(n)n^{-s} + B(s, \chi) \sum_{n \leq y} \overline{\chi(n)} n^{s-1} \\ &+ \sum_{l=1}^q \left[ \frac{\chi(l)e^{-\pi is}\Gamma(1-s)}{2\pi i} \sum_{j=1}^4 \int_{C_j} w^{s-1} \frac{e^{-Pw}}{e^{qw} - 1} dw \right]. \end{aligned} \quad (4.38)$$

Thus the proof will be completed in the case when  $x \leq y$  if we show that the last expression in the RHS of (4.38) is  $\ll q^{1/2}x^{-\sigma} \log(q+2)$ . We will take one  $C_j$ -path at a time. We will write  $w = u + iv = \rho e^{i\varphi}$ ,  $0 < \varphi < 2\pi$ . Then  $|w^{s-1}| = \rho^{\sigma-1}e^{-t\varphi}$ .

On  $C_4$  we have  $\varphi \geq \frac{5\pi}{4}$ ,  $\rho > A\eta$  and  $|e^{qw} - 1| > A$ . Hence

$$\int_{C_4} \ll \eta^{\sigma-1} e^{-5\pi t/4} \int_{-c\eta}^{\infty} e^{-Pu} du \ll 1 \cdot e^{-5\pi t/4} \cdot e^{Pc\eta} \ll e^{Mc\eta - 5\pi t/4} \ll e^{tc - 5\pi t/4}. \quad (4.39)$$

On  $C_3$  we have

$$\begin{aligned}\varphi &\geq \pi/2 + \arctan\left(\frac{c}{1-c}\right) = \pi/2 + \int_0^{\frac{c}{1-c}} \frac{1}{1+x^2} dx \\ &> \pi/2 + A + \int_0^{\frac{c}{1-c}} \frac{1}{(1+x)^2} dx = \pi/2 + A + c.\end{aligned}\quad (4.40)$$

Hence on  $C_3$  we get

$$w^{s-1}e^{-Pw} \ll \eta^{\sigma-1}e^{(-t(\pi/2+A+c)+Pc\eta)} \ll \eta^{\sigma-1}e^{(-\pi t/2-At)}.\quad (4.41)$$

Since  $\operatorname{Re}(qw) = -qc\eta = -2\pi cy \leq -A$ , we find  $|e^{qw} - 1| > A$ . Whence

$$\int_{C_3} \ll \eta^\sigma e^{-\pi t/2-At}.\quad (4.42)$$

On  $C_1$  we have  $|e^{qw} - 1| > Ae^{qu}$ . Thus for the  $C_1$ -part where  $c\eta \leq u \leq \pi\eta$  we have

$$\frac{w^{s-1}e^{-Pw}}{e^{qw} - 1} \ll \eta^{\sigma-1}e^{-t\arctan(\frac{\eta+c\eta}{u})-Pu-qu}.\quad (4.43)$$

Now we first observe that

$$q + P = q + \left[\frac{M-l}{q}\right]q + l \geq q + l + \left(\frac{M-l-(q-1)}{q}\right)q = M+1 \geq x = t/\eta.\quad (4.44)$$

Therefore

$$\frac{w^{s-1}e^{-Pw}}{e^{qw} - 1} \ll \eta^{\sigma-1}e^{-t\{\arctan[\frac{(1+c)\eta}{u}] + \frac{u}{\eta}\}}.\quad (4.45)$$

Next we notice that

$$\frac{d}{dx}\left(\arctan\left(\frac{(1+c)\eta}{u}\right) + \frac{u}{\eta}\right) = -\frac{(1+c)\eta}{u^2 + (1+c)^2\eta^2} + \frac{1}{\eta} > 0,\quad (4.46)$$

so that in the range  $c\eta \leq u \leq \pi\eta$  we have that

$$\begin{aligned}\arctan\left(\frac{(1+c)\eta}{u}\right) + \frac{u}{\eta} &\geq \arctan\left(\frac{(1+c)\eta}{c\eta}\right) + \frac{c\eta}{\eta} \\ &= \arctan\left(\frac{1+c}{c}\right) + c = \pi/2 + c - \arctan\left(\frac{c}{1+c}\right) = \pi/2 + A.\end{aligned}\quad (4.47)$$

Hence here

$$\frac{w^{s-1}e^{-Pw}}{e^{qw} - 1} \ll \eta^{\sigma-1}e^{-t(\pi/2+A)}.\quad (4.48)$$

On the part  $u \geq \pi\eta$  we have  $\frac{w^{s-1}e^{-Pw}}{e^{qw}-1} \ll \eta^{\sigma-1}e^{-xu}$ . Thus

$$\begin{aligned} \int_{C_3} &\ll \eta^{\sigma-1} \int_0^{\pi\eta} e^{-t(\pi/2+A)} du + \eta^{\sigma-1} \int_{\pi\eta}^{\infty} e^{-xu} du \\ &\ll \eta^{\sigma} e^{-t(\pi/2+A)} + \eta^{\sigma-1} e^{-\pi\eta x} \ll \eta^{\sigma} e^{-(\pi/2+A)t}, \end{aligned} \quad (4.49)$$

upon recalling  $\eta x = t$ .

So far we have proved  $\int_{C_j} \ll \eta^{\sigma} e^{-\pi t/2 - At}$  for  $j = 1, 3, 4$ . Noting that

$$e^{-\pi is} \Gamma(1-s) \ll t^{1/2-\sigma} e^{\pi t/2} \quad (4.50)$$

and remembering that we had to sum over  $l$ , we see that the total contribution to (4.38) from when  $j$  goes over 1, 3, 4 is

$$\ll q\eta^{\sigma} e^{-\pi t/2 - At + \pi t/2} t^{1/2-\sigma} = q\eta^{\sigma} t^{1/2-\sigma} e^{-At}. \quad (4.51)$$

This is  $\ll q^{1/2} x^{-\sigma} \log(q+2)$ , since we had a negative exponential in (4.51).

Thus (recall (4.38)) certainly we will have completed the proof of this lemma in the case  $x \leq y$  if we establish that

$$\sum_{l=1}^q \left[ \frac{\chi(l) e^{-\pi is} \Gamma(1-s)}{2\pi i} \int_{C_2} w^{s-1} \frac{e^{-Pw}}{e^{qw}-1} dw \right] \ll q^{1/2} x^{-\sigma} \log(q+2). \quad (4.52)$$

Now on  $C_2$  we have  $w = i\eta + \lambda e^{\pi i/4}$  for  $-\sqrt{2}c\eta \leq \lambda \leq \sqrt{2}c\eta$ . We find

$$\begin{aligned} w^{s-1} &= e^{(s-1)(\frac{\pi i}{2} + \log(\eta + \lambda e^{-\pi i/4}))} \\ &= e^{(s-1)(\frac{\pi i}{2} + \log(\eta) + \frac{\lambda}{\eta} e^{-\pi i/4} - \frac{\lambda^2}{2\eta^2} e^{-\pi i/2} + O(\frac{\lambda^3}{\eta^3}))} \\ &\ll \eta^{\sigma-1} e^{\{-\frac{\pi}{2} + \frac{\lambda}{\sqrt{2}\eta} - \frac{\lambda^2}{2\eta^2} + O(\frac{\lambda^3}{\eta^3})\}t}. \end{aligned} \quad (4.53)$$

We will now prove that the bound in (4.52) holds for the part of the  $C_2$ -contour where  $\operatorname{Re}(w) = u \geq 0$ , the  $u \leq 0$ -case being similar. We will divide  $C_2$  into two parts, namely  $C_{2,I}$  where  $0 \leq u \leq \frac{1}{\sqrt{q}}$  and  $C_{2,II}$  where  $\frac{1}{\sqrt{q}} \leq u \leq c\eta$ . Note that if  $c\eta \leq \frac{1}{\sqrt{q}}$  then the  $II$ -part-contribution is empty, i.e. zero.

Let us first consider the  $C_{2,I}$ -part. Using  $|e^{-xw}| = e^{-\lambda t/(\sqrt{2}\eta)}$  and (4.53) we have

$$w^{s-1} e^{-xw} \ll \eta^{\sigma-1} e^{-(\pi/2 + A\frac{\lambda^2}{\eta^2})t} \ll \eta^{\sigma-1} e^{-\pi t/2}. \quad (4.54)$$

Next we note that since  $y \equiv \frac{1}{2} \pmod{1}$ , we have  $\arg(e^{qw}) \equiv \pi \pmod{2\pi}$ , for  $\lambda = 0$ . Using this we find that

$$\left| \frac{e^{-Pw+xw}}{e^{qw}-1} \right| = \frac{e^{(x-P-q)u}}{|1 - e^{-qw}|} \ll e^{(x-P-q)u}. \quad (4.55)$$

Thus

$$\sum_{l=1}^q \frac{\chi(l)e^{-\pi is}\Gamma(1-s)}{2\pi i} \int_{C_{2,I}} \ll t^{1/2-\sigma} e^{\pi t/2} \eta^{\sigma-1} e^{-\pi t/2} \sum_{l=1}^q \int_0^{1/\sqrt{q}} e^{(x-P-q)u} du. \quad (4.56)$$

If we establish

$$\sum_{l=1}^q \int_0^{1/\sqrt{q}} e^{(x-P-q)u} du \ll \log(q+2), \quad (4.57)$$

then it follows that (4.56) is  $\ll q^{1/2}x^{-\sigma} \log(q+2)$ . And indeed

$$\begin{aligned} \sum_{l=1}^q \int_0^{1/\sqrt{q}} e^{(x-P-q)u} du &\ll \sum_{l=1}^q \int_0^{1/\sqrt{q}} e^{([x]-P-q)u} du = \sum_{l=1}^q \int_0^{1/\sqrt{q}} e^{-lu} du \\ &= \sum_{l=1}^q \frac{1 - e^{-l/\sqrt{q}}}{l} \ll \sum_{l=1}^q \frac{1}{l} \ll \log(q+2). \end{aligned} \quad (4.58)$$

Let us now consider the  $C_{2,II}$ -part. Arguing as before we get

$$\begin{aligned} \int_{C_{2,II}} &\ll \eta^{\sigma-1} e^{-\pi t/2} e^{(x-P-q)/\sqrt{q}} \int_{-c\eta\sqrt{2}}^{c\eta\sqrt{2}} e^{-\frac{A\lambda^2 t}{2\eta^2}} d\lambda \\ &\ll \eta^{\sigma-1} e^{-\pi t/2} e^{([x]-P-q)/\sqrt{q}} \int_{-\infty}^{\infty} e^{-\frac{A\lambda^2 t}{\eta^2}} d\lambda \\ &\ll \eta^{\sigma-1} e^{-\pi t/2} e^{([x]-P-q)/\sqrt{q}} \eta t^{-1/2} \\ &= \eta^\sigma e^{-\pi t/2} t^{-1/2} e^{([x]-P-q)/\sqrt{q}}. \end{aligned} \quad (4.59)$$

Thus we find

$$\begin{aligned} \sum_{l=1}^q \frac{\chi(l)e^{-\pi is}\Gamma(1-s)}{2\pi i} \int_{C_{2,II}} &\ll t^{1/2-\sigma} e^{\pi t/2} \eta^\sigma e^{-\pi t/2} t^{-1/2} \sum_{l=1}^q e^{([x]-P-q)/\sqrt{q}} \\ &\ll q^{1/2} x^{-\sigma} \end{aligned} \quad (4.60)$$

(which is good of course), provided that  $\sum_{l=1}^q e^{([x]-P-q)/\sqrt{q}} \ll \sqrt{q}$ . Indeed the latter sum is equal to

$$\sum_{l=1}^q e^{-l/\sqrt{q}} = e^{-1/\sqrt{q}} \left( \frac{1 - e^{-\sqrt{q}}}{1 - e^{-1/\sqrt{q}}} \right) = \frac{1 - e^{-\sqrt{q}}}{e^{1/\sqrt{q}} - 1} \leq \frac{1}{e^{1/\sqrt{q}} - 1} \leq \frac{1}{(1/\sqrt{q})} = \sqrt{q}. \quad (4.61)$$

This completes the  $x \leq y$ -case [because we are obviously allowed to add the positive term  $q^{1-\sigma}t^{1/2-\sigma}y^{\sigma-1} \log(q+2)$  to the error term in (4.31)].

Now suppose  $y \leq x$ . Using this lemma in the known case we find (change  $s$  into  $(1-s)$  and  $\chi$  into  $\bar{\chi}$ )

$$\begin{aligned} L(1-s, \bar{\chi}) &= \sum_{n \leq y} \overline{\chi(n)} n^{s-1} + B(1-s, \bar{\chi}) \sum_{n \leq x} \chi(n) n^{-s} \\ &\quad + O((q^{1/2}y^{\sigma-1} + q^\sigma t^{\sigma-1/2} x^{-\sigma}) \log(q+2)). \end{aligned} \quad (4.62)$$

Putting this together with (4.29) we obtain

$$\begin{aligned} L(s, \chi) &= B(s, \chi) L(1-s, \bar{\chi}) \\ &= B(s, \chi) \sum_{n \leq y} \overline{\chi(n)} n^{s-1} + B(s, \chi) B(1-s, \bar{\chi}) \sum_{n \leq x} \chi(n) n^{-s} \\ &\quad + O(\{(qt)^{1/2-\sigma}\} \{q^{1/2}y^{\sigma-1} + q^\sigma t^{\sigma-1/2} x^{-\sigma}\} \log(q+2)) \\ &= \sum_{n \leq x} \chi(n) n^{-s} + B(s, \chi) \sum_{n \leq y} \overline{\chi(n)} n^{s-1} \\ &\quad + O((q^{1/2}x^{-\sigma} + q^{1-\sigma}t^{1/2-\sigma}y^{\sigma-1}) \log(q+2)), \end{aligned} \quad (4.63)$$

which is precisely what we wanted.  $\square$

#### 4.4.2 Some more approximate functional equations

Having obtained an approximate functional equation for  $L(s, \chi)$  in Section 4.4.1, one can easily obtain one for  $L'(s, \chi)$  via use of Cauchy's integral trick (see Lemma 4.3). Furthermore, we will then deduce approximate functional equations at  $s = 1/2 + it$  for  $L(s, \chi)^2$ ,  $L(s, \chi)L'(s, \chi)$  and  $L'(s, \chi)^2$ .

**Lemma 4.2.** *For  $s = 1/2 + it$ ,  $\chi$  primitive<sup>3</sup>,  $t \geq A$  we have*

$$B'(s, \chi) = B(s, \chi) \left[ -\log(tq/2\pi) + O\left(\frac{1}{t}\right) \right]. \quad (4.64)$$

*Proof.* Let us begin by noticing the identity  $B(\frac{1}{2} + it, \chi) = e^{-2i\theta(t, \chi)}$ . Now  $\log(B(s, \chi))$  is analytic in a region that covers the points we are interested in. Thus we must have

$$\begin{aligned} B'(s, \chi)/B(s, \chi) &= \frac{d}{ds}(\log(B(s, \chi))) = (1/i) \frac{d}{dt} \left[ \log(B(\tfrac{1}{2} + it, \chi)) \right] \\ &= (1/i) \frac{d}{dt}(-2i\theta(t, \chi)) = -2\theta'(t, \chi) = -\log(tq/2\pi) + O\left(\frac{1}{t}\right), \end{aligned} \quad (4.65)$$

where the last step follows from (4.28).  $\square$

<sup>3</sup>Of course we mean that  $\chi$  is a primitive Dirichlet character modulo  $q$ .

The next lemma is “Cauchy’s integral trick”. It is a useful tool in analytic number theory for bounding the derivatives of an analytic function for which you already have a bound, and we will make use of it many times in this thesis.

**Lemma 4.3** (Cauchy’s integral trick). *Let  $f(z)$  be analytic on a region which includes the (small) circle given by  $|z - s| \leq r$  say. Suppose that  $|f(z)| \leq C$  uniformly for  $|z - s| \leq r$ . Then  $|f^{(n)}(s)| \leq \frac{C \cdot n!}{r^n}$ .*

*Proof.* Cauchy’s integral formula for the  $n$ -th derivative of  $f(z)$  at the point  $s$  gives us

$$f^{(n)}(s) = \frac{n!}{2\pi i} \int_{|w-s|=r} \frac{f(w)}{(w-s)^{n+1}} dw. \quad (4.66)$$

Therefore we trivially have

$$|f^{(n)}(s)| \leq \frac{n!}{2\pi} \int_{|w-s|=r} \frac{|f(w)|}{|w-s|^{n+1}} |dw| \leq \frac{n!}{2\pi} \cdot 2\pi r \cdot \frac{C}{r^{n+1}} = \frac{C \cdot n!}{r^n}. \quad (4.67)$$

□

**Lemma 4.4.** *For  $s = 1/2 + it$ ,  $\chi$  primitive,  $x, y \geq 1$ ,  $t \geq A$ ,  $q \ll t$  and  $2\pi xy = qt$  we have*

$$\begin{aligned} L'(s, \chi) &= - \sum_{n \leq x} \chi(n) (\log n) n^{-s} + B(s, \chi) \sum_{n \leq y} \log \left( \frac{2\pi n}{qt} \right) \overline{\chi(n)} n^{s-1} \\ &\quad + O\left( (q^{1/2} x^{-1/2} + q^{1/2} y^{-1/2}) \log(q+2) \log t \right). \end{aligned} \quad (4.68)$$

*Proof.* We use Cauchy’s integral formula (cf. Lemma 4.3) with a circle of radius  $1/\log t$  around the point  $s = 1/2 + it$ , to find an expression for the derivative of

$$L(s, \chi) - \sum_{n \leq x} \chi(n) n^{-s} + B(s, \chi) \sum_{n \leq y} \overline{\chi(n)} n^{s-1}. \quad (4.69)$$

This gives us the main term in (4.68) for the derivative of  $L(s, \chi)$  together with an error term coming from the error term in (4.64). But this (former) error term does not give us any problems since it is

$$\ll (qt)^{1/2-\sigma} y^\sigma t^{-1} = y^{1/2} t^{-1} \ll q^{1/2} x^{-1/2} \log(q+2) \log t. \quad (4.70)$$

Concerning the error arising in Cauchy’s integral formula, we simply remark that the obvious approach (carefully using error in (4.31)) does work. □

In the next lemma we basically summarise our results so far in this subsection.

**Lemma 4.5.** *Suppose  $P(x)$  is either the polynomial 1 or the polynomial  $-x$ . Then for  $\chi$  primitive,  $x, y \geq 1$ ,  $t \geq A$ ,  $q \ll t$  and  $2\pi xy = qt$  we have*

$$\begin{aligned} P\left(-\frac{d}{ds}\right)(L(s, \chi))\Big|_{s=\frac{1}{2}+it} &= \sum_{n \leq x} \frac{P(\log n)\chi(n)}{n^{1/2+it}} \\ &\quad + B(s, \chi) \sum_{n \leq y} \frac{P(l_2 - \log n)\overline{\chi(n)}}{n^{1/2-it}} + E(x, P, t, q), \end{aligned} \quad (4.71)$$

where

$$E(x, P, t, q) \ll q^{1/2}(x^{-1/2} + y^{-1/2}) \log(q+2)(\log t)^{\deg(P)} \quad (4.72)$$

and

$$l_2 := \log(tq/2\pi). \quad (4.73)$$

*Proof.* We can read off this from Lemmas 4.1 and 4.4.  $\square$

At this point we introduce some new notation. Below the index  $j = 1, 2$ . Let  $P_j(x)$  denote either 1 or  $-x$ . Then define

$$A_j(s, \chi) := P_j\left(-\frac{d}{ds}\right)(L(s, \chi)). \quad (4.74)$$

Also, define

$$a_j(n, t, \chi) := a_j(n, \chi) = \chi(n)P_j(\log n) \quad (4.75)$$

and

$$b_j(n, t, \chi) := \overline{\chi(n)}P_j(l_2 - \log n). \quad (4.76)$$

Finally, let  $U := tq/2\pi$  and  $u := \sqrt{U}$ . When showing the next lemma, we use ideas from Conrey's article [8].

**Lemma 4.6.** *With the same assumptions as in Lemma 4.5 we have*

$$\begin{aligned} (A_1 A_2)(1/2 + it) &= \sum_{n \leq tq/2\pi} \frac{a_1 * a_2(n, t, \chi)}{n^s} + B^2(s, \chi) \sum_{n \leq tq/2\pi} \frac{b_1 * b_2(n, t, \chi)}{n^{1-s}} \\ &\quad + O\left(q^{1/2} \log(q+2)(\log t)^{1+\deg(P_1)+\deg(P_2)}\right). \end{aligned} \quad (4.77)$$

**Remark 4.3.** By  $a_1 * a_2(n, t, \chi)$  we naturally mean  $\sum_{d|n} a_1(d, t, \chi)a_2(n/d, t, \chi)$ .

*Proof.* Throughout this proof we will write  $a_j(n)$  for  $a_j(n, t, \chi)$ ,  $b_j(n)$  for  $b_j(n, t, \chi)$ ,  $A_j(s)$  for  $A_j(s, \chi)$  and  $E(x, P)$  for  $E(x, P, t, \chi)$ .

Then for  $s = 1/2 + it$  we have

$$\begin{aligned}
& \sum_{mn \leq U} \frac{a_1(m)a_2(n)}{m^s n^s} \tag{4.78} \\
&= \sum_{m \leq u} \frac{a_1(m)}{m^s} \left[ \sum_{n \leq U/m} \frac{a_2(n)}{n^s} \right] + \sum_{n \leq u} \frac{a_2(n)}{n^s} \left[ \sum_{m \leq U/n} \frac{a_1(m)}{m^s} \right] - \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m)a_2(n)}{m^s n^s} \\
&= \sum_{m \leq u} \frac{a_1(m)}{m^s} \left[ A_2(s) - B(s, \chi) \sum_{n \leq m} \frac{b_2(n)}{n^{1-s}} - E(U/m, P_2) \right] \\
&\quad + \sum_{n \leq u} \frac{a_2(n)}{n^s} \left[ A_1(s) - B(s, \chi) \sum_{m \leq n} \frac{b_1(m)}{m^{1-s}} - E(U/n, P_1) \right] - \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m)a_2(n)}{m^s n^s},
\end{aligned}$$

where we used (4.71). Similarly we find that

$$\begin{aligned}
& B^2(s, \chi) \sum_{mn \leq U} \frac{b_1(m)b_2(n)}{m^{1-s}n^{1-s}} = B(s, \chi) \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} \left[ A_2(s) - \sum_{n \leq m} \frac{a_2(n)}{n^s} - E(m, P_2) \right] \tag{4.79} \\
&\quad + B(s, \chi) \sum_{n \leq u} \frac{b_2(n)}{n^{1-s}} \left[ A_1(s) - \sum_{m \leq n} \frac{a_1(m)}{m^s} - E(n, P_1) \right] - B^2(s, \chi) \sum_{\substack{m \leq u \\ n \leq u}} \frac{b_1(m)b_2(n)}{m^{1-s}n^{1-s}}.
\end{aligned}$$

We add these two results. Then the main terms of the right-hand sides add up to

$$\begin{aligned}
& A_1(s) \left[ \sum_{n \leq u} \frac{a_2(n)}{n^s} + B(s, \chi) \sum_{n \leq u} \frac{b_2(n)}{n^{1-s}} \right] + A_2(s) \left[ \sum_{m \leq u} \frac{a_1(m)}{m^s} + B(s, \chi) \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} \right] \\
&\quad - B(s, \chi) \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m)b_2(n)}{m^s n^{1-s}} + B(s, \chi) \sum_{m \leq u} \frac{a_1(m)b_2(m)}{m} - B(s, \chi) \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_2(n)b_1(m)}{n^s m^{1-s}} \\
&\quad + B(s, \chi) \sum_{m \leq u} \frac{a_2(m)b_1(m)}{m} - \sum_{\substack{m \leq u \\ n \leq u}} \frac{a_1(m)a_2(n)}{m^s n^s} - B^2(s, \chi) \sum_{\substack{m \leq u \\ n \leq u}} \frac{b_1(m)b_2(n)}{m^{1-s}n^{1-s}} \\
&= A_1(s) [A_2(s) - E(u, P_2)] + A_2(s) [A_1(s) - E(u, P_1)] \\
&\quad - \left[ \sum_{m \leq u} \frac{a_1(m)}{m^s} + B(s, \chi) \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} \right] \cdot \left[ \sum_{m \leq u} \frac{a_1(m)}{m^s} + B(s, \chi) \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} \right] \\
&\quad + 2 \cdot O\left(1 \cdot (\log(qt))^{1+\deg(P_1+\deg(P_2))}\right) \\
&= A_1(s) [A_2(s) - E(u, P_2)] + A_2(s) [A_1(s) - E(u, P_1)] \\
&\quad - [A_1(s) - E(u, P_1)] \cdot [A_2(s) - E(u, P_2)] + O\left(1 \cdot (\log(qt))^{1+\deg(P_1)+\deg(P_2)}\right) \\
&= A_1(s)A_2(s) + E(u, P_1)E(u, P_2) + O\left(1 \cdot (\log(qt))^{1+\deg(P_1)+\deg(P_2)}\right) \\
&= A_1A_2(s) + O\left(1 \cdot (q^{1/2} \log(q+2)(\log t))^{1+\deg(P_1)+\deg(P_2)}\right). \tag{4.80}
\end{aligned}$$

The remaining terms of the right-hand sides equal

$$\begin{aligned}
& - \sum_{m \leq u} \frac{a_1(m)}{m^s} E(U/m, P_2) - \sum_{n \leq u} \frac{a_2(n)}{n^s} E(U/n, P_1) \\
& - B(s, \chi) \sum_{m \leq u} \frac{b_1(m)}{m^{1-s}} E(m, P_2) - B(s, \chi) \sum_{m \leq u} \frac{b_2(n)}{n^{1-s}} E(n, P_1). \tag{4.81}
\end{aligned}$$

Any of these four terms is absorbable into the error term, e.g. the first one is

$$\begin{aligned}
& \ll \sum_{m \leq u} \frac{(\log(qt))^{\deg(P_1)}}{\sqrt{m}} \log(q+2) q^{1/2} (\log t)^{\deg(P_2)} \left( \frac{1}{\sqrt{m}} + \frac{\sqrt{m}}{u} \right) \\
& \ll q^{1/2} \log(q+2) (\log t)^{\deg(P_1) + \deg(P_2)} \sum_{m \leq u} \frac{1}{m} \\
& \ll q^{1/2} \log(q+2) (\log t)^{1 + \deg(P_1) + \deg(P_2)}. \tag{4.82}
\end{aligned}$$

□

**Corollary 4.1.** *For  $\chi$  primitive,  $t \geq A$  and  $q \ll t$  we have*

$$\begin{aligned}
(i) \quad & L^2(1/2 + it, \chi) = \sum_{n \leq tq/2\pi} \frac{\chi(n)d(n)}{n^{1/2+it}} + B^2(1/2 + it, \chi) \sum_{n \leq tq/2\pi} \frac{\overline{\chi(n)}d(n)}{n^{1/2-it}} \\
& + O(q^{1/2} \log(q+2) \log t), \\
(ii) \quad & 2L(1/2 + it, \chi)L'(1/2 + it, \chi) = - \sum_{n \leq tq/2\pi} \frac{\chi(n)d(n) \log n}{n^{1/2+it}} \\
& + B^2(1/2 + it, \chi) \sum_{n \leq tq/2\pi} \frac{\overline{\chi(n)}d(n) \log\left(\frac{4\pi^2 n}{t^2 q^2}\right)}{n^{1/2-it}} + O(q^{1/2} \log(q+2) \log^2 t), \\
(iii) \quad & L'(1/2 + it, \chi)^2 = \sum_{n \leq tq/2\pi} \frac{\chi(n)D(n)}{n^{1/2+it}} \\
& + B^2(1/2 + it, \chi) \sum_{n \leq tq/2\pi} \frac{\overline{\chi(n)}}{n^{1/2-it}} \left\{ D(n) - d(n)l_2 \log n + d(n)l_2^2 \right\} \\
& + O(q^{1/2} \log(q+2) \log^3 t),
\end{aligned}$$

where

$$D(n) = \sum_{d|n} \log d \log\left(\frac{n}{d}\right) \tag{4.83}$$

and as before  $l_2 = \log(tq/2\pi)$ .

*Proof.* The result follows upon choosing the pair  $(P_1, P_2)$  to be  $(1, 1)$ ,  $(1, -x)$  and  $(-x, -x)$  respectively in Lemma 4.6. □

### 4.4.3 Four Lemmas

**Lemma 4.7.** As  $x \rightarrow \infty$ , 
$$\sum_{\substack{m, n \leq x \\ m \neq n}} \frac{d(m)d(n)}{\sqrt{mn} |\log(m/n)|} = O(x \log^3 x).$$

*Proof.* See Lemma B3 in Ingham's article [46]. □

**Lemma 4.8.** Let  $\nu$  be a non-negative integer,  $\lambda \geq 1$  and  $T \geq T_1 \geq \frac{2\pi\lambda}{q}$ . Then

$$\left| \int_{T_1}^T \left( \frac{qt}{2\pi\lambda e} \right)^{2it} \log^\nu \left( \frac{tq}{2\pi} \right) dt \right| \leq \begin{cases} \frac{4 \log^\nu(Tq/2\pi)}{\log(T_1q/2\pi\lambda)} & \text{if } T_1 > \frac{2\pi\lambda}{q}, \\ 8\sqrt{2}\sqrt{T} \log^\nu(Tq/2\pi) & \text{always.} \end{cases} \quad (4.84)$$

*Proof.* Let  $F(t) = 2t(\log t - 1 - \overline{\log}(2\pi\lambda/q))$ , for  $t \in [T_1, T]$ . Then we find that  $F'(t) = 2 \log(\frac{tq}{2\pi\lambda})$ . Also let  $G(t) = \log^\nu(tq/2\pi)$ .

Suppose first that  $T_1 > 2\pi\lambda/q$ . Let  $\varphi(t) = G(t)/F'(t)$ . Our plan is to apply Lemma 4.3 in Titchmarsh [63]. Note that  $\varphi(t) > 0$ . Thus  $\varphi(t)$  is monotonic if and only if  $(1/\varphi)(t) = F'(t)/G(t)$  is monotonic. Let us therefore study  $G(t)F''(t) - G'(t)F'(t)$ . If  $\nu = 0$  or  $1$  then this is easily seen to be positive. And if  $\nu \geq 2$  then it equals  $\left( \frac{2 \log^{\nu-1}(tq/2\pi)}{t} \right) \left( \log(tq/2\pi) - \nu \log(\frac{tq}{2\pi\lambda}) \right)$ . This is zero at say  $t = T_0 := (\frac{2\pi\lambda}{q}) \lambda^{1/(\nu-1)}$ . If  $T_0 \in [T_1, T]$ , then it is easily seen that  $1/\varphi(t)$  has a maximum at  $t = T_0$ . In all cases we can conclude the following two things. Firstly that  $[T_1, T]$  is a union of at most two intervals in which  $\varphi(t)$  is monotonic. Secondly that  $F'(t)/G(t) \geq \min \{ F'(T_1)/G(T_1), F'(T)/G(T) \} \geq F'(T_1)/G(T)$ . Applying Lemma 4.3 in [63] now yields exactly what we wanted in this case.

Suppose now that we only know  $T_1 \geq 2\pi\lambda/q$ . Certainly  $F''(t) = 2/t \geq 2/T$  and  $|G(t)| \leq \log^\nu(Tq/2\pi)$ . If  $T_1 > 2\pi\lambda/q$  we know from above that  $G(t)/F'(t)$  is monotonic so applying (possibly twice) Lemma 4.5 in [63] gives us what we want. Suppose finally that  $T_1 = 2\pi\lambda/q$ . Let  $\delta = 2\sqrt{T}$ . If  $T \leq T_1 + \delta$  then trivially we have the upper bound

$$(T - T_1) \max_{t \in [T_1, T]} |G(t)| \leq \delta \log^\nu(Tq/2\pi) = 2\sqrt{T} \log^\nu(Tq/2\pi), \quad (4.85)$$

which is good. Otherwise

$$\left| \int_{T_1}^{T_1+\delta} \left( \frac{qt}{2\pi\lambda e} \right)^{2it} \log^\nu \left( \frac{tq}{2\pi} \right) dt \right| \leq 2\sqrt{T} \log^\nu(Tq/2\pi) \quad (4.86)$$

and we now turn our attention to the interval  $[T_1 + \delta, T]$ . In this interval we have, since  $F'(T_1) = 0$  and  $F''(t) = 2/t \geq 2/T$ , that  $F'(t) \geq 2\delta/T = 4/\sqrt{T}$ . Exactly

analogously to before we may conclude that  $G(t)/F'(t)$  is monotonic in this interval and that  $F'(t)/G(t) \geq F'(T_1 + \delta)/G(T) \geq \frac{4}{\sqrt{T} \log^\nu(Tq/2\pi)}$ . Applying Lemma 4.3 in [63] we find

$$\left| \int_{T_1 + \delta}^T \left( \frac{qt}{2\pi\lambda e} \right)^{2it} \log^\nu \left( \frac{tq}{2\pi} \right) dt \right| \leq \frac{8}{\left( \frac{4}{\sqrt{T} \log^\nu(Tq/2\pi)} \right)} = 2\sqrt{T} \log^\nu(Tq/2\pi). \quad (4.87)$$

Adding the bounds for the integral over  $[T_1, T_1 + \delta]$  and  $[T_1 + \delta, T]$  we see that the bound given in the second part of this lemma holds in this case too.  $\square$

**Lemma 4.9.** *For fixed  $\sigma \in (\frac{1}{2}, 1)$  we have*

$$\int_1^T |L(\sigma + it, \chi_0) L^{(\alpha)}(\sigma + it, \chi_0) L^{(\beta)}(\sigma + it, \chi_0) L^{(\gamma)}(\sigma + it, \chi_0)| dt = O_{\sigma, \alpha, \beta, \gamma}(Tq^{1/2}), \quad (4.88)$$

where  $\chi_0$  denotes the principal Dirichlet character modulo  $q$ .

*Proof.* The result follows via two successive applications of the Cauchy–Schwarz inequality if we establish

$$\int_1^T |L^{(\alpha)}(\sigma + it, \chi_0)|^4 dt \ll Tq^{1/2}. \quad (4.89)$$

Suppose for the time being that for any small  $\varepsilon > 0$  we have that

$$\int_0^T |L(\sigma + it, \chi_0)|^4 dt \ll Tq^{1/2} \quad (4.90)$$

holds uniformly for  $\sigma \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon]$ . Then since  $\sigma$  is a fixed number belonging to  $(\frac{1}{2}, 1)$ , we can certainly find a small  $\delta$  such that  $\frac{1}{2} < \sigma - \delta < \sigma < \sigma + \delta < 1$  and whence use (4.90) on  $[\sigma - \delta, \sigma + \delta]$ .

From Cauchy's integral formula we then find

$$\begin{aligned} |L^{(\alpha)}(\sigma + it, \chi_0)| &\leq \frac{\alpha!}{2\pi} \int_{|w - (\sigma + it)| = \delta} \frac{|L(w, \chi_0)|}{\delta^{\alpha+1}} |dw| \\ &= \frac{\alpha! \delta}{2\pi \delta^{\alpha+1}} \int_0^{2\pi} |L(\sigma + it + \delta e^{i\theta}, \chi_0)| d\theta. \end{aligned} \quad (4.91)$$

Hölder's inequality gives

$$\begin{aligned} |L^{(\alpha)}(\sigma + it, \chi_0)|^4 &\leq \left(\frac{\alpha!}{2\pi\delta^\alpha}\right)^4 \left(\int_0^{2\pi} |L(\sigma + it + \delta e^{i\theta}, \chi_0)|^4 d\theta\right) \left(\int_0^{2\pi} |1|^{4/3} d\theta\right)^3 \\ &\ll \int_0^{2\pi} |L(\sigma + it + \delta e^{i\theta}, \chi_0)|^4 d\theta. \end{aligned} \quad (4.92)$$

Hence

$$\begin{aligned} \int_1^T |L^{(\alpha)}(\sigma + it, \chi_0)|^4 dt &\ll \int_1^T \int_0^{2\pi} |L(\sigma + it + \delta e^{i\theta}, \chi_0)|^4 d\theta dt \\ &= \int_0^{2\pi} \int_1^T |L(\sigma + it + \delta e^{i\theta}, \chi_0)|^4 dt d\theta \ll \int_0^{2\pi} \int_0^{2T} |L(\sigma_2(\theta) + it, \chi_0)|^4 dt d\theta, \end{aligned} \quad (4.93)$$

where  $\sigma_2(\theta) \in [\sigma - \delta, \sigma + \delta]$ . Therefore, by the discussion above, we obtain that

$$\int_1^T |L^{(\alpha)}(\sigma + it, \chi_0)|^4 dt \ll \int_0^{2\pi} (2T)q^{1/2} d\theta \ll Tq^{1/2}. \quad (4.94)$$

It remains to show the claim that given any small  $\varepsilon > 0$  we have that (4.90) holds uniformly for  $\sigma \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon]$ . Begin by noting that

$$\begin{aligned} |L(\sigma + it, \chi_0)| &= \left| \prod_{p|q} (1 - p^{-s}) \right| \cdot |\zeta(\sigma + it)| \leq \prod_{p|q} (1 + p^{-1/2}) \cdot |\zeta(\sigma + it)| \\ &\leq 2^{\omega(q)} |\zeta(\sigma + it)| \ll q^{1/8} |\zeta(\sigma + it)|. \end{aligned} \quad (4.95)$$

Hence it suffices to prove that

$$\int_0^T |\zeta(\sigma + it)|^4 dt \ll T \quad (4.96)$$

holds uniformly for  $\sigma \in [\frac{1}{2} + \varepsilon, 1 - \varepsilon]$ . In Theorem 7.5 of [63], (4.96) is shown to hold<sup>4</sup> for any  $\sigma \in (\frac{1}{2}, 1)$ . However, that proof is clearly seen to also yield the uniform version that we require.  $\square$

<sup>4</sup>Titchmarsh considers the integral  $\int_1^T |\zeta(\sigma + it)|^4 dt$ , but since  $\zeta(\sigma + it)$  is continuous here for  $t \in [0, 1]$ , the difference is obviously unimportant.

