

TRIGONOMETRIC INTERPOLATION AND QUADRATURE IN PERTURBED POINTS

ANTHONY P. AUSTIN* AND LLOYD N. TREFETHEN†

Abstract. The trigonometric interpolants to a periodic function f in equispaced points converge if f is Dini-continuous, and the associated quadrature formula, the trapezoidal rule, converges if f is continuous. What if the points are perturbed? With equispaced grid spacing h , let each point be perturbed by an arbitrary amount $\leq \alpha h$, where $\alpha \in [0, 1/2)$ is a fixed constant. The Kadec 1/4 theorem of sampling theory suggests there may be trouble for $\alpha \geq 1/4$. We show that convergence of both the interpolants and the quadrature estimates is guaranteed for all $\alpha < 1/2$ if f is twice continuously differentiable, with the convergence rate depending on the smoothness of f . More precisely it is enough for f to have 4α derivatives in a certain sense, and we conjecture that 2α derivatives is enough. Connections with the Fejér–Kalmár theorem are discussed.

Key words. trigonometric interpolation, quadrature, Lebesgue constant, Kadec 1/4 theorem, Fejér–Kalmár theorem, sampling theory

AMS subject classifications. 42A15, 65D32, 94A20

1. Introduction and summary of results. The basic question of robustness of mathematical algorithms is, what happens if the data are perturbed? Yet little literature exists on the effect on interpolants, or on quadratures, of perturbing the interpolation points.

The questions addressed in this paper arise in two almost equivalent settings: interpolation by algebraic polynomials (e.g., in Gauss or Chebyshev points) and periodic interpolation by trigonometric polynomials (e.g., in equispaced points). Although we believe essentially the same results hold in the two settings, this paper deals with just the trigonometric case. Let f be a real or complex function on $[-\pi, \pi)$, which we take to be 2π -periodic in the sense that any assumptions of continuity or smoothness made for f apply periodically at $x = -\pi$ as well as at interior points. For each $N \geq 0$, set $K = 2N + 1$, and consider the centered grid of K equispaced points in $[-\pi, \pi)$,

$$(1.1) \quad x_k = kh, \quad -N \leq k \leq N, \quad h = \frac{2\pi}{K}.$$

There is a unique degree- N trigonometric interpolant through the data $\{f(x_k)\}$, by

*austina@anl.gov, Mathematics and Computer Science Division, Argonne National Laboratory, 9700 S. Cass Ave., Lemont, IL 60439

†trefethen@maths.ox.ac.uk, Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK.

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which we mean a function

$$(1.2) \quad t_N(x) = \sum_{k=-N}^N c_k e^{ikx}$$

with $t_N(x_k) = f(x_k)$ for each k . If I denotes the integral of f ,

$$(1.3) \quad I = \int_{-\pi}^{\pi} f(x) dx,$$

the associated quadrature approximation is the integral of $t_N(x)$, which can be shown to be equal to the result of applying the trapezoidal rule to f :

$$(1.4) \quad I_N = h \sum_{k=-N}^N f(x_k) = \int_{-\pi}^{\pi} t_N(x) dx = 2\pi c_0.$$

It is known that if f is continuous, then

$$(1.5) \quad \lim_{N \rightarrow \infty} |I - I_N| = 0,$$

and if f is Dini-continuous, for which Hölder or Lipschitz continuity are sufficient conditions, then

$$(1.6) \quad \lim_{N \rightarrow \infty} \|f - t_N\| = 0.$$

Moreover, the convergence rates are tied to the smoothness of f , with exponential convergence if f is analytic. Here and throughout, $\|\cdot\|$ is the maximum norm on $[-\pi, \pi)$.

The problem addressed in this paper is the generalization of these results to configurations in which the interpolation points are perturbed. For fixed $\alpha \in (0, 1/2)$, consider a set of points

$$(1.7) \quad \tilde{x}_k = x_k + s_k h, \quad -N \leq k \leq N, \quad |s_k| \leq \alpha.$$

Note that since $\alpha < 1/2$, the \tilde{x}_k are necessarily distinct. Let $\tilde{t}_N(x)$ be the unique degree- N trigonometric interpolant to $\{f(\tilde{x}_k)\}$, and let $\tilde{I}_N = \int \tilde{t}_N(x) dx$ be the corresponding quadrature approximation. As in (1.4), this will be a linear combination of the function values, although no longer with equal weights in general.

