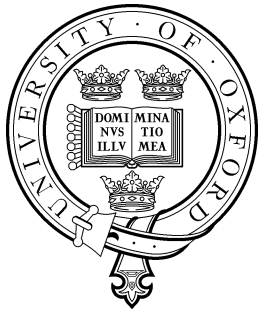
Financial Optimization Problems



Siu Lung Law
St. Anne's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2005

This thesis is dedicated to
my parents
for their continual support throughout the years

Acknowledgements

Firstly, I would like to thank C.V.C.P., St. Anne's College and the Croucher Foundation for providing monetary support. Equally important, I am grateful to my supervisor Dr. Jeff Dewynne, who has given me a lot of useful advice. I am also grateful to Dr. Chiu Fan Lee, who has helped me enormously in the multi-dimensional problem. Dr. Paul Wilmott's initial supervision and Dr. Sam Howison's support at the end are also very much appreciated.

Abstract

The major objective of this thesis is to study optimization problems in finance. Most of the effort is directed towards studying the impact of transaction costs in those problems. In addition, we study dynamic mean-variance asset allocation problems. Stochastic HJB equations, Pontryagin Maximum Principle and perturbation analysis are the major mathematical techniques used.

In Chapter 1, we introduce the background literature. Following that, we use the Pontryagin Maximum Principle to tackle the problem of dynamic mean-variance asset allocation and rediscover the doubling strategy.

In Chapter 2, we present one of the major results of this thesis. In this chapter, we study a financial optimization problem based on a market model without transaction costs first. Then we study the equivalent problem based on a market model with transaction costs. We find that there is a relationship between these two solutions. Using this relationship, we can obtain the solution of one when we have the solution of another.

In Chapter 3, we generalize the results of chapter 2.

In Chapter 4, we use Pontryagin Maximum Principle to study the problem limit of the no-transaction region when transaction costs tend to 0. We find that the limit is the no-transaction cost solution.

Contents

1 Introduction **1**

1.1 Overview of Research 1
1.2 Introduction 2
1.3 Statement of Originality 3
1.4 Weiner Process 3
1.5 Deterministic Control Problem 5
1.5.1 Dynamic Programming 6
1.5.2 Pontryagin Maximum Principle 7
1.6 Stochastic Control 8
1.6.1 Dynamic Programming 9
1.6.2 Pontryagin Maximum Principle 9
1.7 Markowitz Mean-Variance Efficient Frontier 10
1.8 Utility Function 14
1.9 Constant Relative Risk Aversion 14
1.10 Merton's Investment-Consumption Model 16
1.11 Long Term Growth Model 19
1.12 Example of Pontryagin Maximum Principle 20
1.12.1 Market Model Equations 21
1.12.2 Applying the Maximum Principle 22
1.12.3 Kolmogorov Equation and Transition Probability 24
1.12.4 Discussion 27
1.12.5 Dynamic Asset Allocation with Margin Constraints 28
1.13 Investment with Transaction Costs 29
1.13.1 Atkinson, Pliska & Wilmott 29
1.13.2 Atkinson and Al-Ali 31
1.13.3 Atkinson and Mokkhavesa (2001) 32
1.13.4 Mokkhavesa and Atkinson (2002) 32
1.13.5 Atkinson and Mokkhavesa (2004) 33

2 Dynamic Asset Allocation with Transaction Costs **34**
2.1 Introduction . . . 34
2.2 Market Model without Transaction costs . . . 35
2.3 Formulation of Bellman's Equation in the No Transaction Costs Problem 36
2.4 Solving Bellman's Equation in the No Transaction Costs Problem . . 38
2.4.1 Long Term Growth Model . . . 39
2.4.2 Constant Relative Risk Aversion (CRRA) Model . . . 40
2.5 Market Model with Transaction costs . . . 41
2.6 Formulation of Bellman's Equation under Transaction costs . . . 42
2.6.1 Change of Variables . . . 48
2.6.2 Sales and Purchase Regions . . . 49
2.6.3 No-Transaction Region . . . 51
2.6.4 The $O(k^{-2/3})$ Equation . . . 52
2.6.5 The $O(k^{-1/3})$ Equation . . . 52
2.6.6 The $O(1)$ Equation . . . 52
2.6.7 The $O(k^{1/3})$ Equation . . . 53
2.6.8 The $O(k^{2/3})$ Equation . . . 54
2.7 Example . . . 57
2.7.1 Long Term Growth Model . . . 57
2.7.2 Constant Relative Risk Aversion (CRRA) Model . . . 58
2.8 Financial Interpretations . . . 60
2.8.1 Long Term Growth Model . . . 63
2.8.2 CRRA Model . . . 71
2.9 The Choice of Asymptotic Scales . . . 82
2.9.1 H_0, H_1, H_2, $\cdots$, H_{2m-1} . . . 84
2.9.2 H_{2m}, H_{2m+1}, H_{2m+2}, $\cdots$, H_{3m-1} . . . 84
2.9.3 H_{3m}, H_{3m+1}, H_{3m+2}, $\cdots$, H_{4m-1} . . . 85
2.9.4 H_{4m} . . . 86

3 Two Generalizations of the Dynamic Asset Allocation with Transaction Cost Model **88**
3.1 Introduction . . . 88
3.2 Stock Price Dependence . . . 88
3.2.1 Without Transaction Costs . . . 89
3.2.2 Solving Bellman's Equation with No Transaction Costs . . . 90
3.2.3 Modified Long Term Growth Model . . . 91

3.2.4 Modified Constant Relative Risk Aversion (CRRA) Model . . 92
3.2.5 With Transaction Costs 93
3.2.6 Modified Long Term Growth Model 100
3.2.7 Modified Constant Relative Risk Aversion (CRRA) Model . . 101
3.2.8 Financial Interpretations 102
3.2.9 Modified Long Term Growth model 103
3.2.10 Modified CRRA model 104
3.3 Multi-Asset 115
3.3.1 Without Transaction Costs 115
3.3.2 Solving Bellman's Equation with No Transaction Costs 117
3.3.3 Long Term Growth Model 118
3.3.4 Constant Relative Risk Aversion (CRRA) Model 118
3.3.5 With Transaction Costs 119
3.3.6 Financial Interpretations 126
3.3.7 Two Dimensional Case 127
3.3.8 Financial Interpretations 130

4 The Convergence of the No Transaction Region in the Transaction Costs Problem 131
4.1 Introduction 131
4.2 Market Model without Transaction Costs 133
4.3 Applying the Maximum Principle on the No Transaction Costs Problem 134
4.4 Market Model with Transaction Costs 136
4.5 The Order of $\mathbb{E}_t(\tilde{\Psi}_B)$ 140
4.6 Transaction Costs Level Tends to 0 142
4.7 Examples 142

5 Conclusion 143
5.1 Results Achieved 143
5.2 Future Work 143

A Nomenclature 145
A.1 All Chapters 145
A.2 Chapter 1 147
A.3 Chapter 2 147
A.4 Chapter 3 147
A.5 Chapter 4 148

Chapter 1

Introduction

1.1 Overview of Research

This thesis is concerned with mathematics in finance.

In the study of finance, and in fact in science, mathematics can always play an important role. The process of study involves building models, working out the implications or predictions of the models and testing whether they fit with the facts. Mathematics can contribute in the process of working out the implications of the assumptions. Only after obtaining implications, we can compare them with reality, and thus verify the models. Whether a model or a theory is accepted or rejected is solely based on whether its predictions fit the facts better or explain the phenomenon better than other existing models. Mathematics plays a pivotal role in this process.

An example of application of mathematics is this: we want to compare a new option pricing model with the Black-Scholes Model. In order to test whether the new model actually is better than the Black-Scholes model, the first step is to use mathematics to work out the implications predicted by the new model and the Black-Scholes Model under different circumstances. Only after that we can test how good the new model is compared to the Black-Scholes Model.

By comparing the implications worked out by mathematics, we can accept or reject a model. For example, the Black-Scholes Model [10] is preferred to Sprenkle [45] in options pricing solely because the Black-Scholes Model gives predictions more consistent with observation in the same way the theory of relativity is preferred to Newton's theory in predicting the trajectories of different planets. These comparison can never be made if there is no mathematics.

The theme of this thesis is to use mathematics to work out the implication of the introduction of transaction costs in finance. In Chapter 2, we present one of the major results of this thesis, which is about the impact of small proportional transaction costs

in a general, dynamic, finite-horizon financial optimization problem. In other words, suppose we have a solution of an optimization problem based on a market model without transaction costs, and now we want to introduce transaction costs into the market model, what is the relationship between the new solution and the original solution? We successfully find an analytic relationship between these two solutions. Using this relationship, we can obtain the solution of one when we have the solution of another. The mathematical techniques we use include stochastic HJB equations and perturbation analysis.

In Chapter 3, we generalize the result in Chapter 2. Firstly, we consider the case that the reward function depends on the stock price. Secondly, we consider the multi-asset case. The form of the analytic solutions obtained are similar to those in Chapter 2, but of course they are more complicated. Also, we find that many of the financial interpretations may no longer hold in a more general setting.

In Chapter 4, we use Pontryagin Maximum Principle [30] to study the problem of whether the solution of transaction cost problem tends to the solution of the no transaction cost problem as transaction cost tends to zero. We find that under certain conditions, the transaction cost solution does converge.

1.2 Introduction

Firstly, we review the Markowitz Mean-Variance Portfolio problem. Then we introduce stochastic calculus, which is the most important technique in mathematical finance. We next introduce Merton's Investment and Consumption model as well as the long term growth model and the Constant Relative Risk Aversion model. Also, we discuss the Black-Scholes model.

After that, we introduce the major mathematical techniques we use in this thesis, the HJB (Hamilton-Jacobi-Bellman) equation and Pontryagain's Maximum Principle. These are the two major approaches to stochastic control. For the purpose of demonstration, however, we detail the non-stochastic versions of these two techniques and only state the stochastic versions.

Afterwards, we review other important works in portfolio management.

The final part of this chapter is an example of applying the stochastic Pontryagin's Maximum Principle to the problem of dynamic mean-variance portfolio selection. We

find that the resultant 'optimal' strategy is the doubling strategy.

1.3 Statement of Originality

Chapter 2, Chapter 3 and Chapter 4 are mainly new material.

In the last part of Chapter 1, we use the stochastic Maximum Principle to solve the problem of dynamic mean-variance portfolio selection. The discovery that the resultant optimal strategy is the doubling strategy is new. Of course, the doubling strategy itself is not; see [23].

Most of the material in this chapter is a review of other people's work and they only serve as background study.

1.4 Weiner Process

Many models of stock prices involve a Wiener Process to model the random behavior of stock prices. In our later chapters, we need these to model the continuous time price process for dynamic optimization. The following results are largely taken from Bjork [9], Hull [29], Neftci [40] and Wilmott [51].

The first thing we introduce is the Wiener Process, or Brownian motion. A process $X(t)$ is called a Wiener Process if it satisfies:

1. For any t and $\Delta t > 0$,

$$X(t + \Delta t) - X(t) \quad \sim \quad N(0, \Delta t); \tag{1.1}$$

   where $N(\mu, \sigma^2)$ represents a normal distribution with mean μ and variance σ^2.

2. X is continuous in time with probability one; and

3. the increments are independent. That is for any $r < s < t < u$, $X(u) - X(t)$ and $X(s) - X(r)$ are independent.

The Wiener Process allows us to define a stochastic integral, and thus an Itô Process. A process $S(t)$ is called an Itô Process if it can be written as

$$S(t) \quad = \quad S(0) + \int_0^t \mu(\tilde{t}, S(\tilde{t}))d\tilde{t} + \int_0^t \sigma(\tilde{t}, S(\tilde{t}))dX(\tilde{t}) \tag{1.2}$$

for some functions μ and σ of t and S.

In (1.2), the integral with respect to $d\tilde{t}$ can be interpreted as a normal Riemann Integral. The integral with respect to $dX(\tilde{t})$, however, has to be interpreted carefully. In fact, that integral is a random variable. Let $0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} = t$ be a partition of the interval $[0, t]$. Note that there are superscripts (n). This is because the choice of partition for $[0, t]$ depends on n, how many times we want to divide $[0, t]$. The stochastic integral with respect to $dX(\tilde{t})$ is the random variable such that

$$\lim_{n\to\infty} \mathbb{E}_0\left\{ \left[\int_0^t \sigma(\tilde{t}, S(\tilde{t}))dX(\tilde{t}) \right] - \left[\sum_{k=0}^{n-1} \sigma(t_k^{(n)}, S(t_k^{(n)}))[X(t_{k+1}^{(n)}) - X(t_k^{(n)})] \right] \right\} = 0.$$

where $\mathbb{E}_t$ is the expectation operator given information at time t.

Note that the above definition is non-anticipatory. The term

$$\sigma(t_k^{(n)}, S(t_k^{(n)}))[X(t_{k+1}^{(n)}) - X(t_k^{(n)})]$$

means the function σ does not anticipate what $X(t_{k+1}^{(n)})$ will be.

The Itô Process is used frequently to model various items in the financial markets. For example, interest rates, commodity prices and stock prices are often modelled by it.

Usually we use the following shorthand for equation (1.2):

$$dS = \mu(t, S)dt + \sigma(t, S)dX. \tag{1.3}$$

The price of many financial contracts can be expressed as a function of the prices of its underlyings and time. Therefore, it is very important to know how to manipulate functions of the Itô Process. One important mathematical result is Itô's Lemma, which follows.

Suppose there is an Itô Process S which follows equation (1.3). Let $G(S, t)$ be a function of S and t. Then Itô's Lemma states that

$$dG = \{\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial S^2}\}dt + \frac{\partial G}{\partial S}dS. \tag{1.4}$$

In terms of equation (1.2), this is shorthand for

$$\begin{aligned} G(t) &= G(0) + \int_0^t \left\{ \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial S^2} \right\} d\tilde{t} + \int_0^t \frac{\partial G}{\partial S} dS(\tilde{t}) \\ &= G(0) + \int_0^t \left\{ \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial S^2} \right\} d\tilde{t} \\ &\quad + \int_0^t \frac{\partial G}{\partial S} (\mu(\tilde{t}, S) d\tilde{t} + \sigma(\tilde{t}, S) dX) \end{aligned} \tag{1.5}$$

1.5 Deterministic Control Problem

In the next few sections, we describe mathematical techniques in solving control problems. As we can see later, some approaches for dealing with transaction costs and asset allocation involve control problems. Also, our work in later chapters involves stochastic control. Before explaining those techniques, in this section, we explain what a control problem is.

We first consider a deterministic control problem, or a non-stochastic control problem. The usual variables used in controls problems are time t, state variables x_i, control variables u_j, equations of motion, and the value function J.

Time, t, is continuous. We use 0 to denote the initial time, and T is the terminal time. At any time t between 0 and T, we use the state variables, $x_1, x_2, \cdots, x_n$ to denote the state of the system. The state variables are functions of time. At $t = 0$, the state variables are in the initial state $x_1(0), x_2(0), \cdots, x_n(0)$ and at $t = T$, the state variables are in the terminal state $x_1(T), x_2(T), \cdots, x_n(T)$. At any time $0 < t < T$ choices have to be made. Those choices made are denoted by the controls $u_1, u_2, \cdots, u_m$. The state variables evolve according to the equations of motion, which we take to be

$$\begin{aligned} \frac{dx_1}{dt} &= \mu_1(\mathbf{x}(t), \mathbf{u}(t), t) \\ \frac{dx_2}{dt} &= \mu_2(\mathbf{x}(t), \mathbf{u}(t), t) \\ &\vdots \\ \frac{dx_n}{dt} &= \mu_n(\mathbf{x}(t), \mathbf{u}(t), t). \end{aligned} \tag{1.6}$$

where

$$\begin{aligned}\mathbf{x}(t) &= (x_1(t), x_2(t), \cdots, x_n(t)),\\ \mathbf{u}(t) &= (u_1(t), u_2(t), \cdots, u_m(t)).\end{aligned}$$

The aim of the control problem is to choose the control variables $\mathbf{u}$ as function of time in order to maximize a value function J of the form

$$\max_{\mathbf{u}(t)} J = \int_0^T I(\mathbf{x}(t), \mathbf{u}(t), t)dt + F(\mathbf{x}(T), T).$$

We let J^* be the solution of the above equation, that is

$$J^* = \max_{\mathbf{u}(t)} J.$$

There are two approaches in solving the deterministic control problem. One is dynamic programming, and the other is Pontryagin Maximum Principle.

1.5.1 Dynamic Programming

One approach to solve the control problem is dynamic programming. The central idea of dynamic programming is Bellman's Principle of Optimality [7] which asserts that

> An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

We let J^* be the optimal value function for the problem at time t at a state of $\mathbf{x}(t)$. Consider a "small" increment of time Δt. By applying the Principle of Optimality, we have the following fundamental recurrence relation:

$$J^*(\mathbf{x}(t), t) = \max_{\mathbf{u}(t)} \left\{ I(\mathbf{x}(t), \mathbf{u}(t), t)\Delta t + J^*(\mathbf{x}(t) + \Delta\mathbf{x}(t), t + \Delta t) \right\} + o(\Delta t). \tag{1.7}$$

We use a Taylor series expansion

$$J^*(\mathbf{x}(\mathbf{t}) + \Delta\mathbf{x}(t), t + \Delta t) = J^*(\mathbf{x}(t), t) + \sum_{i=1}^{n} \frac{\partial J^*}{\partial x_i}\Delta x_i + \frac{\partial J^*}{\partial t}\Delta t + o(\Delta t) \tag{1.8}$$

and we substitute equation (1.8) into equation (1.7), take the limit as $\Delta t \to 0$, and use equation (1.6) to obtain

$$-\frac{\partial J^*}{\partial t} = \max_{\mathbf{u}(t)} \left\{ I(\mathbf{x}(t), \mathbf{u}(t), t) + \sum_{i=1}^{n} \frac{\partial J^*}{\partial x_i} \mu_i(\mathbf{x}(t), \mathbf{u}(t), t) \right\}, \tag{1.9}$$

which is the Hamilton-Jacobi-Bellman (HJB) equation.

The boundary condition for the HJB equation is the terminal condition

$$J^*(\mathbf{x}(T), T) = F(\mathbf{x}(T), T). \tag{1.10}$$

Usually, the HJB equation is very difficult to solve. Only in very few cases can analytical solutions be found. This is because the equation itself is more complicated than a partial differential equation, as we need to find $\mathbf{u}(\mathbf{t})$ which achieves the maximum. Most of the time, the HJB equation can only be solved numerically.

1.5.2 Pontryagin Maximum Principle

Another common approach to solve the control problems is the Pontryagin Maximum Principle [43]. First we explain what the Maximum Principle is, then we show how it may be deduced.

We define $\mathbf{H}$, the Hamiltonian, as

$$\mathbf{H} = I(\mathbf{x}(\mathbf{t}), \mathbf{u}(t), t) + \sum_{i=1}^{n} \Psi_i \mu_i(\mathbf{x}(t), \mathbf{u}(t), t) \tag{1.11}$$

where Ψ_i are the *adjoint processes*, which are defined by

$$\begin{aligned} \frac{\partial \Psi_i}{\partial t} &= -\frac{\partial \mathbf{H}}{\partial x_i} \\ \Psi_i(T) &= \frac{\partial F}{\partial x_i} \quad , i = 1, \ldots, n. \end{aligned} \tag{1.12}$$

The maximum principle states that if u_i^* are controls that maximize J, they also maximize $\mathbf{H}$.

One way to understand the maximum principle is to notice that actually the Hamiltonian $\mathbf{H}$ and the adjoint processes Ψ_i are

$$\begin{aligned} \mathbf{H} &= I(\mathbf{x}(t), \mathbf{u}(t), t) + \sum_{i=1}^{n} \frac{\partial J^*}{\partial x_i} \mu_i(\mathbf{x}(t), \mathbf{u}(t), t) \\ \Psi_i &= \frac{\partial J^*}{\partial x_i}, i = 1, \ldots, n. \end{aligned} \tag{1.13}$$

In other words, **H** is the function inside the big bracket in equation (1.9). This is very clear if we substitute it into equation (1.12).

For the terminal condition, from equation (1.10), we have

$$\frac{\partial J^*}{\partial x_i}(T) = \frac{\partial F}{\partial x_i}.$$

The above equation can be derived from differentiating (1.9) with respect to x_i.

When we want to apply Pontryagin Principle to solve an optimization problem, usually firstly we solve the adjoint processes, and then we find out the control variables that maximize the Hamiltonian, and thus J.

1.6 Stochastic Control

Studying the deterministic control problem gives us the background to study the more difficult stochastic control problem, which we frequently see in this thesis. In the case of stochastic control problem, we have

$$\begin{aligned} dx_1 &= \mu_1(\mathbf{x}(t), \mathbf{u}(t), t)dt + \sigma_1(\mathbf{x}(t), \mathbf{u}(t), t)dX_1 \\ dx_2 &= \mu_2(\mathbf{x}(t), \mathbf{u}(t), t)dt + \sigma_2(\mathbf{x}(t), \mathbf{u}(t), t)dX_2 \\ &\vdots \\ dx_n &= \mu_n(\mathbf{x}(t), \mathbf{u}(t), t)dt + \sigma_n(\mathbf{x}(t), \mathbf{u}(t), t)dX_n \end{aligned} \tag{1.14}$$

instead of equation (1.6). The X_i are Brownian motions, with correlations ρ_{ij}, that is,

$$\begin{aligned} dX_i dX_j &= \rho_{ij}dt, \\ \rho_{ii} &= 1. \end{aligned}$$

The value function we want to maximize remains

$$\max_{\mathbf{u(t)}} J = \mathbb{E}_0\Big\{ \int_0^T I(\mathbf{x(t)}, \mathbf{u(t)})dt + F(\mathbf{x(T)}, T) \Big\}.$$

1.6.1 Dynamic Programming

The way to derive the HJB equation in stochastic control problems is similar to the deterministic case. The major difference is instead of using Taylor's series expansion, we use Itô's Lemma to expand equation (1.8). Expanding equation (1.8), therefore, gives

$$\begin{aligned} J^*(\mathbf{x(t)}+\Delta\mathbf{x(t)},t+\Delta t) &= J^*(\mathbf{x(t)},t)+\sum_{i=1}^{n}\frac{\partial J^*}{\partial x_i}\Delta x_i+\frac{\partial J^*}{\partial t}\Delta t \\ &\quad +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 J^*}{\partial x_i\partial x_j}\Delta t. \end{aligned} \tag{1.15}$$

Putting this into equation (1.7), the HJB equation becomes

$$\begin{aligned} -\frac{\partial J^*}{\partial t} &= \max_{\mathbf{u(t)}}\Bigg\{I(\mathbf{x(t)},\mathbf{u(t)},t) \\ &\quad +\sum_{i=1}^{n}\frac{\partial J^*}{\partial x_i}\mu_i(\mathbf{x(t)},\mathbf{u(t)},t)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\frac{\partial^2 J^*}{\partial x_i\partial x_j}\Bigg\}. \end{aligned}$$

$$J^*(\mathbf{x(T)},T) = F(\mathbf{x(T)},T). \tag{1.16}$$

For all the technical details regarding the stochastic HJB equation, see Bjork [9].

1.6.2 Pontryagin Maximum Principle

Adapting the Pontryagin Maximum Principle to stochastic control problems is usually very difficult. This is especially true for those control problems in which the volatilities (σ_i) are functions of the control, see Haussmann [25] for details. For necessary condition to achieve the optimum, see Peng [42]. For sufficient conditions, see Zhou [53].

The stochastic control problems we study in this thesis, however, are simpler. This is because the σ terms do not depend on the control. So, for the stochastic Maximum Principle we use, there is only a minor difference with respect to the non-stochastic version. Expectations with respect to the current information should be taken for the

values of the Hamiltonian. In other words, instead of maximizing $\mathbf{H}$, we maximize $\mathbb{E}_t(\mathbf{H})$, which is

$$\begin{aligned}\mathbb{E}_t(\mathbf{H}) &= \mathbb{E}_t\Big\{ I(\mathbf{x(t)}, \mathbf{u(t)}, t) \\ &\quad + \sum_{i=1}^{n} \Psi_i(\mu_i(\mathbf{x(t)}, \mathbf{u(t)}, t) + \sigma_i(\mathbf{x(t)}, \mathbf{u(t)}, t)\frac{dX_i}{dt}) \Big\} \qquad (1.17)\end{aligned}$$

where the Ψ_i are defined, as before, as

$$\begin{aligned}\frac{\partial \Psi_i}{\partial t} &= -\frac{\partial \mathbf{H}}{\partial x_i}, \qquad i = 1, \cdots, n, \\ \Psi_i(T) &= \frac{\partial F}{\partial x_i}, \qquad i = 1, \cdots, n. \qquad (1.18)\end{aligned}$$

$\frac{dX}{dt}$ is not defined in the conventional sense.[1] However, we keep it as it is to use it as a rule of thumb. We find that later on we can use this rule of thumb to solve the adjoint processes Ψ_i as well as the Hamiltonian $\mathbf{H}$. We demonstrate how this can be done in our example later in this chapter. Also, there is an excellent heuristic demonstration of this in Putyatin [44].

The maximum principle can also be used in control problems with constraints by using Lagrange Multipliers. We also demonstrate how this can be done in our example later. For details in applying the Lagrange multiplier in stochastic Maximum Principle, consult Haussmann [25].

1.7 Markowitz Mean-Variance Efficient Frontier

Markowitz's [33] portfolio theory is one of the most important works in mathematical finance. This approach is still very commonly used in asset allocation models.

The major contribution of his work is to define the meaning of efficient. An efficient portfolio is a portfolio such that given a certain, attainable, level of risk, it can achieve the highest expected return or, equivalently, given a certain level of expected return, it has the lowest risk. A rational investor, therefore, prefers to allocate his assets such that his portfolio becomes efficient.

In the Markowitz model, risk is measured in terms of variances (and covariances) of returns, and the investor wants to decide how to invest his total resources, Π, into

[1] In fact, $\frac{dX}{dt}$ is formally defined as white noise.

n different type of stocks. Let A_i denote the amount of resources invested in stock i, so

$$A_1 + \cdots + A_n = \Pi. \tag{1.19}$$

There may be some constraints on A_i. For example, if short sales are not allowed, then we have the constraints $A_i \geq 0$. In this illustration, however, we do not impose any constraints other than equation (1.19).

In the Markowitz model a stock is characterized by its expected return μ_i and standard deviation of returns σ_i. The standard deviation σ_i or variance σ_i^2 is used to quantify the assets risks,

$$\mu_i = \mathbb{E}\left(\frac{S_i' - S_i}{S_i}\right) \tag{1.20}$$

$$\sigma_i^2 = \mathrm{var}\left(\frac{S_i' - S_i}{S_i}\right) \tag{1.21}$$

where S_i and S_i' refer to the price of the security i at the beginning and the end of the investment period respectively. We also assume there are correlations between the returns of the stocks, and let $-1 \leq \rho_{ij} \leq 1$ be the correlation of the returns of stocks i and j.

In reality, it is very difficult to estimate the value of μ_i and ρ_{ij}. A big assumption for the Markowitz theory is the investor is able to estimate these parameters and they are constant over time.

For the sake of convenience, we use the following matrix notation

$$\vec{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix},$$

$$\vec{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix},$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \dots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \sigma_2\sigma_n\rho_{2n} & \dots & \sigma_n^2 \end{pmatrix},$$

and the superscript ^{T} to denote transpose.

The return of the portfolio, μ_Π, is simply the weighted average of the return of the n securities, and the risk of the portfolio is measured by the standard deviation, σ_Π, of the return of the portfolio, so

$$\mu_\Pi = \frac{\vec{A}^T\vec{\mu}}{\Pi}, \tag{1.22}$$

$$\sigma_\Pi = \sqrt{\vec{A}^T\mathbf{\Sigma}\vec{A}}. \tag{1.23}$$

Therefore, the problem of choosing an efficient portfolio is to maximize μ_Π for a given, feasible, σ_Π, or equivalently, to minimize σ_Π for a given μ_Π, by choosing different combinations of A_i. Usually, it is simpler to solve the second version of this problem, that is, to find

$$\min_{A_1,\ldots,A_n} \vec{A}^T\mathbf{\Sigma}\vec{A}$$

given the restriction

$$\frac{\vec{A}^T\vec{\mu}}{\Pi} = \tilde{\mu}_\Pi$$

for some constant $\tilde{\mu}_\Pi$. This problem can be solved by using Lagrange Multipliers [30], which transforms the optimization problem into the following linear equations

$$\begin{aligned} \nabla_A\left((\vec{A}^T\mathbf{\Sigma}\vec{A}) - \lambda_1(\frac{\vec{A}^T\vec{\mu}}{\Pi} - \tilde{\mu}_\Pi) - \lambda_2\vec{A}^T\vec{1}\right) &= 0, \\ \frac{\vec{A}^T\vec{\mu}}{\Pi} &= \tilde{\mu}_\Pi, \\ \vec{A}^T\vec{1} &= \Pi. \end{aligned} \tag{1.24}$$

with λ_1, λ_2 as Lagrange Multipliers and ∇_A as the gradient operator with respect to all the A_i.

For a given $\tilde{\mu}_\Pi$ there is a unique minimum risk $\tilde{\sigma}_\Pi(\tilde{\mu}_\Pi)$. The curve in $\sigma - \mu$ space traced out by $\tilde{\sigma}_\Pi(\tilde{\mu}_\Pi)$ as $\tilde{\mu}_\Pi$ varies is the efficient frontier.

Usually, a risk-free bond is also included in the model. A risk-free bond has a guaranteed return of r, zero variance and thus zero correlation with other stocks. The resulting efficient frontier is a straight line tangent to the original efficient frontier. That portfolio on the original efficient frontier is called the market portfolio $\mathbf{M}$ and the line is called the capital market line. The interpretation of this is the new efficient

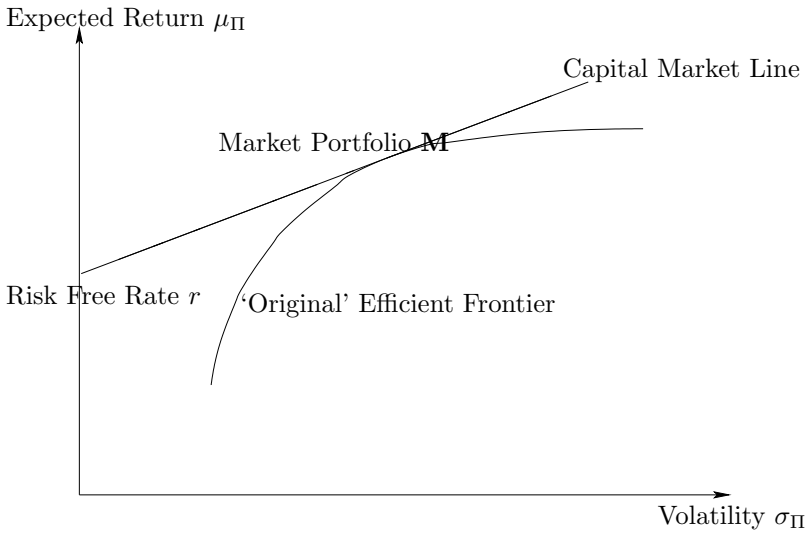


Figure 1.1: Capital Market Line and Efficient Frontier

portfolios are usually a combination of risk-free bond and the market portfolio **M**. Please refer to Figure 1.1 for illustration.

In a world of homogeneous expectations, which means all the investors have the same expectation of return and risk for each stock, **M** is the same for all investors and so all investors can access the same Capital Market Line. Portfolio choice, according to this theory, involves two tasks. The first task is to decide what is **M** and thus the Capital Market Line. The second task is for each individual investor to decide, based on his required return and risk tolerance, which point on the Capital Market Line should be chosen.

This asset allocation process is static. This means the investor allocates the investments at the beginning of the investment period and does not rebalance it until the investment horizon is reached. In reality, the asset allocation process may be dynamic. This means the portfolio may be rebalanced any time before the investment horizon.

1.8 Utility Function

Another approach used in asset allocation is to use utility function. Generally, utility means "usefulness". However, the word utility itself is very confusing as it can mean many different things ranging from the expected-utility hypothesis proposed by Friedman and Savage [21], [22] and [20] and utilitarianism advocated by John Stuart Mill [37]. Here, however, we just assume a utility function is a function so that the investor invests so as to maximize its expectation.

Therefore, the investor allocates his initial investment so as to maximize

$$\mathbb{E}[U(\Pi(T))]$$

where $\Pi(T)$ is the value of the whole portfolio at the end of the investment horizon T. Sometimes we want to have an infinite time horizon. In such a case, we let $T \to \infty$,

In order for U to make sense, U has to be an increasing function. This is because rational investors always prefer more money to less money.

The resultant investment strategy from maximizing the expectation of utility function is generally different to those from Markowitz Mean-Variance Efficient Frontier. There are some utility functions that would favour a more risky investment given the same expected return, which is inconsistent with Markowitz's model. See Section (1.9) for details.

1.9 Constant Relative Risk Aversion

A very common utility function is this constant relative risk aversion function (CRRA). In later chapters, this CRRA function is widely used. Before we can explain what is CRRA, we have to explain what is relative risk aversion.

Let Π denote wealth. Suppose we wishes to pay a fraction ζ of Π to avoid playing a game whose outcome is $(1-\eta)\Pi$, where η is a random variable with $\mathbb{E}[\eta] = 0$, $\mathbb{E}[\eta^2] = \sigma_\eta^2$, which is the risk of the game.[2] Suppose that our risk and return preference can be summarized by assuming we are always indifferent to two games that yield the same expectation from the utility function $U(\Pi)$. Then, we have

$$U((1-\zeta)\Pi) \quad = \quad \mathbb{E}[U((1-\eta)\Pi)]. \tag{1.25}$$

[2] A game is fair if $\mathbb{E}[\eta] = 0$.

Since

$$\begin{aligned} U((1-\zeta)\Pi) &= U(\Pi) - \zeta\Pi U'(\Pi) + \dots \\ U((1-\eta)\Pi) &= U(\Pi) - \eta\Pi U'(\Pi) + \frac{\eta^2\Pi^2 U''(\Pi)}{2} + \dots, \end{aligned}$$

and taking expectations on both sides, for $\eta \ll 1$, we arrive at

$$\zeta = -\frac{1}{2}\sigma_\eta^2 \frac{\Pi U''(\Pi)}{U'(\Pi)}.$$

The relative risk aversion, RRA, is defined as the function $R(\Pi)$,

$$R(\Pi) = -\frac{\Pi U''(\Pi)}{U'(\Pi)}. \tag{1.26}$$

Suppose the relative risk aversion, $R(\Pi)$, is a constant $\tilde{\gamma}$. Then, we have

$$\zeta = \frac{1}{2}\sigma_\eta^2 \tilde{\gamma}$$

where σ_η^2 can be interpreted as risk and ζ as the amount paid to avoid this risk. From this, we can actually find out what is the possible $U(\Pi)$ by solving equation (1.26)w ith $R(\Pi) = \tilde{\gamma}$. In this case we find that

$$U(\Pi) = \begin{cases} \frac{\Pi^{(1-\tilde{\gamma})}-1}{1-\tilde{\gamma}} & \text{if } \tilde{\gamma} \neq 1 \\ \log \Pi & \text{if } \tilde{\gamma} = 1 \end{cases}$$

A function that is in this form is called Constant Relative Risk Aversion(CRRA).

When $\tilde{\gamma} = 0$, it means the investor is risk neutral. This means the investor is indifferent to risk, and he does not pay anything to avoid risk. Hence the investor takes an infinite amount of risk if the risky asset pays out more than the risk-free. When $\tilde{\gamma} > 0$, the investor is risk averse. This means the investor is willing to sacrifice some positive fraction of their wealth to avoid risk. When $\tilde{\gamma} < 0$, the investor is risk seeking. This means the investor is willing to sacrifice some positive fraction of their wealth to seek risk.

Since the RRA of the utility function $U(\Pi)$ is invariant under the transformation

$$U(\Pi) \longrightarrow aU(\Pi) + b$$

where $a > 0$ and b are arbitrary constants. Therefore, a utility function in the form

$$U(\Pi) = \frac{\Pi^\gamma}{\gamma}$$

with $\gamma \neq 0$ is also considered as CRRA. In the case of $\gamma = 0$,

$$U(\Pi) = log\Pi.$$

Here

$$\gamma = 1 - \tilde{\gamma}$$

so $\gamma = 1$ corresponds to risk neutral, $\gamma < 1$ to risk averse and $\gamma > 1$ risk seeking.

1.10 Merton's Investment-Consumption Model

Merton's investment-consumption model [36] is a good example of dynamic asset allocation and stochastic control based on a HJB equation approach.

Merton's model is about a risk averse investor who has an initial amount of resources Π_0 and wants to allocate resources in risky investments, risk-free bonds investment and consumption so as to maximize the expected life-time utility [3]. Here, the utility is an increasing concave function of rate of resources used in consumption, C, in the form

$$U(C) = \frac{C^\gamma}{\gamma}$$

where $\gamma < 1$ is a constant. A sufficient condition for the problem to be solvable is

$$\frac{\partial^2 U(C)}{\partial C^2} < 0,$$

which gives us the restriction $\gamma < 1$. As we can recall from the Section 1.9, this utility function is equivalent to the CRRA function and $\gamma = 1 - \tilde{\gamma}$. So, this γ determines how risk-averse the investor is. The expected life-time utility, which the investor wants to maximize, is

$$\mathbb{E}_0\Big\{ \int_0^\infty e^{\nu t} U(C(t))dt \Big\}$$

[3] For simplicity, we don't consider the bequest function as in Merton [36] and we only consider an infinite, rather than a finite, time horizon.

where $\mathbb{E}_t$ is the expectation operator given information up to time t and $\nu < 0$ is a discounting factor for future utility. [4]

The setting of the market model is as follows. There is only one risky asset and risk-free bond in the market. In the equation describing the risky asset, S is the price, μ and σ are all constants, and they represent its drift and volatility, and X is a standard Brownian motion as described in the Section (1.4). So, the equation is

$$dS = \mu S dt + \sigma S dX. \tag{1.27}$$

In the equation for describing the riks-free bond, r, a constant, represents the risk-free rate, and B is the amount invested. Also, we assume the resources for consumption can only be taken out from the bank rather than from the risky asset. At all times, $C \geq 0$, which means the rate of consumptions cannot be negative.[5] So, we have

$$dB = (rB - C)dt. \tag{1.28}$$

Let Π represents the value of the whole portfolio, that is the sum of resources invested in stocks and risk-free bond, and λ represents the proportion of resources invested in risky assets. The controls, which refers to the choices that the investor can make, are C and λ. Therefore, the model for the value of the portfolio is

$$d\Pi = [(\lambda(\mu - r) + r)\Pi - C]dt + \lambda \Pi \sigma dX. \tag{1.29}$$

In order to apply the stochastic dynamic programming technique, we define

$$J(\Pi, t) = \max_{\lambda, C} \mathbb{E}_t \Big\{ \int_t^\infty e^{\nu\tau} U(C(\tau)) d\tau \Big\}, \tag{1.30}$$

and

$$\tilde{J}(\Pi) = \max_{\lambda, C} \mathbb{E}_0 \Big\{ \int_0^\infty e^{\nu\tau} U(C(\tau)) d\tau \Big\}. \tag{1.31}$$

[4] Strictly speaking, $U(C)$ here is the rate of utility, rather than utility.

[5] As the rate of consumption has to be positive, we find that it is necessary to have

$$-\nu - \gamma r - \frac{\gamma(\mu - r)^2}{2\sigma^2(1 - \gamma)} > 0.$$

We note that the values of

$$\max_{\lambda,C} \mathbb{E}_t\Big\{ \int_t^\infty e^{\nu(\tau-t)} U(C(\tau))d\tau \Big\}$$

are the same for all values of t. So, we have

$$J(\Pi, t) = \tilde{J}(\Pi)e^{\nu t}$$

and thus

$$\frac{\partial \tilde{J}}{\partial t} = \nu \tilde{J}.$$

The corresponding HJB equation[6] becomes

$$0 \;=\; \max_{\lambda,C} \left[-\nu\tilde{J} + r\Pi \frac{d\tilde{J}}{d\Pi} + U(C) + [(\mu - r)\lambda\Pi - C]\frac{d\tilde{J}}{d\Pi} + \frac{\sigma^2\lambda^2\Pi^2}{2}\frac{d^2\tilde{J}}{d\Pi^2} \right] \quad (1.32)$$

We differentiate equation (1.32) with respect to C and λ, and we find that the maximum is obtained when

$$C^* = \tilde{J}_\Pi^{1/(\gamma-1)}$$

and

$$\lambda^* = -\frac{(\mu - r)\tilde{J}_\Pi}{\sigma^2\Pi\tilde{J}_{\Pi\Pi}},$$

where $\tilde{J}_\Pi$ is short hand for $\partial\tilde{J}/\partial\Pi$ and $\tilde{J}_{\Pi\Pi}$ for $\partial^2\tilde{J}/\partial\Pi^2$. There are indeed maxima because, if we differentiate equation (1.32) with respect to C and λ twice we have

$$(\gamma - 1)C^{\gamma-2}$$

and

$$\sigma^2\Pi^2\frac{\partial^2\tilde{J}}{\partial\Pi^2},$$

and both of them are negative.

We put C^* and λ^* into equation (1.32) and find that

$$r\Pi\tilde{J}_\Pi - \frac{(\mu - r)\tilde{J}_\Pi}{2\sigma^2\Pi\tilde{J}_{\Pi\Pi}} + \frac{1-\gamma}{\gamma}\tilde{J}_\Pi^{\frac{\gamma}{\gamma-1}} - \nu\tilde{J} = 0.$$

[6]See section 1.6.1 for more details

It can be shown that the solutions are

$$\tilde{J}(\Pi) = \frac{1}{\gamma} C^{*\gamma-1} \Pi \tag{1.33}$$

$$C^* = \frac{\Pi}{1-\gamma}[-\nu - \gamma r - \frac{\gamma(\mu - r)^2}{2\sigma^2(1-\gamma)}] \tag{1.34}$$

$$\lambda^* = \frac{\mu - r}{\sigma^2(1-\gamma)}. \tag{1.35}$$

It is worth noticing that λ^* is proportional to the *Sharpe Ratio*

$$\frac{\mu - r}{\sigma^2}.$$

The Sharpe Ratio is a common measure of performance, which can be interpreted as "reward per variability". Also, it is proportional to $1/(1-\gamma)$, where $1-\gamma$ is the term relative risk aversion of Section(1.9). When γ is closer to 1, the investor is closer to risk neutral, and we can see that the investor invests more in the risky stock market.

