Problems in Stochastic Analysis. Connections between Rough Paths and Non-Commutative Harmonic Analysis

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Contents

Chapter 1

Abstract

The prevailing aim and flavour of this thesis, is to establish a link between Lyons' theory of Rough Paths and more conventional notions of non commutative harmonic analysis. Three different directions represent the lines of investigation.

The first is a generalization of an algebraic formalism, pertaining to classical pictures of the analysis of functions on groups. The structures of Hopf algebras reveal an algebraic role of geometric rough paths in relation to the shuffle algebra of one-forms.

The second direction examines path-space on Lie Groups and a generalization of the notion of rough paths. It is shown how to approximate the length of a twice differentiable curve in two dimensions through the iterated integral sequence and an analysis of curves in two dimensional hyperbolic space. The formula is given in terms of the asymptotics of the iterated integral sequence.

The third and final direction, was motivated by an attempt to enlarge the definition of the Ito map to non-geometric rough paths. It is shown how random geometric rough paths can be used to interpret backward Ito multiplicative functionals. Some ideas regarding the extent to which the center of mass of a distribution on geometric rough paths actually determines the distribution itself, are presented.

1.1 Notation

The following notation will be defined and used consistently through the thesis and is presented here in an attempt at making referencing easier.

V is unless otherwise stated a finite dimensional vector space

V^* is the dual of V

$T(V)$ is the Tensor algebra built over V, $T(V) = \oplus_{j=0}^{\infty} V^{\otimes j}$

$T^n(V)$ is the truncated Tensor algebra built over V, $T^n(V) = \oplus_{j=0}^{n} V^{\otimes j}$

$T^{(n)}(V) = V^{\otimes n}$, the component of $T(V)$ of homogeneous degree n

$L(V)$ is the Free Lie algebra built over V, $L(V) = V \oplus [V,V] \oplus [V,[V,V]] \oplus \cdots$

$L^n(V)$ is a truncated Free Lie algebra V, $L^n(V) = L(V) \cap T^n(V)$.

$L^{(n)}(V)$ is the component of $L(V)$ of tensorial degree n,

$$L^{(n)}(V) = L(V) \cap T^{(n)}(V)$$

$\mathcal{T}(V^*) = \cup_{n=0}^{\infty} \left(\oplus_{i=0}^{n} (V^*)^{\otimes i} \right)$ is the graded dual of $T(V)$

$\mathcal{P}(M)$ is the collection of piecewise smooth paths $\gamma : [0, T_\gamma] \to M$, where M is C^∞ differentiable

$\mathcal{P}_b(M)$ is the collection of bounded variation paths $\gamma : [0, T_\gamma] \to M$, where M is a metric space

$\mathbf{X}(\gamma) \in T(V)$ will be the signature or full iterated integral sequence of γ (independent of time parameterization)

$\mathbf{X}_{s,t}(\gamma) \in T(V)$ is the iterated integral sequence of the path γ between times s and t for an implicitly chosen time parameterization

$\mathbf{X}_{s,t}^{n}(\gamma) \in T^n(V)$ is the iterated integral sequence of the path γ between times s and t, truncated to $T^n(V)$

$\mathbf{X}_{s,t}^{(i)}(\gamma) \in V^{\otimes i}$ is the i'th iterated integral of the path γ between times s and t

Γ_T is the set of ordered pairs of times,

$$\Gamma_T = \{(s,t) : 0 \leq s \leq t \leq T\}$$

It's topology is that of the Euclidean topology of $\mathbf{R}^2$

$C(\Gamma_T; W)$ is the collection of functions from Γ_T to a topological space W, which are continuous in both time parameters

$\Delta_{s,t}$ is the collection of dissections δ of the time interval $[s,t]$ into finitely many intervals, $\delta = \{s = t_0 < t_1 < \cdots < t_n = t\}$, in which case $|\delta| = n < \infty$ is the size of the dissection

$\Omega(V)^p_n$ is the collection of multiplicative functionals of finite p variation in $T^n(V)$

$\Omega(V)^p$ is the collection of rough paths of finite p variation in a vector space V

$\mathbf{X}^{\lfloor p \rfloor;\infty}$ denotes the element of $\Omega(V)^p_\infty$ that extends some $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p$

$\mathbf{X}^{\lfloor p \rfloor}_{sig} \in T(V)$, the signature of $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p$, is given by $\mathbf{X}^{\lfloor p \rfloor}_{sig} = \mathbf{X}^{\lfloor p \rfloor;\infty}_{0,T_{\mathbf{X}^{\lfloor p \rfloor}}}$

$\Omega G(V)^p$ is the closure of $\mathcal{P}_b(V)$ in $\Omega(V)^p$ under the p variation topology

$\Omega G(V)^{p+} = \cap_{q>p} \Omega G(V)^p$

$\pi_n : (\cup_{m \geq n} T^n(V)) \cup T(V) \to T^n(V)$ is a projection of an element of either a truncated or non-truncated tensor algebra, off the ideal generated by tensors of degree $n+1$ and greater

$\pi_{(n)} : (\cup_{m \geq n} T^n(V)) \cup T(V) \to V^{\otimes n}$ is the projection of tensor onto the component of degree n, so that $\pi_{(n)} = \pi_n - \pi_{n-1}$ where defined

Chapter 2

An Overview of Rough Path Theory

'Atomic Motions '

"An image, a type goes on before our eyes, present each moment; for behold whenever the sun's light and the rays, let in, pour down across dark halls of houses: thou wilt see the many mites in many a manner mixed amid a void in the very light of the rays, and battling on, as in eternal strife, and in battalions contending without halt, in meetings, partings, harried up and down.

....

For thou wilt mark here many a speck, impelled by viewless blows, to change its little course, and beaten backwards to return again, hither and thither in all directions round."

An extract from 'De Rerum Natura' Titus Lucretius Carus, circa 50 B.C.

2.1 The First Points Of Rough Path Theory

This section is largely a review of the work presented by Lyons in [28] and follows the development there, though also draws on parts of [29], particularly for what is known as the 'Universal Limit Theorem'. More emphasis is put on the analytic aspects of the theory, while an algebraic perspective is left to the following chapter.

Perhaps the central achievement in [28] is to link a family of topologies on continuous paths in a vector space which have an associated metric, with the construction of solutions to a family of differential equations. The map from path to solution is known as the Ito functional. The topology on paths is known as the p variation topology and for the constructed solutions, the Ito map is continuous from input path to the output solution path. Subsequent work gives a specific form for p variation metrics that characterize the p variation topologies.

The key objects that are investigated are a class of functions on continuous paths called iterated integrals. It turns out that these functions are intrinsic to the definition or construction of what are rough paths. In the case of smooth paths, or piecewise smooth paths, the definition of these functions is unambiguous and natural, lying firmly within the capabilities of Riemannian Integration. For paths which have less regularity or more 'roughness', this first notion of Riemannian integration is not robust enough. A probabilist will recognize an aspect of this in the difference between Stratonovitch and Ito integrals of semi-martingales. Yet a more subtle approach yields a way of understanding these functions.

2.1.1 Paths in a Vector Space

In the course of the discussion, γ will always be a continuous path in a C^{∞} differentiable space, or a metric space M, $\gamma : [0, T_{\gamma}] \to M$ where $0 \leq T_{\gamma} < \infty$. The set of all continuous, bounded variation paths in a metric space M is denoted by $\mathcal{P}_b(M)$, while $\mathcal{P}(M)$ is the set of all continuous, piecewise smooth paths in M, a C^{∞} space. Let M be a vector space V (with the Euclidean metric). The iterated integrals of elements of $\mathcal{P}_b(V)$ are a collection of functions $\left\{\mathbf{X}^{(n)} : n \in \mathbb{N}_0\right\}$ from $\mathcal{P}_b(V)$ to the collection of continuous functions from the set of pairs of times

$$\Gamma_{T_{\gamma}} = \{(s,t) : 0 \leq s \leq t \leq T_{\gamma}\},$$

to $V^{\otimes n}$. They are defined by

$$\begin{aligned}
\mathbf{X}^{(n)} &: \mathcal{P}_b(V) \to \cup_{T>0} C\left(\Gamma_T, V^{\otimes n}\right) \\
\mathbf{X}^{(n)}(\gamma) &\in C\left(\Gamma_{T_{\gamma}}, V^{\otimes n}\right) \\
\mathbf{X}^{(n)}_{s,t}(\gamma) &= \int \cdots \int_{s<u_1<u_2<\cdots<u_n<t} d\gamma_{u_1} \otimes d\gamma_{u_2} \otimes \cdots \otimes d\gamma_{u_n},
\end{aligned}$$

$V^{\otimes n}$ being the n-fold tensor product of V with itself, always understood to be over $\mathbb{R}$ and where $C(\Gamma_T; W)$ is the collection of continuous functions from Γ_T to a topological space W. Take the notation $T^n(V) = \oplus_{j=0}^{n} V^{\otimes j}$ for the truncated tensor algebra of V of degree n. $T^n(V)$ is an algebra which is inherited from $T(V)$ by quotienting by tensors of degree greater than n.

In $T^n(V)$, the sum of the first n iterated integrals, provides a function

$$\mathbf{X}^n : \mathcal{P}_b(V) \to \cup_{T>0} C\left(\Gamma_T; T^n(V)\right),$$

$$\mathbf{X}^n_{s,t}(\gamma) = \sum_{m=0}^{n} \mathbf{X}^{(m)}_{s,t}(\gamma).$$

In [9], Chen first noted a multiplicative property of the iterated integral sequence of any piecewise smooth path which is known as Chen's identity:

$$\mathbf{X}^n_{s,u}(\gamma) \otimes \mathbf{X}^n_{u,t}(\gamma) = \mathbf{X}^n_{s,t}(\gamma), \tag{2.1}$$

$\forall (s,t), (t,u) \in \Gamma_{T_\gamma}$. A generalization of this property of tensor multiplication and iterated integrals gives a more general definition of a multiplicative functional:

Definition 1 *A Multiplicative Functional of degree n denoted* $\mathbf{X}^n$*, is an element of* $C(\Gamma_{T_{\mathbf{X}^n}}; \mathrm{T}^n(V))$ *that satisfies*

$$\mathbf{X}^n_{s,t} \otimes \mathbf{X}^n_{t,u} = \mathbf{X}^n_{s,u},$$

$\forall (s,t), (t,u) \in \Gamma_{T_{\mathbf{X}^n}}$.

The collection of such multiplicative functionals is denoted $\Omega(V)_n$.

By capturing the multiplicative property of concatenated path segments of elements in $\mathcal{P}_b(V)$ in this way, a larger class of functionals in $T^n(V)$ are admitted to $\Omega(V)_n$ than those due to $\mathcal{P}_b(V)$. For now, these are treated equally, though an important question is how to interpret one set in light of the other.

2.1.2 Rough Paths

In [28], Lyons studied functionals taking values in the truncated tensor algebras that also satisfy Chen's identity. They are associated to paths that are no longer piecewise smooth but according to Wiener (see

[29] for references), are paths of finite p variation for some $1 \leq p < \infty$. While in the case of piecewise smooth paths the higher order iterated integrals are defined according to Riemannian Integration, the notion of an iterated integral for general p variation paths is ambiguous. Lyons looked at multiplicative functionals which extend the first iterated integral processes of p variation paths in an analytically consistent manner. The notion of p variation is extendable to these objects called Rough Paths and induces a topology or sense of continuity for the set. The raison d'être of the whole construction is that one can define a notion of solution to a class of differential equation driven by rough paths, which is continuous with respect to the p variation topology.

p Variation and multiplicative functionals

Wiener's notion of the p variation of a segment of a path $\gamma : [0, T_\gamma] \to V$ between times $0 \leq s \leq t \leq T_\gamma$, where V has a metric $d_V(\cdot, \cdot)$, is

$$\sup_{\delta \in \Delta_{s,t}; |\delta| < \infty} \sum_{j=1}^{|\delta|} d_V\left(\gamma(t_j), \gamma(t_{j-1})\right)^p, \tag{2.2}$$

where $\Delta_{s,t}$ is the set of finite dissections of $[s,t]$,

$$\Delta_{s,t} = \left\{ \delta = \{t_j\}_{j=0}^{|\delta|} \text{ s.t. } s = t_0 < t_1 < \cdots < t_{|\delta|} = t, \ |\delta| < \infty \right\}.$$

Clearly the p variation could be infinite for a given path and given p.

Definition 2 *For $X \in C(\Gamma_{T_X}; W)$ where W is a normed space with norm $\|\cdot\|$, define the function $P_{p;\cdot,\cdot}(X) \in C(\Gamma_{T_X}; \mathbb{R})$*

$$P_{p;s,t}(X) = \sup_{\delta \in \Delta_{s,t}} \sum_{j=1}^{|\delta|} \left\| X_{t_{j-1},t_j} \right\|^p, \tag{2.3}$$

$(s,t) \in \Gamma_{T_X}$.

Remark 3 *If a path has finite $p \geq 1$ variation, then it has finite p' variation for any $p' > p$: Let*

$$x = \sup_{(s,t)\in\Gamma_{T_X}} \|X_{s,t}\|.$$

Since Γ_{T_X} is compact, $x < \infty$ and also $X_{\cdot,\cdot} : \Gamma_{T_X} \to W$ is uniformly continuous, so pick $\tau > 0$ such that $|t-s| < \tau \Rightarrow \|X_{s,t}\| \leq 1$. Then $\forall \delta \in \Delta_{0,T_X}$,

$$\begin{aligned}
\sum_{j=1,t_j\in\delta}^{|\delta|} \left\|X_{t_{j-1},t_j}\right\|^{p'} &= \left\{ \begin{array}{c} \sum_{j=1,t_j\in\delta;|t_j-t_{j-1}|<\tau}^{|\delta|} \left\|X_{t_{j-1},t_j}\right\|^{p'} \\ +\sum_{j=1,t_j\in\delta;|t_j-t_{j-1}|\geq\tau}^{|\delta|} \left\|X_{t_{j-1},t_j}\right\|^{p'} \end{array} \right\} \\
&\leq \sum_{j=1,t_j\in\delta;|t_j-t_{j-1}|<\tau}^{|\delta|} \left\|X_{t_{j-1},t_j}\right\|^{p} + \frac{T_X}{\tau} x^{p'} \\
&\leq P_{p;0,T_X}(X) + \frac{T_X}{\tau} x^{p'}
\end{aligned}$$

which is a bound independent of the dissection $\delta \in \Delta_{0,T_X}$

The picture that Lyons revealed, is that the rough paths, at least for $p \geq 2$, encode non-trivial non-linear information that defines a finer topology associated to p variation paths. The notion of a rough path thus incorporates - again at least when $p \geq 2$ - additional paths that in themselves are not uniquely determined functions of the graph of the path itself. It is not the case either a priori, that such extending rough paths can be associated to a given p variation path and it is also not the case that extensions are unique. Rough paths are multiplicative functionals which determine the whole iterated integral sequence, just as in the case of bounded variation paths where higher iterated integrals can be constructed canonically from the path itself.

Lyons showed that it is possible to uniquely extend a multiplicative functional of degree n if the known iterated integrals satisfy an analytic decay property. To quantify the decay, any system of norms on each $V^{\otimes n}$ can be used that satisfies a consistency condition between differing tensorial degrees:

$$\|\xi \otimes \zeta\| \leq \|\xi\| \, \|\zeta\| \quad \forall \xi \in V^{\otimes n}, \zeta \in V^{\otimes m}. \tag{2.4}$$

Two well known examples of such norms are the injective and projective systems:

$$\|\xi\|_{inj} \equiv \sup_{e_i \in V^*; 1 \leq i \leq n} \frac{(e_1 \otimes e_2 \otimes \cdots \otimes e_n) \circ \xi}{\|e_1\| \, \|e_2\| \cdots \|e_n\|} \quad \forall \xi \in V^{\otimes n} \tag{2.5}$$

$$\|\xi\|_{proj} \equiv \inf_{\xi = \sum_{i=1}^{N} v_{i,1} \otimes v_{i,2} \otimes \cdots \otimes v_{i,n}} \sum_{i=1}^{N} \|v_{i,1}\| \, \|v_{i,2}\| \cdots \|v_{i,n}\| \quad \forall \xi \in V^{\otimes n}. \tag{2.6}$$

Through the thesis, it is assumed that any system of norms will satisfy this property. At this point, no more properties of the norm are necessary, although to define solutions to differential equations, an additional property of symmetry is required, but this will be stated later.

A functional device called a control is used to express the decay.

Definition 4 *A control is a function* $\omega : \Gamma_T \to \mathbb{R}_+$ *which satisfies*

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u),$$

$\forall (s,t), (t,u) \in \Gamma_T$*, that is continuous near the diagonal and such that*

$$\omega(t,t) = 0 \quad \forall 0 \leq t \leq T.$$

The p variation function is analogous to a control since it is also sub-additive:

$$P_{p;s,u}(\gamma) + P_{p;u,t}(\gamma) \leq P_{p;s,t}(\gamma).$$

The formal definition of a multiplicative functional of finite p variation can now be given:

Definition 5 *$\mathbf{X}^n \in \Omega(V)_n$ is said to have finite p variation controlled by ω, if it has the property that $(s,t) \in \Gamma_{T_{\mathbf{X}^n}} \Rightarrow$*

$$\left\|\mathbf{X}_{s,t}^{(i)}\right\| < \frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!},$$

$\forall 1 \leq i \leq n$ where $\mathbf{X}_{s,t}^{(i)}$ is the component of $\mathbf{X}_{s,t}^n$ which lies in $V^{\otimes i}$ and β is a positive constant specified below. A based multiplicative functional also has an initial base point $\mathbf{X}_0^n \in V$ and hence defines a continuous path in V,

$$\begin{aligned}\mathbf{X}_{\cdot}^n &: [0, T_\gamma] \to V \\ \mathbf{X}_t^n &\equiv \mathbf{X}_0^n + \mathbf{X}_{0,t}^{(1)}\end{aligned}$$

Denote the collection of such based multiplicative functionals by $\Omega(V)_n^p$.

The following example is straightforward:

Example 6 *Fix T and let $u \in V^{\otimes n}$. Then $\mathbf{X}_{s,t}^n = 1 + (t-s)u$ is multiplicative in $\mathrm{T}^n(V)$. If $p > n$ then setting*

$$\omega_p(s,t) = \left(\beta(t-s)\|u\|\left(\frac{n}{p}\right)!\right)^{\frac{p}{n}}$$

defines a control and with any base point, defines an element of $\Omega(V)_n^p$ for any $p > n$.

In fact, any tensor $u \in T^n(V) \backslash \mathbb{R}$ defines a multiplicative functional $\mathbf{X}_{s,t}^n = \sum_{j=0}^n (t-s)^j \frac{u^{\otimes j}}{j!} \in T^n(V)$, which has finite p variation if $p > n$. The form for controls are less concise and are given in the appendix as Lemma (119).

Remark 7 *For any $q > p$, $\Omega(V)^p_n \subset \Omega(V)^q_n$. This follows from the facts that the function $x!$ is increasing and that if $\omega(\cdot,\cdot)$ is a control, $\omega(\cdot,t\cdot)^\alpha$ is also a control for any $\alpha > 1$ ($\omega(\cdot,\cdot)^\alpha$ is also sub-additive and regular).*

According to this definition, if $p > n$ and $\mathbf{X} \in \Omega(V)^p_n$, then each $\mathbf{X}^{(i)} \in C(\Gamma_T; V^{\otimes i})$, $1 \leq i \leq n$, has finite $\frac{p}{i}$ variation : if $(s,t) \in \Gamma_{T\mathbf{X}}$, $\delta \in \Delta_{s,t}$, then

$$\begin{aligned}\sum_{\delta:j=1}^{|\delta|} \left\| \mathbf{X}^{(i)}_{t_{j-1},t_j} \right\|^{\frac{p}{i}} &\leq \frac{1}{\left(\beta\left(\frac{i}{p}\right)!\right)^{\frac{p}{i}}} \sum_{\delta:j=1}^{|\delta|} \omega(t_{j-1},t_j) \\ &\leq \frac{\omega(s,t)}{\left(\beta\left(\frac{i}{p}\right)!\right)^{\frac{p}{i}}},\end{aligned}$$

which is an upper bound independent of δ, hence

$$P_{\frac{p}{i};s,t}\left(\mathbf{X}^{(i)}\right) \leq \frac{\omega(s,t)}{\left(\beta\left(\frac{i}{p}\right)!\right)^{\frac{p}{i}}}.$$

The following key theorem reveals why a control is an appropriate device for an element of $\Omega(V)^p_{\lfloor p \rfloor}$. They yield an unique extending multiplicative functional in the full tensor algebra that matches the analytic form for the decay of the homogeneous tensor components or iterated integrals. That is, it produces an unique element of $\Omega(V)^p_\infty$. The theorem thus says that all the essential information about such finite p variation multiplicative functional extensions is bound up in the first $\lfloor p \rfloor$ lower order iterated integrals.

Theorem 8 (Lyons) *Let $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p_{\lfloor p \rfloor}$ be a multiplicative functional of finite p variation with control ω. There exists an unique multiplicative functional $\mathbf{X}^n$ of finite p variation for any $n > \lfloor p \rfloor$ that extends*

$\mathbf{X}^{\lfloor p \rfloor}$. *In particular, the iterated integrals* $\mathbf{X}_{s,t}^{(i)}$ *of the extension satisfy the inequality:*

$$\left\| \mathbf{X}_{s,t}^{(i)} \right\| < \frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}$$

$\forall i > p$ *provided* β *is taken large enough:*

$$\beta > \beta(p) = p^2 \left(1 + 2^{\frac{(1+\lfloor p \rfloor)}{p}} \left(\zeta\left(\frac{(1+\lfloor p \rfloor)}{p}\right) - 1\right)\right) \tag{2.7}$$

suffices.

Naturally, the proof of this theorem is inductive and here a sketch is presented. Taking $\mathbf{X}^n \in \Omega(V)_n^p$ for some $n > p$, Lyons shows that higher order iterated integrals can be induced from the lower ones through Chen's identity. The inclusion map

$$i_n : T^n(V) \to T^{n+1}(V)$$

is an embedding of $\mathbf{X}^n$ into $C\left(\Gamma_{T_{\mathbf{X}^n}}; T^{n+1}(V)\right)$ whose $n+1$'st tensor component is identically zero.

This itself will not produce a multiplicative functional. To do this, take any $(s,t) \in \Gamma_{T_{\mathbf{X}^n}}$. For a given increasing sequence of dissections of (s,t), $\delta_n \subset \delta_{n+1}$, with mesh size approaching zero, Lyons shows that the limit

$$\mathbf{X}_{s,t}^{n+1} = \lim_{n \to \infty} \otimes_{t_j \in \delta_n} i_n\left(\mathbf{X}_{t_j,t_{j+1}}^n\right),$$

exists in $T^{n+1}(V)$. Moreover, a maximal inequality shows that the limit is independent of the sequence of dissections. By construction, $\mathbf{X}^{n+1} \in \Omega(V)_{n+1}^p$, i.e. the same analytic decay or control, is passed on.

The analysis in the proof hinges on establishing an interesting "neo-classical binomial inequality":

Lemma 9 (Lyons) *Let $n \in \mathbb{N}$, $x, y > 0$, $p \geq 1$. Then*

$$\left(\frac{1}{p}\right)^2 \sum_{j=0}^{n} \frac{x^{\frac{j}{p}}}{\left(\frac{j}{p}\right)!} \frac{y^{\frac{n-j}{p}}}{\left(\frac{n-j}{p}\right)!} \leq \frac{(x+y)^{\frac{n}{p}}}{\left(\frac{n}{p}\right)!}.$$

This inequality in turn leads to the form (2.7) for $\beta(p)$ permitting the decay of the iterated integrals in relation to the control ω for the higher levels. However it is thought that a larger value than $\left(\frac{1}{p}\right)^2$ would suffice for the factor in the left hand side, entailing a larger value for $\beta(p)$ and closer control on the decay rate for the iterated integrals.

A formal definition of what is meant by a rough path can now be given together with some notation:

Definition 10 *A Rough Path in a vector space V has a roughness $p \in [1, \infty)$ and is a based multiplicative functional $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p_{\lfloor p \rfloor}$.*

Notation 11 *The set of Rough Paths in a vector space V of roughness p is denoted by $\Omega(V)^p$.*

Remark 12 *Unsurprisingly, products of rough paths are rough paths when the product can be defined. As is with two elements $\gamma^1, \gamma^2 \in \mathcal{P}_b(V)$ where the object $\gamma^1\gamma^2$*

$$\gamma^1\gamma^2 : [0, T_{\gamma_1} + T_{\gamma_2}] \to V$$

$$\left(\gamma^1\gamma^2\right)_t = \left\{ \begin{array}{cc} \gamma^1_t & t \in [0, T_{\gamma_1}] \\ \gamma^2_t & t \in [T_{\gamma_1}, T_{\gamma_1} + T_{\gamma_2}] \end{array} \right\}$$

makes sense as an element of $\mathcal{P}_b(V)$ iff $\gamma^1_{T_{\gamma_1}} = \gamma^2_0$, two elements $\mathbf{X}^{\lfloor p \rfloor}, \mathbf{Y}^{\lfloor p \rfloor} \in \Omega(V)^p$ can be multiplied together when $\mathbf{X}^{\lfloor p \rfloor}_{T_{\mathbf{X}^{\lfloor p \rfloor}}} = \mathbf{Y}^{\lfloor p \rfloor}_0$. Lemma 120 in the Appendix provides a form for the control of the product.

Signatures

Having seen that for an element $\mathbf{X}^n \in \Omega(V)^p_n$, $n \geq \lfloor p \rfloor$ that there is an unique extension $\mathbf{X}^m \in \Omega(V)^p_m$ of $\mathbf{X}^n$, for any $m > n$, the signature of a rough path is defined to be the full iterated integral sequence over the whole time interval, of the extension of the functional to the whole tensor algebra:

Definition 13 *Let $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p$. Set $\mathbf{X}^{\lfloor p \rfloor, \infty}$ to be the unique extension due to Theorem 8 of $\mathbf{X}^{\lfloor p \rfloor}$ to a multiplicative functional taking values in* $\mathrm{T}(V)$,

$$\mathbf{X}^{\lfloor p \rfloor, \infty} \in \Omega(V)^p_\infty .$$

In addition, define the signature $\mathbf{X}^{\lfloor p \rfloor}_{sig} \in \mathrm{T}(V)$:

$$\mathbf{X}^{\lfloor p \rfloor}_{sig} = \mathbf{X}^{\lfloor p \rfloor, \infty}_{0, T_{\mathbf{X}^{\lfloor p \rfloor}}} .$$

In chapter 5 there is a brief discussion as to the capacity and sense of the signature of a rough path to completely define the path itself.

For finite dimensional vector spaces, differing norms affect the control for the decay of the iterated integrals of a rough path, though not actually the objects in the completion. So the value of differing norms relates to the analytical context. In infinite dimensional contexts, the choice of a norm really does seem to affect the analytical existence of higher order iterated integrals, see [7] for example.

2.1.3 The p variation metric

So far, there has been no thought of a notion of continuity or distance related to multiplicative functionals. To achieve this, there is a distance function in $\Omega(V)^p_n$ and in particular on $\Omega(V)^p$.

Definition 14 *Let $\mathbf{X}^n, \mathbf{Y}^n \in \Omega(V)^p_n$. The p variation metric*

$$d_p(\mathbf{X}^n, \mathbf{Y}^n),$$

is defined by

$$d_p(\mathbf{X}^n, \mathbf{Y}^n) = \sum_{i=1}^{n} P_{\frac{p}{i};0,T}\left(\mathbf{X}^{(i)} - \mathbf{Y}^{(i)}\right)^{\frac{i}{p}} + \sup_{u\in[0,T]} \left\| \mathbf{X}^{(1)}_u - \mathbf{Y}^{(1)}_u \right\|, \tag{2.8}$$

where $T = \max(T_{\mathbf{X}^n}, T_{\mathbf{Y}^n})$ and where for any $\mathbf{Z}^n$ such that $T > T_{\mathbf{Z}^n}$, then $\mathbf{Z}^n \in \Omega(V)^p_n$ is defined by augmenting the definition to set $\mathbf{Z}^n_{T_{\mathbf{Z}^n},t} = 0$ for $t > T_{\mathbf{Z}^n}$ (i.e. $\mathbf{Z}^n_t = \mathbf{Z}^n_{T_{\mathbf{Z}^n}}$). (See equation 2.3).

This distance function does specify a metric in $\Omega(V)^p_n$ and hence in $\Omega(V)^p$, though that used in [28] p.253 does not. Both however define the same topology with equivalent Cauchy sequences and according to Lyons, the spaces $\Omega(V)^p_n$ are complete.

Geometrically, a very important subspace of $\Omega(V)^p$ is the closure of the bounded variation paths with respect to the metric d_p. Recall that if $\gamma \in \mathcal{P}_b(V)$, then $\gamma \in \Omega(V)^1$ and hence the functional

$$\mathbf{X}^n(\gamma) \in \Omega(V)^1_n$$

is well defined. Denote this map by

$$\mathbf{X}^n : \mathcal{P}_b(V) \to \Omega(V)^1_n.$$

In the process of extending iterated integrals of elements of $\Omega(V)^p$ to elements of $\Omega(V)^p_\infty$, there is a bound for the modulus of continuity of the difference in norms in terms of the p variation distance. The following corollary is quoted from [28].

Corollary 15 (Lyons) *Suppose $\mathbf{X}^{\lfloor p\rfloor}, \mathbf{Y}^{\lfloor p\rfloor} \in \Omega(V)^p_n$ are controlled by ω and $n \geq \lfloor p\rfloor$. Suppose further that for some $\varepsilon < 1$ that $\forall i \leq n$,*

$$\left\|\mathbf{X}^{(i)}_{s,t} - \mathbf{Y}^{(i)}_{s,t}\right\| \leq \varepsilon \frac{\omega(s,t)^{\frac{i}{p}}}{\alpha\left(\frac{i}{p}!\right)}.$$

Then if $\alpha \geq 3p^2\left(1 + 2^{(\lfloor p\rfloor+1)/p}\left(\zeta\left(\frac{\lfloor p\rfloor+1}{p}\right) - 1\right)\right)$, then $\forall i < \infty$

$$\left\|\mathbf{X}^{(i);\infty}_{s,t} - \mathbf{Y}^{(i);\infty}_{s,t}\right\| \leq \varepsilon \frac{\omega(s,t)^{\frac{i}{p}}}{\alpha\left(\frac{i}{p}!\right)}.$$

Notation 16 *The closure of the image under the map $\mathbf{X}^{\lfloor p\rfloor}$, of $\mathcal{P}_b(V)$ into $\Omega(V)^1_{\lfloor p\rfloor}$ with respect to the p variation topology, is called the collection of geometric rough paths of finite p variation and is denoted $\Omega G(V)^p$.*

This space is examined more closely later. At the moment, a subtle aspect of this definition is worth clarifying.

Remark 17 *Some rough paths belong to each set $\Omega(V)^q$, $q > p$ but not in $\Omega(V)^p$ itself. The typical example is that of Brownian motion with the associated Lévy area process 2.1.4, which defines elements of $\Omega G(V)^p, \forall p > 2$ rather than $\Omega G(V)^2$ itself. In fact, Brownian paths have infinite 2 variation almost surely, however a sequence of dissections which achieve the infinite value, will be path dependent. When $p > 2$, these objects lie in $\Omega G(V)^p$, the completion of $\mathcal{P}_b(V)$ with respect to d_p. In fact, there is a family* *of rough paths indexed by n, called straight rough paths and investigated in section (5.1.4), which lie in* $\Omega(V)^n \cap \Omega G(V)^p\ \forall p > n$ *but do not appear to belong to* $\Omega G(V)^n$ *itself.*

As a consequence, the following notation is used:

Notation 18

$$\Omega G(V)^{p+} = \cap_{q>p} \Omega G(V)^p$$

As far as a path in a vector space of finite p variation is concerned, constructing an element of $\Omega(V)^p_\infty$ to match the first iterated integral process, requires constructing an extension only to $\Omega(V)^p_{\lfloor p \rfloor}$.

2.1.4 A review of processes susceptible to a rough path analysis:

There are many examples of stochastic processes which with probability one define paths with finite p variation. The issue of whether they are amenable to a rough path analysis, concerns finding a regime for associating the $\lfloor p \rfloor$ requisite iterated integrals. The following examples have been established:

Lévy processes Lévy processes with were studied by David Williams [43] and [44] and generically found to be processes of finite p variation for some $p \leq 2$.

Brownian Motion The motivating example of a rough path must surely be Brownian motion. $\forall p > 2$, almost surely, a sample path of Brownian motion has finite p variation. Lévy [27] constructed through Fourier analysis, a process which now generically takes his name, which is the stochastic integral corresponding to the anti-symmetric part of the second iterated integral of Brownian motion. The important point of subtlety is that as geometric rough paths, these multiplicative functionals are in $\Omega G(V)^{2+}$ and not in $\Omega G(V)^{2}$.

The most extensive treatment of this object appears in [26], where Banach Space valued Brownian Motions are considered. The regime of piecewise linear approximation over dyadic partitions is treated to obtain a notion of canonical Lévy area process and an analysis appears of the merits of the injective and projective tensor product norms for differing applications.

Free Brownian Motion was studied in [7] where the Lévy area was successfully constructed in the non-commutative probability setting and leads to a notion of geometric rough paths and solutions of differential equations.

Fractional Brownian motion Coutin and Qian investigated the construction of geometric rough paths for fractional Brownian motion of Hurst parameter $\frac{1}{4} < h < \frac{1}{2}$.

The result, announced in [14] and proved in[15], uses piecewise linear approximation for a dyadic regime, constructing a Lévy area and third iterated integral also. ($\forall p > \frac{1}{h}$, sample paths of F.B.M. almost surely have finite p variation).

Fractals The first study of random processes on fractals with regards to rough paths was in [23], where a Levy area for Brownian motion on the Sierpinski gasket was established. This specific case is subsequently covered in [6], where both nested fractals and Sierpinski carpets fall within the scope of the results presented in the paper.

Reversible Processes In [6], it is shown how for any continuous, reversible Markov process $X : [0,T] \to V$ of finite p variation where $p < 4$ that satisfies the following Hölder condition:

Condition 19 *for some $H > 1/4$, there are constants C_p, $p \geq 1$ such that*

$$\mathbb{E}\left(\|X_t - X_s\|^p\right) \leq C_p \left(t-s\right)^{pH} \quad \forall \left(s,t\right) \in \Gamma_T,$$

then piecewise linear interpolation on dyadic intervals, produces an approximation scheme for an area process that converges almost surely in p variation, providing an element of $\Omega G\left(V\right)_2^p$.

Vaccarro, [45] has extended this work and shown that for any $(s,t) \in \Gamma_T$, there exists a null set off which a piecewise linear regime on dyadic division of the interval $[s,t]$, constructs an area process on $[s,t]$ that agrees with that of [6]'s construction on $[0,T]$ when restricted to $[s,t]$.

2.2 The Itô map

The Itô map is the name given to the functional that solves a differential equation

$$\begin{aligned} dy_t &= f\left(y_t\right) dx_t \\ y_0 &= y, \end{aligned} \tag{2.9}$$

defining a map from a continuous driving signal $x_\cdot : [0,T] \to V$ to a solution $y_\cdot : [0,T] \to W$, where V and W are vector spaces, with

$$f : W \to Hom(V, W).$$

Typically solutions to such a system need not be unique or in fact exist. The principal value of Lyons theory [28], is in using the interpretation of signal and solution as geometric rough paths, to construct new, unique solutions to a particular class of differential equations. Within the context of the p variation metric, this enlarged definition of the Itô map is a uniformly continuous functional.

It is probably impossible to explain concisely and carefully the whole of the route to construct the solutions that Lyons found. The original exposition is in section 4 of [28] but for a *more accessible* source, chapter 6 of [29] is significantly more amenable. In this section, the aim is to explain where the solutions come from, where the solutions are limited and to outline some intuitive concepts about the integration procedure.

2.2.1 Integrals and almost multiplicative functionals

The first step towards defining a solution is to define an integral of a 1-form. Specifically, let

$$\theta : V \to Hom(V, W)$$

and consider the differential equation

$$\begin{aligned} dy_t &= \theta(x_t)\, dx_t \\ y_0 &= a. \end{aligned} \tag{2.10}$$

For Riemannian integration of a bounded variation path $x.$, the required integral, $\int_0^t \theta(x_u)\, dx_u$ can be constructed through an approximation regime. A sequence of dissections of the time interval with successively smaller mesh size, specify approximating paths (which are dependent on points of evaluation).

For example, take a continuous bounded variation path in a vector space $x. : [0,1] \to V$ and let $\delta = \{t_i\}_{i=0}^{|\delta|} \in \Delta_{0,1}$. If $i(t) = \sup\{i : t_i < t\}$ and $\underline{s} = \{s_i\}_{i=0}^{|\delta|-1}$ is a sequence with $t_i \le s_i \le t_{i+1}$, then a candidate approximation to a solution is given by

$$y_t^{\delta,\underline{s}} = a + \sum_{0 \le i < i(t)} \theta(x_{s_i})\left(x_{t_{i+1}} - x_{t_i}\right) + \theta\left(x_{s_{i(t)}}\right)\left(x_t - x_{i(t)}\right).$$

If θ is $Lip(\alpha)$, $\alpha > 0$, then as $mesh(\delta) = \sup_i |t_{i+1} - t_i| \to 0$, the values $y_t^{\delta,\underline{s}}$ converge to y_t, independently of the choices s_i and the function y_t is continuous.

To illuminate the point a little, this convergence works because θ is Lipschitz (in the sense of Stein); $\forall x_1, x_2 \in V$,

$$\theta(x_1) - \theta(x_2) = R(x_1, x_2)(x_1 - x_2),$$

for some $R : V \times V \to Hom(V, Hom(V, W))$, with

$$\|R(x_1, x_2)\| \le M \|x_1 - x_2\|^{\alpha - 1}$$

and some $M < \infty$. As a result, it is possible to show that there is a value y_t such that,

$$y_t^{\delta,\underline{s}} = y_t + o\left(mesh(\delta)^{1+\alpha}\right) \tag{2.11}$$

and thus for a sequence $\delta^n \in \Delta_{0,1}$ with $mesh(\delta^n) \to 0$, the candidate solutions $y_t^{\delta^n,\underline{s}^n}$ converge to y_t which satisfies the o.d.e. They converge uniformly for $t \in [0,1]$ and independently of both the $\underline{s}^n$ and of sequence of dissections δ^n. The crucial point is that if $s < u < t$, then there is the inequality

$$
\begin{aligned}
& \left\| \begin{array}{c} \left(\theta\left(x_s\right)\left(x_u - x_s\right) + \theta\left(x_u\right)\left(x_t - x_u\right)\right) \\ -\theta\left(x_s\right)\left(x_t - x_s\right) \end{array} \right\| \\
= & \left\| \begin{array}{c} \left(\theta\left(x_s\right)\left(x_u - x_s\right) + \theta\left(x_s\right)\left(x_t - x_u\right)\right) + \\ R\left(x_u, x_s\right)\left(x_u - x_s\right)\left(x_t - x_u\right) - \theta\left(x_s\right)\left(x_t - x_s\right) \end{array} \right\| \\
= & \left\| R\left(x_u, x_s\right)\left(x_u - x_s\right)\left(x_t - x_u\right)\right\| \\
\leq & M \sup_{u,v \in [s,t]} \left\| x_v - x_u \right\|^{\alpha+1},
\end{aligned}
\tag{2.12}
$$

from which the relation (2.11) follows eventually.

Why is this important ? The answer is that control of comparative approximations of the form (2.12) alone, guarantees that a solution exists for a bounded variation path x_t. This nugget is the essential point of almost multiplicative functionals.

Definition 20 *An almost multiplicative functional of finite p variation with control ω, is a functional $\mathbf{X}^n : \Gamma_{T_{\mathbf{X}^n}} \to \mathrm{T}^n(V)$ such that $\forall (s,t) \in \Gamma_{T_{\mathbf{X}^n}}$, $j \leq n$,*

$$\left\| \mathbf{X}_{s,t}^{(j)} \right\| \leq \frac{\omega(s,t)^{\frac{j}{p}}}{\beta\left(\frac{j}{p}\right)!}$$

and that there exists $\kappa > 1$ and K such that $\forall (s,u), (u,t) \in \Gamma_{T_{\mathbf{X}^n}}$, $j \leq n$

$$\left\| \left(\mathbf{X}_{s,u}^n \otimes \mathbf{X}_{u,t}^n - \mathbf{X}_{s,t}^n\right)^{(j)} \right\| \leq K\omega(s,t)^{\kappa}$$

The last condition in the definition is the almost multiplicative aspect. Whilst this is an abstract definition, the previous discussion for bounded variation paths contains an example. Namely, take

$$\mathbf{X}_{s,t}^1 = \left(1, \theta\left(x_s\right)\left(\int_s^t dx_u\right)\right)$$

and the control $\omega(s,t) = \sup_{v\in[s,t]} \|x_v - x_s\|$, with $\kappa = 1+\alpha$. Also, any multiplicative functional of finite p variation is clearly an example.

So what now ? Lyons proved that each almost multiplicative functional of finite p variation, is 'close' to an unique element of $\Omega(V)^p$, through an approximation regime analogous to constructing Riemann integrals.

Theorem 21 (Lyons) *Let $\mathbf{X}^n$ be a bounded, almost multiplicative functional with control ω. Then there is an unique multiplicative functional $\tilde{\mathbf{X}}^n \in \Omega(V)^p_n$, controlled by $C_1\omega$ such that $\forall(s,t) \in \Gamma_{T_{\mathbf{X}^n}}$, $\forall i \leq n$,*

$$\left\| \left(\mathbf{X}^n_{s,t} - \tilde{\mathbf{X}}^n_{s,t}\right)^{(i)} \right\| \leq C_2\omega(s,t)^\kappa,$$

for some constants C_1,C_2.

Remark 22 *The objective of the proof is to confirm the existence of the limits*

$$\lim_{\substack{\delta\in\Delta_{s,t}\\ mesh(\delta)\to 0}} \left(\mathbf{X}^n_{s,t_1} \otimes \mathbf{X}^n_{t_1,t_2} \otimes \cdots \otimes \mathbf{X}^n_{t_{|\delta|-1},t}\right)^{(j)} \triangleq \tilde{\mathbf{X}}^{(j)}_{s,t},$$

which in actual fact for $n=1$, amounts to precisely the Riemann integration procedure in the example above, where this time the control ω replaces the h in the error $o(h)$.

There are uniform continuity properties for this map from almost multiplicative functional to multiplicative functionals of finite p variation that are not stated here. Note too that for $n > p$, the multiplicative functionals obtained will be rough paths. The conclusion is thus that to produce an integration scheme with the aim of obtaining a rough path, an appropriate definition of an almost multiplicative functional of finite p variation, will entail the existence of a rough path of finite p variation.

Approximation ideas therefore play an important rôle for this form of integration. To approximate a solution to a differential equation such as (2.10) where θ is a smooth 1-form, an n'th order method of approximation is given by the formula

$$\int_s^t dy_u \approx \theta(x_s)\left(\int_s^t dx_u\right) + \theta^{(1)}(x_s)\left(\iint_{s\le u_1\le u_2\le t} dx_{u_1}\otimes dx_{u_2}\right) + \cdots + \theta^{(j-1)}(x_s)\left(\int\cdots\int_{0\le u_1\le\cdots\le u_j\le t} dx_{u_1}\otimes\cdots\otimes dx_{u_j}\right). \quad (2.13)$$

For a bounded variation path it is correct to order $o\left((t-s)^j\right)$, meaning as before that $j=1$ suffices as an approximation for bounded variation paths . For rougher paths, the formula shows that more terms will be needed in order to get a $o(t-s)$ approximation, which is how Lyons proceeded to construct an almost multiplicative functional. Specifically, take $\mathbf{X}^{\lfloor p\rfloor}\in\Omega G(V)^p$and let θ be a $Lip(\alpha)$ 1-form, $\alpha>p-1$. If (2.13) is used to define the function $\mathbf{Y}^{(1)}:\Gamma_{T_{\mathbf{X}^n}}\to W$ by

$$\mathbf{Y}_{s,t}^{(1)} = \sum_{j=0}^{\lfloor p\rfloor-1}\theta^{(j)}(x_s)\left(\mathbf{X}_{s,t}^{(j+1)}\right), \quad (2.14)$$

then the higher order iterated integrals $\mathbf{Y}_{s,t}^{(j)}$, $1\le j\le\lfloor p\rfloor$ can be defined by using a differential form of (2.14)

$$\begin{aligned}\mathbf{Y}_{s,t}^{(j)} &= \int\cdots\int_{s\le u_1\le\cdots\le u_j\le t} d\mathbf{Y}_{s,u_1}^{(1)}\otimes\cdots\otimes d\mathbf{Y}_{s,u_j}^{(1)} \\ &= \sum_{k_1,\ldots,k_j=0}^{\lfloor p\rfloor-1}\theta^{(k_1)}(x_s)\otimes\cdots\otimes\theta^{(k_j)}(x_s) \\ &\quad \int\cdots\int_{s\le u_1\le\cdots\le u_j\le t} d\mathbf{X}_{s,u_1}^{(k_1+1)}\cdots d\mathbf{X}_{s,u_j}^{(k_j+1)}.\end{aligned}$$

All that remains to be proved is that the function $\mathbf{Y}^{\lfloor p\rfloor}:\Gamma_{T_{\mathbf{X}^{\lfloor p\rfloor}}}\to T^{\lfloor p\rfloor}(W)$ given by

$$\mathbf{Y}_{s,t}^{\lfloor p\rfloor}=\sum_{i=0}^{\lfloor p\rfloor}\mathbf{Y}_{s,t}^{(i)}$$

defines an almost multiplicative functional of finite p variation.

Now the form θ has a Taylor series like approximation, in terms of the derivatives $\theta^{(i)}$, $0\leq i\leq\lfloor p\rfloor-1$:

$$\begin{aligned}\theta^{(i)}(x_2)(v_i,\cdots,v_0) &= \sum_{j=0}^{\lfloor p\rfloor-1-i}\theta^{(i+j)}\left(\frac{(x_2-x_1)^{\otimes j}}{j!},v_i,\cdots,v_0\right)\\ &\quad +R_\theta^j(x_2,x_1)(v_i,\cdots,v_0),\end{aligned}$$

where each $\theta^{(j)}(x)$ and $R_\theta^j(x,y)$ are bounded in operator norm as elements of $Hom\left(\otimes_1^i V,W\right)$. However, provided that each $\frac{(x_2-x_1)^{\otimes j}}{j!}$ is replaced by the j'th iterated integral of a geometric rough path, the operator norm bound of $R_\theta^j(x,y)$ remains the same according to Lyons. This point, although glossed over, is absolutely crucial. It means that different points where each function $\theta^{(i)}$ are evaluated, can be compared in terms of iterated integrals and hence why iterated integrals are good functions to look at. As a result, $\mathbf{Y}_{s,t}^{\lfloor p\rfloor}$ is an almost multiplicative functional controlled by some multiple of ω. Or in other words, for each $1\leq j\leq\lfloor p\rfloor$ and $\forall(s,u),(u,t)\in\Gamma_{T_{\mathbf{X}^n}}$,

$$\left\|\left(\mathbf{Y}_{s,u}^{\lfloor p\rfloor}\otimes\mathbf{Y}_{u,t}^{\lfloor p\rfloor}-\mathbf{Y}_{s,t}^{\lfloor p\rfloor}\right)^{(j)}\right\|\leq K\omega(s,t)^{\kappa}.$$

Excruciatingly, the theory is stuck at the point of handling geometric rough paths only as driving signals for differential equations, since this contractive property only arises because Taylor's theorem is applicable for geometric rough paths.

Remark 23 *A technical point concerns the system used to extend a norm on V to norms on $V^{\otimes n}$. In order to control an important estimate, Lyons found that it is sufficient that the norm of a tensor is* *invariant under permutations : let* $\sigma \in S_n$ *be a permutation of* n *elements and* $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. *Then* σ *defines an element of* $Hom\left(V^{\otimes n}, V^{\otimes n}\right)$ *by*

$$\sigma\left(v_1 \otimes \cdots \otimes v_n\right) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

It extends linearly to the whole of $V^{\otimes n}$ *and the consistency that should be satisfied is* $\forall \sigma \in S_n$, $\forall \underline{v} \in V^{\otimes n}$,

$$\left\|\sigma\left(\underline{v}\right)\right\| = \left\|\underline{v}\right\|.$$

Lyons' integration theorem for geometric rough paths is stated

Theorem 24 (Lyons) *Let* $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G\left(V\right)^p$ *have control* ω *and*

$$\theta \in Lip\left(\gamma - 1\right)\left(V, Hom\left(V, W\right)\right)$$

be a 1-form with $\gamma > p$. *Then the sequence*

$$\mathbf{Y}_{s,t}^{\lfloor p \rfloor} = \left(1, \mathbf{Y}_{s,t}^{(1)}, \cdots \mathbf{Y}_{s,t}^{(\lfloor p \rfloor)}\right)$$

is an almost multiplicative functional of finite p variation and is controlled by a multiple of ω *(the constant is dependent upon* γ, p, θ *and* ω*). The integral is the associated rough path* $\tilde{\mathbf{Y}} \in \Omega G\left(W\right)^p$ *and is written*

$$\begin{aligned} \tilde{\mathbf{Y}}_{s,t} &= \int_{s<u<t} \theta\left(X_u\right)\delta\mathbf{X}_u \\ \delta\tilde{\mathbf{Y}} &= \theta\left(X\right)\delta\mathbf{X} \end{aligned}$$

In addition, the integral which acts on elements of $\Omega G\left(V\right)^p$ and 1-forms that are $Lip\left(\gamma - 1\right)$, is continuous so that the image lies in $\Omega G\left(W\right)^p$.