Let $\sigma > 0$ be any positive real number, and write $\sigma = \nu + \gamma$ with $\gamma \in (0, 1]$. We say that f has σ derivatives if f is ν times continuously differentiable and, moreover, $f^{(\nu)}$ is Hölder continuous with exponent γ . Note that if σ is an integer, then for f to “have σ derivatives” means that f is $\sigma - 1$ times continuously differentiable and $f^{(\sigma-1)}$ is Lipschitz continuous. We will prove the following main theorem, whose central estimate is the bound on $\|f - \tilde{t}_N\|$ in (1.9). The estimates (1.8)–(1.9) are new, whereas (1.10) follows from the work of Kis [11], as discussed in Section 3. Numerical illustrations of these bounds can be found in [1].

THEOREM 1.1. *For any $\alpha \in (0, 1/2)$, if f is twice continuously differentiable, then*

$$(1.8) \quad \lim_{N \rightarrow \infty} |I - \tilde{I}_N| = \lim_{N \rightarrow \infty} \|f - \tilde{t}_N\| = 0.$$

More precisely, if f has σ derivatives for some $\sigma > 4\alpha$, then

$$(1.9) \quad |I - \tilde{I}_N|, \|f - \tilde{t}_N\| = O(N^{4\alpha-\sigma}).$$

If f can be analytically continued to a 2π -periodic function for $-a < \operatorname{Im} x < a$ for some $a > 0$, then for any $\hat{a} < a$,

$$(1.10) \quad |I - \tilde{I}_N|, \|f - \tilde{t}_N\| = O(e^{-\hat{a}N}).$$

Our proofs are based on combining standard estimates of approximation theory, the Jackson theorems, with a new bound on the Lebesgue constants associated with perturbed grids, Theorem 2.1. Our bounds are close to sharp, but not quite. Based on extensive numerical experiments presented in Section 3.3.2 of [1], we conjecture that 4α can be improved to 2α in (1.9) and (2.2); for (2.2) the result would probably then be sharp, but for (1.9) a slight further improvement may still be possible. For the quadrature problem in particular, further experiments presented in Section 3.5.2 of [1] lead us to conjecture that $\tilde{I}_N \rightarrow I$ as $N \rightarrow \infty$ for all continuous functions f for all $\alpha < 1/2$. This conjecture is based on the theory of Pólya in 1933 [14], who showed that such convergence is ensured if and only if the sums of the absolute values of the quadrature weights are bounded as $N \rightarrow \infty$. Experiments indicate that for all $\alpha < 1/2$, these sums are indeed bounded as required. On the other hand, $\tilde{I}_N \rightarrow I$ cannot be guaranteed for any $\alpha \geq 1/2$, since in that case the interpolation points may come together, making the quadrature weights unbounded.

Theorems 1.1 and 2.1 suggest that from the point of view of approximation and quadrature, $\alpha = 1/4$ is not a special value. In Section 4 we comment on the significance of the appearance of this number in the Kadec $1/4$ theorem and more generally on the relationship between approximation theory and sampling theory, two subjects that address closely related questions and yet have little overlap of literatures or experts.

All the estimates reported here were worked out by the first author and presented in his D. Phil. thesis [1]. This work was motivated by work of the second author with Weideman in the review article “The exponentially convergent trapezoidal rule” [16]. It is well known that on an equispaced periodic grid, the trapezoidal rule is exponentially convergent for periodic analytic integrands [4, 16]. With perturbed points, it seemed to us that exponential convergence of a suitably generalized rule should still be expected, and we were surprised to find that there seemed to be no literature on this subject. A preliminary discussion was given in [16, Sec. 9].