1.11 Long Term Growth Model

A small variation of the consumption model is this long term growth model. In this model, instead of considering consumption (putting $C = 0$), the aim is to maximize the long term growth of the portfolio. According to the Kelly Criterion [31], this means maximizing the logarithm of the value of the portfolio. In other words, given an investment period between 0 and T, our goal is to maximize the function

$$\mathbb{E}_0\Big\{ \log\Big\{\Pi(T)\Big\}\Big\}.$$

As in the previous section, we let

$$J(\Pi, t) = \max \mathbb{E}_t\Big\{ \log(\Pi(T))\Big\}. \tag{1.36}$$

The corresponding HJB equation is

$$0 = \max_{\lambda(t)} \left\{ \frac{\partial J}{\partial t} + [r\Pi + (\mu - r)\lambda(t)\Pi(t)]\frac{\partial J}{\partial \Pi} + \frac{\sigma^2\lambda^2\Pi^2}{2}\frac{\partial^2 J}{\partial \Pi^2} \right\}. \tag{1.37}$$

and the final condition is

$$J(\Pi, T) = \log(\Pi),$$

where λ is the portion of resources invested in stock.

We differentiate equation (1.37) with respect to λ, and we find that the maximum is obtained when

$$\lambda = -\frac{(\mu - r)J_{\Pi}}{\sigma^2 \Pi J_{\Pi\Pi}}. \tag{1.38}$$

This point is indeed a maximum because, by differentiating it with respect to λ again, we have

$$\sigma^2 \frac{\partial^2 J}{\partial \Pi^2} < 0, \tag{1.39}$$

if we assume that the backward parabolic HJB equation (1.37) preserves concavity.

If we substitute equation (1.38) into equation (1.37), we have

$$0 = \frac{\partial J}{\partial t} + r\Pi \frac{\partial J}{\partial \Pi} - \frac{(\mu - r)^2 (\frac{\partial J}{\partial \Pi})^2}{2\sigma^2 \frac{\partial^2 \tilde{J}}{\partial \Pi^2}}. \tag{1.40}$$

We find that

$$J(\Pi, t) = \log \Pi + (r + \frac{(\mu - r)^2}{2\sigma^2})(T - t) \tag{1.41}$$

and

$$\lambda = \frac{(\mu - r)}{\sigma^2} \tag{1.42}$$

satisfy equation (1.40) as well as the terminal condition.

So,

$$\mathbb{E}_0 \Big\{ \log \Big\{ \Pi(T) \Big\} \Big\} = \log \Pi_0 + (r + \frac{(\mu - r)^2}{2\sigma^2})T$$

Therefore, the optimal investment is independent of time and the current level of wealth. Interestingly though, the ratio of total resources to the resources invested in stock is exactly the Sharpe Ratio.

1.12 Example of Pontryagin Maximum Principle

In this section, we study an example of the application of the stochastic Pontryagin Maximum Principle. Our example is on the problem of dynamic mean-variance asset allocation without transaction costs. This means we manage a portfolio dynamically,

so as to minimize the variance of the value of the portfolio at some time horizon T, while keeping the expected value of the portfolio equal to a target value ξ. We assume there are no short selling or margin restrictions and trading takes place in continuous time.

The outline of solving the problem is as follows. We firstly apply the stochastic version of the Pontryagin Maximum principle. This helps us to reduce the problem to the problem of maximizing the Hamiltonian. Afterwards, we guess a strategy, and we use Kolmogorov backward equation to establish the transition probabilities of that strategy, which helps us to verify that the strategy indeed maximize the Hamiltonian. Thus, we establish the optimal solution.

The solution we eventually discover is similar to the doubling strategy discovered by Harrison and Kreps [23], which means the optimal solution has no practical value. This is because we do not impose any restrictions on the class of admissible strategies. Still, we consider this study is worthwhile. There are several reasons. Firstly, what we do is completely different to the way Harrison and Kreps discovered their doubling strategy. Secondly, this problem helps us to understand how the Pontryagin maximum principle works and also the possible difficulties it has when it is applied to other problems. Also, it helps us to understand the importance of those finer technical details of the principle. Thirdly, variance minimization is a natural objective to manage a portfolio. Therefore, the problem itself is interesting although the result is not practical. In fact, there is a recent work [44] which attempts to solve this mean-variance problem.

1.12.1 Market Model Equations

The setup of the market model is as follows. [7] Let $S(t)$ be the spot price of a stock at time $0 \leq t \leq T$, where T is the time horizon of the investment period. Let $A(t)$, $B(t)$ and $\Pi(t)$ be the value of assets invested in stocks, risk free bonds and the total value of assets respectively,

$$\Pi(t) \quad = \quad A(t) + B(t). \tag{1.43}$$

[7] Actually the doubling strategy does not need to depend on this particular market model. We just use this market model just as an example to illustrate the problem.

We assume $S(t)$ follows a geometric Brownian motion with growth rate $\mu > 0$ and volatility $\sigma > 0$. The risk free bond, $B(t)$, is compounded continuously with risk free rate r. For simplicity, we assume μ, r, and σ are constants. Cash flows generated from the purchase or sale of stocks, denoted by u, are immediately invested in or withdrawn from the risk free bond account.

Our model can be represented by the following equations

$$\begin{aligned} dS &= \mu S dt + \sigma S dX \\ dA &= (\mu A + u)dt + \sigma A dX \\ dB &= rBdt - udt \\ d\Pi &= r\Pi dt + (\mu - r)Bdt + \sigma(\Pi - B)dX \end{aligned} \tag{1.44}$$

where X is a standard Brownian motion process.

At time $t = 0$, an investor has an initial amount $\Pi(0)$ of resources. The problem is to allocate investments over the given time horizon so as to minimize

$$\mathbb{E}_0\Big\{ \text{var}(\Pi(T)) \Big\}$$

where $\mathbb{E}_t$ is the conditional expectation given information up to time t. At the same time, the strategy has to satisfy

$$\mathbb{E}_0(\Pi(T)) = \xi \tag{1.45}$$

where ξ can be considered as a target we wish to achieve.

For simplicity, we choose to minimize $\mathbb{E}_0(\Pi(T)^2)$ instead of minimizing the variance directly.

1.12.2 Applying the Maximum Principle

According to the stochastic version of the Pontryagin maximum principle, the optimal policy also maximizes the negative expected value of a Hamiltonian **H**, which we proceed to find. The Hamiltonian, **H**, is defined as

$$\begin{aligned} \mathbf{H}(B, A, u) &= \Psi_B(rB - u) \\ &\quad + \Psi_A(\mu A + u + \sigma A \frac{dX}{dt}). \end{aligned} \tag{1.46}$$

Although dX/dt is undefined in the usual sense, we can use it in a formal sense. By doing this we can solve the adjoint process Ψ_A correctly. The adjoint process, Ψ_A, becomes

$$\frac{\partial \Psi_A}{\partial t} = -\frac{\partial \mathbf{H}}{\partial A} = -\Psi_A(\mu + \sigma \frac{dX}{dt}), \tag{1.47}$$

and final value, $\Psi_A(T)$ can be obtained from

$$\begin{aligned} \Psi_A(T) &= \frac{\partial}{\partial B}(-\frac{1}{2}(A+B)^2) + \lambda \frac{\partial (A+B)}{\partial B}\Big|_{B=B(T)} \\ &= -\Pi(T) + \lambda\Big|_{B=B(T)} \end{aligned} \tag{1.48}$$

where λ is the Lagrange multiplier which makes the constraint possible.[8]

Similarly, Ψ_B is defined as

$$\frac{\partial \Psi_B}{\partial t} = -\frac{\partial \mathbf{H}}{\partial B} = -r\Psi_B \tag{1.49}$$

and

$$\Psi_B(T) = -\Pi(T) + \lambda. \tag{1.50}$$

Solving equations (1.47) to (1.50) yields

$$\begin{aligned} \Psi_A &= (-\Pi(T) + \lambda)\frac{S(T)}{S(t)}; \\ \Psi_B &= (-\Pi(T) + \lambda)e^{r(T-t)}. \end{aligned} \tag{1.51}$$

Substituting equation (1.51) into equation (1.46) and dropping the terms that are not dependent on the control u, the problem of maximizing $\mathbb{E}_t(J)$ reduces to the maximization of either side of the expression

$$\begin{aligned} (\mathbb{E}_t(\Psi_A) - \mathbb{E}_t(\Psi_B))u &= \Big[\mathbb{E}_t(-\Pi(T) + \lambda)\frac{S(T)}{S(t)} \\ &\quad -\mathbb{E}_t(-\Pi(T) + \lambda)e^{r(T-t)}\Big]u \end{aligned} \tag{1.52}$$

[8]The Lagrange multiplier is analogous to the Lagrange multiplier we introduced in the section on Mean Variance Portfolio Theory.

By applying the maximum principle, we simplify our original problem. Now, the problem of maximization of equation (1.46) becomes the problem of maximization of either side of equation (1.52).

1.12.3 Kolmogorov Equation and Transition Probability

Maximizing either side of equation (1.52) means

$$u = \begin{cases} +\infty & \text{if } \mathbb{E}_t(\Psi_A) > \mathbb{E}_t(\Psi_B) \\ 0 & \text{if } \mathbb{E}_t(\Psi_A) = \mathbb{E}_t(\Psi_B) \\ -\infty & \text{if } \mathbb{E}_t(\Psi_A) < \mathbb{E}_t(\Psi_B) \end{cases}$$

The above equations mean the solution is to keep on buying or selling until $\mathbb{E}_t(\Psi_A) = \mathbb{E}_t(\Psi_B)$. From now on, we represent our optimal strategy in terms of A and B rather than u. Our problem becomes finding $A(t)$ and $B(t)$ such that $\mathbb{E}_t(\Psi_A) = \mathbb{E}_t(\Psi_B)$. The values of $\mathbb{E}_t(\Psi_A)$ and $\mathbb{E}_t(\Psi_B)$, which dictates the current choice of action, are in some way determined by the choice of $A(t_1)$ and $B(t_1)$ ($t_1 \in [t, T]$), which is the policy we plan to have in the future.

We have an example here to illustrate the problem. Consider the case of $S_0 = 1$ and $\Pi_0 e^{\mu T} < \lambda < \Pi_0 e^{(\mu+\sigma^2)T}$.[9] Suppose there are two future strategies, strategy I and strategy II say. In strategy I, we invest an amount of $\Pi(t)$ in shares and 0 in risk free bonds at time t. In strategy II, we invest $\Pi(t)$ in risk free bonds and 0 in shares at time t. So, in strategy I,

$$\begin{aligned} \mathbb{E}_0(\Psi_A) &= \mathbb{E}_0((-\Pi(T) + \lambda)S(T)/S_0) \\ &= \mathbb{E}_0((-\Pi_0 S(T) + \lambda)S(T)) \\ &= -\Pi_0 e^{(2\mu+\sigma^2)T} + \lambda e^{\mu T} \\ &< 0 \end{aligned} \tag{1.53}$$

and

$$\begin{aligned} \mathbb{E}_0(\Psi_B) &= \mathbb{E}_0((-\Pi(T) + \lambda)e^{rT}) \\ &= \mathbb{E}_0((-\Pi_0 S(T) + \lambda)e^{rT}) \\ &= (-\Pi_0 e^{\mu T} + \lambda)e^{rT}) \end{aligned}$$

[9] The Lagrange multiplier usually cannot be anything we want. However, as we show later, in this particular problem it is possible to have λ take any value.

$$
\begin{aligned}
&> 0 \\
&> \mathbb{E}_0(\Psi_A). \qquad (1.54)
\end{aligned}
$$

So for strategy I it is optimal to sell shares. As for strategy II,

$$
\begin{aligned}
\mathbb{E}_0(\Psi_A) &= \mathbb{E}_0((-\Pi(T)+\lambda)S(T)/S_0) \\
&= (-\Pi_0 e^{rT}+\lambda)e^{\mu T} \\
&> 0 \qquad (1.55)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_0(\Psi_B) &= \mathbb{E}_0((-\Pi(T)+\lambda)e^{rT}) \\
&= (-\Pi_0 e^{rT}+\lambda)e^{rT}) \\
&< (-\Pi_0 e^{rT}+\lambda)e^{\mu T}) \\
&= \mathbb{E}_0(\Psi_A), \qquad (1.56)
\end{aligned}
$$

therefore, it is optimal to buy shares.

The above example illustrates very clearly that we cannot choose a future strategy randomly in calculating $\mathbb{E}_t(\Psi_A)$ and $\mathbb{E}_t(\Psi_B)$. So, how do we decide what values should we use for $A(t_1)$ and $B(t_1)$ ($t_1 \in [t,T]$) in calculating $\mathbb{E}_t(\Psi_A)$ and $\mathbb{E}_t(\Psi_B)$?

According to the maximum principle, in fact the future optimal strategy should be used in calculating $\mathbb{E}_t(\Psi_A)$ and $\mathbb{E}_t(\Psi_B)$. Therefore, the problem of this section becomes finding $A(\Pi,S,t)$ and $B(\Pi,S,t)$, $t \in [0,T]$ such that $\mathbb{E}_t(\Psi_A) = \mathbb{E}_t(\Psi_B)$ for all $t \in [0,T]$.

Now, in order to compute $\mathbb{E}_t(\Psi_A)$ and $\mathbb{E}_t(\Psi_B)$, we use Kolmogorov backward equation [51]for the transition density of $\Pi(T)\frac{S(T)}{S(t)}$ and $\Pi(T)$. This can help us to determine the values of $\mathbb{E}_t(\Psi_A)$ and $\mathbb{E}_t(\Psi_B)$ so that we can determine their values so as to choose our strategy. In the following, the transition probability $p_S(\Pi_0,t_0;\Pi_1,t_1)$ ($t_1 > t_0$) is the conditional probability that $p(\Pi(t_1)=\Pi_1|\Pi(t_0)=\Pi_0)$ and $p_\Pi(\Pi_0,S_0,t_0;\Pi_1,S_1,t_1)$ is the conditional probability that $p(\Pi(t_1)=\Pi_1,S(t_1)=S_1|\Pi(t_0)=\Pi_0,S(t_0)=S_0)$.

We now change the variables of equation (1.44) so that we can apply the Kolmogorov equation to investigate the transition probability.

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dX \\
d\Pi &= (r\Pi + (\mu - r)A)dt + \sigma A dX \qquad (1.57)
\end{aligned}
$$

According to Wilmott *et al.* [52], Kolmogorov backward equation for the probability densities are

$$\begin{aligned}
\frac{\partial p_S}{\partial \tau} &= (r\Pi_0 + (\mu - r)A_0)\frac{\partial p_S}{\partial \Pi_0} + \frac{\sigma^2}{2}\Pi_0^2\frac{\partial^2 p_S}{\partial \Pi_0^2}, && (1.58)\\
\frac{\partial p_\Pi}{\partial \tau} &= (r\Pi_0 + (\mu - r)A_0))\frac{\partial p_\Pi}{\partial \Pi_0} + \mu S_0\frac{\partial p_\Pi}{\partial S_0} \\
&\quad + \frac{\sigma^2}{2}\Big(\Pi_0^2\frac{\partial^2 p_\Pi}{\partial \Pi_0^2} + 2\Pi_0 S_0\frac{\partial^2 p_\Pi}{\partial \Pi_0 \partial S_0} + S_0^2\frac{\partial^2 p_1}{\partial S_0^2}\Big), && (1.59)
\end{aligned}$$

respectively, where $\tau = t_1 - t_0$ and t_1 is considered constant.

Taking expectation on both sides equation (1.58) and equation (1.59) , we find that these two equations admit the solutions

$$p_S = \mathbb{E}_t(-\Pi(T))$$

and

$$p_\Pi = \mathbb{E}_t(-\Pi(T))\frac{S(T)}{S(t)}$$

respectively. The boundary conditions become

$$\mathbb{E}_T(-\Pi(T)) = -\Pi(T)$$

and

$$\mathbb{E}_T(-\Pi(T))\frac{S(T)}{S(T)} = -\Pi(T).$$

The solution of $A(\Pi, S, t)$ is

$$A(\Pi, S, t) = \frac{\lambda - \Pi e^{r(\tau+t_2)}}{e^{\mu(\tau+t_2)} - e^{r(\tau+t_2)}} \qquad (1.60)$$

where t_2 is $T - t_1$, the time between t_1 and expiry.

Putting the above $A(\Pi, S, t)$ to the Kolmogorov equations, equation (1.58) and equation (1.59) become

$$\begin{aligned}
\frac{\partial p_S}{\partial \tau} &= (r\Pi_0 + (\mu - r)\frac{\lambda - \Pi e^{r(\tau+t_2)}}{e^{\mu(\tau+t_2)} - e^{r(\tau+t_2)}})\frac{\partial p_0}{\partial \Pi_0} + \frac{\sigma^2}{2}\Pi_0^2\frac{\partial^2 p_0}{\partial \Pi_0^2} && (1.61)\\
\frac{\partial p_\Pi}{\partial \tau} &= (r\Pi_0 + (\mu - r)\frac{\lambda - \Pi e^{r(\tau+t_2)}}{e^{\mu(\tau+t_2)} - e^{r(\tau+t_2)}}))\frac{\partial p_1}{\partial \Pi_0} + \mu S_0\frac{\partial p_1}{\partial S_0} \\
&\quad + \frac{\sigma^2}{2}\Big(\Pi_0^2\frac{\partial^2 p_1}{\partial \Pi_0^2} + 2\Pi_0 S_0\frac{\partial^2 p_1}{\partial \Pi_0 \partial S_0} + S_0^2\frac{\partial^2 p_1}{\partial S_0^2}\Big) && (1.62)
\end{aligned}$$

respectively.

We find that

$$\mathbb{E}_t(\Pi(\tau)) \quad = \quad A(t)e^{\mu\tau} + (\Pi(t) - A(t))e^{r\tau} \tag{1.63}$$

is a solution of equation (1.61) .

As we want $\mathbb{E}_t(\Pi(\tau)) = \xi$, all we need is to put $\lambda = \xi$.

If we put $t_2 = 0$, which means $t_1 = T$, and consider the boundary condition $\mathbb{E}_T(\Pi(T)) = \lim_{t\to T} \mathbb{E}_t(\Pi(T)) = \xi$, equation (1.63) becomes

$$\mathbb{E}_t(\Pi(\tau)) \quad = \quad \xi \tag{1.64}$$

for any t.

Therefore, our strategy 'guarantees' that the expectation is always equal to ξ. As for the solution of equation (1.62), we have

$$\mathbb{E}_t(\Pi(\tau)S(\tau)) \quad = \quad \xi e^{\mu\tau}. \tag{1.65}$$

Since $\lambda = \xi$, $\mathbb{E}_t(\Psi_A)$ is always equal to $\mathbb{E}_t(\Psi_B)$. So we verify that our choice of $A(t)$ is correct and the resulting strategy is optimal.

1.12.4 Discussion

The strategy we discuss above seems to guarantee that the eventual portfolio value is always equal to ξ. If we calculate the variance of the portfolio value at T using equation (1.58) , we find that it is equal to 0. This means that the strategy seems to make the portfolio another risk free bond, with possibly higher return than r, the risk free interest rate. How is this achieved?

In fact, we have rediscovered a strategy similar to Harrison and Kreps [23] famous doubling strategy. The idea of that doubling strategy can be described as the continuous version of the strategy that wins one dollar for sure from betting on an infinite sequence of coin flips. The coin flipping strategy is as follows. Suppose we win after first flip, we stop. If we lose, then we continue to bet, but doubling the stake, until we win, and then we stop. This strategy "guarantees" we must win one dollar eventually because the probability of losing forever is 0.

The strategy we discover is similar. The strategy is to invest more in stocks if the value of the portfolio is further away from the target, ξ, and invest less in stocks

if the value of the portfolio is closer to the target. This is similar to "doubling" the stakes when we lose and stop when we win.

Mathematically, this optimal strategy is undesirable as well. This is because

$$\mathbb{E}\int_0^T |A|dS$$

is not finite. In most of the commonly used definition of admissible strategy, this integral has to be finite in order to be admissible.

Note that there are many strategies which can minimize the variance and at the same time maintain the expectation of the portfolio as ξ. This strategy is only one of them. It is not difficult to think of another one. For example, we can invest all our resources from the portfolio in risk free bonds until $T/2$ and then start to use the strategy. This also achieves the same result.

The strategy threatens the no arbitrage principle we would expect from equation (1.44). This is because this strategy can create arbitrage opportunities by achieving a risk free return higher than r. Heath and Jarrow [26], therefore, propose the use of short sale restrictions or margin restrictions as they found that such restrictions exclude arbitrage strategy like this.

1.12.5 Dynamic Asset Allocation with Margin Constraints

Heath and Jarrow [26] examined the problem of the doubling strategy within a continuous frictionless market. By frictionless, they mean that there are no transaction costs, no short-sale restrictions, and no taxes and that asset shares are infinitely divisible. They also assume the existence of two securities, described by equation (1.44) . They showed that the introduction of margin restrictions can eliminate the possibility of a doubling strategy. As we see later, the strategy obtained previously no longer works given those restrictions as well.

Heath and Jarrow then went on to examine the impact of margin constraints on options pricing. They showed that while margin constraints impose restrictions on trading, they should have no effect on the price of options in the market and the Black-Scholes value still holds. We do not go into that detail because it is not relevant to our work.

The way Heath and Jarrow modeled the margin constraints is by imposing the following restrictions on trading

$$\Pi = A + B \geq \begin{cases} L_+|A| & \text{if } A > 0 \\ L_-|A| & \text{if } A < 0 \end{cases}$$

where $0 \leq L_+, L_- \leq 1$ are constants.

To understand what the above equations mean, we can consider several cases. Suppose both A and B are positive, then the inequalities impose no restriction. Similarly, the above constraints restrict all the trading strategies with both A and B negative. When A is negative and B is positive, it means short sale. The equations mean on top of the proceedings from the short sales, an extra L_- portion of the stock prices needs to come from the investor's own fund. When B is negative and A is positive, it means the investor is buying the stock on margin and so the investor needs to provide L_+ portion of the stock price.

Imposing such restrictions, as shown by Heath and Jarrow, eliminates the use of the doubling strategies. Heath and Jarrow's proof is very subtle. We refer readers to [26] for the details.

Our strategy obtained in the previous section also no longer works given those restrictions. Here we consider a very simple example. Suppose $L_+ = 1$. This is equivalent to $B \geq 0$ and so no borrowing is allowed. Now let's assume we start with resources $\Pi(0) = \Pi_0$. If the target we have ξ is bigger than $\Pi_0 e^{\mu T}$, even we use the most aggressive strategy available, that is to invest all our resources in stock all the time, the expected value of our portfolio is only equal to $\Pi_0 e^{\mu T}$, which is smaller than ξ. So the previous strategy no longer works.

1.13 Investment with Transaction Costs

Most of our work is related the portfolio management with transaction costs. Therefore, we are going to review some of the more important works here.

1.13.1 Atkinson, Pliska & Wilmott

Atkinson *et al.* [5] make a very successful attempt at solving a problem in portfolio management. Their work is a further study on a model developed by Morton and Pliska [39].

The setup of Morton and Pliska is as follows. An investor has an infinite investment interval in which to invest. The value of stock, $S(t)$, follows a geometric Brownian motion with growth rate $\mu_i > 0$ and volatility $\sigma_i > 0$. The risk free bonds, B, are compounded continuously with risk free rate r. Cash generated by or needed for the purchase or sale of stocks is immediately invested or withdrawn from the risk free bond account. The model is represented by

$$\begin{aligned} dS_i &= \mu_i S_i dt + \sigma_i S_i dX_i, i = 1, \cdots, n, \\ dB &= rBdt \\ &= r(\Pi - \sum_{i=1}^{n} A_i)dt \\ d\Pi &= rBdt + \sum_{i=1}^{n} \mu_i A_i dt + \sum_{i=1}^{n} \sigma_i A_i dX_i \\ &= r(\Pi - \sum_{i=1}^{n} A_i)dt + \sum_{i=1}^{n} \mu_i A_i dt + \sum_{i=1}^{n} \sigma_i A_i dX_i \end{aligned}$$

where the X_is are standard Brownian motions with $dX_i^2 = dt$ and correlated by $dX_i dX_j = \rho_{ij} dt$, $i, j = 1, \cdots, n$. There are no redundant assets and so the covariance matrix.

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \sigma_2\sigma_n\rho_{2n} & \cdots & \sigma_n^2 \end{pmatrix}$$

is non-singular. Let $\Pi(t)$ denote the value of the portfolio at time t. The transaction costs are proportional to the value of the portfolio $\Pi(t)$ at the time of the transaction. Therefore, the transaction cost of a single transaction is

$$k\Pi(t).$$

The problem of the investor is to maximize the asymptotic growth rate

$$\lim_{T\to\infty} \frac{\mathbb{E}[\log \Pi(T)]}{T}.$$

Many works in the study of transaction costs suffer from the following problem. That is the computation of the optimal trading strategies is very difficult. Even

solving a problem with one or two assets is enormously difficult, not to mention 20 to 30. This means the model has no practical value.

Morton and Pliska have the same problem in the work. Thanks to Atkinson and Wilmott [6] and Atkinson *et al.* [5], this problem is overcome by using perturbation analysis to find a solution which is a good approximation to the real solution when k is small. The solution is very easy to compute. Their solutions are even good enough for the case of many risky assets. Therefore, they have effectively solved the problem.

1.13.2 Atkinson and Al-Ali

Atkinson and Al-Ali [1] study the problem of introducing transaction costs into Merton's Investment and Consumption Model. They assume

$$\begin{aligned} dS &= \mu S dt + \sigma S dX \\ dB &= (rB - C)dt - (1 + k_+)d\mathbf{L}(t) + (1 - k_-)d\mathbf{M}(t) \\ d\Pi &= \mu(\Pi - B)dt + rBdt + \sigma(\Pi - B)dX - k_+ d\mathbf{L}(t) - k_- d\mathbf{M}(t) \end{aligned}$$

where $\mathbf{L}(t)$ and $\mathbf{M}(t)$ represent the cumulative purchase and sale of assets A in $[0, t]$, and k_+ and k_- represent the ratio of the transaction costs when risky assets are bought or sold. So, $k_+\mathbf{L}(t)$ and $k_-\mathbf{M}(t)$ represent the total transaction costs paid to purchase stocks and selling risky assets till time t respectively.

Similar to Merton's model, the objective is to maximize [10]

$$\mathbb{E}_0\Big\{\int_0^\infty e^{\nu t}\frac{C^\gamma}{\gamma}dt\Big\}.$$

Perturbation analysis is used in this study. They find that the solution tends to Merton's solution when the transaction costs tend to 0. An explicit solution is obtained for the optimal trading policy.

They then extend the model and consider the case with two and then subsequently many risky assets. They allowed different transaction costs in purchasing and selling risky for each risky assets. In this case, they also successfully solved the optimal trading policy.

This result is consistent with the numerical result of Akian *et al.* [2].

[10] Here, we use the same notation as in the section on Merton's Model.

1.13.3 Atkinson and Mokkhavesa (2001)

Atkinson and Mokkavesa [3] make a study on the utility function. Their study is based on Merton's Investment and Consumption Model. The problem they attempt to solve is to determine the utility function given the investment and consumption behavior of an investor. They were successful in many different cases.

Firstly, they consider the infinite time horizon case. The setup of their model is similar to Merton's Investment Consumption Model we examine. So, they assume

$$dB = (rB - C)dt, \tag{1.66}$$

$$dS = \mu S dt + \sigma S dX, \tag{1.67}$$

$$d\Pi = [(\lambda(\mu - r) + r)\Pi - C]dt + \lambda\Pi\sigma dX. \tag{1.68}$$

Instead of giving a utility function $U(C(t))$ and trying to find out what is the optimal investment and consumption policy(λ^*, C^*), Atkinson *et al.* instead solve $U(C(t))$ when (λ^*, C^*) are given. They find that if

$$C^* = \Pi/\beta_1, \tag{1.69}$$

$$\lambda^* = \beta_2, \tag{1.70}$$

then the governing equation for $U(C)$ is given by

$$0 = U''(C)C\left\{\beta_1\Big[\frac{\beta_2}{2}(\mu - r) + r\Big] - 1\right\} + U'(C)\left\{\beta_1\Big[\frac{\beta_2}{2}(\mu - r) + r\Big] + \nu\beta_1\right\}.$$

In addition to the infinite time horizon problem, in the same paper, Atkinson and Mokkhavesa [3] also solve other cases like two-assets time-dependent, multi-assets time-dependent, and two assets time-dependent with a single stochastic state variable.

1.13.4 Mokkhavesa and Atkinson (2002)

Mokkhavesa and Atkinson [38] extend the results of Atkinson and Al-Ali [1]. They have obtained a result which can applied to any consumption utility function C on a one risky asset framework. The resultant strategy is expressed as a function of the value function.

1.13.5 Atkinson and Mokkhavesa (2004)

Atkinson and Mokkhavesa [4] extends the results of Mokkhavesa and Atkinson [38]. They formulate the problem with more than one uncorrelated risky assets.

Chapter 2

Dynamic Asset Allocation with Transaction Costs

2.1 Introduction

In this chapter, we consider a general problem in dynamic asset allocation with transaction costs. We assume that we are able to solve the equivalent problem without transaction costs first. With the presence of small transaction costs in the new problem, we find that the optimal strategy is to hold a number of assets that is approximately the same as the optimal strategy without transaction costs and the portfolio should only be rebalanced when it is too far away from the optimal number. From this level we find a formula for the position of the free boundaries where transactions should be made in terms of the optimal amount of cash held in the no transaction costs problem. We also find that when the level of transaction costs, k, tend to zero, the band-width of the no-transaction region tends to zero and the no-transaction region converges to the no transaction costs solution. Furthermore, we find that the effect of a transaction cost, say k, in the limit $k \to 0$ to the reward function is $O(k^{2/3})$. Bellman's principle is used to establish the problem as a free boundary boundary, then we use a perturbation analysis to establish the position of the free boundary. For details regarding perturbation analysis, consult Hinch [27].

2.2 Market Model without Transaction costs

The setup of the market model is as follows. Let $S(t)$ be the spot price of a stock at time $0 \leq t \leq T$, where T is the time horizon of the investment period. We assume $S(t)$ follows a geometric Brownian motion with constant growth rate $\mu > 0$ and constant volatility $\sigma > 0$. So, the equation for $S(t)$ is

$$dS = \mu S dt + \sigma S dX$$

where X is a standard Brownian motion. Let $A(t)$ denotes the value of resources invested in stock. So,

$$dA = \mu A dt + \sigma A dX$$

when there is no money put in or withdrawn.

Let $B(t)$ be the value of resources invested in risk-free bond. The risk-free bonds are compounded continuously at the risk free rate r. So, the equation for $B(t)$ is

$$dB = rBdt,$$

when again, there is no transfer of money from the stock account.

Let $\Pi(t)$ be the value of assets invested in stocks, risk-free bonds and the total value of assets respectively,

$$\Pi(t) = A(t) + B(t). \tag{2.1}$$

Cash flows generated from the purchase or sale of stocks are immediately invested or withdrawn from the risk free bond account. Also, we use B as the controller representing the amount of cash invested in the risk-free bonds. The equation for the value of the portfolio, therefore, becomes

$$\begin{aligned} d\Pi &= (\mu A + rB)dt + \sigma A dX \\ &= \mu(\Pi - B)dt + rBdt + \sigma(\Pi - B)dX \end{aligned} \tag{2.2}$$

At time $t = 0$, an investor has an amount Π_0 of resources. The problem is to allocate investments over the given time horizon so as to maximize

$$\mathbb{E}_0\left\{ \int_0^T I(\Pi(\tilde{t}))d\tilde{t} + F(\Pi(T)) \right\},$$

where $\mathbb{E}_t$ is the conditional expectation given information up to time t, I and F are strictly increasing concave differentiable functions. This means the following must hold

$$\begin{aligned} \frac{\partial I}{\partial \Pi} &> 0, \\ \frac{\partial F}{\partial \Pi} &> 0, \\ \frac{\partial^2 I}{\partial \Pi^2} &\leq 0 \\ \frac{\partial^2 F}{\partial \Pi^2} &\leq 0. \end{aligned} \tag{2.3}$$

and one of their second partial derivatives has to be strictly less than 0. The functions I and F can represent anything from utility to the year end bonus of a trader. Of course, the dimensions of

$$\int_0^T I(\Pi(t))dt$$

and

$$F(\Pi(T))$$

have to be the same in order for the problem to make sense. So, for example if $F(\Pi)$ is utility then $I(\Pi)$ is a rate of utility.

We restate the above equation in dynamic programming form so as to apply the Bellman principle of optimality. Therefore, we define the optimal expected value function $J(\Pi, t)$ as

$$J(\Pi, t) = \max_{B(t)} \mathbb{E}_t \left\{ \int_t^T I(\Pi(\tilde{t}))d\tilde{t} + F(\Pi(T)) \right\}. \tag{2.4}$$

2.3 Formulation of Bellman's Equation in the No Transaction Costs Problem

Bellman's principle and Itô's Lemma can be used to derive the Bellman equation for $J(\Pi, t)$ in (2.4), which is

$$0 = \max_{B \in \Theta} \left\{ \frac{\partial J}{\partial t} + I + [rB + \mu(\Pi - B)] \frac{\partial J}{\partial \Pi} + \frac{\sigma^2 (\Pi - B)^2}{2} \frac{\partial^2 J}{\partial \Pi^2} \right\}. \tag{2.5}$$

At $t = T$, we have

$$J(\Pi, T) \quad = \quad F(\Pi). \tag{2.6}$$

If we differentiate the expression inside the expression to be maximized in (2.5) with respect to B, we have

$$-(\mu - r)\frac{\partial J}{\partial \Pi} - \frac{\sigma^2(\Pi - B)}{2}\frac{\partial^2 J}{\partial \Pi^2}.$$

Differentiating it twice, we have

$$\sigma^2 \frac{\partial^2 J}{\partial \Pi^2}.$$

The maximum is achieved when the (optimal) amount invested in the risky assets, B^*, is given by

$$-(\Pi - B^*)\frac{\partial^2 J}{\partial \Pi^2} \quad = \quad \frac{(\mu - r)}{\sigma^2}\frac{\partial J}{\partial \Pi}. \tag{2.7}$$

This equation is very useful later on in determining the transaction boundaries in the proportional transaction cost case.

We assume the backward parabolic partial differential equation (2.5) preserves concavity. This is a property of many parabolic partial differential equations and although we cannot prove it in general for equation (2.5) we have found that concavity is preserved in all the cases for which we have found analytic solutions. So, we assume that J is concave and so

$$\sigma^2 \frac{\partial^2 J}{\partial \Pi^2} \leq 0.$$

Since $F(T)$ and $I(t)$ are strictly increasing functions, so we can suppose J is also a strictly increasing function. [1] This means that the right hand side of equation (2.7) must be positive, and as we assume

$$\frac{\partial^2 J}{\partial \Pi^2} < 0,$$

we have $\Pi - B > 0$, which means the optimal value invested in stocks is always positive. The above equation also guarantees that the optimal value of B is unique.

[1] Suppose $\Pi_1 > \Pi_2$, then we can divide Π_1 into two pools of money: Π_2 and x where x is positive. We can consider the strategy where we invest the pool of Π_2 according to the optimal strategy when the portfolio value is only Π_2 and invest x only in the risk free bond. The value $J(\Pi_1)$ is as least as large as the value function produced by our strategy, which in turn is larger than $J(\Pi_2)$. Therefore, J is a strictly increasing function.

2.4 Solving Bellman's Equation in the No Transaction Costs Problem

In general, the above Bellman equation can only be solved numerically. In this section, however, we try to construct I and F such that analytic solutions exist.

From equation (2.7) we see that

$$B^* = \frac{(\mu - r)}{\sigma^2}\frac{\partial J}{\partial \Pi}/\frac{\partial^2 J}{\partial \Pi^2} + \Pi \tag{2.8}$$

where B^* is the optimal amount invested in risk-free bonds. Putting this back into equation (2.5), we have

$$0 = \frac{\partial J}{\partial t} + I + r\Pi\frac{\partial J}{\partial \Pi} - \frac{(\mu - r)^2}{2\sigma^2}(\frac{\partial J}{\partial \Pi})^2/\frac{\partial^2 J}{\partial \Pi^2}. \tag{2.9}$$

From equation (2.9) , we know that for any function J, as long as we set

$$I = -\{\frac{\partial J}{\partial t} + r\Pi\frac{\partial J}{\partial \Pi} - \frac{(\mu - r)^2}{2\sigma^2}(\frac{\partial J}{\partial \Pi})^2/\frac{\partial^2 J}{\partial \Pi^2}\}, \tag{2.10}$$

we can have exact solutions. Of course, we still have to make sure that the functions J and I thus defined make economic sense and satisfy (2.3).

We can study this in more details if we consider the following change of variables:

$$\begin{aligned} \tau &= T - t \\ x &= \log \Pi. \end{aligned} \tag{2.11}$$

Then we have

$$B^* = \Pi\left\{\frac{(\mu - r)J_x}{\sigma^2(J_{xx} - J_x)} + 1\right\} \tag{2.12}$$

and the Bellman equation (2.9) becomes

$$I = J_\tau - rJ_x + \frac{(\mu - r)^2 J_x^2}{2\sigma^2(J_{xx} - J_x)}. \tag{2.13}$$

We now look for travelling wave solutions of the form

$$J(\Pi, t) = f(x - \nu\tau). \tag{2.14}$$

We find that

$$\begin{aligned}\frac{\partial J}{\partial t} &= -\nu f'(x-\nu\tau)\\ \frac{\partial J}{\partial \Pi} &= \frac{1}{\Pi}f'(x-\nu\tau)\\ \frac{\partial^2 J}{\partial \Pi^2} &= \frac{1}{\Pi^2}(f''(x-\nu\tau)-f'(x-\nu\tau))\end{aligned}$$

and putting all these back into equations (2.8) and (2.10), we have

$$B^* = \Pi\left\{\frac{(\mu-r)f'}{\sigma^2(f''-f')}+1\right\} \tag{2.15}$$

and

$$I = -(\nu+r)f' + \frac{(\mu-r)^2 f'^2}{2\sigma^2(f''-f')}. \tag{2.16}$$

Now, we look at some special cases of f.

2.4.1 Long Term Growth Model

We recall the long term growth model from Section 1.11, the long term growth model is an example of the above class of solution. In the long term growth model,

$$f(x-\nu\tau) = x-\nu\tau, \qquad \nu = -\Big(r+\frac{(\mu-r)^2}{2\sigma^2}\Big). \tag{2.17}$$

Therefore,

$$J(\Pi,t) = \log\Pi + (r+\frac{(\mu-r)^2}{2\sigma^2})(T-t). \tag{2.18}$$

and so the terminal function $F(\Pi(T)) = \log\Pi$, and we find

$$B^* = \Pi\left\{1-\frac{(\mu-r)}{\sigma^2}\right\} \tag{2.19}$$

and

$$I = 0, \tag{2.20}$$

which is exactly the same as those in Section 1.11.

2.4.2 Constant Relative Risk Aversion (CRRA) Model

Here, we consider the case that

$$f(x-\nu\tau) \;=\; \exp(\gamma(x-\nu\tau)), \tag{2.21}$$

which means

$$J(\Pi,t) \;=\; e^{\bar{\nu}(T-t)}\Pi^{\gamma},$$

and the terminal function $F(\Pi(T)) = \Pi^{\gamma}$, which is equivalent to the constant relative risk aversion (CRRA) utility function; refer to Section 1.9 for details.

Therefore, from equation (2.15) and equation (2.16), we have

$$B^{*} \;=\; \Pi\left\{1+\frac{(\mu-r)}{\sigma^{2}(\gamma-1)}\right\} \tag{2.22}$$

and

$$\begin{aligned} I &= \Big(\frac{\gamma(\mu-r)^{2}}{2(\gamma-1)\sigma^{2}}-\gamma r+\bar{\nu}\Big)J \\ &= \Big(\frac{\gamma(\mu-r)^{2}}{2(\gamma-1)\sigma^{2}}-\gamma r+\bar{\nu}\Big)e^{\bar{\nu}(T-t)}\Pi^{\gamma}. \end{aligned} \tag{2.23}$$

Although these are formally solutions of equation (2.10) for any value of γ and $\bar{\nu}$, we must choose their value so that (2.3) are satisfied which gives

$$\gamma < 1$$

and

$$\bar{\nu} \geq \gamma r - \frac{\gamma(\mu-r)^{2}}{2(1-\gamma)\sigma^{2}}.$$

This means we can interpret I as the CRRA utility function of a risk averse investor.

As discussed in Section 1.9, γ represents how risk averse the investor is. The bigger the γ, the more the investor is risk-seeking. The closer γ to 1, the more the investor is close to risk-neutral. This means the investor borrows money from the bank ($B^{*} \to -\infty$) and invests the proceedings in the risky asset as the return of the risky asset is higher. When

$$1-\gamma \;=\; \frac{(\mu-r)}{\sigma^{2}}, \tag{2.24}$$

$B^* = 0$. This means the investor invests all his resources into the risky asset. When

$$1 - \gamma \quad < \quad \frac{(\mu - r)}{\sigma^2}, \tag{2.25}$$

the optimal amount of resources invested in bond, B^*, can be negative. So, the investor is borrowing money to invest in stocks. As γ becomes smaller and smaller, the investor invests less and less in stocks. Nonetheless, no matter how small γ is,

$$B^* \quad < \quad \Pi\left\{1 - \frac{(\mu - r)}{\sigma^2}\right\}. \tag{2.26}$$

This means the investor always invest in stock, no matter how risk averse he is.

2.5 Market Model with Transaction costs

Now we consider the problem with transaction costs. Let $k > 0$ represents the portion of transaction of stocks used as transaction costs. So if the investor buys a number of stocks whose "true" value is S, the investor pays $(1 + k)S$ in cash and if the investor sells the stocks, the investor obtains $(1 - k)S$ in cash.

We begin the study of the transaction costs problem by first stating the market model equations when there are transaction costs. They are

$$\begin{aligned} dS &= \mu S dt + \sigma S dX \\ dB &= rBdt - (1+k)d\mathbf{L}(t) + (1-k)d\mathbf{M}(t) \\ d\Pi &= \mu(\Pi - B)dt + rBdt + \sigma(\Pi - B)dX - kd\mathbf{L}(t) - kd\mathbf{M}(t) \end{aligned} \tag{2.27}$$

where $\mathbf{L}(t)$ and $\mathbf{M}(t)$ represent the cumulative purchase and sale of assets A during $[0, t]$, and which we use as the controls. In the transaction costs problem, B is only used to denote the value of assets invested in risk-free bonds and it is no longer used as a control. Using $\mathbf{L}(t)$ and $\mathbf{M}(t)$ rather than B as controls makes it easier to formulate the optimization problem as a free boundary problem.

2.6 Formulation of Bellman's Equation under Transaction costs

Now we define the optimal expected value function $\tilde{J}(\Pi, B, t)$ as

$$\tilde{J}(\Pi, B, t) \;=\; \max_{\mathbf{L},\mathbf{M}} \mathbb{E}_t\Big\{ \int_t^T I(\Pi(\tilde{t}))d\tilde{t} + F(\Pi(T)) \Big\}. \tag{2.28}$$

The functions I and F here are assumed not to depend on k, and so they are exactly the same as the I and F in equation (2.4). The function $\tilde{J}$ here is different to the J in Section 2.3 as now we have introduced transaction costs, and as a result $\tilde{J}$ depends on B.