Remark 25 *This remark is about the construction of higher order iterated integrals that sit above a finite p variation path in a vector space V. First, a definition.*

Definition 26 *An element $\mathbf{X}^n \in \Omega\left((V_1 \oplus V_2 \oplus \cdots \oplus V_d)\right)_n$ is said to have finite $\underline{p} = (p_1, p_2, \ldots p_d)$ variation controlled by ω providing $\mathbf{X}^{(i_1,\ldots,i_d)}$, the process that is the component of $\mathbf{X}^{(i)}$, $i \leq n$, given by*

$$\mathbf{X}^{(i_1,\ldots,i_d)} \in C\left(\Gamma_{T_{\mathbf{X}^n}}, V_1^{\otimes i_1} \otimes \cdots \otimes V_d^{\otimes i_d}\right)$$

where $i = i_1 + \cdots + i_d$, satisfies

$$\left\|\mathbf{X}_{s,t}^{(i_1,\ldots,i_d)}\right\| \leq \frac{\omega(s,t)^{i_1/p_1+\cdots+i_d/p_d}}{\beta^d\left(\frac{i_1}{p_1}!\right)\cdots\left(\frac{i_d}{p_d}!\right)}$$

Such processes would be denoted by $\Omega\left((V_1 \oplus V_2 \oplus \cdots \oplus V_d)\right)_n^{\underline{p}}$. Uniqueness of extension arguments follow similarly to Theorem 8.

Now let

$$\mathbf{X}^{\lfloor p \rfloor} \in \Omega\left(\left(V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes \lfloor p \rfloor}\right)\right)_{\lfloor p \rfloor}^{(p,p/2,\ldots,p/\lfloor p \rfloor)}.$$

Then, $\mathbf{X}^{\lfloor p \rfloor}$ defines an element $\widetilde{\mathbf{X}}^{\lfloor p \rfloor} \in \Omega\left((V \oplus V \oplus \cdots \oplus V)\right)_{\lfloor p \rfloor}^{p}$, through extending the natural inclusions of $\Omega\left(V^{\otimes i}\right)^{p/i} \in \Omega(V)^p$. Now $\widetilde{\mathbf{X}}^{\lfloor p \rfloor}$ defines an element $\mathbf{Y}^{\lfloor p \rfloor} \in \Omega(V)^p$ through what amounts to taking the integral

$$\begin{aligned}
\delta \mathbf{Y}^{\lfloor p \rfloor} &= \theta\left(\widetilde{X}^{\lfloor p \rfloor}\right) \delta \widetilde{\mathbf{X}}^{\lfloor p \rfloor} \\
\mathbf{Y}_{s,t}^{\lfloor p \rfloor} &= \int_{s<u<t} \theta\left(\widetilde{X}_u^{\lfloor p \rfloor}\right) \delta \widetilde{\mathbf{X}}_u^{\lfloor p \rfloor}
\end{aligned}$$

where θ is the smooth 1 form

$$\theta : \left(\oplus_1^{\lfloor p \rfloor} V\right) \to Hom\left(\oplus_1^{\lfloor p \rfloor} V, V\right)$$

$$\theta\left(v_1,\ldots,v_{\lfloor p\rfloor}\right)\left(dx_1,\ldots,dx_{\lfloor p\rfloor}\right)=\left(v_1+\ldots+v_{\lfloor p\rfloor}\right)\left(dx_1+\ldots+dx_{\lfloor p\rfloor}\right)$$

What relevance does this have for constructing higher order iterated integrals? The element of $\Omega(V)^p$ given by projecting $\widetilde{\mathbf{X}}^{\lfloor p\rfloor}$ onto the V_i, $1\leq i\leq d$ components, are elements of $\Omega(V)^p$. The first iterated integral is not identically zero only for $i=1$ and then it's first iterated integral agrees with that of $\mathbf{Y}^{\lfloor p\rfloor}$. Hence, any $\mathbf{Y}^{\lfloor p\rfloor}$ due to such a construction, is a different extension of the path in V defined by this first iterated integral. As such, there is no way to reconstruct the path $\mathbf{X}^{\lfloor p\rfloor}$ from the $\mathbf{Y}^{\lfloor p\rfloor}$. However, should there exist a regime to construct higher order iterated integrals through, say dyadic approximation procedures utilized in [6] or [26] for example, there would be a bijection between such processes and the decomposition, thus giving a characterization procedure. It should be born in mind that this means that the information of each description is equivalent.

2.2.2 Solving a differential equation

The last part of the theory presented in [28], is the construction of solutions to the differential equation (2.9). This step is achieved by augmenting the solution space to include the driving signal. In this setting it is possible to integrate geometric rough paths against an appropriate 1 form, before embarking on a procedure of Picard iteration which gives convergence to a solution to (2.9) through the boundedness of the integral operator. So let h be the 1-form defined on the Banach space $V\oplus W$, (with a suitable choice of norm on each $(V\oplus W)^{\otimes n}$), so that

$$h(x,y)\begin{pmatrix} dx \\ dy \end{pmatrix}=\begin{pmatrix} dx \\ f(y)\,dx \end{pmatrix}.$$

An element $\mathbf{Z}^{\lfloor p \rfloor} \in \Omega G(V \oplus W)^p$ that satisfies

$$\begin{aligned} \delta \mathbf{Z} &= h(Z)\,\delta \mathbf{Z}^{\lfloor p \rfloor} \\ Z_0 &= \begin{pmatrix} x_0 \\ a \end{pmatrix} \end{aligned}$$

will be said to define a solution to (2.9). The principle is that only knowledge of both $\mathbf{Y}_{s,t}^{\lfloor p \rfloor}$ (the projection of $\mathbf{Z}_{s,t}^{\lfloor p \rfloor}$ onto $T(W)$) and the mixed iterated integrals with the driving signal, will be the whole, real content of a solution. The 'Universal Limit Theorem' as first stated in [28] is

Theorem 27 (Lyons) *Suppose that $f : V \to Lip(\gamma, V, W)$ is a linear map into Lipschitz vector fields. Then consider the Itô map $X \to (X, Y)$ defined for smooth paths by*

$$\begin{aligned} dY_t &= f(Y_t)\,dX_t \\ Y_0 &= a. \end{aligned}$$

Define the 1-form h by

$$\begin{aligned} h(x,y)\begin{pmatrix} dX \\ dY \end{pmatrix} &= h(y)\begin{pmatrix} dX \\ dY \end{pmatrix} \\ &= \begin{pmatrix} dX \\ f(y)\,dX \end{pmatrix}. \end{aligned}$$

For any geometric multiplicative functional $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^p$ with $1 \le p < \gamma$ there is exactly one geometric multiplicative functional extension

$$\mathbf{Z}^{\lfloor p \rfloor} = \begin{pmatrix} \mathbf{X}^{\lfloor p \rfloor} \\ \mathbf{Y}^{\lfloor p \rfloor} \end{pmatrix} \in \Omega G(V \oplus W)^p$$

such that if $Y_t = \mathbf{Y}^{(1)}_{0,t} + y_0$*, then* $\mathbf{Z}^{\lfloor p \rfloor}$ *satisfies the differential equation*

$$\delta \mathbf{Z}^{\lfloor p \rfloor} = h(Y_t)\, \delta \mathbf{Z}^{\lfloor p \rfloor}.$$

Moreover, this solution to the rough differential equation is constructed by Picard iteration. The Itô map is uniformly continuous and the map $\mathbf{X}^{\lfloor p \rfloor} \to \mathbf{Z}^{\lfloor p \rfloor}$ *is the unique continuous extension of the Itô map*

$$\Omega G(V)^p \to \Omega G(V \oplus W)^p .$$

The iteration scheme is quite an involved process and most clearly presented in [29]. For an element $\mathbf{X}^{\lfloor p \rfloor}(\gamma)$ where $\gamma \in \mathcal{P}_b(V)$ it constructs a sequence of elements

$$\mathbf{K}^{\lfloor p \rfloor}(n) = \begin{pmatrix} \mathbf{X}^{\lfloor p \rfloor} \\ \mathbf{Y}^{\lfloor p \rfloor}(n) \\ \mathbf{D}^{\lfloor p \rfloor}(n) \end{pmatrix} \in \Omega G(V \oplus W \oplus W)^p ,$$

starting at the point $\mathbf{K}^{\lfloor p \rfloor}(1) \in \Omega G(V \oplus W \oplus W)^p$ given by the solution to

$$\delta \mathbf{K}^{\lfloor p \rfloor}(1) = \begin{pmatrix} \delta \mathbf{X}^{\lfloor p \rfloor}(\gamma) \\ f(y_0)\, \delta \mathbf{X}^{\lfloor p \rfloor}(\gamma) \\ f(y_0)\, \delta \mathbf{X}^{\lfloor p \rfloor}(\gamma) \end{pmatrix},$$

and proceeds with the iteration scheme

$$\delta \mathbf{K}^{\lfloor p \rfloor}(n+1) = \begin{pmatrix} \delta \mathbf{X}^{\lfloor p \rfloor}(\gamma) \\ f\left(Y^{\lfloor p \rfloor}(n)\right) \delta \mathbf{X}^{\lfloor p \rfloor}(\gamma) \\ g\left(Y^{\lfloor p \rfloor}(n), Y^{\lfloor p \rfloor}(n) - D^{\lfloor p \rfloor}(n)\right)\left(D^{\lfloor p \rfloor}(n)\right) \delta \mathbf{X}^{\lfloor p \rfloor}(\gamma) \end{pmatrix}$$

where $g : W \oplus W \to Lip(\gamma - 1, W \otimes V, W)$ is specially chosen with regards to f in order that $\mathbf{D}^{\lfloor p \rfloor}(n)$ is the difference

$$\mathbf{D}^{\lfloor p \rfloor}(n) = \mathbf{Y}^{\lfloor p \rfloor}(n) - \mathbf{Y}^{\lfloor p \rfloor}(n-1).$$

Lyons shows that at any time point, there is a neighborhood dependent only on f, p, the Lipshitz constant of f and the control of $\mathbf{X}^{\lfloor p \rfloor}(\gamma)$, such that through the iteration scheme, the $\mathbf{K}^{\lfloor p \rfloor}(n)$ converge in the p variation topology on this neighborhood, to what will be an element of $\Omega G(V \oplus W \oplus W)^p$. This limit is shown to be unique and the projection onto the third component shows that the difference $\mathbf{D}^{\lfloor p \rfloor}(n)$ converges to zero. To construct the whole solution for the signal $\mathbf{X}^{\lfloor p \rfloor}(\gamma)$, uniform estimates allow the concatenation of a sequence of solutions, to produce a solution on the whole time interval. The projection of the limiting object onto the element $\mathbf{Z}^{\lfloor p \rfloor}$ of $\Omega G(V \oplus W)^p$ thus satisfies

$$\delta \mathbf{Z}^{\lfloor p \rfloor} = h(Y_t)\, \delta \mathbf{Z}^{\lfloor p \rfloor},$$

and defines a solution to the differential equation 2.9.

Crucially, the map from $\mathcal{P}_b(V)$ to $\Omega G(V \oplus W)^p$ is continuous for the p variation topology. Hence by continuity, it extends to a map of the whole of $\Omega G(V)^p$ to $\Omega G(V \oplus W)^p$ and this function is known as the Itô map.

Remark 28 *The fact that Lyons' integrals are only defined for elements of $\Omega G(V)^p$, mapping into some $\Omega G(V \oplus W)^p$, strikes a chord with conventional geometrical interpretations of integration of 1 forms, which say perhaps that they should always be thought of as developing a path into a Lie group - as will be explained, geometric rough paths are closely related to a particular Lie group.*

Chapter 3

Algebraic Structures

3.1 An introduction

The first section contains a discussion of and an abstract dissection of the tensor algebra associated to a vector space V, the space in which one defines the signature of a rough path. The second section pertains to the object to which the tensor algebra is the dual space, a particular collection of functions on paths in V. It is explained how this linear space in fact can behave like an algebra, where the product is naturally induced from an action on a special class of signatures of rough paths in V. Taking the picture on, the aim is to show how to recover the class of signatures from this space of functions. The procedure is revealed through the machinery of Hopf algebras, providing a classical mathematical picture of duality between groups and functions on the group. Following on is a description of an interpretation of $T(V)$

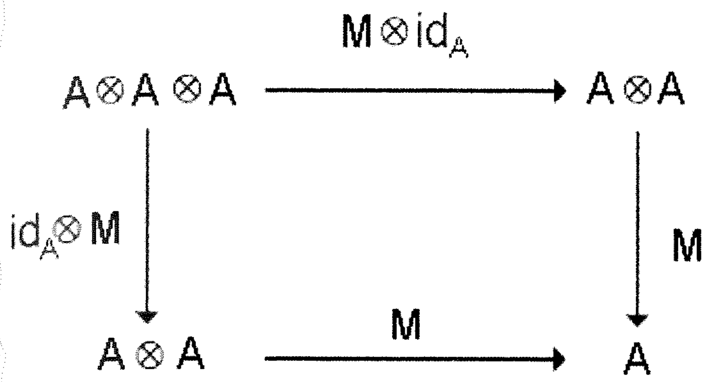


Figure 3.1: **This diagram describes the associativity rule for an algebra**

as a collection of differentiations and a construction that generalizes to provide a space of differential operators. To a significant extent, the ideas presented are heuristic and conceptual - it is an attempt to outline a picture from non-commutative harmonic analysis in the context of rough paths.

To start and with a view towards clarity, recall the abstract definition of an algebra.

Definition 29 *An algebra (A, M, u) over a field $\mathbf{F}$ is a vector space over $\mathbf{F}$ with a multiplication $M : A \otimes_{\mathbf{F}} A \longrightarrow A$ that is associative and an unit $u : \mathbf{F} \longrightarrow \mathbb{A}$ that is unitary:*

$$\left\{ \begin{array}{c} M \circ (id_A \otimes M) = M \circ (M \otimes id_A) \ : \ associative \\ M \circ (id_A \otimes u) = M \circ (u \otimes id_A) = id_A \ : \ uniitary \end{array} \right\}, \tag{3.1}$$

where $id_A : A \hookleftarrow$ is the identity map.

3.2 The Tensor Algebra

3.2.1 Universal Algebra

It was described in (chapter 2), how the truncated tensor algebras $T^n(V)$ are natural spaces for the description of processes that reflect many useful aspects of the evolution of paths in a vector space. The full tensor algebra provides a space for the object known as the signature of a rough path, which is the unique multiplicative object in the tensor algebra to extend a rough path whilst retaining the decay rate of the iterated integrals.

In this section, V will be a finite dimensional vector space.

$T(V)$

The tensor algebra $T(V)$ is the associative algebra, unique up to isomorphism, with the property that for any linear map ϕ from V to an associative algebra A, there exists an unique algebra homomorphism $\widetilde{\phi}$ extending ϕ, that maps $T(V)$ to A so that $\phi = \widetilde{\phi} \circ i$, where i is the inclusion mapping of V into $T(V)$. A model for this object is given by

$$T(V) = \oplus_{n=0}^{\infty} V^{\otimes n},$$

where $V^{\otimes 0} = \mathbb{R}$ and the tensor product is taken over $\mathbb{R}$. The product for the algebra is just the tensor product $\otimes$, while the unit $u : \mathbb{R} \to T(V)$ is the identification of $\mathbb{R}$ with $V^{\otimes 0}$, giving the algebra structure

$(T(V), \otimes, u)$.

To clarify what is meant by the formal infinite direct sum, a sequence $\underline{t} = \sum_{n=0}^{\infty} \underline{t}^{(n)}$ is an element of $T(V)$ where each $\underline{t}^{(n)} \in V^{\otimes n}$. A sequence $\{\underline{t}_m\}_{m=0}^{\infty} \in T(V)$ is said to be summable if for each n, all but a finite number of the $\underline{t}_m^{(n)}$ are zero tensors. The sum $\underline{t} = \sum_{n=0}^{\infty} \underline{t}^{(n)} \in T(V)$ is thus well defined, where $\underline{t}^{(n)} = \sum_{m=0}^{\infty} \underline{t}_m^{(n)}$.

The tensor algebra itself has various relevant structural properties, foremost being a natural relation to the free Lie algebra of the set V.

$L(V)$

A Lie algebra is a vector space L with a product $[\cdot, \cdot]$, which is anti-symmetric:

$$[l_1, l_2] = -[l_2, l_1] \quad \forall l_1, l_2 \in L$$

and satisfies the Jacobi relation

$$[l_1, [l_2, l_3]] + [l_2, [l_3, l_1]] + [l_3, [l_1, l_2]] = 0 \quad \forall l_1, l_2, l_3 \in L.$$

A Lie algebra L and a map $i : V \to L$ is said to be free for V if for any function $f : V \to L_0$ where L_0 is a lie algebra, there is a homomorphism of lie algebras $\psi : L \to L_0$ such that the maps commute.

In $T(V)$, let us define the bracket of two tensors u_1, u_2 to be

$$[u_1, u_2] = u_1 \otimes u_2 - u_2 \otimes u_1.$$

The free Lie algebra of V, $L(V)$ has a model in $T(V)$ characterized as being the smallest subspace of $T(V)$ containing V and closed under the bracket $[\cdot,\cdot]$. The notation for iterated bracketing sequences is:

Notation 30 *Let $u_i \in \mathrm{T}(V)$, $1 \leq i \leq n$, then the iterated bracket is denoted by*

$$\begin{aligned}
[u_1, u_2, u_3, \ldots, u_n] &= [u_1, [u_2, [u_3, \cdots, [u_{n-1}, u_n] \cdots]]] \\
&= [u_1, [u_2, u_3, \cdots, u_{n-1}, u_n]] \\
&= [u_1, u_2, u_3, \cdots, [u_{n-1}, u_n] \cdots].
\end{aligned}$$

An observation due to Dynkin implies that the span of elements in the form $[v_1, v_2, \ldots, v_n]$ is the component of $L(V)$ of homogeneous tensor degree n, denoted by $L^{(n)}(V)$.

Proposition 31 (Dynkin) *For $u \in V^{\otimes n}$ define the linear map $\phi : \mathrm{T}(V) \hookleftarrow$ through*

$$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = [v_1, v_2, \ldots, v_n],$$

for $v_i \in V$, $1 \leq i \leq n$ and the linear extension of ϕ first to the whole of $V^{\otimes n}$ and then to the whole of $\mathrm{T}(V)$. Then

$$u \in \mathrm{L}^{(n)}(V) \Leftrightarrow \phi(u) = nu,$$

so that the subspace $\mathrm{L}^{(n)}(V)$ is the eigenspace of ϕ with eigenvalue n.

This justifies the expression $L^{(n)}(V) = [V, [V, \cdots [V, V] \cdots]]$ where V appears n times and hence the extended expression

$$L(V) = V \oplus [V, V] \oplus [V, [V, V]] \oplus \cdots.$$

$U(L(V))$

The Universal Enveloping algebra of a lie algebra L, $U(L)$, is an associative algebra. It is characterized by the property that to any homomorphism φ of L to an unital associative algebra A, there is an unique extension of φ to an algebra map from $U(L)$ to A. It is unique up to isomorphism and a model for it is given by

$$U(L) = T(L) / J(L),$$

where $J(L)$ is the 2-sided ideal of $T(L)$ generated by elements of the form $l_1 \otimes l_2 - l_2 \otimes l_1 - [l_1, l_2]$ where $l_1, l_2 \in L$.

In fact, $T(V) \cong U(L(V))$ since almost by definition, both associative algebras exhibit the necessary properties for both V and $L(V)$ (there are algebra maps from one to the other which when composed together are the identity).

A theorem of Poincaré, Birkhoff and Witt shows how an ordered basis for L can be used to construct a basis of $U(L)$. However, there is a slightly more natural way to construct a basis system for $U(L(V))$, through a symmetric representation. Given a collection of tensors $l_1, l_2, \ldots, l_n \in L$, denote the symmetrized product in $U(L)$

$$(l_1, l_2, \ldots, l_n) = \frac{1}{n!} \sum_{\sigma \in S_n} l_{\sigma(1)} l_{\sigma(2)} \cdots l_{\sigma(n)}, \tag{3.2}$$

where S_n is the set of permutations of n elements. For each n, there is a subspace of $U(L)$ denoted $U_n(L)$ as the linear span of the $(l_1, l_2, \ldots, l_n)$; $l_i \in L$. $U_n(L)$ is equivalently generated by the n'th powers in $U(L)$ of elements of L (See [33] p.57). Then $U_0(L) = \mathbb{R}$, $U_1(L) = L$ and there is a direct sum decomposition of $U(L)$ (See [33] p.13):

$$U(L) = \oplus_{n \geq 0} U_n(L)$$

In addition, with any ordered basis of L, $\{l_i\}_{i-1}^{\infty}$, this provides a basis system of $U(L)$ in terms of elements of the form $(l_{i_1}, l_{i_2}, \ldots, l_{i_n})$ where $i_1 \leq i_2 \leq \cdots \leq i_n$.

In the case of $T(V)$, the isomorphism $i : U(L(V)) \to T(V)$ given by $i(l_1 l_2 \cdots l_n) = l_1 \otimes l_2 \otimes \cdots \otimes l_n$ induces the corresponding direct sum decomposition $T(V) = \oplus_{n \geq 0} i(U_n(L(V)))$. The decomposition is sometimes referred to as canonical. The projections onto the subspaces $U_n(L(V))$ in $T(V)$ (with an abuse notation suppressing the map i) were first described by Solomon in [38] and can be found in [16] for example, however perhaps more elegantly, [33] relates them to the Hopf algebra setting which is described later.

The next comment concerns basis sets for $T(V)$. With basis sets for each $L^{(n)}(V)$, it is possible to produce an ordered basis for $L(V)$, $\{l_i\}_{i-1}^{\infty}$ say, with $l_i \in L^{(d_i)}(V)$ and $d_i \leq d_{i+1}$ $\forall i$ for simplicity. For the spaces denoted by $U_n^m(L(V)) = U_n(L(V)) \cap V^{\otimes m}$, there is a basis generated by all tensors $(l_{i_1}, l_{i_2}, \ldots, l_{i_m})$ where $i_1 \leq i_2 \leq \cdots \leq i_m$ and $\sum_{j=1}^{m} d_{i_j} = n$.

3.2.2 Exponentials and Logarithms

An intrinsic object in the construction is the collection of exponentials of Lie elements, a generalized group in some sense. Recall the definition of the exponential and logarithmic maps, $\exp(\cdot)$ and $\log(\cdot)$.

Definition 32 *The exponential map is a well defined function*

$$\begin{aligned} \exp &: \mathrm{T}(V) \backslash V^{\otimes 0} \to \mathrm{T}(V) \\ \exp(\underline{v}) &= \sum_{n=0}^{\infty} \frac{\underline{v}^{\otimes n}}{n!}, \end{aligned}$$

where the convergence of the sum is assured since the component of degree zero of $\underline{v}$ is zero, powers $\underline{v}^{\otimes n}$ have a non-zero component of lowest degree at least n, so that only a finite number of terms in the infinite sum contribute to any given tensorial degree.

Definition 33 *The logarithmic map is the function defined by*

$$\begin{aligned} \log &: 1 \oplus \left(\mathrm{T}(V) \backslash V^{\otimes 0}\right) \to \mathrm{T}(V) \backslash V^{\otimes 0} \\ \log(\underline{v}) &= (\underline{v} - 1) - \frac{(\underline{v}-1)^{\otimes 2}}{2} + \frac{(\underline{v}-1)^{\otimes 3}}{3} - \cdots, \end{aligned}$$

which is again well-defined since only a finite number of terms contribute to any given tensor degree.

Each of the maps $\exp(\cdot)$ and $\log(\cdot)$ are bijections for which one can verify algebraically that

$$\exp(\log(\cdot)) : 1 \oplus \left(T(V) \backslash V^{\otimes 0}\right) \hookleftarrow,$$

is the identity map, as is

$$\log(\exp(\cdot)) : T(V) \backslash V^{\otimes 0} \hookleftarrow.$$

The cornerstone of any analysis of the exponential map is the formula that relates products of exponentials, known as the Campbell-Baker-Hausdorff formula 6.1. It expresses how to take the logarithm of the product of two exponentials of elements $\underline{v}_1, \underline{v}_2 \in T(V) \backslash V^{\otimes 0}$. The formula is an infinite series with one well known expression due to Dynkin (see [36] for example), though here the expansion up to fourth order terms is specified:

$$\begin{aligned} & \log\left(\exp\left(\underline{v}_1\right)\otimes\exp\left(\underline{v}_2\right)\right) \\ = \; & \underline{v}_1+\underline{v}_2+\frac{1}{2}\left[\underline{v}_1,\underline{v}_2\right]+\frac{1}{12}\left(\left[\underline{v}_1,\left[\underline{v}_1,\underline{v}_2\right]\right]+\left[\underline{v}_2,\left[\underline{v}_2,\underline{v}_1\right]\right]\right) \\ & +\frac{1}{24}\left[\underline{v}_1,\left[\underline{v}_2,\left[\underline{v}_2,\underline{v}_1\right]\right]\right]+\cdots. \end{aligned}$$

3.3 The Shuffle Algebra

Denote the dual to V by V^* and the set of finite linear combinations of tensors in V^* by $\mathcal{T}(V^*)$, so that

$$\begin{aligned} \mathcal{T}(V^*) &= \cup_{n=0}^{\infty}\left(\oplus_{i=0}^{n}\left(V^*\right)^{\otimes i}\right) \\ &= \cup_{n=0}^{\infty}\left(\oplus_{i=0}^{n}\left(V^{\otimes i}\right)^*\right). \end{aligned}$$

Note that at least when V is finite dimensional, there is the identification $T(V)=\mathcal{T}(V^*)^*$, so that there is the inclusion $\mathcal{T}(V^*)\subset T(V)^*$, $\mathcal{T}(V^*)$ is known as the graded dual of $T(V)$ where the grading is with respect to tensorial degree in $T(V)$.

Hence, to any element $\underline{\omega}\in\mathcal{T}(V^*)$, by letting $\underline{\omega}$ act on the signature of the path $\mathbf{X}_{sig}(\gamma)$, a function is defined on the collection of piecewise smooth paths in V, $\mathcal{P}(V)$: if $\underline{\omega}\in(V^*)^{\otimes i}$ is of the form $\underline{\omega}=\omega_1\otimes\omega_2\otimes\cdots\otimes\omega_i$ with $\omega_j\in V^*$, then

$$\int_\gamma \underline{\omega}\equiv\underline{\omega}\left(\mathbf{X}_{sig}(\gamma)\right)=\underline{\omega}\left(\mathbf{X}_{sig}^{(i)}(\gamma)\right)$$

$$= \int \cdots \int_{0 \le u_1 \le u_2 \le \cdots \le u_i \le 1} \left\{ \begin{array}{c} \omega_1 \left(d\gamma \left(u_1 \right) \right) \ \omega_2 \left(d\gamma \left(u_2 \right) \right) \\ \cdots \ \omega_i \left(d\gamma \left(u_i \right) \right) \end{array} \right\}, \tag{3.4}$$

with an abuse of notation in the sense that the natural inclusion of $\underline{\omega}$ in $T(V)^*$ is considered.

The definition extends naturally by linearity to the whole of $\mathcal{T}(V^*)$. At the same time, these linear functions on $T(V)$ generate a commutative algebra due to the following curious property, occurring when the action of these linear maps is restricted to signatures of piecewise smooth paths. In this case, the product of two elements of $\mathcal{T}(V^*)$ (and also linear functions on $T(V)$) is expressible as another element of $\mathcal{T}(V^*)$ itself. The form of the multiplication is expressed in terms of the so called "shuffle product" and essentially represents an extended integration by parts formula. For $\gamma \in \mathcal{P}_b(V)$, $\gamma : [0,1] \to V$ and an element of the form $\underline{\omega} = \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i$, $\int_\gamma \underline{\omega}$ is an integral over the n-simplex

$$\Delta^n (t_1, \ldots, t_n) = \{(t_1, \ldots, t_n); 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1\},$$

see equation (3.4). The product of two such simplices, $\Delta^n (t_1, \ldots, t_n)$ and $\Delta^m (t_{n+1}, \ldots, t_{n+m})$ is expressible as the union of a collection of $n+m$-simplices in the variables $t_1, \ldots, t_{n+m}$:

$$\begin{aligned} & \Delta^n (t_1, \ldots, t_n) \cap \Delta^m (t_{n+1}, \ldots, t_{n+m}) \\ = \ & \cup_{\sigma \in Sh(n,m)} \Delta^{n+m} \left(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n+m)} \right), \end{aligned}$$

where $Sh(n,m)$ is the subset of the permutations of the set $\{1, 2, \ldots, n+m\}$, such that the relative order of the indices is preserved :

$$\sigma \in Sh(n,m) \Leftrightarrow \left\{ \begin{array}{c} \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n), \\ \sigma^{-1}(n+1) < \sigma^{-1}(n+2) < \cdots < \sigma^{-1}(n+m) \end{array} \right\}.$$

Thus if $\underline{\omega} = \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n$, $\underline{\omega}' = \omega_{n+1} \otimes \omega_2 \otimes \cdots \otimes \omega_{n+m}$

$$\begin{aligned}
&\int_\gamma \underline{\omega} \cdot \int_\gamma \underline{\omega}' \\
&= \int \cdots \int_{\Delta^n(t_1,\ldots,t_n) \cap \Delta^m(t_{n+1},\ldots,t_{n+m})} \left\{ \begin{array}{c} \omega_1 (d\gamma(t_1)) \quad \cdots \\ \cdots \quad \omega_{n+m}(d\gamma(t_{n+m})) \end{array} \right\} \\
&= \sum_{\sigma \in Sh(n,m)} \int \cdots \int_{\Delta^{n+m}\left(t_{\sigma(1)},\ldots,t_{\sigma(n+m)}\right)} \left\{ \begin{array}{c} \omega_1 \left(d\gamma\left(t_{\sigma^{-1}(1)}\right)\right) \quad \cdots \\ \cdots \quad \omega_{n+m}\left(d\gamma\left(t_{\sigma^{-1}(n+m)}\right)\right) \end{array} \right\} \\
&= \sum_{\sigma \in Sh(n,m)} \int \cdots \int_{\Delta^{n+m}(t_1,\ldots,t_{n+m})} \left\{ \begin{array}{c} \omega_{\sigma(1)} (d\gamma(t_1)) \quad \cdots \\ \cdots \quad \omega_{\sigma(n+m)}(d\gamma(t_{n+m})) \end{array} \right\},
\end{aligned}$$

where the second equality is true precisely because the Lebesgue measure of the intersection of any pair of sets in the union is zero and the paths are bounded variation. Probabilists clearly would have an issue of definition of integral for this equality to hold when Brownian paths were the integrators. This equality is an intrinsic property of the geometric rough paths since they are limits of Cauchy sequences of piecewise smooth paths under the $p-$variation metric. These functions are continuous functions of the signature and hence of the paths.

The last equality explains how to define a product on $\mathcal{T}(V^*)$ that describes the effect of multiplying these functions on the space of signatures of bounded variation paths, for if in the case above with the definition

$$sh(\underline{\omega}, \underline{\omega}') = \sum_{\sigma \in Sh(n,m)} \omega_{\sigma(1)} \otimes \omega_{\sigma(2)} \otimes \cdots \otimes \omega_{\sigma(n+m)}$$

then regardless of the piecewise smooth path γ the following equation is an identity

$$\int_\gamma \underline{\omega} \cdot \int_\gamma \underline{\omega}' = \int_\gamma sh(\underline{\omega}, \underline{\omega}').$$

This kind of relation is reminiscent of properties of how characters in group representation theory multiply.

Clearly the definition of $sh(\cdot, \cdot)$ is bilinear and hence can be extended to the whole of $\mathcal{T}(V^*) \otimes \mathcal{T}(V^*)$ as a commutative product. The unit, which is just the inclusion map $v : \mathbb{R} \to \mathcal{T}(V^*)$, completes the algebra structure, $(\mathcal{T}(V^*), sh, v)$.

The reason for the naming as the shuffle product arises when with the picture of $\underline{\omega}$ and $\underline{\omega}'$ as two packs of cards - any riffle shuffle then merges the decks and preserves the order of the cards of $\underline{\omega}$ and $\underline{\omega}'$ respectively. The shuffle product is now the sum over all possible riffle shuffles.

3.4 Coalgebras, Bialgebras and Hopf Algebras

This section sets out the abstract definitions of a Hopf algebra in a self-contained manner, developing an example of the structure being defined for the case of trigonometric functions. The purpose is to clearly explain these algebraic structures and offer some intuition, before utilizing them in the case of rough paths.

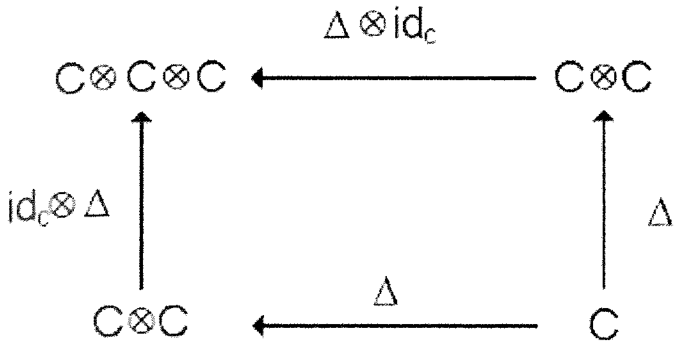


Figure 3.2: **The diagram describes the coassociativity rule for a coalgebra**

A coalgebra is more or less a dual object to an algebra, so in relation to diagram 3.1, it is described by pictures with arrows that 'go the other way'.

Definition 34 *A coalgebra* (C, Δ, ε) *over a field* $\mathbf{F}$ *is a vector space over* $\mathbf{F}$ *with a comultiplication* $\Delta : C \to C \otimes_{\mathbf{F}} C$ *that is coassociative and a co-unit* $\varepsilon : C \to \mathbf{F}$ *that is counitary:*

$$\left\{ \begin{array}{c} (id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta \ : \ coassociative \\ \\ (\varepsilon \otimes id_C) \circ \Delta = (id_C \otimes \varepsilon) \circ \Delta = id_C \ : \ counitary \end{array} \right\}, \tag{3.5}$$

where $id_C : C \hookleftarrow$ *is the identity map.*

Figure 3.2 describes the coassociativity rule - again, the arrows commute.

In general, the dual A^* of an algebra A is not a coalgebra because the map dual to the multiplication, M^* maps $A^* \to (A \otimes A)^* \not\subseteq A^* \otimes A^*$ in general. There is in fact a well defined substitute to A^* often denoted by A^o; [41] ch. 6 is a good reference for details. It is however the case that the co-product always induces an algebra structure on the dual object because $\Delta^*$is defined on $(C \otimes C)^* \supseteq C^* \otimes C^*$.

A simple example of a coalgebra is given by the algebra of trigonometric functions:

Example 35 *Let C be the vector space spanned by the functions $\cos(x)$ and $\sin(x)$. Then the double angle formulae produce the coproduct*

$$\begin{aligned}\Delta(\cos) &= \cos\otimes\cos - \sin\otimes\sin \\ \Delta(\sin) &= \sin\otimes\cos + \cos\otimes\sin\,,\end{aligned}$$

with the co-unit defined by evaluation at 0:

$$\begin{aligned}\varepsilon(\cos) &= 1 \\ \varepsilon(\sin) &= 0\end{aligned}$$

Very often, notation due to Sweedler is used : $\Delta(c) = c_{(1)}\otimes c_{(2)}$ which is taken to mean an expression for the coproduct of an element in terms of a sum over the index i. The coassociativity rule becomes

$$\begin{aligned}c_{(1)}\otimes\left(c_{(2)}\right)_{(1)}\otimes\left(c_{(2)}\right)_{(2)} &= \left(c_{(1)}\right)_{(1)}\otimes\left(c_{(1)}\right)_{(2)}\otimes c_{(1)} \\ &= c_{(1)}\otimes c_{(2)}\otimes c_{(3)},\end{aligned}$$

where the second line is hence unambiguous. A coalgebra is said to be co-commutative when $\tau(\Delta(c)) = \Delta(c)$ where $\tau : C\otimes C \hookleftarrow$ is the twist map $\tau(c_1\otimes c_2) = c_2\otimes c_1$. Clearly trigonometric coalgebra is co-commutative.

Definition 36 *The set of elements in a coalgebra that satisfy the property*

$$\Delta(c) = c\otimes c,$$

are called the group-like elements of the coalgebra.

Remark 37 *Paraphrasing [41], the set of group-like elements in a coalgebra are linearly independent - this in fact is relatively easy to see through linear algebra.*

While not being in a position as of yet to explain any use for this definition, introduce the notion of a bialgebra :

Definition 38 *A Bialgebra* $(H, M, u, \Delta, \varepsilon)$ *is an object which has both an algebra and a coalgebra structure such that one of the following equivalent properties holds :*

1) M *and* u *are coalgebra maps.*

2) Δ *and* ε *are algebra maps.*

Definition 39 *The set of elements of a coalgebra that satisfy*

$$\Delta(c) = c \otimes 1 + 1 \otimes c,$$

are said to be primitive.

Remark 40 *In any Bialgebra, the primitive elements form a Lie algebra with Lie bracket* $[c_1, c_2] = M(c_1, c_2) - M(c_2, c_1)$

Example 41 *The example above generates a bialgebra structure for the algebra* H *of polynomials in the functions* cos *and* sin. *Then the coproduct is an algebra map, so that in general*

$$\Delta\left((\cos)^n (\sin)^m\right) = (\Delta(\cos))^n (\Delta(\sin))^m,$$

which gives for example

$$\Delta\left((\cos)^2\right) = (\cos\otimes\cos - \sin\otimes\sin)^2$$

$$= \cos^2\otimes\cos^2 - 2\cos\sin\otimes\cos\sin + \sin^2\otimes\sin^2$$

and so a functorial expression for $\cos(x+y)^2$*:*

$$\cos(x+y)^2 = \left\{\begin{array}{c} \cos(x)^2\cos(y)^2 - 2\cos(x)\sin(x)\cos(y)\sin(y) \\ +\sin(x)^2\sin(y)^2 \end{array}\right\}.$$

Taking the field to be $\mathbb{C}$*, the linearly independent group-like elements are identifiable as integral powers of the element*

$$\cos + i\sin,$$

giving an algebraic interpretation of Fourier series.

The final structural component of a Hopf algebra is a map called an antipode. The definition requires an examination of an algebra of linear operators with a product which comes from the bialgebra structures. In general, for a coalgebra C and an algebra A, the set of linear maps from C to A, $Hom(C,A)$ has an algebra structure, the convolution product $\star$, which is constructed through the coproduct and product of C and A. If $\varphi,\psi \in Hom(C,A)$, define

$$\varphi\star\psi = M_A\circ(\varphi\otimes\psi)\circ\Delta_C. \tag{3.6}$$

Definition 42 *A Hopf Algebra* $(H,M,u,\Delta,\varepsilon,\eta)$ *over a field* $\mathbb{K}$ *is a bialgebra with an antipode* $\eta \in Hom(H_C,H_A)$ *which is a two-sided inverse in the algebra* $Hom(H_C,H_A)$ *for the identity map* id :

$H_C \to H_A$ *under the product* $\star$, *meaning that*

$$\begin{aligned} id \star \eta &= \eta \star id = \\ M \circ (id \otimes \eta) \circ \Delta &= M \circ (\eta \otimes id) \circ \Delta \\ &= u \circ \varepsilon, \end{aligned}$$

where H_A, H_C *are* H *as an algebra and coalgebra respectively.*

Example 43 *For the algebra* H *of polynomials over the reals in the functions* cos *and* sin *and the unit being the inclusion map to the constant function* 1, *the trigonometric bialgebra becomes a Hopf Algebra whose antipode is defined as*

$$\begin{aligned} \eta(\cos) &= \cos \\ \eta(\sin) &= -\sin \\ \eta(1) &= 1. \end{aligned}$$

Example 44 *The trigonometric Hopf algebra is a particular case of a commutative Hopf algebra of functions on a group, the group being* $[0, 2\pi]$ *with the group multiplication of modular addition.*

Remark 45 *A good reference for examples of Hopf algebras is [21]. Any algebraic group furnishes an example of a Hopf algebra, through looking at what is known as the representative functions on the group. They provide a good analogy for the setting of Rough Paths.*

3.5 Hopf Algebras and Rough Paths

Each of the tensor and the shuffle algebras, have natural Hopf algebra extensions. One is the formal dual of the other and both coproducts are derivable from this property alone, though here a different interpretation is presented.

3.5.1 The Shuffle Hopf Algebra

The analogy to articulate comes from a general kind of Hopf algebra associated to continuous functions on groups. Given a group G such as an algebraic group with a collection $\mathcal{R}(G)$ of continuous functions that separate the points of G, (known as representative functions for an algebraic group), there exists an identity of the form

$$f(g_1 g_2) = \sum_i f_{i,1}(g_1) f_{i,2}(g_2),$$

where $f \in \mathcal{R}(G)$ for some functions $f_{i,j} \in \mathcal{R}(G)$ and any elements $g_1, g_2 \in G$. The induced coproduct $\delta : \mathcal{R}(G) \to \mathcal{R}(G) \otimes \mathcal{R}(G)$ reflects this identity:

$$\begin{aligned} \delta(f) &= \sum_i f_{i,1} \otimes f_{i,2} \\ &= f_{(1)} \otimes f_{(2)}, \end{aligned}$$

using Sweedler's notation. The evaluation of the function at the identity $e \in G$ represents the counit: $\epsilon(f) = f(e)$ and the antipode $\eta : \mathcal{R}(G) \hookleftarrow$ is defined through group inversion:

$$\eta(f)(g) = f(g^{-1}).$$

Take $\gamma_1, \gamma_2 \in \mathcal{P}_b(V)$. By definition, the signature of the concatenation of the paths $\mathbf{X}_{sig}(\gamma_1\gamma_2)$ is just the product of the signatures: $\mathbf{X}_{sig}(\gamma_1\gamma_2) = \mathbf{X}_{sig}(\gamma_1) \otimes \mathbf{X}_{sig}(\gamma_2)$. The algebraic relation is Chen's identity and the signatures of elements of $\mathcal{P}_b(V)$ form a group, though this group is not finite dimensional so not in this case actually algebraic. At the level of iterated integrals,

$$\mathbf{X}_{sig}(\gamma_1\gamma_2)^{(i)} = \sum_{j=0}^{i} \mathbf{X}_{sig}(\gamma_1)^{(j)} \otimes \mathbf{X}_{sig}(\gamma_2)^{(i-j)},$$

so that if $\underline{\omega} = \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i \in (V^*)^{\otimes i}$, here being thought of as a function on the group of signatures, then

$$\begin{aligned}
\underline{\omega}(\mathbf{X}_{sig}(\gamma_1\gamma_2)) &= \underline{\omega}\left(\mathbf{X}_{sig}(\gamma_1\gamma_2)^{(i)}\right) \\
&= \underline{\omega}\left(\sum_{j=0}^{i} \mathbf{X}_{sig}(\gamma_1)^{(j)} \otimes \mathbf{X}_{sig}(\gamma_2)^{(i-j)}\right) \\
&= \sum_{j=0}^{i} \left\{ \begin{array}{c} (\omega_1 \otimes \cdots \otimes \omega_j) \circ \left(\mathbf{X}_{sig}(\gamma_1)^{(j)}\right) \\ \times (\omega_{j+1} \otimes \cdots \otimes \omega_i) \circ \left(\mathbf{X}_{sig}(\gamma_2)^{(i-j)}\right) \end{array} \right\},
\end{aligned}$$

producing the definition of the coproduct on elements $\underline{\omega} = \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i$:

$$\delta(\underline{\omega}) = \sum_{j=0}^{i} (\omega_1 \otimes \cdots \otimes \omega_j) \otimes_H (\omega_{j+1} \otimes \cdots \otimes \omega_i),$$

where $\otimes_H$represents the tensor product of Hopf algebras over $\mathbb{R}$. The definition extends linearly to a non-cocommutative coproduct on the whole of $\mathcal{T}(V^*) = \cup_{n=0}^{\infty}\left(\oplus_{i=0}^{n}(V^*)^{\otimes i}\right)$. The map $\epsilon : \mathcal{T}(V^*) \to \mathbb{R}$ defined by $\epsilon(\underline{\omega}) = \underline{\omega}(1)$ is the evaluation at the identity path (that is to say the path which moves nowhere) and is seen to be the counit. Finally to derive the antipode $\eta : \mathcal{T}(V^*) \hookleftarrow$ again let $\underline{\omega} = \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i$

and $\gamma \in \mathcal{P}_b(V)$. Then

$$\begin{aligned}
& \eta(\underline{\omega})(\mathbf{X}_{sig}(\gamma)) \\
\equiv\ & \underline{\omega}\left(\mathbf{X}_{sig}(\gamma)^{-1}\right) \\
=\ & \underline{\omega}\left(\mathbf{X}_{sig}\left(\gamma^{-1}\right)\right) \\
=\ & \int_{\gamma^{-1}} \underline{\omega} \\
=\ & \int \cdots \int_{0 \leq u_1 \leq u_2 \leq \cdots \leq u_i \leq 1} \omega_1\left(d\gamma^{-1}(u_1)\right) \cdots \omega_i\left(d\gamma^{-1}(u_i)\right) \\
=\ & \int \cdots \int_{0 \leq u_1 \leq u_2 \leq \cdots \leq u_i \leq 1} (-1)^i\ \omega_1(d\gamma(1-u_1)) \cdots \omega_i(d\gamma(1-u_i)) \\
=\ & (-1)^i \int \cdots \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_i \geq 0} \omega_1(d\gamma(t_1)) \cdots \omega_i(d\gamma(t_i)) \\
=\ & (-1)^i \int_\gamma \omega_i \otimes \cdots \otimes \omega_1
\end{aligned}$$

so that η extends linearly to the whole of $\mathcal{T}(V^*)$ through the formula

$$\eta(\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i) = (-1)^i \omega_i \otimes \cdots \otimes \omega_1.$$

The structural axioms (3.1),(3.5),(38) and (42) can clearly be verified and show that $(\mathcal{T}(V^*), sh, \upsilon, \delta, \epsilon, \eta)$ is a Hopf algebra, however that is spared from this discussion.

Remark 46 *The book, Hopf Algebras [41] has much relevant information on the shuffle algebra as regards being the pointed, irreducible component of a dual hopf algebra to* $\mathrm{T}(V)$. *Although there is no interpretation regarding continuous paths, there is much relevant information on general Hopf Algebras.*

A perspective of duality:

$(\mathcal{T}(V^*), sh, v, \delta, \epsilon, \eta)$ is now a well-defined commutative Hopf algebra of functions on bounded variation paths in V. A kind of Hopf algebra duality theorem or principle for the general Hopf algebra of continuous functions on a group, captures how to recover the group from the Hopf algebra H. An element g of the group specifies an algebra map $\phi_g : H \longrightarrow \mathbb{R}$ through evaluation of functions at g, $\phi_g(f) = f(g)$. Considering the set Φ of algebra maps, shows that the coproduct in H provides a way of multiplying these homomorphisms, so inducing a product structure on Φ:

$$\begin{aligned}\left(\phi\hat{\phi}\right)(f) &= \left(\phi \otimes \hat{\phi}\right) \circ \delta(f) \\ &= \phi\left(f_{(1)}\right)\hat{\phi}\left(f_{(2)}\right),\end{aligned}$$

using Sweedler's notation.

The counit by definition provides an algebra map, which is a neutral element for this product. To see that it is a left unit:

$$\begin{aligned}(\epsilon\phi)(f) &= \epsilon\left(f_{(1)}\right)\phi\left(f_{(2)}\right) \\ &= \phi\left(\epsilon\left(f_{(1)}\right)f_{(2)}\right) \\ &= \phi\left((\varepsilon \otimes I) \circ \delta(f)\right) \\ &= \phi(f),\end{aligned}$$

and the right unit property is checked in much the same way.

The fact that there is an antipode ensures that Φ has a group structure: set $\eta(\phi)$ so that $\eta(\phi)(f) =$

$\phi(\eta(f))$. Then

$$\begin{aligned}(\phi \cdot \eta(\phi))(f) &= \phi\left(f_{(1)}\right)\eta(\phi)\left(f_{(2)}\right) \\ &= \phi\left(f_{(1)}\right)\phi\left(\eta\left(f_{(2)}\right)\right) \\ &= \phi\left(f_{(1)}\eta\left(f_{(2)}\right)\right) \\ &= \phi\left((id \otimes \eta) \circ \delta(f)\right) \\ &= \phi\left(u \circ \varepsilon(f)\right) \\ &= \varepsilon(f)\phi(1) = \varepsilon(f),\end{aligned}$$

implying ε is the unit, (1 is the constant function so that as ϕ is an algebra map, $\phi(1) = 1$). Thus Φ has a natural group structure.