Section 2 reduces Theorem 1.1 to a bound on the Lebesgue constant, Theorem 2.1. Sections 3 and 4 are devoted to comments on problems with $\alpha \geq 1/2$ and on the link with sampling theory and Kadec’s theorem, respectively. Section 5 outlines the proof of Theorem 2.1.

2. Reduction to a Lebesgue constant estimate. A fundamental tool of approximation theory is the Lebesgue constant: for any linear projection $L : f \mapsto t$, the Lebesgue constant is the operator norm $\Lambda = \|L\|$. For our problem the operator is the map \tilde{L}_N from a function f to its trigonometric interpolant \tilde{t}_N through the values $\{f(\tilde{x}_k)\}$, and the norm on L is the operator norm induced by $\|\cdot\|$, the ∞ -norm on $[-\pi, \pi)$. We denote the Lebesgue constant by $\tilde{\Lambda}_N$.

Lebesgue constants are linked to quality of approximations by the following well-known bound. If $\tilde{\Lambda}_N$ is the Lebesgue constant associated with the projection \tilde{L}_N :

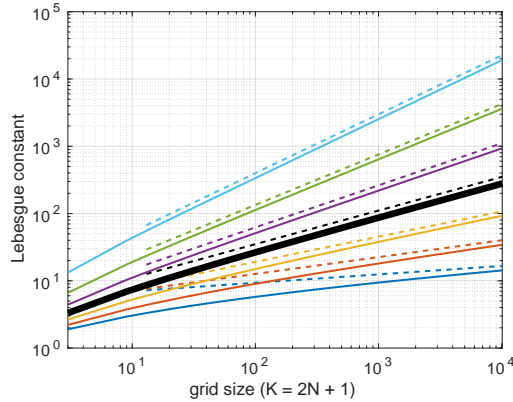


FIG. 2.1. Experimental evidence for the conjecture that (2.2) holds with $N^{2\alpha}$ in place of $N^{4\alpha}$. The solid lines depict the value of $\tilde{\Lambda}_N$ versus the grid size $K = 2N + 1$ for $\alpha = 1/16, 2/16, \dots, 7/16$ for a particular choice of perturbed points \tilde{x}_k that is conjectured in [1] to yield the maximum value of $\tilde{\Lambda}_N$. The bottom line corresponds to $\alpha = 1/16$ and the top line to $\alpha = 7/16$; the thick black line is for $\alpha = 1/4$. The dashed lines depict asymptotic growth rates of $O(N^{2\alpha})$ for their matching solid lines.

$f \mapsto \tilde{t}_N$ and t_N^* is the best approximation to f of degree N , then

$$(2.1) \quad \|f - \tilde{t}_N\| \leq (1 + \tilde{\Lambda}_N) \|f - t_N^*\|.$$

It follows that if $\tilde{\Lambda}_N$ is small, then \tilde{t}_N is a near-optimal approximation to f . If f has a certain smoothness property for which the optimal approximations t_N^* are known to converge at a certain rate, this implies that the interpolants \tilde{t}_N converge at nearly the same rate.

Applying (2.1), we prove Theorem 1.1 by combining a bound on the Lebesgue constants $\tilde{\Lambda}_N$ with bounds on the best approximation errors $\|f - t_N^*\|$. Our estimates of best approximations are standard Jackson theorems, going back to Dunham Jackson in 1911 and 1912. The nonstandard part of the argument, which from a technical point of view is the main contribution of this paper, is the following estimate of the Lebesgue constant, the proof of which is outlined in Section 5.

THEOREM 2.1. *There is a universal constant C such that*

$$(2.2) \quad \tilde{\Lambda}_N \leq \frac{C(N^{4\alpha} - 1)}{\alpha(1 - 2\alpha)}$$

for all $\alpha \in [0, 1/2)$ and $N \geq 2$. For $\alpha = 0$ this bound is to be interpreted by its limiting value given, for example, by L'Hopital's rule, $\tilde{\Lambda}_N \leq 4C \log N$.