The corresponding Bellman equation is

$$\begin{aligned} \max_{\mathbf{l},\mathbf{m}} \Big\{ I + \frac{\partial \tilde{J}}{\partial t} + (rB - (1+k)\mathbf{l} + (1-k)\mathbf{m})\frac{\partial \tilde{J}}{\partial B} \\ +(rB + \mu(\Pi - B) - k\mathbf{l} - k\mathbf{m})\frac{\partial \tilde{J}}{\partial \Pi} + \frac{\sigma^2(\Pi - B)^2}{2}\frac{\partial^2 \tilde{J}}{\partial \Pi^2} \Big\} \;=\; 0 \end{aligned} \tag{2.29}$$

where

$$\mathbf{L}(t) = \int_0^t \mathbf{l}(\tilde{t})d\tilde{t}$$

and

$$\mathbf{M}(t) = \int_0^t \mathbf{m}(\tilde{t})d\tilde{t}.$$

The optimal trading policy, therefore, can be deduced from the following three cases:

1.

$$-(1+k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} < 0 \tag{2.30}$$

and

$$(1-k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} \geq 0, \tag{2.31}$$

where the maximum is achieved by choosing $\mathbf{l} = 0$ and $\mathbf{m} = \infty$[2], which means selling at the maximum rate;

2.
$$-(1+k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} \geq 0 \tag{2.32}$$
and
$$(1-k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} < 0, \tag{2.33}$$
where the maximum is achieved by choosing $\mathbf{l} = \infty$ and $\mathbf{m} = 0$, which means buying at the maximum rate;

3.
$$-(1+k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} < 0 \tag{2.34}$$
and
$$(1-k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} < 0, \tag{2.35}$$
where the maximum is achieved by choosing $\mathbf{l} = 0$ and $\mathbf{m} = 0$, which means neither buying nor selling.

Note that it is impossible to have
$$-(1+k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} > 0 \tag{2.36}$$
and
$$(1-k)\frac{\partial \tilde{J}}{\partial B} - k\frac{\partial \tilde{J}}{\partial \Pi} > 0, \tag{2.37}$$
as we assume $\tilde{J}$ is an increasing function of Π.

The optimal trading strategy, therefore, given t, Π and B, involves three possible regions: a sales region, a purchase region and a no-transaction region. The no transaction region is the region in the middle, see Figure (2.1) for an illustration.

[2]To be more rigorous, we should firstly make the restriction $\mathbf{m} \leq K$ for some constant K. Then we let $K \to \infty$.

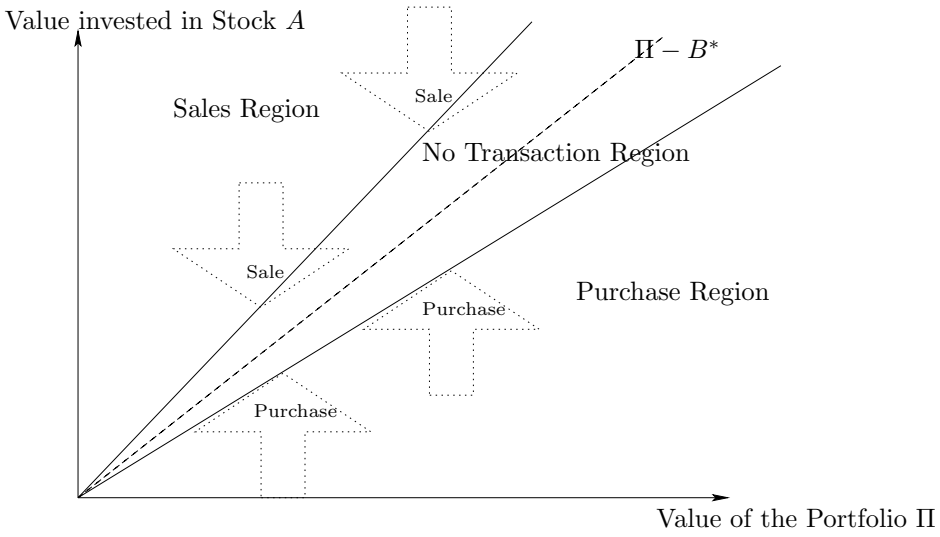


Figure 2.1: This diagram illustrates the three different regions: Purchase Region, Sales Region and no-transaction Region. When the portfolio is in the Sales Region, stock is sold until the portfolio is at the boundary between the no-transaction region and the Sales region. This is in contrast with the case when there is no transaction cost, where stocks are sold until it is at the dotted line. It is similar when the portfolio is at the purchase region. When the portfolio is in the no-transaction, no transaction is made.

Inside the no transaction region, $\mathbf{l}$ and $\mathbf{m}$ are identically zero, which means the optimal strategy is not to make any transaction and the portfolio is allowed to drift freely under the influence of the stock process only. So the value function must satisfy

$$I + \frac{\partial \tilde{J}}{\partial t} + rB(\frac{\partial \tilde{J}}{\partial B} + \frac{\partial \tilde{J}}{\partial \Pi}) + \mu(\Pi - B)\frac{\partial \tilde{J}}{\partial \Pi} + \frac{\sigma^2(\Pi - B)^2}{2}\frac{\partial^2 \tilde{J}}{\partial \Pi^2} = 0. \qquad (2.38)$$

Also, the no-transaction region must exist, if

$$\frac{\partial \tilde{J}}{\partial B} \quad \text{and} \quad \frac{\partial \tilde{J}}{\partial \Pi}$$

are continuous. We proceed to prove this by contradiction, so we assume it doesn't. Assume the sales region exists, but the no-transaction region does not. Since

$$\frac{\partial \tilde{J}}{\partial B} \quad \text{and} \quad \frac{\partial \tilde{J}}{\partial \Pi}$$

are continuous, the purchase region also does not exist. The whole space is sales region. The optimal strategy, therefore, is to sell stocks as quickly as possible without end. For every sales made, however, the portfolio value Π is reduced because of the transaction costs. Eventually, Π becomes $-\infty$. Any strategy which keeps the portfolio finite is a better strategy than this 'without no transaction region' strategy as $\tilde{J}$ is an increasing function. We can argue similarly if we assume only the purchase region exists. So, we can conclude that the no-transaction region must exist.

Both of the sales region and the purchase region must exist as well. If they do not, it is impossible for us to find boundaries beyond which it is optimal to purchase/sell. It turns out that, as we can see later in this chapter, we are able to find such boundaries by asymptotic approximation. We can, thus, conclude their existence.

In the sales region (and at the boundary between sales region and no transaction region), by considering those terms that are dependent on $\mathbf{m}$, we have the equation

$$k\frac{\partial \tilde{J}}{\partial \Pi} = (1-k)\frac{\partial \tilde{J}}{\partial B}. \tag{2.39}$$

The above equation can also be understood by the 'value matching argument'. Suppose the point $(\tilde{\Pi}, \tilde{B}, \tilde{t})$ is at the sales region. When a very small quantity of assets h is sold, the risk-free bonds increase by an amount of $h(1-k)$, while the whole portfolio value is reduced by kh. Therefore, by considering the value of the value function $\tilde{J}$ must be the same after the sales, we have

$$\begin{aligned} \tilde{J}(\tilde{\Pi}+kh, \tilde{B}, \tilde{t}) &= \tilde{J}(\tilde{\Pi}, \tilde{B}+(1-k)h, \tilde{t}) \\ k\frac{\tilde{J}(\tilde{\Pi}+kh, \tilde{B}, \tilde{t}) - \tilde{J}(\tilde{\Pi}, \tilde{B}, \tilde{t})}{kh} &= (1-k)\frac{\tilde{J}(\tilde{\Pi}, \tilde{B}+(1-k)h, \tilde{t}) - \tilde{J}(\tilde{\Pi}, \tilde{B}, \tilde{t})}{(1-k)h}. \end{aligned}$$

As $h \to 0$, the above equation becomes equation (2.39). [3]

From the above arguments, we know that when the portfolio is in the sales region, the optimal strategy is to sell stocks until the portfolio is at the no-transaction region boundary, and thus bring the portfolio back into the no-transaction region.

In the purchase region (and at the boundary between sales region and no transaction region), similarly, we have

$$-k\frac{\partial \tilde{J}}{\partial \Pi} = (1+k)\frac{\partial \tilde{J}}{\partial B}. \tag{2.40}$$

[3] If we consider this problem rigorously and so we make the restriction $\mathbf{m} \leq K$ for some constant K, then the value matching argument as well as equation (2.39) only holds on the boundary between the sales region and the no-transaction region.

At the boundaries between the no-transaction region and sales region, we also have the smooth pasting condition because of optimality. For further details of smooth pasting, please refer to Whally and Wilmott[49] and Morton and Pliska [39]. The smooth pasting condition is

$$(1-k)\frac{\partial^2 \tilde{J}}{\partial B^2} = k\frac{\partial^2 \tilde{J}}{\partial B \partial \Pi}, \tag{2.41}$$

which comes from differentiating equation (2.39) with respect to B and assuming the existence and continuity of the second derivative across the transaction boundary. This can be understood from the following diagrams:

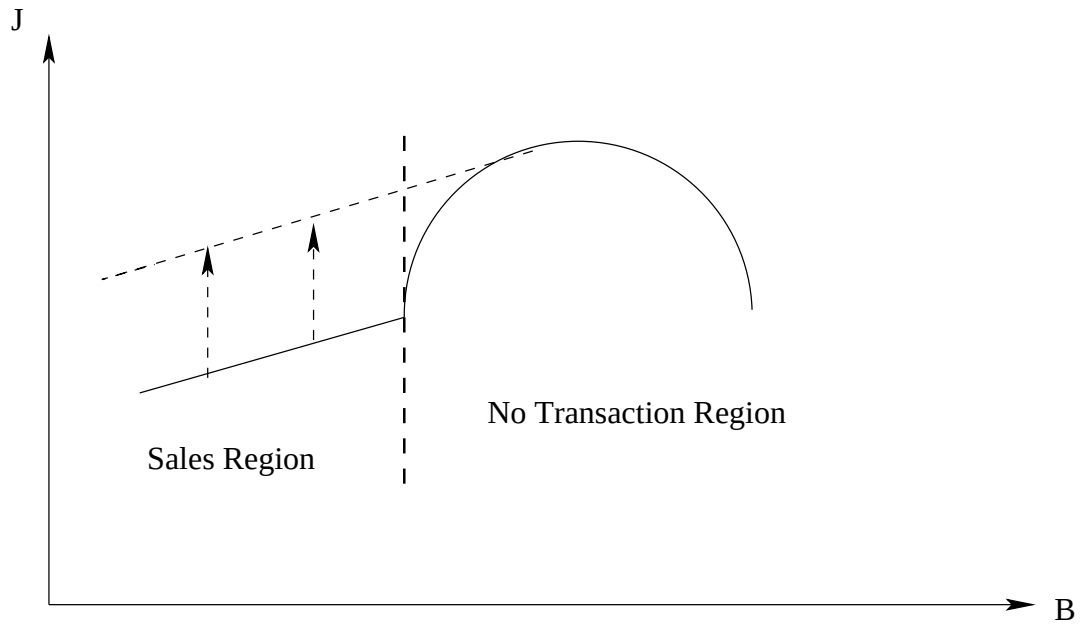


Figure 2.2: At the transaction boundary, if $\frac{\partial J}{\partial B}$ is bigger at the side of sales region, then it is not optimal. A more optimal solution can be found by having a bigger sales region.

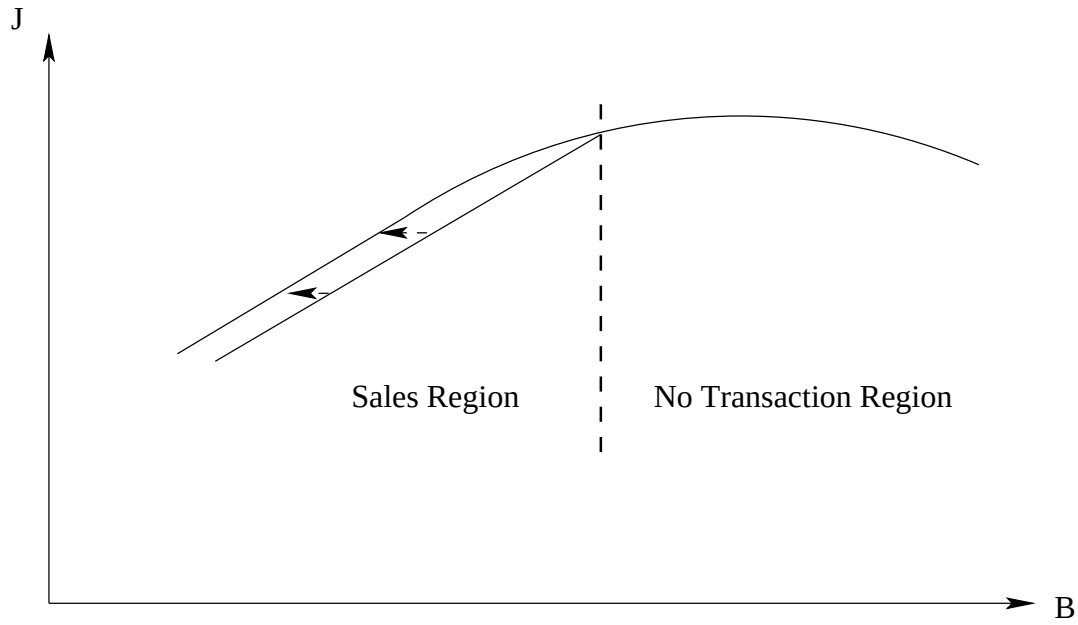


Figure 2.3: At the transaction boundary, if $\frac{\partial J}{\partial B}$ is smaller at the side of sales region, then it is not optimal. A more optimal solution can be found by having a smaller sales region.

From these two diagrams, we can see at the point of transaction boundary, $\frac{\partial \tilde{J}}{\partial B}$ exists and should be continuous across the boundary, or otherwise it is not optimal. Therefore, using the same notation as in the value matching argument, we have

$$\frac{\partial \tilde{J}(\tilde{\Pi} + kh, \tilde{B}, \tilde{t})}{\partial B} = \frac{\partial \tilde{J}(\tilde{\Pi}, \tilde{B} + (1-k)h, \tilde{t})}{\partial B}$$

So, we have

$$\begin{aligned} & k\Big(\frac{\partial \tilde{J}(\tilde{\Pi} + kh, \tilde{B}, \tilde{t})}{\partial B} - \frac{\partial \tilde{J}(\tilde{\Pi}, \tilde{B}, \tilde{t})}{\partial B}\Big)/(kh) \\ = \; & (1-k)\Big(\frac{\partial \tilde{J}(\tilde{\Pi}, \tilde{B} + (1-k)h, \tilde{t})}{\partial B} - \frac{\partial \tilde{J}(\tilde{\Pi}, \tilde{B}, \tilde{t})}{\partial B}\Big)/((1-k)h). \end{aligned}$$

As $h \to 0$, it becomes equation (2.41).

Similarly, at the boundaries between the no-transaction region and purchase region, have the following smooth pasting condition:

$$(1+k)\frac{\partial^2 \tilde{J}}{\partial B^2} = -k\frac{\partial^2 \tilde{J}}{\partial B \partial \Pi}. \tag{2.42}$$

Also, when $t = T$,

$$\tilde{J}(\Pi, B, T) = F(\Pi(T)). \tag{2.43}$$

2.6.1 Change of Variables

We translate the B coordinate according to

$$B = \mathfrak{B}(\Pi, t) + k^{1/3}\beta, \tag{2.44}$$

where $\mathfrak{B}$ is the value of risk-free bonds we have when the level of transaction costs tends to zero, $k \to 0$. As we can see later, this is actually the same as the optimal value of risk free bond held in the no transaction costs problem. We also assume that β is $O(1)$. The choice of the order $k^{1/3}$ in the expansion is inevitable and we explain this choice in Section 2.9.

We set $H(\Pi, \beta, t) = \tilde{J}(\Pi, B, t)$.

We have

$$\begin{aligned}
\frac{\partial \tilde{J}}{\partial B} &= k^{-1/3}\frac{\partial H}{\partial \beta}, \\
\frac{\partial \tilde{J}}{\partial \Pi} &= \frac{\partial H}{\partial \Pi} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial \Pi}, \\
\frac{\partial \tilde{J}}{\partial t} &= \frac{\partial H}{\partial t} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial t}, \\
\frac{\partial^2 \tilde{J}}{\partial \Pi^2} &= \frac{\partial^2 H}{\partial \Pi^2} - k^{-1/3}(2\frac{\partial^2 H}{\partial \beta \partial \Pi}\frac{\partial \mathfrak{B}}{\partial \Pi} + \frac{\partial H}{\partial \beta}\frac{\partial^2 \mathfrak{B}}{\partial \Pi^2}) \\
&\quad + k^{-2/3}\frac{\partial^2 H}{\partial \beta^2}(\frac{\partial \mathfrak{B}}{\partial \Pi})^2.
\end{aligned} \tag{2.45}$$

Also, we let

$$
\begin{aligned}
H(\Pi,\beta,t) &= H_0(\Pi,\beta,t) + k^{1/3}H_1(\Pi,\beta,t) \\
&\quad + k^{2/3}H_2(\Pi,\beta,t) + kH_3(\Pi,\beta,t) \\
&\quad + k^{4/3}H_4(\Pi,\beta,t) + \cdots .
\end{aligned} \tag{2.46}
$$

and assume that $H_0, H_1, H_2, H_3, H_4, \cdots$ as well as all of their derivatives are all of $O(1)$. H_0 is the value function when transaction costs level $k \to 0$. Later, we find that its value is the same as the value function J in equation (2.4).

2.6.2 Sales and Purchase Regions

3.2.5 After the change of coordinates, the sales region equation (2.39), becomes

$$
k\left(\frac{\partial H}{\partial \Pi} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial \Pi}\right) = (1-k)k^{-1/3}\frac{\partial H}{\partial \beta}, \tag{2.47}
$$

and equation (2.40) , the equation at purchase region, becomes

$$
k\left(\frac{\partial H}{\partial \Pi} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial \Pi}\right) = -(1+k)k^{-1/3}\frac{\partial H}{\partial \beta}. \tag{2.48}
$$

The smoothing pasting condition for the sales boundary becomes

$$
k^{2/3}\left(\frac{\partial^2 H}{\partial \Pi \partial \beta} - k^{-1/3}\frac{\partial^2 H}{\partial \beta^2}\frac{\partial \mathfrak{B}}{\partial \Pi}\right) = (1-k)k^{-2/3}\frac{\partial^2 H}{\partial \beta^2}, \tag{2.49}
$$

and the smoothing pasting condition for the purchase boundary becomes

$$
k^{2/3}\left(\frac{\partial^2 H}{\partial \Pi \partial \beta} - k^{-1/3}\frac{\partial^2 H}{\partial \beta^2}\frac{\partial \mathfrak{B}}{\partial \Pi}\right) = -(1+k)k^{-2/3}\frac{\partial^2 H}{\partial \beta^2}. \tag{2.50}
$$

When we substitute equation (2.46) into equation (2.47), the equation at the sales region, and collect terms of the same order in k, we have [4]

$$
\frac{\partial H_0}{\partial \beta} = 0, \tag{2.51}
$$

[4]We assume that all these derivatives are of $O(1)$. In other words, there is no boundary layers exist. This can be justified by the fact that the system is self consistent. Also, k doesn't appear to kill off any high derivatives as $k \to 0$ so singular perturbations are unlikely.

$$\frac{\partial H_1}{\partial \beta} = 0, \tag{2.52}$$

$$\frac{\partial H_2}{\partial \beta} = 0, \tag{2.53}$$

$$\frac{\partial H_3}{\partial \beta} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_0}{\partial \beta} = 0, \tag{2.54}$$

$$\frac{\partial H_4}{\partial \beta} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_1}{\partial \beta} - \frac{\partial H_0}{\partial \Pi} = 0. \tag{2.55}$$

Similarly, if we substitute equation (2.46) into equation (2.48), the equation at the purchase region, and collect terms of the same order in k, we have

$$\frac{\partial H_0}{\partial \beta} = 0, \tag{2.56}$$

$$\frac{\partial H_1}{\partial \beta} = 0, \tag{2.57}$$

$$\frac{\partial H_2}{\partial \beta} = 0, \tag{2.58}$$

$$\frac{\partial H_3}{\partial \beta} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_0}{\partial \beta} = 0, \tag{2.59}$$

$$\frac{\partial H_4}{\partial \beta} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_1}{\partial \beta} + \frac{\partial H_0}{\partial \Pi} = 0. \tag{2.60}$$

If we substitute equation (2.46) into equation (2.49), the smooth pasting equation for the sales boundary, and collect terms of the same order in k, we have

$$\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{2.61}$$

$$\frac{\partial^2 H_1}{\partial \beta^2} = 0, \tag{2.62}$$

$$\frac{\partial^2 H_2}{\partial \beta^2} = 0, \tag{2.63}$$

$$\frac{\partial^2 H_3}{\partial \beta^2} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{2.64}$$

$$\frac{\partial^2 H_4}{\partial \beta^2} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_1}{\partial \beta^2} - \frac{\partial^2 H_0}{\partial \Pi \partial \beta} = 0. \tag{2.65}$$

Similarly, the smooth pasting equation for the purchase boundary becomes

$$\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{2.66}$$

$$\frac{\partial^2 H_1}{\partial \beta^2} = 0, \tag{2.67}$$

$$\frac{\partial^2 H_2}{\partial \beta^2} = 0, \tag{2.68}$$

$$\frac{\partial^2 H_3}{\partial \beta^2} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{2.69}$$

$$\frac{\partial^2 H_4}{\partial \beta^2} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_1}{\partial \beta^2} + \frac{\partial^2 H_0}{\partial \Pi \partial \beta} = 0. \tag{2.70}$$

Finally, equation (3.28), the final condition, becomes

$$H_0(\Pi, \beta, T) = F(\Pi), \tag{2.71}$$

$$H_1(\Pi, \beta, T) = 0, \tag{2.72}$$

$$H_2(\Pi, \beta, T) = 0, \tag{2.73}$$

$$H_3(\Pi, \beta, T) = 0, \tag{2.74}$$

$$H_4(\Pi, \beta, T) = 0. \tag{2.75}$$

2.6.3 No-Transaction Region

After the change of coordinates, the equation in the no transaction region, (2.38), becomes

$$\begin{aligned} I + \frac{\partial H}{\partial t} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial t} \\ +r(\mathfrak{B} + k^{1/3}\beta)(k^{-1/3}\frac{\partial H}{\partial \beta} + \frac{\partial H}{\partial \Pi} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial \Pi}) \\ +\mu(\Pi - (\mathfrak{B} + k^{1/3}\beta))(\frac{\partial H}{\partial \Pi} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial \Pi}) \\ +\frac{\sigma^2(\Pi - (\mathfrak{B} + k^{1/3}\beta))^2}{2}\Big[\frac{\partial^2 H}{\partial \Pi^2} - 2k^{-1/3}\frac{\partial H}{\partial \beta \partial \Pi}\frac{\partial \mathfrak{B}}{\partial \Pi} \\ +k^{-2/3}\frac{\partial^2 H}{\partial \beta^2}(\frac{\partial \mathfrak{B}}{\partial \Pi})^2 - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial^2 \mathfrak{B}}{\partial \Pi^2}\Big] = 0. \end{aligned} \tag{2.76}$$

When we substitute equation (2.46) into equation (2.76), the equation in the no transaction region, and collect terms of the same order in k, we have the following results.

2.6.4 The $O(k^{-2/3})$ Equation

We find that

$$\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi - \mathfrak{B})^2\frac{\partial^2 H_0}{\partial \beta^2} = 0. \tag{2.77}$$

If we combine this with equation (2.51), equation (2.56), equation (2.61) and equation (2.66), we can conclude that H_0 is independent of β. In other words, we have $H_0 = H_0(\Pi, t)$.

2.6.5 The $O(k^{-1/3})$ Equation

If we collect terms of the order $O(k^{-1/3})$ and use the result that H_0 is independent of β, we have

$$\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi - \mathfrak{B})^2\frac{\partial^2 H_1}{\partial \beta^2} = 0. \tag{2.78}$$

If we consider equation (2.78) along with equation (2.52), equation (2.57), equation (2.62) and equation (2.67), we can also conclude that H_1 is independent of β. Therefore, we have $H_1 = H_1(\Pi, t)$.

2.6.6 The $O(1)$ Equation

If we collect those terms of order $O(1)$ and using the result that H_0 and H_1 are independent of β, we have

$$\begin{aligned}\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi - \mathfrak{B})^2\frac{\partial^2 H_2}{\partial \beta^2} &= -\Big\{I + \frac{\partial H_0}{\partial t} + r\mathfrak{B}\frac{\partial H_0}{\partial \Pi} \\ &\quad +\mu(\Pi - \mathfrak{B})\frac{\partial H_0}{\partial \Pi} + \frac{\sigma^2(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_0}{\partial \Pi^2}\Big\}.\end{aligned} \tag{2.79}$$

If we combine equation (2.79) with equations (2.53), (2.58), (2.63), and (2.68), we can conclude that H_2 is independent of β and so the left hand side of equation (2.79) is equal to 0. In other words, we have

$$0 = I + \frac{\partial H_0}{\partial t} + r\mathfrak{B}\frac{\partial H_0}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_0}{\partial \Pi} + \frac{\sigma^2(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_0}{\partial \Pi^2}.$$

Now, we recall that we define $\mathfrak{B}$ as the optimal value invested in the risk-free bonds when the transaction costs k tends to 0. We now show that actually this is equal to the optimal value invested in risk-free bonds in the no transaction costs problem.

We begin our proof by first showing that $J \geq H_0$. Recall from equation (2.5) that

$$\begin{aligned} 0 &= \max_{B \in \Theta} \left\{ I + \frac{\partial J}{\partial t} + rB\frac{\partial J}{\partial \Pi} \right. \\ &\quad \left. + \mu(\Pi - B)\frac{\partial J}{\partial \Pi} + \frac{\sigma^2(\Pi - B)^2}{2}\frac{\partial^2 J}{\partial \Pi^2} \right\} \end{aligned}$$

with

$$J(\Pi, T) = F(\Pi(T)).$$

If we compare the partial differential equation in equation (2.5) with the right hand side of equation (2.79) and the final condition equation (2.71), we find that they are essentially the same and in equation (2.5), B is chosen so as to maximize J. Therefore, we have $J \geq H_0$.[5]

Now, we want to show that $H_0 \geq J$. We recall from equation (2.29) that $\mathbf{L}$ and $\mathbf{M}$ are chosen so as to maximize $\tilde{J}$, and thus H. Also, since H_0 is of the lowest order in the expansion of H, maximizing H means maximizing H_0. So $\mathbf{L}$ and $\mathbf{M}$ are chosen, and so is $\mathfrak{B}$, to maximize H_0. Hence, $H_0 = J$. Since the optimal solution of equation (2.5) is unique, so $\mathfrak{B}$ must be equal to the solution of B in equation (2.5). In other words, $\mathfrak{B}$ is the optimal value invested in risk-free securities in the no transaction costs problem.

2.6.7 The $O(k^{1/3})$ Equation

If we collect those terms of $O(k^{1/3})$ and use the result that H_0, H_1 and H_2 are independent of β, we have

[5]Recall that we assume a solution exists for equation (2.5).

$$\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_3}{\partial \beta^2} = \left\{\sigma^2(\Pi-\mathfrak{B})\frac{\partial^2 H_0}{\partial \Pi^2}+(\mu-r)\frac{\partial H_0}{\partial \Pi}\right\}\beta - \left\{\frac{\partial H_1}{\partial t}+r\mathfrak{B}\frac{\partial H_1}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_1}{\partial \Pi} + \frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_1}{\partial \Pi^2}\right\}. \quad (2.80)$$

If we combine equation (2.80) with equation (2.54), equations (2.59), (2.64) and (2.69), we can conclude that H_3 is also independent of β.

Therefore, we have

$$\frac{\partial H_1}{\partial t}+r\mathfrak{B}\frac{\partial H_1}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_1}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_1}{\partial \Pi^2} = 0. \quad (2.81)$$

From the final condition equation (2.72), we know that $H_1(T)=0$. We can, thus, conclude that actually $H_1(\Pi,\beta,t)=0$.

Also, we have

$$\frac{\partial H_0}{\partial \Pi}/\frac{\partial^2 H_0}{\partial \Pi^2} = -\frac{\sigma^2(\Pi-\mathfrak{B})}{(\mu-r)}. \quad (2.82)$$

This equation is actually the same as equation (2.7), the optimal B^* equation. As we can see later, it is very useful in deducing the transaction boundary.

2.6.8 The $O(k^{2/3})$ Equation

Collecting terms of the order $O(k^{2/3})$ and using the result that H_0, H_2 are independent of β and $H_1=0$, we have

$$\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_4}{\partial \beta^2} = -\frac{1}{2}\sigma^2\frac{\partial^2 H_0}{\partial \Pi^2}\beta^2 - \left\{\frac{\partial H_2}{\partial t}+r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \mu(\Pi-\mathfrak{B})\frac{\partial H_2}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}(\frac{\partial^2 H_2}{\partial \Pi^2})\right\}. \quad (2.83)$$

Therefore,

$$H_4 = H_4^4\beta^4 + H_4^2\beta^2 + H_4^1\beta + H_4^0 \tag{2.84}$$

where

$$H_4^4 = -\frac{1}{12}\frac{\partial^2 H_0}{\partial \Pi^2} / \Big[(\Pi - \mathfrak{B})\frac{\partial \mathfrak{B}}{\partial \Pi}\Big]^2, \tag{2.85}$$

$$\begin{aligned} H_4^2 = & -\Big\{ \frac{\partial H_2}{\partial t} + r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \\ & \mu(\Pi - \mathfrak{B})\frac{\partial H_2}{\partial \Pi} + \frac{\sigma^2(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2}\Big\} / \Big\{ \sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi - \mathfrak{B})^2 \Big\}, \end{aligned} \tag{2.86}$$

H_4^1 and H_4^0 are both functions of Π and t.

Let β_+ and β_- denote the purchase boundary and the sales boundary respectively.

From the smooth pasting equation (2.65) and equation (2.70), we know that at β_+ and β_-,

$$\frac{\partial^2 H_4}{\partial \beta^2} = 0, \tag{2.87}$$

as H_0 and H_1 are independent of β. Therefore, β_+ and β_- are the roots of the equation

$$12H_4^4\beta^2 + 2H_4^2 = 0, \tag{2.88}$$

and thus we have

$$\beta_\pm = \pm\sqrt{-\frac{H_4^2}{6H_4^4}}. \tag{2.89}$$

Now, using equation (2.60), we have

$$4H_4^4\beta_+^3 + 2H_4^2\beta_+ + H_4^1 = -\frac{\partial H_0}{\partial \Pi}. \tag{2.90}$$

We use the value of H_4^2 established in equation (2.88), we have

$$-8H_4^4\beta_+^3 + H_4^1 = -\frac{\partial H_0}{\partial \Pi}. \tag{2.91}$$

Similarly, we can use equation (2.55) to find an equation for β_-. So, we have

$$-8H_4^4\beta_-^3 + H_4^1 = \frac{\partial H_0}{\partial \Pi}. \tag{2.92}$$

If we apply the relationship $\beta_+ = -\beta_-$ to the above equation, we have

$$8H_4^4\beta_+^3 + H_4^1 = \frac{\partial H_0}{\partial \Pi}. \tag{2.93}$$

By considering equation (2.91) and equation (2.93), we have

$$H_4^1 = 0. \tag{2.94}$$

So we have

$$\beta_+ = \frac{1}{2}\left\{\frac{\partial H_0}{\partial \Pi}/H_4^4\right\}^{1/3}. \tag{2.95}$$

If we substitute equation (2.85) and equation (2.82) into the above equation, we have

$$\beta_\pm = \pm(\Pi - \mathfrak{B})\left\{\frac{3\sigma^2}{2(\mu - r)}\left[\frac{\partial \mathfrak{B}}{\partial \Pi}\right]^2\right\}^{1/3}. \tag{2.96}$$

We can use equation (2.96) to find the value of H_4^2. In fact, H_4^2 is

$$H_4^2 = 1/2\frac{\partial^2 H_0}{\partial \Pi^2}\left\{\frac{3\sigma^2}{2(\mu - r)}/\frac{\partial \mathfrak{B}}{\partial \Pi}\right\}^{2/3}. \tag{2.97}$$

So we have

$$\begin{aligned} &\frac{1}{2}\sigma^2\left[\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi - \mathfrak{B})\right]^2\frac{\partial^2 H_0}{\partial \Pi^2}\left\{\frac{3\sigma^2}{2(\mu - r)}/\frac{\partial \mathfrak{B}}{\partial \Pi}\right\}^{2/3} \\ = &-\left\{\frac{\partial H_2}{\partial t} + r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_2}{\partial \Pi} + \frac{\sigma^2(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2}\right\} \end{aligned} \tag{2.98}$$

which can be used to find the value of H_2. Of course,H_2, is the leading order adjustment term for the value function.

2.7 Example

The above formulae can be easily applied to many no transaction costs problem solutions and thus we can obtain the solution for the corresponding small transaction costs problem. Here, we use those exact solutions we obtained for the no-transaction cost problem for illustration.

We recall that in obtaining exact solutions in the no transaction cost problem, we make the following change of variables:

$$\begin{aligned} \tau &= T - t \\ x &= \log \Pi. \end{aligned} \tag{2.99}$$

We have

$$B^* = \Pi\left\{\frac{(\mu - r)J_x}{\sigma^2(J_{xx} - J_x)} + 1\right\}. \tag{2.100}$$

Therefore,

$$\beta_{\pm} = \mp\frac{(\mu - r)J_x}{\sigma^2(J_{xx} - J_x)}\left\{\frac{3\sigma^2}{2(\mu - r)}\left[\frac{(\mu - r)J_x}{\sigma^2(J_{xx} - J_x)} + 1\right]^2\right\}^{1/3}. \tag{2.101}$$

For travelling wave solutions

$$\beta_{\pm} = \mp\frac{(\mu - r)f'}{\sigma^2(f'' - f')}\left\{\frac{3\sigma^2}{2(\mu - r)}\left[\frac{(\mu - r)f'}{\sigma^2(f'' - f')} + 1\right]^2\right\}^{1/3}. \tag{2.102}$$

Now, we study the long term growth model and the CRRA model in more detail.

2.7.1 Long Term Growth Model

We recall from the long term growth model from Section 1.11. We can apply the result from this chapter to that model.

From equation (1.41) and equation (1.42) , we have

$$H_0(\Pi, t) = \log \Pi + (r + \frac{(\mu - r)^2}{2\sigma^2})(T - t) \tag{2.103}$$

and

$$\mathfrak{B} = \Pi(1 - \frac{(\mu - r)}{\sigma^2}). \tag{2.104}$$

So, we have

$$\beta_{\pm} \;=\; \pm\Pi\left\{\frac{3(\sigma^2-\mu+r)^2(\mu-r)^2}{2\sigma^8}\right\}^{1/3},$$

and the leading order adjustment term for the value function, H_2 is given by the equation

$$\begin{aligned}&\frac{1}{2}\left\{\frac{3(\sigma^2-\mu+r)^2(\mu-r)^2}{2\sigma^5}\right\}^{2/3}\\ =\;&\left\{\frac{\partial H_2}{\partial t}+r\mathfrak{B}\frac{\partial H_2}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_2}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2}\right\}\end{aligned}$$

and the boundary condition

$$H_2(\Pi,T) \;=\; 0.$$

Solving gives us the solution

$$H_2(\Pi,t) \;=\; -\frac{T-t}{2}\left\{\frac{3(\sigma^2-\mu+r)^2(\mu-r)^2}{2\sigma^5}\right\}^{2/3}. \tag{2.105}$$

This result is consistent with the result obtained by Davis and Norman [18].

Interpretation of the above results are given in Section 2.8.

2.7.2 Constant Relative Risk Aversion (CRRA) Model

We recall from the CRRA model that

$$\mathfrak{B} \;=\; \Pi\left\{1+\frac{(\mu-r)}{\sigma^2(\gamma-1)}\right\}. \tag{2.106}$$

and

$$H_0 \;=\; e^{\bar{\nu}(T-t)}\Pi^{\gamma}.$$

So, we have

$$\beta_{\pm} \;=\; \mp\Pi\left\{3\frac{(\mu-r+\sigma^2(\gamma-1))^2(\mu-r)^2}{2\sigma^8(\gamma-1)^5}\right\}^{1/3} \tag{2.107}$$

and the leading order adjustment term for the value function, H_2, is given by the equation

$$\begin{aligned}
&\frac{\gamma}{2}\Pi^\gamma \exp\left(\bar{\nu}(T-t)\right)\left\{\frac{9(\sigma^2(\gamma-1)+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^7\sigma^{10}}\right\}^{1/3} \\
= \quad & -\left\{\frac{\partial H_2}{\partial t} + rB\frac{\partial H_2}{\partial \Pi} + \mu(\Pi - B)\frac{\partial H_2}{\partial \Pi} + \frac{\sigma^2(\Pi-B)^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2}\right\}
\end{aligned} \tag{2.108}$$

and the boundary conditions

$$\begin{aligned}
H_2(\Pi, T) &= 0, \\
H_2(0, t) &= 0.
\end{aligned} \tag{2.109}$$

The above equations for H_2 can be solved by expressing H_2 as a series in $(T-t)$, i.e.,

$$H_2(\Pi, t) = \Pi^\gamma \sum_{n=1}^{\infty} H_2^n (T-t)^n \tag{2.110}$$

where $H_2^1, H_2^2, H_2^3, \ldots$ are all constants independent of Π and t, and their values can be obtained by expressing the left hand side of equation (2.108) as

$$\frac{\gamma}{2}\Pi^\gamma \left\{\frac{9(\sigma^2(\gamma-1)+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^7\sigma^{10}}\right\}^{1/3} \sum_{n=0}^{\infty} \frac{(\bar{\nu}(T-t))^n}{n!}$$

and considering the terms $H_2^1, H_2^2, H_2^3 \cdots$ one by one.

So, we have

$$\begin{aligned}
H_2^1 &= \frac{\gamma}{2}\left\{\frac{9(\sigma^2(\gamma-1)+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^7\sigma^{10}}\right\}^{1/3} \\
H_2^{n+1} &= \frac{1}{n+1}\left\{\frac{\nu^n\gamma}{2n}\left(\frac{9(\sigma^2(\gamma-1)+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^7\sigma^{10}}\right)^{1/3}\right. \\
&\quad \left. -\left(\frac{\gamma(\mu-r)^2}{2(\gamma-1)\sigma^2} - \gamma r\right)H_2^n\right\}.
\end{aligned} \tag{2.111}$$

In order to show the above series converges for some regions, we firstly need to show that the sequences $H_2^n, n = 1, 2, 3, \ldots$ is bounded above.

We notice that there exist M_0 such that

$$M_0 > \Big|\frac{\nu^n\gamma}{2n}\bigg(\frac{9(\sigma^2(\gamma-1)^2+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^{17}\sigma^{10}}\bigg)^{1/3}\Big|$$

for all n if we assume $|\nu| \leq 1$. Also, for sufficiently large n^*, we have that

$$\frac{1}{1+n}\Big(\frac{\gamma(\mu-r)^2}{2(\gamma-1)\sigma^2}-\gamma r\Big) < 1/2$$

for all $n > n^*$. So, for all such n, we have

$$\begin{aligned}
H_2^{n+1} &= \frac{1}{1+n}\bigg\{\frac{\nu^n\gamma}{2n}\bigg(\frac{9(\sigma^2(\gamma-1)^2+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^7\sigma^{10}}\bigg)^{1/3} \\
&\quad -\Big(\frac{\gamma(\mu-r)^2}{2(\gamma-1)\sigma^2}-\gamma r\Big)H_2^n\bigg\} \\
&\leq \frac{1}{1+n}\bigg\{\Big|\frac{\nu^n\gamma}{2n}\bigg(\frac{9(\sigma^2(\gamma-1)^2+\mu-r)^4(\mu-r)^4}{4(\gamma-1)^{17}\sigma^{10}}\bigg)^{1/3}\Big| \\
&\quad +\Big|\frac{\gamma(\mu-r)^2}{2(\gamma-1)\sigma^2}-\gamma r\Big|H_2^n\bigg\} \\
&< \frac{1}{2}\Big(M_0+H_2^n\Big) \\
&\leq \max(M_0, H_2^n).
\end{aligned}$$

By a simple induction argument, we know that the whole sequence is bounded by some constant M_1. So, whenever $|T-t| < 1$, the series must converge. [6]

2.8 Financial Interpretations

Our asymptotic analysis gives us some insights on the effect of transaction costs on the portfolio optimization problem. We firstly summarize our findings about portfolio optimization in general, then we illustrate those interpretations that only apply to the Long Term Growth Model and the CRRA model.

1. Transaction costs do not shift the optimal strategy to risk-free bonds or the stocks. In fact, the no transaction-cost optimal strategy is the midpoint of the no-transaction region.

[6] Actually the series converge for all t.

2. The width of the no-transaction region is directly proportional to $k^{1/3}$, and so the larger the transaction costs, the larger the no-transaction region.

3. The width of the no-transaction region depends on the function I and F, through the functions
$$\Pi - \mathfrak{B} \qquad \text{and} \qquad \frac{\partial \mathfrak{B}}{\partial \Pi}.$$

4. We can break down β_+ (or β_-) in equation (2.96) into three terms (Ξ, Υ, and Γ).
$$\begin{aligned} \beta_+ &= \underbrace{(\Pi - \mathfrak{B})}_{\Xi} \underbrace{\left(\frac{\partial \mathfrak{B}}{\partial \Pi}\right)^{2/3}}_{\Upsilon} \underbrace{\left(\frac{3\sigma^2}{2(\mu - r)}\right)^{1/3}}_{\Gamma}, \\ \beta_- &= -\underbrace{(\Pi - \mathfrak{B})}_{\Xi} \underbrace{\left(\frac{\partial \mathfrak{B}}{\partial \Pi}\right)^{2/3}}_{\Upsilon} \underbrace{\left(\frac{3\sigma^2}{2(\mu - r)}\right)^{1/3}}_{\Gamma}. \end{aligned} \tag{2.112}$$

If we assume the optimal strategy in the no transaction cost problem is to invest a fixed ratio of the portfolio into bond, i.e.,
$$\mathfrak{B} = B^* = \Pi b$$
for some constant b, then the term $\Xi = \Pi(1 - b)$ is the optimal value invested in stocks in the no transaction costs problem. The term Υ is $b^{2/3}$, and the term Γ is a constant which is inversely proportional to the cube root of the Sharpe Ratio. Therefore, equation (2.112) becomes
$$\begin{aligned} \beta_+ &= \Pi(1 - b)b^{2/3}\left(\frac{3\sigma^2}{2(\mu - r)}\right)^{1/3} \\ \beta_- &= -\Pi(1 - b)b^{2/3}\left(\frac{3\sigma^2}{2(\mu - r)}\right)^{1/3} \end{aligned} \tag{2.113}$$
and so the no transaction region width starts from 0 when $b = 0$ and increases with b and reaches maximum when
$$b = \frac{2}{5},$$
and afterwards it decreases with b and reaches 0 again when $b = 1$. This can be understood that $b = 0$ means investing all the resources into stock, and so

even with transaction cost, there is no need to rebalance. Similarly, if $b = 1$, the no transaction region width is 0 as this means investing all the resources into bonds and so there is no need to change. When b is close to 0 or 1, it means most of the portfolio is either invested in bond or stock. We conjecture that even with the movement of the stock prices, the ratio between stocks and the value of the portfolio does not change a lot, and so not a lot of rebalancing is needed.[7] So, it is possible to afford a smaller no transaction region. When b is not close to either end, the portfolio's composition changes rapidly and so a bigger no transaction region is needed.

