On the Mossakovskii method for contacts supporting a moment

Matthew Moore*[1] and David Hills[2]
[1]*Mathematical Institute, University of Oxford, Oxford, UK*
[2]*Department of Engineering Science, University of Oxford, Oxford, UK*

Summary The Mossokovskii solution of the half-plane formulation for contacts between elastically-similar materials is extended to contacts supporting a moment. We reduce the Cauchy singular integral formulation of the problem to a non-symmetric Abel integral relating the indenter geometry to the applied normal force. We use the formulation to derive simple expressions the applied normal force and moment as functions of the contact extent and indenter tilt. We also for the coefficients of the square-root terms in the pressure expansion local to the ends of contact, which may be of use in asymptotic analyses of the partial-slip problem.

INTRODUCTION

The usual starting point for analysing contact problems is the Flamant solution for a force applied to the apex of a wedge, specialised to a half-plane, from which a Cauchy integral equation is derived where the primary unknown is the contact pressure distribution [2]. This approach works well for problems in which the normal load is established first, with the shearing force then developed as the normal force is held constant. However, if both the normal load and shearing force are simultaneously varying functions, an alternative incremental method originally formulated by [4] and extended by [3] is more suited. In this analysis, we adapt the method to contacts that support a moment and hence are, in general, non-symmetric.

PROBLEM FORMULATION

Consider the configuration illustrated in figure 1a for a general indenter with profile $y = g(x)$. We suppose the profile is tilted by an angle α with respect to the horizontal, and we shall assume α is known. Our aim will be to derive the necessary applied normal force and applied moment to sustain a contact over $-b < x < a$ with tilt angle α. We assume that the indenter and the material it is indenting are elastically similar and that a half-plane idealisation is suitable. The assumption of elastic similarity uncouples the normal and tangential displacement problems [3]; we shall concentrate on the normal displacement problem here, although the ideas readily extend to consider an applied tangential force and differential remote bulk stresses.

The standard relation between the relative normal displacement gradient and the contact pressure, $p(x)$, is

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\kappa+1}{2\mu\pi} \int_{-b(a)}^{a} \frac{p(s)}{s-x}\,\mathrm{d}s \text{ such that } v'(x) + g'(x) = 0 \text{ for } -b < x < a. \tag{1}$$

The integral above is interpreted in a Cauchy principal value sense, and can be inverted in the usual manner assuming the pressure is bounded at both ends of the contact set. This yields a consistency condition, which, along with the equilibrium condition to balance the applied normal force, yields two equations for the size of the contact set

$$0 = \int_{-b}^{a} \frac{v'(s)}{\sqrt{(a-s)(s+b(a))}}\,\mathrm{d}s, \ P = \int_{-b}^{a} p(s)\,\mathrm{d}s. \tag{2a, b}$$

The form of the consistency condition ($2a$) allows us to say that, if we know a, we can derive $b(a)$. We can then utilise the equilibrium condition ($2b$) to find a, and hence b, as a function of P.

We can use the knowledge of the size of the contact set as a function of the applied normal force to avoid the difficulties associated with singular integrals by approximating the indenter using a series of rectangular punches as we increase P (and hence a), see figure 1b. The contact pressure is then given by

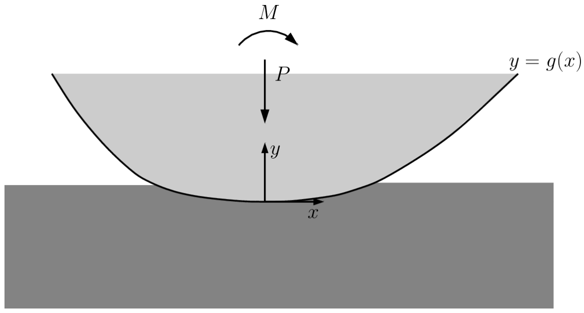
$$p(x,a) = \int_0^a F(s) m_0(x,s)\,\mathrm{d}s, \ m_0(x,a) = \begin{cases} \dfrac{1}{\sqrt{(a-x)(x+b(a))}} & \text{for } -b(a) < x < a, \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

where $m_0(x,s)$ is the contact pressure induced by a flat punch of size $(-b(s), s)$ and $F(\cdot)$ is an unknown coefficient that encodes the geometry of the indenter. We utilise (1) to show that