The $\log N$ bound for an equispaced grid with $\alpha = 0$ is standard, so the substantive result here concerns $\alpha \in (0, 1/2)$. This is what we can prove, but as mentioned in the previous section, based on numerical experiments, we conjecture that (2.2) actually holds with $N^{4\alpha}$ replaced by $N^{2\alpha}$. Figure 2.1, based on computations in Chapter 3 of [1], presents some of this data. We refer the reader to that source for further details.

Given Theorem 2.1, we prove Theorem 1.1 as follows.

Proof of Theorem 1.1, given Theorem 2.1. The Jackson theorems of approximation theory relate the smoothness of a function f to the accuracy of its best approximations [8, 12]. According to one of these theorems, given for example as Theorem 41

of [12], if f is a periodic function on $[-\pi, \pi)$ that has σ derivatives for some $\sigma > 0$ in the sense defined in Section 1, then

$$(2.3) \quad \|f - t_N^*\| = O(N^{-\sigma}).$$

Combining this with Theorem 2.1 gives (1.9). The bound (1.10) follows similarly from the estimate

$$(2.4) \quad \|f - t_N^*\| = O(e^{-\hat{a}N})$$

for any 2π -periodic function f analytic and bounded in the strip of half-width $\hat{a} > 0$ about the real axis; see, for example, eq. (7.17) of [16]. \square

Regarding the constant C in Theorem 2.1, we note that all of our arguments are explicit, and if they are combined together, one finds that for sufficiently large N , C is bounded by $72\pi(3/2 + 5/\log 5) \approx 1042$. This estimate is of course pessimistic. Numerical evidence suggests that $C \approx 0.8$ is sufficient not only for (2.2) as written but also for the same bound with $N^{4\alpha}$ replaced by $N^{2\alpha}$.

3. $\alpha \geq 1/2$, confluent points, and analytic functions. Our framework (1.7) for perturbed points can be generalized to values $\alpha \geq 1/2$. For $\alpha \in [1/2, 1)$, two grid points may coalesce, so one must assume that f' exists in order to ensure that there are appropriate data to define an interpolation problem (in this case, trigonometric Hermite interpolation). Similarly for $\alpha \in [1, 3/2)$, three points may coalesce, so one must assume f'' exists; and so on analogously for any finite value of α . (We wrap grid points around as necessary if the perturbation moves them outside of $[-\pi, \pi)$; equivalently, one could extend f periodically.)

Looking at the statement of Theorem 1.1 but considering values $\alpha \geq 1/2$, one notes that the assumption of $\sigma > 4\alpha$ derivatives is enough to ensure that the necessary derivatives exist for the interpolation problem to make sense; the conjectured sharper condition of $\sigma > 2\alpha$ derivatives is also (just) enough. This coincidence seems suggestive, and we consider it possible that Theorem 1.1 and its conjectured improvement with 2α may in fact be valid for arbitrary $\alpha > 0$, not just $\alpha \in (0, 1/2)$. We have not attempted to prove this, however. As a practical matter, trouble can be expected in floating-point arithmetic as sample points coalesce, so we regard the case $\alpha \geq 1/2$ as somewhat theoretical.

Going further, what if we allow arbitrary perturbations of the interpolation points, so that each \tilde{x}_k may lie anywhere in $[-\pi, \pi)$? Doing so makes sense mathematically if f is infinitely differentiable; so in particular, it makes sense if f is analytic, which implies that it can be analytically continued to a 2π -periodic function on the whole real line. We are now in an area of approximation theory (and potential theory) going back to the work of Runge [15] and Fejér [5], in which a major contributor was Joseph Walsh [6, 17]. For arbitrary x_k , convergence will occur if f is analytic in a sufficiently wide strip around the real axis in the complex x -plane. Repeated points are permitted, with interpolation at such points interpreted in the Hermite sense involving values of both the function and its derivatives. If the points x_k are *uniformly distributed* in the sense that the fraction of points falling in any interval $[a, b) \subseteq [-\pi, \pi)$ converges to $(b - a)/2\pi$ as $N \rightarrow \infty$, then it is enough for f to be analytic in *any* strip around the real axis. Such results were first developed for polynomial approximation on the unit circle of functions analytic in the unit disk, the so-called Fejér–Kalmár theorem [5, 10, 17]. The extension to functions analytic in an annulus was considered by Hlawka [7], and the equivalent problem of trigonometric

