Theory of Specialisations

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I dedicate this thesis to my parents Gül and Mehmet Efem.

Without their support I couldn't have made it this far...

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Abstract

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This thesis aims to develop a theory of specialisations for Zariski structures. The main question this thesis is centred around is when is a specialisation is κ-universal?

In Chapter 3 we look at the case of algebraically closed fields and Zariski structures definable in these. We show that any specialisation of an algebraically closed field defines a residue map of a valuation ring, and moreover we show that any residue map induces a κ-universal specialisation whenever the algebraically closed field is κ-saturated over the residue field.

In Chapters 4 and 5 we look at the κ-universal specialisations of finite and infinite covers of Zariski structures. Under some natural assumptions we show that a specialisation of a cover of a Zariski structure is κ-universal if and only if it extends a κ-universal specialisation of a base Zariski structures and satisfies a simple axiom.

Contents

Chapter 1

Introduction and Overview

1.1 Introduction

In algebraic geometry one considers an algebraically closed field K, and the Zariski topology, which is given by the algebraic varieties; which are determined by the vanishing sets of polynomials over K. In the model theoretic language, algebraic varieties are obviously definable in the structure $(K, +, \cdot)$. A natural way to study these objects is to obtain geometric information by using model theoretic properties of $(K, +, \cdot)$, in which the varieties live. Unfortunately, as Hodges pointed out in [12], this approach does not provide us with new geometric information about varieties.

Hrushovski and Zilber developed another approach [16]. In which, loosely speaking, one forgets about the polynomials and their zero sets and starts with a set M_0 equipped with a collection of subsets of M_0^n for each $n \in \mathbb{N}$, which satisfy certain Zariski type topological axioms. In other words, in this approach one studies the object (M_0, τ) where τ is the mentioned collection of subsets of arbitrary powers of M_0, and for any n the subsets of M_0^n in τ gives a Noetherian topology on M_0^n (therefore τ is often referred as the collection of topologies). The tuple (M_0, τ) is also equipped with a *good notion of dimension* on definable subsets, in the case of algebraic geometry which corresponds to Krull dimension.

A very important tool in this theory is a map called *specialisation*. A *specialisation* is a partial homomorphism $\pi : M \to M_0$ where M is an elementary extension of M with respect to a relational language $\mathcal{L}$ such that the elements of τ are given by positive quantifier free $\mathcal{L}$-formulas. Specialisations are analogous to Robinson's *standard part map* in model theory and *places* or *specialisations* in algebraic geometry. Initially Hrushovski and Zilber used *specialisations* to prove the main theorem of the paper [16], which is known as the Classification Theorem (Theorem 4.0.3).

Later, Onshuus and Zilber attempted to give a first order axiomatisation of the theory of *universal specialisations* [25]. A specialisation is called *κ-universal* if, given any $M' \succeq M \succeq M_0$, any $A \subseteq M'$ with $|A| < \kappa$ and a specialisation $\pi_A : M \cup A \to M_0$ extending π, there is an elementary embedding $\sigma : A \to M$ over $M \cap A$ such that $\pi_A|A = \pi \circ \sigma$.

They proved some preliminary results such as a partial quantifier elimination and model completeness in [25]. Although I will not repeat these results here, and will not try to follow their route to give a first order theory of *universal specialisations*, this preprint is the major motivation behind the main question I tried to answer in this thesis (which I will explain below).

In modern algebraic geometry, specialisations made an early appearance during the formative years of the subject (see e.g. [41, 42]). In classical algebraic geometry over $\mathbb{C}$, continuity, and more generally the usual (analytic) topology of the complex field provide very useful tools. However in abstract algebraic geometry over an arbitrary (algebraically closed) base field, one does not have the luxury of continuity or more generally (analytic) topological arguments one has in the classical case. Specialisations in abstract algebraic geometry provide an arithmetic analogue for the mentioned continuity and (analytic) topological arguments. In other words in algebraic geometry over an arbitrary base field, specialisations can be seen as an arithmetic (and abstract) substitute for the complex topology.

The aim of this thesis is to develop a theory of specialisations for Zariski structures. In particular we are interested in *universal* (and more precisely in *κ-universal*) specialisations. Similar to the situation in algebraic geometry, in the setting of a Zariski structure (M_0, τ), it is a well known fact that any universal specialization of M_0 can be replaced with the collection of topologies τ (see Fact 2.1.64).

In other words, *universal specializations* in the setting of Zariski structures is a model theoretic substitute for continuity and topological arguments such as "taking limits". Therefore it is a meaningful task to develop a theory of specialisations for Zariski structures. For the same reason it is natural for this theory to be centred around *universal specialisations.* This thesis is mainly centred around the question when non-trivial specializations of a Zariski structure is universal?

1.2 Overview

Chapter 2 lays the setting of this thesis and introduces the basic concepts we will be working with. Many of the concepts, definitions and facts recalled in this chapter are already present in the literature, in particular they can be found in [16, 18, 19, 44]. As mentioned in the introduction, the most important notion in this thesis is that of a specialisation and κ-universal specialisation (Definition 2.1.50). Another notion introduced in this chapter, which plays role is *maximal specialisation*:

Definition 2.1.68. Let M_0 be a Zariski structure, $M \succeq M_0$ and $\pi : M \to M_0$ a specialisation. We say that $\mathrm{Dom}(\pi)$ is *maximal* or π *is maximally defined* if there are no specialisations $\pi' : M \to M_0$ extending π non-trivially.

In Section 2.2 Zilber's Ladder Theorem for uncountably categorical structures is recalled. Ladder Theorem does not involve Zariski structures naturally, and is stated in the general context of uncountably categorical structures. It says that any uncountably categorical structure can be constructed in finitely many steps, starting with a strongly minimal structure and taking a cover at each subsequent step (see Theorem 2.2.3).

On the other hand theories of the Zariski structures related to Quantum Harmonic Oscillator, and Quantum Zariski Geometries [35, 46] are uncountably categorical. They are also constructed as covers of strongly minimal Zariski structures.[1] In terms of the Ladder Theorem, they can be seen as single step ladders, where the cover is also a Zariski structure. This motivates the study of Zariski cover structures and their specialisations as the main step in understanding the general case.

In Chapter 3, I consider the *specialisations* of algebraically closed fields and, show that essentially they are residue maps of valuations:

Theorem 3.1.2. *Let k be an algebraically closed field and $K \succeq k$. Given a specialisation $\pi : K \to k$, it defines a residue map, $res : K \to k$ of a valuation ring $\mathcal{O}$.*

Two main result in this chapter are the followings:

Lemma 3.1.15. *Let (N, N_0, res) be a κ-saturated model of $Th(K)^{res}$ (in the language $\mathcal{L}^{\pi,\infty}_{Zar}(P)$), then $res : N \to N_0$ is a κ-universal specialisation.*

Theorem 3.1.16. *Let (M, M_0, res) be a model of $Th(K)^{res}$. Then (M, M_0, res) is elementarily equivalent to a model (N, N_0, res) such that $res : N \to N_0$ is κ-universal.*

[1] in fact, they will be Zariski cover structures.

Theorem 3.1.16 builds on the fact that $Th(K)^{res}$ (which is a theory of algebraically closed valued fields) has quantifier elimination in a certain language (Lemma 3.1.14). Quantifier elimination for various theories of pairs of valued fields is well known in the model theory literature. The main argument used in our proof is very close to Hrushovki and Kazhdan's proof of quantifier elimination in [15, Lemma 6.3]; and also to Leloup [21].