**Lemma 4.10.** *Let  $h, k$  be non-negative integers. Define*

$$W_k(n) := \sum_{d|n} \log^k d. \quad (4.97)$$

Then, as  $x \rightarrow \infty$ , we have

$$\sum_{n \leq x} \chi_0(n) W_h(n) W_k(n) = D_q C_{h,k} x \log^{h+k+3} x + O_{h,k} \left( q x \log^{h+k+2} x \right), \quad (4.98)$$

where

$$D_q = \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} \quad (4.99)$$

and

$$C_{h,k} := \frac{6h!k!}{\pi^2(h+k+3)!} \left\{ \binom{h+k+2}{h+1} - 1 \right\}. \quad (4.100)$$

*Proof.* We start by noting that

$$\sum_{n=1}^{\infty} \frac{\chi_0(n) \sigma_a(n) \sigma_b(n)}{n^s} = \frac{L(s, \chi_0) L(s-a, \chi_0) L(s-b, \chi_0) L(s-a-b, \chi_0)}{L(2s-a-b, \chi_0)}, \quad (4.101)$$

where

$$\sigma_a(n) := \sum_{d|n} d^a. \quad (4.102)$$

This is valid for  $\sigma > 1$  if we have  $a, b < 0$ . For  $\operatorname{Re}(s) > 1$ , define

$$f(s) := \sum_{n=1}^{\infty} W_h(n) W_k(n) \chi_0(n) n^{-s}. \quad (4.103)$$

Then we see that

$$\begin{aligned} f(s) &= \frac{\partial^{h+k}}{\partial a^h \partial b^k} \left[ \sum_{n=1}^{\infty} \frac{\chi_0(n) \sigma_a(n) \sigma_b(n)}{n^s} \right] \Big|_{a=b=0} \\ &= \frac{\partial^{h+k}}{\partial a^h \partial b^k} \left[ \frac{L(s, \chi_0) L(s-a, \chi_0) L(s-b, \chi_0) L(s-a-b, \chi_0)}{L(2s-a-b, \chi_0)} \right] \Big|_{a=b=0}. \end{aligned} \quad (4.104)$$

This last equality can be used to turn  $f(s)$  into a meromorphic function (by analytic continuation) for all  $s$  with  $\operatorname{Re}(s) > 1/2$ .

Next we assume without loss of generality that  $x \equiv \frac{1}{2} \pmod{1}$  and apply Lemma 3.12 in [63]. In the notation used in Titchmarsh we pick  $\psi(n) = n^\epsilon$  (where  $\epsilon > 0$  is a suitable small absolute number),  $c > 1$ ,  $s = 0$  and  $\alpha = h + k + 4$ . We find that

$$\begin{aligned} &\sum_{n \leq x} \chi_0(n) W_h(n) W_k(n) \\ &= \frac{1}{2\pi} \int_{c-iT}^{c+iT} f(w) \frac{x^w}{w} dw + O\left(\frac{x^c}{T(c-1)^{h+k+4}}\right) + O\left(\frac{x^{1+\epsilon} \log x}{T}\right). \end{aligned} \quad (4.105)$$

We move the line of integration from  $\operatorname{Re}(w) = c$  to  $\operatorname{Re}(w) = 1/2 + \epsilon$ . We will first look at the residue at  $w = 1$  and then show that the three integral paths that are introduced do not make any significant contribution.

We thus want the residue of  $f(w)\frac{x^w}{w}$  at  $w = 1$ , where  $f(w)$  is as in (4.104). We look at the Laurent series of  $f(w)$  around  $w = 1$ . The lowest term will be of order  $(w - 1)^{-(h+k+4)}$  and in this case  $L(2w - a - b, \chi_0)^{-1}$  has not been differentiated. Since  $\frac{d^{(h+k+3)}}{dw}(x^w) = (\log x)^{h+k+3}x^w$ , it is here that we find our main term. To find the coefficient of this term, we note that the coefficient of  $(w - 1)^{-(h+k+4)}$  in the Laurent series of  $f(w)$  comes from

$$(-1)^{h+k}L(2w, \chi_0)^{-1} \times \sum_{\alpha=0}^h \sum_{\beta=0}^k \binom{h}{\alpha} \binom{k}{\beta} L(w, \chi_0) L^{(\alpha)}(w, \chi_0) L^{(\beta)}(w, \chi_0) L^{(h+k-\alpha-\beta)}(w, \chi_0), \quad (4.106)$$

so that the coefficient equals

$$\begin{aligned} & \left\{ \prod_{p|q} \frac{(1-p^{-1})^4}{(1-p^{-2})} \right\} \left( \frac{6}{\pi^2} \right) \sum_{\alpha=0}^h \sum_{\beta=0}^k \binom{h}{\alpha} \binom{k}{\beta} \alpha! \beta! (h+k-\alpha-\beta)! \\ &= \frac{6D_q h! k!}{\pi^2} \sum_{\alpha=0}^h \sum_{\beta=0}^k \binom{h+k-\alpha-\beta}{h-\alpha} \\ &= \frac{6D_q h! k!}{\pi^2} \sum_{p=0}^h \sum_{r=0}^k \binom{p+r}{p} \\ &= \frac{6D_q h! k!}{\pi^2} \left\{ \binom{h+k+2}{h+1} - 1 \right\}. \end{aligned} \quad (4.107)$$

Hence the coefficient of  $x(\log x)^{h+k+3}$  is indeed  $C_{h,k}$ .

Let us obtain a crude upper bound for the other residue-terms. Consider once again (4.104). Looking at each of the five  $L$ -functions' Laurent series around  $w = 1$  we note that the product of any four of them cannot contain a term of order lower than  $(w - 1)^{-(h+k+4)}$ . We conclude that we need to study the terms of order at most  $(w - 1)^{h+k+3}$  in the Laurent series of  $L(w, \chi_0)$  and  $\frac{1}{L(2w, \chi_0)}$  around  $w = 1$ , since the terms of higher order cannot contribute to our residue.

Now

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s}) = \zeta(s) U_1(s), \quad (4.108)$$

say. Certainly  $\zeta(s)$  has a pole of order 1 at  $s = 1$ . Also, we may write

$$U_1(s) = \sum_{k|q} \frac{\mu(k)}{k^s}. \quad (4.109)$$

We find

$$U_1^{(m)}(s) = \sum_{k|q} \frac{(-1)^m \mu(k) \log^m k}{k^s} \quad (4.110)$$

and hence

$$U_1^{(m)}(1) = \sum_{k|q} \frac{(-1)^m \mu(k) \log^m k}{k} \leq \log^m q \sum_{k=1}^q \frac{1}{k} \ll \log^{m+1}(q+2). \quad (4.111)$$

Hence all relevant Laurent-coefficients here will be  $\ll \log^{h+k+5}(q+2)$ .

We will similarly deal with

$$\frac{1}{L(2s, \chi_0)} = \frac{1}{\zeta(2s)} \prod_{p|q} (1 - p^{-2s})^{-1} = \frac{1}{\zeta(2s)} U_2(s), \quad (4.112)$$

say. Obviously we have a Laurent series of  $\frac{1}{\zeta(2s)}$  which is in fact a Taylor series. Let

$$U_3(s) := U_2(s) \prod_{p|q} (1 - p^{-2}). \quad (4.113)$$

Then  $U_3(s)$  is an analytic function which is never zero in a region around  $s = 1$  and also  $U_3(1) = 1$ . A moment's thought will yield that we can bound the relevant Laurent-coefficients of  $U_3(s)$  by bounding the coefficients of these orders of the reciprocal function of  $U_3(s)$ , i.e.

$$\prod_{p|q} (1 - p^{-2s}) / \prod_{p|q} (1 - p^{-2}) = U_4(s) / U_5, \quad (4.114)$$

say. The numerator can be dealt with as in the previous paragraph and we obtain  $U_4^{(m)}(s) \ll \log^m(q+2)$ . Also,  $U_5 \geq \prod_{p \leq q} (1 - p^{-1}) \gg \frac{1}{\log(q+2)}$ , by Merten's theorem.

Thus all relevant coefficients in the Laurent series of  $U_2(s)$  are  $\ll \log^{h+k+5}(q+2)$ .

We conclude that the product of any relevant coefficients in the five Laurent series is  $\ll \log^{5(h+k+5)}(q+2) \ll q$ , as desired.

Now we will consider the three integral paths. We will find bounds for the parts where  $t \geq 0$ , an analogous argument may deal with the lower half-plane. We will use some results about the well-known function  $\mu(\sigma)$ , which we discussed in Section 2.1. Firstly that  $\mu(1/2) < 0.165$  (the exact value of our choice, i.e. 0.165, is not important) which tells us that  $\zeta(1/2 + it) \ll t^{0.165}$  (for  $t \geq 1$  say) and secondly that  $\mu(\sigma)$  is continuous, which gives us  $\zeta(1/2 + \epsilon + it) \ll t^{0.166}$  say. Now trivially for  $s = \sigma + it$  with  $\sigma \approx 1/2$  we have that

$$\prod_{p|q} (1 - p^{-s}) \ll \prod_{p|q} (1 + 1) = 2^{\omega(q)} \ll q^{1/10}. \quad (4.115)$$

We may thus certainly conclude that

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s}) \ll t^{0.166} q^{1/10}, \quad (4.116)$$

for  $s = 1/2 + \epsilon + it$ . In fact it is clear that we may use Lemma 4.3 to deduce that

$$L^{(m)}(1/2 + \epsilon + it, \chi_0)/q^{1/10} \ll t^{0.166} \log^m t \ll t^{1/6}. \quad (4.117)$$

Similarly we easily show that  $L^{(m)}(c+it, \chi_0)/q^{1/10} \ll t^{1/6}$ . By the Phragmén–Lindelöf Principle we obtain

$$L^{(m)}(\sigma + it, \chi_0) \ll q^{1/10} t^{1/6}, \text{ for } \sigma \in [1/2 + \epsilon, c]. \quad (4.118)$$

We will also need an upper bound for  $\frac{d^\alpha}{ds^\alpha} \left[ \frac{1}{L(2s, \chi_0)} \right]$  for  $\text{Re}(s) \in [1/2 + \epsilon, c]$  and  $\alpha \leq h + k$ . We get this from upper bounds of  $L^{(\alpha)}(2s, \chi_0)$  and from

$$\begin{aligned} \frac{1}{L(2s, \chi_0)^{2\alpha}} &= \frac{1}{\zeta(2s)^{2\alpha}} \prod_{p|q} (1 - p^{-2s})^{-2\alpha} \ll \zeta(1 + 2\epsilon)^{2\alpha} \prod_{p|q} (1 - p^{-1})^{-2\alpha} \\ &\ll 1 \cdot (\log(q + 2))^{2\alpha}. \end{aligned} \quad (4.119)$$

Now  $\left| \frac{d^\alpha}{ds^\alpha} [L(2s, \chi_0)] \right| = \left| \sum_{n=1}^{\infty} \frac{\chi_0(n) 2^\alpha \log^\alpha n}{n^{2s}} \right| \ll 1$ . Whence

$$\frac{d^\alpha}{ds^\alpha} \left[ \frac{1}{L(2s, \chi_0)} \right] \ll (\log(q + 2))^{2(h+k)} \ll q^{1/10}. \quad (4.120)$$

Thus the horizontal integral is

$$\ll \int_{1/2+\epsilon}^c x^\sigma T^{-1} q^{1/10} (q^{1/10} T^{1/6})^4 d\sigma \ll x^c q T^{-1/3}. \quad (4.121)$$

For the vertical side we write

$$\int_{1/2+\epsilon}^{1/2+\epsilon+iT} = \int_{1/2+\epsilon}^{1/2+\epsilon+i} + \int_{1/2+\epsilon+i}^{1/2+\epsilon+iT}. \quad (4.122)$$

By using the bound in (4.115) the first term is

$$\ll 1 \cdot q^{1/10} \cdot (q^{1/10} \cdot 1)^4 \cdot \frac{x^{1/2+\epsilon}}{1} \ll qx^{1/2+\epsilon}. \quad (4.123)$$

Next, define

$$\Phi(T) := \int_1^T |f(1/2 + \epsilon + it)| dt \ll Tq^{3/5}. \quad (4.124)$$

Then

$$\begin{aligned} \int_1^T |f(1/2+\epsilon+it)| \frac{1}{t} dt &= \int_1^T \Phi'(t)t^{-1} dt = \left[ \frac{\Phi(t)}{t} \right]_1^T + \int_1^T \Phi(t)/t^2 dt \\ &= O(q^{3/5}) + O\left( \int_1^T q^{3/5}t^{-1} dt \right) \ll q^{3/5} \log T \ll qT^\epsilon, \end{aligned} \quad (4.125)$$

using (4.120) and Lemma 4.9. Hence

$$\int_{1/2+\epsilon}^{1/2+\epsilon+iT} f(w) \frac{x^w}{w} dw \ll x^{1/2+\epsilon} q + x^{1/2+\epsilon} qT^\epsilon \ll x^{1/2+\epsilon} qT^\epsilon. \quad (4.126)$$

Thus all-in-all, if all of

$$\frac{x^c}{T(c-1)^{h+k+4}}, \frac{x^{1+\epsilon} \log x}{T}, qx^c T^{-1/3} \text{ and } qx^{1/2+\epsilon} T^\epsilon \quad (4.127)$$

are  $\ll qx(\log x)^{h+k+2}$ , then we are done. Picking  $c = 1 + \epsilon$  and  $T = \sqrt{x}$  we see that all of our terms in (4.127) are  $\ll qx^{5/6+\epsilon} \ll qx(\log x)^{h+k+2}$ .  $\square$

#### 4.4.4 Evaluation of our integrals

**Lemma 4.11.** *For  $\chi$  primitive,  $T \geq Aq$  and  $q \ll \sqrt{\log T}$  we have*

$$\int_{Aq}^T |L(\frac{1}{2} + it, \chi)|^4 dt = \left( \frac{D_q}{2\pi^2} \right) T \log^4 T + O(Tq \log^3 T). \quad (4.128)$$

*Proof.* We follow Ingham's proof of his Theorem B in [46]. Looking at the definition of  $B(s, \chi)$  in (4.30) we see that the  $B(s, \chi)^2$ -term in the approximate functional equation given in part (i) of Corollary 4.1, in which  $s = 1/2 + it$ , may clearly be replaced by

$$\begin{aligned} &[iG(1, \chi)q^{-1/2-it}(2\pi)^{-1/2+it}\chi(-1)\Gamma(\frac{1}{2} - it)e^{-\pi i/4+\pi t/2}]^2 \\ &= iq^{-1}G(1, \chi)^2 \left( \frac{qt}{2\pi e} \right)^{-2it} \left\{ 1 + O\left( \frac{1}{t} \right) \right\}, \end{aligned} \quad (4.129)$$

where we in the last step used Stirling's formula to approximate the gamma function. Indeed the last step allows us to conclude that we may write

$$L^2(1/2 + it, \chi) = S_1 + B_2 \overline{S_1} + E_1, \quad (4.130)$$

where

$$S_1 = S_1(t, \chi) = \sum_{n \leq tq/2\pi} \frac{\chi(n)d(n)}{n^{1/2+it}}, \quad (4.131)$$

$$B_2 = B_2(t, \chi) = q^{-1}G(1, \chi)^2 i \left( \frac{qt}{2\pi e} \right)^{-2it} \quad (4.132)$$

and

$$E_1 = E_1(t, \chi) \ll q^{1/2} \log(q+2) \log t. \quad (4.133)$$

Now

$$|S_1 + B_2 \overline{S_1}|^2 = 2|S_1|^2 + 2\operatorname{Re}(S_1^2 \overline{B_2}) \leq 4|S_1|^2. \quad (4.134)$$

Thus we may write

$$\int_{Aq}^T |L(\tfrac{1}{2} + it, \chi)|^4 dt = 2 \int_{Aq}^T |S_1|^2 dt + 2\operatorname{Re} \left[ \int_{Aq}^T \overline{B_2} S_1^2 dt \right] + R = 2J_1 + 2J_2 + R, \quad (4.135)$$

where

$$R \leq 4 \int_{Aq}^T |S_1 E_1| dt + \int_{Aq}^T |E_1|^2 dt = 4R_1 + R_2 \quad (4.136)$$

say.

Firstly,

$$J_1 = \int_{Aq}^T S_1 \overline{S_1} dt = \int_{Aq}^T \sum_{m, n \leq tq/2\pi} \frac{d(m)d(n)\chi(m)\overline{\chi(n)}}{\sqrt{mn}} \left( \frac{n}{m} \right)^{it} dt = J_{11} + J_{12} \quad (4.137)$$

say, where  $J_{11}$  is the contribution of the terms with  $m = n$ . By Lemma 4.10 and trivial partial summation we have

$$\begin{aligned} J_{11} &= \int_{Aq}^T \sum_{n \leq tq/2\pi} \frac{d(n)^2 \chi_0(n)}{n} dt = \int_{Aq}^T \frac{D_q}{4\pi^2} \log^4(tq/2\pi) + O\left(q \log^3(tq/2\pi)\right) dt \\ &= \frac{D_q}{4\pi^2} \int_{Aq}^T \log^4 t dt + O\left( \int_{Aq}^T q \log^3(tq) dt \right) = \left( \frac{D_q}{4\pi^2} \right) T \log^4 T + O(Tq \log^3 T). \end{aligned} \quad (4.138)$$

Also, by using Lemma 4.7, we see that

$$\begin{aligned} J_{12} &= \sum_{\substack{m, n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n)\chi(m)\overline{\chi(n)}}{\sqrt{mn}} \int_{\max(Aq, 2\pi \max(m, n)/q)}^T \left( \frac{n}{m} \right)^{it} dt \\ &\ll \sum_{\substack{m, n \leq Tq/2\pi \\ m \neq n}} \left\{ \frac{d(m)d(n) \cdot 1}{\sqrt{mn}} \cdot \frac{1}{|\log(m/n)|} \right\} \ll Tq \log^3 T. \end{aligned} \quad (4.139)$$

Secondly,

$$\begin{aligned}
J_2 &= \int_{Aq}^T \overline{B_2} \sum_{m,n \leq tq/2\pi} \frac{d(m)d(n)\chi(m)\chi(n)}{(mn)^{1/2+it}} dt \\
&= \left( -\frac{i\overline{G(1,\chi)}^2}{q} \right) \sum_{m,n \leq Tq/2\pi} \frac{d(m)d(n)\chi(mn)}{\sqrt{mn}} \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left( \frac{tq}{2\pi e\sqrt{mn}} \right)^{2it} dt.
\end{aligned} \tag{4.140}$$

The contribution coming from the terms where both  $m$  and  $n$  are at most  $Aq^2/2\pi$  is trivially seen to be  $\ll q^4 T$ . For the other terms we may use Lemma 4.8 and obtain the upper bound

$$\begin{aligned}
&\sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n)}{\sqrt{mn} |\log(m/n)|} + \sqrt{T} \sum_{n \leq Tq/2\pi} \frac{d(n)^2}{n} \\
&\ll Tq \log^3 T + T^{1/2} \log^4 T \ll Tq \log^3 T.
\end{aligned} \tag{4.141}$$

Thirdly,

$$R_2 = \int_{Aq}^T |E_1|^2 dt \ll Tq \log^2 T \log^2(q+2) \ll Tq^2 \log^2 T. \tag{4.142}$$

And by the Cauchy–Schwarz inequality

$$\begin{aligned}
R_1 &\leq \sqrt{J_1 R_2} \ll \sqrt{T \log^3 T (\log T + q)} \sqrt{Tq^2 \log^2 T} \\
&\leq Tq \log^3 T + Tq^{3/2} \log^{5/2} T \\
&\ll Tq \log^3 T + Tq^2 \log^2 T.
\end{aligned} \tag{4.143}$$

So far we have [notice that we have actually not yet used the assumption that  $q \ll (\log T)^{1/2}$ ] found the main term and three error terms, where by using the inequality of the arithmetic and geometric means, the error term  $Tq^2 \log^2 T$  may be absorbed into the other two error terms. Thus

$$\int_{Aq}^T |L(\tfrac{1}{2} + it, \chi)|^4 dt = \left( \frac{D_q}{2\pi^2} \right) T \log^4 T + O(Tq \log^3 T + q^4 T). \tag{4.144}$$

If we now use  $q \ll \sqrt{\log T}$ , then the theorem trivially follows from (4.144).  $\square$

**Lemma 4.12.** For  $\chi$  primitive,  $T \geq Aq$  and  $q \ll \sqrt{\log T}$  we have

$$\int_{Aq}^T |L(\tfrac{1}{2} + it, \chi)L'(\tfrac{1}{2} + it, \chi)|^2 dt = \left(\frac{2D_q}{15\pi^2}\right) T \log^6 T + O(Tq \log^5 T). \quad (4.145)$$

*Proof.* By an analogous argument to the one in the proof of Lemma 4.11, we see from part (ii) in Corollary 4.1 that we can write

$$2L(1/2 + it, \chi)L'(1/2 + it, \chi) = F_1 + B_2 F_2 + E_2, \quad (4.146)$$

where

$$F_1 = - \sum_{n \leq tq/2\pi} \frac{\chi(n)d(n) \log n}{n^{1/2+it}}, \quad (4.147)$$

$$F_2 = \sum_{n \leq tq/2\pi} \frac{\overline{\chi(n)}d(n) \log(\frac{4\pi^2 n}{t^2 q^2})}{n^{1/2-it}} \quad (4.148)$$

and

$$E_2 \ll q^{1/2} \log(q+2) \log^2 t. \quad (4.149)$$

Now let

$$I_i(T) := \int_{Aq}^T |F_i|^2 dt, \quad i = 1, 2 \quad (4.150)$$

and

$$J(T) := \int_{Aq}^T F_1 \overline{B_2 F_2} dt. \quad (4.151)$$

We begin by writing  $I_1(T) = I_{11}(T) + I_{12}(T)$ , where  $I_{11}$  involves the diagonal terms. Then by Lemma 4.10 and partial summation we have

$$\begin{aligned} I_{11}(T) &= \int_{Aq}^T \sum_{n \leq tq/2\pi} \frac{d(n)^2 \chi_0(n) \log^2 n}{n} dt \\ &= \int_{Aq}^T \left(\frac{D_q}{6\pi^2}\right) \log^6(tq/2\pi) + O\left(q \log^5(tq/2\pi)\right) dt \\ &= \left(\frac{D_q}{6\pi^2}\right) \int_{Aq}^T \log^6 t dt + O\left(\int_{Aq}^T q \log^5(tq) dt\right) = \left(\frac{D_q}{6\pi^2}\right) T \log^6 T + O(Tq \log^5 T). \end{aligned} \quad (4.152)$$

Also,

$$\begin{aligned}
I_{12} &= \sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n) \log m \log n \chi(m) \overline{\chi(n)}}{\sqrt{mn}} \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left(\frac{n}{m}\right)^{it} dt \\
&\ll \sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n) \log m \log n}{\sqrt{mn} |\log(m/n)|} \ll \log^2 T \cdot Tq \log^3 T = Tq \log^5 T, \quad (4.153)
\end{aligned}$$

where we again used Lemma 4.7.

Next we deal with  $I_2(T) = I_{21}(T) + I_{22}(T)$  say, in a similar way. Let us write

$$I_{21}(T) = I_{211}(T) + I_{212}(T) + I_{213}(T) \quad (4.154)$$

with

$$I_{211}(T) = \int_{Aq}^T \sum_{n \leq tq/2\pi} \frac{d(n)^2 \chi_0(n) \log^2 n}{n} dt = I_{11}(T) = \left(\frac{D_q}{6\pi^2}\right) T \log^6 T + O(Tq \log^5 T), \quad (4.155)$$

$$\begin{aligned}
I_{212}(T) &= -4 \int_{Aq}^T \log\left(\frac{qt}{2\pi}\right) \sum_{n \leq tq/2\pi} \frac{d(n)^2 \chi_0(n) \log n}{n} dt \\
&= -4 \int_{Aq}^T \log\left(\frac{qt}{2\pi}\right) \left\{ \frac{D_q}{5\pi^2} \log^5(tq/2\pi) + O\left[q \log^4(tq/2\pi)\right] \right\} dt \\
&= -\frac{4D_q}{5\pi^2} \int_{Aq}^T \log^6 t dt + O\left(\int_{Aq}^T q \log^5(tq) dt\right) \\
&= \left(-\frac{4D_q}{5\pi^2}\right) T \log^6 T + O(Tq \log^5 T) \quad (4.156)
\end{aligned}$$

and

$$\begin{aligned}
I_{213}(T) &= 4 \int_{Aq}^T \log^2\left(\frac{qt}{2\pi}\right) \sum_{n \leq tq/2\pi} \frac{d(n)^2 \chi_0(n)}{n} dt \\
&= 4 \int_{Aq}^T \log^2\left(\frac{qt}{2\pi}\right) \left\{ \frac{D_q}{4\pi^2} \log^4(tq/2\pi) + O\left[q \log^3(tq/2\pi)\right] \right\} dt \\
&= \frac{D_q}{\pi^2} \int_{Aq}^T \log^6 t dt + O\left(\int_{Aq}^T q \log^5(tq) dt\right) \\
&= \left(\frac{D_q}{\pi^2}\right) T \log^6 T + O(Tq \log^5 T). \quad (4.157)
\end{aligned}$$

Thus

$$I_{21}(T) = \left(\frac{11D_q}{30\pi^2}\right)T \log^6 T + O(Tq \log^5 T). \quad (4.158)$$

Also,

$$I_{22}(T) = 4 \sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n)\overline{\chi(m)}\chi(n)}{\sqrt{mn}} I_{220} \quad (4.159)$$

say, where

$$\begin{aligned} I_{220} &= \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left(\frac{m}{n}\right)^{it} \log\left(\frac{tq}{2\pi\sqrt{m}}\right) \log\left(\frac{tq}{2\pi\sqrt{n}}\right) dt \\ &= \frac{1}{i \log(m/n)} \left[ \left(\frac{m}{n}\right)^{it} \log\left(\frac{tq}{2\pi\sqrt{m}}\right) \log\left(\frac{tq}{2\pi\sqrt{n}}\right) \right]_{\max(Aq, 2\pi \max(m,n)/q)}^T \\ &\quad - \frac{1}{i \log(m/n)} \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left(\frac{m}{n}\right)^{it} \left\{ \log\left(\frac{tq}{2\pi\sqrt{m}}\right) + \log\left(\frac{tq}{2\pi\sqrt{n}}\right) \right\} \left(\frac{1}{t}\right) dt \\ &\ll \frac{\log^2 T}{|\log(m/n)|} + \frac{1}{|\log(m/n)|} \int_{\max(Aq, 2\pi \max(m,n)/q)}^T 1 \cdot \log T \cdot t^{-1} dt \\ &\ll \frac{\log^2 T}{|\log(m/n)|}. \end{aligned} \quad (4.160)$$

Therefore

$$I_{22}(T) \ll \log^2 T \sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n)}{\sqrt{mn} |\log(m/n)|} \ll Tq \log^5 T. \quad (4.161)$$

Now we turn our attention to

$$\begin{aligned} J(T) &= q^{-1} \overline{G(1, \chi)}^2 i \sum_{m,n \leq Tq/2\pi} \left\{ \frac{d(m)d(n)\chi(mn) \log m}{\sqrt{mn}} \right. \\ &\quad \times \left. \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left(\frac{qt}{2\pi e\sqrt{mn}}\right)^{2it} \{\log n - 2l_2\} dt \right\} \\ &\ll \log^2 T \sum_{m,n \leq Tq/2\pi} \frac{d(m)d(n)}{\sqrt{mn}} \left| \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left(\frac{qt}{2\pi e\sqrt{mn}}\right)^{2it} dt \right| \\ &\quad + \sum_{m,n \leq Tq/2\pi} \frac{d(m)d(n) \log m}{\sqrt{mn}} \left| \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left(\frac{qt}{2\pi e\sqrt{mn}}\right)^{2it} \log\left(\frac{qt}{2\pi}\right) dt \right|. \end{aligned} \quad (4.162)$$

The first part of the RHS of (4.162) may be dealt with as in the proof of Lemma 4.11, and is  $\ll Tq \log^5 T$ . For the second part of the RHS of (4.162), we notice that the summands are non-negative. Therefore when we seek an upper bound, we may write

$$\sum_{m,n \leq Tq/2\pi} \leq \sum_{m=n \leq Tq/2\pi} + \sum_{m,n \leq Aq^2/2\pi} + \sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n \\ 2\pi \max(m,n)/q > Aq}} = J_a + J_b + J_c. \quad (4.163)$$

Using Lemma 4.8, we have

$$J_a \ll \sqrt{T} \log(Tq/2\pi) \sum_{n \leq Tq/2\pi} \frac{d(n)^2 \log n}{n} \ll Tq \log^5 T. \quad (4.164)$$

Trivially

$$J_b \ll q^4 T \log T \ll Tq \log^5 T. \quad (4.165)$$

And since

$$J_c \ll \sum_{\substack{m,n \leq Tq/2\pi \\ m \neq n}} \frac{d(m)d(n) \log m \log(Tq/2\pi)}{\sqrt{mn} |\log(m/n)|} \ll Tq \log^5 T, \quad (4.166)$$

we may conclude that

$$J(T) \ll Tq \log^5 T. \quad (4.167)$$

Finally, trivially

$$\int_{Aq}^T |E_2|^2 dt \ll Tq \log^4 T \log^2(q+2). \quad (4.168)$$

We can now finish the proof as before by means of the Cauchy–Schwarz inequality.  $\square$

**Lemma 4.13.** *For  $\chi$  primitive,  $T \geq Aq$  and  $q \ll \sqrt{\log T}$  we have*

$$(i) \int_{Aq}^T Y(t, \chi)^4 dt = \left(\frac{D_q}{2\pi^2}\right) T \log^4 T + O(Tq \log^3 T),$$

$$(ii) \int_{Aq}^T Y(t, \chi)^2 Y'(t, \chi)^2 dt = \left(\frac{D_q}{120\pi^2}\right) T \log^6 T + O(Tq \log^5 T).$$

*Proof.* From (4.26) we see that  $Y(t, \chi)^2 = |L(\frac{1}{2} + it, \chi)|^2$ . Therefore the first part of this lemma follows immediately from Lemma 4.11.

Differentiation of (4.26) leads to

$$|L'(\frac{1}{2} + it, \chi)|^2 = Y'(t, \chi)^2 + \theta'(t, \chi)^2 Y(t, \chi)^2. \quad (4.169)$$

Whence

$$|L(\frac{1}{2} + it, \chi)L'(\frac{1}{2} + it, \chi)|^2 = Y(t, \chi)^2 Y'(t, \chi)^2 + \theta'(t, \chi)^2 Y(t, \chi)^4 \quad (4.170)$$

and recalling Lemma 4.12 it remains to show that

$$\int_{Aq}^T \theta'(t, \chi)^2 Y(t, \chi)^4 dt = \left(\frac{D_q}{8\pi^2}\right) T \log^6 T + O(Tq \log^5 T). \quad (4.171)$$

We substitute for  $\theta'(t, \chi)$  the expression given in (4.28). It is easy to show that the “big-oh”-term then leads to a negligible contribution to the LHS of (4.171). Lastly, if we introduce

$$M(T) := \int_{Aq}^T Y(t, \chi)^4 dt = \int_{Aq}^T |L(\frac{1}{2} + it, \chi)|^4 dt, \quad (4.172)$$

then by using (4.144) we find that our (main) contribution to the LHS of (4.171) is

$$\begin{aligned} & \int_{Aq}^T (\frac{1}{2} \log(tq/2\pi))^2 M'(t) dt = \frac{1}{4} \int_{Aq}^T \log^2 t M'(t) dt + O(\log T \log(q+2) M(T)) \\ &= \frac{1}{4} \left[ \log^2 t M(t) \right]_{Aq}^T - \frac{1}{2} \int_{Aq}^T \frac{\log t}{t} M(t) dt + O(Tq \log^5 T) \\ &= \frac{1}{4} M(T) \log^2 T + O\left\{ \int_{Aq}^T \left(\frac{\log t}{t}\right) (t \log^4 t + tq \log^3 t + q^4 t) dt \right\} + O(Tq \log^5 T) \\ &= \left(\frac{D_q}{8\pi^2}\right) T \log^6 T + O(Tq \log^5 T). \end{aligned} \quad (4.173)$$

□

**Lemma 4.14.** *For  $\chi$  primitive,  $T \geq Aq$  and  $q \ll \sqrt{\log T}$  we have*

$$\int_{Aq}^T Y'(t, \chi)^4 dt = \left(\frac{D_q}{1120\pi^2}\right) T \log^8 T + O(Tq \log^7 T). \quad (4.174)$$

*Proof.* From  $Y(t) = e^{i\theta(t, \chi)} L(\frac{1}{2} + it, \chi)$  we get

$$\begin{aligned} Y'(t, \chi)^2 &= \\ &- e^{2i\theta(t, \chi)} \left[ L'(\frac{1}{2} + it, \chi)^2 + 2\theta'(t, \chi) L(\frac{1}{2} + it, \chi) L'(\frac{1}{2} + it, \chi) + \theta'(t, \chi)^2 L(\frac{1}{2} + it, \chi)^2 \right]. \end{aligned} \quad (4.175)$$

We multiply both sides of (4.175) by  $-e^{-2i\theta(t,\chi)}$ , then use Corollary 4.1, remembering (4.28). As before we may replace  $B^2$  by  $B_2$  and we arrive at

$$-e^{-2i\theta(t,\chi)}Y'(t,\chi)^2 = F_3 + B_2\overline{F_3} + E_3, \quad (4.176)$$

where

$$F_3 = \sum_{n \leq tq/2\pi} \frac{\chi(n)}{n^{1/2+it}} \left\{ D(n) - \frac{1}{2}d(n)l_2 \log n + \frac{1}{4}d(n)l_2^2 \right\} \quad (4.177)$$

and

$$E_3 \ll q^{1/2} \log(q+2) \log^3 t. \quad (4.178)$$

We will show that

$$\int_{Aq}^T |F_3|^2 dt = \left( \frac{D_q}{2240\pi^2} \right) T \log^8 T + O(Tq \log^7 T) \quad (4.179)$$

and

$$J_3(T) = \int_{Aq}^T \overline{B_3} F_3^2 dt \ll Tq \log^7 T. \quad (4.180)$$

Then we will be done since trivially

$$\int_{Aq}^T |E_3|^2 dt \ll Tq \log^2(q+2) \log^6 T. \quad (4.181)$$

We have

$$\begin{aligned} J_3(T) &\leq \sum_{m,n \leq Tq/2\pi} \frac{1}{\sqrt{mn}} \left| \int_{\max(Aq, 2\pi \max(m,n)/q)}^T \left( \frac{tq}{2\pi e \sqrt{mn}} \right)^{2it} \right. \\ &\quad \times \left[ \frac{1}{2}d(m) \log^2 m - W_2(m) - \frac{1}{2}d(m)(\log m)l_2 + \frac{1}{4}d(m)l_2^2 \right] \\ &\quad \times \left. \left[ \frac{1}{2}d(n) \log^2 n - W_2(n) - \frac{1}{2}d(n)(\log n)l_2 + \frac{1}{4}d(n)l_2^2 \right] dt \right|. \end{aligned} \quad (4.182)$$

The contribution coming from the terms where both  $m$  and  $n$  are at most  $Aq^2/2\pi$  is trivially seen to be

$$\ll (q^2)^2 \cdot 1 \cdot T \cdot (\log^2 T)^2 \ll Tq \log^7 T. \quad (4.183)$$

For the other terms we have  $2\pi \max(m,n)/q > Aq$  and here the contribution is seen to be  $\ll Tq \log^7 T$  by using Lemma 4.8 and the trivial estimate  $W_2(m) \leq d(m) \log^2 m$ .

We now study  $\int_{Aq}^T |F_3|^2 dt$ . The non-diagonal contribution is

$$\begin{aligned} &\leq \sum_{\substack{m, n \leq Tq/2\pi \\ m \neq n}} \frac{1}{\sqrt{mn}} \left| \int_{\max(Aq, 2\pi \max(m, n)/q)}^T \left(\frac{n}{m}\right)^{it} \right. \\ &\quad \times \left[ \frac{1}{2}d(m) \log^2 m - W_2(m) - \frac{1}{2}d(m)(\log m)l_2 + \frac{1}{4}d(m)l_2^2 \right] \\ &\quad \left. \times \left[ \frac{1}{2}d(n) \log^2 n - W_2(n) - \frac{1}{2}d(n)(\log n)l_2 + \frac{1}{4}d(n)l_2^2 \right] dt \right|. \end{aligned} \quad (4.184)$$

The contribution from the terms where both  $m$  and  $n$  are at most  $Aq^2/2\pi$  is as before easily seen to be  $\ll q^4 T \log^4 T \ll Tq \log^7 T$ . To bound the contribution from the other terms, we simply multiply out the two square brackets in the integrand and notice that we can in all cases first apply Lemma 4.8 and then Lemma 4.7. We find an upper bound of order  $Tq \log^7 T$ .

