Classical and Quantum Aspects of Bosonic Production and Detection

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Abstract

This thesis presents methods for producing and detecting gravitational waves (GWs) and particles within or beyond the Standard Model. Chap. 2 and Chap. 3 describe axion production from an accelerated electron within the WKB approximation. In Chap. 2, we consider an electron accelerated by a time-dependent external potential and consider two trajectories: uniform acceleration and oscillating motion in a standing wave created by two counter-propagating linearly polarized laser beams. We calculate the spectrum and total energy of the emitted particles. Chap. 3 generalizes the results of Chap. 2 for an arbitrary electromagnetic field. We find that the rotation of the electron spin follows the Thomas-BMT equation. We propose an experimental setup for producing axions using laser beams and converting them into photons to impose bounds on the axion-electron coupling constant. We find that the addition of magnetic fields significantly increases particle production. The projected bounds are similar to other laboratory-based experiments for axion masses $m_a \lesssim 10$ keV. Chap. 4 discusses the Unruh effect, which is essential to describe radiation from an accelerating charge in its rest frame. We couple a classical source to a scalar field and derive the emission rate and power. The latter agrees with the classical Larmor result for scalar particles. In Chap. 5, we generalize the results of Chapter Chap. 4 and discuss emission of photons in the context of the Unruh effect. Finally, Chap. 6 is dedicated to the production and detection of GWs using high-energy pulsed lasers through the Gertsenshtein effect. Whereas the strain of produced GWs is too weak to be detected, we find that with today's laser performances, one could detect GWs of astrophysical origin with strains $h \gtrsim 10^{-20}$ and in the future, sensitivities of $h \gtrsim 10^{-26}$ could be reached.
Overall, the thesis examines the plausibility to use high-energy lasers to produce and detect particles of spin 0, 1 and 2 (where for the spin 2 case, we consider classical GWs instead of gravitons).

In defense of this thesis, the following articles are declared as works the author has contributed to.

[1] **G. Vacalis**, G. Marocco, J. Bamber, R. Bingham, and G. Gregori. Detection of high-frequency gravitational waves using high-energy pulsed lasers. Class. Quant. Grav., 40(15):155006, 2023

[2] **G. Vacalis**, A. Higuchi, R. Bingham and G. Gregori. Classical Larmor formula through the Unruh effect for uniformly accelerated electrons. Phys. Rev. D, 109:024044, 2024

[3] J. W. D. Halliday, G. Marocco, K. A. Beyer, C. Heaton, M. Nakatsutsumi, T. R. Preston, C. D. Arrowsmith, C. Baehtz, S. Goede, O. Humphries, A. Laso Garcia, R. Plackett, P. Svensson, **G. Vacalis**, J. Wark, D. Wood, U. Zastrau, R. Bingham, I. Shipsey, S. Sarkar and G. Gregori. Bounds on Heavy Axions with an X-Ray Free Electron Laser. Phys. Rev. Lett., 134:055001, 2025

[4] A. Higuchi, G. E. A. Matsas, D. A. T. Vanzella, R. Bingham, J. P. B. Brito, L. C. B. Crispino, G. Gregori and **G. Vacalis**. Larmor radiation as a witness to the Unruh effect. Phys. Rev. D, 112:065009, 2025

[5] **G. Vacalis**, A. Higuchi, R. Bingham and G. Gregori. Proposal to Use Laser-Accelerated Electrons to Probe the Axion-Electron Coupling. Phys. Rev. Lett., 135:195003, 2025.

Statement of Originality

I hereby declare that this thesis is entirely my own work, except where otherwise indicated. I have clearly identified all material quoted from other sources. Sections based on literature reviews or collaborative work are also explicitly indicated.

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Table of Contents

List of Figures

1 Introduction

1.1 The axion solution to the strong CP problem

The Standard Model (SM) of Particle Physics correctly predicts the outcomes of most laboratory experiments with great precision. It is therefore one of the most successful theories of fundamental physics. However, it possesses a few defects. One of them is known today as the strong CP problem. The quantum chromodynamics (QCD) sector of the SM written as

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\gamma^{\mu} D_{\mu} - \mathcal{M})q - \frac{1}{4} G^{a}_{\mu\nu} G^{a\mu\nu} - \theta \frac{g^2}{32\pi^2} G^{a}_{\mu\nu} \tilde{G}^{a\mu\nu} \tag{1.1}$$

does not exclude the third term which violates the charge-parity (CP) symmetry. q are the quark fields, D_μ is the covariant derivative, $\mathcal{M}$ is the quark mass matrix, and $G_{\mu\nu}$ carries the gauge bosons. $\tilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$ is the dual field strength. The third term in EQ. (1.1) is a total derivative [6] and is therefore not important for perturbation theory. Indeed, $G^a_{\mu\nu}\tilde{G}^{a\mu\nu} = \partial_\mu K^\mu$ and $K^\mu = \varepsilon^{\mu\alpha\beta\gamma}\mathcal{A}^a_\alpha[G^a_{\beta\gamma} - \frac{g}{3}f^{abc}\mathcal{A}^b_\beta\mathcal{A}^c_\gamma]$ with $\mathcal{A}^a_\mu$ the gluon fields. Following the discussion in Ref. [7], we consider for simplicity a theory of two quarks u and d. In the massless limit, in which right- and left-handed quark fields do not mix, QCD seems to be invariant under the symmetry group $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A$. The appearance of quark condensates $\langle\bar{q}q\rangle \neq 0$ breaks the $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_L$ symmetry, and the associated Nambu-Goldstone bosons are the pions.

On the other hand, quark condensates also lead to the breaking of the $\mathrm{U}(1)_A$ symmetry. The latter is the transformation $q \to \mathrm{e}^{i\alpha_A\gamma_5}q$. Contrary to the previous symmetry breaking, there are no associated Nambu-Goldstone bosons. This is what Weinberg named the U(1)-problem [8]. A solution was given by 't Hooft [9, 10] by considering the complicated structure of the true QCD vacuum given by

$$|\theta\rangle \equiv \sum_{n=-\infty}^{+\infty} \mathrm{e}^{in\theta}\,|n\rangle \ , \tag{1.2}$$

where $\theta \in [0, 2\pi)$, $|n\rangle$ are called the n-vacua and the associated integer n is the winding number. The choice of $|\theta\rangle$ as the vacuum gives rise to the third term in EQ. (1.1). Therefore, instead of considering this term as allowed by the symmetry of the theory, one can view it as imposed by the nature of the QCD vacuum. This term violates time reversal (T) and P but conserves C. Thus, CP is not a symmetry.

In addition to QCD, the weak sector of SM contributes to the CP breaking term through the quark mass matrix. We can choose a basis in which, for two quarks, $\mathcal{M} \to \mathrm{diag}(m_u, m_d \mathrm{e}^{i\arg(\det\{\mathcal{M}\})})$. Performing a $\mathrm{U}(1)_A$ rotation for the d quark to obtain real masses gives an additional contribution to the third term in Eq. (1.1). Thus, now the parameter of the CP violation is

$$\bar{\theta} \equiv \theta - \arg(\det\{\mathcal{M}\})\,. \tag{1.3}$$

Eq. (1.1) implies that the neutron carries an electromagnetic dipole moment given by $d_n = 2.4 \times 10^{-16}\bar{\theta}\, e\,\mathrm{cm}$ [11–14]. Recently, in Ref. [15] an upper limit on d_n was found, which results in the bound $\bar{\theta} \leq 7.5 \times 10^{-11}$. A solution to the CP problem would be to explain how two contributions, θ and the quark matrix, coming from two distinct sectors of the SM cancel with precision $\lesssim 10^{-10}$.

Peccei and Quinn (PQ) proposed solving the CP problem elegantly by introducing an additional $\mathrm{U}(1)_{PQ}$ symmetry [16, 17]. Demanding the breaking of the latter at a scale f_a gives rise to a pseudo Nambu-Goldstone boson, the axion [18, 19]. Consider the potential generated by QCD in the presence of pions whose appearance is due to another symmetry breaking as discussed previously [20]

$$V = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2\left(\frac{\bar{\theta}}{2}\right)}\,, \tag{1.4}$$

where f_π is the pion decay constant, m_π, m_u and m_d are the masses of the pion, up and down quark respectively. The minimum of the potential would be at $\bar{\theta} = 0$ but $\bar{\theta}$ is not dynamical. The PQ solution, by introducing the axion field, makes the argument of the potential dynamical. More precisely, through the term

$$\mathcal{L} \supset -\frac{a}{f_a}\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu}\,, \tag{1.5}$$

the argument of EQ. (1.4) is shifted as $\bar{\theta} \to \bar{\theta} + a/f_a$. Then, the axion gets a vacuum expectation value at $-f_a\bar{\theta}$ which sets the neutron electromagnetic dipole moment to $d_n \propto (\bar{\theta} + a/f_a) = 0$, thus solving the CP problem. EQ. (1.5) is a dimension-5 operator and hence defines an effective field theory. The axion's coupling to SM particles is

$$\mathcal{L}_a = \frac{1}{2}(\partial_\mu a)^2 + \frac{a}{f_a}\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{\partial_\mu a}{f_a}\bar{q}c_q^{(0)}\gamma^\mu\gamma_5 q + \frac{1}{4}g^{(0)}_{a\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{1.6}$$

where the last term describes the coupling between the axion and the electromagnetic tensor. Its coupling is $g^{(0)}_{a\gamma} = \frac{\alpha_{\rm em}}{2\pi f_a}E/N$ where E/N is the ratio of the electromagnetic and color anomaly coefficients. These coefficients arise from the current associated with the PQ symmetry [21]

$$\partial_\mu j^{(\rm PQ)\mu} = \frac{Ng^2}{16\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{Ee^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\,. \tag{1.7}$$

$j^{(\rm PQ)\mu}$ is normalized such that N is a non-zero integer. It is possible to have $E = 0$. Under the quark redefinition $q \to \exp\{(i\gamma_5 a\mathcal{Q}_a/(2f_a))\}q$, where $\mathcal{Q}_a$ is a two-dimensional matrix whose trace is equal to unity, the second term in EQ. (1.6) is removed, but the quark mass term is now axion-dependent [22].

The Lagrangian is then

$$\mathcal{L}_a = \frac{1}{2}(\partial_\mu a)^2 + \frac{\partial_\mu a}{f_a}\bar{q}c_q\gamma^\mu\gamma_5 q + \frac{1}{4}g_{a\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu} - \bar{q}_L\mathcal{M}_a q_R + \text{h.c.}\,, \tag{1.8}$$

where

$$c_q = c_q^{(0)} - \mathcal{Q}_a, \quad g_{a\gamma} = g_{a\gamma}^{(0)} - \frac{3\alpha_{\text{em}}}{\pi f_a}\text{Tr}(\mathcal{Q}_a\mathcal{Q}^2), \quad \mathcal{M}_a = \mathrm{e}^{\frac{i\gamma_5 a\mathcal{Q}_a}{2f_a}}\mathcal{M}\mathrm{e}^{\frac{i\gamma_5 a\mathcal{Q}_a}{2f_a}}\,. \tag{1.9}$$

Here, $\mathcal{M}$ is the quark mass matrix and $\mathcal{Q} = \text{diag}(2/3, -1/3)$. Taking $\mathcal{Q}_a = \mathcal{M}^{-1}/\text{Tr}(\mathcal{M}^{-1})$, the axion-photon coupling is [22]

$$g_{a\gamma} = \frac{\alpha_{\text{em}}}{2\pi f_a}\left[\frac{E}{N} - \frac{2}{3}\frac{4m_d + m_u}{m_d + m_u}\right] = \frac{\alpha_{\text{em}}}{2\pi f_a}\left[\frac{E}{N} - 1.92\right]. \tag{1.10}$$

The axion is a massive particle whose mass is related to f_a via [21–23]

$$m_a = 5.7\,\mu\text{eV}\left(\frac{10^{12}\,\text{GeV}}{f_a}\right). \tag{1.11}$$

Different models give different values for some of the parameters discussed above. The most popular models are named Kim-Shifman-Vainshtein-Zakharov (KSVZ) [24, 25] and Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) [26, 27]. For example, in the first model $E/N = 0$, whereas in the second $E/N = 8/3$. Another difference is that the KSVZ model does not predict a coupling between the axion and the electron at tree level. On the other hand, the DFSZ model predicts a coupling of the form

$$\mathcal{L} \supset \frac{g_{ae}}{2m}(\partial_\mu a)\bar{\psi}\gamma^\mu\gamma_5\psi\,. \tag{1.12}$$

The coupling g_{ae} in this model is proportional to the axion mass [21]

$$g_{ae} = 1.8 \times 10^{-10}(2\cos^2\beta)\left(\frac{m_a}{1\text{eV}}\right) \tag{1.13}$$

for 3 generations of quarks and leptons, and where β is a free parameter. The coupling $g_{a\gamma}$ is expressed as a function of the axion mass in a model-independent way (in any standard grand unified theory) as $g_{a\gamma} = 1.45 \times 10^{-10}(m_a/\text{eV})\ \text{GeV}^{-1}$ [21].

To clarify the terminology in what follows, we differentiate QCD axions from axions. The former correspond to solutions to the strong CP problem and for which EQ. (1.13) holds. For the latter, we consider the couplings $g_{a\gamma,}$, g_{ae} and m_a as independent parameters.

So far, there has been no experimental evidence for axions, and experiments are limiting the axion parameter space. The CAST experiment is used to convert hypothetical axions produced in the Sun into photons using a magnetic field [28]. However, predictions on axion production by astrophysical sources can depend on the modeling of the star. Therefore, often purely terrestrial experiments are of interest, such as OSQAR [29]. Other light-shining-through-wall (LSW) experiments have been proposed [30, 31]. Terrestrial experiments have also been conducted to bound to the coupling g_{ae}. This can be done using a nuclear reactor to produce axions [32] or using the anomalous magnetic moment [33]. Another way to produce axions would be through laser-accelerated electrons taking advantage of the progress of laser technology, which is expected to continue in the near future (we note that X-ray free-electron lasers can be used to put bounds on $g_{a\gamma}$ [3], but we will not consider this mechanism in the thesis). The motivation for using this production mechanism is that we expect axion production to increase with the electron's acceleration (assuming a Larmor-type formula for axion radiation), which, using today's laser technology, can be large. In Chap. 2 and Chap. 3, we study axion production through this mechanism using a semi-classical description for the electron. In principle, accelerated electrons can emit axions similarly to Larmor radiation for photons. To study the latter, one can describe the electron through a classical current which takes the form $j^\mu = qv^\mu\delta^{(3)}(\mathbf{x} - \mathbf{x}_{\mathrm{tr}})$, where q, v^μ, $\mathbf{x}_{\mathrm{tr}}$ are the charge, velocity, and position of the electron, respectively. However, contrary to photon radiation, we find that one cannot use a purely classical electron description for the axion case. For this reason, the electron description we adopt will make use of the WKB approximation, which is outlined in the next section. Depending on the interaction Lagrangian used, we either need the zeroth or first order of this approximation.

1.2 The WKB approximation

The WKB approximation is named after Wentzel Kramers and Brillouin, who developed this method [34–36]. It provides an expansion in $\hbar$ to the Schrödinger equation. Concretely, we write the wavefunction as $\psi = \mathrm{e}^{iS(x)/\hbar}$

where for the time-independent case $S(x) = S(\mathbf{x}) - Et$, with E the energy. We then expand $S(\mathbf{x}) = S_0(\mathbf{x}) + \hbar S_1(\mathbf{x}) + ...$ In our case, the expansion in $\hbar$ will be performed for the four component spinors which are solutions to the Dirac equation in the presence of an external potential. For a systematic WKB solution of the Dirac equation, see Ref. [37]. In what follows, the expansion in $\hbar$ can be seen as an expansion in the dimensionless parameter $\hbar k/m \ll 1$, where k is the wave number of the emitted particle and m is the electron mass. We start from the free Dirac Lagrangian of the electron field

$$\mathcal{L} = i\hbar\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi\,, \tag{1.14}$$

where m is the mass of the electron and the slash notation means $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. As usual, $\bar{\psi} = \psi^\dagger\gamma^0$. Here, we write explicitly $\hbar$ as we are using a semi-classical expansion. The gamma matrices have multiple representations. In what follows, we will use the Dirac representation, unless specified otherwise, which is given by the choice

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \mathbb{O} \\ \mathbb{O} & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbb{O} & \sigma^i \\ -\sigma^i & \mathbb{O} \end{pmatrix}, \quad i = 1, 2, 3\,, \tag{1.15}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\mathbb{O}$ is the 2×2 null matrix and $\sigma^i, i = 1, 2, 3$ are the Pauli matrices. We also define the matrix γ_5 by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & \mathbb{O} \\ \mathbb{O} & \mathbb{1} \end{pmatrix}. \tag{1.16}$$

From the Lagrangian, follows the Dirac equation $(i\hbar\partial\!\!\!/ - m)\psi = 0$. The free spinor field is then expanded as

$$\psi(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\frac{m}{p_0}\sum_\alpha \left[b_\alpha(\mathbf{p})\Phi_\alpha(\mathbf{p},x) + d^\dagger_\alpha(\mathbf{p})\Psi_\alpha(\mathbf{p},x)\right], \tag{1.17}$$

where $p_0 = \sqrt{\mathbf{p}^2 + m^2}$ is the energy and the index $\alpha = 1,2$ is the spin index. The operators $b_\alpha(\mathbf{p})$ and $d^\dagger_\alpha(\mathbf{p})$ satisfy the following anti-commutation relations

$$\begin{aligned} \{b_\alpha(\mathbf{p}), b^\dagger_\beta(\mathbf{p}')\} &= \frac{p_0}{m}(2\pi\hbar)^3\delta_{\alpha\beta}\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \\ \{d_\alpha(\mathbf{p}), d^\dagger_\beta(\mathbf{p}')\} &= \frac{p_0}{m}(2\pi\hbar)^3\delta_{\alpha\beta}\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \end{aligned} \tag{1.18}$$

and all other commutators vanish. The modes $\Phi_\alpha(\mathbf{p},x)$ and $\Psi_\alpha(\mathbf{p},x)$ are solutions to the Dirac equation and can be expressed as

$$\Phi_\alpha(\mathbf{p},x) = u_\alpha(\mathbf{p})\mathrm{e}^{-ip\cdot x/\hbar}, \quad \Psi_\alpha(\mathbf{p},x) = v_\alpha(\mathbf{p})\mathrm{e}^{ip\cdot x/\hbar}. \tag{1.19}$$

Using the Dirac equation, we find that the functions u_α and v_α satisfy the Dirac equations in momentum space $(p\!\!\!/ - m)u_\alpha = (p\!\!\!/ + m)v_\alpha = 0$. The solutions to these equations are given by

$$u_\alpha(\mathbf{p}) = \sqrt{\frac{p_0+m}{2m}}\begin{pmatrix} s_\alpha \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}s_\alpha \end{pmatrix}, \quad v_\alpha(\mathbf{p}) = \sqrt{\frac{p_0+m}{2m}}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}s_\alpha \\ s_\alpha \end{pmatrix}. \tag{1.20}$$

If the spin is along the axis x^i, then the vectors s_α are the eigenvectors of σ^i. The choice $\alpha = 1,2$ corresponds to whether the spin in the positive direction of x^i or the negative. Having treated the free spinor field, we now look at the field in the presence of an external classical potential V_μ. In this section we consider either time or space dependent potentials and their effect appears through the covariant derivative

$$D_\mu = \partial_\mu + \frac{i}{\hbar} V_\mu \,. \tag{1.21}$$

The Lagrangian in Eq. (1.14) is modified by the replacement $\partial_\mu \to D_\mu$:

$$\mathcal{L} = i\hbar \bar{\psi} \not{D} \psi - m \bar{\psi} \psi \tag{1.22}$$

and therefore the Dirac equation becomes

$$\left[i\hbar \gamma^\mu \left(\partial_\mu + \frac{i}{\hbar} V_\mu \right) - m \right] \psi = 0 \,, \tag{1.23}$$

We note that taking the complex conjugate of the modified Dirac equation will change the sign of the term involving the potential because of the i factor. This sign change represents the opposite charge for the antiparticle solutions. In what follows, we will not treat the potential perturbatively. Instead, to obtain a semi-classical solution to the modified Dirac equation, we will expand the solution as powers of $\hbar$. The WKB expansion for a time-dependent potential has been treated in detail in Ref. [38]. Therefore, here we only show the main results. We start by expressing Φ_α as

$$\Phi_\alpha(x) = \psi_\alpha(t) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}/\hbar} \tag{1.24}$$

to separate the time and space dependence. By multiplying Eq. (1.23) by $\beta = \gamma^0$ we obtain

$$i\hbar\partial_t\Phi_\alpha - [\boldsymbol{\alpha}\cdot(-i\hbar\boldsymbol{\nabla} - \mathbf{V}) + \beta m]\Phi_\alpha = 0\,, \tag{1.25}$$

where we defined $\alpha^i = \gamma^0\gamma^i$. This leads to

$$i\hbar\partial_t\psi_\alpha - [\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}} + \beta m]\psi_\alpha = 0\,, \quad \tilde{\mathbf{p}}(t) = \mathbf{p} - \mathbf{V}(t)\,. \tag{1.26}$$

The eigenvalues of the matrix $\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}} + \beta m$ are $\pm E(t) = \pm\sqrt{\tilde{\mathbf{p}}^2 + m^2}$. Since we are considering the positive energy solutions we will only take $+E$ as an eigenvalue. The semi-classical approach consists in expanding $\psi(t)$ as powers of $\hbar$ as follows

$$\psi_\alpha(t) = \mathrm{e}^{-\frac{i}{\hbar}S}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \mathrm{e}^{-\frac{i}{\hbar}S}\left[\begin{pmatrix}\varphi^{(0)}\\ \chi^{(0)}\end{pmatrix} + \hbar\begin{pmatrix}\varphi^{(1)}\\ \chi^{(1)}\end{pmatrix} + \ldots\right]. \tag{1.27}$$

To $\mathcal{O}(\hbar)$ [38]

$$\begin{aligned}\Phi_\alpha(x) =& \sqrt{\frac{p_0}{E}}\mathrm{e}^{-\frac{i}{\hbar}S}\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}/\hbar}\\ &\times\left[(1+\hbar g(t))u_\alpha^{(0)} - i\hbar\frac{E+m}{(2E)^2}\sqrt{\frac{E+m}{2m}}\begin{pmatrix}-\Sigma\dot{\Sigma}s_\alpha(t)\\ \dot{\Sigma}s_\alpha(t)\end{pmatrix}\right],\end{aligned} \tag{1.28}$$

where dot means derivative with respect to t, $\dot{g}(t) = \dot{\tilde{p}}^2/(8E^3)$, $\dot{S} = E$, and $u_\alpha^{(0)}$ is the zeroth order spinor, which for constant potential is Eq. (1.20) and in general is

$$u_\alpha^{(0)} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} s_\alpha(t) \\ \Sigma s_\alpha(t) \end{pmatrix} . \tag{1.29}$$

The time dependent two-component spinors are $s_a(t) = U(t)s_\alpha$ with s_α the initial spinor and $U(t)$ is a time-ordered unitary operator defined by

$$U(t) = \mathrm{T}\left(\exp\left[-i \int_0^t \mathrm{d}\zeta \, \frac{\boldsymbol{\sigma} \cdot (\tilde{\mathbf{p}}(\zeta) \times \dot{\tilde{\mathbf{p}}}(\zeta))}{2E(\zeta)(E(\zeta)+m)} \right] \right) . \tag{1.30}$$

Finally, the 2×2 matrix Σ is

$$\Sigma = \frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{E+m} . \tag{1.31}$$

For a purely space-dependent potential, we show the derivation in App. A and show the main results here. As previously, we start from Eq. (1.23). For a mode Φ_α that satisfies the latter, we define

$$\Phi_\alpha = \psi_\alpha(x) \mathrm{e}^{-\frac{i}{\hbar}(p_0 t - p_y y - p_z z)} , \tag{1.32}$$

where here x is one of the space coordinates. The Dirac equation, with gauge choice $V_x = 0$, leads to

$$i\hbar \partial_x \psi_\alpha = (\tilde{p}_0 \gamma^1 \gamma^0 - \tilde{p}_y \gamma^1 \gamma^2 - \tilde{p}_z \gamma^1 \gamma^3 - m\gamma^1)\psi_\alpha , \tag{1.33}$$

where $\tilde{p}_0 = p^0 - V^0$, $\tilde{p}_y = p^y - V^y$ and $\tilde{p}_z = p^z - V^z$. We expand as previously

$$\psi_\alpha(x) = \mathrm{e}^{-iS(x)/\hbar}\left[\begin{pmatrix}\varphi^{(0)}\\ \chi^{(0)}\end{pmatrix} + \hbar\begin{pmatrix}\varphi^{(1)}\\ \chi^{(1)}\end{pmatrix} + \ldots\right]. \tag{1.34}$$

We find that to zeroth order, similarly to the time dependent case

$$\Phi_\alpha = \sqrt{\frac{\tilde{p}_0 + m}{2m}}\begin{pmatrix} s_\alpha(x) \\ \frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0+m} s_\alpha(x)\end{pmatrix}\sqrt{\frac{|p_x|}{\kappa_p}}\exp\left\{\frac{i}{\hbar}\int_0^x \mathrm{d}\zeta\,\kappa_p(\zeta)\right\}\mathrm{e}^{-\frac{i}{\hbar}(p_0 t - p_y y - p_z z)}, \tag{1.35}$$

where $\kappa_p = \sqrt{\tilde{p}_0^2 - \tilde{p}_y^2 - \tilde{p}_z^2 - m^2}$, $\tilde{\mathbf{p}} = (\kappa_p, \tilde{p}_y, \tilde{p}_y)$ and $s_\alpha(x) = U(x)s_\alpha$ with

$$\begin{aligned} &U(x) \\ &= \mathrm{T}_x\left(\exp\left\{-i\int^x \mathrm{d}\tilde{x}\,\frac{\sigma^3(\tilde{p}_y'(\tilde{p}_0+m) - \tilde{p}_y\tilde{p}_0') - \sigma^2(\tilde{p}_z'(\tilde{p}_0+m) - \tilde{p}_z\tilde{p}_0')}{2\kappa_p(\tilde{p}_0+m)}\right\}\right). \end{aligned} \tag{1.36}$$

where T_x and prime mean "x"-ordering and derivative with respect to x, respectively. In Chap. 2, we will use the WKB expansion to describe axion radiation from an electron that follows a one-dimensional trajectory focusing mainly on time-dependent potentials. We generalize the results for the emission amplitude to general electromagnetic fields in Chap. 3.

1.3 Radiation in the rest frame of the charge

After having studied the emission of axions in the laboratory, we will focus on the description of particle radiation as seen in the rest frame of the accelerating electron. For simplicity, we will not consider pseudo-scalars such as axions. Instead, we couple the accelerating particle to a scalar field in Chap. 4 and to the vector field in Chap. 5. Through the equivalence principle, particle production due to acceleration is connected to particle production in a curved spacetime which is a well-understood phenomenon [39–41]. It has led to the study of production mechanisms of dark matter in the early Universe [42–44]. In particular, due to the large Hubble rate, the transition between inflation and radiation domination epoch is a period of particular interest [45–48]. Another example of gravitational production is the radiation emitted by a black hole known as Hawking radiation [49, 50]. A few years after this discovery, in 1976 Unruh [51] showed that in flat spacetime, a uniformly accelerating observer in the usual vacuum "sees" a thermal bath of particles with temperature proportional to the proper acceleration. This result followed two observations. The first one, made by Fulling in 1973 [52], states that the usual vacuum as seen from inertial observers does not coincide with the vacuum associated with an accelerating observer. The second one, made by Davies in 1975 [53], is about the fact that a uniformly accelerating observer in flat spacetime would see an inertial mirror emitting thermal radiation with temperature $T = \hbar a/2\pi c k_B$, where a is the proper acceleration.

The bath of particles seen by an accelerating observer is referred to as the Fulling-Davies-Unruh (FDU) thermal bath. Its temperature is the same as that found by Davies and given by

$$T_{\text{FDU}} = \frac{\hbar a}{2\pi c k_B} \,. \tag{1.37}$$

The Unruh effect is a quantum field theory (QFT) result whose existence has been questioned by some authors [54–56]. One of their arguments is about the fact that the Rindler modes defined in Eq. (4.26) are not defined in all of Minkowski spacetime. We will see how this fact does not compromise the validity of the Unruh effect. In Ref. [57], it is pointed out that another misconception is that the Unruh effect is a statement about the interaction of a detector with the FDU thermal bath. This is not the case, as the Unruh effect does not depend on the introduction of the detector concept [57]. On the other hand, when studying uniformly accelerating detectors in their rest frame, it is essential to account for the FDU thermal bath, which is what is done in what follows. We will calculate the emission rate as well as the power emitted in the rest frame of the charge. We first consider scalar particles in Chap. 4 because it is less laborious to derive the Unruh effect in their case. The results of Chap. 4 can be straightforwardly generalized to other types of particles, which is the subject of Chap. 5. As we shall see, the emission rate and the power grow with the proper acceleration. The electron is the simplest "detector" as large accelerations can be achieved in the laboratory with high-intensity lasers [58–60]. As discussed previously, since we are not considering pseudo-scalars, we can treat the electron classically without the use of the WKB approximation. This is equivalent to taking into account only $\mathcal{O}(\hbar^0)$ terms in the WKB approximation.

1.4 Gravitational waves

A natural extension to the radiation of the aforementioned bosonic particles would be to study the emission of gravitons from an accelerating electron, either in the inertial or accelerated frame. The equivalence between the Minkowski and Rindler descriptions in this case has been established [61]. We expect that gravitational production from particles will be weak due to their small masses and instead consider production from high-energy pulsed lasers. The latter can be described classically, which motivates us to consider gravitational waves (GWs) instead of gravitons.

The first observation of GWs was made in 2015 by the LIGO and Virgo collaborations [62] that measured a signal generated by a black hole (BH) merger with a frequency from a few tenths up to a few hundred Hz. This opened a new path for observing the Universe and obtaining important information about its early stages. Theoretically, there is no restriction on the frequency of GWs. Detection of GWs in the low-frequency regime (which we define as smaller than 10 kHz) is within the capabilities of current detectors [63–66]. The latter cover the range from 10 Hz up to 10 kHz. Frequencies below 10 Hz (and reaching 0.1 mHz) are targeted by LISA [67]. The high frequency regime (defined as above 10 kHz) has not been explored but is important [68]. High-frequency GWs are associated with phenomena in the early Universe [69, 70]. Furthermore, GWs could be the only way to access information about these events since they decouple immediately after being created, maybe even before the emission of cosmic microwave background radiation, contrary to electromagnetic waves. One production mechanism is the evaporation of primordial BHs, which emit GWs in the form of Hawking radiation [71, 72]. The peak of the spectrum in this case is beyond the THz regime. Unfortunately, detecting GWs in the high-frequency regime is challenging due to the strong Big Bang nucleosynthesis (BBN) bound [73]. BBN accurately predicts abundances of deuterium and helium in the Universe without taking into account the energy density in the form of GWs. Therefore, the addition of any form of energy, including the one of GWs, should be constrained to avoid spoiling the predictions of BBN. The BBN constraint on the GW energy density is given by [70, 74]

$$\Omega_{\rm GW} = \frac{4\pi^2}{3H_0^2}\left(\frac{\omega_g}{2\pi}\right)^2 h_c^2 \lesssim 3\times 10^{-6}\,, \tag{1.38}$$

where H_0 is the Hubble rate of expansion today, $\omega_g/2\pi$ is the frequency of the GW and h_c is the characteristic strain, which is a dimensionless parameter associated to the amplitude of the wave. We then see that as the frequency increases, the strain should decrease appropriately. For frequencies beyond 1 THz, $h_c \lesssim 10^{-33}$, which is beyond the sensitivities of today's detectors.

In Chap. 6, we study the production and detection of GWs using high-energy lasers. For the first case, we find the polarization and amplitude of the emitted GWs in terms of the laser parameters. In the second part of the chapter, we study the conversion of an incoming GW into an electromagnetic signal through its interaction with a laser beam. Both processes are described by the Gertsenshtein effect [75](for a simple derivation of the latter, see Ref. [76]). The production and detection using this mechanism have been studied extensively in the literature [77–82]. Existing facilities whose purpose is not related to the detection of GWs can be used to impose exclusion bounds on GWs. In Ref. [83], bounds on GW amplitude were found using the ALPS I facility [84] or the proposed JURA facility [85]. In Ref. [86], it was proposed to use axion haloscopes as GW telescopes. In the last part of Chap. 6, we propose using existing lasers to find exclusion bounds for high-frequency GWs. In particular, since we are interested in GWs whose frequency is of the same order of magnitude as the laser frequency, we propose to use THz, optical, and X-ray lasers, which allow us to study the range $10^{13} - 10^{19}$ Hz roughly. As will be shown, GWs of cosmological origin are too weak to be detected, even with the most powerful lasers, because of the BBN bound. Thus, the sources of interest will be astrophysical sources, and in particular BH mergers.

The thesis is organized in the following way. Chap. 2 discusses the production of axions from an electron accelerated by a time-dependent external electromagnetic potential. In Chap. 3, we consider a general electromagnetic potential to accelerate electrons and propose an experimental setup to impose bounds on g_{ae}. In Chap. 4, we discuss the radiation of scalar particles from an accelerated electron in its rest frame using the Unruh effect and calculate the emission rate and the power recovering the Larmor formula. We generalize these results to the case of vector particles in Chap. 5. Finally, in Chap. 6, we study the emission and detection of GWs using high-energy lasers and discuss the potential sources that emit detectable GWs.

Throughout the thesis, natural units where $c = k_B = \epsilon_0 = \mu_0 = 1$ are used. $\hbar$ is written explicitly in Chap. 2 and the first parts of Chap. 3 since we are using a semi-classical expansion. It is set to 1 in the following chapters. We also use the metric signature $(+,-,-,-)$. Greek letters μ, ν, etc., are used to label spacetime indices, whereas Latin letters i, j, etc., run over space indices.

2 Pseudo-Scalar Production in a Time-Dependent Potential

In this chapter, we study the emission of pseudo-scalar particles from accelerated electrons following a rectilinear motion. We will refer to these particles as axions (and not QCD axions, as they do not necessarily solve the strong CP problem and we do not assume a relation between the couplings and the mass). The acceleration is achieved via a purely time-dependent classical potential, which generates an electric field. We discuss space-dependent potential at the end of this chapter and generalize to an arbitrary electromagnetic potential in the next chapter. We calculate the emission amplitude using a WKB approximation. Depending on the choice of interaction Lagrangian, either the zeroth or first order in the WKB expansion is needed, but it is verified that both lead to the same result. This can be seen by comparing the factors of $\hbar$ in the two interaction terms in EQS. (2.1) and (2.7) and recalling that the WKB approximation is an expansion in powers of $\hbar$. In this chapter, we rigorously derive the axion emission amplitudes for the simple case of one-dimensional trajectories. In CHAP. 3, we find, not fully rigorously, the axion emission amplitude for an arbitrary electromagnetic field. Thus, one of the goals of this chapter is to provide a verification of the general results found in CHAP. 3. This chapter is based on work done by Prof. A. Higuchi and the author.

We consider the coupling of a pseudo-scalar field given by

$$\mathcal{L}_{int}^{\mathrm{ps}} = -\frac{\hbar g_{ae}}{2m}(\partial_\mu a)\bar{\psi}\gamma_5\gamma^\mu\psi\,. \tag{2.1}$$

We use the letter a for the pseudo-scalar field to avoid confusion with the scalar field in the following chapters. g_{ae} is the coupling constant between the pseudo-scalar field and the electron. This is a dimension-5 operator and therefore defines an effective field theory. Higher-order corrections will contain higher-order derivatives. The interaction can be written in terms of a dimension-4 operator using the Dirac equation in the presence of an external potential EQ. (1.23). The first step is to integrate by parts the Lagrangian EQ. (2.1) and obtain

$$\mathcal{L}_{int}^{\mathrm{ps}} = \frac{\hbar g_{ae}}{2m}a\partial_\mu(\bar{\psi}\gamma_5\gamma^\mu\psi) - \partial_\mu\left(\frac{\hbar g_{ae}}{2m}a\bar{\psi}\gamma_5\gamma^\mu\psi\right). \tag{2.2}$$

The quantity of interest is the action which is found by integrating the Lagrangian density over spacetime. Assuming that the fields vanish at the boundaries, the total derivatives can be dropped. Then,

$$\begin{aligned}\mathcal{L}_{int}^{\text{ps}} &= \frac{\hbar g_{ae}}{2m} a(\partial_\mu(\bar{\psi})\gamma_5\gamma^\mu\psi + \bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi) + \text{total derivative} \\ &= \frac{\hbar g_{ae}}{2m} a(-\partial_\mu(\bar{\psi})\gamma^\mu\gamma_5\psi + \bar{\psi}\gamma_5\gamma^\mu\partial_\mu\psi) + \text{total derivative}\,,\end{aligned} \tag{2.3}$$

where we used $\gamma_5\gamma^\mu = -\gamma^\mu\gamma_5$. The Dirac equation with a non-zero potential V_μ is given by

$$\hbar\gamma^\mu\partial_\mu\psi = -i(V_\mu\gamma^\mu + m)\psi\,. \tag{2.4}$$

Taking the complex conjugate of this expression, we find

$$\begin{aligned}&\hbar\partial_\mu\psi^\dagger(\gamma^\mu)^\dagger = i\psi^\dagger\big(V_\mu(\gamma^\mu)^\dagger + m\big)\,,\\ &\hbar\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 = i\psi^\dagger(V_\mu\gamma^0\gamma^\mu\gamma^0 + m)\,,\\ &\hbar\partial_\mu\bar{\psi}\gamma^\mu = i\bar{\psi}(\gamma^\mu V_\mu + m)\,,\end{aligned} \tag{2.5}$$

where in the last step we multiplied by γ^0 on the right. Then, we can replace the derivative terms of the spinor field by ones involving the mass and the potential only. We find

$$\begin{aligned}\mathcal{L}_{int}^{\text{ps}} &= \frac{g_{ae}}{2m} a(-i\bar{\psi}(\gamma^\mu V_\mu + m)\gamma_5\psi + \bar{\psi}\gamma_5(-i)(V_\mu\gamma^\mu + m)\psi) + \text{total derivative} \\ &= \frac{g_{ae}}{2m} a(-2im\bar{\psi}\gamma_5\psi - iV_\mu\bar{\psi}\{\gamma^\mu, \gamma_5\}\,, \psi) + \text{total derivative}\,.\end{aligned} \tag{2.6}$$

But since $\{\gamma^\mu, \gamma_5\} = 0$, we have

$$\mathcal{L}^{\text{ps}}_{int} = -ig_{ae} a\bar{\psi}\gamma_5\psi\,, \tag{2.7}$$

where we drop the total derivative terms. In finding the equivalence between EQS. (2.1) and (2.7) we used the Dirac equation in the presence of the potential but without taking into account the interaction terms (this also removed the $\hbar$ factor). This is justified as we wish to treat perturbatively the electron-axion interaction and EQS. (2.1) and (2.7) are equivalent to leading order in g_{ae}. Removing the axion derivative coupling can also be done by field redefinition. There is a vast literature on field redefinitions in effective field theories; see, for example, Ref. [87]. For perturbative equivalence of pseudo-vector and pseudo-scalar couplings, see Ref. [88] and literature on sigma models for meson-nucleon interactions, such as Ref. [89]. We note that admissible field redefinitions leave the S-matrix invariant. In what follows, we will calculate the interaction probability using both Lagrangians and show that that they are equivalent up to a total derivative as it is expected. Starting first with EQ. (2.7), we notice that since there are no derivatives involved, the interaction Hamiltonian is simply $\mathcal{H}^{\text{ps}}_{int} = -\mathcal{L}^{\text{ps}}_{int}$. The free axion field can be expanded in terms of creation and annihilation operators exactly as a regular massive scalar field :

$$a(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3}\left[a(\mathbf{k})\mathrm{e}^{-ik\cdot x} + a^\dagger(\mathbf{k})\mathrm{e}^{ik\cdot x}\right], \tag{2.8}$$

where $k_0 = \sqrt{k^2 + (m_a/\hbar)^2}$ and the creation and annihilation operators satisfy $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\hbar k_0 \delta^{(3)}(\mathbf{k}-\mathbf{k}')$. The initial state of the electron is given by

$$|i\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{m}{p_0}} f(\mathbf{p}) b^\dagger_\alpha(\mathbf{p})\,|0\rangle \ , \tag{2.9}$$

where $p_0 = \sqrt{\mathbf{p}^2+m^2}$ is the energy of the electron and α is the initial polarization state. The function $f(\mathbf{p})$ is a distribution which we assume to be peaked around some momentum $\bar{\mathbf{p}}$. Thus, we let

$$|f(\mathbf{p})|^2 = (2\pi\hbar)^3 \delta^{(3)}(\mathbf{p}-\bar{\mathbf{p}})\,. \tag{2.10}$$

The final state is given to first order in perturbation theory by

$$|i\rangle \to |i\rangle + |1_{\text{axion}}\rangle = |i\rangle - \frac{i}{\hbar}\int \mathrm{d}^4x\, \mathcal{H}^{\text{ps}}_{int}(x)\,|i\rangle \ , \tag{2.11}$$

and the interaction probability is given by $P = \langle 1_{\text{axion}} | 1_{\text{axion}}\rangle$. Explicitly,

$$\begin{aligned} |1_{\text{axion}}\rangle =& \frac{g_{ae}}{\hbar}\int \mathrm{d}^4x\, \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{p}''}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3}\sqrt{\frac{m}{p_0}\frac{m}{p_0'}\frac{m}{p_0''}} f(\mathbf{p}) \\ & \times \sum_{\beta,\gamma} \bar{\Phi}_\beta(\mathbf{p}',x)\gamma_5 \Phi_\gamma(\mathbf{p},x) \mathrm{e}^{ik\cdot x} b^\dagger_\beta(\mathbf{p}') b_\gamma(\mathbf{p}'') b^\dagger_\alpha(\mathbf{p}) a^\dagger(\mathbf{k})\,|0\rangle \ . \end{aligned} \tag{2.12}$$

For the spinor field we did not take into account the contribution from the modes Ψ_α as we are not interested in positron emission. Using the anti-commutation relations for $b_\gamma(\mathbf{p}'')b^\dagger_\alpha(\mathbf{p})$ results in

$$b_\gamma(\mathbf{p}'')b^\dagger_\alpha(\mathbf{p})\,|0\rangle = \frac{p_0''}{m}(2\pi\hbar)^3 \delta_{\alpha\gamma}\delta^{(3)}(\mathbf{p}-\mathbf{p}'')\,|0\rangle \ , \tag{2.13}$$

and the final state is

$$\begin{aligned} |1_{\text{axion}}\rangle =& \frac{g_{ae}}{\hbar} \int \mathrm{d}^4 x \, \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} \frac{\mathrm{d}^3\mathbf{p}'}{(2\pi\hbar)^3} \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \sqrt{\frac{m}{p_0}\frac{m}{p_0'}} f(\mathbf{p}) \\ & \times \sum_\beta \bar{\Phi}_\beta(\mathbf{p}', x)\gamma_5 \Phi_\alpha(\mathbf{p}, x) \mathrm{e}^{ik\cdot x} b^\dagger_\beta(\mathbf{p}') a^\dagger(\mathbf{k}) |0\rangle \ . \end{aligned} \tag{2.14}$$

Axion emission can also be studied in terms of Feynman diagrams. The relevant diagrams are shown in Fig. 2.1 and the calculation of the amplitude is done in the supplementary material of Ref. [5].

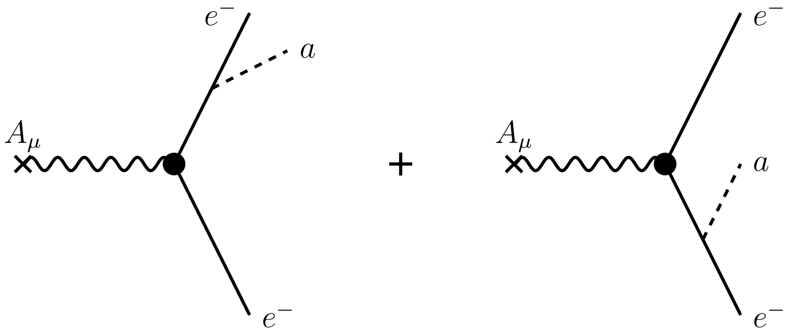

Fig. 2.1: The Feynman diagrams for the axion emission: A_μ is an external potential. The dot represents the interaction of the electron with the external potential.

2.1 Production with time-dependent potential

In the first part of this section, we calculate the axion emission probability. We start by considering a time-dependent potential, for which the modes are $\Phi_\alpha = \psi_\alpha(t, \mathbf{p})\mathrm{e}^{i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar}}$. Then,

$$\bar{\Phi}_\beta(\mathbf{p}', x)\gamma_5 \Phi_\alpha(\mathbf{p}, x)\mathrm{e}^{ik\cdot x} = \bar{\psi}_\beta(\mathbf{p}', t)\gamma_5 \psi_\alpha(\mathbf{p}, t)\mathrm{e}^{ik_0 t}\mathrm{e}^{i\mathbf{x}\cdot\left(\frac{\mathbf{p}}{\hbar} - \frac{\mathbf{p}'}{\hbar} - \mathbf{k}\right)} \ . \tag{2.15}$$

The space integral over the last factor will give $(2\pi\hbar)^3\delta^{(3)}(\mathbf{p}' - \mathbf{p} + \hbar\mathbf{k})$. This assures momentum conservation as it fixes the final momentum of the electron to the initial one minus the momentum of the emitted axion. The final state is then

$$\begin{aligned}|1_{\text{axion}}\rangle =&\frac{g_{ae}}{\hbar}\int \mathrm{d}t\,\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3}\sqrt{\frac{m}{p_0}\frac{m}{p_0'}}f(\mathbf{p})\mathrm{e}^{ik_0t}\\ &\times\sum_\beta\bar{\psi}_\beta(\mathbf{p}-\hbar\mathbf{k},t)\gamma_5\psi_\alpha(\mathbf{p},t)b^\dagger_\beta(\mathbf{p}-\hbar\mathbf{k})a^\dagger(\mathbf{k})\left|0\right\rangle\ ,\end{aligned}\tag{2.16}$$

where now $p_0' = \sqrt{(\mathbf{p}-\hbar\mathbf{k})^2+m^2}$. To find the probability, we compute the quantity $\langle 1_{\text{axion}}|1_{\text{axion}}\rangle$. The factor involving the creation and annihilation operators is given by

$$\begin{aligned}&\left\langle 0\right| b_{\beta'}(\tilde{\mathbf{p}}-\hbar\tilde{\mathbf{k}})b^\dagger_\beta(\mathbf{p}-\hbar\mathbf{k})a(\tilde{\mathbf{k}})a^\dagger(\mathbf{k})\left|0\right\rangle\\ &=2\hbar k_0(2\pi)^3\delta^{(3)}(\tilde{\mathbf{k}}-\mathbf{k})\times\frac{p_0'}{m}(2\pi\hbar)^3\delta_{\beta\beta'}\delta^{(3)}(\tilde{\mathbf{p}}-\mathbf{p})\,.\end{aligned}\tag{2.17}$$

The probability is then

$$\begin{aligned}P^{\text{axion}} =& \frac{g_{ae}^2}{\hbar}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3}\frac{m}{p_0}\frac{m}{p_0'}|f(\mathbf{p})|^2\\ &\times\sum_\beta\left|\int\mathrm{d}t\,\mathrm{e}^{ik_0t}\bar{\psi}_\beta(\mathbf{p}-\hbar\mathbf{k},t)\gamma_5\psi_\alpha(\mathbf{p},t)\right|^2\,.\end{aligned}\tag{2.18}$$

As it will be shown in what follows, the product $\bar{\psi}_\beta(\mathbf{p}-\hbar\mathbf{k},t)\gamma_5\psi_\alpha(\mathbf{p},t)$ will be of order $\hbar$. Then we can let $p_0' = p_0$ to leading order. Moreover we assume that $f(\mathbf{p})$ is peaked around some momentum $\bar{\mathbf{p}}$. Then, the probability is

$$P^{\text{axion}} = \frac{g_{ae}^2}{\hbar} \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \sum_\beta \left| \int \mathrm{d}t \, \mathrm{e}^{ik_0 t} \frac{m}{p_0} \bar{\psi}_\beta(\bar{\mathbf{p}} - \hbar\mathbf{k}, t) \gamma_5 \psi_\alpha(\mathbf{p}, t) \right|^2 . \quad (2.19)$$

Using the WKB approximation, the spinors are given by (see EQ. (1.28))

$$\psi_\alpha = \sqrt{\frac{p_0}{E}} u_\alpha \mathrm{e}^{-i\int \mathrm{d}\zeta \, E(\zeta)} , \quad (2.20)$$

where E is the time dependent energy given by $E = \sqrt{(\mathbf{p} - \mathbf{V}(t))^2 + m^2}$. We note that using exact or WKB solutions for the interacting Dirac equation in S-matrix calculations is referred to as the Furry picture [90]. For plane wave background fields, the WKB solutions become exact (Volkov solution of the Dirac equation becomes "super-integrable") [91]. The zeroth order in $\hbar$ vanishes as

$$\bar{u}_\beta^{(0)}(\mathbf{p}, t)\gamma_5 \bar{u}_\alpha^{(0)}(\mathbf{p}, t) = \frac{E + m}{2m} s_\beta^\dagger U^\dagger(t) \left[\Sigma_p - \Sigma_p^\dagger\right] U(t) s_\alpha = 0 , \quad (2.21)$$

where we used the definition EQ. (1.31) and the fact that the Pauli matrices are hermitian. To simplify the notation, we use $\mathbf{p} = \bar{\mathbf{p}} - \mathbf{V}(t)$ and $\mathbf{p}' = \mathbf{p} - \hbar\mathbf{k}$ from now on. At $\mathcal{O}(\hbar)$, we find

$$\bar{u}_\beta(\mathbf{p} - \hbar k, t)\gamma_5 u_\alpha(\mathbf{p}, t) = \frac{\hbar}{2m} U_p^\dagger(t) s_\beta^\dagger (\mathbf{q} \cdot \boldsymbol{\sigma}) U_p(t) s_\alpha + \mathcal{O}(\hbar^2) , \quad (2.22)$$

where

$$\mathbf{q} = \mathbf{k} - \frac{\mathbf{k}\cdot\mathbf{p}}{E_p(E_p+m)}\mathbf{p} - \frac{i}{E_p}\left(\dot{\mathbf{p}} - \frac{\dot{E}_p}{E_p+m}\mathbf{p}\right). \tag{2.23}$$

Here, dot means derivative with respect to t. This expression can be simplified if $\mathbf{p}$ is rectilinear (we will choose without loss of generality $\mathbf{p}$ to be parallel to the z-axis). Then, by definition, using Eq. (1.30), $U(t)$ is equal to the identity. The interaction probability is then given by

$$P^{\text{axion}} = \frac{\hbar g_{ae}^2}{4}\int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3}\sum_\beta \left|\int \frac{\mathrm{d}t}{E} \mathrm{e}^{ik_0t}\mathrm{e}^{\frac{i}{\hbar}\int \mathrm{d}\zeta\, E_{p'}(\zeta)}\mathrm{e}^{-\frac{i}{\hbar}\int \mathrm{d}\zeta\, E_p(\zeta)} s_\beta^\dagger \mathbf{q}\cdot\boldsymbol{\sigma} s_\alpha\right|^2. \tag{2.24}$$

The momentum $\mathbf{p}(t) = \bar{\mathbf{p}} - \mathbf{V}(t)$ and energy $E_p = \sqrt{\mathbf{p}^2+m^2}$ of the wave can be identified as the classical momentum and energy of a sharply peaked wave packet. The latter is sufficiently localized in space to follow a classical world line but at the same time sufficiently spread out to be peaked in momentum space. The latter condition is necessary to allow us to use Eq. (2.10). Then the momentum of the wave packet can be identified with the one of a classical particle. The exponent is therefore given by

$$\begin{aligned} &ik_0t - \frac{i}{\hbar}\int_0^t \mathrm{d}\zeta\,(E_p - E_{p-\hbar k}) \approx ik_0t - \frac{i}{\hbar}\int_0^t \mathrm{d}\zeta\, E_p\left(1 - \sqrt{1 - \frac{2\hbar\mathbf{k}\cdot\mathbf{p}}{E_p^2}}\right) \\ &\approx ik_0t - i\mathbf{k}\cdot\int \mathrm{d}\zeta\,\frac{\mathbf{p}}{E_p} = ik\cdot x(t)\,, \end{aligned} \tag{2.25}$$

where the velocity of the electron is found as $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{p}/E_p$ and we chose the initial condition $\mathbf{x}(0) = \mathbf{0}$. The proper time of the particle τ is found as $\mathrm{d}t/\mathrm{d}\tau = E_p/m$. The interaction probability is then

$$P^{\text{axion}} = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \sum_\beta \left| \int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x} s_\beta^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} s_\alpha \right|^2 . \tag{2.26}$$

We choose the one-dimensional trajectory of the electron to be along the z axis. Writing $p_z = p$ for simplicity, we find that $q_x = k_x$ and $q_y = k_y$. For the third component we use

$$\begin{aligned} k_z - \frac{k_z p^2}{E_p(E_p+m)} &= k_z \frac{m}{E_p} , \\ \dot{p} - \frac{\dot{E}_p p}{E_p + m} &= \dot{p}\left(1 - \frac{p^2}{E_p(E_p+m)}\right) = \dot{p}\frac{m}{E_p} . \end{aligned} \tag{2.27}$$

Then, we can write $\mathbf{q}$ as

$$\mathbf{q} = \left(k_x, k_y, \frac{m}{E_p}(k_z - i\dot{p}/E_p)\right)^{\mathrm{T}} . \tag{2.28}$$

We also choose the polarization vectors to be eigenvectors of σ^3 with $s_\alpha = (1,0)^{\mathrm{T}}$. The vector s_β can be equal to s_α or the second eigenvector of σ^3 with the eigenvalue -1. The two choices correspond to the cases where the electron emits an axion and does not flip its spin and to the case where it does flip its spin while emitting the axion, respectively. In terms of the S-matrix, the matrix elements are $\langle e', s', a|\, S\, |e, s\rangle$, where $s' = \pm s$. The total probability is the sum of the two cases. The time integral of the spin-flip case gives

$$\begin{aligned}(k_x + ik_y) \int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x} &= (k_x + ik_y) \int \frac{\mathrm{d}\tau}{ik\cdot v} \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{e}^{ik\cdot x} \\ &= (-ik_x + k_y) \int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x} \frac{k\cdot a}{(k\cdot v)^2}\,,\end{aligned} \tag{2.29}$$

where we integrated by parts in the second step and defined $v^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ and $a^\mu = \mathrm{d}^2 x^\mu/\mathrm{d}\tau^2$ the proper velocity and acceleration respectively. The necessity of the integration by parts will be discussed in the following chapters. The non-flip time-integral is

$$\int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x} \frac{m}{E_p}\left(k_z - i\frac{\dot{p}}{E_p}\right) = -im \int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x}\left[-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{k_z}{E(k\cdot v)}\right) + \frac{\dot{p}}{E^2}\right]. \tag{2.30}$$

where we integrated by parts only the term $\propto k_z$ as the second one already involves a derivative of the momentum. Before calculating the integrand, we first connect the derivative of the momentum to the proper acceleration. We calculate

$$-a_\mu a^\mu = \frac{1}{m^2}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}E}{\mathrm{d}\tau}\right)^2\right] = \frac{E^2}{m^4}(\dot{p}^2 - \dot{E}^2) = \frac{\dot{p}^2}{m^2}\,. \tag{2.31}$$

Therefore, in the one-dimensional case, the acceleration is related to the time-derivative of the momentum as $m^2|a|^2 = \dot{p}^2$, where $|a|^2 = -a_\mu a^\mu$. We also find

$$\begin{aligned}k\cdot v &= \frac{1}{m}(k_0 E - k_z p)\,,\\ k\cdot a &= \frac{\dot{p}}{m^2}(k_0 p - k_z E)\,.\end{aligned} \tag{2.32}$$

The integrand of EQ. (2.30) is thus $\dot{p}(k_0^2 - k_z^2)/(m^2(k\cdot v)^2)$. The two probabilities are then given by

$$
\begin{aligned}
P_{\text{flip}} &= \frac{\hbar g_{ae}^2}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} k_\perp^2 \left| \int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x}\frac{k\cdot a}{(k\cdot v)^2}\right|^2 , \\
P_{\text{non-flip}} &= \frac{\hbar g_{ae}^2}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \kappa^4 \left| \int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x}\frac{|a|}{(k\cdot v)^2}\right|^2 ,
\end{aligned}
\tag{2.33}
$$

where $k_\perp^2 = k_x^2 + k_y^2$ and $\kappa^2 = k_\perp^2 + m_a^2/\hbar^2$. We do not write the superscript "axion" from now on to simplify the notation. From these expressions, we can calculate the energy emitted for $m_a = 0$. Let us start with the flip case and multiply the integrand by a factor of $\hbar k$. This gives

$$
\begin{aligned}
E_{\text{flip}} = \frac{\hbar^2 g_{ae}^2}{128\pi^3 m^2}\int \mathrm{d}\Omega \mathrm{d}\tau \mathrm{d}\tau' \,(1-n_z^2)&\frac{n\cdot a(\tau)}{(n\cdot v(\tau))^2}\frac{n\cdot a(\tau')}{(n\cdot v(\tau'))^2} \\
&\times \int_{-\infty}^{+\infty} \mathrm{d}k\, k^2 \mathrm{e}^{ik\cdot(x(\tau)-x(\tau'))} .
\end{aligned}
\tag{2.34}
$$

where we used spherical coordinates, $n^\mu = k^\mu/k = (1,\mathbf{n})$ and extended the $k-$integral to $-\infty$ by multiplying by $1/2$. The last term can be calculated as

$$
\int_{-\infty}^{+\infty} \mathrm{d}k\, k^2 \mathrm{e}^{ik\cdot(x(\tau)-x(\tau'))} = \frac{2\pi}{n\cdot v(\tau') n\cdot v(\tau)} \frac{\mathrm{d}^2}{\mathrm{d}\tau \mathrm{d}\tau'}\frac{\delta(\tau-\tau')}{n\cdot v} . \tag{2.35}
$$

Now we integrate by parts and find

$$
E_{\text{flip}} = \frac{\hbar^2 g_{ae}^2}{64\pi^2 m^2}\int \mathrm{d}\Omega\,(1-n_z^2)\int \frac{\mathrm{d}\tau}{(n\cdot v)^7}\left[n\cdot\dot{a} - \frac{3(n\cdot a)^2}{n\cdot v}\right]^2 , \tag{2.36}
$$

where, from now on, dot means derivative with respect to τ. The same procedure is followed for the case of the non-flip energy. Then using EQ. (2.35), we find

$$E_{\text{non-flip}} = \frac{\hbar^2 g_{ae}^2}{64\pi^2 m^2} \int \mathrm{d}\Omega\,(1-n_z^2)^2 \int \frac{\mathrm{d}\tau}{(n\cdot v)^7}\left[\frac{\mathrm{d}|a|}{\mathrm{d}\tau} - \frac{3|a|(n\cdot a)}{n\cdot v}\right]^2. \quad (2.37)$$

The results derived for the emission probabilities and energies will be used for different electron trajectories. The first one which we will study is the simplest case of uniform acceleration. We note that relations between classical and quantum radiation formulae can be found in Ref. [92].

