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CHARACTER THEORY OF SYMMETRIC GROUPS AND SUBGROUP GROWTH OF SURFACE GROUPS

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ABSTRACT

Results from the character theory of symmetric groups are used to obtain an asymptotic estimate for the subgroup growth of fundamental groups of closed 2-manifolds. The main result implies an affirmative answer, for the class of groups investigated, to a question of Lubotzky's concerning the relationship between the subgroup growth of a one-relator group and that of a free group of appropriately chosen rank. As byproducts, an interesting statistical property of commutators in symmetric groups and the fact that in a 'large' surface group almost all finite index subgroups are maximal are obtained, among other things. The approach requires an asymptotic estimate for the sum $\sum 1/(\chi_\lambda(1))^s$ taken over all partitions λ of n with fixed $s \geqslant 1$, which is also established.

1. *Introduction*

For a group Γ, denote by $s_n(\Gamma)$ the number of subgroups of index n in Γ. If Γ is finitely generated (as will be the case here) or of finite subgroup rank, then $s_n(\Gamma)$ is finite for all n. The natural context for the research reported in this paper is the theory of *subgroup growth*, a fast developing part of what is nowadays called 'asymptotic group theory', which has evolved over the last 15 years in the work of Grunewald, Lubotzky, Mann, Segal and others, including the first author of this paper; its principal objects of study are arithmetic properties of the sequence $\{s_n(\Gamma)\}_{n\geqslant 1}$ or related subgroup counting functions and their connection with the algebraic structure of the underlying group Γ. The original motivation for the latter investigations comes from three main sources: the concept of word growth and, more specifically, Gromov's characterization in [**10**] of finitely generated groups with polynomial word growth, the theory of rings of algebraic integers and their zeta functions, and the work of Marshall Hall and Tibor Radó in the late 1940s on Schreier systems in free groups and their associated subgroups; cf., in particular, [**11–13**]. An account of most of the major results concerning the function $s_n(\Gamma)$ obtained prior to 1992 can be found in Lubotzky's Galway notes [**17**, **18**]. More recent contributions include (in rough chronological order) [**7**, **20**, **19**, **24–26**, **8**, **27–31**].

So far, most of the major developments concerning the theory of subgroup growth have concentrated on one of two classes of discrete groups: finitely generated nilpotent groups and finitely generated groups containing a free subgroup of finite index (that is, fundamental groups of finite graphs of finite groups). On the other hand, almost nothing appears to be known, for instance, about the subgroup growth

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of one-relator groups, that is, groups Γ having a presentation of the form

$$\Gamma \cong \left\langle x_1, x_2, \ldots, x_d \mid w = 1 \right\rangle,$$

where $w = w(x_1, \ldots, x_d)$ is an element of the free group freely generated by the symbols $x_1, \ldots, x_d$. If $d \geqslant 3$, then, by a result of Baumslag and Pride [**1**], Γ contains a subgroup of finite index which can be mapped homomorphically onto a non-abelian free group. Hence, in this case, $s_n(\Gamma)$ grows super-exponentially just like the sequence of subgroup numbers of a non-abelian free group. One might feel, however, that, at least generically, the relationship between the subgroup growth of one-relator groups and that of free groups should be rather more intimate than the latter observation seems to imply. More specifically, one might ask, as Lubotzky does in [**17**], whether the limit

$$\lim_{n\to\infty} \frac{s_n(\Gamma)}{s_n(F_{d-1})} \tag{1}$$

is finite and positive for $d \geqslant 3$, and, if so, what this limit is. A similar type of relationship 'almost' holds when the free subgroup growth of a finitely generated virtually free group is compared with the subgroup growth of the free group of rank $\mu(\Gamma)$, where $\mu(\Gamma) = 1 - m_\Gamma \chi(\Gamma)$ is the free rank of Γ; cf. [**22**, **23**].

The purpose of this paper is to show how the character theory of symmetric groups can be used to obtain precise asymptotic estimates for the function $s_n(\Gamma)$ as $n \to \infty$, when Γ is the fundamental group of a closed 2-manifold, that is, when Γ equals

$$\Gamma_g^+ = \left\langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1][x_2, y_2] \ldots [x_g, y_g] = 1 \right\rangle$$

or

$$\Gamma_h^- = \left\langle x_1, \ldots, x_h \mid x_1^2 x_2^2 \ldots x_h^2 = 1 \right\rangle$$

for some $g \geqslant 2$, respectively $h \geqslant 3$. Our main result provides an asymptotic expansion of the form

$$s_n(\Gamma) \approx 2\, n\, (n!)^{d-2} \left\{ 1 + \sum_{\nu=d-2}^{\infty} \mathscr{C}_\nu(d)\, n^{-\nu} \right\}, \qquad n \to \infty, \tag{2}$$

with explicitly known coefficients $\mathscr{C}_\nu(d)$ and $d = 2g$ or $d = h$, respectively; cf. Theorem 2. Comparing the main term of (2) with the right-hand side of the asymptotic formula

$$s_n(F_r) \sim n\, (n!)^{r-1}, \qquad r \geqslant 2,$$

for $r = d - 1$, we see that Lubotzky's question as to the existence of limit (1) has, for the class of groups investigated here, an affirmative answer, and that the value of this limit is 2. The proof of Theorem 2 depends, among other things, on an asymptotic estimate for the sum

$$\Phi_s(n) := \sum_{\lambda \vdash n} \chi_\lambda(1)^{-s}$$

with fixed $s \geqslant 1$, which is of independent interest. Curiously enough, while certain related character sums have been treated in the literature (cf., for instance, [**36**]), nothing appears to be known about the asymptotics of the functions $\Phi_s(n)$. We show that, for fixed $s \geqslant 1$, $\Phi_s(n)$ has an asymptotic expansion of the form

$$\Phi_s(n) \approx 2\left\{1 + \sum_{\rho=s}^{\infty} \mathscr{A}_\rho(s)\, n^{-\rho}\right\}, \qquad n \to \infty, \tag{3}$$

with appropriate (and explicitly computed) coefficients $\mathscr{A}_\rho(s)$; cf. Theorem 1. Estimate (3) in conjunction with character-theoretic arguments also allows us to establish an interesting statistical property of commutators in symmetric groups: given four random elements $x_1, x_2, x_3, x_4 \in S_n$, we show that the probability that the commutators $[x_1, x_2]$ and $[x_3, x_4]$ generate A_n tends to 1 as n tends to infinity (Theorem 3).

The paper is organized as follows. In Section 2 we derive character formulae for the number of solutions in a finite group G of the equations $[x_1, y_1] \ldots [x_g, y_g] = z$, respectively $x_1^2 \ldots x_h^2 = z$. Proofs of Theorem 2, respectively Theorem 3, that depend on estimate (3) are then provided in Section 3, respectively Section 5, while the proof of Theorem 1 itself occupies Section 6. In Section 4, we use results and formulae from Sections 2 and 3 to establish a number of further facts concerning surface groups; for instance, we show that in a surface group of rank $d \geqslant 3$, almost all finite index subgroups are maximal.