The theorems which are known as Tannaka-Krein duality, concern on the one hand conditions for which it is possible to recover the original group through this construction, while on the other hand, how from starting with a commutative Hopf algebra, it is possible to associate a group in a Hopf algebra that is the group algebra or possibly the measure group algebra, before then recovering the Hopf algebra.

Remark 47 *The issue that the theorem deals with is whether or not the collection of functions separates the points of the group. In the case of compact groups or locally compact groups even, the answer is in the affirmative with the former case equivalent to an application of the Stone-Weierstrass theorem. The situation is closely related to theory of C^* algebras and notions such as the Carrier space. See for example [20] or [37].*

With the aim of identifying the group, here is an examination of the algebra homomorphisms of the shuffle algebra $\mathcal{T}(V^*)$. Each such homomorphism is a linear map and hence defines an element of $T(V)$, the dual of $\mathcal{T}(V^*)$ though due to the induced group law, it will lie in some group $G(\mathcal{T}(V^*))$. Ree showed [32] that a series in $T(V)$ defines a homomorphism of the shuffle algebra if and only if the series has a logarithm which lies in the free Lie algebra $L(V)$. This follows on from Chen's observation that the set of piecewise smooth paths define homomorphisms and have signatures whose logarithms lie in $L(V)$. It is known that all the signatures of geometric rough paths have logarithms in $L(V)$ since they are approximable by smooth paths, so it is possible to interpret more of these homomorphisms within this context, although not all of the homomorphisms : see chapter (5.1). The group multiplication manifests itself through concatenation in $T(V)$ as follows: let $g_i = \exp(\underline{l}_i) \in G(\mathcal{T}(V^*))$, with $\underline{l}_i \in L(V)$, then the Hopf algebra structure implies that the product $g_1 \times g_2$ satisfies

$$g_1 \times g_2 = g_1 \otimes g_2.$$

The Campbell-Baker-Hausdorff formula confirms that the logarithm of this product will lie in $L(V)$ (122). The Hopf algebra structures of $T(V)$ will be explained in the next section.

Remark 48 *Define the element $\rho \in Hom(\mathrm{T}(V), \mathrm{T}(V))$ to be the reversal of tensors map:*

$$\rho(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_n \otimes v_{n-1} \otimes \cdots \otimes v_1. \tag{3.7}$$

The fact that ρ entails a second action of $\mathrm{T}(V)$ on $\mathcal{T}(V^*)$ is not necessarily remarkable in itself, however it defines a map on rough paths themselves: take $\gamma \in \mathcal{P}(V)$. The differential of the 'inverse' path γ^{-1} satisfies

$$d\gamma^{-1}(t) = -d\gamma(T_\gamma - t).$$

The 'transpose' path $\widetilde{\gamma}:[0,T_\gamma]\to V$ *whose differential satisfies*

$$d\widetilde{\gamma}(t)=d\gamma(T_\gamma-t),$$

has signature $\rho(\mathbf{X}(\gamma))$. *With the pairing induced by* ρ:

$$\underline{v}\circ_\rho\underline{\omega}=\underline{\omega}(\rho(\underline{v})),$$

$\forall\underline{\omega}\in\mathcal{T}(V^*)$, $\underline{v}\in\mathrm{T}(V)$, *the group action on the algebra homomorphisms becomes* $g_1\times_\rho g_2=g_2\otimes g_1$. *For the 'sense' of paths, Chen and Lyons utilize the first multiplication to represent following the path due to* g_1 *before the path of* g_2, *although the choice is arbitrary.*

3.5.2 $T(V)$ as a Hopf algebra of differential operators

$T(V)$ as the dual space to $\mathcal{T}(V^*)$ has a dual Hopf algebra structure studied in [33]. The interpretation of this Hopf algebra is less clear than the shuffle algebra, even though the dual pairing makes them equivalent - the multiplication, comultiplication, antipode etc. of each Hopf algebra are encoded in the other (See [41] for example). One route of interpretation is through considering the set of left $G(\mathcal{T}(V^*))$-invariant derivations of the shuffle algebra.

Definition 49 *A derivation of the shuffle algebra is a linear map*

$$d\in Hom(\mathcal{T}(V^*),\mathcal{T}(V^*)),$$

such that $\forall\underline{\omega},\underline{\omega}'\in\mathcal{T}(V^*)$

$$d(sh(\underline{\omega},\underline{\omega}'))=sh(d(\underline{\omega}),\underline{\omega}')+sh(\underline{\omega},d(\underline{\omega}')). \tag{3.8}$$

For an element $g \in G(\mathcal{T}(V^*))$, there is a map $l_g \in End(\mathcal{T}(V^*))$ defined in the sense of functions so that $\forall \underline{\omega} \in \mathcal{T}(V^*)$, $\forall h \in G(\mathcal{T}(V^*))$, $l_g(\underline{\omega})(h) = \underline{\omega}(gh)$. In Sweedler's notation, $l_g(\underline{\omega}) = \underline{\omega}_{(1)}(g)\,\underline{\omega}_{(2)}$.

Definition 50 *A derivation d is left $G(\mathcal{T}(V^*))$-invariant if*

$$d \circ l_g = l_g \circ d \quad \forall g \in G(\mathcal{T}(V^*)).$$

(The multiplication is the usual composition of linear maps, not as might have been thought, that through the product $\star$). Denote the set of left $G(\mathcal{T}(V^*))$-invariant derivations by $D_{\mathrm{L}}(V^*)$.

Reutenauer [33] shows that an element $\underline{l} \in L(V)$ defines a derivation of $\mathcal{T}(V^*)$ through the map $\partial^r_{\underline{l}} \in Hom(\mathcal{T}(V^*), \mathcal{T}(V^*))$,

$$\partial^r_{\underline{l}}(\underline{\omega}) = \underline{\omega}_{(1)}\underline{\omega}_{(2)}(\underline{l}), \tag{3.9}$$

and is seen to equate to the expression,

$$\begin{aligned}
\partial^r_{\underline{l}}(\underline{\omega})(g) &= \lim_{t\to 0} \frac{(\underline{\omega}(g \times \exp(t\underline{l})) - \underline{\omega}(g))}{t} \\
&= \lim_{t\to 0} \frac{\left(\underline{\omega}_{(1)}(g)\,\underline{\omega}_{(2)}(\exp(t\underline{l})) - \underline{\omega}_{(1)}(g)\,\underline{\omega}_{(2)}(1)\right)}{t} \\
&= \underline{\omega}_{(1)}(g) \lim_{t\to 0} \frac{\left(\underline{\omega}_{(2)}(\exp(t\underline{l})) - \underline{\omega}_{(2)}(1)\right)}{t} \\
&= \underline{\omega}_{(1)}(g)\,\underline{\omega}_{(2)}(\underline{l}),
\end{aligned}$$

which is well-defined since $\underline{\omega} \in \mathcal{T}(V^*)$ has finite degree. The maps $\partial^r_{\underline{l}}$ are left $G(\mathcal{T}(V^*))$-invariant due to the cocommutativity property of the coproduct:

$$\begin{aligned}
l_g\left(\partial^r_{\underline{l}}(\underline{\omega})\right) &= l_g\left(\underline{\omega}_{(1)}\right)\underline{\omega}_{(2)}(\underline{l}) \\
&= \left(\underline{\omega}_{(1)}\right)_{(1)}(g)\left(\underline{\omega}_{(1)}\right)_{(2)}\underline{\omega}_{(2)}(\underline{l}) \\
&= \underline{\omega}_{(1)}(g)\,\underline{\omega}_{(2)}\underline{\omega}_{(3)}(\underline{l}) \\
&= \underline{\omega}_{(1)}(g)\left(\underline{\omega}_{(2)}\right)_{(1)}\left(\underline{\omega}_{(2)}\right)_{(2)}(\underline{l}) \quad \textit{coassociativity} \\
&= \underline{\omega}_{(1)}(g)\,\partial^r_{\underline{l}}\left(\underline{\omega}_{(2)}\right) \\
&= \partial^r_{\underline{l}}\left(l_g(\underline{\omega})\right).
\end{aligned}$$

(Defining the map

$$\partial^l_{\underline{l}}(\underline{\omega}) = \underline{\omega}_{(1)}(\underline{l})\,\underline{\omega}_{(2)}, \tag{3.10}$$

entails a right $G(\mathcal{T}(V^*))$-invariant linear operator, relating to the paths of (48))

A set $D_L(V^*)$ of derivations, will generically form a Lie algebra with the standard Lie bracket

$$\left[\partial^r_{\underline{l}}, \partial^r_{\underline{l}'}\right] = \partial^r_{\underline{l}}\partial^r_{\underline{l}'} - \partial^r_{\underline{l}'}\partial^r_{\underline{l}},$$

where the product is just repeated composition of linear maps but it is also the case that the map $\partial^r : L(V) \to D_L(V^*)$ is a Lie algebra isomorphism, i.e. it is not only a Lie algebra map

$$\left[\partial^r_{\underline{l}}, \partial^r_{\underline{l}'}\right] = \partial^r_{[\underline{l},\underline{l}']},$$

but also that $\ker(\partial^r) = 0$ and all elements can be expressed as such i.e. the set of left $G(\mathcal{T}(V^*))$-invariant derivations are isomorphic to $L(V)$ through the map ∂^r. This fact is an extension of Theorem 1.4 [33].

Products of two non-trivial elements of D_L are not derivations since they do not satisfy the Leibnitz type derivation identity (3.8). They are however elements of the associative algebra $Hom\left(\mathcal{T}\left(V^*\right), \mathcal{T}\left(V^*\right)\right)$ that are left $G\left(\mathcal{T}\left(V^*\right)\right)$-invariant with the product that is repeated composition. ∂^r extends to an algebra map $\partial^r : T(V) \to Hom\left(\mathcal{T}\left(V^*\right), \mathcal{T}\left(V^*\right)\right)$ in an unique way, which equates to a similar expression to (3.9):

$$\begin{aligned}
\partial^r_{v_1 \otimes \cdots \otimes v_n}\left(\underline{\omega}\right) &= \partial^r_{v_1} \circ \partial^r_{v_2} \circ \cdots \circ \partial^r_{v_n}\left(\underline{\omega}\right) \\
&= \underline{\omega}_{(1)} \underline{\omega}_{(2)}\left(v_1\right) \cdots \underline{\omega}_{(n+1)}\left(v_n\right) \\
&= \underline{\omega}_{(1)} \underline{\omega}_{(2)}\left(v_1 \otimes \cdots \otimes v_n\right) \\
&= \underline{\omega}_{(1)} \lim_{t \to 0} \left\{ \frac{\underline{\omega}_{(2)}\left(\exp\left(t v_1 \otimes \cdots \otimes v_n\right)\right) - \underline{\omega}_{(2)}\left(1\right)}{t} \right\}
\end{aligned}$$

and more generally for $\underline{v} \in T(V)$,

$$\partial^r_{\underline{v}}\left(\underline{\omega}\right) = \underline{\omega}_{(1)} \lim_{t \to 0} \left\{ \frac{\underline{\omega}_{(2)}\left(\exp\left(t\underline{v}\right)\right) - \underline{\omega}_{(2)}\left(1\right)}{t} \right\} \tag{3.11}$$

In this sense, $T(V) \cong U(L(V))$ can be interpreted as a collection of left $G\left(\mathcal{T}\left(V^*\right)\right)$-invariant differential operators on the shuffle algebra where the decomposition of $\partial^r_{\underline{v}}$ into operators of different orders corresponds to the decomposition of $\underline{v}$ in terms of the components $U_n\left(L\left(V\right)\right)$. It is also a collection of continuous functions on $G\left(\mathcal{T}\left(V^*\right)\right)$ and as a set includes the signatures of geometric rough paths in V.

Hopf algebra structures for $T(V)$

To summarize, $\left(T(V), \otimes, u\right)$ thought of as differential operators is isomorphic to the algebra generated by the derivations of $\mathcal{T}\left(V^*\right)$. Repeated composition of derivations corresponds to tensor multiplication $\otimes$ in

$T(V)$ and the unit $u : \mathbf{R} \to T(V)$ corresponds to the null or constant operator. By definition, derivations obey the identity (3.8) which represents how to differentiate products of functions. For $v \in L^{(1)}(V)$ this induces a map $\Delta : V \to V \otimes V$ defined by

$$\Delta(v) = v \otimes 1 + 1 \otimes v, \tag{3.12}$$

Δ extends to a mapping $T(V) \to T(V) \otimes_H T(V)$ in an unique way as an algebra map due to the universality of $T(V)$ and that $T(V) \otimes_H T(V)$ is an associative algebra with component wise multiplication:

$$\Delta(v_1 v_2 \cdots v_n) = \Delta(v_1)\Delta(v_2)\cdots\Delta(v_n),$$

so that for example:

$$\begin{aligned} \Delta(v_1 v_2) &= (v_1 \otimes_H 1 + 1 \otimes_H v_1)(v_2 \otimes_H 1 + 1 \otimes_H v_2) \\ &= v_1 v_2 \otimes_H 1 + v_1 \otimes_H v_2 + v_2 \otimes_H v_1 + 1 \otimes_H v_1 v_2. \end{aligned}$$

It reflects a rule for how higher order differential operators act on products of functions. In this sense, it ought to be possible to use Δ to characterize the left $G(\mathcal{T}(V^*))$-invariant derivations or elements of $L(V)$ as the set of tensors $\underline{v} \in T(V)$ such that $\Delta(\underline{v}) = \underline{v} \otimes_H 1 + 1 \otimes_H \underline{v}$. This characterization of $L(V)$ is shown in [33], more precisely if $\underline{v} \in T(V)$ then

$$\underline{v} \in L(V) \Leftrightarrow \Delta(\underline{v}) = \underline{v} \otimes_H 1 + 1 \otimes_H \underline{v}.$$

In coalgebraic terms, this is just saying that an element is primitive if and only if it is in $L(V)$.

Remark 51 *It is easy to show that those elements $\underline{v} \in \mathrm{T}(V)$ that satisfy*

$$\Delta(\underline{v}) = \underline{v} \otimes_H 1 + 1 \otimes_H \underline{v},$$

form a Lie algebra and that since $\mathrm{L}(V)$ *is generated by* V *in this manner, one direction is easy. The harder part, proving the converse direction, requires a more involved argument as explained in [33].*

Remark 52 *The identification of* $\mathrm{T}(V)$ *as the universal enveloping algebra of* $\mathrm{L}(V)$ *and the associated decomposition in terms of the* $\mathrm{U}_n(\mathrm{L}(V))$ *has the interpretation as representing elements of* $\mathrm{T}(V)$ *in terms of n'th order differential operators, a grading associated to* $\mathrm{T}(V)$ *different to tensorial degree.*

The map

$$\begin{aligned} \varepsilon &: T(V) \to \mathbf{R} && (3.13) \\ \varepsilon(\underline{v}) &= \varepsilon\left(\sum_{i=0}^{\infty} \underline{v}^{(i)}\right), \underline{v}^{(i)} \in V^{\otimes i} \\ &= \underline{v}^{(0)} && (3.14) \end{aligned}$$

which projects a tensor onto it's degree zero component can be seen to act as a counit. In the context of derivations, the interpretation of $\varepsilon(\underline{v})$ is the application of $\underline{v}$ to the constant function $1 \in \mathcal{T}(V^*)$. Again the computation to confirm that $(T(V), \otimes, u, \Delta, \varepsilon)$ is a bialgebra is omitted, though it appears in [33].

This is the calculation for the antipode $\mu \in Hom(T(V), T(V))$: μ should be defined on $L(V)$ so that if $\underline{l} \in L(V)$ then $\forall j \geq 0$,

$$\begin{aligned} (id \star \mu)\left(\underline{l}^{\otimes j}\right) &= (\mu \star id)\left(\underline{l}^{\otimes j}\right) \\ \otimes \circ \left((id \otimes \mu) \circ \Delta\left(\underline{l}^{\otimes j}\right)\right) &= \otimes \circ \left((\mu \otimes id) \circ \Delta\left(\underline{l}^{\otimes j}\right)\right) \\ &= v\left(\varepsilon\left(\underline{l}^{\otimes j}\right)\right) \\ &= \left\{ \begin{array}{cc} 0 & \forall j > 0 \\ 1 & j = 0 \end{array} \right\}, \end{aligned}$$

since by the symmetric decomposition of $T(V)$ in terms of $U(L(V))$, this defines μ on the whole of $T(V)$.

However

$$\begin{aligned}\Delta\left(\underline{l}^{\otimes j}\right) &= \Delta\left(\underline{l}\right)^{\otimes j} = \left(\underline{l}\otimes_H 1 + 1\otimes_H \underline{l}\right)^{\otimes j} \\ &= \sum_{k=0}^{j}\binom{j}{k}\underline{l}^{\otimes k}\otimes_H \underline{l}^{\otimes(j-k)},\end{aligned}$$

so that if $\mu(\underline{l}) = -\underline{l}$ for any $\underline{l}\in L(V)$, then this implies

$$\begin{aligned}(id\star\mu)\left(\underline{l}^{\otimes j}\right) &= \sum_{k=0}^{j}\binom{j}{k}\underline{l}^{\otimes k}\otimes\mu\left(\underline{l}\right)^{\otimes(j-k)} \\ &= \left(\underline{l}-\underline{l}\right)^{\otimes j} \\ &= (\mu\star id)\left(\underline{l}^{\otimes j}\right) \\ &= \left\{\begin{array}{cc} 1 & j=0 \\ 0 & j>0\end{array}\right\}\end{aligned}$$

as required. μ is defined on simple tensors $v_1\otimes\cdots\otimes v_n$, $v_i\in V$ as the anti-homomorphism of algebras

$$\mu\left(v_1\otimes\cdots\otimes v_n\right) = (-1)^n\, v_n\otimes\cdots\otimes v_1, \tag{3.15}$$

so that $\forall 1\le m\le n$,

$$\mu\left(v_1\otimes\cdots\otimes v_n\right) = \mu\left(v_m\otimes\cdots\otimes v_n\right)\otimes\mu\left(v_1\otimes\cdots\otimes v_m\right).$$

Proposition 53 $\forall\underline{l}\in \mathrm{L}(V)$, $\mu(\underline{l}) = -\underline{l}$.

Proof. Use induction on the statement $\forall \underline{l} \in L^n(V)$, $\mu(\underline{l}) = -\underline{l}$.For $\underline{l} = v_1 \in V$, the statement is clearly true. So assume $\mu(\underline{l}) = -\underline{l}\ \forall \underline{l} \in L^n(V)$ and take $[v_1, v_2, \ldots, v_{n+1}] \in L^{(n+1)}(V)$. Then

$$\begin{aligned}
\mu([v_1, v_2, \ldots, v_{n+1}]) &= \mu(v_1 \otimes [v_2, \ldots, v_{n+1}] - [v_2, \ldots, v_{n+1}] \otimes v_1) \\
&= \left\{ \begin{array}{c} \mu([v_2, \ldots, v_{n+1}]) \otimes \mu(v_1) \\ -\mu(v_1) \otimes \mu([v_2, \ldots, v_{n+1}]) \end{array} \right\} \\
&= [v_2, \ldots, v_{n+1}] \otimes v_1 - v_1 \otimes [v_2, \ldots, v_{n+1}] \\
&= -[v_1, [v_2, \ldots, v_{n+1}]]
\end{aligned}$$

by the induction hypothesis. Since tensors of the form $[v_1, v_2, \ldots, v_{n+1}]$ span $L^{(n+1)}(V)$, $L^{n+!}(V) = L^n(V) \oplus L^{(n+1)}(V)$ and μ is linear, the proposition is true by induction. ■

The conclusion is that μ is the desired antipode to make a Hopf algebra $(T(V), \otimes, u, \Delta, \varepsilon, \mu)$.

Primitives and Group Like Elements As noted above, the primitive elements of $(T(V), \otimes, u, \Delta, \varepsilon, \mu)$ are the Lie elements. The group-like elements, i.e. series S with $\Delta(S) = S \otimes_H S$ are, apart from the zero series, those series which are exponentials of Lie elements (see [33]). In other words, a series is group-like if and only if belongs to $G(\mathcal{T}(V^*))$, i.e. it is a homomorphism of the shuffle algebra. Thus the signature of any geometric rough path is a group-like element of $T(V)$. Though mentioned elsewhere, the remark (37), can be used to infer that the elements of $G(\mathcal{T}(V^*))$ and hence the signatures of geometric rough paths, are either linearly independent or identical. In fact, Chen proved this fact without the machinery of Hopf algebras in [11].

The primitive elements of $\mathcal{T}(V^*)$ are shown in [41] to be just the elements of V^* and are the functions on $\mathcal{P}_b(V)$ that are additive. (Note that this collection does not include a length function, which would be additive if a time parameterization was incorporated to spell out retraced parts of paths).

Hopf Algebra Duality In addition to defining a collection of linear maps of $\mathcal{T}(V^*)$ through derivations, $T(V)$ acts as the dual space of $\mathcal{T}(V^*)$. Unsurprisingly, the two actions are related. The dual action of $\underline{v} \in T(V)$ on $\underline{\omega} \in \mathcal{T}(V^*)$ can be thought of as the composition of the action of $\underline{v}$ as the differential operator $\partial^r_{\underline{v}}$ (3.11) on $\mathcal{T}(V^*)$ with the functional evaluation of $\partial^r_{\underline{v}}(\underline{\omega})$ on the zero path in V, i.e.

$$\underline{v} \circ \underline{\omega} = \epsilon\left(\partial^r_{\underline{v}}(\underline{\omega})\right),$$

where ϵ is the counit of the shuffle algebra.

To evaluate an algebra map of $\mathcal{T}(V^*)$ due to $g = \exp(\underline{l}) \in G(\mathcal{T}(V^*))$, the linear map has the expression

$$\begin{aligned} g \circ \underline{\omega} &= \epsilon\left(l_g(\underline{\omega})\right) \\ &= \epsilon\left(\sum_{k=0}^{\infty} \frac{\left(\partial^l_{\underline{l}}\right)^k}{k!}(\underline{\omega})\right). \end{aligned}$$

As the operators $\partial^l_{\cdot}$ and $\partial^r_{\cdot}$ commute and the derivations are left-$G(\mathcal{T}(V^*))$ invariant, thinking of $\mathcal{T}(V^*)$ as a collection of functions on $G(\mathcal{T}(V^*))$ entails the interpretation of $T(V)$ as a collection of left-$G(\mathcal{T}(V^*))$ invariant distributions on $\mathcal{T}(V^*)$. This heuristic parallels the universal principle due to Berezin for a Lie group G with Lie algebra $\mathfrak{g}$. $U(\mathfrak{g})$, as the collection of operations which are evaluation at the identity, of invariant differential operators on functions on the group, can be interpreted as the collection of distributions on G supported at the identity.

There is no classical Lie group in this setting, though in $G(\mathcal{T}(V^*))$ there is a candidate, which is universal for connected Lie groups of dimension $\dim(V)$ (see [12]).

Reutenauer [33] looks at the Hopf algebras $\mathcal{T}(V)$ and $\mathcal{T}(V^*)$, using the natural pairing to deduce each Hopf algebra's structural maps. To specify the meaning, he obtains the equalities : for any $\underline{v} \in \mathcal{T}(V)$, $\underline{\omega}_1, \underline{\omega}_2 \in \mathcal{T}(V^*)$,

$$\begin{aligned} \underline{v} \circ sh(\underline{\omega}_1, \underline{\omega}_2) &= \Delta(\underline{v}) \circ (\underline{\omega}_1 \otimes_H \underline{\omega}_2) \\ &= \sum \left(\underline{v}_{(1)} \circ \underline{\omega}_1\right) \left(\underline{v}_{(2)} \circ \underline{\omega}_2\right), \end{aligned}$$

so that Δ is the adjoint of sh.

For any $\underline{v}_1, \underline{v}_2 \in \mathcal{T}(V)$, $\underline{\omega} \in \mathcal{T}(V^*)$,

$$\begin{aligned} (\otimes \circ (\underline{v}_1, \underline{v}_2)) \circ (\underline{\omega}) &= (\underline{v}_1 \otimes_H \underline{v}_2) \circ \delta(\underline{\omega}) \\ &= \sum \left(\underline{v}_1 \circ \underline{\omega}_{(1)}\right) \left(\underline{v}_2 \circ \underline{\omega}_{(2)}\right), \end{aligned}$$

so that δ is the adjoint of $\otimes$.

For any $\underline{v} \in \mathcal{T}(V)$ and $\underline{\omega} \in \mathcal{T}(V^*)$,

$$\begin{aligned} \mu(\underline{v}) \circ \underline{\omega} &= \underline{v} \circ \eta(\underline{\omega}) \\ \underline{v} \circ 1 &= \varepsilon(\underline{v}) \\ 1 \circ \underline{\omega} &= \epsilon(\underline{\omega}) \end{aligned}$$

These relations amount to saying that through the non-degenerate pairing

$$\circ : \mathcal{T}(V) \otimes \mathcal{T}(V^*) \to \mathbb{R},$$

$\mathcal{T}(V)$ and $\mathcal{T}(V^*)$ are dually paired Hopf algebras.

3.6 Semi-direct products and vector fields

To understand $T(V)$ as a Hopf algebra, $\mathcal{T}(V^*)$ was thought of as a left $T(V)$ module.

Let A be a unital algebra and B a bialgebra. According to Sweedler [41], if there is a left action of B on A, i.e. A is a left B module, that satisfies the properties

$$b \circ (a_1 a_2) = b_{(1)} a_1 b_{(2)} a_2 \tag{3.16}$$

$$b \circ (1) = \varepsilon(b) 1, \tag{3.17}$$

for any $b \in B$, $a_1, a_2 \in A$, (ε is the counit of B and 1 the unit of A), then the smash or semi-direct product of A with B, $A \sharp B$, is defined to be the algebra which is the set of elements $A \otimes B$ with the product $\times$ given by

$$(a \sharp b) \times \left(\widetilde{a} \sharp \widetilde{b}\right) = a \left(b_{(1)} \circ \widetilde{a}\right) \sharp \left(b_{(2)} \widetilde{b}\right). \tag{3.18}$$

The application of this picture here is described in the following example of differential operators on a Lie group.

Example 54 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and for the formalism above, set $A = C^\infty(G)$, $B = \mathrm{U}(\mathfrak{g})$. Take $f \in C^\infty(G)$ and $v \in \mathfrak{g}$. Then $C^\infty(G)$ is a left $\mathrm{U}(\mathfrak{g})$ module through the action of $\mathrm{U}(\mathfrak{g})$*

as the left G-invariant differential operators on $C^{\infty}(G)$ *defined by*

$$\begin{aligned} (\partial_v^r f)(g) &= \left\{\frac{d}{d\varepsilon} f(g \exp(\varepsilon v))\right\}_{\varepsilon=0} \\ &= f_{(1)}(g)\left\{\frac{d}{d\varepsilon} f_{(2)}(\exp(\varepsilon v))\right\}_{\varepsilon=0} \end{aligned} \tag{3.19}$$

and with the natural algebra extension to $\mathrm{U}(\mathfrak{g})$. *(3.16) is true as it is an expression of Leibnitz's identity while (3.17) refers to (3.5.2), so the semi-direct product exists. To understand the product* $\times$, *note that if* $f\sharp(\partial_{v_1}^r \cdots \partial_{v_n}^r) \in C^{\infty}(G)\sharp\mathrm{U}(\mathfrak{g})$, *then (3.18) implies that*

$$\begin{aligned} f\sharp(\partial_{v_1}^r \cdots \partial_{v_n}^r) &= (f\sharp 1) \times (1\sharp(\partial_{v_1}^r \cdots \partial_{v_n}^r)) \\ &= (f\sharp 1) \times (1\sharp\partial_{v_1}^r) \times \cdots \times (1\sharp\partial_{v_n}^r), \end{aligned}$$

so it is enough to understand $\times$ *for elements of this factored form. Firstly (3.18) implies*

$$(f^1\sharp 1) \times (f^2\sharp 1) = (f^1 f^2\sharp 1),$$

explained as the pointwise multiplication of functions,

$$(1\sharp\partial_v^r) \times (f\sharp 1) = ((\partial_v^r f)\sharp 1) + (f\sharp\partial_v^r),$$

using the coproduct (3.12) to give the application of application of the vector field due to v *on the differential operator* $f\sharp 1$ *(*1 *is the constant differential operator) and*

$$(1\sharp\partial_{v_1}^r) \times (1\sharp\partial_{v_2}^r) = (1\sharp(\partial_{v_1}^r \partial_{v_2}^r)),$$

the composition of left-invariant differential operators. So the product (3.18) for $C^{\infty}(G)\sharp\mathrm{U}(\mathfrak{g})$ *formally expresses how differential operators on G form an algebra - application of two operators is equivalent to the application of the smash product of the operators.*

With regards to the algebraic setting for rough paths, $\mathcal{T}(V^*) \sharp T(V)$ is the analogous object. $(\mathcal{T}(V^*), sh, v)$ takes the rôle of the algebra A, a commutative algebra of functions which are functions on signatures of geometric rough paths. $(T(V), \otimes, u, \Delta, \varepsilon)$ is the bialgebra B, the left $G(\mathcal{T}(V^*))$-invariant differential operators on $\mathcal{T}(V^*)$, so that $\mathcal{T}(V^*)$ is a left-$T(V)$ module according to (3.11), which mirrors the example above and (3.19). $\mathcal{T}(V^*) \sharp T(V)$ is an algebra, not a Hopf algebra, consisting of differential operators and the functions they act on as sub-algebras. However the Hopf algebra structures of $\mathcal{T}(V^*)$ and $T(V)$ can be reconstructed from $\mathcal{T}(V^*) \sharp T(V)$, so it is a fundamental object for the two Hopf algebras presented.

The principles of the smash product construction and the interpretation and application for the case of a specific lie algebra $\mathfrak{g}$ are well known. In [34] for example, a further interesting application is developed. Very briefly, $U(\mathfrak{g})$ is typically subject to a deformation by a parameter q in some way, incorporating a non-commutative aspect and becoming a Hopf algebra that is known as a quantum group. Paired with the dual Hopf algebra, which is thought of as the functions on the quantum group (although it is no-longer commutative), the resulting algebra is thought of as the differential operators on the quantum group and the functions that they act on. The motivation for studying quantum groups through different deformation regimes and parameters, is to construct non-commutative geometries in order to help understand quantum effects i.e. physical phenomena outside the realm of continuum geometry.

3.7 For reference

The material presented comes from a wide source of texts. It is hard to identify precisely which is the best source for the appropriate sections, though the following were useful in a wide context.

Reutenauer [33] provides the most comprehensive discussion of the algebraic structures applicable to Rough Paths. Texts like Dixmier [16], Gelfand and Manin [19], Hochschild [21], Serre [35] and Serre [36] provide context for how these concepts interact with lie algebras generally and information about Hopf algebras. The classical texts on Hopf algebras are Abe [1] and Sweedler [41], giving thorough explanations of the fundamental concepts. For more recent geometrical and physical applications of Hopf algebras, there are sections in [20] and [37].

Chapter 4

Lie Groups and Length

'The Long and Winding Road'

"..., don't keep me waiting here,

lead me to your door.

Yeah yeah yeah, yeah."

The Beatles, 1969

This chapter is an attempt to try to draw together some of the constructions of chapters 2 and 3 in the context of paths in a Lie group. One aim is to make clearer how the collection of signatures in the infinite dimensional Lie group, $\exp(L(V)) \subset U(L(V))$, is a model for paths in connected Lie groups of dimension $\dim(V)$. The full iterated integral sequence of geometric rough paths, define group-like points in the enveloping algebra of the Lie algebra $\mathfrak{g}$ of the Lie group $U(\mathfrak{g})$, at least in the case of matrix lie groups.

The first part concerns a definition of iterated integrals for a connected Lie Group. On one level, the first part is purely an exercise in solving an ordinary differential equation and an application of the material in chapter 2. The idea is therefore to show some natural structures which permit a formulation of rough paths within this context.

The second part contains an example of how realizing a C^2 path in $\mathbb{R}^2$ in a particular Lie group, reveals information about the length of the C^2 path.

4.1 Connected Lie Groups

Iterated integral sequences of paths in a connected Lie group G, can be defined by looking at objects in the tensor algebra of the Lie algebra $\mathfrak{g}$ of the group, or perhaps more accurately, the tensor algebra of $V_{\mathfrak{g}}$, which is $\mathfrak{g}$ as a vector space and forgets the lie algebraic structure. A vector space is a particular example as a commutative Lie group for which the powerful theories defining rough paths were constructed. Any $\gamma \in \mathcal{P}(G)$ defines an iterated integral sequence in $T(V_{\mathfrak{g}})$ through using either of the left or right connections. The left and right Maurer-Cartan forms, $g^{-1} \cdot dg$ and $dg \cdot g^{-1}$ respectively, define piecewise smooth paths (multiplication and inversion are smooth maps) $l_t(\gamma)$ and $r_t(\gamma)$ in $V_{\mathfrak{g}}$ through the differential relations:

$$\begin{aligned} dl_t(\gamma) &= \frac{d}{ds}\left(\gamma_t^{-1}\gamma_{t+s}\right)|_{s=0} \qquad (4.1) \\ &= \gamma_t^{-1} \cdot d\gamma_t \end{aligned}$$

$$\begin{aligned} dr_t(\gamma) &= \frac{d}{ds}\left(\gamma_{t+s}\gamma_t^{-1}\right)|_{s=0} \\ &= d\gamma_t \cdot \gamma_t^{-1}. \end{aligned}$$

For the standard iterated integral sequence and the left connection, the n'th iterated integral $\mathbf{X}_{s,t}^{(n)}$: $\mathcal{P}(G) \to V_{\mathfrak{g}}^{\otimes n}$ is defined

$$\mathbf{X}_{s,t}^{(n)}(\gamma) = \int \cdots \int_{s<u_1<u_2<\cdots<u_n<t} dl_{u_1}(\gamma) \otimes dl_{u_2}(\gamma) \otimes \cdots \otimes dl_{u_n}(\gamma)$$

and Chen's formula applies

$$\mathbf{X}_{s,t}^{(n)}(\gamma) = \sum_{i=0}^{n} \mathbf{X}_{s,u}^{(i)}(\gamma) \otimes \mathbf{X}_{u,t}^{(n-i)}(\gamma) \forall (s,u), (u,t) \in \Gamma_{T_\gamma}.$$

However for the right connection, the reverse iterated integral sequence

$$\mathbf{X}_{s,t}^{\leftarrow(n)}(\gamma) = \int \cdots \int_{s<u_1<u_2<\cdots<u_n<t} dr_{u_n}(\gamma) \otimes \cdots \otimes dr_{u_2}(\gamma) \otimes dr_{u_1}(\gamma),$$

which satisfies the tensor multiplication

$$\mathbf{X}_{s,t}^{\leftarrow(n)}(\gamma) = \sum_{i=0}^{n} \mathbf{X}_{u,t}^{\leftarrow(i)}(\gamma) \otimes \mathbf{X}_{s,u}^{\leftarrow(n-i)}(\gamma) \;\; \forall (s,u), (u,t) \in \Gamma_{T_\gamma},$$

turns out to be more appropriate for purposes of evaluation. Either viewpoint contains the same content.

Since $V_{\mathfrak{g}}$ is a vector space, a norm on $V_{\mathfrak{g}}$ (or $\mathfrak{g}$) with any system of norms on $V_{\mathfrak{g}}^{\otimes i}$ that satisfies the consistency condition (2.4), entails a p variation distance function $D_{T(V_{\mathfrak{g}}),p}$ for multiplicative functionals $\mathbf{X}^n, \mathbf{Y}^n \in \Omega(V_{\mathfrak{g}})_n$

$$D_{T(V_{\mathfrak{g}}),p}(\mathbf{X}^n, \mathbf{Y}^n) = \sum_{i=1}^{n} P_{\frac{p}{i};0,T}\left(\mathbf{X}^{(i)} - \mathbf{Y}^{(i)}\right)^{\frac{i}{p}},$$

which allows the definition of $\Omega(G)^p$ by way of based multiplicative functionals.

In addition, there is an associated metric on G, $d_G(\cdot,\cdot)$, given by

$$d_G(g,h) = \inf_{\gamma\in\mathcal{P}(G):\gamma_0=g,\gamma_{T_\gamma}=h} \int_{0<t<T_\gamma} \left\|\gamma_t^{-1}d\gamma_t\right\|. \tag{4.2}$$

See the appendix 6.4 for details. G is complete for this metric. This means that there is a notion of continuous paths of finite p variation in G according to Wiener (see 2.2), hence in particular bounded variation paths $\mathcal{P}_b(G)$.

Definition 55 *Define the p variation distance in $\Omega(G)^p$ to be*

$$D_{G,p}(\mathbf{X}^n,\mathbf{Y}^n) = \sum_{i=1}^{n} P_{\frac{p}{i};0,T}\left(\mathbf{X}^{(i)}-\mathbf{Y}^{(i)}\right)^{\frac{i}{p}} + d_G\left(\mathbf{X}_0^n,\mathbf{Y}_0^n\right). \tag{4.3}$$

This definition is not entirely satisfactory unless it implies control of the term

$$\sup_{u\in[0,T]} d_G\left(\mathbf{X}_u^{\lfloor p\rfloor},\mathbf{Y}_u^{\lfloor p\rfloor}\right). \tag{4.4}$$

In the vector space situation, to each element $\mathbf{X}^{\lfloor p\rfloor}\in\Omega(V)^p$, there is a continuous path in V defined by

$$\mathbf{X}_t^{\lfloor p\rfloor} = \mathbf{X}_0^{\lfloor p\rfloor} + \mathbf{X}_{0,t}^{(1)} \in V,$$

although an alternative formulation simply relates this additive property to a multiplicative property through the exponential map for Lie groups. Let $\{e_i\}_{i=1}^m$ be a basis for V, and $\{e_i^*\}_{i=1}^m$ a dual basis. For ease put $m = 2$, so that

$$
\begin{aligned}
& \exp \begin{pmatrix} e_1^* \left(\mathbf{X}_t^{\lfloor p \rfloor} \right) & 0 \\ 0 & e_2^* \left(\mathbf{X}_t^{\lfloor p \rfloor} \right) \end{pmatrix} \\
= & \exp \begin{pmatrix} e_1^* \left(\mathbf{X}_0^{\lfloor p \rfloor} \right) & 0 \\ 0 & e_2^* \left(\mathbf{X}_0^{\lfloor p \rfloor} \right) \end{pmatrix} \times \exp \begin{pmatrix} e_1^* \left(\mathbf{X}_{0,t}^{(1)} \right) & 0 \\ 0 & e_2^* \left(\mathbf{X}_{0,t}^{(1)} \right) \end{pmatrix} \\
= & \exp \begin{pmatrix} e_1^* \left(\mathbf{X}_0^{\lfloor p \rfloor} + \mathbf{X}_{0,t}^{(1)} \right) & 0 \\ 0 & e_2^* \left(\mathbf{X}_0^{\lfloor p \rfloor} + \mathbf{X}_{0,t}^{(1)} \right) \end{pmatrix},
\end{aligned}
$$

since the group is commutative.

However, even for a smooth path $\gamma \in \mathcal{P}(G)$, the first iterated integral does not locate the element of the group corresponding to the point of the path at time t through the exponential map (when it is defined). Formally, let I be the identity map

$$I_{\mathfrak{g}} : V_{\mathfrak{g}} \to \mathfrak{g}.$$

Then normally it is the case that

$$\exp \left(I \left(\int_0^t dl_t(\gamma) \right) \right) \neq \gamma_0^{-1} \gamma_t.$$

To recover the path for a general element of $\mathcal{P}(G)$, the requisite information is inextricably bound up in the full iterated integral sequence. Equivalently if logs are taken, it lies in the free Lie algebra $L(V_{\mathfrak{g}})$. An exception are the nilpotent Lie groups of degree n, where $L^n(V_{\mathfrak{g}})$ suffice.

At this point it is possible to see how different choices of extensions of the multiplicative functionals beyond the first iterated integral $\int_0^t dl_t$, that remain within the class of geometric rough paths, relate to different paths in the Lie group. This is not quite the case for a vector space. For example, see the straight rough paths of section 5.1.4. Typical examples of straight rough paths in a vector space V, exhibit behaviour such as accumulating area say, but as paths in V they remain at the base point. The associated behaviour in G is more complicated.

As stated before, to give an intrinsic definition, the formulation of the p variation topology should necessitate control of the term (4.4) which does not appear in (4.3), where $\mathbf{X}_{\cdot}^{\lfloor p \rfloor} \in C\left([0, T_{\mathbf{X}^{\lfloor p \rfloor}}] \to G\right)$ is some associated function. For a vector space this control is clearly a result of the definition of the metric. So, does control of the distance (4.3) entail the additional control (4.4) ?

The issue is resolved through a local analysis of a differential equation. In addition, for the case of connected finite dimensional matrix Lie groups (covering most possibilities) there is a well defined evaluation map. For $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^p$ this map locates points on a path

$$\mathbf{X}_{\cdot}^{\lfloor p \rfloor} : [0, T_{\mathbf{X}^{\lfloor p \rfloor}}] \to G.$$

Primarily, this is a consequence of Lyons theory for paths in the vector space $V_{\mathfrak{g}}$.

4.1.1 A differential equation in $L(V)$

Consider a vector space V with a norm which extends to each $V^{\otimes i}$ in a manner consistent with equation 2.4. Then for any

$$\underline{l} \in L^n(V) \subset \left(\oplus_{i=1}^n V^{\otimes i}\right),$$

$\underline{l}$ has an expression

$$\underline{l} = \sum_{i=1}^{n} \underline{l}^{(i)},$$

where $\underline{l}^{(i)} \in V^{\otimes i}$, so define

$$\|\underline{l}\|_{L^n(V)} = \sum_{i=1}^{n} \left\|\underline{l}^{(i)}\right\|_{V^{\otimes i}}$$

and hence a norm on $\cup_{n=1}^{\infty} L^n(V)$. Then define $L^{\infty}(V)$ to be the completion of $\cup_{n=1}^{\infty} L^n(V)$ with respect to the norm $\|\cdot\|_{\infty}$ on $\cup_{n=1}^{\infty} L^n(V)$ given by

$$\|\underline{l}\|_{\infty} = \lim_{n\to\infty} \|\underline{l}\|_{L^n(V)}.$$

Then $\left(L^{\infty}(V), \|\cdot\|_{L^{\infty}(V)}\right)$ is a Banach space and in addition, a Lie algebra: if $\underline{l}_1, \underline{l}_2 \in L^{\infty}(V)$,

$$[\underline{l}_1, \underline{l}_2]^{(i)} = \sum_{n=1}^{i-1} \left[\underline{l}_1^{(n)}, \underline{l}_2^{(i-n)}\right],$$

so that

$$\begin{aligned} \left\|[\underline{l}_1, \underline{l}_2]^{(i)}\right\|_{V^{\otimes i}} &\leq \sum_{n=1}^{i-1} \left\|\left[\underline{l}_1^{(n)}, \underline{l}_2^{(i-n)}\right]\right\|_{V^{\otimes i}} \\ &\leq 2\sum_{n=1}^{i-1} \left\|\underline{l}_1^{(n)}\right\|_{V^{\otimes n}} \left\|\underline{l}_2^{(i-n)}\right\|_{V^{\otimes(i-n)}}, \end{aligned}$$

by the consistency condition 2.4. Therefore $[\underline{l}_1, \underline{l}_2] \in L^{\infty}(V)$:

$$\begin{aligned} \|[\underline{l}_1, \underline{l}_2]\|_{\infty} &= \sum_{i=1}^{\infty} \left\|[\underline{l}_1, \underline{l}_2]^{(i)}\right\|_{V^{\otimes i}} \\ &\leq 2 \sum_{n,m=1}^{\infty} \left\|\underline{l}_1^{(n)}\right\|_{V^{\otimes n}} \left\|\underline{l}_2^{(m)}\right\|_{V^{\otimes m}} \\ &\leq 2 \|\underline{l}_1\|_{\infty} \|\underline{l}_2\|_{\infty}. \end{aligned}$$

The Campbell Baker Hausdorff formula

$$H(a,b) = \log(\exp(a)\exp(b)),$$

see equation (6.1), has a representation in terms of series of homogeneous degree i in b, $H_i(a,b)$:

$$H(a,b) = \sum_{i=0}^{\infty} H_i(a,b).$$

According to [33] for example, the forms of the first two series are

$$\begin{aligned} H_0(a,b) &= a \\ H_1(a,b) &= b + \frac{1}{2}[a,b] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}(ad(a))^{2n}(b) \\ &= \left(\frac{ad(-a)}{\exp(-a)-1}\right)(b), \end{aligned}$$

where the B_n are the Bernoulli numbers and if $|x| < 2\pi$, satisfy

$$\sum_{n\geq 0}^{\infty} \frac{B_n}{n!}x^n = \frac{x}{\exp(x)-1}.$$

Now if $\zeta(s)$ is the Riemann zeta function, the B_n satisfy (see [5])

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n),$$

hence there is the operator bound on $H_1(a,b)$:

$$\begin{aligned} \|H_1(a,b)\| &\leq \|b\|_\infty + \frac{1}{2}\|[a,b]\|_\infty + \sum_{n=1}^{\infty} \frac{|B_{2n}|}{(2n)!}\left\|(ad(a))^{2n}(b)\right\|_\infty \\ &\leq \|b\|_\infty + \|a\|_\infty \|b\|_\infty + 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \|a\|_\infty^{2n} \|b\|_\infty . \end{aligned}$$

As $n \to \infty$, $\zeta(2n) \to 1$ (again see [5]), hence $\|H_1(a,\cdot)\| < \infty$ for $\|a\|_\infty < \pi$. Indeed, define the vector field $f : L^\infty(V) \to Hom(L^\infty(V), L^\infty(V))$ by

$$f(a)(\cdot) = H_1(a,\cdot).$$

f is then smooth for $\|a\|_\infty < \pi$ (see the appendix Lemma 6.3), hence it is possible to solve this differential equation driven by geometric rough paths in $L^\infty(V)$ in this neighborhood of 0. The idea is to use this formulation to solve the differential equation in $L^\infty(V)$ for the logarithm of the signature of a rough path $\mathbf{X}^{\lfloor p\rfloor} \in \Omega G(V)^p$, a possibility since $V \subset L^\infty(V)$.

For a bounded variation path in a vector space $\gamma \in \mathcal{P}_b(V)$, assume that for t small that the logarithm of $\log(\mathbf{X}_{0,t}(\gamma))$ exists. Then,

$$\begin{aligned}
&\frac{d}{dt}(\log(\mathbf{X}_{0,t}(\gamma)))\\
=\;& \lim_{\delta\to 0}\frac{1}{\delta}(\log(\mathbf{X}_{0,t+\delta}(\gamma)) - \log(\mathbf{X}_{0,t}(\gamma)))\\
=\;& \lim_{\delta\to 0}\frac{1}{\delta}(H(\log(\mathbf{X}_{0,t}(\gamma)),\log(\mathbf{X}_{t,t+\delta}(\gamma))) - \log(\mathbf{X}_{0,t}(\gamma)))\\
=\;& \lim_{\delta\to 0}\frac{1}{\delta}\sum_{n=1}^{\infty} H_n(\log(\mathbf{X}_{0,t}(\gamma)),\log(\mathbf{X}_{t,t+\delta}(\gamma)))\\
=\;& H_1\left(\log(\mathbf{X}_{0,t}(\gamma)),\frac{d\gamma_t}{dt}\right).
\end{aligned}$$

Thus, for $\mathbf{m}_t \in L^\infty(V)$, $\|\mathbf{m}_t\|_\infty < \pi$, the solution to the differential equation

$$\begin{aligned}
\mathbf{m}_0 &= 0\\
d\mathbf{m}_t &= f(\mathbf{m}_t)(d\gamma_t),
\end{aligned}$$

satisfies $\mathbf{m}_t = \log(\mathbf{X}_{0,t}(\gamma))$.

So take $\mathbf{X}^{\lfloor p\rfloor} \in \Omega G(V)^p$, $\mathbf{X}_0^{\lfloor p\rfloor} = 0$ with control $\omega(\cdot,\cdot)$. Then Theorem 27 implies that there is an unique solution to

$$\begin{aligned}\mathbf{m}_0 &= 0 \\ d\mathbf{m}_t &= f(\mathbf{m}_t)\left(d\mathbf{X}_t^{\lfloor p\rfloor}\right),\end{aligned} \tag{4.5}$$

while $\|\mathbf{m}_t\|_\infty < \pi$. In addition, the solution will be a rough path of finite p variation in $L^\infty(V)$ and have control $\kappa_p\omega(\cdot,\cdot)$ for some $0 < \kappa_p < \infty$. Thus,

$$\|\mathbf{m}_t\|_\infty^p < \kappa_p\omega(0,t),$$

where this solution is defined, which can thus be taken to include the set of values of t such that

$$\omega(0,t) \le \frac{\pi^p}{\kappa_p}.$$

On this set, the logarithm of $\mathbf{X}_{0,t}^{\lfloor p\rfloor} = \mathbf{X}_t^{\lfloor p\rfloor}$ is well defined and

$$\log\left(\mathbf{X}_{0,t}^{\lfloor p\rfloor}\right) = \mathbf{m}_t \in L^\infty(V).$$

This clearly does not show that $\log\left(\mathbf{X}_{s,t}^{\lfloor p\rfloor}\right) \in L^\infty(V)$ for any $(s,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p\rfloor}}}$. However locally, in particular if $\omega(s,t) \le \frac{\pi^p}{\kappa_p}$, this statement is true since $\log\left(\mathbf{X}_{s,t}^{\lfloor p\rfloor}\right)$ is then a solution to an equivalent differential equation to 4.5 started at time s.