5. The impact of the risk premium, $\mu - r$, or actually the Sharpe Ratio,

$$\frac{\mu - r}{\sigma^2}$$

to the no-transaction region, is unclear. If b is independent of the Sharpe Ratio, increasing the risk premium or Sharpe Ratio narrows the no-transaction region. This can be understood intuitively. The higher the Sharpe Ratio, the less risk one needs to take on. So it is better, even with the presence of transaction cost, to trade closer to the no transaction cost optimal strategy. However, as illustrated from the long term growth model and the CRRA model, the risk premium and Sharpe Ratio widen the no-transaction cost region. This is because the Sharpe Ratio appears in b and so it overwhelms the term Γ.

6. Similarly, the volatility, σ, usually appears in b. If we hold b constant and assumes it is independent of σ, we find that the higher σ, the wider the no transaction region. It can be understood that that the higher the volatility, the more expensive it is to keep a narrow no transaction region. This can also be illustrated in equation (2.98), the equation for the leading order adjustment term H_2, in where the term of the left hand side increases with σ, if we keep b a constant and let

$$\frac{\partial^2 H_0}{\partial \Pi^2}$$

[7]For example, if the portfolio is worth 100, and 90 is invested in stocks. If the price of the stock goes up 10 percent. Then $1 - b = 99/109 \approx 0.908$. If instead of investing 90 in stocks, 50 is invested instead. If the price of the stock goes up 10 percent, then $1 - b = 55/105 \approx 0.524$.

independent of σ. This in turn increases the magnitude of H_2. In the long term growth model and CRRA model, however, σ, does appear in the terms

$$\Pi - \mathfrak{B} \qquad \text{and} \qquad \frac{\partial \mathfrak{B}}{\partial \Pi},$$

and so σ instead narrows the no transaction region.

7. Transaction costs have an effect of order 2/3 on the value function.

Constantinides [15] studied the effect of introducing transaction costs into Merton's consumption model by considering the various values of the model parameters. Our study, of course, does not incorporate the possibility of consumption. We compare our findings with his and we summarize them below.

1. He found that "transaction costs broaden the region of no transactions", which is consistent with ours.

2. He found that both volatility of the stocks and the transaction costs "shift the region of no transactions toward the risk-free bonds". This is inconsistent with ours. The reason of this inconsistencies, we conjecture, is because in his model, resources used for consumptions are taken from risk-free bonds and so the optimal strategy may have a bias for risk-free bonds.

3. He found that the width of the no-transaction region is insensitive to the variance of the stocks. As shown in our study, whether the no-transaction region is insensitive to the variance of the stocks may depend on what kind of objective functions the investor is maximizing.

Now, we move from the general portfolio optimization and focus on the Long Term Growth Model and the CRRA Model.

2.8.1 Long Term Growth Model

We plot the graph of the no-transaction region with σ, μ, r and t. The results are in Figures 2.4 to 2.7.

We can see from Figures 2.4, 2.5 and 2.6 that the width of the no-transaction region increases with the Sharpe Ratio. We can verify this relationship by differentiating β_+

(or β_-) with the Sharpe Ratio. The width increases with the Sharpe Ratio when it is between 0 and $1/\sqrt{2}$.

As we explain earlier, this is because Sharpe Ratio appears in both

$$\Pi - \mathfrak{B} \qquad \text{and} \qquad \frac{\partial \mathfrak{B}}{\partial \Pi}.$$

We then examine how these variables affect the value function $k^{2/3}H_2$. Firstly, by inspection of the equation for H_2, it is independent of Π, but is directly proportion to time to expiry, see Figure 2.9. Also, an increase of Sharpe Ratio increases its magnitude. This can be confirmed by Figure 2.8.

Again, this is because Sharpe Ratio appears in both

$$\Pi - \mathfrak{B} \qquad \text{and} \qquad \frac{\partial \mathfrak{B}}{\partial \Pi}.$$

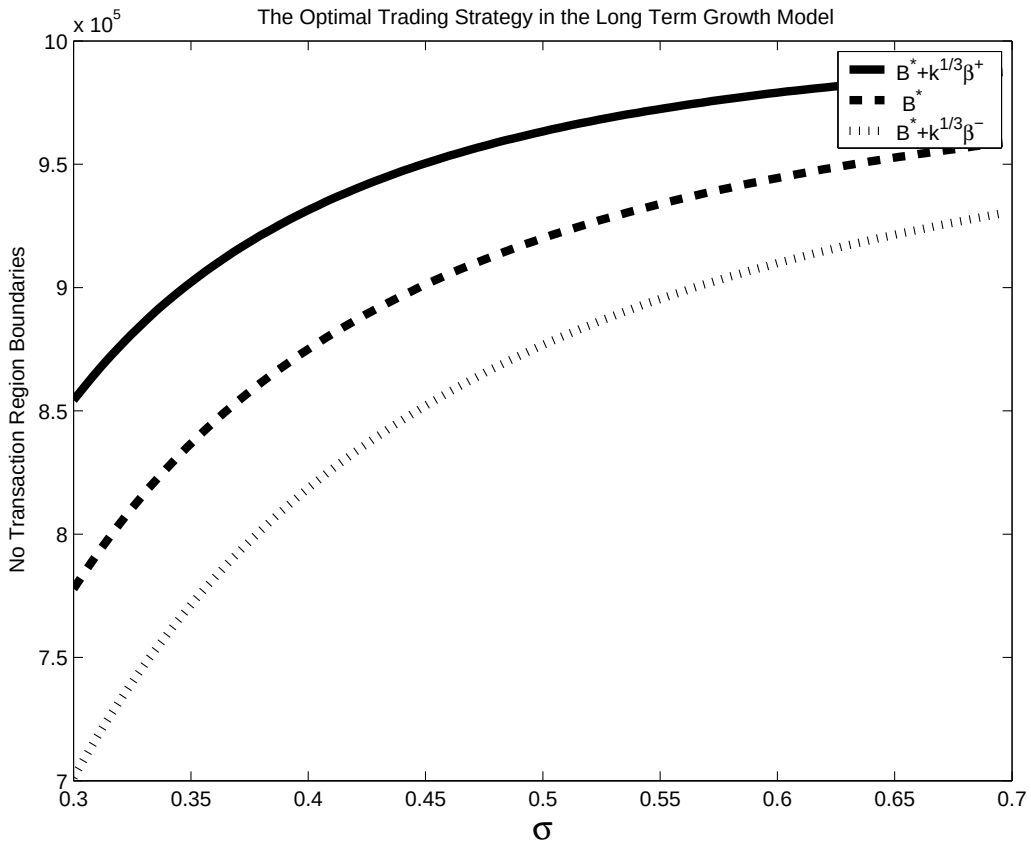


Figure 2.4: The boundaries of the no transaction region as a function of σ in the Long Term Growth Model where $\mu = 0.07, r = 0.05, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. From the graph, we can see that B^* increases with σ.

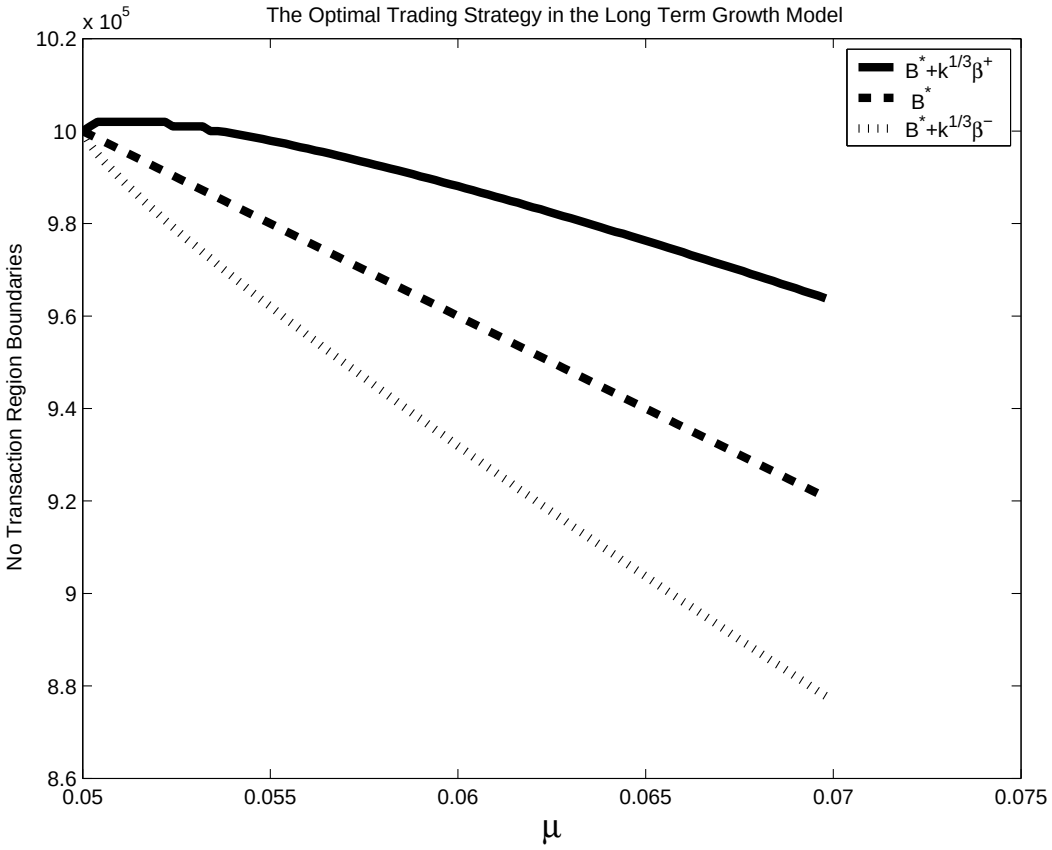


Figure 2.5: The boundaries of the no transaction region as a function of μ in the Long Term Growth Model where $r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. Since μ is directly proportional to the Sharpe Ratio. So, this graph can also seen as an illustration of the relationship between the no transaction region boundaries and the Sharpe Ratio. From this graph, we can see that the no transaction region widens with the increase of μ (or Sharpe Ratio). Also, we find that B^* decreases with μ (or Sharpe Ratio). When $\mu = r$, the Sharpe Ratio is 0 and so we see that the no transaction region width becomes 0 and $B^* = \Pi$.

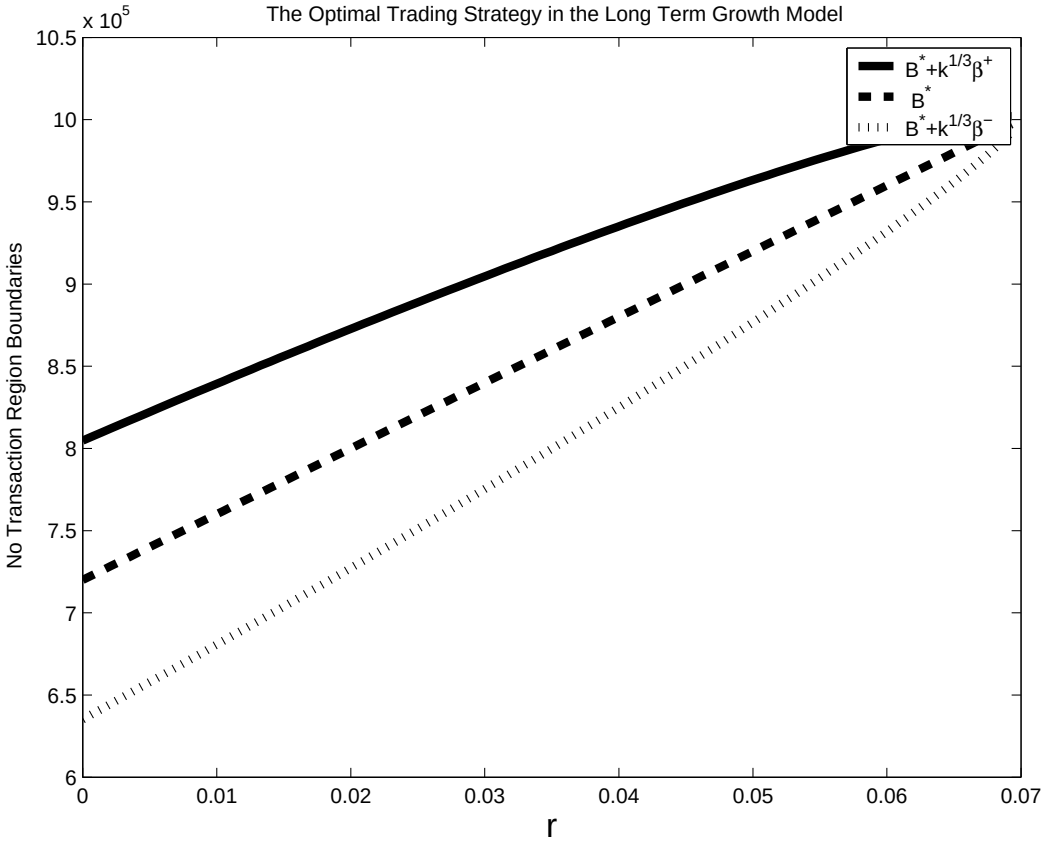


Figure 2.6: The boundaries of the no transaction region in the Long Term Growth Model as a function of r where $\mu = 0.07, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. This graph is equal to a reflection of Figure 2.5. This is because the equations for β^+ and β^- are in terms of $\mu - r$. An increase in r is equivalent to a corresponding decrease in μ.

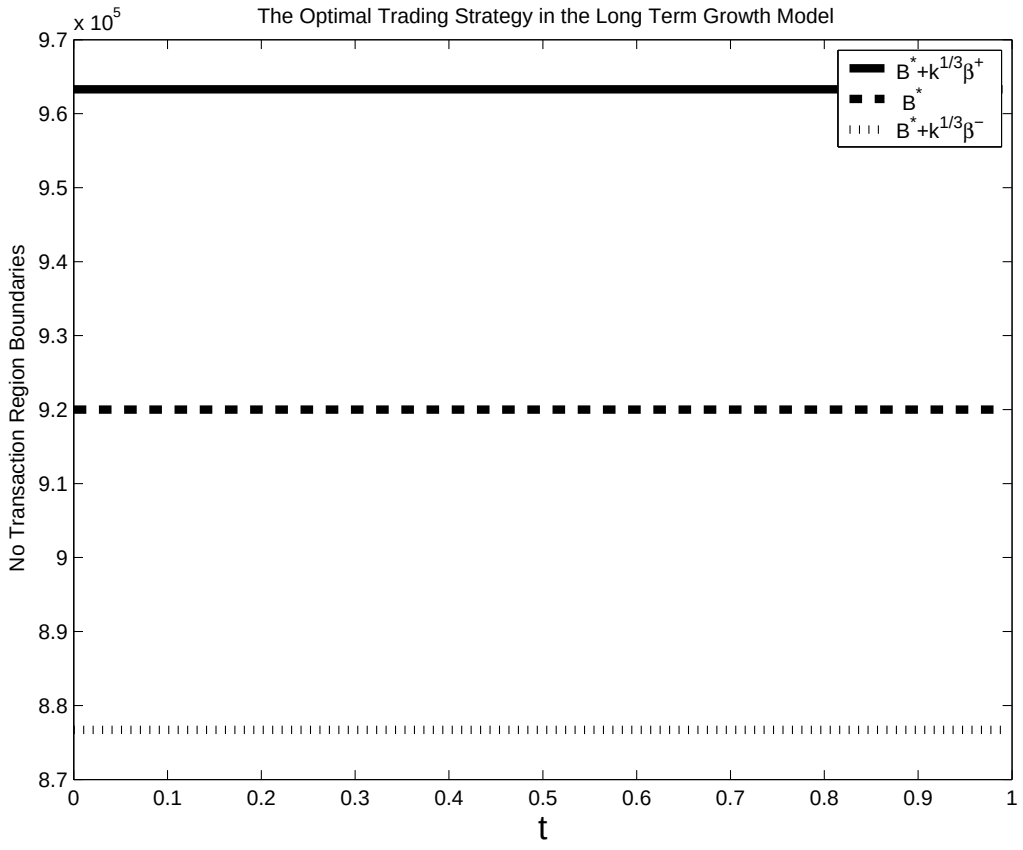


Figure 2.7: The boundaries of the no transaction region in the Long Term Growth Model as functions of t where $\mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, k = 0.01$ and $\Pi = 1000000$. This graph illustrates that just the passage of time without does not change the no transaction region.

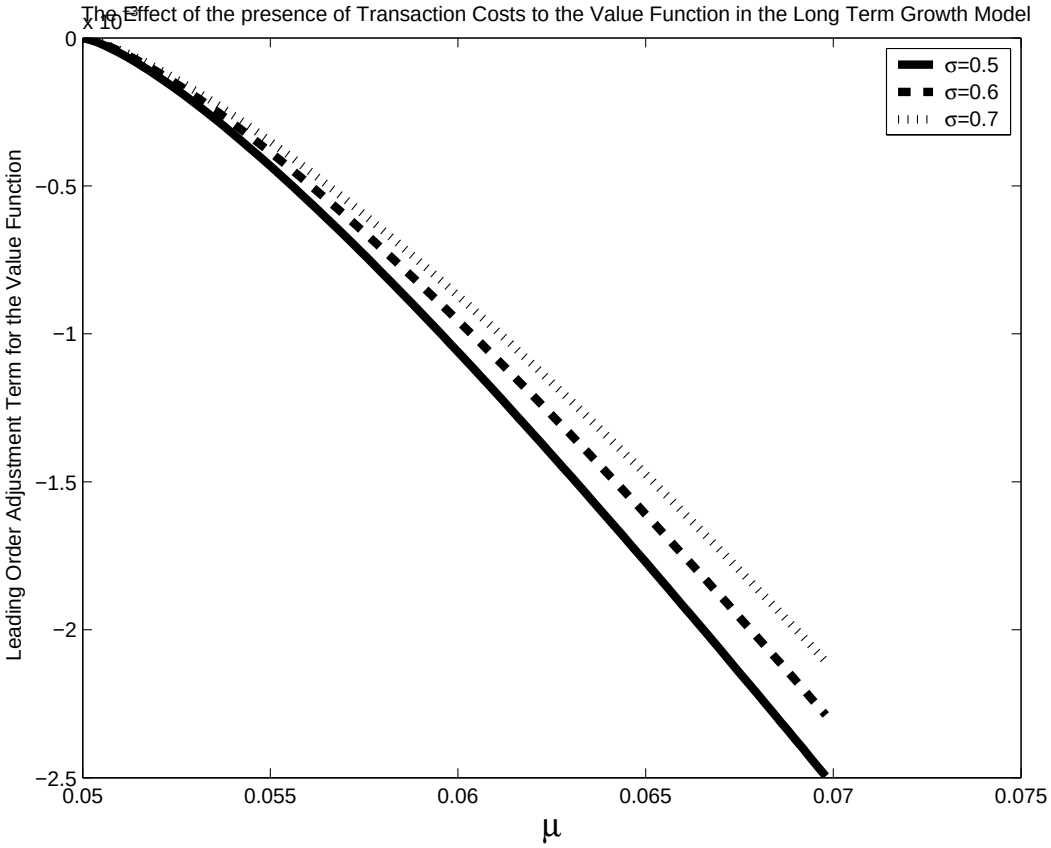


Figure 2.8: The relationship between the adjustment term $k^{2/3}H_2$ and μ (or Sharpe Ratio) for various value of σ for where $r = 0.05, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. We find that the magnitude of the adjustment term increases with μ and decreases with σ. In other words, the magnitude of the adjustment term increases with the Sharpe Ratio.

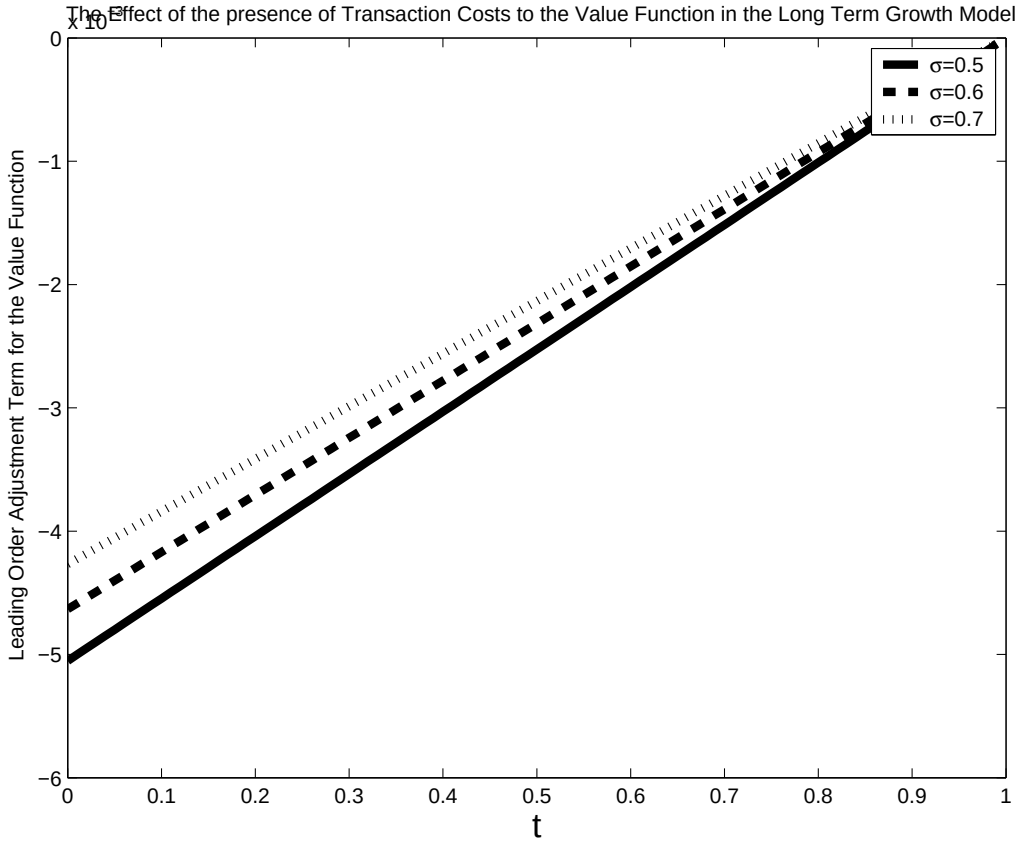


Figure 2.9: The relationship between the adjustment term $k^{2/3}H_2$ and t for various value of σ for where $\mu = 0.07, r = 0.05, S = 100, T = 1, k = 0.01$ and $\Pi = 1000000$. This graphs shows that the magnitude of the adjustment term decreases linearly with time t. When $t = 1$ (expiry), the adjustment term is 0.

2.8.2 CRRA Model

We plot the graph of the no-transaction region as functions of σ, μ, r, t and γ. The results are Figures 2.10 to 2.14.

The relationship between the no-transaction region with σ,μ, r, t are similar to the relationship in the Long Term Growth Model. The difference between the CRRA model and the Long Term Growth Model is the addition of the variables γ and ν.

As we can see from the formulae, ν does not appear in the equations for β_+ and β_-. So, ν does not affect the no transaction region at least not at any order included in our asymptotic.

Regarding γ, from Figure 2.14, we can see that it has a huge effect on $\mathfrak{B}$ as well as β_+ and β_-, and thus the no transaction region. When γ becomes bigger, we find that the optimal amount invested in bonds in the no transaction cost problem, B^* decreases rapidly. When

$$\gamma \;=\; 1 - \frac{\mu - r}{\sigma^2}, \tag{2.114}$$

which means $\gamma = 0.92$ in Figure 2.14, we find that $B^* = 0$. Beyond that, when $\gamma > 0.92$, B^* becomes negative and this means the investor borrows money to invest in stocks. This can be explained intuitively. When γ is closer to 1, the investor is closer to risk neutral, and thus the investor invest more in stocks, as they provide a better expected return, and thus invests less in bonds. Regarding the width of the no transaction region, it increases with γ initially, then it decreases. Finally, when $\gamma > 0.92$, it increases rapidly. This can be explained by our conjectured "b effect" in point 4 of this section.

Next, we investigate the relationship of the leading order adjustment term $k^{2/3}H_2$ with μ, r, t and ν. The results are in Figures 2.15 to 2.18.

The relationship between the leading order adjustment term $k^{2/3}H_2$ and t is very interesting, see Figure 2.17. We look at the function H_2 as in equation (2.110) and find that its relationship with t is very complicated and so it is very difficult to find a clear relationship. Yet, the graph seems to suggest that the relationship may well be a linear one.

The analysis of the graphs for H_2 relative to μ and $r$are equally interesting. Firstly, we can see that γ in general increases the magnitude of H_2. We conjecture that it is because it increases the value of H_0, as it is proportional to Π^γ. Secondly, we find

that the relationship between H_2 and μ, r are no longer linear as in the Long Term Growth Model. H_2 increases with *Sharpe Ratio* initially, and then decreases, and then increases again. We attempt to explain why the shape is like this. When $r = \mu$ (or Sharpe Ratio is equal to 0), all of the resources are invested in bonds. When $1 - \gamma$ is equal to the Sharpe Ratio, all the investments are in stocks. At these two points, the transaction cost trading strategy and the no transaction cost strategy are the same, and so $H_2 = 0$. When the Sharpe Ratio is away from these two points, the transaction cost trading strategy and the no transaction cost strategy differ more, and so the magnitude of H_2 increases correspondingly.

Interestingly, an increase of magnitude in H_2 corresponds to to an increase with the width of the no transaction region. This can be verified from the figures we have here. We conjecture this is because the wider the no transaction region, the transaction cost trading strategy is further away from the no transaction cost optimal trading strategy.

From Figure 2.18, we can see that the effect of ν on H_2 is very small compared to γ.

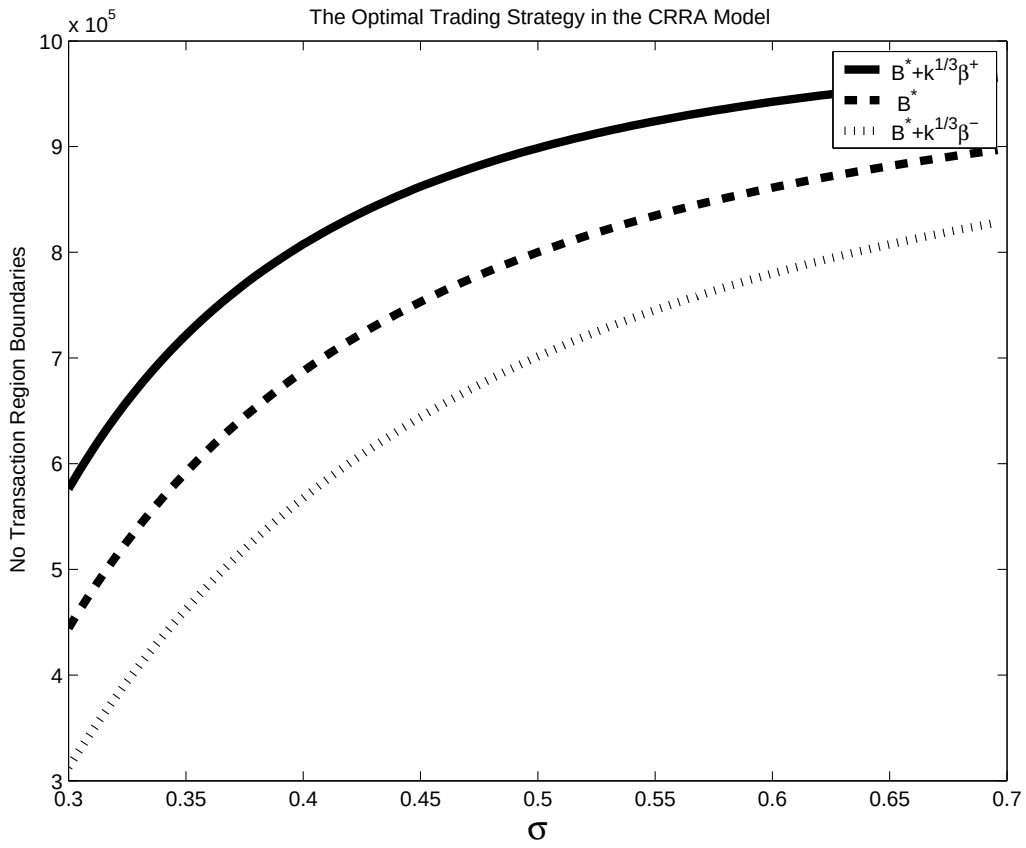


Figure 2.10: The boundaries of the no transaction region as functions of σ in the CRRA Model where $\gamma = 0.6, \nu = 0.07, \mu = 0.07, r = 0.05, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. Similar to the Long Term Growth Model, we can see that B^* increases with σ.

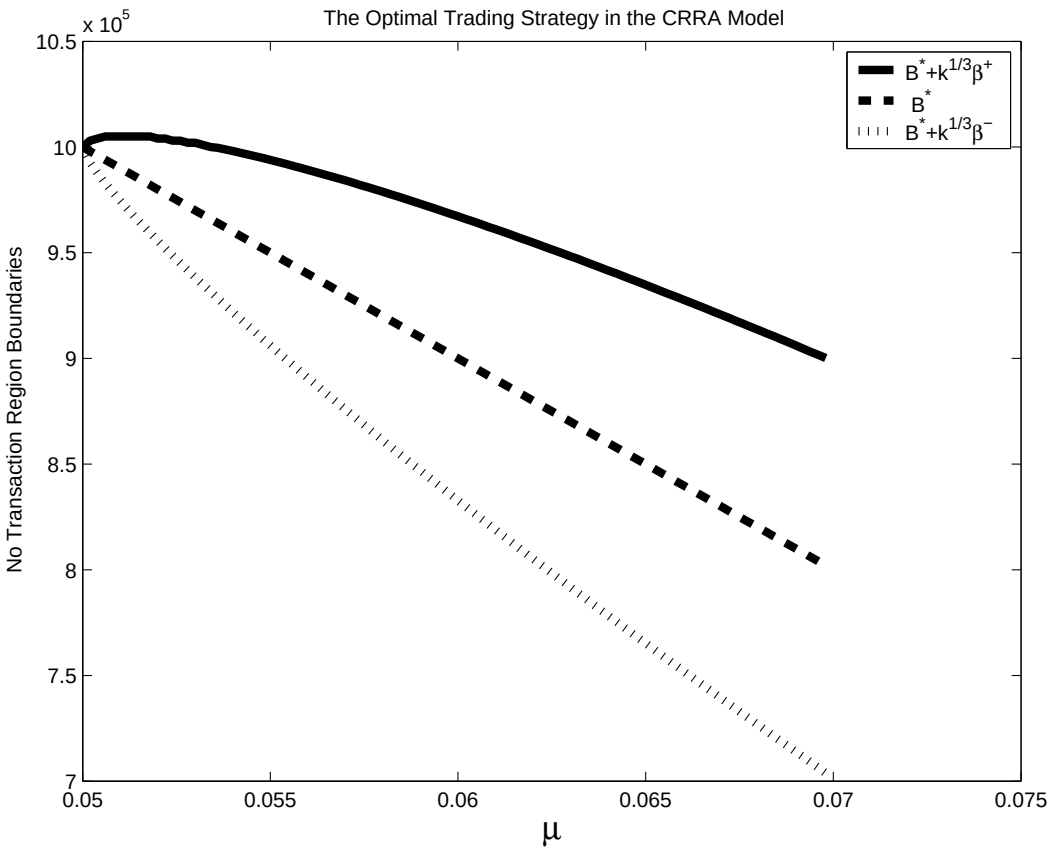


Figure 2.11: The boundaries of the no transaction region as functions of μ in the CRRA Model where $\gamma = 0.6, \nu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. Again, this is similar to the Long Term Growth Model. Since μ is directly proportional to the Sharpe Ratio. So, this graph can also seen as an illustration of the relationship between the no transaction region boundaries and the Sharpe Ratio. From this graph, we can see that the no transaction region widens with the increase of μ (or Sharpe Ratio). Also, we find that B^* decreases with μ (or Sharpe Ratio). When $\mu = r$, the Sharpe Ratio is 0 and so we see that the no transaction region width becomes 0 and $B^* = \Pi$.

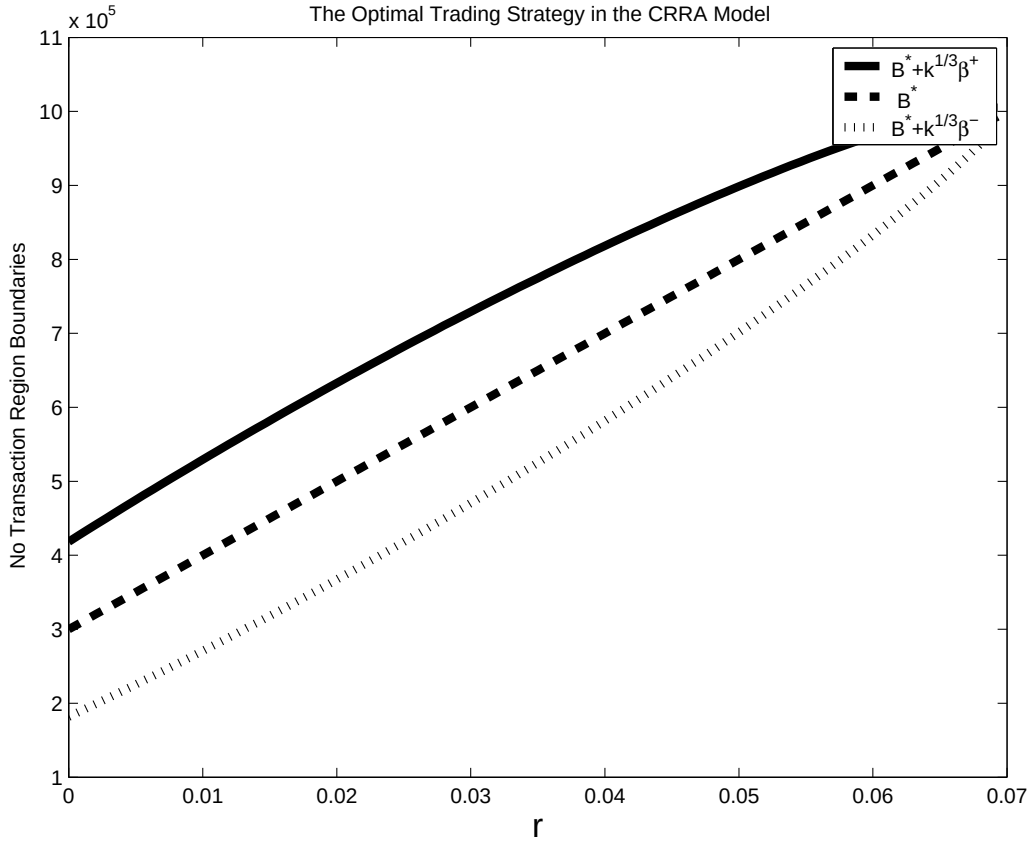


Figure 2.12: The boundaries of the no transaction region as functions of r in the CRRA Model where $\gamma = 0.6, \nu = 0.07, \mu = 0.07, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. Similar to the Long Term Growth Model, this graph is equal to a reflection of Figure 2.11. This is because the equations for β^+ and β^- are in terms of $\mu - r$. An increase in r is equivalent to a corresponding decrease in μ.

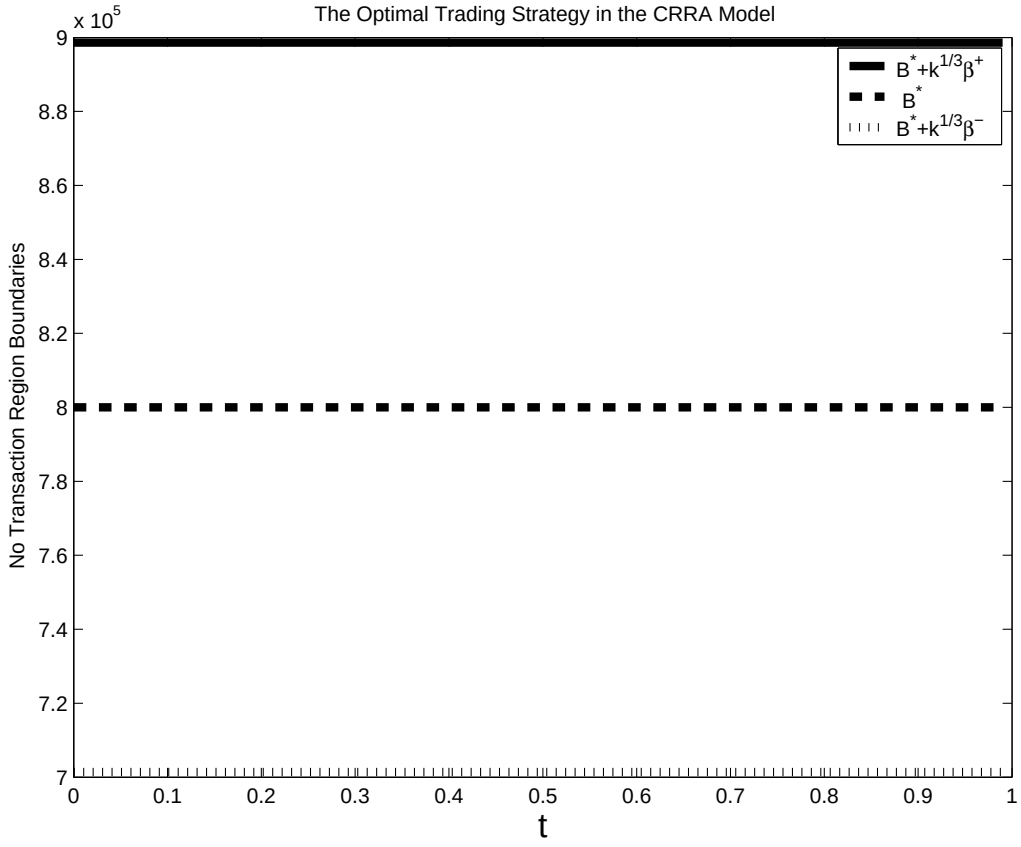


Figure 2.13: The boundaries of the no transaction region in the CRRA Model where $\gamma = 0.6, \nu = 0.07, \mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. Exactly the same as the Long Term Growth Model, this graph illustrates that just the passage of time without does not change the no transaction region.

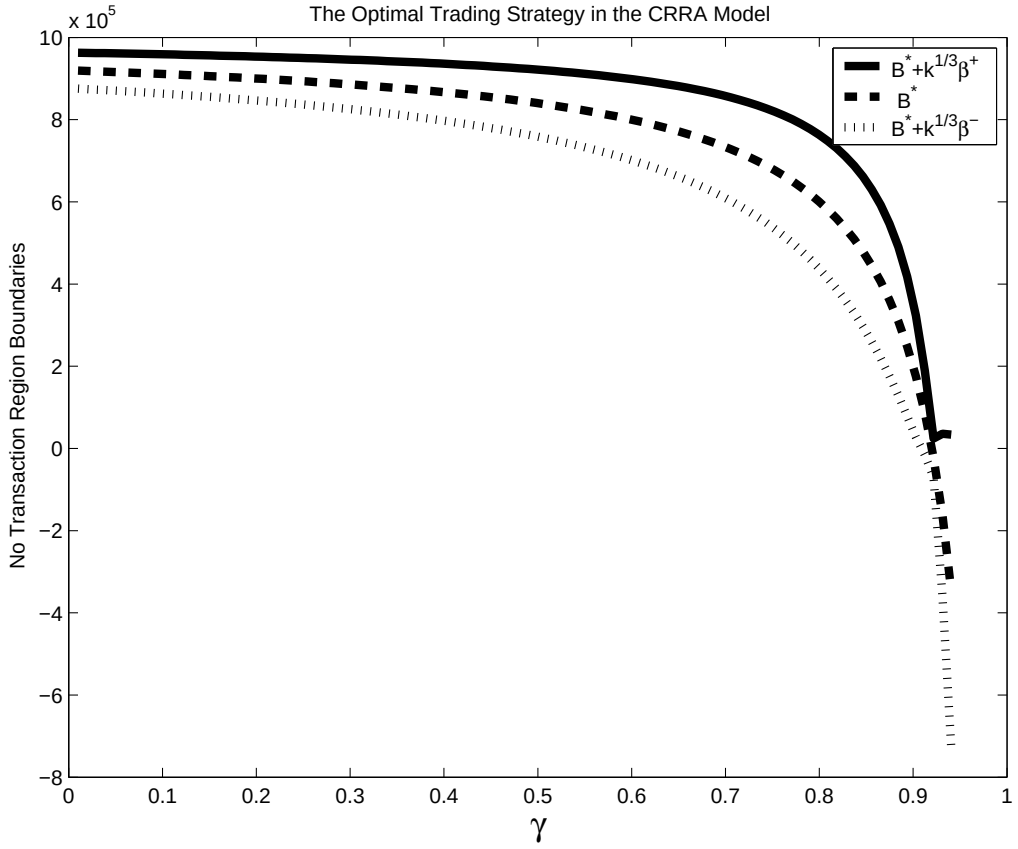


Figure 2.14: The boundaries of the no transaction region as functions of γ in the CRRA Model where $\gamma = 0.6, \nu = 0.07, \mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. When $\gamma = 0.92$, $B^* = 0$ and the width of the no transaction region is 0. When $\gamma > 0.92$, $B^* < 0$ and the no transaction increases again.