2. *Some equations in finite groups*

Let G be a finite group. Denote by $[x, y]$ the commutator $x^{-1}y^{-1}xy$ of elements x and y in G, and, for $z \in G$, let $N^G(z)$ be the number of solutions of the equation $[x, y] = z$ with $x, y \in G$.

LEMMA 1. *For $z \in G$, we have*

$$N^G(z) = |G| \sum_{\chi} \chi(z)/\chi(1),$$

where χ runs through the ordinary irreducible characters of G.

Proof. Fix conjugacy classes C_1 and C_2 of G (not necessarily distinct), and an element $z \in G$. Then the number of solutions of the equation $x \cdot y = z$ with $x \in C_1$ and $y \in C_2$ equals

$$\frac{|C_1||C_2|}{|G|} \sum_{\chi} \frac{\chi(C_1)\chi(C_2)\chi(z^{-1})}{\chi(1)}. \tag{4}$$

This result, or some generalization of it, can be found in various places in the literature; cf., for instance, [**3**, Proposition 9.33] or [**14**, Theorem 6.3.1]. Since $[x, y] = x^{-1} \cdot x^y$, we can obtain the solutions of the equation $[x, y] = z$ with x in a given conjugacy class C of G by first solving the equation $\bar{x} \cdot \bar{x}' = z$ with $\bar{x} \in C^{-1}$ and $\bar{x}' \in C$, and then choosing $y \in G$ in $|C_G(x)| = |G|/|C|$ ways to write a given $\bar{x}'$ as $\bar{x}' = x^y$. Applying (4) and noting that $|C| = |C^{-1}|$, we see that the number of these solutions equals

$$|C| \sum_{\chi} \frac{\chi(C^{-1})\,\chi(C)\,\chi(z^{-1})}{\chi(1)},$$

and hence summing over the conjugacy classes gives

$$N^G(z) = \sum_C |C| \sum_\chi \frac{\chi(C^{-1})\,\chi(C)\,\chi(z^{-1})}{\chi(1)}.$$

Changing the order of summation in the latter equation, and using the fact that characters are class functions plus their orthogonality relations, we find that

$$N^G(z) = \sum_\chi \frac{\chi(z^{-1})}{\chi(1)} \sum_C |C|\,\chi(C^{-1})\,\chi(C) = |G| \sum_\chi \chi(z^{-1})/\chi(1).$$

Our result follows now by conjugation of the last equation, since $\overline{\chi(z)} = \chi(z^{-1})$. □

For $z \in G$, denote by $R^G(z)$ the number of solutions of the equation $x^2 = z$ with $x \in G$. It is known that R^{S_n} is in fact the *model character* of S_n, that is,

$$R^{S_n}(z) = \sum_{\lambda \vdash n} \chi_\lambda(z), \qquad z \in S_n. \tag{5}$$

In general, R^G need not even be a proper character; it is a virtual character of G, and the (integral) coefficients $c_\chi(G)$ in its decomposition

$$R^G(z) = \sum_\chi c_\chi(G)\,\chi(z), \qquad z \in G, \tag{6}$$

satisfy $|c_\chi(G)| \leqslant 1$, and are non-zero if and only if the corresponding character χ is real-valued. The higher root number functions of S_n have more recently also been shown to be proper characters, but apparently no good estimates are known for their coefficients; cf. [**37**] for Scharf's original proof of the latter result. An alternative proof of this last result using symmetric function theory is outlined in the solution to [**38**, Exercise 7.69 c]. A good account of all the facts on root number functions mentioned can be found in [**14**, Chapter 6.2].

For $g, h \geqslant 1$ define

$$N_g^+(G,z) := \left|\left\{(x_1, y_1, \ldots, x_g, y_g) \in G^{2g} : [x_1, y_1][x_2, y_2]\ldots[x_g, y_g] = z\right\}\right|$$

and

$$N_h^-(G,z) := \left|\left\{(x_1, \ldots, x_h) \in G^h : x_1^2 x_2^2 \ldots x_h^2 = z\right\}\right|;$$

in particular, $N_1^+(G,z) = N^G(z)$ and $N_1^-(G,z) = R^G(z)$.

PROPOSITION 1. *Let G be a finite group, and let $z \in G$ be a fixed element. Then we have*

$$N_g^+(G,z) = |G|^{2g-1} \sum_\chi \chi(z)/(\chi(1))^{2g-1}, \tag{7}$$

and

$$N_h^-(G,z) = |G|^{h-1} \sum_\chi c_\chi^h(G)\,\chi(z)/(\chi(1))^{h-1}, \tag{8}$$

where the $c_\chi(G)$ are given by (6).

Proof. We proceed by induction on g, respectively h. If $g = 1$, then (7) holds by virtue of Lemma 1. Suppose that (7) holds for some $g \geqslant 1$. Then

$$\begin{aligned} N_{g+1}^{+}(G,z) &= \sum_{x\in G} N_g^{+}(G,x)\, N^G(x^{-1}z) \\ &= |G|^{2g} \sum_{x\in G} \Big[\sum_{\chi} \chi(x)/(\chi(1))^{2g-1}\Big]\Big[\sum_{\chi} \chi(x^{-1}z)/\chi(1)\Big] \\ &= |G|^{2g} \sum_{\chi_1,\chi_2} (\chi_1(1))^{-(2g-1)}\,(\chi_2(1))^{-1} \sum_{x\in G} \chi_1(x)\,\chi_2(x^{-1}z) \\ &= |G|^{2g+1} \sum_{\chi} \chi(z)/(\chi(1))^{2g+1}, \end{aligned}$$

as required. Here we have used the inductive hypothesis (7) and Lemma 1, as well as the orthogonality relation

$$\sum_{x\in G} \chi_1(x)\,\chi_2((ax)^{-1}) = \chi_1(a^{-1})\,|G|\,\langle \chi_1 \mid \chi_2\rangle\,/\chi_1(1); \tag{9}$$

cf. [**4**, formula 31.16]. In a similar vein, replacing Lemma 1 by formula (6), we obtain (8). □

COROLLARY 1. *For a finite group G and integers $g, h \geqslant 1$, we have*

$$|\mathrm{Hom}(\Gamma_g^{+}, G)| = |G|^{2g-1} \sum_{\chi} (\chi(1))^{-2(g-1)}$$

as well as

$$|\mathrm{Hom}(\Gamma_h^{-}, G)| = |G|^{h-1} \sum_{\chi} c_\chi^h(G)\,(\chi(1))^{-(h-2)};$$

in particular,

$$|\mathrm{Hom}(\Gamma, S_n)| = (n!)^{d-1} \sum_{\lambda\vdash n} (\chi_\lambda(1))^{-(d-2)}$$

if Γ is a surface group of rank $d \geqslant 1$.