interpolation of 2π -periodic functions on $[-\pi, \pi)$ was considered by Kis [11]. All these results may fail in practice because of rounding errors on the computer, however. For example, Figure 3.7 of [1] shows an example with uniformly distributed random interpolation points in $[-\pi, \pi)$, with rounding errors beginning to take over at $N \approx 20$. For the case of interpolation by algebraic polynomials, this kind of effect is familiar in the context of the Runge phenomenon, where polynomial interpolants in equispaced points in $[-1, 1]$ will diverge on a computer as $N \rightarrow \infty$ even for a function like $f(x) = \exp(x)$ for which in principle they should converge.

The importance of the uniform distribution of the interpolation points mentioned above sheds further light on the setting of Theorem 2.1. The maximum possible ratio of the length of the largest interval between two of our perturbed points to the length of the smallest is $(1 + 2\alpha)/(1 - 2\alpha)$; note that the denominator of this expression appears also in the denominator of the right-hand side of (2.2). Having a bound on such a *mesh ratio* ensures the convergence of many numerical algorithms for solving differential equations, and one might accordingly wonder if here, too, a bound on the mesh ratio alone would be enough to ensure good behavior of the interpolants. This is not so. A family of grids with bounded mesh ratio need not be uniformly distributed as $N \rightarrow \infty$, and if it is not, the Lebesgue constants will increase exponentially with N .

4. Sampling theory and the Kadec 1/4 theorem. The field of approximation theory goes back to Borel, de la Vallée Poussin, Fejér, Jackson, Lebesgue, and others at the beginning of the 20th century, and its central question might be characterized like this:

Given a function f of a certain regularity, how fast do its approximations of a given kind converge?

For example, if f is periodic and analytic on $[-\pi, \pi)$, then its equispaced trigonometric interpolants converge exponentially. The same holds if f is analytic in a strip surrounding the whole real line and satisfies a decay condition at ∞ , with trigonometric interpolants generalized to interpolatory series of sinc functions.

The field of sampling theory goes back to Gabor, Kotelnikov, Nyquist, Paley, Shannon, J. M. and E. T. Whittaker, and Wiener a few years later. Its central question might be characterized like this:

Given a function f of a certain regularity, which of its approximations of a given kind are exactly equal to f ?

For example, if f is periodic and analytic on $[-\pi, \pi)$, then its equispaced trigonometric interpolant is exact if f is band-limited (has a Fourier series of compact support) and the grid includes at least two points per wavelength for each wave number present in the series. The same holds if f is a band-limited analytic function on the whole real line, with the Fourier series generalized to the Fourier transform, and again with trigonometric interpolation generalized to sinc interpolation.

Obviously we have worded these characterizations to highlight the similarities between the two fields, which in fact differ in significant ways. Still, it is remarkable how little interaction there has been between the two. What makes this relevant to the present paper is that our theorems and orientation are very much those of approximation theory, whereas most of the scientific interest in perturbed grids in the past has been from the side of sampling theory, and the Kadec 1/4 theorem is the best-known result in this general area.

Kadec's theorem is an answer to a question of sampling theory that originates with Paley and Wiener [13]. The exponentials $\{\exp(i\lambda_k x)\}$, $-\infty < k < \infty$, form an orthonormal basis for $L^2[-\pi, \pi]$ if $\lambda_k = k$ for each k . Thus, the sampling theorist would say that one can recover a function $f \in L^2[-\pi, \pi]$ from its inner products with the functions $\{\exp(i\lambda_k x)\}$. Now suppose these wave numbers are perturbed so that $|\tilde{\lambda}_k - k| \leq \alpha$ for some fixed α . Can one still recover the signal? Specifically, does the family $\{\exp(i\tilde{\lambda}_k x)\}$ form a *Riesz basis* for $L^2[-\pi, \pi]$, that is, a basis that is related to the original one by a bounded transformation with a bounded inverse? Paley and Wiener showed that this is always the case for $\alpha < 1/\pi^2$, and Levinson showed it is not always the case for $\alpha \geq 1/4$. Kadec's theorem shows that Levinson's construction was sharp: for any $\alpha < 1/4$, the family $\{\exp(i\tilde{\lambda}_k x)\}$ forms a Riesz basis [2, 3, 9, 18].