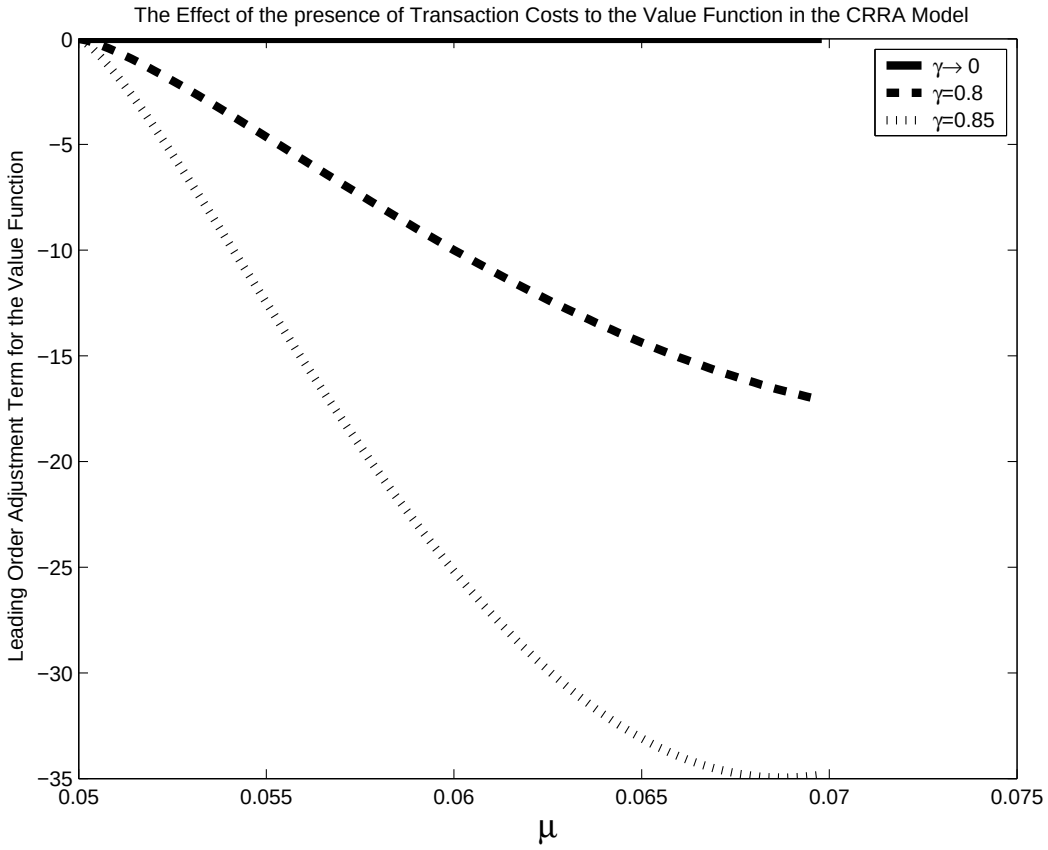


Figure 2.15: The relationship between the adjustment term $k^{2/3}H_2$ and μ in the CRRA Model for various values of γ where $\nu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. This graph shows that the effect of γ to the adjustment term is huge compared to μ (or Sharpe Ratio).

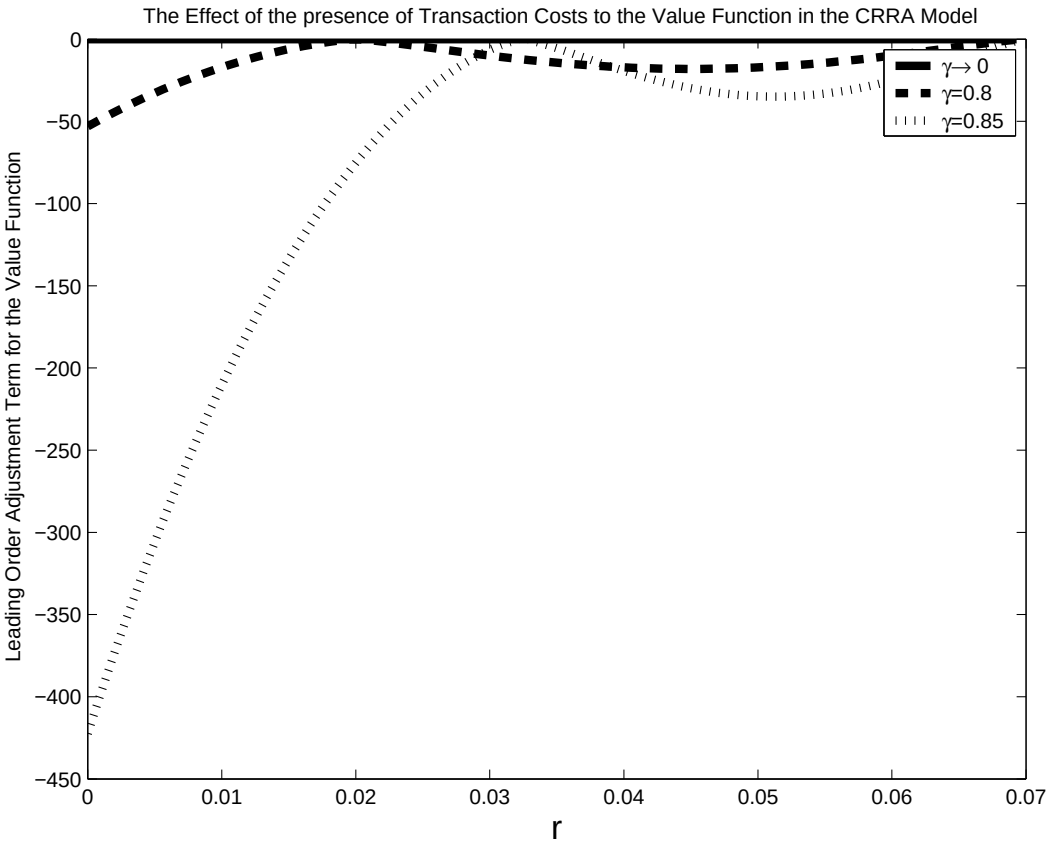


Figure 2.16: The relationship between the adjustment term $k^{2/3}H_2$ and r in the CRRA Model for various values of γ where $\nu = 0.07, \mu = 0.07, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. This graph should be similar to Figure 2.15. In addition, however, this graphs shows that the adjustment term is not a monotone function of the Sharpe Ratio. In the case $\gamma = 0.8$, the adjustment term vanishes when $r = 0.02$. In the case $\gamma = 0.85$, the adjustment term vanishes when $r = 0.0325$.

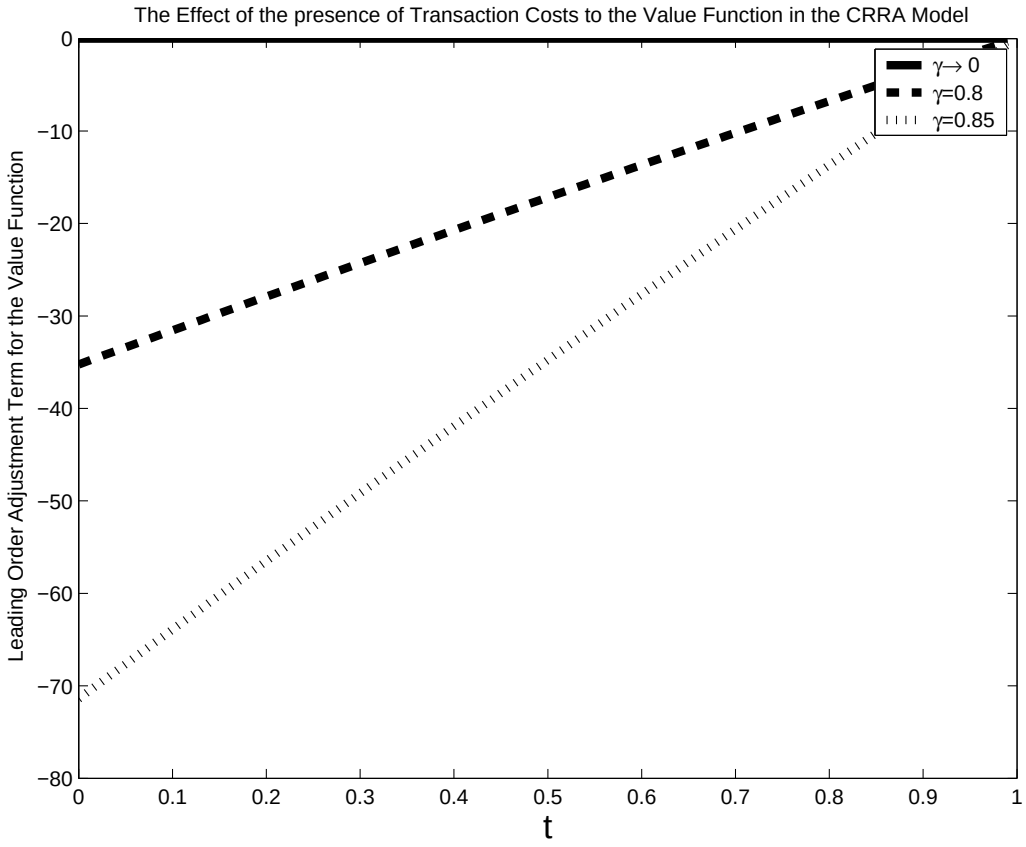


Figure 2.17: The relationship between the adjustment term $k^{2/3}H_2$ and t in the CRRA Model for various values of γ where $\nu = 0.07, \mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, k = 0.01$ and $\Pi = 1000000$. Despite the complicated relationship between t and the adjustment term as shown in equation (2.110), the relationship seems to be linear.

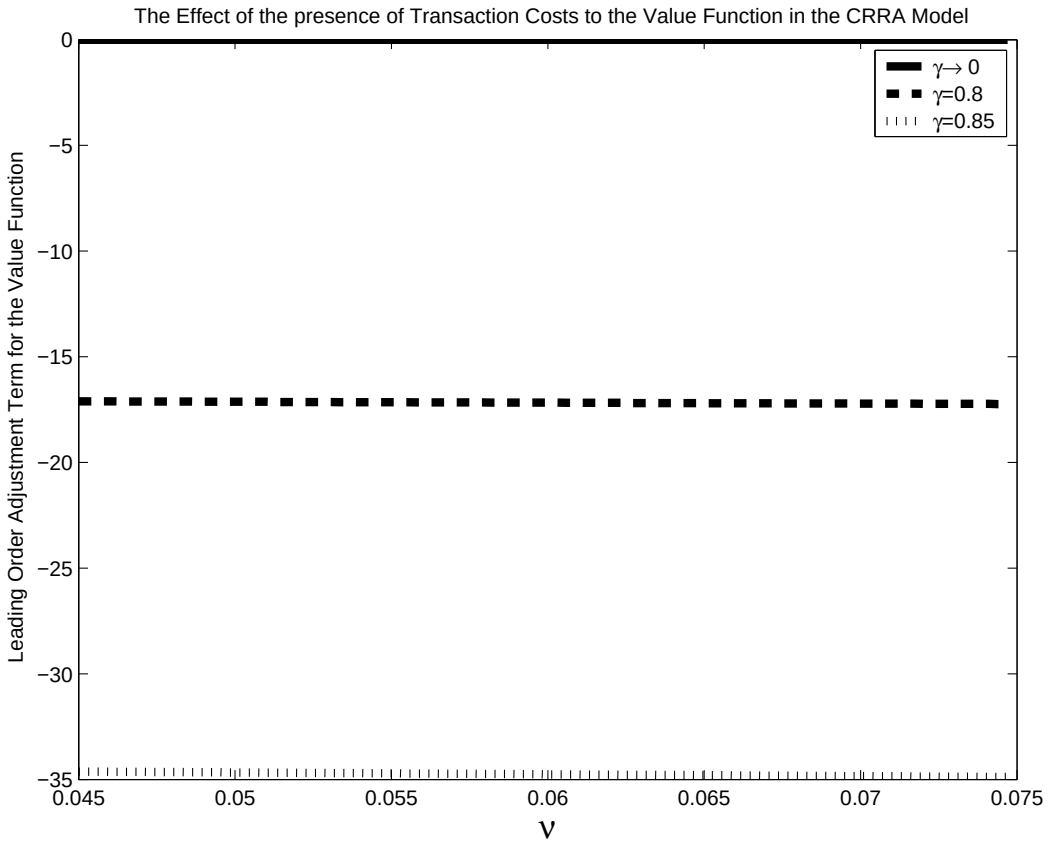


Figure 2.18: The relationship between the adjustment term $k^{2/3}H_2$ and ν in the CRRA Model for various values of γ where $\mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01$ and $\Pi = 1000000$. This graphs shows that the effect of ν is small compared with like γ.

2.9 The Choice of Asymptotic Scales

Here, we justify why we choose $k^{1/3}$ in the coordinate translation

$$B = \mathfrak{B}(\Pi, t) + k^{1/3}\beta,$$

and in the expansion

$$\begin{aligned} H(\Pi, \beta, t) &= H_0(\Pi, \beta, t) + k^{1/3}H_1(\Pi, \beta, t) \\ &\quad + k^{2/3}H_2(\Pi, \beta, t) + kH_3(\Pi, \beta, t) \\ &\quad + k^{4/3}H_4(\Pi, \beta, t) + \cdots. \end{aligned}$$

So, we let

$$B = \mathfrak{B}(\Pi, t) + k^{\alpha}\beta, \tag{2.115}$$

and

$$\begin{aligned} H(\Pi, \beta, t) &= H_0(\Pi, \beta, t) + k^{\epsilon}H_1(\Pi, \beta, t) \\ &\quad + k^{2\epsilon}H_2(\Pi, \beta, t) + k^{3\epsilon}H_3(\Pi, \beta, t) \\ &\quad + k^{4\epsilon}H_4(\Pi, \beta, t) + \cdots. \end{aligned} \tag{2.116}$$

We can safely assume that ϵ is a rational number.

So, we can rewrite equation (2.116) as

$$\begin{aligned} H(\Pi, \beta, t) &= H_0(\Pi, \beta, t) + k^{1/n}H_1(\Pi, \beta, t) \\ &\quad + k^{2/n}H_2(\Pi, \beta, t) + k^{3/n}H_3(\Pi, \beta, t) \\ &\quad + k^{4/n}H_4(\Pi, \beta, t) + \cdots \end{aligned} \tag{2.117}$$

for some integer n. If we consider those terms of $O(1)$, $O(1 + 1/n)$, $O(1 + 2/n)$, $O(1 + 3/n)$, $\cdots$, we must have

$$\alpha = \frac{m}{n}$$

for some integer m. If $\alpha \neq m/n$, we have

$$\frac{\partial H_0}{\partial \Pi} = \frac{\partial H_1}{\partial \Pi} = \frac{\partial H_2}{\partial \Pi} = \frac{\partial H_3}{\partial \Pi} = \cdots = 0,$$

which means H is independent of Π.

Therefore, we also rewrite equation (2.115) as

$$B = \mathfrak{B}(\Pi, t) + k^{m/n}\beta. \tag{2.118}$$

In fact, Π has a first order effect on H. So,

$$\frac{\partial H_0}{\partial \Pi} \neq 0.$$

If we look at those terms of $O(k^{-\frac{m}{n}})$, $O(k^{-\frac{m-1}{n}})$, $O(k^{-\frac{m-2}{n}})$,$O(k^{-\frac{m-3}{n}})$, $\cdots$, $O(1)$, we have

$$\frac{\partial H_0}{\partial \beta} = \frac{\partial H_1}{\partial \beta} = \frac{\partial H_2}{\partial \beta} = \frac{\partial H_3}{\partial \beta} = \cdots = \frac{\partial H_{m+n-1}}{\partial \beta} = 0,$$

$$\frac{\partial H_{m+n}}{\partial \beta} \neq 0. \tag{2.119}$$

We can obtain a similar result for the purchase region boundary.

As for the smooth pasting conditions for the sales region, we have

$$\frac{\partial^2 H_0}{\partial \beta^2} = \frac{\partial^2 H_1}{\partial \beta^2} = \frac{\partial^2 H_2}{\partial \beta^2} = \frac{\partial^2 H_3}{\partial \beta^2} = \cdots = \frac{\partial^2 H_{4m-1}}{\partial \beta^2} = 0,$$

$$\frac{\partial^2 H_{4m}}{\partial \beta^2} = -\frac{\partial^2 H_0}{\partial \beta \partial \Pi}.$$

For the smooth pasting condition of the purchase region, we have

$$\frac{\partial^2 H_0}{\partial \beta^2} = \frac{\partial^2 H_1}{\partial \beta^2} = \frac{\partial^2 H_2}{\partial \beta^2} = \frac{\partial^2 H_3}{\partial \beta^2} = \cdots = \frac{\partial^2 H_{4m-1}}{\partial \beta^2} = 0,$$

$$\frac{\partial^2 H_{4m}}{\partial \beta^2} = \frac{\partial^2 H_0}{\partial \beta \partial \Pi}.$$

Regarding the final condition, we have

$$H_0 = F(T),$$

$$H_1 = H_2 = H_3 = H_4 = \cdots = 0.$$

Now, we look at the equation in the no-transaction region. We have the equations below.

2.9.1 H_0, H_1, H_2, $\cdots$, H_{2m-1}

The equations for H_0, H_1, H_2, $\cdots$, H_{2m-1} are similar to equation (2.77) and equation (2.78) . They are

$$\begin{aligned}
\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_0}{\partial \beta^2} &= 0, && (2.120)\\
\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_1}{\partial \beta^2} &= 0, && (2.121)\\
\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_2}{\partial \beta^2} &= 0, && (2.122)\\
&\vdots\\
\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{2m-1}}{\partial \beta^2} &= 0. && (2.123)
\end{aligned}$$

If we use the same argument as in Sections 2.6.4 and 2.6.5, we can conclude that H_0, H_1, H_2, $\cdots$, H_{2m-1} are all independent of β.

2.9.2 H_{2m}, H_{2m+1}, H_{2m+2}, $\cdots$, H_{3m-1}

As for equation (2.77), the equations for H_{2m}, H_{2m+1}, H_{2m+2}, $\cdots$, H_{3m-1} are as follows

$$\begin{aligned}
&\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{2m}}{\partial \beta^2}\\
=\ &-\left\{I+\frac{\partial H_0}{\partial t}+r\mathfrak{B}\frac{\partial H_0}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_0}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_0}{\partial \Pi^2}\right\}, && (2.124)\\
&\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{2m+1}}{\partial \beta^2}\\
=\ &-\left\{I+\frac{\partial H_1}{\partial t}+r\mathfrak{B}\frac{\partial H_1}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_1}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_1}{\partial \Pi^2}\right\}, && (2.125)\\
&\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{2m+2}}{\partial \beta^2}\\
=\ &-\left\{I+\frac{\partial H_2}{\partial t}+r\mathfrak{B}\frac{\partial H_2}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_2}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2}\right\}, && (2.126)\\
&\vdots
\end{aligned}$$

$$
\begin{aligned}
& \frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{3m-1}}{\partial \beta^2} \\
= \; & -\left\{ I + \frac{\partial H_{m-1}}{\partial t} + r\mathfrak{B}\frac{\partial H_{m-1}}{\partial \Pi} + \mu(\Pi-\mathfrak{B})\frac{\partial H_{m-1}}{\partial \Pi} + \frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_{m-1}}{\partial \Pi^2} \right\}.
\end{aligned}
$$

As in Section 2.6.6, we can also deduce that H_{2m}, H_{2m+1}, H_{2m+2}, $\cdots$, H_{3m-1} are all independent of β. Also, using the argument in Section 2.6.7, we find that H_1, H_2, $\cdots$, H_{m-1} are all identically zero.

2.9.3 H_{3m}, H_{3m+1}, H_{3m+2}, $\cdots$, H_{4m-1}

The equations we have for H_{3m}, H_{3m+1}, H_{3m+2}, $\cdots$, H_{4m-1} are similar to equation (2.80). They are

$$
\begin{aligned}
& \frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{3m}}{\partial \beta^2} \\
= \; & \left\{ \sigma^2(\Pi-\mathfrak{B})\frac{\partial^2 H_0}{\partial \Pi^2} + (\mu-r)\frac{\partial H_0}{\partial \Pi} \right\}\beta \\
& -\left\{ \frac{\partial H_m}{\partial t} + r\mathfrak{B}\frac{\partial H_m}{\partial \Pi} + \mu(\Pi-\mathfrak{B})\frac{\partial H_m}{\partial \Pi} + \frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_m}{\partial \Pi^2} \right\}, \\
& \frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{3m+1}}{\partial \beta^2} \\
= \; & \left\{ \sigma^2(\Pi-\mathfrak{B})\frac{\partial^2 H_1}{\partial \Pi^2} + (\mu-r)\frac{\partial H_1}{\partial \Pi} \right\}\beta \\
& -\left\{ \frac{\partial H_{m+1}}{\partial t} + r\mathfrak{B}\frac{\partial H_{m+1}}{\partial \Pi} + \mu(\Pi-\mathfrak{B})\frac{\partial H_{m+1}}{\partial \Pi} + \frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_{m+1}}{\partial \Pi^2} \right\}, \\
& \frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{3m+2}}{\partial \beta^2} \\
= \; & \left\{ \sigma^2(\Pi-\mathfrak{B})\frac{\partial^2 H_2}{\partial \Pi^2} + (\mu-r)\frac{\partial H_2}{\partial \Pi} \right\}\beta \\
& -\left\{ \frac{\partial H_{m+2}}{\partial t} + r\mathfrak{B}\frac{\partial H_{m+2}}{\partial \Pi} + \mu(\Pi-\mathfrak{B})\frac{\partial H_{m+2}}{\partial \Pi} + \frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_{m+2}}{\partial \Pi^2} \right\}, \\
& \vdots
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{4m-1}}{\partial \beta^2} \\
= &\left\{\sigma^2(\Pi-\mathfrak{B})\frac{\partial^2 H_{m-1}}{\partial \Pi^2}+(\mu-r)\frac{\partial H_{m-1}}{\partial \Pi}\right\}\beta \\
&-\left\{\frac{\partial H_{2m-1}}{\partial t}+r\mathfrak{B}\frac{\partial H_{2m-1}}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_{2m-1}}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_{2m-1}}{\partial \Pi^2}\right\}.
\end{aligned}
$$

We apply the same argument as in Section 2.6.7, and we can conclude that H_{3m}, H_{3m+1}, H_{3m+2}, $\cdots$, H_{4m-1} are all independent of β. Also, H_m, H_{m+1}, $\cdots$, H_{2m-1} are all identically zero.

2.9.4 H_{4m}

The equation for H_{4m} is similar to equation (2.83), which is

$$
\begin{aligned}
&\frac{1}{2}\sigma^2(\frac{\partial \mathfrak{B}}{\partial \Pi})^2(\Pi-\mathfrak{B})^2\frac{\partial^2 H_{4m}}{\partial \beta^2} \\
= &-\frac{1}{2}\sigma^2\frac{\partial^2 H_0}{\partial \Pi^2}\beta^2-\left\{\frac{\partial H_{2m}}{\partial t}+r\mathfrak{B}\frac{\partial H_{2m}}{\partial \Pi}+\mu(\Pi-\mathfrak{B})\frac{\partial H_{2m}}{\partial \Pi}+\frac{\sigma^2(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_{2m}}{\partial \Pi^2}\right\}.
\end{aligned}
\tag{2.127}
$$

Therefore, H_{4m} must be dependent on β as

$$
\frac{\partial^2 H_0}{\partial \Pi^2}\neq 0
$$

in equation (2.127).

We recall from equation (2.119) that H_{m+n} is the first term that is dependent of β. So, we can conclude that $m+n=4m$ and so $n=3m$.

Thus, we justify the coordinate translation

$$
B = \mathfrak{B}(\Pi,t)+k^{1/3}\beta.
$$

Also, we find that H_1, H_2, $\cdots$, H_{2m-1} all are equal to zero. So, our finding that the effect of transaction costs on the value function is of second order is still valid.

It is true that H_{2m+1}, H_{2m+2}, $\cdots$, H_{3m-1} and H_{3m+1}, H_{3m+2}, $\cdots$, H_{4m-1} may not vanish. They, however, have no effect on the transaction boundaries and the leading

order value function adjustment term. Therefore, our choice of expansion for H

$$\begin{aligned} & H(\Pi, \beta, t) \\ = \; & H_0(\Pi, \beta, t) + k^{1/3} H_1(\Pi, \beta, t) + k^{2/3} H_2(\Pi, \beta, t) + k H_3(\Pi, \beta, t) \\ & + k^{4/3} H_4(\Pi, \beta, t) + \cdots \end{aligned}$$

is justified.

Chapter 3

Two Generalizations of the Dynamic Asset Allocation with Transaction Cost Model

3.1 Introduction

In the previous chapter, we worked on a problem in dynamic asset allocation with transaction costs. In this chapter, we extend the study in several different directions. The first generalization we consider is to allow the value function to be dependent on the stock price S. This in principle enable us to study those optimization problems with options in the portfolio. Secondly, we allow more than one stock, but the objective function is as in Chapter 2.

In both of these generalizations, the mathematics is similar to that in Chapter 2, but the results we obtain here, however, are not entirely the same. The formulae for the transaction boundaries in these generalized cases are usually very complicated. The properties of the no-transaction costs region may also be very different.

3.2 Stock Price Dependence

We study the problem when the value function J is dependent of S.

3.2.1 Without Transaction Costs

As in Chapter 2, we study the problem with no transaction costs first. Using the same notation and setup as there, the market model equations are equation (2.2).

$$\begin{aligned} dB &= rBdt \\ dS &= \mu Sdt + \sigma SdX \\ dA &= \mu Adt + \sigma AdX \\ d\Pi &= \mu(\Pi - B)dt + rBdt + \sigma(\Pi - B)dX \end{aligned}$$

At time $t = 0$, an investor has an amount of Π_0 of resources. The problem is to allocate investments over the given time horizon so as to maximize

$$\mathbb{E}_0\left\{ \int_0^T I(\Pi(\tilde{t}), S(\tilde{t}))d\tilde{t} + F(\Pi(T), S(T)) \right\}$$

where $\mathbb{E}_t$ is the conditional expectation given information up to time t. I and F are supposed to be strictly increasing concave differentiable functions of Π and S.

We rewrite the above equation in dynamic programming form so as to apply the Bellman principle of optimality. Therefore, we define the optimal expected value function $J(\Pi, S, t)$ as

$$J(\Pi, S, t) = \max_B \mathbb{E}_t\left\{ \int_t^T I(\Pi(\tilde{t}), S(\tilde{t}))d\tilde{t} + F(\Pi(T), S(T)) \right\}. \tag{3.1}$$

Applying Bellman's Principle and Itô's Lemma gives us

$$\begin{aligned} 0 = \max_B \Bigg\{ & \frac{\partial J}{\partial t} + I + rB\frac{\partial J}{\partial \Pi} + \mu(\Pi - B)\frac{\partial J}{\partial \Pi} + \mu S\frac{\partial J}{\partial S} \\ & + \frac{\sigma^2}{2}\left\{ S^2\frac{\partial^2 J}{\partial S^2} + 2S(\Pi - B)\frac{\partial^2 J}{\partial \Pi \partial S} + (\Pi - B)^2\frac{\partial^2 J}{\partial \Pi^2} \right\}. \end{aligned} \tag{3.2}$$

At $t = T$, we have

$$J(\Pi, S, T) = F(\Pi, S). \tag{3.3}$$

3.2.2 Solving Bellman's Equation with No Transaction Costs

Here, in the same spirit as the previous chapter, we attempt to provide some exact solutions of the no transaction problem.

By differentiating equation (3.2) , we can find B^*, the optimal amount of assets invested in risk-free bonds, which is

$$B^* = \Pi + (\frac{(\mu - r)}{\sigma^2}\frac{\partial J}{\partial \Pi} + S\frac{\partial^2 J}{\partial \Pi \partial S})/\frac{\partial^2 J}{\partial \Pi^2} \tag{3.4}$$

where B^* is the optimal amount invested in risk-free bonds. Putting this back into equation (3.2), we have

$$\begin{aligned} 0 &= \frac{\partial J}{\partial t} + I + r\Pi\frac{\partial J}{\partial \Pi} - \frac{(\mu - r)^2}{2\sigma^2}(\frac{\partial J}{\partial \Pi})^2/\frac{\partial^2 J}{\partial \Pi^2} \\ &\quad +\mu S\frac{\partial J}{\partial S} - (\mu - r)S\frac{\partial J}{\partial \Pi}\frac{\partial^2 J}{\partial \Pi \partial S}/\frac{\partial^2 J}{\partial \Pi^2} + \frac{\sigma^2 S^2 \frac{\partial^2 J}{\partial S^2} - \frac{\partial^2 J}{\partial \Pi \partial S}^2}{2\frac{\partial^2 J}{\partial \Pi^2}}. \end{aligned} \tag{3.5}$$

As for equation (2.9), we know that for any function J, as long as we set the function I accordingly,

$$\begin{aligned} I &= -\Big\{\frac{\partial J}{\partial t} + r\Pi\frac{\partial J}{\partial \Pi} - \frac{(\mu - r)^2}{2\sigma^2}(\frac{\partial J}{\partial \Pi})^2/\frac{\partial^2 J}{\partial \Pi^2} \\ &\quad +\mu S\frac{\partial J}{\partial S} - (\mu - r)S\frac{\partial J}{\partial \Pi}\frac{\partial^2 J}{\partial \Pi \partial S}/\frac{\partial^2 J}{\partial \Pi^2} + \frac{\sigma^2 S^2 \frac{\partial^2 J}{\partial S^2} - \frac{\partial^2 J}{\partial \Pi \partial S}^2}{2\frac{\partial^2 J}{\partial \Pi^2}}\Big\}, \end{aligned} \tag{3.6}$$

we can have exact solutions. As in Chapter 2, we have to make sure that the functions J and I thus defined make economic sense, which means

$$\begin{aligned} \frac{\partial I}{\partial \Pi} &\geq 0, \\ \frac{\partial J}{\partial \Pi} &\geq 0, \\ \frac{\partial^2 J}{\partial \Pi^2} &< 0. \end{aligned} \tag{3.7}$$

We again consider the following change of variables:

$$\begin{aligned} \tau &= T - t \\ x &= \log \Pi \\ y &= \log S. \\ \bar{J}(x, y, \tau) &= J(\Pi, S, t) \end{aligned} \tag{3.8}$$

We have

$$B^* = \Pi\left\{\frac{(\mu-r)\bar{J}_x + \sigma^2\bar{J}_{xy}}{\sigma^2(\bar{J}_{xx}-\bar{J}_x)} + 1\right\} \tag{3.9}$$

and

$$I = \bar{J}_\tau - r\bar{J}_x - \mu\bar{J}_y + \frac{\sigma^2\bar{J}_y}{2} - \frac{\sigma^2\bar{J}_{yy}}{2} + \frac{((\mu-r)\bar{J}_x + \sigma^2\bar{J}_{xy})^2}{2\sigma^2(\bar{J}_x x - \bar{J}_x)}. \tag{3.10}$$

Now, we look at some special cases of $\bar{J}(\tau, x, y)$.

3.2.3 Modified Long Term Growth Model

We recall the long term growth model from Section 1.11. Now, we want to modify the long term growth model so that it depends on S. We let [1]

$$\bar{J}(x, y, \tau) = x + \beta y - \nu\tau. \tag{3.11}$$

Therefore,

$$J(\Pi, S, t) = \log\Pi + \beta\log S - \nu(T-t). \tag{3.12}$$

and so the terminal function

$$F(\Pi(T), S(T)) = \log\Pi + \beta\log S,$$

and we find

$$B^* = \Pi\left\{1 - \frac{(\mu-r)}{\sigma^2}\right\} \tag{3.13}$$

and

$$I = -\nu - \frac{(\mu-r)^2}{2\sigma^2} - r - \mu\beta + \frac{\beta\sigma^2}{2}. \tag{3.14}$$

We notice that I in fact is a constant. Therefore, it does not affect our problem at all. It is included here so that we can have a very tidy J. Therefore, this optimization problem is just to maximize the expectation of the terminal function $F(\Pi(T), S(T))$.

[1] Note β here is not the β from Chapter 2.

The amount of resources invested in stock is the same as in our previous Long Term Growth Model in Section 1.11. There is a reason. Those terms involving $S(T)$ in the terminal function $F(\Pi(T), S(T))$ can be separated from those terms involving $\Pi(T)$. Therefore,

$$\begin{aligned} J(\Pi, S, t) &= \max_B \mathbb{E}_t \Big\{ \log \Pi(T) + \beta \log S(T) \Big\} \\ &= \max_B \Big\{ \mathbb{E}_t \log \Pi(T) + \mathbb{E}_t \beta \log S(T) \Big\} \\ &= \max_B \Big\{ \mathbb{E}_t \log \Pi(T) \Big\} + \mathbb{E}_t \Big\{ \beta \log S(T) \Big\} \\ &= \max_B \Big\{ \mathbb{E}_t \log \Pi(T) \Big\} + \beta(\mu(T-t) - \frac{1}{2}\sigma^2(T-t)). \end{aligned} \tag{3.15}$$

The control B does not affect $\mathbb{E}_t(\beta \log S(T))$. Therefore, the addition of S dependence in this case has no effect at all on the investment policy.

3.2.4 Modified Constant Relative Risk Aversion (CRRA) Model

Here, we consider the case where

$$\bar{J}(x, y, \tau) = exp\,(\gamma x + \alpha y - \nu \tau) \tag{3.16}$$

which means

$$J(\Pi, S, t) = e^{\bar{\nu}(T-t)} \Pi^\gamma S^\alpha,$$

with the terminal function $F(\Pi(T)) = \Pi^\gamma S^\alpha$.

Therefore, we have

$$B^* = \Pi \Big\{ 1 - \frac{\alpha}{1-\gamma} - \frac{\mu - r}{\sigma^2(1-\gamma)} \Big\} \tag{3.17}$$

and

$$\begin{aligned} I &= \Big(\frac{\gamma(\mu-r)^2 - 2r\gamma\alpha\sigma^2 + 2\mu\alpha\sigma^2 + \sigma^4\alpha^2}{2(\gamma-1)\sigma^2} - \gamma r + \bar{\nu} + \frac{\sigma^2\alpha}{2} \Big) J \\ &= \Big(\frac{\gamma(\mu-r)^2 - 2r\gamma\alpha\sigma^2 + 2\mu\alpha\sigma^2 + \sigma^4\alpha^2}{2(\gamma-1)\sigma^2} - \gamma r + \bar{\nu} + \frac{\sigma^2\alpha}{2} \Big) e^{\bar{\nu}(T-t)} \Pi^\gamma S^\alpha. \end{aligned} \tag{3.18}$$

Although these are formally solutions of equation (3.2) for any values of γ and $\bar{\nu}$, we must choose the value of γ, $\bar{\nu}$ and α properly so that the criteria in (3.7) are satisfied. This means choosing γ and $\bar{\nu}$ such that

$$\gamma < 1$$

and

$$\bar{\nu} > \gamma r - \frac{\gamma(\mu - r)^2 - 2r\gamma\alpha\sigma^2 + 2\mu\alpha\sigma^2 + \sigma^4\alpha^2}{2(\gamma - 1)\sigma^2} - \frac{\sigma^2\alpha}{2}.$$

The amount invested in stocks in this modified CRRA model is different to our previous CRRA model, and it is no longer proportional the Sharpe ratio, but it is still inversely proportional to $1 - \gamma$, the relative risk aversion factor. The value function J is also different.

We notice that when

$$\alpha = (1 - \gamma) - \frac{\mu - r}{\sigma^2}, \tag{3.19}$$

$B^* = 0$. This means the investor does not invest in bonds at all. He puts all his investments in stocks.

3.2.5 With Transaction Costs

Now we consider the problem with transaction costs. As in the previous chapter, let $k > 0$ represents the portion of of stock price used as transaction costs.

The market model equations become

$$\begin{aligned} dS &= \mu S dt + \sigma S dX_1 \\ dB &= rBdt - (1 + k)d\mathbf{L}(t) + (1 - k)d\mathbf{M}(t) \\ d\Pi &= \mu(\Pi - B)dt + rBdt + \sigma(\Pi - B)d\beta_t - kd\mathbf{L}(t) - kd\mathbf{M}(t) \end{aligned} \tag{3.20}$$

where $\mathbf{L}(t)$ and $\mathbf{M}(t)$ represent the cumulative purchase and sale of assets A in $[0, t]$, which we use as the controls. As in Chapter 2, B is only used to denote the value of assets invested in stocks here and it is no longer used as a control.

The optimal expected value function $\tilde{J}(\Pi, B, t)$ is

$$\tilde{J}(\Pi, B, S, t) = \max_{\mathbf{L},\mathbf{M}} \mathbb{E}_t \left\{ \int_t^T I(\Pi(\tilde{t}), S(\tilde{t}))d\tilde{t} + F(\Pi(T), S(T)) \right\}. \tag{3.21}$$

Applying Bellman's Principle and Itô's Lemma , we have

$$\begin{aligned}\max_{\mathbf{l},\mathbf{m}\in\Theta}\Bigg\{ I + \frac{\partial \tilde{J}}{\partial t} + (rB - (1+k)d\mathbf{l} + (1-k)\mathbf{m})\frac{\partial \tilde{J}}{\partial B} \\ +(rB + \mu(\Pi - B) - k\mathbf{l} - k\mathbf{m})\frac{\partial \tilde{J}}{\partial \Pi} + \mu S\frac{\partial J}{\partial S} \\ +\frac{\sigma^2}{2}\Bigg\{ S^2\frac{\partial^2 \tilde{J}}{\partial S^2} + 2S(\Pi - B)\frac{\partial^2 \tilde{J}}{\partial \Pi \partial S} + (\Pi - B)^2\sigma^2\frac{\partial^2 \tilde{J}}{\partial \Pi^2}\Bigg\} &= 0. \end{aligned} \tag{3.22}$$

In the no-transaction region, therefore, the value function $\tilde{J}$ satisfies

$$\begin{aligned} I + \frac{\partial \tilde{J}}{\partial t} + rB(\frac{\partial \tilde{J}}{\partial B} + \frac{\partial \tilde{J}}{\partial \Pi}) + \mu(\Pi - B)\frac{\partial \tilde{J}}{\partial \Pi} + \mu S\frac{\partial \tilde{J}}{\partial S} \\ +\frac{\sigma^2}{2}\Bigg\{ S^2\frac{\partial^2 \tilde{J}}{\partial S^2} + 2S(\Pi - B)\frac{\partial^2 \tilde{J}}{\partial \Pi \partial S} + (\Pi - B)^2\frac{\partial^2 \tilde{J}}{\partial \Pi^2}\Bigg\} &= 0. \end{aligned} \tag{3.23}$$

In the sales region (and at the boundary between sales region and no transaction region), $\tilde{J}$ satisfies

$$-k\frac{\partial \tilde{J}}{\partial \Pi} = (1+k)\frac{\partial \tilde{J}}{\partial B}. \tag{3.24}$$

In the purchase region (and at the boundary between sales region and no transaction region), $\tilde{J}$ satisfies

$$k\frac{\partial \tilde{J}}{\partial \Pi} = (1-k)\frac{\partial \tilde{J}}{\partial B}. \tag{3.25}$$

The smooth pasting equation at the boundaries between the no-transaction region and sales region is

$$(1-k)\frac{\partial^2 \tilde{J}}{\partial B^2} = k\frac{\partial^2 \tilde{J}}{\partial B \partial \Pi}, \tag{3.26}$$

and at the boundaries between the no-transaction region and purchase region is

$$(1+k)\frac{\partial^2 \tilde{J}}{\partial B^2} = -k\frac{\partial^2 \tilde{J}}{\partial B \partial \Pi}. \tag{3.27}$$

Also, when $t = T$, we have the final condition

$$\tilde{J}(\Pi, T) = F(\Pi). \tag{3.28}$$

We translate the B coordinate according to

$$B = \mathfrak{B}(\Pi, S, t) + k^{1/3}\beta, \tag{3.29}$$

where $\mathfrak{B}$ is the value of risk-free bonds we have when the level of transaction costs tends to zero, $k \to 0$.

We let $H(\Pi, \beta, S, t) = \tilde{J}(\Pi, B, S, t)$. We have

$$\begin{aligned}
\frac{\partial \tilde{J}}{\partial B} &= k^{-1/3}\frac{\partial H}{\partial \beta}, \\
\frac{\partial \tilde{J}}{\partial \Pi} &= \frac{\partial H}{\partial \Pi} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial \Pi}, \\
\frac{\partial \tilde{J}}{\partial t} &= \frac{\partial H}{\partial t} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial t}, \\
\frac{\partial \tilde{J}}{\partial S} &= \frac{\partial H}{\partial S} - k^{-1/3}\frac{\partial H}{\partial \beta}\frac{\partial \mathfrak{B}}{\partial S}, \\
\frac{\partial^2 \tilde{J}}{\partial \Pi^2} &= \frac{\partial^2 H}{\partial \Pi^2} - k^{-1/3}(2\frac{\partial^2 H}{\partial \beta \partial \Pi}\frac{\partial \mathfrak{B}}{\partial \Pi} + \frac{\partial H}{\partial \beta}\frac{\partial^2 \mathfrak{B}}{\partial \Pi^2}) + k^{-2/3}\frac{\partial^2 H}{\partial \beta^2}(\frac{\partial \mathfrak{B}}{\partial \Pi})^2, \\
\frac{\partial^2 \tilde{J}}{\partial S^2} &= \frac{\partial^2 H}{\partial S^2} - k^{-1/3}(2\frac{\partial^2 H}{\partial \beta \partial S}\frac{\partial \mathfrak{B}}{\partial S} + \frac{\partial H}{\partial \beta}\frac{\partial^2 \mathfrak{B}}{\partial S^2}) + k^{-2/3}\frac{\partial^2 H}{\partial S^2}(\frac{\partial \mathfrak{B}}{\partial S})^2, \\
\frac{\partial^2 \tilde{J}}{\partial S \partial \Pi} &= \frac{\partial^2 H}{\partial S \partial \Pi} - k^{-1/3}(\frac{\partial^2 H}{\partial \beta \partial S}\frac{\partial \mathfrak{B}}{\partial \Pi} + \frac{\partial^2 H}{\partial \beta \partial \Pi}\frac{\partial \mathfrak{B}}{\partial S} + \frac{\partial H}{\partial \beta}\frac{\partial^2 \mathfrak{B}}{\partial S \partial \Pi}) \\
&\quad + k^{-2/3}\frac{\partial^2 H}{\partial \beta^2}\frac{\partial \mathfrak{B}}{\partial S}\frac{\partial \mathfrak{B}}{\partial \Pi}.
\end{aligned} \tag{3.30}$$

Also, we let

$$\begin{aligned}
H(\Pi, \beta, S, t) &= H_0(\Pi, \beta, S, t) + k^{1/3}H_1(\Pi, \beta, S, t) \\
&\quad + k^{2/3}H_2(\Pi, \beta, S, t) + kH_3(\Pi, \beta, S, t) \\
&\quad + k^{4/3}H_4(\Pi, \beta, S, t) + \cdots
\end{aligned} \tag{3.31}$$

and assume that $H_0, H_1, H_2, H_3, H_4, \ldots$ and their derivatives are all of $O(1)$. H_0 is the value function when transaction costs level $k \to 0$.