3. *Subgroup growth of surface groups*

Here, we are going to establish an asymptotic expansion for the function $s_n(\Gamma)$ in the case where Γ is a surface group of rank $d \geqslant 3$. By [**6**, Proposition 1], the functions $s_n(\Gamma)$ and $h_n(\Gamma) := |\mathrm{Hom}(\Gamma, S_n)|/(n!)$ are related via the transformation formula

$$\sum_{k=1}^{n} s_k(\Gamma)\,h_{n-k}(\Gamma) = n\,h_n(\Gamma), \qquad n \geqslant 1. \tag{10}$$

Corollary 1 furnishes the explicit formula

$$h_n(\Gamma) = (n!)^{d-2}\,\Phi_{d-2}(n), \tag{11}$$

where, for a fixed integer s,

$$\Phi_s(n) := \sum_{\lambda\vdash n} (\chi_\lambda(1))^{-s}.$$

We shall use the following estimate for the functions $\Phi_s(n)$ with $s \geqslant 1$. Its proof occupies Section 6.

Theorem 1. *For each fixed integer $s \geqslant 1$, we have the asymptotic expansion*

$$\Phi_s(n) \approx 2 \sum_{\rho=0}^{\infty} \mathscr{A}_\rho(s)\, n^{-\rho}, \qquad n \to \infty,$$

where the coefficients $\mathscr{A}_\rho(s)$ are given by

$$\mathscr{A}_\rho(s) := \sum_{r \geqslant 0} \sum_{\mu \vdash r} H[\mu]^s \sum_{(\kappa_\nu)} \prod_{\nu \in N(\mu,r)} \left[\binom{s+\kappa_\nu - 1}{\kappa_\nu} (2r-\nu)^{\kappa_\nu} \right], \qquad \rho \geqslant 0.$$

Here, $H[\mu]$ is the hook product of μ,

$$N(\mu, r) := [2r] - \left\{ j + \sum_{i=0}^{j-1} m_{r-i}(\mu) \,:\, 1 \leqslant j \leqslant r \right\},$$

and $\sum_{(\kappa_\nu)}$ denotes the sum over the family of discrete variables $\{\kappa_\nu \,:\, \nu \in N(\mu,r)\}$ satisfying $\kappa_\nu \geqslant 0$ for $\nu \in N(\mu,r)$ and $\sum \kappa_\nu = \rho - rs$.

Corollary 2. *For fixed $s \geqslant 1$, we have*

$$\Phi_s(n) = 2 \,+\, \mathcal{O}(n^{-s}), \qquad n \to \infty.$$

Proof. This is immediate from Theorem 1, since, for $1 \leqslant \rho < s$ and every $r \geqslant 0$, the summation over the κ_ν in the definition of the $\mathscr{A}_\rho(s)$ is empty, while $\mathscr{A}_0(s) = 1$. □

Formula (11), when combined with Theorem 1, provides an explicit asymptotic expansion for the function $h_n(\Gamma)$, provided that $d \geqslant 3$; however, we are still faced with the question of how to transfer asymptotic information from $h_n(\Gamma)$ to the function $s_n(\Gamma)$. To deal with the latter problem, we use an asymptotic method for divergent power series due to Wright [**39**] and, in greater generality, to Bender [**2**]. Adopt, for a moment, the following more formal point of view. Consider two sequences $1 = h_0, h_1, h_2, \ldots$ and $s_1, s_2, s_3, \ldots$ of real numbers satisfying a relation

$$\sum_{k=1}^{n} h_{n-k}\, s_k = c\, n\, h_n, \qquad n \geqslant 1, \tag{12}$$

with some constant $c > 0$. We require $s_n \geqslant 0$ and $h_n > 0$ for all $n \geqslant 1$. Define the triangle $\Delta = (H_k^n)_{0 \leqslant k \leqslant n}$ associated with transformation (12) by

$$H_k^n := \frac{h_n}{h_k\, h_{n-k}}, \qquad 0 \leqslant k \leqslant n,$$

and, for each fixed integer $K \geqslant 1$, put

$$\mathscr{S}_n^{(K)} := \sum_{k=K}^{n-K} (H_k^n)^{-1}.$$

If the sequence $\{h_n\}_{n \geqslant 0}$ is growing in an appropriately rapid fashion, then the following criterion can be used to transfer information about the asymptotic behaviour of the h_n to the sequence s_k.

Lemma 2. *Suppose that, for some integer $K \geqslant 1$ as $n \to \infty$,*

(i) $h_{n-1} = o(h_n)$;

(ii)$_K$ $\mathscr{S}_n^{(K)} = \mathcal{O}(h_{n-K}/h_n)$.

Then the sequence s_n satisfies the relation

$$\frac{s_n}{c\,n} = \sum_{k=0}^{K-1} \mathfrak{c}_k\, h_{n-k} \;+\; \mathcal{O}(h_{n-K}), \qquad n \to \infty,$$

where $\mathfrak{c}_k$ is the coefficient of z^k in the series $(\sum_{n\geqslant 0} h_n z^n)^{-1}$.

This is a consequence of the main results in [**2**]; cf. also [**23**, Section 2]. Now assume that $d \geqslant 3$, so that Γ contains a subgroup of finite index which maps homomorphically onto a non-abelian free group. The sequences $\{h_n(\Gamma)\}_{n\geqslant 0}$ and $\{s_n(\Gamma)\}_{n\geqslant 1}$ are related via the transformation formula (12) with $c = 1$, and (11) in conjunction with Corollary 2 implies in particular that

$$h_{n-1}(\Gamma)/h_n(\Gamma) \sim n^{-(d-2)}, \qquad n \to \infty.$$

Hence, under the assumption $d \geqslant 3$, Lemma 2(i) is satisfied. Also, for fixed $K \geqslant 1$,

$$h_{n-K}(\Gamma)/h_n(\Gamma) = \prod_{k=1}^{K} h_{n-K+k-1}(\Gamma)/h_{n-K+k}(\Gamma) \sim n^{-K(d-2)},$$

and condition (ii)$_K$ takes the (equivalent) form

$$\mathcal{S}_n^{(K)}(\Gamma) := \sum_{k=K}^{n-K} \frac{h_k(\Gamma)\, h_{n-k}(\Gamma)}{h_n(\Gamma)} = \mathcal{O}(n^{-K(d-2)}), \qquad n \to \infty.$$

However, as our next result shows, the latter estimate holds for every fixed positive integer K, provided that $d \geqslant 3$.

LEMMA 3. *For $d \geqslant 3$ and fixed $K \geqslant 1$, we have*

$$\mathcal{S}_n^{(K)}(\Gamma) = \mathcal{O}(n^{-K(d-2)}), \qquad n \to \infty.$$

Proof. By Corollary 2, there exists a constant $C \geqslant 1$ such that

$$1 \leqslant \Phi_{d-2}(n) \leqslant C, \qquad n \geqslant 1.$$

Taking $n > 2K$, we find from the unimodal property of Pascal's triangle that for $d \geqslant 3$ and fixed K,

$$\begin{aligned}
0 \leqslant \mathcal{S}_n^{(K)}(\Gamma) &\leqslant C^2 \sum_{k=K}^{n-K} \binom{n}{k}^{-(d-2)} \\
&\leqslant 2C^2 \binom{n}{K}^{-(d-2)} \;+\; C^2\,(n-2K-1) \binom{n}{K+1}^{-(d-2)} \\
&= 2C^2 \binom{n}{K}^{-(d-2)} \;+\; C^2\,(n-2K-1) \left(\frac{n-K}{K+1}\right)^{-(d-2)} \binom{n}{K}^{-(d-2)} \\
&\leqslant 2C^2 \binom{n}{K}^{-(d-2)} \;+\; C^2\,(n-2K-1) \left(\frac{n-K}{K+1}\right)^{-1} \binom{n}{K}^{-(d-2)} \\
&< C^2\,(K+3) \binom{n}{K}^{-(d-2)} = \mathcal{O}(n^{-K(d-2)}).
\end{aligned}$$

□

In view of Lemma 3 and the previous discussion, Lemma 2 applies in our situation for every $K \geqslant 1$, and we get the following.