As a next step, it is natural to consider non-locally modular Zariski geometries and ask the following question: Let M_0 be a non-locally modular Zariski geometry and $M \succeq M_0$. Assume M is κ-saturated over M_0, and let $\pi : M \to M_0$ be a specialisation. Is it true that π is a κ-universal specialization?

If M_0 is one dimensional, by the Classification theorem it is a finite fibration of $\mathbb{P}^1(k)$ for some algebraically closed field k.

In Chapter 4, we consider the specialisations of finite fibrations of Zariski structures, and investigate the relation between the specialisations of the finite fibration and specialisations of the base. In Section 4.1, we consider finite fibrations with the following natural topological structure:

Let M_0 be a Zariski structure and $\mathrm{pr} : C_0 \to M_0$ be a surjective n to one map. For any closed relation R of M_0, denote by R_{pr} its pull back via pr (i.e. $R_{\mathrm{pr}} = \mathrm{pr}^{-1}(R)$). Then, consider C_0 as a structure of the language $\mathcal{L}_{\mathrm{pr}}$ whose basic relations are R_{pr} for each closed relation R of M_0. Declare the sets definable by positive quantifier free formulas with parameters to be closed. We showed that this defines a Noetherian topology on all Cartesian powers of C_0 (Lemma 4.1.2 and Lemma 4.1.6).

The aim was to prove that if a specialisation $\pi_C : C \to C_0$ of a finite fibration $\mathrm{pr} : C_0 \to M_0$ of M_0, projects to a κ-universal specialisation $\pi_M : M \to M_0$ of the Zariski structure M_0, then it must be κ-universal. However, the desired result is not true for arbitrary finite fibrations, and a family of counterexamples is constructed in Section 4.2 (Propositions 4.2.1, 4.2.2).

These counterexamples seems rather pathological, and are not encountered naturally in the literature. The reason such counterexamples can be constructed is the lack of structure on the fibres. However, in natural examples where the fibration is also a Zariski structure there is a rich structure on the fibres. See the first example of a non-algebraic ample Zariski geometry constructed by Hrushovski and Zilber in [16], Quantum Harmonic Oscillator considered in [35] and the Quantum Zariski Geometries considered by Zilber in [46]. These examples will be Zariski cover structures (see Definition 5.1.1).

In Subsection 4.3.1, we show that an arbitrary finite fibration can be enriched in such a way that the conclusion of Theorem 4.4.7 holds. In order to do it we introduce a new predicate, called the "restricted inequality predicate", which induces additional topological structure on the fibres. The new language which contains the restricted inequality predicate as a basic relation is denoted by $\mathcal{L}_{\text{ineq}}$. I show that finite fibrations enriched with the restricted inequality predicate are Noetherian topological structures (more precisely, they become $\mathcal{L}_{\text{ineq}}$-topological structures).

Theorem 4.3.3. *Consider C_0 as an $\mathcal{L}_{ineq}$-topological structure. If $\pi_C : C \to C_0$ is a maximal specialisation which projects to a specialisation $\pi : M \to M_0$ then π_C is κ-universal if π is.*

Although we have considered fibrations of Zariski structures as topological structures in this section, a more natural approach would have been considering them as Zariski structures. A natural way of defining the dimension of definable sets of the fibration is as follows:

Let $\text{pr} : C_0 \to M_0$ be a finite fibration and C_0 be an $\mathcal{L}_{pr}$-topological structure. Let $S \subseteq C_0^m$ be closed, we define the dimension of S as $\dim S := \dim \text{pr}(S)$. Then we extend dim to constructible sets in the usual way: For a constructible $Q \subseteq C_0^n$ define $\dim Q := \dim \overline{Q}$. By the quantifier elimination dim is defined on all definable sets.

Note that however, this notion of definition does not satisfy the Strong Irreducibility property of the Zariski structures. The inverse image of the diagonal in M_0^2 is an irreducible and closed subset of C_0^2, and the diagonal in C_0^2 is a closed and irreducible proper subset of the inverse image of the diagonal in M_0^2. But both of these sets have dimension one. Hence Strong Irreducibility fails.

In fact this argument shows us that we need the restricted inequality predicate added to the language $\mathcal{L}_{\text{pr}}$ as a part of the topological structure on C_0 if we wish to consider it as a Zariski structure with the above notion of dimension.

Section 4.4 is a case study of the Zariski structures constructed in relation to the first example of a non-algebraic Zariski geometry in [16] by Hrushovki and Zilber. We show that this pathology does not occur for these structures:

Theorem 4.4.7. *Let $\pi_C : C \to C_0$ be a maximal specialisation. If the specialisation $\pi : C/{\sim} \to M_0$ induced by π_C is κ-universal, then π_C is κ-universal.*

Where C_0 is a Zariski structure constructed from M_0 by the method described in the process of constructing the first example of a non-algebraic ample Zariski geometry constructed by Hrushovski and Zilber in [16].

Section 4.4 and in particular the proof of Theorem 4.4.7 should be seen as a prelude to Chapter 5, in the sense that we aim to extend Theorem 4.4.7 to certain covers of Zariski structures. Which is achieved for a special case; i.e. under some additional assumptions (see Theorem 5.2.14 below). Before this, in Section 5.1, we also extend the analysis carried out in Section 4.4 to Zariski cover structures.

The Ladder Theorem and the Zariski structures constructed in [16, 35, 46] in relation to the first example of a non-algebraic Zariski geometry, Quantum Harmonic Oscillator, and Quantum Zariski geometries are the main motivations for considering covers of Zariski structures, and their specialisations. In this chapter we consider specialisations of a special type of cover of Zariski a structure. Namely *Zariski cover structures.*

Definition 5.1.1. Let $\mathcal{C} := (C, M, \mathrm{pr})$ be a two-sorted Zariski structure such that

(i) There is a Zariski topological group G in $\mathcal{C}$ acting morphically and freely on C with Zariski continuous bijections

(ii) M is interpretable in C as a topological sort and $\mathrm{pr} : C \to M$ denotes the canonical quotient-map. It is a $\emptyset$-definable surjection.

(iii) For each $m \in M$, the fibre $\mathrm{pr}^{-1}(m)$ is an orbit of an element in C, i.e. $\mathrm{pr}^{-1}(m) = G \cdot x$ for some $x \in C$.

(iv) The group G is a Zariski topological group in M (in particular G is interpretable as a topological sort in M)

Then $\mathcal{C} := (C, M, \mathrm{pr})$ is said to be a *(two sorted) Zariski cover structure.*

The covers of Zariski structures possibly arising from the Ladder Theorem are not always expected be Zariski cover structures. However, as the known natural examples (see [35, 16, 46]) of covers of Zariski structures are actually Zariski cover structures, the study of their specialisations is meaningful.

Zariski cover structures capture the nice -but often complicated- structure of the covers constructed in relation to the first example of a non-algebraic Zariski geometry, Quantum Harmonic Oscillator, and Quantum Zariski geometries (respectively in [16, 35, 46]).

In this chapter we start with an analysis of Zariski cover structures. Most importantly, we study the (closed) relations between fibres. The study concludes in showing that these relations (or connections) are functions from the cover structure to certain imaginary sorts (Theorem 5.1.16 and Corollary 5.1.17). We call these functions *connecting functions*.

While dealing with arbitrary fibrations of Zariski structures, we realised that one may need strong assumptions in order to ensure the specialisations behave nicely. The notion of Zariski cover structure and the Continuous Connections assumption of Chapter 5 (introduced in paragraph 5.1.18) provides us with natural assumptions which help to ensure the behaviour of the specialisations of the cover is nice. A valuable achievement of this chapter is the identification of Zariski cover structures and Continuous Connections property for dealing with specialisations of covers. Although Quantum Harmonic Oscillator and Quantum Zariski Structures motivated the notion of Zariski cover structures, unfortunately at this point we are unable to prove whether they satisfy Continuous Connections assumption or not.