The equation in $\mathfrak{g}$

In a particular lie group G, the equivalent formulation provides a local solution to equation 4.5, describes the evolution of the logarithm of a curve (where the logarithm is defined). The following lemma is a standard part of Lie group theory, see for example Höchschild [22].

Lemma 56 *If G is a finite dimensional lie group, the exponential map, $Exp : \mathfrak{g} \to G$, is an isomorphism of some open connected neighborhood of $0 \in \mathfrak{g}$, $W_{\mathfrak{g}}$ say, onto an open connected neighborhood U_G of $1 \in G$.*

So let $\mathfrak{g}$ be the lie algebra of a finite dimensional lie group and pick a norm on $\mathfrak{g}$, $\|\cdot\|_{\mathfrak{g}}$. The linear map $ad : \mathfrak{g} \to Hom(\mathfrak{g}, \mathfrak{g})$ is bounded since $\mathfrak{g}$ is finite dimensional: there exists $0 < \|ad\| < \infty$ such that for all $a, b \in \mathfrak{g}$,

$$\|[a,b]\|_{\mathfrak{g}} \leq \|ad\| \, \|a\|_{\mathfrak{g}} \, \|b\|_{\mathfrak{g}} .$$

As before, the 1 form $f : \mathfrak{g} \to Hom(\mathfrak{g}, \mathfrak{g})$

$$f(a)(b) = b + \frac{1}{2}[a,b] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (ad(a))^{2n}(b)$$

is smooth, this time on a neighborhood $V_{\mathfrak{g}}$ of $0 \in \mathfrak{g}$,

$$V_{\mathfrak{g}} = \left\{ \|a\|_{\mathfrak{g}} < \frac{2\pi}{\|ad\|} \right\}.$$

Hence again, solutions to the differential equation in $\mathfrak{g}$,

$$\begin{aligned} \mathbf{y}_0^{\mathfrak{g}} &= 0 \\ d\mathbf{y}_t^{\mathfrak{g}} &= f\left(\mathbf{y}_t^{\mathfrak{g}}\right)\left(d\mathbf{X}_t^{\lfloor p \rfloor}\right). \end{aligned}$$

exist and are unique for elements of $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V_{\mathfrak{g}})^p$ in $V_{\mathfrak{g}}$ and have finite p variation with control $\kappa_{\mathfrak{g},p}\omega(\cdot,\cdot)$ for some $\kappa_{\mathfrak{g},p} > 0$, (where $\omega(\cdot,\cdot)$ is a control for $\mathbf{X}^{\lfloor p \rfloor}$). The solutions are also continuous with respect to the p variation metric on $\Omega G(V_{\mathfrak{g}})^p$, hence the solutions converge uniformly on $V_{\mathfrak{g}}$.

To map these solutions to G, restrict them to the neighborhood $W_{\mathfrak{g}} \cap V_{\mathfrak{g}}$ (see Lemma 56) where the exponential map is well defined. Let $N_{\mathfrak{g}}^{\epsilon} = \left\{y : \|y\|_{\mathfrak{g}} < \epsilon\right\}$ and define

$$\delta_{\mathfrak{g}} = \sup_{N_{\mathfrak{g}}^{\epsilon} \subset W_{\mathfrak{g}} \cap V_{\mathfrak{g}}} \epsilon$$

To which end, the continuous paths

$$Exp\left(\mathbf{y}_{\cdot}^{\mathfrak{g}}\right) : [0,\tau) \to G$$

where $\tau = \inf_{\mathbf{y}_t^{\mathfrak{g}} \notin N_{\mathfrak{g}}^{\delta_{\mathfrak{g}}}} \{t > 0\}$, provide solutions to the differential equation in G

$$\begin{aligned} Exp\left(\mathbf{y}_0^{\mathfrak{g}}\right) &= e \\ d\left(Exp\left(\mathbf{y}_t^{\mathfrak{g}}\right)\right) &= Exp\left(\mathbf{y}_t^{\mathfrak{g}}\right) \circ d\mathbf{X}_t^{\lfloor p \rfloor} \end{aligned} \tag{4.6}$$

for $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V_{\mathfrak{g}})^p$. Since $\left\|\mathbf{y}_t^{\mathfrak{g}}\right\|_{\mathfrak{g}} \leq \kappa_{\mathfrak{g},p}\omega(0,t)$, this is the case for $t \in [0, T_{\mathbf{X}^{\lfloor p \rfloor}}]$ such that

$$\omega(0,t) < \frac{\delta_{\mathfrak{g}}}{\kappa_{\mathfrak{g},p}}.$$

In the case of $\gamma \in \mathcal{P}_b(G)$, the solution of (4.6) $Exp\left(\mathbf{y}_t^{\mathfrak{g}}\right)$ coincides with the ordinary differential equation

$$\begin{aligned} g_0 &= e \\ dg_t &= g_t \circ d\gamma_t. \end{aligned}$$

In general, the solutions can be extended beyond the interval $[0, \tau_0)$ where

$$\tau_s = \sup\left(t > s : \omega(s,t) < \frac{1}{\kappa_{\mathfrak{g},p}} \min\left(\delta_{\mathfrak{g}}, \frac{2\pi}{\|ad\|}\right)\right),$$

to a global solution by concatenating a finite number of local solutions. Since, the solutions in $\mathfrak{g}$ have finite p variation locally, the form of the metric in G, equation (4.2), implies that the local solutions in G have finite p variation, which is in turn implies the global solution has finite p variation since it is the concatenation of a finite number of local solutions (see Lemma 120). Indeed, the local uniform convergence at the level of the lie algebra, entails uniform convergence for the concatenated global solution in G with respect to the p variation topology. In other words, the control (4.4) is guaranteed, as required.

4.1.2 Evaluation maps I

The name evaluation is for the following reasons. Signatures of geometric rough paths in a connected lie group G, are group-like elements of the Hopf algebra $T(V_{\mathfrak{g}})$, which can also be thought of as $U(L(V_{\mathfrak{g}}))$, the universal enveloping algebra of the free lie algebra $L(V_{\mathfrak{g}})$. As such, freeness of $L(V_{\mathfrak{g}})$ means that the formal identification of $V_{\mathfrak{g}}$ in $\mathfrak{g}$,

$$I : V_{\mathfrak{g}} \to \mathfrak{g},$$

entails an unique lie algebra map of $L(V_{\mathfrak{g}})$ to $\mathfrak{g}$ and indeed an unique map

$$I : U(L(V_{\mathfrak{g}})) \to U(\mathfrak{g}),$$

which is a Hopf algebra itself. The image of group like elements of $T(V_{\mathfrak{g}}) \cong U(L(V_{\mathfrak{g}}))$, are group like in $U(\mathfrak{g})$. As previously remarked, the role of $U(\mathfrak{g})$ as relates to G, can be similar to the object that would be the group algebra. The idea is that the evaluation map is a composition of maps on certain group like elements of $U(L(V_{\mathfrak{g}}))$, to group like elements of $U(\mathfrak{g})$ and onwards into the group G itself.

So in this section, $\mathfrak{g} \subset M_{n\times n}$ (the $n \times n$ real matrices) will be the lie algebra of a connected matrix lie group $G \subset M_{n\times n}$. The aim is to look at an evaluation map that recovers a path in G from the iterated integrals and initial point alone, corresponding to a given geometric rough path of finite $p-$variation. This evaluation map thus locates the element of G that pertains to the endpoint of a path (c.f. [12]). For $\mathbf{X}^{\lfloor p\rfloor} \in \Omega G\,(G)^{p}$, this group-like element $\mathbf{X}_{s,t}^{\lfloor p\rfloor}$ can be mapped into G to give an incremental group element $g_{s,t}\left(\mathbf{X}^{\lfloor p\rfloor}\right)$.

Matrix Lie Groups

Take $I : V_{\mathfrak{g}} \to \mathfrak{g} \subset M_{n\times n}$ to be the embedding of $V_{\mathfrak{g}}$ into $M_{n\times n}$(an associative algebra). The freeness of $L(V_{\mathfrak{g}})$ implies I extends to a lie map from $L(V_{\mathfrak{g}}) \to M_{n\times n}$ uniquely. It is not clear however, that this map is defined analytically with a degree of confidence on infinite series. In the case of a geometric rough path $\mathbf{X}^{\lfloor p\rfloor} \in \Omega G\,(G)^{p}$, it is certainly true that $\mathbf{X}_{s,t}^{\lfloor p\rfloor;\infty} = \exp\left(\underline{l}_{s,t}\right)$ for some $\underline{l}_{s,t} \in L(V_{\mathfrak{g}})$. However, typically there is no analytic control on the homogeneous components of $\underline{l}_{s,t}$ from rough path theory to understand how I acts on the tail of these lie elements. Known methods of extending norms beyond V to $V^{\otimes n}$ that are consistent with (2.4), do not globally provide tractable bounds on the lie components.

Instead, it is possible to look at I acting on the whole iterated integral sequence at once. Through associating a differential equation, Lyons' result (Theorem 8) applies and confirms that the map of the incremental full iterated integral sequence of a geometric rough path, is into G.

Definition 57 *For $m \in M_{n\times n}$, define the algebra norm*

$$\|m\|_{M_{n\times n}} = \sup_{\|e\|_{\mathbb{R}^n}\leq 1} \|m(e)\|_{\mathbb{R}^n} . \tag{4.7}$$

where $\|\cdot\|_{\mathbb{R}^n}$ is the Euclidean norm on $\mathbb{R}^n$ and for $v \in V_{\mathfrak{g}}$, set

$$\|v\|_{V_{\mathfrak{g}}} = \|I(v)\|_{M_{n\times n}} .$$

Let $V_{\mathfrak{g}}$ have a norm $\|\cdot\|_{V_{\mathfrak{g}}}$ and endow each component $V_{\mathfrak{g}}^{\otimes n}$ with the projective norm $\|\cdot\|_{proj}$ (2.6). For an infinite series $\underline{S} \in T(V_{\mathfrak{g}})$, define the L^1 type extension of the projective norm :

$$\|\underline{S}\|_{proj} = \sum_{n=0}^{\infty} \left\|\underline{S}^{(n)}\right\|_{proj} .$$

The next Lemma shows that when a series $\underline{S}$ has finite norm, $\|\underline{S}\|_{proj} < \infty$, the object $I(\underline{S})$ is a well defined element of the associative algebra $M_{n\times n}$.

Lemma 58 *Let G be a connected matrix lie group with lie algebra $\mathfrak{g} \subset M_{n\times n}$. If $\underline{S} \in \mathrm{T}(V_{\mathfrak{g}})$ satisfies $\|\underline{S}\|_{proj} < \infty$, then $I(\underline{S})$ exists and $\|I(\underline{S})\|_{M_{n\times n}} < \infty$.*

Proof. $M_{n\times n}$ is a Banach algebra for this norm, hence for any decomposition of the component of $\underline{S}$ of degree n, $\underline{S}^{(n)} = \sum_{j=1}^{N} s_{j,1} \otimes s_{j,2} \otimes \cdots \otimes s_{j,n}$,

$$\begin{aligned} \left\|I\left(\underline{S}^{(i)}\right)\right\|_{M_{n\times n}} &= \left\|\sum_{j=1}^{N} I(s_{j,1})\, I(s_{j,2}) \cdots I(s_{j,i})\right\|_{M_{n\times n}} \\ &\leq \sum_{j=1}^{N} \|I(s_{j,1})\|_{M_{n\times n}} \|I(s_{j,2})\|_{M_{n\times n}} \cdots \|I(s_{j,i})\|_{M_{n\times n}} . \end{aligned}$$

But $\|I(v)\|_{M_{n\times n}} = \|v\|_{V_{\mathfrak{g}}}$ and hence $\left\|I\left(\underline{S}^{(i)}\right)\right\|_{M_{n\times n}} \leq \left\|\underline{S}^{(i)}\right\|_{proj}$, so that $\sum_{i=1}^{\infty} \left\|I\left(\underline{S}^{(i)}\right)\right\|_{M_{n\times n}} < \infty$ and $I(\underline{S})$ is a well defined element of $M_{n\times n}$. ∎

A full iterated integral sequence of any rough path $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(G)^p$ with the projective norm, is of this type because the norms of the tails of the sequence decay in a pseudo-exponential manner. The following Lemma describes and confirms this and that the evaluation map is well defined.

Lemma 59 *Let G be a connected matrix lie group with lie algebra $\mathfrak{g} \subset M_{n\times n}$ and $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(G)^p$. Then,*

$$\forall (s,t) \in \Gamma_{T_\gamma}, \left\| \mathbf{X}_{s,t}^{\lfloor p \rfloor,\infty} \right\|_{proj} < \infty$$

and

$$I\left(\mathbf{X}_{\cdot,\cdot}^{\lfloor p \rfloor,\infty}\right) \in C\left(\Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}; M_{n\times n}\right).$$

Proof. Since there is a control ω such that $\forall (s,t)$, $\left\| \mathbf{X}_{s,t}^{\lfloor p \rfloor,(n)} \right\|_{proj} \leq \frac{\omega(s,t)^{\frac{n}{p}}}{\beta\left(\frac{n}{p}\right)!}$, define the sequence $x_n = \frac{\omega(s,t)^{\frac{n}{p}}}{\beta\left(\frac{n}{p}\right)!}$, $n \geq 1$. Stirling's formula gives the asymptotic approximation, $x! \sim x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi}$ giving the relation

$$\frac{x_{n+1}}{x_n} \sim \left(\frac{p}{n}\omega(s,t)\right)^{\frac{1}{p}},$$

so that $\frac{x_{n+1}}{x_n} \to 0$ as $n \to \infty$ and so the sequence $\{x_n\}_{n=1}^{\infty}$ is summable and thus $\left\| \mathbf{X}_{s,t}^{\lfloor p \rfloor,\infty} \right\|_{proj} < \infty$. The map I is an algebra map so since $\mathbf{X}_{\cdot,\cdot}^{\lfloor p \rfloor,\infty}$ satisfies Chen's identity, the map is multiplicative : i.e. $I\left(\mathbf{X}_{s,t}^{\lfloor p \rfloor,\infty}\right) = I\left(\mathbf{X}_{s,u}^{\lfloor p \rfloor,\infty}\right) \times I\left(\mathbf{X}_{u,t}^{\lfloor p \rfloor,\infty}\right) \forall (s,u),(u,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}$. The regularity of the control $\omega(u,t) \to 0$ as $t \downarrow u$, implies that $I\left(\mathbf{X}_{u,t}^{\lfloor p \rfloor,\infty}\right) \to id_{n\times n}$, the identity matrix as $t \downarrow u$, hence the map is continuous. ■

This does not quite mean that the paths $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(G)^p$ define incremental elements of G, rather in $M_{n\times n}$. First consider I applied to the full iterated integral sequence of an element $\gamma \in \mathcal{P}_b(G)$.

Lemma 60 *Let $\gamma \in \mathcal{P}_b(G)$. Assume that it is parameterized at unit speed and pick $0 \leq s < T_\gamma$. Consider* *the continuous paths* $g_s(\cdot) : [s, T_\gamma] \to M_{n\times n}$

$$g_s(t) = I\left(\mathbf{X}^{\infty}_{s,t}(\gamma)\right).$$

Then $\forall t \in [s, T_\gamma]$, $g_s(t) = \gamma_s^{-1}\gamma_t \in G$.

Proof. Clearly the equality holds for $t = s$. To show it holds $\forall (s,t) \in \Gamma_{T_\gamma}$, it suffices to check that $g_s(t)$ is the solution (which will be unique), to the differential equation

$$\begin{aligned} dg_s(t) &= g_s(t)\, dl_t \\ g_s(s) &= id_{n\times n} \end{aligned}$$

First :

$$I\left(\mathbf{X}^{\infty}_{s,t}(\gamma)\right) = \sum_{j=0}^{\infty} I\left(\mathbf{X}^{(j)}_{s,t}(\gamma)\right). \tag{4.8}$$

Then

$$\begin{aligned} &\lim_{\delta\to 0} \frac{I\left(\mathbf{X}^{\infty}_{s,t+\delta}(\gamma)\right) - I\left(\mathbf{X}^{\infty}_{s,t}(\gamma)\right)}{\delta} \\ =\ & I\left(\mathbf{X}^{\infty}_{s,t}(\gamma)\right) \lim_{\delta\to 0} \frac{I\left(\mathbf{X}^{\infty}_{t,t+\delta}(\gamma)\right) - I\left(\mathbf{X}^{\infty}_{t,t}(\gamma)\right)}{\delta} \\ =\ & \left\{ \begin{array}{c} g_s(t) \lim_{\delta\to 0} \frac{1}{\delta} \left\{ \sum_{j=0}^{\infty} I\left(\mathbf{X}^{(j)}_{t,t+\delta}(\gamma)\right) - I(\mathbf{1}) - I\left(\int_t^{t+\delta} dl_u\right)\right\} \\ +\beta_s(t) \lim_{\delta\to 0} \frac{1}{\delta} I\left(\int_t^{t+\delta} dl_u\right) \end{array} \right\} \\ =\ & g_s(t) \lim_{\delta\to 0} \frac{1}{\delta} \left\{ \sum_{j=2}^{\infty} I\left(\mathbf{X}^{(j)}_{t,t+\delta}(\gamma)\right)\right\} + \beta_s(t) \frac{dl_t}{dt} \\ =\ & g_s(t) \frac{dl_t}{dt}, \end{aligned}$$

since as $l_t \in \mathcal{P}(V_{\mathfrak{g}})$, there is the bound on the iterated integrals

$$\forall j, \left\| I\left(\mathbf{X}^{(j)}_{t,t+\delta}(\gamma)\right)\right\|_{M_{n\times n}} \le \frac{(c\delta)^j}{j!},$$

giving

$$\lim_{\delta\to 0}\frac{1}{\delta}\left\{\sum_{j=2}^{\infty}\left\|I\left(\mathbf{X}^{(j)}_{t,t+\delta}(\gamma)\right)\right\|_{M_{n\times n}}\right\}=0.$$

■

So, for any $s\in[0,T_\gamma)$, the function $g_{s,\cdot}(\gamma)\in C([s,T_\gamma]\to G)$ is the solution to the differential equation in the vector space $M_{n\times n}$:

$$\begin{aligned} g_{s,s}(\gamma) &= I_{n\times n} \\ dg_{s,t}(\gamma) &= g_s(t)\,dl_t. \end{aligned}$$

Lyons theorem (27) now says that for the p variation topology, the map from driving path to solution is continuous for the p variation topology.

Corollary 61 *Let* $\mathbf{X}^{\lfloor p\rfloor}\in\Omega G(G)^p$. *Then* $I\left(\mathbf{X}^{\lfloor p\rfloor,\infty}_{\cdot,\cdot}\right)\in C\left(\Gamma_{T_{\mathbf{X}^{\lfloor p\rfloor}}};G\right)$.

Proof. One route to show this is rather crude and makes use of the estimate of the modulus of continuity (15). It shows that if $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{P}_b(G)$, Cauchy with respect to $D_{G,p}$ and converging to $\mathbf{X}^{\lfloor p\rfloor}$, then in $M_{n\times n}$ for any $(s,t)\in\Gamma_{T_{\mathbf{X}^{\lfloor p\rfloor}}}$

$$\lim_{n\to\infty} I\left(\mathbf{X}^{\infty}_{s,t}(\gamma_n)\right)=I\left(\mathbf{X}^{\lfloor p\rfloor,\infty}_{s,t}\right).$$

Since G is closed, $I\left(\mathbf{X}^{\lfloor p\rfloor,\infty}_{s,t}\right)\in G$.

Alternatively, section 4.1.1 shows that for connected matrix lie groups, the map I is uniformly continuous with respect to the p variation topology. Hence if $\{\gamma_n\}_{n=1}^{\infty}$ is as before, the sequence converges uniformly to a continuous path in G of finite p variation and hence the limiting path takes values in G.

■

As a conclusion, elements $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(G)^p$ where G is a finite dimensional connected lie group, define signature paths $\mathbf{X}^{\lfloor p \rfloor, \infty} \in C\left(\Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}, T(V_{\mathfrak{g}})\right)$ which when evaluated through the function I, give elements of $C\left(\Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}, G\right)$ of finite p variation. If two elements of $\Omega G(G)^p$ are close for the p variation topology, their extended iterated integral sequence of their signature, are close in the way quantified by Lyons for paths in a vector space. At one level, the conclusion is a confirmation of a version of Lyons quoted result Theorem 27 for solving a differential equation in a Banach algebra. However there is also a natural way to develope the rough path in the lie algebra into the lie group. The reason for this approach is to discuss how the iterated integrals are somehow elements of the tensor product of a tangent space, which in the case of lie groups has a global interpretation through the natural left (or right) connection.

For the right connection and the iterated integral sequence $\mathbf{X}_{s,t}^{\leftarrow(n)}(\gamma)$ of a path $\gamma \in \mathcal{P}(G)$, the existence of the evaluation map I is checked in exactly the same way. However, the evaluation is now of the form

$$I\left(\mathbf{X}_{s,t}^{\leftarrow\infty}(\gamma)\right) = \gamma_t \gamma_s^{-1},$$

in accordance with the reverse tensor multiplication. It is not the case that this reversal of tensors of the iterated integrals is the only reason why the evaluation map produces $\gamma_t \gamma_s^{-1}$ as opposed to $\gamma_s^{-1}\gamma_t$ for the left connection. The evaluation would be related to the transpose path (Remark 48), a different path in G.

Remark 62 *Given that any rough path can be developed into any connected matrix Lie group G as a well* *defined function of the iterated integrals, a natural question to ask is how a distribution on rough paths interacts with the distribution of the developments of the rough paths in the Lie groups. For example, is there a natural way to characterize fractional Brownian motion in a Lie group. What implications are there for choices of higher order iterated integrals say for Wiener motion in order that the developed paths correspond with a notion of Wiener motion in the lie group ? In particular, there could be specific information to say that area process should concur with Lévy's distribution [27].*

Remark 63 *For the case of a connected matrix lie group $G \subset M_{n\times n}$, the evaluation map applied to $\Omega(G)^p$ generally produces a path in $M_{n\times n}$, not in G. In the case of a vector space, the first iterated integral alone maps to a path of finite $p-$variation in V, however it seems a hard problem to construct a well defined map that associates an unique element of $\Omega G(G)^p$ to non-geometric rough paths. One method of projecting lie algebras is mentioned in the appendix and shown not to work, section 6.6.*

4.2 The shuffle algebra revisited

Parallel to the construction above, there is an analogous shuffle algebra analysis for paths in a connected Lie group. There is a canonical way of constructing a left-invariant 1-form on G from an element of $\mathfrak{g}^*$. The subset of the cotangent bundle generated by these canonical extensions, define a collection of functions which relate to paths in G in the same sense as the shuffle algebra structure of section 3.3.

Notation 64 *Take $\omega \in V_{\mathfrak{g}}^* \simeq \mathrm{T}_e^*(G)$ the dual space to $V_{\mathfrak{g}}$ identified with the cotangent space of G at the* *identity e. Then ω defines an element $\widetilde{\omega}$ in the whole of the cotangent bundle of G by*

$$\widetilde{\omega}(v) = \omega\left(g^{-1} \circ v\right),$$

where $v \in \mathrm{T}_g(G)$, which is the tangent space to $g \in G$. Denote the set of 1-forms in the image of this map by $\mathrm{T}^{*,\sim}(G)$.

Now take $\underline{\omega} \in \left(V_{\mathfrak{g}}^{*}\right)^{\otimes n}$ of the form

$$\underline{\omega} = \omega_1 \otimes \cdots \otimes \omega_n$$

with $\omega_i \in V_{\mathfrak{g}}^{*}$. If $\gamma \in \mathcal{P}_b(G)$, then

$$\begin{aligned}
&\underline{\omega}\left(\mathbf{X}_{sig}(\gamma)\right) \\
&= \int \cdots \int_{0 \le u_1 \le u_2 \le \cdots \le u_n \le T_\gamma} \omega_1\left(dl_{u_1}(\gamma)\right) \cdots \omega_n\left(dl_{u_n}(\gamma)\right) \\
&= \int \cdots \int_{0 \le u_1 \le u_2 \le \cdots \le u_n \le T_\gamma} \omega_1\left(\gamma_{u_1}^{-1} d\gamma_{u_1}\right) \cdots \omega_n\left(\gamma_{u_n}^{-1} d\gamma_{u_n}\right) \\
&= \int \cdots \int_{0 \le u_1 \le u_2 \le \cdots \le u_n \le T_\gamma} \widetilde{\omega}_1\left(d\gamma_{u_1}\right) \cdots \widetilde{\omega}_n\left(d\gamma_{u_n}\right).
\end{aligned}$$

The action of such $\underline{\omega}$ on $\Omega G(G)^1$, extends to an action for the whole of $\mathcal{T}\left(V_{\mathfrak{g}}^{*}\right)$ on $\Omega G(G)^1$, which at the same time is mirrored by an action on $\mathcal{P}_b(G)$ by $\mathcal{T}\left(\mathrm{T}^{*,\sim}(G)\right)$.

As with $\mathcal{T}(V^*)$, the shuffle product applies in $\mathcal{T}\left(V_{\mathfrak{g}}^{*}\right)$ to represent how this collection of linear functions on $\Omega G(G)^1$ multiply together to produce in exactly the same fashion, or equivalently the multiplication of $\mathcal{T}\left(\mathrm{T}^{*,\sim}(G)\right)$ as functions on $\mathcal{P}_b(G)$.

An aspect to discuss is related to the set of signatures that can appear in the tensor algebra $T(V)$ for rough paths in a vector space as compared to $T(V_{\mathfrak{g}})$ and rough paths in a Lie group G under an isomorphism between V and $V_{\mathfrak{g}}$. The set is the same because the elements of $\mathcal{P}_b(G)$ are thought of as paths in $V_{\mathfrak{g}}$ through a smooth correspondence : the map of differentials from $\mathrm{T}_{\gamma_t}(G)$ to $\mathrm{T}_e(G)$ produces a bounded variation path $l_t = \int_0^t dl_s \in V_{\mathfrak{g}}$. Given this fact, the paths $l_t \in V_{\mathfrak{g}}$ that could possibly appear as signatures through isomorphic correspondence of the $V_{\mathfrak{g}}$'s for different Lie algebras, are the same. Hence the objects in the completion with respect to the p variation topologies are the same.

For the right-connection and right invariant vector-fields, the appropriate extension of $\omega \in \mathfrak{g}^*$ to an element of $\mathrm{T}^*(G)$ is given by $\widehat{\omega}(dx) = \omega\left(dxg^{-1}\right)$.

The duality ideas of $\mathcal{T}\left(V_{\mathfrak{g}}^*\right)$ and $T(V_{\mathfrak{g}})$ carry through also - this time, $T(V_{\mathfrak{g}})$ is thought of as the tensor algebra of left-invariant vector fields on G.

Generally, there is a shuffle algebra based on $\Lambda(G)$, the smooth 1-forms on G, $\mathcal{T}(\Lambda(G))$. As in fact with the above construction, any element of $\mathcal{P}_b(G)$ defines a homomorphism of the shuffle algebra of forms $\mathcal{T}(\Lambda(G))$. In fact, the construction exists for $\mathcal{T}(\Lambda(M))$ for any smooth manifold. The set of algebra homomorphisms though, contain objects which are no longer identifiable with paths though. For example, let M have a Riemannian metric, $\gamma \in \mathcal{P}_b(M)$ and f be a bounded measurable function on M. Define $\phi_{f,\gamma} : \mathcal{T}(\Lambda(G)) \to \mathbb{R}$ by

$$\begin{aligned}
&\phi_{f,\gamma}(\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_n) \\
&= \int_{0 \le u_1 \le u_2 \le \cdots \le u_n \le T_\gamma} \cdots \int f(\gamma_{u_1})\, \omega_1(\gamma_{v_1}; d\gamma_{v_1}) \cdots f(\gamma_{u_n})\, \omega_n(\gamma_{v_n}; d\gamma_{v_n}),
\end{aligned}$$

where $\omega_i(x;\cdot) : T_x(M) \to \mathbb{R}$. $\phi_{f,\gamma}$ is an algebra homomorphism of the shuffle algebra built over $\mathcal{T}(\Lambda(G))$, though does not correspond to a path itself. For the case of a Lie group and $\gamma \in \mathcal{P}(G)$, this action of multiplying the differential $\gamma^{-1}d\gamma$ by such an element f, does produce another path, that will be of bounded variation. Tavares examines a similar construction, a space of generalized loops on a smooth manifold, in [42]. The Hopf algebra formalism induces a group structure for these loops which have applications to construct a formalism for gauge theories and quantum gravity.

4.3 Paths in 2-dimensional hyperbolic space

Let G be an n-dimensional connected matrix lie group with lie algebra $\mathfrak{g}$ and let $M = G/K$, where K is a compact subgroup of G. Extending further the picture of rough paths of finite p-variation as elements of $\Omega G\left(V_{\mathfrak{g}}\right)^{p}$, the graphs of rough paths on manifolds $M = G/K$, can be identified as being the graphs of the image in M of the signature process of rough paths on G under the quotient map

$$q : G \to G/K.$$

Let $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G\left(G\right)^{p}$ start from the point

$$g = \mathbf{X}_0^{\lfloor p \rfloor} \in G$$

and let I be the evaluation map. Then the curve

$$\gamma_t = q\left(g \cdot I\left(\mathbf{X}_{sig;0,t}^{\lfloor p \rfloor}\right)\right)$$

defines a path in G/K which is the image of the geometric rough path in G. Riemannian manifolds that are globally symmetric are diffeomorphic to spaces which fall into this category. A family of such examples are the hyperbolic spaces.

This section is an analysis how a C^2 curve in $\mathbb{R}^2$ can be identified with a path in 2−dimensional hyperbolic space. The aim is to show how the end point is a function of the iterated integral sequence and the start point. Through a scaling procedure, it is possible to use the curvature of hyperbolic space to deduce length information about the original curve. It is a function of the limit of the scaling and the hyperbolic distance between the beginning point and the end point. Hence, using the relationship to the iterated integral sequence, length information is obtained through a kind of non-commutative harmonic analysis of the iterated integrals of the path.

The first objective is to show that in $\mathfrak{U}$, the upper half-plane model of hyperbolic space, curves of constant curvature are segments of circles which intersect the real axis at an angle which is a function of the curvature of the curve alone.

The second objective is to show that if the magnitude of the curvature of a C^2 curve, is bounded above by a constant $\kappa \leq 1$, then the image of the developed path in $\mathfrak{U}$ is restricted to a given subset. As $\kappa \to 0$, the subset centers around a geodesic from $z = i$, so confining the development.

The final part involves developing scaled versions of a C^2 curve γ to deduce information about it's length from it's iterated integral sequence.

4.3.1 Cartan development

The Upper half-plane model of hyperbolic space, $\mathfrak{U}$

The following stated properties of $\mathfrak{U}$ are all well known - they are used in the calculations afterwards. $SL(2,\mathbb{R})/SO(2)$ is isomorphic to the upper-half plane model of hyperbolic space

$$\mathfrak{U} = \{z = x + iy : x, y \in \mathbb{R}, y > 0\},$$

under the continuous action $\widetilde{q} : SL(2,\mathbb{R}) \to \mathfrak{U} \simeq SL(2,\mathbb{R})/SO(2)$

$$\widetilde{q}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{ai+b}{ci+d}$$

With the metric $g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$, $\mathfrak{U}$ is a Riemannian manifold with constant negative curvature $\kappa = -1$. If $\gamma(t) = x(t) + iy(t)$, $t \in [0,1]$ is a C^1 curve in $\mathfrak{U}$, then the length of the curve $l(\gamma)$ is thus given by the formula

$$l(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y} dt = \int_0^1 \frac{\left|\frac{dz}{dt}\right|}{y} dt.$$

Geodesics are given by curves which are semi-circles centered on the real axis. Denote by w_θ the set of points on the geodesics that pass through $z = i$ that subtend an angle θ with the imaginary axis (see diagram 4.4). Straight lines parallel to the imaginary axis are included in this description, being semi-circles whose center is at infinity.

Suppose $\gamma : [0, T_\gamma] \to \mathfrak{U}$ is a C^2 curve, $\gamma_t = x_t + iy_t \in \mathfrak{U}$, that is parameterized by arc length.

According to [18], the vector

$$\begin{aligned}\mu_t &= (\mu_t^x, \mu_t^y)\\ &= \left(\ddot{x}_t - \frac{2\dot{x}_t\dot{y}_t}{y_t}, \ddot{y}_t + \frac{\left(\dot{x}_t^2 - \dot{y}_t^2\right)}{y_t}\right)\end{aligned}$$

is orthogonal to the curve at each point in the sense that the inner product $\mu_t \mathrm{g} \begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix} = 0$. Then the generalization of the Euclidean curvature of the curve is given in relation to the inner product by $\sqrt{\mu_t \mathrm{g} \mu_t^T}$.

For two points $z_1, z_2 \in \mathfrak{U}$ the hyperbolic distance $d(z_1, z_2)$ can be expressed:

$$\cosh\left(d\left(z_1, z_2\right)\right) = 1 + \frac{\left|z_1 - z_2\right|^2}{2\Im\left(z_1\right)\Im\left(z_2\right)} \tag{4.9}$$

The subgroup $PSL(2,\mathbb{R})$ defined as

$$\begin{aligned}PSL(2,\mathbb{R}) &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{array}{c} a,b,c,d \in \mathbb{R} \\ ad - bc = 1 \\ a \geq 0 \end{array}\right)\right\}\\ &\simeq SL(2,\mathbb{R}) / \{\pm I_2\},\end{aligned}$$

has lie algebra

$$psl(2,\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \quad a,b,c,d \in \mathbb{R}\right\}.$$

$PSL(2,\mathbb{R})$ acts on $\mathfrak{U}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longrightarrow \frac{az+b}{cz+d}.$$

With this action, $PSL(2,\mathbb{R})$ is the set of isometries of $\mathfrak{U}$. The following two important subgroups of $PSL(2,\mathbb{R})$ $S = \{s_\lambda : \lambda \in \mathbb{R}\}$,

$$R = \{r_\theta : \theta \in [0,\pi)\} = SO(2) / \{\pm I_2\},$$

generate the whole group:

$$s_\lambda = \exp \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix} = \begin{pmatrix} \exp(\lambda/2) & 0 \\ 0 & \exp(-\lambda/2) \end{pmatrix} \quad \lambda \in \mathbb{R},$$

acts by on $\mathfrak{U}$ by multiplication, $z \to \exp(\lambda) z$, fixing the geodesic which is the imaginary axis and moving points on it a distance $|\lambda|$ either away from or towards the origin if λ is either positive or negative. Diagram (4.4) shows $s_\lambda(i) = ie^\lambda$ for some $\lambda > 0$.

$$r_\theta = \exp \begin{pmatrix} 0 & -\theta/2 \\ \theta/2 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \quad \theta \in [0,\pi),$$

acts on $\mathfrak{U}$ by "rotation" about the point $z = i$: the point $z = i$ is fixed while a geodesic which subtends an angle α with the imaginary axis in a clockwise sense, is shifted by an angle θ in a clockwise sense at the point i. The diagram also shows how w_0 is transformed by r_θ to w_θ.

Note that $R \simeq SO(2) / \{\pm I_2\}$, that $r_\theta(i) = i \ \forall r_\theta \in R$ and that

$$PSL(2,\mathbb{R}) / (SO(2) / \{\pm I_2\}) \simeq U.$$

So define the quotient map $q : PSL(2,\mathbb{R}) \to U$ which effects this isomorphism by

$$q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{ai+b}{ci+d}. \tag{4.10}$$

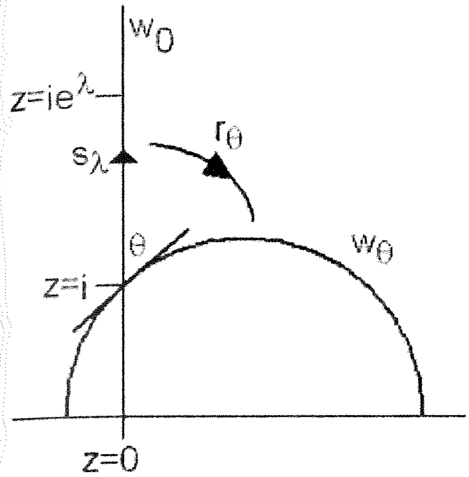


Figure 4.1: **The diagram shows the geodesics w_0 and w_θ in U and indicates the action of s_λ and r_θ.**

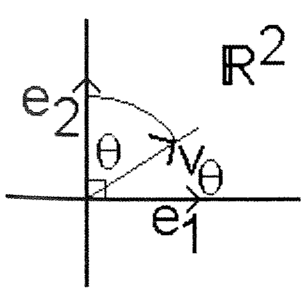


Figure 4.2: **A description of the orientation of the vectors e_1, e_2 and v_θ in $\mathbb{R}^2$.**

4.4 Paths in $\mathfrak{U}$

Let $\gamma \in P_b(\mathbb{R}^2)$ where $\mathbb{R}^2$ has orthonormal basis e_1, e_2 for the Euclidean metric with the orientation of diagram 4.2. The family of lie algebra maps

$$\phi_\lambda : \mathbb{R}^2 \to psl(2, \mathbb{R}),$$

by

$$\phi_\lambda(xe_1 + ye_2) = \frac{\lambda}{2}\begin{pmatrix} y & x \\ x & -y \end{pmatrix}. \tag{4.11}$$

Using the Lemmas of section 4.1.2, the family of evaluation maps

$$\{I \circ \phi_\lambda | \lambda > 0\},$$

define paths in $PSL(2, \mathbb{R})$,

$$I \circ \phi_\lambda : P(\mathbb{R}^2) \to P(PSL(2, \mathbb{R})).$$

Application of the continuous function $q : PSL(2, \mathbb{R}) \to \mathfrak{U}$ hence gives a family of continuous paths $\gamma^{q,\lambda} \in P(\mathfrak{U})$. The aim of the following section is to show that the paths in $\mathfrak{U}$ are the so called developments of the path $\gamma \in P(\mathbb{R}^2)$ into hyperbolic space under different scalings within $\mathbb{R}^2$.

4.4.1 Cartan development in $\mathfrak{U}$

For an abstract discussion of developing paths into a Riemannian manifold, [40] provides detailed description of how to realise the motion in the orthonormal frame bundle. This particular type of manifold requires less technology through the additional geometric properties of being a two dimensional symmetric space.

Consider elements $\gamma \in P_b^0\left(\mathbb{R}^2\right) = \left\{\gamma \in P_b\left(\mathbb{R}^2\right) : \gamma_0 = 0\right\}$, with the Euclidean metric with orthonormal basis e_1,e_2 as in diagram 4.2. To develop the simple path moving in a straight line at unit speed in a direction

$$\begin{aligned}\underline{v}_\theta &= \sin(\theta)\, e_1 + \cos(\theta)\, e_2 \in \mathbb{R}^2 \\ \gamma_t(\theta) &= t\underline{v}_\theta,\end{aligned}$$

consider the iterated integral sequence $\mathbf{X}_{s,t}^{\infty}(\gamma(\theta)) = \exp\left((t-s)\,\underline{v}_\theta\right)$ and the identification ϕ_λ of $\mathbb{R}^2$ with $psl(2,\mathbb{R})$ as in the paragraph above. With the norm $\|\cdot\|_{M_{2\times 2}}$ on $psl(2,\mathbb{R})$, ϕ_λ is continuous and $\|\phi_\lambda\| = \lambda$, so $\forall p \geq 1$ extends to a map

$$\phi_\lambda : \Omega^0 G\left(\mathbb{R}^2\right)^p \to \Omega^e G\left(PSL(2,\mathbb{R})\right)^p, \tag{4.12}$$

where

$$\Omega^e G\left(PSL(2,\mathbb{R})\right)^p = \left\{\mathbf{X}^{\lfloor p \rfloor} \in \Omega^g G(G)^p : \mathbf{X}_0^{\lfloor p \rfloor} = g\right\}.$$

The evaluation map

$$I : \Omega^e G\left(PSL(2,\mathbb{R})\right)^p \to \cup_{T>0} C\left(\Gamma_T, PSL(2,\mathbb{R})\right), \tag{4.13}$$

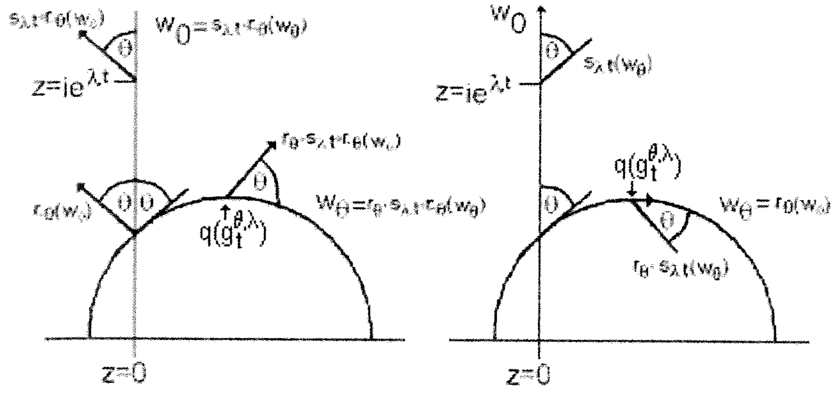
turns tensors in $T(psl(2,\mathbb{R}))$ to matrix multiplication in $M_{2\times 2}$ and hence provides the form for the incremental group elements:

$$
\begin{aligned}
g_{s,t}^{\theta,\lambda} &= I\left(\phi_\lambda\left(\mathbf{X}_{s,t}^{\infty}\left(\gamma\left(\theta\right)\right)\right)\right) \\
&= I\left(\phi_\lambda\left(\exp\left(\left(t-s\right)\underline{v}_\theta\right)\right)\right) \\
&= \sum_{j=0}^{\infty}\frac{\left(\lambda\left(t-s\right)\right)^j}{j!}I\left(\phi_1\left(\underline{v}_\theta^{\otimes j}\right)\right) \\
&= \sum_{j=0}^{\infty}\frac{\left(\lambda\left(t-s\right)/2\right)^j}{j!}\begin{pmatrix}\cos\left(\theta\right) & \sin\left(\theta\right) \\ \sin\left(\theta\right) & -\cos\left(\theta\right)\end{pmatrix}^j \\
&= \begin{pmatrix}\cosh\left(\lambda\left(t-s\right)/2\right)+\cos\left(\theta\right)\sinh\left(\lambda\left(t-s\right)/2\right) & \sin\left(\theta\right)\sinh\left(\lambda\left(t-s\right)/2\right) \\ \sin\left(\theta\right)\sinh\left(\lambda\left(t-s\right)/2\right) & \cosh\left(\lambda\left(t-s\right)/2\right)-\cos\left(\theta\right)\sinh\left(\lambda\left(t-s\right)/2\right)\end{pmatrix} \\
&= \begin{pmatrix}\cos\left(\theta/2\right) & -\sin\left(\theta/2\right) \\ \sin\left(\theta/2\right) & \cos\left(\theta/2\right)\end{pmatrix}\begin{pmatrix}\exp\left(\lambda\left(t-s\right)/2\right) & 0 \\ 0 & \exp\left(-\lambda\left(t-s\right)/2\right)\end{pmatrix} \\
&\quad\times\begin{pmatrix}\cos\left(\theta/2\right) & \sin\left(\theta/2\right) \\ -\sin\left(\theta/2\right) & \cos\left(\theta/2\right)\end{pmatrix} \\
&= r_\theta s_{\lambda(t-s)} r_{-\theta}. \qquad (4.14)
\end{aligned}
$$

Clearly this element of $C\left(\Gamma_{T_\gamma}, PSL\left(2,\mathbb{R}\right)\right)$ satisfies

$$
g_{t,s}^{\theta,\lambda}\circ g_{s,r}^{\theta,\lambda} = g_{t,r}^{\theta,\lambda} \quad \forall\left(t,s\right),\left(s,r\right)\in\Gamma_{T_\gamma}.
$$

The action of evaluation at $z=i$, given by the quotient map $q: PSL\left(2,\mathbb{R}\right)\to\mathfrak{U}$ (4.10), applied to the path $g_t^{\theta,\lambda}=g_{0,t}^{\theta,\lambda}$, describes a curve in $\mathfrak{U}$ from the point i. From the decomposition (4.14), the effect is to



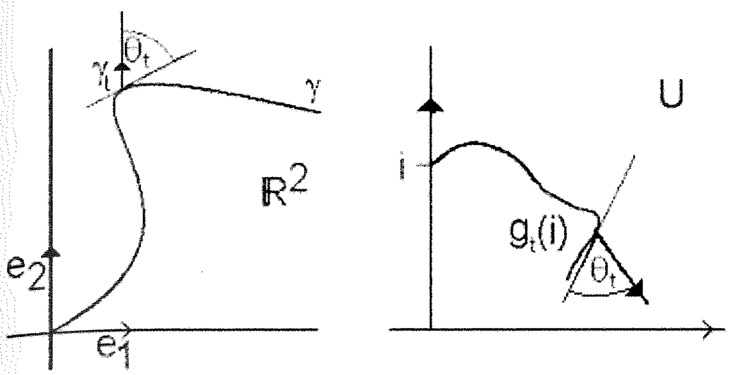
preserve the geodesic w_θ and to shift points on it through a distance λt.

Remark 65 *Note however that since $r_{-\theta} \in R$, $r_{-\theta}(i) = i$, so the image of the curves $q\left(g_t^{\theta,\lambda}\right) = q\left(r_\theta s_{\lambda t} r_{-\theta}\right)$ and $q\left(r_\theta s_{\lambda t}\right)$ are the same in $\mathfrak{U}$. The difference is described in figure 4.4.1 which shows that the orientation of the frames (denoted by the tangential arrow) at the point $q\left(g_t^{\theta,\lambda}\right)$ is correct for the left hand picture but wrongly aligned for the right hand picture.*

This demonstrates that it is necessary to take a little care to find the correct path in $\mathfrak{U}$ to describe the path development, which is really a path in the orthonormal frame bundle of $\mathfrak{U}$ (see [40]). A heuristic for thinking about the 3 dimensional lie group $PSL(2,\mathbb{R})$ as the collection of isometries of $\mathfrak{U}$, is as the set of rotations R which parameterize the orthonormal frames at $z = i$, with the translations to different points of $\mathfrak{U}$. The next proposition demonstrates the evaluation map is in fact correct for this purpose.

Proposition 66 *Let $\gamma \in \mathrm{P}(\mathbb{R}^2)$. The path $g_t(i) = q(g_t)$ where*

$$g_t = I \circ \phi_1\left(\mathbf{X}_{0,t}^{\infty}(\gamma)\right), \tag{4.15}$$



is the path of the development of γ into $\mathfrak{U}$ from the point $z = i$, where the tangent spaces at γ_0 and $z = i$ are identified according to 4.11 and figure 4.2.

Proof. Certainly $g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $g_0(i) = i$. Without loss of generality, assume that γ is parameterized at unit speed and denote by θ_t the angle that γ_t makes in a clockwise sense with e_2 (see figure 4.3).

Elements of $PSL(2, \mathbb{R})$ map geodesics to geodesics and are conformal maps, hence preserve angles. Being isometries, they preserve the metric so if for any $t \in [0, T_\gamma]$, the differential of the curve $g_t(i)$ is infinitesimally tangent to the geodesic $g_t(w_{\theta_t})$ then at all times, $g_t(i)$ is moving at an angle θ_t with respect to the frame defined by g_t and also at unit speed. In other words it is enough to check that the differential equation

$$\frac{dg_t\left(I_1\left(\exp\left(\delta \underline{v}_{\theta_t}\right)\right)(i)\right)}{d\delta} \mid_{\delta=0} = \frac{dg_{t+\delta}(i)}{d\delta} \mid_{\delta=0},$$

since $p_\delta = I_1\left(\exp\left(\delta \underline{v}_{\theta_t}\right)\right)(i)$ is a path moving along w_{θ_t} at unit speed. But $g_{t+\delta} = g_t \circ I_1\left(\mathbf{X}^{\infty}_{t,t+\delta}(\gamma)\right)$ and

$$\frac{d}{d\delta}\left(I_1\left(\mathbf{X}^{\infty}_{t,t+\delta}(\gamma)\right)\right) \mid_{\delta=0} = I_1\left(\underline{v}_{\theta_t}\right)$$

$$= \frac{d}{d\delta} \left(I_1 \left(\exp \left(\delta \underline{v}_{\theta_t} \right) \right) \right) |_{\delta=0} ,$$

by definition of θ_t, hence the equality holds. (c.f. Lemma 60) The curve $g_t(i)$ thus corresponds to the curve γ rolled onto $\mathfrak{U}$ without any slipping, i.e. it is the development of γ into $\mathfrak{U}$. ■

As a corollary to the proposition, there is a description of the development of scaled paths.