PROPOSITION 2. *For Γ as above, we have the asymptotic expansion*

$$s_n(\Gamma) \approx n\,h_n(\Gamma)\left\{1 + \sum_{k=1}^{\infty} \mathfrak{c}_k(d)\,\frac{h_{n-k}(\Gamma)}{h_n(\Gamma)}\right\}, \qquad n \to \infty,$$

where the $\mathfrak{c}_k(d)$ are the coefficients of the series $\left(\sum_{n\geqslant 0}(n!)^{d-2}\Phi_{d-2}(n)z^n\right)^{-1}$.

Combining formula (11) with Theorem 1 and Proposition 2, we obtain, after some trivial manipulations, the main result of this paper.

THEOREM 2. *Let Γ be a surface group of rank $d \geqslant 3$. Then the function $s_n(\Gamma)$ satisfies the asymptotic expansion*

$$s_n(\Gamma) \approx 2\,n\,(n!)^{d-2}\left\{1 + \sum_{v=1}^{\infty} \mathscr{C}_v(d)\,n^{-v}\right\}, \qquad n \to \infty, \tag{13}$$

where, for $v \geqslant 1$,

$$\mathscr{C}_v(d) = \sum_{k=1}^{\lfloor v/(d-2)\rfloor} \sum_{\eta_1+\ldots+\eta_{k-1}+\eta+\rho+k(d-2)=v} \mathfrak{c}_k(d)k^{\eta}\binom{\eta+\rho-1}{\eta}\mathscr{A}_\rho(d-2) \times \prod_{i=1}^{k-1}\left[i^{\eta_i}\binom{d+\eta_i-3}{\eta_i}\right].$$

COROLLARY 3. *For Γ as in Theorem 2, we have*

$$s_n(\Gamma) = 2\,n\,(n!)^{d-2}\left\{1 + \mathscr{O}(n^{-(d-2)})\right\}; \tag{14}$$

in particular, $\lim_{n\to\infty}(s_n(\Gamma)/s_n(F_{d-1})) = 2$.

Proof. The first assertion is immediate from Theorem 2, while the second statement, concerning the quotient $s_n(\Gamma)/s_n(F_{d-1})$, follows from (14) and the fact that for $r \geqslant 2$,

$$s_n(F_r) \sim n\,(n!)^{r-1}, \qquad n \to \infty;$$

cf. [**33**, Theorem 2]. □

4. *Miscellaneous results*

4.1. *A representation-theoretic formula for $s_n(\Gamma)$*

Rewriting equation (10), which holds for every finitely generated group Γ, in terms of generating functions, we find that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\left(1 + \sum_{n\geqslant 1} h_n(\Gamma)\,z^n\right)\right) = \sum_{n\geqslant 0} s_{n+1}(\Gamma)\,z^n.$$

Integrating the latter equation, expanding the logarithm, and comparing coefficients, we find that

$$s_n(\Gamma) = n\sum_{v=1}^{n}\frac{(-1)^{v+1}}{v}\sum_{\substack{n_1,\ldots,n_v\geqslant 1\\ n_1+\ldots+n_v=n}} h_{n_1}(\Gamma)\ldots h_{n_v}(\Gamma).$$

If we now take Γ to be a surface group of rank $d \geqslant 1$, then the last equation, when combined with (11), yields the identity

$$s_n(\Gamma) = n \sum_{\nu=1}^{n} \frac{(-1)^{\nu+1}}{\nu} \sum_{\substack{n_1,\ldots,n_\nu \geqslant 1 \\ n_1+\ldots+n_\nu = n}} \prod_{j=1}^{\nu} \Big[(n_j!)^{d-2}\, \Phi_{d-2}(n_j) \Big], \tag{15}$$

which explicitly expresses the subgroup numbers of Γ in representation-theoretic terms.

4.2. *Fundamental groups of torus and Klein bottle*

It follows immediately from Corollary 1 and formula (10) that the groups Γ_g^+ and Γ_{2g}^-, while not isomorphic, are *isospectral*, that is, they have the same number of index n subgroups for every n; in particular, the torus group $\Gamma_1^+ \cong \mathbb{Z}^2$ and the fundamental group of the Klein bottle

$$\Gamma_2^- \cong \mathbb{Z} \underset{2\mathbb{Z}}{*} \mathbb{Z}$$

have the same subgroup numbers. However,

$$|\mathrm{Hom}(\Gamma_1^+, S_n)| = \sum_{\pi \in S_n} |C_{S_n}(\pi)| = \sum_{\pi \in S_n/\sim} \big(S_n : C_{S_n}(\pi)\big) \cdot |C_{S_n}(\pi)| = n!\, p(n),$$

where $p(n)$ is the number of partitions of n. Comparing equation (10) for Γ_1^+ with the well known identity (cf., for instance, [**16**, Satz 7.3])

$$n\, p(n) = \sum_{k=1}^{n} \sigma(k)\, p(n-k)$$

relating the partition function to the arithmetic function $\sigma(n) = \sum_{\nu | n} \nu$, we find that

$$s_n(\Gamma_1^+) = s_n(\Gamma_2^-) = \sigma(n), \qquad n \geqslant 1. \tag{16}$$

Formulae (15) and (16) are due to Mednykh; cf. [**21**, Theorem A, Theorem 1, Theorem 2].