After the change of coordinates and collecting terms of the same order the sales region equation, as in Section , we have

$$\frac{\partial H_0}{\partial \beta} = 0, \tag{3.32}$$

$$\frac{\partial H_1}{\partial \beta} = 0, \tag{3.33}$$

$$\frac{\partial H_2}{\partial \beta} = 0, \tag{3.34}$$

$$\frac{\partial H_3}{\partial \beta} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_0}{\partial \beta} = 0, \tag{3.35}$$

$$\frac{\partial H_4}{\partial \beta} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_1}{\partial \beta} - \frac{\partial H_0}{\partial \Pi} = 0. \tag{3.36}$$

Similarly, we have the following equations for the purchase region,

$$\frac{\partial H_0}{\partial \beta} = 0, \tag{3.37}$$

$$\frac{\partial H_1}{\partial \beta} = 0, \tag{3.38}$$

$$\frac{\partial H_2}{\partial \beta} = 0, \tag{3.39}$$

$$\frac{\partial H_3}{\partial \beta} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_0}{\partial \beta} = 0, \tag{3.40}$$

$$\frac{\partial H_4}{\partial \beta} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial H_1}{\partial \beta} + \frac{\partial H_0}{\partial \Pi} = 0. \tag{3.41}$$

As for the smooth pasting equation for the sales boundary, we have

$$\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{3.42}$$

$$\frac{\partial^2 H_1}{\partial \beta^2} = 0, \tag{3.43}$$

$$\frac{\partial^2 H_2}{\partial \beta^2} = 0, \tag{3.44}$$

$$\frac{\partial^2 H_3}{\partial \beta^2} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{3.45}$$

$$\frac{\partial^2 H_4}{\partial \beta} - (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_1}{\partial \beta^2} - \frac{\partial^2 H_0}{\partial \Pi \partial \beta} = 0. \tag{3.46}$$

Similarly, the smooth pasting equation for the purchase boundary becomes

$$\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{3.47}$$

$$\frac{\partial^2 H_1}{\partial \beta^2} = 0, \tag{3.48}$$

$$\frac{\partial^2 H_2}{\partial \beta^2} = 0, \tag{3.49}$$

$$\frac{\partial^2 H_3}{\partial \beta^2} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_0}{\partial \beta^2} = 0, \tag{3.50}$$

$$\frac{\partial^2 H_4}{\partial \beta} + (1 - \frac{\partial \mathfrak{B}}{\partial \Pi})\frac{\partial^2 H_1}{\partial \beta^2} + \frac{\partial^2 H_0}{\partial \Pi \partial \beta} = 0. \tag{3.51}$$

The final condition becomes

$$H_0(\Pi, \beta, S, T) = F(\Pi, S) \tag{3.52}$$

$$H_1(\Pi, \beta, S, T) = 0 \tag{3.53}$$

$$H_2(\Pi, \beta, S, T) = 0 \tag{3.54}$$

$$H_3(\Pi, \beta, S, T) = 0 \tag{3.55}$$

$$H_4(\Pi, \beta, S, T) = 0. \tag{3.56}$$

As in Chapter 2, we now consider the equation at the no-transaction region. After translating the coordinates, expanding H according to equation (3.31) , and collecting terms of the same order in k, we have the equations below.

1. $O(k^{-2/3})$ Equation

$$\frac{1}{2}\sigma^2\Big\{\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi - \mathfrak{B}) + S\frac{\partial \mathfrak{B}}{\partial S}\Big\}^2 \frac{\partial^2 H_0}{\partial \beta^2} = 0. \tag{3.57}$$

   Using the same argument as in Section 2.6.4, we can conclude that H_0 is independent of β.

2. $O(k^{-1/3})$ Equation

$$\frac{1}{2}\sigma^2\Big\{S\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi - \mathfrak{B}) + S\frac{\partial \mathfrak{B}}{\partial S}\Big\}^2 \frac{\partial^2 H_1}{\partial \beta^2} = 0. \tag{3.58}$$

   Similarly, we can conclude that H_1 is independent of β.

3. $O(1)$ Equation

$$\begin{aligned} &-\frac{1}{2}\sigma^2\Big\{S\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi - \mathfrak{B}) + S\frac{\partial \mathfrak{B}}{\partial S}\Big\}^2 \frac{\partial^2 H_2}{\partial \beta^2} \\ = \; & I + \frac{\partial H_0}{\partial t} + r\mathfrak{B}\frac{\partial H_0}{\partial \Pi} + \mu S\frac{\partial H_0}{\partial S} + \mu(\Pi - \mathfrak{B})\frac{\partial H_0}{\partial \Pi} \\ & + \sigma^2\Big[\frac{S^2}{2}\frac{\partial^2 H_0}{\partial S^2} + \frac{(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_0}{\partial \Pi^2} + S(\Pi - \mathfrak{B})\frac{\partial^2 H_0}{\partial S \partial \Pi}\Big]. \end{aligned} \tag{3.59}$$

As in Section 2.6.6, H_2 here is independent of β. Also, we have $H_0 = J$.

4. $O(k^{1/3})$ Equation

$$\begin{aligned}
&\frac{1}{2}\sigma^2\Big\{\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi - \mathfrak{B}) + S\frac{\partial \mathfrak{B}}{\partial S}\Big\}^2\frac{\partial^2 H_3}{\partial \beta^2} \\
= \;& \Big\{\sigma^2(\Pi - \mathfrak{B})\frac{\partial^2 H_0}{\partial \Pi^2} + \sigma^2 S\frac{\partial^2 H_0}{\partial \Pi \partial S} + (\mu - r)\frac{\partial H_0}{\partial \Pi}\Big\}\beta \\
& -\Big\{\frac{\partial H_1}{\partial t} + r\mathfrak{B}\frac{\partial H_1}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_1}{\partial \Pi} + \mu S\frac{\partial H_1}{\partial S} \\
& +\sigma^2\Big[\frac{S^2}{2}\frac{\partial^2 H_1}{\partial S^2} + \frac{(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_1}{\partial \Pi^2} + S(\Pi - \mathfrak{B})\frac{\partial^2 H_1}{\partial S \partial \Pi}\Big]\Big\}.
\end{aligned} \tag{3.60}$$

Like Section 2.6.7, we know that H_3 is independent on β. Also, we have $H_1 = 0$.

Unlike Section 2.6.7, we have

$$\sigma^2(\Pi - \mathfrak{B})\frac{\partial^2 H_0}{\partial \Pi^2} + \sigma^2 S\frac{\partial^2 H_0}{\partial \Pi \partial S} + (\mu - r)\frac{\partial H_0}{\partial \Pi} \;=\; 0 \tag{3.61}$$

instead of equation (2.82). This equation, though, reduces to equation (2.82) when H_0 is independent of S.

5. $O(k^{2/3})$ Equation

$$\begin{aligned}
&\frac{1}{2}\sigma^2\Big\{\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi - \mathfrak{B}) + S\frac{\partial \mathfrak{B}}{\partial S}\Big\}^2\frac{\partial^2 H_4}{\partial \beta^2} \\
= \;& -\frac{1}{2}\sigma^2\frac{\partial^2 H_0}{\partial \Pi^2}\beta^2 - \Big\{\frac{\partial H_2}{\partial t} + r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_2}{\partial \Pi} + \mu S\frac{\partial H_2}{\partial S} \\
& +\sigma^2\Big[\frac{S^2}{2}\frac{\partial^2 H_2}{\partial S^2} + \frac{(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2} + S(\Pi - \mathfrak{B})\frac{\partial^2 H_2}{\partial S \partial \Pi}\Big]\Big\}.
\end{aligned} \tag{3.62}$$

As in Section 2.6.8, we can let

$$H_4 \;=\; H_4^4\beta^4 + H_4^2\beta^2 + H_4^1\beta + H_4^0. \tag{3.63}$$

Instead of equation (2.85), however, we have

$$H_4^4 \;=\; -\frac{1}{12}\frac{\partial^2 H_0}{\partial \Pi^2}\Big/\Big[(\Pi - \mathfrak{B})\frac{\partial \mathfrak{B}}{\partial \Pi} + S\frac{\partial \mathfrak{B}}{\partial S}\Big]^2, \tag{3.64}$$

which reduces to equation (2.85) when $\mathfrak{B}$ is independent of S, and

$$\begin{aligned} H_4^2 &= -\Big\{ \frac{\partial H_2}{\partial t} + r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_2}{\partial \Pi} + \mu\frac{\partial H_2}{\partial S} \\ &\quad +\sigma^2\Big[\frac{S^2}{2}\frac{\partial^2 H_2}{\partial S^2} + \frac{(\Pi-\mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2} \\ &\quad +S(\Pi-\mathfrak{B})\frac{\partial^2 H_2}{\partial S \partial \Pi}\Big]\Big\} \Big/ \sigma^2\Big\{\frac{\partial \mathfrak{B}}{\partial \Pi}(\Pi-\mathfrak{B}) + S\frac{\partial \mathfrak{B}}{\partial S}\Big\}^2 \end{aligned} \tag{3.65}$$

in general.

Therefore, H_4^1 and H_4^0 are both functions of Π, S and t, but not β.

Using the same argument as in Section 2.6.8, we have

$$H_4^1 = 0 \tag{3.66}$$

and

$$\begin{aligned} \beta_+ &= -\left\{ \frac{3\frac{\partial H_0}{\partial \Pi}\Big[(\Pi-\mathfrak{B})\frac{\partial \mathfrak{B}}{\partial \Pi} + S\frac{\partial \mathfrak{B}}{\partial S}\Big]^2}{2\frac{\partial^2 H_0}{\partial \Pi^2}} \right\}^{1/3} \\ \beta_- &= \left\{ \frac{3\frac{\partial H_0}{\partial \Pi}\Big[(\Pi-\mathfrak{B})\frac{\partial \mathfrak{B}}{\partial \Pi} + S\frac{\partial \mathfrak{B}}{\partial S}\Big]^2}{2\frac{\partial^2 H_0}{\partial \Pi^2}} \right\}^{1/3}. \end{aligned} \tag{3.67}$$

However, unlike Section 2.6.8, where we could use equation (2.82) to simplify the expression

$$\frac{\partial H_0}{\partial \Pi} \Big/ \frac{\partial^2 H_0}{\partial \Pi^2},$$

where we can no longer use the analogous equation (3.61) to simplify equation (3.67). So, equation (3.67) is the simplest form we can get for the boundaries.

The value H_4^2 can be obtained by the relationship

$$H_4^2 = -6H_4^4\beta^2, \tag{3.68}$$

which we can put it into equation (3.65) to find H_2, which is the leading order adjustment term for H.

3.2.6 Modified Long Term Growth Model

We now apply the transaction cost formulae to the modified long term growth model. We recall that

$$H_0(\Pi, S, t) \;=\; \log \Pi + \alpha \log S - \nu(T-t) \tag{3.69}$$

and

$$\mathfrak{B} \;=\; \Pi(1 - \frac{(\mu - r)}{\sigma^2}). \tag{3.70}$$

So, we have

$$\beta_{\pm} \;=\; \mp\Pi\left\{\frac{3(\sigma^2 - \mu + r)^2(\mu - r)^2}{2\sigma^8}\right\}^{1/3}, \tag{3.71}$$

and the leading order adjustment term for the value function, H_2 is given by the equation

$$\begin{aligned} & \frac{1}{2}\left\{\frac{3(\sigma^2 - \mu + r)^2(\mu - r)^2}{2\sigma^5}\right\}^{2/3} \\ = \;& \left\{\frac{\partial H_2}{\partial t} + r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_2}{\partial \Pi} + \mu S\frac{\partial H_2}{\partial S}\right. \\ & \left. +\sigma^2\Big[\frac{S^2}{2}\frac{\partial^2 H_2}{\partial S^2} + \frac{(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2} + S(\Pi - \mathfrak{B})\frac{\partial^2 H_2}{\partial S \partial \Pi}\Big]\right\}. \end{aligned} \tag{3.72}$$

and the boundary condition

$$H_2(\Pi, T) \;=\; 0.$$

Solving gives us the solution

$$H_2(\Pi, T) \;=\; -\frac{T-t}{2}\left\{\frac{3(\sigma^2 - \mu + r)^2(\mu - r)^2}{2\sigma^5}\right\}^{2/3}. \tag{3.73}$$

Comparing this with the corresponding terms with the long term growth model in the previous chapter which does not depend on S, they are exactly the same.

3.2.7 Modified Constant Relative Risk Aversion (CRRA) Model

We recall from the modified CRRA model that

$$H_0(\Pi, S, t) = e^{\bar{\nu}(T-t)}\Pi^\gamma S^\alpha,$$

and

$$\mathfrak{B} = \Pi\left\{1 + \frac{(\sigma^2\alpha + \mu - r)}{\sigma^2(\gamma - 1)}\right\}. \tag{3.74}$$

So, we have

$$\begin{aligned}\beta_+ &= -\Pi\left\{\frac{3(\mu - r + \sigma^2\gamma - \sigma^2 + \sigma^2\alpha)^2(\mu - r + \sigma^2\alpha)^2}{2\sigma^8(\gamma-1)^5}\right\}^{1/3} \\ \beta_- &= \Pi\left\{\frac{3(\mu - r + \sigma^2\gamma - \sigma^2 + \sigma^2\alpha)^2(\mu - r + \sigma^2\alpha)^2}{2\sigma^8(\gamma-1)^5}\right\}^{1/3}.\end{aligned} \tag{3.75}$$

and the leading order adjustment term for the value function, H_2 is given by the equation

$$\begin{aligned}&\frac{1}{2}\gamma(\gamma-1)^{-7/3}\Pi^\gamma S^\alpha \exp\bar{\nu}(T-t)\left\{\frac{3(\mu - r + \sigma^2\gamma - \sigma^2 + \sigma^2\alpha)^2(\mu - r + \sigma^2\alpha)^2}{2\sigma^5}\right\}^{2/3} \\ = &-\left\{\frac{\partial H_2}{\partial t} + r\mathfrak{B}\frac{\partial H_2}{\partial \Pi} + \mu(\Pi - \mathfrak{B})\frac{\partial H_2}{\partial \Pi} + \mu S\frac{\partial H_2}{\partial S}\right. \\ &\left.+\sigma^2\Big[\frac{S^2}{2}\frac{\partial^2 H_2}{\partial S^2} + \frac{(\Pi - \mathfrak{B})^2}{2}\frac{\partial^2 H_2}{\partial \Pi^2} + S(\Pi - \mathfrak{B})\frac{\partial^2 H_2}{\partial S \partial \Pi}\Big]\right\}\end{aligned} \tag{3.76}$$

and the boundary conditions

$$\begin{aligned}H_2(\Pi, T) &= 0 \\ H_2(0, t) &= 0.\end{aligned} \tag{3.77}$$

As in the previous CRRA model, the above equations for H_2 can be solved by expressing H_2 as a series of $(T-t)$, i.e.

$$H_2(\Pi, t) = \Pi^\gamma S^\alpha \sum_{n=0}^{\infty} H_2^n (T-t)^n \tag{3.78}$$

where $H_2^1, H_2^2, H_2^3 \cdots$ are all constants independent of Π and t, and their values can be obtained by expressing the left hand side of equation (3.76) as

$$\frac{1}{2}\gamma(\gamma-1)^{-7/3}\Pi^\gamma S^\alpha\left\{\frac{3(\mu - r + \sigma^2\gamma - \sigma^2 + \sigma^2\alpha)^2(\mu - r + \sigma^2\alpha)^2}{2\sigma^5}\right\}^{2/3}\sum_{n=1}^{\infty}\frac{(\bar{\nu}(T-t))^n}{n!}$$

and considering the terms $H_2^1, H_2^2, H_2^3 \cdots$ one by one.

So, we have

$$\begin{aligned} H_2^1 &= \frac{1}{2}\gamma(\gamma-1)^{-7/3}\left\{\frac{3(\mu-r+\sigma^2\gamma-\sigma^2+\sigma^2\alpha)^2(\mu-r+\sigma^2\alpha)^2}{2\sigma^5}\right\}^{2/3} \\ H_2^{n+1} &= \frac{1}{n+1}\left\{\frac{\nu^n\gamma(\gamma-1)^{-7/3}}{2n}\left(\frac{3(\mu-r+\sigma^2\gamma-\sigma^2+\sigma^2\alpha)^2(\mu-r+\sigma^2\alpha)^2}{2\sigma^5}\right)^{2/3}\right. \\ &\quad \left.-\Big(\frac{\gamma(\mu-r)(\mu-r+\alpha\sigma^2)}{2(\gamma-1)\sigma^2}-\gamma r-\mu\alpha-\frac{\sigma^2\alpha(\alpha-1)}{2}\Big)H_2^n\right\}. \end{aligned} \tag{3.79}$$

3.2.8 Financial Interpretations

Many of the financial interpretations on the model in Chapter 2 still applies to our new model. For clarity, we list all the interpretations that apply to the current model, even though they may look repetitive to those in previous chapter.

1. The width of the no-transaction region is dependent on whether the terms

$$\frac{\partial \mathfrak{B}}{\partial \Pi}$$

and

$$\frac{\partial \mathfrak{B}}{\partial S}$$

are of the same sign. If they are of the same sign, the no-transaction region is wider.

2. Transaction costs do not shift the optimal strategy to risk-free bonds or the stocks. In fact, the no transaction costs optimal strategy is the midpoint of the no transaction region.

3. The width of the no-transaction region is proportional to the optimal value invested in stocks in the no transaction costs problem.

4. The width of the no-transaction region is directly proportional to $k^{1/3}$, and so the larger the transaction costs, the larger the no-transaction region.

5. The width of the no-transaction region depends on the functions I and F.

6. As illustrated in the long term growth model and CRRA model, the volatility, σ, usually appears in the terms

$$\frac{\partial \mathfrak{B}}{\partial S}, \qquad \frac{\partial \mathfrak{B}}{\partial \Pi}, \frac{\partial H_0}{\partial \Pi} \qquad \text{and} \qquad \frac{\partial^2 H_0}{\partial \Pi^2}$$

and so it is impossible for us to just inspect equation (3.67) and decide the effect of volatility on the bandwidth of the no-transaction region. In fact, the exact effect of volatility, σ, on the bandwidth of the no-transaction region is unclear. The impact of volatility in the no-transaction regions in the long term growth model and CRRA model varies.

7. The impact of the risk premium, $\mu - r$, or actually the Sharpe Ratio,

$$\frac{\mu - r}{\sigma^2},$$

to the no-transaction region is unclear. From equation (3.67) , it seems that the Sharpe Ratio narrows the no-transaction cost region. In the long term growth model, the Sharpe Ratio widens the no-transaction cost region. In the CRRA model, the no-transaction cost boundary equations do not relate directly to risk premium and Sharpe Ratio.

8. Transaction costs have an effect of order $k^{2/3}$ on the value function.

9. In the Long Term Growth Model, H_2, the leading adjustment term of the value function, is in fact independent of Π and S, but is directly proportion to time before expiry. Also, an increase in Sharpe Ratio increases H_2.

10. In the Modified CRRA Model, the effect of the transaction costs on the value function is proportional to Π^γ and S^β. Also, the effect Sharpe Ratio to H_2, the leading adjustment term of the value function is also unclear, as H_2 does not seem to relate directly with it. The effect of γ and β on H_2 is unknown, and it depends on its values compared with other values μ, r, σ.

3.2.9 Modified Long Term Growth model

This Modified Long Term Growth Model is essentially the same as the one in the previous chapter. This is because the difference between the objective function our these two Long Term Growth Model is $\alpha \log S$, which cannot be controlled. Therefore, we do not discuss it in details here.

3.2.10 Modified CRRA model

We plot the graph of the no-transaction region with σ,μ, r, t ,γ and α. The results are shown in Figures 3.5 to Figure 3.4.

The relationship between the no-transaction region in this modified CRRA model with σ, μ, r, t are similar to the relationship in the CRRA model. The difference between these two models is the addition of the variable α.

Indeed, there are a few interesting points in Figure 3.4, the graph for α. Firstly, we notice that the width of the no transaction region becomes 0 at some point. If we look at the formula for β_+ and β_- (the position of the no transaction region), we find that actually when

$$\alpha \;=\; (1-\gamma) - \frac{\mu - r}{\sigma^2},$$

which in that particular graph is when $\alpha = 0.32$, both β_+ and β_- are equal to 0. Also, $\mathfrak{B} = B^* = 0$, this means all the resources are invested in stocks. Hence, the no transaction region does not really matter. When $\alpha > 0.32$, $\mathfrak{B} > 0$, which means it is optimal to borrow money and invest in stock. This is similar to the relationship between the no transaction region width and γ in the CRRA model in the previous chapter.

Now, we investigate the relationship of the leading order adjustment term $k^{2/3}H_2$ with μ, r, t and ν. The results are in Figures 3.6 to 3.9.

Analogous to the findings in the CRRA model in the previous chapter, from figure 3.9, we can see that the effect of ν to H_2 is very small compared to γ. Also, the relationship between the leading order adjustment term $k^{2/3}H_2$ and t looks linear, despite of the complex relationship between in the formula. See Figure 3.8.

The analysis of the graphs for H_2 relative to μ and r are also similar to the CRRA model. Firstly, we can see that α in general (not always though) increases the magnitude of H_2. We conjecture that it is because it increases the value of H_0, as it is proportional to S^α. Secondly, we find that the relationship between H_2 and μ, r are not linear, which is similar to the CRRA model. H_2 increases with r initially, and then decreases, and then increases again. Our explanation is similar to the one in the CRRA model, which we repeat here. When $r = \mu$, all of the resources are invested in bonds. When $1-\gamma-\alpha$ is equal to the Sharpe Ratio, all the investments are in stocks. In these two points, the transaction cost trading strategy and the no transaction cost

strategy are of no difference, and so $H_2 = 0$. When the Sharpe Ratio is further away from these two points, the transaction cost trading strategy and the no transaction cost strategy differ more, and so the magnitude of H_2 increases correspondingly.

Similar to the CRRA model we investigated in the previous chapter, an increase of magnitude in H_2 corresponds to to an increase with the width of the no transaction region. This can be verified from the figures we have here. We conjecture this is because the wider the no transaction region, it means the transaction cost trading strategy is further away from the no transaction cost optimal trading strategy.

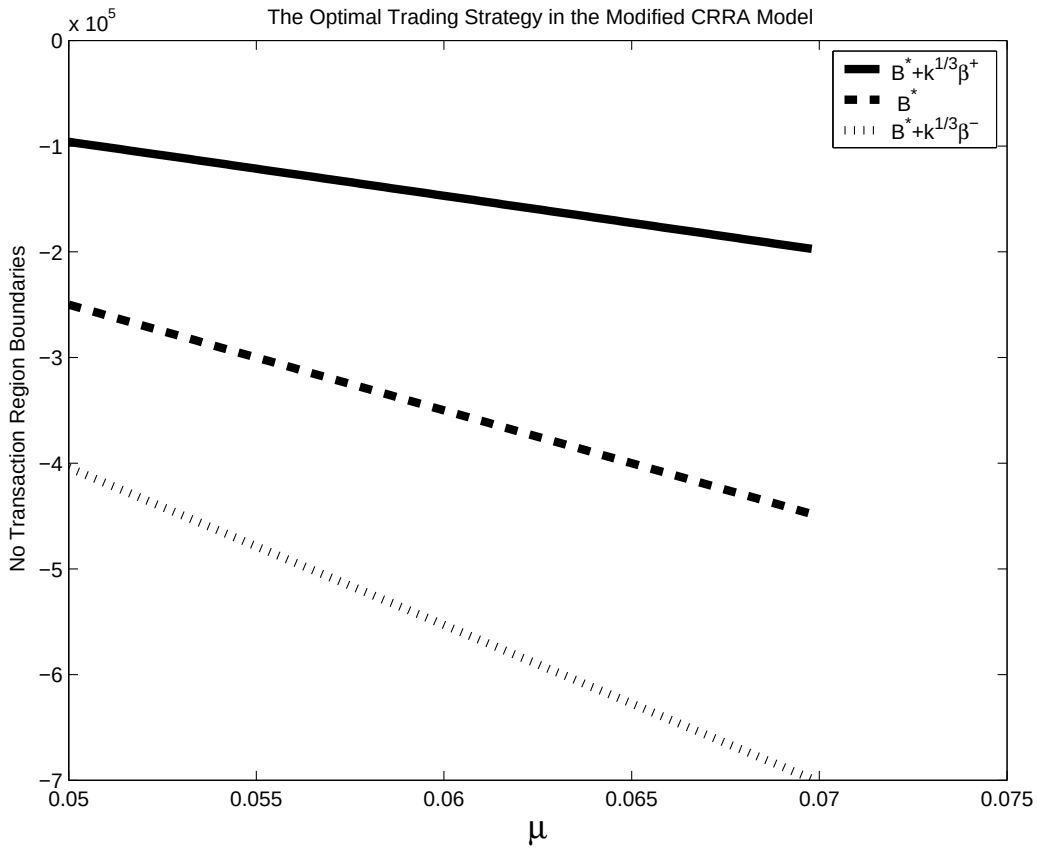


Figure 3.1: The boundaries of the no transaction region as a function of μ in the Modified CRRA Model where $\nu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6, \beta = 0.5$ and $\Pi = 1000000$. The behavior of the no-transaction region in this modified CRRA model is similar to those in the CRRA model. Since μ is directly proportional to the Sharpe Ratio. So, this graph can also seen as an illustration of the relationship between the no transaction region boundaries and the Sharpe Ratio. From this graph, we can see that the no transaction region widens with the increase of μ (or Sharpe Ratio). Also, we find that B^* decreases with μ (or Sharpe Ratio).

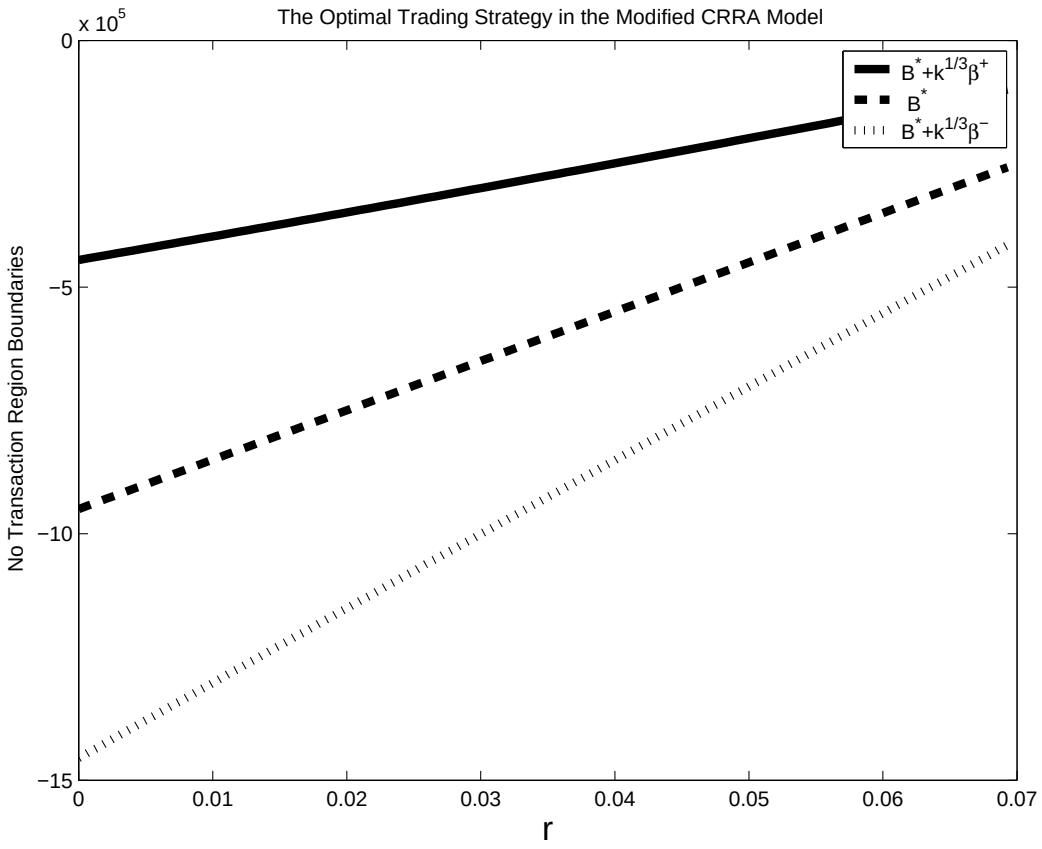


Figure 3.2: The boundaries of the no transaction region as a function of r in the modified CRRA Model where $\nu = 0.07, \mu = 0.07, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6, \alpha = 0.5$ and $\Pi = 1000000$. The behavior of the no-transaction region in this modified CRRA model is similar to those in the CRRA model. Since r decreases with the Sharpe Ratio. So, this graph can also seen as an illustration of the relationship between the no transaction region boundaries and the Sharpe Ratio. From this graph, we can see that the no transaction region narrows with the increase of r (or Sharpe Ratio). Also, we find that B^* increases with r (or decreases with Sharpe Ratio).

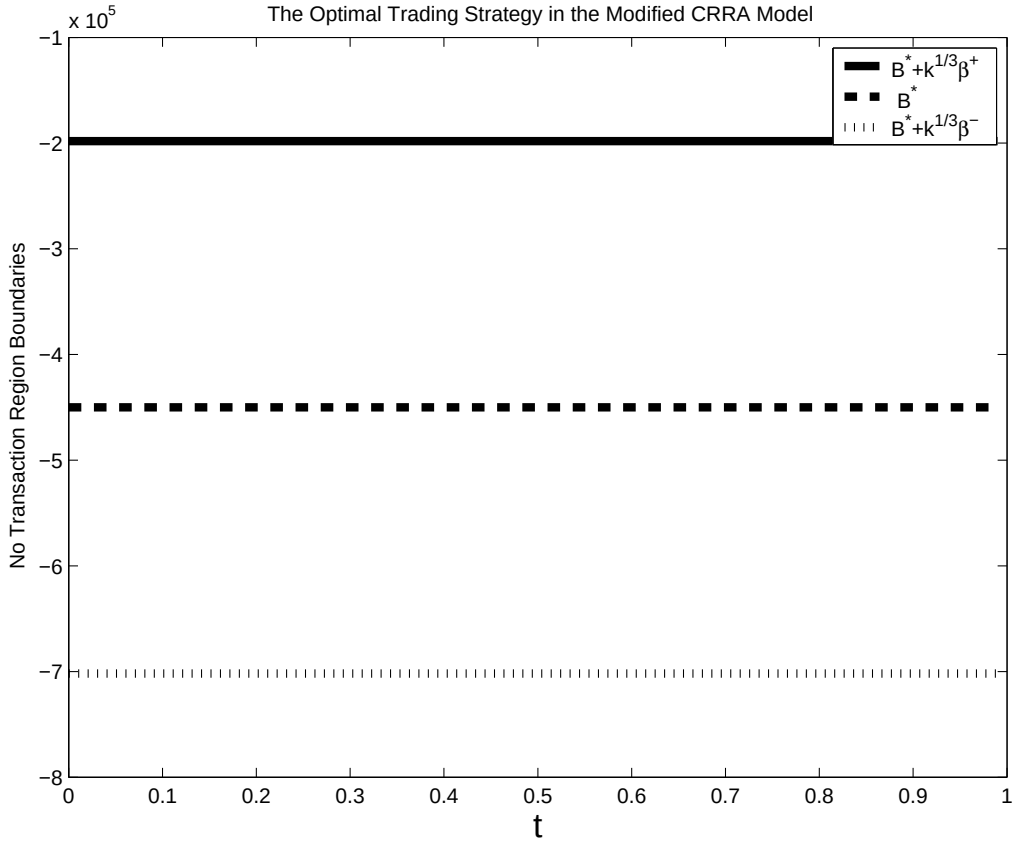


Figure 3.3: The boundaries of the no transaction region as a function of t in the Modified CRRA Model where $\nu = 0.07, \mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, k = 0.01, \gamma = 0.6, \alpha = 0.5$ and $\Pi = 1000000$. Similar to the CRRA model, this graph illustrates that just the passage of time without does not change the no transaction region.

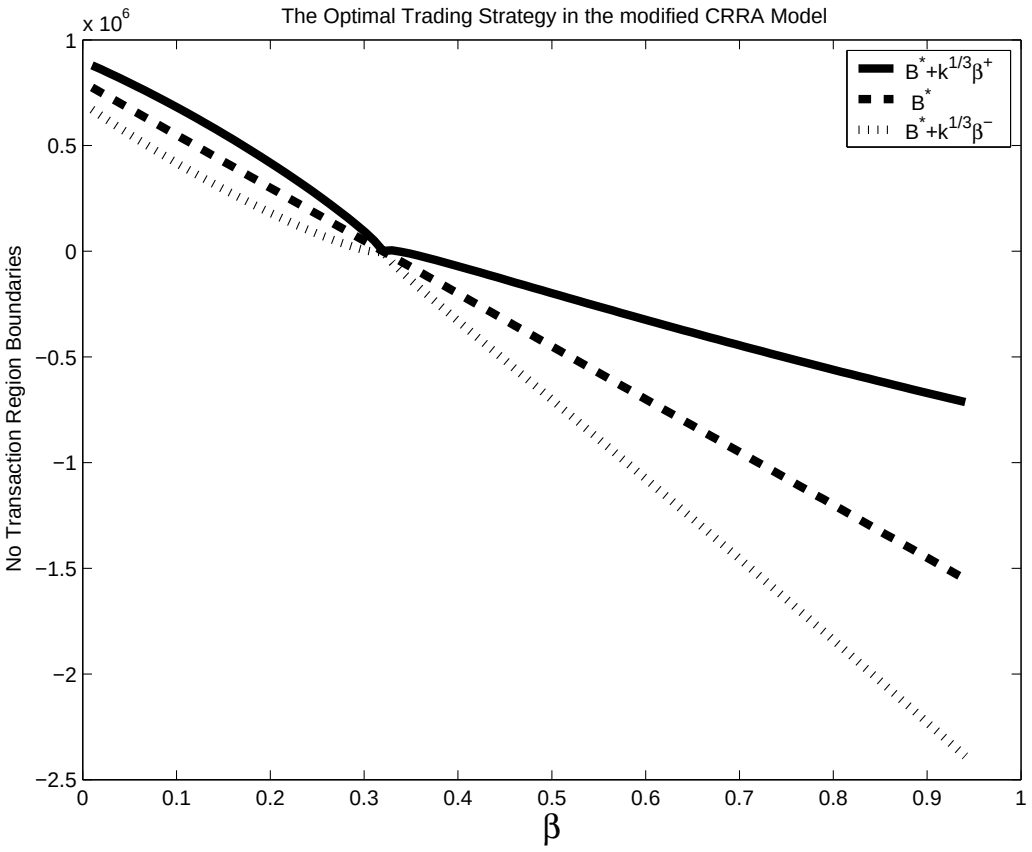


Figure 3.4: The boundaries of the no transaction region as a function of α in the Modified CRRA Model where $\nu = 0.07, \mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, \gamma = 0.6, k = 0.01$ and $\Pi = 1000000$. When $\alpha = 0.32$, both β_+ and β_- are equal to 0. Also, $\mathfrak{B} = B^* = 0$, this means all the resources are invested in stocks. Hence, the no transaction region does not really matter. When $\alpha > 0.32$, $\mathfrak{B} > 0$, which means it is optimal to borrow money and invest in stock. This is similar to the relationship between the no transaction region width and γ in the CRRA model.

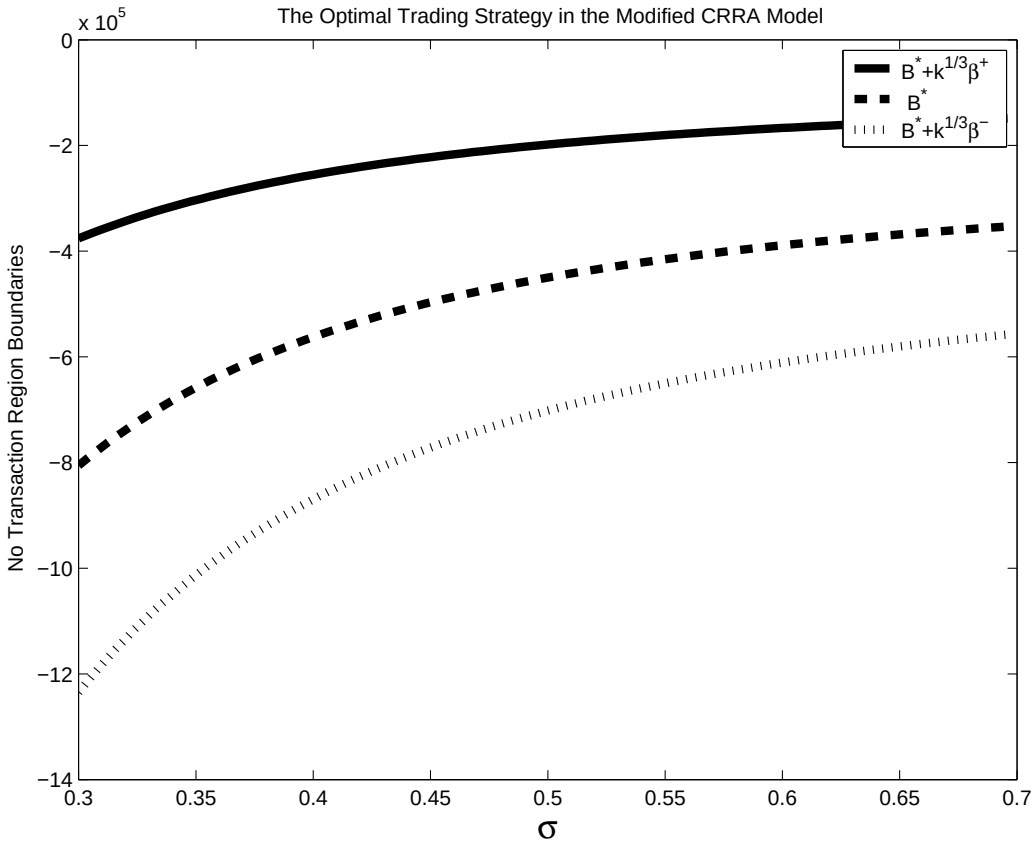


Figure 3.5: The boundaries of the no transaction region as a function of σ in the CRRA Model where $\nu = 0.07, \mu = 0.07, r = 0.05, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. Similar to CRRA, we can see that B^* increases with σ.

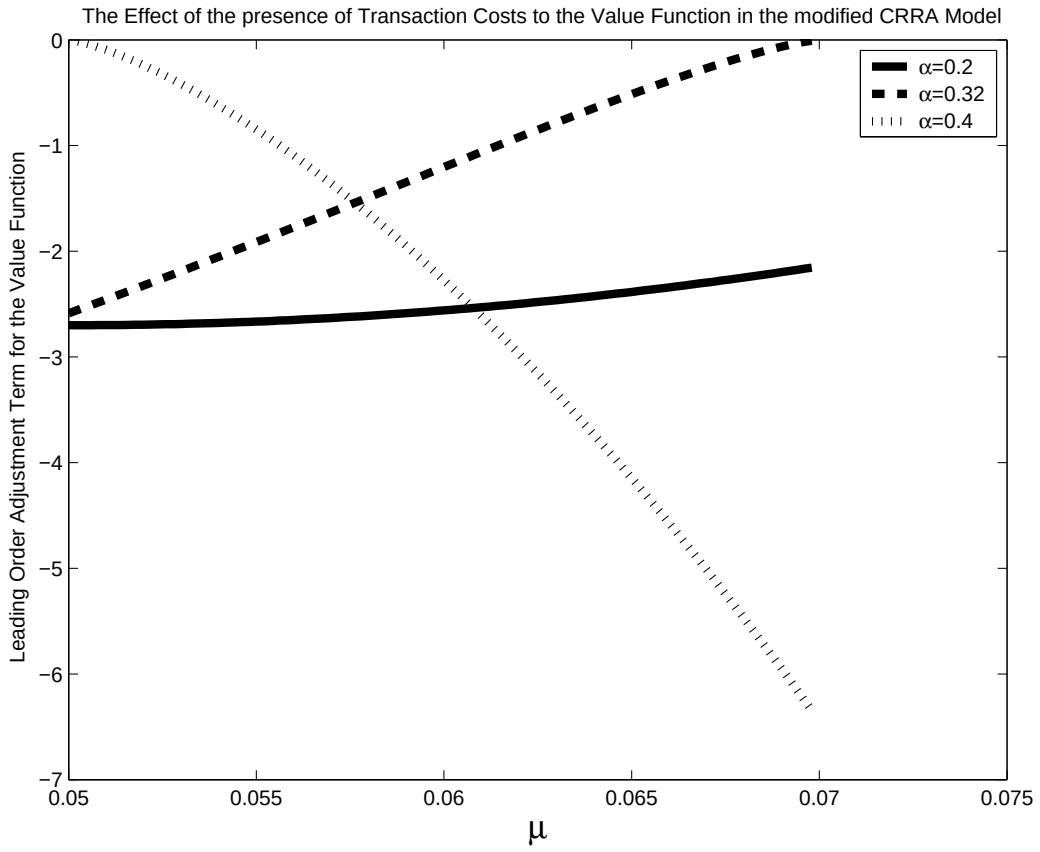


Figure 3.6: The relationship between the adjustment term $k^{2/3}H_2$ and μ in the Modified CRRA Model for various values of β where $\nu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. The analysis of the H_2 relative to μ is also similar to the CRRA model. Firstly, we can see that α in general (not always though) increases the magnitude of H_2. Secondly, we find that the relationship between H_2 and μ are not linear, which is similar to the CRRA model. H_2 increases with initially, and then decreases, and then increases again.