After the study of Zariski cover structures, we analyse their specialisations. The main result of this section (Section 5.2) is a characterisation of universal specialisations Zariski cover structures (under some additional assumptions, given in detail in this chapter), which explains its relation to the specialisations of the base structure.

Theorem 5.2.14. *Let $\mathcal{C}_0 = (C_0, M_0, \mathrm{pr}) \preceq \mathcal{C} = (C, M, \mathrm{pr})$ be two Zariski cover structures satisfying the Continuous Connections Assumption. Let $\pi : \mathcal{C} \to \mathcal{C}_0$ be a specialisation. Then the following are equivalent:*

(i) *π is $\aleph_0$-universal;*

(ii) *the restriction $\pi_M : M \to M_0$ is $\aleph_0$-universal and $\pi : \mathcal{C} \to \mathcal{C}_0$ is a maximal specialisation;*

(iii) *the restriction $\pi_M : M \to M_0$ is $\aleph_0$-universal, $\pi : \mathcal{C} \to \mathcal{C}_0$ is a maximal specialisation, and the following holds*

$$\forall m \in \mathrm{Dom}(\pi)\ \exists c \in \mathrm{Dom}(\pi)\ \ \mathrm{pr}(c) = m$$

An important aspect of Zariski cover structures is that the “binding groups” are the same for each fibre. Which is an essential part of our passage from the Ladder Theorem to Zariski cover structures, along with adding the topology. In more general notions of cover (such as the ones defined in [14] and [1]) one allows different groups to act on different fibres. Also the mode of interpretation of the “binding group” in Zariski cover structures is rather specific. We assume they are interpreted as Zariski topological groups. Which is stronger then saying they are Zariski groups (so manifolds). For this reason it is a natural question whether one can prove the results of Chapter 5 when the "binding groups" are Zariski groups (manifolds) rather than Zariski topological groups (topological sorts)? We will not attempt to make this generalisation in the thesis; and it will be considered as future work.

The main Theorems 4.3.3, 4.4.7, mentioned above can be seen to have a "downward direction" in the sense that in all of them one starts with a specialisation on the cover and shows that to decide its universality it is enough to project it to specialisation of the base structure and show that it is universal.

The other direction, I.e. starting with a κ-universal specialisation $\pi : M \to M_0$ of the base and showing that one can lift it to a κ-universal specialisation $\pi_C : C \to C_0$ of the cover is equally important. The results in this direction, in a similar sense as above, can be thought to have an "upward direction".

In this direction, I proved the following "upward direction" statement:

Proposition 4.4.6. *Let $\pi : M \to M_0$ be a specialisation. Then π can be lifted to a specialisation $\pi_C : C \to C_0$ such that if $c \in \mathrm{Dom}(\pi_C)$, then the orbit of c under the action of the finite group H is contained in $\mathrm{Dom}(\pi_C)$. Moreover, if π is a maximal specialisation then so is π_M.*

Note that Proposition 4.4.6 covers the case of finite covers which arises from the construction of the first example of a non algebraic Zariski geometry given by Hrushovki and Zilber in [16].

Note that Proposition 4.4.6 is proved for *maximal specialisations.* As it is proved in Proposition 2.1.69, universal specialisations are maximal. Therefore, the results mentioned above are also valid for lifting κ-universal specialisations. In other words one, can replace maximal with universal in the result mentioned above. In their current form, these statements are more general.

These results resonate well with an important theme in the theory of specialisations in abstract algebraic geometry. Namely the extension of specialisations. The main result for this theme in abstract algebraic geometry is equivalent to the existence of resultants in the elimination theory. These results can be seen as a natural attempt to extend this theme to Zariski structures.

Although there is a very well established theory of specialisations and valuations in algebraic geometry, in non-commutative algebraic geometry the issue of specialisations and valuations is far from being well understood. Some non-commutative algebro-geometric objects can be seen as Zariski geometries, as one can see at [35, 45, 46].

Developing a theory of specialisations for Zariski geometries will contribute in particular to understanding specialisations and valuations of the non-commutative algebro-geometric objects considered in [35, 45, 46]. In a broader perspective, an important aim is to represent non-commutative geometry in terms of Zariski geometries. From this perspective, a theory of specialisations for Zariski geometries has the potential to improve our understanding of specialisations and valuations in non-commutative geometry.

Chapter 2

Preliminaries

2.1 Zariski Structures

Let $\mathcal{L}$ be a language and M be an $\mathcal{L}$-structure. The closed subsets of M^n for all $n \in \mathbb{N}$ are defined as follows: The subsets defined by primitive relations of $\mathcal{L}$ are (definably) *closed*, and moreover the sets which are given by positive quantifier free $\mathcal{L}$-formulas are *closed*.
Equivalently they can be defined by the following topological axioms:

1. Finite intersection of closed sets is closed;
2. Finite unions of closed sets are closed;
3. M is closed;
4. The graph of equality is closed;
5. Any singleton in M is closed;
6. Cartesian products of closed sets are closed;
7. The image of a closed set under a permutation of coordinates is closed;
8. For $a \in M^k$ and closed $S \subseteq M^{k+l}$, the set $S(a, M^l)$ is closed.

In the rest of this thesis I will refer to these axioms as the topological axioms, and when referring to a specific one I will state its number clearly in the form "topological axiom x" where $x = 1, \ldots, 8$. An $\mathcal{L}$-structure that satisfies the topological axioms will be called an *$\mathcal{L}$-topological structure*. I will sometimes drop $\mathcal{L}$, and say topological structure when it is clear from the context.

A *constructible set* is defined to be a boolean combination of closed sets. In other words they are nothing but sets defined by quantifier free $\mathcal{L}$-formulas. A set S is said to be *irreducible* if there are no proper relatively closed subsets $S_1, S_2 \subset S$ such that $S = S_1 \cup S_2$. A topological structure is called *Noetherian* if it satisfies the descending chain condition (DCC) for closed sets.

Remark that in Noetherian topological structures any closed set S can be written as $S = S_1 \cup \ldots \cup S_k$ where $S_1, \ldots, S_k \subset S$ are distinct and relatively closed; moreover they are unique up to ordering. They are called *irreducible components* of S.

A topological structure is said to be semi-proper if it satisfies the following condition:

(SP) Semi Properness: For a closed irreducible $S \subset M^n$ and a projection $\mathrm{pr} : M^n \to M^m$, there is a proper closed subset $F \subset \overline{\mathrm{pr}(S)}$ such that $\overline{\mathrm{pr}(S)} \setminus F \subset \mathrm{pr}(S)$

The *dimension*, denoted by dim is a function from definable sets of Noetherian topological structure to natural numbers which satisfies:

(DP) Dimension of a Point: $\dim(a) = 0$ for all $a \in M$;

(DU) Dimension of Unions: $\dim(S_1 \cup S_2) = \max(\dim(S_1), \dim(S_2))$ for closed S_1, and S_2;

(SI) Strong irreducibility: For any irreducible $S \subseteq_{cl} U \subseteq_{op} M^n$ and any closed $S_1 \subsetneq S$, $\dim(S_1) < \dim(S)$;

(AF) Addition Formula: For any irreducible closed $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection $\mathrm{pr} : M^n \to M^m$,

$$\dim(S) = \dim(\mathrm{pr}(S)) + \min_{a \in \mathrm{pr}(S)} (\mathrm{pr}^{-1}(a) \cap S)$$

(FC) Fibre Condition: Given $S \subseteq_{cl} U \subseteq_{op} M^n$ and a projection $\mathrm{pr} : M^n \to M^m$, there is a relatively open $V \subseteq_{op} \mathrm{pr}(S)$ such that, for any $v \in V$

$$\min_{a \in \mathrm{pr}(S)} (\dim(\mathrm{pr}^{-1}(a) \cap S)) = \dim(\mathrm{pr}^{-1}(v) \cap S)$$

A Noetherian, semi-proper topological structure together with a dimension function which satisfies the above conditions is said to be a *Noetherian $\mathcal{L}$-Zariski Structure*. When the language $\mathcal{L}$ is clear from the context, I will usually drop it; moreover, since all Zariski structures in this thesis are Noetherian I will also omit this, and simply write *Zariski structure* when there is no ambiguity.