4.3. *Uniform asymptotics*

If we restrict attention to the main term in Theorem 2, then uniformity in d and n can be achieved. Indeed, if Γ_d is a surface group of rank d, then for $n > 1$ and $d > 2$,

$$\begin{aligned}\frac{h_n(\Gamma_d)}{(n!)^{d-2}} &= \sum_{\lambda \vdash n} \big(\chi_\lambda(1)\big)^{-(d-2)} \\ &= 2 + \mathcal{O}\Bigg(\Bigg[\sum_{\substack{\lambda \vdash n \\ \lambda \neq (1^n),(n)}} \big(\chi_\lambda(1)\big)^{-1} \Bigg] \Bigg[\max_{\substack{\lambda \vdash n \\ \lambda \neq (1^n),(n)}} \big(\chi_\lambda(1)\big)^{-(d-3)} \Bigg] \Bigg) \\ &= 2 + \mathcal{O}\big(n^{-1} 2^{-(d-3)}\big),\end{aligned}$$

and

$$\begin{aligned} n\, h_n(\Gamma_d) - s_n(\Gamma_d) &\leqslant \sum_{k=1}^{n-1} k\, h_k(\Gamma_d)\, h_{n-k}(\Gamma_d) \\ &\leqslant C\, n \sum_{k=1}^{n-1} (k!)^{d-2} \big((n-k)!\big)^{d-2},\end{aligned}$$

where

$$C := \left(\sup_{n,d} \frac{h_n(\Gamma_d)}{(n!)^{d-2}}\right)^2,$$

the supremum existing by the previous computation. Hence, dividing by $nh_n(\Gamma_d)$, we find that

$$\left|\frac{s_n(\Gamma_d)}{nh_n(\Gamma_d)} - 1\right| \leqslant C \sum_{k=1}^{n-1} \binom{n}{k}^{-(d-2)} = \mathcal{O}\big(n^{-(d-2)}\big).$$

Consequently,

$$\frac{s_n(\Gamma_d)}{2n(n!)^{d-2}} = \frac{s_n(\Gamma_d)}{nh_n(\Gamma_d)} \frac{h_n(\Gamma_d)}{2(n!)^{d-2}} = 1 + \mathcal{O}\big(n^{-1}2^{-(d-3)}\big). \tag{17}$$

In particular, we find from (17) that, for $n > 1$ fixed and $d \to \infty$,

$$s_n(\Gamma_d) \sim 2n(n!)^{d-2}.$$

The latter result can also be obtained by combining formula (15) with the observation that $\Phi_s(n) = 2 + \mathcal{O}(1)\mathcal{O}(n^{-s}) \to 2$ for $n > 1$ fixed and $s \to \infty$; cf. [**21**, Corollary D].

4.4. *Maximal subgroups*

For a finitely generated group Γ, let $m_n(\Gamma)$ be the number of maximal subgroups of index n in Γ. Here we will use (17) to prove that for a surface group $\Gamma = \Gamma_d$ of rank $d \geqslant 3$,

$$\frac{m_n(\Gamma)}{s_n(\Gamma)} = 1 + \mathcal{O}\big(e^{-cn}\big)$$

with some constant $c > 0$; that is, almost all finite index subgroups of a 'large' surface group are maximal. As is well known, a subgroup of index n in Γ_d is isomorphic to a surface group of rank $n(d-2)+2$. The number of non-maximal subgroups of index n in Γ is bounded above by the number of pairs of subgroups (Δ, Δ') with $\Delta < \Delta' < \Gamma$ and $(\Gamma : \Delta) = n$. Hence, using (17), we find that

$$\begin{aligned} s_n(\Gamma) - m_n(\Gamma) &\leqslant \sum_{\substack{\nu \mid n \\ 1<\nu<n}} s_\nu(\Gamma)\, s_{n/\nu}(\Gamma_{\nu(d-2)+2}) \\ &\leqslant Cn \sum_{\substack{\nu \mid n \\ 1<\nu<n}} (\nu!)^{d-2}\big((n/\nu)!\big)^{\nu(d-2)}. \end{aligned}$$

Approximating the factorials occurring in the latter expression by Stirling's formula, dividing by $s_n(\Gamma)$, and using (17) again, we find that

$$\begin{aligned} 1 - \frac{m_n(\Gamma_d)}{s_n(\Gamma_d)} &\leqslant C' \sum_{\substack{\nu \mid n \\ 1<\nu<n}} \left(\left(\frac{2\pi}{e^2}\right)^{\nu/2} \frac{\nu^{(\nu+1)/2}\, n^{(\nu-1)/2}}{\nu^n}\right)^{d-2} \\ &\leqslant C' \sum_{\substack{\nu \mid n \\ 1<\nu<n}} \left(\frac{n^{\nu/n}}{\nu}\right)^{n(d-2)} \end{aligned}$$

with some constant $C' > 0$. Hence it suffices to show that

$$\frac{n^{v/n}}{v} \leqslant \frac{2}{3}, \qquad v \mid n,\ 1 < v < n,\ n > n_0.$$

In order to see this, we distinguish the cases $2 \leqslant v \leqslant 2\sqrt{n}$ and $2\sqrt{n} < v \leqslant n/2$. In the first case,

$$\frac{n^{v/n}}{v} \leqslant \frac{n^{2/\sqrt{n}}}{2} \to \frac{1}{2} \qquad \text{as } n \to \infty,$$

while in the second case,

$$\frac{n^{v/n}}{v} \leqslant \frac{n^{1/2}}{2\sqrt{n}} = \frac{1}{2}.$$

4.5. *Parity patterns of surface groups*

Formula (11) is sufficiently explicit to yield information about the parity patterns of surface groups, that is, the sequences $(s_n(\Gamma) \bmod 2)_{n \geqslant 1} \in \mathrm{GF}(2)^{\mathbb{N}}$. Call an irreducible character χ of S_n a 2-*core* character if $n!/\chi(1)$ is odd. By the hook formula, this condition is equivalent to requiring that all hook lengths of the partition associated with χ are odd. It is easy to see from this that an irreducible character χ_λ is 2-core if and only if λ is of the form $\lambda = (k, k-1, \ldots, 1)$ for some $k \geqslant 1$; hence S_n has a 2-core character if and only if n is triangular, that is, $n = k(k+1)/2$ for some positive integer k, in which case there is precisely one such character.

PROPOSITION 3. *For a surface group Γ of rank $d \geqslant 3$, we have the recurrence relation*

$$s_n(\Gamma) \equiv \sum_{\substack{k \geqslant 1 \\ k(k+1) < 2n}} s_{n-k(k+1)/2}(\Gamma) + \delta(n) \mod 2, \qquad n \geqslant 1,$$

where

$$\delta(n) := \begin{cases} 1 & n \text{ odd and triangular} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the parity of $s_n(\Gamma)$ does not depend on Γ.

Proof. Rewrite formula (11) as

$$h_n(\Gamma) = \sum_{\lambda \vdash n} \left(\frac{n!}{\chi_\lambda(1)} \right)^{d-2}.$$

Then, by the definition of a 2-core character, the fact that $d \geqslant 3$, and the remarks above, we find that $h_n(\Gamma)$ is odd if and only if n is triangular. Using this information in equation (10), we obtain our claim, the contribution $\delta(n)$ coming from the term $nh_n(\Gamma)$. □

Combining Proposition 3 with classical results of Legendre and Gauss concerning representation numbers of binary quadratic forms, it can be deduced, for instance, that for a surface group Γ of rank at least 3, the function $s_n(\Gamma)$ is odd if and only if n is a square or twice a square; cf. [**32**].

5. *A statistical property of commutators in symmetric groups*

For $z \in S_n$, denote by $N(z)$ the number of solutions of the equation $[x, y] = z$ with $x, y \in S_n$, that is, $N(z) = N^{S_n}(z)$. For $z \in A_n$, define $R(z)$ via the equation

$$N(z) = 2n! + R(z).$$

The key property of this remainder term $R(z)$ turns out to be the following.