2.1.1 Uniformly accelerated electrons

In this section, we consider uniformly accelerated electrons that emit axions through Larmor-type radiation. All quantities will be calculated in the inertial frame. For uniform acceleration, we have $t = a^{-1}\sinh a\tau$ and $z = a^{-1}\cosh a\tau$, while $x(\tau) = y(\tau) = 0$. $a > 0$ is a constant. Therefore, $-a_\mu a^\mu = a^2$ and $\mathrm{d}|a|/\mathrm{d}\tau = 0$. We define the rapidity ϑ as $k_0 = \kappa\cosh a\vartheta$ and $k^z = \kappa\sinh a\vartheta$. The flip probability for fixed transverse momentum $k_\perp$ is given by

$$\begin{aligned} P_{\text{flip}}^{\text{uni-acc}}(k_\perp) &= \frac{g^2\hbar a^3}{64\pi^3 m^2}\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\,\frac{k_\perp^2}{\kappa^2}\left|\int_{-\infty}^{+\infty}\mathrm{d}\tau\,\frac{\sinh a(\vartheta-\tau)}{\cosh^2 a(\vartheta-\tau)}\mathrm{e}^{-i\frac{\kappa}{a}\sinh a(\vartheta-\tau)}\right|^2 \\ &= \frac{\hbar g_{ae}^2 a^3 k_\perp^2}{64\pi^3 m^2\kappa^2}\int_{-\infty}^{+\infty}\mathrm{d}\vartheta\mathrm{d}\tau\mathrm{d}\sigma\,\frac{\mathrm{e}^{2i\frac{\kappa}{a}\cosh a(\vartheta-\tau)\sinh a\sigma/2}}{(\cosh^2 a(\vartheta-\tau)+\sinh^2 a\sigma/2)^2} \\ &\qquad \times\left(\cosh^2 a(\vartheta-\tau) - \cosh^2 a\sigma/2\right), \end{aligned} \quad (2.38)$$

where $\sigma = \tau' - \tau''$ and $\tau = (\tau'+\tau'')/2$ and τ', τ'' are the proper times after expanding the integrand. We notice that the integrand becomes $\tau-$independent after the change of variables $\vartheta \to \bar{\vartheta} = \vartheta - \tau$. This corresponds to using the rapidity in the rest frame of the electron as an integration variable. Then, the rate is found by dividing the resulting probability by the total (infinite) acceleration time $T = \int_{-\infty}^{+\infty} \mathrm{d}\tau$. Then, we can define $s_\pm = \bar{\vartheta} \pm \sigma/2$ and find

$$\begin{aligned} R^{\text{uni-acc}}_{\text{flip}}(k_\perp) &= \frac{g^2\hbar a^3}{64\pi^3 m^2}\frac{k_\perp^2}{\kappa^2}\left|\int_{-\infty}^{+\infty} \mathrm{d}s\, \frac{\sinh as\, \mathrm{e}^{i\frac{\kappa}{a}\sinh as}}{\cosh^2 as}\right|^2 \\ &= \frac{\hbar g_{ae}^2}{16\pi^3 m^2 a} k_\perp^2 \left|K_0\left(\frac{\kappa}{a}\right)\right|^2, \end{aligned} \tag{2.39}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind. The same procedure can be followed for the non-flip case and the rate is

$$R^{\text{uni-acc}}_{\text{non-flip}}(k_\perp) = \frac{\hbar g_{ae}^2}{16\pi^3 m^2 a}\kappa^2\left|K_1\left(\frac{\kappa}{a}\right)\right|^2. \tag{2.40}$$

We note that in both case, the total rate, found by integrating the above expressions over $\mathbf{k}_\perp$ is finite. This is also true when $m_a = 0$. This is because although the Bessel function $K_1(x)$ diverges for small x, when multiplied by x^3, the overall result is finite. We find explicitly for $m_a = 0$,

$$R^{\text{uni-acc}}_{\text{tot}} = \frac{\hbar g_{ae}^2 a^3}{8\pi^2 m^2}\int_0^{+\infty} \mathrm{d}x\, x^3\big(|K_0(x)|^2 + |K_1(x)|^2\big) = \frac{\hbar g_{ae}^2 a^3}{8\pi^2 m^2}, \quad m_a = 0\,, \tag{2.41}$$

where $x = k_\perp/a$. To find the power emitted, we use the results for the energies. For the flip case, we have that $n \cdot v = \cosh a\tau - n_z \sinh a\tau$, $n \cdot a =$

$a(\sinh a\tau - n_z \cosh a\tau)$, $n \cdot \dot{a} = a^2(n \cdot v)$. Thus,

$$E_{\text{flip}}^{\text{uni-acc}} = \frac{\hbar^2 g_{ae}^2 a^4}{42\pi m^2} \int \mathrm{d}\tau \cosh a\tau \,, \quad E_{\text{non-flip}}^{\text{uni-acc}} = \frac{3\hbar^2 g_{ae}^2 a^4}{70\pi m^2} \int \mathrm{d}\tau \cosh a\tau \,. \tag{2.42}$$

The factor $\cosh a\tau$ is the Lorentz factor γ. We note that $\mathrm{d}\tau \cosh a\tau = \mathrm{d}t$. The total power emitted is then

$$S_{\text{tot}}^{\text{uni-acc}} = \frac{\hbar^2 g_{ae}^2 a^4}{15\pi m^2} \,. \tag{2.43}$$

This is a Larmor-type formula for axion emission with suppression factor $(\hbar a/m)^2$. Although it seems that the stronger dependence on proper acceleration (a^4 factor instead of a^2 for scalars and photons) would allow one to increase significantly the power detected in an experiment, even the highest accelerations that can be achieved today with laser beams are much smaller than the electron mass. Moreover, the presence of $\hbar$ in the emitted power is due to the fact that the leading order in the WKB expansion vanished exactly, making the emission of pseudo-scalar particles from a charge a quantum process. For scalars and photons, the power emitted can be found using either a classical or quantum treatment. The scaling a^4/m^2 appears similar to the correction to Larmor for photons calculated in the context of the Unruh effect [93]. The Larmor formula for photons can be written in terms of the Thomson scattering cross section as $S = \sigma_{\text{Th}} \times E^2$, where $\sigma_{\text{Th}} = (8\pi/3)\alpha^2/m^2$ and E is the electric field related to the acceleration by $a = eE/m$. We note that σ_{Th} is akin to the prefactor g_{ae}^2/m^2. It is known [94] that the cross section in an intense laser background is of the form $\sigma_{\rm NLO} = \sigma_{\rm Th} f(a_0) = \sigma_{\rm Th}(1+\mathcal{O}(a_0^2))$, where $a_0 = eE/m\omega$, with ω the laser frequency.

Uniform acceleration is not the only electron motion that can be achieved in the laboratory. We will now look at the case where electrons are accelerated by two counter-propagating laser beams.

2.1.2 Oscillating motion in laser fields

In this section, we consider the experimental proposal described in Ref. [95]. Electrons are accelerated by counter-propagating laser beams. The latter are assumed to be plane waves with an angular frequency ω_0. The total electromagnetic fields are given by

$$\begin{aligned} E_z &= E_0[\cos\omega_0(t-x) + \cos\omega_0(t+x)]\,, \\ B_y &= -E_0[\cos\omega_0(t-x) - \cos\omega_0(t+x)]\,. \end{aligned} \tag{2.44}$$

Here, we assume that the beams are propagating in the x direction and that the electric and magnetic fields are along z and y, respectively. We note that the electromagnetic potential in this case is not only time but also space dependent and therefore the results derived previously do not apply directly. However, for electrons placed at the nodes $\omega_0 x = 2n\pi, n \in \mathbb{N}$, the magnetic field vanishes and the electric field becomes $E_z = 2E_0\cos\omega_0 t$. The classical equation of motion for an electron placed at $x = 0$ is

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = -eE_z\,, \tag{2.45}$$

and all other components of the momentum vanish. The solution to this equation is given by

$$\gamma\beta_z = -2a_0 \sin\omega_0 t\,, \quad \gamma = \sqrt{1 + 4a_0^2 \sin^2\omega_0 t}\,, \tag{2.46}$$

where $a_0 = eE_0/m\omega_0$ is the laser strength parameter. Today's lasers can achieve $a_0 \gg 1$, which implies that when $\sin\omega_0 t \sim 1$, the electron becomes relativistic. There is a period during each oscillation where the electron is uniformly accelerating. For $a_0 \gg 1$ and $4a_0^2 \gg \cosh^2 2a_0\omega_0 t$ [95]

$$\sin\omega_0 t = \frac{1}{2a_0}\sinh 2a_0\omega_0\tau\,, \quad \cos\omega_0 z = \frac{1}{2a_0}\cosh 2a_0\omega_0\tau\,. \tag{2.47}$$

Within this approximation, the motion corresponds to uniform acceleration with proper acceleration $a = 2a_0\omega_0$ (we note that for $\omega_0 z$ close to $-\pi/2$, we have $\omega_0 z \approx -\pi/2 + \cosh(2a_0\omega_0\tau)/2a_0$). The additional additive constant $-\pi/2$ does not change the results derived in the previous section because the space dependence of the emission probabilities is in the factor $\mathrm{e}^{ik\cdot x}$. Then, the additional phase does not change the result because of the modulus square of the time integrals.

Calculation of the energy

Although the authors in Ref. [95] were interested in uniform acceleration, here we consider the complete oscillating motion of the electron. We will first calculate the energy emitted in one oscillation of period $2\pi/\omega_0$. Starting with the flip case, we use Eq. (2.36). To ease notation we will use $\beta \equiv \beta_z$, since the velocity is one-dimensional. Then,

$$\begin{aligned} n\cdot v &= \frac{1-n_z\beta}{\sqrt{1-\beta^2}}\,, \quad n\cdot a = \frac{(\beta-n_z)\beta'}{(1-\beta^2)^2}\,, \\ n\cdot\dot{a} &= \frac{1+3\beta^2-4n_z\beta}{(1-\beta^2)^{7/2}}\beta'^2 + \frac{\beta-n_z}{(1-\beta^2)^{5/2}}\beta''\,, \end{aligned} \tag{2.48}$$

where prime means derivative with respect to t. The integrand of Eq. (2.36) is then calculated (changing the integration variable to inertial time)

$$\begin{aligned} &\int \frac{\mathrm{d}\tau}{(n\cdot v)^7}\left[n\cdot\dot{a} - \frac{3(n\cdot a)^2}{n\cdot v}\right]^2 \\ &= \int \frac{\mathrm{d}t}{(1-\beta^2)(1-n_z\beta)^7}\left[(\beta-n_z)\beta'' + \frac{1+(1-3\beta^2)\beta n_z+(4\beta^2-3)n_z^2}{(1-\beta^2)(1-n_z\beta)}\beta'^2\right]^2 . \end{aligned} \tag{2.49}$$

In order to calculate the total energy, it is more convenient to calculate the angular integrals first. Since the integrand does not depend on the angle ϕ, we simply have to calculate the integral over $n_z=\cos\theta$. After squaring the expression above, we obtain for the terms β''^2 and β'^4 respectively

$$\begin{aligned} &\int_{-1}^{1}\mathrm{d}n_z\,\frac{(1-n_z^2)(\beta-n_z)^2}{(1-\beta^2)(1-n_z\beta)^7} = \frac{4}{15(1-\beta^2)^4}\,, \\ &\int_{-1}^{1}\mathrm{d}n_z\,\frac{(1-n_z^2)(1+(1-3\beta^2)\beta n_z+(4\beta^2-3)n_z^2)^2}{(1-\beta^2)^3(1-n_z\beta)^9} = \frac{4(15\beta^2+4)}{21(1-\beta^2)^6}\,. \end{aligned} \tag{2.50}$$

For the cross term involving $\beta''\beta'^2$, we find

$$\begin{aligned} &\int_{-1}^{1}\mathrm{d}n_z\,\frac{2(1-n_z^2)(\beta-n_z)((1+(1-3\beta^2)\beta n_z+(4\beta^2-3)n_z^2))}{(1-\beta^2)^2(1-n_z\beta)^8} \\ &= \frac{184\beta}{105(1-\beta^2)^5}\,. \end{aligned} \tag{2.51}$$

Then, the flip energy is

$$E_{\text{flip}} = \frac{\hbar^2 g_{ae}^2}{24\pi m^2} \int \frac{\mathrm{d}t}{(1-\beta^2)^4} \left[\frac{\beta''^2}{5} + \frac{46\beta''\beta'^2\beta}{35(1-\beta^2)} + \frac{(15\beta^2+4)\beta'^4}{7(1-\beta^2)^2} \right] . \tag{2.52}$$

The non-flip energy is found in a similar manner. We first note that

$$\begin{aligned} |a| &= \frac{\beta'}{(1-\beta^2)^{3/2}} \,, \\ \frac{\mathrm{d}|a|}{\mathrm{d}\tau} &= \frac{\beta''}{(1-\beta^2)^2} + \frac{3\beta\beta'^2}{(1-\beta^2)^3} \,, \end{aligned} \tag{2.53}$$

which implies

$$E_{\text{non-flip}} = \frac{\hbar^2 g_{ae}^2}{10\pi m^2} \int \frac{\mathrm{d}t}{(1-\beta^2)^4} \left[\frac{\beta''^2}{3} + \frac{16\beta''\beta'^2\beta}{7(1-\beta^2)} + \frac{3(9\beta^2+1)\beta'^4}{7(1-\beta^2)^2} \right] . \tag{2.54}$$

The expressions for the energy expressed in this way hold for any one-dimensional trajectory. We will apply them for the oscillatory trajectory described in Eq. (2.46). We define

$$w(t) = -2a_0 \sin \omega_0 t \,, \tag{2.55}$$

such that

$$\beta = \frac{w}{\sqrt{1+w^2}} \,, \quad \beta' = \frac{w'}{(1+w^2)^{3/2}} \,, \quad \beta'' = \frac{w''}{(1+w^2)^{3/2}} - \frac{3ww'^2}{(1+w^2)^{5/2}} \,. \tag{2.56}$$

Then, in terms of w, the energies are

$$\begin{aligned} E_{\text{flip}} &= \frac{\hbar^2 g_{ae}^2}{24\pi m^2}\int_0^{2\pi/\omega_0} \mathrm{d}t \left[\frac{1}{5}(1+w^2)w''^2 + \frac{4}{35}ww'^2w'' + \frac{4}{7}w'^4\right], \\ E_{\text{non-flip}} &= \frac{\hbar^2 g_{ae}^2}{10\pi m^2}\int_0^{2\pi/\omega_0} \mathrm{d}t \left[\frac{1}{5}(1+w^2)w''^2 + \frac{2}{7}ww'^2w'' + \frac{3}{7}w'^4\right], \end{aligned} \tag{2.57}$$

where we integrated the emitted power over one period of oscillation. The time-interals are now straightforward to compute. We find

$$E_{\text{flip}} = \frac{\hbar^2 g_{ae}^2}{30m^2}\omega_0^3\left(a_0^2 + 11a_0^4\right), \quad E_{\text{non-flip}} = \frac{2\hbar^2 g_{ae}^2}{15m^2}\omega_0^3\left(a_0^2 + 6a_0^4\right), \tag{2.58}$$

and the total energy is simply the sum of the two

$$E_{\text{tot}}^{\text{1d}} = \frac{\hbar^2 g_{ae}^2}{6m^2}\omega_0^3\left(a_0^2 + 7a_0^4\right). \tag{2.59}$$

The average power is found by multiplying this expression by $\omega_0/2\pi$. The proper acceleration is given by $|a| = 2a_0|\cos\omega_0 t|$. Then, for $a_0 \gg 1$, the average power scales as $\sim \hbar^2 g_{ae}^2 |a|^4/m^2$, which is the same scaling as for uniform acceleration (see Eq. (2.43)).

Axion spectrum

Although we calculated the total energy emitted, we have yet to find either the typical axion energy (or momentum since we assume that $m_a = 0$ in these calculations) or the axion emission rate. We expect the typical axion energy in the rest frame of the electron to be of the order of the proper acceleration, which scales as $\sim 2a_0$. Since the Lorentz factor is $\gamma \sim 2a_0$, we expect the energy in the laboratory frame to be $\sim 4a_0^2\omega_0$. Since we are considering a periodic motion, we will be able to calculate the emission rate using Fourier series. We start by expressing the flip probability in spherical coordinates

$$\frac{\mathrm{d}P_{\text{flip}}}{\mathrm{d}\Omega} = \frac{\hbar g_{ae}^2}{64\pi^3 m^2}\int_0^{+\infty} \mathrm{d}k\, k(1-n_z^2)|I_{\text{flip}}(k)|^2\,, \tag{2.60}$$

where we defined

$$I_{\text{flip}}(k) = \int \mathrm{d}\tau\, \frac{n\cdot a}{(n\cdot v)^2}\mathrm{e}^{ik\cdot x} = -\int \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{n\cdot v}\right)\mathrm{e}^{ikt}\mathrm{e}^{-ikn_z z(t)}\,, \tag{2.61}$$

where $z(t)$ is the position of the electron. Here, and in what follows, we denote the frequency by $k_0 \equiv k$. The function multiplying e^{ikt} is periodic with period $2\pi/\omega_0$. The integral is then proportional to $\delta(k-n\omega_0)$, $n\in\mathbb{N}$. Then, assuming that the time interval of integration is a multiple of $2\pi/\omega_0$, the boundary term will vanish after integration by parts. Thus,

$$I_{\text{flip}}(k) = ik\int \mathrm{d}t\, \frac{\mathrm{d}\tau}{\mathrm{d}t}\mathrm{e}^{ik\cdot x} = ik\int \mathrm{d}\xi\, \frac{\mathrm{d}\tau}{\mathrm{d}\xi}\mathrm{e}^{ik\xi}\,, \tag{2.62}$$

where $\xi = t - n_z z(t)$. For radiation integrals of this type, their infrared divergences and Coulomb tails, see also Ref. [96]. The function $\mathrm{d}\tau/\mathrm{d}\xi$ is periodic in ξ with period $2\pi/\omega_0$ as well. Thus, we write

$$\frac{\mathrm{d}\tau}{\mathrm{d}\xi} = \sum_{n=-\infty}^{+\infty} a_n \mathrm{e}^{in\omega_0\xi}\,, \tag{2.63}$$

where the Fourier coefficients are given by

$$a_n = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \mathrm{d}\xi\, \frac{\mathrm{d}\tau}{\mathrm{d}\xi} \mathrm{e}^{-in\omega_0\xi} = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \mathrm{d}t\, \frac{\mathrm{e}^{-in\omega_0(t-n_z z(t))}}{\sqrt{1+4a_0^2\sin^2\omega_0 t}}\,. \tag{2.64}$$

Going back to the time integral, we can now express the integrand using the Fourier expansion

$$I_{\text{flip}}(k) = ik\sum_{n=-\infty}^{+\infty} a_n \int \mathrm{d}\xi\, \mathrm{e}^{i\xi(k+n\omega_0)} = 2\pi ik \sum_{n=1}^{+\infty} a_{-n}\delta(k-n\omega_0)\,, \tag{2.65}$$

where we used $k > 0$. To calculate the probability, we must square $I_{\text{flip}}(k)$. The square of the delta function in the expression above gives a factor of $\delta(0)$. The divergence is expected as we consider an infinite time of interaction. However, we are interested in the rate, not the probability itself. Therefore, we divide the resulting expression by the total time of interaction $T = 2\pi\delta(0)$. Then,

$$\frac{\mathrm{d}R_{\text{flip}}}{\mathrm{d}\Omega} = \frac{\hbar g_{ae}^2\omega_0^3}{32\pi^2 m^2}(1-n_z^2)\sum_{n=1}^{+\infty} n^3|a_n|^2\,, \tag{2.66}$$

where we used $|a_n| = |a_{-n}|$. The position of the electron is

$$z(t) = -\frac{1}{\omega_0}\arccos\left(\frac{2a_0}{\sqrt{1+4a_0^2}}\cos\omega_0 t\right) + \frac{1}{\omega_0}\arccos\left(\frac{2a_0}{\sqrt{1+4a_0^2}}\right), \tag{2.67}$$

where the constant was chosen such that $\xi(0) = 0$ and $\xi(2\pi/\omega_0) = 2\pi/\omega_0$. In this way, the integral bounds in EQ. (2.64) are consistent. EQ. (2.66), apart from giving the total rate, can also be used to find the spectrum for discrete energies. As said previously, we expect a peak in the spectrum around $4a_0^2\omega_0$.

Now examining the non-flip rate we have

$$\frac{\mathrm{d}P_{\text{non-flip}}}{\mathrm{d}\Omega} = \frac{\hbar g_{ae}^2}{64\pi^3 m^2} \int_0^{+\infty} \mathrm{d}k\, k(1-n_z^2)^2 |I_{\text{non-flip}}(k)|^2 \,, \tag{2.68}$$

where

$$I_{\text{non-flip}}(k) = \int \mathrm{d}\tau\, \frac{|a|}{(n\cdot v)^2} \mathrm{e}^{ik\cdot x} = \int \mathrm{d}\xi\, \frac{\mathrm{d}\tau}{\mathrm{d}\xi} \frac{|a|}{(n\cdot v)^2} \mathrm{e}^{ik\xi} \,, \tag{2.69}$$

where again $\xi = t - n_z z(t)$. The function $\frac{\mathrm{d}\tau}{\mathrm{d}\xi}\frac{|a|}{(n\cdot v)^2}$ is periodic in t with period $2\pi/\omega_0$. Following the same procedure as for the flip-probability, the function is also periodic in ξ with the same period and therefore $I_{\text{non-flip}}$ is peaked at $k = n\omega_0$. We write

$$\frac{\mathrm{d}\tau}{\mathrm{d}\xi} \frac{|a|}{(n\cdot v)^2} = \omega_0 \sum_{n=-\infty}^{+\infty} n b_n \mathrm{e}^{in\omega_0\xi} \,, \tag{2.70}$$

where we multiplied by a factor of ω_0 to make the coefficients b_n dimensionless. The Fourier coefficients are

$$b_n = \frac{\omega_0}{2\pi n} \int_0^{2\pi/\omega_0} \mathrm{d}\xi\, \frac{\mathrm{d}\tau}{\mathrm{d}\xi} \frac{|a|}{(n\cdot v)^2} \mathrm{e}^{-in\omega_0\xi} \frac{1}{\omega_0} \,. \tag{2.71}$$

Using t as the integration variable, we can write b_n as

$$\begin{aligned} b_n &= \frac{1}{2\pi n} \int_0^{2\pi/\omega_0} \mathrm{d}t\, \frac{w'}{\sqrt{1+w^2}(\sqrt{1+w^2} - n_z w)^2} \mathrm{e}^{-in\omega_0(t-n_z z(t))} \\ &= \frac{1}{2\pi n} \int_0^{2\pi/\omega_0} \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{w}{\sqrt{1+w^2} - n_z w} \right) \mathrm{e}^{-in\omega_0(t-n_z z(t))} \\ &= \frac{i\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \mathrm{d}t\, \frac{w}{\sqrt{1+w^2}} \mathrm{e}^{-in\omega_0(t-n_z z(t))} \,, \end{aligned} \tag{2.72}$$

where w was defined in Eq. (2.55), and the boundary term vanishes since $w(2\pi/\omega_0) = w(0) = 0$. The time-integral is then given by

$$I_{\text{non-flip}}(k) = \omega_0 \sum_{n=-\infty}^{+\infty} n b_n \int \mathrm{d}\xi\, \mathrm{e}^{i\xi(k+n\omega_0)} = -2\pi\omega_0 \sum_{n=1}^{+\infty} n b_{-n} \delta(k - n\omega_0)\,, \tag{2.73}$$

where again $k > 0$. Dividing the probability by $2\pi\delta(0)$, we find that the rate is

$$\frac{\mathrm{d}R_{\text{non- flip}}}{\mathrm{d}\Omega} = \frac{\hbar g_{ae}^2 \omega_0^3}{32\pi^2 m^2}(1-n_z^2)^2 \sum_{n=1}^{+\infty} n^3 |b_n|^2\,. \tag{2.74}$$

From the emission rates expressions, the power is found by multiplying the integrand by a factor of $\hbar k$. Here the energy is discrete and given by $k = n\omega_0$. Then,

$$\begin{aligned}
\frac{\mathrm{d}S_{\text{flip}}}{\mathrm{d}\Omega} &= \frac{\hbar^2 g_{ae}^2 \omega_0^4}{32\pi^2 m^2}(1-n_z^2) \sum_{n=1}^{+\infty} n^4 |a_n|^2\,,\\
\frac{\mathrm{d}S_{\text{non-flip}}}{\mathrm{d}\Omega} &= \frac{\hbar^2 g_{ae}^2 \omega_0^4}{32\pi^2 m^2}(1-n_z^2)^2 \sum_{n=1}^{+\infty} n^4 |b_n|^2\,.
\end{aligned} \tag{2.75}$$

We verify that the energy obtained by multiplying the above expression by $2\pi/\omega_0$ which is the duration of a cycle is the same as Eq. (2.58). We use Parseval's theorem which states that if f is a periodic function of period 2π and if

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, f(x) \mathrm{e}^{inx}\,, \tag{2.76}$$

Then,

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}x \, |f(x)|^2 \,. \tag{2.77}$$

Since inside the sums we are interested in, the Fourier coefficients are multiplied by n^4, we first observe that

$$n^2 c_n = -\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}x \, f''(x) \mathrm{e}^{inx} \,, \tag{2.78}$$

which then implies

$$\sum_{n=1}^{+\infty} n^4 |c_n|^2 = \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}x \, |f''(x)|^2 \,, \tag{2.79}$$

if $|c_n| = |c_{-n}|$, which is the case for the Fourier coefficients a_n and b_n that we calculated. Starting with the flip case,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}u}{\sqrt{1+w^2} - n_z w} \mathrm{e}^{inu} \,, \quad \text{up to a phase,} \tag{2.80}$$

where $u = \omega_0 (t - n_z z(t))$. Then,

$$\begin{aligned}
&\int_{-1}^{1} \mathrm{d}n_z \, (1 - n_z^2) \sum_{n=1}^{+\infty} n^4 |a_n|^2 \\
&= \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}x \int_{-1}^{1} \mathrm{d}n_z \, (1 - n_z^2) \left| \frac{\mathrm{d}^2}{\mathrm{d}u^2} \frac{1}{\sqrt{1+w^2} - n_z w} \right|^2 \\
&= \frac{\omega_0}{105\pi} \int_0^{2\pi/\omega_0} \mathrm{d}t \left(20 w'^4 + 4 w w'^2 w'' + 7(1+w^2) w''^2 \right) \\
&= \frac{4}{15} a_0^2 (1 + 11 a_0^2) \,,
\end{aligned} \tag{2.81}$$

where in the third line we changed the integration variable to t and integrated over n_z. Replacing this result in Eq. (2.75) and multiplying by $2\pi/\omega_0$ gives the first expression in Eq. (2.58) as expected. The same can be done for the non-flip case. There, the Fourier coefficients are given by

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}u\, \frac{w}{\sqrt{1+w^2} - n_z w} \mathrm{e}^{inu}\,, \quad \text{up to a phase.} \tag{2.82}$$

Then,

$$\begin{aligned}
&\int_{-1}^{1} \mathrm{d}n_z\,(1-n_z^2)^2 \sum_{n=1}^{+\infty} n^4 |b_n|^2 \\
&= \frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}x \int_{-1}^{1} \mathrm{d}n_z\,(1-n_z^2)^2 \left|\frac{\mathrm{d}^2}{\mathrm{d}u^2}\frac{w}{\sqrt{1+w^2}-n_z w}\right|^2 \\
&= \frac{4\omega_0}{105\pi}\int_0^{2\pi/\omega_0} \mathrm{d}t\,\left(9w'^4 + 6ww'^2w'' + 7(1+w^2)w''^2\right) \\
&= \frac{16}{15}a_0^2(1+6a_0^2)\,.
\end{aligned} \tag{2.83}$$

Inserting this expression in Eq. (2.75) and multiplying by $2\pi/\omega_0$ gives the second expression of Eq. (2.58) as expected. Thus, we verified that the expansion of the emission rate and power into Fourier series is consistent with the expression for the energy derived previously.

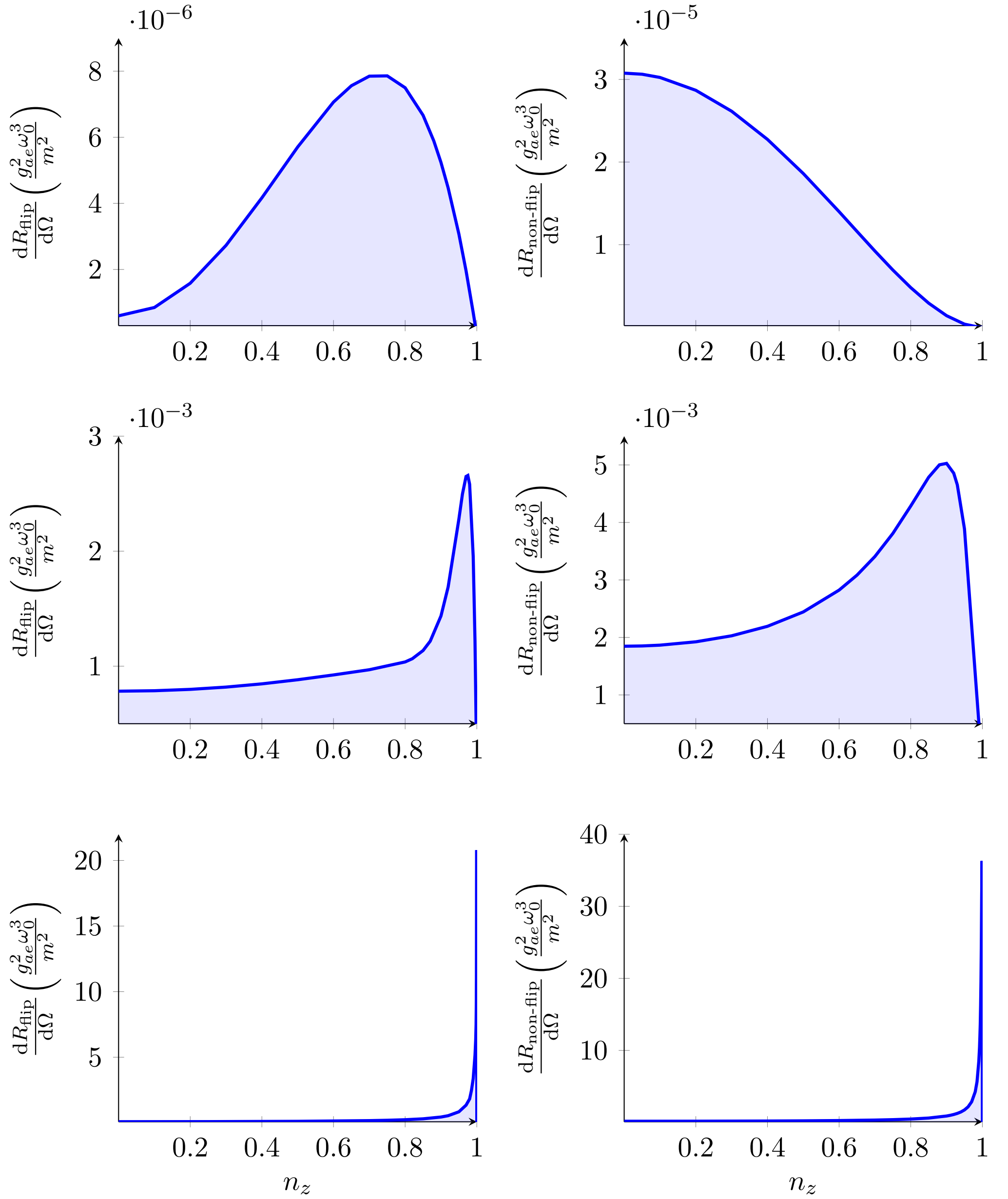

Fig. 2.2: Angular distribution of the flip and non-flip rates for one-dimensional electron-trajectory for $a_0 = 0.1$ (top), $a_0 = 1$ (middle) and $a_0 = 10$ (bottom).

We now look at the angular distribution of the emission rate. Firstly, we note that since both emission rates do not depend on the angle ϕ, we only need to consider the dependence on $n_z = \cos\theta$. Moreover, both differential rates are symmetric under exchange $n_z \to -n_z$. To show this, it is sufficient to show that it is the case for the Fourier coefficients a_n and b_n. Consider a_n as a function of n_z. Then,

$$\begin{aligned}
|a_n(-n_z)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} \mathrm{d}y\, \frac{\exp\left\{-in\left(y - n_z \arccos\left(\frac{2a_0}{\sqrt{1+4a_0^2}}\cos y\right)\right)\right\}}{\sqrt{1+4a_0^2\sin^2 y}}\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \mathrm{d}y\, \frac{\exp\left\{-in\left(y + n_z \arccos\left(\frac{2a_0}{\sqrt{1+4a_0^2}}\cos y\right)\right)\right\}}{\sqrt{1+4a_0^2\sin^2 y}}\right| \\
&= \frac{1}{2\pi}\left|\int_0^{2\pi} \mathrm{d}y\, \frac{\exp\left\{-in\left(y + n_z \arccos\left(\frac{2a_0}{\sqrt{1+4a_0^2}}\cos y\right)\right)\right\}}{\sqrt{1+4a_0^2\sin^2 y}}\right| \\
&= |a_n(n_z)|\,,
\end{aligned} \tag{2.84}$$

where in the second line we shifted $y \to y - \pi$ and in the third line we took the complex conjugate and shifted back $y \to y + \pi$. The same is true for b_n. Then, we only need to consider the interval $0 \leq n_z \leq 1$. In the next chapter, we consider a circular motion and the Fourier coefficients will be written in terms of Bessel functions as expected for synchrotron radiation [97]. In Fig. 2.2, we show the rate distribution for the flip and non-flip cases. When a_0 increases, the emission becomes centered around $n_z = 1$ or $\theta = 0$. This is expected as when $a_0 \gg 1$, the electron is relativistic (see Eq. (2.46)) and axions are emitted along its trajectory.

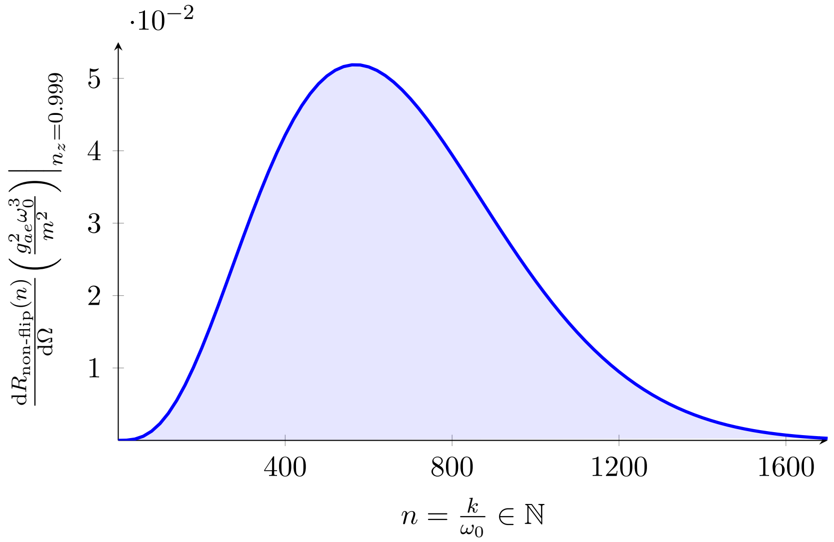

FIG. 2.3: Spectrum of the non-flip rate for one-dimensional trajectory for $a_0 = 10$ along the direction where emission is maximal ($n_z = 0.999$).

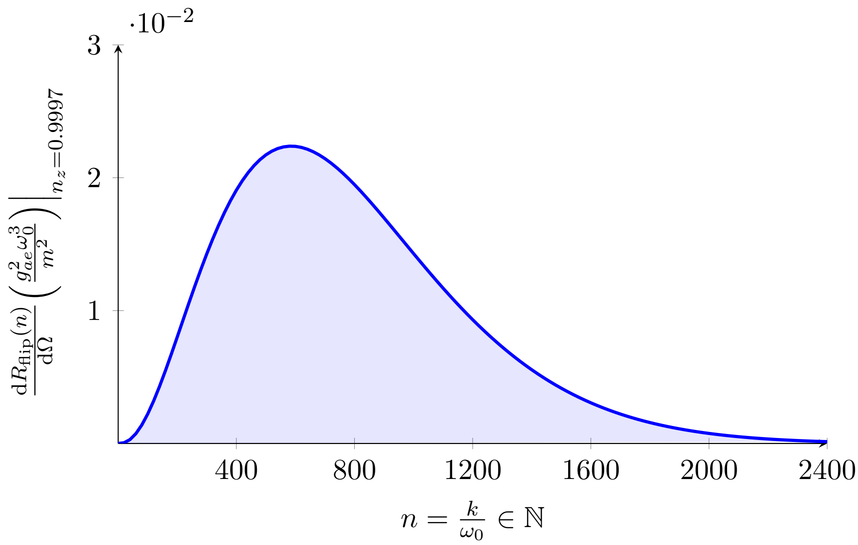

FIG. 2.4: Spectrum of the flip rate for one-dimensional trajectory for $a_0 = 10$ along the direction where emission is maximal ($n_z = 0.9997$).

By summing over $n = k/\omega_0$ and integrating over the solid angle, the total rate can be calculated as a function of a_0. For large a_0, we found that the energy scales as $\propto a_0^4$. Moreover, the typical energy of emitted axions is peaked around around $4a_0^2\omega_0$. Then we expect the rate (and the number of axions) to scale as a power law of a_0 with the power being around 2. Numerically, we find that the number of axions during one oscillations is well described by

$$P_{\text{flip}} \approx 0.15\frac{g_{ae}^2\omega_0^2}{m^2}a_0^{2.2}\,, \quad P_{\text{non-flip}} \approx 0.32\frac{g_{ae}^2\omega_0^2}{m^2}a_0^{2.2}\,, \quad a_0 \gg 1\,, \tag{2.85}$$

where $P = R \times 2\pi/\omega_0$. As expected, the power is close to 2. This can also be seen by looking at the peak in Fig. 2.3 and Fig. 2.4.

2.1.3 Different choice of initial polarization vectors

We now return to the general formula for the emission probability in Eq. (2.26). After finding this expression, we assumed that the initial polarization vector was $(1,0)^{\mathrm{T}}$ (the eigenvalue of the Pauli matrix σ^3 with eigenvalue 1) for simplicity, since one component vanishes. Would the result of the flip and non-flip probabilities have been the same if we had chosen a different polarization vector? We expect that a different choice should not change the final result. To show this, let us first consider that the initial polarization vector is $s_a = (1,i)^{\mathrm{T}}/\sqrt{2}$ (the normalized eigenvector of the matrix σ^2). Starting from Eq. (2.26), and expanding the modulus square of the time-integrals we find

$$\begin{aligned}P &= \frac{\hbar g_{ae}^2}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\int \mathrm{d}\tau\mathrm{d}\tau'\,\mathrm{e}^{ik\cdot(x(\tau)-x(\tau'))}q^{i*}(\tau')q^j(\tau)s_\alpha^\dagger\sigma^i\sum_\beta s_\beta s_\beta^\dagger\sigma^j s_\alpha\\ &= \frac{\hbar g_{ae}^2}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\,\mathbf{q}\cdot\boldsymbol{\sigma}s_\alpha\right|^2,\end{aligned} \tag{2.86}$$

where we used $\sum_\beta s_\beta s_\beta^\dagger = \mathbb{I}_{2\times 2}$. Using

$$\mathbf{q}\cdot\boldsymbol{\sigma}s_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} q_z + i(k_x - ik_y)\\ -iq_z + k_x + ik_y\end{pmatrix}, \tag{2.87}$$

the time integral is

$$\begin{aligned}\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\,\mathbf{q}\cdot\boldsymbol{\sigma}s_\alpha\right|^2 &= \frac{1}{2}\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\,(q_z + i(k_x - ik_y))\right|^2\\ &\quad+\frac{1}{2}\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\,(q_z - i(-k_x - ik_y))\right|^2,\end{aligned} \tag{2.88}$$

where in the second term we multiplied by an overall factor of i. Now, since this expression is integrated over $\mathbf{k}$, we can make the substitution $k_x \to -k_x$ for the second term. The term $\mathrm{e}^{ik\cdot x}$ is unaffected by this change since the trajectory is along z. Finally by using the identity $\frac{1}{2}|a+b| + \frac{1}{2}|a-b|^2 = |a|^2 + |b|^2$ for $a, b \in \mathbb{C}$, we find

$$\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\,\mathbf{q}\cdot\boldsymbol{\sigma}s_\alpha\right|^2 = \left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\,q_z\right|^2 + k_\perp^2\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}\right|^2, \tag{2.89}$$

which is the expression found in Eq. (2.26). Therefore, we showed that the total probability and thus the total radiated power do not depend on the choice of initial and final polarization vector. On the other hand, this is not true separately for the flip and non-flip probabilities. We find

$$\begin{aligned} P^{y-\mathrm{pol}}_{\text{non-flip}} &= \frac{\hbar g_{ae}^2}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} k_y^2 \left|\int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x}\right|^2 , \\ P^{y-\mathrm{pol}}_{\text{flip}} &= \frac{\hbar g_{ae}^2}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \left[k_x^2 \left|\int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x}\right|^2 + \left|\int \mathrm{d}\tau\, \mathrm{e}^{ik\cdot x} q_z\right|^2 \right] . \end{aligned} \tag{2.90}$$

2.1.4 Interaction Lagrangian equivalence

In this section, we verify that the interaction probability is the same as the one obtained by using the interaction Lagrangian in Eq. (2.1) which involves a derivative of the axion field. We note that since this term involves a factor of $\hbar$ we expect that the zeroth order in the WKB approximation will be sufficient in the following calculation. Also, in this section we do not assume that the trajectory is uni-dimensional. The first step is to find the interaction Hamiltonian. The full Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu a \partial^\mu a + \frac{1}{2\hbar^2} m_a^2 a^2 - \frac{\hbar g_{ae}}{2m}\partial_\mu a \bar{\psi}\gamma_5\gamma^\mu\psi \, , \tag{2.91}$$

where here a is the axion field not to be confused with the acceleration four-vector. The canonical conjugate momentum is given by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial(\partial_0 a)} = \partial_0 a - \frac{\hbar g_{ae}}{2m}\bar{\psi}\gamma_5\gamma^0\psi \, . \tag{2.92}$$

Then, defining $\lambda = \hbar g_{ae}/2m$ and $J^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi$, the Hamiltonian density is becomes

$$\begin{aligned}\mathcal{H} &= \pi_a \partial_0 a - \mathcal{L} \\ &= \pi_a\left(\pi_a + \lambda J^0\right) - \frac{1}{2}\left(\pi_a + \lambda J^0\right)^2 + \lambda J^0\left(\pi_a + \lambda J^0\right) + \lambda\partial_i a J^i - \frac{m_a^2 a^2}{2\hbar^2} \\ &= \frac{1}{2}\pi_a^2 + \lambda\pi_a J^0 + \lambda\partial_i a J^i + \frac{\lambda^2}{2}(J^0)^2 - \frac{m_a^2 a^2}{2\hbar^2}\,.\end{aligned} \tag{2.93}$$

In Hamiltonian perturbation theory, we replace π_a by its free-field expression, i.e. $\pi_a = \partial_0 a$. The interaction Hamiltonian is then

$$\mathcal{H}^{\text{ps}}_{\text{int}} = \lambda\partial_\mu J^\mu + \frac{\lambda^2}{2}(J^0)^2\,. \tag{2.94}$$

We neglect the four-fermi interaction term because it is $\mathcal{O}(g_{ae}^2)$. Therefore,

$$\mathcal{H}^{\text{ps}}_{\text{int}} = \frac{\hbar g_{ae}}{2m}\partial_\mu a\bar{\psi}\gamma_5\gamma^\mu\psi\,, \tag{2.95}$$

which is just $-\mathcal{L}^{\text{ps}}_{\text{int}}$, although the interaction has a derivative of the axion field (see also Ref. [88]). Also, since the two interaction Lagrangians differ only by a total derivative, we expect the interaction amplitude found using this Hamiltonian to be the same as the previous one up to a total derivative. The final state after the emission of a photon is found by making the substitution $\bar{\Phi}_\beta\gamma_5\Phi_\alpha \to \hbar\bar{\Phi}_\beta\gamma_5\not{k}\Phi_\alpha/2m$ in Eq. (2.14), where $\not{k} = \gamma^\mu k_\mu$. Then the interaction probability is calculated in the same way as before, and we find

$$P = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \sum_\beta \left| \int \mathrm{d}t \, \mathrm{e}^{ik_0 t} \frac{m}{p_0} \bar{\psi}_\beta(\mathbf{p} - \hbar\mathbf{k}, t) \gamma_5 \not{k} \psi_\alpha(\mathbf{p}, t) \right|^2 . \tag{2.96}$$

We calculate the time integrand. As expected only the first order in the WKB approximation is needed and we can safely set $\mathbf{p} - \hbar\mathbf{k} \approx \mathbf{p}$. We find

$$\begin{aligned} &\mathrm{e}^{ik_0 t} \frac{m}{p_0} \bar{\psi}_\beta(\mathbf{p} - \hbar\mathbf{k}, t) \gamma_5 \not{k} \psi_\alpha(\mathbf{p}, t) \\ &= \frac{E+m}{2E} \mathrm{e}^{ik\cdot x(t)} s_\beta^\dagger(t) \left[\mathbf{k}\cdot\boldsymbol{\sigma} - 2\frac{k_0(\mathbf{p}\cdot\boldsymbol{\sigma})}{E+m} + \frac{(\mathbf{p}\cdot\boldsymbol{\sigma})(\mathbf{k}\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma})}{(E+m)^2} \right] s_\alpha(t) \,, \end{aligned} \tag{2.97}$$

where $s_{\alpha,\beta}(t) = U(t) s_{\alpha,\beta}$ and $x^\mu(t) = (t, \mathbf{x}(t))$. For the last term we note

$$(\mathbf{p}\cdot\boldsymbol{\sigma})(\mathbf{k}\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma}) = 2(\mathbf{k}\cdot\mathbf{p})(\mathbf{p}\cdot\boldsymbol{\sigma}) - p^2(\mathbf{k}\cdot\boldsymbol{\sigma}) \,. \tag{2.98}$$

Then,

$$\begin{aligned} &\mathrm{e}^{ik_0 t} \frac{m}{p_0} \bar{\psi}_\beta(\mathbf{p} - \hbar\mathbf{k}, t) \gamma_5 \not{k} \psi_\alpha(\mathbf{p}, t) \\ &= \frac{m}{E} \mathrm{e}^{ik\cdot x(t)} s_\beta^\dagger(t) \left[\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k_0(\mathbf{p}\cdot\boldsymbol{\sigma})}{m} + \frac{(\mathbf{k}\cdot\mathbf{p})(\mathbf{p}\cdot\boldsymbol{\sigma})}{E+m} \right] s_\alpha(t) \,. \end{aligned} \tag{2.99}$$

We compare this last equation to the integrand of Eq. (2.24). Since the two interaction Hamiltonians are equivalent, the difference between the two integrands should be either zero or a total time derivative. Since the difference does not vanish, it must be the second case. We define the difference as

$$\begin{aligned}\Delta_t &= \frac{\mathrm{e}^{ik\cdot x}}{E} s_\beta^\dagger(t) \left[\frac{(\mathbf{k}\cdot\mathbf{p})(\mathbf{p}\cdot\boldsymbol{\sigma})}{mE} - \frac{k_0(\mathbf{p}\cdot\boldsymbol{\sigma})}{m} + \frac{i}{E}\left(\dot{\mathbf{p}} - \frac{\dot{E}}{E+m}\mathbf{p}\right)\cdot\boldsymbol{\sigma}\right] s_\alpha(t) \\ &= \frac{i}{m}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{E} s_\beta^\dagger(t)(\mathbf{p}\cdot\boldsymbol{\sigma}) s_\alpha(t)\mathrm{e}^{ik\cdot x}\right],\end{aligned} \tag{2.100}$$

as expected, where the second line can be verified using

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} s_{\alpha,\beta}(t) &= -\frac{i}{E(E+m)}\left(\mathbf{p}\times\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}\right)\cdot\boldsymbol{\sigma} s_{\alpha,\beta}(t)\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{ik\cdot x} &= i\left(k_0 - \frac{\mathbf{k}\cdot\mathbf{p}}{E}\right)\mathrm{e}^{ik\cdot x}\,.\end{aligned} \tag{2.101}$$