Corollary 67 *If $\gamma \in \mathrm{P}(\mathbb{R}^2)$, then scaling $\mathbb{R}^2$ by a constant factor $\lambda > 0$ produces a curve in $\mathbb{R}^2$, which developed into $\mathfrak{U}$ follows the curve $g_t^\lambda(i)$ where $g_t^\lambda = I \circ \phi_\lambda \left(\mathbf{X}_{0,t}^\infty(\gamma)\right)$.*

Proof. This is immediate from the proposition above and the definition of ϕ_λ 4.12. ■

Developing a C^2 path in Hyperbolic Space

Having established how to develop paths from $\mathbb{R}^2$ into $\mathfrak{U}$, the aim now is to investigate the following effect. Developing a path into negatively curved spaces, has the effect of straightening the path because the curvature offers resistance to the path curving back on itself. As the strength of the curvature increases, the resistance is more successful at inhibiting the curving effect.

Suppose one has a C^2 path γ in $\mathbb{R}^2$ of finite length $l(\gamma)$ starting from the origin. There are various ways to parameterize the curve but think of a pedal parameterization, in other words, move along the curve at unit speed and let θ_t be the angle that a tangent line to the curve at time t, makes in a clockwise sense with the e_2 vector as in figure 4.3. The path $\theta_\cdot : [0, l(\gamma)] \to [0, 2\pi)$ parameterizes γ:

$$\gamma_t = \int_0^t \left(\sin(\theta_s) e_1 + \cos(\theta_s) e_2\right) ds,$$

as does knowledge of θ_0 and $\dot{\theta}_t = \ddot{x}_t \dot{y}_t - \ddot{y}_t \dot{x}_t$ the curvature at γ_t. This is the parameterization used in what follows.

The next proposition describes curves of constant curvature in $\mathfrak{U}$.

Proposition 68 *In $\mathfrak{U}$, curves with constant curvature $\dot{\theta}$ are circles or segments of circles whose center lies in $\mathfrak{U} \cup \mathbb{R}$. If $\left|\dot{\theta}\right| > 1$ the circles do not touch the real axis, if $\left|\dot{\theta}\right| = 1$ the circles touch the real axis once and if $\left|\dot{\theta}\right| < 1$ the circles intersect the real axis twice. In the latter case, the angle $\alpha\left(\dot{\theta}\right)$ which is exterior to the circle and between the circle and the real axis is an invariant of the curvature. It is given by $\cos\left(\alpha\left(\dot{\theta}\right)\right) = \left|\dot{\theta}\right|$ so that $0 \leq \left|\alpha\left(\dot{\theta}\right)\right| \leq \pi/2$ and the center of the circle is in $\mathfrak{U}$.*

Proof. Take the following differential equation in terms of $\dot{\theta}$ for an element of $PSL(2,\mathbb{R})$:

$$dh_t^{\dot{\theta}} = h_t^{\dot{\theta}} \circ \frac{dt}{2} \begin{pmatrix} 1 & -\dot{\theta} \\ \dot{\theta} & -1 \end{pmatrix}. \tag{4.16}$$

The solution for $h^{\dot{\theta}}_{\cdot} : [0,T] \to PSL(2,\mathbb{R})$, is $h_t^{\dot{\theta}} = \exp\left(\frac{t}{2}\begin{pmatrix} 1 & -\dot{\theta} \\ \dot{\theta} & -1 \end{pmatrix}\right)$. The curve $z_t^{\dot{\theta}} = h_t^{\dot{\theta}}(i)$ has the form:

$$z_t^{\dot{\theta}} = \frac{\dot{\theta}\left(\cosh\left(t\sqrt{1-\dot{\theta}^2}\right) - 1\right) + i\left(1-\dot{\theta}^2\right)}{\left(\cosh\left(t\sqrt{1-\dot{\theta}^2}\right) - \dot{\theta}^2 - \sqrt{1-\dot{\theta}^2}\sinh\left(t\sqrt{1-\dot{\theta}^2}\right)\right)} \tag{4.17}$$

and after computation, $z_t^{\dot{\theta}}$ satisfies $\left|z_t^{\dot{\theta}} - \left(\frac{1}{\dot{\theta}} + i\right)\right| = \frac{1}{\left|\dot{\theta}\right|}$.

In fact $z_t^{\dot{\theta}}$ is a curve of constant curvature : elements of $PSL(2,\mathbb{R})$ preserve the metric, hence since $h_{s+t}^{\dot{\theta}}(i) = h_s^{\dot{\theta}}\left(h_t^{\dot{\theta}}(i)\right)$, the curvature at any two points is the same. To check that this curve has curvature

$\dot{\theta}$, it is only necessary to compute the curvature at $t = 0$, i.e. to show that $\sqrt{\mu_0^{\dot{\theta}} g \mu_0^{\dot{\theta}t}} = \left|\dot{\theta}\right|$ and that the sense of curving matches that of the circle in $\mathbb{R}^2$ with curvature $\dot{\theta}$.

$$z_t = x_t + iy_t,$$

so the form (4.17) enables the evaluations

$$x_0 = 0 \quad \dot{x}_0 = 0 \quad \ddot{x}_0 = \dot{\theta}$$

$$y_0 = 1 \quad \dot{y}_0 = 1 \quad \ddot{y}_0 = 1$$

and the calculation $\mu_0 = \left(\dot{\theta}, 0\right)$ which gives $\sqrt{\mu_0 g \mu_0^t} = \left|\dot{\theta}\right|$. The fact that the center of the circle that z_t lies on is $\frac{1}{\dot{\theta}} + i$ ensures that the sense of curving is preserved.

In the case that $\left|\dot{\theta}\right| > 1$, the circle does not touch the real axis. If $\left|\dot{\theta}\right| \leq 1$ however, it does. Since elements of $PSL(2, \mathbb{R})$ are Mobius transformations of $\mathbb{C}$ which preserve angles between lines and fix the real axis, the exterior angle subtended with the real axis and a circle (when $\left|\dot{\theta}\right| \leq 1$), remains constant under the action of $PSL(2, \mathbb{R})$. The circle given by z_t above subtends the angle α with the real axis, $\cos\left(\alpha\left(\dot{\theta}\right)\right) = \left|\dot{\theta}\right|$ as $t \to \infty$, clearly only when $\left|\dot{\theta}\right| \leq 1$, hence this is an invariant for when $\left|\dot{\theta}\right| \leq 1$. ∎

Remark 69 *It is interesting to tie up why a different form to (4.15) for the evolution of the C^2 curve γ comes from the proposition. The form of (4.15) is*

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$dg_t = g_t \circ I_1 (d\gamma_t)$$

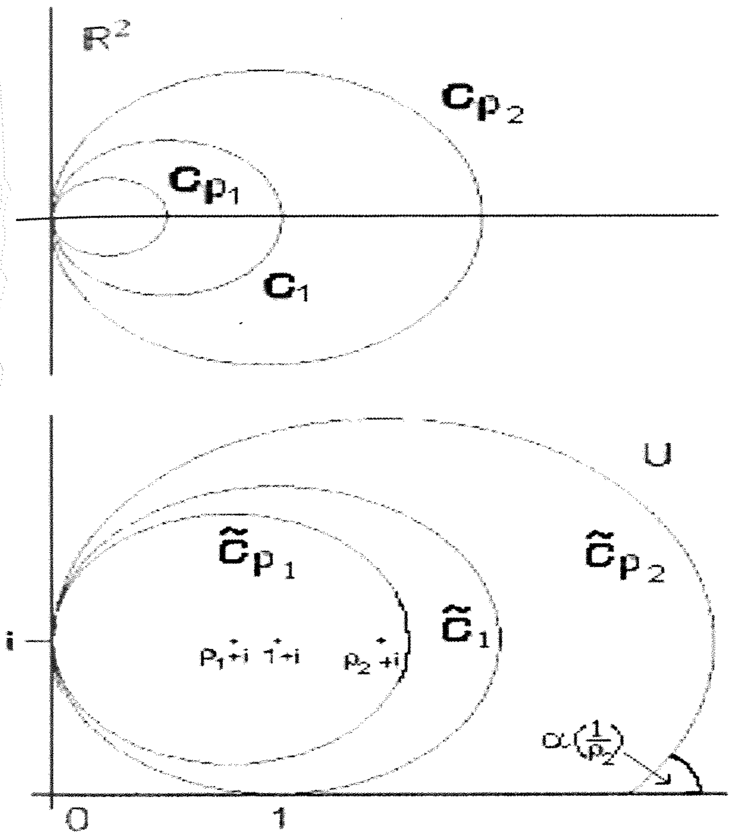


Figure 4.3: **3 circles in $\mathbb{R}^2$ are developed into $\mathfrak{U}$, C_{ρ_1},C_1 and $C_{\rho_2}$of radii $\rho_1 < 1$, 1 and $\rho_2 > 1$ respectively, have images $\widetilde{C}_{\rho_1}$,$\widetilde{C}_1$ and $\widetilde{C}_{\rho_2}$. The latter make angle $\alpha\left(\frac{1}{\rho_2}\right)$ with the real axis and satisfies $\cos\left(\alpha\left(\frac{1}{\rho_2}\right)\right) = \frac{1}{\rho_2}$.**

and hence $g_t(i)$ is the developed curve, whereas the form of (4.16) is

$$h_0 = \begin{pmatrix} \cos(\theta_0/2) & -\sin(\theta_0/2) \\ \sin(\theta_0/2) & \cos(\theta_0/2) \end{pmatrix}$$

$$dh_t = h_t \circ \frac{dt}{2} \begin{pmatrix} 1 & -\dot{\theta}_t \\ \dot{\theta}_t & -1 \end{pmatrix}$$

and $h_t(i)$ is also the developed curve. The relationship between the two curves is mentioned in remark 65 and boils down to a difference in frames. In particular, $g_t^{-1}h_t = \begin{pmatrix} \cos(\theta_t/2) & -\sin(\theta_t/2) \\ \sin(\theta_t/2) & \cos(\theta_t/2) \end{pmatrix} = r_{\theta_t} \in R$ *which reflects how 4.14 leads to the differential equation 4.16: the initial point for h_0 is taken care of but the final adjusting rotation is not.*

The next lemma uses the knowledge of curves of constant curvature in $\mathfrak{U}$ to describe a bounded set in $\mathfrak{U}$ of points through which curves of low bounded curvature can pass through.

Lemma 70 *Let $0 \leq \kappa \leq 1$ be a bound on the curvature of a C^2 curve γ in $\mathbb{R}^2$ which moves in direction $-e_2$ at $t = 0$. Define $\mathfrak{U}_\kappa(z, \beta) \subset \mathfrak{U}$ to be the closed set that is bordered by the real axis and the pair of circles centered which have constant curvature κ in $\mathfrak{U}$ and pass through the point z making a tangential angle of β with the imaginary axis (in a downwards sense), so that $\mathfrak{U}_\kappa(z, \beta)$ is bounded by the circles centered at $\pm\kappa^{-1} + i$ of radius $\kappa^{-1}$and includes the points μi, $0 \leq \mu \leq 1$. Then, if γ is developed into $\mathfrak{U}$ from z at the initial angle β according to the identification ϕ_1 (4.11), the image of the development lies in $\mathfrak{U}_\kappa(z, \beta)$. The two bounding curves themselves, parameterized by unit speed, describe the points which minimise the hyperbolic distance from $z = i$ for curves developed for the same amount of time under the*

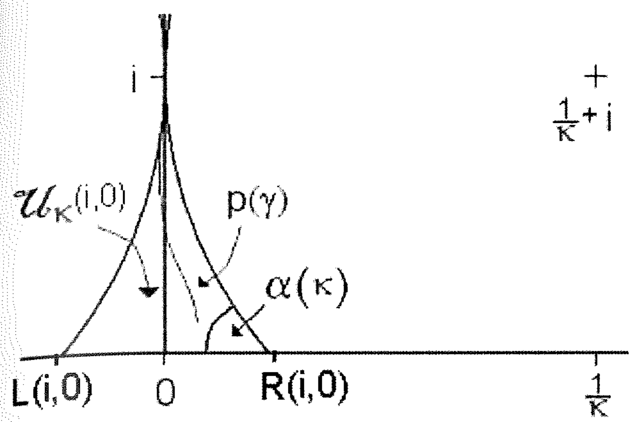


Figure 4.4: **A description of $\mathfrak{U}_\kappa(i,0)$ and a curve γ developed to path $p(\gamma)$.**

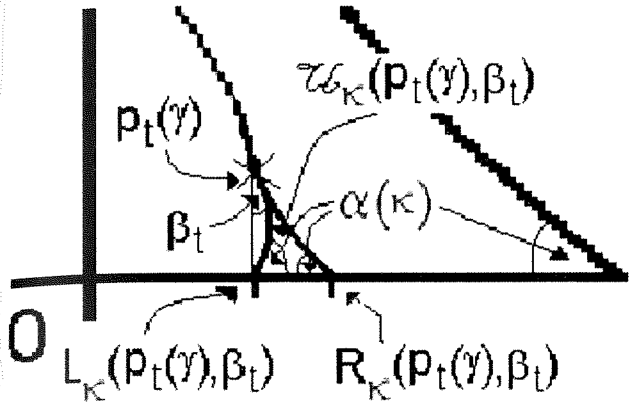
bounded curvature condition. This distance $\rho_\kappa(t)$ is given by

$$\cosh(\rho_\kappa(t)) = \frac{\cosh\left(t\sqrt{1-\kappa^2}\right)-\kappa^2}{1-\kappa^2}.$$

Proof. When $\kappa \leq 1$ the previous proposition states that a curve of constant curvature $\dot{\theta} = \kappa$ intersects the real axis at an angle $\alpha(\kappa)$ where $\cos(\alpha(\kappa)) = \kappa$. The development of γ, $p_t(\gamma) \in \mathfrak{U}$ satisfies $p_0(\gamma) = i$ and $\dot{p}_0(\gamma) = -1$, remaining in $\mathfrak{U}_\kappa(i,0)$.

To show that $p_t(\gamma) = x_t(\gamma) + iy_t(\gamma) \in \mathfrak{U}_\kappa(i,0)\ \forall t \in [0, T_\gamma]$, consider the two points $L_\kappa(z,\beta) \leq R_\kappa(z,\beta)$ where $z \in \mathfrak{U}_M$ and $\beta \in [-\pi,\pi]$ is an angle representing a tangent direction for curves at z, the angle being measured anti-clockwise from the imaginary axis towards zero. $L_\kappa(z,\beta)$ and $R_\kappa(z,\beta)$ denote the left and right intersection points with the real axis, of curves of maximal curvature developed from the point z at the angle β. Simple trigonometry implies

$$\begin{aligned} R_\kappa(x+iy,\beta) &= x + \frac{y(\cos(\beta)-\sin(\alpha(\kappa)))}{(\cos(\alpha(\kappa))-\sin(\beta))} \\ L_\kappa(x+iy,\beta) &= x - \frac{y(\cos(\beta)-\sin(\alpha_\kappa))}{(\cos(\alpha(\kappa))+\sin(\beta))}. \end{aligned}$$



Using the fact that $\cos(\alpha(\varkappa)) = \varkappa$ for a curve developed with constant curvature $\varkappa$, the angle $\beta_t = \beta_t(\gamma)$ satisfies the differential equation

$$\dot{\beta}_t = \left(\dot{\theta}_t - \sin(\beta_t)\right).$$

With κ fixed, the aim is to show that the functions

$$\begin{aligned} R_t &= R_\kappa(p_t(\gamma), \beta_t) \\ L_t &= L_\kappa(p_t(\gamma), \beta_t), \end{aligned}$$

respectively satisfy

$$\begin{aligned} \left(\dot{R}_t \leq 0\right) &\text{ with equality } &\Leftrightarrow \quad \left(\dot{\theta}_t = \kappa\right) \\ \left(\dot{L}_t \geq 0\right) &\text{ with equality } &\Leftrightarrow \quad \left(\dot{\theta}_t = -\kappa\right), \end{aligned}$$

Figure 4.4.1 represents how the region $\mathfrak{U}_\kappa(p_t(\gamma), \beta_\gamma)$ and the functions R_t and L_t, depend on the point $p_t(\gamma)$ and the angle β_t.

Take the calculation for R_t, L_t being a similar computation:

$$\dot{R}_t = \dot{x}_t(\gamma) + \frac{\dot{y}_t(\gamma)(\cos(\beta_t) - \sin(\alpha(\kappa)))}{(\cos(\alpha(\kappa)) - \sin(\beta_t))} + \frac{\dot{\beta}_t y_t(\gamma)(1 - \sin(\alpha(\kappa) + \beta_t))}{(\cos(\alpha(\kappa)) - \sin(\beta_t))^2}.$$

From the metric $\dot{y}_t(\gamma)^2 + \dot{x}_t(\gamma)^2 = y_t(\gamma)^2$ and hence $\dot{x}_t(\gamma) = y_t(\gamma)\sin(\beta_t)$, $\dot{y}_t(\gamma) = -y_t(\gamma)\cos(\beta_t)$ which implies

$$\begin{aligned}
\dot{R}_t &= y_t(\gamma)\left\{\begin{array}{c} \sin(\beta_t) + \frac{-\cos(\beta_t)(\cos(\beta_t)-\sin(\alpha(\kappa)))}{(\cos(\alpha(\kappa))-\sin(\beta_t))} \\ + \frac{(\dot{\theta}_t - \sin(\beta_t))(1-\sin(\alpha(\kappa))\cos(\beta_t)-\cos(\alpha(\kappa))\sin(\beta_t))}{(\cos(\alpha(\kappa))-\sin(\beta_t))^2} \end{array}\right\} \\
&= \frac{y_t(\gamma)\left(\dot{\theta}_t - \cos(\alpha(\kappa))\right)(1 - \sin(\alpha(\kappa) + \beta_t))}{(\cos(\alpha(\kappa)) - \sin(\beta_t))^2} \leq 0.
\end{aligned}$$

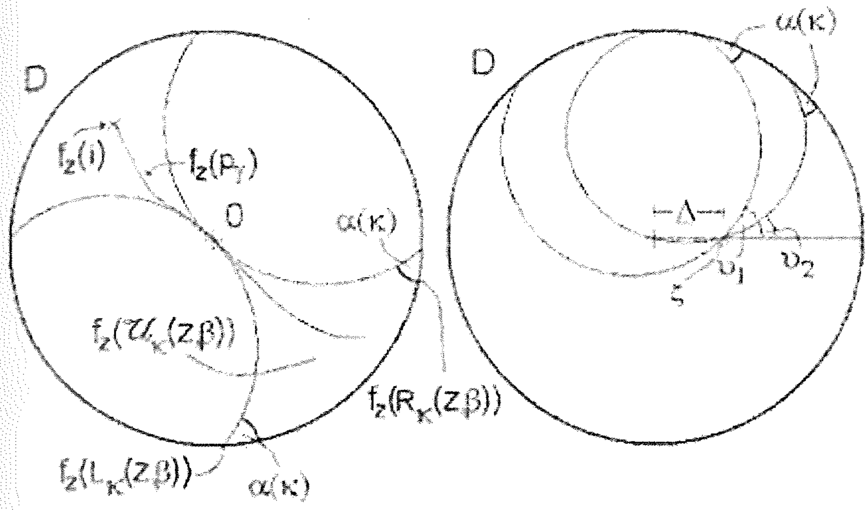
From a similar calculation

$$\dot{L}_t = \frac{y_t(\gamma)\left(\dot{\theta}_t + \cos(\alpha(\kappa))\right)(1 - \sin(\alpha(\kappa) - \beta_t))}{(\cos(\alpha(\kappa)) + \sin(\beta_t))^2} \geq 0,$$

since $-\kappa = -\cos(\alpha(\kappa)) \leq \dot{\theta}_t \leq \cos(\alpha(\kappa)) = \kappa$.

Therefore, when $\kappa \leq 1$, a developed path $p_{\cdot}(\gamma)$ cannot move out of $\mathfrak{U}_\kappa(i, 0)$: if one did, then clearly one of the functions L_t or R_t would decrease or increase respectively at some time, which from the differential inequalities is impossible. So indeed, the developed path from a point z at an angle β as described with curvature bounded by $\kappa \leq 1$, will lie in the region $\mathfrak{U}_\kappa(z, \beta)$.

It remains to be proved that for a given time/length t, out of all possible C^2 curves, the minimising distance from the starting point after time t is given by travelling along a curve of maximal curvature. For this, consider the unit disc D as a model for hyperbolic space as it is easier to understand the metric. Precisely the same characterization is true of curves of a given curvature $\dot{\theta}$ going through any point in D, namely they subtend the angle $\alpha\left(\left|\dot{\theta}\right|\right)$ at the boundary (the transformation of $\mathfrak{U}$ to D is conformal). In



the left part of Figure 4.4.1, the region $\mathfrak{U}_\kappa(z,\beta)$ has been mapped conformally from $\mathfrak{U}$ to $f_z(\mathfrak{U}_\kappa(z,\beta)) \subset D$ where f_z is one of the Mobius transformations for which $f_z(z) = 0$.

For a point $\varsigma \in D$, the hyperbolic distance δ from the origin to ς is given by $|\varsigma| = \tanh(\delta)$. A contradiction is obtained if it is supposed that there is a C^2 curve which develops into D so that at time t it is closer to the origin than for the curve of constant maximal curvature. For a given point of a development $\varsigma \in D$ with $\Delta = |\varsigma|$, there is a set of feasible angles that can be subtended with the radial line from the origin. By symmetry, these angles are characterized as precisely those which permit development back to the origin by a path of curvature bounded by κ. That is, the image of $\mathfrak{U}_\kappa$ in D for development from z initially at the given angle, must contain the origin.

In figure 4.4.1, the right hand disc shows how at a point ς, it is not feasible that a developed path can pass through ς at angle υ_1, since the previous part of the development is restricted from the origin, whereas υ_2 is the critical angle of feasibility.

For $t \in [0, T_\gamma]$, let

$$\begin{aligned} \varsigma_t &= f_i(p_t(\gamma)) \\ \Delta_t &= |\varsigma_t| \\ \delta_t &= \tanh^{-1}(\Delta_t), \end{aligned}$$

so that δ_t is the hyperbolic distance between the start and endpoint of the development $p_t(\gamma)$.

Consider a path $\varrho_{\kappa,\cdot} : [0,\infty) \to D$ of unit speed and constant curvature κ with $\varrho_{\kappa,0} = 0$. Then $|\varrho_{\kappa,\tau}| \nearrow 1$ as $\tau \to \infty$. Take $t \in [0, T_\gamma]$ and let $\tau(t)$ be the unique value such that

$$\left|\varrho_{\kappa,\tau(t)}\right| = \Delta_t.$$

Then

$$\dot{\Delta}_t \geq \frac{\partial |\varrho_{\kappa,\tau}|}{\partial \tau}|_{\tau=\tau(t)},$$

since if not, the angle at ς_t on the developed curve made with the straight line from the origin is greater than that which the curve $\varrho_{\kappa,\tau}$ makes at $\tau = \tau(t)$, which means that this angle is infeasible for the development ς_t to have come from the origin. This implies $\tau(t) \geq t$ and hence that $|\varrho_{\kappa,t}| \leq \Delta_t \; \forall t \in [0, T_\gamma]$. That is

$$|\varrho_{\kappa,t}| \leq |f_i(p_t(\gamma))| = \tanh(\delta_t) \;\; \forall t \in [0, T_\gamma].$$

Hence to calculate the minimum distance $\rho_\kappa(t) = \tanh^{-1}(|\varrho_{\kappa,t}|)$ using the form (4.9), it suffices (according to (4.17)) to evaluate and simplify the equation to produce the desired form:

$$\begin{aligned}
&\cosh\left(\rho_\kappa(t)\right) \\
=\ & 1+\frac{\left|z^\kappa_{-t}-i\right|^2}{2\Im\left(z^\kappa_{-t}\right)\Im(i)} \\
=\ & 1+\frac{\left\{\begin{array}{c}\left(\kappa\left(\cosh\left(t\sqrt{1-\kappa^2}\right)-1\right)\right)^2+ \\ \left(1-\cosh\left(t\sqrt{1-\kappa^2}\right)+\sqrt{1-\kappa^2}\sinh\left(t\sqrt{1-\kappa^2}\right)\right)^2\end{array}\right\}}{2\left(\cosh\left(t\sqrt{1-\kappa^2}\right)-\kappa^2-\sqrt{1-\kappa^2}\sinh\left(t\sqrt{1-\kappa^2}\right)\right)} \\
=\ & 1+\frac{\cosh\left(t\sqrt{1-\kappa^2}\right)-1}{1-\kappa^2} \\
=\ & \frac{\cosh\left(t\sqrt{1-\kappa^2}\right)-\kappa^2}{1-\kappa^2}.
\end{aligned}$$

■

4.4.2 Relating motion in $\mathbb{R}^2$, $PSL(2,\mathbb{R})$ and $\mathfrak{U}$

The aim of this section is to connect together the understanding of motion of a twice differentiable path in $\mathbb{R}^2$, with the motion in $PSL(2,\mathbb{R})$ given by (4.12) and (4.13) on the one hand and the development of the path into $\mathfrak{U}$ on the other. The result is that the iterated integral sequence can be used to find the length of the path.

Take a C^2 path in $\mathbb{R}^2$ of finite length parameterized at unit speed, $\gamma_\cdot : [0, l(\gamma)] \to \mathbb{R}^2$ with $\gamma_0 = 0$. Use the orientation of figure 4.3 and the description

$$\gamma_t = x_t(\gamma)\,e_1 + y_t(\gamma)\,e_2.$$

Then $\tan(\theta_t) = \frac{\dot{x}_t(\gamma)}{\dot{y}_t(\gamma)}$ and the curvature $\dot{\theta}_t(\gamma)$ has the form

$$\dot{\theta}_t(\gamma) = \ddot{x}_t(\gamma)\,\dot{y}_t(\gamma) - \ddot{y}_t(\gamma)\,\dot{x}_t(\gamma)$$

and is bounded in magnitude since $l(\gamma) < \infty$. Set

$$\kappa_\gamma = \inf\left\{\kappa \middle| \quad \kappa \geq \left|\dot{\theta}_t(\gamma)\right|, 0 \leq t \leq l(\gamma)\right\}.$$

Define the set of scaled paths $\gamma^\lambda : [0, \lambda l(\gamma)] \to \mathbb{R}^2$ by

$$\gamma_t^\lambda = \lambda \gamma_{t/\lambda}, \tag{4.18}$$

so that

$$\begin{aligned} l\left(\gamma^\lambda\right) &= \lambda l(\gamma) \\ \dot{\theta}_t\left(\gamma^\lambda\right) &= \lambda^{-1}\dot{\theta}_{t/\lambda}(\gamma) \end{aligned}$$

and thus

$$\kappa_{\gamma^\lambda} = \lambda_\gamma^{-1}\kappa_\gamma.$$

Proposition 71 *Let γ be a C^2 path of length $0 < l(\gamma) < \infty$ in $\mathbb{R}^2$ and γ^λ be the paths defined in equation (4.18). Define d_γ^λ to be the hyperbolic distance between the start and the endpoint of the development of the path γ^λ. If $\lambda \geq \kappa_\gamma$, then*

$$l(\gamma)\sqrt{1 - \left(\frac{\kappa_\gamma}{\lambda}\right)} \leq \frac{d_\gamma^\lambda}{\lambda} \leq l(\gamma),$$

therefore

$$l(\gamma) = \lim_{\lambda \to \infty} \frac{d_\gamma^\lambda}{\lambda}.$$

Proof. We may assume that γ is parameterized by unit speed. Since $\kappa_\gamma < \infty$, when $\lambda \geq \kappa_\gamma$, $\kappa_{\gamma^\lambda} = \frac{\kappa_\gamma}{\lambda} \leq 1$. By rotational symmetry, we may assume that $\dot{\gamma}_0^\lambda = -e_2$ so that lemma (70) applies. This gives the lower bound for the distance d_γ^λ, of

$$\rho_{\kappa_\gamma \lambda}\left(\lambda l\left(\gamma\right)\right) \leq d_\gamma^\lambda \leq \lambda l\left(\gamma\right),$$

where

$$\begin{aligned}\cosh\left(\rho_{\kappa_\gamma \lambda}\left(\lambda l\left(\gamma\right)\right)\right) &= \frac{\cosh\left(\lambda l\left(\gamma\right)\sqrt{1-\left(\frac{\kappa_\gamma}{\lambda}\right)^2}\right)-\left(\frac{\kappa_\gamma}{\lambda}\right)^2}{1-\left(\frac{\kappa_\gamma}{\lambda}\right)^2} \\ &\geq \cosh\left(\lambda l\left(\gamma\right)\sqrt{1-\left(\frac{\kappa_\gamma}{\lambda}\right)^2}\right)\end{aligned}$$

and the upper bound is due to the length of the developed path being $\lambda l\left(\gamma\right)$. Therefore for $\lambda \geq \kappa_\gamma$

$$l\left(\gamma\right)\sqrt{1-\left(\frac{\kappa_\gamma}{\lambda}\right)^2} \leq \frac{d_\gamma^\lambda}{\lambda} \leq l\left(\gamma\right), \tag{4.19}$$

as required. The final statement of the proposition follows on. ■

In order to relate d_γ^λ to the iterated integral sequence $\mathbf{X}_{0,l(\gamma)}^\infty\left(\gamma\right)$, consider a general element g of $PSL\left(2,\mathbb{R}\right)$. Then there is the following form for the distance between i and $g\left(i\right)$:

Lemma 72 *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}\left(2,\mathbb{R}\right)$. *The hyperbolic distance between* i *and* $g\left(i\right)$, $\rho\left(i,g\left(i\right)\right)$ *has the form :*

$$\rho\left(i,g\left(i\right)\right) = \log\left(\alpha+\sqrt{\alpha^2-1}\right), \tag{4.20}$$

where

$$\alpha = \frac{\left(a^2+b^2+c^2+d^2\right)}{2}. \tag{4.21}$$

Proof. Set $z = i$ and $w = \frac{ai+b}{ci+d} = \frac{(ac+bd)+i}{c^2+d^2}$ in equation (4.9) to obtain

$$\begin{aligned}
\cosh\rho(z,w) &= 1 + \frac{(bd+ac)^2 + \left(1-\left(c^2+d^2\right)\right)^2}{2\left(c^2+d^2\right)} \\
&= \frac{(bd+ac)^2 + 1 + \left(c^2+d^2\right)^2}{2\left(c^2+d^2\right)} \\
&= \frac{a^2c^2 + b^2d^2 + 2abcd + 1 + \left(c^2+d^2\right)^2}{2\left(c^2+d^2\right)} \\
&= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2 + \left(c^2+d^2\right)^2}{2\left(c^2+d^2\right)} \\
&= \frac{\left(a^2+b^2+c^2+d^2\right)}{2},
\end{aligned}$$

to which end, the given form for $\rho(i, g(i)) \geq 0$ is obtained. ■

Remark 73 *In the context of $\mathfrak{U}$ being a symmetric space and with reference to [30], the following proposition is more widely applicable as an interpretation of the distance between the quotient of the identity and the quotient of a group element g. Without saying any more, it concurs with the formula (4.20).*

Proposition 74 *The distance formula (4.20) also has the form*

$$\rho(e,g) = \sqrt{\frac{tr\left(\left(\log\left(gg^t\right)\right)^2\right)}{2}}.$$

Proof. Take the decomposition

$$\begin{aligned}
gg^t &= \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} \\
&= U \begin{pmatrix} \alpha+\sqrt{\alpha^2-1} & 0 \\ 0 & \alpha-\sqrt{\alpha^2-1} \end{pmatrix} U^{-1},
\end{aligned}$$

in terms of the eigenvalues of the symmetric matrix gg^t, where U is an orthogonal matrix and α is given by (4.21).Then

$$\alpha - \sqrt{\alpha^2 - 1} = \frac{1}{\alpha + \sqrt{\alpha^2 - 1}}$$

and so

$$\begin{aligned} tr\left(\left(\log\left(gg^t\right)\right)^2\right) &= tr\left(\begin{pmatrix} \log\left(\alpha + \sqrt{\alpha^2-1}\right) & 0 \\ 0 & \log\left(\alpha - \sqrt{\alpha^2-1}\right) \end{pmatrix}^2\right) \\ &= 2\log\left(\alpha + \sqrt{\alpha^2-1}\right)^2, \end{aligned}$$

which proves the proposition. ■

Remark 75 *In this example, it is possible to see that evaluating a path γ at it's endpoint to an element $g \in G$ say and taking its transpose matrix g^t relate the transpose paths of remark 48: let γ^t be the transpose of γ, then g^t is the evaluation of γ^t. An interesting problem may be to see how these actions on paths of evaluation and transposition are generally related for the setting of symmetric spaces.*

As a corollary to the previous proposition and lemma, there is a form for the length of a C^2 path in terms of the iterated integral sequence:

Corollary 76 *Let γ be C^2 path of length $0 < l(\gamma) < \infty$ in $\mathbb{R}^2$ and set*

$$\begin{aligned} g_\gamma^\lambda &= I \circ \phi_\lambda \left(\mathbf{X}_{0,l(\gamma)}^\infty (\gamma) \right) \\ &= \sum_{j=0}^\infty \lambda^j I \circ \phi_1 \left(\mathbf{X}_{0,l(\gamma)}^{(j)} (\gamma) \right) \\ &= \begin{pmatrix} a_\gamma^\lambda & b_\gamma^\lambda \\ c_\gamma^\lambda & d_\gamma^\lambda \end{pmatrix}, \end{aligned} \tag{4.22}$$

where $I \circ \phi_\lambda$ is defined by (4.13) and (4.12). Then $\rho\left(i, g_\gamma^\lambda (i)\right) = d_\gamma^\lambda$ and $l(\gamma)$,the length of γ satisfies

$$l(\gamma) = \lim_{\lambda \to \infty} \frac{\log \left(\left(a_\gamma^\lambda\right)^2 + \left(b_\gamma^\lambda\right)^2 + \left(c_\gamma^\lambda\right)^2 + \left(d_\gamma^\lambda\right)^2 \right)}{\lambda}.$$

If $\lambda \geq \kappa_\gamma$ then

$$\exp\left(\lambda l(\gamma) \sqrt{1 - \left(\frac{\kappa_\gamma}{\lambda}\right)^2} \right) \leq \alpha_\gamma^\lambda + \sqrt{\left(\alpha_\gamma^\lambda\right)^2 - 1} \leq \exp\left(\lambda l(\gamma)\right),$$

where

$$\alpha_\gamma^\lambda = \frac{\left(a_\gamma^\lambda\right)^2 + \left(b_\gamma^\lambda\right)^2 + \left(c_\gamma^\lambda\right)^2 + \left(d_\gamma^\lambda\right)^2}{2}.$$

Proof. d_γ^λ is the distance between the point i and $g_\gamma^\lambda (i)$, the endpoint of the development of γ into $\mathfrak{U}$ so the lemma (72) implies

$$d_\gamma^\lambda = \log\left(\alpha_\gamma^\lambda + \sqrt{\left(\alpha_\gamma^\lambda\right)^2 - 1} \right).$$

Then proposition 71 implies that as $\lambda \to \infty$, $d_\gamma^\lambda \to \infty$, so $\alpha_\gamma^\lambda \to \infty$ also. Proposition 71 also states

$$\begin{aligned} l(\gamma) &= \lim_{\lambda \to \infty} \frac{d_\gamma^\lambda}{\lambda} \\ &= \lim_{\lambda \to \infty} \frac{\log\left(\alpha_\gamma^\lambda + \sqrt{\left(\alpha_\gamma^\lambda\right)^2 - 1} \right)}{\lambda} \\ &= \lim_{\lambda \to \infty} \frac{\log\left(2\alpha_\gamma^\lambda\right)}{\lambda}, \end{aligned}$$

as required and the inequality follows from equation (4.19). ■

4.4.3 Evaluating $I \circ \phi_\lambda$ for an iterated integral sequence

Having seen that $l(\gamma)$ is a function of the full iterated sequence if γ is a C^2 path, it remains to show how to evaluate the form of $g_\gamma^\lambda = \begin{pmatrix} a_\gamma^\lambda & b_\gamma^\lambda \\ c_\gamma^\lambda & d_\gamma^\lambda \end{pmatrix}$. The following computation indicates how to make the evaluation for any element of the tensor algebra $T(\mathbb{R}^2)$ and in particular for $\gamma \in \mathcal{P}(\mathbb{R}^2)$. It is not the most revealing computation.

With the choice of orthonormal basis of $\mathbb{R}^2$, e_1, e_2, a basis of $T(\mathbb{R}^2)$ can be constructed from the collection of all 'words' in e_1, e_2, i.e. tensors of the form

$$W_{\underline{i}} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n},$$

with $\underline{i} = \{i_1, i_2, \ldots, i_n\} \in \{1,2\}^n$, $n \geq 0$. A basis for $(\mathbb{R}^2)^{\otimes n}$ is given by the set of tensors

$$W^n = \{W_{\underline{i}} | \quad \underline{i} \in \{1,2\}^n\}.$$

Proposition 77 *If $W_{\underline{i}} \in W^n$, then $W_{\underline{i}}$ has an unique minimal representation $W_{\underline{i}} = e_1^{\otimes j_1} \otimes e_2^{\otimes k_1} \otimes \cdots \otimes e_2^{\otimes k_m}$ where $j_h, k_h > 0$. Set*

$$\sigma(\underline{i}) = \sum_{m_1 < m_2} j_{m_1} k_{m_2}$$

and $\|\underline{j}\| = \sum_{p=1}^m j_p$, $\|\underline{k}\| = \sum_{p=1}^m k_p$. Then with $I \circ \phi_1$ defined by (4.13) and (4.12)

$$I \circ \phi_1(W_{\underline{i}}) = (-1)^{\sigma(\underline{i})} I_1\left(e_1^{\|\underline{j}\|}\right) I_1\left(e_2^{\|\underline{k}\|}\right)$$

and hence

$$
\begin{aligned}
& I \circ \phi_1 \left(W_{\underline{i}}\right) \\
= \; & (-1)^{\sigma(\underline{i})} \left(\frac{1}{2}\right)^{\|\underline{j}\|+\|\underline{k}\|} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{if } \|\underline{j}\| \textit{ is even, } \|\underline{k}\| \textit{ is even} \\
= \; & (-1)^{\sigma(\underline{i})} \left(\frac{1}{2}\right)^{\|\underline{j}\|+\|\underline{k}\|} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{if } \|\underline{j}\| \textit{ is even, } \|\underline{k}\| \textit{ is odd} \\
= \; & (-1)^{\sigma(\underline{i})} \left(\frac{1}{2}\right)^{\|\underline{j}\|+\|\underline{k}\|} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \textit{if } \|\underline{j}\| \textit{ is odd, } \|\underline{k}\| \textit{ is odd} \\
= \; & (-1)^{\sigma(\underline{i})} \left(\frac{1}{2}\right)^{\|\underline{j}\|+\|\underline{k}\|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{if } \|\underline{j}\| \textit{ is odd, } \|\underline{k}\| \textit{ is even.}
\end{aligned}
$$

Proof. Since

$$
\begin{aligned}
I \circ \phi_1 \left(e_1 \otimes e_2\right) &= \frac{1}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= -I \circ \phi_1 \left(e_2 \otimes e_1\right),
\end{aligned}
$$

the general form $I \circ \phi_1 \left(e_2^{\otimes i} \otimes e_1^{\otimes j}\right) = (-1)^{ij} \, I \circ \phi_1 \left(e_1^{\otimes j} \otimes e_2^{\otimes i}\right)$ applies and induction implies that

$$
I \circ \phi_1 \left(W_{\underline{i}}\right) = (-1)^{\sigma(\underline{i})} \, I \circ \phi_1 \left(e_1^{\|\underline{j}\|}\right) I_1 \left(e_2^{\|\underline{k}\|}\right).
$$

Observe that

$$I \circ \phi_1 \left(e_1^{\otimes 2}\right) = I \circ \phi_1 \left(e_2^{\otimes 2}\right) = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so the 4 identities above follow. ■

Take $\gamma \in \mathcal{P}_b\left(\mathbb{R}^2\right)$ with a representation $\gamma_t = \gamma_t^1 e_1 + \gamma_t^2 e_2$ to provide the following representation of the n'th iterated integral:

$$\mathbf{X}_{0,T_\gamma}^{(n)}(\gamma) = \sum_{W_{\underline{i}} \in W^n} \int \cdots \int_{0 \leq u_1 \leq \cdots \leq u_n \leq T_\gamma} d\gamma_{u_1}^{i_1} \cdots d\gamma_{u_n}^{i_n} W_{\underline{i}}.$$

W^n splits into two subsets according to whether the index set $\underline{i}$ satisfies $\left\|\underline{j}\right\| = \#\{m | i_m = 1\}$ is odd or even. The map I_1 acts on $W_{\underline{i}} \in W$ accordingly, so denote the following subsets:

Notation 78 *Define the subsets:*

$$W^{n,e} = \left\{W_{\underline{i}} \in W^n | \left\|\underline{j}\right\| \text{ is even}\right\}$$

$$W^{n,o} = \left\{W_{\underline{i}} \in W^n | \left\|\underline{j}\right\| \text{ is odd}\right\}.$$

Then the evaluation of the n'th iterated integral defines a collection of co-ordinates:

Proposition 79 *For $\gamma \in \mathcal{P}_b\left(\mathbb{R}^2\right)$ define the co-ordinates $\epsilon_e^n(\gamma)$,$\epsilon_o^n(\gamma)$:*

$$\epsilon_e^n(\gamma) = \sum_{W_{\underline{i}} \in W^{n,e}} \frac{(-1)^{\sigma(\underline{i})}}{2^n} \int \cdots \int_{0 \leq u_1 \leq \cdots \leq u_n \leq T_\gamma} d\gamma_{u_1}^{i_1} \cdots d\gamma_{u_n}^{i_n}$$

$$\epsilon_o^n(\gamma) = \sum_{W_{\underline{i}} \in W^{n,o}} \frac{(-1)^{\sigma(\underline{i})}}{2^n} \int \cdots \int_{0 \leq u_1 \leq \cdots \leq u_n \leq T_\gamma} d\gamma_{u_1}^{i_1} \cdots d\gamma_{u_n}^{i_n}.$$

Then the iterated integrals satisfy

$$\begin{aligned}
I\circ\phi_\lambda\left(\mathbf{X}_{0,T_\gamma}^{(2n)}(\gamma)\right) &= \lambda^{2n}\begin{pmatrix} \epsilon_e^{2n}(\gamma) & -\epsilon_o^{2n}(\gamma) \\ \\ \epsilon_o^{2n}(\gamma) & \epsilon_e^{2n}(\gamma)\end{pmatrix} \\
I\circ\phi_\lambda\left(\mathbf{X}_{0,T_\gamma}^{(2n+1)}(\gamma)\right) &= \lambda^{2n+1}\begin{pmatrix} \epsilon_e^{2n+1}(\gamma) & \epsilon_o^{2n+1}(\gamma) \\ \\ \epsilon_o^{2n+1}(\gamma) & -\epsilon_e^{2n+1}(\gamma)\end{pmatrix}.
\end{aligned}$$

Proof. The proof is immediate from Proposition (77). ■

Corollary 80 *For $\gamma\in\mathcal{P}_b\left(\mathbb{R}^2\right)$, $g_\gamma^\lambda = I\circ\phi_\lambda\left(\mathbf{X}_{0,T_\gamma}^{\infty}(\gamma)\right)$ has the form*

$$g_\gamma^\lambda = \begin{pmatrix} e_e(\lambda,\gamma)+o_e(\lambda,\gamma) & -e_o(\lambda,\gamma)+o_o(\lambda,\gamma) \\ \\ e_o(\lambda,\gamma)+o_o(\lambda,\gamma) & e_e(\lambda,\gamma)-o_e(\lambda,\gamma)\end{pmatrix},$$

where the series are given by

$$\begin{aligned}
e_e(\lambda,\gamma) &= \sum_{n=0}^{\infty}\lambda^{2n}\epsilon_e^{2n}(\gamma) \\
e_o(\lambda,\gamma) &= \sum_{n=0}^{\infty}\lambda^{2n}\epsilon_o^{2n}(\gamma) \\
o_e(\lambda,\gamma) &= \sum_{n=0}^{\infty}\lambda^{2n+1}\epsilon_e^{2n+1}(\gamma) \\
o_o(\lambda,\gamma) &= \sum_{n=0}^{\infty}\lambda^{2n+1}\epsilon_o^{2n+1}(\gamma).
\end{aligned}$$

Proof. The statements follow from the summation

$$g_\gamma^\lambda = I_\lambda\left(\mathbf{X}_{0,T_\gamma}^{\infty}(\gamma)\right) = \sum_{n=0}^{\infty} I_\lambda\left(\mathbf{X}_{0,T_\gamma}^{(n)}(\gamma)\right)$$

■

Finally, it is possible to tie in this to the computation in Corollary (76) to give an estimate for the length $l(\gamma)$ when γ is twice differentiable:

Corollary 81 *Let $\gamma \in \mathcal{P}_b(\mathbb{R}^2)$ be twice differentiable. Then*

$$\alpha_\gamma^\lambda = (e_e(\lambda,\gamma))^2 + (e_o(\lambda,\gamma))^2 + (o_e(\lambda,\gamma))^2 + (o_o(\lambda,\gamma))^2 \tag{4.23}$$

$$l(\gamma) = \lim_{\lambda\to\infty} \frac{\log\left(2\left(\begin{array}{c}(e_e(\lambda,\gamma))^2 + (e_o(\lambda,\gamma))^2 \\ +(o_e(\lambda,\gamma))^2 + (o_o(\lambda,\gamma))^2\end{array}\right)\right)}{\lambda}$$

and for $\lambda \geq \kappa_\gamma$,

$$\exp\left(\lambda l(\gamma)\sqrt{1-\left(\frac{\kappa_\gamma}{\lambda}\right)^2}\right) \leq \alpha_\gamma^\lambda + \sqrt{(\alpha_\gamma^\lambda)^2 - 1} \leq \exp(\lambda l(\gamma)). \tag{4.24}$$

Proof. α_γ^λ is given by $g_\gamma^\lambda = \begin{pmatrix} a_\gamma^\lambda & b_\gamma^\lambda \\ c_\gamma^\lambda & d_\gamma^\lambda \end{pmatrix}$ and

$$\alpha_\gamma^\lambda = \frac{(a_\gamma^\lambda)^2 + (b_\gamma^\lambda)^2 + (c_\gamma^\lambda)^2 + (d_\gamma^\lambda)^2}{2}.$$

But matching this with the form of Corollary (76) implies the formula (4.23). The last pair of statements come from Corollary (80) ■

Conclusion 82 *To conclude the section briefly, it has been shown how to use iterated integrals of C^2 paths in $\mathbb{R}^2$ to obtain length information. There are many criticisms of the conclusions and ways in which it would be desirable to improve it with more time and motivation. To state some of them, one is to ascertain how the formula extends to bounded variation paths, or piecewise C^2 paths. Another regards* *extending the picture to paths in* $\mathbb{R}^n$. *Another would be to examine more closely the inequality (4.24) and extract finer information about the magnitude of the higher order iterated integrals of these paths. In addition, many directions clearly exist for developing the principles for p variation rough paths.*

Chapter 5

Rough Paths and $T(V)$

'L'Homme révolté'

"What is a rebel ? A man who says no : but whose refusal does not imply a renunciation. He is also a man who says yes as soon as he begins to think for himself. A slave who has taken orders all his life, suddenly decides he cannot obey some new command. What does he mean by saying 'no' ?"

Albert Camus, 1951.

This final chapter is concerned with the rôle of the group-like elements of the Hopf algebra $T(V)$ with regards to rough paths. It contains a collection of observations which seem to gather together as part of a discussion on the free lie algebra and analytic properties of signatures of geometric rough paths. The first section concerns some group aspects and observations about the free lie algebra. The second is about measures on geometric rough paths and how they relate firstly to a kind of moment generating function or Laplace transform and secondly how they relate to non-geometric rough paths. It is hoped that the latter leads to a route to extend and interpret an Itô integration theory for a class of non-geometric rough paths.

5.1 Groups and Signatures in $T(V)$

In this section the idea is to look at objects in the full tensor algebra instead of truncated tensor algebras. Initially there are some definitions related to the signature map. Some results regarding the analytic meaning of the logarithm of the signature of a geometric rough path are presented, before a discussion of a particular kind of geometric rough paths, christened 'Straight rough paths'.