LEMMA 4. *We have*

$$\sum_{z \in A_n} |R(z)| \ll n^{-1}(n!)^2. \tag{18}$$

Proof. By Lemma 1, we have

$$N(z) = n! \sum_{\lambda \vdash n} \chi_\lambda(z)/\chi_\lambda(1).$$

An orthogonality argument similar to those occurring in the proof of Proposition 1 now shows that

$$\sum_{z \in A_n} (N(z))^2 = (n!)^3 \sum_{\lambda \vdash n} (\chi_\lambda(1))^{-2},$$

which, when combined with Corollary 2, yields the estimate

$$\sum_{z \in A_n} (N(z))^2 = 2(n!)^3 \{1 + \mathcal{O}(n^{-2})\}, \qquad n \to \infty. \tag{19}$$

It follows that

$$\begin{aligned}\sum_{z \in A_n} (R(z))^2 &= \sum_{z \in A_n} \big(N(z) - 2n!\big)^2 \\ &= \sum_{z \in A_n} (N(z))^2 - 4n! \sum_{z \in A_n} N(z) + 2(n!)^3 \\ &\ll n^{-2}(n!)^3.\end{aligned}$$

Here, we have estimated the first sum via (19), while the second sum equals $(n!)^2$, since every pair $(x, y) \in S_n^2$ gives rise to a solution of the equation $[x, y] = z$ for some $z \in A_n$. An application of the Cauchy–Schwarz inequality now gives

$$\left(\sum_{z \in A_n} |R(z)|\right)^2 \leqslant \frac{n!}{2} \sum_{z \in A_n} (R(z))^2 \ll n^{-2}(n!)^4,$$

whence the lemma. □

In order to get some feeling for the quality of the estimate provided by Lemma 4, note that for individual z the quantity $R(z)$ can be quite large in modulus, for instance

$$R(1) = n! \big(p(n) - 2\big),$$

which is roughly by a factor $e^{c\sqrt{n}}$ larger than the average obtained from (18). With Lemma 4 in hand, we can now establish the following statistical property of commutators in S_n.

THEOREM 3. *Choose four elements x_1, x_2, x_3, x_4 from S_n at random. Then the probability that the commutators $[x_1, x_2]$ and $[x_3, x_4]$ generate A_n tends to 1 as n tends to infinity.*

Proof. It is well known that two randomly chosen elements of A_n generate A_n with probability tending to 1 as n tends to infinity; cf., for instance, [**5**]. Choose $\delta > 0$. Then, for sufficiently large n,

$$|\{(\sigma, \tau) \in A_n \times A_n : \langle \sigma, \tau \rangle \neq A_n\}| \leqslant \left(\frac{\delta}{2} n!\right)^2.$$

Consequently, there exists a set $M_0 \subseteq A_n$ such that (i) $|M_0| \leqslant (\delta/2)n!$, and (ii) for every $\sigma \in A_n - M_0$ there are at most $(\delta/2)n!$ elements $\tau \in A_n$ with $\langle \sigma, \tau \rangle \neq A_n$. Moreover, if $M \subseteq A_n$ is any set of size at most $(\delta/2)n!$, then, by Lemma 4,

$$\sum_{z \in M} N(z) \leqslant \delta\,(n!)^2 + \sum_{z \in A_n} |R(z)| = \big(\delta + \mathcal{O}(n^{-1})\big)(n!)^2 \leqslant 2\delta\,(n!)^2.$$

Hence the number of quadruples $(x_1, x_2, x_3, x_4) \in S_n^4$ such that the commutators $[x_1, x_2]$ and $[x_3, x_4]$ do not generate A_n is at most $4\delta\,(n!)^4$, that is, the probability p_n that four elements x_1, x_2, x_3, x_4 chosen from S_n at random have the property that

$$\langle [x_1, x_2], [x_3, x_4] \rangle = A_n$$

satisfies $p_n \geqslant 1 - 4\delta$ for n sufficiently large. □

6. *Proof of Theorem* 1

Recall that the irreducible representations of S_n and their associated characters are explicitly parametrized by the partitions of n, that is, sequences

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\ell$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$. The number $\ell = ||\lambda||$ of summands is called the *length* or *norm* of λ, and, as usual, we write $\lambda \vdash n$ to indicate that λ is a partition of n. Denote by χ_λ the irreducible character corresponding to the partition λ. Then $\chi_\lambda(1)$, the dimension of the irreducible representation associated with λ, can be computed via the formula

$$\chi_\lambda(1) = \frac{n!}{\prod_{(i,j) \in [\lambda]} h_{i,j}}, \tag{20}$$

where $h_{i,j}$ is the *hook length* determined by the node (i, j) in the Ferrers diagram $[\lambda]$ of λ, that is, the number of points contained in the hook

$$H_{i,j} := \{(i, j') \in [\lambda] : j' \geqslant j\} \cup \{(i', j) \in [\lambda] : i' \geqslant i\}$$

corresponding to (i, j). The quantity $\prod_{(i,j)} h_{i,j}$ is called the *hook product* of λ, and is denoted as $H[\lambda]$. Formula (20) is the celebrated hook length formula of Frame, Robinson and Thrall, first established in [**9**], and subsequently re-proved in a number of different ways; cf., for instance, [**15**, **34**, **35**].

In order to prove Theorem 1, it suffices (by symmetry) to establish, for every fixed integer $R \geqslant 1$, the following two facts:

(i) $$\sum_{\substack{\lambda\vdash n\\ \lambda_1>n-R/s}} \chi_\lambda(1)^{-s} = 1 + \sum_{\rho=1}^{R-1}\mathscr{A}_\rho(s)n^{-\rho} + \mathcal{O}(n^{-R}), \qquad n\to\infty,$$

(ii) $$\sum_{\substack{\lambda\vdash n\\ \|\lambda\|\leqslant\lambda_1\leqslant n-R/s}} \chi_\lambda(1)^{-s} = \mathcal{O}(n^{-R}), \qquad n\to\infty.$$

In order to prove (i), consider a partition λ of n with $\lambda_1 = n-r$ for some fixed $r\geqslant 0$. By the hook formula (20), we have

$$\chi_\lambda(1) = H[\mu]^{-1}\prod_{v\in N(\mu,r)}(n-2r+v),$$

where $\mu = (\lambda_2,\lambda_3,\ldots,\lambda_\ell)$ is the partition of r obtained by deleting the first part of λ, and where $N(\mu,r)$ is as defined in Theorem 1. Hence, by the binomial theorem,

$$\sum_{\substack{\lambda\vdash n\\ \lambda_1=n-r}}\chi_\lambda(1)^{-s} = n^{-rs}\sum_{\mu\vdash r}H[\mu]^s\prod_{v\in N(\mu,r)}\left[\sum_{\kappa=0}^{\infty}\binom{s+\kappa-1}{\kappa}(2r-v)^\kappa n^{-\kappa}\right],$$

and the right-hand side of this equation, being a convergent series, also provides an asymptotic expansion for the left-hand side as $n\to\infty$. Summing this asymptotic expansion over $0\leqslant r<R/s$, we obtain (i).