2.2 Production with space dependent potential

In this section, we calculate the interaction probability for the emission of an axion by an electron for a purely space dependent electromagnetic potential. The procedure is similar to the time-dependent potential. The main difference is that the we will need to use the solutions to the Dirac equation in the presence of a space-dependent potential. We will also use the interaction Hamiltonian in Eq. (2.95) because we only need the leading order term in the WKB approximation. The one-particle final state is given by

$$|1_{\text{axion}}\rangle = \frac{g_{ae}}{2m}\int \mathrm{d}x\,\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\frac{m}{p_0'}\sqrt{\frac{m}{p_0}}f(\mathbf{p})\mathrm{e}^{-ik_x x}\sum_\beta \bar{\psi}_\beta\gamma_5\not{k}\psi_\alpha$$
$$\times\int \mathrm{d}t\mathrm{d}y\mathrm{d}z\,\mathrm{e}^{\frac{it}{\hbar}(\hbar k_0 - p_0 + p_0')}\mathrm{e}^{\frac{iy}{\hbar}(p_y - p_y' - \hbar k_y)}\mathrm{e}^{\frac{iz}{\hbar}(p_z - p_z' - \hbar k_z)}a^\dagger(\mathbf{k})b_\beta^\dagger(\mathbf{p}')\,|0\rangle\;. \tag{2.102}$$

The last factor gives

$$\begin{aligned}&\int \mathrm{d}t\mathrm{d}y\mathrm{d}z\,\mathrm{e}^{\frac{it}{\hbar}(\hbar k_0 - p_0 + p_0')}\mathrm{e}^{\frac{iy}{\hbar}(p_y - p_y' - \hbar k_y)}\mathrm{e}^{\frac{iz}{\hbar}(p_z - p_z' - \hbar k_z)}\\ &= (2\pi\hbar)^3\delta(p_y' - p_y + \hbar k_y)\delta(p_z' - p_z + \hbar k_z)\delta(p_0' - p_0 + \hbar k_0)\,.\end{aligned} \tag{2.103}$$

We would like to write the last factor as a delta function in terms of p_x. For this we need to solve the equation $p_0' = p_0 - \hbar k_0$. Since the other two delta functions fix $p_y' = p_y - \hbar k_y$ and $p_z' = p_z - \hbar k_z$, at zeroth order in $\hbar$ we have

$$p_x' = \pm p_x + \mathcal{O}(\hbar)\,. \tag{2.104}$$

Since the electron moves in either the positive or negative x direction, we can disregard one of the solutions. Then,

$$\delta(p_0' - p_0 + \hbar k_0) \to \frac{p_0}{|p_x|}\delta(p_x' - p_x + \mathcal{O}(\hbar))\,. \tag{2.105}$$

Moreover, except in the factor $\bar{\psi}_\beta \gamma_5 \not{k} \psi_a$ we do not need the first order in $\hbar$. Therefore, we have

$$\begin{aligned} |1_{\text{axion}}\rangle = \frac{g_{ae}}{2m} \int \mathrm{d}x \, \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} \frac{m}{|p_x|} \sqrt{\frac{m}{p_0}} f(\mathbf{p}) \mathrm{e}^{-ik_x x} \\ \times \sum_\beta \bar{\psi}_\beta(\mathbf{p}) \gamma_5 \not{k} \psi_\alpha(\mathbf{p}) a^\dagger(\mathbf{k}) b^\dagger_\beta(\mathbf{p'}) \, |0\rangle \; . \end{aligned} \tag{2.106}$$

Then, the probability is

$$P = \langle 1_{\text{axion}} | 1_{\text{axion}} \rangle = \frac{g_{ae}^2 \hbar}{4m^2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \sum_\beta \left| \int \mathrm{d}x \frac{m}{|p_x|} \bar{\psi}_\beta \gamma_5 \not{k} \psi_a \mathrm{e}^{-ik_x x} \right|^2 . \tag{2.107}$$

The WKB solution in the case of a space-dependent potential was found in the introduction. We copy the result here for convenience

$$\psi_\alpha(p) = u_\alpha^{(0)} \sqrt{\frac{|p_x|}{\kappa_p}} \exp\left\{ \frac{i}{\hbar} \int_0^x \mathrm{d}\zeta \, \kappa_p(\zeta) \right\}, \quad u_\alpha^{(0)} = \sqrt{\frac{\tilde{p}_0 + m}{2m}} \begin{pmatrix} s_\alpha(x) \\ \frac{\tilde{\mathbf{p}} \cdot \boldsymbol{\sigma}}{\tilde{p}_0 + m} s_\alpha(x) \end{pmatrix}, \tag{2.108}$$

where $s_\alpha(x) = U(x) s_a$. From the delta functions $\tilde{p}'_0 = \tilde{p}_0 - \hbar k_0$, $\tilde{p}'_y = \tilde{p}_y - \hbar k_y$, $\tilde{p}'_z = \tilde{p}_z - \hbar k_z$. Then,

$$\kappa_p - \kappa_{p'} = \hbar \frac{\tilde{p}_0 k_0 - \tilde{p}_y k_y - \tilde{p}_z k_z}{\kappa_p} + \mathcal{O}(\hbar^2) \, . \tag{2.109}$$

We now use the point-particle description. We can thus substitute $\mathrm{d}x/\kappa_p = \mathrm{d}t/E$ where $E = \tilde{p}_0$. The exponential factors in the modulus square are

$$\begin{aligned}
&\mathrm{e}^{-ik_x x}\exp\left\{\frac{i}{\hbar}\int_0^x \mathrm{d}\zeta\,(\kappa_p - \kappa_{p'})\right\} = \mathrm{e}^{-ik_x x}\exp\left\{i\int_0^x \mathrm{d}\zeta\,\frac{\tilde{p}_0 k_0 - \tilde{p}_y k_y - \tilde{p}_z k_z}{\kappa_p}\right\}\\
&= \exp\left\{i\int_0^t \mathrm{d}\tilde{t}\,\frac{Ek_0 - k_x p_x - k_y p_y - k_z p_z}{E}\right\} = \exp\left\{ik_0 t - i\int_0^t \mathrm{d}\tilde{t}\,\mathbf{k}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tilde{t}}\right\}\\
&= \mathrm{e}^{ik\cdot x}
\end{aligned} \tag{2.110}$$

Then, the probability is

$$P = \frac{g_{ae}^2\hbar}{4m^2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\sum_\beta\left|\int \mathrm{d}\tau\,\mathrm{e}^{ik\cdot x}s_\beta^\dagger\left[\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k_0}{m}\mathbf{p}\cdot\boldsymbol{\sigma} + \frac{(\mathbf{k}\cdot\mathbf{p})(\mathbf{p}\cdot\boldsymbol{\sigma})}{m(E+m)}\right]s_\alpha\right|^2. \tag{2.111}$$

Thus, the result for the probability is the same as in the time-dependent case apart from the spin rotation because the matrices $U(x)$ and $U(t)$ are not the same at first sight. Using EQS. (1.30) and (1.36), the two phases are

$$\begin{aligned}
&-\frac{i}{2m}\int_0^\tau \mathrm{d}\tau'\,\mathcal{F}^{(1)} = -\frac{i}{2m}\int_0^\tau \mathrm{d}\tau'\,\frac{\boldsymbol{\sigma}\cdot\left(\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\right)}{E+m},\\
&-\frac{i}{2m}\int_0^\tau \mathrm{d}\tau'\,\mathcal{F}^{(2)}\\
&= -\frac{i}{2m}\int_0^\tau \mathrm{d}\tau'\left(\sigma^z\partial_x\tilde{p}_y - \sigma^y\partial_x\tilde{p}_z - \sigma^z\frac{\tilde{p}_y\partial_x E}{E+m} + \sigma^y\frac{\tilde{p}_z\partial_x E}{E+m}\right),
\end{aligned} \tag{2.112}$$

where $\tilde{p}^\mu = p^\mu - V^\mu$ is the time (or space) dependent potential. To see how they are connected, we will write the phases using the electromagnetic fields.

Firstly, if $-e < 0$ is the charge of the electron,

$$V^0 = -e\phi\,, \quad \mathbf{V} = -e\mathbf{A}\,, \tag{2.113}$$

where A^μ is the electromagnetic classical potential. We recall that the electric and magnetic fields are found from the potential as

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}\,, \quad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{2.114}$$

Then, we note

$$\mathcal{F}^{(1)} = e\frac{\boldsymbol{\sigma}}{E+m} \cdot \left(\mathbf{p} \times \frac{\partial \mathbf{A}}{\partial t}\right) = e\boldsymbol{\sigma} \cdot \left(\mathbf{B} - \frac{\mathbf{p} \times \mathbf{E}}{E+m}\right), \tag{2.115}$$

where in the last line, we used $\mathbf{B} = \mathbf{0}$ and $\boldsymbol{\nabla}\phi = \mathbf{0}$. For $\mathcal{F}^{(2)}$, we write,

$$\begin{aligned} \mathcal{F}^{(2)} &= e\boldsymbol{\sigma} \cdot \left(\boldsymbol{\nabla} \times \mathbf{A} + \frac{\mathbf{p} \times \boldsymbol{\nabla} E}{E+m}\right) = e\boldsymbol{\sigma} \cdot \left(\boldsymbol{\nabla} \times \mathbf{A} + \frac{\mathbf{p} \times (\partial_t \mathbf{A} + \boldsymbol{\nabla}\phi)}{E+m}\right) \\ &= \boldsymbol{\sigma} \cdot \left(\mathbf{B} - \frac{\mathbf{p} \times \mathbf{E}}{E+m}\right), \end{aligned} \tag{2.116}$$

where we used $\partial_t \mathbf{A} = \mathbf{0}$ for a space-dependent potential. Therefore, the two phases are derived from the same expression. The spin rotation for these two cases is

$$\frac{\mathrm{d}s_\alpha}{\mathrm{d}\tau} = -\frac{ie}{2m}\boldsymbol{\sigma} \cdot \left(\mathbf{B} - \frac{\mathbf{p} \times \mathbf{E}}{E+m}\right)s_\alpha\,. \tag{2.117}$$

This corresponds to the Thomas-BMT equation for relativistic spin dynamics [98, 99] (see also Refs. [37, 100] and Refs. [101, 102] for an interpretation of the cross-product term). Since this expression is valid for both t and x dependent potentials, we expect it to be true for arbitrary electromagnetic fields. This is what is shown in the next chapter.

3 Pseudo-Scalar Production and Experimental Proposal

Continuing the discussion on axion production, in this chapter, we consider a general electromagnetic field as a source for electron acceleration. We find the WKB approximation for an arbitrary external potential and use it to calculate the emission probability, energy, and spectrum of emitted axions. For a systematic WKB solution of the Dirac equation, see Ref. [37]. We use the results to propose an experimental setup to impose bounds on the coupling g_{ae}. Sec. 3.1 is based on calculations performed by Prof. A. Higuchi. The remainder of the chapter is based on Ref. [5]. The latter contains contributions from all coauthors.

3.1 Dirac equation for the general electromagnetic field and axion probability amplitude

In this section, we find the solutions to the Dirac equation within the WKB approximation for an arbitrary potential. In terms of A_μ, the Dirac equation is given by

$$i\gamma^\mu(\hbar\partial_\mu - ieA_\mu)\psi - m\psi = 0\,. \tag{3.1}$$

Applying $[i\gamma^\mu(\hbar\partial_\mu - ieA_\mu) + m]$ on the left gives

$$[(i\hbar\partial^\mu + eA^\mu)(i\hbar\partial_\mu + eA_\mu) - m^2]\psi + \frac{i}{2}e\hbar\gamma^{\mu\nu}F_{\mu\nu}\psi = 0\,, \tag{3.2}$$

where we defined $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. We let

$$\psi = \Psi\exp\left(-\frac{i}{\hbar}S\right). \tag{3.3}$$

Eq. (3.2) thus becomes

$$\begin{aligned}&\left[(\partial^\mu S + eA^\mu)(\partial_\mu S + eA_\mu) - m^2\right]\Psi \\ &+ i\hbar\left\{2(\partial^\mu S + eA^\mu)\partial_\mu\Psi + \left[\partial^\mu(\partial_\mu S + eA_\mu) + \frac{e}{2}\gamma^{\mu\nu}F_{\mu\nu}\right]\Psi\right\} = 0\,,\end{aligned}$$

where we neglected $\mathcal{O}(\hbar^2)$ terms. We solve this equation order by order in $\hbar$. To zeroth order,

$$(\partial^\mu S + eA^\mu)(\partial_\mu S + eA_\mu) - m^2 = 0\,. \tag{3.4}$$

To show that the solution S exists, we construct the classical world lines

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = -\frac{e}{m} F^{\mu\nu} \frac{\mathrm{d}x_\nu}{\mathrm{d}\tau}\,. \tag{3.5}$$

We assume that $A_\mu = 0$ in the past and on the hypersurface $t = t_i$. Then we consider the world lines with uniform velocity emanating from this hypersurface towards the future. We define τ as the proper time along these world lines measured from the $t = t_i$ hypersurface. We also assume that these world lines do not cross each other. Following the procedure for time and space-dependent potentials, we write the following ansatz

$$\partial_\mu S + eA_\mu = m v_\mu\,, \tag{3.6}$$

where $v^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Then, Eq. (3.4) is satisfied because $v^\mu v_\mu = 1$. Eq. (3.6) can be solved for S if and only if $eF_{\mu\nu} - m(\partial_\mu v_\nu - \partial_\nu v_\mu) = 0$ or

$$\partial_\mu v_\nu - \partial_\nu v_\mu - \frac{e}{m} F_{\mu\nu} = 0\,. \tag{3.7}$$

Firstly, we note that Eq. (3.7) is satisfied at $t = t_i$, (or $\tau = 0$) because $F_{\mu\nu} = 0$ and v^μ is constant there. Thus, what we need to show is that the proper time derivative of Eq. (3.7) is zero. To show this, we expand

($v^\lambda \partial_\lambda = \mathrm{d}/\mathrm{d}\tau$)

$$\begin{aligned}
&v^\lambda \partial_\lambda \Big(\partial_\mu v_\nu - \partial_\nu v_\mu - \frac{e}{m} F_{\mu\nu}\Big) \\
&= (\partial_\mu v^\lambda)\Big(\partial_\nu v_\lambda - \partial_\lambda v_\nu - \frac{e}{m} F_{\nu\lambda}\Big) - (\partial_\nu v^\lambda)\Big(\partial_\mu v_\lambda - \partial_\lambda v_\mu - \frac{e}{m} F_{\mu\lambda}\Big) \\
&\quad - \frac{e}{m} v^\lambda (\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu})\,. \qquad (3.8)
\end{aligned}$$

The last line vanishes by the Bianchi identity. Then, the unique solution to this equation with the initial condition Eq. (3.7) at $\tau = 0$ is Eq. (3.7) for all τ. This implies that the solution S to Eq. (3.6) exists. Then, by inserting Eq. (3.3) in Eq. (3.1) and using Eq. (3.6) we find to leading order in $\hbar$,

$$(\gamma^\mu p_\mu - m)\psi = 0\,, \qquad (3.9)$$

where $p_\mu = m v_\mu$. Then,

$$\psi \propto \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0 + m}\varphi \end{pmatrix} \qquad (3.10)$$

is the solution, where φ is a two-component spinor. To first order in $\hbar$, using the zeroth-order result,

$$2\frac{\mathrm{d}\Psi}{\mathrm{d}\tau} + \Big(\partial_\mu v^\mu + \frac{e}{2m}\gamma^{\mu\nu}F_{\mu\nu}\Big)\Psi = 0\,. \qquad (3.11)$$

We let

$$\Psi = \widetilde{\Psi}\exp\Big(-\frac{1}{2}\int_0^\tau \mathrm{d}\xi\, \partial_\mu v^\mu(\xi)\Big)\,. \qquad (3.12)$$

This definition allows one to remove the second term in the last equation. Then,

$$\frac{\mathrm{d}\widetilde{\Psi}}{\mathrm{d}\tau} = -\frac{e}{4m}\gamma^{\mu\nu}F_{\mu\nu}\widetilde{\Psi} = -\frac{e}{2m}\begin{pmatrix} i\boldsymbol{\sigma}\cdot\mathbf{B} & \boldsymbol{\sigma}\cdot\mathbf{E} \\ \boldsymbol{\sigma}\cdot\mathbf{E} & i\boldsymbol{\sigma}\cdot\mathbf{B}\end{pmatrix}. \tag{3.13}$$

Thus, if we write

$$\widetilde{\Psi} = \begin{pmatrix} \sqrt{p_0+m}\,s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\sqrt{p_0+m}}s \end{pmatrix}, \tag{3.14}$$

which satisfies Eq. (3.10), then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{p_0+m}\,s\right) &= -i\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}\sqrt{p_0+m}\,s - \frac{e}{2m}\frac{(\boldsymbol{\sigma}\cdot\mathbf{E})(\boldsymbol{\sigma}\cdot\mathbf{p})}{\sqrt{p_0+m}}\,s\,, \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\sqrt{p_0+m}}\,s\right) &= -\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{E}\sqrt{p_0+m}\,s - i\frac{e}{2m}\frac{(\boldsymbol{\sigma}\cdot\mathbf{B})(\boldsymbol{\sigma}\cdot\mathbf{p})}{\sqrt{p_0+m}}\,s\,.
\end{aligned} \tag{3.15}$$

From the first equation, using $\dot{p}_0 = -(e/2m)\mathbf{p}\cdot\mathbf{E}$, we find

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = -\frac{ie}{2m}\left(\mathbf{B} - \frac{\mathbf{p}\times\mathbf{E}}{p_0+m}\right)\cdot\boldsymbol{\sigma}\,s\,, \tag{3.16}$$

which is exactly the spin rotation we expected from the time- and space-dependent calculations and corresponds to the Thomas-BMT equation [98, 99].cSee Ref. [100] for a simple derivation of the Thomas-BMT equation based on covariance. The solution to this differential equation is

$$s = \mathrm{T}\left(\exp\left\{-\frac{ie}{2m}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\left(\mathbf{B}(\tau') - \frac{\mathbf{p}(\tau')\times\mathbf{E}(\tau')}{p_0(\tau')+m}\right)\cdot\boldsymbol{\sigma}\right\}\right)s(\tau_0)\,, \tag{3.17}$$

where T here is the proper time ordering operator. Since $\dot{\mathbf{p}} = -(e/m)(p_0\mathbf{E} + \mathbf{p} \times \mathbf{B})$, the second line in EQ. (3.15) can be verified using the first line and the Thomas-BMT equation. Thus, the WKB solution to first order can be written as

$$\psi = \sqrt{p_0 + m} \begin{pmatrix} s \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p_0+m}\, s \end{pmatrix} \exp\left(-\frac{1}{2} \int_0^\tau \mathrm{d}\xi \, \partial_\mu v^\mu(\xi) \right) \exp\left(-\frac{i}{\hbar} S \right) , \tag{3.18}$$

where $\partial_\mu S = mv_\mu - eA_\mu$ and $mv_\mu = p_\mu$. We note that

$$\overline{\psi}\gamma^\mu\psi = 2mv^\mu \exp\left(- \int_0^\tau \mathrm{d}\xi \, \partial_\mu v^\mu(\xi) \right) . \tag{3.19}$$

Then, to first order in the WKB approximation, the fermion current is conserved $\partial_\mu(\overline{\psi}\gamma^\mu\psi) = 0$. Fermion current conservation holds for the full solution of the Dirac equation as well. We expand the electron field as

$$\psi(x) = \sum_{\mathbf{p},\alpha} \left[u_{(\mathbf{p},\alpha)}(x) b_{(\mathbf{p},\alpha)} + v_{(\mathbf{p},\alpha)}(x) d^\dagger_{(\mathbf{p},\alpha)} \right] , \tag{3.20}$$

where

$$\begin{aligned} &\int \mathrm{d}^3\mathbf{x} \; u^\dagger_{(\mathbf{p},\alpha)} u_{(\mathbf{p},\beta)} = \int \mathrm{d}^3\mathbf{x} \; v^\dagger_{(\mathbf{p},\alpha)} v_{(\mathbf{p},\beta)} = \delta_{\alpha\beta} \, , \\ &\int \mathrm{d}^3\mathbf{x} \; u^\dagger_{(\mathbf{p},\alpha)} v_{(\mathbf{p},\beta)} = 0 \, , \end{aligned} \tag{3.21}$$

and $\left\{ b_{(\mathbf{p},\alpha)}, b^\dagger_{(\mathbf{p},\beta)} \right\} = \left\{ d_{(\mathbf{p},\alpha)}, d^\dagger_{(\mathbf{p},\beta)} \right\} = \delta_{\alpha\beta}$, with all other anti-commutators vanishing. We choose a wave-packet solution, $u_{(\mathbf{p},\alpha)}(t, \mathbf{x})$ peaked near the classical world line but approximately with a definite momentum and spin, as one of these basis states. (Thus, this wave packet behaves like a classical electron). $u_{(\mathbf{p},\alpha)}(t,\mathbf{x})$ can be written as (see Eq. (3.10))

$$u_{(\mathbf{p},\alpha)}(t,\mathbf{x}) = \sqrt{\frac{p_0+m}{2p_0}} \begin{pmatrix} s_\alpha \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p_0+m}\, s_\alpha \end{pmatrix} G(t,\mathbf{x})\,, \tag{3.22}$$

where the normalization was chosen in accordance with $\psi^\dagger\psi = 2p_0$ and s_α satisfies Eq. (3.16). The function $G(t,\mathbf{x})$ is peaked about a classical world line and satisfies

$$\int \mathrm{d}^3\mathbf{x}\, |G(t,\mathbf{x})|^2 = 1\,. \tag{3.23}$$

That is, although $G(t,\mathbf{x})$ is a smooth function, we may assume that $|G(t,\mathbf{x})|^2$ is well approximated by $\delta^{(3)}(\mathbf{x}-\mathbf{x}(t))$. $G(t,\mathbf{x})$ contains the exponential factors in Eq. (3.18). Let $b_{(\mathbf{p},\alpha)}$ be the annihilation operator corresponding to the wave-packet solution $u_{(\mathbf{p},\alpha)}(t,\mathbf{x})$. The initial and final states are $b^\dagger_{(\mathbf{p},\alpha)}\,|0\rangle$, and $a^\dagger_{\mathbf{k}} b^\dagger_{(\mathbf{p},\beta)}\,|0\rangle$, where $a^\dagger_{\mathbf{k}}$ is the creation operator for the axion with momentum $\mathbf{k}$. We assume that the solutions corresponding to $b^\dagger_{(\mathbf{p},\alpha)}\,|0\rangle$ and $b^\dagger_{(\mathbf{p},\beta)}\,|0\rangle$ are the same except possibly for the spin states, (the momentum stays approximately the same). That is, s_α and s_β may or may not be the same, but they have the same spatial wave function. The final state is (dropping the momentum index)

$$
\begin{aligned}
|1_{\text{axion}}\rangle &= -\frac{i}{\hbar}\int \mathrm{d}^4x\, \mathcal{H}_{\text{int}}(x) b^\dagger_\alpha |0\rangle \\
&= \frac{g_{ae}}{2m}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \sum_\beta \int \mathrm{d}^4x\, \overline{u_\beta}\gamma_5 \not{k} u_\alpha \mathrm{e}^{ik\cdot x} a^\dagger_{\mathbf{k}} b^\dagger_\beta |0\rangle \\
&= \frac{g_{ae}}{4m^2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \sum_\beta \int \mathrm{d}\tau\, \overline{\Phi_\beta}\gamma_5 \not{k} \Phi_\alpha\, \mathrm{e}^{ik\cdot x(\tau)} a^\dagger_{\mathbf{k}} b^\dagger_\beta |0\rangle \ ,
\end{aligned}
\tag{3.24}
$$

where

$$
\Phi_\alpha \equiv \Phi_{(\mathbf{p},\alpha)} = \sqrt{p_0+m}\begin{pmatrix} s_\alpha \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{p_0+m}\, s_\alpha \end{pmatrix} . \tag{3.25}
$$

The spinor Φ_α transforms covariantly under Lorentz transformations (see App. B). We have

$$
\overline{\Phi_\beta}\gamma_5 \not{k} \Phi_\alpha = 2 s^\dagger_\beta \left[m\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k\cdot p + mk}{p_0+m}\mathbf{p}\cdot\boldsymbol{\sigma} \right] s_\alpha \, , \tag{3.26}
$$

Since $\langle 0| b_{(\beta)} b^\dagger_{(\beta)} |0\rangle = 1$ and $[a_{\mathbf{k}}, a_{\mathbf{k}'}] = (2\pi)^3 2\hbar k_0 \delta^{(3)}(\mathbf{k}-\mathbf{k}')$ we find the probability for emission to be

$$
P_{\text{em}} = \hbar \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \sum_\beta \left|\mathcal{A}_{(\mathbf{p},\mathbf{k},\beta,\alpha)}\right|^2 , \tag{3.27}
$$

where

$$
\begin{aligned}
\mathcal{A}_{(\mathbf{p},\mathbf{k},\beta,\alpha)} &= \frac{g_{ae}}{2m}\int \mathrm{d}\tau\, s_\beta^\dagger \left[\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k\cdot p + mk}{m(p_0+m)}\mathbf{p}\cdot\boldsymbol{\sigma}\right] s_\alpha\, \mathrm{e}^{ik\cdot x(\tau)} \\
&= \frac{ig_{ae}}{2m}\int \mathrm{d}\tau\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left\{(k\cdot v)^{-1} s_\beta^\dagger \left[\mathbf{k}\cdot\boldsymbol{\sigma} - \frac{k\cdot p + mk}{m(p_0+m)}\mathbf{p}\cdot\boldsymbol{\sigma}\right] s_\alpha\right\} \mathrm{e}^{ik\cdot x(\tau)} \\
&= \frac{ig_{ae}}{2m}\int_{\tau_0}^{\tau} \mathrm{d}\tau\, \frac{\mathrm{e}^{ik\cdot x(\tau)}}{(k\cdot v)^2} s_\beta^\dagger(\tau)\mathbf{Q}(\tau)\cdot\boldsymbol{\sigma} s_\alpha(\tau)\,.
\end{aligned}
\tag{3.28}
$$

During the derivation we defined

$$
\mathbf{Q} = \mathbf{V} - (k\cdot a)\mathbf{k} - [V_0 - (k\cdot a)k_0]\frac{\mathbf{p}}{p_0+m}\,, \tag{3.29}
$$

where

$$
V^\mu = \frac{e}{m}(k\cdot v)F^{\mu\nu}k_\nu \quad \leftrightarrow \quad
\begin{aligned}
V_0 &= \frac{e}{m}(k\cdot v)\mathbf{k}\cdot\mathbf{E}\,, \\
\mathbf{V} &= \frac{e}{m}(k\cdot v)(k_0\mathbf{E} + \mathbf{k}\times\mathbf{B})\,.
\end{aligned}
\tag{3.30}
$$

We also define

$$
\mathbf{F} = \frac{e}{2m}\left(\mathbf{B} - \frac{\mathbf{p}\times\mathbf{E}}{p_0+m}\right), \tag{3.31}
$$

such that the spin rotation can be written as $\mathrm{d}s_\alpha/\mathrm{d}\tau = -i\mathbf{F}\cdot\boldsymbol{\sigma}s_\alpha$. We also used

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathbf{p}}{p_0+m}\right) = \frac{2\mathbf{F}\times\mathbf{p} - e\mathbf{E}}{p_0+m}\,, \tag{3.32}
$$

which is obtained using the equations of motion for the Lorentz-force

$$\begin{aligned}\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} &= -\frac{e}{m}(p_0\mathbf{E} + \mathbf{p}\times\mathbf{B})\,,\\ \frac{\mathrm{d}p_0}{\mathrm{d}\tau} &= -\frac{e}{m}\mathbf{p}\cdot\mathbf{E}\,.\end{aligned} \tag{3.33}$$

Having found a general expression for the emission amplitude, it is useful to verify that the previous results (of uni-dimensional electron motion) are recovered. In this case, consider an electric field $\mathbf{E}$ parallel to the $z-$axis and $\mathbf{B} = \mathbf{0}$. This implies that the electron's momentum is parallel to the electric field and therefore from Eq. (3.16) the spin is time independent. As usual, we choose the initial spin to be in the $z-$direction, i.e. $s_\alpha = (1, 0)^{\mathrm{T}}$. The spin non flip amplitude is found using Eq. (3.28):

$$\mathcal{A}^{\mathrm{nf}}_{(p_z,\mathbf{k})} = \frac{ig_{ae}}{2m}\int \mathrm{d}\tau\, \frac{\mathrm{e}^{ik\cdot x}}{(k\cdot v)^2}Q_z(\tau)\,. \tag{3.34}$$

Q_z is found from equation (3.29). We have $(k\cdot v) = (p_0k_0 - k_zp_z)/m$ and $(k\cdot a) = \dot{p}_z(k_0p_z - k_zp_0)/mp_0$, where dot means derivative with respect to the proper time. Moreover from the Lorentz-force equations $\dot{p}_z = -ep_0E_z/m$. Then,

$$Q_z(\tau) = -\kappa^2 a(\tau)\,, \tag{3.35}$$

where the acceleration is defined as $a(\tau) = -eE_z/m$ as we are taking the sign of the charge to be negative and $\sqrt{-a_\mu a^\mu} = e|E_z|/m$. The emission amplitude for the flip case is given by

$$\mathcal{A}^{\mathrm{f}}_{(p_z,\mathbf{k})} = \frac{ig_{ae}}{2m}\int \mathrm{d}\tau\, \frac{\mathrm{e}^{ik\cdot x}}{(k\cdot v)^2}[Q_x(\tau) + iQ_y(\tau)]\,, \tag{3.36}$$

where using Eq. (3.29)

$$Q_x(\tau) + iQ_y(\tau) = -(k\cdot a)(k_x + ik_y) \tag{3.37}$$

as for this configuration $V_x = V_y = 0$ and $p_x = p_y = 0$. Inserting Eqs. (3.34) and (3.36) into Eq. (3.27) gives Eq. (2.33).

We now extend our description to a two-dimensional trajectory. Consider an electric field in the (xz) plane (i.e. with vanishing y-component) and a magnetic field in the $y-$direction. The motion of the electron is thus confined in the (xz) plane. Moreover, the vector $\mathbf{F}$ defined in Eq. (3.31) is parallel to the $y-$axis. Then, considering Eq. (3.17), we see that the proper-time ordering operator will contain only the matrix σ^2. If we choose the initial spin to be an eigenvector of the matrix σ^2 then the form of both s_α, s_β will be significantly simplified. Labelling the eigenvectors of σ^2 as $s_\pm$ such that $\sigma^2 s_\pm = \pm s_\pm$, the solution to Eq. (3.16) is

$$s_\pm(\tau) = \exp\left\{\mp\frac{ie}{2m}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\left(B_y - \frac{v_z E_x - v_x E_y}{v_0 + 1}\right)\right\}s_\pm(\tau_0)\,. \tag{3.38}$$

For the spin non-flip case, this exponential factor will not appear since this integrand of the emission amplitude is proportional to $s_\beta^\dagger \mathbf{Q}\cdot\boldsymbol{\sigma} s_\alpha$. The emission probability is thus similar to the flip emission probability for the one-dimensional case and given by

$$P_{\text{em}}^{\text{nf}(2d)} = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} k_y^2 \left| \int \mathrm{d}\tau \, \mathrm{e}^{ik\cdot x} \frac{k \cdot a}{(k \cdot v)^2} \right|^2 . \tag{3.39}$$

Here, we notice that the factor $k_\perp^2$ is replaced by k_y^2 since the perpendicular direction to the electron trajectory is along the $y-$axis. When the spin of the electron flips, the emission probability is

$$P_{\text{em}}^{\text{f}(2d)} = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \left| \int \mathrm{d}\tau \, \mathrm{e}^{ik\cdot x} \mathrm{e}^{\mp i f(\tau)} \frac{Q_z \pm iQ_x}{(k \cdot v)^2} \right|^2 , \tag{3.40}$$

where $f(\tau) = 2\int_{\tau_0}^{\tau} \mathrm{d}\tau' \, F_y(\tau')$ and

$$\begin{aligned} Q_z \pm iQ_x = & -\frac{e}{m}(k\cdot v)[k_0(E_z \pm iE_x) + (k_x \mp ik_z)B_y] - (k\cdot a)(k_z \pm ik_x) \\ & - \left[\frac{e}{m}(k\cdot v)\mathbf{k}\cdot\mathbf{E} - (k\cdot a)k_0\right] \frac{v_z \pm iv_x}{v_0 + 1} . \end{aligned}$$

In what follows, we will consider an electron accelerating by two counter-propagating laser beams as described in EQ. (2.44). The electron is confined in the (xz) plane. Therefore, the results derived here are applicable to this case. Moreover, we can take $E_x = 0$.

3.2 TOTAL ENERGY OF EMITTED AXIONS

In this section we calculate the total energy of emitted axions using the generalized emission probability found in EQ. (3.27). In this derivation, we will take $m_a = 0$. We define

$$\widetilde{\mathbf{Q}} = k_0^{-2}\mathbf{Q}\,, \quad n^\mu = \frac{k^\mu}{k_0}\,, \quad \widetilde{V}^\mu = (k_0(k\cdot v)e/m)^{-1}V^\mu = F^{\mu\nu}n_\nu\,. \tag{3.41}$$

Thus,

$$\widetilde{\mathbf{Q}} = \frac{e}{m}(n\cdot v)\widetilde{\mathbf{V}} - (n\cdot a)\mathbf{n} - \left[\frac{e}{m}(n\cdot v)\widetilde{V}^0 - (n\cdot a)\right]\frac{\mathbf{p}}{p_0+m}\,. \tag{3.42}$$

The normalization is used to remove the dependence on the magnitude of the momentum $\mathbf{k}$ from $\widetilde{\mathbf{Q}}$. The energy is found from Eq. (3.27) by multiplying the integrand by a factor $\hbar k$. Then energy with initial spinor s_α and final spinor s_β is given by

$$\begin{aligned} E_{\beta\alpha} = \frac{\hbar^2 g_{ae}^2}{64\pi^3 m^2}\int \mathrm{d}\Omega \int_0^{+\infty}\mathrm{d}k\,k^2 \int \frac{\mathrm{d}\tau\mathrm{d}\tau'}{(n\cdot v(\tau))^2(n\cdot v(\tau'))^2} \\ \times\, s_\beta^\dagger(\tau)\widetilde{\mathbf{Q}}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)s_\alpha^\dagger(\tau')\widetilde{\mathbf{Q}}(\tau')\cdot\boldsymbol{\sigma}s_\beta(\tau')\mathrm{e}^{ik\cdot[x(\tau)-x(\tau')]}\,. \end{aligned} \tag{3.43}$$

Again, we define $\xi = t - \mathbf{n}\cdot\mathbf{x}$ to write the last term as $\mathrm{e}^{ik(\xi-\xi')}$. Then $\mathrm{d}\xi/\mathrm{d}\tau = n\cdot v$. We also have $k^2\mathrm{e}^{ik(\xi-\xi')} = \frac{\mathrm{d}^2}{\mathrm{d}\xi\mathrm{d}\xi'}\mathrm{e}^{ik(\xi-\xi')}$. Thus integrating by parts

$$\begin{aligned} E_{\beta\alpha} = \frac{\hbar^2 g_{ae}^2}{64\pi^3 m^2}\int \mathrm{d}\Omega \int \mathrm{d}\xi\mathrm{d}\xi'\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{s_\beta^\dagger(\tau)\widetilde{\mathbf{Q}}(\tau)\cdot\boldsymbol{\sigma}s_\alpha(\tau)}{(n\cdot v(\tau))^3}\right] \\ \times\,\frac{\mathrm{d}}{\mathrm{d}\xi'}\left[\frac{s_\alpha^\dagger(\tau')\widetilde{\mathbf{Q}}(\tau')\cdot\boldsymbol{\sigma}s_\beta(\tau')}{(n\cdot v(\tau'))^3}\right]\int_0^{+\infty}\mathrm{d}k\,\mathrm{e}^{ik[\xi-\xi']}\,. \end{aligned} \tag{3.44}$$

Previously, we extended the bounds of the $k-$integral to $-\infty$ by multiplying by $1/2$. This could be done because of the symmetry of the integrand under the exchange $\xi \leftrightarrow \xi'$. Here in the case where $\beta \neq \alpha$ the integrand is not symmetric under this exchange. Another way to extend the integral bounds is to send $k \to -k$ and then take the complex conjugate. This method works if the factor multiplying $\mathrm{e}^{ik(\xi-\xi')}$ is real, which is not the case here because of the spin rotation in Eq. (3.16). On the other hand, summing over the final spin and averaging over the initial one, the imaginary part vanishes and the factor multiplying $\mathrm{e}^{ik(\xi-\xi')}$ becomes real. Thus, the average energy is

$$\langle E \rangle = \frac{\hbar^2 g_{ae}^2}{128\pi^2 m^2} \int \mathrm{d}\Omega \int \frac{\mathrm{d}\tau}{n \cdot v} \sum_{\alpha,\beta} \left| \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{1}{[n \cdot v(\tau)]^3} s_\beta^\dagger(\tau) \widetilde{\mathbf{Q}}(\tau) \cdot \boldsymbol{\sigma} s_\alpha(\tau) \right] \right|^2, \tag{3.45}$$

where extending the range to $(-\infty, +\infty)$ was achieved by multiplying by $1/2$ and integrating over k gave $2\pi\delta(\xi - \xi')$. Using $\mathrm{d}s/\mathrm{d}\tau = -i\mathbf{F} \cdot \boldsymbol{\sigma} s$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} \Big(s_\beta^\dagger(\tau) \widetilde{\mathbf{Q}}(\tau) \cdot \boldsymbol{\sigma} s_\alpha(\tau) \Big) &= s_\beta^\dagger(\tau) \dot{\widetilde{\mathbf{Q}}}(\tau) \cdot \boldsymbol{\sigma} s_\alpha(\tau) + i s_\beta^\dagger(\tau) \Big[\mathbf{F} \cdot \boldsymbol{\sigma}, \widetilde{\mathbf{Q}} \cdot \boldsymbol{\sigma} \Big] s_\alpha(\tau) \\ &= s_\beta^\dagger(\tau) \Big[\dot{\widetilde{\mathbf{Q}}}(\tau) - 2\mathbf{F} \times \widetilde{\mathbf{Q}} \Big] \cdot \boldsymbol{\sigma} s_\alpha(\tau) \,. \end{aligned} \tag{3.46}$$

where we used $\left[\mathbf{F}\cdot\boldsymbol{\sigma}, \widetilde{\mathbf{Q}}\cdot\boldsymbol{\sigma}\right] = 2i(\mathbf{F}\times\widetilde{\mathbf{Q}})\cdot\boldsymbol{\sigma}$. Then, the integrand of the average energy is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{[n\cdot v(\tau)]^3}s^\dagger_\beta\widetilde{\mathbf{Q}}\cdot\boldsymbol{\sigma}s_\alpha\right] = \frac{s^\dagger_\beta}{(n\cdot v)^3}\left[-\frac{3(n\cdot a)}{(n\cdot v)}\widetilde{\mathbf{Q}} + \dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F}\times\widetilde{\mathbf{Q}}\right]\cdot\boldsymbol{\sigma}s_\alpha$$
$$\equiv s^\dagger_\beta(\mathbf{S}\cdot\boldsymbol{\sigma})s_\alpha\,. \tag{3.47}$$

The modulus square of this expression contains the factor $\sum_\alpha s_\alpha(\tau)s^\dagger_\alpha(\tau) = \mathbb{1}$ for all τ. Additionally, using $(\mathbf{S}\cdot\boldsymbol{\sigma})^2 = \mathbf{S}^2\mathbb{1}$, the energy yields

$$\begin{aligned}\langle E\rangle &= \frac{\hbar^2 g_{ae}^2}{128\pi^2 m^2}\int\mathrm{d}\Omega\int\frac{\mathrm{d}\tau}{n\cdot v}\mathbf{S}^2\sum_\beta s^\dagger_\beta(\tau)s_\beta(\tau)\\ &= \frac{\hbar^2 g_{ae}^2}{64\pi^2 m^2}\int\mathrm{d}\Omega\int\frac{\mathrm{d}\tau}{(n\cdot v)^7}\\ &\qquad\times\left[\frac{9(n\cdot a)^2}{(n\cdot v)^2}\widetilde{\mathbf{Q}}^2 - \frac{3(n\cdot a)}{(n\cdot v)^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{\mathbf{Q}}^2 + \|\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F}\times\widetilde{\mathbf{Q}}\|^2\right].\end{aligned} \tag{3.48}$$

We now verify that the term between brackets is Lorentz invariant. We start with $\widetilde{\mathbf{Q}}^2$ and define

$$U^\mu = (n\cdot a)n^\mu - \frac{e}{m}(n\cdot v)\widetilde{V}^\mu = \frac{e}{m}n_\alpha v_\nu(F^{\nu\alpha}n^\mu - F^{\mu\alpha}n^\nu)\,, \tag{3.49}$$

where we used

$$a^\mu = -\frac{e}{m}F^{\mu\nu}v_\nu\,. \tag{3.50}$$

Note that the last factor is antisymmetric in the indices $(\mu\nu)$. Therefore $U^\mu v_\mu = 0$. Then, we can write $\widetilde{\mathbf{Q}}$ in terms of U^μ i.e.

$$\widetilde{\mathbf{Q}} = -\mathbf{U} + \frac{m\mathbf{v}}{p_0 + m}U_0 \quad \rightarrow \quad \widetilde{\mathbf{Q}}^2 = \mathbf{U}^2 + \frac{p_0 - m}{p_0 + m}(U_0)^2 - 2\frac{mU_0\mathbf{U}\cdot\mathbf{v}}{p_0 + m}\,. \tag{3.51}$$

From $U^\mu v_\mu = 0$, we deduce $m\mathbf{U}\cdot\mathbf{v} = U_0 p_0$. Thus showed that

$$\widetilde{\mathbf{Q}}^2 = -U_\mu U^\mu\,, \tag{3.52}$$

which is manifestly Lorentz invariant. Now we consider the term the third term in the expression of the energy, i.e. $\|\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F}\times\widetilde{\mathbf{Q}}\|^2$. Explicitly,

$$\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F}\times\widetilde{\mathbf{Q}} = -\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} + \frac{\mathbf{p}}{p_0 + m}\frac{\mathrm{d}U_0}{\mathrm{d}\tau} - \frac{e\mathbf{E}U_0}{p_0 + m} + 2\mathbf{F}\times\mathbf{U}\,, \tag{3.53}$$

where we expressed $\widetilde{\mathbf{Q}}$ in terms of U_μ and used EQ. (3.32). Following the definition of $\mathbf{F}$ in EQ. (3.31) and using $\mathbf{U}\cdot\mathbf{p} = U_0 p_0$, we find

$$\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F}\times\widetilde{\mathbf{Q}} = -\left[\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} + \frac{e}{m}(\mathbf{U}\times\mathbf{B} + U_0\mathbf{E})\right] + \frac{\mathbf{p}}{p_0 + m}\left(\frac{\mathrm{d}U_0}{\mathrm{d}\tau} + \frac{e}{m}\mathbf{U}\cdot\mathbf{E}\right). \tag{3.54}$$

As we did previously, we define

$$H^\mu = \frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} + \frac{e}{m}F^{\mu\nu}U_\nu\,. \tag{3.55}$$

We can write $\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F} \times \widetilde{\mathbf{Q}}$ only in terms of H^μ. Explicitly

$$\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F} \times \widetilde{\mathbf{Q}} = -\mathbf{H} + \frac{\mathbf{p}}{p_0 + m} H_0 \,, \tag{3.56}$$

where we used $F^{0i}U_i = \mathbf{E} \cdot \mathbf{U}$ and $F^{i\mu}U_\mu = U_0 E^i + (\mathbf{U} \times \mathbf{B})^i$. Explicit calculation leads to

$$\begin{aligned} H^\mu = {} & \frac{e}{m}\Big(\dot{F}^{\beta\alpha} v_\beta n_\alpha - \dot{F}^{\mu\nu} n_\nu (n \cdot v)\Big) \\ & + \frac{e^2}{m^2}\big(n_\alpha F^{\alpha\beta} F_{\beta\gamma} v^\gamma n^\mu - (n \cdot v) F^{\mu\nu} F_{\nu\alpha} n^\alpha\big) \,. \end{aligned} \tag{3.57}$$

Therefore contracting with v_μ gives $H^\mu v_\mu = 0$. Or, in other words, $H_0 p_0 = \mathbf{H} \cdot \mathbf{p}$. Then,

$$\|\dot{\widetilde{\mathbf{Q}}} - 2\mathbf{F} \times \widetilde{\mathbf{Q}}\|^2 = \mathbf{H}^2 - (H^0)^2 = -H_\mu H^\mu \,. \tag{3.58}$$

which is Lorentz invariant as well. Thus the energy can be written as

$$\begin{aligned} \langle E \rangle = \frac{\hbar^2 g_{ae}^2}{64\pi^2 m^2} \int \mathrm{d}\Omega \int & \frac{\mathrm{d}\tau}{(n \cdot v)^7} \\ & \times \left[-H_\mu H^\mu - \frac{9(n \cdot a)^2}{(n \cdot v)^2} U_\mu U^\mu + \frac{3(n \cdot a)}{(n \cdot v)^2} \frac{\mathrm{d}}{\mathrm{d}\tau}(U_\mu U^\mu) \right] . \end{aligned} \tag{3.59}$$

Because of $n_\mu n^\mu = 0$ as we are considering a massless axion and $\widetilde{V}^\mu n_\mu = 0$, we find,

$$U_\mu U^\mu = -\frac{e^2}{m^2}(n \cdot v)^2 n_\mu F^{\mu\nu} F_{\nu\rho} n^\rho \,. \tag{3.60}$$

Similarly for $H_\mu H^\mu$, we find

$$\begin{aligned} H_\mu H^\mu = &-\frac{2e^2}{m^2}(n\cdot v)\frac{d(n\cdot a)}{\mathrm{d}\tau}n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda \\ &+\frac{e^2}{m^2}(n\cdot v)^2\Big(\dot{F}_{\mu\alpha}+\frac{e}{m}F_{\mu\beta}F^{\beta}{}_{\alpha}\Big)n^\alpha\Big(\dot{F}^{\mu\nu}+\frac{e}{m}F^{\mu\alpha}F_\alpha{}^\nu\Big)n_\nu\,. \end{aligned} \tag{3.61}$$

We can verify that this result agrees with the formula for the energy in a one-dimensional trajectory. We choose the electric field to be along the z-axis. Then, $F_{03}=-F_{30}$ are the only non-zero components of the electromagnetic tensor. We find

$$\begin{aligned} &n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda = E_z^2(1-n_z^2)\,, \\ &n_\mu \dot{F}^{\mu\nu}F_{\nu\lambda}F^{\lambda\kappa}n_\kappa = 0\,, \\ &n_\mu F^{\mu\nu}F_{\nu\lambda}F^{\lambda\kappa}F_{\kappa\sigma}n^\sigma = E_z^4(1-n_z^2)\,. \end{aligned} \tag{3.62}$$

We insert these expressions into U_μ and H_μ to find the average energy. Defining $\mathcal{E}_z = eE_z/m$,

$$\begin{aligned} \langle E\rangle = \frac{\hbar^2 g_{ae}^2}{64\pi^2 m^2}\int \frac{\mathrm{d}\tau\mathrm{d}\Omega}{(n\cdot v)^7}(1-n_z^2)\Bigg[&3(n\cdot a)^2\mathcal{E}_z^2-6(n\cdot a)(n\cdot v)\mathcal{E}_z\frac{\mathrm{d}\mathcal{E}_z}{\mathrm{d}\tau} \\ &+2(n\cdot v)\frac{\mathrm{d}(n\cdot a)}{\mathrm{d}\tau}\mathcal{E}_z^2+(n\cdot v)^2\left(\left(\frac{\mathrm{d}\mathcal{E}_z}{\mathrm{d}\tau}\right)^2-\mathcal{E}_z^4\right)\Bigg]\,. \end{aligned} \tag{3.63}$$

We parametrize the velocity as $v^\mu = (v^0, v^z) = (\sqrt{1+w^2}, w)$. Then the acceleration and its derivative are

$$a^\mu = \begin{pmatrix} ww' \\ \sqrt{1+w^2}w' \end{pmatrix}, \qquad \dot{a}^\mu = \begin{pmatrix} \sqrt{1+w^2}(ww''+(w')^2) \\ (1+w^2)w''+w(w')^2 \end{pmatrix}, \tag{3.64}$$

where as usual prime and dot mean derivative with respect to t and τ respectively. The two are related via $\mathrm{d}t/\mathrm{d}\tau = \sqrt{1+w^2}$. Writing Eq. (3.50) explicitly as

$$\begin{aligned} a^0 &= -\frac{e}{m}\mathbf{v}\cdot\mathbf{E}\,, \\ \mathbf{a} &= -\frac{e}{m}(v_0\mathbf{E} + \mathbf{v}\times\mathbf{B})\,, \end{aligned} \tag{3.65}$$

we find using the first line $\mathcal{E}_z = -w'$. Performing the angular integrals first we find

$$\langle E\rangle = \frac{g_{ae}^2}{24\pi m^2}\int \mathrm{d}t\left[\frac{8}{5}w'^4 + \frac{4}{5}ww'^2w'' + (1+w^2)w''^2\right]. \tag{3.66}$$

To compare with the oscillating field, we let $w = -2a_0\sin\omega_0 t$ and after integrating between 0 and $2\pi/\omega_0$ we recover Eq. (2.59). For the case of uniform acceleration, $w = \sinh a\tau$, with $a > 0$ a constant, $w' = a$ which implies $w'' = 0$. Thus, we also find Eq. (2.43).

3.3 Motion in a constant magnetic field

We apply the results for the total energy to the case of a constant magnetic field. So far, we have not studied a trajectory in a plane, and this is the simplest case to consider, since relativistic motion in a constant magnetic field is "super-integrable" [91]. The results we derive in this section, such as the typical axion momentum, will also be used qualitatively for a trajectory inside laser fields. We assume that the magnetic field is parallel to the y-axis. Using Eq. (3.65), $a^0 = 0$. Moreover, using Eq. (3.33), we see that the energy p_0 is constant. As a result, the Lorentz factor $v^0 \equiv a_0$ is constant and the proper time is related to the physical time via $t = a_0 \tau$. We denote the Lorentz factor by a_0 because the solutions to the equations of motions are given by $v^x \propto \cos \omega_0 t$ and $v^z \propto \sin \omega_0 t$, where $\omega_0 = eB_y/(mv^0)$. Therefore, here a_0, as previously defined, coincides with the Lorentz factor, whereas in the case of lasers a_0 is the strength parameter and can be less than one. By definition, here $a_0 \geq 1$. The electron trajectory is (by choosing appropriate initial conditions)

$$
\begin{aligned}
z(t) &= \frac{\sqrt{1 - a_0^{-2}}}{\omega_0} \cos \omega_0 t \,, \\
x(t) &= \frac{\sqrt{1 - a_0^{-2}}}{\omega_0} \sin \omega_0 t \,.
\end{aligned}
\tag{3.67}
$$

The integrand in Eq. (3.59), for constant electromagnetic fields is

$$
\left(3 \frac{(n \cdot a)^2}{(n \cdot v)^7} + 2 \frac{n \cdot \dot{a}}{(n \cdot v)^6} \right) \frac{e^2}{m^2} n_\mu F^{\mu\nu} F_{\nu\lambda} n^\lambda - \frac{e^4}{m^4} \frac{1}{(n \cdot v)^5} F^{\mu\alpha} F_\alpha{}^\nu n_\nu F_{\mu\beta} F^\beta{}_\lambda n^\lambda \,.
\tag{3.68}
$$

The only non-zero components of the electromagnetic tensor are $F_{13} = -F_{31} = B_y$. Performing the angular integral (see App. C), we find

$$\begin{aligned}
-\frac{1}{4\pi}\int \mathrm{d}\Omega\, \frac{e^4}{m^4}\frac{1}{(n\cdot v)^5}F^{\mu\alpha}F_\alpha{}^\nu n_\nu F_{\mu\beta}F^\beta{}_\lambda n^\lambda &= 2a_0^5\omega_0^4\left(a_0^2-\frac{2}{3}\right), \\
\frac{1}{4\pi}\int \mathrm{d}\Omega\, \frac{3e^2}{m^2}\frac{(n\cdot a)^2}{(n\cdot v)^7}n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda &= 3a_0^5\omega_0^4(a_0^2-1)\left(\frac{8}{15}a_0^2-\frac{4}{15}\right), \\
\frac{1}{4\pi}\int \mathrm{d}\Omega\, \frac{2e^2}{m^2}\frac{n\cdot \dot{a}}{(n\cdot v)^6}n_\mu F^{\mu\nu}F_{\nu\lambda}n^\lambda &= 2a_0^5\omega_0^4(a_0^2-1)\left(\frac{16}{5}a_0^2-\frac{8}{5}\right).
\end{aligned} \tag{3.69}$$

We note that after integrating over the solid angle, the result is time-independent. Therefore, the emitted power is constant. The energy emitted during a cycle of $2\pi/\omega_0$ is found after summing all the contributions and yields

$$\langle E\rangle_{\text{circular}} = \frac{\hbar^2 g_{ae}^2\omega_0^3}{m^2}\left(a_0^8-\frac{5}{4}a_0^6+\frac{1}{3}a_0^4\right). \tag{3.70}$$

We note that for $a_0 \gg 1$ (i.e. a relativistic electron), the energy grows as a_0^8 which is much larger compared the energy emitted using an electric field which only grows as a_0^4. Therefore, the presence of the magnetic field in the laser beams will increase significantly particle production.