The diagonal contribution equals

$$\sum_{n \leq Tq/2\pi} \frac{\chi_0(n)}{n} \int_{\max(Aq, 2\pi n/q)}^T \left[ \frac{1}{2}d(n) \log^2 n - W_2(n) - \frac{1}{2}d(n)(\log n)l_2 + \frac{1}{4}d(n)l_2^2 \right]^2 dt. \quad (4.185)$$

We now wish to change the lower limit of integration to 0. When we do that there is no problem coming from that  $|\log x| \rightarrow \infty$  as  $x \rightarrow 0$ , however, we wish to find an upper bound for the change. For the terms where  $n \leq Aq^2/2\pi$  we may change the lower integration limit from  $Aq$  to 0 with a total error which is

$$\ll (Aq^2/2\pi) \cdot 1 \cdot (Aq) \cdot \log^4(q+2) \ll Tq \log^7 T. \quad (4.186)$$

For the terms where  $n \geq Aq^2/2\pi$  we may change the lower integration limit from  $2\pi n/q$  to 0 with an error which is

$$\ll \sum_{n \leq Tq/2\pi} \left(\frac{1}{n}\right) \left(\frac{2\pi n}{q}\right) d(n)^2 \log^4 T \ll Tq \log^7 T, \quad (4.187)$$

where we in the last estimate used Lemma 4.10 (with  $q = 1$ ). Hence we study

$$\sum_{n \leq Tq/2\pi} \frac{\chi_0(n)}{n} \int_0^T \left[ \frac{1}{2}d(n) \log^2 n - W_2(n) - \frac{1}{2}d(n)(\log n)l_2 + \frac{1}{4}d(n)l_2^2 \right]^2 dt. \quad (4.188)$$

By integration by parts we find that

$$\int_0^T \log^v \left(\frac{tq}{2\pi}\right) dt = T \log^v T + O(Tq \log^{v-1} T). \quad (4.189)$$

Now we simply multiply out the 16 ( $= 4 \cdot 4$ ) parts in (4.188) and see what each one of them contribute. When doing the latter we use that Lemma 4.10 implies, after partial summation, that

$$\left[ \sum_{n \leq Tq/2\pi} \frac{\chi_0(n)W_h(n)W_k(n) \log^r n}{n} - \frac{D_q C_{h,k}}{h+k+r+4} \log^{h+k+r+4} T \right] \ll q \log^{h+k+r+3} T, \quad (4.190)$$

where  $h+k+r+4 = 8-v$ . An elementary calculation thus now yields that the diagonal contribution equals

$$\begin{aligned} & D_q \left( \frac{59}{6720} C_{0,0} - \frac{11}{168} C_{0,2} + \frac{1}{8} C_{2,2} \right) T \log^8 T + O(Tq \log^7 T) \\ &= \frac{D_q}{2240\pi^2} T \log^8 T + O(Tq \log^7 T), \end{aligned} \quad (4.191)$$

where  $C_{h,k}$  above was as defined in (4.98). This completes the proof.  $\square$

## 4.5 A Wirtinger-like inequality

**Theorem 4.2.** *Let  $v \geq 0$  and put*

$$\lambda_0(v) = \frac{1}{8} \{1 + 4v + \sqrt{1 + 8v}\}. \quad (4.192)$$

*Suppose that  $y = y(x) \in C^1[0, \pi/2]$  and  $y(0) = 0$ . Then we have*

$$\int_0^{\pi/2} y'(x)^4 + 6vy(x)^2 y'(x)^2 - 3\lambda_0(v)y(x)^4 dx \geq 0. \quad (4.193)$$

*Proof.* We will treat this as a problem in the calculus of variations. There will not really be any ideas from number theory here, but for the sake of completeness we sketch a proof of this theorem. Note that in the case  $v = 0$  this is Theorem 256 in Hardy, Littlewood and Pólya's [38]. When we treat the case  $v > 0$ , we will be inspired by their proof and also of course by Hall. The basic plan is to solve the Euler equation and then to show that this solution gives a minimum (for the LHS) in (4.193).

By e.g. applying the Weierstrass approximation theorem to  $y'$ , one can go from considering continuously differentiable functions to considering smooth functions. We may thus assume below that  $y$  is smooth.

First note that since the problem is homogenous, we may suppose that  $y(x)$  takes the value 1 or 0 at  $x = \pi/2$ . We first suppose that  $y(\pi/2) = 1$ . Let

$$F(x, y, z) := z^4 + 6vy^2z^2 - 3\lambda y^4, \quad (4.194)$$

so that  $F(x, y, y') = y^4 + 6vy^2y'^2 - 3\lambda y^4$ . The Euler equation is given by

$$12y'^2y'' + 12vy^2y'' + 12vyy'^2 + 12\lambda y^3 = 0. \quad (4.195)$$

We multiply by  $y'/3$  and integrate to find that

$$y'^4 + 2vy^2y'^2 + \lambda y^4 = \text{Constant}. \quad (4.196)$$

When we search for a suitable extremal (solution to Euler's equation), we want a function satisfying  $\frac{\partial F}{\partial y'} = 0$  at  $x = \pi/2$  (from free end-point argument). This is equivalent to  $y'(\pi/2) = 0$ . Looking at (4.196) we are therefore led to study the function, call it  $y_0(x)$ , coming from choosing the Constant to be equal to  $\lambda$ . But we need that choice to give  $y_0(\pi/2) = 1$ . Fortunately, it does as we will show now. From our Euler equation we deduce that

$$y_0'^2 = -vy_0^2 + \sqrt{\lambda - (\lambda - v^2)y_0^4}. \quad (4.197)$$

We may take  $y_0'(0)$  to be positive. Then  $y_0'$  is positive until  $x = P$  say, where then  $y_0(P) = 1$ . Next, let  $F_2(u) : [0, \infty] \rightarrow [\lambda, \infty]$  be the increasing function

$$F_2 : u \mapsto u^4 + 2vu^2 + \lambda. \quad (4.198)$$

Let its inverse be  $\phi$ . Thus  $y_0'/y_0 = \phi(\lambda/y_0^4)$ . Then

$$\begin{aligned} P &= \int_0^1 \frac{1}{y_0'(x)} dy_0 = \int_0^1 \frac{1}{\phi(\lambda/y_0^4)} \frac{1}{y_0} dy_0 = [F_2(u) = \lambda/y_0^4] \\ &= \frac{1}{4} \int_0^\infty \frac{F_2'(u)}{uF_2(u)} du = \frac{1}{2} \int_{-\infty}^\infty \frac{u^2 + v}{u^4 + 2vu^2 + \lambda} du. \end{aligned} \quad (4.199)$$

Using standard complex analysis it is possible to show that if  $\lambda = \lambda_0(v)$  then the above expression for  $P$  equals  $\pi/2$ , as required. Note that the above also tells us that  $y_0'(x) > 0$  on  $[0, \pi/2)$ .

The next thing to do is to show that for our minimum candidate  $y_0$  we indeed get

equality in (4.193). We have

$$\begin{aligned}
\int_0^{\pi/2} y_0^4 dx &= \int_0^1 \frac{y_0^4}{y_0'} dy_0 = \int_0^1 \frac{y_0^4}{y_0 \phi(\lambda_0/y_0^4)} dy_0 = \frac{\lambda_0}{4} \int_0^\infty \frac{F_2'(u)}{u F_2^2(u)} du \\
&= \frac{\lambda_0}{4} \lim_{a \rightarrow 0} \int_a^\infty \frac{F_2'(u)}{u F_2^2(u)} du = \frac{\lambda_0}{4} \lim_{a \rightarrow 0} \left\{ \left[ -\frac{1}{u F_2(u)} \right]_a^\infty - \int_a^\infty \frac{1}{u^2 F_2(u)} du \right\} \\
&= \frac{1}{4} \lim_{a \rightarrow 0} \left\{ \frac{\lambda_0}{a F_2(a)} - \int_a^\infty \frac{1}{u^2} du + \int_a^\infty \frac{F_2(u) - \lambda_0}{u^2 F_2(u)} du \right\} \\
&= \frac{1}{4} \lim_{a \rightarrow 0} \left\{ \frac{\lambda_0}{a F_2(a)} - \frac{1}{a} \right\} + \frac{1}{4} \int_0^\infty \frac{F_2(u) - \lambda_0}{u^2 F_2(u)} du = \frac{1}{8} \int_{-\infty}^\infty \frac{F_2(u) - \lambda_0}{u^2 F_2(u)} du \\
&= \frac{1}{8} \int_{-\infty}^\infty \frac{u^2 + v}{u^4 + 2vu^2 + \lambda_0} du + \frac{1}{8} \int_{-\infty}^\infty \frac{v}{u^4 + 2vu^2 + \lambda_0} du = \frac{\pi}{8} + \frac{\pi v}{16\lambda_0}, \quad (4.200)
\end{aligned}$$

where the very last integral can be found using standard complex analysis. Similarly we have

$$\begin{aligned}
\int_0^{\pi/2} y_0^2 y_0' dx &= \int_0^1 y_0^3 \left( \frac{y_0'}{y_0} \right) dy_0 = \int_0^1 y_0^3 \phi(\lambda_0/y_0^4) dy_0 = \frac{\lambda_0}{4} \int_0^\infty \frac{u F_2'(u)}{F_2^2(u)} du \\
&= \frac{\lambda_0}{4} \left\{ \left[ -\frac{u}{F_2(u)} \right]_0^\infty + \int_0^\infty \frac{1}{F_2(u)} du \right\} = \frac{\lambda_0}{8} \int_{-\infty}^\infty \frac{1}{u^4 + 2vu^2 + \lambda_0} du = \frac{\pi}{16}. \quad (4.201)
\end{aligned}$$

When we evaluate the third integral we use the following relation (from the Euler equation):

$$y_0'^4 + 2vy_0^2 y_0'^2 + \lambda_0 y_0^4 = \lambda_0. \quad (4.202)$$

We find that

$$\int_0^{\pi/2} y_0^4 dx = \frac{\pi}{2} \cdot \lambda_0 - 2v(\pi/16) - \lambda_0 \left( \frac{\pi}{8} + \frac{\pi v}{16\lambda_0} \right) = \frac{3\pi\lambda_0}{8} - \frac{3\pi v}{16}. \quad (4.203)$$

Whence

$$\int_0^{\pi/2} F(x, y_0, y_0') dx = \left[ \frac{3\pi\lambda_0}{8} - \frac{3\pi v}{16} \right] + 6v \left[ \frac{\pi}{16} \right] - 3\lambda_0 \left[ \frac{\pi}{8} + \frac{\pi v}{16\lambda_0} \right] = 0. \quad (4.204)$$

We will complete the proof in this case by showing that  $y_0$  is a minimal solution to our problem, in the sense that for any admissible  $y$ , not identically equal to  $y_0$ , we

have that

$$\int_0^{\pi/2} F(x, y, y') dx > \int_0^{\pi/2} F(x, y_0, y'_0) dx. \quad (4.205)$$

To do this we will need to use general theory from calculus of variations. For this background theory the reader is referred to Chapter 6 of [4] (or Chapter VII of [38]). Below we will look at what happens in our specific case.

The idea is to build the field of extremals from the family  $\alpha y_0(x)$ , where  $\alpha \in \mathbb{R}$ . We may conclude that for any small  $\delta > 0$  we have that for any  $C^1$ -curve passing through  $(\delta, y_0(\delta))$  and  $(\pi/2, 1)$  we have

$$\int_{\delta}^{\pi/2} F(x, y, y') - F(x, y_0, y'_0) dx = \int_{\delta}^{\pi/2} E(x, y, p(x, y), y') dx, \quad (4.206)$$

where the Weierstrass excess-function is given by

$$E(x, y, p(x, y), y') = F(x, y, y') - F(x, y, p) - (y' - p)F_{y'}(x, y, p) \quad (4.207)$$

and where  $p(x, y)$  is the slope of the (unique) extremal passing through  $(x, y)$ . Since we are dealing with functions in  $C^1[0, \pi/2]$  we may let  $\delta \rightarrow 0$  and argue that (4.206) continues to hold with the lower limit of integration being 0. Thus

$$\int_0^{\pi/2} F(x, y, y') - F(x, y_0, y'_0) dx = \int_0^{\pi/2} E(x, y, p(x, y), y') dx. \quad (4.208)$$

In our special case

$$E(x, y, p(x, y), y') = (y' - p)^2[(y' + p)^2 + 2p^2 + 6vy^2]. \quad (4.209)$$

This clearly gives us (4.193) in this case, since  $E$  thus is non-negative. Moreover, equality in (4.193) holds precisely when  $E$  is identically zero on the interval  $[0, \pi/2]$ . A careful analysis yields that this occurs if and only if  $y = y_0$ .

Let us finally consider the case in which  $y(\pi/2) = 0$ . Now we embed our minimum-candidate  $y \equiv 0$  into the same field of extremals. By proceeding in a similar way as in the first case, we conclude that (4.193) holds (and equality holds in this case precisely when  $y \equiv 0$ ).  $\square$

**Theorem 4.3.** *Let  $v \geq 0$  and*

$$\lambda_0(v) = \frac{1}{8}\{1 + 4v + \sqrt{1 + 8v}\}. \quad (4.210)$$

Suppose that  $y = y(x) \in C^1[a, b]$  and  $y(a) = y(b) = 0$ . Then we have

$$\int_a^b \left(\frac{b-a}{\pi}\right)^4 y'(x)^4 + 6v \left(\frac{b-a}{\pi}\right)^2 y(x)^2 y'(x)^2 dx \geq \int_a^b 3\lambda_0(v) y(x)^4 dx. \quad (4.211)$$

*Proof.* We may assume that  $a = 0$  and  $b = \pi$ , since the general case then follows easily from a linear transformation.

The idea is to apply Theorem 4.2 twice, first to  $y(x)$  and then to  $y(\pi - x)$ . Adding the resulting two inequalities yields this theorem.  $\square$

**Remark 4.4.** Extending a function  $y_0(x)$  which satisfied equality in (4.193) by defining  $y_0(x) := y_0(\pi - x)$  on  $[\pi/2, \pi]$  does yield an admissible function which satisfies equality in (4.211) (when  $a = 0$  and  $b = \pi$ ).

# Chapter 5

## Using an amplifier in Hall's method

### 5.1 Initial discussion

Hall's method for finding large gaps between consecutive zeros of a function requires establishing a useful inequality and then estimating the involved integrals. In Chapter 3 we used second moment integrals and in Chapter 4 we used fourth moment integrals. The latter gave a better result, that is to say a greater value for the (relevant) lim sup. Hall establishes in [32, 33, 35] a continuation of this pattern, that using asymptotics<sup>1</sup> for

$$\int_0^T |Z(t)|^{2k-2h} |Z'(t)|^{2h} dt, \quad h = 0, 1, \dots, k \quad (5.1)$$

for bigger natural numbers  $k$  leads to longer gaps between consecutive zeros, on the critical line, of  $\zeta(s)$ .

As a side-note, let us briefly discuss one possible and naive thought as to why higher moments give better gap-results. Assume that (cf. (2.75)) for any fixed  $k \in \mathbb{N}$  we have that

$$\int_0^T |Z(t)|^{2k} dt \asymp T(\log T)^{k^2}. \quad (5.2)$$

Then elementary arguments yield that essentially all of the contribution to the integral in (5.2) for large integers  $k$  come from the relatively short subintervals of  $[0, T]$  where  $|Z(t)|$  is big. For example, denote the union of the subintervals of  $[0, T]$  in which  $|Z(t)| \geq (\log T)^{2.5}$  by  $I_{2.5}$  say. Then

$$|I_{2.5}| \cdot (\log T)^{15} \leq \int_0^T |Z(t)|^6 dt \ll T \log^9 T, \quad (5.3)$$

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<sup>1</sup>Hall predicts his integral-estimates via random matrix theory (see [43]).

so that  $|I_{2.5}| \ll T(\log T)^{-6}$ . Also, denote the subinterval of  $[0, T]$  in which  $|Z(t)| \leq (\log T)^{2.49}$  by  $J_{2.49}$  say. Then

$$\int_{J_{2.49}} |Z(t)|^6 dt \leq (\log T)^{4.98} \int_0^T |Z(t)|^4 dt \ll T(\log T)^{8.98}, \quad (5.4)$$

so that essentially all of the contribution to  $\int_0^T |Z(t)|^6 dt$  comes from integrating over the complement of  $J_{2.49}$  in  $[0, T]$ . Thus the contribution to  $\int_0^T |Z(t)|^{2k} dt$  comes, roughly speaking, only from the parts of  $[0, T]$  where  $|Z(t)|$  is unusually big. And if one thinks of  $Z(t)$  as an irregular “sinusoidal” function, it seems plausible that large gaps between consecutive zeros of  $Z(t)$  in  $[0, T]$  often go hand-in-hand with subintervals of  $[0, T]$  in which  $Z(t)$  is big in modulus. And since Hall’s method depends on integral-estimates, maybe large gaps more easily will be detected with Hall’s strategy if one considers higher moments.

Today sixth moment results for  $Z(t)$  have not been proved (cf. Section 2.4.1), so naturally we can not use these to prove gap-results. However, in this chapter an amplifier is used to show the following:

**Theorem 5.1.** *For any sufficiently large  $T$ , there exists a subinterval of  $[T, 2T]$  of length at least  $2.766 \times \frac{2\pi}{\log T}$ , in which the function  $t \mapsto \zeta(\frac{1}{2} + it)$  has no zeros.*

**Remark 5.1.** Notice that if we assume the RH, then Theorem 5.1 implies that  $\lambda \geq 2.766$ .

## 5.2 Building-stones in the proof of Theorem 5.1

### 5.2.1 Introducing the function $P(t, u, v, \kappa)$

**Definition 5.1.** Define

$$P(t, u, v, \kappa) = P(t, u, v, \kappa, T) := \exp(vi\theta(t))M(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T}), \quad (5.5)$$

where

$$\theta(t) = \operatorname{Im}\left(\log\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right)\right) - \frac{t}{2}\log\pi \quad (5.6)$$

and<sup>2</sup>

$$M(s) := \sum_{h \leq T^u} \frac{1}{h^s}, \quad (5.7)$$

with  $0 < u < 1/11$ .

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<sup>2</sup>For simplicity we write  $M(s)$  rather than say  $M_u(s)$ .

## 5.2.2 Main Assumption 5.1

We will now make an “assumption”.

**Main Assumption 5.1.** Suppose that all the gaps between consecutive zeros of the function  $t \mapsto P(t, u, v, \kappa)$  with  $t \in [T, 2T - \frac{\kappa}{\log T}]$  are<sup>3</sup> at most  $\frac{\kappa}{\log T}$ .

**Remark 5.2.** For a suitable choice of  $\kappa$ ,  $u$  and  $v$ , we will eventually prove that Main Assumption 5.1 leads to a contradiction.

## 5.2.3 Immediate consequences of Main Assumption 5.1

Denote the zeros of  $P(t, u, v, \kappa)$  with  $T \leq t \leq 2T - \frac{\kappa}{\log T}$  by  $t_1, t_2, \dots, t_N$ , ordered in non-decreasing order. Main Assumption 5.1 implies that

$$t_{n+1} - t_n \leq \frac{\kappa}{\log T}, \quad (5.8)$$

for  $n = 1, 2, \dots, N - 1$ .

**Remark 5.3.** For future need we note here that Main Assumption 5.1 implies that

$$t_1 \leq T + \frac{\kappa}{\log T} \text{ and } t_N \geq 2T - \frac{2\kappa}{\log T}. \quad (5.9)$$

## 5.2.4 Wirtinger’s inequality and an application of it

We begin with the statement of the simplest version of Wirtinger’s inequality.

**Theorem 5.2.** *Suppose that  $y(t)$  is a continuously differentiable function which satisfies  $y(0) = y(\pi) = 0$ . Then*

$$\int_0^\pi |y(t)|^2 dt \leq \int_0^\pi |y'(t)|^2 dt. \quad (5.10)$$

*Proof.* For the case when  $y(t)$  is a real-valued function, the reader is referred to Theorem 256 in [38].

Say now that  $y(t) = y_1(t) + iy_2(t)$ , with  $y_1$  and  $y_2$  thus being real-valued continuously differentiable functions. Then clearly  $y'(t) = y_1'(t) + iy_2'(t)$ . What we want to show is

$$\int_0^\pi y_1(t)^2 + y_2(t)^2 dt \leq \int_0^\pi y_1'(t)^2 + y_2'(t)^2 dt, \quad (5.11)$$

but this immediately follows from the known (real) case.  $\square$

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<sup>3</sup>For either of the two zeros near the endpoints of the interval, we will here mean the distance from them to the respective endpoint.

**Remark 5.4.** In Chapter 3 we did not need the above inequality for complex-valued functions, since  $Z(t)$  is real-valued for real  $t$ . The reason we worked with  $Z(t)$  rather than  $t \mapsto \zeta(1/2 + it)$  in Chapter 3 is simply that the former gives a better gap-result.

**Corollary 5.1.** *For  $n = 1, 2, \dots, N - 1$  we have*

$$\begin{aligned} \int_{t_n}^{t_{n+1}} |P(t, u, v, \kappa)|^2 dt &\leq \left( \frac{t_{n+1} - t_n}{\pi} \right)^2 \int_{t_n}^{t_{n+1}} |P'(t, u, v, \kappa)|^2 dt \\ &\leq \left( \frac{\kappa}{\pi \log T} \right)^2 \int_{t_n}^{t_{n+1}} |P'(t, u, v, \kappa)|^2 dt. \end{aligned} \quad (5.12)$$

*Proof.* One may make a linear substitution in Theorem 5.2 to obtain a similar result if the function  $y(t)$  has zeros at two general points  $a$  and  $b$ . We do so for the function  $P(t, u, v, \kappa)$ , which is continuously differentiable, and this gives us the first inequality. The latter inequality follows immediately from (5.8).  $\square$

Simply summing up the inequalities in Corollary 5.1 for  $n = 1, 2, \dots, N - 1$ , we obtain

$$\int_{t_1}^{t_N} |P(t, u, v, \kappa)|^2 dt \leq \left( \frac{\kappa}{\pi \log T} \right)^2 \int_{t_1}^{t_N} |P'(t, u, v, \kappa)|^2 dt. \quad (5.13)$$

### 5.2.5 Choosing weight-functions

**Definition 5.2.** With  $\eta > 0$  being any suitably small (fixed) constant, we define

$$h(x) := \begin{cases} \exp(-\eta T_0/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \quad (5.14)$$

with

$$T_0 = T^{1-\epsilon}. \quad (5.15)$$

Then clearly  $h(x)$  is  $C^\infty$  and  $h(x) \leq 1$ . Also,

$$h'(x) := \begin{cases} \eta T_0^{-1} (T_0/x)^2 \exp(-\eta T_0/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \quad (5.16)$$

which is seen to imply  $h'(x) \ll T_0^{-1}$ . And more generally one finds that  $h^{(j)}(x) \ll_j T_0^{-j}$ .

Now take

$$w_-(x) = h(x - T - T_0)h(2T - T_0 - x) \quad (5.17)$$

and

$$w_+(x) = \exp(2\eta)h(x - T + T_0)h(2T + T_0 - x). \quad (5.18)$$

Remembering Remark 5.3, it is easily seen that (5.13) implies

**Corollary 5.2.**

$$\int_{-\infty}^{\infty} w_-(t)|P(t, u, v, \kappa)|^2 dt \leq \left(\frac{\kappa}{\pi \log T}\right)^2 \int_{-\infty}^{\infty} w_+(t)|P'(t, u, v, \kappa)|^2 dt. \quad (5.19)$$

### 5.3 Going from Corollary 5.2 to Theorem 5.1

In this section we will write down asymptotic estimates for the LHS and the RHS in (5.19), and use these to obtain an inequality in terms of  $\kappa$ .

#### 5.3.1 Giving names to some integrals

Recall that  $P(t, u, v, \kappa) = \exp(vi\theta(t))M(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T})$ . Obviously

$$|P(t, u, v, \kappa)|^2 = |M(\frac{1}{2} + it)|^2 |\zeta(\frac{1}{2} + it)|^2 |\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T})|^2. \quad (5.20)$$

Next, using

$$M'(s) = - \sum_{h \leq T^u} \frac{\log h}{h^s} = N(s) - \log(T^u)M(s), \quad (5.21)$$

where

$$N(s) := \sum_{h \leq T^u} \frac{\log(T^u/h)}{h^s}, \quad (5.22)$$

we find that

$$\begin{aligned} \frac{P'(t, u, v, \kappa)}{i \exp(vi\theta(t))} &= \{v\theta'(t) - \log(T^u)\}M(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T}) \\ &+ N(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T}) \\ &+ M(\frac{1}{2} + it)\zeta'(\frac{1}{2} + it)\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T}) \\ &+ M(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)\zeta'(\frac{1}{2} + it + \frac{i\kappa}{\log T}). \end{aligned} \quad (5.23)$$

Thus

$$\begin{aligned}
|P'(t, u, v, \kappa)|^2 &= \{v\theta'(t) - \log(T^u)\}^2 |M(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \\
&+ 2\{v\theta'(t) - \log(T^u)\} \operatorname{Re} \left[ M(\tfrac{1}{2} + it) N(\tfrac{1}{2} - it) |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \right] \\
&+ |N(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \\
&+ 2\{v\theta'(t) - \log(T^u)\} \operatorname{Re} \left[ |M(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + it) \zeta(\tfrac{1}{2} - it) |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \right] \\
&+ 2\{v\theta'(t) - \log(T^u)\} \operatorname{Re} \left[ |M(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + it + \frac{i\kappa}{\log T}) \zeta(\tfrac{1}{2} - it - \frac{i\kappa}{\log T}) \right] \\
&+ |M(\tfrac{1}{2} + it)|^2 |\zeta'(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \\
&+ |M(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 |\zeta'(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \\
&+ 2\operatorname{Re} \left[ |M(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + it) \zeta(\tfrac{1}{2} - it) \zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T}) \zeta'(\tfrac{1}{2} - it - \frac{i\kappa}{\log T}) \right] \\
&+ 2\operatorname{Re} \left[ M(\tfrac{1}{2} + it) N(\tfrac{1}{2} - it) \zeta'(\tfrac{1}{2} + it) \zeta(\tfrac{1}{2} - it) |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \right] \\
&+ 2\operatorname{Re} \left[ M(\tfrac{1}{2} + it) N(\tfrac{1}{2} - it) |\zeta(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + it + \frac{i\kappa}{\log T}) \zeta(\tfrac{1}{2} - it - \frac{i\kappa}{\log T}) \right].
\end{aligned} \tag{5.24}$$

For future need we now introduce some notation. Let

$$A_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) |M(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 dt, \tag{5.25}$$

$$B_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) \operatorname{Re} \left[ M(\tfrac{1}{2} + it) N(\tfrac{1}{2} - it) |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \right] dt, \tag{5.26}$$

$$C_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) |N(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 dt, \tag{5.27}$$

$$D_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) \operatorname{Re} \left[ |M(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + it) \zeta(\tfrac{1}{2} - it) |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \right] dt, \tag{5.28}$$

$$E_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) \operatorname{Re} \left[ |M(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + it + \frac{i\kappa}{\log T}) \zeta(\tfrac{1}{2} - it - \frac{i\kappa}{\log T}) \right] dt, \tag{5.29}$$

$$F_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) |M(\tfrac{1}{2} + it)|^2 |\zeta'(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 dt, \tag{5.30}$$

$$G_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) |M(\tfrac{1}{2} + it)|^2 |\zeta(\tfrac{1}{2} + it)|^2 |\zeta'(\tfrac{1}{2} + it + \frac{i\kappa}{\log T})|^2 dt, \tag{5.31}$$

$$H_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) \operatorname{Re} \left[ |M(\frac{1}{2} + it)|^2 \zeta'(\frac{1}{2} + it) \zeta(\frac{1}{2} - it) \zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T}) \zeta'(\frac{1}{2} - it - \frac{i\kappa}{\log T}) \right] dt, \quad (5.32)$$

$$I_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) \operatorname{Re} \left[ M(\frac{1}{2} + it) N(\frac{1}{2} - it) \zeta'(\frac{1}{2} + it) \zeta(\frac{1}{2} - it) |\zeta(\frac{1}{2} + it + \frac{i\kappa}{\log T})|^2 \right] dt \quad (5.33)$$

and

$$J_{\pm} = \int_{-\infty}^{\infty} w_{\pm}(t) \operatorname{Re} \left[ M(\frac{1}{2} + it) N(\frac{1}{2} - it) |\zeta(\frac{1}{2} + it)|^2 \zeta'(\frac{1}{2} + it + \frac{i\kappa}{\log T}) \zeta(\frac{1}{2} - it - \frac{i\kappa}{\log T}) \right] dt. \quad (5.34)$$

### 5.3.2 Evaluation of the integrals defined in Section 5.3.1

The reader is referred to Section 5.4 for details on how to evaluate (asymptotically) the integrals (5.25)-(5.34). Below we give the answers.

$$A_{\pm} = A_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^9 T + O(T \log^8 T), \quad (5.35)$$

$$B_{\pm} = B_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{10} T + O(T \log^9 T), \quad (5.36)$$

$$C_{\pm} = C_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{11} T + O(T \log^{10} T), \quad (5.37)$$

$$D_{\pm} = D_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{10} T + O(T \log^9 T), \quad (5.38)$$

$$E_{\pm} = E_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{10} T + O(T \log^9 T), \quad (5.39)$$

$$F_{\pm} = F_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{11} T + O(T \log^{10} T), \quad (5.40)$$

$$G_{\pm} = G_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{11} T + O(T \log^{10} T), \quad (5.41)$$

$$H_{\pm} = H_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{11} T + O(T \log^{10} T), \quad (5.42)$$

$$I_{\pm} = I_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{11} T + O(T \log^{10} T) \quad (5.43)$$

and

$$J_{\pm} = J_{\kappa} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_{\pm}(t) dt \right] \cdot \log^{11} T + O(T \log^{10} T), \quad (5.44)$$

with

$$a_3 = \prod_p \left\{ \left( 1 + \frac{4}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p} \right)^4 \right\} \quad (5.45)$$

and where the coefficients are given by

$$\begin{aligned} A_\kappa = & \frac{(-10)}{\kappa^8} + \frac{(2u - 2u^2 + \frac{u^3}{3})}{\kappa^6} + \frac{(\frac{u^3}{3} - \frac{u^4}{4})}{\kappa^4} + \frac{8 \sin(\kappa u)}{\kappa^9} \\ & + \frac{(10 - 8u) \cos(\kappa u)}{\kappa^8} + \frac{(-4 + 10u - 4u^2) \sin(\kappa u)}{\kappa^7} + \frac{(2u - 3u^2 + u^3) \cos(\kappa u)}{\kappa^6} \\ & + \frac{(-8) \sin \kappa}{\kappa^9} + \frac{(\frac{-u^3}{3}) \cos \kappa}{\kappa^6} + \frac{8 \sin(\kappa(1-u))}{\kappa^9} + \frac{(8u) \cos(\kappa(1-u))}{\kappa^8} \\ & + \frac{(-4u^2) \sin(\kappa(1-u))}{\kappa^7} + \frac{(-u^3) \cos(\kappa(1-u))}{\kappa^6}, \end{aligned} \quad (5.46)$$

$$\begin{aligned} B_\kappa = & \frac{(1-2u)}{\kappa^8} + \frac{(\frac{-u^3}{3} + \frac{u^4}{6})}{\kappa^6} + \frac{(\frac{u^4}{8} - \frac{u^5}{12})}{\kappa^4} + \frac{(-1+2u) \cos(\kappa u)}{\kappa^8} \\ & + \frac{(-u + 2u^2 - \frac{u^3}{3}) \sin(\kappa u)}{\kappa^7} + \frac{(\frac{u^2}{2} - \frac{2u^3}{3} + \frac{u^4}{6}) \cos(\kappa u)}{\kappa^6} + \frac{(\frac{u^3}{3}) \sin \kappa}{\kappa^7} + \frac{(\frac{-u^4}{6}) \cos \kappa}{\kappa^6} \\ & + \frac{(\frac{-u^3}{3}) \sin(\kappa(1-u))}{\kappa^7} + \frac{(\frac{-u^4}{6}) \cos(\kappa(1-u))}{\kappa^6}, \end{aligned} \quad (5.47)$$

$$\begin{aligned} C_\kappa = & \frac{(-20)}{\kappa^{10}} + \frac{(2u - 2u^2)}{\kappa^8} + \frac{(\frac{-u^4}{6} + \frac{u^5}{15})}{\kappa^6} + \frac{(\frac{u^5}{20} - \frac{u^6}{36})}{\kappa^4} + \frac{12 \sin(\kappa u)}{\kappa^{11}} \\ & + \frac{(20 - 12u) \cos(\kappa u)}{\kappa^{10}} + \frac{(-6 + 20u - 6u^2) \sin(\kappa u)}{\kappa^9} + \frac{(4u - 8u^2 + 2u^3) \cos(\kappa u)}{\kappa^8} \\ & + \frac{(u^2 - \frac{4u^3}{3} + \frac{u^4}{3}) \sin(\kappa u)}{\kappa^7} + \frac{(-12) \sin \kappa}{\kappa^{11}} + \frac{(\frac{u^4}{6}) \sin \kappa}{\kappa^7} + \frac{(\frac{-u^5}{15}) \cos \kappa}{\kappa^6} \\ & + \frac{12 \sin(\kappa(1-u))}{\kappa^{11}} + \frac{(12u) \cos(\kappa(1-u))}{\kappa^{10}} + \frac{(-6u^2) \sin(\kappa(1-u))}{\kappa^9} \\ & + \frac{(-2u^3) \cos(\kappa(1-u))}{\kappa^8} + \frac{(\frac{u^4}{3}) \sin(\kappa(1-u))}{\kappa^7}, \end{aligned} \quad (5.48)$$

$$D_\kappa = -\frac{A_\kappa}{2}, \quad (5.49)$$

$$E_\kappa = -\frac{A_\kappa}{2}, \quad (5.50)$$

$$\begin{aligned}
F_\kappa = & \frac{(-66)}{\kappa^{10}} + \frac{(-\frac{11}{3} + 8u - 8u^2 + \frac{4u^3}{3})}{\kappa^8} + \frac{(\frac{2u}{3} - \frac{2u^2}{3} + \frac{5u^3}{6} - \frac{2u^4}{3} - \frac{u^5}{60})}{\kappa^6} \\
& + \frac{(\frac{u^3}{9} - \frac{u^4}{8} + \frac{u^5}{20} - \frac{u^6}{72})}{\kappa^4} + \frac{84 \sin(\kappa u)}{\kappa^{11}} + \frac{(66 - 84u) \cos(\kappa u)}{\kappa^{10}} \\
& + \frac{(-16 + 66u - 42u^2) \sin(\kappa u)}{\kappa^9} + \frac{(\frac{11}{3} + 8u - 25u^2 + \frac{38u^3}{3}) \cos(\kappa u)}{\kappa^8} \\
& + \frac{(-\frac{4}{3} + \frac{11u}{3} - \frac{11u^3}{3} + \frac{11u^4}{6}) \sin(\kappa u)}{\kappa^7} + \frac{(\frac{2u}{3} - \frac{7u^2}{6} + \frac{u^3}{2} + \frac{u^4}{12} - \frac{u^5}{12}) \cos(\kappa u)}{\kappa^6} \\
& + \frac{(-84) \sin \kappa}{\kappa^{11}} + \frac{26 \cos \kappa}{\kappa^{10}} + \frac{(-\frac{4u^3}{3}) \cos \kappa}{\kappa^8} + \frac{(-\frac{2u^3}{3} + \frac{u^4}{3}) \sin \kappa}{\kappa^7} + \frac{(-\frac{u^4}{12} + \frac{u^5}{60}) \cos \kappa}{\kappa^6} \\
& + \frac{84 \sin(\kappa(1-u))}{\kappa^{11}} + \frac{(-26 + 84u) \cos(\kappa(1-u))}{\kappa^{10}} + \frac{(26u - 42u^2) \sin(\kappa(1-u))}{\kappa^9} \\
& + \frac{(13u^2 - \frac{38u^3}{3}) \cos(\kappa(1-u))}{\kappa^8} + \frac{(-\frac{11u^3}{3} + \frac{11u^4}{6}) \sin(\kappa(1-u))}{\kappa^7} \\
& + \frac{(-\frac{u^4}{3} + \frac{u^5}{12}) \cos(\kappa(1-u))}{\kappa^6},
\end{aligned} \tag{5.51}$$

$$\begin{aligned}
G_\kappa = & \frac{(-148)}{\kappa^{10}} + \frac{(-\frac{14}{3} + 18u - 18u^2 + 3u^3)}{\kappa^8} + \frac{(\frac{2u}{3} - u^2 + \frac{11u^3}{6} - \frac{7u^4}{6})}{\kappa^6} + \frac{(\frac{u^3}{9} - \frac{u^4}{12})}{\kappa^4} \\
& + \frac{152 \sin(\kappa u)}{\kappa^{11}} + \frac{(148 - 152u) \cos(\kappa u)}{\kappa^{10}} + \frac{(-40 + 148u - 76u^2) \sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{14}{3} + 22u - 56u^2 + \frac{67u^3}{3}) \cos(\kappa u)}{\kappa^8} + \frac{(-\frac{4}{3} + \frac{14u}{3} + 2u^2 - \frac{26u^3}{3} + \frac{10u^4}{3}) \sin(\kappa u)}{\kappa^7} \\
& + \frac{(\frac{2u}{3} - \frac{4u^2}{3} + \frac{u^3}{2} + \frac{u^4}{3} - \frac{u^5}{6}) \cos(\kappa u)}{\kappa^6} + \frac{(-152) \sin \kappa}{\kappa^{11}} + \frac{36 \cos \kappa}{\kappa^{10}} + \frac{(-3u^3) \cos \kappa}{\kappa^8} \\
& + \frac{(-u^3) \sin \kappa}{\kappa^7} + \frac{152 \sin(\kappa(1-u))}{\kappa^{11}} + \frac{(-36 + 152u) \cos(\kappa(1-u))}{\kappa^{10}} \\
& + \frac{(36u - 76u^2) \sin(\kappa(1-u))}{\kappa^9} + \frac{(18u^2 - \frac{67u^3}{3}) \cos(\kappa(1-u))}{\kappa^8} \\
& + \frac{(-5u^3 + \frac{10u^4}{3}) \sin(\kappa(1-u))}{\kappa^7} + \frac{(-\frac{u^4}{2} + \frac{u^5}{6}) \cos(\kappa(1-u))}{\kappa^6},
\end{aligned} \tag{5.52}$$