The proof of (ii) proceeds in several steps. We begin by providing three different bounds for $\chi_\lambda(1)$, valid for various ranges of λ_1. We then estimate the left-hand side of (ii) as

$$\sum_{\substack{\lambda\vdash n\\ \|\lambda\|\leqslant\lambda_1\leqslant n-R/s}}\chi_\lambda(1)^{-s}\leqslant S_1+S_2+S_3,$$

where

$$S_1 := \sum_{\substack{\lambda\vdash n\\ 2n/3\leqslant\lambda_1\leqslant n-R/s}}\chi_\lambda(1)^{-s},$$

$$S_2 := \sum_{\substack{\lambda\vdash n\\ n/4\leqslant\lambda_1\leqslant 2n/3}}\chi_\lambda(1)^{-s},$$

and

$$S_3 := \sum_{\substack{\lambda\vdash n\\ \|\lambda\|\leqslant\lambda_1\leqslant n/4}}\chi_\lambda(1)^{-s},$$

and use the previously established inequalities to estimate the sums S_1, S_2, and S_3.

CLAIM 1. *We have*

$$\chi_\lambda(1)\geqslant\binom{\lambda_1}{n-\lambda_1}, \qquad \lambda_1\geqslant n/2. \tag{21}$$

Proof. Consider the hook numbers $h_{1,j}$ corresponding to the nodes in the first row of $[\lambda]$. These numbers are mutually distinct, $h_{1,j} \leqslant n$ for all $1 \leqslant j \leqslant \lambda_1$, and

$$\prod_{j=n-\lambda_1+1}^{\lambda_1} h_{1,j} = (2\lambda_1 - n)!.$$

Hence

$$\prod_{j=1}^{\lambda_1} h_{1,j} \leqslant (2\lambda_1 - n)!\,(n)_{n-\lambda_1} = \frac{n!\,(2\lambda_1 - n)!}{\lambda_1!}.$$

Combining the latter inequality with the trivial estimate

$$H[(\lambda_2, \ldots, \lambda_\ell)] \leqslant (n - \lambda_1)!$$

for the remaining nodes, our claim follows from the hook formula (20). □

CLAIM 2. *We have*

$$\chi_\lambda(1) \geqslant \frac{(n + \lfloor n/8 \rfloor - \lambda_1)!}{(n - \lambda_1)!\,(\lfloor n/8 \rfloor + 5)!}, \qquad \lambda_1 \geqslant n/4. \tag{22}$$

Proof. Consider the $\lfloor n/8 \rfloor$ nodes in the first row of $[\lambda]$ which are farthest to the right. Below each of these nodes there are at most five other nodes; hence

$$\prod_{j=\lambda_1-\lfloor n/8 \rfloor+1}^{\lambda_1} h_{1,j} \leqslant (\lfloor n/8 \rfloor + 5)!,$$

while for the remaining nodes in the first row we use the estimate

$$\prod_{j=1}^{\lambda_1-\lfloor n/8 \rfloor} h_{1,j} \leqslant (n)_{\lambda_1-\lfloor n/8 \rfloor} = \frac{n!}{(n + \lfloor n/8 \rfloor - \lambda_1)!}.$$

Combining these inequalities with the trivial bound

$$H[\lambda'] \leqslant (n - \lambda_1)!$$

for the hook product of the partition $\lambda' = (\lambda_2, \ldots, \lambda_\ell)$, our claim follows from the hook formula (20). □

CLAIM 3. *We have*

$$\chi_\lambda(1) \geqslant \left(\tfrac{3}{2}\right)^{n/4}, \qquad ||\lambda|| \leqslant \lambda_1 \leqslant n/4. \tag{23}$$

Proof. Think of the hook lengths $h_{i,j}$ with $(i, j) \in [\lambda]$ as written down in increasing order (with multiplicities), and denote by $h(i)$ the ith member of this sequence. Then $h(i) \leqslant i$ for all $1 \leqslant i \leqslant n$. To see this, consider a node of $[\lambda]$ representing $h(i)$. The hook corresponding to this node contains $h(i) - 1$ nodes of hook length strictly less than $h(i)$. Thus there are at least $h(i) - 1$ entries preceding $h(i)$. For $||\lambda|| \leqslant \lambda_1 \leqslant n/4$, the number $h_{1,1}$, the largest hook length associated with $[\lambda]$, satisfies $h_{1,1} \leqslant n/2$. Hence

$$H[\lambda] = \prod_{i=1}^{n} h(i) \leqslant \prod_{i=1}^{n} \min(i, n/2),$$

and therefore, by (20),

$$\chi_\lambda(1) \geqslant \prod_{i=1}^{n} \frac{i}{\min(i, n/2)} = \prod_{i=\lfloor n/2 \rfloor+1}^{n} \frac{i}{n/2} \geqslant \prod_{i=\lceil 3n/4 \rceil}^{n} \frac{i}{n/2} \geqslant \left(\frac{3}{2}\right)^{n/4}. \qquad \square$$

Claim 4. *We have* $S_1 = \mathcal{O}(n^{-R})$, $n \to \infty$.

Proof. Consider a partition $\lambda \vdash n$ with greatest part λ_1 satisfying $2n/3 \leqslant \lambda_1 \leqslant n - R/s$. Then we have

$$\binom{\lambda_1}{n-\lambda_1} \geqslant 2^{n-\lambda_1-\lceil R/s\rceil} \left(\frac{n}{3}\right)^{\lceil R/s\rceil} / \lceil R/s\rceil !,$$

and hence, by (21),

$$\chi_\lambda(1)^{-s} \leqslant 6^{s\lceil R/s\rceil} (\lceil R/s\rceil !)^s \, 2^{-s(n-\lambda_1)} \, n^{-R}. \tag{24}$$

We now make use of the estimate

$$p(n) \leqslant 2\,\mathrm{e}^{c_0\sqrt{n}}, \quad c_0 := \pi\sqrt{2/3},$$

for the partition function, which holds for all $n \geqslant 1$, and can be proved by elementary means; cf., for instance, [**16**, Satz 7.6]. Multiplying the right-hand side of (24) by $2\mathrm{e}^{c_0\sqrt{n-\lambda_1}}$ and summing over $2n/3 \leqslant \lambda_1 \leqslant n - R/s$, we find that

$$0 \leqslant S_1 \leqslant 2 \cdot 6^{s\lceil R/s\rceil} (\lceil R/s\rceil !)^s \, n^{-R} \, \tilde{S},$$

where

$$\tilde{S} := \sum_{2n/3 \leqslant \lambda_1 \leqslant n-R/s} \mathrm{e}^{c_0\sqrt{n-\lambda_1}} \, 2^{-s(n-\lambda_1)}.$$

For $\lambda_1 \leqslant n - (2c_0/(s\log 2))^2$, the summands of $\tilde{S}$ are bounded above by $2^{-s(n-\lambda_1)/2}$; hence, splitting $\tilde{S}$ accordingly, we obtain the inequality

$$\tilde{S} \leqslant 2^{-R}\, \mathrm{e}^{2c_0^2/(s\log 2)} \left[\left(\frac{2c_0}{s\log 2}\right)^2 - R/s\right] + \frac{1-(2^{-s/2})^{\lfloor n/3\rfloor+1}}{1-2^{-s/2}},$$

whose right-hand side remains bounded as $n \to \infty$. □

Claim 5. *We have* $S_2 = \mathcal{O}(\mathrm{e}^{-\mathrm{sn}/8})$, $n \to \infty$.