Axion spectrum

In this section, we find the spectrum of emitted axions in the case of a constant magnetic field. We proceed as for a purely electric field. This is possible because the motion of the electrons is periodic with period $2\pi/\omega_0$. Then the rate can be written in terms of Fourier coefficients, and the task will be to calculate the latter. We start with the non-flip case. The amplitude can be found from the first line of Eq. (3.28) (this is before integration by parts) by taking the initial spin to be along the positive y-direction. Then

$$\mathcal{A}^{\text{nf}}_{(\mathbf{p},\mathbf{k},\beta=\alpha)} = \frac{g_{ae}}{2m} k_y \int \mathrm{d}\tau \, \mathrm{e}^{ik\cdot x(\tau)} \,. \tag{3.71}$$

We use t as an integration variable. Then, defining $k\xi = k \cdot x$ (where we wrote $k\xi \equiv k_0\xi$), the infinite probability amplitude is

$$P^{\text{nf}}_{\text{em}} = \frac{\hbar g_{ae}^2}{4m^2} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} k_y^2 \left| \frac{1}{a_0} \int \mathrm{d}\xi \, \mathrm{e}^{ik\xi} \frac{\mathrm{d}t}{\mathrm{d}\xi} \right|^2 . \tag{3.72}$$

Since the motion of the electron is periodic, we can write the time integrand as

$$\frac{1}{a_0} \frac{\mathrm{d}t}{\mathrm{d}\xi} = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{-in\omega_0 \xi} \,. \tag{3.73}$$

Integrating this sum over time will result $2\pi\delta(k - n\omega_0) \sum_n c_n$. After squaring we note the appearance of a factor $2\pi\delta(0)$. We interpret this divergent factor as the total time of emission T. Finally, we divide by T to find the rate. The explicit expression of $\mathrm{d}t/\mathrm{d}\xi$ is not needed to find Fourier coefficients, but we show it for completeness. We start by using spherical coordinates and define the angles θ and φ as

$$(k_z, k_x, k_y) = (k \sin\theta \cos\varphi, k \sin\theta \sin\varphi, k \cos\theta) \,. \tag{3.74}$$

ξ can thus be expressed using Eq. (3.67). We find

$$\begin{aligned}\xi &= t - \frac{\sqrt{1-a_0^{-2}}}{\omega_0}\sin\theta\cos(\omega_0 t-\varphi)\,,\\ \frac{\mathrm{d}t}{\mathrm{d}\xi} &= \frac{1}{1+\sqrt{1-a_0^{-2}}\sin\theta\sin(\omega_0 t-\varphi)}\,.\end{aligned}\tag{3.75}$$

The Fourier coefficients then are

$$\begin{aligned}c_n &= \frac{\omega_0}{2\pi a_0}\int_0^{2\pi/\omega_0}\mathrm{d}\xi\,\mathrm{e}^{in\omega_0\xi}\frac{\mathrm{d}t}{\mathrm{d}\xi}\\ &= \frac{1}{2\pi a_0}\int_0^{2\pi}\mathrm{d}s\,\mathrm{e}^{in[s-\beta\sin\theta\cos(s-\varphi)]}\quad (s=\omega_0 t)\\ &= \frac{\mathrm{e}^{in\varphi-in\pi/2}}{a_0}J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\,.\end{aligned}\tag{3.76}$$

where in the last step, we used equations 3.715.13 and 3.715.18 of Ref. [103], and $J_n(x)$ is the Bessel function of order n. We note that the appearance of Bessel function is typical for circular motion, i.e. synchrotron radiation [97]. The emission rate is

$$R_{\mathrm{em}}^{\mathrm{nf}} = \frac{\hbar g_{ae}^2\omega_0^3}{16\pi m^2}\sum_{n=1}^{\infty}\frac{n^3}{a_0^2}\int_0^{\pi}\mathrm{d}\theta\left|J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2\cos^2\theta\sin\theta\,.\tag{3.77}$$

We also would like to have an expression for the emitted power to verify that our treatment is consistent with Eq. (3.70). Since the energy is found from the probability with an additional factor of $\hbar k$ and the delta function fixes k to $n\omega_0, n\in\mathbb{N}$, we only to multiply the last expression by $\hbar n\omega_0$ to find

$$S_{\mathrm{em}}^{\mathrm{nf}} = \frac{\hbar^2 g_{ae}^2\omega_0^4}{16\pi m^2}\sum_{n=1}^{\infty}\frac{n^4}{a_0^2}\int_0^{\pi}\mathrm{d}\theta\left|J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2\cos^2\theta\sin\theta\,.\tag{3.78}$$

Using the integral representation of the Bessel function and Parseval's theorem as for the electric field case, we find

$$\sum_{n=1}^{\infty} n^4 |J_n(n\alpha)|^2 = \frac{27\alpha^6 + 472\alpha^4 + 592\alpha^2 + 64)\alpha^2}{256(1-\alpha^2)^{13/2}} \,. \tag{3.79}$$

Now, we can substitute $\alpha = \sqrt{1 - a_0^{-2}}\sin\theta$ and perform the angular integral in Eq. (3.78). Finally multiplying the latter by $2\pi/\omega_0$, the period of one oscillation, we arrive at

$$E^{\mathrm{nf}}_{\mathrm{circular}} = \frac{g_{ae}^2 \hbar^2}{8m^2}\left(\frac{1}{3}a_0^8 - \frac{3}{5}a_0^6 + \frac{4}{15}a_0^4\right)\omega_0^3 \,. \tag{3.80}$$

Now we look at the spin flip probability. Using $n_z \pm i n_x = \sin\theta\, e^{\pm i\varphi}$ and $p_z \pm i p_x = \pm i m \sqrt{a_0^2 - 1} e^{\pm i \omega_0 t}$, the emission amplitude in this case is

$$\mathcal{A}^{\mathrm{f}}_{(\mathbf{p},\mathbf{k},\mp,\pm)} = \frac{g_{ae}}{2m}\int \mathrm{d}\tau \left[k \sin\theta e^{\mp i(\omega_0 t - \varphi)} \mp i \sqrt{a_0^2 - 1}\frac{k\cdot p + mk}{p_0 + m}\right]\mathrm{e}^{ik\cdot x} \,. \tag{3.81}$$

where $\pm$ refers to the spin being in the positive or negative direction of the y-axis. We notice that $p_0 = m a_0$ is constant and

$$\int \mathrm{d}\tau\, k\cdot p\, \mathrm{e}^{ik\cdot x} = -im \int \mathrm{d}\tau \, \frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{e}^{ik\cdot x} = 0 \,, \tag{3.82}$$

by periodicity. Then, the amplitude simplifies to

$$\mathcal{A}^{\mathrm{f}}_{(\mathbf{p},\mathbf{k},\mp,\pm)} = \mp i \frac{g_{ae} k}{2m}\int \left[\sqrt{\frac{a_0 - 1}{a_0 + 1}} \pm i \sin\theta\, e^{\mp i(\omega_0 t - \varphi)}\right]\mathrm{e}^{ik\cdot x}\,\mathrm{d}\tau \,. \tag{3.83}$$

We now change variables to ξ and define the Fourier coefficients as

$$\left[\sqrt{\frac{a_0-1}{a_0+1}} \pm i\sin\theta e^{\mp i(\omega_0 t-\varphi)}\right]\frac{\mathrm{d}t}{\mathrm{d}\xi} = \sum_{n=-\infty}^{\infty} c_n^{\pm}\mathrm{e}^{-in\omega_0\xi}\,. \tag{3.84}$$

Inverting this relation and using the integral representation of the Bessel function as previously

$$\begin{aligned} a_0 c_n^{\pm}\mathrm{e}^{-in\varphi} &= \mathrm{e}^{-i\frac{n\pi}{2}}\sqrt{\frac{a_0-1}{a_0+1}}J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right) \\ &\quad \pm i\sin\theta\mathrm{e}^{-i(n\mp1)\pi/2}J_{n\mp1}\left(n\sqrt{1-a_0^{-2}}\sin\theta\right). \end{aligned} \tag{3.85}$$

Thus, the rate and the power are given by

$$\begin{aligned} R_{\mathrm{em}}^{\mathrm{f},\pm} = \frac{\hbar g_{ae}^2\omega_0^3}{16\pi m^2}\sum_{n=1}^{\infty}\frac{n^3}{a_0^2}\int_0^{\pi}\mathrm{d}\theta\left|\sqrt{\frac{a_0-1}{a_0+1}}J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right. \\ \left. -\sin\theta J_{n\mp1}\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2\sin\theta\,, \end{aligned} \tag{3.86}$$

and

$$\begin{aligned} S_{\mathrm{em}}^{\mathrm{f},\pm} = \frac{\hbar^2 g_{ae}^2\omega_0^4}{16\pi m^2}\sum_{n=1}^{\infty}\frac{n^4}{a_0^2}\int_0^{\pi}\mathrm{d}\theta\left|\sqrt{\frac{a_0-1}{a_0+1}}J_n\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right. \\ \left. -\sin\theta J_{n\mp1}\left(n\sqrt{1-a_0^{-2}}\sin\theta\right)\right|^2\sin\theta\,. \end{aligned} \tag{3.87}$$

We can use Parseval's theorem again to calculate the energy emitted. We cannot calculate separately the energy due to the flips $(+\to-)$ and $(-\to+)$ but we can take the average. This is because $|c_n^{(\pm)}| \neq |c_{-n}^{(\pm)}|$, but $|c_{-n}^{(+)}| = |c_n^{(-)}|$.

Then,

$$\sum_{n=1}^{\infty} n^4(|c_n^{(+)}|^2 + |c_n^{(-)}|^2) = \sum_{n=-\infty}^{\infty} n^4|c_n^{(+)}|^2 = -\frac{27\alpha^8 + 472\alpha^6 + 592\alpha^4 + 64\alpha^2}{256(1-\alpha^2)^{13/2}} - \frac{45\alpha^8 + 560\alpha^6 - 480\alpha^4 - 1152\alpha^2 - 128}{256(1-\alpha^2)^{13/2}}\sin^2\theta \tag{3.88}$$

where again $\alpha = \sqrt{1 - a_0^{-2}}\sin\theta$. Then, the total energy is

$$E_{\text{circular}}^{(\text{f,av})} = \frac{\hbar^2 g_{ae}^2}{8m^2}\left(\frac{23}{3}a_0^8 - \frac{47}{5}a_0^6 + \frac{12}{5}a_0^4\right)\omega_0^4\,. \tag{3.89}$$

Summing Eqs. (3.89) and (3.80) we recover Eq. (3.70) as expected. The latter was derived by averaging over initial and final spin states and therefore we lost information about separate contributions to the total energy. Using the method of Fourier coefficients we find that the major part to the energy (23/24 for $a_0 \gg 1$) comes from the spin flip case. This could be because the typical axion energy is larger for this case. In order to verify this we must study the spectrum of produced axions by calculating numerically Eqs. (3.77) and (3.86) for different values of k/ω_0 at the angle which maximizes axion production as was done for a purely electric field.

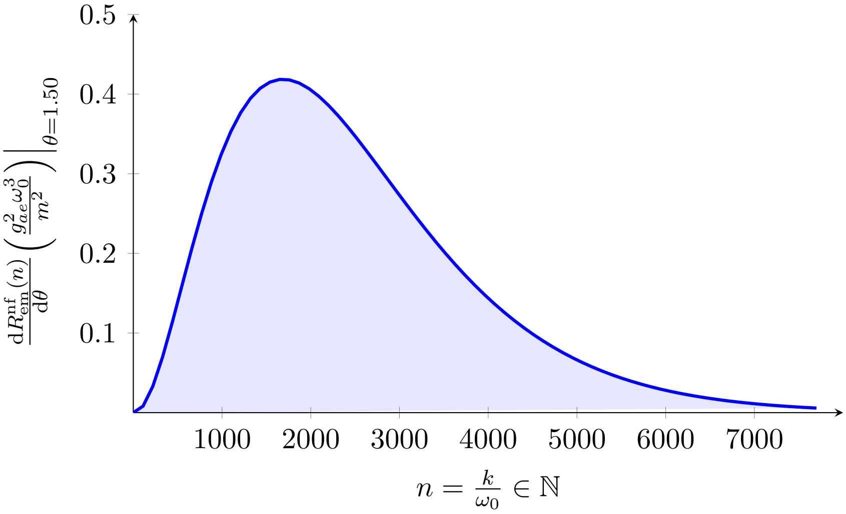

Fig. 3.1: Spectrum of the non-flip rate in the case of a constant magnetic field (two-dimensional trajectory) for $a_0 = 10$ at the angle $\theta = 1.50$ rad.

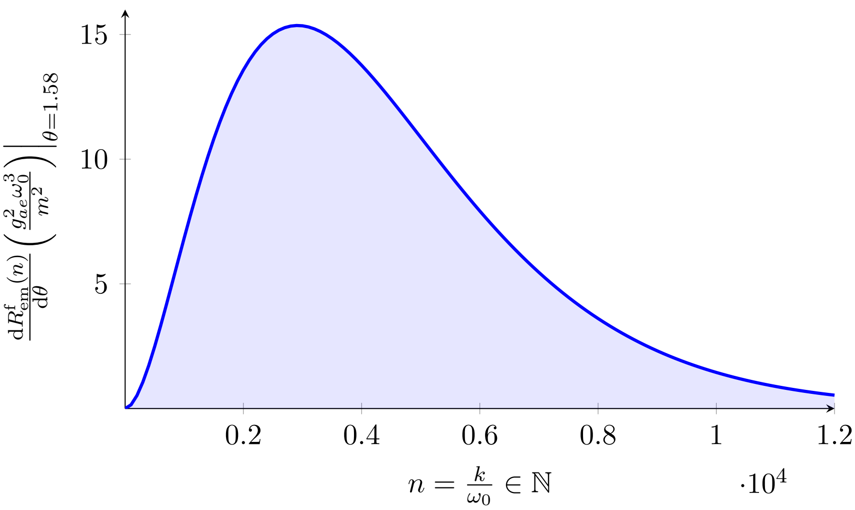

Fig. 3.2: Spectrum of the spin flip (from positive to negative direction with respect to the $y-$axis) rate in the case of a constant magnetic field (two-dimensional trajectory) for $a_0 = 10$ at the angle $\theta = 1.58$ rad.

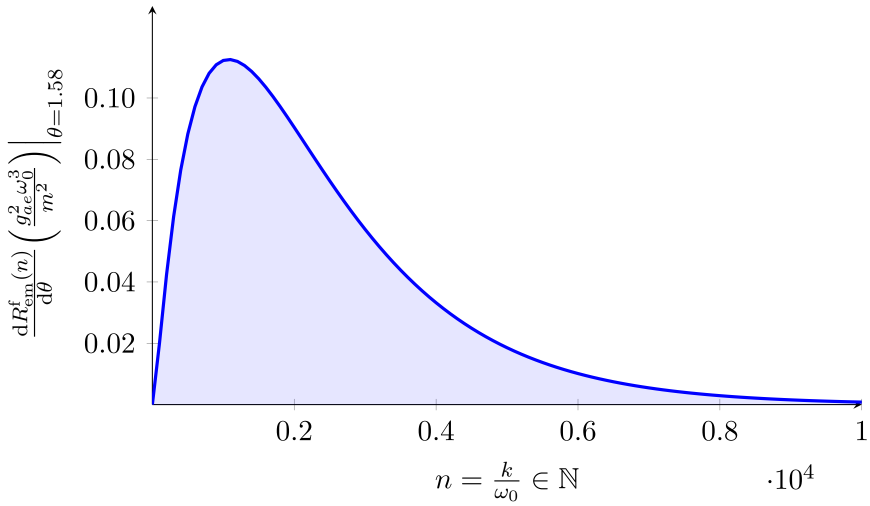

Fig. 3.3: Spectrum of the spin flip (from negative to positive direction with respect to the $y-$axis) rate in the case of a constant magnetic field (two-dimensional trajectory) for $a_0 = 10$ at the angle $\theta = 1.58$ rad.

We plot the differential rates as functions of the discrete energy $n = k/\omega_0$ for the different spin cases. For large n, the rate decays exponentially. When the spin of the electron flips, we must differentiate between the two cases $(+ \to -), (- \to +)$. Qualitatively, we notice that the peak of the distribution is around $\sim a_0^3$ for all cases. We also notice that the case that maximizes the production of axions is the spin-flip case $(+ \to -)$. To find more precisely the typical energy of the emitted axions, we divide Eq. (3.78) by Eq. (3.77) and Eq. (3.87) by Eq. (3.86) for different values of a_0. We find numerically

$$\frac{S^{\mathrm{nf}}_{\mathrm{em}}}{R^{\mathrm{nf}}_{\mathrm{em}}} \approx 2.9 a_0^3 \omega_0 \, , \qquad \frac{S^{\mathrm{f}(-\to+)}_{\mathrm{em}}}{R^{\mathrm{f}(-\to+)}_{\mathrm{em}}} \approx 1.9 a_0^3 \omega_0 \, , \qquad \frac{S^{\mathrm{f}(+\to-)}_{\mathrm{em}}}{R^{\mathrm{f}(+\to-)}_{\mathrm{em}}} \approx 3.6 a_0^3 \omega_0 \, . \tag{3.90}$$

Therefore, in fact, the typical axion energy scales as $\sim a_0^3$ for all three cases. By comparing this result with the acceleration induced by a purely electric field, we observe that the difference in total energy is due to the difference in typical axion energy ($\sim a_0^3$ and $\sim a_0^{1.8}$ in the magnetic and electric case respectively) and the axion number ($\sim a_0^5$ and $\sim a_0^{2.2}$ in the magnetic and electric cases respectively). In the next section, we will consider axion emission from laser fields given by Eq. (2.44). Even if the magnetic field is non-constant but oscillating, we take the typical axion energy of the same order as the one for a constant magnetic field. From the above calculations, $E_a \approx k_a \sim 4a_0^3\omega_0$.

3.4 Experimental proposal

In this section, we use the results of the production of axions from accelerated electrons to propose an experimental setup which will be used to impose limits on the coupling constant g_{ae}. In particular, we consider an electron gas jet accelerated by two counter-propagating laser beams whose electromagnetic fields have the form of Eq. (2.44). Once the axions are produced, they cannot be detected directly and need to be reconverted into photons first. Therefore, following LSW experiments, we propose to shield any form of radiation other than axions (the latter pass through the wall as they are assumed to be weakly interacting). Then, they interact with the electrons of a solid material and are reconverted into photons through the Compton-like process $a + e^- \to \gamma + e^-$. The produced photons can then be detected (see Fig. 3.4).

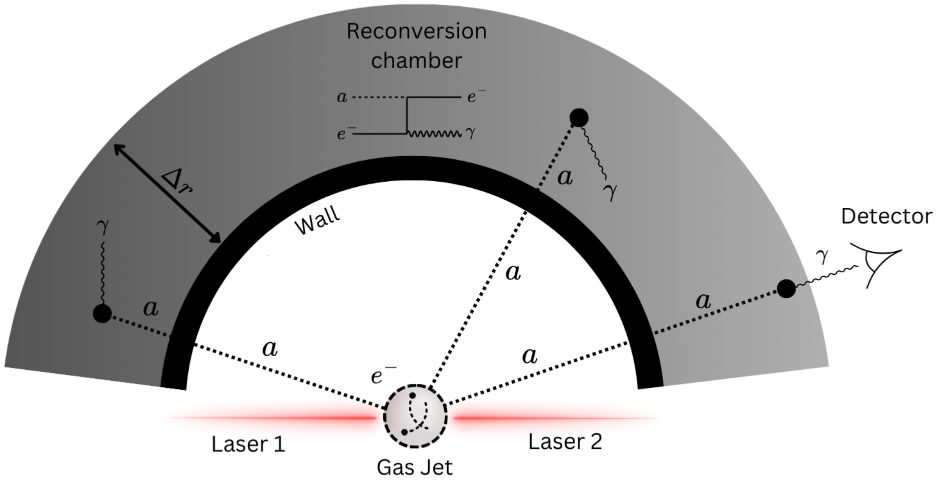

Fig. 3.4: Diagram of the experimental proposal adapted from Ref. [5]. We assume that there are enough detectors to cover a large solid angle.

Throughout this section, we put $\hbar = 1$. The results of the previous section are applicable only if the axion mass m_a is much smaller than the typical axion momentum.

The motion of the electron in the laser beams cannot be found analytically except if they are at the nodes. Therefore, in order to find the energy emitted during one period of $2\pi/\omega_0$ we calculate the trajectories numerically and use Eq. (3.59) to find the energy. This expression depends only on the fields and the velocity and acceleration of the particle. We write the axion energy emitted by one electron during one cycle as $\langle E \rangle_{(2\text{d})} = g_{ae}^2 \omega_0^3 m^{-2} \mathcal{N}(a_0)$ where $\mathcal{N}(a_0)$ is a numerically found number that depends only on a_0 and the initial position of the electron. One can verify numerically that different initial positions give similar values of $\mathcal{N}(a_0)$ except if the electron is at the nodes where $\mathcal{N}(a_0) = a_0^2(7a_0^2 + 1)/6$ using Eq. (2.59).

It is convenient to calculate the total number of particles, but this cannot be done as for a constant magnetic field because the motion of the electron is no longer periodic. As explained in the previous section, we take the typical axion momentum to be $k_a \sim 4a_0^3\omega_0$. Thus, the axion number is estimated as $N_a^{(1)} = \langle E \rangle_{(2\mathrm{d})}/(4a_0^3\omega_0)$. where the superscript (1) indicates that it is the number for one cycle. We can account for multiple electron cycles as follows. Consider that the laser beams meet at the point $x = 0$ and one electron is placed at some point x. The number of cycles of duration $2\pi/\omega_0$ it will feel is given by $\nu(\tau_p - 2|x|)$ where $\nu = \omega_0/2\pi$ and τ_p is the duration of the pulse (we can think of a laser beam as having length τ_p). This implies that electrons located at $|x| > \tau_p/2$ will not feel the effect of the two beams at the same time. Then, instead of one electron, consider a number n_e located near the point x. This number is given by $\rho_e A \mathrm{d}x$ where ρ_e is the electron density and A the beam cross section. To find the total number of cycles, we multiply $\nu(\tau_p - 2|x|)$ by n_e and integrate over x. This gives

$$\rho_e A \nu \int_{-\tau_p/2}^{\tau_p/2} \mathrm{d}x\,(\tau_p - 2|x|) = \frac{(\rho_e A \tau_p)(\nu\tau_p)}{2}\,. \tag{3.91}$$

Multiplying this result by the number of laser shots n_s, we find the total number of axions emitted as

$$N_a^{tot} = \frac{1}{2}(\rho_e A \tau_p)(\nu\tau_p) n_s N_a^{(1)}\,. \tag{3.92}$$

Since for relativistic electrons, axions are emitted along the trajectory, we expect the latter to be in the (xz) plane. In this estimation, we assumed that emission does not depend on the position of electrons, which can be verified numerically. The axions then interact with the electrons of the reconversion material through the Compton-like process $a + e \to \gamma + e^-$ with differential cross section [104]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z g_{ae}^2 \alpha E_\gamma}{8\pi m^2 k_a}\left(1 + \frac{4m^2 E_\gamma^2}{y^2} - \frac{4mE_\gamma}{y} - \frac{4m_a^2 k^2 m E_\gamma \sin^2\theta}{y^3}\right). \tag{3.93}$$

where $y = 2mE_a + m_a^2$. α is the fine structure constant, E_a is the energy of the axion, m_a its mass, and k_a its momentum. The energy of the outgoing photon is

$$E_\gamma = \frac{y}{2(m + E_a - k_a\cos\theta)} \xrightarrow{m_a \to 0} \frac{mk_a}{m + k_a(1-\cos\theta)}\,. \tag{3.94}$$

The total cross section in the case $m_a \approx 0$ is given by

$$\sigma = \frac{Z\alpha g_{ae}^2}{4m^2}\left(\frac{m}{k_a}\ln\left(1 + \frac{2k_a}{m}\right) - 2\frac{1 + 3k_a/m}{(1 + 2k_a/m)^2}\right), \quad m_a \to 0\,. \tag{3.95}$$

We define the function $f(x)$ as

$$f(x) = \frac{1}{x}\ln(1 + 2x) - 2\frac{1 + 3x}{(1 + 2x)^2}\,, \tag{3.96}$$

to write the cross section as $\sigma = Z\alpha g_{ae}^2 f(k_a/m)/4m^2$. For small x, f behaves as $f(x) \approx \frac{8x^2}{3}$. For large x, f is suppressed as $\ln x/x$. Therefore, f reaches its highest value when $x = k_a/m \sim 1$. We calculate the number of photons emitted by the interaction of the produced axions with the electrons of the material by considering an infinitesimal (spherical) volume

$\mathrm{d}V = r^2 \sin\theta \mathrm{d}r \mathrm{d}\phi \mathrm{d}\theta$. The number of photons is related to the cross section via

$$\mathrm{d}N_\gamma = \frac{\sigma \mathrm{d}N_m \mathrm{d}N_a}{\mathrm{d}\Sigma}\,, \tag{3.97}$$

where $\mathrm{d}N_m$ and $\mathrm{d}N_a$ are respectively the number of atoms of the material in $\mathrm{d}V$ and the number of axions. $\mathrm{d}\Sigma = r^2 \sin\theta \mathrm{d}\phi \mathrm{d}\theta$ is the surface of interaction. Writing $\mathrm{d}N_m = \rho_m \mathrm{d}V$ where ρ_m is the density of the material (considered to be constant), we can write the number of photons as

$$\mathrm{d}N_\gamma = \sigma \rho_m \mathrm{d}r \mathrm{d}N_a\,. \tag{3.98}$$

We also write $\mathrm{d}N_a = \mathrm{d}\Omega \frac{\mathrm{d}N_a}{\mathrm{d}\Omega}$. Taking the integral over a sphere, we have

$$N_\gamma = \int \mathrm{d}\Omega \mathrm{d}r\, \sigma \rho_m \frac{\mathrm{d}N_a}{\mathrm{d}\Omega} = \sigma \rho_m \int_{r_{min}}^{r_{max}} \mathrm{d}r \int \mathrm{d}\Omega\, \frac{\mathrm{d}N_a}{\mathrm{d}\Omega}\,. \tag{3.99}$$

The last factor is just the total number of axions emitted (which we calculated numerically). Then, defining $\Delta r = r_{max} - r_{min}$, we have

$$N_\gamma = \sigma \rho_m \Delta r N_a^{tot}\,. \tag{3.100}$$

The distance Δr should be at most of the order of the attenuation length of the photons in the material so they can escape without being absorbed. Inserting Eq. (3.92) into Eq. (3.100), we can express the total number of outgoing photons as

$$N_\gamma = \frac{Z\alpha^2 g_{ae}^4}{8a_0^5}\mathcal{N}(a_0) f\left(\frac{4a_0^3\omega_0}{m}\right) n_s \frac{\rho_m \rho_e \tau_p \Delta r \omega_0 E_{\text{las}}}{m^6}\,, \tag{3.101}$$

where E_{las} is the laser energy per pulse found using $A\tau_p = 8\pi\alpha E_{\text{las}}/(m^2\omega_0^2 a_0^2)$. This number corresponds to the number of detected particles only if the detectors can cover a solid angle of 4π which is very challenging to achieve in experimentally. On the other hand, if an important part of the solid angle is covered by detectors, this difference account for an $\mathcal{O}(1)$ factor. The bounds on g_{ae} are found by requiring $N_\gamma \geq 1$ or $N_\gamma \leq 1$ in the presence or absence of photon detection, respectively. Then, taking the fourth root (since $N_\gamma \propto g_{ae}^4$) of this $\mathcal{O}(1)$ factor will have little impact on the final result. We note that photons emitted inwards also will not reach the detector. Again, we assume that this accounts for a $\mathcal{O}(1)$ factor.

The properties of the lasers are the following. The energy, wavelength, pulse and beam diameter are given by $E_{\text{las}} = 1$ kJ, $\lambda \approx 1\,\mu\text{m}$, $\tau_p = 10^{-12}$ s, $D \sim 10\lambda$. We also assume that the repetition rate is 10 Hz. Although a laser of this type is not currently available, this is within current technological capabilities. Then, $a_0 \approx 30$, and the photon and axion energies are given by $E_\gamma \approx k_a \approx 133$ keV. We also find numerically $\mathcal{N}(a_0 = 30) \approx 10^{13}$. We verified this numerical result by averaging over ten oscillation periods and with multiple initial electron positions. We note that for an electric field $\mathcal{N}(a_0 = 30) \approx 10^6$, while for a pure magnetic field $\mathcal{N}(a_0 = 30) \approx 7 \cdot 10^{11}$. Therefore, indeed the presence of the magnetic field significantly increases particle production. The source of electrons is a hydrogen gas with density

$\rho_e = 10^{20}$ cm^{-3}.

For reconversion, we use aluminum, and the attenuation length for these photon energies is around $\Delta r \sim 1$ cm. After one week of measurement, in the absence of photon detection, we find $g_{ae} \lesssim 4.1 \times 10^{-5}$. Next-generation lasers could reach $E_{\text{las}} = 10^2$ kJ, $\tau_p = 10^{-10}$ and a repetition rate of 1 MHz. This is not currently available, but with the rapid development of diode laser technology, it could become in the future. Keeping the remaining parameters identical and considering one year of measurements, $g_{ae} \lesssim 8.5 \times 10^{-8}$. These results are valid as long as $m_a \ll k_a \sim 10^2$ keV.

We compare these results with other purely laboratory experiments in Fig. 3.5. The authors in Ref. [33] used anomalous magnetic moment, whereas the authors in Ref. [32] assumed axion production using a nuclear reactor for neutrino experiments. The bounds on g_{ae} are valid for m_a up to 10 keV. Current laser technology allows us to find bounds similar to those found in the above-mentioned techniques. Furthermore, a future laser system could probe the parameter space of the DFSZ axion [26, 27] (see Fig. 3.5).

The axions can also be converted into photons through the process $a + N_T \to \gamma + N_T$ where N_T is an atomic target. This process involves the coupling $g_{a\gamma}$. In the absence of photon detection, it allows us to impose upper bounds on the product $g_{ae}g_{a\gamma}$. We do not consider this conversion process.

The axions can also decay into photons before interacting with the electrons of the material. The decay rate for this process is given by $\Gamma(a \to \gamma\gamma) = g_{a\gamma}^2 m_a^3/(64\pi)$ [105]. The survival probability for the axion to reach the recon-

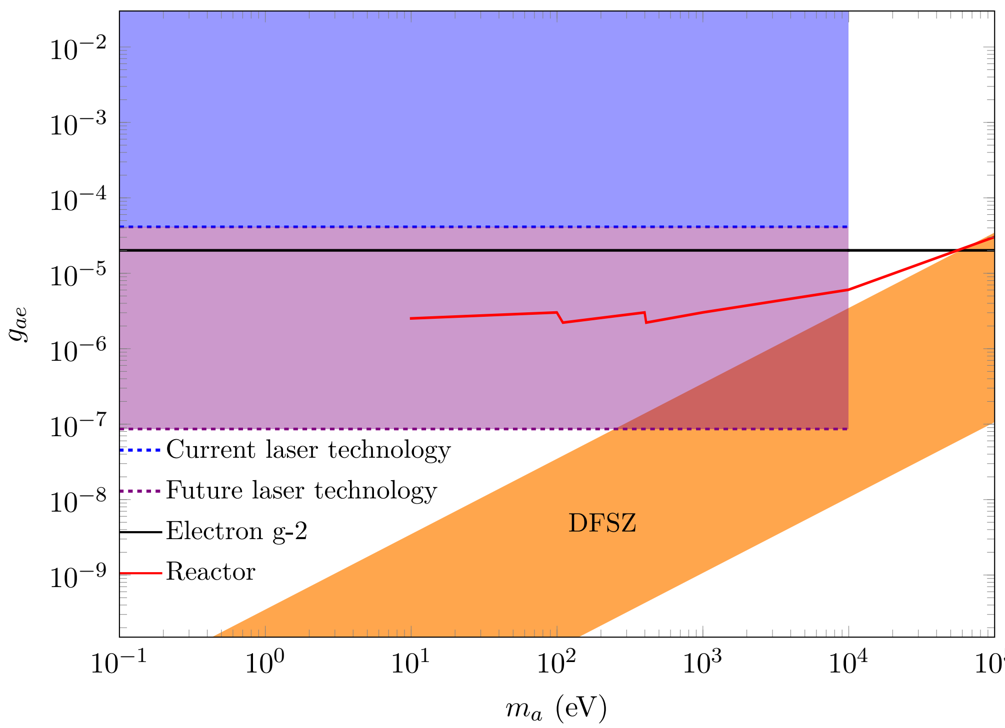

FIG. 3.5: Bounds on g_{ae} of laboratory based experiments. The figure is adapted from Ref. [5]. The red and black curves correspond to exclusion bounds imposed in Ref. [32] and Ref. [33] respectively. For the next generation lasers, the duration of measurement is one year instead of one week. A part of the parameter space of the DFSZ axion [26, 27] is also shown.

version chamber at a distance ℓ is $P_{\text{sur}} = \exp[-\ell m_a \Gamma/k_a]$ [32]. For $\ell \lesssim 1$ m, $m_a \lesssim 10$ keV and $g_{a\gamma} \lesssim 10^{-4}$ GeV^{-1} which follows from Ref. [32], $P_{\text{sur}} \approx 1$. Therefore, all axions reach the conversion chamber. The bound we used for $g_{a\gamma}$ is model-independent.

4 The Unruh Effect and Scalar Particle Emission

After discussing particle emission from accelerating charges in the inertial frame, we now look at radiation in the accelerated frame. To simplify calculations, we will consider scalar fields in this chapter and vector fields in Chap. 5. The electron trajectory is uniform acceleration, which is well described by the Rindler coordinates. To understand radiation from an accelerated charge in this context, it is necessary to take into account the Unruh effect. This chapter is organized as follows. Firstly, we will look at general concepts of QFT in curved space as they will be useful to understand the Unruh effect itself despite the fact that it takes place in flat space. Later, we show parts of the derivation of the Unruh effect, and finally we calculate the emission rate of scalars by the charge as well as the power in the rest frame of the charge. Sec. 4.1 and Sec. 4.3 are reviews based on Refs. [39, 57].

4.1 QFT in curved space

Before we discuss QFT in a curved space, we will briefly describe the special case of flat space. Our discussion will use scalar fields as an illustration, but the results can be generalized to other types of fields. The fields are quantum operators and are expanded in terms of modes. The modes are solutions to the equations of motion and form a complete set with respect to a scalar product that will be defined in what follows. We define positive frequency modes as the ones that are proportional to $\mathrm{e}^{-ik_0 t}$, while negative frequency modes are proportional to $\mathrm{e}^{+ik_0 t}$ where $k_0 > 0$ is the energy of the given mode and t is the time parameter. The reason why we can always define the Minkowski modes in this way is the following. Time translation is a symmetry in flat space, which implies that the time parameter does not appear explicitly in the equations of motion. However, they contain only a second derivative of time. Therefore, in Minkowski spacetime, we can always choose the modes to be proportional to $\mathrm{e}^{-ik_0 t}$ or to $\mathrm{e}^{+ik_0 t}$. The coefficient operators of the positive-frequency modes are called annihilation operators, whereas the ones of the negative-frequency modes are called creations operators. The vacuum, which can be thought of as the absence of particles, is defined as the state annihilated by all annihilation operators.

In a general spacetime with no isometries, one cannot choose the modes as in flat space. By choosing different sets of modes, one chooses different operators and, therefore, different vacuum states. As a consequence, the concept of particles becomes difficult to define. There is, however, a special case of curved space where one can still define a preferred vacuum state: a spacetime which is globally hyperbolic (i.e. it possesses a Cauchy surface) and whose metric is of the form [57]

$$\mathrm{d}s^2 = N(x)^2\mathrm{d}t^2 - G_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j, \tag{4.1}$$

where the functions N and G do not depend on the time parameter t. The spacetime is said to be static. In this case one can choose the modes to be proportional to $\mathrm{e}^{-i\omega_i t}$ where $\omega_i > 0$ is interpreted as the energy of the mode. Let us consider a general space time with metric $g_{\mu\nu}$ and a free massive scalar field. The Lagrangian of this theory is

$$\frac{\mathcal{L}_{\text{scalar}}}{\sqrt{-g}} = \frac{1}{2}\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{2}m^2\Phi^2. \tag{4.2}$$

The equations of motion are given by

$$\frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\big) + m^2\Phi = 0, \tag{4.3}$$

where $g = \det(g_{\mu\nu})$. We define the Klein-Gordon inner product for two solutions f_1 and f_2 to the equations of motion as [39, 57]

$$(f_1, f_2)_{\text{KG}} = i\int_\Sigma \mathrm{d}^3\mathbf{x}\,\sqrt{g_\Sigma}n_\mu(f_1^*(x)\nabla^\mu f_2(x) - f_2(x)\nabla^\mu f_1^*(x)), \tag{4.4}$$

where Σ is a hypersurface of constant time parameter. g_Σ is the metric determinant restricted to Σ and n_μ is a future-directed unit vector normal to Σ. We assume that there is a complete set of solutions $\{f_i, f_i^*\}$ that satisfy

$$(f_i, f_j)_{\mathrm{KG}} = -\left(f_i^*, f_j^*\right)_{\mathrm{KG}} = \delta_{ij} \qquad \left(f_i, f_j^*\right)_{\mathrm{KG}} = (f_i^*, f_j)_{\mathrm{KG}} = 0. \tag{4.5}$$

We labelled the modes using discrete indices for simplicity. The scalar field can be expanded in terms of the set $\{f_i, f_i^*\}$ as

$$\Phi = \sum_i \left[a_i f_i + a_i^\dagger f_i^*\right] \tag{4.6}$$

and we impose the commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \quad [a_i, a_j] = \left[a_i^\dagger, a_j^\dagger\right] = 0. \tag{4.7}$$

The operators a_i and $a_i^\dagger$ are the coefficients mentioned above that are associated to the modes and are known as the annihilation and creation operators, respectively. The state annihilated by a_i for all i is the vacuum state denoted by $|0\rangle$. Since the choice of f_i determines the operators a_i, the vacuum state depends on the choice of modes. Moreover, since in a general spacetime there is no preferred way to choose the modes f_i as is the case in Minkowski spacetime, a given vacuum state associated with a choice of modes may appear to be a state containing particles for another choice of modes. To see this, consider two complete sets of modes $\{f_i^{(1)}\}$ and $\{f_I^{(2)}\}$. Since the two sets are complete, there exist coefficients α_{Ii} and β_{Ii} known as the Bogoliubov coefficients such that [39]

$$f_I^{(2)} = \sum_i \left[\alpha_{Ii} f_i^{(1)} + \beta_{Ii} f_i^{(1)*} \right]. \tag{4.8}$$

The Bogoliubov coefficients are found explicitly as

$$\begin{aligned} \alpha_{Ii} &= \left(f_i^{(1)}, f_I^{(2)} \right)_{\mathrm{KG}} = \left(f_I^{(2)}, f_i^{(1)} \right)^*_{\mathrm{KG}}, \\ \beta_{Ii} &= -\left(f_i^{(1)*}, f_I^{(2)} \right)_{\mathrm{KG}} = \left(f_I^{(2)*}, f_i^{(1)} \right)_{\mathrm{KG}}. \end{aligned} \tag{4.9}$$

The relation between the modes of the two different sets leads to a relation between the two sets of operators. To see this, we expand the scalar field as

$$\Phi = \sum_i \left[a_i^{(1)} f_i^{(1)} + a_i^{(1)\dagger} f_i^{(1)*} \right] = \sum_I \left[a_I^{(2)} f_I^{(2)} + a_I^{(2)\dagger} f_I^{(2)*} \right]. \tag{4.10}$$

By taking the inner product of Eq. (4.10) with $f_i^{(1)}$ we find

$$a_i^{(1)} = \sum_I \left[\alpha_{Ii} a_I^{(2)} + \beta_{Ii}^* a_I^{(2)\dagger} \right]. \tag{4.11}$$

Similarly, by taking the inner product of Eq. (4.10) with $f_I^{(2)}$ we find

$$a_I^{(2)} = \sum_i \left[\alpha_{Ii}^* a_i^{(1)} - \beta_{Ii}^* a_i^{(1)\dagger} \right]. \tag{4.12}$$

If the Bogoliubov coefficients β_{Ii} are non-zero, then creation and annihilation operators of the two sets mix. We label the vacua of the two sets $|0_{(1)}\rangle$ and $|0_{(2)}\rangle$. Then by definition $a_i^{(1)} |0_{(1)}\rangle = 0$ and $a_I^{(2)} |0_{(2)}\rangle = 0$ for all i and I. The number operators are defined as $N_i^{(1)} = a_i^{(1)\dagger} a_i^{(1)}$ and $N_I^{(2)} = a_I^{(2)\dagger} a_I^{(2)}$ with no summation over the indices. One can calculate the following expectation values

$$\langle 0_{(2)}| \, N_i^{(1)} \, |0_{(2)}\rangle = \sum_I |\beta_{Ii}|^2, \quad \langle 0_{(1)}| \, N_I^{(2)} \, |0_{(1)}\rangle = \sum_i |\beta_{Ii}|^2. \tag{4.13}$$

Thus, if the Bogoliubov coefficients β_{Ii} do not vanish then the expectation value of the number operator does not vanish if it is evaluated using the vacuum state which is associated with a different choice of modes. However as mentioned above, if the spacetime is globally hyperbolic and static, i.e. with metric of the form Eq. (4.1), then there is a preferred vacuum, called static, which is associated to the modes proportional to $\mathrm{e}^{-i\omega_i t}$ with $\omega_i > 0$.

4.2 Rindler wedges

The concepts we introduced in the previous section apply to the usual (flat) Minkowski spacetime whose metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2. \tag{4.14}$$

Being a static and globally hyperbolic spacetime, it has a natural static vacuum state. This is the familiar vacuum state used in QFT. Here, it is denoted by $|0_\mathrm{M}\rangle$.

To discuss QFT in the case of accelerated observers, we introduce the following coordinate transformation

$$t = a^{-1}\mathrm{e}^{a\xi}\sinh a\tau, \quad z = a^{-1}\mathrm{e}^{a\xi}\cosh a\tau, \tag{4.15}$$

where $a > 0$ is a constant. The Minkowski metric in Eq. (4.14) then takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{2a\xi}\left(\mathrm{d}\tau^2 - \mathrm{d}\xi^2\right) - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{4.16}$$

The coordinates $(\tau, \xi) \in \mathbb{R}^2$ only cover the part of Minkowski spacetime defined by $z > |t|$ known as the right Rindler wedge (RRW) (see Fig. 4.1). Rindler coordinates are suitable for describing uniformly accelerating observers. In fact, the proper acceleration of world lines with ξ, x, y being constants is given by $a\mathrm{e}^{-a\xi}$. We note that a world line for which $\xi = 0$ has proper acceleration a. The non-vanishing Christoffel symbols in the RRW are

$$\Gamma^{\tau}_{\ \tau\xi} = \Gamma^{\xi}_{\ \xi\xi} = \Gamma^{\xi}_{\ \tau\tau} = a\,. \tag{4.17}$$

Similarly, one can define the coordinates $(\bar{\tau}, \bar{\xi}) \in \mathbb{R}^2$ as

$$t = a^{-1}\mathrm{e}^{a\bar{\xi}}\sinh a\bar{\tau}, \quad z = -a^{-1}\mathrm{e}^{a\bar{\xi}}\cosh a\bar{\tau}. \tag{4.18}$$

The coordinates $(\bar{\tau}, \bar{\xi}) \in \mathbb{R}^2$ cover the part of Minkowski spacetime defined by $z < -|t|$ known as the left Rindler wedge (LRW). Both the RRW and the LRW are static globally hyperbolic spacetimes. Therefore, they also have a natural vacuum state denoted by $|0_{\mathrm{R}}\rangle$. The regions $t > |z|$ and $t < -|z|$ are called expanding and contracting Kasner universes, respectively. We will not discuss them in what follows.

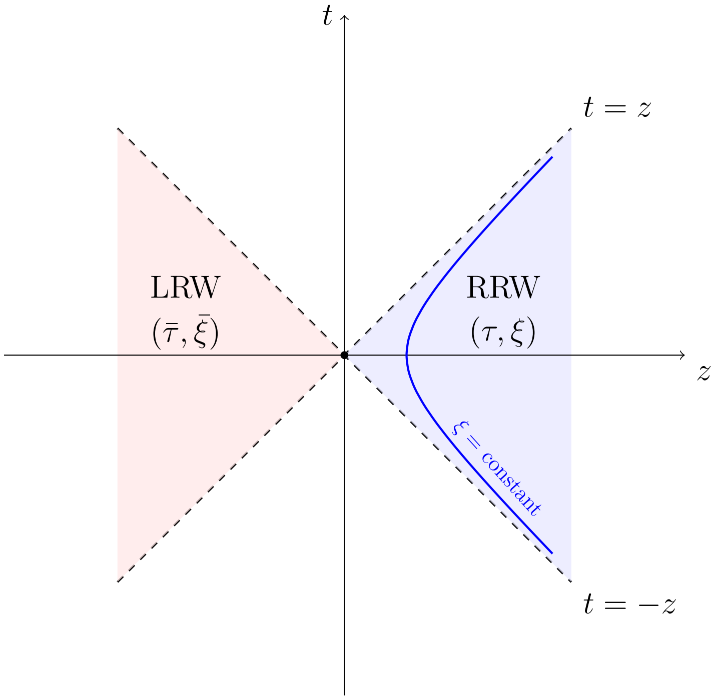

Fig. 4.1: Spacetime diagram showing the Left and Right Rindler wedges (LRW and RRW, respectively), bounded by the lightlike lines $t = \pm z$.