Note that classically the dimension is taken to be the Krull dimension in [16]. This axiomatic definition of dimension makes it possible to work with other notions of dimension. However, unless specifically stated otherwise, I always assume that I am working with the Krull dimension.

A (Noetherian) Zariski structure is said to be *complete* if it satisfies:

(P) Properness: projections of closed sets are closed. I.e. For any closed $S \subseteq M$, $pr_{i_1,\ldots,i_m}(S)$ is closed.

It is called *quasi-compact* if:

(QC) Quasi-compactness: M is complete and for a finitely consistent family $\mathcal{C}$ of closed subsets of M^n, $\cap\mathcal{C}$ is not empty.

It is *pre-smooth* if:

(PS) Pre-Smoothness: For any closed, irreducible $S_1, S_2 \subseteq M^n$, any irreducible component of $S_1 \cap S_2$ has the dimension at least $\dim S_1 + \dim S_2 - n$.

Example 2.1.1.

An algebraically closed field K in language $\mathcal{L}_{\text{Zar}}$ for algebraic varieties (which only has predicates for Zariski closed subsets of K^n for $n \in \mathbb{N}$) together with the Krull dimension is a Zariski structure.

Furthermore, let $\mathbb{P}^1(K)$ be the projective line over K in the language which only has predicates for projective Zariski closed subsets of $(\mathbb{P}^1(K))^n$ for all $n \in \mathbb{N}$. together with the Krull dimension is a Zariski structure.

Details for both cases can be found in [34, chap. 1-4].

Definition 2.1.2. Let M be a Zariski structure, and A be a constructible set. A is said to be *pre-smooth (with M)* if for any relatively closed irreducible subsets $S_1, S_2 \subseteq_{cl} A^n \times M^m$, any irreducible component of $S_1 \cap S_2$ has dimension at least $\dim S_1 + \dim S_2 - \dim(A^n \times M^m)$.

A Zariski structure M is said to be a Zariski geometry if it satisfies

(SPS) Strong Pre-Smoothness: For any constructible irreducible $A \subseteq M^n$, there is a definable open subset $A^0 \subseteq_{op} A$ which is pre-smooth.

Fact 2.1.3. *A Zariski Structure M has quantifier elimination. In more geometric words, every definable set is constructible.*

2.1.1 Manifolds

Following [16] it is now convenient to give definitions of "manifolds" and "morphisms". Let M be a Zariski structure and $D \subseteq M^n$ be an irreducible definable (i.e. constructible) subset. Let $E \subseteq D \times D$ be a closed equivalence relation on D. Then D/E is an imaginary sort. By $\mathrm{p} : D \to D/E$ we will denote canonical quotient map (sometimes we will add a subscript to p). One can naturally consider the Cartesian powers $(D/E)^m$, moreover observe that $(D/E)^m$ can be identified with D^m/E^m where $(a_1, \ldots, a_m)E^m(b_1, \ldots, b_m)$ if and only if a_iEb_i for all i. Now we can equip D/E together with all its Cartesian powers $(D/E)^m$ with the corresponding quotient topologies via the natural projection maps from D and its Cartesian powers D^m.

Definition 2.1.4. The imaginary sorts D/E, where $D \subseteq M^n$ is an irreducible definable subset and E is a closed equivalence relation on $D \times D$ as described above, together with the collection of closed subsets of its Cartesian powers is called a *topological sort*.

Fact 2.1.5 (Lemma 3.7.2 in [44])**.** *Topological sorts satisfy the topological axioms 1-8, hence they are topological structures. Moreover they are Noetherian, and satisfy (SP). If M is complete then so are its topological sorts.*

Observe that if N_1/E_1 and N_2/E_2 are two topological sorts in a Zariski structure M, then $N_1 \times N_2/E_1 \times E_2$ is also a topological sort. It is identified with $N_1/E_1 \times N_2/E_2$. Hence product of two topological sorts in M is a topological sort in M.

Therefore a Zariski structure M together with all the topological sorts in it will form a multi-sorted Zariski structure.

Proposition 2.1.6. *If E is a closed equivalence relation on D and E' a closed equivalence relation on the topological sort $T = D/E$, then T/E' is a topological sort which can be represented also as D/E'' for some closed equivalence relation E'' on D.*

Proof. Let $T = D/E$ be a topological sort and $E' \subseteq T \times T$ be a closed equivalence relation on T. So $E' \subseteq D \times D/E \times E = D/E \times D/E$ is closed.

Let $E'' := \mathrm{p}_T^{-1}(E')$. It is clear that $E'' \subseteq D \times D$ is a closed subset. We will show that E'' is an equivalence relation.

Let $d \in D$. Since E' is an equivalence relation on T, $[d]E'[d]$ where $[d]$ denotes the E-equivalence class of d. Then $\mathrm{p}_T^{-1}(d) \times \mathrm{p}_T^{-1}(d) \subseteq E''$. In particular $(d, d) \in E''$.

Via similar arguments one can show E'' is symmetric and transitive. Since the arguments are straightforward (and only a little more tedious), we shall skip them. $\square$

Dimension of closed sets of a topological sort D/E can be defined as follows: Let $T \subseteq (D/E)^k$ be a closed irreducible set and $S = \mathrm{p}^{-1}(T)$. Define

$$\dim(T) = \dim(S) - \min\{\dim(\mathrm{p}^{-1}(t)) : t \in T\}$$

For a closed set, define its dimension as usual to be maximum of the dimensions of its irreducible components.

Remark 2.1.7. Lemma 3.7.5 in [44] shows that S in definition of dimension above can be taken to be any closed irreducible set such that $\mathrm{p}(S) = T$.

Fact 2.1.8 (Lemma 3.7.7 in [44])**.** *A topological sort with the dimension notion as defined above satisfies (DU), (SI), and (DP).*

Note that a topological sort with the dimension defined as above does not necessarily satisfy (AF) and (FC). If a topological sort is a Zariski structure it will be called a *Zariski set* or a *Z-set*.

Topological sorts does not necessarily satisfy (PS). An example of a topological sort not satisfying (PS) is given in [16]: Let M be a Zariski geometry and E be the closed equivalence relation which has a unique equivalence class of cardinality two and all other classes are singletons. Therefore the natural projection $M \to (M/E)$ is generically one to one, with a single fibre of size two. Then (PS) fails for $M^2 \times M/E$.