$$\begin{aligned}
H_\kappa = & \frac{117}{\kappa^{10}} + \frac{(-\frac{5}{2} - 14u + 14u^2 - \frac{7u^3}{3})}{\kappa^8} + \frac{(\frac{u}{2} - \frac{u^2}{2} - \frac{7u^3}{6} + \frac{25u^4}{24})}{\kappa^6} + \frac{(\frac{u^3}{12} - \frac{u^4}{16})}{\kappa^4} \\
& + \frac{(-130) \sin(\kappa u)}{\kappa^{11}} + \frac{(-117 + 130u) \cos(\kappa u)}{\kappa^{10}} + \frac{(38 - 117u + 65u^2) \sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{5}{2} - 24u + \frac{89u^2}{2} - \frac{58u^3}{3}) \cos(\kappa u)}{\kappa^8} + \frac{(-1 + \frac{5u}{2} - 5u^2 + 7u^3 - \frac{17u^4}{6}) \sin(\kappa u)}{\kappa^7} \\
& + \frac{(\frac{u}{2} - \frac{3u^2}{4} + \frac{u^3}{2} - \frac{5u^4}{12} + \frac{u^5}{6}) \cos(\kappa u)}{\kappa^6} + \frac{130 \sin \kappa}{\kappa^{11}} + \frac{(-31) \cos \kappa}{\kappa^{10}} + \frac{(-4) \sin \kappa}{\kappa^9} \\
& + \frac{(\frac{7u^3}{3}) \cos \kappa}{\kappa^8} + \frac{(\frac{5u^3}{6} - \frac{u^4}{4}) \sin \kappa}{\kappa^7} + \frac{(-\frac{u^3}{6} + \frac{u^4}{24}) \cos \kappa}{\kappa^6} + \frac{(-130) \sin(\kappa(1-u))}{\kappa^{11}} \\
& + \frac{(31 - 130u) \cos(\kappa(1-u))}{\kappa^{10}} + \frac{(4 - 31u + 65u^2) \sin(\kappa(1-u))}{\kappa^9} \\
& + \frac{(4u - \frac{31u^2}{2} + \frac{58u^3}{3}) \cos(\kappa(1-u))}{\kappa^8} + \frac{(-2u^2 + \frac{13u^3}{3} - \frac{17u^4}{6}) \sin(\kappa(1-u))}{\kappa^7} \\
& + \frac{(-\frac{u^3}{2} + \frac{5u^4}{12} - \frac{u^5}{6}) \cos(\kappa(1-u))}{\kappa^6},
\end{aligned} \tag{5.53}$$

$$\begin{aligned}
I_\kappa = & \frac{(-35)}{\kappa^{10}} + \frac{(-\frac{1}{2} + 5u - 4u^2 + \frac{2u^3}{3})}{\kappa^8} + \frac{(\frac{5u^3}{12} - \frac{u^4}{4} + \frac{u^5}{40})}{\kappa^6} + \frac{(-\frac{u^4}{16} + \frac{u^5}{20} - \frac{u^6}{144})}{\kappa^4} \\
& + \frac{32 \sin(\kappa u)}{\kappa^{11}} + \frac{(35 - 32u) \cos(\kappa u)}{\kappa^{10}} + \frac{(-11 + 35u - 16u^2) \sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{1}{2} + 6u - \frac{27u^2}{2} + \frac{14u^3}{3}) \cos(\kappa u)}{\kappa^8} + \frac{(\frac{u}{2} + \frac{u^2}{2} - \frac{13u^3}{6} + \frac{3u^4}{4}) \sin(\kappa u)}{\kappa^7} \\
& + \frac{(-\frac{u^2}{4} + \frac{u^3}{4} + \frac{u^4}{24} - \frac{u^5}{24}) \cos(\kappa u)}{\kappa^6} + \frac{(-32) \sin \kappa}{\kappa^{11}} + \frac{5 \cos \kappa}{\kappa^{10}} + \frac{(-\frac{2u^3}{3}) \cos \kappa}{\kappa^8} \\
& + \frac{(-\frac{u^3}{3} - \frac{u^4}{12}) \sin \kappa}{\kappa^7} + \frac{(\frac{u^4}{8} - \frac{u^5}{40}) \cos \kappa}{\kappa^6} + \frac{32 \sin(\kappa(1-u))}{\kappa^{11}} + \frac{(-5 + 32u) \cos(\kappa(1-u))}{\kappa^{10}} \\
& + \frac{(5u - 16u^2) \sin(\kappa(1-u))}{\kappa^9} + \frac{(\frac{5u^2}{2} - \frac{14u^3}{3}) \cos(\kappa(1-u))}{\kappa^8} \\
& + \frac{(-\frac{u^3}{2} + \frac{3u^4}{4}) \sin(\kappa(1-u))}{\kappa^7} + \frac{(\frac{u^5}{24}) \cos(\kappa(1-u))}{\kappa^6}
\end{aligned} \tag{5.54}$$

and finally

$$\begin{aligned}
J_\kappa = & \frac{63}{\kappa^{10}} + \frac{(-\frac{1}{2} - 6u + 7u^2 - \frac{7u^3}{6})}{\kappa^8} + \frac{(-\frac{u^3}{4} + \frac{u^4}{8})}{\kappa^6} + \frac{(-\frac{u^4}{16} + \frac{u^5}{24})}{\kappa^4} \\
& + \frac{(-50) \sin(\kappa u)}{\kappa^{11}} + \frac{(-63 + 50u) \cos(\kappa u)}{\kappa^{10}} + \frac{(20 - 63u + 25u^2) \sin(\kappa u)}{\kappa^9} \\
& + \frac{(\frac{1}{2} - 14u + \frac{49u^2}{2} - \frac{43u^3}{6}) \cos(\kappa u)}{\kappa^8} + \frac{(\frac{u}{2} - 4u^2 + \frac{14u^3}{3} - \frac{7u^4}{6}) \sin(\kappa u)}{\kappa^7} \\
& + \frac{(-\frac{u^2}{4} + \frac{7u^3}{12} - \frac{5u^4}{12} + \frac{u^5}{12}) \cos(\kappa u)}{\kappa^6} + \frac{50 \sin \kappa}{\kappa^{11}} + \frac{(-5) \cos \kappa}{\kappa^{10}} + \frac{(\frac{7u^3}{6}) \cos \kappa}{\kappa^8} + \frac{(\frac{u^4}{4}) \sin \kappa}{\kappa^7} \\
& + \frac{(\frac{u^4}{24}) \cos \kappa}{\kappa^6} + \frac{(-50) \sin(\kappa(1-u))}{\kappa^{11}} + \frac{(5 - 50u) \cos(\kappa(1-u))}{\kappa^{10}} \\
& + \frac{(-5u + 25u^2) \sin(\kappa(1-u))}{\kappa^9} + \frac{(-\frac{5u^2}{2} + \frac{43u^3}{6}) \cos(\kappa(1-u))}{\kappa^8} \\
& + \frac{(\frac{5u^3}{6} - \frac{7u^4}{6}) \sin(\kappa(1-u))}{\kappa^7} + \frac{(\frac{u^4}{6} - \frac{u^5}{12}) \cos(\kappa(1-u))}{\kappa^6}.
\end{aligned} \tag{5.55}$$

**Remark 5.5.** If the above ten coefficients are seen as Laurent series in terms of  $\kappa$ , then numerical calculations show that all coefficients for negative  $\kappa$ -powers equal zero. This had to be the case since our expressions are analytic in  $\kappa$ . The latter can be seen from the fact that e.g. the LHS of (5.35) remains bounded if we let  $\kappa \rightarrow 0$ .

**Remark 5.6.** When<sup>4</sup> we put  $u = 1$ , the limit of  $A_\kappa$  as  $\kappa \rightarrow 0$  equals  $\frac{42}{9!}$ . If we let the weight-function  $w(t)$  be an approximation to the characteristic function on  $[T, 2T]$ , then (5.35) is seen to be consistent with the conjecture that

$$\int_T^{2T} |\zeta(\frac{1}{2} + it)|^6 dt \sim \frac{42}{9!} \cdot a_3 \cdot T \log^9 T \tag{5.56}$$

which of course is good news<sup>5</sup>.

### 5.3.3 Obtaining an inequality in terms of $\kappa$

It is now time to investigate both sides of Corollary 5.2. Focusing on the RHS, we are led to recall (5.24). Since  $w_+(t)$  is supported in  $[\frac{T}{2}, 4T]$  we may use that

$$\theta'(t) = \frac{\log T}{2} + O(1). \tag{5.57}$$

<sup>4</sup>Although the results in this chapter (via [44]) only are shown for  $u < 1/11$ , it may be that they hold for  $u < 1$ .

<sup>5</sup>However, putting  $u = 1/2$  and letting  $\kappa \rightarrow 0$  does not give half of the sixth power moment.

The contribution to the integral in the RHS of (5.19) coming from the error term in (5.57) can be seen to be  $\ll T \log^{10} T$ . To do this we simply use the Cauchy–Schwarz inequality

$$\left| \int f(x)g(x) dx \right|^2 \leq \int |f(x)|^2 dx \int |g(x)|^2 dx. \quad (5.58)$$

We can thus via (5.20) and (5.24) convert both sides of (5.19) into expressions involving the integrals in (5.25)-(5.34). The latter are of course evaluated using (5.35)-(5.44). Explicitly this procedure yields

$$\begin{aligned} & A_\kappa \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_-(t) dt \right] \cdot \log^9 T + O(T \log^8 T) \\ & \leq \left( \frac{\kappa}{\pi} \right)^2 \cdot \left\{ (v/2 - u)^2 A_\kappa + (v - 2u) B_\kappa + C_\kappa + (v - 2u) D_\kappa + (v - 2u) E_\kappa \right. \\ & \quad \left. + F_\kappa + G_\kappa + 2H_\kappa + 2I_\kappa + 2J_\kappa \right\} \cdot a_3 \cdot \left[ \int_{-\infty}^{\infty} w_+(t) dt \right] \cdot \log^9 T + O(T \log^8 T). \end{aligned} \quad (5.59)$$

Immediately from the definitions (5.17) and (5.18) one sees

$$w_-(t) \geq \exp(-2\eta) \chi_{[T+2T_0, 2T-2T_0]}(t) \quad (5.60)$$

and

$$w_+(t) \leq \exp(2\eta) \chi_{[T-T_0, 2T+T_0]}(t). \quad (5.61)$$

Therefore for sufficiently large  $T$  we have (recall (5.15))

$$\int_{-\infty}^{\infty} w_-(t) dt \geq C_-(\eta) T \quad (5.62)$$

and

$$\int_{-\infty}^{\infty} w_+(t) dt \leq C_+(\eta) T, \quad (5.63)$$

for some (fixed) constants  $C_\pm(\eta)$ , which can be chosen as close to 1 as we like.

Summarizing, we conclude that *if*

$$\begin{aligned} A_\kappa & > \left( \frac{\kappa}{\pi} \right)^2 \cdot \left\{ (v/2 - u)^2 A_\kappa + (v - 2u) B_\kappa + C_\kappa + (v - 2u) D_\kappa + (v - 2u) E_\kappa \right. \\ & \quad \left. + F_\kappa + G_\kappa + 2H_\kappa + 2I_\kappa + 2J_\kappa \right\}, \end{aligned} \quad (5.64)$$

then we have a contradiction to Main Assumption 5.1. That would imply the existence of a subinterval of  $[T, 2T - \frac{\kappa}{\log T}]$  of length at least  $\frac{\kappa}{\log T}$ , in which the function  $t \mapsto P(t, u, v, \kappa)$  has no zeros. A simple proof by contradiction shows that this implies that there must be a subinterval of  $[T, 2T]$  of length at least  $\frac{2\kappa}{\log T}$ , in which the function  $t \mapsto \zeta(\frac{1}{2} + it)$  has no zeros.

Using Mathematica, the inequality (5.64) is seen to hold with  $u = 0.0909$ ,  $v = 2.13$  and  $\kappa = 8.69$ . Thus Theorem 5.1 holds since  $2 \cdot 8.69 > 2.766 \cdot 2\pi$ .

**Remark 5.7.** If one studies the  $\kappa$ -inequality (5.64) as  $u \rightarrow 0$ , one sees that (5.64) is satisfied for  $\kappa = 8.264$  (with  $v = 2$ ), yielding gaps of length at least 2.63 times the average. This is effectively what Hall did in [34] (he did not use any amplifier).

**Remark 5.8.** For what it is worth, note that if the results (again via [44]) would remain valid for any  $u < 1/2$  (whether this is the case or not is unknown), then one could take  $u = 0.4999$ ,  $v = 2.68$  and  $\kappa = 10.23$  and see that (5.64) holds. This would imply the existence of gaps of length at least 3.25 times the average. Moreover,  $u = 0.55$  and  $v = 2.74$  would yield gaps of length at least 3.26 times the average and  $u = 0.9999$  and  $v = 3$  would yield gaps of length at least 3.05 times the average<sup>6</sup>.

**Remark 5.9.** As a side-note, it is likely that replacing  $M(\frac{1}{2} + it)$  in the definition of our function  $P(t, u, v, \kappa)$  in (5.5) by

$$\sum_{h \leq T^u} \frac{A + B \frac{\log(T^u/h)}{\log(T^u)}}{h^{1/2+it}}, \quad (5.65)$$

with some suitable choice of  $A$  and  $B$ , would have led to a slightly better gap-result. However, such calculations would be very long.

## 5.4 Evaluation of our integrals

### 5.4.1 On the article “The twisted fourth moment of the Riemann zeta function”

We will make use of the main theorem in the article “The twisted fourth moment of the Riemann zeta function”, written by Hughes and Young [44]. Before we reproduce their result, we must introduce a little bit of notation.

Define

$$A_{\alpha, \beta, \gamma, \delta}(s) = \frac{\zeta(1 + s + \alpha + \gamma)\zeta(1 + s + \alpha + \delta)\zeta(1 + s + \beta + \gamma)\zeta(1 + s + \beta + \delta)}{\zeta(2 + 2s + \alpha + \beta + \gamma + \delta)}. \quad (5.66)$$

Let

$$\sigma_{\alpha, \beta}(n) = \sum_{n_1 n_2 = n} n_1^{-\alpha} n_2^{-\beta}. \quad (5.67)$$

---

<sup>6</sup>Being unable to explain why using  $u = 1/2$  leads to bigger gaps than  $u = 1$ , let me just mention that this was also the case when I (admittedly on rough paper and using ratios conjectures) looked at how amplifying the second moment of the Riemann zeta-function improved Hall’s method for finding large gaps.

Next, suppose  $(h, k) = 1$ ,  $p^{h_p} || h$  and  $p^{k_p} || k$ , and define

$$B_{\alpha, \beta, \gamma, \delta, h, k}(s) = \prod_{p|h} \left( \frac{\sum_{j=0}^{\infty} \sigma_{\alpha, \beta}(p^j) \sigma_{\gamma, \delta}(p^{j+h_p}) p^{-j(s+1)}}{\sum_{j=0}^{\infty} \sigma_{\alpha, \beta}(p^j) \sigma_{\gamma, \delta}(p^j) p^{-j(s+1)}} \right) \quad (5.68)$$

$$\times \prod_{p|k} \left( \frac{\sum_{j=0}^{\infty} \sigma_{\alpha, \beta}(p^{j+k_p}) \sigma_{\gamma, \delta}(p^j) p^{-j(s+1)}}{\sum_{j=0}^{\infty} \sigma_{\alpha, \beta}(p^j) \sigma_{\gamma, \delta}(p^j) p^{-j(s+1)}} \right).$$

Then we write

$$Z_{\alpha, \beta, \gamma, \delta, h, k}(s) = A_{\alpha, \beta, \gamma, \delta}(s) B_{\alpha, \beta, \gamma, \delta, h, k}(s). \quad (5.69)$$

**Theorem 5.3** (Main theorem in [44]). *Let*

$$I(h, k) = \int_{-\infty}^{\infty} \left( \frac{h}{k} \right)^{-it} \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta + it\right) \zeta\left(\frac{1}{2} + \gamma - it\right) \zeta\left(\frac{1}{2} + \delta - it\right) w(t) dt, \quad (5.70)$$

where  $w(t)$  is a smooth, non-negative function with support contained in  $[\frac{T}{2}, 4T]$ , satisfying  $w^{(j)}(t) \ll_j T_0^{-j}$  for all  $j = 0, 1, 2, \dots$ , where  $T^{\frac{1}{2}+\epsilon} \ll T_0 \ll T$ . Suppose  $(h, k) = 1$ ,  $hk \leq T^{\frac{2}{11}-\epsilon}$ , and that  $\alpha, \beta, \gamma, \delta$  are complex numbers  $\ll (\log T)^{-1}$ . Then

$$I(h, k) = \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \left( Z_{\alpha, \beta, \gamma, \delta, h, k}(0) + \left( \frac{t}{2\pi} \right)^{-\alpha-\beta-\gamma-\delta} Z_{-\gamma, -\delta, -\alpha, -\beta, h, k}(0) \right. \quad (5.71)$$

$$+ \left( \frac{t}{2\pi} \right)^{-\alpha-\gamma} Z_{-\gamma, \beta, -\alpha, \delta, h, k}(0) + \left( \frac{t}{2\pi} \right)^{-\alpha-\delta} Z_{-\delta, \beta, \gamma, -\alpha, h, k}(0)$$

$$+ \left. \left( \frac{t}{2\pi} \right)^{-\beta-\gamma} Z_{\alpha, -\gamma, -\beta, \delta, h, k}(0) + \left( \frac{t}{2\pi} \right)^{-\beta-\delta} Z_{\alpha, -\delta, \gamma, -\beta, h, k}(0) \right) dt$$

$$+ O(T^{\frac{3}{4}+\epsilon} (hk)^{\frac{7}{8}} (T/T_0)^{\frac{9}{4}}).$$

**Brief comment on the proof:** The proof is very complicated and the reader is referred to [44]. What follows is just a very brief outline.

The starting point is an approximate functional equation. Let  $G(s)$  be an even, entire function of rapid decay as  $|s| \rightarrow \infty$  in any fixed strip  $|\operatorname{Re}(s)| \leq C$  and let

$$V_{\alpha, \beta, \gamma, \delta, t}(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} g_{\alpha, \beta, \gamma, \delta}(s, t) x^{-s} ds, \quad (5.72)$$

where

$$g_{\alpha, \beta, \gamma, \delta}(s, t) = \frac{\Gamma\left(\frac{\frac{1}{2} + \alpha + s + it}{2}\right) \Gamma\left(\frac{\frac{1}{2} + \beta + s + it}{2}\right) \Gamma\left(\frac{\frac{1}{2} + \gamma + s - it}{2}\right) \Gamma\left(\frac{\frac{1}{2} + \delta + s - it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2} + \alpha + it}{2}\right) \Gamma\left(\frac{\frac{1}{2} + \beta + it}{2}\right) \Gamma\left(\frac{\frac{1}{2} + \gamma - it}{2}\right) \Gamma\left(\frac{\frac{1}{2} + \delta - it}{2}\right)}. \quad (5.73)$$

Then

$$\begin{aligned}
& \zeta\left(\frac{1}{2} + \alpha + it\right)\zeta\left(\frac{1}{2} + \beta + it\right)\zeta\left(\frac{1}{2} + \gamma - it\right)\zeta\left(\frac{1}{2} + \delta - it\right) \\
&= \sum_{m,n \geq 1} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{-it} V_{\alpha,\beta,\gamma,\delta,t}(\pi^2 mn) \\
&\quad + X_{\alpha,\beta,\gamma,\delta,t} \sum_{m,n \geq 1} \frac{\sigma_{-\gamma,-\delta}(m)\sigma_{-\alpha,-\beta}(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{-it} V_{-\gamma,-\delta,-\alpha,-\beta,t}(\pi^2 mn) \\
&\quad + O((1 + |t|)^{-2007}),
\end{aligned} \tag{5.74}$$

where

$$X_{\alpha,\beta,\gamma,\delta,t} := \pi^{\alpha+\beta+\gamma+\delta} \frac{\Gamma\left(\frac{\frac{1}{2}-\alpha-it}{2}\right) \Gamma\left(\frac{\frac{1}{2}-\beta-it}{2}\right) \Gamma\left(\frac{\frac{1}{2}-\gamma+it}{2}\right) \Gamma\left(\frac{\frac{1}{2}-\delta+it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha+it}{2}\right) \Gamma\left(\frac{\frac{1}{2}+\beta+it}{2}\right) \Gamma\left(\frac{\frac{1}{2}+\gamma-it}{2}\right) \Gamma\left(\frac{\frac{1}{2}+\delta-it}{2}\right)}. \tag{5.75}$$

Using the approximate functional equation (5.74) in the definition of  $I(h, k)$  (see (5.70)) yields

$$\begin{aligned}
I(h, k) &= \sum_{m,n \geq 1} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^{1/2}} \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} V_{\alpha,\beta,\gamma,\delta,t}(\pi^2 mn) w(t) dt \\
&\quad + \sum_{m,n \geq 1} \left\{ \frac{\sigma_{-\gamma,-\delta}(m)\sigma_{-\alpha,-\beta}(n)}{(mn)^{1/2}} \right. \\
&\quad \left. \times \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} X_{\alpha,\beta,\gamma,\delta,t} V_{-\gamma,-\delta,-\alpha,-\beta,t}(\pi^2 mn) w(t) dt \right\} + O(1).
\end{aligned} \tag{5.76}$$

The two main terms in (5.76) can be treated similarly. Denoting the first one by  $I^{(1)}(h, k)$ , then upon opening up the integral formula for  $V$ , one finds that

$$\begin{aligned}
I^{(1)}(h, k) &= \sum_{m,n \geq 1} \left\{ \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^{1/2}} \right. \\
&\quad \left. \times \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} (\pi^2 mn)^{-s} \int_{-\infty}^{\infty} \left(\frac{hm}{kn}\right)^{-it} g_{\alpha,\beta,\gamma,\delta}(s, t) w(t) dt ds \right\}.
\end{aligned} \tag{5.77}$$

The authors of [44] split the sum in (5.77) into the diagonal part corresponding to the terms for which  $hm = kn$  and the non-diagonal part (the other terms). Whereas it is relatively simple to treat the diagonal contribution, it was a nice achievement to be able to treat the more complicated off-diagonal terms (results from the article [16] by Duke, Friedlander and Iwaniec are used and the latter part of the proof of Theorem 5.3 involves a lot of simplifying).

**Remark 5.10.** We will always use the choice  $T_0 = T^{1-\epsilon}$  and the practice of letting  $\epsilon$  stand for a small positive number, not necessarily always the same.

**Remark 5.11.** It is not immediately obvious that the main term in (5.71) is an analytic function in terms of the shifts (e.g.  $Z_{\alpha,\beta,\gamma,\delta}(0)$  has singularities at  $\alpha = -\gamma$ ,  $\alpha = -\delta$ ,  $\beta = -\gamma$ ,  $\beta = -\delta$ ), however, due to nice cancellation, analyticity holds. Lemma 2.5.1 in the article [10] by Conrey, Farmer, Keating, Rubinstein and Snaith is very helpful when showing this. In Section 5.4.4 we will carry out the details in a similar situation.

### 5.4.2 Initial step in using Theorem 5.3

Suppose that  $w(t)$  is a function satisfying the assumptions in Theorem 5.3, given the choice  $T_0 = T^{1-\epsilon}$ . With

$$M_1(s) = \sum_{h \leq T^u} \frac{a_1(h)}{h^s} \quad (5.78)$$

and

$$M_2(s) = \sum_{k \leq T^u} \frac{a_2(k)}{k^s}, \quad (5.79)$$

where  $0 < u < 1/11$  and the  $a_i$ -coefficients are real, we will want to asymptotically evaluate expressions such as

$$\int_{-\infty}^{\infty} M_1(\tfrac{1}{2}+it)M_2(\tfrac{1}{2}-it)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta+it)\zeta(\tfrac{1}{2}+\gamma-it)\zeta(\tfrac{1}{2}+\delta-it)w(t) dt. \quad (5.80)$$

By expanding out  $M_1$  and  $M_2$  we obtain

$$\sum_{h,k \leq T^u} \frac{a_1(h)a_2(k)}{\sqrt{hk}} \int_{-\infty}^{\infty} \left(\frac{h}{k}\right)^{-it} \zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta+it)\zeta(\tfrac{1}{2}+\gamma-it)\zeta(\tfrac{1}{2}+\delta-it)w(t) dt. \quad (5.81)$$

Let us write (5.71) as

$$I(h, k) = J(h, k) + E(h, k). \quad (5.82)$$

Then (5.81) equals

$$\begin{aligned} & \sum_{h,k \leq T^u} \frac{a_1(h)a_2(k)}{\sqrt{hk}} I(h, k) \\ &= \sum_{m \leq T^u} \frac{1}{m} \sum_{\substack{h,k \leq T^u/m \\ (h,k)=1}} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} I(h, k) \\ &= \sum_{m \leq T^u} \frac{1}{m} \sum_{\substack{h,k \leq T^u/m \\ (h,k)=1}} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} J(h, k) + \sum_{m \leq T^u} \frac{1}{m} \sum_{\substack{h,k \leq T^u/m \\ (h,k)=1}} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} E(h, k). \end{aligned} \quad (5.83)$$

The first term in (5.83) clearly equals

$$\begin{aligned}
& \sum_{m \leq T^u} \frac{1}{m} \sum_{h,k \leq T^u/m} \frac{a_1(hm)a_2(km)}{\sqrt{hk}} J(h,k) \sum_{d|(h,k)} \mu(d) \\
&= \sum_{m \leq T^u} \frac{1}{m} \sum_{d \leq T^u/m} \frac{\mu(d)}{d} \sum_{h,k \leq T^u/md} \frac{a_1(hmd)a_2(kmd)}{\sqrt{hk}} J(hd, kd) \\
&= \sum_{d \leq T^u} \frac{\mu(d)}{d} \sum_{m \leq T^u/d} \frac{1}{m} \sum_{h,k \leq T^u/md} \frac{a_1(hmd)a_2(kmd)}{\sqrt{hk}} J(hd, kd). \tag{5.84}
\end{aligned}$$

The second term in (5.83) is

$$\ll T^{\frac{3}{4}+\epsilon} (T/T_0)^{\frac{9}{4}} \sum_{m \leq T^u} \left\{ \frac{1}{m} \sum_{h \leq T^u/m} |a_1(hm)| h^{3/8} \sum_{k \leq T^u/m} |a_2(km)| k^{3/8} \right\}. \tag{5.85}$$

### 5.4.3 Specialising on the most standard case

Let us now investigate the expression in (5.80) when  $a_1(h) = a_2(k) = 1$  for all values of  $h$  and  $k$ . Our goal is to simplify the main term (which will turn out to be of order  $T \log^9 T$ ) as much as possible, treating anything which is  $\ll T \log^8 T$  as an error term.

As noticed in Section 5.4.2, (5.80) splits up into a main term (5.84) and an error term (5.85). The latter is

$$\ll T^{\frac{3}{4}+\epsilon} (T/T_0)^{\frac{9}{4}} \sum_{m \leq T^u} \left\{ \frac{1}{m} \cdot (T^u/m)^{11/8} \cdot (T^u/m)^{11/8} \right\} \ll T^{\frac{3}{4}+\epsilon} (T)^{\frac{9\epsilon}{4}} T^{\frac{11u}{4}} \ll T, \tag{5.86}$$

recalling Remark 5.10 for the second step and the last step being true upon choosing  $\epsilon$  to be sufficiently small. The main term (5.84) is

$$\begin{aligned}
& \sum_{d \leq T^u} \frac{\mu(d)}{d} \sum_{m \leq T^u/d} \frac{1}{m} \sum_{h,k \leq T^u/md} \frac{J(hd, kd)}{\sqrt{hk}} \\
&= \sum_{d \leq T^u} \frac{\mu(d)}{d^2} \sum_{m \leq T^u/d} \frac{1}{m} \int_{-\infty}^{\infty} w(t) \sum_{h,k \leq T^u/md} \frac{1}{hk} \{ \dots \} dt, \tag{5.87}
\end{aligned}$$

where the expression  $\{ \dots \}$  in (5.87) stands for

$$\begin{aligned}
& Z_{\alpha, \beta, \gamma, \delta, hd, kd}(0) + \left( \frac{t}{2\pi} \right)^{-\alpha-\beta-\gamma-\delta} Z_{-\gamma, -\delta, -\alpha, -\beta, hd, kd}(0) \\
&+ \left( \frac{t}{2\pi} \right)^{-\alpha-\gamma} Z_{-\gamma, \beta, -\alpha, \delta, hd, kd}(0) + \left( \frac{t}{2\pi} \right)^{-\alpha-\delta} Z_{-\delta, \beta, \gamma, -\alpha, hd, kd}(0) \\
&+ \left( \frac{t}{2\pi} \right)^{-\beta-\gamma} Z_{\alpha, -\gamma, -\beta, \delta, hd, kd}(0) + \left( \frac{t}{2\pi} \right)^{-\beta-\delta} Z_{\alpha, -\delta, \gamma, -\beta, hd, kd}(0). \tag{5.88}
\end{aligned}$$

#### 5.4.4 Studying $\mathcal{Q}_{A,B}(T_1, T_2, f_1, f_2)$

As we shall see in the Sections 5.4.5 and 5.4.6, it is possible to simplify our main term (5.87) considerably. However, we first need to introduce and become familiar with a bit of new notation.

Suppose that  $f_1(x_1, x_2, T_2)$  and  $f_2(x_1, x_2, T_2)$  are functions which are analytic and symmetric in their complex variables  $x_1$  and  $x_2$ . Also, let  $A$  and  $B$  be sets of complex numbers with  $|A| = |B| = 2$  and write them as

$$A := \{\alpha_1, \alpha_2\} \quad (5.89)$$

and

$$B := \{\alpha_3, \alpha_4\}. \quad (5.90)$$

We then define

$$\mathcal{Q}_{A,B}(T_1, T_2, f_1, f_2) := \sum_{\substack{R \subset A \\ S \subset B \\ |R|=|S|}} \mathcal{Q}((A \setminus R) \cup (-S), (B \setminus S) \cup (-R), T_1, T_2, f_1, f_2), \quad (5.91)$$

where we by  $-U$  mean  $\{-u : u \in U\}$  and

$$\mathcal{Q}(X, Y, T_1, T_2, f_1, f_2) := T_1^{(\delta_X + \delta_Y)/2} F_1(X, T_2) F_2(Y, T_2) \prod_{\substack{x \in X \\ y \in Y}} \frac{1}{(x+y)}, \quad (5.92)$$

with

$$\delta_X := \sum_{x \in X} x \quad (5.93)$$

and where for  $X = \{x_1, x_2\}$  we let

$$F_i(X, T_2) := f_i(x_1, x_2, T_2), \quad i = 1, 2. \quad (5.94)$$

Next let  $\Xi$  denote the set of  $\binom{4}{2}$  permutations  $\sigma \in S_4$  satisfying  $\sigma(1) < \sigma(2)$  and  $\sigma(3) < \sigma(4)$ . Let us for  $\sigma \in \Xi$  define

$$\begin{aligned} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}) &= K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}, T_1, T_2, f_1, f_2) \\ &:= \mathcal{Q}(\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}\}, \{-\alpha_{\sigma(3)}, -\alpha_{\sigma(4)}\}, T_1, T_2, f_1, f_2). \end{aligned} \quad (5.95)$$

Then one has that

$$\sum_{\sigma \in \Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}) = \mathcal{Q}_{A,-B}(T_1, T_2, f_1, f_2). \quad (5.96)$$

We now show that the LHS of (5.96) is an analytic function of the  $\alpha$ -shifts. The problem is where any of the relevant  $K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)})$  has a singularity. However, we will now show that these singularities must be removable. Suppose that

$$\alpha_i \neq \alpha_j, \text{ for } i \neq j. \quad (5.97)$$

From Lemma 2.5.1 in [10] we then have the following formula:

$$\begin{aligned} & \sum_{\sigma \in \Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)}) \\ &= \frac{1}{(2!)^2} \frac{1}{(2\pi i)^4} \int \dots \int \frac{K(z_1, z_2; z_3, z_4) \Delta(z_1, \dots, z_4)^2}{\prod_{i=1}^4 \prod_{j=1}^4 (z_i - \alpha_j)} dz_1 \dots dz_4, \end{aligned} \quad (5.98)$$

where

$$\Delta(z_1, \dots, z_4) := \prod_{1 \leq i < j \leq 4} (z_j - z_i), \quad (5.99)$$

and where one integrates about circles enclosing the  $\alpha_i$ 's. By choosing the radii of the circles to be suitably large<sup>7</sup>, we obtain an upper bound for the RHS of (5.98). The function  $\sum_{\sigma \in \Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}; \alpha_{\sigma(3)}, \alpha_{\sigma(4)})$  thus remains bounded whenever (5.97) is satisfied. This allows us to conclude<sup>8</sup> that the possible singularities must be removable. Hence  $\mathcal{Q}_{A,-B}(T_1, T_2, f_1, f_2)$  is analytic by (5.96), which in turn implies that  $\mathcal{Q}_{A,B}(T_1, T_2, f_1, f_2)$  is an analytic function of the shifts.

### 5.4.5 Initial simplification of Theorem 5.3 in the most standard case

**Theorem 5.4.** *Suppose that  $w(t)$  is a smooth, non-negative function with support contained in  $[\frac{T}{2}, 4T]$  satisfying  $w^{(j)}(t) \ll_j (T^{1-\epsilon})^{-j}$  for all  $j = 0, 1, 2, \dots$ , that  $\alpha, \beta, \gamma, \delta$  are complex numbers  $\ll (\log T)^{-1}$  and let*

$$M(s) = \sum_{h \leq T^u} \frac{1}{h^s}, \quad (5.100)$$

with  $0 < u < 1/11$ . Then

$$\begin{aligned} & \int_{-\infty}^{\infty} |M(\tfrac{1}{2} + it)|^2 \zeta(\tfrac{1}{2} + \alpha + it) \zeta(\tfrac{1}{2} + \beta + it) \zeta(\tfrac{1}{2} + \gamma - it) \zeta(\tfrac{1}{2} + \delta - it) w(t) dt \\ &= \left[ \int_{-\infty}^{\infty} w(t) dt \right] \cdot a_3 \cdot T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot \sum_{m \leq T^u} \frac{1}{m} \mathcal{Q}_{A,B}(T, T^u/m, f, f) + O(T \log^8 T), \end{aligned} \quad (5.101)$$

<sup>7</sup>If analyticity is to be shown for say  $|\alpha_i| \leq C$ , then we can pick the radii of the circles to be  $3C$  since then  $|z_i - \alpha_j|^{-1} \leq 1/C$ .

<sup>8</sup>By Riemann's Extension Theorem — see for example [29].

with

$$a_3 = \prod_p \left\{ \left( 1 + \frac{4}{p} + \frac{1}{p^2} \right) \left( 1 - \frac{1}{p} \right)^4 \right\} \quad (5.102)$$

and where we are here taking (see also definition given in (5.92))

$$A := \{\alpha, \beta\}, \quad (5.103)$$

$$B := \{\gamma, \delta\} \quad (5.104)$$

and<sup>9</sup>

$$f(x_1, x_2, T_2) := \frac{1}{x_1 x_2} - \frac{T_2^{-x_1}}{x_1(x_2 - x_1)} - \frac{T_2^{-x_2}}{x_2(x_1 - x_2)}. \quad (5.105)$$

**Remark 5.12.** It is easy to show that  $f(x_1, x_2, T_2)$  is analytic in  $x_1$  and  $x_2$ . Assume (without loss of generality) that  $|x_1|, |x_2| \leq C$  say. Consider

$$\frac{1}{2\pi i} \int_R \frac{T_2^s}{s(s+x_1)(s+x_2)} ds, \quad (5.106)$$

with  $R$  denoting a counter-clockwise integral-contour around a square with vertices at  $\pm 2C \pm 2Ci$ . Suppose that  $x_1$  and  $x_2$  are different and non-zero. Then (5.106) equals (5.105) by Cauchy's Residue Theorem, and the integral in (5.106) is obviously bounded. It follows that any possible singularities of  $f(x_1, x_2, T_2)$  must be removable.

**Remark 5.13.** By Section 5.4.4 we thus know that  $\mathcal{Q}_{A,B}(T_1, T_2, f, f)$  is analytic in terms of the shifts. Hence the main term in the RHS of (5.101) is analytic in the shifts.

*Proof.* We will first proceed under the assumption that for some fixed constant  $C > 0$  we have that

$$|\alpha_i| \geq C/\log T, \quad |\alpha_i + \alpha_j| \geq C/\log T \quad \text{and} \quad |\alpha_i - \alpha_j| \geq C/\log T, \quad (5.107)$$

where  $\alpha_i$  and  $\alpha_j$  (with  $i$  and  $j$  distinct) stand for any of the shifts.