4.3 Derivation of the Unruh effect for scalar particles

The theory of a free massive scalar field is described by Eq. (4.2). In Minkowski coordinates we have $\sqrt{-g} = 1$ and $\nabla_\mu \to \partial_\mu$. In these coordinates, the equation of motion is the usual Klein-Gordon equation.

$$\left(\partial^\mu \partial_\mu + m^2\right)\Phi = 0. \tag{4.19}$$

The quantized scalar field can therefore be expanded as

$$\Phi(x) = \int \mathrm{d}^3\mathbf{k}\, \Big[a_{\mathbf{k}}^M \phi_{\mathbf{k}} + a_{\mathbf{k}}^{M\dagger} \phi_{\mathbf{k}}^*\Big], \tag{4.20}$$

where the modes $\phi_{\mathbf{k}}$ are solutions to the equation of motion and are given by

$$\phi_{\mathbf{k}}(x) = [(2\pi)^3 2k_0]^{-1/2} \mathrm{e}^{-ik_0 t + i\mathbf{k}\cdot\mathbf{x}}. \tag{4.21}$$

The normalization is found using the Klein-Gordon inner product in flat space

$$(f, g)_{\mathrm{KG}} = i \int \mathrm{d}^3\mathbf{x}\, (f^*(x)\partial_t g(x) - g(x)\partial_t f^*(x)) \tag{4.22}$$

and imposing for the Minkowski modes

$$(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}) = \delta^{(3)}(\mathbf{k} - \mathbf{k}')\,. \tag{4.23}$$

The operators $a_{\mathbf{k}}^M$ annihilate the Minkowski vacuum $a_{\mathbf{k}}^M \left|0_{\mathrm{M}}\right\rangle = 0$ and satisfy the commutation relations

$$[a_{\mathbf{k}}^M, a_{\mathbf{k}'}^{M\dagger}] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')\,, \tag{4.24}$$

with all other commutators vanishing. In Rindler coordinates defined in Eq. (4.15), $\sqrt{-g} = \mathrm{e}^{2a\xi}$, and the equation of motion is of the form

$$\big(\partial_\tau^2 - \partial_\xi^2 - \mathrm{e}^{2a\xi}(\partial_x^2 - \partial_y^2 - m^2)\big)\Phi = 0\,. \tag{4.25}$$

Since $\partial_\tau, \partial_\xi$ and ∂_y are Killing vectors, the positive-frequency modes will be proportional to $\mathrm{e}^{-i\omega\tau + i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}$ where $\mathbf{k}_\perp = (k_x, k_y) \in \mathbb{R}^2$, $\mathbf{x}_\perp = (x, y)$ and ω is a positive constant. It corresponds to the Rindler energy and is not related to the momentum variables via a dispersion relation. Explicitly the modes are [57]

$$v^R_{\omega\mathbf{k}_\perp}(\tau,\xi,\mathbf{x}_\perp) = \sqrt{\frac{\sinh \pi\omega/a}{4\pi^4 a}} K_{i\omega/a}\left(\frac{\kappa \mathrm{e}^{a\xi}}{a}\right)\mathrm{e}^{-i\omega\tau + i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\,, \tag{4.26}$$

where $\kappa = \sqrt{k_\perp^2 + m^2}$ with $k_\perp = |\mathbf{k}_\perp|$ and $K_\nu(z)$ is the modified Bessel function of the second kind. The prefactor guarantees that

$$\left(v^R_{\omega\mathbf{k}_\perp}, v^R_{\omega'\mathbf{k}'_\perp}\right)_{\mathrm{KG}} = \delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\,. \tag{4.27}$$

The scalar field can be expanded in the RRW as

$$\Phi^R(x) = \int_0^{+\infty} \mathrm{d}\omega \int \mathrm{d}^2\mathbf{k}_\perp \Big[a^R_{\omega\mathbf{k}_\perp} v^R_{\omega\mathbf{k}_\perp} + a^{R\dagger}_{\omega\mathbf{k}_\perp} v^{R*}_{\omega\mathbf{k}_\perp}\Big], \tag{4.28}$$

where the operators satisfy the commutation relation

$$\Big[a^R_{\omega\mathbf{k}_\perp}, a^{R\dagger}_{\omega'\mathbf{k}'_\perp}\Big] = \delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp) \tag{4.29}$$

and all other commutators vanish. The quantization of the scalar field in the LRW proceeds in the same way. The orthonormal modes $v^L_{\omega,\mathbf{k}_\perp}(\bar{\tau},\bar{\xi},\mathbf{x}_\perp)$ are obtained from the right ones $v^R_{\omega,\mathbf{k}_\perp}(\tau,\xi,\mathbf{x}_\perp)$ in Eq. (4.26) by replacing $\tau\to\bar{\tau}$ and $\xi\to\bar{\xi}$. The field can be expanded in the two Rindler wedges as

$$\begin{aligned}\Phi &= \Phi^R+\Phi^L\\ &= \int_0^{+\infty}\mathrm{d}\omega\int\mathrm{d}^2\mathbf{k}_\perp\left[a^R_{\omega\mathbf{k}_\perp}v^R_{\omega\mathbf{k}_\perp}+a^{R\dagger}_{\omega\mathbf{k}_\perp}v^{R*}_{\omega\mathbf{k}_\perp}+a^L_{\omega\mathbf{k}_\perp}v^L_{\omega\mathbf{k}_\perp}+a^{L\dagger}_{\omega\mathbf{k}_\perp}v^{L*}_{\omega\mathbf{k}_\perp}\right],\end{aligned} \tag{4.30}$$

where the left operators also satisfy

$$\left[a^L_{\omega\mathbf{k}_\perp},a^{L\dagger}_{\omega'\mathbf{k}'_\perp}\right]=\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp-\mathbf{k}'_\perp)\,. \tag{4.31}$$

The Rindler vacuum is annihilated by the operators $a^R_{\omega\mathbf{k}_\perp}, a^L_{\omega\mathbf{k}_\perp}$: $a^R_{\omega\mathbf{k}_\perp}\left|0_\mathrm{R}\right\rangle = a^L_{\omega\mathbf{k}_\perp}\left|0_\mathrm{R}\right\rangle=0$. for all ω and all $\mathbf{k}_\perp$. The operator in Eq. (4.30) is defined only in the two Rindler wedges. Nonetheless, the modes $v^R_{\omega\mathbf{k}_\perp}$ and $v^L_{\omega\mathbf{k}_\perp}$ can naturally be expanded to all of Minkowski spacetime as will be shown in what follows. Then Eq. (4.30) is a mode expansion equivalent to Eq. (4.20). To derive the Unruh effect we will express the Rindler modes as functions of Minkowski modes.

$$v^R_{\omega\mathbf{k}_\perp}=\int\mathrm{d}^3\mathbf{k}'\left[\alpha^R_{\omega\mathbf{k}'\mathbf{k}_\perp}\phi_{\mathbf{k}'}+\beta^R_{\omega\mathbf{k}'\mathbf{k}_\perp}\phi^*_{\mathbf{k}'}\right], \tag{4.32}$$

where $\alpha^R_{\omega\mathbf{k}'\mathbf{k}_\perp}$ and $\beta^R_{\omega\mathbf{k}'\mathbf{k}_\perp}$ are the Bogoliubov coefficients. Since $v^R_{\omega\mathbf{k}_\perp}\propto \mathrm{e}^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}$, we must have $\alpha^R_{\omega\mathbf{k}'\mathbf{k}_\perp}\propto\delta^{(2)}(\mathbf{k}_\perp-\mathbf{k}'_\perp)$ and $\beta^R_{\omega\mathbf{k}'\mathbf{k}_\perp}\propto\delta^{(2)}(\mathbf{k}_\perp+\mathbf{k}'_\perp)$. By naming the proportionality coefficients as $\alpha^R_{\omega k'_z k_\perp}$ and $\beta^R_{\omega k'_z k_\perp}$ respectively, we arrive at

$$v^R_{\omega \mathbf{k}_\perp} = \int \frac{\mathrm{d}k_z}{\sqrt{(2\pi)^3 2k_0}} \mathrm{e}^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \left[\alpha^R_{\omega k_z k_\perp} \mathrm{e}^{-ik_0 t + ik_z z} + \beta^R_{\omega k_z k_\perp} \mathrm{e}^{ik_0 t - ik_z z} \right] , \tag{4.33}$$

where we renamed the integration variable $k'_z \to k_z$ to simplify notation. As we will see the Bogoliubov coefficients do not depend on the direction of $\mathbf{k}_\perp$ but on its modulus. The same can be done for the left Rindler modes

$$v^L_{\omega \mathbf{k}_\perp} = \int \frac{\mathrm{d}k_z}{\sqrt{(2\pi)^3 2k_0}} \mathrm{e}^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \left[\alpha^L_{\omega k_z k_\perp} \mathrm{e}^{-ik_0 t + ik_z z} + \beta^L_{\omega k_z k_\perp} \mathrm{e}^{ik_0 t - ik_z z} \right] . \tag{4.34}$$

The left and the right Bogoliubov coefficients are related to each other. To see this, we make the observation that from EQS. (4.15) and (4.18) it follows that the relation between (τ, ξ) and (t, z) is the same as the relation between $(\bar{\tau}, \bar{\xi})$ and $(t, -z)$. Therefore $v^L_{\omega \mathbf{k}_\perp}$ can be obtained from $v^R_{\omega \mathbf{k}_\perp}$ by the transformation $z \to -z$. It follows that

$$\alpha^L_{\omega k_z k_\perp} = \alpha^R_{\omega - k_z k_\perp} , \quad \beta^L_{\omega k_z k_\perp} = \beta^R_{\omega - k_z k_\perp} . \tag{4.35}$$

Then, it is sufficient to calculate only the right Bogoliubov coefficients. The Bogoliubov coefficients are coordinate independent allowing us to find a convenient surface on which to evaluate the modes to simplify calculations. This surface is the Killing horizon $t = z, t > 0$ which corresponds in Rindler coordinates to $\xi \to -\infty$. The derivation of the Bogoliubov coefficients can be found in Ref. [57]. The result is

$$\alpha^R_{\omega k_z k_\perp} = \alpha^L_{\omega -k_z k_\perp} = \frac{\mathrm{e}^{-i\omega\vartheta(k_z)}}{\sqrt{2\pi a k_0(1-\mathrm{e}^{-2\pi\omega/a})}}\,, \tag{4.36}$$

where we introduced the rapidity

$$\vartheta(k_z) = \frac{1}{2a}\ln\frac{k_0+k_z}{k_0-k_z}\,. \tag{4.37}$$

Inverting this relation gives

$$k_z = \kappa \sinh a\vartheta\,, \quad k_0 = \kappa\cosh a\vartheta\,. \tag{4.38}$$

Similarly, the β-coefficients are

$$\beta^R_{\omega k_z k_\perp} = \beta^L_{\omega -k_z k_\perp} = -\frac{\mathrm{e}^{-\pi\omega/a}\mathrm{e}^{-i\omega\vartheta(k_z)}}{\sqrt{2\pi a k_0(1-\mathrm{e}^{-2\pi\omega/a})}}. \tag{4.39}$$

What is important here is not the explicit expressions themselves but the following relation between the α and β coefficients

$$\beta^R_{\omega k_z k_\perp} = -\mathrm{e}^{-\pi\omega/a}\alpha^{L*}_{\omega k_z k_\perp}\,, \quad \beta^L_{\omega k_z k_\perp} = -\mathrm{e}^{-\pi\omega/a}\alpha^{R*}_{\omega k_z k_\perp}\,. \tag{4.40}$$

Since $\alpha_{\omega k_z k_\perp}$ and $\beta_{\omega k_z k_\perp}$ are linked to positive and negative frequency modes respectively and because of the linear relations of Eq. (4.40), there exists two linear combinations of $v^R_{\omega\mathbf{k}_\perp}$ and $v^L_{\omega\mathbf{k}_\perp}$ that do not have a negative frequency contributions. These combination are

$$w_{-\omega\mathbf{k}_\perp} = \frac{v^R_{\omega\mathbf{k}_\perp} + \mathrm{e}^{-\pi\omega/a} v^{L*}_{\omega-\mathbf{k}_\perp}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,, \quad w_{+\omega\mathbf{k}_\perp} = \frac{v^L_{\omega\mathbf{k}_\perp} + \mathrm{e}^{-\pi\omega/a} v^{R*}_{\omega-\mathbf{k}_\perp}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,. \tag{4.41}$$

where the factor $(1-\mathrm{e}^{-2\pi\omega/a})^{-1/2}$ is for normalization:

$$\begin{aligned} \left(w_{\pm\omega\mathbf{k}_\perp}, w_{\pm\omega'\mathbf{k}'_\perp}\right)_{\mathrm{KG}} &= \delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\,, \\ \left(w^*_{\pm\omega\mathbf{k}_\perp}, w^*_{\pm\omega'\mathbf{k}'_\perp}\right)_{\mathrm{KG}} &= -\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\,. \end{aligned} \tag{4.42}$$

The modes $w_{\pm\omega\mathbf{k}_\perp}$ are called purely positive-frequency because their expansion involves only the Minkowski modes $\phi_{\mathbf{k}}$ and not their complex conjugate $\phi^*_{\mathbf{k}}$. Explicitly

$$w_{\pm\omega\mathbf{k}_\perp} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}} \mathrm{e}^{\pm i\vartheta(k_z)\omega} \phi_{\mathbf{k}}\,. \tag{4.43}$$

This means that the full scalar field Φ can be expanded in terms of these modes as

$$\Phi(x) = \int_0^{+\infty} \mathrm{d}\omega \int \mathrm{d}^2\mathbf{k}_\perp \left[w_{-\omega\mathbf{k}_\perp} a_{(-,\omega,\mathbf{k}_\perp)} + w_{+\omega\mathbf{k}_\perp} a_{(+,\omega,\mathbf{k}_\perp)} + \mathrm{h.c.}\right]\,, \tag{4.44}$$

with the operators $a_{(-,\omega,\mathbf{k}_\perp)}$ and $a_{(+,\omega,\mathbf{k}_\perp)}$ annihilating the Minkowski vacuum $|0_{\mathrm{M}}\rangle$. They are related to Rindler operators as

$$a^R_{\omega\mathbf{k}_\perp} = \frac{a_{(-,\omega,\mathbf{k}_\perp)} + \mathrm{e}^{-\pi\omega/a} a^\dagger_{(+,\omega,-\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}, \quad a^L_{\omega\mathbf{k}_\perp} = \frac{a_{(+,\omega,\mathbf{k}_\perp)} + \mathrm{e}^{-\pi\omega/a} a^\dagger_{(-,\omega,-\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,. \tag{4.45}$$

These relations can be inverted:

$$a_{(-,\omega,\mathbf{k}_\perp)} = \frac{a^R_{\omega\mathbf{k}_\perp} - \mathrm{e}^{-\pi\omega/a} a^{L\dagger}_{\omega-\mathbf{k}_\perp}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} , \quad a_{(+,\omega,\mathbf{k}_\perp)} = \frac{a^L_{\omega\mathbf{k}_\perp} - \mathrm{e}^{-\pi\omega/a} a^{R\dagger}_{\omega-\mathbf{k}_\perp}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} . \tag{4.46}$$

From here we can deduce that

$$\left(a^R_{\omega\mathbf{k}_\perp} - \mathrm{e}^{-\pi\omega/a} a^{L\dagger}_{\omega-\mathbf{k}_\perp}\right) |0_\mathrm{M}\rangle = 0 , \quad \left(a^L_{\omega\mathbf{k}_\perp} - \mathrm{e}^{-\pi\omega/a} a^{R\dagger}_{\omega-\mathbf{k}_\perp}\right) |0_\mathrm{M}\rangle = 0 . \tag{4.47}$$

Thus, the expectation values of the number operators can be calculated explicitly using Eq. (4.47) and the commutation relations Eqs. (4.29) and (4.31):

$$\langle 0_\mathrm{M}| \, a^{R\dagger}_{\omega\mathbf{k}_\perp} a^R_{\omega'\mathbf{k}'_\perp} \, |0_\mathrm{M}\rangle = \frac{1}{\mathrm{e}^{2\pi\omega/a} - 1} \delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp) , \tag{4.48}$$

and the same holds for $a^{L\dagger}_{\omega\mathbf{k}_\perp} a^L_{\omega'\mathbf{k}'_\perp}$. This result summarizes the Unruh effect. The Minkowski vacuum appears to be a thermal state of temperature $T_U = a/2\pi$ in the Rindler wedges. The first factor in Eq. (4.48) is the Bose-Einstein distribution with temperature T_U. The relation between the Minkowski and Unruh creation operators is the same as for the modes in Eq. (4.43) (see App. D). Therefore,

$$a^\dagger_{(\pm,\omega,\mathbf{k}_\perp)} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}} \mathrm{e}^{\pm i\vartheta(k_z)\omega} a^{M\dagger}_{\mathbf{k}} . \tag{4.49}$$

We also note that the modes the modes $w_{\pm\omega\mathbf{k}_\perp}$ were defined initially only on the RRW and LRW. Thus they were note defined on the plane $t = z = 0$. But using Eq. (4.43), we see that they can be expressed as distributions over all of Minkowski spacetime. Then, the quantity defined by integrating the product of $w_{\pm\omega\mathbf{k}_\perp}$ with a compactly supported smooth function over all of Minkowski spacetime is well defined. Moreover, the Wightman two-point function is correctly reproduced if we use Rindler modes instead of Minkowksi modes [57].

4.4 Interaction of a charge with the FDU thermal bath

So far we have only discussed the presence of the FDU thermal bath for accelerated observers. An accelerating charge, with proper acceleration a, can interact with this bath by absorbing or emitting a Rindler particle (at tree level). The interaction in the RRW can be described by the action

$$S_I^{\text{scalar}} = -\int \mathrm{d}^4x\,\sqrt{-g}j(x)\Phi^R(x)\,, \tag{4.50}$$

where $\Phi^R(x)$ is given by Eq. (4.28) and the scalar classical "current" is $j(x) = q\delta(\xi)\delta^{(2)}(\mathbf{x}_\perp)$ where q is the charge of the particle. The factors $\delta(\xi)$ and $\delta^{(2)}(\mathbf{x}_\perp)$ ensure that the particle is uniformly accelerating along the $z-$axis and located at the origin of the (xy) plane, respectively. In its current state, the interaction Eq. (4.50) describes a charge that accelerates and interacts with the FDU forever. It would be ideal to consider this interaction to be of finite duration, but this would complicate the calculations. Instead, one can consider a charging and de-charging process through a wire from $\xi = 0$ to $+\infty$. To achieve that, we can make the charge time-dependent through the replacement $q \to qF(\tau)$ where $F(\tau)$ is a smooth function. The scalar current then takes the form

$$j(x) = qF(\tau)\delta(\xi)\delta^{(2)}(\mathbf{x}_\perp)\,. \tag{4.51}$$

$F(\tau)$ has the following properties. We let $F(\tau) = 1$ for $|\tau| < T$ where $2T \gg 1/a$ and $F(\tau) = 0$ for $|\tau| > T + b$ where $1/a \ll b \ll T$. The period $T < |\tau| < T + b$ corresponds to a smooth transition period between the two. The behavior of $F(\tau)$ is shown in Fig. 4.2. We note that considering a finite period of acceleration is essential to derive the Larmor formula classically [106].

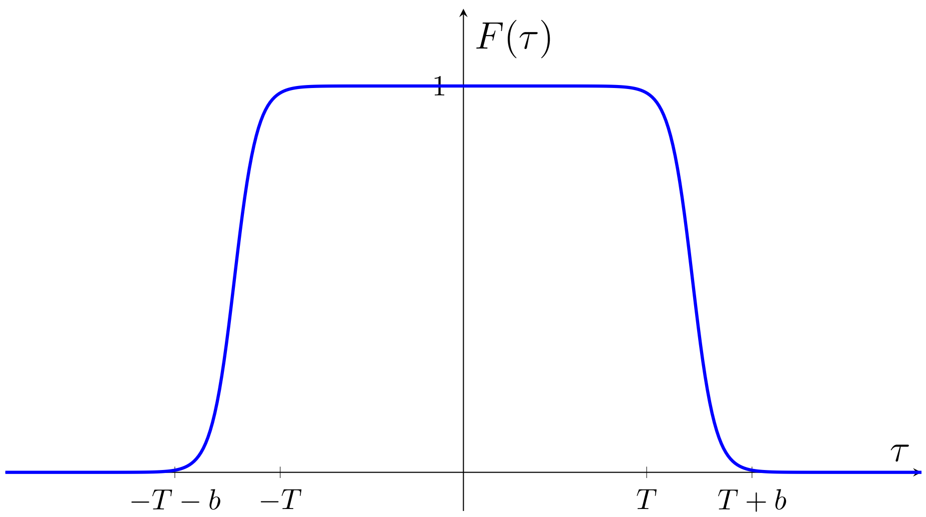

Fig. 4.2: $F(\tau)$. At the end of the calculation, we take $T \to +\infty$ but keep b constant.

At tree level, the emission amplitude of a Rindler particle of energy ω and transverse momentum $\mathbf{k}_\perp$ is given by

$$\mathcal{A}^e_{(\omega,\mathbf{k}_\perp)} = i \left\langle \omega, \mathbf{k}_\perp \right| S_I^{\text{scalar}} \left| 0_{\mathrm{R}} \right\rangle \,, \tag{4.52}$$

where $|\omega, \mathbf{k}_\perp\rangle = a^{R\dagger}_{\omega\mathbf{k}_\perp} |0_\mathrm{R}\rangle$. Explicitly, using EQS. (4.50), (4.51) and (4.52), we arrive at

$$\mathcal{A}^e_{(\omega,\mathbf{k}_\perp)} = -iq\hat{F}(\omega)\sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}}K_{i\omega/a}\left(\frac{\kappa}{a}\right), \tag{4.53}$$

where $\hat{F}(\omega)$ is the Fourier transform of $F(\tau)$ defined as

$$\hat{F}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, F(\tau)\mathrm{e}^{i\omega\tau}. \tag{4.54}$$

The amplitude for the absorption of a particle of energy ω and transverse momentum $-\mathbf{k}_\perp$ is

$$\mathcal{A}^a_{(\omega,-\mathbf{k}_\perp)} = i\,\langle 0_\mathrm{R}|\, S_I\, |\omega, -\mathbf{k}_\perp\rangle = -iq\hat{F}(-\omega)\sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}}K_{i\omega/a}\left(\frac{\kappa}{a}\right). \tag{4.55}$$

We note that the amplitudes are related via the relation

$$\frac{\mathcal{A}^a_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{\mathrm{e}^{2\pi\omega/a}-1}} = \frac{\mathcal{A}^e_{(-\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{2\pi\omega/a}}}. \tag{4.56}$$

The total one-particle probability is found by integrating the squares of the amplitudes over ω and $\mathbf{k}_\perp$ while taking into account the FDU thermal bath. Explicitly,

$$P^{\mathrm{scalar}}_{\mathrm{tot}} = \int_0^{+\infty}\mathrm{d}\omega\int \mathrm{d}^2\mathbf{k}_\perp\left[\left|\mathcal{A}^e_{(\omega,\mathbf{k}_\perp)}\right|^2\left(1+\frac{1}{\mathrm{e}^{2\pi\omega/a}-1}\right)+\frac{\left|\mathcal{A}^a_{(\omega,-\mathbf{k}_\perp)}\right|^2}{\mathrm{e}^{2\pi\omega/a}-1}\right]. \tag{4.57}$$

The first term multiplying the emission amplitude corresponds to spontaneous emission whereas the second corresponds to the induced one. The factor $(e^{2\pi\omega/a}-1)^{-1}$ is the Bose-Einstein distribution of temperature $T_U = a/2\pi$ (see EQ. (4.48)). We notice that the total interaction probability can be written as the norm squared of a one-particle final state given by

$$|1_{\rm scalar}\rangle = \int_0^{+\infty} d\omega \int d^2\mathbf{k}_\perp \left[\frac{\mathcal{A}^e_{(\omega,\mathbf{k}_\perp)} a^\dagger_{(-,\omega,\mathbf{k}_\perp)}}{\sqrt{1-e^{-2\pi\omega/a}}} + \frac{\mathcal{A}^a_{(\omega,-\mathbf{k}_\perp)} a^\dagger_{(+,\omega,\mathbf{k}_\perp)}}{\sqrt{e^{2\pi\omega/a}-1}} \right] |0_{\rm M}\rangle \ . \tag{4.58}$$

It is straightforward to check that $P^{\rm scalar}_{\rm tot} = \langle 1_{\rm scalar}|1_{\rm scalar}\rangle$. We can use EQ. (4.49) to write the total probability in a more convenient way:

$$|1_{\rm scalar}\rangle = \int_{-\infty}^{+\infty} d\omega \int \frac{d^3\mathbf{k}}{\sqrt{2\pi a k_0}} \frac{e^{-i\vartheta(k_z)\omega} \mathcal{A}^e_{(\omega\mathbf{k}_\perp)}}{\sqrt{1-e^{-2\pi\omega/a}}} a^{M\dagger}_{\mathbf{k}} |0_{\rm M}\rangle \, , \tag{4.59}$$

where we also used EQ. (4.56). Then,

$$\begin{aligned} P^{\rm scalar}_{\rm tot} &= \langle 1_{\rm scalar}|1_{\rm scalar}\rangle \\ &= \int \frac{d^3\mathbf{k}}{2\pi a k_0} \left| \int_{-\infty}^{+\infty} d\omega \, \frac{e^{-i\vartheta(k_z)\omega} \mathcal{A}^e_{(\omega\mathbf{k}_\perp)}}{\sqrt{1-e^{-2\pi\omega/a}}} \right|^2 \\ &= \frac{q^2}{16\pi^5 a} \int d^2\mathbf{k}_\perp d\vartheta \left| \int_{-\infty}^{+\infty} d\omega \, e^{-i\vartheta\omega} e^{\pi\omega/2a} K_{i\omega/a}(\kappa/a) \hat{F}(\omega) \right|^2 , \end{aligned} \tag{4.60}$$

where we made the change of variables $d\vartheta = dk_z/ak_0$. The modulus squared can be written as

$$\begin{aligned} &\left| \int_{-\infty}^{+\infty} d\omega \, e^{-i\vartheta\omega} e^{\pi\omega/2a} K_{i\omega/a}(\kappa/a) \hat{F}(\omega) \right|^2 \\ &= \left| \int_{-\infty}^{+\infty} d\tau \, F(\tau) \int_{-\infty}^{+\infty} d\omega \, e^{-i\omega(\vartheta-\tau)} e^{\pi\omega/2a} K_{i\omega/a}(\kappa/a) \right|^2 . \end{aligned} \tag{4.61}$$

From equation 6.796 of Ref. [103], we have that

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \mathrm{e}^{-i\omega y} \mathrm{e}^{\pi\omega/2a} K_{i\omega/a}(z) = \pi a \mathrm{e}^{-iz \sinh ay} \,. \tag{4.62}$$

Therefore,

$$P_{\text{tot}}^{\text{scalar}} = \frac{a}{16\pi^3} \int \mathrm{d}^2\mathbf{k}_\perp \mathrm{d}\vartheta \left| \mathcal{A}^{\text{scalar}}(\mathbf{k}) \right|^2 , \tag{4.63}$$

where we defined the amplitude

$$\mathcal{A}^{\text{scalar}}(\mathbf{k}) = q \int_{-\infty}^{+\infty} \mathrm{d}\tau \, F(\tau) \mathrm{e}^{-i\kappa/a \sinh a(\vartheta - \tau)} \,. \tag{4.64}$$

As it stands, this expression is not convenient for identifying the power emitted by the charge during the period of uniform acceleration. Firstly, we write

$$F(\tau)\mathrm{e}^{-i\kappa/a \sinh a(\vartheta-\tau)} = \frac{-iF(\tau)}{\kappa \cosh a(\theta - \tau)} \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{e}^{-i\kappa/a \sinh a(\vartheta-\tau)} \,. \tag{4.65}$$

We then integrate by parts to obtain

$$\begin{aligned} \mathcal{A}^{\text{scalar}}(\mathbf{k}) = \frac{iqa}{\kappa} \int_{-\infty}^{+\infty} \mathrm{d}\tau \Bigg(& \frac{F(\tau) \sinh a(\vartheta - \tau)}{\cosh^2 a(\vartheta - \tau)} \\ & + \frac{F'(\tau)}{a \cosh a(\vartheta - \tau)} \Bigg) \mathrm{e}^{-i(\kappa/a) \sinh a(\vartheta-\tau)} \,. \end{aligned} \tag{4.66}$$

The second term, which is proportional to $F'(\tau)$, is subdominant in the limit

$T \to +\infty$. To show this, we write

$$\mathrm{e}^{-i(\kappa/a)\sinh a(\vartheta-\tau)} = i\sinh a(\vartheta-\tau)\int_{\frac{\kappa}{a}}^{\infty} \mathrm{d}z\, \mathrm{e}^{-iz\sinh a(\vartheta-\tau)}\,, \tag{4.67}$$

where we assumed a convergence term in the exponent $\sinh a(\vartheta-\tau) \to \sinh a(\vartheta-\tau) - i\epsilon, \epsilon \to 0^+$. We then use the following identity, with $g(\tau)$ a compactly supported smooth function and n a natural number

$$\begin{aligned}&\int_{-\infty}^{+\infty} \mathrm{d}\tau\, g(\tau)\int_{\frac{\kappa}{a}}^{\infty} \frac{\mathrm{d}z}{z^n}\mathrm{e}^{-iz\sinh a(\vartheta-\tau)}\\ &= \frac{i}{a}\int_{-\infty}^{+\infty} \mathrm{d}\tau\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{g(\tau)}{\cosh a(\vartheta-\tau)}\right]\int_{\frac{\kappa}{a}}^{+\infty} \frac{\mathrm{d}z}{z^{n+1}}\mathrm{e}^{-iz\sinh a(\vartheta-\tau)}\end{aligned} \tag{4.68}$$

to write the term proportional to $F'(\tau)$ as

$$\begin{aligned}&\frac{-i}{a^3}\int_{-\infty}^{+\infty} \mathrm{d}\tau\, \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\cosh a(\vartheta-\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{F'(\tau)\sinh a(\vartheta-\tau)}{\cosh^2 a(\vartheta-\tau)}\right)\right)\\ &\qquad\times\int_{\frac{\kappa}{a}}^{+\infty} \frac{\mathrm{d}z}{z^2}\mathrm{e}^{-iz\sinh a(\vartheta-\tau)}.\end{aligned} \tag{4.69}$$

The last factor is bounded as

$$\left|\int_{\frac{\kappa}{a}}^{\infty} \frac{\mathrm{d}z}{z^2}\mathrm{e}^{-iz\sinh a(\vartheta-\tau)}\right| \le \int_{\frac{\kappa}{a}}^{\infty} \frac{\mathrm{d}z}{z^2} = \frac{a}{\kappa}\,. \tag{4.70}$$

Therefore, because of the exponential suppression in Eq. (4.69) and the form of $F'(\tau)$, this term is subdominant if $|\vartheta| < T$. The amplitude $\mathcal{A}^{\mathrm{scalar}}$ is then given by

$$\mathcal{A}^{\text{scalar}}(\mathbf{k}) \approx \frac{iqa}{\kappa}\int_{-\infty}^{+\infty} \mathrm{d}\tau\, \frac{F(\tau)\sinh a(\vartheta-\tau)}{\cosh^2 a(\vartheta-\tau)} \mathrm{e}^{-i(\kappa/a)\sinh a(\vartheta-\tau)}, \quad \text{if } |\vartheta| < T\,. \tag{4.71}$$

$\mathcal{A}^{\text{scalar}}(\mathbf{k}) \approx 0$ if $|\vartheta| > T+b$ because of the exponential decay of the integrand. We now consider the range $T < |\vartheta| < T+b$. We notice that since $\mathcal{A}^{\text{scalar}}$ is a continuous function, integrating $|\mathcal{A}^{\text{scalar}}|^2$ with respect to ϑ between T and $T+b$ gives a finite result. The emission rate is then

$$\begin{aligned} R^{\text{scalar}}(k_\perp) &= \lim_{T\to+\infty} \frac{P^{\text{scalar}}(k_\perp)}{2T} \\ &= \lim_{T\to+\infty} \frac{1}{2T}\frac{q^2a^3}{16\pi^3\kappa^2}\int \mathrm{d}\vartheta \left| \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \frac{F(\tau)\sinh a(\vartheta-\tau)}{\cosh^2 a(\vartheta-\tau)} \mathrm{e}^{-i(\kappa/a)\sinh a(\vartheta-\tau)} \right|^2 . \end{aligned} \tag{4.72}$$

Expanding the modulus square in this expression, we obtain

$$\int \mathrm{d}\tau'\mathrm{d}\tau''\, \frac{F(\tau')F(\tau'')\sinh a(\vartheta-\tau')\sinh a(\vartheta-\tau'')}{\cosh^2 a(\vartheta-\tau')\cosh^2 a(\vartheta-\tau'')} \mathrm{e}^{-i\frac{\kappa}{a}(\sinh a(\vartheta-\tau')-\sinh a(\vartheta-\tau''))} . \tag{4.73}$$

Making the change of variables $\tau = (\tau'+\tau'')/2$ and $\sigma = \tau'-\tau''$, we find

$$\begin{aligned} &\int \mathrm{d}\tau\mathrm{d}\sigma\, \frac{\mathrm{e}^{2i\kappa/a\cosh a(\vartheta-\tau)\sinh a\sigma/2}}{(\cosh^2 a(\vartheta-\tau)+\sinh^2 a\sigma/2)^2} \\ &\qquad \times F(\tau+\sigma/2)F(\tau-\sigma/2)(\cosh^2 a(\vartheta-\tau)-\cosh^2 a\sigma/2)\,. \end{aligned} \tag{4.74}$$

For large T, the integral above is well-approximated by limiting the integral bounds by $|\tau| < T$ where $F(\tau+\sigma/2)F(\tau-\sigma/2) = 1$ as long as $|\vartheta| < T$

and $||\vartheta| - T| \gg 1/a$. Then, we can make the change of variables introducing $\bar{\vartheta} = \vartheta - \tau$ (the rapidity in the rest frame of the charge) and the integrand becomes τ-independent. The τ-integral results in a factor of $2T$. Therefore, the emission rate is

$$\begin{aligned} R^{\text{scalar}}(k_\perp) = \frac{q^2 a^3}{16\pi^3 \kappa^2} \int \mathrm{d}\bar{\vartheta}\mathrm{d}\sigma \, & \frac{\mathrm{e}^{2i\kappa/a \cosh a\bar{\vartheta} \sinh a\sigma/2}}{(\cosh^2 a\bar{\vartheta} + \sinh^2 a\sigma/2)^2} \\ & \times (\cosh^2 a\bar{\vartheta} - \cosh^2 a\sigma/2) \,. \end{aligned} \tag{4.75}$$

We verify that the emission rate agrees with previous results. Introducing the variables $s_\pm = \bar{\vartheta} \pm \sigma/2$, we can simplify the above expression as

$$R^{\text{scalar}}(k_\perp) = \frac{q^2 a^3}{16\pi^3 \kappa^2} \left| \int_{-\infty}^{+\infty} \mathrm{d}s \, \frac{\sinh as}{\cosh^2 as} \mathrm{e}^{i\kappa/a \sinh as} \right|^2 = \frac{q^2}{4\pi^3 a} \left| K_0\left(\frac{\kappa}{a}\right) \right|^2 , \tag{4.76}$$

which agrees with Ref. [107]. The total emission rate is found by integrating the result over $\mathbf{k}_\perp$. The result can be found analytically in the case $m = 0$ and is

$$R^{\text{scalar}}_{\text{tot}} = \frac{q^2 a}{4\pi^2} \,, \quad m = 0 \,. \tag{4.77}$$

When $m \neq 0$, the emission rate can be found numerically.

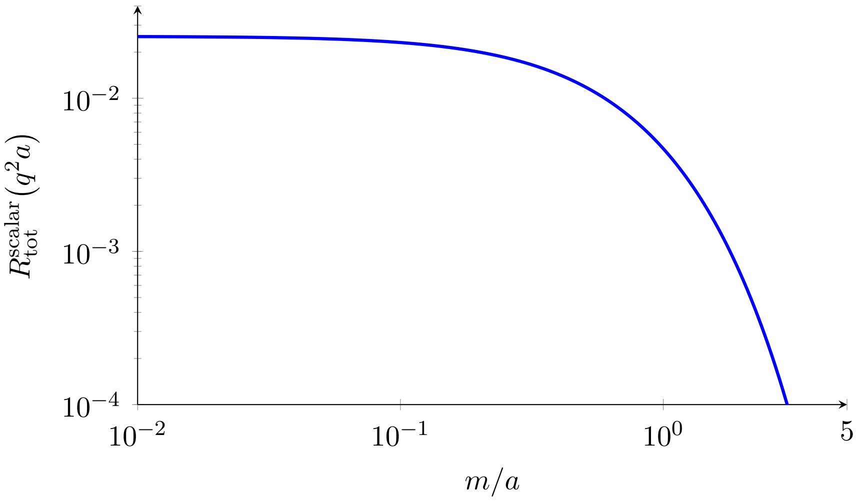

Fig. 4.3: Scalar emission rate as a function of the mass. The rate is suppressed when the mass of the particle becomes similar to the proper acceleration. The suppression is $\sim \mathrm{e}^{-2m/a}$.

In Fig. 4.3, we show the behavior of the emission rate for different masses. We see that heavy-particle production is exponentially suppressed and the rate is given by $R^{\text{scalar}}_{\text{tot}} = q^2 a(8\pi)^{-1}\mathrm{e}^{-2m/a}$. To find the emitted power, it is convenient to go back to Eq. (4.75). The energy and momentum in the rest frame of the charge are

$$\bar{k}_0 = \kappa \cosh a\bar{\vartheta}\,, \quad \bar{k}_z = \kappa \sinh a\bar{\vartheta}\,. \tag{4.78}$$

To see why this is the case, we use the definition of $\bar{\vartheta} = \vartheta - \tau$ where τ is the average proper time. Then

$$\bar{k}_0 = \gamma k_0 - \gamma\beta k_z\,, \quad \bar{k}_z = \gamma k_z - \gamma\beta k_0\,, \tag{4.79}$$

where we used Eq. (4.38) and recalled that $\gamma = \cosh a\tau$, $\beta = \tanh a\tau$ are the Lorentz factor and the velocity, respectively, in the case of a uniformly accelerating trajectory. Thus, $\bar{k}_0$ and $\bar{k}_z$ are the energy and momentum boosted in the rest frame of the charge. In spherical coordinates, the total emission rate is then

$$\begin{aligned}\frac{\mathrm{d}R^{\mathrm{scalar}}}{\mathrm{d}\bar{\Omega}} &= \frac{q^2a^2}{16\pi^3}\int_0^{+\infty}\mathrm{d}\bar{k}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\frac{\bar{k}^2(\bar{k}^2\sin^2\bar{\theta}+m^2)}{(\bar{k}^2+m^2)^{5/2}}\\ &\times\frac{\mathrm{e}^{2i\frac{\sqrt{\bar{k}^2+m^2}}{a}\sinh a\sigma/2}}{(1+\frac{\bar{k}^2\sin^2\bar{\theta}+m^2}{\bar{k}^2+m^2}\sinh^2 a\sigma/2)^2}\left(\frac{\bar{k}^2+m^2}{\bar{k}^2\sin^2\bar{\theta}+m^2}-\cosh^2 a\sigma/2\right),\end{aligned}\tag{4.80}$$

where $\bar{k}=\sqrt{\bar{k}_z^2+k_\perp^2}$ is the total momentum in the rest frame of the charge, the angle $\bar{\theta}$ is defined as $k_\perp=\bar{k}\sin\bar{\theta}$ and $\mathrm{d}\bar{\Omega}$ is the solid angle element. From here it is straightforward to find the emitted power by the accelerating charge by multiplying the integrand by a factor of energy $\bar{k}_0=\sqrt{\bar{k}^2+m^2}$.

$$\begin{aligned}\frac{\mathrm{d}S^{\mathrm{scalar}}}{\mathrm{d}\bar{\Omega}} &= \frac{q^2a^2}{16\pi^3}\int_0^{+\infty}\mathrm{d}\bar{k}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\frac{\bar{k}^2(\bar{k}^2\sin^2\bar{\theta}+m^2)}{(\bar{k}^2+m^2)^{2}}\\ &\times\frac{\mathrm{e}^{2i\frac{\sqrt{\bar{k}^2+m^2}}{a}\sinh a\sigma/2}}{(1+\frac{\bar{k}^2\sin^2\bar{\theta}+m^2}{\bar{k}^2+m^2}\sinh^2 a\sigma/2)^2}\left(\frac{\bar{k}^2+m^2}{\bar{k}^2\sin^2\bar{\theta}+m^2}-\cosh^2 a\sigma/2\right).\end{aligned}\tag{4.81}$$

The power can be calculated analytically in the massless case as

$$\begin{aligned}\frac{\mathrm{d}S^{\mathrm{scalar}}}{\mathrm{d}\bar{\Omega}} &= \frac{q^2a^2}{32\pi^3}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\frac{1-\sin^2\bar{\theta}\cosh^2 a\sigma/2}{(1+\sin^2\bar{\theta}\sinh^2 a\sigma/2)^2}\int_{-\infty}^{+\infty}\mathrm{d}\bar{k}\,\mathrm{e}^{2i\frac{\bar{k}}{a}\sinh a\sigma/2}\\ &= \frac{q^2a^2}{16\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\frac{1-\sin^2\bar{\theta}\cosh^2 a\sigma/2}{(1+\sin^2\bar{\theta}\sinh^2 a\sigma/2)^2}\delta(\sigma)\\ &= \frac{q^2a^2}{16\pi^2}\cos^2\bar{\theta}\,.\end{aligned}\tag{4.82}$$

The total power is

$$S^{\text{scalar}} = \frac{q^2 a^2}{12\pi}, \quad m = 0\,. \tag{4.83}$$

We note that the emitted scalar power is half the value of the electromagnetic power. This is due to the fact that scalar emission is longitudinal (factor $\cos^2\bar{\theta}$ in the angular distribution), while electromagnetic emission is transverse, which allows for two polarization directions [108]. This result was also found for a classical calculation in Ref. [107]. The main difference with Ref. [107] is that the calculation of the power presented here is quantum. The total power for massive scalars can be calculated numerically.

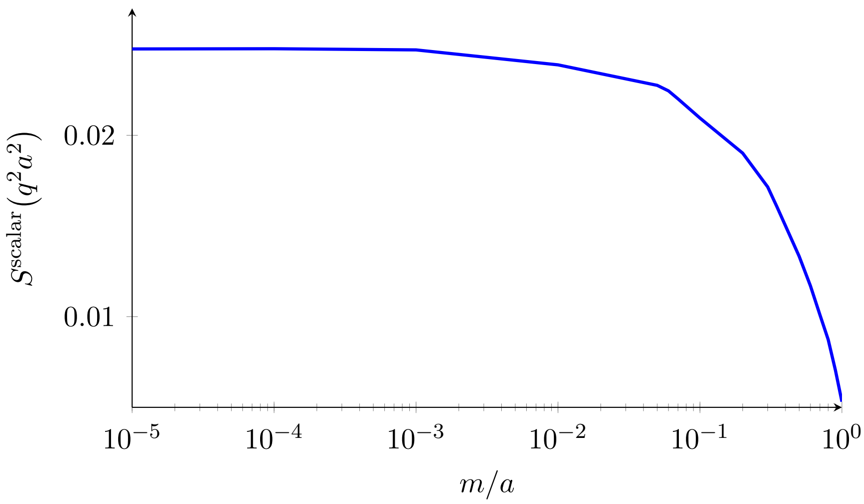

Fig. 4.4: Scalar emitted power as a function of the mass.

Similarly to the emission rate, the total power decreases when the mass of the scalar particle increases for a given proper acceleration. This is the case because, although the energy of each particle increases when the mass increases, the total number of particles decreases more rapidly. The overall result can be seen in Fig. 4.4.

4.5 Minkowski amplitudes

The results derived using the Rindler description can also be found using a Minkowski description instead. The emission amplitude of a usual (Minkowski) particle of momentum $\mathbf{k}$ is given by

$$\mathcal{A}^M_{\mathbf{k}} = \langle \mathbf{k}| \, i \int \mathrm{d}^4 x \, j(x) \Phi(x) \, |0_{\mathrm{M}}\rangle \ , \tag{4.84}$$

where $|\mathbf{k}\rangle = a^{M\dagger}_{\mathbf{k}} |0_{\mathrm{M}}\rangle$ is a one particle state with momentum $\mathbf{k}$. Using the commutation relations of the Minkowski operators we can write the amplitude as

$$\mathcal{A}^M_{\mathbf{k}} = i \int \mathrm{d}^4 x \, j(x) \phi^*_{\mathbf{k}} \, . \tag{4.85}$$

In Minkowski coordinates, the scalar current is given by

$$j(x) = q \tilde{F}(t) \delta^{(2)}(\mathbf{x}_\perp) \frac{\delta(z - \sqrt{t^2 + a^{-2}})}{a\sqrt{t^2 + a^{-2}}} \, , \tag{4.86}$$

where $\tilde{F}(t) = F(\tau)$. The total probability of interaction is found by integrating the emission amplitude over all the momenta.

$$\begin{aligned} P^{\text{scalar}}_{\text{Min}} &= \int \mathrm{d}^3\mathbf{k} \left|\mathcal{A}^M_{\mathbf{k}}\right|^2 \\ &= q^2 \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0 (2\pi)^3} \left| \int_{-\infty}^{+\infty} \frac{\mathrm{d}t \, \tilde{F}(t)}{a\sqrt{t^2 + a^{-2}}} \mathrm{e}^{i(k_0 t - k_z \sqrt{t^2 + a^{-2}})} \right|^2 . \end{aligned} \tag{4.87}$$

We make the change of variables $t = a^{-1}\sinh a\tau$ which physically corresponds to changing from the inertial time to the proper time of the particle. We then find with $\mathrm{d}t = a\sqrt{t^2 + a^{-2}}\mathrm{d}\tau = \cosh a\tau \mathrm{d}\tau$

$$P^{\text{scalar}}_{\text{Min}} = q^2 \int \frac{\mathrm{d}^3\mathbf{k}}{2k_0(2\pi)^3} \left| \int_{-\infty}^{+\infty} \mathrm{d}\tau\, F(\tau) \mathrm{e}^{i(k_0 \sinh a\tau - k_z \cosh a\tau)/a} \right|^2 . \tag{4.88}$$

To recover our results found using the Rindler modes, we use the ϑ as an integration variable we arrive at

$$P^{\text{scalar}}_{\text{Min}} = \frac{q^2 a}{16\pi^3} \int \mathrm{d}^2\mathbf{k}_\perp \mathrm{d}\vartheta \left| \int_{-\infty}^{+\infty} \mathrm{d}\tau\, F(\tau) \mathrm{e}^{-i\frac{\kappa}{a}\sinh a(\vartheta - \tau)} \right|^2 . \tag{4.89}$$

By comparing this equation with Eq. (4.63) we see that the total probability is the same whether we use the Minkowski or Rindler amplitudes. The equivalence between the Minkowski and Rindler descriptions holds for an *arbitrary* function $j(x)$ whose support is in the RRW. It is straightforward to show this using the identity in Eq. (4.62). We will show it explicitly for the electromagnetic case in the next section.

In [109], it was shown that the emission of a particle in the inertial frame corresponds to the emission or absorption of a particle to or from the FDU thermal bath. We verified this result here using an explicit calculation by finding the total probability in the two cases. In the accelerated frame, it was necessary to sum the contributions from both the emission and absorption amplitudes. Higher order processes include the absorption of a Rindler particle followed by an emission. In the limit where the time between absorption and emission becomes arbitrarily small, the accelerated electron acts as a scatterer. In the inertial frame this results in the emission of two particles (or double Compton scattering for photons) [93, 110–112].

In this chapter, we studied the Unruh effect, which states that the usual inertial vacuum is a thermal state for uniformly accelerated observers, and focused on a free massive scalar field as an illustration. We showed by calculating explicitly the Bogoliubov coefficients how one can define purely positive-frequency modes as a linear combination of Rindler modes which was essential in deriving the Unruh effect itself. We found, as expected, that the temperature of the FDU bath was proportional to the proper acceleration.

Using the Unruh effect, we then considered a uniformly accelerating particle that interacts with the FDU bath. At tree level, the interaction consists of two processes, the emission and absorption of Rindler particles to and from the FDU bath. These processes are equivalent to the emission of a Minkowski particle in the inertial frame. We verified that this is the case by explicitly calculating the interaction probability in two different ways. The first was to sum the emission and absorption amplitudes of Rindler particles in the accelerating frame. The second was to consider only the emission amplitude of Minkowski particles in the inertial frame. We find that both probabilities are equal.

In finding the probability, the emission rate and power, we assumed that the particle is interacting with the FDU only for a finite time. After having identified the part of the radiation coming from uniform acceleration, we took the infinite time limit. The final result for the emission rate agrees with previous studies. Moreover, we found that the power is given by a Larmor-type formula when the scalar particle is massless, thereby verifying a classical result. We also found that when the mass of the particle becomes comparable to the proper acceleration, both the emission rate and the power are suppressed.

Although this analysis was done for scalar particles, the methods developed here hold also for other types of particles, as will be seen in the next chapter.

5 Radiation of Spin-1 Particles

In this chapter, we generalize the results we found for scalar particles to the more interesting case of spin-1 particles. Firstly, the photon field will be quantized in the Rindler wedges. Then, we will couple a point particle via a classical current to the photon field to calculate the emission and absorption probabilities. Since we will be working in the rest frame of the particle, it will be necessary to take into account the FDU thermal bath. We will verify that the emission or absorption of a Rindler photon to or from the thermal bath corresponds to the emission of a Minkowski photon in the inertial frame, as was done for scalar particles. The additional difficulty that arises when dealing with the photon field is the presence of different polarizations. However, of the four polarizations only two are physical. Additionally, it will be shown that only one physical polarization will couple to the current of interest. Sec. 5.1 which discusses the quantization of the photon field is a review and is based on Refs. [113, 114]. Sec. 5.2 and Sec. 5.3, which discuss the derivation of the Larmor formula, is based on Ref. [2] which contains contributions from all co-authors. Sec. 5.5 shows the equivalence between the Rindler and Minkowski descriptions and contains some elements of Ref. [4]. Finally, Sec. 5.6, which discusses particle emission in Minkowski space, is based on work done by Prof. A. Higuchi and the author.

5.1 Quantization of the vector field in the RRW

We consider a massless vector field in the Feynman gauge described by

$$\mathcal{L} = -\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\nabla_\alpha A^\alpha)^2\right), \tag{5.1}$$

where g is the determinant of the metric (in usual Minkowski spacetime $g=-1$). The last term in the Lagrangian fixes the gauge and allows for the equations of motion to be

$$\nabla_\mu \nabla^\mu A_\nu = 0\,, \tag{5.2}$$

where we used the fact that $\nabla_\mu\nabla_\nu A^\mu = \nabla_\nu\nabla_\mu A^\mu$ in a space with vanishing Ricci tensor. To quantize the vector field in the RRW, we use the expansion

$$\hat{A}^R_\mu(x) = \int \mathrm{d}^2\mathbf{k}_\perp \int_0^{+\infty} \mathrm{d}\omega \sum_{\lambda=1}^{4}\left[a^R_{(\lambda,\omega,\mathbf{k}_\perp)} A^{R(\lambda,\omega,\mathbf{k}_\perp)}_\mu(x) + \text{h.c.}\right], \tag{5.3}$$

where, $\mathbf{k}_\perp = (k_x, k_y)$, the index λ runs over the different polarizations and $a^R_{(\lambda,\omega,\mathbf{k}_\perp)}$ are the annihilation operators with respect to the Rindler vacuum $|0_\mathrm{R}\rangle$. Since the equations of motion of the vector and the scalar fields are very similar, we can express $A_\mu^{R(\lambda,\omega,\mathbf{k}_\perp)}$ in terms of the scalar RRW mode solution $v^R_{\omega\mathbf{k}_\perp}$ defined in Eq. (4.26) with $m = 0$. A possible choice of independent normal modes is given by [113, 114]

$$\begin{aligned}
A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)} &= C^{(\mathrm{I},\omega,\mathbf{k}_\perp)}(0, 0, k_y v^R_{\omega\mathbf{k}_\perp}, -k_x v^R_{\omega\mathbf{k}_\perp})\,,\\
A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)} &= C^{(\mathrm{II},\omega,\mathbf{k}_\perp)}(\partial_\xi v^R_{\omega\mathbf{k}_\perp}, \partial_\tau v^R_{\omega\mathbf{k}_\perp}, 0, 0)\,,\\
A_\mu^{R(G,\omega,\mathbf{k}_\perp)} &= C^{(G,\omega,\mathbf{k}_\perp)}\nabla_\mu v^R_{\omega\mathbf{k}_\perp}\,,\\
A_\mu^{R(L,\omega,\mathbf{k}_\perp)} &= C^{(L,\omega,\mathbf{k}_\perp)}(0, 0, k_x v^R_{\omega\mathbf{k}_\perp}, k_y v^R_{\omega\mathbf{k}_\perp})\,,
\end{aligned} \tag{5.4}$$

where the constants $C^{(\lambda,\omega,\mathbf{k}_\perp)}$ will be fixed in what follows. In this notation, $A_\mu = (A_\tau, A_\xi, A_x, A_y)$. It is straightforward to see that the modes $A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)}$ and $A_\mu^{R(L,\omega,\mathbf{k}_\perp)}$ are solutions to the equations of motion: their τ and ξ components are zero and their x and y components must satisfy the scalar equation of motion, since the Christoffel symbols involving x or y coordinates are zero (see Eq. (4.17)). But since these coordinates are proportional to $v^R_{\omega\mathbf{k}_\perp}$, they are automatically solutions to Eq. (5.2). We notice that the mode $A_\mu^{R(G,\omega,\mathbf{k}_\perp)}$ is a pure gauge. Therefore,

$$\nabla^\alpha\nabla_\alpha\nabla_\mu v^R_{\omega\mathbf{k}_\perp} = \nabla_\mu\nabla^\alpha\nabla_\alpha v^R_{\omega\mathbf{k}_\perp} = 0\,. \tag{5.5}$$

Finally, for the mode $\lambda = \mathrm{II}$, we note that $A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)} \propto \varepsilon_{\mu\nu}\nabla^\nu v^R_{\omega\mathbf{k}_\perp}$ where $\varepsilon_{\mu\nu}$ is the antisymmetric tensor of the plane of Minkowski where x and y

coordinates are fixed and with metric

$$ds^2 = dt^2 - dz^2 = e^{2a\xi}(d\tau^2 - d\xi^2)\,. \tag{5.6}$$

We choose, in Minkowski coordinates, $\varepsilon_{zt} = -\varepsilon_{tz} = 1$. In Rindler coordinates, we have $\varepsilon_{\xi\tau} = -\varepsilon_{\tau\xi} = e^{2a\xi}$. Then, as previously,

$$\nabla^\alpha \nabla_\alpha \varepsilon_{\mu\nu} \nabla^\nu v^R_{\omega\mathbf{k}_\perp} = \varepsilon_{\mu\nu} \nabla^\nu \nabla^\alpha \nabla_\alpha v^R_{\omega\mathbf{k}_\perp} = 0\,. \tag{5.7}$$

We define a physical mode as a mode that is not a pure gauge and satisfies the Lorenz conditions $\nabla^\mu A_\mu = 0$. Only $A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)}$ and $A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)}$ are physical modes, since $A_\mu^{R(G,\omega,\mathbf{k}_\perp)}$ is a pure gauge and $A_\mu^{R(L,\omega,\mathbf{k}_\perp)}$ does not satisfy the Lorenz condition. One can verify that both $A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)}$ and $A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)}$ satisfy the Lorenz condition. Firstly

$$\nabla^\mu A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)} = \partial^x A_x^{R(\mathrm{I},\omega,\mathbf{k}_\perp)} + \partial^y A_y^{R(\mathrm{I},\omega,\mathbf{k}_\perp)} \propto (k_y \partial_x - k_x \partial_y) v^R_{\omega\mathbf{k}_\perp} = 0\,, \tag{5.8}$$

because $v^R_{\omega\mathbf{k}_\perp} \propto e^{i\mathbf{k}_\perp \cdot \mathbf{x}_\perp}$. For $A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)}$, we use EQ. (4.17). Then

$$\nabla^\mu A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)} \propto e^{-2a\xi}(\partial_\tau A_\tau^{R(\mathrm{II},\omega,\mathbf{k}_\perp)} - \partial_\xi A_\xi^{R(\mathrm{II},\omega,\mathbf{k}_\perp)}) = 0\,, \tag{5.9}$$

where we used EQ. (5.4). Our next goal is to find the normalization for the vector modes. We define the Klein-Gordon inner product as in [113, 114] by

$$\big(A^{R(i)}, A^{R(j)}\big) = \int_\Sigma d\Sigma_\mu\, \Xi^\mu[A^{R(i)}, A^{R(j)}]\,, \tag{5.10}$$

where $(i)=(\lambda,\omega,\mathbf{k}_\perp)$ and $(j)=(\lambda',\omega',\mathbf{k}'_\perp)$. Σ is a hypersurface of constant τ. The vector Ξ^μ is given by

$$\Xi^\mu[A^{R(i)},A^{R(j)}]=\frac{i}{\sqrt{-g}}\big(A_\nu^{R(i)*}\pi^{R(j)\mu\nu}-A_\nu^{R(j)}\pi^{R(i)*\mu\nu}\big)\,, \tag{5.11}$$

where the conjugates of the modes are given by

$$\pi^{R(i)\mu\nu}=\left.\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right|_{A_\mu^{R(i)}}=\sqrt{-g}\big(\nabla^\nu A^{R(i)\mu}-\nabla^\mu A^{R(i)\nu}-g^{\mu\nu}\nabla_\alpha A^{R(i)\alpha}\big)\,. \tag{5.12}$$

Calculating the divergence of the vector Ξ^μ we find for modes satisfying the Lorenz condition

$$\nabla_\mu\Xi^\mu[A^{R(i)},A^{R(j)}]=i\big[A_\nu^{(j)}\nabla_\mu\nabla^\mu A^{(i)\nu*}-A_\nu^{(i)*}\nabla_\mu\nabla^\mu A^{(j)\nu}\big]=0\,, \tag{5.13}$$

where in the last step we used the equations of motion. Therefore, the inner Klein-Gordon product is $\tau-$ independent. Imposing the commutation relations for the physical modes to be

$$\left[a^R_{(\lambda,\omega,\mathbf{k}_\perp)},a^{R\dagger}_{(\lambda',\omega',\mathbf{k}'_\perp)}\right]=\delta_{\lambda\lambda'}\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}-\mathbf{k}')\,, \tag{5.14}$$

leads to the following normalization of the physical modes

$$\left(A^{R(\lambda,\omega,\mathbf{k}_\perp)},A^{R(\lambda',\omega',\mathbf{k}'_\perp)}\right)=\delta^{\lambda\lambda'}\delta(\omega-\omega')\delta^{(2)}(\mathbf{k}_\perp-\mathbf{k}'_\perp)\,, \tag{5.15}$$

for $\lambda, \lambda' = \mathrm{I}, \mathrm{II}$. The infinitesimal element is defined as $\mathrm{d}\Sigma_\mu = \mathrm{d}\Sigma\, n_\mu$ where in Rindler coordinates $\mathrm{d}\Sigma \equiv \mathrm{d}x\mathrm{d}y\mathrm{d}\xi$ and n^μ is a future directed unit vector: $n^\mu = (1,0,0,0)$. For $\lambda = \lambda' = \mathrm{I}$, we find

$$\mathrm{d}\Sigma_\mu \Xi^\mu[A^{R(i)}, A^{R(j)}] = \mathrm{d}\Sigma \left|C^{(\mathrm{I},\omega,\mathbf{k}_\perp)}\right|^2 (\omega' + \omega) v^{R*}_{\omega\mathbf{k}_\perp} v^{R}_{\omega'\mathbf{k}'_\perp} (k_x k'_x + k_y k'_y)\,. \quad (5.16)$$

Integrating this expression over the coordinates x and y results in the appearance of a factor $\delta^{(2)}(\mathbf{k}_\perp - \mathbf{k}'_\perp)$ because $v^R_{\omega\mathbf{k}_\perp} \propto \mathrm{e}^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}$. Then, the normalization condition reduces to

$$\left|C^{(\mathrm{I},\omega,\mathbf{k}_\perp)}\right|^2 k_\perp^2 (\omega' + \omega) \sqrt{\frac{\sinh\frac{\pi\omega}{a}\sinh\frac{\pi\omega'}{a}}{\pi^4 a^2}}\, \mathrm{e}^{i\tau(\omega-\omega')} \times \int_{-\infty}^{+\infty} \mathrm{d}\xi\, K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) K_{i\omega'/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) = \delta(\omega - \omega')\,, \quad (5.17)$$

where we used the fact that the Bessel function $K_{i\nu}(z)$ is real if ν and z are real and positive. Then, we use the result [114]

$$\int_{-\infty}^{+\infty} \mathrm{d}\xi\, K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) K_{i\omega'/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) = \frac{\pi^2 a}{2\omega \sinh\frac{\pi\omega}{a}} \delta(\omega - \omega')\,. \quad (5.18)$$

Then, we obtain the following normalization for the first physical mode

$$\left|C^{(\mathrm{I},\omega,\mathbf{k}_\perp)}\right| = k_\perp^{-1}\,, \quad (5.19)$$

Choosing the overall phase as $C^{(\mathrm{I},\omega,\mathbf{k}_\perp)} = ik_\perp^{-1}$ is convenient in what follows. The constant $C^{(\mathrm{II},\omega,\mathbf{k}_\perp)}$ was found in Refs. [113, 114] and is $C^{(\mathrm{II},\omega,\mathbf{k}_\perp)} = k_\perp^{-1}$.