However, one can still talk about definable (and actually interpretable) sets in M satisfying (PS):

Definition 2.1.9. Let M be a Zariski structure and T a topological sort in M. Let A be a constructible set of T. We say that A is *pre-smooth with* T if for any relatively closed irreducible $S_1, S_2 \subseteq A^k \times T^m$, and any irreducible component S of $S_1 \cap S_2$

$$\dim S \geq \dim S_1 + \dim S_2 - \dim(A^k \times T^m)$$

Remark 2.1.10. This is a minor generalization of the definition of a set *pre-smooth with* an ambient Zariski structure given in [44]. In the original definition the topological sort T is not involved. However, since we will be considering imaginary sets (and in particular topological sorts) it is important to make this distinction and give the definition as it is stated above, although in practice it will not make any difference.

In the definition above if one is to take T to be a subset of M then one gets the definition of a set *pre-smooth with* M as it is given in [44].

Lemma 2.1.11 (Closed Graph Theorem)**.** *Let U and V be pre-smooth topological sorts. Let $f : U \to V$ be a function such that its graph $\Gamma \subseteq U \times V$ is closed. Then f is continuous.*

Lemma 2.1.11 is originally proved by Hrushovski and Zilber in [16] (Lemma 5.5). Their formulation is for one dimensional Zariski geometries and, U and V are assumed to be irreducible regular subsets of special sorts. However, these differences in the setting does not affect the proof. The proof given in [16] is also valid for the above lemma.

Definition 2.1.12. Let T_1 and T_2 be two topological sorts of a Zariski structure M. A *Zariski morphism* (*Z-morphism* or simply *morphism* for short) is a map $f : T_1 \to T_2$ such that $f \times \mathrm{id} : T_1 \times M^n \to T_2 \times M^n$ is continuous for all $n \in \mathbb{N}$.

Defining Zariski morphisms this way is due to Junker [18]. Initially Hrushovski and Zilber [16] (and later Zilber [44]) defined Zariski morphisms as maps between topological sorts with closed graphs. Without pre-smoothness maps with closed graphs are not necessarily continuous. In the original approach in [16] the pre-smoothness is taken for granted. In which case two definitions coincide (see Lemma 5.5 in [16]).

Remark 2.1.13. Let T_1 and T_2 be two topological sorts; and $f : T_1 \to T_2$ a morphism. Let $\psi(y, \bar{z})$ be a positive quantifier free formula defining a closed subset of $T \times M^n$. Since the map $f \times \mathrm{id} : T_1 \times M^n \to T_2 \times M^n$ is continuous, $\exists y\, f(x) = y \wedge \psi(y, \bar{z})$ is equivalent to a positive quantifier formula; as it defines the pre-image of the closed set defined by $\psi(y, z)$, which is closed. We will often write $\psi(f(x), \bar{z})$ to mean $\exists y\, f(x) = y \wedge \psi(y, \bar{z})$.

The following facts are immediate:

Fact 2.1.14. *Graphs of Zariski morphisms are closed.*

Fact 2.1.15. *Let $T := D/E$ be a topological sort. The quotient map $\mathrm{p}_T : D \to T$ is a morphism.*

Fact 2.1.16. *Let $f : T_1 \to T_2$ and $g : T_2 \to T_3$ be morphisms, then their composition $g \circ f : T_1 \to T_3$ is a morphism.*

Fact 2.1.17. *Let T_1, T_2, T_3 and T_4 be topological sorts, and $f : T_1 \to T_2$ and $g : T_3 \to T_4$ be morphisms, then $f \times g : T_1 \times T_3 \to T_2 \times T_4$ is a morphism.*

Fact 2.1.18. *Let $f : T_1 \to T_2$ be a morphism between the topological sorts T_1, T_2. Let E_f be the closed equivalence relation given by the pre-image of $=$ under f, and $Im(f) \subseteq T_2$ be the image of T_1 under f. Then f factorises through T_1/E_f and $Im(f)$ as follows:*

$$T_1 \xrightarrow{p} T_1/E_f \xrightarrow{\tilde{f}} Im(f) \xhookrightarrow{i} T_2$$

where $\tilde{f}$ is a bijective morphism, and i is the inclusion map.

Fact 2.1.19. *Let T_1 and T_2 be two topological sorts, and $f : T_1 \to T_2$ be a morphism.*

(i) If $V \subseteq T_1$ is a definable irreducible subset, then $f_{|V} : V \to T_2$ is a morphism.

(ii) If E is a closed equivalence relation on T_1 such that f is constant on equivalence classes of E, then f induces a morphism $f/E : T_1/E \to T_2$.

Corollary 2.1.20. *Let T_1 and T_2 be topological sorts, and $f : T_1 \to T_2$ a surjective morphism. Let $E(x_1, x_2)$ be the equivalence relation $f(x_1) = f(x_2)$ on T_1. Then T_1/E is a topological sort and f induces a homeomorphism between T_1/E and T_2.*

Definition 2.1.21. Let $T = N/E$ be a topological sort. If E is an equivalence relation with finite fibres, then T is called and *eq-fold*.

Definition 2.1.22. Let $T = N/E$ be an eq-fold, if there is a finite group H acting on N and if E is given by the action of H (i.e. for $x, y \in N$, xEy if and only if there is an $h \in H$ such that $h \cdot x = y$), then T is called an *orbifold*. Furthermore, if N is pre-smooth with M, then T is called a *special sort*.

Remark 2.1.23. In [16] special sorts defined in a different way: Let N be a one dimensional Zariski geometry. Let H be a subgroup of the symmetric group Sym_n. Consider the natural action of H on N^n. Then N^n/H is a special sort in the sense of [16].

Note that the natural action of H on N^n is by Zariski continuous bijections. In our setting the action of H defines the closed equivalence relation

$$xE_Hy \text{ if and only if } \exists h \in H(h \cdot x = y)$$

Then $N^n/H = N^n/E_H$. Hence special sorts in the sense of [16] are also special sorts in the sense above.

It is also relevant and necessary to discuss finite sorts; although briefly, they will appear in Chapter 5. Let M be a Zariski structure, and D be a definable subset of M^n. Let $E \subseteq D \times D$ be a closed equivalence relation. To construct a finite sort as a quotient D/E one needs to allow D to reducible. A detailed discussion on the sorts of the form D/E where D is not necessarily irreducible is given in [19, Sect. 3.2]. Here we will only discuss the case when D/E is finite.

For a reducible definable $D \subset M^n$, and a closed equivalence relation $E \subset D \times D$, let $\mathrm{p} : D \to D/E$ be the canonical quotient map, and suppose that D/E is finite. Then the corresponding quotient topologies on the Cartesian powers $(D/E)^m$ are discrete. All the results we proved and recalled above for topological sorts immediately extend to these finite sorts. In particular, a finite sort D/E where D is definable and E is closed satisfy topological axioms 1-8, and (SP), and they are Noetherian.

One can similarly introduce dimension on D/E. Let $T \subseteq (D/E)^m$ be a closed irreducible set, and $S = \mathrm{p}^{-1}(T)$. Then $T = \{t\}$ is a singleton set, because of the discrete topology, and $S = \mathrm{p}^{-1}(t)$. Now, as before we can define

$$\dim(T) = \dim(S) - \min\{\dim(\mathrm{p}^{-1}(t)) : t \in T\}$$

Clearly, $\dim(S) - \min\{\dim(\mathrm{p}^{-1}(t)) : t \in T\} = 0$, and hence we get $\dim(T) = 0$ as needed. One can also see that the dimension axioms (DP), (DU), (SI), (AF), and (FC) hold trivially.

For this, we wish to consider such finite sorts as topological sorts, but will still make a distinction and call them *finite topological sorts*. This distinction is to emphasize that they are not irreducible.