5.1.1 Some Definitions

Recall definition 13, the definition of the signature of a rough path in a Banach space:

Definition 83 *Let* $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p$. *Set* $\mathbf{X}^{\lfloor p \rfloor, \infty}$ *to be the unique extension due to Theorem 8 of* $\mathbf{X}^{\lfloor p \rfloor}$ *to a multiplicative functional taking values in* $\mathrm{T}(V)$,

$$\mathbf{X}^{\lfloor p \rfloor, \infty} \in \Omega(V)^p_{\infty}.$$

In addition, define the signature $\mathbf{X}^{\lfloor p \rfloor}_{sig} \in \mathrm{T}(V)$:

$$\mathbf{X}^{\lfloor p \rfloor}_{sig} = \mathbf{X}^{\lfloor p \rfloor, \infty}_{0, T_{\mathbf{X}^{\lfloor p \rfloor}}}.$$

The signature thus defines a function *sig* from the collection of rough paths into $T(V)$. It is independent of the base point of the path, so that in effect, *sig* acts on baseless multiplicative functionals of finite p variation:

$$\begin{aligned} sig &: \cup_{p\geq 1}\Omega(V)^p_{\lfloor p\rfloor} \to T(V) \\ sig\left(\mathbf{X}^{\lfloor p\rfloor}\right) &= \mathbf{X}^{\lfloor p\rfloor}_{sig}. \end{aligned}$$

In addition, an analytic property holds for any $\mathbf{X}^{\lfloor p\rfloor} \in \Omega(V)^p_{\lfloor p\rfloor}$ which has control ω:

$$\left\|\mathbf{X}^{(i)}_{sig}\right\| \leq \frac{\omega\left(0, T_{\mathbf{X}^{\lfloor p\rfloor}}\right)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}\ i \geq 1.$$

Define the following collections of subsets of $T(V)$:

Definition 84 *For any $p \geq 1$, the sets $\mathfrak{G}_p(V)$ and $\mathrm{G}_p(V)$ are given by*

$$\mathfrak{G}_p(V) = \left\{ \begin{array}{c} S \in \mathrm{T}(V) \text{ such that there exists a } C < \infty \text{ with} \\ \\ \|S_i\| \leq \frac{C^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!} \text{ for all } i \geq 1 \\ \\ \text{where } S = \sum_{i=0}^{\infty} S_i \text{ with } S_0 = 1,\ S_i \in V^{\otimes i} \\ \\ \text{and } \beta \geq p^2 \end{array} \right\},$$

$$\mathrm{G}_p(V) = \left\{S \in \mathfrak{G}_p(V) \text{ s.t. } \log(S) \in \mathrm{L}(V)\right\}.$$

Thus the image of $\Omega(V)^p_{\lfloor p\rfloor}$ under the map *sig* is a subset of $\mathfrak{G}_p(V)$. Define the image in $\mathfrak{G}_p(V)$ and $\mathrm{G}_p(V)$ as follows:

Definition 85 *For any $p \geq 1$, define the sets $\mathfrak{G}_{p,R}(V)$ and $\mathrm{G}_{p,R}(V)$*

$$\begin{aligned} sig\left(\Omega(V)^{p}_{\lfloor p \rfloor}\right) &= \mathfrak{G}_{p,R}(V) \\ sig\left(\Omega G(V)^{p}_{\lfloor p \rfloor}\right) &= \mathrm{G}_{p,R}(V) \end{aligned}$$

There is a kind of generalized p variation function $P_p(\cdot)$, which quantifies the rate at which the norms on the tensorial component of elements of $\mathfrak{G}_p(V)$ behave overall and represents a lower bound for any p variation control pertinent to the signature:

Definition 86 *For $S \in \mathfrak{G}_p(V)$, define the function $P_p : \mathfrak{G}_p(V) \to \mathbb{R}^+$,*

$$P_p(S) = \sup_{i \geq 1} \left(\beta\left(\frac{i}{p}\right)! \left\| S^{(i)} \right\|\right)^{\frac{p}{i}}.$$

Thus if $S \in \mathfrak{G}_p(V)$ then,

$$\left\| S^{(i)} \right\| \leq \frac{P_p(S)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}\ i \geq 1.$$

Elements of three out of four of these families of sets typically constitute a group, while the fourth is a semi-group. This is expressed more precisely in the next lemma:

Lemma 87 *For any $p \geq 1$, $\mathfrak{G}_p(V)$ and $\mathrm{G}_p(V)$ are semigroups. If $\|\cdot\|$ satisfies*

$$\begin{aligned} \|v_1 \otimes \cdots \otimes v_n\| &= \|v_n \otimes \cdots \otimes v_1\| \qquad (5.1) \\ &= \|\rho(v_1 \otimes \cdots \otimes v_n)\| \end{aligned}$$

$\forall v_1, \ldots, v_n \in V$ then $\mathrm{G}_p(V)$ is a group. In addition, both of $\mathfrak{G}_{p,R}(V)$ and $\mathrm{G}_{p,R}(V)$ are groups.

Proof. Take $S,\tilde{S}\in\mathfrak{G}_p(V)$, then $S\otimes\tilde{S}$ satisfies $\left(S\otimes\tilde{S}\right)_0=1$ and using Lemma (9),

$$\begin{aligned}
\left\|\left(S\otimes\tilde{S}\right)_i\right\| &\leq \sum_{j=0}^{i}\|S_j\|\left\|\tilde{S}_{i-j}\right\| \\
&\leq \frac{P_p(S)^{\frac{i}{p}}+P_p\left(\tilde{S}\right)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}+\frac{1}{\beta^2}\sum_{j=1}^{i-1}\frac{P_p(S)^{\frac{j}{p}}P_p\left(\tilde{S}\right)^{\frac{i-j}{p}}}{\left(\frac{j}{p}\right)!\left(\frac{i-j}{p}\right)!} \\
&\leq \frac{P_p(S)^{\frac{i}{p}}+P_p\left(\tilde{S}\right)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}\left(1-\frac{1}{\beta}\right)+\frac{p^2}{\beta^2}\frac{\left(P_p(S)+P_p\left(\tilde{S}\right)\right)^{\frac{i}{p}}}{\left(\frac{n}{p}\right)!} \\
&\leq \frac{P_p(S)^{\frac{i}{p}}+P_p\left(\tilde{S}\right)^{\frac{i}{p}}+\left(P_p(S)+P_p\left(\tilde{S}\right)\right)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!} \\
&\leq \frac{\left(P_p(S)^{\frac{1}{p}}+P_p\left(\tilde{S}\right)^{\frac{1}{p}}+\left(P_p(S)+P_p\left(\tilde{S}\right)\right)^{\frac{1}{p}}\right)^i}{\beta\left(\frac{i}{p}\right)!}
\end{aligned}$$

and the first statement follows. To show $\mathrm{G}_p(V)$ is a group, consider $S=\exp(l)\in\mathrm{G}_p(V)$, $l=\sum_{i=0}^{\infty}l_i$, $l_i\in L^{(i)}(V)$. Then

$$\begin{aligned}
S^{-1} &= \exp(-l) \\
&= \exp(\mu(l)) \\
&= \mu(\exp(l))
\end{aligned}$$

where μ is the anti-homomorphism of algebras of (3.15), so that $\left(S^{-1}\right)_i=\mu(S_i)=(-1)^i\rho(S_i)$ where ρ is the reversal of tensors map (3.7). Therefore, if the system of norms satisfies (5.1), then $\left\|\left(S^{-1}\right)_i\right\|=\|S_i\|$ and $S^{-1}\in\mathrm{G}_p(V)$ as required. Clearly $1\in\mathrm{G}_p(V)$, so $\mathrm{G}_p(V)$ is a group.

Both $\mathfrak{G}_p^R(V)$ and $\mathrm{G}_p^R(V)$ are groups by Corollary 121 in the appendix: the issue is to show that inverse multiplicative functional have finite p variation, which entails that the inverse of the signature has the required control of the norms of the various components. ■

Remark 88 *The sets $\mathfrak{G}_p(V)$ contain elements which are without inverses in $\mathfrak{G}_p(V)$. Simply take $v \in V$ and observe that $1 - v \in \mathfrak{G}_p(V)$ for each p. The inverse is then formally given by $\sum_{n=0}^{\infty} v^{\otimes n}$. The n'th level component is then $v^{\otimes n}$ and for typical norms systems, like the projective or injective systems, the decay matches no exponential rate.*

The final definition is of a family of subsets of the free lie algebra, the sets $L_p(V)$ and $L_{p,R}(V)$.

Definition 89 *Let $\mathrm{L}_p(V)$ and $\mathrm{L}_{p,R}(V)$ be the subsets of the free lie algebra whose exponential series lie in $\mathrm{G}_p(V)$ and $\mathrm{G}_{p,R}(V)$ respectively:*

$$
\begin{aligned}
l &\in \mathrm{L}_p(V) \Leftrightarrow \exp(l) \in \mathrm{G}_p(V) \\
l &\in \mathrm{L}_{p,R}(V) \Leftrightarrow \exp(l) \in \mathrm{G}_{p,R}(V).
\end{aligned}
$$

It is not known whether $L_{p,R}(V)$ is a strict subset of $L_p(V)$, although it does seem unlikely that the sets are equal. The question essentially concerns whether the exponential of a general element of $L_p(V)$ (i.e. a typical element of $\mathrm{G}_p(V)$), can be factorized in $\mathrm{G}_p(V)$ in some 'sensible' way or not. If this is possible, then Lyons' theorem (8) implies that the first $\lfloor p \rfloor$ tensorial components define a geometric rough path and hence the log of the signature is in $L_{p,R}(V)$.

A different possibility is that an element of $L_p(V)$ not in $L_{p,R}(V)$, could belong to $L_{q,R}(V)$ for some $q > p$. This possibility is intriguing : it is demonstrated later on however, that for the injective system of norms, for any $n > 0$, there exist elements of $L_n(V)$ which belong to $L_{p,R}(V)$ for every $p > n$. Whether the function $P_p(\cdot)$ plays a characterizing rôle in the matter is unclear.

5.1.2 The kernel of *sig*

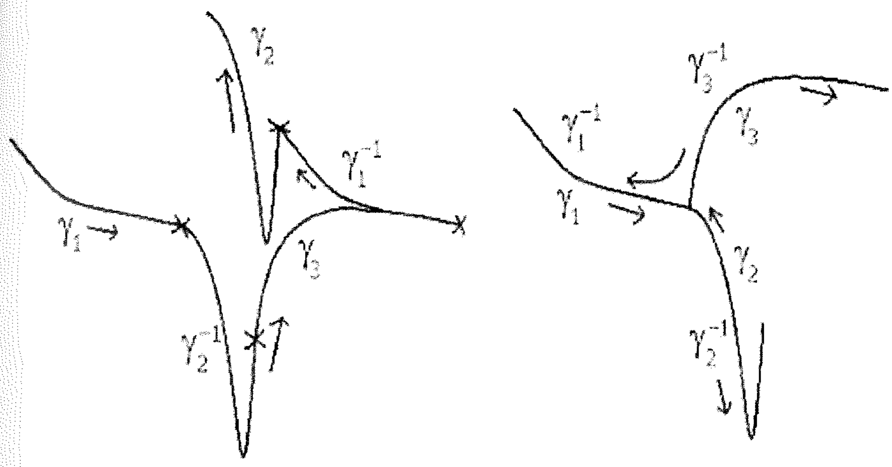
While working with signatures, it is important to understand the kernel of the map of multiplicative functionals to signatures. For each value of $p \geq 1$, the restricted action of *sig* to either of $\Omega(V)^p_{\lfloor p \rfloor}$ or $\Omega G(V)^p_{\lfloor p \rfloor}$, is a homomorphism into $\mathfrak{G}_{p,R}(V)$ or $\mathrm{G}_{p,R}(V)$ respectively. The kernel, being the subgroup of $\Omega(V)^p_{\lfloor p \rfloor}$, or $\Omega G(V)^p_{\lfloor p \rfloor}$ such that $sig\left(\mathbf{X}^{\lfloor p \rfloor}\right) = 1$, essentially defines the collection of rough paths of finite p variation which cannot be distinguished by the collection of functions generated by the canonical 1 forms on V (or on a lie group if that is the context).

Recently, in [24], Lyons and Hambly extended a result in [10] of Chen, related to signatures of piecewise smooth paths on lie groups, to bounded variation paths. They show that the kernel of $sig : \Omega G(V)^1 \to \mathrm{G}_1(V)$ is a collection of so called treelike paths.

This set is described in [24] with analogy to free groups as follows. Consider a collection of bounded variation paths in a vector space, $\{\gamma_i\}_{i=1}^n$ and their inverses $\left\{\gamma_i^{-1}\right\}_{i=1}^n$. Formally, products of these paths generate paths which can be described by 'words' in the letters which are the indeterminates $\{\gamma_i\}_{i=1}^n \cup \left\{\gamma_i^{-1}\right\}_{i=1}^n$, for example,

$$\gamma_1\gamma_2^{-1}\gamma_3\gamma_1^{-1}\gamma_2 \text{ or } \gamma_1\gamma_2^{-1}\gamma_2\gamma_3\gamma_3^{-1}\gamma_1^{-1}.$$

In the free group, $\gamma_i\gamma_i^{-1} = 1$ for $1 \leq i \leq n$, so the second of these paths is formally 'contractible' to 1, while the first is not. Note the representation of these two paths in figure 5.1.2. The pertinent difference between the left and right hand picture, is that in the right hand one, the curve retraces itself exactly. Thus, the graph of the path resembles a tree whose branches and leaves are visited in turn before returning



to the trunk and eventually the base of the tree at the end.

Chen's Lemma and the fact that $sig\,(\gamma_1\gamma_2) = sig\,(\gamma_1)\,sig\,(\gamma_2)$, imply that if in the free group, a word $\gamma_{i_1}^{\pm1}\gamma_{i_2}^{\pm1}\cdots\gamma_{i_n}^{\pm1}$ is contractible to 1, then

$$sig\left(\gamma_{i_1}^{\pm1}\gamma_{i_2}^{\pm1}\cdots\gamma_{i_n}^{\pm1}\right) = 1,$$

in other words, these paths are in the kernel of *sig*. The authors of [24] show that the kernel of *sig* is completely characterized by a continuous version of this phenomena. This set of paths are named treelike.

For general rough paths, there are clearly examples of this type of tree-like phenomena. It is necessary to take into account the higher order processes, such as the area process, which can make their own excursions or treelike behaviour. As a path in a vector space, this type of excursion is not possible to detect in the graph of the path (see section 5.1.4). However in a Lie group, the excursions can be exhibited (see chapter 4). It is possible that within the context of canonical 1 forms on nilpotent lie groups, an extension of an uniqueness theorem exists for the iterated integral sequences of geometric rough paths. An intrinsic and different difficulty to handle is that the p variation of these rougher paths, seems to entail a different flavour of argument than that used in [24].

The conclusion is that at least in the case of the set $G_{1,R}(V)$, elements can be uniquely associated to a reduced path of minimal length and without treelike excursions, which is parameterized at unit speed.

5.1.3 The Free Lie Algebra

Very little is known about how to identify the collection of elements of the free lie algebra that correspond to geometric rough paths of finite p variation, i.e. how to characterize the sets $L_p(V)$ in $L(V)$ itself. It is possible to exploit the differential equation in $L^{\infty}(V)$, (4.5) for the logarithm of a geometric rough path of finite p variation $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^p$, to show that in $L^{\infty}(V)$, any such series behaves something like the series expansion of an analytic function. A suitable weighting ensures that the series always has analytic meaning within $L^{\infty}(V)$. Consider again the differential equation of section 4.1.1:

$$\begin{aligned} \mathbf{m}_0 &= 0 \\ d\mathbf{m}_t &= f(\mathbf{m}_t)\left(d\mathbf{X}_t^{\lfloor p \rfloor}\right). \end{aligned} \tag{5.2}$$

The vector field

$$f(\mathbf{m})(\cdot) \in Hom\left(L^{\infty}(V), L^{\infty}(V)\right)$$

is smooth in a neighborhood of the origin, $\|\mathbf{m}\| < \pi$ (see section 6.3), hence Theorem 27 implies that there are local solutions in $L^{\infty}(V)$ to the differential equation (5.2) for the logarithm of the signature of a geometric rough path of finite p variation.

In particular, pick $\delta < \pi$, $p \geq 1$ and let M_δ be the $\lfloor p \rfloor + 1$ Lipschitz norm of f on

$$U_\delta = \{\mathbf{m} \in L^\infty(V) \mid \|\mathbf{m}\|_\infty \leq \delta\}.$$

According to chapter 6 of [29], this means precisely that for any $C > 0$, there exists a constant $\kappa = \kappa_{C,p,M_\delta}$ such that if $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^p$ has control ω and if $\omega(0,t) \leq C$, then there is an unique solution $\mathbf{m}_t$ to equation (5.2) which is a geometric rough path of finite p variation in $L^\infty(V)$ while it remains in the set U_δ and it has control $\kappa\omega$. Immediately this implies that

$$\|\mathbf{m}_t\|_\infty^p \leq \frac{\kappa\omega(0,t)}{\beta\left(\frac{1}{p}\right)!}, \tag{5.3}$$

while $\mathbf{m}_t \in U_\delta$, that is

$$\|\mathbf{m}_t\|_\infty \leq \delta.$$

So for a given choice of δ and C, inequality (5.3) implies that the solution $\mathbf{m}_t = \log\left(\mathbf{X}_{0,t}^{\lfloor p \rfloor;sig}\right)$ remains in U_δ at least for the set of times $t \geq 0$ such that

$$\omega(0,t) \leq \min\left(C, \frac{\delta^p \beta\left(\frac{1}{p}\right)!}{\kappa_{C,p,M_\delta}}\right).$$

While it is possible to maximize the right hand side over C for a given choice of δ, the content of the conclusion of the calculation remains the same. It is, that is that the logarithm of the signature of a geometric rough path of finite p variation in V, is itself locally a geometric rough path of finite p variation in $L^\infty(V)$.

Proposition 90 *Let $p \geq 1$ be fixed. Then for any $\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^p$, there exists constants $C_p > 0$, $\kappa_{C_p,p} > 0$ such that for pairs of times $(s,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}$ with $\omega(s,t) \leq C_p$ where ω is a control for $\mathbf{X}^{\lfloor p \rfloor}$, the* *logarithm of the signature of* $\mathbf{X}^{\lfloor p \rfloor}$*, is a geometric rough path* $\log\left(\mathbf{X}^{\lfloor p \rfloor;\infty}\right) \in \Omega G\left(\mathrm{L}^{\infty}\left(V\right)\right)^{p}$ *with control* $\kappa_{C_p,p}\omega$.

Proof. From the preceding discussion, this is true for $s = 0$ and extends to any time point $s \in [0, T_{\mathbf{X}^{\lfloor p \rfloor}}]$ through finding a solution to equation (5.3) starting from time s. ■

While these solutions produce local estimates for the logarithm, it is possible to think about the global solution for the logarithm, as a series solution in $L^{\infty}(V)$ with a quantifiable radius of convergence. First observe that scaling a rough path in V by some real factor $\lambda > 0$, provides a new rough path with an associated scaled control:

Lemma 91 *If* $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^{p}$ *has control* ω*, for* $\lambda > 0$*, define the scaled rough path* ${}^{\lambda}\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^{p}$ *by*

$$ {}^{\lambda}\mathbf{X}_{s,t}^{(i)} = \lambda^{i}\mathbf{X}_{s,t}^{(i)} \; 1 \leq i \leq \lfloor p \rfloor . $$

Then ${}^{\lambda}\mathbf{X}^{\lfloor p \rfloor}$ *has control* ${}^{\lambda}\omega$ *where*

$$ {}^{\lambda}\omega(s,t) = \lambda^{p}\omega(s,t) \;\; \forall (s,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}. $$

Proof. This is immediate from the following : if $(s,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}$,

$$ \begin{aligned} \left\| {}^{\lambda}\mathbf{X}_{s,t}^{(i)} \right\|^{\frac{p}{i}} &= \left(\lambda^{i} \left\| \mathbf{X}_{s,t}^{(i)} \right\|\right)^{\frac{p}{i}} \\ &\leq \frac{\lambda^{p}\omega(s,t)}{\beta\left(\frac{i}{p}\right)!}. \end{aligned} $$

■

Now it is clear how to proceed to obtain a weighted series solution. Through a scaling, the weighted solution will exist for all times as a geometric rough path in $L^{\infty}(V)$ and the scaling interacts simply with the logarithms according to tensorial degree.

Proposition 92 *Fix $p \geq 1$. Then for any geometric rough path*

$$\mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^{p}$$

that has control ω, there exists a constant $\lambda > 0$ such that for any $(s,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}$,

$$^{\lambda}\mathbf{m}_{s,t} = \sum_{i=1}^{\infty} \lambda^{i} \log\left(\mathbf{X}_{s,t}^{\lfloor p \rfloor;\infty}\right)^{(i)} \in \mathrm{L}^{\infty}(V).$$

In addition, $^{\lambda}\mathbf{m}_{s,t}$ is a geometric rough path of finite p variation that belongs to $\Omega G(\mathrm{L}^{\infty}(V))^{p}$ and has control $\lambda^{p}\kappa_{C_p,p}\omega$, where C_p and $\kappa_{C_p,p}$ are as in Proposition 90.

Proof. Set

$$\lambda = \left(\frac{C_p}{\omega\left(0, T_{\mathbf{X}^{\lfloor p \rfloor}}\right)}\right)^{\frac{1}{p}},$$

so that $^{\lambda}\omega\left(0, T_{\mathbf{X}^{\lfloor p \rfloor}}\right) = C_p$. Then Proposition 90 implies that

$$^{\lambda}\mathbf{m}_{s,t} = \log\left(^{\lambda}\mathbf{X}_{s,t}^{\lfloor p \rfloor;\infty}\right)$$

is a rough path of finite p variation that lies in $\Omega G(L^{\infty}(V))^{p}$ with control $\kappa_{C_p,p}^{\lambda}\omega = \lambda^{p}\kappa_{C_p,p}\omega$ and it is clear that

$$\log\left(^{\lambda}\mathbf{X}_{s,t}^{\lfloor p \rfloor;\infty}\right)^{(i)} = \lambda^{i} \log\left(\mathbf{X}_{s,t}^{\lfloor p \rfloor;\infty}\right)^{(i)} \quad i \geq 1.$$

In addition $^{\lambda}\mathbf{m}_{s,t}$ is a rough path of finite p variation so that for all $(s,t) \in \Gamma_{T_{\mathbf{X}^{\lfloor p \rfloor}}}$,

$$\left\|^{\lambda}\mathbf{m}_{s,t}\right\|_{\infty}^{p} = \left(\sum_{i=1}^{\infty} \lambda^{i} \left\|\log\left(\mathbf{X}_{s,t}^{\lfloor p \rfloor;\infty}\right)^{(i)}\right\|\right)^{p} \leq \lambda^{p}\kappa_{C_p,p}\omega(s,t)$$

■

5.1.4 Straight Rough Paths

The set of formal series, $L_{1,R}(V)$ does not form a vector space in the conventional sense. The Campbell Baker Hausdorff formula, (see theorem 122), expresses a product which makes $L_{p,R}(V)$ into a group, but the natural scaling of $L_{p,R}(V)$ is by a factor λ^i, $i \geq 1$ for the component of tensor degree i. The reasoning that $L_{1,R}(V)$ is not a vector space is the following. If it was, then for $l \in L_{1,R}(V)$ say, scalar multiples λl, for all $\lambda \in \mathbb{R}$ correspond to bounded variation paths through the exponential map. Thus for any $n \geq 1$, such a path has an n'th root for any n, so it's graph should be an n fold repetition of some sub path. Now, as described in [24], signatures of bounded variation paths are unique modulo an equivalence relation for treelike excursions. So for a bounded variation path whose graph clearly cannot be factorized n times into reduced paths, an n'th root cannot exist. A similar deduction about $L_{p,R}(V)$ is contingent on extending the result of [24] to finite p variation geometric rough paths.

The bounded variation paths which do have this property of n'th roots for any $n \geq 1$, are straight lines. Straight rough paths are a similar nature of construction. This section mostly uses simple analytic manipulations of the definitions of rough paths. It is shown that exponentials of elements of $L(V)$ which have finite tensorial degree, correspond to the signatures of a particular nature of geometric rough paths. These paths are analogous to straight lines for various reasons. One reason is that when developed into a Lie group, the associated path in the Lie algebra follows a straight line and hence the path in the group follows a straight line. Another is that it is possible to take any real power of the path. The third relates to the theorem on establishing the uniqueness of extension of the iterated integral sequence and requires a little elaboration. The heuristic is that geometric multiplicative functionals in any truncated tensor algebras $T^n(V)$ can be thought of as defining paths in the truncated Lie algebras $L^n(V)$. At an abstract level, to approximate to a geometric rough path, with a given time partition, the appropriate regime is to use straight rough paths between time points. This corresponds to the regime for linear approximation to bounded variation signals and that it works is an interpretation of Theorem 8 for the case of geometric rough paths.

Construction of straight rough paths

There is a class of rough paths whose logarithms of signatures form finite dimensional vector subspaces of the free Lie algebra, namely those paths with logarithm which is of finite degree. This section constructively shows it is possible to construct corresponding geometric rough paths with these logarithms.

Recall the Campbell-Baker-Hausdorff formula which expresses the group structure that the free Lie algebra inherits from the group structure of exponentials (see 122): if $\mathfrak{g}$ is a Lie algebra and $a, b \in \mathfrak{g}$, then

$$\begin{aligned} H(a,b) &= \log(\exp(a) \otimes \exp(b)) \\ &= a + b + \frac{1}{2}[a,b] + \frac{1}{12}([a,[a,b]] + [b,[b,a]]) + \ldots . \end{aligned}$$

The following simple extension is used in what follows.

Lemma 93 *Let $H(\cdot,\cdot)$ be the Campbell-Baker-Hausdorff product. Then*

$$\begin{aligned} & H(H(a,b), H(-a,-b)) \\ = {} & \log(\exp(a) \otimes \exp(b) \otimes \exp(-a) \otimes \exp(-b)) \\ = {} & [a,b] + O(3) \end{aligned}$$

where $O(3)$ *means tensors of order 3 and higher in* a *and* b *combined.*

Proof. Just compute:

$$\begin{aligned}
& H\left(H\left(a,b\right),H\left(-a,-b\right)\right) \\
= \; & H\left(a+b+\frac{1}{2}\left[a,b\right]+O\left(3\right),-a-b+\frac{1}{2}\left[-a,-b\right]+O\left(3\right)\right) \\
= \; & a+b+\frac{1}{2}\left[a,b\right]-a-b+\frac{1}{2}\left[a,b\right]+\frac{1}{2}\left[a+b,-a-b\right]+O\left(3\right) \\
= \; & \left[a,b\right]+O\left(3\right)
\end{aligned}$$

■

The aim is to show that it is possible to associate a geometric rough path for a signature whose logarithm follows a straight line in a truncated free lie algebra. Since the Lie bracket of two such elements will have finite degree, there is also a sense of a lie algebra. The Lemma above enables the process to be initiated - through concatenating particular sequences of paths, it is possible to achieve certain values of logarithm in a truncated lie algebra. A scaling routine then enables the construction of a Cauchy sequence for the p variation metric, which converges to the desired value of logarithm.

V will be a finite dimensional vector space. To start, two preliminary propositions offer firstly an analytic control of the p variation of a repetitively looping path and secondly a way to relate particular bounded variation paths to algebraic quantities.

Proposition 94 *For a bounded variation loop* $\gamma:[0,1]\rightarrow V$, $\gamma(0)=\gamma(1)$, *where* V *has norm* $\|\cdot\|$, *the* p *variation of* γ^n, *the loop traversed* n *times, grows asymptotically in a linear fashion for any* $p\geq 1$. *In* *particular there exist* $c_1(\gamma), c_2(\gamma) > 0$ *such that for all* n,

$$c_1(\gamma) \leq \frac{P_{p;0,n}\left(X^{(1)}(\gamma^n)\right)}{n} \leq c_2(\gamma)$$

Proof. Denote by γ^n the loop γ traversed n times,

$$\begin{aligned} \gamma^n &: [0,n] \to V \\ \gamma^n(t) &= \gamma(t-m) \quad t \in [m, m+1]. \end{aligned}$$

$P_{p;0,n}\left(X^{(1)}(\gamma^n)\right)$ is the $p-$variation of such a n-fold loop.

Set $D_p(\gamma) = \sup_{(s,t)\in\Gamma_1} \left\| X^{(1)}_{s,t}(\gamma) \right\|^p$ and consider a dissection $\delta \in \Delta_{0,n}$, $\delta = \{t_j\}_{j=0}^{|\delta|}$.

For $m = 1, \ldots, n-1$, set $j_m = \inf_{t_j > m} j$. Then

$$\begin{aligned} \sum_{i=1}^{|\delta|} \left\| X^{(1)}_{t_{i-1},t_i}(\gamma^n) \right\|^p &= \left\{ \begin{array}{c} \sum_{i=1;i\neq j_m, m\in\{1,\ldots n-1\}}^{|\delta|} \left\| X^{(1)}_{t_{i-1},t_i}(\gamma^n) \right\|^p \\ + \sum_{m=1}^{n-1} \left\| X^{(1)}_{t_{j_m-1},t_{j_m}}(\gamma^n) \right\|^p \end{array} \right\} \\ &\leq nP_{p;0,1}\left(X^{(1)}(\gamma)\right) + (n-1) D_p(\gamma) \\ &\leq n\left(P_{p;0,1}\left(X^{(1)}(\gamma)\right) + D_p(\gamma)\right) \end{aligned}$$

since each term in the second sum is bounded above by $D_p(\gamma)$ and the first sum is separable into n dissections of γ. This bound is independent of the dissection chosen, hence it bounds the supremum over all dissections and the p variation of the loop. The lower bound $P_{p;0,n}\left(X^{(1)}(\gamma^n)\right) \geq nP_{p;0,1}\left(X^{(1)}(\gamma)\right)$ is trivial, resulting in the inequalities

$$nP_{p;0,1}\left(X^{(1)}(\gamma)\right) \leq P_{p;0,n}\left(X^{(1)}(\gamma^n)\right) \leq n\left(P_{p;0,1}\left(X^{(1)}(\gamma)\right) + D_p(\gamma)\right),$$

which concludes the proposition. ■

Now it is possible to start looking at paths and algebra interacting.

Proposition 95 *For any* $1 \leq n < \infty$ *and any sequence* $v_i \in V$, $i = 1, \ldots, n$, $n < \infty$, *there exists* $\gamma_{[v_1, v_2, \ldots, v_n]} \in \mathcal{P}_b(V)$ *whose signature*

$$\mathbf{X}_{sig}\left(\gamma_{[v_1, v_2, \ldots, v_n]}\right)$$

has logarithm which projects onto $\mathrm{L}^n(V)$ *to be* $[v_1, v_2, \ldots, v_n]$.

Proof. Use a simple inductive argument..

For $n = 1$, the straight line $\gamma_{v_1} : [0, 1] \to V$, $\gamma_{v_1}(t) = tv_1$ suffices. It has first iterated integral process $\mathbf{X}^{(1)}_{s,t}(\gamma_{v_1}) = (t - s) v_1$ and in fact extends to

$$\mathbf{X}^{\infty}_{s,t}(\gamma_{v_1}) = \exp((t - s) v_1)$$

as an element of $\Omega G(V)^1_{\infty}$.

For general n, take a piecewise smooth path $\gamma_{[v_2, \ldots, v_{n+1}]}$, of which the projection of the log of the signature

$$\mathbf{X}_{sig}\left(\gamma_{[v_2, \ldots, v_{n+1}]}\right)$$

into $L^{n+1}(V)$ is

$$[v_2, \ldots, v_{n+1}] + e(v_2, \ldots, v_{n+1}),$$

where $e(v_2, \ldots, v_{n+1}) \in L^{(n+1)}(V)$. Take the straight line γ_{v_1} and concatenate paths to form the path

$$\gamma_{[v_1, \ldots, v_{n+1}]} \triangleq \gamma_{v_1} \gamma_{[v_2, \ldots, v_{n+1}]} (\gamma_{v_1})^{-1} \left(\gamma_{[v_2, \ldots, v_{n+1}]}\right)^{-1}.$$

The signature of $\gamma_{[v_1,\ldots,v_{n+1}]}$ in $T^{n+1}(V)$ is then :

$$\begin{aligned}
& \mathbf{X}^{n+1}\left(\gamma_{[v_1,\ldots,v_{n+1}]}\right) \\
= & \mathbf{X}^{n+1}\left(\gamma_{v_1}\right) \otimes \mathbf{X}^{n+1}\left(\gamma_{[v_2,\ldots,v_{n+1}]}\right) \\
& \otimes \mathbf{X}^{n+1}\left(\gamma_{v_1}\right)^{-1} \otimes \mathbf{X}^{n+1}\left(\gamma_{[v_2,\ldots,v_{n+1}]}\right)^{-1} \\
= & \left(\exp\left(v_1\right)\right)^{n+1} \otimes \left(\exp\left(\left[v_2,\ldots,v_{n+1}\right] + e\left(v_2,\ldots,v_{n+1}\right)\right)\right)^{n+1} \\
& \otimes \left(\exp\left(-v_1\right)\right)^{n+1} \otimes \left(\exp\left(-\left[v_2,\ldots,v_{n+1}\right] - e\left(v_2,\ldots,v_{n+1}\right)\right)\right)^{n+1}
\end{aligned}$$

Lemma (93) implies

$$\begin{aligned}
\mathbf{X}^{n+1}\left(\gamma_{[v_1,\ldots,v_{n+1}]}\right) &= \left(\exp\left(\left[v_1,\left[v_2,\ldots,v_{n+1}\right]\right]\right)\right)^{n+1} \\
&= \left(\exp\left(\left[v_1,v_2,\ldots,v_{n+1}\right]\right)\right)^{n+1}
\end{aligned}$$

as required, completing the inductive step of the proposition. ■

From here, the next step is to use these propositions to show the existence of an element of $\Omega G(V)^{n+}$ whose signature has a logarithm which is a straight line in a component of $L^{(n)}(V)$.

Proposition 96 *Let V be a finite dimensional vector space and*

$$\underline{l}^{(n)} \in \mathrm{L}^{(n)}(V).$$

There is a geometric rough path $\mathbf{X}^n\left(\underline{l}^{(n)}\right) \in \Omega G(V)^{n+}$ with $T_{\mathbf{X}^n} = 1$ say, whose extension to $\Omega G(V)^{n+}_{\infty}$, $\mathbf{X}^{n;\infty}\left(\underline{l}^{(n)}\right)$, has the form

$$\mathbf{X}^{n;\infty}_{s,t}\left(\underline{l}^{(n)}\right) = \exp\left((t-s)\,\underline{l}^{(n)}\right)$$

$\forall (s,t) \in \Gamma_1$.

Proof. First of all use the proposition 31 to express $\underline{l}^{(n)}$ as a sum of lie elements of the form $[v_1, v_2, \ldots, v_n]$,

$$\underline{l}^{(n)} = \sum_{i=1}^{N(\underline{l})} [v_{i_1}, v_{i_2}, \ldots, v_{i_n}]$$

where $v_{i_j} \in V$ and $N(\underline{l}) < \infty$ - this is possible since V is finite dimensional which implies $L^{(n)}(V)$ is finite dimensional. Denote by γ a key path that is the concatenation of the $N\left(\underline{l}^{(n)}\right)$ paths $\gamma_i = \gamma_{[v_{i_1}, v_{i_2}, \ldots, v_{i_n}]}$ given by proposition 95 and where a concatenation order is chosen once and for all; what follows is independent of this initial choice however. Without loss of generality, we may assume that the time parameterization has $T_\gamma = 1$. Then Chen's identity, equation (2.1), implies that the signature of γ is the product of the signatures of the $N\left(\underline{l}^{(n)}\right)$ paths γ_i:

$$\mathbf{X}(\gamma) = \otimes_{i=1}^{N(\underline{l})} \mathbf{X}(\gamma_i)$$

The log of this signature projected onto $L^n(V)$, is $\underline{l}^{(n)}$ by the Campbell-Baker-Hausdorff formula (6.1), since when restricted to tensors of degree n or less, in this case $H(\cdot, \cdot)$ is additive.

Take the following sequence of paths, $\gamma^k \in \mathcal{P}_b(V)$, $k = 0, 1, 2, \ldots$ which are based on this key path and which have the property that the log of their signatures converge to $\underline{l}^{(n)}$ in $\mathcal{L}(V)$. First set $\gamma^0 = \gamma$ and consider the maps $\phi_n : \mathcal{P}_b(V) \hookleftarrow$ which effect a scaling and concatenation:

$$\phi_n(\gamma)(t) = \left\{ \begin{array}{cc} \frac{1}{2^{\frac{1}{n}}}\gamma(2t) : & t \in \left[0, \frac{1}{2}\right) \\ \\ \frac{1}{2^{\frac{1}{n}}}\gamma(2t-1) : & t \in \left[\frac{1}{2}, 1\right] \end{array} \right\}$$

So define the sequence of paths $\gamma^k = \phi_n\left(\gamma^{k-1}\right)$.

As a consequence, the iterated integrals of the γ^k are given recursively by:

$$\mathbf{X}^{(i)}_{0,t}\left(\gamma^k\right) = \mathbf{X}^{(i)}_{\frac{1}{2},\frac{1}{2}+t}\left(\gamma^k\right) = \frac{1}{2^{\frac{i}{n}}}\mathbf{X}^{(i)}_{0,2t}\left(\gamma^{k-1}\right) : \quad t \in \left[0, \frac{1}{2}\right], 1 \le i \le n$$

and more generally:

$$\mathbf{X}^{(i)}_{\frac{j}{2^k},\frac{j}{2^k}+t}\left(\gamma^k\right)=\frac{1}{2^{\frac{ki}{n}}}\mathbf{X}^{(i)}_{0,2^kt}\left(\gamma\right):\quad t\in\left[0,\frac{1}{2^k}\right],0\le j\le 2^k-1,1\le i\le n \tag{5.4}$$

Also, since if $1\le h\le n-1$, $\mathbf{X}^{(h)}_{0,1}\left(\gamma^0\right)=0$, then for $t\in\left[0,\frac{1}{2^k}\right)$

$$\begin{aligned}\mathbf{X}^{(i)}_{0,\frac{j}{2^k}+t}\left(\gamma^k\right) &= \sum_{h=0}^{i}\mathbf{X}^{(h)}_{0,\frac{j}{2^k}}\left(\gamma^k\right)\otimes\mathbf{X}^{(i-h)}_{\frac{j}{2^k},\frac{j}{2^k}+t}\left(\gamma^k\right)\\ &= \mathbf{X}^{(i)}_{\frac{j}{2^k},\frac{j}{2^k}+t}\left(\gamma^k\right)\\ &= \mathbf{X}^{(i)}_{0,t}\left(\gamma^k\right)\end{aligned} \tag{5.5}$$

What remains to be proved and is less clear is that $\left\{\gamma^k\right\}_{k=0}^{\infty}$ is Cauchy with respect to d_p $\forall p>n$, and that the iterated integrals converge as $k\to\infty$ according to :

$$\mathbf{X}^{(i)}_{s,t}\left(\gamma^k\right)\to 0\quad 1\le i\le n-1$$

$$\mathbf{X}^{(n)}_{s,t}\left(\gamma^k\right)\to(t-s)\,\underline{l}^{(n)}.$$

The point is to show that the sequence $d_p\left(\mathbf{X}\left(\gamma^k\right),\mathbf{X}\left(\gamma^{k+1}\right)\right)$ is bounded by a summable geometric series.

Recall the definition of d_p :

$$d_p\left(\mathbf{X}^{\lfloor p\rfloor},\mathbf{Y}^{\lfloor p\rfloor}\right)=\sum_{i=1}^{\lfloor p\rfloor}P_{\frac{p}{i};0,T}\left(\mathbf{X}^{(i)}-\mathbf{Y}^{(i)}\right)^{\frac{i}{p}}+\sup_{u\in[0,T]}\left\|\mathbf{X}^{(1)}_u-\mathbf{Y}^{(1)}_u\right\|$$

Compare the i'th iterated integrals of the two paths γ^k and γ^{k+1}. For $i<n$, each iterated integral $\mathbf{X}\left(\gamma^k\right)^{(i)}_{0,t}$ is a path in $V^{\otimes i}$ which loops 2^k times (at least) according to (5.4),(5.5) and the fact that $\mathbf{X}\left(\gamma^1\right)^{i}_{0,1}=0$ for $i<n$. This implies also that the difference $\mathbf{X}^{(i)}_{0,t}\left(\gamma^k\right)-\mathbf{X}^{(i)}_{0,t}\left(\gamma^{k+1}\right)$ is also a path in

$V^{\otimes i}$ which loops at least 2^k times for $i < n$. In addition for $1 \leq i \leq n$:

$$\begin{aligned}
&\mathbf{X}^{(i)}_{0,\frac{j}{2^k}+t}\left(\gamma^k\right) - \mathbf{X}^{(i)}_{0,\frac{j}{2^k}+t}\left(\gamma^{k+1}\right) \\
&= 2^{\frac{-ik}{n}}\left(\mathbf{X}^{(i)}_{0,2^k t}\left(\gamma^0\right) - \mathbf{X}^{i}_{0,2^k t}\left(\gamma^1\right)\right): \quad t \in \left[0, \frac{1}{2}\right], 0 \leq j \leq 2^k - 1
\end{aligned}$$

Each difference of an n'th iterated integral, $\mathbf{X}^{(n)}_{0,t}\left(\gamma^k\right) - \mathbf{X}^{(n)}_{0,t}\left(\gamma^{k+1}\right) \in V^{\otimes n}$ is also a path which loops : the path $\gamma^0 = \gamma$ is constructed so that $\mathbf{X}^{(n)}_{0,1}\left(\gamma^0\right) = \underline{l}^{(n)}$ and the iteration defined in order that $\mathbf{X}^{(n)}_{0,1}\left(\gamma^1\right) = \underline{l}^{(n)}$ too.

Therefore, for $1 \leq i \leq n$, $\mathbf{X}^{(i)}_{0,t}\left(\gamma^k\right) - \mathbf{X}^{(i)}_{0,t}\left(\gamma^{k+1}\right)$ is identifiable as the loop $\mathbf{X}^{(i)}_{0,t}\left(\gamma^0\right) - \mathbf{X}^{(i)}_{0,t}\left(\gamma^1\right)$ repeatedly run 2^k times with the space scaling in V of $2^{\frac{-k}{n}}$ or in $V^{\otimes i}$ of $2^{\frac{-ik}{n}}$. Now let $p > n$ and turn to look at $P_{p/i;0,1}$ of these differences of i'th iterated integrals. Set $\varkappa_i = \sup_{(s,t)\in\Gamma_1}\left\|\mathbf{X}^{(i)}_{s,t}\left(\gamma^0\right) - \mathbf{X}^{(i)}_{s,t}\left(\gamma^1\right)\right\|$. Then from the bound of lemma (94)

$$\begin{aligned}
&P_{p/i;0,1}\left(\mathbf{X}^{(i)}\left(\gamma^k\right) - \mathbf{X}^{(i)}\left(\gamma^{k+1}\right)\right)^{\frac{i}{p}} \\
\leq\ & \left\{2^k\left(2^{\frac{-pk}{n}} P_{p/i;0,1}\left(\mathbf{X}^{(i)}\left(\gamma^0\right) - \mathbf{X}^{(i)}\left(\gamma^1\right)\right) + \left(2\varkappa_i 2^{\frac{-k}{n}}\right)^p\right)\right\}^{\frac{i}{p}} \\
=\ & 2^{ki\left(\frac{1}{p}-\frac{1}{n}\right)}\left(P_{p/i;0,1}\left(\mathbf{X}^{(i)}\left(\gamma^0\right) - \mathbf{X}^{(i)}\left(\gamma^1\right)\right) + \left(2\varkappa_i\right)^p\right) \\
=\ & 2^{ki\left(\frac{1}{p}-\frac{1}{n}\right)} C_i \qquad (5.6)
\end{aligned}$$

where the values $C_i = P_{p/i;0,1}\left(\mathbf{X}^{(i)}\left(\gamma^0\right) - \mathbf{X}^{(i)}\left(\gamma^1\right)\right) + \left(2\varkappa_i\right)^p$ are constants and $p > n$, so that $2^{i\left(\frac{1}{p}-\frac{1}{n}\right)} < 1$. For each $1 \leq i \leq n$, equation (5.6) thus defines a summable geometric series over k.

Looking at the bound

$$d_p\left(\mathbf{X}^{\lfloor p\rfloor}\left(\gamma^k\right), \mathbf{X}^{\lfloor p\rfloor}\left(\gamma^{k+1}\right)\right) = \sup_{t\in[0,T]}\left\|\mathbf{X}^{(1)}_t\left(\gamma^k\right) - \mathbf{X}^{(1)}_t\left(\gamma^{k+1}\right)\right\| \leq \frac{1}{2^{\frac{k}{n}}}\varkappa_1$$

provides the inequality

$$d_p\left(\mathbf{X}^{\lfloor p\rfloor}\left(\gamma^k\right),\mathbf{X}^{\lfloor p\rfloor}\left(\gamma^{k+1}\right)\right)\le\frac{1}{2^{\frac{k}{n}}}\varkappa_1+\sum_{i=1}^{\lfloor p\rfloor}2^{ki\left(\frac{1}{p}-\frac{1}{n}\right)}C_i$$

which as required is finitely summable.

To check that the limit $\mathbf{X}^n\left(\underline{l}^{(n)}\right)\in\Omega G(V)^{n+}$ of γ^k with respect to d_p, has the form $\mathbf{X}^n_{s,t}\left(\underline{l}^{(n)}\right)=\left(\exp\left((t-s)\,\underline{l}^{(n)}\right)\right)^n$ for $(s,t)\in\Gamma_1$, (5.4) implies that the $1\le i<n$ iterated integrals $\mathbf{X}^{(i)}_{s,t}\left(\gamma^k\right)$ converge to 0 uniformly over time pairs $0\le s\le t\le 1$ as $k\to\infty$. The n'th iterated integral $\mathbf{X}^{(n)}_{\frac{j_1}{2^k},\frac{j_2}{2^k}}\left(\gamma^k\right)$ by construction equals $\frac{(j_2-j_1)}{2^k}\underline{l}^{(n)}$ for s and t dyadic rationals and as in the case of the lower order iterated integrals, the relation of equation 5.4 implies uniform convergence. To conclude, the limiting rough path of finite n^+ variation, has signature in $T^n(V)$, $\mathbf{X}^n_{s,t}\left(\underline{l}^{(n)}\right)=\left(\exp\left((t-s)\,\underline{l}^{(n)}\right)\right)^n$, which by the uniqueness of extension has the form

$$\mathbf{X}^{n,\infty}_{s,t}\left(\underline{l}^{(n)}\right)=\left(\exp\left((t-s)\,\underline{l}^{(n)}\right)\right)\in T(V).$$

■

As a helpful device, a sequence of paths are constructed to facilitate the final step in this section which is to combine a collection of homogeneous lie elements together in the free lie algebra to get a geometric rough path, the logarithm of whose signature moves at a constant rate in the direction of the sum of the lie elements.

Lemma 97 *Define the subsets of* $[0,1]$, $A_m=\cup_{i=0}^{2^m-1}\left\{t:\frac{i}{2^m}\le t\le:\frac{2i+1}{2^{m+1}}\right\}$, *the paths*

$$\alpha_{s,t}(m)=2\int_s^t 1_{A_m}(u)\,du$$

and $\alpha_{s,t} = (t-s)$. *Then for any* $\delta > 1$, $\alpha(m) \to \alpha$ *in* δ *variation.*

Proof. So take $\delta > 1$,

$$\lim_{m\to\infty}\left\{\sup_{|\Delta|<\infty}\sum_{\Delta}\left|\alpha_{s,t}(m)-\alpha_{s,t}\right|^{\delta}\right\}=0$$

Now

$$\alpha_{s,t}(m)-\alpha_{s,t}=(\alpha_{0,t}(m)-\alpha_{0,t})-(\alpha_{0,s}(m)-\alpha_{0,s})$$

and the curve $(\alpha_{0,t}(m)-\alpha_{0,t})$, $t\in[0,1]$ has just a saw-tooth diagram of height $\frac{1}{2^m}$ repeated 2^m times. Since $\delta > 1$, the maximizing dissection picks out the 2^{m+1} largest increments, so

$$\sup_{|\Delta|<\infty}\sum_{\Delta}\left|\alpha_{s,t}(m)-\alpha_{s,t}\right|^{\delta}=2^{m+1}\frac{1}{2^{m\delta}}$$

and the lemma is proved. ∎

Corollary 98 *For any* $p > 1$, *the functions*

$$\alpha_{s,t}^{(j)}(m)=\frac{(\alpha_{s,t}(m))^j}{j!}$$

with $\alpha_{s,t}(m)$ *as above, converge in* p *variation to* $\frac{(t-s)^j}{j!}$.

Proof. To see this, use the following inequality:

$$\begin{aligned}\left|\alpha_{s,t}^{(j)}(m)-\frac{(t-s)^j}{j!}\right|^p &= \left(\frac{1}{j!}\right)^p\left|\alpha_{s,t}(m)^j-(t-s)^j\right|^p\\ &= \left(\frac{1}{j!}\right)^p\left|\alpha_{s,t}(m)-(t-s)\right|^p\left|\sum_{k=0}^{j-1}\alpha_{s,t}(m)^k(t-s)^{j-k-1}\right|^p\\ &\le \left(\frac{1}{(j-1)!}\right)^p\left|\alpha_{s,t}(m)-(t-s)\right|^p\end{aligned}$$

since for $0 \leq s \leq t \leq 1$, $\alpha_{s,t}(m) \leq 1$, $(t-s) \leq 1$. So the convergence of $\alpha^{(j)}_{s,t}(m)$ to $\frac{(t-s)^j}{j!}$ in 1^+ variation, is implied by the convergence of $\alpha_{s,t}(m)$ to $(t-s)$ in 1^+ variation. ■

To round off the section, it is possible to combine homogeneous lie elements of differing tensorial degrees and produce a geometric rough path.