Let us begin by noticing that there is an obvious identification to be made between the terms in (5.87) and in the main term of (5.101) (indeed both expressions involve the same number of terms, namely six)<sup>10</sup>. We shall prove Theorem 5.4 under the assumption (5.107) by treating each term in (5.87) individually. Below we will focus

<sup>9</sup>When comparing (5.103) with (5.89), the reader will probably find it easiest to just identify  $\alpha$  and  $\beta$  with  $\alpha_1$  and  $\alpha_2$  respectively. Note however that due to symmetry, it does not matter if the roles are reversed.

<sup>10</sup>The reader may find it helpful to factor out  $(\frac{t}{2\pi})^{-(\alpha+\beta+\gamma+\delta)/2}$  in (5.88).

on the third term in (5.87), which will correspond (recalling (5.91)) to the case  $R = \{\alpha\}$  and  $S = \{\gamma\}$  in (5.101). The other terms can be treated analogously.

Our starting point will thus be given by

$$\left\{ \int_{-\infty}^{\infty} w(t) \left( \frac{t}{2\pi} \right)^{-\alpha-\gamma} dt \right\} \sum_{d \leq T^u} \frac{\mu(d)}{d^2} \sum_{m \leq T^u/d} \frac{1}{m} \sum_{h,k \leq T^u/md} \frac{Z_{-\gamma,\beta,-\alpha,\delta,hd,kd}(0)}{hk}. \quad (5.108)$$

Let us write (recall (5.69))

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(0) = A_{\alpha,\beta,\gamma,\delta}(0) B_{\alpha,\beta,\gamma,\delta}(h) E_{\alpha,\beta,\gamma,\delta}(k), \quad (5.109)$$

with

$$B_{\alpha,\beta,\gamma,\delta}(h) := \prod_{p|h} \left( \frac{\sum_{j=0}^{\infty} \sigma_{\alpha,\beta}(p^j) \sigma_{\gamma,\delta}(p^{j+h_p}) p^{-j}}{\sum_{j=0}^{\infty} \sigma_{\alpha,\beta}(p^j) \sigma_{\gamma,\delta}(p^j) p^{-j}} \right) \quad (5.110)$$

and

$$E_{\alpha,\beta,\gamma,\delta}(k) := \prod_{p|k} \left( \frac{\sum_{j=0}^{\infty} \sigma_{\alpha,\beta}(p^{j+k_p}) \sigma_{\gamma,\delta}(p^j) p^{-j}}{\sum_{j=0}^{\infty} \sigma_{\alpha,\beta}(p^j) \sigma_{\gamma,\delta}(p^j) p^{-j}} \right). \quad (5.111)$$

Using this notation, (5.108) becomes

$$\left\{ \int_{-\infty}^{\infty} w(t) \left( \frac{t}{2\pi} \right)^{-\alpha-\gamma} dt \right\} A_{-\gamma,\beta,-\alpha,\delta}(0) \times \sum_{d \leq T^u} \frac{\mu(d)}{d^2} \sum_{m \leq T^u/d} \left\{ \frac{1}{m} \sum_{h \leq T^u/md} \frac{B_{-\gamma,\beta,-\alpha,\delta}(hd)}{h} \sum_{k \leq T^u/md} \frac{E_{-\gamma,\beta,-\alpha,\delta}(kd)}{k} \right\}. \quad (5.112)$$

It would be desirable to have more information about the two innermost sums in (5.112). They are similar. Let us study  $\sum_{h \leq T^u/md} \frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h}$ . First note that

$$B_{\alpha,\beta,\gamma,\delta}(w) = \prod_{p|w} B_{\alpha,\beta,\gamma,\delta}(p^{w_p}) \ll \prod_{p|w} w_p p^{\epsilon w_p} \ll \prod_{p|w} p^{\epsilon w_p} \cdot p^{\epsilon w_p} = w^{2\epsilon}. \quad (5.113)$$

Now define for  $\text{Re}(s) > 1$ ,

$$F(s, \alpha, \beta, \gamma, \delta, d) := \sum_{h=1}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h^s}. \quad (5.114)$$

Suppose for the time being that  $\frac{T^u}{md}$  is a half-integer, say  $\frac{T^u}{md} = M + \frac{1}{2}$  for some positive integer  $M$ . We claim that then

$$\sum_{h \leq T^u/md} \frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h} = \frac{1}{2\pi i} \int_{\epsilon-iW}^{\epsilon+iW} \frac{F(1+s, \alpha, \beta, \gamma, \delta, d) (T^u/md)^s}{s} ds + O(d^\epsilon), \quad (5.115)$$

with (somewhat arbitrary choice)

$$W = 10 \times (T^u/md)^{1.1}. \quad (5.116)$$

It is easy to show (this is one version of Perron's formula) that

$$\frac{1}{2\pi i} \int_{\epsilon-iW}^{\epsilon+iW} \frac{x^s}{s} ds = H(x) + O\left(\frac{x^\epsilon}{W|\log x|}\right), \quad (5.117)$$

for  $x > 0$ ,  $x \neq 1$ , where

$$H(x) = \begin{cases} 1 & \text{if } x > 1, \\ 0 & \text{if } x < 1. \end{cases} \quad (5.118)$$

Expanding out  $F(1+s, \alpha, \beta, \gamma, \delta, d)$  in (5.115), we thus get

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\epsilon-iW}^{\epsilon+iW} \frac{F(1+s, \alpha, \beta, \gamma, \delta, d)(T^u/md)^s}{s} ds \\ &= \sum_{h \leq T^u/md} \frac{B_{\alpha, \beta, \gamma, \delta}(hd)}{h} + O\left(\sum_{h=1}^{\infty} \frac{|B_{\alpha, \beta, \gamma, \delta}(hd)|}{h} \cdot \frac{(T^u/mdh)^\epsilon}{W|\log(T^u/mdh)|}\right). \end{aligned} \quad (5.119)$$

If  $h \notin [\frac{T^u}{2md}, \frac{3T^u}{2md}]$ , then  $|\log(T^u/mdh)|^{-1} \ll 1$ . The part of the error in (5.119) corresponding to such values of  $h$  is thus

$$\ll \sum_{h=1}^{\infty} \frac{(hd)^{\epsilon/2}}{h} \cdot \frac{(T^u/mdh)^\epsilon}{W} \ll d^\epsilon. \quad (5.120)$$

For  $M + \frac{1}{2} = \frac{T^u}{md} < h \leq \frac{3T^u}{2md}$ , we write

$$h = M + R, \quad R = 1, \dots, \left[\frac{3T^u}{2md}\right] - M \quad (5.121)$$

and spot that here

$$|\log(T^u/mdh)|^{-1} \ll \frac{M}{R}, \quad (5.122)$$

so that the part of the error in (5.119) corresponding to these values of  $h$  is certainly

$$\ll \frac{d^\epsilon}{W} \sum_{R=1}^M \frac{M}{R} \ll \frac{d^\epsilon}{W} \cdot M \log(M+1) \ll d^\epsilon. \quad (5.123)$$

A very similar argument applies when  $\frac{T^u}{2md} \leq h \leq \frac{T^u}{md}$ , which concludes the proof of the claim in the case when  $\frac{T^u}{md}$  is a half-integer. If this is not the case, then certainly for some  $0 < \mu < 1$  we have that  $\frac{T^u}{md} + \mu$  is a half-integer. We obtain<sup>11</sup>

$$\begin{aligned} & \sum_{h \leq T^u/md} \frac{B_{\alpha, \beta, \gamma, \delta}(hd)}{h} = \sum_{h \leq T^u/md + \mu} \frac{B_{\alpha, \beta, \gamma, \delta}(hd)}{h} + O(d^\epsilon) \\ &= \frac{1}{2\pi i} \int_{\epsilon-iW}^{\epsilon+iW} \frac{F(1+s, \alpha, \beta, \gamma, \delta, d)(T^u/md + \mu)^s}{s} ds + O(d^\epsilon). \end{aligned} \quad (5.124)$$

<sup>11</sup>By looking at the derivation of (5.115), it is obvious that introducing  $\mu$  does not necessitate a change in our choice of  $W$ .

We are thus led to investigating the RHS of (5.124). By contemplating the definition of  $B_{\alpha,\beta,\gamma,\delta}(hd)$ , one concludes (see (5.128)-(5.130)) that one may write

$$F(s, \alpha, \beta, \gamma, \delta, d) =: \zeta(s + \gamma)\zeta(s + \delta)G(s, \alpha, \beta, \gamma, \delta, d), \quad (5.125)$$

with  $G(s, \alpha, \beta, \gamma, \delta, d)$  being an analytic function for  $\operatorname{Re}(s) > 1/2$ . Let us in the following discussion restrict ourselves to when  $0.9 \leq \operatorname{Re}(s) \leq 1.1$ . We have

$$\begin{aligned} G(s, \alpha, \beta, \gamma, \delta, d) &= \prod_{p|d} \{(1 - p^{-s-\gamma})(1 - p^{-s-\delta})\} \sum_{\substack{h=1 \\ p|h \Rightarrow p|d}}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(hd)}{h^s} \\ &\times \prod_{p \nmid d} \left\{ (1 - p^{-s-\gamma})(1 - p^{-s-\delta}) \sum_{M=0}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(p^M)}{p^{Ms}} \right\}. \end{aligned} \quad (5.126)$$

Let us write this as

$$G(s, \alpha, \beta, \gamma, \delta, d) = G_1(s, \alpha, \beta, \gamma, \delta, d) \times G_2(s, \alpha, \beta, \gamma, \delta, d). \quad (5.127)$$

We notice that (uniformly)

$$B_{\alpha,\beta,\gamma,\delta}(1) = 1, \quad (5.128)$$

$$B_{\alpha,\beta,\gamma,\delta}(p) = p^{-\gamma} + p^{-\delta} + O(p^{-0.9+2\epsilon}) \quad (5.129)$$

and

$$B_{\alpha,\beta,\gamma,\delta}(p^N) = O((N+1)p^{N\epsilon}), \quad N \geq 2. \quad (5.130)$$

Thus

$$\sum_{M=0}^{\infty} \frac{B_{\alpha,\beta,\gamma,\delta}(p^M)}{p^{Ms}} = 1 + p^{-s-\gamma} + p^{-s-\delta} + O(p^{-1.8+2\epsilon}) \quad (5.131)$$

and hence

$$G_2(s, \alpha, \beta, \gamma, \delta, d) = \prod_{p \nmid d} (1 + O(p^{-1.8+2\epsilon})). \quad (5.132)$$

In particular we obtain

$$G_2(s, \alpha, \beta, \gamma, \delta, d) \ll 1. \quad (5.133)$$

The main thing to notice in the above argument is that the terms of order (roughly)  $p^{-\operatorname{Re}(s)}$  exactly cancel in the factors of  $G_2(s, \alpha, \beta, \gamma, \delta, d)$ . Keeping this in mind, one can show

$$\frac{\partial G_2(s, \alpha, \beta, \gamma, \delta, d)}{\partial x} \ll \sum_{p \nmid d} \log p \cdot p^{-1.8+2\epsilon} \cdot 1 \ll 1, \quad (5.134)$$

where  $x$  here shall mean any of  $\alpha, \beta, \gamma, \delta, s$ .

Now we study  $G_1(s, \alpha, \beta, \gamma, \delta, d)$ . First of all,

$$G_{11}(s, \alpha, \beta, \gamma, \delta, d) := \prod_{p|d} \{(1 - p^{-s-\gamma})(1 - p^{-s-\delta})\} \ll \prod_{p|d} \{2 \cdot 2\} \ll d^\epsilon. \quad (5.135)$$

Also, logarithmic differentiation yields

$$\frac{\partial G_{11}(s, \alpha, \beta, \gamma, \delta, d)}{\partial x} \ll d^\epsilon. \quad (5.136)$$

Next, we study

$$G_{12}(s, \alpha, \beta, \gamma, \delta, d) := \sum_{\substack{h=1 \\ p|h \Rightarrow p|d}}^{\infty} \frac{B_{\alpha, \beta, \gamma, \delta}(hd)}{h^s}. \quad (5.137)$$

Recalling (5.113) we find

$$\begin{aligned} G_{12}(s, \alpha, \beta, \gamma, \delta, d) &\ll d^\epsilon \sum_{\substack{h=1 \\ p|h \Rightarrow p|d}}^{\infty} h^{-0.9+\epsilon} \ll d^\epsilon \prod_{p|d} \sum_{j=0}^{\infty} p^{j(-0.9+\epsilon)} \\ &\ll d^\epsilon \prod_{p|d} (1 + O(p^{-0.9+\epsilon})) \ll d^\epsilon. \end{aligned} \quad (5.138)$$

Also, although the details are somewhat more delicate, one can in a straightforward direct way do a similar upper bound calculation as above in order to deduce that

$$\frac{\partial G_{12}(s, \alpha, \beta, \gamma, \delta, d)}{\partial x} \ll d^\epsilon. \quad (5.139)$$

Putting things together we have

$$G(s, \alpha, \beta, \gamma, \delta, d) \ll d^\epsilon \quad (5.140)$$

and

$$\frac{\partial G(s, \alpha, \beta, \gamma, \delta, d)}{\partial x} \ll d^\epsilon. \quad (5.141)$$

Now it is time to go back to (5.124) which tells us that

$$\begin{aligned} &\sum_{h \leq T^u/md} \frac{B_{-\gamma, \beta, -\alpha, \delta}(hd)}{h} \\ &= \frac{1}{2\pi i} \int_{\epsilon-iW}^{\epsilon+iW} \frac{\zeta(1-\alpha+s)\zeta(1+\delta+s)(T^u/md+\mu)^s G(1+s, -\gamma, \beta, -\alpha, \delta, d)}{s} ds \\ &+ O(d^\epsilon). \end{aligned} \quad (5.142)$$

Using a rectangular path, we move the line of integration from  $\operatorname{Re}(s) = \epsilon$  to  $\operatorname{Re}(s) = -0.05$ . The contribution along any of the two horizontal line-segments is

$$\ll \int_{-0.05}^{\epsilon} \frac{(W^{0.1 \times \frac{1}{6} + \epsilon})^2 (T^u/md + \mu)^\epsilon d^\epsilon}{W} dx \ll d^\epsilon. \quad (5.143)$$

And the contribution along the new vertical line-segment is

$$\begin{aligned}
\int_{-0.05-iW}^{-0.05+iW} &= \int_{-0.05-iW}^{-0.05-2i} + \int_{-0.05-2i}^{-0.05+2i} + \int_{-0.05+2i}^{-0.05+iW} \\
&\ll 2 \int_2^W \frac{(t^{0.1 \times \frac{1}{6} + \epsilon})^2 (T^u/md + \mu)^{-0.05} d^\epsilon}{t} dt + \int_{-2}^2 \frac{(T^u/md + \mu)^{-0.05} d^\epsilon}{0.05} dt \\
&\ll (T^u/md)^{-0.05} d^\epsilon \left\{ \int_2^W t^{-\frac{29}{30} + \epsilon} dt + 1 \right\} \\
&\ll d^\epsilon.
\end{aligned} \tag{5.144}$$

By Cauchy's Residue Theorem

$$\sum_{h \leq T^u/md} \frac{B_{-\gamma, \beta, -\alpha, \delta}(hd)}{h} = \text{Res}_{s=0} + \text{Res}_{s=\alpha} + \text{Res}_{s=-\delta} + O(d^\epsilon), \tag{5.145}$$

where of course we are referring to the residues of the integrand in (5.142).

Finally we have (almost) got all the puzzle-pieces needed to conclude the proof under the assumption (5.107). We consider the various factors in (5.112). First of all

$$\begin{aligned}
\int_{-\infty}^{\infty} w(t) \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} dt &= T^{-\alpha-\gamma} \int_{T/2}^{4T} w(t) \exp\left\{(-\alpha-\gamma) \log\left(\frac{t}{2\pi T}\right)\right\} dt \\
&= T^{-\alpha-\gamma} \int_{T/2}^{4T} w(t) \left\{1 + O\left(\frac{1}{\log T}\right)\right\} dt \\
&= T^{-\alpha-\gamma} \int_{-\infty}^{\infty} w(t) dt + O\left(1 \cdot \int_{T/2}^{4T} \frac{|w(t)|}{\log T} dt\right) \\
&= T^{-\alpha-\gamma} \int_{-\infty}^{\infty} w(t) dt + O\left(\frac{T}{\log T}\right) \\
&= T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot T^{(\beta-\gamma+\delta-\alpha)/2} \cdot \int_{-\infty}^{\infty} w(t) dt + O\left(\frac{T}{\log T}\right),
\end{aligned} \tag{5.146}$$

where we for future need also remark that the main term in (5.146) trivially is  $\ll T$ .

Secondly,

$$\begin{aligned}
A_{-\gamma, \beta, -\alpha, \delta}(0) &= \frac{\zeta(1-\gamma-\alpha)\zeta(1-\gamma+\delta)\zeta(1+\beta-\alpha)\zeta(1+\beta+\delta)}{\zeta(2-\gamma+\beta-\alpha+\delta)} \\
&= \frac{1}{\zeta(2)} \cdot \frac{1}{(-\gamma-\alpha)} \cdot \frac{1}{(-\gamma+\delta)} \cdot \frac{1}{(\beta-\alpha)} \cdot \frac{1}{(\beta+\delta)} + O(\log^3 T),
\end{aligned} \tag{5.147}$$

where we remark that the main term in (5.147) obviously is of order  $\log^4 T$ .

Thirdly, the  $m$ -summands in (5.112) contain two sums like in (5.145). Each of those sums will be evaluated by using (5.145) and hence give rise to residues. We will

now explain how to treat the residue at  $s = \alpha$  in (5.145). Explicitly the residue is

$$\frac{\zeta(1 + \delta + \alpha)(T^u/md + \mu)^\alpha G(1 + \alpha, -\gamma, \beta, -\alpha, \delta, d)}{\alpha}. \quad (5.148)$$

Repeatedly using the Fundamental Theorem of Calculus

$$F(\gamma(b)) - F(\gamma(a)) = \int_{\gamma} F'(z) dz, \quad (5.149)$$

we obtain, recalling (5.141), that

$$G(1 + \alpha, -\gamma, \beta, -\alpha, \delta, d) = G(1, 0, 0, 0, 0, d) + O\left(\frac{d^\epsilon}{\log T}\right). \quad (5.150)$$

We easily deduce that (5.148) is

$$\frac{(T^u/md)^\alpha G(1, 0, 0, 0, 0, d)}{(\delta + \alpha)\alpha} + O(d^\epsilon \log T), \quad (5.151)$$

where clearly the main term in (5.151) is

$$\ll d^\epsilon \log^2 T. \quad (5.152)$$

The other five residues are treated similarly. Then we put in our new expressions for  $\sum_{h \leq T^u/md} \frac{B_{-\gamma, \beta, -\alpha, \delta}(hd)}{h}$  and  $\sum_{k \leq T^u/md} \frac{E_{-\gamma, \beta, -\alpha, \delta}(kd)}{k}$  into (5.112). Since the outer sum over  $d$  in (5.112) always will be convergent (due to the presence of  $d^2$  in the denominator), an inspection yields two things. First of all that if we ever choose to take the error-part of any of the discussed expressions, we will in (5.112) end up with something that is  $\ll T \log^8 T$  and which thus can be relegated to the error term in (5.101). And secondly that we may replace the  $(T^u/md)^\alpha$  in (5.151) by  $(T^u/m)^\alpha$  and change the range of summation over  $m$  in (5.112) from  $m \leq T^u/d$  to  $m \leq T^u$ , since the change introduced by these is  $\ll T \log^8 T$ .

All-in-all we get that (5.112) equals

$$\left[ \int_{-\infty}^{\infty} w(t) dt \right] \cdot \frac{1}{\zeta(2)} \cdot \sum_{d \leq T^u} \frac{\mu(d) G(1, 0, 0, 0, 0, d)^2}{d^2} \quad (5.153)$$

$$\cdot T^{-(\alpha + \beta + \gamma + \delta)/2} \cdot \sum_{m \leq T^u} \frac{1}{m} \mathcal{Q}(\{\beta, -\gamma\}, \{\delta, -\alpha\}, T, T^u/m, f, f) + O(T \log^8 T).$$

We may extend the finite sum over  $d$  in (5.153) to an infinite one, since

$$\sum_{d > T^u} \frac{\mu(d) G(1, 0, 0, 0, 0, d)^2}{d^2} \ll \sum_{d > T^u} \frac{1 \cdot d^{2\epsilon}}{d^2} \ll \frac{1}{\log T}. \quad (5.154)$$

It then remains to establish the identity

$$\frac{1}{\zeta(2)} \sum_{d=1}^{\infty} \frac{\mu(d)G(1, 0, 0, 0, 0, d)^2}{d^2} = \prod_p \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^4 \right\}. \quad (5.155)$$

By an explicit calculation

$$G(1, 0, 0, 0, 0, 1) = \prod_p \left\{ (1 - p^{-1})^2 \sum_{M=0}^{\infty} \frac{B_{0,0,0,0}(p^M)}{p^M} \right\} = \prod_p \left\{ \frac{(1 - p^{-1})(1 + 2p^{-1})}{(1 + p^{-1})} \right\}. \quad (5.156)$$

It is natural to define

$$g(1, 0, 0, 0, 0, d) := G(1, 0, 0, 0, 0, d)/G(1, 0, 0, 0, 0, 1) \quad (5.157)$$

and next we show that  $g(1, 0, 0, 0, 0, d)$  is a multiplicative function. Let  $D_1, D_2 \in \mathbb{N}$  be relatively prime. Recall (5.127) and note that  $B_{0,0,0,0}(n)$  is a multiplicative function. By expanding out the terms involved in the relation

$$g(1, 0, 0, 0, 0, D_1) \cdot g(1, 0, 0, 0, 0, D_2) = g(1, 0, 0, 0, 0, D_1 D_2), \quad (5.158)$$

this equality is, after cancellation, seen to be equivalent to

$$\sum_{\substack{h=1 \\ p|h \Rightarrow p|D_1}}^{\infty} \frac{B_{0,0,0,0}(hD_1)}{h} \cdot \sum_{\substack{k=1 \\ p|k \Rightarrow p|D_2}}^{\infty} \frac{B_{0,0,0,0}(kD_2)}{k} = \sum_{\substack{m=1 \\ p|m \Rightarrow p|D_1 D_2}}^{\infty} \frac{B_{0,0,0,0}(mD_1 D_2)}{m}. \quad (5.159)$$

This identity can be seen to hold by equalling denominators (use multiplicativity of the function  $B_{0,0,0,0}(n)$ ).

Therefore the LHS of (5.155) equals

$$\frac{G(1, 0, 0, 0, 0, 1)^2}{\zeta(2)} \sum_{d=1}^{\infty} \frac{\mu(d)g(1, 0, 0, 0, 0, d)^2}{d^2}. \quad (5.160)$$

Using multiplicativity we are led to studying

$$\sum_{M=0}^{\infty} \frac{\mu(p^M)g(1, 0, 0, 0, 0, p^M)^2}{p^{2M}} = 1 - \frac{g(1, 0, 0, 0, 0, p)^2}{p^2}. \quad (5.161)$$

An explicit calculation gives that

$$g(1, 0, 0, 0, 0, p) = \frac{2 + p^{-1}}{1 + 2p^{-1}}. \quad (5.162)$$

Upon using this, the result in (5.155) follows, since

$$\frac{1}{\zeta(2)} = \prod_p (1 - p^{-2}) \quad (5.163)$$

and this completes the proof of Theorem 5.4 under the assumption of (5.107).

Let us write  $\alpha = \alpha_1$ ,  $\beta = \alpha_2$ ,  $\gamma = \alpha_3$  and  $\delta = \alpha_4$ . Suppose that Theorem 5.4 is to be proved for all  $|\alpha_i| \leq C/\log T$ . Consider now (without the extra assumption (5.107)) any  $|\alpha_i| \leq C/\log T$ . The idea is to use Cauchy's integral formula in order to go from the previous "easier" case to the general case.

Both the LHS and the main term in the RHS of Theorem 5.4 are analytic functions of the complex variables  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$  (recall Remarks 5.11 and 5.13), let us denote them by  $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  and  $R(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  respectively. Let  $D$  be the polydisc defined as the Cartesian product of the open discs  $D_i$ , i.e.  $D = \prod_{i=1}^4 D_i$ , where

$$D_i := \{s \in \mathbb{C} : |s - \alpha_i| < r_i\}, \quad (5.164)$$

with

$$r_i = \frac{2^{i+1}C}{\log T}. \quad (5.165)$$

An application of Cauchy's integral formula yields that

$$\begin{aligned} & L(\alpha_1, \alpha_2, \alpha_3, \alpha_4) - R(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \frac{1}{(2\pi i)^4} \int \cdots \int \int_{\partial D_1 \times \cdots \times \partial D_4} \frac{L(\beta_1, \beta_2, \beta_3, \beta_4) - R(\beta_1, \beta_2, \beta_3, \beta_4)}{(\beta_1 - \alpha_1) \cdots (\beta_4 - \alpha_4)} d\beta_1 \cdots d\beta_4. \end{aligned} \quad (5.166)$$

Now we notice that the  $\beta_i$  satisfy  $|\beta_i - \alpha_i| = r_i$ , which is easily seen to imply that  $\beta_i \ll 1/\log T$  and that

$$|\beta_i| \geq 2C/\log T, \quad |\beta_i + \beta_j| \geq 2C/\log T \quad \text{and} \quad |\beta_i - \beta_j| \geq 2C/\log T. \quad (5.167)$$

This theorem thus applies if the  $\beta_i$ -terms are seen as shifts, so that we have

$$L(\beta_1, \beta_2, \beta_3, \beta_4) - R(\beta_1, \beta_2, \beta_3, \beta_4) \ll T \log^8 T. \quad (5.168)$$

By using (5.168) and considering the trivial upper bound for (5.166), the latter is  $\ll T \log^8 T$ . This finally completes the proof of Theorem 5.4.  $\square$

### 5.4.6 Further simplification of Theorem 5.3 in the most standard case

Let us finish the discussion of how to evaluate the integral in (5.25). We take<sup>12</sup>

$$\{\alpha, \beta, \gamma, \delta\} = \left(\frac{i}{\log T}\right) \{\kappa + \lambda, -\lambda, -\lambda, -\kappa + \lambda\} \quad (5.169)$$

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<sup>12</sup>We will later let  $\lambda \rightarrow 0$ .

and apply Theorem 5.4. One obtains an expression involving sums over  $m$  (see (5.101)). We now show that such sums can be dealt with by using<sup>13</sup>

$$\sum_{m \leq T^u} \frac{(T^u/m)^\alpha}{m} \approx \frac{(T^{\alpha u} - 1)}{\alpha}, \quad (5.170)$$

that is to say that after having done these replacements in all of the sums in the RHS of Theorem 5.4, the arising version of (5.101) is true.

We first prove this claim under the extra condition (5.107). By partial summation we have for  $t \ll T$  that

$$\sum_{m \leq t} \frac{1}{m^{1+\alpha}} = [t] \cdot t^{-(1+\alpha)} + (1 + \alpha) \int_1^t [w] \cdot w^{-2-\alpha} dw = \frac{(1 - t^{-\alpha})}{\alpha} + O(1). \quad (5.171)$$

It follows immediately that

$$\sum_{m \leq T^u} \frac{(T^u/m)^\alpha}{m} = T^{\alpha u} \sum_{m \leq T^u} \frac{1}{m^{1+\alpha}} = \frac{(T^{\alpha u} - 1)}{\alpha} + O(1). \quad (5.172)$$

Using (5.172) and trivial estimates, we reach our conclusion (i.e. the error-parts in (5.172) can be absorbed into the error term in (5.101)).

In order to retrieve the general case from this special case, we work exactly as in the proof of Theorem 5.4. However, in order to apply Cauchy's integral formula we must first show analyticity of the new RHS of (5.101). To this end, we apply Lemma 2.5.1 in [10]. Ensuring that the conditions for applying the latter are met here essentially boils down to checking that what one gets after using<sup>14</sup> (5.170) on

$$\begin{aligned} & \sum_{m \leq T^u} \frac{f(x_1, x_2, T^u/m) f(x_3, x_4, T^u/m)}{m} \\ &= \sum_{m \leq T^u} \left\{ \frac{1}{m} \left( \frac{1}{x_1 x_2} - \frac{(T^u/m)^{-x_1}}{x_1(x_2 - x_1)} - \frac{(T^u/m)^{-x_2}}{x_2(x_1 - x_2)} \right) \right. \\ & \quad \left. \times \left( \frac{1}{x_3 x_4} - \frac{(T^u/m)^{-x_3}}{x_3(x_4 - x_3)} - \frac{(T^u/m)^{-x_4}}{x_4(x_3 - x_4)} \right) \right\} \end{aligned} \quad (5.173)$$

is an analytic function in terms of shifts  $x_1, x_2, x_3$  and  $x_4$ . Explicitly one ends up

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<sup>13</sup>To make sense of the RHS in the formula in the case  $\alpha = 0$ , use the  $\alpha^0$ -coefficient in the Taylor series.

<sup>14</sup>Read this as doing the relevant replacements.

with

$$\begin{aligned}
& \frac{\log(T^u)}{x_1 x_2 x_3 x_4} + \frac{(T^{-u x_3} - 1)}{x_1 x_2 x_3^2 (x_4 - x_3)} + \frac{(T^{-u x_4} - 1)}{x_1 x_2 x_4^2 (x_3 - x_4)} \\
& + \frac{(T^{-u x_1} - 1)}{x_1^2 (x_2 - x_1) x_3 x_4} + \frac{(T^{-u x_2} - 1)}{x_2^2 (x_1 - x_2) x_3 x_4} \\
& - \frac{(T^{-u(x_1+x_3)} - 1)}{x_1 x_3 (x_2 - x_1)(x_4 - x_3)(x_1 + x_3)} - \frac{(T^{-u(x_1+x_4)} - 1)}{x_1 x_4 (x_2 - x_1)(x_3 - x_4)(x_1 + x_4)} \\
& - \frac{(T^{-u(x_2+x_3)} - 1)}{x_2 x_3 (x_1 - x_2)(x_4 - x_3)(x_2 + x_3)} - \frac{(T^{-u(x_2+x_4)} - 1)}{x_2 x_4 (x_1 - x_2)(x_3 - x_4)(x_2 + x_4)}, \quad (5.174)
\end{aligned}$$

which admittedly does not look too pleasant at first sight. However, there is a clever and natural strategy to employ to realise why (5.174) has to be analytic in  $x_1, x_2, x_3$  and  $x_4$ .

Consider<sup>15</sup>

$$\Psi(x_1, x_2, x_3, x_4) := \frac{1}{(2\pi i)^2} \int_{R_2} \int_{R_1} \frac{(T^{u(s+w)} - 1)}{(s+x_1)(s+x_2)s(w+x_3)(w+x_4)w(s+w)} ds dw, \quad (5.175)$$

where  $R_1$  and  $R_2$  denote counter-clockwise rectangular paths, with vertices at  $\pm 2 \pm 2i$  and  $\pm 1 \pm i$  respectively. Whenever

$$x_i \neq 0 \quad \text{and} \quad x_i \neq \pm x_j \quad (5.176)$$

is satisfied, one notices that  $\Psi(x_1, x_2, x_3, x_4)$  equals (5.174), by carefully applying Cauchy's Residue Theorem twice. Also, trivial estimates give that  $\Psi(x_1, x_2, x_3, x_4)$  is bounded. Therefore we may conclude that all the possible singularities of (5.174) are removable.

In order to evaluate (5.25) we thus use Theorem 5.4 and proceed as explained above by using (5.170). We then substitute in (5.169) and view our answer as a Laurent series in terms of  $\lambda$ . Since the LHS of (5.101) remains bounded as  $\lambda \rightarrow 0$ , we must have cancellation so that our Laurent series actually is a Taylor series. Since we are letting  $\lambda \rightarrow 0$  anyway, what all this means is that in practice one focuses term-wise on finding just the  $\lambda^0$ -coefficients<sup>16</sup>.

Doing this gives us an answer in terms of  $\kappa$ . By seeing various symmetries in the calculations one can both save time and simplify the answer<sup>17</sup>. For example in

<sup>15</sup>To arrive at (5.175), essentially one first uses Remark 5.12 on both the expressions in round brackets in (5.173), then moves the sum over  $m$  inside the double-integral and finally applies (5.170).

<sup>16</sup>Here the use of Mathematica was helpful.

<sup>17</sup>On a related note, by Remark 5.5 one could focus on finding just the analytic part of each term. Although this would save time, one advantage of keeping track of the negative  $\kappa$ -powers is that if all their coefficients cancel in the (total) answer, then it is "likely" that one has not made any calculation-errors!

the present integral-calculation one can spot that the contributions from the terms originating from the first and second term in (5.88) are complex conjugates<sup>18</sup>. This will mean that via use of Euler's formula

$$\exp(ix) = \cos x + i \sin x, \quad (5.177)$$

one obtains nice trigonometric terms in the answer (see (5.46)).

### 5.4.7 Simplified versions of Theorem 5.3 in two other cases

Let us recall the notation

$$M(s) = \sum_{h \leq T^u} \frac{1}{h^s} \quad (5.178)$$

and

$$N(s) = \sum_{h \leq T^u} \frac{\log(T^u/h)}{h^s}. \quad (5.179)$$

Let us also recall the notation

$$f(x_1, x_2, T_2) = \frac{1}{x_1 x_2} - \frac{T_2^{-x_1}}{x_1(x_2 - x_1)} - \frac{T_2^{-x_2}}{x_2(x_1 - x_2)} \quad (5.180)$$

and define

$$g(x_1, x_2, T_2) := \frac{\log T_2}{x_1 x_2} - \frac{1}{x_1^2 x_2} - \frac{1}{x_2^2 x_1} + \frac{T_2^{-x_1}}{x_1^2(x_2 - x_1)} + \frac{T_2^{-x_2}}{x_2^2(x_1 - x_2)}. \quad (5.181)$$

The following two theorems can be proved very similarly to Theorem 5.4.

**Theorem 5.5.** *With the same assumptions as in Theorem 5.4, we have*

$$\begin{aligned} & \int_{-\infty}^{\infty} M\left(\frac{1}{2} + it\right) N\left(\frac{1}{2} - it\right) \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta + it\right) \zeta\left(\frac{1}{2} + \gamma - it\right) \zeta\left(\frac{1}{2} + \delta - it\right) w(t) dt \\ &= \left[ \int_{-\infty}^{\infty} w(t) dt \right] \cdot a_3 \cdot T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot \sum_{m \leq T^u} \frac{1}{m} \mathcal{Q}_{A,B}(T, T^u/m, g, f) + O(T \log^9 T), \end{aligned} \quad (5.182)$$

where we still use (5.103) and (5.104).

**Theorem 5.6.** *With the same assumptions as in Theorem 5.4, we have*

$$\begin{aligned} & \int_{-\infty}^{\infty} |N\left(\frac{1}{2} + it\right)|^2 \zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta + it\right) \zeta\left(\frac{1}{2} + \gamma - it\right) \zeta\left(\frac{1}{2} + \delta - it\right) w(t) dt \\ &= \left[ \int_{-\infty}^{\infty} w(t) dt \right] \cdot a_3 \cdot T^{-(\alpha+\beta+\gamma+\delta)/2} \cdot \sum_{m \leq T^u} \frac{1}{m} \mathcal{Q}_{A,B}(T, T^u/m, g, g) + O(T \log^{10} T), \end{aligned} \quad (5.183)$$

where we again still use (5.103) and (5.104).

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<sup>18</sup>Similarly one here pairs together the terms corresponding to the third and sixth term, and the fourth and fifth term in (5.88).

Similarly to how (5.170) was used, one may handle the above sums over  $m$  by using

$$\sum_{m \leq T^u} \frac{(T^u/m)^\alpha \log(T^u/m)}{m} \approx \frac{1}{\alpha^2} - \frac{T^{\alpha u}}{\alpha^2} + \frac{T^{\alpha u} \log(T^u)}{\alpha} \quad (5.184)$$

and

$$\sum_{m \leq T^u} \frac{(T^u/m)^\alpha \log^2(T^u/m)}{m} \approx -\frac{2}{\alpha^3} + \frac{2T^{\alpha u}}{\alpha^3} - \frac{2T^{\alpha u} \log(T^u)}{\alpha^2} + \frac{T^{\alpha u} \log^2(T^u)}{\alpha}, \quad (5.185)$$

these two formulas arising by applying partial summation to (5.171).