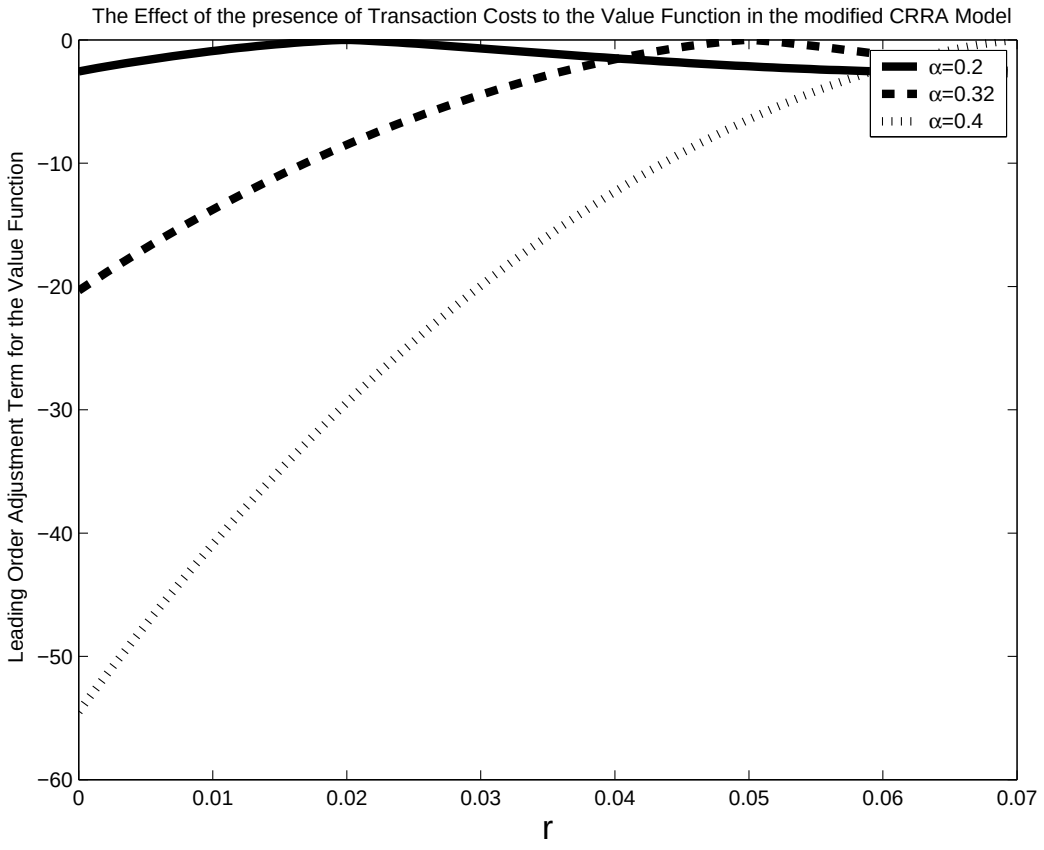


Figure 3.7: The relationship between the adjustment term $k^{2/3}H_2$ and r in the Modified CRRA Model for various values of α where $\nu = 0.07, \mu = 0.07, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. The analysis of the graphs for H_2 relative to r is also similar to the CRRA model. Firstly, we can see that α in general (not always though) increases the magnitude of H_2. Secondly, we find that the relationship between H_2 and r are not linear, which is similar to the CRRA model. H_2 increases with initially, and then decreases, and then increases again.

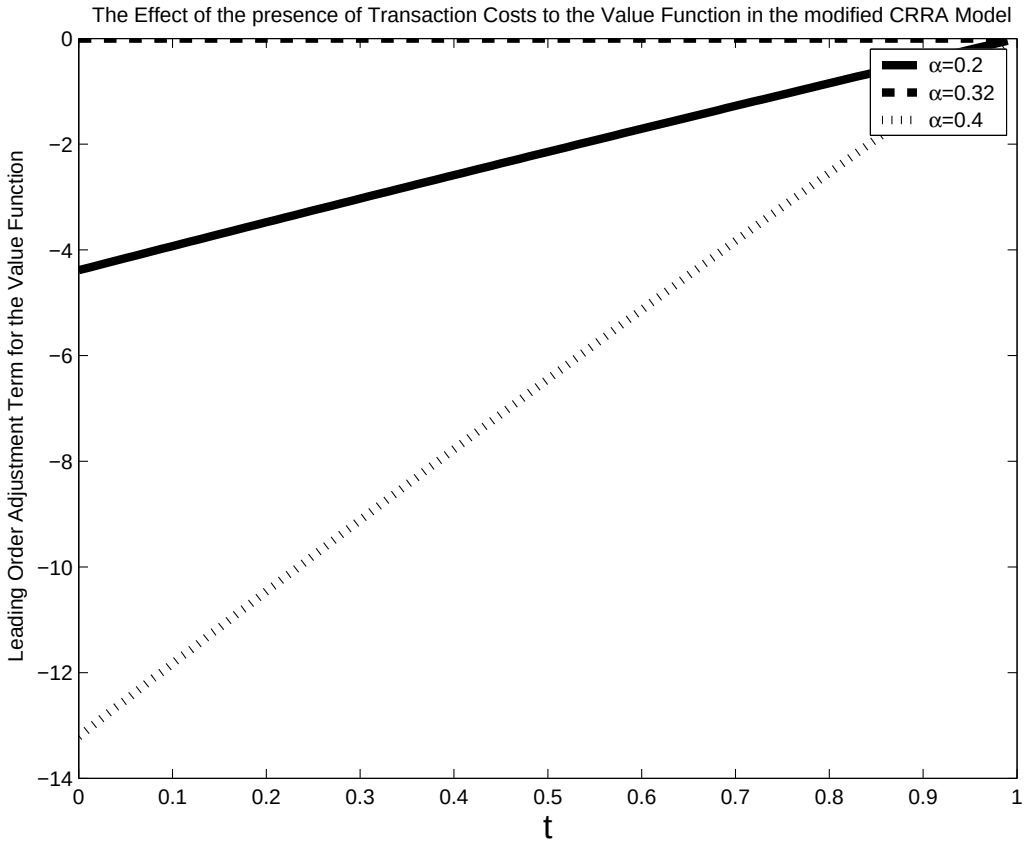


Figure 3.8: The relationship between the adjustment term $k^{2/3}H_2$ and t in the Modified CRRA Model for various values of α where $\nu = 0.07, \mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. Despite of the very complicated formula of the adjustment term, the relationship between the adjustment term and t looks linear.

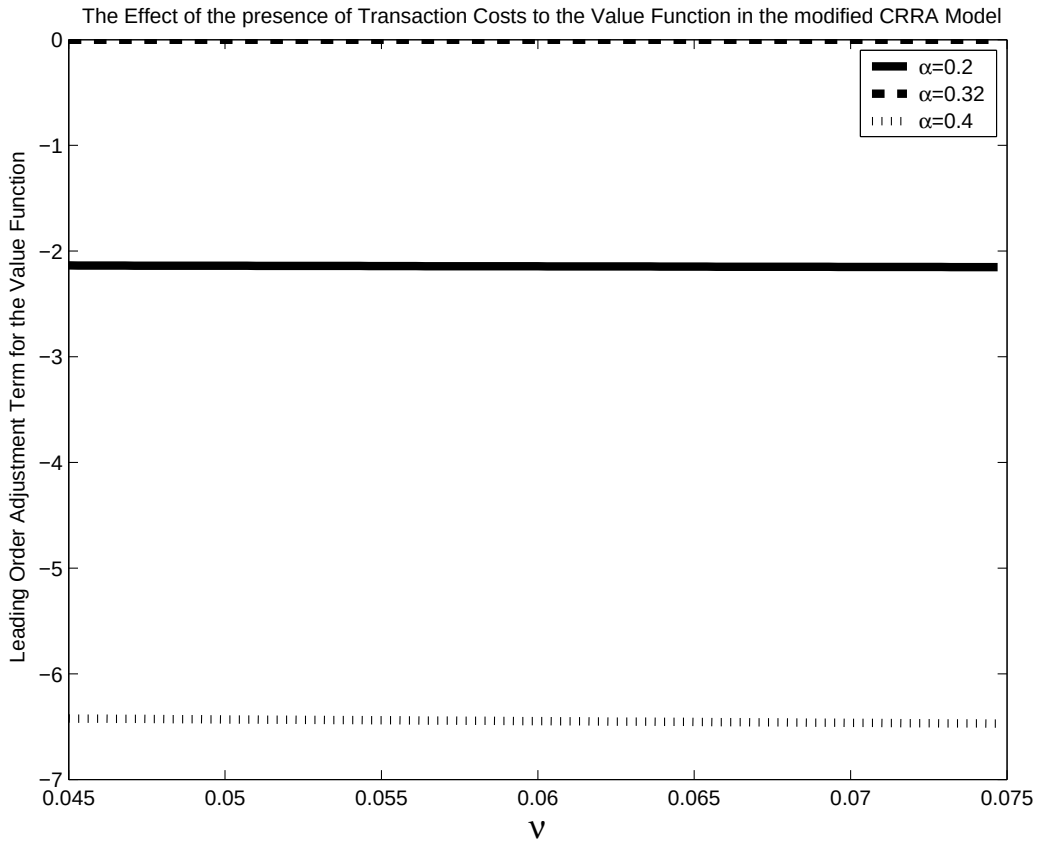


Figure 3.9: The relationship between the adjustment term $k^{2/3}H_2$ and ν in the Modified CRRA Model for various values of α where $\mu = 0.07, r = 0.05, \sigma = 0.5, S = 100, T = 1, t = 0.5, k = 0.01, \gamma = 0.6$ and $\Pi = 1000000$. we can see that the effect of ν to H_2 is very small compared to α

3.3 Multi-Asset

In this section, we study how to extend the result of Chapter 2 into a multi-asset version. The notation we use here is the same as in Chapter 2, and not related to that of Section 3.2.

The extension here is not as straightforward as in the previous section. The major problem is here we can no longer use B as the control in the no-transaction costs problem and thus use it as the variable to perturb in the transaction costs problem. Instead, here we will use A_i, the value invested in stock i. This makes the analysis more complicated.

3.3.1 Without Transaction Costs

As always, we study the no-transaction costs problem first.

We use the notation and a setup similar to the previous sections. Now, however, we consider a market with investment opportunities of n stocks and a risk free bond. Let $S_i(t)$ be the spot price of stock i at time t. Similar to previous chapters, we assume $S_i(t)$ follows a geometric Brownian motion with growth rate $\mu_i > 0$ and volatility $\sigma_i > 0$. The risk free bonds, B, compounds continuously with risk free rate r. The volatilities σ_i, growth rates μ_i and interest rate r are constants.

The models for S_i and B are

$$\begin{aligned} dS_i &= \mu_i S_i dt + \sigma_i S_i dX_i, i = 1, \cdots, n, \\ dB &= rBdt \end{aligned}$$

where X_i , $i = 1, \ldots, n$, are standard Brownian motions whose correlations $-1 \leq \rho_{ij} \leq 1$ are constants, ρ_{ij}, representing the correlation coefficient between Brownian motions X_i and X_j. There are no redundant assets and so the covariance matrix, $\mathbf{\Sigma}$,

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \ldots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \sigma_2\sigma_n\rho_{2n} & \ldots & \sigma_n^2 \end{pmatrix}$$

is non-singular.

As usual, let $A_i(t)$ be the value of assets invested in stock i. Let $\Pi(t)$ be the value of the whole portfolio,

$$\Pi = B + \sum_{i=1}^{n} A_i.$$

We assume cash generated or needed from the purchase or sale of stocks is immediately invested or withdrawn from the risk free bonds. So, Π can be described as

$$\begin{aligned} d\Pi &= rBdt + \sum_{i=1}^{n} \mu_i A_i dt + \sum_{i=1}^{n} \sigma_i A_i dX_i \\ &= r(\Pi - \sum_{i=1}^{n} A_i)dt + \sum_{i=1}^{n} \mu_i A_i dt + \sum_{i=1}^{n} \sigma_i A_i dX_i \end{aligned}$$

The A_i, $i = 1, \ldots, n$, are the non-anticipating controller representing the value being invested in stock i.

As in Chapter 2, at time $t = 0$, an investor has an amount Π_0 of resources. The problem is to allocate investments over the given time horizon so as to maximize

$$\mathbb{E}_0\Big\{ \int_0^T I(\Pi(\tilde{t}))d\tilde{t} + F(\Pi(T)) \Big\}.$$

We restate the above equation in dynamic programming form so as to apply Bellman principle of optimality. Therefore, we define the optimal expected value function $J(\Pi, t)$ as

$$J(\Pi, t) = \max_{B} \mathbb{E}_t\Big\{ \int_t^T I(\Pi(\tilde{t}))d\tilde{t} + F(\Pi) \Big\}. \tag{3.80}$$

Applying the Bellman Principle and Itô's Lemma gives us the Bellman Equations, which is

$$\begin{aligned} 0 = \max_{A_1, A_2, \ldots, A_n} \Big\{ & \frac{\partial J}{\partial t} + I + r(\Pi - \sum_{i=1}^{n} A_i)\frac{\partial J}{\partial \Pi} \\ & + \sum_{i=1}^{n} \mu_i A_i \frac{\partial J}{\partial \Pi} + \Big[\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} \sigma_i \sigma_j \rho_{ij} A_i A_j \Big] \frac{\partial^2 J}{\partial \Pi^2} \Big\}, \end{aligned} \tag{3.81}$$

with $\rho_{ij} = 1$ for $i = 1, \ldots, n$.

At $t = T$, we have

$$J(\Pi, T) = F(\Pi(T)). \tag{3.82}$$

3.3.2 Solving Bellman's Equation with No Transaction Costs

As in the previous section, we can solve the Bellman Equation by differentiating equation (3.81) with the controls.

$$\frac{\partial}{\partial A_k}\left\{\frac{\partial J}{\partial t} + I + r(\Pi - \sum_{i=1}^{n} A_i)\frac{\partial J}{\partial \Pi} + \sum_{i=1}^{n}\mu_i A_i \frac{\partial J}{\partial \Pi} + \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\sigma_i\sigma_j\rho_{ij}A_iA_j\right]\frac{\partial^2 J}{\partial \Pi^2}\right\}$$

$$= \quad (\mu - r)J_\Pi + \sum_{i=1}^{n}\sigma_{ki}A_iJ_{\Pi\Pi}.$$

If we define

$$\vec{\hat{\mu}} = \begin{pmatrix} \mu_1 - r \\ \vdots \\ \mu_n - r \end{pmatrix},$$

$$\vec{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix},$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \dots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \sigma_2\sigma_n\rho_{2n} & \dots & \sigma_n^2 \end{pmatrix},$$

and the superscripts T and -1 to denote transpose and inverse, respectively. Therefore, we have

$$J_\Pi\vec{\hat{\mu}} + \mathbf{\Sigma}\vec{A}J_{\Pi\Pi} \quad = \quad 0. \tag{3.83}$$

or equivalently

$$\vec{A} \quad = \quad -\frac{J_\Pi}{J_{\Pi\Pi}}\mathbf{\Sigma}^{-1}\vec{\hat{\mu}}. \tag{3.84}$$

We rewrite equation (3.81) in matrix notation as

$$0 \quad = \quad \max_{\vec{A}}\left\{\frac{\partial J}{\partial t} + I + r\Pi\frac{\partial J}{\partial \Pi} + \vec{\hat{\mu}}^T\vec{A} + \frac{1}{2}\vec{A}^T\mathbf{\Sigma}\vec{A}\frac{\partial^2 J}{\partial \Pi^2}\right\}. \tag{3.85}$$

If we put equation (3.84) back into equation (3.81), we have

$$0 \quad = \quad \frac{\partial J}{\partial t} + I + r\Pi\frac{\partial J}{\partial \Pi} - \frac{1}{2}\vec{\hat{\mu}}^T\mathbf{\Sigma}^{-1}\vec{\hat{\mu}}\frac{J_\Pi^2}{J_{\Pi\Pi}}. \tag{3.86}$$

When $n = 1$, equation (3.86) becomes

$$0 \quad = \quad \frac{\partial J}{\partial t} + I + r\Pi\frac{\partial J}{\partial \Pi} - \frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2\frac{J_\Pi^2}{J_{\Pi\Pi}}. \tag{3.87}$$

This is consistent with the result in the previous chapter.

3.3.3 Long Term Growth Model

We can apply this multi-asset optimization to the Long Term Growth Model as in Section 1.11, which is to maximize $\mathbb{E}[\log \Pi]$.

We have

$$J(\Pi, t) = \log \Pi + (r + \frac{1}{2}\vec{\hat{\mu}}^T \mathbf{\Sigma}^{-1}\vec{\hat{\mu}})(T-t). \tag{3.88}$$

and

$$\vec{A}^* = \Pi \mathbf{\Sigma}^{-1}\vec{\hat{\mu}} \tag{3.89}$$

So, when $n = 1$, we have

$$A^* = \frac{(\mu - r)^2}{\sigma}\Pi. \tag{3.90}$$

The result here is consistent to those in Section 1.11.

3.3.4 Constant Relative Risk Aversion (CRRA) Model

As for the CRRA model in Chapter 2, the objective functions of the this multi-asset version are

$$\begin{aligned} I(\Pi, t) &= \Big(\frac{\gamma(\mu - r)^2}{2(\gamma - 1)\sigma^2} - \gamma r + \bar{\nu}\Big)e^{\bar{\nu}(T-t)}\Pi^\gamma \\ F(\Pi(T)) &= \Pi(T)^\gamma. \end{aligned} \tag{3.91}$$

This gives us

$$J(\Pi, t) = e^{\bar{\nu}(T-t)}\Pi^\gamma,$$

and

$$\vec{A} = \frac{\Pi \mathbf{\Sigma}^{-1}\vec{\hat{\mu}}}{1 - \gamma}. \tag{3.92}$$

3.3.5 With Transaction Costs

Now we consider the problem with transaction costs. As in the previous chapter, let $k > 0$ represents the portion of transaction of any stock used as transaction costs. For simplicity, we also assume $\rho_{ij} = 0$ whenever $i \neq j$.

The market model equations become

$$\begin{aligned}
dS_i &= \mu_i S_i dt + \sigma_i S_i dX_i, \\
dB &= rBdt - (1+k)d\mathbf{L}_i(t) + (1-k)d\mathbf{M}_i(t) \\
&= r(\Pi - \sum_{i=1}^{n} A_i)dt - (1+k)d\mathbf{L}_i(t) + (1-k)d\mathbf{M}_i(t), \\
dA_i &= \mu_i A_i dt + d\mathbf{L}_i(t) - d\mathbf{M}_i(t)dt + \sigma_i A_i dX_i, i = 1, \cdots, n, \\
d\Pi &= r(\Pi - \sum_{i=1}^{n} A_i)dt + \sum_{i=1}^{n} \mu_i A_i dt + \sum_{i=1}^{n} \sigma_i A_i dX_i \\
&\quad - \sum_{i=1}^{n} k d\mathbf{L}_i(t) - \sum_{i=1}^{n} k d\mathbf{M}_i(t),
\end{aligned} \tag{3.93}$$

where $\mathbf{L}_i(t)$ and $\mathbf{M}_i(t)$ represent the cumulative purchase and sale of assets A_i in $[0, t]$, which we use as the controls. Here, A_i are only used to denote the value of assets invested in risk-free bonds here and it is no longer used as a control.

The optimal expected value function $\tilde{J}(\Pi, A_1, A_2, \cdots, A_n, t)$ is

$$\tilde{J}(\Pi, A_1, A_2, \cdots, A_n, t) = \max_{\mathbf{L}_i, \mathbf{M}_i} \mathbb{E}\Big\{ \int_t^T I(\Pi(\tilde{t}))d\tilde{t} + F(\Pi(T)) \Big\}. \tag{3.94}$$

Applying Bellman's Principle, we have

$$\begin{aligned}
&\max_{\mathbf{l}_i, \mathbf{m}_i} \Big\{ I + \frac{\partial \tilde{J}}{\partial t} + \sum_{i=1}^{n} (\mu_i A_i + \mathbf{l}_i - \mathbf{m}_i) \frac{\partial \tilde{J}}{\partial A_i} + \\
&\Big(r(\Pi - \sum_{i=1}^{n} A_i) + \sum_{i=1}^{n} (\mu_i A_i - k d\mathbf{l}_i - k\mathbf{m}_i) \Big) \frac{\partial \tilde{J}}{\partial \Pi} \\
&+ \sum_{i=1}^{n} \Big(\frac{1}{2}\sigma_i^2 A_i^2 \frac{\partial^2 \tilde{J}}{\partial A_i^2} + \frac{1}{2}\sigma_i^2 A_i^2 \frac{\partial^2 \tilde{J}}{\partial \Pi^2} + \sigma_i^2 A_i^2 \frac{\partial^2 \tilde{J}}{\partial \Pi \partial A_i} \Big) = 0.
\end{aligned} \tag{3.95}$$

In the no-transaction region, therefore, the value function $\tilde{J}$ satisfies

$$
\begin{aligned}
0 &= I + \frac{\partial \tilde{J}}{\partial t} + \Big(r(\Pi - \sum_{i=1}^{n} A_i) + \sum_{i=1}^{n} \mu_i A_i\Big)\frac{\partial \tilde{J}}{\partial \Pi} \\
&\quad + \sum_{i=1}^{n} \Big(\frac{1}{2}\sigma_i^2 A_i^2 \frac{\partial^2 \tilde{J}}{\partial A_i^2} + \frac{1}{2}\sigma_i^2 A_i^2 \frac{\partial^2 \tilde{J}}{\partial \Pi^2} + \sigma_i^2 A_i^2 \frac{\partial^2 \tilde{J}}{\partial \Pi \partial A_i}\Big). \qquad (3.96)
\end{aligned}
$$

In the sales region for stock i, $i = 1, \ldots, n$, (and at the boundary between sales region and no transaction region), $\tilde{J}$ satisfies

$$
k\frac{\partial \tilde{J}}{\partial \Pi} = -\frac{\partial \tilde{J}}{\partial A_i}. \qquad (3.97)
$$

In the purchase region for stock i, $i = 1, \ldots, n$, (and at the boundary between sales region and no transaction region), $\tilde{J}$ satisfies

$$
k\frac{\partial \tilde{J}}{\partial \Pi} = \frac{\partial \tilde{J}}{\partial A_i}. \qquad (3.98)
$$

The smooth pasting equation at the boundaries between the no-transaction region and sales region for stock i, $i = 1, \ldots, n$, is

$$
\frac{\partial^2 \tilde{J}}{\partial A_i^2} = -k\frac{\partial^2 \tilde{J}}{\partial A_i \partial \Pi}, \qquad (3.99)
$$

and at the boundaries between the no-transaction region and purchase region is

$$
\frac{\partial^2 \tilde{J}}{\partial A_i^2} = k\frac{\partial^2 \tilde{J}}{\partial A_i \partial \Pi}. \qquad (3.100)
$$

Also, when $t = T$, we have the final equation

$$
\tilde{J}(\Pi, A_1, A_2, \cdots, A_n, T) = F(\Pi). \qquad (3.101)
$$

For $i = 1, \ldots, n$, we translate the A_i coordinate according to

$$
A_i = \mathfrak{A}_i(\Pi, t) + k^{1/3}\alpha_i, \qquad (3.102)
$$

where $\mathfrak{A}$ is the value of stock i we have when the level of transaction costs, k_i, tends to zero.

We let $H(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, t) = \tilde{J}(\Pi, A_1, A_2, \cdots, A_n, t)$.

We have

$$\begin{aligned}
\frac{\partial \tilde{J}}{\partial A_i} &= k^{-1/3}\frac{\partial H}{\partial \alpha_i}, \\
\frac{\partial \tilde{J}}{\partial \Pi} &= \frac{\partial H}{\partial \Pi} - \sum_{i=1}^{n} k^{-1/3}\frac{\partial H}{\partial \alpha_i}\frac{\partial \mathfrak{A}_i}{\partial \Pi}, \\
\frac{\partial \tilde{J}}{\partial t} &= \frac{\partial H}{\partial t} - \sum_{i=1}^{n} k^{-1/3}\frac{\partial H}{\partial \alpha_i}\frac{\partial \mathfrak{A}_i}{\partial t}, \\
\frac{\partial^2 \tilde{J}}{\partial A_i^2} &= k^{-2/3}\frac{\partial^2 H}{\partial \alpha_i^2}, \\
\frac{\partial^2 \tilde{J}}{\partial A_i \partial A_j} &= k^{-2/3}\frac{\partial^2 H}{\partial \alpha_i \partial \alpha_j}, i \neq j, \\
\frac{\partial^2 \tilde{J}}{\partial \Pi^2} &= \frac{\partial^2 H}{\partial \Pi^2} - \sum_{i=1}^{n} k^{-1/3}(2\frac{\partial^2 H}{\partial \alpha_i \partial \Pi}\frac{\partial \mathfrak{A}_i}{\partial \Pi} + \frac{\partial H}{\partial \alpha_i}\frac{\partial^2 \mathfrak{A}_i}{\partial \Pi^2}) + \sum_{i=1}^{n}\sum_{j=1}^{n} k^{-2/3}\frac{\partial^2 H}{\partial \alpha_i \partial \alpha_j}\frac{\partial \mathfrak{A}_i}{\partial \Pi}\frac{\partial \mathfrak{A}_j}{\partial \Pi}, \\
\frac{\partial^2 \tilde{J}}{\partial \Pi \partial A_i} &= k^{-1/3}\frac{\partial^2 H}{\partial \Pi \partial \alpha_i} - k^{-2/3}\sum_{j=1}^{n}\frac{\partial^2 H}{\partial \alpha_i \partial \alpha_j}\frac{\partial \mathfrak{A}_j}{\partial \Pi}.
\end{aligned}$$

Also, we let

$$\begin{aligned}
H(\Pi, , \alpha_1, \alpha_2, \cdots, \alpha_n, t) &= H_0(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, t) + k^{1/3}H_1(\Pi, , \alpha_1, \alpha_2, \cdots, \alpha_n, t) \\
&\quad + k^{2/3}H_2(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, t) + kH_3(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, t) \\
&\quad + k^{4/3}H_4(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, t) + \cdots . \qquad (3.103)
\end{aligned}$$

$H_0, H_1, H_2, H_3, H_4, \cdots$ are all of $O(1)$. H_0 is the value function when transaction costs level $k \to 0$.

After the change of coordinates and collecting terms of the same order the sales region equation, for $i = 1, \ldots, n$, we have

$$\frac{\partial H_0}{\partial \alpha_i} = 0 \qquad (3.104)$$

$$\frac{\partial H_1}{\partial \alpha_i} = 0 \qquad (3.105)$$

$$\frac{\partial H_2}{\partial \alpha_i} = 0 \qquad (3.106)$$

$$\frac{\partial H_3}{\partial \alpha_i} - \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial H_0}{\partial \alpha_j} = 0 \tag{3.107}$$

$$\frac{\partial H_4}{\partial \alpha_i} + \frac{\partial H_0}{\partial \Pi} - \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial H_1}{\partial \alpha_j} = 0. \tag{3.108}$$

Similarly, we have the following equations for the purchase region,

$$\frac{\partial H_0}{\partial \alpha_i} = 0 \tag{3.109}$$

$$\frac{\partial H_1}{\partial \alpha_i} = 0 \tag{3.110}$$

$$\frac{\partial H_2}{\partial \alpha_i} = 0 \tag{3.111}$$

$$\frac{\partial H_3}{\partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial H_0}{\partial \alpha_j} = 0 \tag{3.112}$$

$$\frac{\partial H_4}{\partial \alpha_i} - \frac{\partial H_0}{\partial \Pi} + \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial H_1}{\partial \alpha_j} = 0. \tag{3.113}$$

As for the smooth pasting equation for the sales boundary, we have

$$\frac{\partial^2 H_0}{\partial \alpha_i^2} = 0 \tag{3.114}$$

$$\frac{\partial^2 H_1}{\partial \alpha_i^2} = 0 \tag{3.115}$$

$$\frac{\partial^2 H_2}{\partial \alpha_i^2} = 0 \tag{3.116}$$

$$\frac{\partial H_3}{\partial \alpha_i^2} - \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial^2 H_0}{\partial \alpha_j^2} = 0 \tag{3.117}$$

$$\frac{\partial H_4}{\partial \alpha_i^2} + \frac{\partial^2 H_0}{\partial \Pi \partial \alpha_i} - \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial^2 H_1}{\partial \alpha_i \partial \alpha_j} = 0. \tag{3.118}$$

Similarly, the smooth pasting equation for the purchase boundary becomes

$$\frac{\partial^2 H_0}{\partial \alpha_i^2} = 0 \tag{3.119}$$

$$\frac{\partial^2 H_1}{\partial \alpha_i^2} = 0 \tag{3.120}$$

$$\frac{\partial^2 H_2}{\partial \alpha_i^2} = 0 \tag{3.121}$$

$$\frac{\partial H_3}{\partial \alpha_i^2} + \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial^2 H_0}{\partial \alpha_j^2} = 0 \tag{3.122}$$

$$\frac{\partial H_4}{\partial \alpha_i^2} - \frac{\partial^2 H_0}{\partial \Pi \partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \frac{\partial^2 H_1}{\partial \alpha_i \partial \alpha_j} = 0. \tag{3.123}$$

The final condition equation becomes

$$H_0(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, T) = F(\Pi) \tag{3.124}$$

$$H_1(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, T) = 0 \tag{3.125}$$

$$H_2(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, T) = 0 \tag{3.126}$$

$$H_3(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, T) = 0 \tag{3.127}$$

$$H_4(\Pi, \alpha_1, \alpha_2, \cdots, \alpha_n, T) = 0. \tag{3.128}$$

Like what we have done before, we now consider the equation at the no-transaction region. After translating the coordinates, expanding H according to equation (3.103), and collecting terms of the same order in k, we have the equations below.

1. $O(k^{-2/3})$ Equation

$$\mathfrak{D}(H_0) = 0 \tag{3.129}$$

where $\mathfrak{D}$ is an operator defined as

$$\begin{aligned}
\mathfrak{D} &= \frac{1}{2}\left(\sum_{i=1}^{n} \sigma_i^2 \mathfrak{A}_i^2\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \frac{\partial \mathfrak{A}_i}{\partial \Pi} \frac{\partial \mathfrak{A}_j}{\partial \Pi}\right) \\
&\quad + \frac{1}{2}\left(\sum_{i=1}^{n} \sigma_i^2 \mathfrak{A}_i^2 \frac{\partial^2}{\partial \alpha_i^2}\right) \\
&\quad - \sum_{i=1}^{n} \left(\sigma_i^2 \mathfrak{A}_i^2\right)\left(\sum_{j=1}^{n} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \frac{\partial \mathfrak{A}_j}{\partial \Pi}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \left\{ \frac{1}{2} \frac{\partial \mathfrak{A}_i}{\partial \Pi} \frac{\partial \mathfrak{A}_j}{\partial \Pi} \left(\sum_{l=1}^{n} \sigma_l^2 \mathfrak{A}_l^2\right) \right. \\
&\quad \left. + \frac{1}{2} \sigma_i^2 \mathfrak{A}_i^2 - \frac{\partial \mathfrak{A}_i}{\partial \Pi} \sigma_i^2 \mathfrak{A}_i^2 \right\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \mathfrak{D}_{ij},
\end{aligned} \tag{3.130}$$

where

$$\mathfrak{D}_{ij} = \frac{1}{2}\frac{\partial \mathfrak{A}_i}{\partial \Pi}\frac{\partial \mathfrak{A}_j}{\partial \Pi}(\sum_{l=1}^{n}\sigma_l^2\mathfrak{A}_l^2) + \frac{1}{2}\sigma_i\sigma_j\rho_{ij}\mathfrak{A}_i\mathfrak{A}_j - \frac{\partial \mathfrak{A}_i}{\partial \Pi}\sigma_i^2\mathfrak{A}_i^2 \quad (3.131)$$

Note that in the definition for $\mathfrak{D}_{ij}$, the term ρ_{ij} vanishes whenever $i \neq j$. Using the same argument as in Section 2.6.4, we can conclude that H_0 is independent of α_i for $i = 1, \ldots, n$.

2. $O(k^{-1/3})$ Equation

$$\mathfrak{D}(H_1) = 0. \quad (3.132)$$

Similarly, we can conclude that H_1 is independent of α_i for $i = 1, \ldots, n$.

3. $O(1)$ Equation

$$\mathfrak{D}(H_2) = -\mathfrak{M}(H_0), \quad (3.133)$$

with $\mathfrak{M}$ is an operator defined as

$$\begin{aligned}\mathfrak{M} &= \frac{\partial}{\partial t} + I + r(\Pi - \sum_{i=1}^{n}\mathfrak{A}_i)\frac{\partial}{\partial \Pi} \\ &\quad + \sum_{i=1}^{n}\mu_i\mathfrak{A}_i\frac{\partial}{\partial \Pi} + \left(\sum_{i=1}^{n}\frac{1}{2}\sigma_i^2\mathfrak{A}_i^2\right)\frac{\partial^2}{\partial \Pi^2}. \quad (3.134)\end{aligned}$$

Similar to Section 2.6.6, H_2 here is independent of α_i, $i = 1, \ldots, n$. Also, we have $H_0 = J$.

4. $O(k^{1/3})$ Equation

$$\mathfrak{D}(H_3) = -\sum_{i=1}^{n}\frac{\partial \mathfrak{M}(H_0)}{\partial \mathfrak{A}_i}\alpha_i - \mathfrak{M}(H_1). \quad (3.135)$$

As in Section 2.6.7, we know that H_3 is independent of α_i for $i = 1, \ldots, n$. Also, we have $H_1 = 0$.

Unlike Section 2.6.7, we have

$$\sum_{i=1}^{n}\frac{\partial \mathfrak{M}(H_0)}{\partial \alpha_i}\mathfrak{A}_i = 0 \quad (3.136)$$

instead of equation (2.82).

5. $O(k^{2/3})$ Equation

$$\mathfrak{D}H_4 = -\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\frac{\partial^2 H_0}{\partial\Pi^2}\alpha_i^2 - \mathfrak{M}H_2. \tag{3.137}$$

In other words, it is

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 H_4}{\partial\alpha_i\partial\alpha_j}\mathfrak{D}_{ij} = -\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2\frac{\partial^2 H_0}{\partial\Pi^2}\alpha_i^2 - \mathfrak{M}H_2. \tag{3.138}$$

Again, the ρ_{ij} above vanishes whenever $i \neq j$.

We notice that the no transaction boundary for stock i is dependent on the holdings of other stocks (α_j). Here, we study the case for the transaction boundary for stock 1 with $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$.

We let α_+ be the boundary between the purchase region and the no transaction region for α_1 when $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$. Similarly, we let α_- be the boundary between the sales region and the no transaction region. Therefore, from equation (3.138), equation (3.108) and equation (3.118) we know that α_- satisfy

$$\begin{aligned}\frac{\partial H_4}{\partial\alpha_1} + \frac{\partial H_0}{\partial\Pi} &= 0\\ \frac{\partial H_4}{\partial\alpha_1^2} &= 0.\end{aligned} \tag{3.139}$$

Similarly, from equation (3.138), equation (3.113) and equation (3.123) we know that α_+ satisfy

$$\begin{aligned}\frac{\partial H_4}{\partial\alpha_1} - \frac{\partial H_0}{\partial\Pi} &= 0\\ \frac{\partial H_4}{\partial\alpha_1^2} &= 0.\end{aligned} \tag{3.140}$$

We let

$$H_4 = H_4^4\alpha_1^4 + H_4^2\alpha_1^2 + H_4^1\alpha_1 + H_4^0 + \alpha^*, \tag{3.141}$$

where α^* vanishes when $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$.

From equation (3.118) and equation (3.123), we know that α_+ and α_- satisfy

$$\frac{\partial H_4}{\partial \alpha_1^2} = 0. \tag{3.142}$$

So, α_+ and α_- are the roots of

$$0 = 6H_4^4\alpha_1^2 + H_4^2. \tag{3.143}$$

So, we know that $\alpha_+ = -\alpha_-$ because the sum of roots vanishes.

Now, we use equation (3.108) and equation (3.113). This gives us α_- satisfies

$$-\frac{\partial H_0}{\partial \Pi} = 4H_4^4\alpha_-^3 + 2H_4^2\alpha_- + H_4^1. \tag{3.144}$$

Similarly, α_+ satisfies

$$\frac{\partial H_0}{\partial \Pi} = 4H_4^4\alpha_+^3 + 2H_4^2\alpha_+ + H_4^1. \tag{3.145}$$

Adding equation (3.144) with equation (3.145), we have

$$H_4^1 = 0. \tag{3.146}$$

Putting equation (3.143) into equation (3.145), we have

$$\frac{\partial H_0}{\partial \Pi} = 16H_4^4\alpha_+^3 \tag{3.147}$$

and

$$H_4^2 = -\Big(\frac{27H_4^4}{8}\Big(\frac{\partial H_0}{\partial \Pi}\Big)^2\Big)^{1/3}. \tag{3.148}$$

3.3.6 Financial Interpretations

Although we have not solved the above equations precisely, still it gives us some insights on the effect of transaction costs on the portfolio optimization problem. We summarize our findings below.

1. The midpoint of the no-transaction region for stock i may no longer be the optimal solution of the no transaction costs problem. In fact, the midpoint depends on the value of other stocks held. This is because if we solve equation (3.138),

$$H_4 \;=\; H_4^4\alpha_i^4 + H_4^3\alpha_i^3 + H_4^2\alpha_i^2 + H_4^1\alpha_i + H_4^0, \tag{3.149}$$

with H_4^4, H_4^3, H_4^2 and H_4^1 are all functions of α_j with $j \neq i$. This is because H_4^3 may not vanish. From equation (3.118) and equation (3.123), we know that $6H_4^3$ is the sum of α_i^+ and α_i^-, which are the position of the sales boundary and the purchase boundary of stock i. So, it is possible

$$\alpha_i^+ + \alpha_i^- \;\neq\; 0. \tag{3.150}$$

2. The width of the no-transaction region for every stock i is still directly proportional to $k^{1/3}$, and so the larger the transaction costs, the larger the no-transaction region.

3. The effect of transaction costs on the value function is of the order $k^{2/3}$.

3.3.7 Two Dimensional Case

Now we look in detail at the case where there are two risky assets and the bond. In other words, $n = 2$.

We let

$$\begin{aligned} H_4 \;=\;& a\alpha_1^4 + b\alpha_1^3\alpha_2 + c\alpha_1^2\alpha_2^2 + d\alpha_1\alpha_2^3 + e\alpha_2^4 + f\alpha_1^2 + g\alpha_1\alpha_2 + h\alpha_2^2 \\ & + p\alpha_1 + q\alpha_2 + r. \end{aligned} \tag{3.151}$$

From the previous section, we know that $p = q = 0$.

So, we have

$$\begin{aligned} \mathfrak{D}H_4 \;=\;& \alpha_1^2(12a\mathfrak{D}_{11} + 3b(\mathfrak{D}_{12} + \mathfrak{D}_{21}) + 2c\mathfrak{D}_{22}) \\ & + \alpha_1\alpha_2(6b\mathfrak{D}_{11} + 4c(\mathfrak{D}_{12} + \mathfrak{D}_{21}) + 6d\mathfrak{D}_{22}) \\ & + \alpha_2^2(2c\mathfrak{D}_{11} + 3d(\mathfrak{D}_{12} + \mathfrak{D}_{21}) + 12e\mathfrak{D}_{22}) \\ & + 2f\mathfrak{D}_{11} + g(\mathfrak{D}_{12} + \mathfrak{D}_{21}) + 2h\mathfrak{D}_{22}. \end{aligned} \tag{3.152}$$

Using equation (3.138), so the solution for (a, b, c, d, e) is

$$\begin{aligned}(a,b,c,d,e) &= (-\frac{\sigma_1^2(\mathfrak{D}_{12}+\mathfrak{D}_{21})}{24\mathfrak{D}_{11}(\mathfrak{D}_{12}+\mathfrak{D}_{21})}\frac{\partial^2 H_0}{\partial \Pi^2}, 0, 0, 0, -\frac{\sigma_2^2(\mathfrak{D}_{12}+\mathfrak{D}_{21})}{24\mathfrak{D}_{22}(\mathfrak{D}_{12}+\mathfrak{D}_{21})}\frac{\partial^2 H_0}{\partial \Pi^2}) \\ &+x(-\frac{1}{4\mathfrak{D}_{11}^2}, \frac{1}{\mathfrak{D}_{11}(\mathfrak{D}_{12}+\mathfrak{D}_{21})}, 0, -\frac{1}{\mathfrak{D}_{22}(\mathfrak{D}_{12}+\mathfrak{D}_{21})}, \frac{1}{4\mathfrak{D}_{22}^2}) \\ &+y(-\frac{\mathfrak{D}_{22}}{2\mathfrak{D}_{11}(\mathfrak{D}_{12}+\mathfrak{D}_{21})}+\frac{\mathfrak{D}_{12}+\mathfrak{D}_{21}}{4\mathfrak{D}_{11}^2}, -\frac{1}{\mathfrak{D}_{11}}, \frac{3}{\mathfrak{D}_{12}+\mathfrak{D}_{21}}, -\frac{1}{\mathfrak{D}_{22}}, \\ &-\frac{\mathfrak{D}_{11}}{2\mathfrak{D}_{22}(\mathfrak{D}_{12}+\mathfrak{D}_{21})}+\frac{\mathfrak{D}_{12}+\mathfrak{D}_{21}}{4\mathfrak{D}_{22}^2}) \end{aligned} \quad (3.153)$$

where x and y are constants.

Obviously, $x = 0$ or otherwise H_4 is not symmetric over α_1 and α_2.

Now, we consider the transaction boundary for stock 1 (α^+, α^-) when $\alpha_2 = \alpha_*$.

From equation (3.118) and equation (3.123), we know that α_+ and α_- are the roots of α_1 in the equation of

$$12a\alpha_1^2 + 6b\alpha_1\alpha_* + 2c\alpha_*^2 + 2f = 0. \quad (3.154)$$

Therefore, we have

$$\begin{aligned}\alpha_+ + \alpha_- &= -\frac{b\alpha_*}{2a} \\ \alpha_+\alpha_- &= \frac{c\alpha_*^2 + f}{6a}\end{aligned} \quad (3.155)$$

Now, we use equation (3.108) and equation (3.113). This gives us an equation for α_+ and α_-,

$$\begin{aligned}4a(\alpha_+^3+\alpha_-^3) + 3b(\alpha_+^2+\alpha_-^2)\alpha_* + 2c(\alpha_++\alpha_-)\alpha_*^2 \\ +2d\alpha_*^3 + 2f(\alpha_++\alpha_-) + 2g\alpha_* &= 0,\end{aligned} \quad (3.156)$$

which means

$$\begin{aligned}4a(\alpha_++\alpha_-)((\alpha_++\alpha_-)^2 - 3\alpha_+\alpha_-) \\ +3b((\alpha_++\alpha_-)^2 - 2\alpha_+\alpha_-)\alpha_* \\ +2c(\alpha_++\alpha_-)\alpha_*^2 + 2d\alpha_*^3 + 2f(\alpha_++\alpha_-) + 2g\alpha_* &= 0.\end{aligned} \quad (3.157)$$

Therefore, we have

$$\begin{aligned} 4a(-\frac{b\alpha_*}{2a})((-\frac{b\alpha_*}{2a})^2 - 3\frac{c\alpha_2^2+f}{6a}) \\ +3b((-\frac{b\alpha_*}{2a})^2 - 2\frac{c\alpha_2^2+f}{6a})\alpha_* \\ +2c(-\frac{b\alpha_*}{2a})\alpha_*^2 + 2d\alpha_*^3 + 2f(-\frac{b\alpha_*}{2a}) + 2g\alpha_* &= 0. \end{aligned} \tag{3.158}$$

Since the above relationship holds for all α_*, so the coefficient for α_*^3 and α_* have to be both identically zero. In other words, we have

$$\frac{b^3}{4a^2} - \frac{bc}{a} + 2d = 0 \tag{3.159}$$

and

$$-\frac{bf}{a} + 2g = 0. \tag{3.160}$$

By considering the transaction boundary for α_2, similarly we have

$$\frac{d^3}{4e^2} - \frac{cd}{e} + 2b = 0 \tag{3.161}$$

and

$$-\frac{dh}{e} + 2g = 0. \tag{3.162}$$

We can use equation (3.159) and equation (3.161) to solve for y. Indeed, we find that there is a unique solution of $y = 0$.