Proposition 99 *For any $1 \leq n < \infty$ and any $\underline{l}^n \in L^n(V)$, i.e. $\underline{l}^n$ is an element of the truncated free Lie Algebra to tensorial level n, there exists $\mathbf{X}^n(\underline{l}^n) \in \Omega G(V)^{n+}$, whose extension to $\Omega G(V)^{n+}_{\infty}$, $\mathbf{X}^{n;\infty}(\underline{l}^n)$ satisfies*

$$\mathbf{X}^{n;\infty}_{s,t}(\underline{l}^n) = \exp((t-s)\underline{l}^n) : \quad (s,t) \in \Gamma_1$$

Proof. For an inductive hypothesis use the statement of the lemma for fixed n to obtain such paths at level $n+1$. To approximate a level $n+1$ signature, the time interval $[0,1]$ is divided into dyadic segments at a given depth m in order to define a sequence of elements in $\Omega G(V)^{(n+1)+}$. This sequence is shown to be Cauchy for the metric d_p where p is any $p > n+1$, hence since each of these $\Omega G(V)^p$ are complete metric spaces, the sequence is convergent in $\Omega G(V)^p$ for all $p > n+1$.

When $n = 1$, Lemma 96 applies.

So in general, it is sufficient to show that for

$$\underline{l}^{n+1} = \sum_{i=1}^{n+1} \underline{l}^{(i)} : \underline{l}^{(i)} \in L^{(i)}(V),$$

there is a Cauchy sequence of geometric rough paths that converge with respect to the p variation metric (2.8) for any $p > n+1$ to the rough path $\mathbf{X}^{n+1}(\underline{l}^{n+1})$ given the inductive hypothesis for n.

From the hypothesis, take the geometric rough path $\mathbf{X}^n(\underline{l}^n) \in \Omega G(V)^{n+}$ where $\underline{l}^n = \sum_{i=1}^n \underline{l}^{(i)}$ and from Lemma 96, the element

$$\mathbf{X}^{n+1}\left(\underline{l}^{(n+1)}\right) \in \Omega G(V)^{(n+1)+}$$

to construct a sequence $\mathbf{X}^{n+1}(\underline{l}^{n+1}; m) \in \Omega G(V)^{(n+1)+}$, $m \geq 1$ defined as follows:

$$\mathbf{X}^{n+1}_{\frac{i}{2^m},t}\left(\underline{l}^{n+1}; m\right) = \exp\left(2\left(t - \frac{i}{2^m}\right)\left(\underline{l}^{(1)} + \cdots + \underline{l}^{(n)}\right)\right) : \frac{i}{2^m} \leq t \leq \frac{2i+1}{2^{m+1}}$$

$$\mathbf{X}^{n+1}_{\frac{2i+1}{2^{m+1}},t}\left(\underline{l}^{n+1}; m\right) = \exp\left(2\underline{l}^{(n+1)}\right) : \frac{2i+1}{2^{m+1}} \leq t \leq \frac{i+1}{2^m}$$

for $t \in [0,1]$, so alternating on dyadic times $\frac{j}{2^{m+1}}$ between following each of the two paths at twice their speed. Lemma 120 implies $\forall m \geq 1$, $\mathbf{X}^{n+1}(\underline{l}^{n+1}; m) \in \Omega G(V)^{(n+1)+}$.

Take the sets A_m of Lemma 97. Then in the free nilpotent lie group of degree $n+1$, $g_{s,t} = \mathbf{X}^{n+1}_{s,t}(\underline{l}^{n+1}; m)$ satisfies the differential equation

$$g_{s,t}^{-1} dg_{s,t} = 2\left(\left(1_{A_m}(t)\left(\underline{l}^{(1)} + \cdots + \underline{l}^{(n)}\right)\right) + \left(1_{[0,1]/A_m}(t)\,\underline{l}^{(n+1)}\right)\right) dt$$

Thus the form of the sequence in $\Omega G(V)^{(n+1)+}$ is

$$\mathbf{X}^{n+1}_{s,t}\left(\underline{l}^{n+1}; m\right) = \left(\exp\left(\begin{array}{c} \left(2\int_s^t 1_{A_m}(u)\,du\right)\left(\underline{l}^{(1)} + \cdots + \underline{l}^{(n)}\right) \\ +\left(2\int_s^t 1_{[0,1]/A_m}(u)\,du\right)\underline{l}^{(n+1)} \end{array}\right)\right)^{n+1}$$

and this exists purely by construction - the contribution in V^{n+1} is a mixture of that inherited from lower order lie elements due to the path for $\underline{l}^n$, with that due to the path for the lie element $\underline{l}^{(n+1)} \in V^{n+1}$. In general, this additive property is not true since the Campbell-Baker-Hausdorff formula introduces additional brackets to contribute. However these non-commuting terms start with $\left[\underline{l}^{(1)} + \cdots + \underline{l}^{(n)}, \underline{l}^{(n+1)}\right]$ and only contribute to tensors of higher order than $n+1$.

Define the collection of tensors $y_{i,j} \in U_j^i(L(V)) = V^{\otimes i} \cap U_j(L(V))$, $0 \leq j \leq i \leq n+1$ as follows :

$$
\begin{aligned}
y_{i,j} &= \pi_{(i)}\left(\left(\underline{l}^{(1)} + \cdots + \underline{l}^{(n+1)}\right)^{\otimes j}\right) \\
&= \sum_{\underline{k}:\|\underline{k}\|=i;|\underline{k}|=j} \underline{l}^{(k_1)} \otimes \cdots \otimes \underline{l}^{(k_j)}
\end{aligned}
$$

The iterated integrals are now expressible in terms of the tensors $y_{i,j}$ as follows:

$$
\begin{aligned}
\mathbf{X}_{s,t}^{(i)}(m) &= \sum_{j=1}^{i} \alpha_{s,t}^{(j)}(m)\, y_{i,j} \\
&= \pi_{(i)}\left(\exp\left(\begin{array}{c} \left(2\int_s^t 1_{A_m}(u)\,du\right)\left(\underline{l}^{(1)} + \cdots + \underline{l}^{(n)}\right) \\ + \left(2\int_s^t 1_{[0,1]/A_m}(u)\,du\right)\underline{l}^{(n+1)} \end{array}\right)\right)
\end{aligned}
$$

for $1 \leq i \leq n$ where the expression for $\alpha_{s,t}^{(j)}(m)$ is:

$$
\begin{aligned}
&\alpha_{s,t}^{(j)}(m) \\
&= 2^j \int \cdots \int_{s \leq u_1 \leq u_2 \leq \cdots \leq u_j \leq t} 1_{A_m}(u_1) \cdots 1_{A_m}(u_j)\, du_1 du_2 \cdots du_j \\
&= \frac{\left(\alpha_{s,t}^{(1)}(m)\right)^j}{j!}
\end{aligned}
$$

and the $(n+1)$'st iterated integral is

$$
\mathbf{X}_{s,t}^{(n+1)}(m) = \sum_{j=2}^{n+1} \alpha_{s,t}^{(j)}(m)\, y_{n+1,j} + \widetilde{\alpha}_{s,t}^{(1)}(m)\, \underline{l}^{(n+1)}
$$

where

$$
\widetilde{\alpha}_{s,t}^{(1)}(m) = 2\int_s^t 1_{[0,1]/A_m}(u)\,du
$$

Thus the difference between the i'th iterated integrals between consecutive steps is as follows. For

$1 \leq i \leq n$

$$\begin{aligned}\mathbf{X}_{s,t}^{(i)}(m) - \mathbf{X}_{s,t}^{(i)}(m+1) &= \sum_{j=1}^{i} \left(\alpha_{s,t}^{(j)}(m) - \alpha_{s,t}^{(j)}(m+1)\right) y_{i,j} \\ &= \sum_{j=1}^{i} \left\{ \frac{\left(\alpha_{s,t}^{(1)}(m)\right)^{j} - \left(\alpha_{s,t}^{(1)}(m+1)\right)^{j}}{j!} \right\} y_{i,j}\end{aligned}$$

and for $i = n+1$

$$\begin{aligned}&\mathbf{X}_{s,t}^{(n+1)}(m) - \mathbf{X}_{s,t}^{(n+1)}(m+1) \\ &= \sum_{j=2}^{n+1} \left\{ \frac{\left(\alpha_{s,t}^{(1)}(m)\right)^{j} - \left(\alpha_{s,t}^{(1)}(m+1)\right)^{j}}{j!} \right\} y_{n+1,j} \\ &\quad + \left(\widetilde{\alpha}_{s,t}^{(1)}(m) - \widetilde{\alpha}_{s,t}^{(1)}(m+1)\right) \underline{l}^{(n+1)}\end{aligned}$$

So now the aim is to show that the geometric multiplicative functionals $\mathbf{X}_{s,t}(m)$ converge in p variation if $(n+1) \leq p$. The decomposition above implies the following control for $1 \leq i \leq n$:

$$\begin{aligned}&P_{\frac{p}{i};0,1}\left(\mathbf{X}_{s,t}^{(i)}(m) - \mathbf{X}_{s,t}^{(i)}(m+1)\right)^{\frac{i}{p}} \\ \leq\ & \sum_{j=1}^{i} P_{\frac{p}{i};0,1}\left(\left\{ \frac{\left(\alpha^{(1)}(m)\right)^{j} - \left(\alpha^{(1)}(m+1)\right)^{j}}{j!} \right\} y_{i,j}\right)^{\frac{i}{p}} \\ \leq\ & \sum_{j=1}^{i} P_{\frac{p}{i};0,1}\left(\frac{\left(\alpha^{(1)}(m)\right)^{j} - \left(\alpha^{(1)}(m+1)\right)^{j}}{j!}\right)^{\frac{i}{p}} \left\|y_{i,j}\right\|\end{aligned}$$

and a similar estimate for $P_{\frac{p}{n+1};0,1}\left(\mathbf{X}^{(n+1)}(m) - \mathbf{X}^{(n+1)}(m+1)\right)^{\frac{n+1}{p}}$. Now, $\mathbf{X}_0^{(1)}(m) = \mathbf{X}_0^{(1)}(0)\ \forall m$, so that there is the crude estimate:

$$\begin{aligned}\sup_{0\leq t\leq 1}\left\|\mathbf{X}_t^{(1)}(m) - \mathbf{X}_t^{(1)}(m+1)\right\| &= \sup_{0\leq t\leq 1}\left\|\mathbf{X}_{0,t}^{(1)}(m) - \mathbf{X}_{0,t}^{(1)}(m+1)\right\| \\ &\leq P_{p;0,1}\left(\mathbf{X}^{(1)}(m) - \mathbf{X}^{(1)}(m+1)\right)^{\frac{1}{p}}\end{aligned}$$

Since $p > n+1 \geq i$, Lemma 98 implies the $\left(\frac{p}{i}\right)$ variation convergence of each $\left(\alpha^{(1)}(m)\right)^{j}$ and hence the finite summability of

$$d_p\left(\mathbf{X}^{n+1}(m),\mathbf{X}^{n+1}(m+1)\right),$$

proving the convergence of the sequence $\mathbf{X}^{n+1}(m)$ in $\Omega G(V)^p$ $\forall p>n+1$ with respect to d_p. Clearly the limit is the multiplicative functional $\mathbf{X}^{n+1}\left(\underline{l}^{n+1}\right)$.

Note that a control for such geometric rough paths is given in the appendix in Lemma 119. ■

Remark 100 *Although this is not actually proved here, it is believed that the straight rough paths define elements of* $\Omega(V)^n$ *and* $\Omega G(V)^{n+}$ *but not* $\Omega G(V)^n$ *and hence relates to remark 17.*

It is possible to return briefly to the discussion of whether the function $P_p(\cdot)$ typically characterizes the p variation of a rough path. For straight rough paths it does:

Corollary 101 *For any Banach space* V *and for any* $n>0$ *there exist geometric rough paths* $\mathbf{X}\in\Omega G(V)^{n+}$ *for the injective norm system (see equation 2.5) and constants* c *such that for infinitely many* i,

$$\left\|\mathbf{X}_{s,t}^{(i)}\right\|\geq\frac{c^{\frac{i}{n}}}{\left(\frac{i}{n}\right)!}.$$

Proof. Pick $\underline{l}^{(n)}\in L^{(n)}\left(\widetilde{V}\right)$ where $\widetilde{V}\subset V$ is a finite dimensional vector subspace of V. Then the element $\mathbf{X}^n\left(\underline{l}^{(n)}\right)$ of Proposition 96 is in $\Omega G(V)^{n+}$ and

$$\begin{aligned}
\left\|\mathbf{X}_{s,t}^{(ni)}\left(\underline{l}^{(n)}\right)\right\|_{inj} &= \left\|\frac{(t-s)^i}{i!}\left(\underline{l}^{(n)}\right)^{\otimes i}\right\|_{inj} \\
&= \frac{(t-s)^i}{i!}\sup_{e_j\in V^*;1\le j\le ni}\frac{(e_1\otimes e_2\otimes\cdots\otimes e_{ni})\circ\left(\underline{l}^{(n)}\right)^{\otimes i}}{\|e_1\|\,\|e_2\|\cdots\|e_{ni}\|} \\
&= \frac{(t-s)^i}{i!}\left(\sup_{e_j\in V^*;1\le j\le n}\frac{(e_1\otimes e_2\otimes\cdots\otimes e_n)\circ\left(\underline{l}^{(n)}\right)}{\|e_1\|\,\|e_2\|\cdots\|e_n\|}\right)^i \\
&= \frac{\left((t-s)\left\|\underline{l}^{(n)}\right\|_{inj}\right)^i}{i!},
\end{aligned}$$

as required. ■

5.2 Measures and $T(V)$

The motivation for looking at rough paths stems largely from the fact that they provide pathwise solutions to differential equations. Not only did Lyons extend the Itô map by constructing solutions to the class of differential equations driven by rough paths (Theorem 27) but he also showed that the map from driving path to solution is continuous for the p variation metric topology. Since a large class of continuous random processes in a vector space V have finite p variation, a task for probabilists is to lift the process by constructing higher order iterated integral processes and so produce a measure on elements of $\Omega G(V)^p$. The questions examined here are related to how much the expectation of the signature random variable specifies the distribution of the signature itself.

As previously mentioned, Chen first remarked that in $T(V)$, the iterated integral sequences of piecewise smooth paths are linearly independent or the same. This is a statement also about grouplike elements of a Hopf algebra as was mentioned in section 3.5.2 and hence this property extends to the collection of signatures of geometric rough paths.

This chapter concerns issues about when there are uniqueness statements concerning the center of mass of the measure on the signature of geometric rough paths and some related issues for rough paths. In particular some possible ideas about extending the Itô map. It is shown how to cast the center of mass problem in terms of a moment uniqueness question, a problem that classically has motivated much research and has strong connections to classical harmonic analysis. It seems however that there are additional issues entailed that muddy these waters.

A motivating example is a Wiener process in $\mathbb{R}^d$. For any $2 < p < 3$, there exists an extension of Wiener space $\left(C\left([0,1], \mathbb{R}^d\right), \mathcal{B}\left(C\left([0,1], \mathbb{R}^d\right)\right), \mu^{\otimes d}\right)$, to a probability space $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right), \mu_{d,G}\right)$ such that the expectation of the signature has the form,

$$\begin{aligned} \mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;\infty}\right) &= \exp\left(\frac{(t-s)}{2d}\sum_{i=1}^{d} e_i^{\otimes 2}\right) \\ &= \sum_{n=0}^{\infty} \mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;(n)}\right) \end{aligned}$$

where $(s,t) \in \Gamma_1$ and where e_i, $1 \leq i \leq d$ form an orthonormal basis for $\mathbb{R}^d$. In fact this means that $\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2}\right)$ itself is a rough path, an element of $\Omega\left(V\right)^p$ $\forall p > 2$, which permits an interpretation as a Backward Itô multiplicative functional associated to driftless path. It would be useful to be able to say, possibly through interpreting a moment problem, that there is an unique distribution on group-like elements of $T(V)$ as an Hopf algebra, that gives this center of mass. It appears that the techniques of the moment problem say that it could be a false claim and do not seem to resolve the issue. The aim of this chapter is to outline the issues involved in this kind of problem.

5.2.1 Some Measure spaces

The idea is to look at a measure space of signatures and logarithms of signatures due to geometric rough paths of finite p variation. So consider $\mathrm{G}_{p,R}(V)$ and $L_{p,R}(V)$ with topologies due to a distance function induced by the p variation metric in the following manner:

Definition 102 *Let $p \geq 1$ and $S_1, S_2 \in \mathrm{G}_{p,R}(V)$. Then define the distance $d_{\mathrm{G}_{p,R}(V)}(S_1, S_2)$ to be*

$$d_{\mathrm{G}_{p,R}(V)}(S_1, S_2) = \inf_{\mathbf{X}_i \in I_p(S_i)} d_p(\mathbf{X}_1, \mathbf{X}_2),$$

where d_p is the p variation distance function and $I_p(S)$ is the set

$$I_p(S) = \left\{ \mathbf{X}^{\lfloor p \rfloor} \in \Omega G(V)^p \text{ s.t. } \mathbf{X}^{\lfloor p \rfloor}_{sig} = S \text{ and } \mathbf{X}^{\lfloor p \rfloor,(1)}_0 = 0 \right\}.$$

In addition, define the distance $d_{\mathrm{L}^R_p(V)}(\cdot,\cdot)$ on $\mathrm{L}_{p,R}(V)$ by

$$d_{\mathrm{L}_{p,R}(V)}(l_1, l_2) = d_{\mathrm{G}_{p,R}(V)}(\exp(l_1), \exp(l_2)).$$

Remark 103 *In the case of bounded variation paths, there is an unique path of minimal length that is parameterized at unit speed associated to some $S = \exp(l) \in \mathrm{G}_{1,R}(V)$ where $l \in \mathrm{L}_{1,R}(V)$ (see [24]). By taking this path to define the distance function, the distance in fact defines a metric that is complete, so in this way, $\mathrm{G}_{1,R}(V)$ and $\mathrm{L}_{1,R}(V)$ have the possibility of a complete metric space topology.*

Using these distance functions, there are measure spaces associated to the sigma algebras generated by the topologies due to the open sets.

Notation 104 *If $O(\mathrm{G}_{p,R}(V))$ and $O(\mathrm{L}_{p,R}(V))$ denote the open sets associated to the topologies due to*

$d_{\mathrm{G}_{p,R}(V)}(\cdot,\cdot)$ and $d_{\mathrm{L}_{p,R}(V)}(\cdot,\cdot)$ in $\mathrm{G}_{p,R}(V)$ and $\mathrm{L}_{p,R}(V)$ respectively then let

$$\begin{aligned}\mathcal{B}(\mathrm{G}_{p,R}(V)) &= \sigma(O(\mathrm{G}_{p,R}(V)))\\ \mathcal{B}(\mathrm{L}_{p,R}(V)) &= \sigma(O(\mathrm{L}_{p,R}(V)))\end{aligned}$$

be the Borel extension sigma algebras generated by these classes of open sets.

For any $p \geq 1$, $(L_{p,R}(V), \mathcal{B}(L_{p,R}(V)))$ and $(\mathrm{G}_{p,R}(V), \mathcal{B}(\mathrm{G}_{p,R}(V)))$ are both well defined measure spaces.

To construct a sigma algebra for each $\Omega G(V)^p$, recall that the spaces of geometric multiplicative functionals of finite p variation are a subsets of the continuous functions from some simplex Γ_T to $T^{\lfloor p \rfloor}(V)$,

$$\Omega G(V)^p_{\lfloor p \rfloor} \subset C\left(\cup_{T>0}\Gamma_T; T^{\lfloor p \rfloor}(V)\right).$$

So for $m \geq 1$ let $\mathbf{C}_m(V)$ be the $\pi-$system of cylinder sets generated by such continuous functionals:

$$\mathbf{C}_m(V) = \left\{\begin{array}{c}\{\mathbf{X}^m_0 \in A_0\} \cap \left(\cap_{i=1}^n \left\{\mathbf{X}^m_{t_i,t_{i+1}} \in A_i\right\}\right)\\ \text{s.t. } n < \infty,\ 0 < t_1 < t_2 < \cdots < t_n = T_{\mathbf{X}^m},\\ \forall 0 \leq i \leq n,\ A_i \in \mathcal{B}(T^m(V))\end{array}\right\},$$

where $\mathcal{B}(T^n(V))$ is the standard Borel sigma algebra on the vector space $T^n(V)$. Define the following sigma algebra:

Notation 105 *For $p \geq 1$, denote by $\mathcal{B}(\Omega G(V)^p)$ the Borel extension of the class of sets $\mathbf{C}_{\lfloor p \rfloor}(V)$*

$$\mathcal{B}(\Omega G(V)^p) = \sigma\left(\mathbf{C}_{\lfloor p \rfloor}(V)\right).$$

Then for any $p \geq 1$, $(\Omega G(V)^p, \mathcal{B}(\Omega G(V)^p))$ is a well defined measure space.

Example 106 *If $2 < p < 3$, then $\left(C([0,1];\mathbb{R}^d), \mathcal{C}(C([0,1];\mathbb{R}^d)), \mu^{\otimes d}\right)$, the probability space of Wiener measure, extends to a probability space denoted by $\left(\Omega G(\mathbb{R}^d)^p, \mathcal{B}(\Omega G(\mathbb{R}^d)^p), \mu_{d,G}\right)$. The route of the extension is to define an area process through a regime which takes piecewise linear approximations at dyadic time points, on an element $\omega \in C([0,1];\mathbb{R}^d)$. $\mu^{\otimes d}$ almost surely, such a routine converges. (See [26] for examples of related work).*

Remark 107 *The action of $sig(\cdot)$ (see definition (13)) when it is restricted to $\Omega G(V)^p$, is continuous with respect to the p variation topology and hence is a measurable function:*

$$sig : (\Omega G(V)^p, \mathcal{B}(\Omega G(V)^p)) \to (\mathrm{G}_{p,R}(V), \mathcal{B}(\mathrm{G}_{p,R}(V))),$$

since the sigma algebra generated by the p variation topology, is contained in $\mathcal{B}(\Omega G(V)^p)$. In addition, the function $P_p : \mathrm{G}_{p,R}(V) \to \mathbb{R}$ given by definition (86) is measurable with respect to $\mathcal{B}(\mathrm{G}_{p,R}(V))$ due to Corollary 15.

5.2.2 A Moment problem

For any $p \geq 1$, the composition of functions $\log(sig(\cdot))$ is continuous from of $(\Omega G(V)^p, \mathcal{B}(\Omega G(V)^p))$ to $(L_{p,R}(V), \mathcal{B}(L_{p,R}(V)))$, because sig itself is continuous. So, with a slight abuse of notation, define the measurable function $\mathbf{l}$, (it will always be clear which p is taken at a particular point):

Definition 108

$$\begin{aligned}\mathbf{l} &: \left(\Omega G(V)^p, \mathcal{B}(\Omega G(V)^p)\right) \to \left(\mathrm{L}_{p,R}(V), \mathcal{B}(\mathrm{L}_{p,R}(V))\right) \\ \mathbf{l}(\omega) &= \log(sig(\omega)).\end{aligned}$$

Take a probability space $(\Omega G(V)^p, \mathcal{B}(\Omega G(V)^p), \mathbb{P})$. Then $\mathbb{E}_{\mathbb{P}}(sig(\omega))$ is a version of the moment generating function of $\mathbf{l}$:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(sig(\omega)) &= \mathbb{E}_{\mathbb{P}}(\exp(\mathbf{l}(\omega))) \\ &= \sum_{j=0}^{\infty} \frac{\mathbb{E}_{\mathbb{P}}\left(\mathbf{l}(\omega)^{\otimes j}\right)}{j!}.\end{aligned}$$

Since both maps log and exp are bijections, the random variable $\mathbf{l}(\omega)$ seems to be the best description of $sig(\omega)$ as far as information content is concerned (certainly it is true algebraically). $\mathbf{l}(\omega)$ represents how, for want of a better description, a rough path ω contributes to differing geometric aspects of its course. The component in V is a total linear increment, or displacement from start to finish, while the component in $[V, V]$ has an interpretation as a tensorial representation of the area the curve makes when projected onto two dimensional subspaces of V. A full description of what all these components represent remains unknown however. It does not appear that the components are entirely independent though, as discussed in section 5.1. These issues aside, the random variable $\mathbf{l}(\omega)$ seems to be a pertinent description of the signature $sig(\omega)$.

For a probabilist, a natural question relating to moment generating functions, is whether the point $\mathbb{E}_{\mathbb{P}}(sig(\omega)) \in T(V)$ is a defining point of the measure. When can there exist two probability measures,

$\mathbb{P}_1, \mathbb{P}_2$ with

$$\mathbb{E}_{\mathbb{P}_1}(sig(\omega)) = \mathbb{E}_{\mathbb{P}_2}(sig(\omega)),$$

but whose distribution functions are different ? The problem is a form of what is known as the Hamburger moment problem and as such, a complete solution is not known. It is clearly related to Laplace transforms and so to Choquet simplicies. Take for example a set K of signatures that is metrizable and compact. Then Choquet's theory of simplicies [13], says that points in $\overline{K}$, the convex hull of K, are uniquely specified by a measure supported on K, the extreme points of $\overline{K}$. In other words, the barycenter of the measure specifies the measure itself. The measures involved here typically are not supported on some compact set unfortunately.

The Hamburger Moment Problem

There are two conventional issues of probability related to moment sequences with the following data. Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and set $m_k = \mathbb{E}_P\left(X^k\right)$. The first issue is to ascertain when there exists a random variable with moments that are a given sequence of numbers m_k. The second issue concerns uniqueness and when can random variables with different distributions have the same moment sequence. Or to put it another way, if a moment sequence can arise from different distributions. While the first issue is resolved, the second and relevant problem is not. Perhaps the most useful or powerful criteria currently known comes from research in harmonic analysis in the 1930's by Torsten Carleman. It is stated as follows for what is known as the Hamburger case, where the support of the distribution is the whole of the real line:

Criterion 109 *Let $F(x)$ be a probability distribution function of a random variable on $\mathbb{R}$ and set $m(k) = \int_{-\infty}^{\infty} x^k dF(x)$. If*

$$\sum_{k=1}^{\infty} \frac{1}{m(2k)^{\frac{1}{2k}}} = \infty,$$

then there is no other distribution function than F with moment sequence $m(k)$.

The gap in the understanding is to find a necessary condition for the conclusion to hold although other sufficient criteria do exist. For example a condition of Krein exists for when there is knowledge of the form of the density function.

For a joint distribution function $F(x_1, \ldots, x_n)$ of a random variable on $\mathbb{R}^n$, the Carleman condition for uniqueness of the joint distribution becomes

$$\sum_{k=1}^{\infty} \frac{1}{(m_1(2k) + \cdots + m_n(2k))^{\frac{1}{2k}}} = \infty \tag{5.7}$$

where $m_i(2k) = \int_{\mathbb{R}^K} x_i^k dF(x)$, see Prohorov and Rozanov [31].

A distribution on an infinite dimensional random variable is harder to characterize in terms of it's moments. The tack here is to examine finite collections of linear functions of the random variable, to see if these joint distributions can be characterized by the moment sequence.

Next is a description of how to obtain such moment sequences from a point $\mathbb{E}_{\mathbb{P}}(sig(\omega)) \in T(V)$.

Symmetric Decomposition of $T(V)$.

The relationship between $T(V)$ and $U(L(V))$ provides a route to identify the mixed moments of a distribution on some $L_p(V)$ by way of the symmetric decomposition of $U(L(V))$ described in section

3.2.1. Consider $\mathcal{L}^*(V)$, the graded dual of $L(V)$.

Definition 110 *The graded dual of* $\mathrm{L}(V)$ *is the collection of linear functions on* $\mathrm{L}(V)$ *that belong to the dual space of some truncated free lie algebra:*

$$\mathcal{L}^*(V) = \cup_{n\geq 1} (\mathrm{L}^n(V))^* .$$

If $\{\nu_i\}_{i=1}^n \in \mathcal{L}^*(V)$ and $l \in L_{p,R}(V)$, the values

$$\nu_1(l)^{j_1} \cdots \nu_n(l)^{j_n} \quad (j_1, \ldots, j_n) \in \mathbb{N}^+$$

are linear functions of $\exp(l) \in \mathrm{G}_{p,R}(V)$. The next proposition illustrates this and shows how using the symmetric decomposition, enables the extraction of the random variables $\{\nu_i(\mathbf{l})\}_{i=1}^n \in \mathbb{R}^n$ from $\mathbb{E}_{\mathbb{P}}(\exp(\mathbf{l}))$, if $\mathbf{l}$ is a random variable taking values in some $L_{p,R}(V)$.

First recall the convolution product $\star$ for the algebra $Hom(H,H)$ of a Hopf algebra H (3.6), the coproduct $\Delta : T(V) \to T(V) \otimes_H T(V)$ of equation (3.12) and the counit ε defined by equation (3.13). Additionally, in section 6.5, the projection $\Pi_1 : U(L(V)) \to L(V)$ is specified as an element of $Hom(T(V), T(V))$.

Proposition 111 *Take a probability space* $(\Omega, \mathcal{F}, P)$ *and a measurable function*

$$m: (\Omega, \mathcal{F}) \to (\mathrm{L}_{p,R}(V), \mathcal{B}(\mathrm{L}_{p,R}(V))) .$$

Suppose that $(\nu_1, \nu_2, \ldots, \nu_n) \in (\mathrm{L}^*(V))^n$ *and take* $\underline{i} = (i_1, \ldots, i_n)$ *with* $\|\underline{i}\| = \sum_{j=1}^n i_j$*. By viewing the map*

$$\nu_1 \circ \Pi_1 \in Hom(\mathrm{T}(V), \mathbb{R})$$

as an element of $Hom(\mathrm{T}(V), \mathrm{T}(V))$, *it's possible to express the mixed moments of* $(\nu_1(\mathbf{m}(\omega)), \ldots, \nu_n(\mathbf{m}(\omega)))$ *by*

$$\begin{aligned}
& \mathbb{E}_P\left(\nu_1^{i_1}(\mathbf{m}(\omega)) \cdots \nu_n^{i_n}(\mathbf{m}(\omega))\right) \\
= \; & \varepsilon\left(\left(\nu_1 \circ \Pi_1\right)^{\star i_1} \star \cdots \star \left(\nu_n \circ \Pi_1\right)^{\star i_n} \circ \mathbb{E}_P(\exp(\mathbf{m}(\omega)))\right) \\
= \; & \varepsilon\left\{\begin{array}{c} \left(\left(\nu_1 \circ \Pi_1\right)^{\otimes_H i_1} \otimes_H \cdots \otimes_H \left(\nu_n \circ \Pi_1\right)^{\otimes_H i_n}\right) \circ \\ \left(\Delta^{(\|\underline{i}\|-1)} \circ \mathbb{E}_P(\exp(\mathbf{m}(\omega)))\right) \end{array}\right\}.
\end{aligned}$$

Proof. Section 6.5 implies that the linear maps

$$\Pi_1^{\otimes_H j} \circ \Delta^{(j-1)} : T(V) \to T(V)^{\otimes_H j}$$

obey

$$\Pi_1^{\otimes_H j} \circ \Delta^{(j-1)}\left(m^{\otimes k}\right) = \left\{\begin{array}{c} j!m^{\otimes_H j} \; j = k \\ 0 \; j \neq k \end{array}\right\},$$

for any $m \in L(V)$. Hence,

$$\begin{aligned}
& \Pi_1^{\otimes_H j} \circ \Delta^{(j-1)}\left(\mathbb{E}_P(\exp(\mathbf{m}(\omega)))\right) \\
= \; & \mathbb{E}_P\left(\sum_{k=0}^{\infty} \Pi_1^{\otimes_H j} \circ \Delta^{(j-1)}\left(\frac{\mathbf{m}(\omega)^{\otimes k}}{k!}\right)\right) \\
= \; & \mathbb{E}_P\left(\mathbf{m}(\omega)^{\otimes_H j}\right) \in T(V)^{\otimes_H j},
\end{aligned}$$

so that

$$\begin{aligned}
& \varepsilon\left(\left(\nu_1 \circ \Pi_1\right)^{\star i_1} \star \cdots \star \left(\nu_n \circ \Pi_1\right)^{\star i_n} \circ \mathbb{E}_P(\exp(\mathbf{m}(\omega)))\right) \\
= \; & \varepsilon\left(\left(\nu_1{}^{\otimes i_1} \otimes \cdots \otimes \nu_n{}^{\otimes i_n}\right) \circ \Pi_1^{\otimes_H j} \circ \Delta^{(j-1)}\left(\mathbb{E}_P(\exp(\mathbf{m}(\omega)))\right)\right) \\
= \; & \mathbb{E}_P\left(\nu_1(\mathbf{m}(\omega))^{\otimes i_1} \otimes \cdots \otimes \nu_n(\mathbf{m}(\omega))^{\otimes i_n}\right)
\end{aligned}$$

and the proposition is concluded. ■

This proposition confirms the rôle of $\mathbb{E}_{\mathbb{P}}(sig(\omega))$ as the moment generating function of $\mathbf{l}(\omega)$ for a measure space $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right), \mathbb{P}\right)$. The methodology of this proposition is not particularly easy to use in practice to evaluate the moment sequences or examine it's behaviour. The following example avoids this problem because the moments are known in the literature. The example is that of the distribution on $L_{p,R}(V)$ for $2 < p < 3$ generated by geometric rough paths associated to Wiener processes with the Lévy area process.

Proposition 112 *Take $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right), \mu_{d,G}\right)$ to be the probability space of example 106. Then the distribution of the random variable*

$$\mathbf{l}^2(\omega) = \pi_2(\mathbf{l}(\omega)) \in \mathrm{L}^2\left(\mathbb{R}^d\right)$$

is determined by

$$\mathbb{E}_{\mu_{d,G}}(sig(\omega)) = \mathbb{E}_{\mu_{d,G}}(\exp(\mathbf{l}(\omega))).$$

Proof. Proposition 111 explains how to obtain the mixed moments of the random variables. Now $\mathbf{l}^2(\omega)$ decomposes in terms of an orthonormal basis $\{e_i\}_{i=1}^d$ as follows

$$\mathbf{l}^2(\omega) = \sum_{1\leq i\leq d} \frac{W_i(\omega)}{\sqrt{d}} e_i + \sum_{1\leq i<k\leq d} A_{i,k}(\omega) \frac{[e_i, e_k]}{2d}.$$

For $1 \leq i \leq d$, each W_i has the same distribution as a normal random variable $W \sim N(0,1)$. For $1 \leq i < k \leq d$, $A_{i,k}$ has the distribution of the total area of the process which is the almost sure limit of the area processes associated to piecewise linear approximations at dyadic time intervals to a 2 dimensional Brownian motion. So all $A_{i,k}$ are identically distributed according to the same Lévy area random variable A. The even moments of W and A are given by

$$\begin{aligned}\mathbb{E}\left(W^{2j}\right) &= \frac{(2j)!}{j!2^j}\\ \mathbb{E}\left(A^{2j}\right) &= \frac{4\,(2j)!}{(2j+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2j+1}}.\end{aligned}$$

The first form is well known while the second is due to Lévy [27], in which the characteriztic function of the area random variable is found.

According to equation (5.7), it is sufficient to show that $\sum_{j\geq 1}\alpha_j = \infty$, where α_j are given by

$$\begin{aligned}\alpha_j &= \left(\sum_{1\leq i\leq d}\mathbb{E}_{\mu_{d,G}}\left(W_i^{2j}\right)+\sum_{1\leq i<j\leq d}\mathbb{E}_{\mu_{d,G}}\left(A_{i,k}^{2j}\right)\right)^{-\frac{1}{2j}}\\ &= \left(\frac{d\,(2j)!}{j!2^j}+\frac{2d\,(d-1)\,(2j)!}{(2j+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2j+1}}\right)^{-\frac{1}{2j}}\end{aligned}$$

There is an uniform bound on the sequence of sums $C_j = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2j+1}}$. Denote it C, so that it gives the inequality

$$\begin{aligned}\alpha_j &\geq \left((2j)!\left(\frac{d}{j!2^j}+\frac{2d\,(d-1)}{(2j+1)}C\right)\right)^{-\frac{1}{2j}}\\ &\geq \frac{1}{((2j)!)^{\frac{1}{2j}}}\left(\frac{d}{2^j}+\frac{2d\,(d-1)}{(2j+1)}C\right)^{-\frac{1}{2j}}\\ &\geq \frac{\widetilde{C}}{((2j)!)^{\frac{1}{2j}}}\end{aligned}$$

for some constant $\widetilde{C} > 0$. Stirling's formula then says that as $j \to \infty$,

$$\begin{aligned}((2j)!)^{\frac{1}{2j}} &\sim \left((2j)^{2j+\frac{1}{2}}\,e^{-2j}\sqrt{2\pi}\right)^{\frac{1}{2j}}\\ &\sim \frac{2j}{e},\end{aligned}$$

so that

$$\sum_{j=1}^{\infty}\alpha_j \geq \widehat{C}\sum_{j=1}^{\infty}\frac{1}{j}=\infty,$$

for some other constant $\widehat{C}>0$. Thus equation (5.7) is satisfied and the distribution of $\mathbf{l}^2$ is determined by it's moments which in turn are functions of $\mathbb{E}_{\mu_{d,G}}(\exp(\mathbf{l}(\omega)))$ by Proposition 111. ■

It is not known whether components of tensorial degree greater than 2 of the distribution of $\mathbf{l}(\omega)$ due to $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right), \mu_{d,G}\right)$ are determined by theirs moment sequences. For $i=1,2$, $\nu\in\left(L^{(i)}\left(\mathbb{R}^d\right)\right)^*$, $m_{\nu(\mathbf{l})}(2j)^{\frac{1}{2j}}\sim j^{\frac{i}{2}}$. If this continues for $i>2$, then Carleman's criteria will not hold. It is worth bearing in mind that if $\nu_1\in\left(L^{(1)}\left(\mathbb{R}^d\right)\right)^*$, then $Y=\nu_1(\mathbf{l})^3$ has a moment sequence that is matched by a random variable with a different distribution. This means that if $\nu_3\in\left(L^{(3)}\left(\mathbb{R}^d\right)\right)^*$, then if the moments of $\nu_3(\mathbf{l})$ behave like those of Y, they will not satisfy Carleman's criteria. However, just as the distribution of $\nu_1(\mathbf{l})$ is determined and hence that of Y is also, possibly a similar effect holds for $\nu_3(\mathbf{l})$ also: the random variables for different components themselves are not independent and only through knowing the whole sequence of $\mathbf{l}(\omega)$ is it possible to tell if $\mathbf{l}(\omega)\in L_{p,G}\left(\mathbb{R}^d\right)$.

A different way of looking at the problem is through the random variable $P_p(\cdot)$ of Definition 86. Consider a probability space $(\mathrm{G}_p(V), \mathcal{B}(\mathrm{G}_p(V)), \mathbb{P})$. If the moments of P_p satisfy Carleman's condition, then so is the distribution of the first $\lfloor p\rfloor$ components of $\mathcal{L}^*(V)$. To start, a preliminary lemma is proved:

Lemma 113 *Suppose that $X\geq 0$ is a random variable on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ whose moments $m_X(k)=\mathbb{E}_{\mathbb{P}}\left(X^k\right)$ satisfy Carleman's criteria (109). Then if*

$$|Y(\omega)|\leq\sum_{i=1}^{n}\beta_i X(\omega)^{\alpha_i}\ \textit{almost surely,}$$

where $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 1$*and* $\beta_i \geq 0$ *for* $1 \leq i \leq n$*, then the moments of* Y*,* $m_Y(k) = \mathbb{E}_{\mathbb{P}}\left(Y^k\right)$ *also satisfies Carleman's criteria (109). In addition the constant* $C = n\sum_{i=1}^{n}|\beta_i|$ *is such that if for all* $k \geq 1$*,* $m_X(2k) \leq 1$*, then*

$$m_Y(2k)^{\frac{1}{2k}} \leq C,$$

or if not, then there exists $k_0 > 0$ *such that* $k \geq k_0$*,*

$$m_Y(2k)^{\frac{1}{2k}} \leq Cm_X(2k)^{\frac{1}{2k}}.$$

Proof. Observe the inequality:

$$\begin{aligned} m_Y(2k) &\leq \mathbb{E}_{\mathbb{P}}\left(\left(\sum_{i=1}^{n}\beta_i X(\omega)^{\alpha_i}\right)^{2k}\right) \\ &\leq n^{2k}\sum_{i=1}^{n}\beta_i^{2k}\mathbb{E}_{\mathbb{P}}\left(X(\omega)^{2k\alpha_i}\right) \\ &\leq n^{2k}\sum_{i=1}^{n}\beta_i^{2k}m_X(2k)^{\alpha_i}. \end{aligned}$$

Suppose now that for all $k \geq 1$, $m_X(2k) \leq 1$. Then there is a bound independent of k:

$$\begin{aligned} m_Y(2k)^{\frac{1}{2k}} &\leq n\left(\sum_{i=1}^{n}\beta_i^{2k}\right)^{\frac{1}{2k}} \\ &\leq n\sum_{i=1}^{n}|\beta_i|, \end{aligned}$$

which immediately implies Y's moments satisfy Carleman's criteria.

If not then observe that for $1 \leq j \leq k$,

$$m_X(2k)^{\frac{1}{2k}} \leq m_X(2j)^{\frac{1}{2j}},$$

so if it isn't true that for all $k \geq 1$, $m_X(2k) \leq 1$, then set

$$k_0 = \inf\{k|\ m_X(2k) \geq 1\},$$

so that $k \geq k_0$ means $m_X(2k) \geq 1$. Then, since $0 \leq \alpha_i \leq 1$, for $1 \leq i \leq n$, if $k \geq k_0 = \inf\{k|\ m_X(2k) \geq 1\}$, then $m_X(2k)^{\alpha_i} \leq m_X(2k)$ and so

$$\begin{aligned} m_Y(2k)^{\frac{1}{2k}} &\leq n\left(\sum_{i=1}^{n} \beta_i^{2k} m_X(2k)\right)^{\frac{1}{2k}} \\ &\leq n\sum_{i=1}^{n} |\beta_i|\, m_X(2k)^{\frac{1}{2k}} \end{aligned}$$

and hence

$$\sum_{k\geq k_0} \frac{1}{m_Y(2k)^{\frac{1}{2k}}} \geq \left(n\sum_{i=1}^{n}|\beta_i|\right)^{-1} \sum_{k\geq k_0} \frac{1}{m_X(2k)^{\frac{1}{2k}}} = \infty$$

so that Y also satisfies Carleman's criteria (109). ■

Proposition 114 *Let V be a Banach space and $(\mathrm{G}_{p,R}(V), \mathcal{B}(\mathrm{G}_{p,R}(V)), \mathbb{P})$ a probability space such that the random variable $P_p(\omega)$ satisfies Carleman's criteria (109). Then for any*

$$(\nu_1, \nu_2, \ldots, \nu_n) \in \left(\left(\mathrm{L}^{\lfloor p \rfloor}(V)\right)^*\right)^n,$$

the joint distribution of the random variable Ψ given by

$$\Psi(\omega) = (\nu_1(\log(\omega)), \nu_2(\log(\omega)), \ldots, \nu_n(\log(\omega))) \in \mathbb{R}^n,$$

has a distribution determined by $\mathbb{E}_P(\omega)$.

Proof. Consider the continuous linear maps

$$\Pi_1|_{V^{\otimes i}} : V^{\otimes i} \to L^{(i)}(V)\ \ \forall i \geq 1,$$

due to section 6.5. They are bounded linear maps, hence have operator norms $0 < \rho_i < \infty$:

$$\|\Pi_1|_{V^{\otimes i}}(\underline{v})\| \leq \rho_i \|\underline{v}\|\ \ \forall \underline{v} \in V^{\otimes i},\ \forall i \geq 1.$$

By definition the components of $\omega \in \mathrm{G}_{p,R}(V)$ satisfy

$$\left\|\omega^{(i)}\right\| \leq \frac{P_p(\omega)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}\ \forall i \geq 1,$$

so that

$$\begin{aligned}\left\|\log(\omega)^{(i)}\right\| &= \left\|\Pi_1|_{V^{\otimes i}}\left(\omega^{(i)}\right)\right\| \\ &\leq \rho_i \frac{P_p(\omega)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}\end{aligned}$$

Now if $\nu \in \left(L^{\lfloor p \rfloor}(V)\right)^*$, there is the decomposition

$$\nu = \sum_{i=1}^{\lfloor p \rfloor} \nu^{(i)} \text{ where } \nu^{(i)} \in \left(L^{(i)}(V)\right)^*$$

and $\nu^{(i)}$ has norm $\left\|\nu^{(i)}\right\| < \infty$. Combining this,

$$\begin{aligned}|\nu(\log(\omega))| &\leq \sum_{i=1}^{\lfloor p \rfloor} \left\|\nu^{(i)}\left(\log(\omega)^{(i)}\right)\right\| \\ &\leq \sum_{i=1}^{\lfloor p \rfloor} \rho_i \left\|\nu^{(i)}\right\| \frac{P_p(\omega)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!},\end{aligned}$$

so if $\beta_i(\nu) = \frac{\rho_i\left\|\nu^{(i)}\right\|}{\beta\left(\frac{i}{p}\right)!}$, $\alpha_i = \frac{i}{p}$ $1 \leq i \leq \lfloor p \rfloor$, then Lemma 113 says that $m_\nu(k)$, the moments of $\nu(\log(\omega))$, satisfy Carleman's criteria.

If $m_{\nu_i}(k) = \mathbb{E}_P\left(\nu_i(\log(\omega))^k\right)$, $1 \leq i \leq \lfloor p \rfloor$ then by condition (5.7), to prove that the joint distribution of Ψ is determined by $\mathbb{E}_P(\omega)$, it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{1}{\left(m_{\nu_1}(2k) + \cdots + m_{\nu_n}(2k)\right)^{\frac{1}{2k}}} = \infty.$$

Lemma (113) also guarantees this since if for all $k \geq 1$, $m_X(2k) \leq 1$, then there exist constants C_i such that $m_{\nu_i}(2k) \leq C_i^{2k}$ for $1 \leq i \leq n$ and so

$$\begin{aligned}
\left(m_{\nu_1}(2k) + \cdots + m_{\nu_n}(2k)\right)^{\frac{1}{2k}} &\leq \left(C_1^{2k} + \cdots + C_n^{2k}\right)^{\frac{1}{2k}} \\
&\leq C_1 + \cdots + C_n,
\end{aligned}$$

or if not, then for $k \geq k_0$,

$$\begin{aligned}
\left(m_{\nu_1}(2k) + \cdots + m_{\nu_n}(2k)\right)^{\frac{1}{2k}} &\leq \left(C_1^{2k} + \cdots + C_n^{2k}\right)^{\frac{1}{2k}} m_X(2k)^{\frac{1}{2k}} \\
&\leq (C_1 + \cdots + C_n)\, m_X(2k)^{\frac{1}{2k}},
\end{aligned}$$

Hence either way, the inequalities ensure that the mixed moments of Ψ guarantee that there is an unique joint distribution with Ψ's moment sequence. ■

The constraint on the random variable P_p in this proposition seems a reasonable property to check for distributions, however there has been no attempt to use it in fact. The first distribution to examine would be do due Wiener measure, though this has not been achieved. If true, it does not add to the conclusion of Proposition 112.

One thought to bear in mind if investigating this problem further, is that a non-uniqueness of measure of the distribution at any finite dimensional level may be prohibited either in the infinite limit or if not, by insisting that the measure is supported on $\mathrm{G}_{p.R}(V)$, which is hard to determine.

5.2.3 Other developments

The original purpose of looking at measures on geometric rough paths was that they can provide an interpretation of some non-geometric rough paths. One hope is that for rough paths which are expressible in an unique way as an expectation of geometric rough paths, a corresponding Itô integral can be defined and interpreted in some way as an expectation of integrals of geometric rough paths.

While the above discussion concerns looking at the full iterated integral sequence in one chunk, a weakened version of an uniqueness statement does exist for the case of the whole rough path. To start, take the simple case of Wiener motion in $\mathbb{R}^d$ and calculate the expectation of the first two iterated integrals. By independence of increments, the result should be a multiplicative functional. In addition, it has finite 2 variation.

Lemma 115 *Take again* $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right), \mu_{d,G}\right)$ *to be the probability space of Example 106. For* $(s,t) \in \Gamma_1$ *set* $\mathbf{X}^2_{s,t} = \mathbb{E}_{\mu_{d,G}}\left(\omega^2_{s,t}\right)$. *Then* $\mathbf{X}^2 \in \Omega\left(V\right)^2$ *and* $\forall (s,t) \in \Gamma_1$

$$\begin{aligned} \mathbf{X}^2_{s,t} &= \mathbb{E}_{\mu_{d,G}}\left(\omega^2_{s,t}\right) \\ &= \left(1, 0, \frac{(t-s)}{2d}\sum_{i=1}^{d} e_i^{\otimes 2}\right) \end{aligned}$$

where $e_1, \ldots e_d$ *form an orthonormal basis for* $\mathbb{R}^d$.