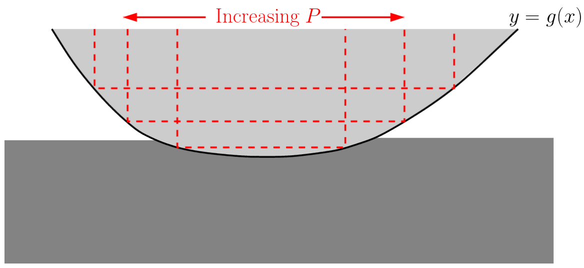
$$\frac{2\mu g'(x)}{\kappa+1} = \int_0^x \frac{F(s)}{\sqrt{(x-s)(x+b(s))}}\,\mathrm{d}s \text{ for } 0 < x < a, \tag{4}$$

which is a non-symmetric Abel integral equation for $F(\cdot)$, and a direct analogue to the moment-free case considered in [3].

*Corresponding author. E-mail: moorem@maths.ox.ac.uk.



(a) $a)$



(b) $b)$

Figure 1: $a)$ An indenter of profile $y = g(x)$ contacts an elastically similar half-space under an applied normal force, P, and an applied moment, M. $b)$ A schematic showing the Mossakovskii method. We approximate the contact by considering an infinite number of flat punches that conform to the geometry of the indenter, enforced by (2a) and (4).

GENERAL RESULTS

For a general geometry, (4) must be tackled numerically. However, we can use (4) to derive a number of useful results that have applications in considerations of the partial slip problem for contacts where the applied normal and shearing forces are varying functions of time [3, 4].

Firstly, integrating (3) and, recalling the equilibrium condition (2a), yields

$$P = \pi \int_0^a F(s)\,\mathrm{d}s, \text{ that is, } F(a) = \frac{1}{\pi}\frac{\mathrm{d}P}{\mathrm{d}a}, \tag{5}$$

so that we can remove $F(\cdot)$ from the problem entirely by replacing it with the applied normal force in (3)–(4). Secondly, the applied moment M necessary to sustain the contact is thus given by

$$M = \frac{1}{2}\left[P(a)(a - b(a)) - \int_0^a P(s)(1 - b'(s))\,\mathrm{d}s\right]. \tag{6}$$

Hence, given that the tilt angle was encoded in the original geometry, (4)–(6) can be viewed as a way to find P and M as functions of a and α (or vice versa).

Finally, we can expand the contact pressure solution (3) locally at the contact ends to find the coefficients of the square-root terms, $K_{n,a}$ and $K_{n,b}$, which gives

$$p = \frac{2}{\pi}\frac{P'(a)}{\sqrt{a + b(a)}}\sqrt{a - x} + o(\sqrt{a - x}) \text{ as } x \to a^-, \text{ so that } K_{n,a} = \frac{2}{\pi}\frac{P'(a)}{\sqrt{a + b(a)}}, \tag{7}$$

while

$$p = \frac{2}{\pi}\frac{P'(a)}{b'(a)\sqrt{a + b(a)}}\sqrt{b(a) + x} + o(\sqrt{b(a) + x}) \text{ as } x \to -b(a)^+, \text{ so that } K_{n,b} = \frac{2}{\pi}\frac{P'(a)}{b'(a)\sqrt{a + b(a)}}. \tag{8}$$

In particular, we note that $K_{n,b} = K_{n,a}/b'(a)$. Such simple relations for the K_n-coefficients are of use when considering asymptotic models of contacts for particular geometries, where we only consider one end of the contact and consider the tractions there. This has applications in problems where the differential bulk tensions are large enough to reverse the direction of slip at one end of the contact, so that the Ciavarella-Jäger theorem is no longer applicable [1].

CONCLUSION

We have extended the Mossakovskii method to half-plane contact problems that allow for an applied moment. The singular integral formulation is reduced to a non-symmetric Abel integral equation by assuming that the left-hand contact point is known as a function of the right-hand contact point using the standard consistency condition. The resulting solution has been used to derive general expressions for the K_n-coefficients in the contact pressure expansion local to the contact edges, as well as relating the applied normal force and moment, to the contact extent and tilt of indenter.

References

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