Definition 2.1.24. Let X be a set interpretable in a Zariski structure M. Call X a *manifold* if there are finitely many sets $U_i, \ldots, U_k$ such that:

1. Each U_i is an irreducible open subset of some topological sort T_i and, each U_i is pre-smooth with T_i.

2. For each $i = 1, \ldots, k$ there is an injective map $f_i : U_i \to X$ and, $X = f_1(U_1) \cup \ldots \cup f_k(U_k)$.

3. For each i, j the set $\{(x, y) \in U_i \times U_j : f_i(x) = f_j(y)\} \subseteq U_i \times U_j$ is closed and irreducible. Moreover, its projections to U_i and U_j are non-empty open subsets.

On a manifold X one can define a topology in the natural way: A set $S \subseteq X^n$ is said to be closed if $f_i^{-1}(S)$ is closed in U_i for each i. Condition 3 in the definition of manifold and Lemma 2.1.11 guarantees that each f_i is a homeomorphism. It is easy to see that this topology is Noetherian.

Remark 2.1.25. This definition agrees with the definition of manifold in [16]. The definition of manifold given in [16] is actually a special case of the definition of manifold given here. I.e. manifolds as defined in [16] are also manifolds in the sense of the thesis. It is enough to show that an irreducible regular subset U of a special sort N in the sense of [16] is an irreducible open subset of a topological sort T in the sense of the thesis, and that U is pre-smooth with T.

Since U is regular, it is open in its closure C. Since U is irreducible, C is also irreducible. Recall that the special sort (in the sense of [16]) is a quotient $N = D^n/H$ where $H = Sym(n)$, and D is a Zariski geometry. Consider the natural map $\mathrm{pr} : D^n \to N$. Since C is closed in N, by definition, $\mathrm{pr}^{-1}(C)$ is closed in D^n. Put $S := \mathrm{pr}^{-1}(C)$, then S is closed in D^n. Also, $C = S/H$. Therefore C is a topological sort in the sense of Definition 2.1.4 (In particular C is a special sort in the sense of Definition 2.1.22). So, U is an open, irreducible subset of a topological sort (in the sense of Definition 2.1.22).

Take $T := C$. Now it follows from Lemma 5.4 in [16] that U is pre-smooth with T.

Fact 2.1.26 (Lemma 3.7.16 in [44]). *Let X be a manifold, then X is an irreducible Z-set satisfying (PS).*

Definition 2.1.27. Let X_1 and X_2 be manifolds. A map $f : X_1 \to X_2$ is said to be a *Zariski morphism* (or a *Z-morphism*) if its graph is a closed subset of $X_1 \times X_2$.

Remark 2.1.28. One can define morphisms of manifolds in a similar way to the morphism of topological sorts, as given in Definition 2.1.12. Since manifolds satisfy pre-smoothness (see Fact 2.1.26), Fact 2.1.11 implies that morphisms between manifolds are continuous.

As there is no danger of confusion, we will often drop the prefix as we did in the above remark, and say morphism instead of Zariski morphism both for topological sorts and manifolds.

As before the following fact is immediate.

Fact 2.1.29. *Let X_1, X_2, X_3 and X_4 be manifolds and $f : X_1 \to X_2$ and $g : X_3 \to X_4$ be morphisms. Then:*

1. *their product $f \times g : X_1 \times X_3 \to X_2 \times X_4$ is a morphism, and*

2. if $X_2 \subseteq X_3$, their composition $g \circ f : X_1 \to X_4$ is also a morphism.

For a one dimensional Zariski geometry M, in [16], Hrushovski and Zilber showed that any set X interpretable in M can be piecewise given the structure of a manifold such that the sets U_i are from special sorts. This property is called "elimination of imaginaries" in [16], we will keep this terminology.

It is desirable to have a similar "elimination of imaginaries" in higher dimensional Zariski geometries. More explicitly one would hope for the following claim to hold.

Claim 2.1.30. Let X be set interpretable in a Zariski geometry M. Then X can be piecewise given a manifold structure. In particular, the sets U_i described in the definition of manifold can be chosen to be subsets of special sorts.

"Elimination of imaginaries" for one dimensional Zariski geometries depends on the following unpublished result by Pillay and Lascar:

Fact 2.1.31. *Let M be a strongly minimal set such that* $\mathrm{acl}(\emptyset) \subset M$ *is infinite, and let S be an imaginary sort. Then there is a partition of S into finitely many $\emptyset$-definable sets X_i and, there are special sorts S_i and $\emptyset$-definable injective maps $f_i : X_i \to S_i$.*

The hypothesis M is strongly minimal is essential. Therefore it is not immediate to generalise "elimination of imaginaries" to higher dimensional Zariski geometries as they are not necessarily strongly minimal.

2.1.2 Multi-Sorted Zariski Structures

Multi-sorted Zariski geometries are already considered in [16] and also implicitly in [18]. Later Zilber considered them in [43] where one of the aims is to view Zariski structures from a category theoretic perspective.

Indeed in latter chapters (especially Chapters 4 and 5) of this thesis I will work in the setting of multi-sorted Zariski structures. I will follow both [16, 43] and recall some basics about multi-sorted Zariski geometries. One should also note that the category theoretic perspective will not be of any concern in this thesis.

Most of the multi-sorted Zariski structures considered in this thesis can be given as H^{eq} constructions for a suitable Zariski structure H. Therefore, unless otherwise is stated, multi-sorted Zariski structure we consider contains a "home sort" H in the sense that any other sort M_i is interpretable in H.

Definition 2.1.32. Let H be a Zariski structure. A multi-sorted Zariski structure M with the home sort H consists of a collection of sorts $(M_i)_{i \in I}$ such that:

(i) M is a multi-sorted structure with a multi-sorted language $\mathcal{L}$.

(ii) Each M_i is a topological sort in H; and there is an i such that $M_i = H$.

(iii) Each M_i is an $\mathcal{L}_i$-Zariski structure where $\mathcal{L}_i$ is the natural language for the sort M_i and $\mathcal{L}_i \subset \mathcal{L}$.

(iv) If M_i and M_j are sorts in M then their product $M_i \times M_j$ is also a sort in M.

If all sorts M_i of M are pre-smooth, then M is called a *multi-sorted Zariski geometry*.

Remark 2.1.33. (i) As the sorts M_i are topological sorts in H, the dimension on each of them is induced from M as discussed for topological sorts in general in the previous subsection.

(ii) In the same discussion about topological sorts, we showed that for any two topological sorts M_i and M_j, their product $M_i \times M_j$ is also naturally a topological sort. For any two sorts in M we are also including their product as sort in M. Therefore, for any sorts $M_1, \ldots, M_k$ of M, the product $\prod_{i=1}^{k} M_k \subset M$ is a sort in M.

(iii) Although we called M a multi-sorted Zariski structure (or a multi-sorted Zariski geometry if all sorts are pre-smooth), it is not necessarily an $\mathcal{L}$-Zariski structure!

As $\mathcal{L}$ is a multi-sorted language, all of the syntactic conventions about multi-sorted languages are in force. Most of the arguments in this thesis do not rely on the syntax of $\mathcal{L}$, but it is important to keep this in mind.

More importantly, I will use multi-sorted versions of notions such as definability, type, elementary embedding, parameter set, algebraic and definable closure in the natural way.