### 5.4.8 Differentiation

Some of our integrals in (5.25)-(5.34) involve differentiation. This is no problem though, as it is possible to differentiate our Theorems 5.4, 5.5 and 5.6 with respect to any of the shifts. To do this, we simply use Cauchy's integral trick (see Lemma 4.3) and the result follows immediately. An illustration of this is now done, namely in the case when we differentiate Theorem 5.4 once with respect to  $\alpha$ . The conclusion in this case is as follows:

**Theorem 5.7.** *With notation and assumptions as in Theorem 5.4,*

$$\begin{aligned} & \int_{-\infty}^{\infty} |M(\tfrac{1}{2} + it)|^2 \zeta'(\tfrac{1}{2} + \alpha + it) \zeta(\tfrac{1}{2} + \beta + it) \zeta(\tfrac{1}{2} + \gamma - it) \zeta(\tfrac{1}{2} + \delta - it) w(t) dt \\ &= \left[ \int_{-\infty}^{\infty} w(t) dt \right] \cdot a_3 \cdot \frac{\partial}{\partial \alpha} \left[ T^{-(\alpha+\beta+\gamma+\delta)/2} \sum_{m \leq T^u} \frac{1}{m} \mathcal{Q}_{A,B}(T, T^u/m, f, f) \right] \\ &+ O(T \log^9 T). \end{aligned} \quad (5.186)$$

*Proof.* Beginning with Theorem 5.4, we know that the LHS and the main term in the RHS of (5.101) are analytic functions of the complex variables  $\alpha, \beta, \gamma, \delta$ . Take the first one and subtract the latter and we get an analytic function, let us call it  $D(\alpha, \beta, \gamma, \delta)$ . Theorem 5.4 tells us that

$$D(\alpha, \beta, \gamma, \delta) \ll T \log^8 T. \quad (5.187)$$

Now using Cauchy's integral formula for the derivative with a radius  $r = 1/\log T$ , i.e.

$$\frac{\partial D(\alpha, \beta, \gamma, \delta)}{\partial \alpha} = \frac{1}{2\pi i} \int_{|w-\alpha|=r} \frac{D(w, \beta, \gamma, \delta)}{(w-\alpha)^2} dw, \quad (5.188)$$

we see that

$$\frac{\partial D}{\partial \alpha} \ll T \log^9 T. \quad (5.189)$$

To finish the proof, remember that clearly the derivative of the difference of two analytic functions equals the difference of the derivatives of those two functions.  $\square$

# Chapter 6

## Further results concerning Dirichlet $L$ -functions

### 6.1 Statement of the result

The aim of this chapter is to prove the following theorem:

**Theorem 6.1.** *Let  $T > 0$  be fixed. For sufficiently large  $Q$  there exist  $q \in (\frac{5Q}{4}, \frac{7Q}{4}]$  for which there exists an even primitive Dirichlet character  $\chi(\bmod q)$  such that the function  $t \mapsto L(\frac{1}{2} + it, \chi)$  has no zeros in some subinterval of  $[T, 2T]$  of length at least  $3.54 \times \frac{2\pi}{\log Q}$ .*

**Remark 6.1.** Assuming the GRH, the above can be compared with the discussion in Section 2.3.3, where we noted that if  $T$  is fixed and large, then the relevant average gap-length is  $\frac{2\pi}{\log Q}$ .

### 6.2 Overview of the remainder of this chapter

We suppose (see Main Assumption 6.1) that for all  $\frac{5Q}{4} < q \leq \frac{7Q}{4}$  and all even primitive Dirichlet characters  $\chi(\bmod q)$  we have that all the gaps lying in the interval  $[T, 2T]$  between consecutive zeros of the function  $t \mapsto L(\frac{1}{2} + it, \chi)$  are at most  $\frac{3\kappa}{\log Q}$  ( $\kappa$  will in fact be *defined* to be the smallest number satisfying this). We will prove Theorem 6.1 by showing that it is necessary that  $\kappa > 1.18 \times \frac{2\pi}{\log Q}$ .

Similarly to how we in Chapter 3 introduced the function  $Z(t) : \mathbb{R} \rightarrow \mathbb{R}$  whose zeros coincided with the zeros of  $t \mapsto \zeta(1/2 + it)$ , we introduce in Section 6.7.2 the function  $W(t, \chi) : \mathbb{R} \rightarrow \mathbb{R}$  whose<sup>1</sup> zeros coincide with the zeros of  $t \mapsto L(1/2 + it, \chi)$ . The latter implies that all the gaps lying in the interval  $[T, 2T]$  between consecutive

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<sup>1</sup>The constructions of  $W(t, \chi)$  and the function  $Y(t, \chi)$  introduced in Section 4.3 are similar.

zeros of the function  $W(t, \chi)$  are at most  $\frac{3\kappa}{\log Q}$ . A simple proof by contradiction<sup>2</sup> shows that basically all the gaps lying in the interval  $[T, 2T]$  between consecutive zeros of the function defined by

$$f(t, \chi, \kappa) := W\left(t - \frac{\kappa}{\log Q}, \chi\right)W(t, \chi)W\left(t + \frac{\kappa}{\log Q}, \chi\right) \quad (6.1)$$

are at most  $\frac{\kappa}{\log Q}$ .

In analogy with Chapter 3, we proceed by applying the simplest version of Wirtinger's inequality to  $f(t, \chi, \kappa)$ . The end-result is Corollary 6.4, which says that

$$\begin{aligned} & \sum_{\frac{5Q}{4} < q \leq \frac{7Q}{4}} \sum_{\chi \pmod{q}}^b \int_{t_{1,\chi}}^{t_{N_{\chi,\chi}}} f(t, \chi, \kappa)^2 dt \\ & \leq \left(\frac{\kappa}{\pi \log Q}\right)^2 \sum_{\frac{5Q}{4} < q \leq \frac{7Q}{4}} \sum_{\chi \pmod{q}}^b \int_{t_{1,\chi}}^{t_{N_{\chi,\chi}}} f'(t, \chi, \kappa)^2 dt. \end{aligned} \quad (6.2)$$

Here  $t_{1,\chi}$  (which is approximately  $T$ ) and  $t_{N_{\chi,\chi}}$  (which is approximately  $2T$ ) are roughly speaking the first respectively the last zero of  $f(t, \chi, \kappa)$  in the interval  $[T, 2T]$  and we have summed up the resulting inequality over all even primitive Dirichlet characters modulo  $q$  and then have also in the outer sum summed over all  $\frac{5Q}{4} < q \leq \frac{7Q}{4}$ .

Basically we estimate both sides<sup>3</sup> of (6.2). However, as is to be expected keeping in mind that our function  $f(t, \chi, \kappa)$  depends on  $\kappa$ , these results are expressions in terms of  $\kappa$ . By substituting in these into (6.2) we obtain an inequality in terms of  $\kappa$ . In Section 6.9 we note that this inequality requires that  $\kappa > 1.18 \times \frac{2\pi}{\log Q}$ , as desired.

Much of this chapter concerns the treatment of both sides of (6.2). Very roughly speaking what we want to do here is to be able to evaluate sixth power moments of Dirichlet  $L$ -functions. No one has yet been successful in proving the sixth power moment of the Riemann zeta-function or Dirichlet  $L$ -functions. However, Conrey, Iwaniec and Soundararajan [14] have found a nice expression for the sixth power moment of Dirichlet  $L$ -functions, *but* only when one averages over all even (or odd) primitive Dirichlet characters modulo  $q$  over a range of values of  $q$  and they also need to average over an interval in  $t$ . For the statement of their main result, see Theorem 6.2 in Section 6.3.

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<sup>2</sup>One needs to be careful close to the endpoints. For strict correctness, the reader is directed to Section 6.7.2.

<sup>3</sup>Actually we only find a lower bound for the LHS and an upper bound for the RHS, however, there is no real loss here.

Their main result is the starting block for the evaluations in this chapter. In Section 6.4 we discuss how to interpret their result and go through some simple consequences of it. In Section 6.5 we obtain a simpler version of their theorem (see Theorem 6.4). This has two purposes. Firstly, the calculations of the expressions in terms of  $\kappa$  described above are long and complicated and it is simpler to explain them if we start from a simpler theorem. Secondly, we also want to evaluate sixth power moments with differentiated terms in the integrand. Here it is important to us that we have first simplified some terms featuring in their main theorem (we make some terms independent of the “shifts” and can therefore treat these as constants during differentiation). In Section 6.6 our simplified theorem is used together with Cauchy’s integral trick in order to obtain a differentiated version of our simplified theorem. In Section 6.8 we go through the details of how our previous results in practice enable us to use Corollary 6.4 to obtain an inequality in terms of  $\kappa$ . Finally, in Section 6.10 we carefully illustrate the simplest of a few similar calculations, whose answers are given in Section 6.8.3.

### 6.3 On the article “The sixth power moment of Dirichlet $L$ -functions”

In this chapter we will thus make use of the main theorem in the article “The sixth power moment of Dirichlet  $L$ -functions”, written by Conrey, Iwaniec and Soundararajan (see Theorem 1 in [14]). This theorem will be reproduced as Theorem 6.2 in this section. However, first there is some notation to be introduced.

Let  $A$  and  $B$  be sets of complex numbers, with cardinality equal to 3. The elements of these two sets are the so called “shifts”. Suppose that the modulus of the real part of  $\alpha$  and  $\beta$  is  $< 1/4$  for  $\alpha \in A$  and  $\beta \in B$ . If we let

$$\Lambda(s, \chi) := (q/\pi)^{(s-1/2)/2} \Gamma(s/2) L(s, \chi), \quad (6.3)$$

then

$$\begin{aligned} \Lambda_{A,B}(\chi) &:= \prod_{\alpha \in A} \Lambda(1/2 + \alpha, \chi) \prod_{\beta \in B} \Lambda(1/2 + \beta, \bar{\chi}) \\ &= \left(\frac{q}{\pi}\right)^{\delta_{A,B}} G_{A,B} \mathcal{L}_{A,B}(\chi), \end{aligned} \quad (6.4)$$

where

$$\mathcal{L}_{A,B}(\chi) := \prod_{\alpha \in A} L(1/2 + \alpha, \chi) \prod_{\beta \in B} L(1/2 + \beta, \bar{\chi}), \quad (6.5)$$

$$G_{A,B} := \prod_{\alpha \in A} \Gamma\left(\frac{1/2 + \alpha}{2}\right) \prod_{\beta \in B} \Gamma\left(\frac{1/2 + \beta}{2}\right) \quad (6.6)$$

and<sup>4</sup>

$$\delta_{A,B} := \frac{1}{2} \left( \sum_{\alpha \in A} \alpha + \sum_{\beta \in B} \beta \right). \quad (6.7)$$

Further, let

$$\mathcal{Z}(A, B) := \prod_{\substack{\alpha \in A \\ \beta \in B}} \zeta(1 + \alpha + \beta) \quad (6.8)$$

and

$$\mathcal{A}(A, B) := \prod_p \mathcal{B}_p(A, B) \mathcal{Z}_p(A, B)^{-1}, \quad (6.9)$$

with

$$\mathcal{Z}_p(A, B) := \prod_{\substack{\alpha \in A \\ \beta \in B}} \zeta_p(1 + \alpha + \beta), \quad (6.10)$$

where

$$\zeta_p(x) := \left(1 - \frac{1}{p^x}\right)^{-1} \quad (6.11)$$

and

$$\mathcal{B}_p(A, B) := \int_0^1 \prod_{\alpha \in A} z_{p,\theta}(1/2 + \alpha) \prod_{\beta \in B} z_{p,-\theta}(1/2 + \beta) d\theta, \quad (6.12)$$

with  $z_{p,\theta}(x) := 1/(1 - e(\theta)/p^x)$ , where  $e(\theta) := \exp(2\pi i\theta)$ . The conditions on the real parts of the elements of  $A$  and  $B$  ensure that the Euler product for  $\mathcal{A}$  converges absolutely<sup>5</sup>. Let

$$\mathcal{B}_q := \prod_{p|q} \mathcal{B}_p. \quad (6.13)$$

Next let

$$\mathcal{Q}_{A,B}(q) := \sum_{\substack{S \subset A \\ T \subset B \\ |S|=|T|}} \mathcal{Q}(\bar{S} \cup (-T), \bar{T} \cup (-S); q), \quad (6.14)$$

where  $\bar{S}$  denotes the complement of  $S$  in  $A$  and by the set  $-S$  we mean  $\{-s : s \in S\}$ , and

$$\mathcal{Q}(X, Y; q) := \left(\frac{q}{\pi}\right)^{\delta_{X,Y}} G_{X,Y} \left(\frac{\mathcal{AZ}}{\mathcal{B}_q}\right)(X, Y). \quad (6.15)$$

For future need we notice the obvious fact that

$$\mathcal{Q}(X, Y; q) = \mathcal{Q}(Y, X; q). \quad (6.16)$$

---

<sup>4</sup>Cf. (5.93).

<sup>5</sup>See the proof of Lemma 6.4 and in particular (6.121).

For future convenience we recall three more definitions from [14]. Let<sup>6</sup>

$$a_3 = \prod_p \left\{ \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right\}, \quad (6.17)$$

$$a_3(\mathcal{L}) := \prod_p \left\{ \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{5}{p} - \frac{5}{p^2} + \frac{14}{p^3} - \frac{15}{p^4} + \frac{5}{p^5} + \frac{4}{p^6} - \frac{4}{p^7} + \frac{1}{p^8}\right) \right\} \quad (6.18)$$

and

$$A_t := \{\alpha + t : \alpha \in A\}. \quad (6.19)$$

They prove the following:

**Theorem 6.2** (Main theorem in [14]). *Suppose that  $|A| = |B| = 3$  and that  $\alpha, \beta \ll 1/\log Q$ , for  $\alpha \in A$ ,  $\beta \in B$ . Suppose that  $\Psi$  is smooth on  $\mathbb{R}$  and compactly supported in  $[1, 2]$  and  $\Phi(t)$  is an entire function of  $t$  which decays rapidly as  $t \rightarrow \infty$  in any fixed horizontal strip. Then*

$$\begin{aligned} & \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(t) \sum_{\chi(\bmod q)}^b \Lambda_{A_{it}, B_{-it}}(\chi) dt \\ &= \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(t) \phi^b(q) \mathcal{Q}_{A_{it}, B_{-it}}(q) dt + O(Q^{7/4+\epsilon}), \end{aligned} \quad (6.20)$$

uniformly in  $\alpha$  and  $\beta$ , where the  $\sum^b$  indicates that the sum is restricted to even primitive characters and  $\phi^b(q)$  denotes the number of primitive even Dirichlet characters modulo  $q$ .

**Remark 6.2.** Note that here the “big oh”-constant may depend on  $\Psi$  and  $\Phi$ .

**Brief sketch of proof:** For a complete proof of Theorem 6.2, the reader is referred to [14]. Below is a brief sketch of the proof. We will here assume that  $q > 1$ , which can basically be justified by recalling that  $\Psi$  is compactly supported in  $[1, 2]$ .

Consider

$$\frac{1}{2\pi i} \int_{(1)} \Lambda_{A_s, B_s}(\chi) \frac{H(s)}{s} ds, \quad (6.21)$$

where

$$H(s) := \prod_{\substack{\alpha \in A \\ \beta \in B}} \left( s^2 - \left( \frac{\alpha + \beta}{2} \right)^2 \right)^3. \quad (6.22)$$

---

<sup>6</sup>We have here chosen to keep the same notation as in [14], although the usage of  $a_3(\mathcal{L})$  is confusing in that there is no dependence on “ $\mathcal{L}$ ”.

Expanding the  $L$ -functions into their Dirichlet series we see that (6.21) is

$$\Lambda_{A,B}^0(\chi) := \left(\frac{q}{\pi}\right)^{\delta_{A,B}} \sum_{m,n=1}^{\infty} \frac{\sigma_{-A}(m)\sigma_{-B}(n)}{\sqrt{mn}} \chi(m)\bar{\chi}(n)W_{A,B}(mn\pi^3/q^3), \quad (6.23)$$

where the generalised sum-of-divisors function is defined<sup>7</sup> by

$$\sigma_{s_1,\dots,s_R}(m) := \sum_{m=m_1\cdots m_R} m_1^{s_1} \cdots m_R^{s_R} \quad (6.24)$$

and where

$$W_{A,B}(\xi) := \frac{1}{2\pi} \int_{(1)} G_{A_s,B_s} \xi^{-s} \frac{H(s)}{s} ds. \quad (6.25)$$

For future need, we here note that

$$\sigma_{A_s}(n) = \sigma_A(n)n^s. \quad (6.26)$$

Also,

$$\sum_{g=0}^{\infty} \frac{\sigma_{-A}(p^g)\sigma_{-B}(p^g)}{p^g} = \mathcal{B}_p(A, B), \quad (6.27)$$

which tells us that for  $\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$  we have

$$\sum_{(n,q)=1} \frac{\sigma_{-A}(n)\sigma_{-B}(n)}{n} = \left(\frac{\mathcal{AZ}}{\mathcal{B}_q}\right)(A, B). \quad (6.28)$$

From the usual functional equation for Dirichlet  $L$ -functions it follows that

$$\Lambda_{A,B}(\chi) = \Lambda_{-B,-A}(\chi). \quad (6.29)$$

Returning to (6.21), we now move the line of integration to  $\operatorname{Re}(s) = -1$  and, by using (6.29), obtain (recall that  $q > 1$ ) that

$$H(0)\Lambda_{A,B}(\chi) = \Lambda_{A,B}^0(\chi) + \Lambda_{-B,-A}^0(\chi). \quad (6.30)$$

If  $\Phi$  is as in the statement of Theorem 6.2, then we have

$$\begin{aligned} \Lambda_{A,B}^1(\chi) &:= \int_{-\infty}^{\infty} \Phi(t)\Lambda_{A_{it},B_{-it}}^0(\chi) dt \\ &= \sum_{m,n=1}^{\infty} \frac{\sigma_{-A}(m)\sigma_{-B}(n)}{\sqrt{mn}} \chi(m)\bar{\chi}(n)V_{A,B}(m, n; q), \end{aligned} \quad (6.31)$$

---

<sup>7</sup>With  $R = 2$ , this definition does not give exactly the same as the one used in Chapter 5 (see (5.67)), however, this ought not to give the reader any trouble.

where

$$V_{A,B}(\xi, \eta; \mu) := \left(\frac{\mu}{\pi}\right)^{\delta_{A,B}} \int_{-\infty}^{\infty} \Phi(t) (\xi/\eta)^{-it} W_{A_{it}, B_{-it}}(\xi\eta\pi^3/\mu^3) dt. \quad (6.32)$$

We will want to study asymptotically

$$\mathcal{I}_{A,B} = \mathcal{I}_{A,B}(\Psi, \Phi, Q) := H(0) \sum_{q=1}^{\infty} \sum_{\chi(\bmod q)}^b \Psi(q/Q) \int_{-\infty}^{\infty} \Phi(t) \Lambda_{A_{it}, B_{-it}}(\chi) dt, \quad (6.33)$$

where of course  $\Psi$  is a fixed function satisfying the criteria in the statement of Theorem 6.2. Using (6.30) we obtain

$$\mathcal{I}_{A,B} = \Delta_{A,B} + \Delta_{-B,-A}, \quad (6.34)$$

where

$$\begin{aligned} \Delta_{A,B} &:= \sum_{q=1}^{\infty} \sum_{\chi(\bmod q)}^b \Psi(q/Q) \Lambda_{A,B}^1(\chi) \\ &= \sum_{q=1}^{\infty} \sum_{\chi(\bmod q)}^b \Psi(q/Q) \sum_{m,n=1}^{\infty} \frac{\sigma_{-A}(m)\sigma_{-B}(n)}{\sqrt{mn}} \chi(m)\bar{\chi}(n) V_{A,B}(m, n; q). \end{aligned} \quad (6.35)$$

Let us next study  $\Delta_{A,B}$ . There is a diagonal contribution coming from the terms  $m = n$ , which we call  $\mathcal{D}_{A,B}$ . Clearly

$$\mathcal{D}_{A,B} = \sum_{\substack{n,q \\ (n,q)=1}} \frac{\sigma_{-A}(n)\sigma_{-B}(n)}{n} \phi^b(q) \Psi\left(\frac{q}{Q}\right) V_{A,B}(n, n; q). \quad (6.36)$$

Recalling (6.32) and also (6.26) and (6.28), we see that (6.36) equals

$$\begin{aligned} &\sum_{q=1}^{\infty} \left\{ \phi^b(q) \Psi\left(\frac{q}{Q}\right) \right. \\ &\quad \times \left. \int_{-\infty}^{\infty} \Phi(t) \frac{1}{2\pi i} \int_{(1)} G_{A_{s+it}, B_{s-it}}(q/\pi)^{3s+\delta_{A,B}} \left(\frac{\mathcal{AZ}}{\mathcal{B}_q}\right)(A_s, B_s) \frac{H(s)}{s} ds dt \right\}. \end{aligned} \quad (6.37)$$

By moving the line of integration to  $\text{Re}(s) = -1/2 + \epsilon$  and noting that the integrand has a pole at  $s = 0$ , one may obtain that (6.37) equals

$$\begin{aligned} &H(0) \sum_{q=1}^{\infty} \Psi\left(\frac{q}{Q}\right) \phi^b(q) (q/\pi)^{\delta_{A,B}} \left(\frac{\mathcal{AZ}}{\mathcal{B}_q}\right)(A, B) \int_{-\infty}^{\infty} \Phi(t) G_{A_{it}, B_{-it}} dt + O(Q^{3/2+\epsilon}) \\ &= H(0) \sum_{q=1}^{\infty} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \int_{-\infty}^{\infty} \Phi(t) \mathcal{Q}(A_{it}, B_{-it}; q) dt + O(Q^{3/2+\epsilon}). \end{aligned} \quad (6.38)$$

We shall see that, as one may guess at this point,  $\mathcal{D}_{A,B}$  does indeed lead to the term corresponding to  $S = T = \emptyset$  in (6.14). Similarly  $\mathcal{D}_{-B,-A}$  leads to the term corresponding to  $S = A, T = B$ . The other terms in (6.14) arise from the non-diagonal contribution.

Consider once again the expression in (6.35). Start by looking at the sum over  $\chi$ . If  $(mn, q) = 1$  then

$$\sum_{\chi(\bmod q)}^b \chi(m)\bar{\chi}(n) = \sum_{\chi(\bmod q)}^* \frac{1 + \chi(-1)}{2} \chi(m)\bar{\chi}(n) = \frac{1}{2} \sum_{\substack{q=dr \\ r|(m\pm n)}} \mu(d)\varphi(r), \quad (6.39)$$

where we by  $\pm$  mean

$$\sum_{\pm} = \sum_{+} + \sum_{-}. \quad (6.40)$$

Thus we obtain that

$$\Delta_{A,B} = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\sigma_{-A}(m)\sigma_{-B}(n)}{\sqrt{mn}} \sum_{\substack{d,r \\ (dr,mn)=1 \\ r|(m\pm n)}} \varphi(r)\mu(d)\Psi\left(\frac{dr}{Q}\right)V_{A,B}(m, n; dr). \quad (6.41)$$

For the terms  $m \neq n$ , we divide the sum in (6.41) into two parts, namely  $\mathcal{S}_{A,B}$  corresponding to  $d > D$  and  $\mathcal{G}_{A,B}$  corresponding to  $d \leq D$ . A suitable choice turns out to be  $D = Q^{1/4}$ . We thus have

$$\Delta_{A,B} = \mathcal{D}_{A,B} + \mathcal{S}_{A,B} + \mathcal{G}_{A,B}. \quad (6.42)$$

When investigating  $\mathcal{S}_{A,B}$  and  $\mathcal{G}_{A,B}$ , Conrey, Iwaniec and Soundararajan express the condition  $r|(m \pm n)$  (or a similar condition) in terms of characters  $\psi(\bmod r)$  and then single out the contribution coming from the principal character  $\psi = \psi_0$ . Overall the treatment of the off-diagonal contribution is more complicated than the diagonal one. It was a nice achievement by the three authors named above to be able to handle these terms as well, especially considering the fact that the off-diagonal terms actually give a contribution to the main term in our answer. We will below state the outcome of their investigation. Given  $\alpha \in A$  and  $\beta \in B$ , let us define

$$A^*(\alpha, \beta) := (A - \{\alpha\}) \cup \{-\beta\}. \quad (6.43)$$

Then we may explicitly write down what their method yields as follows:

$$\begin{aligned} & \mathcal{G}_{A,B} + \mathcal{S}_{A,B} \\ &= H(0) \sum_{\substack{\alpha \in A \\ \beta \in B}} \sum_{q=1}^{\infty} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \int_{-\infty}^{\infty} \Phi(t) \mathcal{Q}(A^*(\alpha, \beta)_{it}, B^*(\alpha, \beta)_{-it}; q) dt + O(Q^{7/4+\epsilon}). \end{aligned} \quad (6.44)$$

Of course we may find a similar expression to (6.44) for  $\mathcal{G}_{-B,-A} + \mathcal{S}_{-B,-A}$ . Remembering (6.16), we put these two together with the diagonal contribution and conclude that

$$\begin{aligned} & \mathcal{D}_{A,B} + \mathcal{S}_{A,B} + \mathcal{G}_{A,B} + \mathcal{D}_{-B,-A} + \mathcal{S}_{-B,-A} + \mathcal{G}_{-B,-A} \\ &= H(0) \sum_{q=1}^{\infty} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \int_{-\infty}^{\infty} \Phi(t) \mathcal{Q}_{A_{it}, B_{-it}}(q) dt + O(Q^{7/4+\epsilon}). \end{aligned} \quad (6.45)$$

By using (6.34), we establish that

$$\begin{aligned} & H(0) \sum_{q=1}^{\infty} \sum_{\chi \pmod{q}}^b \Psi(q/Q) \int_{-\infty}^{\infty} \Phi(t) \Lambda_{A_{it}, B_{-it}}(\chi) dt \\ &= H(0) \sum_{q=1}^{\infty} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \int_{-\infty}^{\infty} \Phi(t) \mathcal{Q}_{A_{it}, B_{-it}}(q) dt + O(Q^{7/4+\epsilon}). \end{aligned} \quad (6.46)$$

We would like to divide through by  $H(0)$  in (6.46), however, there is a conceivable problem when  $\alpha + \beta = 0$  for  $\alpha \in A$  and  $\beta \in B$ . To tackle this difficulty, our first step will be to show that  $\mathcal{Q}_{A,B}(q)$  is an analytic function of the shifts. The problem is where any of the relevant  $\mathcal{Z}(X, Y)$  has a pole. However, we will now show that these singularities must be removable. To do this, we use Lemma 2.5.1 in the article [10] by Conrey, Farmer, Keating, Rubinstein and Snaith. In the notation used in the latter we take  $k = 3$ ,  $f(s) = 1/s$  and

$$K(\alpha_1, \alpha_2, \alpha_3; \alpha_4, \alpha_5, \alpha_6) = \mathcal{Q}(\{\alpha_1, \alpha_2, \alpha_3\}, \{-\alpha_4, -\alpha_5, -\alpha_6\}; q). \quad (6.47)$$

We find that

$$\mathcal{Q}_{\{\alpha_1, \alpha_2, \alpha_3\}, \{-\alpha_4, -\alpha_5, -\alpha_6\}}(q) = \sum_{\sigma \in \Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}; \alpha_{\sigma(4)}, \alpha_{\sigma(5)}, \alpha_{\sigma(6)}), \quad (6.48)$$

where<sup>8</sup>  $\Xi$  is the set of  $\binom{6}{3}$  permutations of  $\sigma \in S_6$  such that  $\sigma(1) < \sigma(2) < \sigma(3)$  and  $\sigma(4) < \sigma(5) < \sigma(6)$ . Now suppose that

$$\alpha_i \neq \alpha_j, \text{ for } i \neq j. \quad (6.49)$$

Then we may use the following formula from [10]:

$$\begin{aligned} & \sum_{\sigma \in \Xi} K(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}; \alpha_{\sigma(4)}, \alpha_{\sigma(5)}, \alpha_{\sigma(6)}) \\ &= \frac{-1}{(3!)^2} \frac{1}{(2\pi i)^6} \int \dots \int \frac{K(z_1, z_2, z_3; z_4, z_5, z_6) \Delta(z_1, \dots, z_6)^2}{\prod_{i=1}^6 \prod_{j=1}^6 (z_i - \alpha_j)} dz_1 \dots dz_6, \end{aligned} \quad (6.50)$$

---

<sup>8</sup>Obviously  $\Xi$  here denotes a different set than what it did in Chapter 5.

where

$$\Delta(z_1, \dots, z_6) := \prod_{1 \leq i < j \leq 6} (z_j - z_i), \quad (6.51)$$

and where one integrates about (small) circles enclosing the  $\alpha_i$ 's. By choosing the radii of the circles to be  $C_i/\log Q$  ( $i = 1, \dots, 6$ ), for any suitably large constants  $C_i$ , we may obtain an upper bound (depending on  $Q$ ) for the RHS of (6.50). By recalling (6.48), we therefore see that the function  $\mathcal{Q}_{\{\alpha_1, \alpha_2, \alpha_3\}, \{-\alpha_4, -\alpha_5, -\alpha_6\}}(q)$  remains bounded whenever (6.49) is satisfied. This allows us to conclude that the possible singularities must be removable. We may conclude that  $\mathcal{Q}_{A,B}(q)$  is an analytic function of the shifts. Let us for future need remark here that it obviously follows that  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$  is an analytic function of the shifts.

Now let  $\alpha_1, \alpha_2, \alpha_3$  denote the shifts in  $A$  and let  $\alpha_4, \alpha_5, \alpha_6$  denote the shifts in  $B$  and let us now return to (6.46) by rewriting it as follows:

$$H(0)l(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = H(0)r(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + O(Q^{7/4+\epsilon}), \quad (6.52)$$

where we for future convenience recall that  $H(0)$  by definition clearly is a function of the shifts. From uniform convergence of the relevant integrals (given any value of  $Q$ ) one may deduce that  $l(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$  is an analytic function of the shifts. As remarked above,  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$  is an analytic function of the shifts and hence in particular it is bounded. This implies that  $r(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$  is an analytic function of the shifts.

Thus  $(l - r)(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$  is an analytic function of the shifts for  $\alpha_i \ll 1/\log Q$ ,  $i = 1, \dots, 6$ . Now consider any  $\alpha_i \leq C/\log Q$ ,  $i = 1, \dots, 6$ . We may apply Cauchy's integral formula to the above function in order to obtain that

$$\begin{aligned} & (l - r)(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \\ &= \frac{1}{(2\pi i)^6} \int \cdots \int \int_{\partial D_1 \times \cdots \times \partial D_6} \frac{(l - r)(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)}{(\beta_1 - \alpha_1) \cdots (\beta_6 - \alpha_6)} d\beta_1 \cdots d\beta_6, \end{aligned} \quad (6.53)$$

where<sup>9</sup>  $D = \prod_{i=1}^6 D_i$  (i.e.  $D$  is the polydisc defined as the Cartesian product of the open discs  $D_i$ ), where

$$D_i := \{s \in \mathbb{C} : |s - \alpha_i| < r_i\}, \quad (6.54)$$

with

$$r_i = \frac{2^{i+1}C}{\log Q}. \quad (6.55)$$

---

<sup>9</sup>Obviously  $D$  will here thus not be what it was in Chapter 5.

By this construction we find that

$$|\beta_i + \beta_j| \geq 2C/\log Q, \text{ for any } i \neq j \quad (6.56)$$

and in particular

$$1/H(0) \ll \log^{54} Q. \quad (6.57)$$

Using this together with (6.52), we deduce that the numerator in the integrand in (6.53) is

$$\ll Q^{7/4+\epsilon} \log^{54} Q \ll Q^{7/4+\epsilon'}. \quad (6.58)$$

By estimating the integral in (6.53) trivially, we finally obtain that

$$l(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = r(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + O(Q^{7/4+\epsilon'}), \quad (6.59)$$

which completes our sketch of the proof (by “rewriting”  $\epsilon'$  as  $\epsilon$ ).  $\square$

We now make the following definition here:

**Definition 6.1.** Let  $N_1, N_2$  and  $N_3$  be large positive real numbers.  $\Psi(t)$  will be said to be reasonable if it is smooth, is compactly supported in  $[1, 2]$  and if we also have that  $\Psi(t) \leq N_1$  and  $\Psi'(t) \leq N_2$  for  $t \in [1, 2]$ .  $\Phi(z)$  will be said to be reasonable if it is an entire function which decays rapidly as  $|z| \rightarrow \infty$  in any fixed horizontal strip and if  $\Phi(t) \leq N_3 \exp(-t^2)$  for  $t \in \mathbb{R}$ .

**Remark 6.3.** Notice that if  $\Psi$  and  $\Phi$  are reasonable then they satisfy the conditions in Theorem 6.2.

**Remark 6.4.** From here on it will always be assumed that  $\Psi$  and  $\Phi$  are reasonable.

## 6.4 Interpretation of Theorem 6.2

Our first step towards understanding what Theorem 6.2 says will be to study  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$ . In particular, looking at (6.14) and (6.15), we see the importance of examining

$$G_{X_{it}, Y_{-it}}, \quad (6.60)$$

$$\mathcal{B}_p(X_{it}, Y_{-it}), \quad (6.61)$$

$$\mathcal{Z}_p^{-1}(X_{it}, Y_{-it}) \quad (6.62)$$

and

$$\mathcal{A}(X_{it}, Y_{-it}). \quad (6.63)$$

Clearly (6.60) depends on  $t$  and is equal to

$$\left| \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \right|^6 \quad (6.64)$$

when all the shifts in the sets  $X$  and  $Y$  are zero. Let us denote the shifts in  $X$  by  $\alpha_1, \alpha_2$  and  $\alpha_3$  and the shifts in  $Y$  by  $\alpha_4, \alpha_5$  and  $\alpha_6$ . Then by looking at the definition given in (6.12) we observe that  $\mathcal{B}_q(X_{it}, Y_{-it})$  is equal to the  $e(0 \cdot \theta)$ -coefficient in the following product:

$$\left(1 + \frac{e(\theta)}{p^{1/2+it+\alpha_1}} + \frac{e(2\theta)}{p^{1+2it+2\alpha_1}} + \dots\right) \times \dots \times \left(1 + \frac{e(-\theta)}{p^{1/2-it+\alpha_6}} + \frac{e(-2\theta)}{p^{1-2it+2\alpha_6}} + \dots\right). \quad (6.65)$$

This observation immediately tells us that  $\mathcal{B}_q(X_{it}, Y_{-it})$  is independent of  $t$ . Moreover, a straightforward calculation shows that when all the shifts in the sets  $X$  and  $Y$  are zero, then (6.61) equals

$$\left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^{-5}. \quad (6.66)$$

By going back to (6.10) we see that (6.62) clearly is independent of  $t$  and when all the shifts in the sets  $X$  and  $Y$  are zero then (6.62) equals

$$\left(1 - \frac{1}{p}\right)^9. \quad (6.67)$$

Thus  $\mathcal{A}(X_{it}, Y_{-it})$  is independent of  $t$  and (6.63) equals

$$\prod_p \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^4 \right\} \quad (= a_3) \quad (6.68)$$

when all the shifts in the sets  $X$  and  $Y$  are zero.

Our next goal will be to explore what Theorem 6.2 says when all the shifts are zero.

**Corollary 6.1** (Corollary to Theorem 6.2). *Letting all the shifts in Theorem 6.2 equal*

zero, we obtain that

$$\begin{aligned}
& \sum_{q \geq 1} \sum_{\chi \pmod{q}}^b \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(t) |L(1/2 + it, \chi)|^6 |\Gamma((1/2 + it)/2)|^6 dt \\
&= a_3 \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \phi^b(q) \left( \frac{42 \log^9 q}{9!} + O(\log^8 q) \right) \\
&\quad \times \int_{-\infty}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \\
&= \frac{42 a_3(\mathcal{L})}{9!} Q^2 \log^9 Q \int_0^{\infty} \frac{\Psi(x)x}{2} dx \int_{-\infty}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \\
&\quad + O(Q^2 \log^8 Q). \tag{6.69}
\end{aligned}$$

*Proof.* We apply Theorem 6.2 with  $A = \{\delta, 2\delta, 4\delta\}$  and  $B = \{\delta, 2\delta, 4\delta\}$  and then let  $\delta \rightarrow 0$ . The  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$  in the integrand of Theorem 6.2 is a sum of 20 terms of type  $\mathcal{Q}(X, Y; q)$ . If we treat  $\delta$  as a complex variable then all the factors of (6.15) are analytic functions of  $\delta$ , except  $\mathcal{Z}(X, Y)$  which is a meromorphic function of  $\delta$ , with a pole of order 9 at  $\delta = 0$ . Therefore each of those 20 terms will have a Laurent series in  $\delta$ . However, when we take the sum of them, we must get an analytic function of  $\delta$ , since the LHS of Theorem 6.2 is continuous with respect to  $\delta$  (as  $\delta \rightarrow 0$ ). As we will let  $\delta \rightarrow 0$ , we are not interested in the coefficients of the positive  $\delta$ -powers, but rather want to find the  $\delta^0$ -coefficient. The key here is to focus on the contribution coming from taking the  $\delta^9$ -coefficient from the term  $(\frac{q}{\pi})^{\delta x, Y}$ , since this yields the highest power of  $\log q$ . Note that this part of the  $\delta^0$ -coefficient is equal to  $42 \log^9 q / 9!$ . This establishes the first equality-statement of this corollary.

Now we will show the second equality-statement. Suppose for the moment that we have managed to prove that the main term on the second line of this corollary is equal to the third line. Then by treating the error term in a similar way (one may want to notice that  $|\Psi(x)| \ll 1$ ), we obtain the desired result.