Proof. Consider a partition λ with largest part λ_1 satisfying $n/4 \leqslant \lambda_1 \leqslant 2n/3$. Observe that the right-hand side of (22) decreases as λ_1 increases, that is, the estimate (22) becomes worst at the upper limit. Hence, for $n \geqslant 6477$,

$$\chi_\lambda(1) \geqslant \frac{(n+\lfloor n/8\rfloor - \lfloor 2n/3\rfloor)!}{(n-\lfloor 2n/3\rfloor)!\,(\lfloor n/8\rfloor+5)!} \geqslant 3^{\lfloor n/8\rfloor}, \qquad n/4 \leqslant \lambda_1 \leqslant 2n/3.$$

Consequently, for sufficiently large n,

$$S_2 \leqslant 2\,\mathrm{e}^{c_0\sqrt{n}}\, 3^{-s\lfloor n/8\rfloor} = \mathcal{O}(\mathrm{e}^{-sn/8}).$$ □

Claim 6. *We have* $S_3 = \mathcal{O}(\mathrm{e}^{-\mathrm{cn}})$, $n \to \infty$, *with some constant* $c > 0$.

Proof. By (23),

$$S_3 \leqslant 2\,\mathrm{e}^{c_0\sqrt{n}} \left(\tfrac{3}{2}\right)^{-sn/4} = \mathcal{O}(\mathrm{e}^{-cn}).$$ □

Assertion (ii) follows from Claims 4–6, and the proof of Theorem 1 is complete. □

References

1. B. Baumslag and S. Pride, 'Groups with two more generators than relators', *J. London Math. Soc.* 17 (1978) 425–426.
2. E. Bender, 'An asymptotic expansion for the coefficients of some formal power series', *J. London Math. Soc.* 9 (1975) 451–458.
3. C. Curtis and I. Reiner, *Methods of representation theory* (John Wiley, New York, 1981).
4. C. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (John Wiley, New York, 1988).
5. J. Dixon, 'The probability of generating the symmetric group', *Math. Z.* 110 (1969) 199–205.
6. A. Dress and T. Müller, 'Decomposable functors and the exponential principle', *Adv. Math.* 129 (1997) 188–221.
7. M. du Sautoy, 'Finitely generated groups, *p*-adic analytic groups and Poincaré series', *Ann. of Math.* 137 (1993) 639–670.
8. M. du Sautoy and F. Grunewald, 'Analytic properties of zeta functions and subgroup growth', *Ann. of Math.* 152 (2000) 793–833.
9. J. Frame, G. de B. Robinson and R. Thrall, 'The hook graphs of the symmetric groups', *Canad. J. Math.* 6 (1954) 316–325.
10. M. Gromov, 'Groups of polynomial growth and expanding maps', *Inst. Hautes Études Sci. Publ. Math.* 53 (1981) 53–78.
11. M. Hall, 'Coset representations in free groups', *Trans. Amer. Math. Soc.* 67 (1949) 421–432.
12. M. Hall, 'Subgroups of finite index in free groups', *Canad. J. Math.* 1 (1949) 187–190.
13. M. Hall and T. Radó, 'On Schreier systems in free groups', *Trans. Amer. Math. Soc.* 64 (1948) 386–408.
14. A. Kerber, *Algebraic combinatorics via finite group actions* (BI-Wissenschaften, Mannheim, 1991).
15. C. Krattenthaler, 'Bijective proofs of the hook formulas for the number of standard Young tableaux, ordinary and shifted', *J. Combin.* 2 (1995) 199–207.
16. E. Krätzel, *Zahlentheorie* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1981).
17. A. Lubotzky, 'Subgroup growth', Groups '93 Galway/St Andrews, University College Galway (lecture notes).
18. A. Lubotzky, 'Counting finite index subgroups', *Groups* '93 *Galway/St Andrews*, LMS Lecture Note Series 212 (Cambridge University Press, 1995) 368–404.
19. A. Lubotzky, 'Subgroup growth and congruence subgroups', *Invent. Math.* 119 (1995) 267–295.
20. A. Lubotzky, A. Mann and D. Segal, 'Finitely generated groups of polynomial subgroup growth', *Israel J. Math.* 82 (1993) 363–371.
21. A. Mednykh, 'On the number of subgroups in the fundamental group of a closed surface', *Comm. Algebra* 16 (1988) 2137–2148.
22. T. Müller, 'Combinatorial aspects of finitely generated virtually free groups', *J. London Math. Soc.* (2) 44 (1991) 75–94.
23. T. Müller, 'Counting free subgroups of finite index', *Archiv. Math.* 59 (1992) 525–533.
24. T. Müller, 'Subgroup growth of free products', *Invent. Math.* 126 (1996) 111–131.
25. T. Müller, 'Combinatorial classification of finitely generated virtually free groups', *J. Algebra* 195 (1997) 285–294.
26. T. Müller, 'Enumerating representations in finite wreath products', *Adv. Math.* 153 (2000) 118–154.
27. T. Müller, 'Modular subgroup arithmetic and a theorem of Philip Hall', *Bull. London Math. Soc.*, to appear.
28. T. Müller, 'Modular subgroup arithmetic', submitted for publication.
29. T. Müller, 'Modular subgroup arithmetic in free products', submitted for publication.
30. T. Müller, 'Representations in finite wreath products and subgroup growth', in preparation.
31. T. Müller, 'Poincaré's problem for free products', in preparation.
32. T. Müller and J.-C. Puchta 'Parity patterns in one-relator groups', *J. Group Theory*, to appear.
33. M. Newman, 'Asymptotic formulas related to free products of cyclic groups', *Math. Comp.* 136 (1976) 838–846.
34. J.-C. Novelli, I. Pak and A. Stoyanovskii, 'A direct bijective proof of the hook-length formula', *Discrete Math. Theoret. Comput. Sci.* 1 (1997) 53–67.
35. I. Pak, 'Hook length formula and geometric combinatorics', *Seminaire Lotharingien de Combinatoire* (Université Louis Pasteur, Strasbourg, to appear).
36. A. Regev, 'Asymptotic values for degrees associated with strips of Young diagrams', *Adv. Math.* 41 (1981) 115–136.

37. T. Scharf, 'Die Wurzelanzahlfunktion in symmetrischen Gruppen', *J. Algebra* 139 (1991) 446–457.
38. R. Stanley, *Enumerative combinatorics – Vol.* 2 (Cambridge University Press, New York, 1999).
39. E. Wright, 'Asymptotic relations between enumerative functions in graph theory', *Proc. London Math. Soc.* 20 (1970) 558–572.

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