When M is a multi-sorted Zariski geometry with a home sort, in [16] it is assumed that for any closed S of any sort M_i there is n and a closed set $C \subseteq H^n \times S$ such that for any two points a, b in a dense open subset of S, there is $e \in H^n$ with $C(e)$ passes through a but not b. I.e. C separates a and b. I will call this property *multi-sorted ampleness.*

Proposition 2.1.34. *Let M be a multi-sorted Zariski structure with a home sort H. Then any sort M_i of M is stably embedded.*

Proof. Since H is stable, H^{eq} is also stable. Therefore any sort interpretable in H is stably embedded. $\square$

A simple corollary of the proposition above is the following:

Corollary 2.1.35. *Let M be a multi-sorted Zariski geometry with the home sort H. Then any sort M_i is an ample Zariski geometry.*

Example 2.1.36. An immediate example will be quasi-projective varieties. Let $V \subseteq \mathbb{P}^{n_V}$ and $W \subseteq \mathbb{P}^{n_W}$ be quasi-projective varieties with $dimV = n_V$ and $\dim W = n_W$ in their natural Zariski languages $\mathcal{L}_V$ and $\mathcal{L}_W$. One embeds $V \times W$ into $\mathbb{P}^N$ via the Segre embedding (where $N = (n_V + 1)(n_W + 1) - 1$). Then $V \times W$ becomes a variety with the induced Zariski topology from $\mathbb{P}^N$. Therefore $V \times W$ is a Zariski structure in the natural Zariski language $\mathcal{L}_{V\times W}$.

Let $M = \bigsqcup_{k,l\in\mathbb{N}} V^k \times W^l$, where $V^k \times W^0 = V^k$ and $V^0 \times W^l = W^l$. In other words $V, W, V \times W, \ldots, V^k \times W^l, \ldots$ are the sorts of M and the language of M is $\mathcal{L} \supseteq \bigcup_{k,l\in\mathbb{N}} \mathcal{L}_{V^k\times W^l}$.

Remark 2.1.37. Since the product of two abstract algebraic varieties are again an abstract algebraic variety, the same example can be repeated more generally with abstract algebraic varieties.

Most definitions in the thesis are given for one sorted Zariski structures, however they can be considered for multi-sorted Zariski structures in a natural way. I will clearly state when working with multi-sorted Zariski structures. If it is not stated, it will always be a one sorted Zariski structure.

2.1.3 Groups Interpretable in Zariski Structures

Let G be a group interpretable in a Zariski structure M. The following is proved in [16] and the proof is similar to Weil's Group Chunk Theorem.

Fact 2.1.38 (Lemma 5.11 in [16] and Theorem 2.6 in [24]). *Let M be a one dimensional Zariski geometry and G a group interpretable in M. Then G can be given a manifold structure such that multiplication and inversion are Z-morphisms.*

Such groups are called *Zariski groups* (or *Z-groups* for short). More precisely:

Definition 2.1.39. A manifold G together with a closed ternary relation $P \subseteq G^3$ such that P is the graph of a binary group operation on G is said to be a *Zariski group* (or a *Z-group*).

Therefore, Fact 2.1.38 says that groups definable in one dimensional Zariski geometries are Zariski groups.

In fact for a simple group G which is non orthogonal to a strongly minimal non-locally modular Zariski geometry one can say more:

Fact 2.1.40. *Let M be a Zariski geometry, let G be a simple group definable in M such that G is non-orthogonal to a strongly minimal non-locally modular Zariski geometry. Then G is an algebraic group.*

2.1.4 Groups Interpretable as Topological Sorts

In Section 5, we will consider a stronger mode of interpretation. Specifically we will work with groups interpretable as topological sorts. In this subsection we will consider some general theory of such groups.

Definition 2.1.41. Let C be a Zariski structure, and G be a group that is a topological sort or a finite topological sort in C such that multiplication $m : G \times G \to G$ and inversion $^{-1} : G \to G$ are morphisms. Such a group G will be called a *Zariski topological group in C* (or *topological group* for short). Whenever the underlying sort is a finite topological sort, we often say *finite topological group* to emphasize this.

Remark 2.1.42. The main difference when the underlying sort is a topological sort or a finite topological sort is that the resulting group will be connected (for topological sorts) or that it will be not connected.

It is possible more generally to consider non-connected groups as topological groups. This will require a more detailed study of topological sorts which are built on definable sets which are not necessarily irreducible. A detailed discussion on both, can be found in [19].

Although this would be more general, and give a more uniform approach, in our considerations (especially in Chapter 5), non-connected groups only appear as finite groups. For this reason we choose only to include finite groups as topological groups.

Fact 2.1.43 (Proposition 5.6 in [19])**.** *Let G be a Zariski topological group, and $H < G$ be a subgroup (not necessarily definable!). Then the closure $\overline{H}$ is also a subgroup. If H is normal, then so is $\overline{H}$. Moreover, a definable submonoid of G is a closed subgroup.*

Definition 2.1.44. Let C be a Zariski structure and G be Zariski topological group in C. Let $A \subseteq C$ be a constructible set.

(i) We say that G *acts morphically on* A if the action

$$\begin{array}{rcl}\Theta : G \times A & \to & A \\ (g, a) & \mapsto & g \cdot a\end{array}$$

is a morphism. Frequently we will use $\cdot$ to denote the action instead of Θ.

(ii) We say that the action is *proper* if the equivalence relation

$$E_G(x, y) \text{ defined by } \exists g \in G\, (y = g \cdot x)$$

is closed in $C \times C$.

(iii) We will say that the action of G is *free*, if it is proper, and the action $G \times A \to A$ is invertible in the sense that there is a morphism

$$\begin{array}{ll}E_G & \to G \\ (a, g \cdot a) & \mapsto g\end{array}$$

Unless stated otherwise, we will always assume that Zariski topological groups act morphically.

Fact 2.1.45 (Lemme 5.16 in [19]). *Let T be a topological sort and let G be Zariski topological group acting on T. Let $S \subseteq T$ be a closed subset and let $X \subseteq T$ be any subset (not even necessarily definable).*

(i) $\{g \in G : g \cdot X \subseteq S\}$ is a closed subset of G.

(ii) For all $t \in T$ the stabilizer subgroup $G_t = \{g \in G : g \cdot t = t\}$ is a closed subgroup. In particular, $G_X := \{g \in G : g \cdot t = t \text{ for all } t \in T\}$

(iii) For all $g \in G$, the set $\mathrm{Fix}_T(g) := \{t \in T : g \cdot t = t\}$ of all elements of T fixed by g is a closed subset of T. Moreover, $\mathrm{Fix}_T(G) := \{t \in T : g \cdot t = t \text{ for all } g \in G\}$ is closed.

Fact 2.1.46 (Propositions 5.17 and Corollaire 5.18 in [19]). *Let T be a topological sort and G be a Zariski topological group acting on T. Then*

(i) The orbits of the action of G are locally closed.

(ii) The closure of an orbit $X := G \cdot x$, and its border ∂X are unions of orbits.

(iii) The orbits of minimal dimension are closed.

Lemma 2.1.47. *Suppose G is a Zariski topological group acting freely on a definable set D and $H \lhd G$ is a definable normal subgroup. Then G/H is a Zariski topological group, D/H is a topological sort, and G/H acts morphically, properly and freely on D/H.*

Proof. By the Fact 2.1.43 we can assume H is closed. First let us show that the quotient D/H is a topological sort. Since the action is free there is a morphism $E_G \to G$ defined by $(a, g \cdot a) \mapsto g$. Let E_H be the pre-image of H under this map. Clearly, E_H is a definable equivalence relation on D given by

$$xE_Hy :\Leftrightarrow \exists h \in H\, (h \cdot x = y)$$

Since H is closed its inverse image E_H under this morphism is closed in E_G. Hence E_H is a closed equivalence relation on D. Therefore D/H is a topological sort.