Note that the standard setting of Kadec's theorem involves perturbation of wave numbers from equispaced values, in contrast to the results of this paper, which involve perturbation of interpolation points from equispaced values. In view of the Fourier transform, however, these settings are related, so one might imagine, based on Kadec's theorem, that $\alpha = 1/4$ might be a critical value for trigonometric interpolation in perturbed points. Instead, we have found that the critical value is $\alpha = 1/2$.

We explain this apparent discrepancy as follows. The Paley–Wiener theory and Kadec's theorem are results concerning the L^2 norm, which in many applications would represent energy. In our application of trigonometric interpolation, something related to the L^2 norm does indeed happen at $\alpha = 1/4$. Suppose we look at a 2-norm Lebesgue constant $\tilde{\Lambda}_N^{(2)}$ for the perturbed grid interpolation problem, defined as the operator norm on $L : f \mapsto \tilde{t}_N$ induced by the discrete ℓ^2 -norm on the data $\{f(\tilde{x}_k)\}$ and on the Fourier coefficients of the interpolant \tilde{t}_k . Numerical experiments reported in Section 3.4.3 of [1] indicate that whereas the usual ∞ -norm Lebesgue constant is unbounded for all α , $\tilde{\Lambda}_N^{(2)}$ is bounded as $N \rightarrow \infty$ for any $\alpha < 1/4$ but not always bounded for $\alpha \geq 1/4$. (Indeed Kadec's theorem may imply this result.) For $\alpha \in (1/4, 1/2)$, we conjecture $\tilde{\Lambda}_N^{(2)} = O(N^{4\alpha-1})$.

Thus a sampling theorist might say that for $\alpha \in [1/4, 1/2)$, trigonometric interpolation is *unstable* in the sense that it may amplify signals unboundedly in ℓ^2 as $N \rightarrow \infty$. On the other hand the approximation theorist might note that the instability is very weak, involving not even one power of N . Assuming that the conjectured sharpening of the estimate (1.9) of Theorem 1.1 is valid, one derivative of smoothness of f is enough to suppress the instability, ensuring $\|f - \tilde{t}_N\| \rightarrow 0$ as $N \rightarrow \infty$ for all $\alpha < 1/2$. The numerical analyst might add that on a computer, amplification of rounding errors by $o(N)$ is unlikely to cause trouble. For $\alpha \geq 1/2$, in strong contrast, the amplification is unbounded in any norm even for finite N , and trouble is definitely to be expected.

5. Proof of the Lebesgue constant estimate, Theorem 2.1. A full proof of Theorem 2.1, filling 20 pages, is the subject of Chapter 4 of the first author's D. Phil. thesis [1]. Many detailed trigonometric estimates are involved, and we do not know how to shorten it significantly. For readers interested in full details, that chapter has been made available in the Supplementary Materials attached to this paper.

Here, we outline the argument. To prove the bound (2.2) on the Lebesgue constant,

$$(5.1) \quad \tilde{\Lambda}_N \leq \frac{C(N^{4\alpha} - 1)}{\alpha(1 - 2\alpha)},$$

we begin by noting that $\tilde{\Lambda}_N$ is given by

$$(5.2) \quad \tilde{\Lambda}_N = \max_{x \in [-\pi, \pi]} \tilde{L}(x),$$

where \tilde{L} is the Lebesgue function

$$(5.3) \quad \tilde{L}(x) = \sum_{k=-N}^N |\tilde{\ell}_k(x)|,$$

where $\tilde{\ell}_k$ is the k th Lagrange cardinal trigonometric polynomial for the perturbed grid,

$$(5.4) \quad \tilde{\ell}_k(x) = \prod_{j \neq k} \sin\left(\frac{x - \tilde{x}_j}{2}\right) \bigg/ \sin\left(\frac{\tilde{x}_k - \tilde{x}_j}{2}\right).$$

The function $\tilde{\ell}_k(x)$ takes the values 1 at \tilde{x}_k and 0 at the other grid points \tilde{x}_j , and the sum (5.3) adds up contributions at a point x from all the $2N + 1$ cardinal functions associated with grid points to its left and right.