Proof. For any $(s,t) \in \Gamma_1$,

$$\omega_{s,t} = \left(\begin{array}{c} 1 + \sum_{1\leq i\leq d} \frac{W_{i;s,t}(\omega)}{\sqrt{d}} e_i + \frac{1}{2}\left(\sum_{1\leq i\leq d} \frac{W_{i;s,t}(\omega)}{\sqrt{d}} e_i\right)^{\otimes 2} \\ + \sum_{1\leq i<k\leq d} \frac{A_{i,k;s,t}(\omega)}{d} \frac{[e_i, e_k]}{2} \end{array}\right)$$

where $W_{i;s,t} \sim N(0,(t-s))$, $1 \leq i \leq d$ are independent normal random variables and $A_{i,k;s,t}(\omega)$ are Lévy Area random variables (see [27]) associated to the pair $(W_{i;s,t}, W_{j;s,t})$ $1 \leq i < j \leq d$ for a time $(t-s)$. Hence

$$\begin{aligned}
\mathbb{E}_{\mu_{d,G}}\left(\sum_{1\leq i\leq d} W_{i;s,t}(\omega)\, e_i\right) &= 0 \\
\mathbb{E}_{\mu_{d,G}}\left(\left(\sum_{1\leq i\leq d} W_{i;s,t}(\omega)\, e_i\right)^{\otimes 2}\right) &= (t-s)\sum_{i=1}^{d} e_i^{\otimes 2} \\
\mathbb{E}_{\mu_{d,G}}\left(\sum_{1\leq i<k\leq d} A_{i,k;s,t}(\omega)\,\frac{[e_i,e_k]}{2}\right) &= 0
\end{aligned}$$

for $1 \leq i < k \leq d$ and so

$$\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2}\right) = \left(1, 0, \frac{(t-s)}{2d}\sum_{i=1}^{d} e_i^{\otimes 2}\right).$$

Clearly the functional $\mathbf{X}^2$ is multiplicative and if

$$\omega(s,t) = \beta\frac{(t-s)}{2d}\left\|\sum_{i=1}^{d} e_i^{\otimes 2}\right\|,$$

then ω is a control for $\mathbf{X}^2$ when $p \geq 2$, so that $\mathbf{X}^2 \in \Omega(V)^2$ ∎

This extends to give a form for the expectation of the signature process with regards to the extension of $\mathbf{X}^2$ due to Theorem 8.

Proposition 116 *The extension $\mathbf{X}^{2;\infty} \in \Omega(V)^2_\infty$ of $\mathbf{X}^2$ from the previous lemma has the form*

$$\begin{aligned}
\mathbf{X}_{s,t}^{2;\infty} &= \exp\left(\frac{(t-s)}{2d}\sum_{i=1}^{d} e_i^{\otimes 2}\right) \\
&= \mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;\infty}\right)
\end{aligned}$$

Proof. Clearly $\mathbf{X}^{2;\infty}$ is multiplicative and agrees with $\mathbf{X}^2$ when projected to $\Omega(V)^2$. Theorem 8 implies that the extension of $\mathbf{X}^2$ to $\Omega(V)^2_\infty$ will be unique and that the behaviour of the norms of higher order iterated integrals will be influenced by the control of $\mathbf{X}^2$. Hence all that needs to be checked is the control of the norms of the higher order iterated integrals. If n is odd, the n'th iterated integral of $\mathbf{X}^{2;\infty}_{s,t}$ is zero. So consider the norm of the $2n$'th iterated integral:

$$\begin{aligned}\left\|\mathbf{X}^{2;(2n)}_{s,t}\right\| &= \frac{1}{n!}\left(\frac{(t-s)}{2}\right)^n \left\|\left(\sum_{i=1}^d e_i^{\otimes 2}\right)^{\otimes n}\right\| \\ &\le \frac{1}{n!}\left(\frac{(t-s)}{2}\right)^n \left\|\left(\sum_{i=1}^d e_i^{\otimes 2}\right)\right\|^n \\ &\le \frac{\omega(s,t)^n}{\beta^n n!} < \frac{\omega(s,t)^n}{\beta n!},\end{aligned}$$

which is as required since $\beta \ge \beta(2) \ge 1$.

Now Chen's identity for geometric rough paths, equation (2.1), means that the functional

$$\mathbb{E}_{\mu_{d,G}}\left(\omega^{2;\infty}_{\cdot,\cdot}\right) : \Gamma_1 \to T(V),$$

is multiplicative: for all $(s,r),(r,t) \in \Gamma_1$,

$$\begin{aligned}\mathbb{E}_{\mu_{d,G}}\left(\omega^{2;\infty}_{s,t}\right) &= \sum_{n=0}^{\infty} \mathbb{E}_{\mu_{d,G}}\left(\omega^{2;(n)}_{s,t}\right) \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \mathbb{E}_{\mu_{d,G}}\left(\omega^{2;(j)}_{s,r} \otimes \omega^{2;(n-j)}_{r,t}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathbb{E}_{\mu_{d,G}}\left(\omega^{2;(n)}_{s,r}\right) \otimes \mathbb{E}_{\mu_{d,G}}\left(\omega^{2;(m)}_{r,t}\right) \\ &= \mathbb{E}_{\mu_{d,G}}\left(\omega^{2;\infty}_{s,r}\right) \otimes \mathbb{E}_{\mu_{d,G}}\left(\omega^{2;\infty}_{r,t}\right).\end{aligned}$$

Since $\mathbb{E}_{\mu_{d,G}}\left(\omega^{2;2}_{\cdot,\cdot}\right)$ agrees with $\mathbf{X}^{2;\infty}_{\cdot,\cdot}$, the multiplicative extensions will agree if $\mathbb{E}_{\mu_{d,G}}\left(\omega^{2;\infty}_{s,t}\right)$ has finite

2 variation, which is confirmed by Lemma 117. Hence the equality

$$\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;\infty}\right) = \exp\left(\frac{(t-s)}{2d}\sum_{i=1}^{d} e_i^{\otimes 2}\right).$$

■

The following lemma confirms that the norms of the tensorial components of $\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;\infty}\right)$ behave like a finite $p \geq 2$ rough path.

Lemma 117 *Consider the probability space* $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right), \mu_{d,G}\right)$. *For any* $n > 0$ *and any* $(s,t) \in \Gamma_1$,

$$\begin{aligned} \mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;(2n-1)}\right) &= 0 \\ \left\|\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;(2n)}\right)\right\| &\leq \frac{\left\|\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;(2)}\right)\right\|^n}{n!} = \frac{(t-s)^n\left\|\mathbb{E}_{\mu_{d,G}}\left(\omega_{0,1}^{2;(2)}\right)\right\|^n}{n!}. \end{aligned}$$

Proof. First observe that the distribution of the n'th iterated integral scales with time according to $t^{\frac{n}{2}}$. Therefore, for all $n > 0$,

$$\left\|\mathbb{E}_{\mu_{d,G}}\left(\omega_{s,t}^{2;(n)}\right)\right\| = (t-s)^{\frac{n}{2}}\left\|\mathbb{E}_{\mu_{d,G}}\left(\omega_{0,1}^{2;(n)}\right)\right\|,$$

so it is enough to prove the statement for $(s,t) = (0,1)$. So for $n \geq 0$ set

$$\left\|\mathbb{E}_{\mu_{d,G}}\left(\omega_{0,1}^{2;(n)}\right)\right\| = C_n < \infty.$$

The statement is true $m = 1$, so assume it is true for any $m \leq n$. Then using the scaling property of the distribution, Chen's identity, the independence of increments for the measure $\mu_{d,G}$ and the induction hypothesis,

$$\begin{aligned}
C_{2n+1} &= \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{0,1}^{2;(2n+1)} \right) \right\| \\
&= \left\| \sum_{j=0}^{2n+1} \mathbb{E}_{\mu_{d,G}} \left(\omega_{0,\frac{1}{2}}^{2;(j)} \right) \otimes \mathbb{E}_{\mu_{d,G}} \left(\omega_{\frac{1}{2},1}^{2;(2n+1-j)} \right) \right\| \\
&\leq \sum_{j=0}^{2n+1} \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{0,\frac{1}{2}}^{2;(j)} \right) \right\| \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{\frac{1}{2},1}^{2;(2n+1-j)} \right) \right\| \\
&\leq \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{0,\frac{1}{2}}^{2;(2n+1)} \right) \right\| + \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{\frac{1}{2},1}^{2;(2n+1)} \right) \right\| \\
&\leq 2^{1-n} C_{2n+1},
\end{aligned}$$

which gives $C_{2n+1} = 0$. Then for the $2(n+1)$'st iterated integral,

$$\begin{aligned}
C_{2n+2} &= \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{0,1}^{2;(2n+2)} \right) \right\| \\
&\leq \sum_{j=0}^{n+1} \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{0,\frac{1}{2}}^{2;(2j)} \right) \right\| \left\| \mathbb{E}_{\mu_{d,G}} \left(\omega_{\frac{1}{2},1}^{2;(2(n-j))} \right) \right\| \\
&\leq 2^{-n} C_{2n+2} + 2^{-(n+1)} \sum_{j=1}^{n} \frac{C_2^{n+1}}{j!\,(n-j)!} \\
&\leq 2^{-n} C_{2n+2} + \frac{2^{-(n+1)} C_2^{n+1}}{(n+1)!} \left(2^{n+1} - 2 \right),
\end{aligned}$$

which can be rearranged to give the sought after

$$C_{2n+2} \leq \frac{C_2^{n+1}}{(n+1)!},$$

completing the induction. ■

An uniqueness of measure statement is now possible for the non geometric rough path $\mathbf{X}^2_{\cdot,\cdot}$ of Lemma 115. Among all measures η on $\Omega G(V)^{2+}$, the following statement says that there is only one which satisfies the condition

$$\mathbb{E}_\eta \left(\omega_{s,t}^{2;\infty} \right) = \mathbf{X}_{s,t}^{2;\infty} \, \forall (s,t) \in \Gamma_1,$$

in other words, knowledge of all the moments over all times and that the functional $\mathbf{X}^2_{\cdot,\cdot}$ extends to an element of $\Omega(V)^2$, means that there is an unique measure on geometric rough paths of finite $p > 2$ variation whose center of mass simulates the functional $\mathbf{X}^2_{\cdot,\cdot}$.

Proposition 118 *Let* $\mathbf{X}^2_{\cdot,\cdot} \in \Omega(V)^2_\infty$ *be as in Lemma 115. There is an unique measure* $\eta = \mu_{d,G}$ *on the measure space* $\left(\Omega G\left(\mathbb{R}^d\right)^p, \mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right)\right)$ *for any* $2 < p < 3$, *such that* η *almost surely,* $\omega \in \Omega G(V)^{2+}$, $\omega_0 = (1,0,0)$ *and* $\forall (s,t) \in \Gamma_1$

$$\mathbb{E}_\eta\left(\omega^{2;\infty}_{s,t}\right) = \mathbf{X}^{2;\infty}_{s,t}$$

Proof. For $2 < p < 3$, the sigma algebra $\mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right)$ is generated by the π system of cylinder sets $\mathbf{C}_2\left(\mathbb{R}^d\right)$ (see section 5.2.1). Hence, if a measure η on a cylinder set is uniquely specified by the functional $\mathbf{X}^2_{\cdot,\cdot}$, it is uniquely specified on the whole of $\mathcal{B}\left(\Omega G\left(\mathbb{R}^d\right)^p\right)$. So pick $\underline{A} \in \mathbf{C}_2\left(\mathbb{R}^d\right)$,

$$\underline{A} = \left((A_0), (A_1, (0,t_1)), \ldots, (A_{n-1}, (t_{n-1}, 1))\right).$$

Then

$$\eta\left(\underline{A}\right) = \mu\left(\left\{\left((1,0,0), \omega_{0,t_1}, \ldots, \omega_{t_{n-1},1}\right) \in (A_0, A_1, \ldots, A_{n-1})\right\}\right),$$

so that if the multidimensional random variable $\left((1,0,0), \omega_{0,t_1}, \ldots, \omega_{t_{n-1},1}\right)$ has a distribution determined by it's moment sequence, then $\eta\left(\underline{A}\right)$ is specified. If $e_1, \ldots, e_d$ is an orthonormal basis of $\mathbb{R}^d$, then for $j = 1, \ldots, n$,

$$\mathbf{1}^2\left(\omega_{t_{j-1},t_j}\right) = \sum_{1\le i\le d} \sqrt{\frac{t_j - t_{j-1}}{d}} W_{i,j} e_i + \sum_{1\le i<k\le d} \frac{(t_j - t_{j-1})}{2d} A_{i,k,,j}\left[e_i, e_k\right]$$

where each $W_{i,j}$ has a $N(0,1)$ distribution and each $A_{i,k,,j}$ has a Lévy Area distribution. The calculation in Proposition 112 now applies in exactly the same way to specify that the distribution of

$\left(\mathbf{l}^2\left(\omega_{0,t_1}\right),\ldots,\mathbf{l}^2\left(\omega_{t_{n-1},1}\right)\right)$ and hence $\left((1,0,0),\omega_{0,t_1},\ldots,\omega_{t_{n-1},1}\right)$, is uniquely determined by the moment sequences of the $\mathbf{l}^2\left(\omega_{s,t}\right)$ in the evaluations $\mathbf{X}^2_{s,t}$. The measure $\mu_{d,G}$ concurs on $\mathbf{C}_2\left(\mathbb{R}^d\right)$, so by the previous argument, $\eta = \mu_{d,G}$. ■

The proposition amounts to a characterization of Wiener measure along the same lines as Lévy's classical characterization, as being a martingale whose quadratic variation is the time parameter. The additional content is that it says that among all geometric rough paths with independent increments, there is only one whose area increment has zero expectation. In this context, it is a characterization of the Lévy area random variable.

There are several ways to extend this Proposition, increasing it's generality. The germ of the idea came from an observation about rough paths of finite $2 \leq p < 3$ variation. For such paths, it is always possible to associate a geometric rough path of finite p variation as follows: if $\mathbf{Y}^2 \in \Omega\left(V\right)^p$, then for all times $(s,t) \in \Gamma_{T_{\mathbf{Y}^2}}$,

$$\log_2\left(\mathbf{Y}^2_{s,t}\right) = \mathbf{Y}^{(1)}_{s,t} + \mathbf{Y}^{(2)}_{s,t} - \frac{1}{2}\mathbf{Y}^{(1)}_{s,t} \otimes \mathbf{Y}^{(1)}_{s,t} \in V \oplus V^{\otimes 2}.$$

This element projects to an element of $L^2\left(V\right)$ by ignoring it's symmetric component in $V^{\otimes 2}$. The result is an element of $L^2\left(V\right)$, whose exponential in fact defines a geometric rough path of finite p variation. The symmetric component can be understood however, as a sum

$$\sum_i v^{\otimes 2}_{s,t;i} - u^{\otimes 2}_{s,t;i},$$

where each $v_{s,t;i}, u_{s,t;i} \in V$. The positive part can now be interpreted as a diffusive component, while the negative component as an anti-diffusive component (the difference is a sense of backward in time or forward in time diffusions). Suppose now there is no forward diffusive component, then the question is does the rough path locally define an unique measure on geometric rough paths, whose expectation matches the extended rough path $\mathbf{Y}^{2;\infty}$.

So to extend the idea, one direction to pursue is to whether for any $\mathbf{X}^2 \in \Omega G\left(\mathbb{R}^d\right)^p$ where $2 \leq p < 3$ (and assume $T_{\mathbf{X}^2} = 1$), there exists $\mu_{d,G}$ almost surely, a perturbed version $\mathbf{X}^2(\omega) \in \Omega G\left(\mathbb{R}^d\right)^p$, such that for all $(s,t) \in \Gamma_1$,

$$\mathbb{E}\left(\mathbf{X}^2_{s,t}(\omega)\right) = \mathbf{X}^2_{s,t} + \frac{(t-s)}{2d}\sum_{i=1}^{d} e_i^{\otimes 2}.$$

Another qualitative investigation would be into perturbing simple geometric rough paths of finite $p > 3$ variation in this manner, like straight rough paths. Indeed, it is possible to think of diffusions in any direction which defines a straight rough path and so define higher order measures.

With regards to an algebraic context, geometric rough paths have signature whose logarithm lies in the free lie algebra, which according to section 3.5.2, have an interpretation as first order derivations of the shuffle algebra. The interpretation of the element $\sum_{i=1}^{d} e_i^{\otimes 2}$ is as a simple second order derivation, but it would be interesting to see how far the viewpoint can be taken, particularly for higher order operators and indeed if the current understanding of rough paths is sufficient. In [46], there already exists interesting progress in exploiting this very viewpoint to construct approximate solutions to such problems.

Chapter 6

Appendix

6.1 Some calculations for controls

Lemma 119 *Let $\underline{u} \in \mathrm{T}^n(V) \backslash \mathbb{R}$. Then for $\mathbf{X}^n(\underline{u}) \in \Omega(V)_n$, the multiplicative functional given by*

$$\mathbf{X}^n_{s,t}(\underline{u}) = \sum_{j=0}^{n} (t-s)^j \frac{\underline{u}^{\otimes j}}{j!} \in \mathrm{T}^n(V)$$

for $(s,t) \in \Gamma_{T_{\mathbf{X}^n(\underline{u})}}$, has finite p variation if $p \geq n$ and thus belongs to $\Omega(V)^n_n$. (Note that $\mathrm{T}^n(V) = \pi_n(\mathrm{T}(V))$ where π_n quotients $\mathrm{T}(V)$ by the ideal generated by tensors of degree $m > n$, so that $\mathbf{X}^n_{s,t}(\underline{u}) = \pi_n(\exp((t-s)\underline{u}))$).

Proof. Let $\underline{u} = \sum_{j=1}^{n} \underline{u}^{(j)}$ where $\underline{u}^{(j)} \in V^{\otimes j}$. Then,

$$\mathbf{X}^{(j)}_{s,t}(\underline{u}) = \sum_{k=1}^{j} (t-s)^k \underline{u}_{j,k}$$

for some constant vectors, $\underline{u}_{j,k}$, $1 \leq k \leq j \leq n$. So for $1 \leq j \leq n$ define the controls $\omega^{(j)}(\cdot) \in \Gamma_{T_{\mathbf{X}^n(\underline{u})}} \to \mathbb{R}$

$$\omega^{(j)}(s,t) = \left\{\beta\left(\frac{j}{n}\right)!\left(\sum_{k=1}^{j}\left\|\underline{u}_{j,k}\right\|\right)\max\left\{(t-s),(t-s)^j\right\}\right\}^{\frac{n}{j}}$$

so that clearly

$$\left\|\mathbf{X}_{s,t}^{(j)}(\underline{u})\right\| \leq \frac{\omega^{(j)}(s,t)^{\frac{j}{n}}}{\beta\left(\frac{j}{n}\right)!}.$$

So set $\omega(\cdot) \in \Gamma_{T_{\mathbf{X}^n(\underline{u})}} \to \mathbb{R}$ to be

$$\omega(\cdot) = \sum_{k=1}^{j} \omega^{(j)}(\cdot)$$

and then ω is a control for $\mathbf{X}^n(\underline{u})$ in n variation, hence $\mathbf{X}^n(\underline{u}) \in \Omega(V)_n^n$. ■

Lemma 120 *The concatenation of two rough paths* $\mathbf{X}^{\lfloor p \rfloor}, \widetilde{\mathbf{X}}^{\lfloor p \rfloor} \in \Omega(V)^p$ *with controls* $\omega, \widetilde{\omega}$ *respectively, also lies in* $\Omega(V)^p$ *and is denoted by* $\left(\mathbf{X}\widetilde{\mathbf{X}}\right)^{\lfloor p \rfloor}$. *Let* $T = T_{\mathbf{X}^{\lfloor p \rfloor}}$, $\widetilde{T} = T_{\widetilde{\mathbf{X}}^{\lfloor p \rfloor}}$. *Then* $\left(\mathbf{X}\widetilde{\mathbf{X}}\right)^{\lfloor p \rfloor}$ *is controlled by* ω' *as follows:*

$$\begin{aligned}
\omega'(s,t) &= \omega(s,t) \quad 0 \leq s \leq t \leq T \\
\omega'(s,t) &= \left\{\begin{array}{c} \omega(s,T)^{\frac{1}{p}} + \widetilde{\omega}(0,t-T)^{\frac{1}{p}} \\ +\left(\omega(s,T) + \widetilde{\omega}(0,t-T)\right)^{\frac{1}{p}} \end{array}\right\}^{p} \quad 0 \leq s \leq T \leq t \leq T + \widetilde{T} \\
\omega'(s,t) &= \widetilde{\omega}(s-T, t-T) \quad T \leq s \leq t \leq T + \widetilde{T}
\end{aligned}$$

Proof. The product is defined

$$\begin{aligned}
\left(\mathbf{X}\widetilde{\mathbf{X}}\right)_{s,t} &= \mathbf{X}_{s,t} \quad 0 \leq s \leq t \leq T \\
\left(\mathbf{X}\widetilde{\mathbf{X}}\right)_{s,t} &= \mathbf{X}_{s,T} \otimes \widetilde{\mathbf{X}}_{0,t-T} \quad 0 \leq s \leq T \leq t \leq T + \widetilde{T} \\
\left(\mathbf{X}\widetilde{\mathbf{X}}\right)_{s,t} &= \widetilde{\mathbf{X}}_{s-T,t-T} \quad T \leq s \leq t \leq T + \widetilde{T}
\end{aligned}$$

$\mathbf{X}\widetilde{\mathbf{X}}$ is clearly a multiplicative functional so all we are required to check is that $\omega'(s,t)$ is a control for time pairs (s,t) with $0 \leq s \leq T \leq t \leq T+\widetilde{T}$, since for all other time pairs, the original controls suffice. For $0 \leq s \leq T \leq t \leq T+\widetilde{T}$,

$$\left(\mathbf{X}\widetilde{\mathbf{X}}\right)_{s,t} = \mathbf{X}_{s,T} \otimes \widetilde{\mathbf{X}}_{0,t-T}$$

so that

$$\begin{aligned}
\left\|\left(\mathbf{X}\widetilde{\mathbf{X}}\right)_{s,t}^{(i)}\right\| &= \left\|\sum_{j=0}^{i} \mathbf{X}_{s,T}^{(j)} \otimes \widetilde{\mathbf{X}}_{0,t-T}^{i-j}\right\| \\
&\leq \sum_{j=0}^{i} \left\|\mathbf{X}_{s,T}^{(j)}\right\| \left\|\widetilde{\mathbf{X}}_{0,t-T}^{i-j}\right\| \\
&\leq \frac{\left(\omega(s,T)^{\frac{i}{p}} + \widetilde{\omega}(0,t-T)^{\frac{i}{p}}\right)}{\beta\left(\frac{i}{p}\right)!} + \sum_{j=1}^{i-1} \frac{\omega(s,T)^{\frac{j}{p}}\,\widetilde{\omega}(0,t-T)^{\frac{i-j}{p}}}{\beta^2\left(\frac{j}{p}\right)!\left(\frac{i-j}{p}\right)!}
\end{aligned}$$

(by the neo-classical inequality Lemma (9)) and continuing on,

$$\begin{aligned}
\left\|\left(\mathbf{X}\widetilde{\mathbf{X}}\right)_{s,t}^{(i)}\right\| &\leq \frac{\left(\omega(s,T)^{\frac{i}{p}} + \widetilde{\omega}(0,t-T)^{\frac{i}{p}}\right)}{\beta\left(\frac{i}{p}\right)!} + \frac{\left(\omega(s,T) + \widetilde{\omega}(0,t-T)\right)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!} \\
&\leq \frac{\left(\left(\omega(s,T)^{\frac{1}{p}} + \widetilde{\omega}(0,t-T)^{\frac{1}{p}}\right) + \left(\omega(s,T) + \widetilde{\omega}(0,t-T)\right)^{\frac{1}{p}}\right)^{i}}{\beta\left(\frac{i}{p}\right)!} \\
&\leq \frac{\omega'(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!}.
\end{aligned}$$

Thus ω' is a control for $\left(\mathbf{X}\widetilde{\mathbf{X}}\right)^{\lfloor p \rfloor}$ as claimed since it is sub-additive, continuous near the diagonal and zero on the diagonal. ■

Corollary 121 *For any $p \geq 1$, the signatures of the collection $\Omega(V)^{p}_{\lfloor p \rfloor}$ of baseless multiplicative functionals of finite p variation forms a group $\mathfrak{G}_p^R(V)$.*

Proof. Lemma 120 shows that $\Omega(V)^p_{\lfloor p\rfloor}$ is a semi-group which clearly has a unit. To see that the set contains inverses, let $\mathbf{x}\in 1\oplus\left(\oplus_{j=1}^{n}V^{\otimes j}\right)$. The inverse function

$$inv_n : 1\oplus\left(\oplus_{j=1}^{n}V^{\otimes j}\right)\to 1\oplus\left(\oplus_{j=1}^{n}V^{\otimes j}\right)$$

is given by composition of functions

$$inv_n(\mathbf{x}) = \exp_n\left(-\log_n(\mathbf{x})\right).$$

where $\log_n$ and $\exp_n$ are polynomial functions:

$$\begin{aligned}
\log_n &: 1\oplus\left(\oplus_{j=1}^{n}V^{\otimes j}\right)\to \oplus_{j=1}^{n}V^{\otimes j}\\
\log_n(\mathbf{x}) &= \log_n\left(1+\mathbf{x}^{(1)}+\cdots\mathbf{x}^{(n)}\right)\\
&= \sum_{j=1}^{n}(-1)^j\frac{\pi_n\left(\left(\mathbf{x}^{(1)}+\cdots+\mathbf{x}^{(n)}\right)^{\otimes j}\right)}{j}.
\end{aligned}$$

$$\begin{aligned}
\exp_n &: \oplus_{j=1}^{n}V^{\otimes j}\to 1\oplus\left(\oplus_{j=1}^{n}V^{\otimes j}\right)\\
\exp_n(\mathbf{x}) &= \exp_n\left(1+\mathbf{x}^{(1)}+\cdots\mathbf{x}^{(n)}\right)\\
&= \sum_{j=0}^{n}\frac{\pi_n\left(\left(\mathbf{x}^{(1)}+\cdots+\mathbf{x}^{(n)}\right)^{\otimes j}\right)}{j!}.
\end{aligned}$$

Hence if for $1\le i\le n$,

$$\left\|\mathbf{x}^{(i)}\right\|\le\frac{\omega(s,t)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!},$$

and then there exist constants κ_i,μ_i, $1\le i\le n$,

$$\left\|\log_n(\mathbf{x})^{(i)}\right\|\le\kappa_i\omega(s,t)^{\frac{i}{p}}$$

and then

$$\left\|\exp_n\left(-\log_n(\mathbf{x})\right)^{(i)}\right\|\le\mu_i\omega(s,t)^{\frac{i}{p}}.$$

Thus there is a constant ν_n such that

$$\left\|\exp_n\left(-\log_n(\mathbf{x})\right)^{(i)}\right\| \leq \frac{\left(\nu_n\omega(s,t)\right)^{\frac{i}{p}}}{\beta\left(\frac{i}{p}\right)!},\ 1 \leq i \leq n.$$

Now if $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^p_{\lfloor p \rfloor}$, just define ${}^{inv}\mathbf{X}^{\lfloor p \rfloor}$ by ${}^{inv}\mathbf{X}^{\lfloor p \rfloor}_0 = \mathbf{X}^{\lfloor p \rfloor}_{T_{\mathbf{X}^{\lfloor p \rfloor}}}$, $T_{{}^{inv}\mathbf{X}^{\lfloor p \rfloor}} = T_{\mathbf{X}^{\lfloor p \rfloor}}$,

$${}^{inv}\mathbf{X}^{\lfloor p \rfloor}_{s,t} = inv_n\left(\mathbf{X}^{\lfloor p \rfloor}_{T_{\mathbf{X}^{\lfloor p \rfloor}}-t,T_{\mathbf{X}^{\lfloor p \rfloor}}-s}\right)\ \forall (s,t) \in \Gamma_{T_{{}^{inv}\mathbf{X}^{\lfloor p \rfloor}}}.$$

Then, if $\omega(s,t)$ is a control for $\mathbf{X}^{\lfloor p \rfloor}$, then $\nu_n\omega(s,t)$ is a control for ${}^{inv}\mathbf{X}^{\lfloor p \rfloor}$ and ${}^{inv}\mathbf{X}^{\lfloor p \rfloor}$ has finite p variation. Moreover for all $(s,t) \in \Gamma_{T_{{}^{inv}\mathbf{X}^{\lfloor p \rfloor}}}$,

$$\mathbf{X}^{\lfloor p \rfloor}_{s,t} \otimes^{inv} \mathbf{X}^{\lfloor p \rfloor}_{T_{\mathbf{X}^{\lfloor p \rfloor}}-t,T_{\mathbf{X}^{\lfloor p \rfloor}}-s} = 1$$

and ${}^{inv}\mathbf{X}^{\lfloor p \rfloor}$ is thus a form of inverse to $\mathbf{X}^{\lfloor p \rfloor}$ (though not unique in an important respect). It means that both $sig\left(\mathbf{X}^{\lfloor p \rfloor}\right)$ and $sig\left(\mathbf{X}^{\lfloor p \rfloor}\right)^{-1}$ belong to $\mathfrak{G}^R_p(V)$ as

$$sig\left(\mathbf{X}^{\lfloor p \rfloor}\right)^{-1} = sig\left({}^{inv}\mathbf{X}^{\lfloor p \rfloor}\right).$$

■

6.2 Campbell-Baker-Hausdorff

A formulation of the Campbell Baker Hausdorff formula due to Dynkin [17], in terms of the operation

$ad(x)(y) = xy - yx$

Theorem 122 (Campbell-Baker-Hausdorff-Dynkin)

$$\begin{aligned} H(a,b) &= \log\left(\exp(a) \otimes \exp(b)\right) \qquad (6.1) \\ &= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} c(p_1,\cdots,q_k)\,(ad(a))^{p_1}(ad(b))^{q_1}\cdots(ad(b))^{q_k} \end{aligned}$$

where

$$c(p_1,\cdots,p_k,q_1,\cdots,q_k) = \frac{1}{(p_1+\cdots+p_k+q_1+\cdots+q_k)}\frac{1}{p_1!\cdots p_k!q_1!\cdots q_k!}$$

and where $p_1,\cdots p_k,q_1,\cdots,q_k \geq 0$ *and* $p_i+q_i>0$ *for* $i=1,\ldots,k$. *(Hence either* $q_k=1$ *or* $p_k=1$ *and* $q_k=0$.*)*

The first five terms are given in [39] by:

$$\begin{aligned} H(a,b) &= a+b+\frac{1}{2}[a,b]+\frac{1}{12}\{[a,[a,b]]+[b,[b,a]]\}+\frac{1}{24}[a,[b,[b,a]]] \\ &\quad -\frac{1}{720}\left\{\begin{array}{c} [a,[a,[a,[a,b]]]]-2[b,[a,[a,[a,b]]]] \\ -6[a,[a,[b,[b,a]]]]-6[b,[b,[a,[a,b]]]] \\ -2[a,[b,[b,[b,a]]]]+[b,[b,[b,[b,a]]]] \end{array}\right\}+\cdots \end{aligned}$$

Nicolas Victoir passed on the reference [39] in which there is calculated a large number of the terms for the series. It is a beautiful object to study and from the point of view of quantization, has been interpreted in terms of Bernoulli numbers, graphs and weights in [2] for example.

6.3 A Smooth Vector Field

For a Banach space V, take a system of compatible norms on the tensor products $V^{\otimes i}$, $i > 1$, according to (2.4). Define the norm $\|\cdot\|_{\infty}$ on $\cup_{n>1} L^n(V)$ through for $\mathbf{l}^n = \sum_{j=1}^{n} \mathbf{l}^{(j)}$, $\mathbf{l}^{(j)} \in L^{(j)}(V)$,

$$\|\mathbf{l}^n\|_{\infty} = \sum_{j=1}^{n} \left\|\mathbf{l}^{(j)}\right\|$$

and define the Banach space $L^{\infty}(V)$ to be the completion of $\cup_{n>1} L^n(V)$ with respect to the norm $\|\cdot\|_{\infty}$.

Lemma 123 *Let $H_1(a,b)$ be the component of $H(a,b)$ (6.1) of degree 1 in b. Define the vector field $f : \mathrm{L}^{\infty}(V) \to Hom(\mathrm{L}^{\infty}(V), \mathrm{L}^{\infty}(V))$ by*

$$f(a)(b) = H_1(a,b).$$

Then f is smooth for $\|a\|_{\infty} < \pi$.

Proof. According to [33] for example,

$$\begin{aligned} f(a)(b) &= H_1(a,b) \\ &= b + \frac{1}{2}[a,b] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} (ad(a))^{2n}(b) \\ &= \sum_{n=1}^{\infty} f_n(a)(b) \end{aligned}$$

where

$$f_n(\lambda a)(b) = \lambda^n f_n(a)(b)$$

and for $n \geq 1$, $|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n)$. Therefore

$$\begin{aligned}
\|f_{2n}(a)(b)\|_\infty &= \left\| \frac{B_{2n}}{(2n)!} (ad(a))^{2n} (b) \right\|_\infty \\
&= \frac{2\zeta(2n)}{(2\pi)^{2n}} \left\| (ad(a))^{2n} (b) \right\|_\infty \\
&\leq \frac{2\zeta(2n)}{\pi^{2n}} \|a\|_\infty^{2n} \|b\|_\infty .
\end{aligned}$$

Now for $1 \leq i \leq j$ let $v_i \in L^\infty(V)$ and set $u_i = v_i$. Then put $u_i = a$ for $j+1 \leq i \leq 2n$. This gives the following form for derivatives of the vector fields f_{2n}, $j \geq 1$:

$$f_{2n}^{(j)}(a; v_1, \cdots, v_j)(b) = \frac{B_{2n}}{(2n)!(2n-j)!} \sum_{\sigma \in S_{2n}} \left[u_{\sigma(1)}, \cdots, u_{\sigma(n)}, b \right],$$

so that

$$\begin{aligned}
\left\| f_{2n}^{(j)}(a; v_1, \cdots, v_j)(b) \right\|_\infty &\leq \frac{2\zeta(2n)}{(2\pi)^{2n}(2n-j)!} \sum_{\sigma \in S_{2n}} \left\| \left[u_{\sigma(1)}, \cdots, u_{\sigma(n)}, b \right] \right\|_\infty \\
&\leq \frac{2\zeta(2n)(2n)!}{\pi^{2n}(2n-j)!} \|a\|_\infty^{2n-j} \|v_1\|_\infty \cdots \|v_j\|_\infty \|b\|_\infty .
\end{aligned}$$

By setting

$$c_{2n,j}(\alpha) = \frac{2\zeta(2n)(2n)!}{\pi^{2n}(2n-j)!} \alpha^{2n-j},$$

there is firstly a bound on the norm of the $f_{2n}^{(j)}$, $n \geq 1$,

$$f_{2n}^{(j)}(a; \cdot, \cdots, \cdot) : L^\infty(V) \to Hom\left(L^\infty(V)^{\otimes j}, Hom(L^\infty(V), L^\infty(V)) \right)$$

$$\left\| f_{2n}^{(j)}(a; \cdot, \cdots, \cdot) \right\| \leq c_{2n,j}(\|a\|_\infty).$$

Using d'Alembert's ratio test,

$$\begin{aligned}\lim_{n\to\infty}\frac{c_{2n+2,j}(\alpha)}{c_{2n,j}(\alpha)} &= \lim_{n\to\infty}\frac{2\zeta(2n+2)(2n+2)!\alpha^{2n+2-j}}{\pi^{2n+2}(2n+2-j)!}\frac{\pi^{2n}(2n-j)!}{2\zeta(2n)(2n)!\alpha^{2n-j}} \\ &= \lim_{n\to\infty}\frac{(2n+2)(2n+1)\alpha^2}{(2n+2-j)(2n+1-j)\pi^2} \\ &= \frac{\alpha^2}{\pi^2},\end{aligned}$$

the sums of these norms are finite if $\|a\| < \pi$:

$$C_j\left(\|a\|_\infty\right) = \sum_{n=1}^{\infty} c_{2n,j}\left(\|a\|_\infty\right) < \infty.$$

Therefore, since $f_0^{(1)} = 0$, $f_1^{(1)}(a; v_1) = \frac{1}{2}ad(v_1)$ and $f_1^{(j)} = 0$ otherwise,

$$\begin{aligned}\left\|f^{(j)}(a; v_1, \cdots, v_j)(b)\right\|_\infty &\le \sum_{n=1}^{\infty}\left\|f_n^{(j)}(a; v_1, \cdots, v_j)(b)\right\|_\infty \\ &< \left\{\begin{array}{cc}(1 + C_1(\alpha))\|v_1\|_\infty\|b\|_\infty & j = 1 \\ C_j(\alpha)\|v_1\|_\infty\cdots\|v_j\|_\infty\|b\|_\infty & j > 1\end{array}\right\}.\end{aligned}$$

and hence f is C^∞ when $\|a\| < \pi$. ■

6.4 A metric for a connected Lie group

Let G be a finite dimensional Lie group with lie algebra $\mathfrak{g}$. Suppose there is a norm on the lie algebra, $\|\cdot\|_{\mathfrak{g}}$. For any two points $g, h \in G$, denote

$$\mathcal{P}_{g,h}(G) = \left\{\gamma \in \mathcal{P}(V) \text{ s.t. } \gamma_0 = g, \gamma_{T_\gamma} = h\right\}$$

and define the length of such paths,

$$l(\gamma) = \int_{0<u<T_\gamma}\left\|\gamma_t^{-1}d\gamma_t\right\|_{\mathfrak{g}}.$$

Then there is the following Lemma

Lemma 124 *Let G be a connected finite dimensional lie group whose lie algebra $\mathfrak{g}$ has norm $\|\cdot\|_{\mathfrak{g}}$. The function $d_G(\cdot,\cdot) : G \times G \to \mathbb{R}$*

$$d_G(g,h) = \inf_{\gamma \in \mathcal{P}_{g,h}(G)} l(\gamma)$$

defines a metric on G.

Proof. Clearly $d_G(g,h) \geq 0$ for all $g,h \in G$.

To check that $d_G(g,h) = 0$ if and only if $g = h$, suppose that there are elements $g \neq h$ such that $d_G(g,h) = 0$. Then without loss of generality, $g = e$, the identity since the definition implies that $d_G(g,h) = d_G(e, g^{-1}h)$ for all $g,h \in G$. Now in a neighborhood V of 0, $\log : G \to \mathfrak{g}$ is a well defined analytic bijection (see section 3 of [22] for example) with inverse $Exp : \mathfrak{g} \to G$. Since $g \neq e$, there exists $\delta > 0$ such that if $V_\delta = \left\{\|l\|_{\mathfrak{g}} \leq \delta\right\}$, then $V_\delta \subset V$ and $g \notin Exp(V_\delta)$. Then since $Exp(V_\delta)$ is a closed set in G,

$$\begin{aligned} \Upsilon &: G \to Hom(\mathfrak{g},\mathfrak{g}) \\ \Upsilon(g)(l) &= g^{-1}(l), \end{aligned}$$

Υ is a smooth function, whose image restricted to V_δ is bounded below in norm. i.e. there exists $\kappa_\delta > 0$ such that $\forall g \in Exp(V_\delta)$, $\left\|g^{-1}(l)\right\| \geq \kappa_\delta \|l\|$.

Take any $\gamma \in \mathcal{P}_{e,g}(G)$ and consider the path $m_t = \log(\gamma_t)$ for $\gamma_t \in Exp(V_\delta)$. Then

$$\begin{aligned}
\int_{0<u<T_\gamma} \left\|\gamma_t^{-1} d\gamma_t\right\|_{\mathfrak{g}} &\geq \int_{t:\{\gamma_t \in V_\delta\}} \left\|\gamma_t^{-1} d\gamma_t\right\|_{\mathfrak{g}} \\
&\geq \int_{t:\{\gamma_t \in V_\delta\}} \left\|\gamma_t^{-1} dm_t\right\|_{\mathfrak{g}} \\
&\geq \kappa_\delta \int_{t:\{\gamma_t \in V_\delta\}} \left\|dm_t\right\|_{\mathfrak{g}} \\
&\geq \delta\kappa_\delta > 0.
\end{aligned}$$

Thus $d_G(e,g) > 0$ as required.

The triangle inequality, follows immediately since if $g_1, g_2, g_3 \in G$, any paths from g_1 to g_2 can be concatenated with a path from g_2 to g_3 to give a path from g_1 to g_3. ■

Lemma 125 *For such a metric $d_G(\cdot,\cdot)$ as above, G is complete.*

Proof. Using a similar technique to the lemma above, the issue of completeness can be reduced to an issue in the lie algebra for the norm $\|\cdot\|_{\mathfrak{g}}$ which is complete. In fact it is possible to show that there exists a constant $c > 1$ such that in a neighborhood of $0 \in \mathfrak{g}$,

$$c^{-1} d_G(e, Exp(l)) \leq \|l\|_{\mathfrak{g}} \leq c d_G(e, Exp(l)).$$

Since $Exp : \mathfrak{g} \rightarrow G$ is continuous, G is complete. ■

6.5 Some More Algebra

Define the elements $id, 1, \in Hom\left(T(V), T(V)\right)$ by

$$\begin{aligned} id\left(\underline{v}\right) &= \underline{v} \\ 1\left(\underline{v}\right) &= v \circ \varepsilon\left(\underline{v}\right) \end{aligned}$$

$\forall \underline{v} \in T(V)$ so that 1 is just the projection operator of $T(V)$ onto $V^{\otimes 0}$. In addition, consider $T(V)$ as the universal enveloping algebra $U(L(V))$ and let Π_j be the projections $\Pi_j : U(L(V)) \longrightarrow U_j(L)$. According to [33], the maps Π_j as elements of $Hom\left(T(V), T(V)\right)$, satisfy the relation

$$\Pi_j = \frac{\Pi_1^{\star j}}{j!}$$

with respect to the product $\star$ (3.6), with the deduction that

$$\begin{aligned} I &= \sum_{j=0}^{\infty} \Pi_j \\ &= \sum_{j=0}^{\infty} \frac{\Pi_1^{\star j}}{j!} \\ &= \exp\left(\Pi_1\right). \end{aligned}$$

Consequently, there is the following form for $\Pi_1$in $Hom\left(T(V), T(V)\right)$

$$\begin{aligned} \Pi_1 &= \log\left(I\right) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(I - 1\right)^{\star k}. \end{aligned}$$

In [38] it is shown that Π_1 evaluates on simple tensors to

$$\Pi_1\left(x_1 \otimes \cdots \otimes x_n\right) = \sum_{m=1}^{n} (-1)^{m-1} \frac{1}{nm} \binom{n}{m}^{-1} \sum_{\sigma \in G_{n,m}} \left[x_{\sigma 1}, \ldots, x_{\sigma n}\right]$$

for any $x_1, \ldots x_n \in L(V)$ in fact, where the $G_{n,m}$ are particular subsets of permutations of n elements as follows. Take σ to be a permutation of $\{1, \ldots, n\}$ and denote the set of intervals $\{p, p+1, \ldots, q\}$ of $\{1, \ldots, n\}$ where $\sigma p < \sigma(p+1) < \cdots < \sigma q$, by C_σ. Then let D_σ be the maximal elements of C_σ. Set $G_{n,j}$ to be the permutations σ of $\{1, \ldots, n\}$ where $Card\ D_\sigma = j$.

In general, let $B_{n,j}$ be the set of sequences $(P_1, \ldots, P_j)$ of disjoint non-empty subsets of $\{1, \ldots, n\}$ with union $\{1, \ldots, n\}$. Solomon showed that Π_j evaluates to have the form

$$\Pi_j (x_1 \otimes \cdots \otimes x_n) = \frac{1}{j!} \sum_{(P_1, \ldots, P_j) \in B_{n,j}} \Pi_1 (P_1 \underline{x}) \otimes \cdots \otimes \Pi_1 (P_j \underline{x})$$

where if $P = \{i_1, \ldots, i_r\}$then

$$P\underline{x} = P(x_1 \otimes \cdots \otimes x_n) = x_{i_1} \otimes \cdots \otimes x_{i_r}.$$

Using this formalism, it is possible to express a remark about the tail of the signature of a geometric rough path.

Lemma 126 *Let $\underline{l} \in \mathrm{L}(V)$ so that $\exp(\underline{l}) \in \mathrm{T}(V)$. Then it is possible to reconstruct the lower order iterated integrals purely from knowing the tail of $\exp(\underline{l})$, so that the information of a signature of a rough path is completely contained in the tail.*

Proof. Let the given information be the sequence $\left\{ \exp(\underline{l})^{(i)} \right\}_{i=n}^{\infty}$ where $\exp(\underline{l})^{(i)} = \pi_{(i)}(\exp(\underline{l}))$. Then to identify the element

$$\underline{l} = \sum_{i=1}^{\infty} \underline{l}^{(i)},$$

the notation above gives simply if $i \geq n$ that

$$\begin{aligned} \underline{l}^{(i)} &= \Pi_1 \circ \exp(\underline{l})^{(i)} \\ &= \Pi_1 \circ \pi_{(i)} (\exp(\underline{l})) . \end{aligned}$$

So it remains to show how to identify the lower order lie components. For $1 \leq i < n$, it is possible to identify $\left(\underline{l}^{(i)}\right)^{\otimes m}$ for any $m \geq \frac{n}{i}$:

$$\frac{\left(\underline{l}^{(i)}\right)^{\otimes m}}{m!} = \left(\Pi_1 \circ \pi_{(i)}\right)^{\star m} \circ \left(\exp(\underline{l})^{(im)}\right),$$

where $\star$ is the convolution product of (3.6). If m is odd, there is an unique m'th real root of $\left(\underline{l}^{(i)}\right)^{\otimes m}$ which can be identified using a basis system such as that due to Poincaré, Birkhoff and Witt. Hence the tail of the iterated integral sequence contains the information of the head of the sequence also. ■

6.6 No 'Lie' map

Non-geometric rough paths have signature processes which do not have logarithms that lie in a free Lie algebra but a larger Lie algebra

$$\mathfrak{L}(V) = \oplus_{n=1}^{\infty} V^{\otimes n}.$$

As such, it is tempting to try to find a geometric rough path by projecting the path of the logarithm in $\mathfrak{L}(V)$ into $L(V)$ and ascertaining whether the exponential of this path defines a geometric rough path. Unfortunately, this procedure appears not to be a group homomorphism. To expand on this putative procedure, take $\mathbf{X}^{\lfloor p \rfloor} \in \Omega(V)^{\lfloor p \rfloor}$. Then the hoped for geometric rough path $\Phi\left(\mathbf{X}^{\lfloor p \rfloor}\right) \in \Omega G(V)^{\lfloor p \rfloor}$ would take the form

$$\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{s,t}=\exp\left(\Pi_1\left(\log\left(\mathbf{X}^{\lfloor p\rfloor}_{s,t}\right)\right)\right)$$

(in the algebra $T^{\lfloor p\rfloor}(V)$). However this map is not multiplicative in general, that is

$$\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{s,t}\otimes\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{t,u}\neq\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{s,u}.$$

To see this, consider $4<p<5$, $x_1,x_2\in V$ and

$$\begin{aligned}\mathbf{X}^{\lfloor p\rfloor}_{s,t} &= \exp\left((t-s)\,x_1^{\otimes 2}\right)\\ \mathbf{X}^{\lfloor p\rfloor}_{t,u} &= \exp\left((u-t)\,x_2^{\otimes 2}\right)\end{aligned}$$

Then $\Pi_1\left(x_i^{\otimes 2}\right)=0$, so

$$\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{s,t}=\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{t,u}=1.$$

But

$$\mathbf{X}^{\lfloor p\rfloor}_{s,u}=\exp\left((t-s)\,x_1^{\otimes 2}+(u-t)\,x_2^{\otimes 2}+\frac{(t-s)(u-t)}{2}\left[x_1^{\otimes 2},x_2^{\otimes 2}\right]\right)$$

and so

$$\begin{aligned}\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{s,u} &= \exp\left(\Pi_1\left(\begin{array}{c}(t-s)\,x_1^{\otimes 2}+(u-t)\,x_2^{\otimes 2}\\ +\frac{(t-s)(u-t)}{2}\left[x_1^{\otimes 2},x_2^{\otimes 2}\right]\end{array}\right)\right)\\ &= \exp\left(\Pi_1\left(\frac{(t-s)(u-t)}{2}\right)\right).\end{aligned}$$

So it is necessary to evaluate $\Pi_1\left(\left[x_1^{\otimes 2},x_2^{\otimes 2}\right]\right)$. Sparing the algebra of the calculation, it evaluates to

$$\Pi_1\left(\left[x_1^{\otimes 2},x_2^{\otimes 2}\right]\right)=\frac{1}{3}\left[\left[x_2,\left[x_1,x_2\right]\right],x_1\right].$$

Thus $\Phi\left(\mathbf{X}^{\lfloor p\rfloor}\right)_{s,u}=\exp\left(\frac{(t-s)(u-t)}{6}\left[\left[x_2,\left[x_1,x_2\right]\right],x_1\right]\right)\neq 1$ and the procedure does not extend in the required fashion.

Chapter 7

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