5.2 Vector Unruh modes

After having quantized the vector field in the RRW, it is straightforward to quantize it in the LRW. We know from the scalar case that the left modes $v^L_{\omega\mathbf{k}_\perp}$, which vanish in RRW, can be obtained from the right ones by the flip $z \to -z$. This implies that by replacing τ and ξ by $\bar{\tau}$ and $\bar{\xi}$ respectively, where $(\bar{\tau}, \bar{\xi})$ are the left Rindler coordinates, one can obtain $v^L_{\omega\mathbf{k}_\perp}$ from $v^R_{\omega\mathbf{k}_\perp}$. Then the left vector modes can simply be obtained from the right ones by replacing $v^R_{\omega\mathbf{k}_\perp} \to v^L_{\omega\mathbf{k}_\perp}$. The Unruh modes are defined very similarly to the scalar case by

$$\begin{aligned} W_\mu^{(\lambda,-,\omega,\mathbf{k}_\perp)} &= \frac{A_\mu^{R(\lambda,\omega,\mathbf{k}_\perp)} + \mathrm{e}^{-\pi\omega/a} A_\mu^{L(\lambda,\omega,-\mathbf{k}_\perp)*}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,, \\ W_\mu^{(\lambda,+,\omega,\mathbf{k}_\perp)} &= \frac{A_\mu^{L(\lambda,\omega,\mathbf{k}_\perp)} + \mathrm{e}^{-\pi\omega/a} A_\mu^{R(\lambda,\omega,-\mathbf{k}_\perp)*}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,. \end{aligned} \tag{5.20}$$

Both left and right Rindler physical modes can be written as

$$\begin{aligned} A_\mu^{L(\mathrm{I},\omega,\mathbf{k}_\perp)} &= i k_\perp^{-1}(0, k_y v^L_{\omega\mathbf{k}_\perp}, -k_x v^L_{\omega\mathbf{k}_\perp}, 0)\,, \\ A_\mu^{L(\mathrm{II},\omega,\mathbf{k}_\perp)} &= k_\perp^{-1}(\partial_z v^L_{\omega\mathbf{k}_\perp}, 0, 0, \partial_t v^L_{\omega\mathbf{k}_\perp})\,, \\ A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)} &= i k_\perp^{-1}(0, k_y v^R_{\omega\mathbf{k}_\perp}, -k_x v^R_{\omega\mathbf{k}_\perp}, 0)\,, \\ A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)} &= k_\perp^{-1}(\partial_z v^R_{\omega\mathbf{k}_\perp}, 0, 0, \partial_t v^R_{\omega\mathbf{k}_\perp})\,, \end{aligned} \tag{5.21}$$

in the notation $A_\mu = (A_t, A_x, A_y, A_z)$. The first physical mode stays unchanged when going from Rindler to Minkowski coordinates as the τ and ξ

components are zero, whereas for the second we used the fact that

$$A_\mu^{L,R(\mathrm{II},\omega,\mathbf{k}_\perp)} = k_\perp^{-1}\varepsilon_{\mu\nu}\partial^\nu v_{\omega\mathbf{k}_\perp}^{L,R}\,. \tag{5.22}$$

Therefore, the physical modes can be written in Minkowski coordinates as

$$\begin{aligned} W_\mu^{(\mathrm{I},\pm,\omega,\mathbf{k}_\perp)} &= ik_\perp^{-1}(0, k_y w_{\pm\omega\mathbf{k}_\perp}, -k_x w_{\pm\omega\mathbf{k}_\perp}, 0)\,,\\ W_\mu^{(\mathrm{II},\pm,\omega,\mathbf{k}_\perp)} &= k_\perp^{-1}(\partial_z w_{\pm\omega\mathbf{k}_\perp}, 0, 0, \partial_t w_{\pm\omega\mathbf{k}_\perp})\,, \end{aligned} \tag{5.23}$$

where $w_{\pm\omega\mathbf{k}_\perp}$ are the scalar Unruh modes defined in Eq. (4.41). Then, using the expansion of $w_{\pm\omega\mathbf{k}_\perp}$ in terms of the Minkowski scalar modes $\phi_\mathbf{k}$, we write the vector modes as

$$\begin{aligned} W_\mu^{(\mathrm{I},\pm,\omega,\mathbf{k}_\perp)} &= i\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}}\mathrm{e}^{\pm i\vartheta(k_z)\omega}\varepsilon_\mu^{\mathrm{I}}(\mathbf{k})\phi_\mathbf{k}\,,\\ W_\mu^{(\mathrm{II},\pm,\omega,\mathbf{k}_\perp)} &= i\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}}\mathrm{e}^{\pm i\vartheta(k_z)\omega}\varepsilon_\mu^{\mathrm{II}}(\mathbf{k})\phi_\mathbf{k}\,, \end{aligned} \tag{5.24}$$

where $\vartheta(k_z)$ is the rapidity defined in Eq. (4.37) and $\varepsilon_\mu^{\mathrm{I,II}}(\mathbf{k})$ are the polarization vectors defined by

$$\begin{aligned} \varepsilon_\mu^{\mathrm{I}}(\mathbf{k}) &= \left(0, \frac{k_y}{k_\perp}, -\frac{k_x}{k_\perp}, 0\right),\\ \varepsilon_\mu^{\mathrm{II}}(\mathbf{k}) &= \left(\frac{k_z}{k_\perp}, 0, 0, -\frac{k_0}{k_\perp}\right), \end{aligned} \tag{5.25}$$

Both polarizations satisfy $k\cdot\varepsilon^\lambda = 0$, and $\varepsilon^\lambda\cdot\varepsilon^{\lambda'} = -\delta^{\lambda\lambda'}$. The relation between the Minkowski and Unruh modes is the same as the relation between

Minkowksi and Unruh creation operators:

$$
\begin{aligned}
a^{\dagger}_{(\mathrm{I},\pm,\omega,\mathbf{k}_\perp)} &= i\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}}\mathrm{e}^{\pm i\vartheta(k_z)\omega}b^{\mathrm{I}\dagger}_{\mathbf{k}}\,,\\
a^{\dagger}_{(\mathrm{II},\pm,\omega,\mathbf{k}_\perp)} &= i\int_{-\infty}^{+\infty}\frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}}\mathrm{e}^{\pm i\vartheta(k_z)\omega}b^{\mathrm{II}\dagger}_{\mathbf{k}}\,,
\end{aligned}
\tag{5.26}
$$

where $b^{\mathrm{I}\dagger}_{\mathbf{k}}, b^{\mathrm{II}\dagger}_{\mathbf{k}}$ are the Minkowski creation operators with momentum $\mathbf{k}$ and polarizations $\varepsilon^{\mathrm{I}}_{\mu}, \varepsilon^{\mathrm{II}}_{\mu}$ respectively. They satisfy the commutation relations

$$
\left[b^{\mathrm{I}}_{\mathbf{k}}, b^{\mathrm{I}\dagger}_{\mathbf{k}'}\right] = \left[b^{\mathrm{II}}_{\mathbf{k}}, b^{\mathrm{II}\dagger}_{\mathbf{k}'}\right] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,,
\tag{5.27}
$$

with all the other commutators vanishing. These relations will be used to express the one-photon interaction probability starting from the Rindler modes going to Minkowski modes. We now find the relation between the Unruh and Rindler operator. The full vector field can be written in terms of Unruh modes as

$$
\begin{aligned}
\hat{A}_\mu = \int \mathrm{d}^2\mathbf{k}_\perp \int_0^{+\infty} \mathrm{d}\omega \sum_\lambda \big[& W_\mu^{(\lambda,-,\omega,\mathbf{k}_\perp)} a_{(\lambda,-,\omega,\mathbf{k}_\perp)} \\
& + W_\mu^{(\lambda,+,\omega,\mathbf{k}_\perp)} a_{(\lambda,+,\omega,\mathbf{k}_\perp)} + \text{h.c.}\big]\,,
\end{aligned}
\tag{5.28}
$$

where the operators $a_{(\lambda,\pm,\omega,\mathbf{k}_\perp)}$ annihilate the Minkowski vacuum. The full vector field can also be written in terms of the left and right Rindler modes as

$$
\hat{A}_\mu = \int \mathrm{d}^2\mathbf{k}_\perp \int_0^{+\infty} \mathrm{d}\omega \sum_\lambda \left[A_\mu^{R(\lambda,\omega,\mathbf{k}_\perp)} a^R_{(\lambda,\omega,\mathbf{k}_\perp)} + A_\mu^{L(\lambda,\omega,\mathbf{k}_\perp)} a^L_{(\lambda,\omega,\mathbf{k}_\perp)} + \text{h.c.}\right].
\tag{5.29}
$$

From these two expansions, using the Klein-Gordon inner product, we deduce that

$$\begin{aligned} a^R_{(\lambda,\omega,\mathbf{k}_\perp)} &= \frac{a_{(\lambda,-,\omega,\mathbf{k}_\perp)} + \mathrm{e}^{-\pi\omega/a} a^\dagger_{(\lambda,+,\omega,-\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,, \\ a^L_{(\lambda,\omega,\mathbf{k}_\perp)} &= \frac{a_{(\lambda,+,\omega,\mathbf{k}_\perp)} + \mathrm{e}^{-\pi\omega/a} a^\dagger_{(\lambda,-,\omega,-\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\,, \end{aligned} \tag{5.30}$$

for $\lambda = \mathrm{I}, \mathrm{II}$. As in the scalar case, the Rindler annihilation operators are a superposition of annihilation and creation Minkowski operators. This implies that $|0_\mathrm{M}\rangle$ and $|0_\mathrm{R}\rangle$ do not coincide and the Minkowski vacuum appears as a state filled with particles to an accelerated observer.

5.3 Radiation of vector particles in the rest frame

In this section we will discuss the radiation of vector particles from an accelerated charge (an electron for example) in the same way as it was done for scalar particles. The interaction between the charge located at $\xi = x = y = 0$ in the RRW and the vector field is achieved via the following action

$$S_I^{\mathrm{vec}} = -\int \mathrm{d}^4x\,\sqrt{-g} j^\mu \hat{A}^R_\mu\,, \tag{5.31}$$

where j^μ is a classical current associated to the charge. If the charge accelerates for an infinite duration, the form of the current is $j^\tau = q\delta(\xi)\delta^{(2)}(\mathbf{x}_\perp)$, $j^\xi = j^x = j^y = 0$. As shown previously for scalar particles, in order to find the correct formula for radiation, one needs to consider a finite time of acceleration. This is achieved by smoothly turning on and off the charge. We therefore make the substitution $q \to qF(\tau)$ where $F(\tau)$ is of the form shown in Fig. 4.2. We assume that the current is still conserved, i.e. $\nabla_\mu j^\mu = 0$. The component j^ξ can no longer be zero and the current then becomes

$$\begin{aligned} j^\tau &= qF(\tau)\delta(\xi)\delta^{(2)}(\mathbf{x}_\perp)\,, \\ j^\xi &= -qF'(\tau)\mathrm{e}^{-2a\xi}\theta(\xi)\delta^{(2)}(\mathbf{x}_\perp)\,, \\ j^x &= 0\,, \\ j^y &= 0\,. \end{aligned} \tag{5.32}$$

Since the x and y components of the current are zero, it will not couple to $A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)}$ and $A_\mu^{R(L,\omega,\mathbf{k}_\perp)}$. On the other hand, because of current conservation, the current will not couple to $A_\mu^{R(G,\omega,\mathbf{k}_\perp)}$ either. Current conservation implies that

$$\nabla_\mu j^\mu = \partial_\mu j^\mu + \left(\Gamma^\tau_{\tau\xi} + \Gamma^\xi_{\xi\xi}\right) j^\xi = \partial_\mu j^\mu + 2aj^\xi = 0\,. \tag{5.33}$$

Then by integration by parts

$$\begin{aligned} &\int \mathrm{d}^4x\,\sqrt{-g}j^\mu A_\mu^{R(G,\omega,\mathbf{k}_\perp)} \propto \int \mathrm{d}^4x\,\sqrt{-g}j^\mu \partial_\mu v^R_{\omega\mathbf{k}_\perp} \\ &= -\int \mathrm{d}^4x\,\partial_\mu(\sqrt{-g}j^\mu)v^R_{\omega\mathbf{k}_\perp} = -\int \mathrm{d}^4x\,\sqrt{-g}(\partial_\mu j^\mu + 2aj^\xi)v^R_{\omega\mathbf{k}_\perp} = 0\,, \end{aligned} \tag{5.34}$$

where we used $\sqrt{-g} = \mathrm{e}^{2a\xi}$. Therefore, we only need to consider the emission and absorption of vector particles with polarization $\lambda = \mathrm{II}$. The emission amplitude is given by

$$\mathcal{A}^{(e,\mathrm{II})}_{(\omega,\mathbf{k}_\perp)} = i \langle \mathrm{II}, \omega, \mathbf{k}_\perp | \, S_I^{\mathrm{vec}} \, |0_\mathrm{R}\rangle \,, \tag{5.35}$$

where $|\mathrm{II}, \omega, \mathbf{k}_\perp\rangle = a^{R}_{(\mathrm{II},\omega,\mathbf{k}_\perp)} |0_\mathrm{R}\rangle$. We find explicitly using Eqs. (5.31) and (5.32)

$$\begin{aligned}\mathcal{A}^{(e,\mathrm{II})}_{(\omega,\mathbf{k}_\perp)} = -iq\hat{F}(\omega)\sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} \\ \times \left[K'_{i\omega/a}\left(\frac{k_\perp}{a}\right) - \frac{\omega^2}{k_\perp}\int_0^{+\infty} \mathrm{d}\xi \, K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right)\right],\end{aligned} \tag{5.36}$$

where prime means derivative with respect to the argument and $\hat{F}(\omega)$ is the Fourier transform of $F(\tau)$ as defined in Eq. (4.54). The absorption amplitude is given by

$$\mathcal{A}^{(a,\mathrm{II})}_{(\omega,-\mathbf{k}_\perp)} = i \langle 0_\mathrm{R} | \, S_I^{\mathrm{vec}} \, |\mathrm{II}, \omega, -\mathbf{k}_\perp\rangle \ . \tag{5.37}$$

Using the same procedure as for the emission, we find

$$\begin{aligned}\mathcal{A}^{(a,\mathrm{II})}_{(\omega,-\mathbf{k}_\perp)} = -iq\hat{F}(-\omega)\sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} \\ \times \left[K'_{i\omega/a}\left(\frac{k_\perp}{a}\right) - \frac{\omega^2}{k_\perp}\int_0^{+\infty} \mathrm{d}\xi \, K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right)\right].\end{aligned} \tag{5.38}$$

The emission and absorption amplitudes are related by the following equation

$$\frac{\mathcal{A}^{(a,\mathrm{II})}_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{\mathrm{e}^{2\pi\omega/a}-1}} = \frac{\mathcal{A}^{(e,\mathrm{II})}_{(-\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{2\pi\omega/a}}} \,. \tag{5.39}$$

As in the scalar case, the total interaction probability is given by integrating the modulus square of the emission and absorption probabilities while taking into account the FDU thermal bath. Explicitly,

$$P_{\text{tot}}^{\text{vec}} = \int \mathrm{d}^2\mathbf{k}_\perp \int_0^{+\infty} \mathrm{d}\omega \left[\frac{\left|\mathcal{A}^{(e,\text{II})}_{(\omega,\mathbf{k}_\perp)}\right|^2}{1-\mathrm{e}^{-2\pi\omega/a}} + \frac{\left|\mathcal{A}^{(a,\text{II})}_{(\omega,-\mathbf{k}_\perp)}\right|^2}{\mathrm{e}^{2\pi\omega/a}-1} \right] . \tag{5.40}$$

The factor associated to the emission amplitude is the sum of the thermal factor of the FDU thermal bath and 1 which accounts for spontaneous emission. We notice that the probability can be also written as the norm square of a one-vector final state, i.e. $P_{\text{tot}}^{\text{vec}} = \langle 1_{\text{vec}}|1_{\text{vec}}\rangle$, where

$$|1_{\text{vec}}\rangle = \int \mathrm{d}^2\mathbf{k}_\perp \int_0^{+\infty} \mathrm{d}\omega \left[\mathcal{A}^{(e,\text{II})}_{(\omega,\mathbf{k}_\perp)} a^{R\dagger}_{(\text{II},\omega,\mathbf{k}_\perp)} + \mathcal{A}^{(a,\text{II})}_{(\omega,-\mathbf{k}_\perp)} a^{R}_{(\text{II},\omega,-\mathbf{k}_\perp)} \right] |0_{\text{M}}\rangle \ . \tag{5.41}$$

Expressing the Rindler operators as linear combinations of Unruh operators using Eq. (5.30), we find

$$\begin{aligned} |1_{\text{vec}}\rangle = \int \mathrm{d}^2\mathbf{k}_\perp \int_0^{+\infty} \mathrm{d}\omega \Bigg[& \frac{\mathcal{A}^{(e,\text{II})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} a^{\dagger}_{(\text{II},-,\omega,\mathbf{k}_\perp)} \\ & + \frac{\mathcal{A}^{(a,\text{II})}_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{\mathrm{e}^{2\pi\omega/a}-1}} a^{\dagger}_{(\text{II},+,\omega,+\mathbf{k}_\perp)} \Bigg] |0_{\text{M}}\rangle \ . \end{aligned} \tag{5.42}$$

Now are able to express the one-vector final state in terms of regular Minkowski momentum using Eq. (5.26).

$$|1_{\text{vec}}\rangle = i \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{2\pi a k_0}} \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{\mathcal{A}^{(e,\text{II})}_{(\omega,\mathbf{k}_\perp)} \mathrm{e}^{-i\vartheta(k_z)\omega}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} b^{\text{II}\dagger}_{\mathbf{k}} |0_{\text{M}}\rangle \ . \tag{5.43}$$

The probability is then expressed as

$$P_{\text{tot}}^{\text{vec}} = \int \frac{\mathrm{d}^3\mathbf{k}}{2\pi a k_0} \left| \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{\mathcal{A}^{(e,\mathrm{II})}_{(\omega,\mathbf{k}_\perp)} \mathrm{e}^{-i\vartheta(k_z)\omega}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} \right|^2 . \tag{5.44}$$

The calculation to derive the rate and the emitted power will be analogous to the scalar case. We show it here for completeness while using some useful results we derived in the scalar case. Using EQ. (5.36), we can rewrite the total probability as

$$P_{\text{tot}}^{\text{vec}} = \frac{a}{16\pi^3} \int \mathrm{d}^2\mathbf{k}_\perp \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \, \left|\mathcal{A}^{\text{vec}}(\mathbf{k})\right|^2 , \tag{5.45}$$

using $\mathrm{d}\vartheta = \mathrm{d}k_z/(ak_0)$ and where

$$\begin{aligned} \mathcal{A}^{\text{vec}}(\mathbf{k}) = -\frac{q}{\pi a} \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \hat{F}(\omega) \mathrm{e}^{-i\omega\vartheta} \mathrm{e}^{\pi\omega/2a} \\ \times \left[K'_{i\omega/a}\left(\frac{k_\perp}{a}\right) - \frac{\omega^2}{k_\perp} \int_0^{+\infty} \mathrm{d}\xi \, K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) \right] . \end{aligned} \tag{5.46}$$

We express the integrand in terms of the variable τ instead of ω by using EQ. (4.62). For the first term we differentiate EQ. (4.62) with respect to the variable z to obtain

$$\begin{aligned} \mathcal{A}^{\text{vec}}(\mathbf{k}) = q \int_{-\infty}^{+\infty} \mathrm{d}\tau \, \Bigg[& iF(\tau) \sinh a(\vartheta-\tau) \mathrm{e}^{-i(k_\perp/a)\sinh a(\vartheta-\tau)} \\ & - \frac{F''(\tau)}{ak_\perp} \int_{k_\perp/a}^{+\infty} \frac{\mathrm{d}z}{z} \mathrm{e}^{-iz\sinh a(\vartheta-\tau)} \Bigg] . \end{aligned} \tag{5.47}$$

As in the scalar case the first term is not suited for identifying the period of uniform acceleration. We will integrate it by parts by using Eq. (4.65). Then, for the resulting terms involving a derivative of $F(\tau)$ we use Eqs. (4.67) and (4.68) to write the amplitude as

$$\mathcal{A}^{\text{vec}}(\mathbf{k}) = \frac{qa}{k_\perp}\int_{-\infty}^{+\infty} \mathrm{d}\tau \left[\frac{F(\tau)\mathrm{e}^{-i(k_\perp/a)\sinh a(\vartheta-\tau)}}{\cosh^2 a(\vartheta-\tau)} \right.$$
$$\left. - \frac{i}{a^3}\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{1}{\cosh a(\vartheta-\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{F'(\tau)}{\cosh^2 a(\vartheta-\tau)}\right]\right\}\int_{k_\perp/a}^{+\infty}\frac{\mathrm{d}z}{z^2}\mathrm{e}^{-iz\sinh a(\vartheta-\tau)}\right]. \tag{5.48}$$

The second term is exponentially decaying as $|\vartheta-\tau| \to +\infty$. Therefore it is subdominant for $|\vartheta| < T$ where we recall that $2T$ is the period of uniform acceleration. The rate for fixed transverse momentum, is given by (see the discussion for the scalar case)

$$R^{\text{vec}}(k_\perp) = \frac{q^2a^3}{16\pi^3k_\perp^2}\int_{-\infty}^{+\infty}\mathrm{d}\overline{\vartheta}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\,\frac{\mathrm{e}^{2i(k_\perp/a)\cosh a\overline{\vartheta}\sinh a\sigma/2}}{[\cosh^2 a\overline{\vartheta}+\sinh^2 a\sigma/2]^2}, \tag{5.49}$$

where $\bar{\vartheta} = \vartheta - \tau$ is the rapidity in the rest frame of the charge (τ in this case is the average proper time). This expression will be used to find the emitted power. It can be calculated analytically as

$$R^{\text{vec}}(k_\perp) = \frac{q^2a^3}{16\pi^3\kappa^2}\left|\int_{-\infty}^{+\infty}\mathrm{d}s\,\frac{\mathrm{e}^{i(k_\perp/a)\sinh as}}{\cosh^2 as}\right|^2 = \frac{q^2}{4\pi^3a}\left|K_1\left(\frac{k_\perp}{a}\right)\right|^2. \tag{5.50}$$

We note that the total rate (obtained by integrating the above expression over the transverse momenta $\mathbf{k}_\perp$) is divergent due to the contribution of small $\mathbf{k}_\perp$. The rate with fixed transverse momentum is similar to the one for scalar particles. The only difference is the index of the Bessel function. As we will see later, the index corresponds to the spin of the quantum field. The total power is found by multiplying the integrand of the rate by a factor of $\bar{k}$, which is the energy in the rest frame of the charge. It is then

$$S^{\mathrm{vec}} = \frac{q^2 a^2}{16\pi^3} \int \frac{\mathrm{d}^2\mathbf{k}_\perp \mathrm{d}\bar{k}_z}{k_\perp^2} \int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \frac{\mathrm{e}^{2i(\kappa/a)\cosh a\bar{\vartheta} \sinh a\sigma/2}}{[\cosh^2 a\bar{\vartheta} + \sinh^2 a\sigma/2]^2} \, . \tag{5.51}$$

We use $\bar{k} = k_\perp \cosh a\bar{\vartheta}$ and $\mathrm{d}^2\mathbf{k}_\perp = \mathrm{d}\Omega \mathrm{d}\bar{k}\bar{k}^2$ where $\bar{k}^2 = k_\perp^2 + \bar{k}_z^2$ to write

$$\frac{\mathrm{d}S^{\mathrm{vec}}}{\mathrm{d}\Omega} = \frac{q^2 a^2}{16\pi^3} \int_0^{+\infty} \mathrm{d}\bar{k} \, \frac{k_\perp^2}{\bar{k}^2} \int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \frac{\mathrm{e}^{2i(\bar{k}/a)\sinh a\sigma/2}}{[1 + (\kappa^2/\bar{k}_0^2)\sinh^2 a\sigma/2]^2} \, . \tag{5.52}$$

We extend the bounds of the momentum integral to $-\infty$ by multiplying by $1/2$ and find

$$\begin{aligned} \frac{\mathrm{d}S^{\mathrm{vec}}}{\mathrm{d}\Omega} &= \frac{q^2 a^2}{32\pi^3} \int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \frac{\sin^2 \bar{\theta}}{[1 + \sin^2 \bar{\theta} \sinh^2 a\sigma/2]^2} \int_{-\infty}^{+\infty} \mathrm{d}\bar{k} \, \mathrm{e}^{2i(\bar{k}/a)\sinh a\sigma/2} \\ &= \frac{q^2 a^2}{16\pi^2} \sin^2 \bar{\theta} \, . \end{aligned} \tag{5.53}$$

where the angle $\bar{\theta}$ was defined as $\bar{k} = k_\perp \sin \bar{\theta}$. We find the usual Larmor formula by integrating over the solid angle:

$$S^{\mathrm{vec}} = \frac{q^2 a^2}{6\pi} \, . \tag{5.54}$$

The same result can be derived by doing the calculation in the inertial frame. More precisely, it means that the interaction probabilities of the two reference frames are the same. This was shown for scalar particles previously, whereas it was shown in the vector case in [113, 114] for the case of uniform acceleration, i.e. for the current choice Eq. (5.32) with $F(\tau) = 1$. In Sec. 5.5, it will be shown that the equivalence between the two reference frames holds for any choice of conserved current. We discuss the case of the massive vector field in the next section.

5.4 Radiation of massive vector particles in the rest frame

In this section, we discuss radiation of massive vector particles from an accelerating charge. Since part of this discussion is similar to the massless case, we will refer to results of the previous sections without giving all the details of the derivations. The main goal of this section is to study the effect of the mass of the vector field on particle emission. The theory of a free massive vector field can be written as

$$\mathcal{L} = \sqrt{-g}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_A^2 A_\mu A^\mu\Big)\,, \tag{5.55}$$

where m_A is the mass of the field. The equations of motion are given by $\nabla_\mu F^{\mu\nu} + m_A^2 A^\nu = 0$. Taking the divergence leads to $\nabla_\mu A^\mu = 0$, since $m_A^2 \neq 0$ and $\nabla_\mu \nabla_\nu F^{\mu\nu} = 0$. The last identity holds because the Christoffel symbols are constant in Rindler space. As a consequence, the equations of motion are

$$\big(\nabla_\mu \nabla^\mu + m_A^2\big) A^\nu = 0\,, \tag{5.56}$$

where we used the fact that $\nabla_\mu \nabla^\nu A^\mu = \nabla^\nu \nabla_\mu A^\mu$ for a metric with vanishing Ricci tensor. Due to the constraint $\nabla_\mu A^\mu = 0$, the massive vector field has three degrees of freedom. We expand the field in the RRW as in EQ. (5.3) but only sum over the three physical polarizations. Using EQ. (5.4), the physical polarizations can be written as

$$
\begin{aligned}
A_\mu^{R(\mathrm{I},\omega,\mathbf{k}_\perp)} &= C^{(\mathrm{I},\omega,\mathbf{k}_\perp)}(0,0,k_y v^R_{\omega\mathbf{k}_\perp}, -k_x v^R_{\omega\mathbf{k}_\perp})\,, \\
A_\mu^{R(\mathrm{II},\omega,\mathbf{k}_\perp)} &= C^{(\mathrm{II},\omega,\mathbf{k}_\perp)}(\partial_\xi v^R_{\omega\mathbf{k}_\perp}, \partial_\tau v^R_{\omega\mathbf{k}_\perp}, 0, 0)\,, \\
A_\mu^{R(G,\omega,\mathbf{k}_\perp)} &= C^{(G,\omega,\mathbf{k}_\perp)}(\nabla_\mu v^R_{\omega\mathbf{k}_\perp} + i m_A^2 (0,0,\alpha v^R_{\omega\mathbf{k}_\perp}/k_x, \beta v^R_{\omega\mathbf{k}_\perp}/k_y))\,,
\end{aligned}
\tag{5.57}
$$

where now $v^R_{\omega\mathbf{k}_\perp}$ is the solution to scalar Klein-Gordon equation with non-zero mass $(\nabla_\mu \nabla^\mu + m_A^2) v^R_{\omega\mathbf{k}_\perp} = 0$ and α and β are some parameters to be determined. Then, following the same discussion as in the massless case, the three polarizations are solutions to the equations of motions. We added a second term for $A_\mu^{R(G,\omega,\mathbf{k}_\perp)}$, since a pure gauge no longer satisfies the Lorenz condition as $\nabla^\mu \nabla_\mu v^R_{\omega\mathbf{k}_\perp} = -m_A^2 v^R_{\omega\mathbf{k}_\perp} \neq 0$. Requiring $\nabla^\mu A_\mu^{R(G,\omega,\mathbf{k}_\perp)} = 0$, results in the condition $\alpha + \beta = 1$. Additionally, imposing $\left(A^{R(\mathrm{I},\omega,\mathbf{k}_\perp)}, A^{R(G,\omega',\mathbf{k}'_\perp)}\right) = 0$, we find $k_y^2 \alpha = k_x^2 \beta$. Then, $\alpha = k_x^2/k_\perp^2$ and $\beta = k_y^2/k_\perp^2$. We note that since the mass term in the Lagrangian does not depend on a derivative of the vector field, the conjugate is the same both massless and massive cases. $\nabla_\mu \Xi^\mu = 0$, also holds in the massive case. It is straightforward to verify that all polarizations are orthogonal to each other. Then, by imposing EQ. (5.14), EQ. (5.15) holds for $\lambda = \mathrm{I}, \mathrm{II}, G$. This allows us to find the normalization constants which are given by

$$C^{(\mathrm{I},\omega,\mathbf{k}_\perp)} = ik_\perp^{-1}\,, \quad C^{(\mathrm{II},\omega,\mathbf{k}_\perp)} = \kappa^{-1}\,, \quad C^{(G,\omega,\mathbf{k}_\perp)} = \frac{k_\perp}{m_A\kappa}\,. \tag{5.58}$$

We quantize the field in the LRW and define the Unruh modes as in EQ. (5.20). They can be expanded in terms of Minkowski modes. For the polarizations $\lambda = \mathrm{I,II}$, the expansion is given in EQ. (5.24) and for $\lambda = G$,

$$W_\mu^{(G,\pm,\omega,\mathbf{k}_\perp)} = -i\int_{-\infty}^{+\infty} \frac{\mathrm{d}k_z}{\sqrt{2\pi a k_0}} \mathrm{e}^{\pm i\vartheta(k_z)\omega} \varepsilon_\mu^G(\mathbf{k})\phi_{\mathbf{k}}\,, \tag{5.59}$$

where the polarizations are (in Minkowski coordinates)

$$\varepsilon^{\mathrm{II}\mu}(\mathbf{k}) = \left(\frac{k_z}{\kappa}, 0, 0, \frac{k_0}{\kappa}\right), \quad \varepsilon^{G\mu}(\mathbf{k}) = \frac{k_\perp}{\kappa m_A}\left(k_0, \frac{k_x\kappa^2}{k_\perp^2}, \frac{k_y\kappa^2}{k_\perp^2}, k_z\right), \tag{5.60}$$

and the polarization vector $\varepsilon^{\mathrm{I}\mu}(\mathbf{k})$ is the same as in the massless case. We note that $\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = -\delta^{\lambda\lambda'}$, $\varepsilon^\lambda \cdot k = 0$ and

$$\sum_\lambda \varepsilon_\mu^{\lambda *}(\mathbf{k})\varepsilon_\nu^\lambda(\mathbf{k}) = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m_A^2}\,, \tag{5.61}$$

as expected where the sum is taken over $\lambda = \mathrm{I,II}, G$. We now couple the vector field to the current j^μ via EQ. (5.31). As for the massless case, only the polarization $\lambda = \mathrm{II}$ couples to the current. We note that the term added to the polarization $\lambda = G$ for the massive case, has only x and y non-zero components and therefore does not couple to the current. Carrying out the calculation exactly as for the massless vector case we find that the rate is

$$\begin{aligned} R^{\text{vec}}(k_\perp) &= \frac{q^2 a^3}{16\pi^3 \kappa^2} \int_{-\infty}^{+\infty} \mathrm{d}\overline{\vartheta} \int_{-\infty}^{+\infty} \mathrm{d}\sigma \, \frac{\mathrm{e}^{2i(\kappa/a)\cosh a\overline{\vartheta} \sinh a\sigma/2}}{[\cosh^2 a\overline{\vartheta} + \sinh^2 a\sigma/2]^2} \\ &= \frac{q^2}{4\pi^3 a} \left| K_1\left(\frac{\kappa}{a}\right)\right|^2 . \end{aligned} \tag{5.62}$$

Contrary to the massless case, the total rate found by integrating over $\mathbf{k}_\perp$ is finite as the infrared divergence is removed by the presence of a non-zero mass. For small masses the rate diverges logarithmically with leading term $\propto -\ln m_A/a$. In Fig. 5.1, the total rate is plotted as a function of the particle mass. We note that when the mass is large, i.e. $m_A \gg a$, the rate is suppressed exponentially. More precisely $R^{\text{vec}}_{\text{tot}} = q^2 a (8\pi)^{-1} \mathrm{e}^{-2m_A/a}$ for $m_A \gg a$ exactly as for scalar particles. This is due to the fact that heavy particles are less likely to be produced.

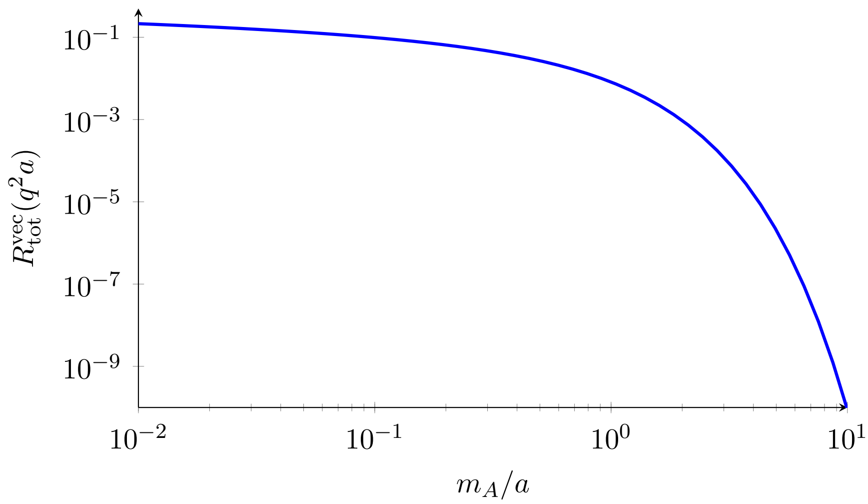

Fig. 5.1: Total emission rate as a function of the mass of the vector particle.

As previously, we multiply the integrand of the rate by a factor of energy to find the power. It is given by

$$\frac{\mathrm{d}S^{\mathrm{vec}}}{\mathrm{d}\Omega} = \frac{q^2 a^2}{16\pi^3} \int_0^{+\infty} \mathrm{d}\bar{k}\, \frac{\bar{k}^2 \kappa^2}{\bar{k}_0^4} \int_{-\infty}^{+\infty} \mathrm{d}\sigma\, \frac{\mathrm{e}^{2i(\bar{k}_0/a)\sinh a\sigma/2}}{[1 + (\kappa^2/\bar{k}_0^2)\sinh^2 a\sigma/2]^2} \,. \tag{5.63}$$

The smooth limit $m_A \to 0$ can be taken in this integral, and we recover the Larmor formula. This is because the difference between the theories of massless and massive vector fields is that the latter describes an additional degree of freedom given by the physical polarization $\lambda = G$, which does not couple to the current j^μ. The only polarization that couples to the current is $\lambda = \mathrm{II}$, for which the limit $m_A \to 0$ can be taken to recover the massless limit. Similarly to the emission of scalar particles, the power is also suppressed when the mass of the vector particles becomes comparable to the proper acceleration.

5.5 Equivalence between inertial and accelerated frames for an arbitrary conserved current

Here we consider a general conserved current j^μ and a massless vector field for simplicity. The results can be generalized straightforwardly to the massive case. The conservation of the current implies immediately it does not couple to the nonphysical pure-gauge mode $A_\mu^{R(G,\omega,\mathbf{k}_\perp)}$. We additionally assume that the current does not couple to the mode $A_\mu^{R(L,\omega,\mathbf{k}_\perp)}$. Unlike for the special case of uniform acceleration, in general, the first physical mode couples to the general current. Then, we find that the absorption and emission amplitudes for the two physical polarizations are given by

$$\begin{aligned}
\mathcal{A}^{(e,\lambda)}_{(\omega,\mathbf{k}_\perp)} &= i\,\langle\lambda,\omega,\mathbf{k}_\perp|\,S^{\mathrm{vec}}_I\,|0_\mathrm{R}\rangle = -i\int \mathrm{d}^4x\,\sqrt{-g}j^\mu A^{(\lambda,\omega,\mathbf{k}_\perp)*}_\mu\,,\\
\mathcal{A}^{(a,\lambda)}_{(\omega,-\mathbf{k}_\perp)} &= i\,\langle 0_\mathrm{R}|\,S^{\mathrm{vec}}_I\,|\lambda,\omega,-\mathbf{k}_\perp\rangle = -i\int \mathrm{d}^4x\,\sqrt{-g}j^\mu A^{(\lambda,\omega,-\mathbf{k}_\perp)}_\mu\,,
\end{aligned} \tag{5.64}$$

for $\lambda = \mathrm{I},\mathrm{II}$. The one-vector final state is given by

$$|1_\mathrm{vec}\rangle = \sum_{\lambda=\mathrm{I,II}}\int \mathrm{d}^2\mathbf{k}_\perp\int_0^{+\infty}\mathrm{d}\omega\left[\mathcal{A}^{(e,\lambda)}_{(\omega,\mathbf{k}_\perp)}a^{R\dagger}_{(\lambda,\omega,\mathbf{k}_\perp)} + \mathcal{A}^{(a,\lambda)}_{(\omega,-\mathbf{k}_\perp)}a^{R}_{(\lambda,\omega,-\mathbf{k}_\perp)}\right]|0_\mathrm{M}\rangle\ . \tag{5.65}$$

Then using Eq. (5.30) and the fact that the Unruh annihilator operators annihilate the Minkowski vacuum, we obtain

$$\begin{aligned}
|1_\mathrm{vec}\rangle = \int \mathrm{d}^2\mathbf{k}_\perp\int_0^{+\infty}\mathrm{d}\omega\Bigg[&\frac{\mathcal{A}^{(e,\mathrm{I})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}a^{\dagger}_{(\mathrm{I},-,\omega,\mathbf{k}_\perp)} + \frac{\mathcal{A}^{(a,\mathrm{I})}_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{\mathrm{e}^{2\pi\omega/a}-1}}a^{\dagger}_{(\mathrm{I},+,\omega,+\mathbf{k}_\perp)}\\
&+\frac{\mathcal{A}^{(e,\mathrm{II})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}a^{\dagger}_{(\mathrm{II},-,\omega,\mathbf{k}_\perp)} + \frac{\mathcal{A}^{(a,\mathrm{II})}_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{\mathrm{e}^{2\pi\omega/a}-1}}a^{\dagger}_{(\mathrm{II},+,\omega,+\mathbf{k}_\perp)}\Bigg]|0_\mathrm{M}\rangle\ .
\end{aligned} \tag{5.66}$$

Then the total probability can be found as

$$\begin{aligned}
P^{\mathrm{vec}}_{\mathrm{tot}} &= \langle 1_\mathrm{vec}|1_\mathrm{vec}\rangle\\
&= \sum_{\lambda=\mathrm{I,II}}\int \mathrm{d}^2\mathbf{k}_\perp\int_0^{+\infty}\mathrm{d}\omega\left[\frac{\left|\mathcal{A}^{(e,\lambda)}_{(\omega,\mathbf{k}_\perp)}\right|^2}{1-\mathrm{e}^{-2\pi\omega/a}} + \frac{\left|\mathcal{A}^{(a,\lambda)}_{(\omega,-\mathbf{k}_\perp)}\right|^2}{\mathrm{e}^{2\pi\omega/a}-1}\right],
\end{aligned} \tag{5.67}$$

where we used the inverse relation Eq. (5.30) given by

$$\left[a_{(\lambda,-,\omega,\mathbf{k}_\perp)},a^\dagger_{(\lambda',-,\omega',\mathbf{k}'_\perp)}\right]=\left[a_{(\lambda,+,\omega,\mathbf{k}_\perp)},a^\dagger_{(\lambda',+,\omega',\mathbf{k}'_\perp)}\right]=\delta_{\lambda\lambda}\delta(\omega-\omega')\delta(\mathbf{k}_\perp-\mathbf{k}'_\perp)\,,$$
$$\left[a_{(\lambda,-,\omega,\mathbf{k}_\perp)},a^\dagger_{(\lambda',+,\omega',\mathbf{k}'_\perp)}\right]=0\,. \tag{5.68}$$

The total probability is the sum of emission and absorption amplitude for both polarizations. By computing explicitly the amplitudes, we find

$$\frac{\mathcal{A}^{(a,\lambda)}_{(\omega,-\mathbf{k}_\perp)}}{\sqrt{\mathrm{e}^{2\pi\omega/a}-1}}=\frac{\mathcal{A}^{(e,\lambda)}_{(-\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{2\pi\omega/a}}},\qquad \lambda=\mathrm{I},\mathrm{II}\,. \tag{5.69}$$

For the polarization $\lambda=\mathrm{I}$, this equality is achieved via the proper choice of phase in Eq. (5.19). This allows us to write the probability in terms of emission probabilities only as

$$P^{\mathrm{vec}}_{\mathrm{tot}}=\sum_{\lambda=\mathrm{I},\mathrm{II}}\int\mathrm{d}^2\mathbf{k}_\perp\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\frac{\left|\mathcal{A}^{(e,\lambda)}_{(\omega,\mathbf{k}_\perp)}\right|^2}{1-\mathrm{e}^{-2\pi\omega/a}}\,. \tag{5.70}$$

We repeat the calculation of uniform acceleration to obtain an integral over all the physical momenta. We find

$$P^{\mathrm{vec}}_{\mathrm{tot}}=\sum_{\lambda=\mathrm{I},\mathrm{II}}\int\frac{\mathrm{d}^3\mathbf{k}}{2\pi ak_0}\left|\int_{-\infty}^{+\infty}\mathrm{d}\omega\,\frac{\mathrm{e}^{-i\vartheta\omega}\mathcal{A}^{(e,\lambda)}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}}\right|^2\,. \tag{5.71}$$

Explicitly, the amplitude of the first polarization is given by

$$\mathcal{A}^{(e,\mathrm{I})}_{(\omega,\mathbf{k}_\perp)} = -i\sqrt{\frac{\sinh \pi\omega/a}{4\pi^4 a}} \int \mathrm{d}^4x \,\sqrt{-g}k_\perp^{-1}(k_y j^x - k_x j^y)\mathrm{e}^{i\omega\tau - i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right). \tag{5.72}$$

Then,

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \,\frac{\mathrm{e}^{-i\vartheta\omega}\mathcal{A}^{(e,\mathrm{I})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} = \frac{-i}{\sqrt{8\pi^4 a}} \int \mathrm{d}^4x \,\sqrt{-g}j^\mu \varepsilon^{\mathrm{I}}_\mu \mathrm{e}^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp} \int \mathrm{d}\omega \,\mathrm{e}^{-i\omega(\vartheta-\tau)}\mathrm{e}^{\pi\omega/2a} K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right). \tag{5.73}$$

The last integral can be calculated as

$$\int \mathrm{d}\omega \,\mathrm{e}^{-i\omega(\vartheta-\tau)}\mathrm{e}^{\pi\omega/2a} K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) = \pi a \mathrm{e}^{i(k_0 t - k_z z)}\,, \tag{5.74}$$

where we used the coordinate transformation $t = a^{-1}\mathrm{e}^{a\xi}\sinh a\tau$, $z = a^{-1}\mathrm{e}^{a\xi}\cosh a\tau$ and the definition of the rapidity $k_z = k_\perp \sinh a\vartheta$, $k_0 = k_\perp \cosh a\vartheta$. Then, we find

$$\left| \int_{-\infty}^{+\infty} \mathrm{d}\omega \,\frac{\mathrm{e}^{-i\vartheta\omega}\mathcal{A}^{(e,\mathrm{I})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} \right|^2 = \frac{a}{8\pi^2}\left| \int \mathrm{d}^4x \, j^\mu \varepsilon^{\mathrm{I}}_\mu \mathrm{e}^{ik\cdot x} \right|^2, \tag{5.75}$$

where, by switching to Minkowski coordinates, $\sqrt{-g} = 1$. We now calculate the contribution of $\lambda = \text{II}$ in EQ. (5.71). The amplitude is given by

$$\begin{aligned}\mathcal{A}^{(e,\text{II})}_{(\omega,\mathbf{k}_\perp)} &= -\frac{i}{\sqrt{8\pi^4 a}} \int \mathrm{d}^4 x \, \sqrt{-g} k_\perp^{-1} \mathrm{e}^{-i\mathbf{k}_\perp \cdot \mathbf{x}_\perp} \\ &\times \left[j^\tau k_\perp \mathrm{e}^{a\xi} \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \mathrm{e}^{-i\omega(\vartheta-\tau)} \mathrm{e}^{\pi\omega/2a} K'_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) \right. \\ &\left. + i j^\xi \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \omega \mathrm{e}^{-i\omega(\vartheta-\tau)} \mathrm{e}^{\pi\omega/2a} K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) \right],\end{aligned} \tag{5.76}$$

and prime means derivative with respect to the argument. The integrals can be calculated as follows

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \mathrm{e}^{-i\omega(\vartheta-\tau)} \mathrm{e}^{\pi\omega/2a} K'_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) &= -i\pi a \sinh a(\vartheta - \tau) \mathrm{e}^{i(k_0 t - k_z z)} \,, \\ \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \omega \mathrm{e}^{-i\omega(\vartheta-\tau)} \mathrm{e}^{\pi\omega/2a} K_{i\omega/a}\left(\frac{k_\perp \mathrm{e}^{a\xi}}{a}\right) &= \pi a k_\perp \mathrm{e}^{a\xi} \cosh a(\vartheta - \tau) \mathrm{e}^{i(k_0 t - k_z z)} \,.\end{aligned} \tag{5.77}$$

Then, we can write

$$\begin{aligned}&\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-i\vartheta\omega} \mathcal{A}^{(e,\text{II})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1 - \mathrm{e}^{-2\pi\omega/a}}} \\ &= \frac{a^{3/2}}{\sqrt{8}\pi} \int \mathrm{d}^4 x \, \sqrt{-g} \mathrm{e}^{ik\cdot x} k_\perp^{-1} \big(j^\tau (k_0 t - k_z z) + j^\xi (k_0 z - k_z t) \big) \,.\end{aligned} \tag{5.78}$$

As with the $\lambda = \text{I}$ polarization, we wish to express the this integral using the polarization $\varepsilon^{\text{II}}_\mu$. In order to do this, we express the current in Minkowski coordinates. j^μ transforms as a vector. Thus,

$$\begin{aligned} j^t &= azj^\tau + atj^\xi \\ j^z &= atj^\tau + azj^\xi \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} j^\tau &= \frac{zj^t - tj^z}{a(z^2-t^2)} \\ j^\xi &= \frac{tj^t - zj^z}{a(z^2-t^2)} \end{aligned} . \tag{5.79}$$

We then find

$$\left| \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{\mathrm{e}^{-i\vartheta\omega} \mathcal{A}^{(e,\mathrm{II})}_{(\omega,\mathbf{k}_\perp)}}{\sqrt{1-\mathrm{e}^{-2\pi\omega/a}}} \right|^2 = \frac{a}{8\pi^2} \left| \int \mathrm{d}^4x \, \mathrm{e}^{ik\cdot x} j^\mu \varepsilon^{\mathrm{II}}_\mu \right|^2 , \tag{5.80}$$

where as for the first polarization, we switched to Minkowski coordinates where $\sqrt{-g} = 1$. The total probability is then given by

$$P^{\mathrm{vec}}_{\mathrm{tot}} = \sum_{\lambda=\mathrm{I,II}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \left| \int \mathrm{d}^4x \, \mathrm{e}^{ik\cdot x} j^\mu \varepsilon^{(\lambda)}_\mu \right|^2 . \tag{5.81}$$

A similar result for the emitted power was found in Ref. [92]. This probability can be obtained by a Minkowski-frame calculation. The Minkowski emission amplitude of a vector particle of polarization λ and momentum $\mathbf{k}$ is given by

$$\mathcal{A}^{(\lambda,\mathbf{k})}_{\mathrm{Min}} = \langle \mathbf{k}, \lambda | \, i \int \mathrm{d}^4x \, j^\mu(x) A_\mu(x) \, |0_{\mathrm{M}}\rangle \ , \tag{5.82}$$

where $|\mathbf{k}, \lambda\rangle = a^{\lambda\dagger}(\mathbf{k}) \, |0_{\mathrm{M}}\rangle$ and $a^{(\lambda)}(\mathbf{k})$ is the Minkowski annihilation operator of momentum $\mathbf{k}$ and polarization λ. Here A_μ is the full vector field in

Minkowski space. It can be written as

$$A_\mu = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{4} \left(a^\lambda(\mathbf{k}) \varepsilon_\mu^\lambda(\mathbf{k}) \mathrm{e}^{-ik\cdot x} + \mathrm{h.c.} \right) . \tag{5.83}$$

The operators $a^\lambda(\mathbf{k})$ satisfy $a^\lambda(\mathbf{k}) \left|0_\mathrm{M}\right\rangle = 0$ as well as

$$\left[a^\lambda(\mathbf{k}), a^{\lambda' \dagger}(\mathbf{k}') \right] = 2k_0 (2\pi)^3 \delta^{\lambda\lambda'} \delta(\mathbf{k} - \mathbf{k}') . \tag{5.84}$$

For the physical polarizations, $a^\lambda(\mathbf{k})$ is the same as $b_\mathbf{k}^\lambda$ defined in Eq. (5.26) up to a factor $\sqrt{(2\pi)^3 2k_0}$. Then, the Minkowski amplitude is

$$\mathcal{A}_{\mathrm{Min}}^{(\lambda,\mathbf{k})} = i \int \mathrm{d}^4x \, j^\mu \varepsilon_\mu^\lambda(\mathbf{k}) \mathrm{e}^{ik\cdot x} . \tag{5.85}$$

By defining the total probability as

$$P_{\mathrm{tot}}^{\mathrm{Min}} = \sum_{\lambda=\mathrm{I,II}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \left| \mathcal{A}_{\mathrm{Min}}^{(\lambda,\mathbf{k})} \right|^2 , \tag{5.86}$$

we precisely recover Eq. (5.81). We verified explicitly that the interaction probability between an accelerating charge inducing an *arbitrary* conserved current and the vector field is the same for both the inertial and accelerated frames. In the case of the accelerated frame two effects had to be considered. The first is the presence of the Unruh thermal bath and the second is the fact that the absorption *and* the emission of Rindler particles correspond to the emission of Minkowski particles. We define the Fourier transform of j^μ, as

$$j^\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}E}{(2\pi)^4}\widehat{j}^\mu(\mathbf{k},E)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}-iEt}\,. \tag{5.87}$$

Due to current conservation $\partial_\mu j^\mu = 0$, we have

$$k_\mu \hat{j}^\mu(\mathbf{k},k_0) = 0\,. \tag{5.88}$$

Then, expanding Eq. (5.81), we have

$$P_{\mathrm{tot}}^{\mathrm{vec}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\int \mathrm{d}^4x'\mathrm{d}^4x''\, j^\mu(x')j^{\nu*}(x'')\mathrm{e}^{ik\cdot(x'-x'')}\sum_{\lambda=\mathrm{I,II}}\varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)}\,. \tag{5.89}$$

For the second polarization vector we can make the substitution

$$\varepsilon_\mu^{\mathrm{II}} \to \varepsilon_\mu^{\mathrm{II}} - \frac{k_z}{k_\perp k_0}k_\mu\,, \tag{5.90}$$

where because $k\cdot k = 0$, the properties $\varepsilon^{\mathrm{II}}\cdot k = 0$ and $\varepsilon^{\mathrm{II}}\cdot\varepsilon^{\mathrm{II}} = -1$ still hold. Since the shift is proportional to k_μ the spacetime integral in Eq. (5.81) is invariant due to current conservation:

$$\int \mathrm{d}^4x\,\mathrm{e}^{ik\cdot x}j^\mu k_\mu = -i\int \mathrm{d}^4x\,\partial_\mu \mathrm{e}^{ik\cdot x}j^\mu = i\int \mathrm{d}^4x\,\mathrm{e}^{ik\cdot x}\partial_\mu j^\mu = 0\,. \tag{5.91}$$

For massless vector field the following equation holds

$$\sum_{\lambda=\mathrm{I,II}}\varepsilon_\mu^{(\lambda)}\varepsilon_\nu^{(\lambda)} = -\varepsilon_\mu^{(0)}\varepsilon_\nu^{(3)} - \varepsilon_\mu^{(3)}\varepsilon_\nu^{(0)} - \eta_{\mu\nu}\,, \tag{5.92}$$

where $\varepsilon_\nu^{(3)}$ and $\varepsilon_\nu^{(0)}$ are the nonphysical polarizations. We note that $\varepsilon_\nu^{(3)} \propto k_\nu$. After integrating over the spacetime variables, the total probability becomes

$$P_{\text{tot}}^{\text{vec}} = -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\left(\varepsilon_\mu^{(0)}\varepsilon_\nu^{(3)} + \varepsilon_\mu^{(3)}\varepsilon_\nu^{(0)} + \eta_{\mu\nu}\right)\widehat{j}^\mu(\mathbf{k},k_0)\widehat{j}^{*\nu}(\mathbf{k},k_0)\,. \tag{5.93}$$

Since $\varepsilon_\nu^{(3)} \propto k_\nu$, the first two terms vanish, and we arrive at the known formula (see for example Ref. [92])

$$P_{\text{tot}}^{\text{vec}} = -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\widehat{j}^\mu(k)\widehat{j}_\mu^*(k)\,, \tag{5.94}$$

where we write $\widehat{j}^\mu(\mathbf{k},k_0) = \widehat{j}^\mu(k)$.