The argument begins by showing that on the interval $[x_{-(k+1)}^*, x_{-k}^*]$, $\tilde{\ell}_0$ satisfies the bound

$$(5.5) \quad |\tilde{\ell}_0(x)| \leq M_k, \quad x \in [x_{-(k+1)}^*, x_{-k}^*], \quad 0 \leq k \leq N,$$

for certain numbers M_0, \dots, M_N , independently of the choice of perturbed points $\{\tilde{x}_k\}$. The points x_k^* are defined by $x_0^* = 0$, $x_{-(N+1)}^* = -\pi$, and

$$x_k^* = 2 \arctan \left(\frac{\cos(kh) - \cos(\alpha h) + \tan(\tilde{x}_0/2) \sin(kh)}{\tan(\tilde{x}_0/2) (\cos(kh) + \cos(\alpha h)) - \sin(kh)} \right), \quad -N \leq k \leq N, k \neq 0;$$

the most important fact about them is that they satisfy the inequalities

$$(k - \alpha)h \leq x_k^* \leq (k + \alpha)h, \quad -N \leq k \leq N.$$

Thus, (5.5) bounds $\tilde{\ell}_0$ on certain subintervals of $[-\pi, 0]$. By exploiting symmetry, these bounds yield similar bounds on $\tilde{\ell}_0$ on similar subintervals of $[0, \pi]$ as well as bounds on the other $2N$ contributions to \tilde{L} in (5.3). We are eventually led to the estimate

$$\tilde{L}(x) \leq 9 \sum_{k=0}^N M_k,$$

which holds uniformly for $x \in [-\pi, \pi]$. The factor of 9 on the right-hand emerges due to the particular way in which the symmetry of the problem is exploited; see Lemmas 16 and 17 in the supplementary materials for details.

For sufficiently large N , the M_k satisfy

$$(5.6) \quad M_k \leq \frac{10\pi}{1 - 2\alpha}, \quad k = 0, 1$$

and

$$(5.7) \quad M_k \leq \frac{3\pi(k+1)^{2\alpha}}{(1 - 2\alpha)(k-1)^{1-2\alpha}}, \quad 2 \leq k \leq N.$$

The bound (5.1) follows for sufficiently large values of N by an estimation of the sums of (5.6) and (5.7) over all k ; small values of N are finite in number and thus can be handled by adjusting the constant C . The numbers M_k are defined by

$$(5.8) \quad M_k = \max_{x \in [-\pi, 0] \cap R_k} \frac{P_k(x)}{Q_k}, \quad 0 \leq k \leq N,$$

with

$$P_k(x) = \prod_{i=1}^N \left| \sin \left(\frac{x - (i - \alpha)h}{2} \right) \right| \times \prod_{i=1}^k \left| \sin \left(\frac{x + (i - \alpha)h}{2} \right) \right| \times \prod_{i=k+1}^N \left| \sin \left(\frac{x + (i + \alpha)h}{2} \right) \right|$$

and

$$Q_k = \prod_{i=1}^N \left| \sin \left(\frac{(2\alpha - i)h}{2} \right) \right| \times \prod_{i=1}^k \left| \sin \left(\frac{ih}{2} \right) \right| \times \prod_{i=k+1}^N \left| \sin \left(\frac{(2\alpha + i)h}{2} \right) \right|.$$

The set R_k in the definition of the range of the maximum in (5.8) is the interval

$$R_k = [(-k - 1 - \alpha)h, (-k + \alpha)h].$$

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