Thus we are done if we can show that

$$\begin{aligned}
& a_3 \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \phi^b(q) \log^9 q \\
&= a_3(\mathcal{L}) Q^2 \log^9 Q \int_0^{\infty} \frac{\Psi(x)x}{2} dx + O(Q^2 \log^8 Q). \tag{6.70}
\end{aligned}$$

Let us make two definitions.

$$G(q) := \frac{\phi^{\flat}(q) \log^9 q}{q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \quad (6.71)$$

and

$$H(P) := \sum_{q \leq P} G(q). \quad (6.72)$$

Assume for the moment that

$$H(t) = \frac{a_3(\mathcal{L})}{2a_3} t \log^9 t + O(t \log^8 t). \quad (6.73)$$

Then by partial summation we have that (6.70) equals

$$\begin{aligned} a_3 \sum_{q \geq 1} q \Psi\left(\frac{q}{Q}\right) G(q) &= a_3 \sum_{Q < q \leq 2Q} q \Psi\left(\frac{q}{Q}\right) G(q) \\ &= -a_3 \int_Q^{2Q} \frac{d}{dt} \left( t \Psi\left(\frac{t}{Q}\right) \right) H(t) dt \\ &= -a_3 \int_Q^{2Q} \frac{d}{dt} \left( t \Psi\left(\frac{t}{Q}\right) \right) \left\{ \frac{a_3(\mathcal{L})}{2a_3} t \log^9 t + O(t \log^8 t) \right\} dt \\ &= \frac{a_3(\mathcal{L})}{2} \int_Q^{2Q} t \Psi\left(\frac{t}{Q}\right) \left\{ \log^9 t + O(\log^8 t) \right\} dt + O\left( \int_Q^{2Q} \left| \frac{d}{dt} \left( t \Psi\left(\frac{t}{Q}\right) \right) \right| t \log^8 t dt \right) \\ &= a_3(\mathcal{L}) Q^2 \log^9 Q \int_0^{\infty} \frac{\Psi(x)x}{2} dx + O(Q^2 \log^8 Q). \end{aligned} \quad (6.74)$$

To complete this proof we must prove the assumption (6.73). Trivially

$$\sum_{q \leq P} \frac{\log^9 q}{q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \ll \sum_{q \leq P} \frac{\log^9 P}{q} \cdot 1 \ll \log^{10} P \ll P^{13/16}. \quad (6.75)$$

Therefore, since  $\phi^{\flat}(q) = \frac{\phi^*(q)}{2} + O(1)$ , where  $\phi^*(q)$  is denoting the number of primitive Dirichlet characters modulo  $q$ , (6.73) will follow if we show that

$$\sum_{q \leq t} \frac{\phi^*(q) \log^9 q}{q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} = \frac{a_3(\mathcal{L})}{a_3} t \log^9 t + O(t \log^8 t). \quad (6.76)$$

Now  $\phi^*(q)$  is a well-known multiplicative function given by  $\phi^*(p) = p - 2$  for primes  $p$  and  $\phi^*(p^k) = p^k(1 - 1/p)^2$  for  $k \geq 2$ . By standard partial summation, (6.76) follows

from that, with  $C$  being some constant, the following result holds:

$$\sum_{q \leq x} \frac{\phi^*(q)}{q^2} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} = \frac{a_3(\mathcal{L})}{a_3} \log x + C + O(x^{-3/16}). \quad (6.77)$$

Now we prove (6.77). Note that we can without loss of generality assume that  $x \equiv \frac{1}{2} \pmod{1}$ . For  $\operatorname{Re}(s) > 1$ , introduce

$$\begin{aligned} K(s) &:= \sum_{q \geq 1} \frac{\phi^*(q)}{q^{1+s}} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \\ &= \prod_p \left( 1 + \frac{\phi^*(p)}{p^{1+s}} \cdot \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} + \frac{(1 - \frac{1}{p})^7}{(1 + \frac{4}{p} + \frac{1}{p^2})} \cdot \left( \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots \right) \right) \\ &= \prod_p \left\{ (1 - p^{-s})^{-1} \right\} D(s) \end{aligned} \quad (6.78)$$

say, where  $D(s)$  certainly converges absolutely and uniformly for  $\operatorname{Re}(s) > 1/4$  and

$$D(s)|_{s=1} = \frac{a_3(\mathcal{L})}{a_3}. \quad (6.79)$$

Now we use Lemma 3.12 in Titchmarsh [63]. In the notation used there we take  $f(s) = K(s)$ ,  $c$  to be a small fixed positive constant and then also  $\sigma = 1$  and  $T = x$ . Then we obtain that

$$\sum_{q \leq x} \frac{\phi^*(q)}{q^2} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} = \frac{1}{2\pi i} \int_{c-ix}^{c+ix} K(1+s) \frac{x^s}{s} ds + O(x^{c-1}). \quad (6.80)$$

Next we move the line of integration to  $\operatorname{Re}(s) = -1/2$ . The main term in our answer comes from the residue at  $s = 0$ . To bound the contribution from the two horizontal and the one vertical line of integration we may for example work along the lines of Chapter 12 in [63]. The contribution from the horizontal lines is  $\ll x^{c-5/6}$ . The contribution from the vertical line is certainly  $\ll x^{c-1/2}$ . Thus picking  $c = 5/16$  we obtain that

$$\begin{aligned} \sum_{q \leq x} \frac{\phi^*(q)}{q^2} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} &= \operatorname{Residue}_{s=0} \left\{ K(1+s) \frac{x^s}{s} \right\} + O(x^{-3/16}) \\ &= \frac{a_3(\mathcal{L})}{a_3} \log x + C + O(x^{-3/16}), \end{aligned} \quad (6.81)$$

where we use (6.79) when evaluating the residue. □

**Remark 6.5.** As a side-note, we note for future convenience that partial summation applied to (6.77) implies that

$$\sum_{q \leq t} \frac{\phi^*(q)}{q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} = \frac{a_3(\mathcal{L})}{a_3} t + O(t^{13/16}) \quad (6.82)$$

and hence that

$$H_2(t) := \sum_{q \leq t} \frac{\phi^b(q)}{q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} = \frac{a_3(\mathcal{L})}{2a_3} t + O(t^{13/16}). \quad (6.83)$$

Let

$$G_2(q) = \frac{\phi^b(q)}{q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})}. \quad (6.84)$$

By working as in the proof of Corollary 6.1 we get that

$$\begin{aligned} a_3 \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} &= a_3 \sum_{q \geq 1} q \Psi\left(\frac{q}{Q}\right) G_2(q) = a_3 \sum_{Q < q \leq 2Q} q \Psi\left(\frac{q}{Q}\right) G_2(q) \\ &= -a_3 \int_Q^{2Q} \frac{d}{dt} \left( t \Psi\left(\frac{t}{Q}\right) \right) H_2(t) dt = -a_3 \int_Q^{2Q} \frac{d}{dt} \left( t \Psi\left(\frac{t}{Q}\right) \right) \left\{ \frac{a_3(\mathcal{L})}{2a_3} t + O(t^{13/16}) \right\} dt \\ &= \frac{a_3(\mathcal{L})}{2} \int_Q^{2Q} t \Psi\left(\frac{t}{Q}\right) dt + O\left( \int_Q^{2Q} \left| \frac{d}{dt} \left( t \Psi\left(\frac{t}{Q}\right) \right) \right| t^{13/16} dt \right) \\ &= a_3(\mathcal{L}) Q^2 \int_0^\infty \frac{\Psi(x)x}{2} dx + O(Q^{29/16}). \end{aligned} \quad (6.85)$$

**Lemma 6.1.** *Suppose that all shifts are zero in Theorem 6.2. Then the LHS of Theorem 6.2 is  $\ll Q^2 \log^9 Q$ .*

*Proof.* We use Corollary 6.1 and then it remains to show that

$$42a_3(\mathcal{L})Q^2 \frac{\log^9 Q}{9!} \int_0^\infty \frac{\Psi(x)x}{2} dx \int_{-\infty}^\infty \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \ll Q^2 \log^9 Q. \quad (6.86)$$

Clearly we have that

$$\left| \int_0^\infty \frac{\Psi(x)x}{2} dx \right| \leq \int_1^2 \frac{|\Psi(x)|x}{2} dx \ll \int_1^2 \frac{1 \cdot x}{2} dx \ll 1. \quad (6.87)$$

Also, since  $\Gamma(\sigma + it)$  is a bounded function on the line  $\sigma = 1/4$ , we find that

$$\left| \int_{-\infty}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \right| \ll \int_{-\infty}^{\infty} |\Phi(t)| dt \ll \int_{-\infty}^{\infty} \exp(-t^2) dt \ll 1. \quad (6.88)$$

This completes the proof.  $\square$

**Remark 6.6.** Notice that the “tail” of the integral

$$\int_{-\infty}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \quad (6.89)$$

is unimportant. Indeed, by using the following Chernoff bound (see [7]) for the complimentary error function

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2) dt \leq \exp(-x^2), \quad x \geq 0 \quad (6.90)$$

we see that

$$\begin{aligned} \int_{\log Q}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt &\ll \int_{\log Q}^{\infty} \exp(-t^2) \cdot 1 dt \\ &\ll \exp(-(\log Q)^2) = Q^{-\log Q} \ll Q^{-1}, \end{aligned} \quad (6.91)$$

and similarly for the integral-part between  $-\infty$  and  $-\log Q$ . In particular we could change the integral involving  $\Phi(t)$  in Corollary 6.1 so that it just goes from  $-\log Q$  to  $\log Q$  instead of from  $-\infty$  to  $\infty$ .

## 6.5 Simplification of Theorem 6.2 to obtain Theorem 6.4

For future need we will in the next three lemmas basically show that the expressions in (6.64), (6.66), (6.67) and (6.68) for (6.60)-(6.63) respectively remain rather accurate when we allow small shifts.

**Lemma 6.2.** *Suppose that  $|A| = |B| = 3$  and that  $\alpha, \beta \ll 1/\log Q$ , for  $\alpha \in A, \beta \in B$ . Then  $G_{A_{it}, B_{-it}} = |\Gamma((1/2 + it)/2)|^6 \left(1 + O\left(\frac{\log \log Q}{\log Q}\right)\right)$ , for  $|t| \leq \log Q$ .*

*Proof.* By definition

$$G_{A_{it}, B_{-it}} = \prod_{\alpha \in A} \Gamma\left(\frac{1}{4} + \frac{it}{2} + \frac{\alpha}{2}\right) \prod_{\beta \in B} \Gamma\left(\frac{1}{4} - \frac{it}{2} + \frac{\beta}{2}\right). \quad (6.92)$$

We thus need to show that

$$\Gamma\left(\frac{1}{4} + \frac{it}{2} + \frac{\alpha}{2}\right) = \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \left(1 + O\left(\frac{\log \log Q}{\log Q}\right)\right), \quad (6.93)$$

for  $\alpha \ll 1/\log Q$  and  $|t| \leq \log Q$ . In order to do this we use the Fundamental Theorem of Calculus

$$F(\gamma(b)) - F(\gamma(a)) = \int_{\gamma} F'(z) dz, \quad (6.94)$$

with  $F(z) = \log \Gamma(z)$ . We have

$$F'(z) = \frac{\Gamma'(z)}{\Gamma(z)} \ll \log(2 + |t|) \ll \log \log Q. \quad (6.95)$$

Therefore we trivially obtain that

$$F\left(\frac{1}{4} + \frac{it}{2} + \frac{\alpha}{2}\right) - F\left(\frac{1}{4} + \frac{it}{2}\right) \ll \frac{\log \log Q}{\log Q}. \quad (6.96)$$

Taking exponentials, we obtain the desired result.  $\square$

**Lemma 6.3.** *For  $q \ll Q$  and shifts  $\alpha, \beta \ll 1/\log Q$ , there exists  $M \in \mathbb{N}$  such that*

$$\mathcal{B}_q(A_{it}, B_{-it}) = \prod_{p|q} \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^{-5} \right\} \cdot \left(1 + O\left(\frac{(\log \log Q)^M}{\log Q}\right)\right). \quad (6.97)$$

*Proof.* Assume for the moment that we know that

$$\mathcal{B}_q(A_{it}, B_{-it}) \ll (\log \log Q)^M \quad (6.98)$$

and that

$$\frac{\partial \mathcal{B}_p}{\partial \alpha_i} \ll \frac{\log p}{p}, \quad (6.99)$$

where in (6.99) we have  $p$  denoting a prime and where we mean the partial derivative with respect to any of the shifts in  $A$  or  $B$ .

Then since  $\mathcal{B}_q = \prod_{p|q} \mathcal{B}_p$ , we find that

$$\begin{aligned} \frac{\partial \mathcal{B}_q}{\partial \alpha_i} &= \sum_{p|q} \left\{ \mathcal{B}_{q/p} \cdot \frac{\partial \mathcal{B}_p}{\partial \alpha_i} \right\} \ll \sum_{p|q} \left\{ (\log \log Q)^M \cdot \frac{\log p}{p} \right\} \\ &= (\log \log Q)^M \sum_{p|q} \frac{\log p}{p} \ll (\log \log Q)^{M+1}. \end{aligned} \quad (6.100)$$

Therefore (remember (6.66)), by the Fundamental Theorem of Calculus, we trivially get that

$$\mathcal{B}_q(A_{it}, B_{-it}) - \prod_{p|q} \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^{-5} \right\} \ll (\log \log Q)^{M+1} (\log Q)^{-1} \quad (6.101)$$

and since

$$\prod_{p|q} \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^{-5} \right\} \geq 1 \quad (6.102)$$

we get the result (by “rewriting”  $M + 1$  as  $M$ ).

It thus remains to show (6.98) and (6.99). We start with the first statement. Suppose that

$$\mathcal{B}_p = 1 + O(p^{-1}), \text{ for prime } p \leq q (\ll Q). \quad (6.103)$$

Then

$$\begin{aligned} \mathcal{B}_q &:= \prod_{p|q} \mathcal{B}_p \ll \prod_{p|q} (1 + c/p) \leq \prod_{p|q} \left(1 + \frac{1 + [c]}{p}\right) \leq \prod_{p|q} (1 + 1/p)^{1+[c]} = \\ &= \left\{ \prod_{p|q} (1 + 1/p) \right\}^{1+[c]} \ll (\log \log Q)^{1+[c]}. \end{aligned} \quad (6.104)$$

Let us denote the shifts in  $A$  by  $\alpha_1, \alpha_2$  and  $\alpha_3$  and the shifts in  $B$  by  $\alpha_4, \alpha_5$  and  $\alpha_6$ . Now in order to prove (6.103) we recall that  $\mathcal{B}_p$  is the  $e(0 \cdot \theta)$ -coefficient in the following product:

$$\left(1 + \frac{e(\theta)}{p^{1/2+\alpha_1}} + \frac{e(2\theta)}{p^{1+2\alpha_1}} + \dots\right) \times \dots \times \left(1 + \frac{e(-\theta)}{p^{1/2+\alpha_6}} + \frac{e(-2\theta)}{p^{1+2\alpha_6}} + \dots\right). \quad (6.105)$$

It is easy to spot that the answer will begin with

$$1 + \left(\sum_{i=1}^3 p^{-1/2-\alpha_i}\right) \left(\sum_{i=4}^6 p^{-1/2-\alpha_i}\right). \quad (6.106)$$

This is clearly  $1 + O(p^{-1})$ . Let us finish this part of the proof by showing why the rest of the contribution in (6.105) is  $O(p^{-2})$ . To do this we consider the worst possible scenario, which clearly is when all the shifts are equal to  $-\frac{c}{\log Q}$ . Let here  $k := \frac{1}{2} - \frac{c}{\log Q}$ . We are thus considering

$$\left(1 + \frac{e(\theta)}{p^k} + \frac{e(2\theta)}{p^{2k}} + \dots\right) \times \dots \times \left(1 + \frac{e(-\theta)}{p^k} + \frac{e(-2\theta)}{p^{2k}} + \dots\right). \quad (6.107)$$

Now the  $e(j\theta)/p^{jk}$ -coefficient in  $\left(1 + \frac{e(\theta)}{p^k} + \frac{e(2\theta)}{p^{2k}} + \dots\right)^3$  is  $1 + 2 + \dots + (j+1) \ll j^2$  and similarly for the  $e(-j\theta)/p^{jk}$ -coefficient in  $\left(1 + \frac{e(-\theta)}{p^k} + \frac{e(-2\theta)}{p^{2k}} + \dots\right)^3$ . Thus ignoring

the contribution mentioned in (6.106), the rest is then

$$\ll \sum_{j=2}^{\infty} \frac{j^4}{p^{2jk}} \ll \frac{1}{p^{4k}} = p^{-2+4c/\log Q} \ll p^{-2}. \quad (6.108)$$

Let us finally show (6.99). This is similar to the above discussion. Suppose without loss of generality that the partial differentiation is with respect to  $\alpha_1$ . Again we do the worst case scenario and the  $e(-j\theta)/p^{jk}$ -coefficient is of course unchanged. To get the  $e(j\theta)/p^{jk}$ -coefficient we consider

$$\left(1 + 2\frac{e(\theta)}{p^k} + 3\frac{e(2\theta)}{p^{2k}} + \dots\right) \left(\frac{e(\theta) \log p}{p^k} + 2\frac{e(2\theta) \log p}{p^{2k}} + \dots\right). \quad (6.109)$$

The  $e(j\theta)/p^{jk}$ -coefficient becomes  $(\log p)(1 \cdot j + \dots + j \cdot 1) \ll j^3 \log p$ . Hence the relevant  $e(0 \cdot \theta)$ -coefficient is

$$\ll \sum_{j=1}^{\infty} \frac{j^2 \cdot j^3 \cdot \log p}{p^{2jk}} \ll \frac{\log p}{p^{2k}} \ll \frac{\log p}{p}. \quad (6.110)$$

□

When we prove the next lemma, it is important to keep (6.67) and (6.68) in mind.

**Lemma 6.4.**

$$\mathcal{A}(A_{it}, B_{-it}) = \prod_p \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^4 \right\} \cdot \left(1 + O\left(\frac{(\log \log Q)^2}{\log Q}\right)\right) \quad (6.111)$$

for shifts  $\alpha, \beta \ll 1/\log Q$ .

*Proof.* We have that

$$\begin{aligned} \mathcal{A}(A_{it}, B_{-it}) &:= \\ &= \prod_{p \leq \log Q} \mathcal{B}_p(A_{it}, B_{-it}) \cdot \prod_{p \leq \log Q} \mathcal{Z}_p^{-1}(A_{it}, B_{-it}) \cdot \prod_{p > \log Q} (\mathcal{B}_p \mathcal{Z}_p^{-1})(A_{it}, B_{-it}) \\ &=: F_1 \cdot F_2 \cdot F_3. \end{aligned} \quad (6.112)$$

By a very similar proof to the proof of Lemma 6.3 we see that

$$F_1 = \left( \prod_{p \leq \log Q} \left\{ \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^{-5} \right\} \right) \cdot \left(1 + O\left(\frac{(\log \log Q)^M}{\log Q}\right)\right). \quad (6.113)$$

Obviously, since all the shifts are assumed to be  $\ll 1/\log Q$ , this is true also for the sum of any two of the shifts. Therefore, letting  $\alpha_{i,j}$  below stand for a complex number  $\ll 1/\log Q$ , we will study

$$\begin{aligned} \prod_{p \leq \log Q} \frac{(1 - \frac{1}{p^{1+\alpha_{i,j}}})}{(1 - p^{-1})} &= \prod_{p \leq \log Q} \left\{ (1 - p^{-1})^{-1} \left( 1 - \frac{1}{p} \left\{ 1 + O\left(\frac{\log \log Q}{\log Q}\right) \right\} \right) \right\} \\ &= \prod_{p \leq \log Q} \left\{ 1 + O\left(\frac{\log \log Q}{p \log Q}\right) \right\}. \end{aligned} \quad (6.114)$$

Taking logarithms in (6.114) we get

$$\sum_{p \leq \log Q} \log \left\{ 1 + O\left(\frac{\log \log Q}{p \log Q}\right) \right\}, \quad (6.115)$$

which is

$$\ll \left(\frac{\log \log Q}{\log Q}\right) \sum_{p \leq \log Q} \frac{1}{p} \ll \frac{(\log \log Q)^2}{\log Q}. \quad (6.116)$$

Taking exponentials this shows that

$$F_2 = \left( \prod_{p \leq \log Q} \left(1 - \frac{1}{p}\right)^9 \right) \cdot \left(1 + O\left(\frac{(\log \log Q)^2}{\log Q}\right)\right). \quad (6.117)$$

Since we may choose to let all the shifts equal zero in the following result, we will complete the proof of this lemma by showing that

$$F_3 = 1 + O\left(\frac{1}{\log Q}\right). \quad (6.118)$$

Recall from (6.106) and the discussion thereafter that

$$\mathcal{B}_p(A_{it}, B_{-it}) = 1 + \left(\sum_{i=1}^3 p^{-1/2-\alpha_i}\right) \left(\sum_{i=4}^6 p^{-1/2-\alpha_i}\right) + O(p^{-2+4c/\log Q}). \quad (6.119)$$

Working from the definition it is easy to show that

$$\mathcal{Z}_p^{-1}(A_{it}, B_{-it}) = 1 - \left(\sum_{i=1}^3 p^{-1/2-\alpha_i}\right) \left(\sum_{i=4}^6 p^{-1/2-\alpha_i}\right) + O(p^{-2+4c/\log Q}). \quad (6.120)$$

It follows that

$$\mathcal{B}_p(A_{it}, B_{-it}) \mathcal{Z}_p^{-1}(A_{it}, B_{-it}) = 1 + O(p^{-2+4c/\log Q}). \quad (6.121)$$

At this point it is then easy to show (for example by first taking logarithms and then taking exponentials) that

$$\begin{aligned} F_3 &:= \prod_{p > \log Q} (\mathcal{B}_p \mathcal{Z}_p^{-1})(A_{it}, B_{-it}) \\ &= \prod_{p > \log Q} \left( 1 + O\left(p^{-2+4c/\log Q}\right) \right) = 1 + O\left(\frac{1}{\log Q}\right). \end{aligned} \quad (6.122)$$

□

In the next theorem, we have replaced the  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$ -terms featuring in Theorem 6.2 by an expression that is more independent of the shifts and we have also cut off the “tail” of the integral involving  $\Phi(t)$ . Basically the result continues to hold, albeit our new error term is bigger.

**Definition 6.2.** Define

$$\mathcal{P}_{A,B}(q) := \sum_{\substack{S \subset A \\ T \subset B \\ |S|=|T|}} \mathcal{P}(\bar{S} \cup (-T), \bar{T} \cup (-S); q), \quad (6.123)$$

where

$$\mathcal{P}(X, Y; q) := \left(\frac{q}{\pi}\right)^{\delta_{X,Y}} \mathcal{Z}(X, Y). \quad (6.124)$$

**Theorem 6.3.** *Suppose that  $|A| = |B| = 3$  and that  $\alpha, \beta \ll 1/\log Q$ , for  $\alpha \in A$ ,  $\beta \in B$ . Suppose also that  $\Psi$  and  $\Phi$  are reasonable. Then there exists  $M \in \mathbb{N}$  such that*

$$\begin{aligned} &\sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(t) \sum_{\chi \pmod{q}}^b \Lambda_{A_{it}, B_{-it}}(\chi) dt \\ &= a_3 \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \mathcal{P}_{A,B}(q) \int_{-\log Q}^{\log Q} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \\ &\quad + O\left(Q^2 \log^8 Q (\log \log Q)^M\right), \end{aligned} \quad (6.125)$$

with  $a_3$  as given in (6.17).

*Proof.* Let us denote the shifts in  $A$  by  $\alpha_1, \alpha_2$  and  $\alpha_3$  and the shifts in  $B$  by  $\alpha_4, \alpha_5$  and  $\alpha_6$ . Let  $C$  be a positive constant. We will first prove the above claim under the extra assumption that

$$|\alpha_i \pm \alpha_j| \geq 2C/\log Q, \text{ for any } i \neq j, \quad (6.126)$$

where  $\pm$  means that the condition holds both with the plus sign and with the minus sign. Also, we are of course still assuming that all the shifts are  $\ll 1/\log Q$ .

We apply Theorem 6.2 to the LHS of this theorem. In the RHS of (6.20) we split the integral involving  $\Phi(t)$  as follows:

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{-\log Q} + \int_{-\log Q}^{\log Q} + \int_{\log Q}^{\infty}. \quad (6.127)$$

Let us denote the contributions from these three parts by  $P_1$ ,  $P_2$  and  $P_3$ . We will first treat the main term  $P_2$ . Then we will show that the term  $P_3$  can be absorbed into the error term of this theorem and the situation is the same with  $P_1$ .

By putting together Lemmas 6.2, 6.3 and 6.4 we have that

$$\begin{aligned} & G_{A_{it}, B_{-it}}\left(\frac{\mathcal{A}}{\mathcal{B}_q}\right)(A_{it}, B_{-it}) \\ &= |\Gamma((1/2 + it)/2)|^6 a_3 \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \left(1 + O\left(\frac{(\log \log Q)^M}{\log Q}\right)\right), \end{aligned} \quad (6.128)$$

for  $|t| \leq \log Q$ .

In the RHS of (6.20),  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$  is a sum of 20 terms. In each of the latter we substitute in (6.128). The part of  $P_2$  coming from the contribution coming from picking out the main term in (6.128) in all of the 20 cases is equal to the main term of this theorem. Next we show that the contribution from each of the 20 error terms coming from the substitution of (6.128) can be absorbed into the error term of this theorem.

Consider one of the 20 error-parts described above. Since the shifts are assumed to be at most of order  $1/\log Q$ , we clearly have that

$$\left(\frac{q}{\pi}\right)^{\delta_{X,Y}} \ll 1. \quad (6.129)$$

Also, by the extra assumption (6.126), we have that

$$\mathcal{Z}(X, Y) \ll \log^9 Q, \quad (6.130)$$

for all relevant sets  $X$  and  $Y$ . Therefore by using Remark 6.5, we obtain the required upper bound.

In order to complete the proof of this theorem under the extra assumption we have to show that  $P_3$  is sufficiently small to be absorbed into the error term of this

theorem. Of course this follows if we are able to find a suitable upper bound for the contribution of any individual of the 20 terms that  $\mathcal{Q}_{A_{it}, B_{-it}}(q)$  constitutes.

We will again make use of (6.129) and (6.130). However, as we can no longer use (6.128), we will instead simply find an upper bound for

$$G_{A_{it}, B_{-it}}\left(\frac{\mathcal{A}}{\mathcal{B}_q}\right)(A_{it}, B_{-it}). \quad (6.131)$$

As the gamma function  $\Gamma(s)$  is bounded for  $2/5 \leq \operatorname{Re}(s) \leq 3/5$ , we have

$$G_{A_{it}, B_{-it}} \ll 1. \quad (6.132)$$

We can deduce from Lemma 6.3 that

$$\mathcal{B}_q^{-1}(A_{it}, B_{-it}) \ll \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \quad (6.133)$$

and Lemma 6.4 yields that

$$\mathcal{A}(A_{it}, B_{-it}) \ll 1. \quad (6.134)$$

Using all these upper bounds we reach our conclusion upon remembering Remarks 6.5 and 6.6.

Suppose that the theorem is to be proved for all  $|\alpha_i| \leq C/\log Q$ . Consider now (without the extra assumption) any  $|\alpha_i| \leq C/\log Q$ . The idea is to use Cauchy's integral formula in order to go from the above "easier" case to the general case. Certainly the LHS of this theorem is an analytic function of the complex variables  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  and  $\alpha_6$ , let us denote it by  $L(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ . We can therefore replace it by a 6-fold integral involving  $L(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ . Explicitly, let  $D$  be the polydisc defined as the Cartesian product of the open discs  $D_i$ , i.e.  $D = \prod_{i=1}^6 D_i$ , where

$$D_i := \{s \in \mathbb{C} : |s - \alpha_i| < r_i\}, \quad (6.135)$$

with

$$r_i = \frac{2^{i+1}C}{\log Q}. \quad (6.136)$$

An application of Cauchy's integral formula yields that

$$\begin{aligned} & L(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \\ &= \frac{1}{(2\pi i)^6} \int \cdots \int \int_{\partial D_1 \times \cdots \times \partial D_6} \frac{L(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)}{(\beta_1 - \alpha_1) \cdots (\beta_6 - \alpha_6)} d\beta_1 \cdots d\beta_6. \end{aligned} \quad (6.137)$$

Now we notice that the  $\beta_i$  satisfy  $|\beta_i - \alpha_i| = r_i$ . But this is easily seen to imply that  $\beta_i \ll 1/\log Q$  and that  $|\beta_i \pm \beta_j| \geq 2C/\log Q$ , so that this theorem applies if the  $\beta_i$ -terms are seen as shifts. We substitute in the outcome for  $L(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$  into (6.137). The error term we get in the integrand goes through and stays of course of the same order. The main term that we now have in the numerator of our integrand, let us denote it by  $R(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ , is an analytic function of the complex variables  $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$  and  $\beta_6$ , which means that we can undo the Cauchy-process and retrieve  $R(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ . The former can be seen by using Lemma 2.5.1 in the article [10] (by working in a similar way as in the paragraph after (6.46)). This completes the proof.  $\square$

Theorem 6.3 is simpler than its “earlier” version Theorem 6.2 in the sense that some terms in the former are independent of the shifts. We will next find a simpler version of Theorem 6.3.

**Definition 6.3.** Define

$$\mathcal{R}_{A,B}(Q) := \sum_{\substack{S \subset A \\ T \subset B \\ |S|=|T|}} \mathcal{R}(\bar{S} \cup (-T), \bar{T} \cup (-S); Q), \quad (6.138)$$

where

$$\mathcal{R}(X, Y; Q) := Q^{\delta_{X,Y}} \prod_{\substack{x \in X \\ y \in Y}} \frac{1}{x+y}. \quad (6.139)$$

**Theorem 6.4.** *Suppose that  $|A| = |B| = 3$  and that  $\alpha, \beta \ll 1/\log Q$ , for  $\alpha \in A$ ,  $\beta \in B$ . Suppose also that  $\Psi$  and  $\Phi$  are reasonable. Then there exists  $M \in \mathbb{N}$  such that*

$$\begin{aligned} & \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(t) \sum_{\chi \pmod{q}}^b \Lambda_{A_{it}, B_{-it}}(\chi) dt \\ &= a_3(\mathcal{L})w(\Phi) \int_0^{\infty} \frac{\Psi(x)x}{2} dx Q^2 \mathcal{R}_{A,B}(Q) + O\left(Q^2 \log^8 Q (\log \log Q)^M\right), \end{aligned} \quad (6.140)$$

where we have introduced the notation

$$w(\Phi) := \int_{-\infty}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt. \quad (6.141)$$

*Proof.* We will below prove this theorem under the extra assumption (6.126). The general case then follows exactly as in the proof of Theorem 6.3.

Let us focus on  $\mathcal{P}_{A,B}(q)$  in the RHS of Theorem 6.3 which is a sum of 20 terms. We will next look at what such a term looks like (recall (6.124)).

Now  $\mathcal{Z}(X, Y)$  is a product of nine factors of the form  $\zeta(1 + \lambda)$ , where  $\lambda$  is a linear combination of the shifts (naturally we here do not mean that all  $\lambda$  are exactly the same). For each factor we may write

$$\zeta(1 + \lambda) = \frac{1}{\lambda} + O(1). \quad (6.142)$$

Notice that due to our assumption (6.126), we have that

$$1/\lambda \ll \log Q. \quad (6.143)$$

Suppose we multiply together the nine factors of  $\mathcal{Z}(X, Y)$ . We may choose the term  $1/\lambda$  or the term  $O(1)$  in (6.142). If we do not in each case choose the former, the corresponding product would be

$$\ll \log^8 Q. \quad (6.144)$$

If we then multiply this by  $(\frac{q}{\pi})^{\delta_{X,Y}}$  we would, since (6.129) holds, still obtain something that is

$$\ll \log^8 Q. \quad (6.145)$$

As we have seen before (i.e. via use of the fact that (6.85)  $\ll Q^2$ ), one may then conclude that this will give a total contribution of order

$$\ll Q^2 \log^8 Q \quad (6.146)$$

which thus may be absorbed into the error term of this theorem. Thus (under the assumption of (6.126)) we may replace  $\mathcal{Z}(X, Y)$  (in Theorem 6.3) by

$$\prod_{\substack{x \in X \\ y \in Y}} \frac{1}{x + y}. \quad (6.147)$$

Next, we consider

$$\left(\frac{q}{\pi}\right)^{\delta_{X,Y}} = Q^{\delta_{X,Y}} \cdot \exp(\delta_{X,Y} \log(q/Q\pi)). \quad (6.148)$$

Notice that since  $\Psi$  is compactly supported in  $[1, 2]$  we may assume that  $Q \leq q \leq 2Q$ . Also, clearly  $\delta_{X,Y} \ll 1/\log Q$ . Therefore

$$\left(\frac{q}{\pi}\right)^{\delta_{X,Y}} = Q^{\delta_{X,Y}} \cdot \left(1 + O\left(\frac{1}{\log Q}\right)\right). \quad (6.149)$$

Since

$$Q^{\delta_{x,Y}} \ll 1, \quad (6.150)$$

the same reasoning as in the previous paragraph tells us that we may replace  $(\frac{q}{\pi})^{\delta_{x,Y}}$  by  $Q^{\delta_{x,Y}}$  (again under the assumption (6.126)).

At this point our main term looks like

$$a_3 \mathcal{R}_{A,B}(Q) \int_{-\log Q}^{\log Q} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \phi^b(q) \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})}. \quad (6.151)$$

Using (6.85) and (6.91), it follows immediately that (6.151) is

$$\begin{aligned} & a_3(\mathcal{L}) \times \mathcal{R}_{A,B}(Q) \times \left\{ \int_{-\infty}^{\infty} \Phi(t) |\Gamma((1/2 + it)/2)|^6 dt + O(Q^{-1}) \right\} \\ & \times \left\{ Q^2 \int_0^{\infty} \frac{\Psi(x)x}{2} dx + O(Q^{29/16}) \right\}. \end{aligned} \quad (6.152)$$

Recalling that our assumption implies that  $\mathcal{R}_{A,B}(Q) \ll \log^9 Q$ , it is easily seen that all but the main term in (6.152) can be relegated to the error term of this theorem, leaving us with the desired main term.  $\square$

## 6.6 Differentiated version of Theorem 6.4

**Theorem 6.5.** *Suppose that  $|A| = |B| = 3$  and that  $\alpha, \beta \ll 1/\log Q$ , for  $\alpha \in A$ ,  $\beta \in B$ . Suppose also that  $\Psi$  and  $\Phi$  are reasonable. Denote the shifts in  $A$  by  $\alpha_1, \alpha_2$  and  $\alpha_3$  and the shifts in  $B$  by  $\alpha_4, \alpha_5$  and  $\alpha_6$ . Then there exists  $M \in \mathbb{N}$  such that*

$$\begin{aligned} & \sum_{q \geq 1} \Psi\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi(t) \sum_{\chi(\bmod q)}^b \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \Lambda_{A_{it}, B_{-it}}(\chi) dt \\ & = a_3(\mathcal{L}) w(\Phi) \int_0^{\infty} \frac{\Psi(x)x}{2} dx Q^2 \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \mathcal{R}_{A,B}(q) + O\left(Q^2 \log^{10} Q (\log \log Q)^M\right). \end{aligned} \quad (6.153)$$

*Proof.* This is simple and we work as in the proof of Theorem 5.7. Thus we carefully use Cauchy's integral formula and bound the error term trivially.  $\square$

## 6.7 Building-stones in the proof of Theorem 6.1

### 6.7.1 Main Assumption 6.1

We will now make an “assumption”.

**Main Assumption 6.1.** Suppose that for all  $\frac{5Q}{4} < q \leq \frac{7Q}{4}$  and all even primitive Dirichlet characters  $\chi(\bmod q)$  we have that all the gaps with  $t \in [T, 2T]$  between consecutive zeros of  $L(\frac{1}{2} + it, \chi)$  are<sup>10</sup> at most  $\frac{3\kappa}{\log Q}$ .

**Definition 6.4** (Remark on Main Assumption 6.1). We will in fact take  $\kappa$  to be the smallest real number such that Main Assumption 6.1 is satisfied.

**Remark 6.7.** Thus  $\kappa$  will depend on  $Q$  and will obviously be well-defined.

### 6.7.2 Introducing the function $f(t, \chi, \kappa)$

In Section 4.3 we introduced a function  $H(s, \chi)$  with the properties that  $H(1/2 + it, \chi)$  is real (for  $t \in \mathbb{R}$ ) and that  $H(1/2 + it, \chi) = 0$  if and only if  $L(\frac{1}{2} + it, \chi) = 0$ .

**Definition 6.5.** For  $\chi$  being an even primitive Dirichlet character, define

$$\begin{aligned} W(t, \chi) &:= \left(\frac{q}{\pi}\right)^{-\frac{1}{4}} H(1/2 + it, \chi) \\ &= K(\chi)^{1/2} \left(\frac{q}{\pi}\right)^{\frac{1}{2}(\frac{1}{2} + it) - \frac{1}{4}} \Gamma\left(\frac{\frac{1}{2} + it}{2}\right) L(\frac{1}{2} + it, \chi). \end{aligned} \quad (6.154)$$

**Remark 6.8.** Using the comment before Definition 6.5, it is obvious that  $W(t, \chi)$  is real (for  $t \in \mathbb{R}$ ) and that  $W(t, \chi) = 0$  if and only if  $L(\frac{1}{2} + it, \chi) = 0$ .

We next introduce one more function.

**Definition 6.6.**

$$f(t, \chi, \kappa) = f(t, \chi, \kappa, Q) := W\left(t - \frac{\kappa}{\log Q}, \chi\right) W(t, \chi) W\left(t + \frac{\kappa}{\log Q}, \chi\right). \quad (6.155)$$

**Remark 6.9.** Denote the zeros of  $f(t, \chi, \kappa)$  with  $T + \frac{\kappa}{\log Q} \leq t \leq 2T - \frac{\kappa}{\log Q}$  by  $t_{1,\chi}, t_{2,\chi}, \dots, t_{N_\chi,\chi}$ , ordered in non-decreasing order. Then keeping Remark 6.8 in mind, it is simple<sup>11</sup> to deduce that Main Assumption 6.1 implies that

$$t_{n+1,\chi} - t_{n,\chi} \leq \frac{\kappa}{\log Q}, \quad (6.156)$$

for  $n = 1, 2, \dots, N_\chi - 1$ .