5.6 Production of vector particles in Minkowski space

5.6.1 The massless case

In this section, we will look at Larmor radiation using only a Minkowski description. The main goal of this section is to give some physical intuition behind the necessity of integration by parts for the interaction amplitude in Eq. (5.47). Again, for simplicity, we consider a massless vector field given by

$$A_\mu(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\left(a_\mu(\mathbf{k})\mathrm{e}^{-ik\cdot x} + a_\mu^\dagger(\mathbf{k})\mathrm{e}^{ik\cdot x}\right). \tag{5.95}$$

We note that this expansion is equivalent to EQ. (5.83). EQ. (5.95) is more convenient for the following discussion. The commutation relations for the operators are given by

$$\left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] = -2k_0(2\pi)^3\eta_{\mu\nu}\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,. \tag{5.96}$$

We note that quantization of all four polarizations induces a negative norm (see Gupta-Bleuer quantization for example in Ref. [115]). We consider a point charge q which follows the world line $x^\mu(\tau)$, where τ is the proper time. The current $j^\mu(x)$ induced by the charge is

$$\begin{aligned} j^0(x) &= q\delta^{(3)}(\mathbf{x}-\mathbf{x}(\tau))\,, \\ j^i(x) &= q\mathrm{v}^i\delta^{(3)}(\mathbf{x}-\mathbf{x}(\tau))\,, \end{aligned} \tag{5.97}$$

where $\mathrm{v}^i = \mathrm{d}x^i/\mathrm{d}t$ and $t = x^0(\tau)$. Here we put $F(\tau) = 1$ and will discuss this choice at the end of the section. The current couples to the vector field via the interaction Hamiltonian density $\mathcal{H}_I = j^\mu(x)A_\mu(x)$. The state at time t is given to first order in perturbation theory by

$$|t\rangle = |0\rangle + |t_1\rangle = |0\rangle - i\int_{-\infty}^{t}\mathrm{d}t'\int \mathrm{d}^3\mathbf{x}\, j^\mu(x)A_\mu\,|0\rangle\ . \tag{5.98}$$

Here $|0\rangle$ is the Minkowski vacuum. We drop the subscript M since we are not considering the Rindler vacuum anymore. We define the amplitude of photon emission with momentum $\mathbf{k}$ at time t as

$$\begin{aligned}\mathcal{A}^\mu(\mathbf{k},t) &= i\int_{-\infty}^{t} \mathrm{d}^4x\, j^\mu(x)\mathrm{e}^{ik\cdot x} \\ &= iq\int_{-\infty}^{t} \mathrm{d}t'\, \frac{\mathrm{d}x^\mu}{\mathrm{d}t'}\mathrm{e}^{ik_0(t'-\mathbf{n}\cdot\mathbf{x}(\tau))}\,,\end{aligned} \tag{5.99}$$

where the unit vector $\mathbf{n}$ is defined as $\mathbf{n} = \mathbf{k}/k_0$. Since we are considering a massless vector, the energy is simply $k_0 = |\mathbf{k}|$. With this definition, we can express the state $|t_1\rangle$ as

$$\begin{aligned}|t_1\rangle &= -iq\int_{-\infty}^{t} \mathrm{d}t' \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{\mathrm{d}x^\mu}{\mathrm{d}t'}\mathrm{e}^{ik_0(t'-\mathbf{n}\cdot\mathbf{x}(\tau))}a_\mu^\dagger(\mathbf{k})\,|0\rangle \\ &= -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}^\mu(\mathbf{k},t)a_\mu^\dagger(\mathbf{k})\,|0\rangle\ .\end{aligned} \tag{5.100}$$

Next, we calculate the expectation value $\langle t|\, A_\mu(x)\,|t\rangle$ to first order. Out of the four terms from the perturbative expansion, only two are non-zero. The term $\langle 0|\, A_\mu(x)\,|0\rangle$ is trivially zero whereas the term $\langle t_1|\, A_\mu(x)\,|t_1\rangle$ is zero because it involves the terms of the form $\langle 0|\, a_\rho a_\mu a_\nu^\dagger\,|0\rangle$ and $\langle 0|\, a_\rho a_\mu^\dagger a_\nu^\dagger\,|0\rangle$ which vanish after using the commutation relations in Eq. (5.96). Therefore,

$$\begin{aligned}\langle t|\, A_\mu(x)\,|t\rangle &= \langle 0|\, A_\mu(x)\,|t_1\rangle + \langle t_1|\, A_\mu(x)\,|0\rangle \\ &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\left[\mathcal{A}_\mu(\mathbf{k},t)\mathrm{e}^{-ik\cdot x} + \mathcal{A}_\mu^*(\mathbf{k},t)\mathrm{e}^{ik\cdot x}\right].\end{aligned} \tag{5.101}$$

This result will be useful in what follows. We write the amplitude $\mathcal{A}_\mu(\mathbf{k},t)$ as

$$\mathcal{A}^\mu(\mathbf{k},t) = q\int_{-\infty}^{t} \mathrm{d}t'\, \frac{\mathrm{v}^\mu}{k_0(1-\mathbf{n}\cdot\mathbf{v})}\frac{\mathrm{d}}{\mathrm{d}t'}\mathrm{e}^{ik_0(t'-\mathbf{n}\cdot\mathbf{x}(\tau))}\,, \tag{5.102}$$

and then we integrate by parts assuming a convergence factor in the infinite past to remove the past boundary term. We then have

$$\mathcal{A}^\mu(\mathbf{k},t) = \frac{q\mathrm{v}^\mu}{k_0(1-\mathbf{n}\cdot\mathbf{v})}\mathrm{e}^{ik_0(t-\mathbf{n}\cdot\mathbf{x}(\tau))} - \frac{q}{k_0}\int_{-\infty}^{t}\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{v}^\mu}{1-\mathbf{n}\cdot\mathbf{v}}\right)\mathrm{e}^{ik_0(t'-\mathbf{n}\cdot\mathbf{x}(\tau))}\,. \tag{5.103}$$

We choose the motion to be one of constant velocity with respect to t, i.e. $\mathbf{x}(t) = \mathbf{v}t$, where $\mathbf{v}$ is constant with respect to t. Then the expectation value of the first term only using EQ. (5.101) is

$$\begin{aligned}
\langle t|\,A^\mu(x)\,|t\rangle_{\text{boundary}} &= \frac{1}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{q\mathrm{v}^\mu\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)}}{k_0^2(1-\mathbf{n}\cdot\mathbf{v})}+\text{h.c.}\right]\\
&= \frac{1}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{q\mathrm{v}^\mu}{k_0^2}\left[\frac{1}{1-\mathbf{n}\cdot\mathbf{v}}+\frac{1}{1+\mathbf{n}\cdot\mathbf{v}}\right]\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)}\\
&= \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{q\mathrm{v}^\mu}{k_0^2(1-(\mathbf{n}\cdot\mathbf{v})^2)}\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)}\,.
\end{aligned} \tag{5.104}$$

If we choose additionally a static charge i.e. $\mathbf{v} = \mathbf{0}$, then the above integral can be calculated in spherical coordinates as

$$\begin{aligned}
\langle t|\,A^0_{\mathbf{v}=\mathbf{0}}(x)\,|t\rangle_{\text{boundary}} &= \frac{q}{4\pi^2}\int_0^{+\infty}\mathrm{d}k\int_{-1}^{1}\mathrm{d}\!\cos\theta\,\mathrm{e}^{ikr\cos\theta}\\
&= \frac{q}{2\pi^2 r}\int_0^{+\infty}\frac{\mathrm{d}k}{k}\sin kr\\
&= \frac{q}{4\pi r}\,,
\end{aligned} \tag{5.105}$$

which is the Coulomb potential. Therefore, the integration by parts and the omission of the boundary term correspond to removing the Coulomb potential, which is not relevant for the radiation process we described previously in the context of the Unruh effect. For a discussion on Coulomb terms, see also Ref. [96]. If the third component of $\mathbf{v}$ is non-zero we have, writing $\mathrm{v}^z = v$,

$$k_0^2(1-(\mathbf{n}\cdot\mathbf{v})^2) = k_x^2 + k_y^2 + \gamma^{-2}k_z^2\,, \tag{5.106}$$

where, as usual $\gamma = (1-v^2)^{-1/2}$. The expectation value is then found to be

$$\begin{aligned}\langle t|\, A^0_{\mathrm{v}^z=v}(x)\,|t\rangle_{\text{boundary}} &= \gamma\,\langle t|\, A^0_{\mathbf{v}=\mathbf{0}}(x,y,\gamma(z-vt))\,|t\rangle_{\text{boundary}}\ ,\\ \langle t|\, A^z_{\mathrm{v}^z=v}(x)\,|t\rangle_{\text{boundary}} &= \gamma v\,\langle t|\, A^0_{\mathbf{v}=\mathbf{0}}(x,y,\gamma(z-vt))\,|t\rangle_{\text{boundary}}\ .\end{aligned} \tag{5.107}$$

The result is the Lorentz transformed Coulomb potential. Since we are interested in radiation processes we can safely ignore the Coulomb potential term in the interaction amplitude. We define the total amplitude by letting $t\to+\infty$:

$$\mathcal{A}^\mu(\mathbf{k}) = \lim_{t\to+\infty}\mathcal{A}^\mu(\mathbf{k},t) = -\frac{q}{k_0}\int_{-\infty}^{+\infty}\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x^\mu/\mathrm{d}t}{1-\mathbf{n}\cdot\mathbf{v}}\right)\mathrm{e}^{ik_0(t-\mathbf{n}\cdot\mathbf{x}(\tau))}\,. \tag{5.108}$$

We define $\xi = t - \mathbf{n}\cdot\mathbf{x}(t)$. Then, the amplitude is can be simply written as

$$\mathcal{A}^\mu(\mathbf{k}) = -\frac{q}{k_0}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\xi^2}\mathrm{e}^{ik_0\xi}\,. \tag{5.109}$$

Furthermore it is convenient to write the amplitude using the proper time of the particle. The proper four-velocity v^μ is defined as $v^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. To change integration variables, it is useful to note that $\mathrm{d}\xi/\mathrm{d}\tau = n \cdot v$ where $n^\mu = k^\mu/k_0 = (1, \mathbf{n})$. The amplitude is thus,

$$\begin{aligned}
\mathcal{A}^\mu(\mathbf{k}) &= -\frac{q}{k_0}\int_{-\infty}^{+\infty} \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\xi}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)\mathrm{e}^{ik\cdot x} \\
&= -q\int_{-\infty}^{+\infty} \mathrm{d}\tau\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{v^\mu}{k\cdot v}\right)\mathrm{e}^{ik\cdot x} \\
&= -q\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{k\cdot v}\left(a^\mu - \frac{k\cdot a}{k\cdot v}v^\mu\right)\mathrm{e}^{ik\cdot x}\,,
\end{aligned} \tag{5.110}$$

where a^μ is the proper four-acceleration defined as $a^\mu = \mathrm{d}v^\mu/\mathrm{d}\tau = \mathrm{d}^2x^\mu/\mathrm{d}\tau^2$. We calculate the total emission probability as

$$\begin{aligned}
P = \lim_{t\to\infty}\langle t_1|t_1\rangle &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3 2k_0'}\,\langle 0|\, a_\nu(\mathbf{k}')a_\mu^\dagger(\mathbf{k})\,|0\rangle\,\mathcal{A}^{\nu*}(\mathbf{k}')\mathcal{A}^\mu(\mathbf{k}) \\
&= -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}_\mu^*(\mathbf{k})\mathcal{A}^\mu(\mathbf{k})\,,
\end{aligned} \tag{5.111}$$

where is the second line we used the commutation relations in Eq. (5.96). The total energy emitted is defined as usual by multiplying the integrand of the probability by a factor of energy. Thus

$$E = -\frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{A}_\mu^*(\mathbf{k})\mathcal{A}^\mu(\mathbf{k})\,. \tag{5.112}$$

The total emitted energy can be expressed in spherical coordinates follows

$$E = -\frac{q^2}{16\pi^3}\int \mathrm{d}\Omega \int \mathrm{d}\xi\mathrm{d}\bar{\xi}\,\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\xi^2}\frac{\mathrm{d}^2x_\mu}{\mathrm{d}\bar{\xi}^2}\int_0^{+\infty}\mathrm{d}k_0\,\mathrm{e}^{ik_0(\xi-\bar{\xi})}\,. \tag{5.113}$$

As before, it is convenient to extend the lower bound of the energy integral to $-\infty$. For this, we notice that the exchange $\xi \leftrightarrow \bar{\xi}$ is equivalent to $k_0 \to -k_0$. To compensate we have to multiply the result by 1/2. The integral over k_0 is then simply $2\pi\delta(\xi - \bar{\xi})$ and we find

$$\begin{aligned}
E &= -\frac{q^2}{32\pi^3}\int \mathrm{d}\Omega \int \mathrm{d}\xi \mathrm{d}\bar{\xi}\, \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\xi^2}\frac{\mathrm{d}^2 x_\mu}{\mathrm{d}\bar{\xi}^2}\int_{-\infty}^{+\infty} \mathrm{d}k_0 \mathrm{e}^{ik_0(\xi-\bar{\xi})} \\
&= -\frac{q^2}{16\pi^2}\int \mathrm{d}\Omega \int \mathrm{d}\xi\, \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\xi^2}\frac{\mathrm{d}^2 x_\mu}{\mathrm{d}\xi^2} \\
&= -\frac{q^2}{16\pi^2}\int \mathrm{d}\Omega \int \frac{\mathrm{d}\tau}{(n\cdot v)^3}\left(a^\mu - \frac{(n\cdot a)v^\mu}{n\cdot v}\right)\left(a_\mu - \frac{(n\cdot a)v_\mu}{n\cdot v}\right) \\
&= \frac{q^2}{16\pi^2}\int \mathrm{d}\Omega \int \frac{\mathrm{d}\tau}{(n\cdot v)^3}\left[-a_\mu a^\mu - \frac{(n\cdot a)^2}{(n\cdot v)^2}\right],
\end{aligned} \tag{5.114}$$

where in the last step we used the fact that $v \cdot v = 1$ which implies that $a\cdot v = 0$. We can apply this result to recover the Larmor formula. Assuming a uniformly accelerating particle with trajectory $t(\tau) = a^{-1}\sinh a\tau$ and $z(\tau) = a^{-1}\cosh a\tau$, we find $v^0 = \cosh a\tau$, $v^z = \sinh a\tau$ and $a^0 = a\sinh a\tau$, $a^z = a\cosh a\tau$ where a here is the proper acceleration which is a constant. Then using $-a_\mu a^\mu = a^2$, we find

$$\begin{aligned}
E &= \frac{q^2 a^2}{8\pi}\int \mathrm{d}\tau \int_{-1}^{1} \mathrm{d}n_z \left[\frac{1}{(\cosh a\tau - n_z \sinh a\tau)^3} - \frac{(\sinh a\tau - n_z\cosh a\tau)^2}{(\cosh a\tau - n_z \sinh a\tau)^5}\right] \\
&= \frac{q^2 a^2}{6\pi}\int \mathrm{d}\tau\, \cosh a\tau\,,
\end{aligned} \tag{5.115}$$

which is the Larmor formula as expected. The presence of the factor $\cosh a\tau$ in the proper time integral is due to the boost. We can write $\mathrm{d}\tau \cosh a\tau = \mathrm{d}(a^{-1}\sinh a\tau) = \mathrm{d}t$. $\cosh a\tau$ is the Lorentz factor $\gamma = v^0$ for the special case of uniform acceleration. More generally, it is straightforward to show that for a trajectory along the $z-$axis

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \frac{\gamma q^2 a^2}{6\pi}\,, \tag{5.116}$$

where we defined $-a_\mu a^\mu = a^2$. We now will link the vector amplitude $\mathcal{A}^\mu$ using Eq. (5.110) with the amplitude scalar amplitude $\mathcal{A}^{\text{vec}}$ found in Eq. (5.48) in the context of the Unruh effect. Recalling that for a massless scalar the the momentum can be written using the rapidity as $k^0 = k_\perp \cosh a\vartheta$ and $k^z = k_\perp \sinh a\vartheta$, we obtain

$$\begin{aligned}
k\cdot x &= \frac{k_\perp}{a}(\cosh a\vartheta \sinh a\tau - \sinh a\vartheta \cosh a\tau) = -\frac{k_\perp}{a}\sinh a(\vartheta-\tau)\,,\\
k\cdot v &= k_\perp(\cosh a\vartheta \cosh a\tau - \sinh a\vartheta \sinh a\tau) = k_\perp \cosh a(\vartheta-\tau)\,,\\
k\cdot a &= k_\perp a(\cosh a\vartheta \sinh a\tau - \sinh a\vartheta \cosh a\tau) = -k_\perp a \sinh a(\vartheta-\tau)\,.
\end{aligned} \tag{5.117}$$

The non-zero components of $\mathcal{A}^\mu$ read

$$\begin{aligned}
\mathcal{A}^0(\mathbf{k}) &= -q\int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau}{(k\cdot v)^2}[(k\cdot v)a^0 - (k\cdot a)v^0]\mathrm{e}^{ik\cdot x}\\
&= -\frac{qa\sinh a\vartheta}{k_\perp}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau\,\mathrm{e}^{-i(k_\perp/a)\sinh a(\vartheta-\tau)}}{\cosh^2 a(\vartheta-\tau)}\,,
\end{aligned} \tag{5.118}$$

and

$$\begin{aligned}
\mathcal{A}^z(\mathbf{k}) &= -q\int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau}{(k\cdot v)^2}[(k\cdot v)a^z - (k\cdot a)v^z]\mathrm{e}^{ik\cdot x}\\
&= -\frac{qa\cosh a\vartheta}{k_\perp}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau\,\mathrm{e}^{-i(k_\perp/a)\sinh a(\vartheta-\tau)}}{\cosh^2 a(\vartheta-\tau)}\,.
\end{aligned} \tag{5.119}$$

Since $(k_0/k_\perp, k_z/k_\perp) = (\cosh a\vartheta, \sinh a\vartheta)$, we can write

$$\mathcal{A}_\mu(\mathbf{k}) = -\frac{qa}{k_\perp}\varepsilon^{\mathrm{II}}_\mu(\mathbf{k}) \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau\, \mathrm{e}^{-i(k_\perp/a)\sinh a(\vartheta-\tau)}}{\cosh^2 a(\vartheta-\tau)}\,, \tag{5.120}$$

where $\varepsilon^{\mathrm{II}}_\mu(\mathbf{k})$ is the polarization vector defined in Eq. (5.25). We can establish that using Eq. (5.48), $\mathcal{A}_\mu(\mathbf{k}) = -\varepsilon^{\mathrm{II}}_\mu(\mathbf{k})\mathcal{A}^{\mathrm{vec}}(\mathbf{k})$ with $F(\tau) = 1$. To arrive at the Larmor formula for both the Minkowski and Rindler point of view, two different methods were used. In the non-inertial case we turned on and off the interaction using the function $F(\tau)$. In the inertial calculation, we identified the boundary term as the energy contribution coming from the Coulomb potential and omitted it, as it is not related to radiation (see also Ref. [96]). In the Rindler case, the function $F(\tau)$ allowed us to make the integrals convergent. Since we did not introduce a function of this sort in the inertial calculation, one might ask how the integrals are convergent in this case. Taking the limit $t \to +\infty$ in Eq. (5.99) does not give a finite result. To regularize the integral, in Ref. [116], a compactly supported cutoff function was introduced, $\chi(x)$ which has exactly the same form as the function $F(\tau)$. We multiply the integrand of $\mathcal{A}^\mu(\mathbf{k})$ by a factor of $\chi(r\xi)$ where $0 < r \leq 1$ is a parameter that we will send to 0. Then,

$$\begin{aligned}\mathcal{A}^\mu(\mathbf{k}) &= iq\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\mathrm{e}^{ik_0\xi} \to iq\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\chi(r\xi)\mathrm{e}^{ik_0\xi}\\ &= -\frac{q}{k_0}\int_{-\infty}^{+\infty}\mathrm{d}\xi\left[\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\xi^2}\chi(r\xi) + r\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\xi^2}\chi'(r\xi)\right]\mathrm{e}^{ik_0\xi}\,.\end{aligned} \tag{5.121}$$

Since $\chi(r\xi)$ is 1 for $\xi \in [-c/r, c/r]$ where c is some parameter, by sending $r \to 0$, then second term in the above expression vanishes and we recover

Eq. (5.109).

5.6.2 The massive case

We now look at radiation of massive vector particles in Minkowski space. The theory of the free vector field with mass m_A in Minkowski spacetime is given by Eq. (5.55), where $\sqrt{-g} \to 1, \nabla_\mu \to \partial_\mu$. The equations of motion are $\partial_\mu F^{\mu\nu} + m_A^2 A^\nu = 0$. Taking the divergence of this equation leads to $\partial_\mu A^\mu = 0$ since $m_A \neq 0$ and $F^{\mu\nu}$ is antisymmetric. Inserting this result back into the equation of motions gives

$$\left(\Box + m_A^2\right) A^\mu = 0\,. \tag{5.122}$$

The last equation could had been obtained immediately by adding the Feynman gauge fixing term $-(\partial_\mu A^\mu)^2/2$ in the Lagrangian. The massive vector field can be expanded as

$$A_\mu = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \sum_\lambda \left[\varepsilon_\mu^\lambda a^\lambda(\mathbf{k}) \mathrm{e}^{-ik\cdot x} + \text{h.c.}\right], \tag{5.123}$$

where now $k_0 = \sqrt{\mathbf{k}^2 + m_A^2}$ and the sum is taken over the three physical polarizations. The polarization vectors satisfy Eq. (5.61) as well as $\varepsilon^{\mu\lambda}\varepsilon_\mu^{*\lambda'} = -\delta^{\lambda\lambda'}$. The non-zero commutation relations between the operators are

$$\left[a^\lambda(\mathbf{k}), a^{\lambda'\dagger}(\mathbf{k}')\right] = \delta^{\lambda\lambda'}(2\pi)^3 2k_0 \delta^{(3)}(\mathbf{k} - \mathbf{k}')\,. \tag{5.124}$$

Similarly to the massless case we couple the massive vector field to the current given by Eq. (5.97). Then to first order in perturbation theory, the one particle state at time t is

$$|t_1\rangle = -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \mathcal{A}^\mu(\mathbf{k},t) \sum_\lambda \varepsilon_\mu^{\lambda *} a^{\lambda\dagger}(\mathbf{k}) |0\rangle \ , \tag{5.125}$$

where $\mathcal{A}^\mu(\mathbf{k},t)$ is given by the first line in Eq. (5.99) (we note that since the particles have a mass, $\mathbf{k}/k_0$ is not a unit vector). Then, using the completeness relation of the polarization vectors, with $|t\rangle = |0\rangle + |t_1\rangle$

$$\langle t| A_\mu |t\rangle = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2k_0} \left[\left(\mathcal{A}_\mu(\mathbf{k},t) - \frac{k_\mu k_\nu \mathcal{A}^\nu(\mathbf{k},t)}{m_A^2} \right) \mathrm{e}^{-ik\cdot x} + \text{h.c.} \right] . \tag{5.126}$$

The amplitude $\mathcal{A}^\mu(\mathbf{k},t)$ is an spacetime integral over $j^\mu \mathrm{e}^{ik\cdot x}$. By integration by parts of $k_\nu j^\mu \mathrm{e}^{ik\cdot x}$, due to current conservation, only the boundary term remains, which vanishes when $t \to +\infty$. Since in the end, we wish to take this limit, we ignore the term proportional to $k_\nu \mathcal{A}^\nu(\mathbf{k},t)$. We now wish to identify the physical motivation for integrating the amplitude by parts. $\mathcal{A}^\mu(\mathbf{k},t)$ is given by Eq. (5.103) with replacement $\mathbf{n} \to \mathbf{k}/\sqrt{\mathbf{k}^2 + m_A^2}$. Then, for a world line given by $\mathbf{x}(t) = \mathbf{0}$, the expectation value of the boundary term is

$$\begin{aligned} \langle t| A^0_{\mathbf{v}=\mathbf{0}}(x) |t\rangle_{\text{boundary}} &= \frac{q}{4\pi^2} \int_0^{+\infty} \mathrm{d}k \, \frac{k^2}{k^2 + m_A^2} \int_{-1}^{1} \mathrm{d}\cos\theta \, \mathrm{e}^{ikr\cos\theta} \\ &= \frac{q}{2\pi^2 r} \int_0^{+\infty} \frac{\mathrm{d}k\, k}{k^2 + m_A^2} \sin kr \\ &= \frac{q \mathrm{e}^{-m_A r}}{4\pi r} , \end{aligned} \tag{5.127}$$

which is the Yukawa potential. This is expected, since in the massless case the boundary term corresponded to the Coulomb potential. The interaction probability given by taking the limit as $t \to +\infty$ of $\langle t_1 | t_1 \rangle$ is the same as in the massless case (see the second line of Eq. (5.111)). This holds for the energy as well and the amplitude is given by Eq. (5.110). Because of the presence of a non-zero mass, we cannot integrate over the momentum as we did in Eq. (5.114) to obtain a delta function. However, taking the limit as $m_A \to 0$, in the expression of the energy gives the Larmor formula.

6 Gravitational Wave Production and Detection

In this chapter, we study the production and detection of gravitational waves using high-energy laser beams. Contrary to the approach in previous chapters, we do not quantize the field that describes the emitted particles, nor do we consider emission from an accelerating charge. For spin-2 particles, this was done in the context of the Unruh effect in Ref. [61], where the authors explicitly verified the agreement between the interaction rate calculated in the uniformly accelerated frame and the emission rate in the inertial frame. The treatment is similar to what was discussed in the case of spin-0 and spin-1 particles. For accelerated- and inertial-frame calculations to agree, the FDU thermal bath needs to be taken into account. Moreover, both emission and absorption of a Rindler graviton correspond to emission of a Minkowski particle. The emission rate to first order in perturbation theory is given by [61]

$$R^{(\text{spin-2})}(\mathbf{k}_\perp) = \frac{\mu^2}{8\pi^3 a}\left|K_2\left(\frac{k_\perp}{a}\right)\right|^2, \tag{6.1}$$

which is similar to the cases of spin 0 and spin 1 particles (see EQS. (4.76) and (5.50)) and where μ is the mass of the particle. In fact, the three results can be combined for massless fields in the form

$$R^s(k_\perp) = \frac{q^2}{4\pi^3 a s!}\left|K_s\left(\frac{k_\perp}{a}\right)\right|^2, \tag{6.2}$$

where $s = 0, 1, 2$ is the integer spin of the particle and q is the coupling between the particle and the quantum field. Similarly to the electromagnetic case, the total rate is infrared divergent because $K_2(z) \sim z^{-2}$ for small z.

Gravitational radiation is more commonly discussed in the context of BH binaries. In Ref. [117], the graviton field was quantized and coupled to an Unruh-DeWitt detector which is able to make internal energy transitions from E_i to E_f while simultaneously emitting a graviton. The purpose was to study the recoil for gravitational radiation. The Lagrangian interaction is schematically given by $h_{\mu\nu}T^{\mu\nu}$ where $h_{\mu\nu}$ is the graviton field and $T^{\mu\nu}$ is the stress energy tensor. The system in consideration (the Unruh-DeWitt detector) is a BH binary, and the resulting power agrees with the classical Peters-Mathews formula [118].

In this chapter, we will consider the production and detection of classical waves. SEC. 6.1 is a review based on Ref. [119]. SEC. 6.3, which describes the detection of GWs using high-energy lasers, is based on Ref. [1] which contains contributions from all coauthors.

6.1 GW FORMALISM

The full General Relativity (GR) action includes the Einstein-Hilbert action as well as a matter part which contains the stress energy tensor. The former is given by

$$S_{\mathrm{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}R\,, \tag{6.3}$$

where R is the Ricci scalar. Varying the action with respect to the metric, we find the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\,T_{\mu\nu}\,. \tag{6.4}$$

GWs are studied within the context of linearized gravity. This means that the metric is written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)\,, \qquad |h_{\mu\nu}| \ll 1\,. \tag{6.5}$$

All relevant quantities are calculated to leading order in h. The first of them is the Riemann tensor given by

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})\,. \tag{6.6}$$

Eq. (6.4) can be more easily written in terms of the tensor

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,, \tag{6.7}$$

where the trace is $h = \eta^{\mu\nu}h_{\mu\nu}$ because indices are raised or lowered using the Minkowski metric. Then, Eq. (6.4) is

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -16\pi G\,T_{\mu\nu}\,. \tag{6.8}$$

To proceed with the resolution of these equations of motion, we must choose a reference frame. Choosing a frame breaks the invariance of GR under coordinate transformation and allows one to get rid of nonphysical degrees of freedom. Assuming that we chose a reference frame where Eq. (6.5) holds, there is a residual gauge freedom. Indeed, GR is invariant under the symmetry group of coordinate transformations

$$x^\mu \to x'^\mu(x)\,, \tag{6.9}$$

where the new set of coordinates x'^μ is a diffeomorphism, meaning that it is bijective, invertible, differentiable and has a differentiable inverse. Under the transformation EQ. (6.9), the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x) \,. \tag{6.10}$$

We now choose a particular set of transformations given by

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \,, \tag{6.11}$$

where $|\partial_{\mu}\xi^{\nu}| = \mathcal{O}(h)$. Using EQS. (6.10) and (6.11), the perturbation $h_{\mu\nu}$ transforms as

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} \,, \tag{6.12}$$

which is the residual gauge freedom we were looking for. One can see that the slow varying diffeormorphism condition $|\partial_{\mu}\xi^{\nu}| = \mathcal{O}(h)$ was imposed to preserve the smallness of the perturbation under the coordinate transformation. We now use the gauge freedom to choose the Lorenz gauge

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0 \tag{6.13}$$

in the linearized field EQ. (6.8). The reason why it is possible to impose this condition is the following. Using EQS. (6.7) and (6.12) we find that

$$\partial^{\nu}\bar{h}_{\mu\nu} \to (\partial^{\nu}\bar{h}_{\mu\nu})' = \partial^{\nu}\bar{h}_{\mu\nu} - \Box\xi_{\mu} \,. \tag{6.14}$$

Therefore, if the initial configuration was such that $\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x)$, where f_μ is some function of the coordinates, we choose the coordinate transformation such that $\Box\xi_\mu(x) = f_\mu(x)$. This is always possible because the d'Alembertian operator is invertible. In this new frame, therefore, the Lorenz condition in EQ. (6.13) is satisfied. The Einstein equations now read

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,, \tag{6.15}$$

or in terms of the original perturbation,

$$\Box h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right), \tag{6.16}$$

where $T = T^\mu_\mu$ is the trace of $T_{\mu\nu}$. Before discussing the different frames one can choose to study GWs, we link the theory of linearized gravity with that of a free spin-2 field. Expanding the action EQ. (6.3) to second order in $h_{\mu\nu}$, one obtains

$$S_{\rm EH} = -\frac{1}{64\pi G}\int \mathrm{d}^4x\,[\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \partial_\mu h\partial^\mu h + 2\partial_\mu h^{\mu\nu}\partial_\nu h - 2\partial_\mu h^{\mu\nu}\partial_\rho h^\rho_\nu]\,. \tag{6.17}$$

In the case of a spin 1 massless particle, for example the photon, one is interested in a theory which is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu\alpha$, where α is some function of the spacetime coordinates. In the case of a spin 2 particle, the generalization comes from assigning a Lorentz index to the function α and requiring that the field $h_{\mu\nu}$ is symmetric after the transformation. Therefore $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \,, \tag{6.18}$$

which is exactly the symmetry of linearized gravity Eq. (6.12). We want to write a gauge invariant action quadratic in the spin-2 field. Since we consider massless gravitons we do not write terms $h^2, h_{\mu\nu}h^{\mu\nu}$. The most general action is

$$S_{\rm FP} = \int \mathrm{d}^4x \left[a_1 \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + a_2 \partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho} + a_3 \partial_\nu h^{\mu\nu} \partial_\mu h + a_4 \partial_\mu h \partial^\mu h\right]. \tag{6.19}$$

All terms of the form $h\partial\partial h$ are related to the ones above by integration by parts. Requiring that the action is invariant under Eq. (6.18), the coefficients a_i are all fixed apart from an overall normalization. One finds $a_1 = -\frac{1}{2}a_2 = \frac{1}{2}a_3 = -a_4$. Choosing $a_1 = -1/2$ for the energy to be positive definite,

$$S_{\rm FP} = -\frac{1}{2}\int \mathrm{d}^4x \left[\partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} - \partial_\mu h \partial^\mu h + 2\partial_\nu h^{\mu\nu} \partial_\mu h - 2\partial_\rho h_{\mu\nu} \partial^\nu h^{\mu\rho}\right]. \tag{6.20}$$

This is the Fierz-Pauli action [120] (ignoring the mass term). We see that after rescaling

$$h_{\mu\nu} \to \frac{1}{\sqrt{32\pi G}} h_{\mu\nu} \,, \tag{6.21}$$

Eq. (6.20) is equal to Eq. (6.17) (the last term in Eq. (6.20) is the same as the last in Eq. (6.17) after two integrations by parts). The theory of a massless spin 2 field is equivalent to the geometric approach of the linearized Einstein-Hilbert action. The coupling of the graviton field with other fields is achieved via the term:

$$S_{\text{int}} = \frac{\kappa}{2} \int \mathrm{d}^4x \, h_{\mu\nu} T^{\mu\nu} \,. \tag{6.22}$$

To recover the Einstein equations, the coupling constant must be $\kappa = \sqrt{32\pi G}$.

6.1.1 The TT frame

To study the propagation of GWs, we are interested in the region where $T_{\mu\nu} = 0$, which is outside the source. In this case, Eq. (6.16) becomes

$$\Box h_{\mu\nu} = 0 \,. \tag{6.23}$$

Therefore, the form of the perturbation can be simplified by noticing that the Lorentz gauge in Eq. (6.13) is not uniquely fixed. Indeed, to impose this gauge condition, we chose the coordinate transformation in Eq. (6.11) such that $\Box\xi_\mu(x) = \partial^\nu \bar{h}_{\mu\nu}$. But the choice $\xi'^\mu = \xi^\mu + \zeta^\mu$, with $\Box\zeta^\mu = 0$ would still impose the Lorentz gauge. Therefore, by performing an additional coordinate transformation $x^\mu \to x^\mu + \zeta^\mu$ with $\Box\zeta^\mu = 0$, the metric perturbation $h_{\mu\nu}$ in Eq. (6.23) transforms as $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu$. Therefore, the new metric still satisfies $\Box h'_{\mu\nu} = 0$. We can now conveniently choose ζ^μ, to impose four conditions on $h'_{\mu\nu}$ (for convenience we drop the prime in what follows). ζ^0 can be chosen to fix $h = 0$ and the remaining ζ^i are chosen to fix $h^{0i} = 0$. Since $h = 0$, $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and therefore the Lorentz gauge condition Eq. (6.13) is valid also for $h_{\mu\nu}$:

$$\partial^0 h_{00} + \partial^i h_{0i} = 0 \rightarrow \partial^0 h_{00} = 0. \tag{6.24}$$

Therefore, h_{00} is constant in time. Since the GW is time-dependent, this means that $h_{00} = 0$. Overall we have:

$$h^{0\mu} = 0, \qquad h^i_i = 0, \qquad \partial^i h_{ij} = 0, \qquad \Box h_{ij} = 0\,. \tag{6.25}$$

This defines the traceless-transverse (TT) frame. Of the 10 initial degrees of freedom, 6 remained after imposing the Lorentz gauge. Furthermore, conveniently fixing the residual gauge, we removed 4 additional degrees of freedom. We notice that the TT frame cannot be chosen inside the source where $T_{\mu\nu} \neq 0$. Eq. (6.25) allows for plane-wave solutions $h_{ij} \propto \mathrm{e}^{ik\cdot x}, k = \omega(1, \hat{n})$, where $\hat{n}$ is the direction of propagation of the wave. The condition $\partial^i h_{ij} = 0$ becomes $n^i h_{ij} = 0$. Therefore, the non-zero components of h_{ij} are in the plane orthogonal to $\hat{n}$.

Given a plane wave $h_{\mu\nu}$ propagating in the direction $\hat{n}$ in the Lorentz frame but not yet in the TT frame, we can find $h^{\mathrm{TT}}_{\mu\nu}$, by introducing the following transverse projectors:

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j\,, \qquad \Lambda_{ij,kl}(\hat{n}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\,. \tag{6.26}$$

The Λ tensor is traceless. The projectors satisfy

$$P_{ik}P_{kj} = P_{ij}, \quad n^i P_{ij} = 0, \quad \Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}, \quad n^a \Lambda_{ij,kl} = 0\,, \tag{6.27}$$

where $a = i, j, k, l$. h_{ij}^{TT} and h_{ij} (Lorentz frame) are related by

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl}\,, \tag{6.28}$$

since by construction, the right-hand side is transverse and traceless in the indices (i, j) and they satisfy the same equations of motion $\Box h_{ij}^{\mathrm{TT}} = \Box h_{ij} = 0$.

Physical interpretation of the TT frame

We consider a test particle in a curved background with metric $g_{\mu\nu}$, in the absence of external non-gravitational forces. Its equation of motion is the geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho}(x)\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0\,. \tag{6.29}$$

where τ is the proper time of the particle. We assume it is at rest at $\tau = 0$. Using EQ. (6.29),

$$\left.\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2}\right|_{\tau=0} = -\Gamma^i_{\nu\rho}(x)\left.\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right|_{\tau=0} = -\Gamma^i_{00}(x)\left.\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\right|_{\tau=0}, \tag{6.30}$$

where in the second step we used the fact that the particle is initially at rest. The linearized Christoffel symbol is:

$$\Gamma^i_{00}(x) = \frac{1}{2}(\partial_0 h_{0i} - \partial_i h_{00}) = 0\,, \tag{6.31}$$

as it can be seen from EQ. (6.25). Therefore, if the test particle is at rest at $\tau = 0$, which implies $\mathrm{d}x^i/\mathrm{d}\tau = 0$, then $\mathrm{d}^2x^i/\mathrm{d}\tau^2 = 0$ at $\tau = 0$ as well. One can verify by induction that this is the case for higher order derivatives of $\mathrm{d}x^i/\mathrm{d}\tau = 0$ at $\tau = 0$ as well. Therefore, $\mathrm{d}x^i/\mathrm{d}\tau = 0$ at all times. This means that particles that were at rest before the arrival of the wave remain at rest after its passage as well. Physically, the coordinates in the TT frame are stretched, such that particles that were at rest remain so even in the presence of a GW. This does not mean that the GW had no physical effect, rather it shows that we chose the coordinates in such a way that masses at rest initially remain so. Instead, physical effects can be measured using proper distances and proper times.

6.1.2 Proper detector frame

Although the TT frame is convenient because of the simple form of the metric perturbation, it is not usually used for experiments. A natural choice of frame is the Proper detector (PD) frame which generalizes the idea of an inertial observer to curved space-time. Moreover, the electromagnetic fields are defined in the PD frame. The metric in this frame is [82, 121, 122]

$$
\begin{aligned}
g_{00} &= 1 - 2a_i x^i - (a_i x^i)^2 + (\boldsymbol{\omega} \times \mathbf{x})^2 - \gamma_{00} - 2(\boldsymbol{\omega} \times \mathbf{x})_i \gamma_{0i} \\
&\qquad\qquad\qquad\qquad - (\boldsymbol{\omega} \times \mathbf{x})_i (\boldsymbol{\omega} \times \mathbf{x})_j \gamma_{ij} \,, \\
g_{0i} &= (\boldsymbol{\omega} \times \mathbf{x})_i - \gamma_{0i} - (\boldsymbol{\omega} \times \mathbf{x})_j \gamma_{ij} \,, \\
g_{ij} &= -\delta_{ij} - \gamma_{ij} \,,
\end{aligned}
\tag{6.32}
$$

with a^i being the acceleration (due to Earth's gravity) and ω^i the angular velocity with respect to local gyroscopes [119]. The coefficients $\gamma_{\mu\nu}$ are

$$
\begin{aligned}
\gamma_{00} &= \sum_{n=0}^{+\infty} \frac{2}{(n+3)!} \hat{R}_{0k0l,m_1 \dots m_n} x^k x^l x^{m_1} \dots x^{m_n} \\
&\qquad\qquad \times \left[(n+3) + 2(n+2) a_i x^i + (n+1)(a_i x^i)^2 \right] , \\
\gamma_{0i} &= \sum_{n=0}^{+\infty} \frac{2}{(n+3)!} \hat{R}_{0kil,m_1 \dots m_n} x^k x^l x^{m_1} \dots x^{m_n} \left[(n+2) + (n+1) a_i x^i \right] , \\
\gamma_{ij} &= \sum_{n=0}^{+\infty} \frac{2(n+1)}{(n+3)!} \hat{R}_{ikjl,m_1 \dots m_n} x^k x^l x^{m_1} \dots x^{m_n} \,,
\end{aligned}
\tag{6.33}
$$

where

$$
\hat{R}_{\mu\nu\rho\sigma,m_1 \dots m_n} = \left. \frac{\partial^n R_{\mu\nu\rho\sigma}}{\partial x^{m_1} \dots \partial x^{m_1}} \right|_{x=0} . \tag{6.34}
$$

For the Riemann tensor we use the linearized form in EQ. (6.6). For the interactions considered, $a_i x^i \ll 1$, so we can keep only the first term in the first two lines of EQ. (6.33). Additionally, $(\boldsymbol{\omega} \times \mathbf{x})_i \ll 1$ and we will be interested in the part of the metric related to the Riemann tensor which contains the GW contribution. Therefore, using EQ. (6.5), the metric perturbation in the

PD frame is given by $h_{\mu\nu} = -\gamma_{\mu\nu}$.

6.2 GW production in the laboratory using high energy lasers

The equations of motions for the graviton field in the Lorentz gauge are:

$$\Box h^{\mathrm{G}}_{\mu\nu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) . \tag{6.35}$$

The electromagnetic stress energy tensor in curved space reads

$$T_{\mu\nu} = F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu} - \frac{1}{4}g_{\mu\nu}F_{\sigma\alpha}g^{\alpha\beta}F_{\beta\rho}g^{\rho\sigma} , \tag{6.36}$$

and $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h^{\mathrm{G}}_{\mu\nu}$. Expanding, one has

$$\Box h^{\mathrm{G}}_{\mu\nu} = -\frac{\kappa}{2}\left(T^0_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^0\right) + \mathcal{O}(\kappa^2) , \tag{6.37}$$

where the expression for $T^0_{\mu\nu}$ is obtained by using Eq. (6.36) and replacing $g_{\mu\nu}$ with $\eta_{\mu\nu}$. Neglecting next to leading orders in κ and noticing that $T^0 = 0$, one has:

$$\Box h^{\mathrm{G}}_{\mu\nu} = -\frac{\kappa}{2}T^0_{\mu\nu} , \tag{6.38}$$

where both the metric perturbation and the stress energy tensor are in the Lorentz frame. The electromagnetic tensor in the Lorentz frame is related to the one in the laboratory frame by the transformation

$$F_{\mu\nu} = \frac{\partial x'^{\delta}}{\partial x^{\mu}} \frac{\partial x'^{\gamma}}{\partial x^{\nu}} F'_{\delta\gamma} \,, \tag{6.39}$$

where the prime quantities are the laboratory frame. The two coordinate systems are related by EQ. (6.11). Expanding, we obtain

$$F_{\mu\nu} = F'_{\mu\nu} + F'_{\mu\gamma}\partial_{\nu}\xi^{\gamma} + F'_{\nu\gamma}\partial_{\nu}\xi^{\gamma} + F'_{\delta\gamma}\partial_{\nu}\xi^{\gamma}\partial_{\mu}\xi^{\delta} \,. \tag{6.40}$$

We assume it is a slow varying diffeomorphism, i.e. $\partial_{\mu}\xi^{\nu} \sim \mathcal{O}(h)$ as in EQ. (6.11) in terms of the metric perturbation and therefore $\partial_{\mu}\xi^{\nu} \sim \mathcal{O}(\kappa)$ in terms of the graviton field. Since we are only keeping leading order terms in κ in the computation, we can use the stress-energy tensor in the laboratory frame in EQ. (6.38). In terms of the metric perturbation, EQ. (6.38) reads

$$\Box h_{\mu\nu} = -16\pi G T^{0}_{\mu\nu} \,, \tag{6.41}$$

where now the right hand side is evaluated in the laboratory frame. We consider the production of GWs from the interaction of two laser beams assumed to be counter-propagating plane waves aligned with the $\hat{x}$ axis of frequencies ω_1, ω_2. We denote the electric fields by $\mathbf{E_1}, \mathbf{E_2}$. The electromagnetic tensor $F_{\mu\nu}$ has two contributions, each coming from one laser: $F_{\mu\nu} = F^{(1)}_{\mu\nu} + F^{(2)}_{\mu\nu}$. Therefore, schematically, the stress energy tensor will have two square terms in $(F^{(1)})^2$ and $(F^{(2)})^2$ and one interference term $F^{(1)}F^{(2)}$. Since the production of GWs comes from the interaction of the two lasers, we consider only the contribution of the last term.

$$\begin{aligned} T_{22}^{(12)} &= -T_{33}^{(12)} = (E_{1z}E_{2z} - E_{1y}E_{2y})\mathrm{e}^{i(\omega_1-\omega_2)t - i(\omega_1+\omega_2)x}, \\ T_{23}^{(12)} &= (E_{1z}E_{2y} + E_{1y}E_{2z})\mathrm{e}^{i(\omega_1-\omega_2)t - i(\omega_1+\omega_2)x}, \end{aligned} \tag{6.42}$$

where we are interested in wave frequencies $\omega_1 - \omega_2$. All other components vanish. The metric perturbation is (solving Eq. (6.41)):

$$h_{22}(t, \mathbf{x}) = 16\pi G(E_{1y}E_{2y} - E_{1z}E_{2z}) \int \mathrm{d}^3\mathbf{y}\, \frac{\mathrm{e}^{i(\omega_1-\omega_2)(t-|\mathbf{x}-\mathbf{y}|) - i(\omega_1+\omega_2)y_1}}{4\pi|\mathbf{x}-\mathbf{y}|}, \tag{6.43}$$

where the integral is taken over the overlapping volume of the two lasers assumed to be a cuboid of cross section b^2 and length L. The value of the perturbation at $r = |\mathbf{x}| \gg |\mathbf{y}|$ is:

$$h_{22}(t, \mathbf{x}) = \frac{4G\mathrm{e}^{i(\omega_1-\omega_2)(t-r)}}{r}(E_{1y}E_{2y} - E_{1z}E_{2z}) \int \mathrm{d}^3\mathbf{y}\, \mathrm{e}^{i(\omega_1-\omega_2)\mathbf{n}\cdot\mathbf{y} - i(\omega_1+\omega_2)y_1}, \tag{6.44}$$

where $\mathbf{n} = \mathbf{x}/r$. Therefore

$$\begin{aligned} h_{22}(t, \mathbf{x}) = {}& \frac{4G\mathrm{e}^{i(\omega_1-\omega_2)(t-r)}}{r}(E_{1y}E_{2y} - E_{1z}E_{2z}) b^2 L \mathrm{sinc}\left((\omega_1-\omega_2)\frac{b}{2}n_2\right) \\ & \times \mathrm{sinc}\left((\omega_1-\omega_2)\frac{b}{2}n_3\right) \mathrm{sinc}\left(((\omega_1-\omega_2)n_1 - (\omega_1+\omega_2))\frac{L}{2}\right). \end{aligned} \tag{6.45}$$

h_{22} is maximal for $n_2 = n_3 = 0$ and $n_1 = 1$. Choosing $L\omega_2 = (2n+1) \times \frac{\pi}{2}, n \in \mathbb{N}$, one has

$$h_{22}(t, \mathbf{x}) = -h_{33}(t, \mathbf{x}) = \frac{4G}{r}(E_{1y}E_{2y} - E_{1z}E_{2z})V\mathrm{e}^{i(\omega_1-\omega_2)(t-r)}\frac{1}{L\omega_2}\,, \qquad (6.46)$$

and $V = b^2 L$. Similarly

$$h_{23}(t, \mathbf{x}) = -\frac{4G}{r}(E_{1z}E_{2y} + E_{1y}E_{2z})V\mathrm{e}^{i(\omega_1-\omega_2)(t-r)}\frac{1}{L\omega_2}\,. \qquad (6.47)$$

Using Eq. (6.28), we can express the metric perturbation in the TT frame:

$$h_{22}^{\mathrm{TT}} = h_{22}\,, \quad h_{23}^{\mathrm{TT}} = h_{23}\,, \quad h_{33}^{\mathrm{TT}} = h_{33}\,, \qquad (6.48)$$

and every other component vanishes. One notices that the polarization of the laser beams allows one to control the polarization of the GW. If the two beams are linearly polarized along the same direction, for example $E_{1z} = E_{2z} = 0$, then $h_{23}^{\mathrm{TT}} = 0$ and the GW is + polarized. On the other hand, if the two beams are polarized along different directions, for example $E_{1z} = E_{2y} = 0$, then $h_{22}^{\mathrm{TT}} = h_{33}^{\mathrm{TT}} = 0$ and the GW is $\times$ polarized. The strain in Eq. (6.46) scales as:

$$h_{22} \sim 3.3 \cdot 10^{-43}\left(\frac{10\mathrm{cm}}{r}\right)\left(\frac{E_{\mathrm{las}}}{1\mathrm{J}}\right)\frac{1}{L\omega_2}\,, \qquad (6.49)$$

where E_{las} is the energy per pulse of either laser. The result is proportional to the laser energy, as $T_{\mu\nu}$ is quadratic in $F_{\mu\nu}$ which contains the electromagnetic fields. The dependence on r comes from solving the equations of motion. For

$E_{\text{las}} = 1\text{MJ}, L\omega_2 \sim 10^2$, we find the strain $h_{22} \sim 10^{-39}$. The strain of the GW (i.e. the amplitude of $h_{\mu\nu}$) is thus very small to detect (as we will see in the next section, detection of a GW using a high-energy laser can be achieved today if $h \gtrsim 10^{-20}$). We note that the production of GWs has also been studied in Ref. [123]. The GW amplitude obtained by the authors was stronger than the one shown here by some orders of magnitude but too weak to be detected as well.

6.3 Detection of high-frequency GWs using high-energy lasers

Since the strain of a GW produced in the laboratory is weak, we now investigate the plausibility of using high-energy lasers to convert GWs from cosmological or astrophysical sources into a detectable electromagnetic signal. We adopt the geometric approach and treat $h_{\mu\nu}$ as a dimensionless parameter. The incoming GW is a monochromatic plane wave of frequency ω_g and interacts with a laser beam of frequency ω. We assume that $\omega_g \sim \omega$. The two waves are counter-propagating, and we choose the coordinate system such that they are aligned along the x-axis (see Fig. 6.1).

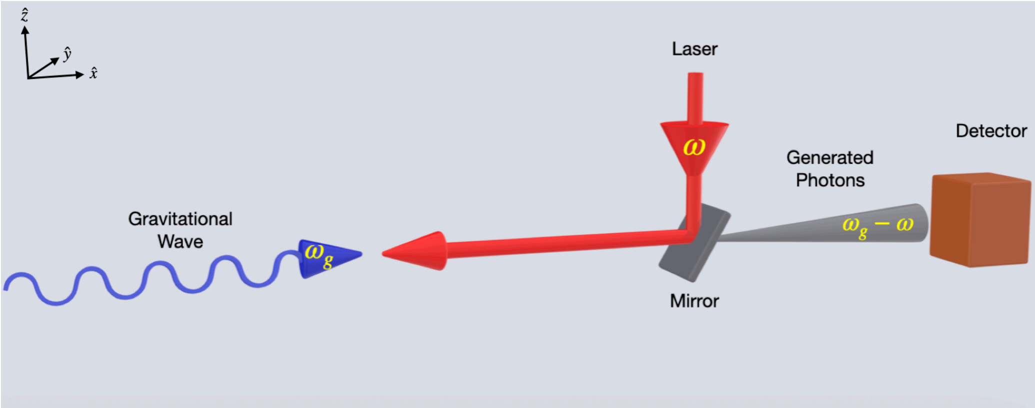

Fig. 6.1: Diagram of the experimental proposal adapted from Ref. [1]. An incoming GW interacts with the linearly polarized laser beam of frequency ω. The scattered electromagnetic wave propagates through the mirror and enters a detector capable of single-photon counting.