So, we have $b = 0$, $c = 0$ and $d = 0$, which implies $g = 0$.

Using equation (3.148) , we have

$$f = -\Big(\frac{27a}{8}\Big(\frac{\partial H_0}{\partial \Pi}\Big)^2\Big)^{1/3} \tag{3.163}$$

and

$$h = -\Big(\frac{27e}{8}\Big(\frac{\partial H_0}{\partial \Pi}\Big)^2\Big)^{1/3}. \tag{3.164}$$

So, α_+ and α_- are

$$\alpha_\pm = \pm\sqrt{-\frac{f}{6a}}. \tag{3.165}$$

Also, H_2 can be solved by using the equation

$$\mathfrak{M}H_2 = -2f\mathfrak{D}_{11} - 2h\mathfrak{D}_{22}. \tag{3.166}$$

3.3.8 Financial Interpretations

The above equations are very long and complex. Still, we have find some pattern about the behavior or the solution.

1. The midpoint of the no-transaction region for stock i is the same as the optimal solution of the no transaction costs problem. This is because $b = d = 0$.

2. The distance of the holding of a stock to the no transaction cost optimum solution (α_*) has no effect on the no transaction region of another stock. This is very clear from equation (3.165).

3. The width of the no-transaction region for every stock i is still directly proportional to $k^{1/3}$, and so the larger the transaction costs, the larger the no-transaction region.

4. The effect of transaction costs on the value function is of the order $k^{2/3}$.

5. The relations between μ, r or σ to the transaction boundary or the effect of transaction costs on the value function is very complex and not clear cut.

Chapter 4

The Convergence of the No Transaction Region in the Transaction Costs Problem

4.1 Introduction

In this chapter, we use the Pontryagin Maximum Principle to demonstrate a mathematical relationship between the solution for an optimization problem without transaction costs and the solution for the otherwise same optimization problem with transaction costs. The optimization problem we are considering here is more general than those in previous chapters.

We consider the problem without transaction costs first. In that problem, an investor has a fixed time investment interval $[0, T]$. The investor can invest his resources in risk free bonds or in stock i, where $i = 1, 2, \cdots n$. At time t, the investor has an amount of $A_i(t)$ in stock i and $B(t)$ in the risk free bond. His problem is to allocate investments over the given time horizon so as to maximize a certain objective function of Π where

$$\Pi \quad = \quad B + \sum_{i=1}^{n} A_i.$$

We assume in the above optimization problem there is a unique interior solution A_i^* for stocks $i = 1, \ldots, n$.

Now we consider the transaction costs problem. In this transaction costs problem, however, there are two difference. Firstly, of course, there are transaction costs. By

transaction costs we mean that a cost has to be paid when the investor buys or sells stocks. Let $k_i > 0$ represents the portion of transaction of stocks i used as transaction costs. So if the investor buys a number of stocks i whose "true" value is $\tilde{A}_i$, the investor pays $(1 + k_i)\tilde{A}_i$ in cash. If the investor sells the stocks, the investor obtains $(1 - k_i)\tilde{A}_i$ in cash. Secondly, instead of maximizing $J(\Pi)$, the problem is to maximize $J(\tilde{\Pi})$, where

$$\tilde{\Pi} \;=\; B + \sum_{i=1}^{n} \kappa_i A_i$$

with $\kappa_i A_i$ represents the liquidation value of A_i; $\kappa_i A_i$ depends on k_i and the sign of A_i such that for $i = 1, \cdots, n$, if $A_i > 0$

$$1 - k_i < \kappa_i \leq 1$$

and if $A_i < 0$

$$1 \leq \kappa_i < 1 + k_i.$$

In this chapter, we demonstrate the following. In the optimization problem with transaction costs, for every stock $i = 1, \ldots, n$, the optimal strategy is not to transact when the value of investments in stock i is close to the no transaction costs optimal strategy A_i^*. Let us call the region where it is optimal not to transact at all the no-transaction region. When the portfolio is too far away from A_i^*, it is optimal to buy or sell so as to bring it back to the no-transaction region. Also as the level of transaction costs k_i tends to 0, the no-transaction region tends to A_i^*.

We recall that one of the important results in our analysis at Chapter 3 is that the no-transaction region in the transaction cost problem converges to the optimal no transaction cost solution when the transaction costs level $k \to 0$. That result is similar to the result in this chapter. This chapter's result, however, is more general. This is because the value function J we want to maximize can depend on the transaction costs k_i through the variables κ_i. Also, we allow different transaction costs for different stocks.

Many transaction costs problems usually involve HJB equations which lead to a free boundary problem. Perturbation analysis usually is very useful in determining the free boundary. A common difficulty, however, in applying perturbation analysis in such problems is the determination of the limit of the no transaction region when the

transaction costs tend to zero. Failure to resolve this difficulty makes the perturbation analysis less robust than desired. The result in this chapter, of course, solves precisely this problem.

4.2 Market Model without Transaction Costs

We now proceed by first considering the case of no transaction costs.

The setup of the market model is similar to Chapter 3 multi asset Section. We consider a market with investment opportunities of n stocks and risk free bonds. Let $S_i(t)$ be the spot prices of stock i at time t. We assume that $S(t)$ follows a geometric Brownian motion with growth rate $\mu_i > 0$ and variance $\sigma_i^2 > 0$. The risk free bonds, B, are compounded continuously with risk free rate r. The variance σ_i^2, drift μ_i of asset i, and the risk free rate r are assumed to be constants. Assuming there is no transfer of money between stocks and the risk free bond, the model is represented by

$$\begin{aligned} dS_i &= \mu_i S_i dt + \sigma_i S_i dX_i, \qquad i = 1, \cdots, n, \\ dB &= rBdt. \end{aligned}$$

Here, X_i are standard Brownian motions with $dX_i^2 = dt$ and $dX_i dX_j = \rho_{ij} dt$, $i, j = 1, \cdots, n$, where $-1 \leq \rho_{ij} \leq 1$ is the correlation coefficient between Brownian motions X_i and X_j. There are no redundant assets and so the covariance matrix, $\mathbf{\Sigma}$,

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \cdots & \sigma_1\sigma_n\rho_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1\sigma_n\rho_{1n} & \sigma_2\sigma_n\rho_{2n} & \cdots & \sigma_n^2 \end{pmatrix}$$

is non-singular. The correlation coefficients ρ_{ij} are constants.

Let $A_i(t)$ be the value of assets invested in stocks i. Let $\Pi(t)$ be the value of the whole portfolio, so

$$\Pi = B + \sum_{i=1}^{n} A_i.$$

Let u_i be the non-anticipating controller representing the amount of cash being transferred between the risk free bonds B and the stock account i. Cash generated or needed from the purchase or sale of stocks are immediately invested or withdrawn from the risk free bonds.

The model is represented by

$$\begin{aligned} dB &= (rBdt - \sum_{i=1}^{n} u_i)dt \\ dA_i &= \mu_i A_i dt + u_i dt + \sigma_i A_i dX_i, \qquad i = 1, \cdots, n. \end{aligned}$$

The investor has an amount of A_i^0 in stocks i and B^0 in risk free bonds at time $t = 0$. The problem is to allocate investments over the given time horizon $[0, T]$ so as to maximize the expected value function $J(\Pi_0; t = 0)$, which is

$$J(\Pi_0; t = 0) = \max_{u_1, u_2, \cdots, u_n} \mathbb{E}_0 \Big\{ \int_0^T I(\Pi(t), t)dt + F(\Pi(T)) \Big\}. \tag{4.1}$$

The functions I and F represents the objectives whose expectation the investor wants to maximize. They are continuously differentiable functions. Of course, the dimensions of

$$\int_0^T I(\Pi(t))dt$$

and

$$F(\Pi(T))$$

are the same. Thus, J is a positive increasing functions of Π. F is also a concave function of Π. Also, we assume the portfolio Π has to be positive at all times. If it is equal to 0, the investment process ends.

4.3 Applying the Maximum Principle on the No Transaction Costs Problem

According to the stochastic version of the the Pontryagin maximum principle, the optimal policy also maximizes the expected value of a Hamiltonian **H**, which we proceed to find.

The Hamiltonian, **H**, is defined as

$$\begin{aligned} & \mathbf{H}(B, A_1, \cdots, A_n, V_1, \cdots, V_n, u_1, \cdots, u_n) \\ = \; & I + \Psi_B(rB - \sum_{i=1}^{n} u_i) + \sum_{i=1}^{n} \Psi_i^A(\mu A_i + u_i + \sigma A_i \frac{dX_i}{dt}) \end{aligned} \tag{4.2}$$

where $dX_i/dt, i = 1, \cdots, n$, are the formal derivative of the Brownian motions, i.e., white noises, Ψ_i^A and Ψ_B are the adjoint processes and Ψ_B is defined by

$$\frac{\partial \Psi_B}{\partial t} = -\frac{\partial \mathbf{H}}{\partial B} = -r\Psi_B \tag{4.3}$$

with the boundary value, $\Psi_B(T)$, obtained from the transversality condition

$$\begin{aligned} \Psi_B(T) &= \frac{\partial F(\Pi(T))}{\partial B(T)} \\ &= \frac{dF(\Pi(T))}{d\Pi(T)}. \end{aligned} \tag{4.4}$$

Similarly, Ψ_i^A, $i = 1, \cdots, n$, are defined by

$$\frac{\partial \Psi_i^A}{\partial t} = -\frac{\partial \mathbf{H}}{\partial A_i} = -\Psi_i^A(\mu_i + \sigma_i \frac{dX_i}{dt}), \tag{4.5}$$

and

$$\begin{aligned} \Psi_i^A(T) &= \frac{\partial F(\Pi(T))}{\partial A_i(T)} \\ &= \frac{dF(\Pi(T))}{d\Pi(T)}. \end{aligned} \tag{4.6}$$

Solving equations (4.3) to (4.6) yields

$$\begin{aligned} \Psi_B &= \frac{\partial F(\Pi(T))}{\partial \Pi(T)} e^{r(T-t)}; \\ \Psi_i^A &= \frac{dF(\Pi(T))}{d\Pi(T)} \frac{S_i(T)}{S_i(t)}. \end{aligned} \tag{4.7}$$

Substituting equation (4.7) into 4.2 and dropping the terms that are not dependent on the controls u, the problem of maximizing $\mathbb{E}_t(J)$ reduces to the maximization of the following expression:

$$\sum_{i=1}^{n} \Big(\mathbb{E}_t(\Psi_i^A - \Psi_B) u_i \Big) = \sum_{i=1}^{n} \mathbb{E}_t \Big\{ \frac{\partial F(\Pi(T))}{\partial \Pi(T)} \frac{S(T)}{S(t)} - \frac{\partial F(\Pi(T))}{\partial \Pi(T)} e^{r(T-t)} \Big\} u_i,$$

where the expectations are taken on every possible stocks movement path with optimal decisions taken.

Maximizing the expression in 4.8 means choosing[1] $u_i = +\infty$ when $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B) > 0$, i.e., buying stocks i as quickly as possible when $\mathbb{E}_t(\Psi_i^A) > \mathbb{E}_t(\Psi_B)$, choosing $u_i = -\infty$ when $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B) < 0$, i.e., selling as many stocks i as possible when $\mathbb{E}_t(\Psi_i^A) < \mathbb{E}_t(\Psi_B)$, and choosing $u_i = 0$ when $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B) = 0$, i.e., do not buy or sell stocks i when $\mathbb{E}_t(\Psi_i^A) = \mathbb{E}_t(\Psi_B)$. Achieving $\mathbb{E}_t(\Psi_i^A) = \mathbb{E}_t(\Psi_B)$ means that the right number of stocks i and risk free bonds is held.

If there is a unique interior optimal solution as we have assumed, the expressions in equation (4.8) is always equals zero if the optimal strategy is used. This can be seen as follows. Suppose there are more stocks i in the portfolio than optimal, the maximum principle instructs to sell, and so $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B) < 0$. And suppose there are less stocks i in the portfolio than optimal, the maximum principle will instruct to buy, and so $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B) > 0$. Since the function $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B)$ is continuous, at the optimal point $\mathbb{E}_t(\Psi_i^A) - \mathbb{E}_t(\Psi_B) = 0$.

Therefore finding the optimal strategy is equivalent to the problem of finding a strategy such that $\mathbb{E}_t(\Psi_A) - \mathbb{E}_t(\Psi_B) = 0$.

From now on, we call the optimal holding value of stock i in the no transaction costs problem A_i^*.

4.4 Market Model with Transaction Costs

Now we consider the problem with transaction costs. Let $k_i > 0$ represent the portion of transaction of stocks i used as transaction costs. So if the investor buys a number of stocks i whose "true" value is $\tilde{A}_i$, the investor pays $(1 + k_i)\tilde{A}_i$ in cash. If the investor sells the stocks, the investor obtains $(1 - k_i)\tilde{A}_i$ in cash.

So our market model equations become

$$\begin{aligned} dS_i &= \mu_i S_i dt + \sigma_i S_i dX_i, i = 1, \cdots, n, \\ dB &= \Big(rBdt - \sum_{i=1}^{n}(u_i + k_i|u_i|)\Big)dt \\ dA_i &= \mu_i A_i dt + u_i dt + \sigma_i A_i dX_i, i = 1, \cdots, n. \end{aligned} \tag{4.8}$$

[1]To be more rigorous, we should firstly make the restriction $|u_i| < M$ for some constant M. Then we let $M \to \infty$.

Instead of maximizing $J(\Pi)$, this problem is to maximize $J(\tilde{\Pi})$, where $\tilde{\Pi}$ is the 'liquidation' value of Π with

$$\tilde{\Pi} \;=\; B + \sum_{i=1}^{n} \kappa_i A_i$$

and κ_i depends on k_i and the sign of A_i such that for $i = 1, \cdots, n$, if $A_i > 0$,

$$1 - k_i < \kappa_i \leq 1$$

and if $A_i < 0$,

$$1 \leq \kappa_i < 1 + k_i.$$

Rigorously speaking, we cannot define $\tilde{\Pi}$ as above. This is because $J(\tilde{\Pi})$ is not continuously differentiable at $A_i = 0$, which is a necessary condition for the application of the Maximum Principle. A solution for this is to change the definition of $\tilde{\Pi}$ for those values close to $A_i = 0$ so as to smooth out the corner. The effect of this on the values of $\tilde{\Pi}$ and $J(\tilde{\Pi})$ and thus to the optimization problem is minimal.

In a manner similar to the problem without transaction costs, the problem reduces to the maximization of the following expression

$$\sum_{i=1}^{n} \Big(\underbrace{(\mathbb{E}_t(\tilde{\Psi}_i^A) - \mathbb{E}_t(\tilde{\Psi}_B))}_{\Phi_i^1} u_i - k_i \underbrace{\mathbb{E}_t(\tilde{\Psi}_B)}_{\Phi_i^2} |u_i| \Big) \tag{4.9}$$

where $\tilde{\Psi}_i^A$ and $\tilde{\Psi}_B$ are the adjoint processes of the Hamiltonian of the transaction costs problem. Since the value function J now depends on $\tilde{\Pi}$ rather than Π, $\tilde{\Psi}_i^A$ are not the same as Ψ_i^A. They are given now by

$$\tilde{\Psi}_i^A \;=\; \kappa_i \frac{\partial F(\tilde{\Pi}(T))}{\partial \tilde{\Pi}(T)} \frac{S_i(T)}{S_i(t)}. \tag{4.10}$$

As before, $\tilde{\Psi}_B$ is equal to

$$\tilde{\Psi}_B \;=\; \frac{\partial F(\tilde{\Pi}(T))}{\partial \tilde{\Pi}(T)} e^{r(T-t)}. \tag{4.11}$$

Also, note that in this problem the expectations of $\tilde{\Psi}_i^A$ and $\tilde{\Psi}_B$ are different to the no transaction cost problem. This is because with transaction costs the optimal strategy is different and also the portfolio value changes with each transaction.

Depending on the sign and magnitude of Φ_i^1 and Φ_i^2, u_i can be either $+\infty$, $-\infty$ or 0. For every stock i, $i = 1, \cdots, n$, there are three possibilities:

1. $u_i = +\infty$. This is if and only if $\Phi_i^1 > 0$ and $\Phi_i^1 > k_i \Phi_i^2$. In other words, this is if and only if $\mathbb{E}_t(\tilde{\Psi}_i^A) > (1 + k_i)\mathbb{E}_t(\tilde{\Psi}_B)$. Under this condition, the optimal strategy is to buy stocks at a rate which is as high as possible;

2. $u_i = -\infty$. This happens if and only if $\Phi_i^1 < 0$ and $\Phi_i^1 < -k_i \Phi_i^2$. In other words, this is if and only if $\mathbb{E}_t(\tilde{\Psi}_i^A) < (1 - k_i)\mathbb{E}_t(\tilde{\Psi}_B)$. Under this condition, the optimal strategy is to sell stocks at a rate which is as high as possible; and

3. $u_i = 0$. This is if and only if $0 \leq |\Phi_i^1| \leq k_i \Phi_i^2$. In other words, this is if and only if

$$\frac{|\mathbb{E}_t(\tilde{\Psi}_i^A) - \mathbb{E}_t(\tilde{\Psi}_B)|}{k_i} \quad \leq \quad \mathbb{E}_t(\tilde{\Psi}_B),$$

   which is equivalent to

$$(1 - k_i)\mathbb{E}_t(\tilde{\Psi}_B) \leq \mathbb{E}_t(\tilde{\Psi}_i^A) \leq (1 + k_i)\mathbb{E}_t(\tilde{\Psi}_B). \tag{4.12}$$

   Under this condition, the optimal strategy is not to buy or sell any stocks.

Therefore, the trading policy, given t, S_i, A_i and B, can be summarized by the following table. There are three possible regions: sales region, purchase region and no-transaction region. The no-transaction region is represented by inequalities 4.12 and is the region in the middle. As long as the point $\mathbb{E}_t(\Psi_i^A)$ is inside the no-transaction region, the optimal strategy is not to trade. If the point $\mathbb{E}_t(\Psi_i^A)$ is below $(1-k_i)\mathbb{E}_t(\Psi_B)$ (moving out of the no-transaction region), the optimal strategy is to purchase stocks, which then move the point $\mathbb{E}_t(\Psi_i^A)$ as well as the no-transaction region boundary, and thus bring the point back i, and the sale region works in similar fashion.

	$\Phi_i^1 \geq 0$	$\Phi_i^1 < 0$
$\Phi_i^2 \geq 0$	No Transaction if $\Phi_i^1 < k_i \Phi_i^2$ otherwise buy	No Transaction if $\lvert\Phi_i^1\rvert < k_i \Phi_i^2$ otherwise sell
$\Phi_i^2 < 0$	Buy	Sell

Another way to understand the optimal trading strategy is to look at Figure (4.1). Whenever Φ_i^1 is positive, it is either buy or not to transact. Whenever Φ_i^1 is negative, it is either sell or not to transact. The no transaction region is when the point (Φ_i^1, Φ_i^2) is within the two lines with slope $1/k_i$ and $-1/k_i$.

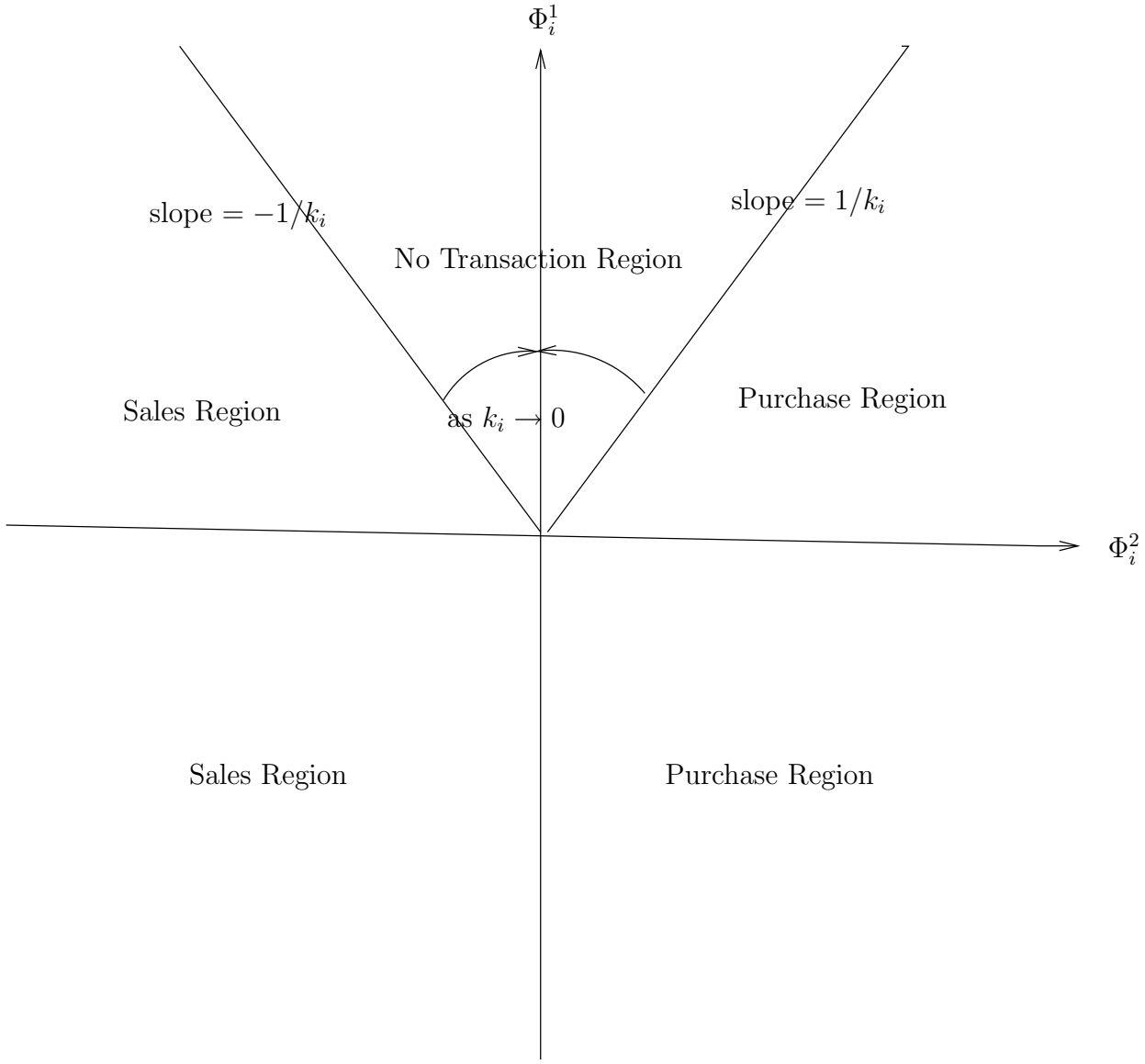


Figure 4.1: The table in the previous page can be visualised by this figure. The no transaction region is within the two lines with slope $1/k_i$ and $-1/k_i$. As $k_i \to 0$, these two lines become closer and closer to the Y-axis and so the no-transaction region becomes smaller and smaller.

If the portfolio is at the purchase region, the optimal strategy is to purchase stocks as quickly as possible. It is inevitable that the portfolio enters the no-transaction region eventually. We proceed to prove this by contradiction so we assume it doesn't. As both Ψ_i^A and Ψ^B are continuous, the portfolio does not enter the sales region as well. So, the optimal strategy is to purchase stocks as quickly as possible without end. For every purchase made, however, the portfolio value $\tilde{\Pi}$ is reduced because of the transaction costs. Eventually, $\tilde{\Pi}$ is equal to 0, the lower limit allowed for the portfolio value and the investment process ends. Any strategy which keeps the portfolio $\tilde{\Pi} > 0$ is a better strategy than this strategy as both I and F are increasing functions. So, we can conclude that this strategy is not optimal and thus it cannot be the optimal solution.

A similar conclusion holds for the sales region.

We can, therefore, conclude that the no-transaction region exists.

4.5 The Order of $\mathbb{E}_t(\tilde{\Psi}_B)$

We now proceed to prove that if given that the portfolio is at the no transaction-region,

$$\mathbb{E}_t(\tilde{\Psi}_B) \quad = \quad o(1). \tag{4.13}$$

This result, as we can see later, is pivotal in obtaining the conclusion of this chapter.

We recall from equation (4.11) that

$$\tilde{\Psi}_B \quad = \quad \frac{\partial F}{\partial \tilde{\Pi}} e^{r(T-t)}.$$

For $i = 1, \cdots, n$, therefore,

$$\mathbb{E}_t(\tilde{\Psi}_B | \Pi(t) = \mathbf{\Pi}) \quad = \quad \mathbb{E}_t\Big\{ \frac{\partial F}{\partial \tilde{\Pi}} e^{r(T-t)} | \Pi(t) = \mathbf{\Pi} \Big\}$$

We find that the above function is bounded above by another function $M(k_1, k_2, \cdots, k_n)$, which we define as

$$M(k_1, k_2, \cdots, k_n) \quad = \quad \hat{\mathbb{E}}_t\Big\{ \frac{\partial F}{\partial \tilde{\Pi}} (\tilde{\Pi}(T) + \mathbf{T}) e^{r(T-t)} | k_1, k_2, \cdots, k_n, \Pi(t) = \mathbf{\Pi} \Big\}, \tag{4.14}$$

where $\hat{\mathbb{E}}$ is the expectation operator taken for every possible stock movement path and the strategy used is the same as those strategy used in equation (4.9). We define $\mathbf{T}$ as the total value accumulated at time T if the money spent in transaction costs in $[0, T]$ has been invested in risk-free bonds instead.

For every possible stock movement path, therefore, the portfolio value $\tilde{\Pi}(T) + \mathbf{T}$ in equation (4.14) is larger than the portfolio value $\tilde{\Pi}(T)$ in equation (4.9). As $F(\tilde{\Pi}(T))$ is a concave function,

$$\frac{\partial F(\tilde{\Pi}(T))}{\partial \tilde{\Pi}(T)}$$

is an increasing function. It follows that

$$\mathbb{E}_t(\tilde{\Psi}_B | k_1, k_2, \cdots, k_n, \Pi(t) = \mathbf{\Pi}) \leq M(k_1, k_2, \cdots, k_n)$$

for any level of transaction costs $k_1, k_2, \cdots, k_n$.

Now, let $\mathbf{M}$, a constant, defined as

$$\mathbf{M} = \sup_{u_1, u_2, \cdots, u_n} \mathbb{E}_t \left\{ \frac{\partial F(\tilde{\Pi})}{\partial \tilde{\Pi}} e^{r(T-t)} \right\}, \tag{4.15}$$

where the expectation is taken for every possible stock movement path and the strategy used is the optimal strategy to maximize

$$\frac{\partial F(\tilde{\Pi}(T))}{\partial \tilde{\Pi}(T)}.$$

Therefore,

$$M(k_1, k_2, \cdots, k_n) \leq \mathbf{M}$$

for all level of transaction costs $k_1, k_2, \cdots, k_n$.

Since $\mathbf{M}$ is a constant independent of $k_1, k_2, \cdots, k_n$, and

$$\mathbb{E}_t(\tilde{\Psi}_B) \leq \mathbf{M}$$

and thus

$$\mathbb{E}_t(\tilde{\Psi}_B) = O(1). \tag{4.16}$$

4.6 Transaction Costs Level Tends to 0

Now, we show that when the transaction costs level tends to 0, the no transaction region converges to the optimal strategy without transaction costs.

When $k_i \to 0$, $i = 1, \ldots, n$, $k_i\Phi_i^2$ of equation (4.9) must converge to 0. This is because $\Phi_i^2 = O(1)$. So, Φ_i^1 converges to 0. This means the limit of the corresponding strategy must be an optimal strategy for the no transaction cost problem. Since such a strategy is unique, we can conclude that the limit must be the same as the no transaction cost strategy. The result follows.

4.7 Examples

We can apply the results in this chapter on the long term growth model and the CRRA model. In the long term growth model, we know from Section 3.3.3 that the no transaction costs problem has a unique solution. Similarly, in the CRRA model, it has been shown in Section 3.3.4 the no transaction cost solution exist and is unique. Therefore, the result of this chapter can be applied to these two problems. In other words, the solution of the transaction costs problem converge to the no transaction costs problem when $k \to 0$.

Chapter 5

Conclusion

5.1 Results Achieved

We have studied the impact of transaction costs on the problems of portfolio optimizations for a general function. Firstly, we studied the case of of only one risky asset. In such a problem, we find that the transaction-region converges to the no transaction costs optimal solution when the level of transaction costs tend to 0. We have found that the width of the boundaries are of the order $k^{1/3}$ while the adjustment term for the value function is of order $k^{2/3}$. Also, we have successfully obtained the approximate formulae for the boundaries of the no-transaction region and the leading order adjustment term for the value functions.

We have also applied the formulae we have obtained on some functions. In particularly, we look at the long term growth model and the CRRA model. The result on the long term growth model is consistent with Davis and Norman [18].

In the later chapters, we extend the results to the cases where (1) the payoff function is dependent on S, the price of the stock; and (2) there are more than one uncorrelated risky assets. In these cases, we have also established similar results to the case with one risky asset, although the formulae for for the no transaction region boundary are different.

5.2 Future Work

There are several directions in which our work can be extended. For instance,

1. The study of dynamic mean-variance asset allocation with presence of short sales restrictions or margin restrictions;

2. Extend the work in Chapter 2 and Chapter 3 to study the case of stochastic volatility, discrete rebalancing, and Multi-Asset with different transaction costs for different stocks.

3. Extend the first part of Chapter 3 to develop a theory to price options with the consideration of transaction costs.

Appendix A

Nomenclature

The followings are some of the notations we often use in our thesis:

A.1 All Chapters

* Optimal solution

$A(t)$ Value invested at stocks at time t (used as controls in chapter 3 in the multi-asset no transaction costs problem)

$\mathfrak{A}_i$ The optimal value of stock i held in the when $k \to 0$

α_i $A_i = \mathfrak{A}_i + k^{1/3}\alpha_i$

$B(t)$ Risk free bonds held at time t (used as controls in Chapter 2 in the no transaction costs problem)

$\mathfrak{B}$ The optimal value of risk free bond held when $k \to 0$

β $B = \mathfrak{B} + k^{1/3}\beta$

$\mathbb{E}$ Expectation Operator

$F(\Pi(T)$ Reward function at time T

γ A parameter in the Constant Relative Risk Aversion Model (CRRA), and $1-\gamma$ represents the relative risk aversion

H The value function, after the change of coordinates, in the transaction costs problem we sought to maximize

$\mathbf{H}$ Hamiltonian

$I(\Pi(T))$ The rate of reward during the whole investment period

J the value function in the no-transaction cost problem we sought to maximize

$\tilde{J}$ the value function in the transaction cost problem we sought to maximize

H_n the value function in the transaction cost problem after change of coordinates we sought to maximize

k kS transaction cost for a single transaction of 1 stock at price S, and k is assumed to be constant

μ growth rate of the stock

n the number of stocks in the market

$\Pi(t)$ is the value of the portfolio, which equals to $B(t) + A(t)$

Ψ adjoint processes of the Hamiltonian

r interest rate and assumed to be constant (given)

ρ_{ij} Correlation between stock i and stock j

$S(t)$ Value of stock at time t (decided by geometric Brownian motion)

σ volatility of the stock

$\mathbf{\Sigma}$ Covariance Matrix

t Time

T Time for the Investment Horizon

X Brownian Motion

A.2 Chapter 1

C Consumption

δt a small increment of time

λ the portion of resources invested in stock in the long term growth model, Merton's Investment Consumption Model, or the Langrange multiplier in the Pontryagin Maximum Principle illustration

ν discount factor in Merton's Investment Consumption Model

p_0 condition probability of $p(\Pi(t_1) = \Pi_1 | \Pi(t_0) = \Pi_0)$

p_1 condition probability of $p(\Pi(t_1) = \Pi_1, S(t_1) = S_1 | \Pi(t_0) = \Pi_0, S(t_0) = S_0)$

u controller representing the cash flows used to buy stocks

U utility function

ξ target return of the portfolio

A.3 Chapter 2

$\mathbf{L}$ cumulative purchase of assets A during $[0, t]$

$\mathbf{M}$ cumulative sale of assets A during $[0, t]$

A.4 Chapter 3

α a factor in the modified CRRA model

$\mathfrak{D}$ an operator defined as

$$\begin{aligned}\mathfrak{D} &= \frac{1}{2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\mathfrak{A}_i\mathfrak{A}_j\Big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}\frac{\partial\mathfrak{A}_i}{\partial\Pi}\frac{\partial\mathfrak{A}_j}{\partial\Pi}\Big) \\ &+\frac{1}{2}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\mathfrak{A}_i\mathfrak{A}_j\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}\Big) \\ &-\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}\mathfrak{A}_i\mathfrak{A}_j\Big)\Big(\sum_{j=1}^{n}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}\frac{\partial\mathfrak{A}_j}{\partial\Pi}\Big)\end{aligned}$$

$\mathfrak{D}_{ij}$ an operator defined as

$$\begin{aligned}\mathfrak{D}_{ij} &= \frac{1}{2}\frac{\partial \mathfrak{A}_i}{\partial \Pi}\frac{\partial \mathfrak{A}_j}{\partial \Pi}(\sum_{k=1}^{n}\sum_{l=1}^{n}\sigma_k\sigma_l\rho_{kl}\mathfrak{A}_k\mathfrak{A}_l) \\ &\quad +\frac{1}{2}\sigma_i\sigma_j\rho_{ij}\mathfrak{A}_i\mathfrak{A}_j - \frac{\partial \mathfrak{A}_i}{\partial \Pi}(\sum_{k=1}^{n}\sigma_i\sigma_k\rho_{ik}\mathfrak{A}_i\mathfrak{A}_k) \qquad \text{(A.1)}\end{aligned}$$

$\mathfrak{M}$ an operator defined as

$$\begin{aligned}\mathfrak{M} &= \frac{\partial}{\partial t} + I + r(\Pi - \sum_{i=1}^{n}\mathfrak{A}_i)\frac{\partial}{\partial \Pi} \\ &\quad +\sum_{i=1}^{n}\mu_i\mathfrak{A}_i\frac{\partial}{\partial \Pi} + \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}\sigma_i\sigma_j\rho_{ij}\mathfrak{A}_i\mathfrak{A}_j\right)\frac{\partial^2}{\partial \Pi^2} \qquad \text{(A.2)}\end{aligned}$$

A.5 Chapter 4

κ_i Ratio that stock i is worth when liquidated, and so $\kappa_i A_i$ epresents the liquidation value of A_i

Φ_i^1 $\mathbb{E}_t(\tilde{\Psi}_i^A) - \mathbb{E}_t(\tilde{\Psi}_B)$

Φ_i^2 $\mathbb{E}_t(\tilde{\Psi}_B)$

Ψ_i^A Ajoint processes of asset i in the transaction cost problem

Ψ_B Ajoint processes of B in the transaction cost problem

$\tilde{(\Pi)}$ Liquidation value of Π in the transaction cost problem

u_i Control for asset i

Bibliography

[1] Atkinson, C. & Al-Ali, B. (1997) *On an Investment-Consumption Model with Transaction Costs: An Asymptotic Analysis,* Applied Mathematical Finance **4** 109-133

[2] Akian, M., Menaldi, J. & Sulem, A. (1996) *On an Investment Consumption Model with Transaction Costs,* SIAM Journal of Control and Optimization **34** 329-364

[3] Atkinson, C. & Mokkhavesa, S. (2001) *Towards the Determination of Utility Preference from Optimal Portfolio Selections,* Applied Mathematical Finance **8** 1-26

[4] Atkinson, C. & Mokkhavesa, S. (2004) *Multi-asset Portfolio Optimization with Transaction Cost,* Applied Mathematical Finance **11** 95-123

[5] Atkinson, C., Pliska, S. & Wilmott, P. (1997) *Portfolio Management with Transaction Costs,* Proceedings of the Royal Society: Mathematical, Physical and Engineering Sceinces **453** 551-562

[6] Atkinson, C. & Wilmott, P. (1995) *Portfolio Management with Transaction Costs: An Asymptotic Analysis of the Morton and Pliska Model,* Mathematical Finance **5** 357-367

[7] Bellman R. & Dreyfus, S. (1962) Applied Dynamic Programming, Princeton Univesity Press

[8] Bensaid, B., Lesne, J., Pages, H. & Scheinkman, J. (1992) *Derivative Asset Pricing with Transaction Costs,* Mathematical Finance **2** 63-86

[9] Bjork, T. (1999) Arbitrage Theory in Continuous Time, Oxford University Press

[10] Blacks, F & Scholes, M. (1973) *The Pricing of Options and Corporate Liabilities,* Journal of Finance **81** 637-654

[11] Boyle, P. & Emanuel, D. (1980) *Discretely Adjusted Option Hedges,* Journal of Financial Economics **8** 259-282

[12] Boyle, P. & Vorst, T. (1992) *Option Replication in Discrete Time with Transaction Costs,* Journal of Finance **47** 271-293

[13] Brealey, R. & Myers, S. (2000) Principles of Corporate Finance, McGraw-Hill

[14] Clewlow, L. & Hodges, S. (1997) *Optimal Delta-Hedging under Transactions Costs,* Journal of Economic Dynamics and Control **21** 1353-1376

[15] Constantinides, G. (1986) *Capital Market Equlibrium with Transaction Costs,* The Journal of Political Economy **94** 842-862

[16] Cox, J., Ross, S. & Rubinstein, M. (1979) *Option Pricing: A Simplified Approach,* Journal of Financial Economics **7** 229-263

[17] Davis, H., Panas, V. & Zariphopoulou, T. (1993) *European Option Pricing with Transaction Costs,* SIAM Journal of Control and Optimization **31** 470-493

[18] Davis, H. & Norman, A. R. (1990) *Portfolio Selection with Transaction Costs,* Mathematics of Operations Research **15** 676-713

[19] Edirsinghe, C., Vasanttilak, N & Uppal R. (1993) *Optimal Replication of Options with Transations Costs and Trading Restrictions,* Journal of Financial and Quantitative Analysis **28** 117-138

[20] Friedman, M. (1955) *What All is Utility?* The Economic Journal **65** 405-409

[21] Friedman, M. & Savage, L. J. (1948) *The Utility Analysis of Choices Involving Risk,* The Journal of Political Economy **56** 279-304

[22] Friedman, M. & Savage, L. J. (1952) *The Expected-Utility Hypothesis and the Measurability of Utility,* The Journal of Political Economy **60** 463-474

[23] Harrison, J. & Kreps, D. (1979) *Martingales and Arbitrage in Multiperiod Securities Markets,* Journal of Economic Theory **20** 381-408

[24] Harrison, J. & Pliska S. (1981) *Martingales and Stochastic Integrals in the Theory of Continuous Trading,* Stochastic Processes and their Applications **11** 215-260

[25] Haussmann, U. G. (1986) *A stochastic maximum principle for the optimal control of diffusion processs,* Pitman Research Notes in Mathematics, no. 151. Harlow, UK: Longman Scientific and Technical, 1986.

[26] Heath, D. & Jarrow, R. (1987) *Arbitrage, Continuous Trading, and Margin Requirements,* Journal of Finance **42** 1129-1142

[27] Hinch, E. (1991) Perturbation Methods, Cambridge University Press

[28] Hodges, S. & Neuberger A. (1989) *Optimal Replication of Contingent Claims under Transaction Costs,* The Review of Futures Markets **8** 222-239

[29] Hull, J. (1996) Options, Futures, and Other Derivatives, Prentice Hall International

[30] Intriligator, M. D. (2002) Mathematical Optimization and Economic Theory (Classics in Applied Mathematics, 39), Society for Industrial and Applied Mathematics

[31] Kelly, J. L. (1956) *A new interpretation of information rate,* Bell System Technical Journals **35** 917-926

[32] Leland, H. (1985) *Option Pricing and Replication with Transaction Costs,* Journal of Finance **40** 1283-1301

[33] Markowitz, H. (1952)*Portfolio Selection,* Journal of Finance **7** 77 - 91

[34] Merton, R. (1973) *Theory of Rational Option Pricing,* Bell Journal of Economic and Management Science **4** 141-183

[35] Merton, R. (1989) *On the Application of the Continuous Time Theory of Finance to Financial Intermediation and Insurance,* The Geneva Papers on Risk and Insurance225-261

[36] Merton, R. (1992) Continuous Time Finance, Blackwell

[37] Mill, J. S. (1993) On Liberty, J. M. Dent

[38] Mokkhavesa, S. & Atkinson, C.(2002) *Perturbation Solution of Optimal Portfolio Theory with Transaction Costs for Any Utility Function,* IMA Journal of Management Mathematics **13** 131-151

[39] Morton, A. & Pliska, S. (1995) *Optimal Portfolio Management with Fixed Transaction Costs,* Mathematical Finance **5** 337-356

[40] Neftci, S. N. (2000) An Introduction to the Mathematics of Financial Derivatives, Academic Press

[41] Oksendal, Brent (1998) Stochastic Differential Equations : An Introduction With Applications, Springer

[42] Peng, S.(1990)*A General Stochastic Maximum Principle for Optimal Control Problems,* SIAM Journal of Control and Optimization **23** 966-979

[43] Pontryagin, L. S., Bolytanskii, V. G., Gamkrelidze, R. V., & Mischenko, E. F. The mathematical Theory of Optimal Processes, Pergamon

[44] Putyatin, V.(1998)*Mathematical models of securities markets,* University of Southampton Ph. D. Thesis

[45] Sprenkle, C. (1961) *Warrant Prices as Indications of Expectations,* Yale Economics Essays 179-232

[46] Taleb, N. (1996) Dynamic Hedging : Managing Vanilla and Exotic Options, Wiley

[47] Toft, K. (1996) *On the Mean-Variance Tradeoff in Option Replication with Transaction Costs,* Journal of Financial and Quantitative Analysis **31** 233-263

[48] Whalley, A. E. & Wilmott P. (1994) *Optimal Hedging for Options with Small but Arbitrary Transaction Cost Structure,* Ociam Working Paper, Mathematical Institue, Oxford

[49] Whalley, A. E. & Wilmott P. (1997) *An Asymptotic Analysis of an the Davis, Panas & Zariphopoulou Model for Option Pricing with Transaction Costs,* Mathematical Finance **7** 307-324

[50] Whalley, A. E. & Wilmott P. (Oct-94) *Hedge with an Edge,* Risk Magazine **7** 82-85

[51] Wilmott, P. (1998) Derivatives : The Theory and Practice of Financial Engineering, John Wiley

[52] Wilmott, P., Dewynne, J. & Howison, S (1993) Option Pricing: Mathematical Models and Computation, Oxford Financial Press

[53] Zhou, X. Y. (1996) *Sufficient Conditions of Optimality for Stochastic Systems with Controllable Diffusions,* IEEE Transactions on Automatic Control **41** 1176-1179