<sup>10</sup>For either of the two zeros near the endpoints of the interval we will here mean the distance from them to the respective endpoint.

<sup>11</sup>The reader can, for example by drawing a simple diagram, easily do an argument by contradiction to reach the desired conclusion.

**Remark 6.10.** For future reference we note here that Main Assumption 6.1 implies that

$$t_{1,\chi} \leq T + \frac{2\kappa}{\log Q} \text{ and } t_{N_\chi,\chi} \geq 2T - \frac{2\kappa}{\log Q}. \quad (6.157)$$

### 6.7.3 Wirtinger's inequality and an application of it

By using the simplest version of Wirtinger's inequality (see (3.3)) and working exactly as in Chapter 3, we get the following:

**Corollary 6.2.** For  $n = 1, 2, \dots, N_\chi - 1$  we have that

$$\begin{aligned} \int_{t_{n,\chi}}^{t_{n+1,\chi}} f(t, \chi, \kappa)^2 dt &\leq \left( \frac{t_{n+1,\chi} - t_{n,\chi}}{\pi} \right)^2 \int_{t_{n,\chi}}^{t_{n+1,\chi}} f'(t, \chi, \kappa)^2 dt \\ &\leq \left( \frac{\kappa}{\pi \log Q} \right)^2 \int_{t_{n,\chi}}^{t_{n+1,\chi}} f'(t, \chi, \kappa)^2 dt. \end{aligned} \quad (6.158)$$

**Corollary 6.3.** For any  $\frac{5Q}{4} < q \leq \frac{7Q}{4}$  and all even primitive Dirichlet characters  $\chi(\bmod q)$  we have that

$$\int_{t_{1,\chi}}^{t_{N_\chi,\chi}} f(t, \chi, \kappa)^2 dt \leq \left( \frac{\kappa}{\pi \log Q} \right)^2 \int_{t_{1,\chi}}^{t_{N_\chi,\chi}} f'(t, \chi, \kappa)^2 dt. \quad (6.159)$$

*Proof.* We simply sum up the inequalities in Corollary 6.2 for  $n = 1, 2, \dots, N_\chi - 1$ .  $\square$

**Corollary 6.4.**

$$\begin{aligned} \sum_{\frac{5Q}{4} < q \leq \frac{7Q}{4}} \sum_{\chi(\bmod q)}^b \int_{t_{1,\chi}}^{t_{N_\chi,\chi}} f(t, \chi, \kappa)^2 dt \\ \leq \left( \frac{\kappa}{\pi \log Q} \right)^2 \sum_{\frac{5Q}{4} < q \leq \frac{7Q}{4}} \sum_{\chi(\bmod q)}^b \int_{t_{1,\chi}}^{t_{N_\chi,\chi}} f'(t, \chi, \kappa)^2 dt. \end{aligned} \quad (6.160)$$

*Proof.* We simply sum up the inequalities in Corollary 6.3 over all the relevant Dirichlet characters.  $\square$

## 6.8 Investigating Corollary 6.4

Our goal in this section is basically to find estimates for the LHS and the RHS in Corollary 6.4. This shall then give us an inequality in terms of  $\kappa$  which we will consider in Section 6.9.

### 6.8.1 Lower bound for the LHS of Corollary 6.4

Let us begin by finding a nice expression for a lower bound for the LHS in Corollary 6.4. By Remark 6.10, the integration limits are very close to  $T$  and  $2T$  respectively. Let  $\epsilon$  be a small (fixed) positive number. Then clearly the LHS of Corollary 6.4 is

$$\geq \sum_{q \geq 1} \chi_{(\frac{5}{4}, \frac{7}{4}]} \left( \frac{q}{Q} \right) \sum_{\chi \pmod{q}}^b \int_{T+\epsilon}^{2T-\epsilon} f(t, \chi, \kappa)^2 dt. \quad (6.161)$$

Next we notice that if we let the shifts in  $A$  and  $B$  be  $\frac{\kappa i}{\log Q}$ ,  $0$  and  $-\frac{\kappa i}{\log Q}$  (and we do that!), then

$$f(t, \chi, \kappa)^2 = \Lambda_{A_{it}, B_{-it}}(\chi). \quad (6.162)$$

Substituting this into (6.161) we have an expression that looks a lot like the LHS of Theorem 6.2. However, in order to proceed we will need to approximate the characteristic functions by suitable weight-functions  $\Psi$  and  $\Phi$  that satisfy the conditions of Theorem 6.4. Below  $u$  will denote a suitably small (fixed) positive number.

Construction of  $\Psi_1$  : First we will define

$$g(x) := \begin{cases} \exp(-u^2/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \quad (6.163)$$

Then  $g(x)$  is smooth and approximates  $\chi_{[0, \infty)}(t)$ . Now let

$$\Psi_1(t) := g(t - 5/4)g(7/4 - t). \quad (6.164)$$

Then clearly

$$\Psi_1(t) \leq \chi_{(5/4, 7/4]}(t). \quad (6.165)$$

Construction of  $\Phi_1$  : We define

$$\Phi_1(z) := \frac{1}{\sqrt{\pi}u} \int_{T+\epsilon+\sqrt{u}}^{2T-\epsilon-\sqrt{u}} \exp(-(v-z)^2/u^2) dv - \exp\left(\left((3T)^2 - \frac{1}{u}\right) \exp(-z^2)\right). \quad (6.166)$$

Then  $\Phi_1(z)$  is reasonable and is pretty “close” to  $\chi_{[T, 2T]}(t)$ , if we see it as a function from  $\mathbb{R}$  to  $\mathbb{R}$ . Indeed for  $t \in \mathbb{R}$  we have the following:

$$\Phi_1(t) \leq \chi_{[T+\epsilon, 2T-\epsilon]}(t). \quad (6.167)$$

One possible way to show (6.167) is to simply consider what happens when  $t \in [T+\epsilon, 2T-\epsilon]$ , when this fails but  $t$  is somewhat close to that interval (say  $t \in [0, 3T]$ ) and finally when  $t \notin [0, 3T]$ , keeping in mind that the Gaussian integral satisfies

$$\int_{-\infty}^{\infty} \exp(-t^2) dt = \sqrt{\pi} \quad (6.168)$$

and the Chernoff bound (6.90) for the complimentary error function. Explicitly, in the first of the three above cases one trivially has

$$\Phi_1(t) \leq \frac{2}{\sqrt{\pi}u} \int_0^{\infty} \exp(-v^2/u^2) dv \leq 1. \quad (6.169)$$

If  $t \in [0, T + \epsilon]$  or  $t \in [2T - \epsilon, 3T]$  then one has

$$\begin{aligned} \Phi_1(t) &\leq \frac{1}{\sqrt{\pi}u} \int_{\sqrt{u}}^{\infty} \exp(-v^2/u^2) dv - \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-(3T)^2) \\ &= \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{1/\sqrt{u}}^{\infty} \exp(-v^2) dv - \exp(-1/u) \\ &\leq 1 \cdot \exp(-1/u) - \exp(-1/u) = 0. \end{aligned} \quad (6.170)$$

If  $t < 0$  then one has

$$\begin{aligned} \Phi_1(t) &\leq \frac{1}{\sqrt{\pi}u} \int_{\sqrt{u}-t}^{\infty} \exp(-v^2/u^2) dv - \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-t^2) \\ &= \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{(\sqrt{u}-t)/u}^{\infty} \exp(-v^2) dv - \exp\left((3T)^2 - \frac{1}{u} - t^2\right) \\ &\leq 1 \cdot \exp\left(-\{(\sqrt{u}-t)/u\}^2\right) - \exp\left(-\frac{1}{u} - t^2\right) \\ &\leq \exp\left(-1/u - (t/u)^2\right) - \exp\left(-\frac{1}{u} - t^2\right) \leq 0. \end{aligned} \quad (6.171)$$

Finally, if  $t > 3T$  then one has

$$\begin{aligned} \Phi_1(t) &\leq \frac{1}{\sqrt{\pi}u} \int_{\sqrt{u}+t-2T}^{\infty} \exp(-v^2/u^2) dv - \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-t^2) \\ &= \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{(\sqrt{u}+t-2T)/u}^{\infty} \exp(-v^2) dv - \exp\left((3T)^2 - \frac{1}{u} - t^2\right) \\ &\leq 1 \cdot \exp\left(-\{(\sqrt{u}+t-2T)/u\}^2\right) - \exp\left(-\frac{1}{u} - t^2\right) \\ &\leq \exp\left(-\{(\sqrt{u}+t/3)/u\}^2\right) - \exp\left(-\frac{1}{u} - t^2\right) \\ &\leq \exp\left(-1/u - (t/3u)^2\right) - \exp\left(-\frac{1}{u} - t^2\right) \leq 0. \end{aligned} \quad (6.172)$$

Now going back to (6.161), then clearly the LHS of Corollary 6.4 is

$$\geq \sum_{q \geq 1} \Psi_1\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi_1(t) \sum_{\chi \pmod{q}}^b \Lambda_{A_{it}, B_{-it}}(\chi) dt. \quad (6.173)$$

We apply Theorem 6.4 to this expression and obtain

$$a_3(\mathcal{L})w(\Phi_1) \int_0^{\infty} \frac{\Psi_1(x)x}{2} dx Q^2 \mathcal{R}_{A,B}(Q) + O\left(Q^2 \log^8 Q (\log \log Q)^M\right). \quad (6.174)$$

At this point we simply claim that (6.174) equals

$$\begin{aligned} & C_0(\kappa) a_3(\mathcal{L}) Q^2 \log^9 Q \int_0^{\infty} \frac{\Psi_1(x)x}{2} dx \int_{-\infty}^{\infty} \Phi_1(t) |\Gamma((1/2 + it)/2)|^6 dt \\ & + O\left(Q^2 \log^8 Q (\log \log Q)^M\right), \end{aligned} \quad (6.175)$$

where  $C_0(\kappa)$  is a coefficient<sup>12</sup> which naturally depends on the shift  $\kappa$ . We will return to this in Section 6.10.

We now need to find lower bounds for  $\Psi_1$  and  $\Phi_1$ . One easily shows that

$$\Psi_1(t) \geq \exp(-2u) \chi_{[5/4+u, 7/4-u]}(t) \quad (6.176)$$

and

$$\begin{aligned} \Phi_1(t) & \geq \left(1 - 2 \exp\left((3T)^2 - \frac{1}{u}\right)\right) \chi_{[T+\epsilon+2\sqrt{u}, 2T-\epsilon-2\sqrt{u}]}(t) \\ & - \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-t^2) \chi_{\mathbb{R} \setminus [T+\epsilon+2\sqrt{u}, 2T-\epsilon-2\sqrt{u}]}(t). \end{aligned} \quad (6.177)$$

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<sup>12</sup>For the explicit expression of  $C_0(\kappa)$ , see (6.197).

Substituting these two into (6.175), we conclude that (6.175) is

$$\begin{aligned}
&\geq C_0(\kappa)a_3(\mathcal{L})Q^2 \log^9 Q \int_{5/4+u}^{7/4-u} \frac{x \exp(-2u)}{2} dx \\
&\times \int_{-\infty}^{\infty} \left\{ \left(1 - 2 \exp\left((3T)^2 - \frac{1}{u}\right)\right) \chi_{[T+\epsilon+2\sqrt{u}, 2T-\epsilon-2\sqrt{u}]}(t) \right. \\
&\quad \left. - \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-t^2) \chi_{\mathbb{R} \setminus [T+\epsilon+2\sqrt{u}, 2T-\epsilon-2\sqrt{u}]}(t) \right\} |\Gamma((1/2 + it)/2)|^6 dt \\
&+ O\left(Q^2 \log^8 Q (\log \log Q)^M\right) \\
&\geq C_1(u, \epsilon)C_0(\kappa)a_3(\mathcal{L})Q^2 \log^9 Q \int_{5/4}^{7/4} \frac{x}{2} dx \int_T^{2T} |\Gamma((1/2 + it)/2)|^6 dt \\
&+ O\left(Q^2 \log^8 Q (\log \log Q)^M\right), \tag{6.178}
\end{aligned}$$

where  $C_1(u, \epsilon)$  is some constant which tends to 1 as  $u$  and  $\epsilon$  both tend<sup>13</sup> to 0.

We can now conclude that the LHS of Corollary 6.4 is

$$\begin{aligned}
&\geq C_1(u, \epsilon)C_0(\kappa)a_3(\mathcal{L})Q^2 \log^9 Q \int_{5/4}^{7/4} \frac{x}{2} dx \int_T^{2T} |\Gamma((1/2 + it)/2)|^6 dt \\
&\quad + O\left(Q^2 \log^8 Q (\log \log Q)^M\right). \tag{6.179}
\end{aligned}$$

## 6.8.2 Upper bound for the RHS of Corollary 6.4

Now let us turn our attention to the RHS of Corollary 6.4. We will deal with this in a similar way. The inequalities will go the other way around this time and we will make use of Theorem 6.5 rather than Theorem 6.4, but the basic strategy will be the same as in the treatment in Section 6.8.1 of the LHS.

The function  $f'(t, \chi, \kappa)^2$  features in the RHS of Corollary 6.4. Using the definition given in (6.6) we find that

$$\begin{aligned}
f'(t, \chi, \kappa) &:= W'\left(t - \frac{2\pi\kappa}{\log Q}, \chi\right) W(t, \chi) W\left(t + \frac{2\pi\kappa}{\log Q}, \chi\right) \\
&\quad + W\left(t - \frac{2\pi\kappa}{\log Q}, \chi\right) W'(t, \chi) W\left(t + \frac{2\pi\kappa}{\log Q}, \chi\right) \\
&\quad + W\left(t - \frac{2\pi\kappa}{\log Q}, \chi\right) W(t, \chi) W'\left(t + \frac{2\pi\kappa}{\log Q}, \chi\right). \tag{6.180}
\end{aligned}$$

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<sup>13</sup>This is to be interpreted as that we can choose to use as small fixed  $u$  and  $\epsilon$  as we like.

Hence we get nine different parts to consider. Now, forgetting for the moment that the integral in Corollary 6.4 does not quite go from  $T$  to  $2T$  and that Theorem 6.5 requires  $\Psi$  and  $\Phi$  to be reasonable, let us see what the big picture is. Denoting the shifts in  $A$  and  $B$  by  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4, \alpha_5, \alpha_6$ , we may apply partial differentiation in Theorem 6.5 to whichever pair of those six variables we like. Let us decide to after differentiation put  $\alpha_1 = \alpha_4 = \frac{\kappa i}{\log Q}$ ,  $\alpha_2 = \alpha_5 = 0$  and  $\alpha_3 = \alpha_6 = -\frac{\kappa i}{\log Q}$ . Say for example that we wanted to do the case of

$$W'(t - \frac{\kappa}{\log Q}, \chi)^2 W(t, \chi)^2 W(t + \frac{\kappa}{\log Q}, \chi)^2. \quad (6.181)$$

Then we differentiate with respect to  $\alpha_3$  and  $\alpha_4$ . Naturally we use that differentiated case of Theorem 6.5. Just as we in Corollary 6.1 got the coefficient  $42/9!$ , we will here get coefficients depending on  $\kappa$ . The calculations of these  $\kappa$ -coefficients are elementary but take a lot of time. Indeed, they are similar to the calculation in Section 6.10, where  $C_0(\kappa)$  is found. However, before we go into this, we will now first explicitly show that effectively Theorem 6.5 may be applied.

We will now construct suitable functions  $\Psi_2$  and  $\Phi_2$ .

Construction of  $\Psi_2$  : Recall the definition of the function  $g(x)$  given in (6.163). Now let

$$\Psi_2(t) := \exp(2u)g(t - (5/4 - u))g((7/4 + u) - t). \quad (6.182)$$

Then clearly  $\Psi_2(t)$  is compactly supported in  $[1, 2]$  and

$$\Psi_2(t) \geq \chi_{(5/4, 7/4]}(t). \quad (6.183)$$

Construction of  $\Phi_2$  :

$$\Phi_2(z) := \frac{u^{-1}}{\sqrt{\pi}(1 - \exp(-1/u))} \int_{T-\sqrt{u}}^{2T+\sqrt{u}} \exp(-(v - z)^2/u^2) dv. \quad (6.184)$$

For  $t \in \mathbb{R}$  we have the following:

$$\Phi_2(t) \geq \chi_{[T, 2T]}(t). \quad (6.185)$$

Thus the RHS of Corollary 6.4 is

$$\leq \left(\frac{\kappa}{\pi \log Q}\right)^2 \sum_{q \geq 1} \Psi_2\left(\frac{q}{Q}\right) \int_{-\infty}^{\infty} \Phi_2(t) \sum_{\chi(\bmod q)}^b f'(t, \chi, \kappa)^2 dt. \quad (6.186)$$

We write out the nine terms we get from expanding out  $f'(t, \chi, \kappa)^2$  (cf. (6.180)). In each case we may then apply Theorem 6.5. Then it is crucial to find the  $\kappa$ -depending

main-term-coefficients mentioned above. Before exploring what these coefficients are, we must conclude the present discussion. Thus after having done the above in any of the nine cases we are left with an expression of the type (ignoring for now the constant  $(\frac{\kappa}{\pi \log Q})^2$ )

$$C_i(\kappa)a_3(\mathcal{L})Q^2 \log^{11} Q \int_0^\infty \frac{\Psi_2(x)x}{2} dx \int_{-\infty}^\infty \Phi_2(t)|\Gamma((1/2 + it)/2)|^6 dt + O\left(Q^2 \log^{10} Q(\log \log Q)^M\right), \quad (6.187)$$

where  $C_i(\kappa)$  is the relevant  $\kappa$ -constant in the  $i$ -th case out of the total nine cases.

We now need to find upper bounds for  $\Psi_2$  and  $\Phi_2$ . For  $t \in \mathbb{R}$  we have the following:

$$\Psi_2(t) \leq \exp(2u)\chi_{[5/4-u, 7/4+u]}(t) \quad (6.188)$$

and

$$\Phi_2(t) \leq \frac{1}{(1 - \exp(-1/u))} \chi_{[T-2\sqrt{u}, 2T+2\sqrt{u}]}(t) + \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-t^2) \chi_{\mathbb{R} \setminus [T-2\sqrt{u}, 2T+2\sqrt{u}]}(t). \quad (6.189)$$

Substituting these two into (6.187), we conclude that (6.187) is

$$\begin{aligned} &\leq C_i(\kappa)a_3(\mathcal{L})Q^2 \log^{11} Q \int_{5/4-u}^{7/4+u} \frac{x \exp(2u)}{2} dx \\ &\times \int_{-\infty}^\infty \left\{ \frac{1}{(1 - \exp(-1/u))} \chi_{[T-2\sqrt{u}, 2T+2\sqrt{u}]}(t) + \exp\left((3T)^2 - \frac{1}{u}\right) \exp(-t^2) \chi_{\mathbb{R} \setminus [T-2\sqrt{u}, 2T+2\sqrt{u}]}(t) \right\} |\Gamma((1/2 + it)/2)|^6 dt \\ &+ O\left(Q^2 \log^{10} Q(\log \log Q)^M\right). \\ &\leq C_2(u, \epsilon)C_i(\kappa)a_3(\mathcal{L})Q^2 \log^{11} Q \int_{5/4}^{7/4} \frac{x}{2} dx \int_T^{2T} |\Gamma((1/2 + it)/2)|^6 dt \\ &+ O\left(Q^2 \log^{10} Q(\log \log Q)^M\right), \quad (6.190) \end{aligned}$$

where  $C_2(u, \epsilon)$  is some constant which tends to 1 as  $u$  and  $\epsilon$  both tend to 0.

### 6.8.3 $\kappa$ -coefficients

We finish this section by stating what these  $\kappa$ -coefficients are. For simplicity let us use the notation  $\text{Macl}(f(z)) = f(z) - \text{Princ}(f(z))$ , where  $\text{Princ}(f(z))$  is the principal part of  $f(z)$ . An example is  $\text{Macl}(\frac{\cos z}{z^2}) = \frac{\cos z}{z^2} - \frac{1}{z^2}$ . Let us introduce the following notation for the  $\kappa$ -coefficients:

$C_1(\kappa)$  in the case of ...

$$W(t - \frac{2\pi\kappa}{\log Q}, \chi)^2 W'(t, \chi)^2 W(t + \frac{2\pi\kappa}{\log Q}, \chi)^2, \quad (6.191)$$

$C_2(\kappa) = C_3(\kappa)$  in the case of ...

$$W'(t - \frac{2\pi\kappa}{\log Q}, \chi) W(t - \frac{2\pi\kappa}{\log Q}, \chi) W(t, \chi)^2 W'(t + \frac{2\pi\kappa}{\log Q}, \chi) W(t + \frac{2\pi\kappa}{\log Q}, \chi), \quad (6.192)$$

$C_4(\kappa) = C_5(\kappa)$  in the case of ...

$$W'(t - \frac{2\pi\kappa}{\log Q}, \chi)^2 W(t, \chi)^2 W(t + \frac{2\pi\kappa}{\log Q}, \chi)^2 \quad (6.193)$$

and

$$W(t - \frac{2\pi\kappa}{\log Q}, \chi)^2 W(t, \chi)^2 W'(t + \frac{2\pi\kappa}{\log Q}, \chi)^2 \quad (6.194)$$

and finally  $C_6(\kappa) = C_7(\kappa) = C_8(\kappa) = C_9(\kappa)$  in the case of ...

$$W'(t - \frac{2\pi\kappa}{\log Q}, \chi) W(t - \frac{2\pi\kappa}{\log Q}, \chi) W(t, \chi) W'(t, \chi) W(t + \frac{2\pi\kappa}{\log Q}, \chi)^2 \quad (6.195)$$

and

$$W(t - \frac{2\pi\kappa}{\log Q}, \chi)^2 W(t, \chi) W'(t, \chi) W'(t + \frac{2\pi\kappa}{\log Q}, \chi) W(t + \frac{2\pi\kappa}{\log Q}, \chi). \quad (6.196)$$

Thus the nine cases actually only give rise to four different  $\kappa$ -coefficients.

The calculations needed to find our  $\kappa$ -coefficients are long, but similar. We present the details of the simplest case, namely  $C_0(\kappa)$ , in Section 6.10. We have

$$C_0(\kappa) = \text{Macl} \left\{ \frac{\cos \kappa}{\kappa^8} + \frac{\sin \kappa}{\kappa^9} + \frac{\cos(2\kappa)}{8\kappa^8} - \frac{\sin(2\kappa)}{2\kappa^9} \right\}, \quad (6.197)$$

$$C_1(\kappa) = \text{Macl} \left\{ -\frac{\cos \kappa}{4\kappa^8} + \frac{3 \sin \kappa}{4\kappa^9} + \frac{\cos \kappa}{\kappa^{10}} - \frac{\sin \kappa}{\kappa^{11}} \right. \\ \left. + \frac{\cos(2\kappa)}{96\kappa^8} - \frac{\sin(2\kappa)}{8\kappa^9} - \frac{\cos(2\kappa)}{2\kappa^{10}} + \frac{\sin(2\kappa)}{2\kappa^{11}} \right\}, \quad (6.198)$$

$$C_2(\kappa) = \text{Macl} \left\{ -\frac{3 \sin \kappa}{4\kappa^9} - \frac{11 \cos \kappa}{4\kappa^{10}} - \frac{21 \sin \kappa}{4\kappa^{11}} \right. \\ \left. + \frac{\cos(2\kappa)}{32\kappa^8} - \frac{3 \sin(2\kappa)}{8\kappa^9} - \frac{53 \cos(2\kappa)}{32\kappa^{10}} + \frac{21 \sin(2\kappa)}{8\kappa^{11}} \right\}, \quad (6.199)$$

$$C_4(\kappa) = \text{Macl} \left\{ -\frac{\cos \kappa}{12\kappa^8} + \frac{5 \sin \kappa}{4\kappa^9} + \frac{3 \cos \kappa}{\kappa^{10}} + \frac{5 \sin \kappa}{\kappa^{11}} \right. \\ \left. - \frac{\cos(2\kappa)}{32\kappa^8} + \frac{3 \sin(2\kappa)}{8\kappa^9} + \frac{13 \cos(2\kappa)}{8\kappa^{10}} - \frac{5 \sin(2\kappa)}{2\kappa^{11}} \right\} \quad (6.200)$$

and

$$C_6(\kappa) = \text{Macl} \left\{ \frac{\cos \kappa}{8\kappa^8} - \frac{3 \sin \kappa}{8\kappa^9} - \frac{5 \cos \kappa}{4\kappa^{10}} - \frac{3 \sin \kappa}{4\kappa^{11}} - \frac{\cos(2\kappa)}{16\kappa^{10}} + \frac{3 \sin(2\kappa)}{8\kappa^{11}} \right\}. \quad (6.201)$$

**Remark 6.11.** Note that the  $\kappa^0$ -coefficient in the Taylor series of  $C_0(\kappa)$  is  $\frac{42}{9!}$  and that the  $\kappa^0$ -coefficient in the Taylor series of  $C_i(\kappa)$  for  $1 \leq i \leq 9$  is  $\frac{3}{10!}$ . These are namely the expected results if we let  $\kappa \rightarrow 0$ .

## 6.9 Conclusion of the proof of Theorem 6.1

Having done the hard work in Section 6.8, we now just have to put things together. We have shown that the LHS of Corollary 6.4 is (see (6.179))

$$\geq C_1(u, \epsilon) C_0(\kappa) a_3(\mathcal{L}) Q^2 \log^9 Q \int_{5/4}^{7/4} \frac{x}{2} dx \int_T^{2T} |\Gamma((1/2 + it)/2)|^6 dt \\ + O\left(Q^2 \log^8 Q (\log \log Q)^M\right). \quad (6.202)$$

We have shown that the RHS of Corollary 6.4 is (see (6.190) and the comment just above (6.197) and also remember to once again include the constant  $(\frac{\kappa}{\pi \log Q})^2$  which we temporarily left out from our discussion before)

$$\leq \left(\frac{\kappa}{\pi \log Q}\right)^2 C_2(u, \epsilon) \left\{ C_1(\kappa) + 2C_2(\kappa) + 2C_4(\kappa) + 4C_6(\kappa) \right\} a_3(\mathcal{L}) Q^2 \log^{11} Q \\ \times \int_{5/4}^{7/4} \frac{x}{2} dx \int_T^{2T} |\Gamma((1/2 + it)/2)|^6 dt + O\left(Q^2 \log^8 Q (\log \log Q)^M\right). \quad (6.203)$$

The above clearly implies that

$$C_0(\kappa) \leq \left(\frac{\kappa}{\pi}\right)^2 \left\{ C_1(\kappa) + 2C_2(\kappa) + 2C_4(\kappa) + 4C_6(\kappa) \right\}. \quad (6.204)$$

Equality holds in (6.204) when  $\kappa \approx 7.42 \approx 1.18 \cdot 2\pi$ . If  $\kappa$  is smaller than that value, then we do get a contradiction to (6.204) and hence to Main Assumption 6.1. In other words we must have that  $\kappa > 7.42 \approx 1.18 \cdot 2\pi$ . But this (a priori) immediately gives us Theorem 6.1.

## 6.10 How to calculate the $\kappa$ -coefficients

Here we show (6.197), the calculation of the other  $C_i(\kappa)$ -coefficients being similar although slightly more difficult.

We begin by considering (6.174), keeping in mind that

$$\mathcal{R}_{A,B}(Q) = \sum_{\substack{S \subset A \\ T \subset B \\ |S|=|T|}} \mathcal{R}(\bar{S} \cup (-T), \bar{T} \cup (-S); Q), \quad (6.205)$$

where

$$\mathcal{R}(X, Y; Q) = Q^{\delta_{X,Y}} \prod_{\substack{x \in X \\ y \in Y}} \frac{1}{x+y} \quad (6.206)$$

and that we essentially want the shifts in the sets  $A$  and  $B$  to be  $\frac{\kappa i}{\log Q}$ ,  $0$  and  $-\frac{\kappa i}{\log Q}$ .

One way to explicitly proceed is as follows. Let  $A = \{ai + \delta, bi + 2\delta, ci + 4\delta\}$  and  $B = \{di + \delta, ei + 2\delta, fi + 4\delta\}$ , where  $a = d = \frac{\kappa}{\log Q}$ ,  $b = e = 0$ ,  $c = f = \frac{-\kappa}{\log Q}$  and  $\delta$  is a small number which we will let tend to  $0$ . There are in total 20 choices for the sets  $S$  and  $T$ . It will turn out that we only have to look at 10 of them in detail, by a symmetry-argument. To those 10 pairs of  $S$  and  $T$  which we must consider the corresponding pairs of  $\bar{S} \cup (-T)$  and  $\bar{T} \cup (-S)$  are as follows:

$$\{ai + \delta, bi + 2\delta, ci + 4\delta\} \& \{di + \delta, ei + 2\delta, fi + 4\delta\}, \quad (6.207)$$

$$\{bi + 2\delta, ci + 4\delta, -di - \delta\} \& \{ei + 2\delta, fi + 4\delta, -ai - \delta\}, \quad (6.208)$$

$$\{bi + 2\delta, ci + 4\delta, -ei - 2\delta\} \& \{di + \delta, fi + 4\delta, -ai - \delta\}, \quad (6.209)$$

$$\{bi + 2\delta, ci + 4\delta, -fi - 4\delta\} \& \{di + \delta, ei + 2\delta, -ai - \delta\}, \quad (6.210)$$

$$\{ai + \delta, ci + 4\delta, -di - \delta\} \& \{ei + 2\delta, fi + 4\delta, -bi - 2\delta\}, \quad (6.211)$$

$$\{ai + \delta, ci + 4\delta, -ei - 2\delta\} \& \{di + \delta, fi + 4\delta, -bi - 2\delta\}, \quad (6.212)$$

$$\{ai + \delta, ci + 4\delta, -fi - 4\delta\} \& \{di + \delta, ei + 2\delta, -bi - 2\delta\}, \quad (6.213)$$

$$\{ai + \delta, bi + 2\delta, -di - \delta\} \& \{ei + 2\delta, fi + 4\delta, -ci - 4\delta\}, \quad (6.214)$$

$$\{ai + \delta, bi + 2\delta, -ei - 2\delta\} \& \{di + \delta, fi + 4\delta, -ci - 4\delta\} \quad (6.215)$$

and

$$\{ai + \delta, bi + 2\delta, -fi - 4\delta\} \& \{di + \delta, ei + 2\delta, -ci - 4\delta\}. \quad (6.216)$$

In each of the above listed cases (and in the other 10 cases although we never have to do the latter ones in “practice”) we will treat the relevant  $\mathcal{R}(X, Y; Q)$  by representing both  $Q^{\delta_{X,Y}}$  and  $\prod_{\substack{x \in X \\ y \in Y}} \frac{1}{x+y}$  by their Laurent series and multiply them together, seeing  $\delta$  as our variable. We then proceed by summing up all the 20 Laurent series to obtain one final Laurent series. Since (6.173) is a continuous function of  $\delta$  as  $\delta \rightarrow 0$ , so must (6.174) be. This means that the Laurent series of  $\mathcal{R}_{A,B}(Q)$  which we had obtained can not have any negative  $\delta$ -powers. Furthermore, we are uninterested in the positive  $\delta$ -powers as we will let  $\delta \rightarrow 0$ . We therefore focus on identifying the  $\delta^0$ -coefficient and therefore in turn on finding the  $\delta^0$ -coefficient in each of the 20 cases. Below we will discuss how to do the 10 cases listed above, before explaining why we then get the other 10 cases for “free”.

These  $\delta^0$ -coefficients will be expressions in terms of  $\kappa$ . We will treat them as Laurent series. Each individual such Laurent series may have negative  $\kappa$ -powers but the sum of all 20 Laurent series is an analytic expression in  $\kappa$ , since (6.173) and therefore (6.174) is a continuous function of  $\kappa$  as  $\kappa \rightarrow 0$ . We here remark that this sum will be some constant in terms of  $\kappa$ , namely  $C_0(\kappa)$  by definition, multiplied by  $\log^9 Q$ . We will then substitute this expression for  $\mathcal{R}_{A,B}(Q)$  into (6.174). Because of the cancellation of possible negative  $\kappa$ -powers we will below in each of the 20 cases focus on finding only the analytic part of the  $\kappa$ -expressions. The sum of these is thus guaranteed to equal  $C_0(\kappa)$ .

Upon inspection of (6.207)-(6.216) we notice that (6.207), (6.210), (6.212) and (6.214) will then not give any contribution to  $C_0(\kappa)$ , since in those cases  $\kappa$  will not feature in  $\delta_{X,Y}$  (the latter means that we will only get negative  $\kappa$ -powers). Moreover, clearly the contribution from the terms (6.209) and (6.211) will be equal and likewise the contribution from (6.213) will be the same as from (6.215).

Let us now in some detail find the contribution to  $C_0(\kappa)$  from the term (6.208) above. The basic plan is to write out what we get from multiplying the Laurent series of  $Q^{\delta_{X,Y}}$  and  $\prod_{\substack{x \in X \\ y \in Y}} \frac{1}{x+y}$  and then let  $\delta \rightarrow 0$ . This will give us an expression in terms

of  $\kappa$  and by eliminating any negative  $\kappa$ -powers, we obtain the contribution in this case to  $C_0(\kappa)$ . Here are the details:

$$\begin{aligned}
& \exp\left(\left(\frac{2\kappa i}{\log Q} - 5\delta\right) \log Q\right) \cdot \frac{1}{\left(\frac{2\kappa i}{\log Q} + 2\delta\right)} \cdots \frac{1}{\left(\frac{2\kappa i}{\log Q} - 8\delta\right)} \\
&= \exp(2\kappa i) \cdot \exp(-5\delta \log Q) \cdot \frac{\log^8 Q}{-64\delta\kappa^8} \cdot \frac{1}{\left(1 - \frac{20\delta \log Q}{i\kappa} + \dots\right)} \\
&= \exp(2\kappa i) \cdot \exp(-5\delta \log Q) \cdot \frac{\log^8 Q}{-64\delta\kappa^8} \cdot \left(1 + \frac{20\delta \log Q}{i\kappa} + \dots\right). \tag{6.217}
\end{aligned}$$

Recall that the pair of sets corresponding to (6.208) comes from a pair of  $S$  and  $T$ . Had we instead done the above calculation in the case corresponding to the pair of sets coming from the pair  $\bar{S}$  and  $\bar{T}$  we would have obtained the same expression, but with a minus-sign in front and with negative exponentials. Putting these two together<sup>14</sup> and focusing on the  $\delta^0$ -coefficient we obtain

$$\begin{aligned}
& \frac{2 \cdot \cos(2\kappa) \cdot (-5) \cdot \log^9 Q}{(-64) \cdot \kappa^8} + \frac{2 \cdot \sin(2\kappa) \cdot 20 \cdot \log^9 Q}{(-64) \cdot \kappa^9} \\
&= \log^9 Q \cdot \left\{ \frac{5 \cos(2\kappa)}{32\kappa^8} - \frac{5 \sin(2\kappa)}{8\kappa^9} \right\}. \tag{6.218}
\end{aligned}$$

Again, recall that our strategy is to ignore any negative  $\kappa$ -powers, hence the introduction of the ‘‘Macl’’-notation used in this text in Section 6.8.3. Thus our final answer (and to make notation here coherent with the one in Section 6.8.3 we exclude the ‘‘ $\log^9 Q$ ’’-term) for the contribution from these two (out of the total 20) pairs equals

$$\text{Macl} \left\{ \frac{5 \cos(2\kappa)}{32\kappa^8} - \frac{5 \sin(2\kappa)}{8\kappa^9} \right\}. \tag{6.219}$$

It is possible to handle the contribution coming from (6.209), (6.213) and (6.216) in an analogous way and one finds that the contribution in each of these three cases is respectively

$$\text{Macl} \left\{ \frac{2 \cos \kappa}{5\kappa^8} - \frac{19 \sin \kappa}{20\kappa^9} \right\}, \tag{6.220}$$

$$\text{Macl} \left\{ \frac{\cos \kappa}{10\kappa^8} + \frac{29 \sin \kappa}{20\kappa^9} \right\} \tag{6.221}$$

and

$$\text{Macl} \left\{ -\frac{\cos(2\kappa)}{32\kappa^8} + \frac{\sin(2\kappa)}{8\kappa^9} \right\}. \tag{6.222}$$

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<sup>14</sup>One can for each pair of sets in the list (6.207)-(6.216) make the analogous choice of a pair of sets among the ‘‘other’’ 10 pairs of sets. Then they will together make a contribution that can be expressed in trigonometric functions.

Hence, putting things together we have that

$$\begin{aligned}
C_0(\kappa) &= \text{Macl} \left\{ \left( \frac{5 \cos(2\kappa)}{32\kappa^8} - \frac{5 \sin(2\kappa)}{8\kappa^9} \right) + 2 \cdot \left( \frac{2 \cos \kappa}{5\kappa^8} - \frac{19 \sin \kappa}{20\kappa^9} \right) \right. \\
&\quad \left. + 2 \cdot \left( \frac{\cos \kappa}{10\kappa^8} + \frac{29 \sin \kappa}{20\kappa^9} \right) + \left( -\frac{\cos(2\kappa)}{32\kappa^8} + \frac{\sin(2\kappa)}{8\kappa^9} \right) \right\} \\
&= \text{Macl} \left\{ \frac{\cos \kappa}{\kappa^8} + \frac{\sin \kappa}{\kappa^9} + \frac{\cos(2\kappa)}{8\kappa^8} - \frac{\sin(2\kappa)}{2\kappa^9} \right\}. \tag{6.223}
\end{aligned}$$

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