As will be shown in what follows, a resonance occurs only for the scattered waves of frequency $\omega_g - \omega$ and more precisely for $\omega_g = 2\omega$. To study the resonance, we assume $\omega_g > \omega$. The Maxwell action in curved space is

$$S_{\text{Maxwell}} = \int \mathrm{d}^4x \sqrt{-g}\left(-\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}\, F_{\alpha\beta}\right). \tag{6.50}$$

We linearize the action using Eq. (6.5) and find

$$S_{\text{Maxwell}} = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}\left[F^{\mu\nu} + \frac{1}{2} h F^{\mu\nu} + h^{\nu}{}_{\alpha} F^{\alpha\mu} - h^{\mu}{}_{\alpha} F^{\alpha\nu}\right]. \tag{6.51}$$

By noticing the antisymmetry of the term between brackets, we can integrate by parts, and find the equations of motion. They are given by

$$S_{\text{Maxwell}} = -\frac{1}{2}\int \mathrm{d}^4x\, A_\mu(\partial_\nu F^{\mu\nu} + j^\mu_{\text{eff}}) \xrightarrow[\frac{\delta S}{\delta A_\mu}=0]{} \partial_\nu F^{\nu\mu} = j^\mu_{\text{eff}}\,, \tag{6.52}$$

which are the Maxwell equations in curved space and where we defined the effective four-current

$$j^\mu_{\text{eff}} = \partial_\nu\left(\frac{1}{2}hF^{\mu\nu} + h^\nu{}_\alpha F^{\alpha\mu} - h^\mu{}_\alpha F^{\alpha\nu}\right). \tag{6.53}$$

Combining the inhomogeneous Maxwell equations with Gauss' law $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, and Faraday's law $\boldsymbol{\nabla}\times\mathbf{E} + \partial_t\mathbf{B} = \mathbf{0}$ we obtain

$$\Box\mathbf{E} = -\partial_t\mathbf{j}_{\text{eff}} - \boldsymbol{\nabla} j^0_{\text{eff}}\,, \quad \Box\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{j}_{\text{eff}}\,. \tag{6.54}$$

We note that the current j^μ_{eff} is not invariant between frames even at $\mathcal{O}(h)$ order. Therefore, a choice of frame must be made for the calculation of the current and a convenient choice seems to be the PD frame which is the preferred frame defined by the laboratory. The metric perturbation in this frame is given by $h_{\mu\nu} = -\gamma_{\mu\nu}$ where $\gamma_{\mu\nu}$ is given by Eq. (6.33). Using the fact that the Riemann tensor is invariant between frames at $\mathcal{O}(h)$, we compute Eq. (6.34) in the TT frame for convenience for $R_{\mu\nu\rho\sigma} \propto \mathrm{e}^{i\omega_g(t-x)}$. Computing the series in Eq. (6.33) to all orders, assuming $a_ix^i \ll 1$, the metric perturbation is given by

$$
\begin{aligned}
h_{00} &= -\omega_g^2 h_{ab}^{\mathrm{TT}} x^a x^b \left(-\frac{i}{\omega_g x} + \frac{1-\mathrm{e}^{-i\omega_g x}}{(\omega_g x)^2} \right) , \\
h_{0i} &= -\omega_g^2 \left[h_{ia}^{\mathrm{TT}} x x^a - \delta_{ix} h_{ab}^{\mathrm{TT}} x^a x^b \right] \left(-\frac{i}{2\omega_g x} - \frac{\mathrm{e}^{-i\omega_g x}}{(\omega_g x)^2} - i \frac{1-\mathrm{e}^{-i\omega_g x}}{(\omega_g x)^3} \right) , \\
h_{ij} &= \omega_g^2 \left[(\delta_{ix} h_{ja}^{\mathrm{TT}} + \delta_{jx} h_{ia}^{\mathrm{TT}}) x x^a - h_{ij}^{\mathrm{TT}} x^2 - \delta_{ix}\delta_{jx} h_{ab}^{\mathrm{TT}} x^a x^b \right] \\
&\qquad\qquad\qquad \times \left(-\frac{1+\mathrm{e}^{-i\omega_g x}}{(\omega_g x)^2} - 2i \frac{1-\mathrm{e}^{-i\omega_g x}}{(\omega_g x)^3} \right) ,
\end{aligned}
\tag{6.55}
$$

where $a, b = y, z$ and $h^{\mathrm{TT}} \propto \mathrm{e}^{i\omega_g t}$ is the metric perturbation in the TT frame. This result was first derived in Ref. [82]. For simplicity, we choose a linearly polarized laser such that $E_z = 0$. The only non-zero components of the electromagnetic tensor are $F^{20} = -F^{21} = E_0 \mathrm{e}^{-i\omega(t+x)}$. Replacing Eq. (6.55) into Eq. (6.53), one finds

$$
j^\mu_{\mathrm{eff}} = E_0 \mathrm{e}^{i(\omega_g - \omega)t - i\omega x} \omega_g \begin{pmatrix} (h_+ y + h_\times z)(\omega_g f_1(u) + i\omega f_2(u)) \\ -i(\omega - \omega_g)(h_+ y + h_\times z) f_2(u) \\ h_+ (f_2(u) + (\partial_u - i)(u^2 f_1(u))) \\ h_\times (f_2(u) + (\partial_u - i)(u^2 f_1(u))) \end{pmatrix} , \tag{6.56}
$$

where $u = \omega_g x$ and

$$
f_1(x) = -\frac{1+\mathrm{e}^{-ix}}{x^2} - 2i \frac{1-\mathrm{e}^{-ix}}{x^3} , \qquad f_2(x) = -i \frac{1-\mathrm{e}^{-ix}}{x^2} - \frac{1}{x} + \frac{i}{2} . \tag{6.57}
$$

The Maxwell equations read

$$
\begin{aligned}
\Box E_x &= E_0 \mathrm{e}^{i(\omega_g-\omega)t-i\omega x}\omega_g^3(h_+ y + h_\times z)\left[2i\frac{\omega}{\omega_g}f_1(u) + \left(1-\frac{2\omega}{\omega_g}\right)f_2(u) - f_1'(u)\right], \\
\Box E_y &= E_0 \mathrm{e}^{i(\omega_g-\omega)t-i\omega x}\omega_g^2 h_+\left[-f_1(u) - i f_2(u) - i\left(1-\frac{\omega}{\omega_g}\right)(\partial_u - i)(u^2 f_1(u))\right], \\
\Box E_z &= E_0 \mathrm{e}^{i(\omega_g-\omega)t-i\omega x}\omega_g^2 h_\times\left[-f_1(u) - i f_2(u) - i\left(1-\frac{\omega}{\omega_g}\right)(\partial_u - i)(u^2 f_1(u))\right], \\
\Box B_x &= 0\,, \\
\Box B_y &= E_0 \mathrm{e}^{i(\omega_g-\omega)t-i\omega x}\omega_g^2 h_\times\left[i f_2(u) + \left(\frac{i\omega}{\omega_g} - \partial_u\right)(\partial_u - i)(u^2 f_1(u)) + f_1(u)\right], \\
\Box B_z &= -E_0 \mathrm{e}^{i(\omega_g-\omega)t-i\omega x}\omega_g^2 h_+\left[i f_2(u) + \left(\frac{i\omega}{\omega_g} - \partial_u\right)(\partial_u - i)(u^2 f_1(u)) + f_1(u)\right].
\end{aligned}
\tag{6.58}
$$

To simplify calculations, we consider an interaction region, which is a rectangular cuboid with sides of length $b \times b \times L$, with $b \leq L$ and $b\omega_g \gg 1$. The side of length L is parallel to the x-axis. The solutions to the Maxwell equations are of the form

$$
E_y(t, \mathbf{x}) = \int \mathrm{d}^3\mathbf{y}\, \mathrm{e}^{i(\omega_g-\omega)(t-|\mathbf{x}-\mathbf{y}|)-i\omega y_1}\frac{f_{E_2}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\,, \tag{6.59}
$$

where the integral is taken over the interaction region. The function f_{E_2} is the the right hand side of the second line in Eq. (6.58) without the exponential factor. We place the detector far from the interaction region such that $|\mathbf{x}| \gg |\mathbf{y}|$. We can simply replace the denominator in the integrand by $4\pi|\mathbf{x}|$. For the exponent, assuming $\omega_g - \omega \sim \omega$,

$$\omega|\mathbf{x}-\mathbf{y}| = \omega R - \omega \mathbf{n}\cdot\mathbf{y} + \frac{\omega y^2}{2R} - \frac{\omega(\mathbf{n}\cdot\mathbf{y})^2}{2R} + \frac{\omega y^2 \mathbf{n}\cdot\mathbf{y}}{2R^2} - \frac{\omega(\mathbf{n}\cdot\mathbf{y})^3}{2R^2} + \mathcal{O}\left(\frac{\omega y^4}{R^3}\right), \quad y \leq L\,, \tag{6.60}$$

where $R = |\mathbf{x}|$ is the distance between the interaction region and the detector. We assume that R is large enough to keep only the first two terms in this expansion. In this case, taking the E_y component as an example,

$$E_y(t,\mathbf{x}) = -E_0\omega_g h_+ \frac{\mathrm{e}^{i(\omega_g-\omega)(t-r)}}{4\pi r} b^2 \mathrm{sinc}\left(\frac{b(\omega_g-\omega)n_2}{2}\right)\mathrm{sinc}\left(\frac{b(\omega_g-\omega)n_3}{2}\right) \times \int_{-\frac{L\omega_g}{2}}^{\frac{L\omega_g}{2}} \mathrm{d}u\, \mathrm{e}^{i(1-\frac{\omega}{\omega_g})n_1 u - i\frac{\omega}{\omega_g}u}\left[f_1(u) + i f_2(u) + i\left(1-\frac{\omega}{\omega_g}\right)(\partial_u - i)(u^2 f_1(u))\right], \tag{6.61}$$

where the components of the unit vector in spherical coordinates are given as usual by $n_1 = \cos\phi\sin\theta, n_2 = \sin\phi\sin\theta, n_3 = \cos\theta$. Under the assumption $b(\omega_g - \omega) \gg 1$, the sinc functions are maximized when $n_2, n_3 \approx 0$. We choose to place the detector along this direction. We therefore expand around $\theta = \pi/2$ and $\phi = 0, 2\pi$ neglecting second order terms. Then, $n_1 \approx 1$ and the integral in Eq. (6.61) reads (for $a = \omega/\omega_g \neq 1/2$)

$$\frac{2(1-a)\sin(aL\omega_g)}{a} + \frac{3-2a}{2a-1}\sin\left((2a-1)\frac{L\omega_g}{2}\right) + 2i(a-1)(-2i\pi \pm 2i\pi)\,, \tag{6.62}$$

where $\pm$ corresponds to $a > 1/2$ or $a < 1/2$ respectively. The above expression is $\mathcal{O}(1)$ for $a \neq 1/2$. When $a \to 1/2$, the integral becomes $\mathcal{O}(L\omega_g)$ which is large by assumption. This is due to the second term in the last expression. In what follows we assume $\omega_g = 2\omega$ to study the resonance. Then,

$$E_y(t, \mathbf{x}) = E_0 \frac{\mathrm{e}^{i\omega(t-r)}}{4\pi r} \omega_g^2 h_+ L b^2 \mathrm{sinc}\left(\frac{b\omega n_2}{2}\right) \mathrm{sinc}\left(\frac{b\omega n_3}{2}\right), \tag{6.63}$$

and the same can be done for the other electromagnetic components. We find

$$\begin{aligned}
E_z &= E_0 \frac{\mathrm{e}^{i\omega(t-r)}}{4\pi r} \omega_g^2 h_\times L b^2 \mathrm{sinc}\left(\frac{b\omega n_2}{2}\right) \mathrm{sinc}\left(\frac{b\omega n_3}{2}\right), \\
B_y &= -E_0 \frac{\mathrm{e}^{i\omega(t-r)}}{4\pi r} \omega_g^2 h_\times L b^2 \mathrm{sinc}\left(\frac{b\omega n_2}{2}\right) \mathrm{sinc}\left(\frac{b\omega n_3}{2}\right), \\
B_z &= E_0 \frac{\mathrm{e}^{i\omega(t-r)}}{4\pi r} \omega_g^2 h_+ L b^2 \mathrm{sinc}\left(\frac{b\omega n_2}{2}\right) \mathrm{sinc}\left(\frac{b\omega n_3}{2}\right), \\
E_x &= E_0 \frac{\mathrm{e}^{i\omega(t-r)}}{4\pi r} b^3 \omega_g^2 \frac{1}{L\omega_g} \left(h_+ \mathrm{sinc}\left(\frac{b\omega n_3}{2}\right) \frac{\cos\left(\frac{b\omega n_2}{2}\right) - \mathrm{sinc}\left(\frac{b\omega n_2}{2}\right)}{\frac{b\omega n_2}{2}} \right. \\
&\left. + h_\times \mathrm{sinc}\left(\frac{b\omega n_2}{2}\right) \frac{\cos\left(\frac{b\omega n_3}{2}\right) - \mathrm{sinc}\left(\frac{b\omega n_3}{2}\right)}{\frac{b\omega n_3}{2}} \right).
\end{aligned} \tag{6.64}$$

The power reaching the detector is

$$P = \int \mathrm{d}\Omega\, R^2 \mathbf{n} \cdot \mathbf{S}, \tag{6.65}$$

where $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ is the Poynting vector. The integrand is highly peaked around the direction $\mathbf{n} = (1, 0, 0)$. Therefore, assuming the detector is large enough, we can integrate over the whole solid angle. We use the following expansion to leading order in $(b\omega)^{-1}$

$$\int \mathrm{d}\Omega\; \sin\theta \cos\phi\, \mathrm{sinc}^2\left(\frac{b\omega}{2}\cos\theta\right) \mathrm{sinc}^2\left(\frac{b\omega}{2}\cos\phi \sin\theta\right) \approx \frac{4\pi^2}{(b\omega)^2}. \tag{6.66}$$

The same can be done for the longitudinal part. The power is then

$$P = 2E_0^2 \cos^2(\omega(t-R)) b^2 (h_+^2 + h_\times^2)(2L^2\omega^2 + 1)\,. \tag{6.67}$$

To find the energy emitted during one laser pulse we integrate this expression from $t = 0$ to $t = L$. Thus

$$E_{pp} = \int_0^L \mathrm{d}t\, P \approx E_0^2 b^2 L (h_+^2 + h_\times^2)(2L^2\omega^2 + 1)\,, \tag{6.68}$$

where we used $L\omega \gg 1$. The second term in the last factor can be dropped because of this condition. The laser energy per pulse is $E_{\mathrm{las}} = E_0^2 b^2 L/2$. Thus

$$E_{pp} = 4E_{\mathrm{las}}(h_+^2 + h_\times^2) L^2\omega^2\,. \tag{6.69}$$

For $\omega_g \neq \omega$, the energy of the scattered wave scales as $E_{pp} \sim E_{\mathrm{las}}(h_+^2 + h_\times^2)$. The enhancement factor for the resonance is $(L\omega)^2$. The total number of photons entering the detector is $N_\gamma = n_s E_{pp}/\omega$ where n_s is the number of shots during the experiment since the energy of the outgoing photon is $\omega_g - \omega = \omega$. Assuming single photon counting is possible we require that $N_\gamma \geq 1$, which for $h_\times = h_+ \equiv h$ gives

$$\begin{aligned} h_{\min} &= \frac{1}{\sqrt{8 n_s \omega L^2 E_{\mathrm{las}}}}\,, \quad \omega_g = 2\omega\,, \\ h_{\min} &\sim \sqrt{\frac{\omega_g - \omega}{2 n_s E_{\mathrm{las}}}}\,, \quad \omega_g \neq 2\omega\,. \end{aligned} \tag{6.70}$$

6.3.1 Projected bounds

We can use the result in Eq. (6.70) for lasers operating in the THz, optical, and X-ray regimes. For all, we assume that the duration of the experiment is one day.

THz laser: We use the THz free electron laser [124] for which $\tau = 1$ ps, $n_s = 1.7 \times 10^{10}$ (repetition rate of 200 kHz), $E_{\text{las}} = 100\,\mu$J, and the frequency is 1 THz $< \omega/2\pi <$ 30 THz.

Optical laser: We use the National Ignition Facility (NIF) laser [125] which has 192 beamlines. For each, $\tau = 20$ ns, $n_s = 4$, $E_{\text{las}} = 9.4$ kJ. The NIF laser can operate at three wavelengths, 1053, 527 and 351 nm.

X-ray laser: We use the free-electron laser at the European XFEL or the one at the SLAC National Accelerator Laboratory [126]. $\tau = 0.1$ ps, $n_s = 8.6 \times 10^5$ (repetition rate of 10 Hz), and the frequency is in the range 5.8 keV $< \omega/2\pi <$ 24 keV. Here, the energy depends on the frequency. Between 5.8 and 9.3 keV, $E_{\text{las}} = 2$ mJ, between 9.3 and 12 keV, $E_{\text{las}} = 1$ mJ, and between 12 and 24 keV $E_{\text{las}} = 0.5$ mJ.

The main experimental challenge with the proposed setup in the resonant case is making certain that photons coming from the laser do not enter the detector as they cannot be differentiated from photons of the generated electromagnetic signal since the two have the same frequency.

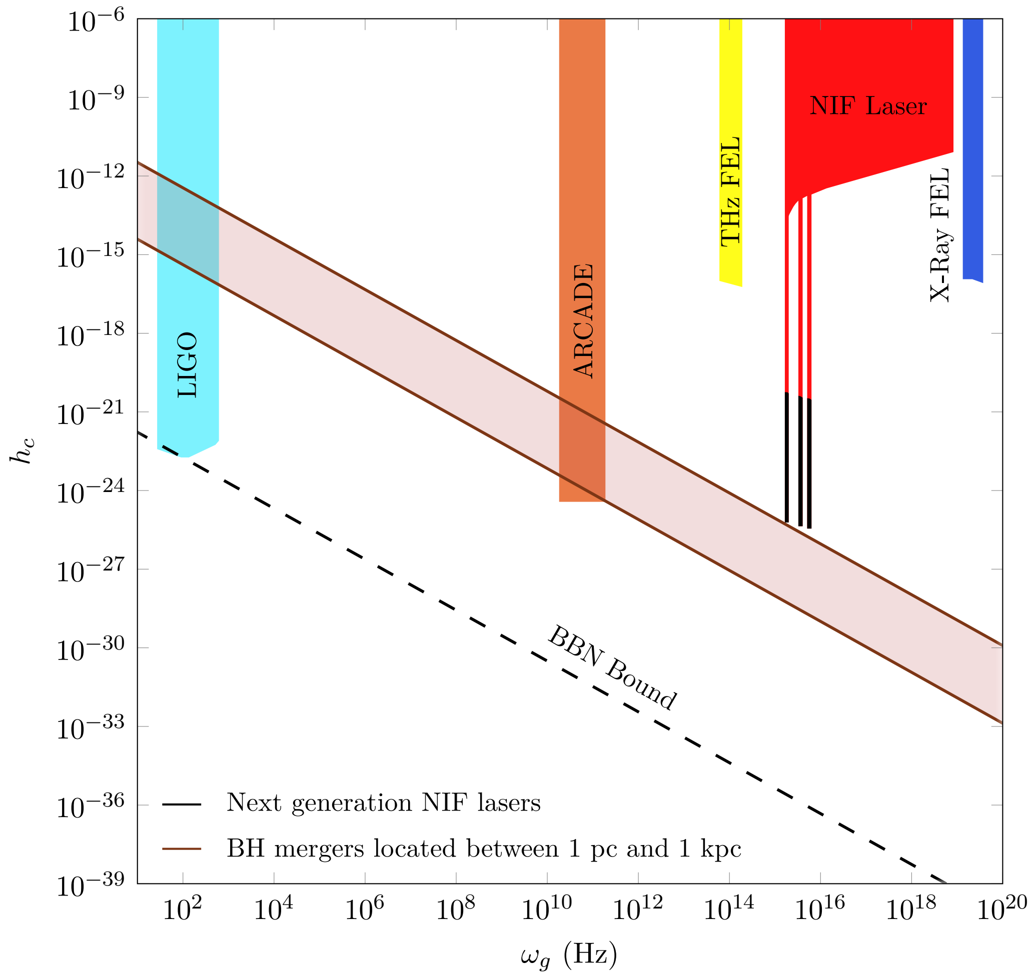

Fig. 6.2: Sensitivity of existing detectors and the laser detectors discussed in this section. The figure is adapted from Ref. [1]. LIGO [63] covers the low frequency regime whereas ARCADE [127] impose exclusion bounds in the GHz regime. The different laser sensitivities are shown. For the NIF laser, only three frequencies are resonant. Regarding the sources, the BBN bound does not allow for GW detection from cosmological sources. The sources shown here and BH mergers located at the Innermost Stable Circular Orbit (ISCO). For the latter, the characteristic strain and the GW amplitude are related by a factor of $\sqrt{2\dot{f}/f^2}$ [128] which is of $\mathcal{O}(1)$.

The bounds for each laser described are given by

$$
\begin{aligned}
h_{\rm min}^{\rm THz} &= 2.3\times 10^{-16}\left(\frac{1.7\times 10^{10}}{n_s}\right)^{1/2}\left(\frac{30\,{\rm THz}}{\omega_g/2\pi}\right)^{1/2}\left(\frac{1\,{\rm ps}}{\tau}\right)\left(\frac{100\,\mu{\rm J}}{E_{\rm las}}\right)^{1/2}, \\
h_{\rm min}^{\rm opt} &= 1.8\times 10^{-20}\left(\frac{4}{n_s}\right)^{1/2}\left(\frac{8.5\times 10^{14}\,{\rm Hz}}{\omega_g/2\pi}\right)^{1/2}\left(\frac{20\,{\rm ns}}{\tau}\right)\left(\frac{9.4\,{\rm kJ}}{E_{\rm las}}\right)^{1/2}, \\
h_{\rm min}^{\rm Xray} &= 1.3\times 10^{-16}\left(\frac{8.6\times 10^{5}}{n_s}\right)^{1/2}\left(\frac{1.4\times 10^{19}\,{\rm Hz}}{\omega_g}\right)^{1/2}\left(\frac{0.1\,{\rm ps}}{\tau}\right)\left(\frac{2\,{\rm mJ}}{E_{\rm las}}\right)^{1/2}.
\end{aligned}
\tag{6.71}
$$

We compare the bounds of the laser detectors with other experiments in Fig. 6.2. The NIF operates at fixed frequencies. Thus, the enhancement in sensitivity coming from the resonance can be applied only to these frequencies. This corresponds to the red peaks shown in Fig. 6.2. We can find future bounds for optical lasers by assuming the 192 beams of the NIF can be used for a total energy of 1.8 MJ. Taking a future repetition rate of 10 kHz, the bound becomes

$$
h_{\rm min}^{\rm opt,fut} = 6.3\times 10^{-26}\left(\frac{8.6\times 10^{8}}{n_s}\right)^{1/2}\left(\frac{8.5\times 10^{14}{\rm Hz}}{\omega_g/2\pi}\right)^{1/2}\left(\frac{20\,{\rm ns}}{\tau}\right)\left(\frac{1.8\,{\rm MJ}}{E_{\rm las}}\right)^{1/2}.
\tag{6.72}
$$

The sensitivity of the X-ray laser can be increased as well. Suppose that the incoming laser beam enters a region separated by two crystal planes and is reflected each time it reaches a plane. At the resonance, the scattered electromagnetic wave has the same frequency as the incoming one and propagates in the opposite direction, being reflected by the crystal planes. The energy entering the detector is Eq. (6.70) multiplied by the number of reflections given by $n_{\rm ref} = l/(\tau\cos\theta_B)$ with θ_B the Bragg angle. The distance separating the two planes is $\tau \sin\theta_B$ and $l \sim 1$ m. The lower bound is then $h_{\text{min}}^{\text{Xray}} = 1.3 \times 10^{-18}$.

6.3.2 Sources of GWs

Having found the bounds on the GW strain, we now consider the nature of possible GW sources. The BBN bound takes the form [70, 74] $h_c \lesssim 10^{-33}(2\pi\,\text{THz})/\omega_g$. This is out of reach for today's laser capacities. One of the reasons for this is the small interaction time between the laser beam of the GWs. For these sources, it is convenient to use constant magnetic fields as in Ref. [83]. On the other hand, one can consider coherent sources such as BH mergers provided that they are close enough to the detector, since the GW strain is inversely proportional to the distance between the source and the detector. The frequency of emitted GWs increases close to coalescence and reaches its largest value on the Innermost Stable Circular Orbit (ISCO) and is $f_{\text{ISCO}} \propto m^{-1}$, where m is the total mass of the binary [119]. For radial distances smaller than the ISCO radius, the trajectories are not longer circular, and GW emission ceases. As a consequence, the highest GW frequency is bound by the inverse of the sum of the BH masses. Therefore, detecting high-frequency GWs can lead to the discovery of light binaries.

In Eq. (6.70), we assumed that the frequency of the GW is constant for the duration of the experiment, partly to satisfy the resonance condition $\omega_g = 2\omega$. On the other hand, for binary systems close to coalescence, GW frequency is time-dependent. Assuming a small departure from the resonance condition, the GW frequency is $\omega_g + \delta\omega_g$. From Eq. (6.62), it is clear that for the resonant condition to be valid, we must have

$$\frac{\delta\omega_g}{\omega_g} \ll \frac{1}{L\omega} \ll 1\,. \tag{6.73}$$

The GW frequency is

$$\omega_{\rm gw}(t) = 2\left(\frac{5}{256(t_{\rm coal}-t)}\right)^{3/8}(GM_c)^{-5/8}\,,\quad t \leq t_{\rm coal}\,, \tag{6.74}$$

where M_c is the chirp mass and $t_{\rm coal}$ is the time to coalescence. We define Δt as the time interval for the frequency to change from 2ω to $2\omega+\delta\omega_g$. Then, using Eqs. (6.73) and (6.74),

$$\Delta t \ll \frac{5}{24(Gm)^{5/3}L\omega^{11/3}} \sim 10\,\mathrm{days}\left(\frac{1\,\mathrm{ps}}{\tau}\right)\left(\frac{30\,\mathrm{THz}}{2\omega/2\pi}\right)^{11/3}\left(\frac{10^{-22}\,M_\odot}{m}\right)^{5/3}\,, \tag{6.75}$$

where we considered the frequency of the THz regime and m is the total mass of the binary (assuming equal masses). In this case, the binary mass must be $m \sim 10^{-22}M_\odot$ for the frequency to be constant during the duration of the experiment.

7 Conclusion

In this thesis, we studied the emission and detection of various particles and waves. For particles of spin 0 or 1, the production mechanism we examined was emission from accelerated charges. Starting with the case of spin 0 particles, we studied axion production from accelerated electrons in the inertial frame. The goal of the two dedicated chapters (Chap. 2 and Chap. 3) was to describe the radiation from the electrons in a possible experiment. Today, one of the used techniques in electron acceleration is laser beams, which are described by both electric and magnetic fields. Laser-accelerated electrons can have complicated trajectories (except if they are accelerated by one laser, which is a plane wave [91]). As a consequence, we first considered acceleration from a purely time-dependent potential. Then, we generalized the results to an arbitrary electromagnetic potential. Unlike scalar particles, axions couple to electrons via the Yukawa interaction term in Eq. (2.7). The presence of the γ_5 matrix in the dimension-4 operator cancels exactly the leading order term in the WKB expansion for the one-axion final state in Eq. (2.14) (see

EQ. (2.21)), making the process quantum, which can also be seen by the presence of a factor $\hbar$ for the interaction probability. We also verified that the interaction term EQ. (2.7) is equivalent to EQ. (2.1) to lowest order in g_{ae}. We derived a Larmor-type formula for the uniform acceleration given in EQ. (2.43) which exhibits a suppression factor of $(\hbar a/m)^2$. The results for a time-dependent potential are also applicable for an electron in the presence of a standing wave created by two counter-propagating laser beams.

We generalized the results of one-dimensional trajectories to an arbitrary electromagnetic potential. The first observation is that the initial spin of the electron rotates according to the Thomas-BMT equation. Then, we derived the axion-electron interaction amplitude from which we expressed the total energy emitted. We compared the cases of electric and magnetic fields and found that axion production is significantly increased in the second case for relativistic electrons. We propose an experimental setup shown schematically in FIG. 3.4 in which emitted axions are reconverted into photons. Assuming that single-photon counting is possible and that a large proportion of the total solid angle can be covered by detectors, we impose bounds on the coupling constant g_{ae} for axion masses $m_a \lesssim 10$ keV. We find that for today's laser performances the bounds are comparable to other earth-based experiments. However, as laser performances are subject to important enhancements, the imposed bounds could become more stringent in the near future. If this is the case, then the parameter space of the QCD axion could be explored for purely laboratory-based experiments (we note that the bounds imposed by other experiments where the axions are not produced in the laboratory are significantly more stringent).

We then studied particle emission from the point of view of the accelerating charge. Instead of working with pseudo-scalar particles, we considered the case of scalars for simplicity. Studying radiation in the accelerated frame naturally led us to the Unruh effect. The latter states that the usual inertial vacuum is a thermal state for uniformly accelerated observers, and we focused on a free massive scalar field as an illustration. We showed using a Bogoliubov coefficient description how one can define purely positive-frequency modes as a linear combination of Rindler modes, which was essential in deriving the Unruh effect itself. We found, as expected, that the temperature of the FDU bath was proportional to the proper acceleration. We then showed how the quantization in the Rindler wedges can be extended to the electromagnetic field as well.

Using the Unruh effect, we then considered a uniformly accelerating particle that was interacting with the FDU bath. At tree level, the interaction consists of two processes, the emission and absorption of Rindler particles to and from the FDU bath. These processes are equivalent to the emission of a Minkowski particle in the inertial frame (and scattering of Rindler particles corresponds to the emission of two Minkowski particles [93, 110–112]). We verify that this is the case by explicitly calculating the interaction probability to first order in perturbation theory in two different ways. The first was to sum the emission and absorption amplitudes of Rindler particles in the accelerating frame. The second was to consider only the emission amplitude of Minkowski particles in the inertial frame. We find by explicit calculation that both probabilities are equal for an *arbitrary* classical current (in the electromagnetic case, the only assumption was current conservation). In a recent manuscript sent for publication [4], this equivalence is extended to the gravitational case. Since calculations were performed to first order in perturbation theory, the radiation described in the inertial frame is classical. Then, as expected, for the special case of a point charge, we find the Larmor formula in Eqs. (4.83) and (5.54) (we find that production is suppressed exponentially for massive particles when the mass becomes similar to the proper acceleration). Therefore, measuring classical radiation can be seen as an indirect observation of the Unruh effect since taking into account the FDU thermal bath was necessary to recover the Larmor formula. During the calculation, a finite period of uniform acceleration was assumed by turning on and off the coupling of the charge to the quantum field. The finite interaction is also necessary in a purely classical description [106]. We note that recovering the classical limit of observables from scattering amplitudes has been studied recently, see e.g. Ref. [129]. Observing the Unruh effect using classical electrodynamics has also been proposed in Ref. [130].

In the last chapter of the thesis, we discussed the production and detection of GWs in the laboratory. Firstly, through the Gertsenshtein effect, GWs can be generated by the interaction of two high-energy counter-propagating laser beams. We find that even for the most energetic lasers, the produced GWs would be too weak to be detectable. The inverse process (conversion of a GW into an electromagnetic signal using a high-energy laser) is also possible and has the advantage of not having to rely on weak GW strains. Instead, requiring that at least one photon is produced during the experiment, we are able to impose bounds on the strain. We found that the signal is resonant when the GW frequency is twice the laser frequency and the enhancement factor was of the order $(\omega\tau)^2$ where τ is the laser pulse. The regime of GWs with frequencies above 1 GHz can thus be explored. We derived the expression for the minimum strain and applied it to THz, optical and X-ray lasers. Even if $h_{\rm min} \propto \omega^{-1/2}$, the optical laser was the most sensitive with $h_{\rm min} \sim 10^{-20}$ because of the large energy and pulse. If the repetition rate of the NIF laser increases significantly in the future (10 kHZ), the sensitivities can reach 10^{-26}. However, the inconvenience with this type of laser is that it operates at three fixed frequencies. We find that GWs produced by cosmological sources are not accessible because of the strong BBN bound. For astrophysical sources, only light BH mergers or exotic compact objects [131, 132] can emit in this frequency range, and we considered the former because of the stronger GW strain. The highest frequency emitted by BH merges is proportional to the inverse of its total mass. Moreover, the frequency is constant only in the early inspiral phase. Thus, only very light BH binaries must be considered, which in turn decreases the astrophysical reach of the detector.

To conclude, the present thesis discusses the feasibility of using high-energy lasers to explore physics within or beyond the SM through particle production from accelerating charges and detection of GWs in the laboratory. In the first case, radiation in the accelerated frame is discussed through the Unruh effect as it is essential to reproduce the Larmor formula.

Appendices

A WKB Approximation for a Purely Space-Dependent Potential

In this section we assume that the potential is only space dependent. We start from the Dirac Eq. (1.23). For a mode Φ_α satisfying the latter, we define

$$\Phi_\alpha = \psi_\alpha(x) \mathrm{e}^{-\frac{i}{\hbar}(p_0 t - p_y y - p_x x)}, \tag{A.1}$$

where here x is the space coordinate not to be confused with $x^\mu = (t, \mathbf{x})$. Then,

$$(\gamma^0 p_0 - \gamma^2 p_y - \gamma^3 p_z + i\hbar\gamma^1\partial_x - \gamma^\mu V_\mu - m)\psi_\alpha(x) = 0\,. \tag{A.2}$$

We make the gauge choice $V_x = 0$. Thus,

$$(i\hbar\gamma^1\partial_x + (p_0 - V^0)\gamma^0 - (p_y - V^y)\gamma^2 - (p_z - V^z)\gamma^3 - m)\psi(x) = 0\,. \tag{A.3}$$

We define $\tilde{p}_0 = p_0 - V^0, \tilde{p}_y = p_y - V^y, \tilde{p}_z = p_z - V^z$. We also mean $p_z \equiv p^z$ and $p_y \equiv p^y$. Multiplying by $-\gamma^1$ and using the fact that $(\gamma^1)^2 = -\mathbb{1}_{4\times 4}$

$$i\hbar\partial_x\psi_\alpha = (\tilde{p}_0\gamma^1\gamma^0 - \tilde{p}_y\gamma^1\gamma^2 - \tilde{p}_z\gamma^1\gamma^3 - m\gamma^1)\psi_\alpha\,. \tag{A.4}$$

We now expand $\psi_\alpha(x)$ as

$$\psi_\alpha(x) = \exp\left\{-\frac{i}{\hbar}S(x)\right\}\left[\begin{pmatrix}\varphi^{(0)}\\ \chi^{(0)}\end{pmatrix} + \hbar\begin{pmatrix}\varphi^{(1)}\\ \chi^{(1)}\end{pmatrix} + \hbar^2\begin{pmatrix}\varphi^{(2)}\\ \chi^{(2)}\end{pmatrix} + \ldots\right], \tag{A.5}$$

which when inserted into the Dirac equation leads to

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix}S' + i\hbar\begin{pmatrix}\varphi'\\ \chi'\end{pmatrix} = \begin{pmatrix} i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 & -(\tilde{p}_0 + m)\sigma^1 \\ -(\tilde{p}_0 - m)\sigma^1 & i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \tag{A.6}$$

where prime means derivative with respect to x. The zeroth order in $\hbar$ gives

$$\begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} S' = \begin{pmatrix} i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 & -(\tilde{p}_0 + m)\sigma^1 \\ -(\tilde{p}_0 - m)\sigma^1 & i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 \end{pmatrix} \begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} . \tag{A.7}$$

Therefore S' is an eigenvalue of the matrix. But the two eigenvalues of the matrix above are $\pm\kappa_p = \pm\sqrt{\tilde{p}_0^2 - \tilde{p}_y^2 - \tilde{p}_z^2 - m^2}$. We choose the negative value, i.e. $S' = -\kappa_p$, because in the absence of potential $\kappa = p_x, S = -p_x x$ and we recover $\Phi \propto \mathrm{e}^{-\frac{i}{\hbar}p\cdot x}$. Therefore

$$S = -\int_0^x \mathrm{d}x\, \kappa_p = -\int_0^x \mathrm{d}x\, \sqrt{\tilde{p}_0^2 - \tilde{p}_y^2 - \tilde{p}_z^2 - m^2}\, . \tag{A.8}$$

We now wish to find the eigenvector. For that, we use

$$(i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 + \kappa_p)\varphi^{(0)} = (\tilde{p}_0 + m)\sigma^1\chi^{(0)}\, . \tag{A.9}$$

We multiply both sides by σ^1 and use $\sigma^1\sigma^3 = -i\sigma^2$ and $\sigma^1\sigma^2 = i\sigma^3$ to obtain

$$\chi^{(0)} = \frac{1}{\tilde{p}_0 + m}(\kappa_p\sigma^1 + \tilde{p}_y\sigma^2 + \tilde{p}_z\sigma^3)\varphi^{(0)}\, . \tag{A.10}$$

We define for convenience $\tilde{\mathbf{p}} = (\kappa_p, \tilde{p}_y, \tilde{p}_z)$ in order to write

$$\chi^{(0)} = \frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0 + m}\varphi^{(0)}\, . \tag{A.11}$$

We also have

$$i\begin{pmatrix}(\varphi^{(n)})' \\ (\chi^{(n)})'\end{pmatrix} = \begin{pmatrix} i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 + \kappa_p & -(\tilde{p}_0+m)\sigma^1 \\ -(\tilde{p}_0-m)\sigma^1 & i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 + \kappa_p \end{pmatrix}\begin{pmatrix}\varphi^{(n+1)} \\ \chi^{(n+1)}\end{pmatrix}, \tag{A.12}$$

for $n \in \mathbb{N}$. We would like to find a matrix such that when applied to the right-hand side of the equation above gives the null vector. This matrix is given by

$$\begin{pmatrix} i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 - \kappa_p & -(\tilde{p}_0+m)\sigma^1 \\ -(\tilde{p}_0-m)\sigma^1 & i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 - \kappa_p \end{pmatrix}. \tag{A.13}$$

Note that this matrix is similar to the previous one with a sign flip for κ_p. Therefore,

$$\begin{pmatrix} i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 - \kappa_p & -(\tilde{p}_0+m)\sigma^1 \\ -(\tilde{p}_0-m)\sigma^1 & i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 - \kappa_p \end{pmatrix}\begin{pmatrix}(\varphi^{(n)})' \\ (\chi^{(n)})'\end{pmatrix} = \mathbf{0}\,. \tag{A.14}$$

In particular, for $n = 0$,

$$\sigma^1(\varphi^{(0)})' = \frac{1}{\tilde{p}_0 - m}(i\tilde{p}_y\sigma^3 - i\tilde{p}_z\sigma^2 - \kappa_p)(\chi^{(0)})'\,. \tag{A.15}$$

Multiplying both sides by σ^1 we obtain

$$\begin{aligned}(\varphi^{(0)})' &= \frac{1}{\tilde{p}_0 - m}(\tilde{p}_y\sigma^2 + \tilde{p}_z\sigma^3 - \kappa_p\sigma^1)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0 + m}\varphi^{(0)}\right) \\ &= \frac{\tilde{p}_y^2 + \tilde{p}_z^2 - \kappa_p^2 + \tilde{p}_y\kappa_p[\sigma^2, \sigma^1] + \kappa_p\tilde{p}_z[\sigma^3, \sigma^1]}{\tilde{p}_0^2 - m^2}(\varphi^{(0)})' \\ &+ \frac{\tilde{p}_y\sigma^2 + \tilde{p}_z\sigma^3 - \kappa_p\sigma^1}{\tilde{p}_0 - m}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0 + m}\right)\varphi^{(0)}\,.\end{aligned} \tag{A.16}$$

Therefore,

$$\frac{i\tilde{p}_y\sigma^3 - i\tilde{p}_y\sigma^2 + \kappa_p}{\tilde{p}_0 + m}(\varphi^{(0)})' = \frac{\tilde{p}_y\sigma^2 + \tilde{p}_z\sigma^3 - \kappa_p\sigma^1}{2\kappa}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0 + m}\right)\varphi^{(0)}\,. \tag{A.17}$$

We now apply $i\tilde{p}_y\sigma^3 - i\tilde{p}_y\sigma^2 - \kappa_p$ on both sides and obtain

$$\begin{aligned}&(\varphi^{(0)})' = -\frac{\tilde{p}_0 + m}{2\kappa_p}\sigma^1\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0 + m}\right)\varphi^{(0)} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{\frac{\tilde{p}_0 + m}{\kappa_p}}\right) - \frac{(i\sigma^3\tilde{p}_y' - i\sigma^2\tilde{p}_z')(\tilde{p}_0 + m) - (i\sigma^3\tilde{p}_y - i\sigma^2\tilde{p}_z)\tilde{p}_0'}{2\kappa_p(\tilde{p}_0 + m)}\right]\varphi^{(0)}\,.\end{aligned} \tag{A.18}$$

The solution to this differential equation is given by

$$\begin{aligned}\varphi^{(0)} = C\sqrt{\frac{\tilde{p}_0 + m}{\kappa_p}}\mathrm{T}_x\exp\Bigg[-i\int^x \mathrm{d}\tilde{x} \\ \times\frac{\sigma^3(\tilde{p}_y'(\tilde{p}_0 + m) - \tilde{p}_y\tilde{p}_0') - \sigma^2(\tilde{p}_z'(\tilde{p}_0 + m) - \tilde{p}_z\tilde{p}_0')}{2\kappa_p(\tilde{p}_0 + m)}\Bigg]s_\alpha\,,\end{aligned} \tag{A.19}$$

where C is some constant we will fix. Defining $U(x)$ in Eq. (1.36) and $s_\alpha(x) = U(x)s_\alpha$, the solution to the Dirac equation to zeroth order is

$$\Phi_\alpha = C\sqrt{\frac{\tilde{p}_0 + m}{\kappa_p}} \begin{pmatrix} s_\alpha(x) \\ \frac{\tilde{\mathbf{p}}\cdot\boldsymbol{\sigma}}{\tilde{p}_0+m} s_\alpha(x) \end{pmatrix} \exp\left\{\frac{i}{\hbar}\int_0^x \mathrm{d}\zeta\, \kappa_p(\zeta)\right\} \mathrm{e}^{-\frac{i}{\hbar}(p_0 t - p_y y - p_z z)} \,. \tag{A.20}$$

To match the free solution, we choose the coefficient C to be $C = \sqrt{|p_x|/2m}$ and we arrive at Eq. (1.35).

B The Lorentz Transformation of the Thomas-BMT Equation

In this Appendix, it will be shown that the spinor $\Phi_{(\mathbf{p},\alpha)}$ defined in Eq. (3.25) transforms covariantly under Lorentz transformations. It is straightforward to show that it is covariant under rotations. Thus, it is sufficient to show that this holds also for boosts along one direction. We choose the x-direction without loss of generality. Under this boost, the transformation takes the form

$$\begin{aligned}\Phi_{(\mathbf{p},\alpha)} &\to \exp\left\{\left(\frac{\lambda}{2}\gamma^0\gamma^1\right)\right\}\Phi_{(\mathbf{p},\alpha)} \\ &= \sqrt{p_0+m}\begin{pmatrix}\left(\cosh\frac{\lambda}{2}+\frac{\sigma_1\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\sinh\frac{\lambda}{2}\right)s_\alpha \\ \left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\cosh\frac{\lambda}{2}+\sigma_1\sinh\frac{\lambda}{2}\right)s_\alpha\end{pmatrix},\end{aligned} \tag{B.1}$$

where λ is the boost parameter and s_α satisfies the Thomas-BMT equation. We write the transformed spinor as

$$\exp\left(\tfrac{\lambda}{2}\gamma^0\gamma^1\right)\Phi_{(\mathbf{p},\alpha)} = \sqrt{p_0' + m}\begin{pmatrix} s' \\ \frac{\mathbf{p}'\cdot\boldsymbol{\sigma}}{p_0'+m}s' \end{pmatrix}, \tag{B.2}$$

where the transformed 4−momentum is $(p_0', p_1', \mathbf{p}_\perp)$ with

$$\begin{aligned} p_0' &= p_0 \cosh\lambda + p_1 \sinh\lambda\,, \\ p_1' &= p_0 \sinh\lambda + p_1 \cosh\lambda\,. \end{aligned} \tag{B.3}$$

p_1 and p_1' are the components of the contravariant vectors. By comparing the top two components of Eqs. (B.1) and (B.2), we find

$$s' = \sqrt{\frac{p_0+m}{p_0'+m}}\left(\cosh\tfrac{\lambda}{2} + \frac{\sigma_1\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\sinh\tfrac{\lambda}{2}\right)s_\alpha\,. \tag{B.4}$$

We must verify that this relation is in agreement with the equation for the lower components of Eqs. (B.1) and (B.2) which is

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{p_0'+m}s' = \sqrt{\frac{p_0+m}{p_0'+m}}\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\cosh\tfrac{\lambda}{2} + \sigma_1\sinh\tfrac{\lambda}{2}\right)s_\alpha\,. \tag{B.5}$$

We first verify that $s'^\dagger s' = s_\alpha^\dagger s_\alpha$. The inverse of Eq. (B.4) is

$$s_\alpha = \sqrt{\frac{p_0+m}{p_0'+m}}\left(\cosh\tfrac{\lambda}{2} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}\sigma_1}{p_0+m}\sinh\tfrac{\lambda}{2}\right)s'\,. \tag{B.6}$$

Then, replacing s_α into the right-hand side of Eq. (B.5) we verify the equality. What is left to show is that if s_α satisfies the Thomas-BMT equation, then so does s':

$$\frac{\mathrm{d}s_\alpha}{\mathrm{d}\tau} = -i\mathbf{F}\cdot\boldsymbol{\sigma}\, s_\alpha \quad \rightarrow \quad \frac{\mathrm{d}s'}{\mathrm{d}\tau} = -i\mathbf{F}'\cdot\boldsymbol{\sigma}\, s'\,, \tag{B.7}$$

where $\mathbf{F}'$ is composed of the Lorentz-transformed electromagnetic fields

$$\begin{aligned}
E_1' &= E_1\,, \quad B_1' = B_1\,,\\
E_2' &= E_2\cosh\lambda + B_3\sinh\lambda\,,\\
B_3' &= B_3\cosh\lambda + E_2\sinh\lambda\,,\\
E_3' &= E_3\cosh\lambda - B_2\sinh\lambda\,,\\
B_2' &= B_2\cosh\lambda - E_3\sinh\lambda\,.
\end{aligned} \tag{B.8}$$

We define

$$W = \sqrt{\frac{p_0+m}{p_0'+m}}\left(\cosh\tfrac{\lambda}{2} + \frac{\sigma_1\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\sinh\tfrac{\lambda}{2}\right), \tag{B.9}$$

such that $s' = W s_\alpha$. In the notation $(\mathbf{X}\cdot\boldsymbol{\sigma})_k = X_k$, what we need to show is

$$\left(i\frac{\mathrm{d}W}{\mathrm{d}\tau}W^\dagger\right)_k + (W\boldsymbol{\sigma}\cdot\mathbf{F}W^\dagger)_k = F_k'\,,\ \ k = 1,2,3. \tag{B.10}$$

Eq. (B.10) can be verified explicitly by using the transformed fields in the right hand-side and the definition of W.

C Angular Integration of the Total Axion Energy

The goal of this appendix is to derive Eq. (3.69) starting with Eq. (3.59) for the case of a constant magnetic field. The trajectory of the electron is described by

$$
\begin{aligned}
v^\mu &= \sqrt{a_0^2 - 1}\left(\frac{a_0}{\sqrt{a_0^2-1}}, \cos\omega_0 t, 0, -\sin\omega_0 t\right), \\
a^\mu &= a_0\omega_0\sqrt{a_0^2-1}(0, -\sin\omega_0 t, 0, -\cos\omega_0 t)\,, \\
\dot{a}^\mu &= a_0^2\omega_0^2\sqrt{a_0^2-1}(0, -\cos\omega_0 t, 0, \sin\omega_0 t)\,.
\end{aligned}
\tag{C.1}
$$

The angular integral can be performed starting with the general result

$$
\frac{1}{4\pi}\int \mathrm{d}\Omega\, \frac{n_\mu n_\nu}{(n\cdot v)^5} = 2v^0 v_\mu v_\nu - \frac{1}{3}\delta^0_\mu v_\nu + \delta^0_\nu v_\mu + v^0\eta_{\mu\nu}\,, \tag{C.2}
$$

where we used $v \cdot v = 1$. Then, for any tensor $N_{\mu\nu}$,

$$\frac{1}{4\pi}\int \mathrm{d}\Omega\, \frac{n^\mu n^\nu N_{\mu\nu}}{(n\cdot v)^5} = 2v^0 v^\mu v^\nu N_{\mu\nu} - \frac{1}{3}(N_{0\nu}v^\nu + v^\mu N_{\mu 0} + v^0 N_\mu{}^\mu)\,. \tag{C.3}$$

To derive the first identity in Eq. (3.69), we choose $N_{\mu\nu} = F^{\rho\alpha}F_{\alpha\mu}F_{\rho\beta}F^\beta_{\ \nu}$. The only non-zero component of $F_{\mu\nu}$ is $F_{13} = -F_{13}$. Thus, $N_{\mu 0} = N_{0\mu} = 0$. We also have $a^0 = 0$ for a constant magnetic field. We calculate $v^\mu v^\nu N_{\mu\nu} = -B_y^2(a_0^2 - 1)$ and $N^\mu_\mu = 2B_y^4$.

Differentiating Eq. (C.2) twice gives

$$\begin{aligned}
&\int \frac{\mathrm{d}\Omega}{4\pi}\,\frac{(n\cdot a)^2}{(n\cdot v)^7} n^\mu n^\nu N_{\mu\nu} = (a\cdot a)\left[-\frac{8}{15}v^0 v^\mu v^\nu N_{\mu\nu}\right.\\
&\left.+\frac{1}{15}(N_{0\nu}v^\nu + v^\mu N_{\mu 0} + v^0 N_\mu{}^\mu)\right] + \frac{2}{15}[a^0 a^\mu v^\nu + a^0 v^\mu a^\nu + v^0 a^\mu a^\nu]N_{\mu\nu}\,,\\
&\int \frac{\mathrm{d}\Omega}{4\pi}\,\frac{n\cdot \dot a}{(n\cdot v)^6} n^\mu n^\nu N_{\mu\nu}\\
&= (v\cdot\dot a)\left[\frac{16}{5}v^0 v^\mu v^\nu N_{\mu\nu} - \frac{2}{5}(N_{0\nu}v^\nu + v^\mu N_{\mu 0} + v^0 N_\mu{}^\mu)\right]\\
&\quad - \frac{2}{5}[\dot a^0 v^\mu v^\nu + v^0 \dot a^\mu v^\nu + v^0 v^\mu \dot a^\nu]N_{\mu\nu} + \frac{1}{15}(N^{0\nu}\dot a_\nu + \dot a_\mu N^{\mu 0} + \dot a^0 N_\mu{}^\mu)\,,
\end{aligned} \tag{C.4}$$

where we used $v \cdot a = 0$. In both cases, we choose $N_{\mu\nu} = F_{\mu\lambda}F^\lambda_{\ \nu}$. Again, $N_{0\mu} = N_{\mu 0} = 0$. For the first case, we have $a\cdot a = -a_0^2(a_0^2-1)\omega_0^2$, $v^\mu v^\nu N_{\mu\nu} = B_y^2(a_0^2-1)$, $N_\mu{}^\mu = -2B_y^2$ and finally $a^\mu a^\nu N_{\mu\nu} = a_0^2(a_0^2-1)\omega_0^2 B_y^2$. For the second case, $\dot a^0 = 0$, $v\cdot\dot a = a_0^2(a_0^2-1)\omega_0^2$ and $\dot a^\mu v^\nu N_{\mu\nu} = -a_0^2(a_0^2-1)B_y^2\omega_0^2$. We then find the second and third identities of Eq. (3.69).

D Modes and Operators

Let us suppose that the positive frequency part of a scalar field can be expanded in two different ways

$$\phi^+(x) = \sum_i a_i f_i(x) = \sum_I b_I g_I(x)\,, \tag{D.1}$$

where a_i and b_I are annihilation operators. We chose a discrete expansion for simplicity, but the results can be generalized to a continuous case. The operators satisfy the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [b_I, b_J^\dagger] = \delta_{IJ}. \tag{D.2}$$

Then, the sets $\{f_i\}$ and $\{g_I\}$ are orthonormal with respect to the same Klein-Gordon inner product.

$$\langle f_i, f_j\rangle_{\mathrm{KG}} = \delta_{ij}, \quad \langle g_I, g_J\rangle_{\mathrm{KG}} = \delta_{IJ}\,. \tag{D.3}$$

The product is linear in the second argument and anti-linear in the first. Suppose that the second set of modes can be expanded in terms of the first as

$$g_I = \sum_i U_{Ii} f_i \,, \tag{D.4}$$

for some operator U. We also assume that U is invertible. Then U is unitary:

$$\begin{aligned}\delta_{IJ} = \langle g_I, g_J \rangle_{\mathrm{KG}} &= \sum_{i,j} \langle U_{Ii} f_i, U_{Jj} f_j \rangle_{\mathrm{KG}} = \sum_{i,j} U^*_{Ii} U_{Jj} \langle f_i, f_j \rangle_{\mathrm{KG}} \\ &= \sum_i U^*_{Ii} U_{Ji} \,.\end{aligned} \tag{D.5}$$

In other words $U^*_{Ii} = (U^{-1})_{iI}$. Then, from Eqs. (D.1) and (D.4)

$$\sum_i a_i f_i(x) = \sum_{I,i} b_I U_{Ii} f_i(x) \,. \tag{D.6}$$

Applying Eq. (D.3),

$$a_i = \sum_I b_I U_{Ii} \quad \rightarrow \quad b_I = \sum_i a_i (U^{-1})_{iI} = \sum_i U^*_{Ii} a_i \,. \tag{D.7}$$

Then finally

$$b_I^\dagger = \sum_i U_{Ii} a_i^\dagger \,. \tag{D.8}$$

By comparing Eqs. (D.4) and (D.8), we see that the relation satisfied by the modes is the same as the one satisfied by the creation operators.

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