Next, we sill how that G/H is a topological group. The closed normal subgroup H defines the equivalence relation

$$a \sim b :\Leftrightarrow ab^{-1} \in H$$

for all $a, b \in G$. Observe that $\sim$ is the pre-image of H under the morphism

$$\begin{array}{rcl} G \times G & \to & G \\ (a, b) & \mapsto & ab^{-1} \end{array}$$

Since H is closed, $\sim$ is closed. Hence $G/\!\sim\, = G/H$ is a topological sort in G. To see that multiplication on G/H is a morphism observe that $\mathrm{p}_H \circ m : G \times G \to G \to G/H$ is a morphism. Moreover $H \times H$ is a closed equivalence relation on $G \times G$, such that the morphism $\mathrm{p}_H \circ m$ is constant on its classes. By Fact 2.1.19, multiplication is a morphism. A similar argument will show that inversion is also a morphism.

By Proposition 2.1.6, G/H is also a topological sort in C, and the group operations are again morphisms.

Next, let us consider the action of G/H on D/H. The action $\Theta : G \times D \to D$ and the quotient map $\mathrm{p}_H : D \to D/H$ are morphisms. Therefore their composition

$$\mathrm{p}_H \circ \Theta : G \times D \to D \to D/H$$

is a morphism. Moreover $\mathrm{p}_H \circ \Theta$ is constant on equivalence classes of the closed equivalence relation defined by $H \times H$ on $G \times D$. Therefore

$$\mathrm{p}_H \circ \Theta / H \times H : G \times D/(H \times H) \to D/H$$

is a morphism. Since $G \times D/(H \times H) \simeq G/H \times D/H$, the action of G/H on D/H is a morphism. $\square$

Lemma 2.1.48. *Let H be a finite Zariski topological group acting freely on a definable set D and $T = D/H$ be a topological sort, let $\mathrm{p}_T : D \to T$ be the canonical quotient map. Let $Q \subseteq D$ be a closed subset. Then $\mathrm{p}_T(Q) \subseteq T$ is also closed.*

Proof. Note that

$$\mathrm{p}_T^{-1}(\mathrm{p}_T(Q)) = H \cdot Q = \bigcup_{h \in H} h \cdot Q$$

Since subsets $h \cdot Q$ are closed, the statement follows. $\square$

Remark 2.1.49. Let G be a Zariski topological group acting morphically on D. We will also assume that the action is proper. I.e. the definable equivalence relation

$$E_G(x, y) :\Leftrightarrow \exists g \in G \, (y = g \cdot x)$$

on D will be closed. However, in general E_G is not necessarily closed in $D \times D$.

Junker proved that the equivalence relation E_G has the following nice property [19, Lemme 5.20]: For any constructible subset $Q \subseteq D$, which is E_G-saturated [1], its closure $\overline{Q}$ is also E_G-saturated.

Junker calls such (definable) equivalence relations "admissible". Although the imaginary sort D/E_G where E_G is admissible, may not be a topological sort, it has many properties in common with topological sorts. But certainly not all. In particular for the graph of $=$ to be closed in the quotient topology on D/E_G, it is sufficient and necessary that E_G is closed.

2.1.5 Specialisations

Definition 2.1.50. Let M_0 be a Noetherian topological structure and M an elementary extension of M_0. Let π be a (partial) function $\pi : M \to M_0$ such that $\pi(m) = m$ for all $m \in M_0$; and for every formula $S(\overline{x})$ over $\emptyset$, defining an M_0-closed set and for every $\overline{a} \in M^n$ (which is also in the domain of π), $M \models S(\overline{a})$ implies $M_0 \models S(\pi\overline{a})$. Such a function is said to be a *specialisation*. We will also call the tuple (M, π) a *specialisation*.

A specialisation is said to be *κ-universal* if, given any $M' \succeq M \succeq M_0$, any $A \subseteq M'$ with $|A| < \kappa$ and a specialisation $\pi_A : M \cup A \to M_0$ extending π, there is an elementary embedding $\sigma : A \to M$ over $M \cap A$ such that $\pi_A | A = \pi \circ \sigma$.

An $\aleph_0$-universal specialisation is said to be *universal.*

[1] the word saturated here is not related to the model theoretic use of saturated. For an equivalence relation E on a set D, a subset $Q \subseteq D$ is said to be E-saturated if $a \in Q$ and bEa, then $b \in Q$.

In other words specialisations are $\mathcal{L}$-homomorphisms from substructures of M to M_0. The following lemma, originally due van den Dries [38] and proved more generally for first order structures, gives a description of M_0-closed sets in terms of specialisations.

Lemma 2.1.51. *For each model M of $Th(M_0)$, and each specialisation $\pi : A \to M_0$ with $A \subseteq M$, if $T(x)$ is an $\mathcal{L}$-formula such that $M \models T(a)$ implies $M_0 \models T(\pi(a))$ then, there is a positive quantifier free formula $S(x)$ such that $Th(M_0) \vdash T(x) \longleftrightarrow S(x)$.*

Remark 2.1.52. Let $\pi : M \to M_0$ be a specialisation and let $a \in \mathrm{Dom}(\pi)$. Instead of $\pi(a)$, I will sometimes write a^π to mean the same thing (i.e. $a^\pi = \pi(a)$). If $\overline{a} = (a_1, \ldots, a_n)$ is a tuple from $\mathrm{Dom}(\pi)$, then $\overline{a}^\pi := (a_1^\pi, \ldots, a_n^\pi)$.

We defined specialisations and κ-universal specialisations for one sorted Zariski structures however it makes sense to consider these notions for multi-sorted Zariski structures. (even for the ones without a home sort, but I will assume they have a home sort).

Definition 2.1.53. Let M be a multi-sorted Zariski structure with sorts S. Let N be an elementary extension of M. A (partial) specialisation $\pi = (\pi_{s_1}, \ldots, \pi_{s_n}) : N_{s_1} \times \ldots \times N_{s_n} \to M_{s_1} \times \ldots \times M_{s_n}$ is a map such that each $\pi_{s_i} : N_{s_i} \to M_{s_i}$ is a specialisation.

It is said to be κ-*universal* if, given any $N' \succ N \succ M$, any $A_{s_1} \subseteq N'_{s_1}, \ldots, A_{s_n} \subseteq N'_{s_n}$ with $|A_{s_i}| < \kappa$ for each i and a specialisation $\pi_A = (\pi_{A_{s_1}}, \ldots, \pi_{A_{s_n}}) : N_{s_1} \cup A_{s_1} \times \ldots \times N_{s_n} \cup A_{s_n} \to M_{s_1} \times \ldots \times M_{s_n}$ extending $\pi = (\pi_{s_1}, \ldots, \pi_{s_n})$, there is an embedding $\sigma = (\sigma_{s_1}, \ldots, \sigma_{s_n}) : A_{s_1} \times \ldots \times A_{s_n} \to M$ over $(N_{s_1} \times \ldots \times N_{s_n}) \cap (A_{s_1} \times \ldots \times A_{s_n})$ such that $\pi_{A_{s_i}} | A_{s_i} = \pi_{s_i} \circ \sigma_{s_i}$ for each i.

In the rest of this subsection I will give the key properties and definitions about specialisations. All of them are given for a one sorted Zariski structure. However, it is clear how one can generalize them to multi-sorted case when needed.

Let $\mathcal{C}$ be a Zariski structure and $\mathcal{C}' \succeq \mathcal{C}$. Let D, and R be definable sets in $\mathcal{C}$. By $D(\mathcal{C}')$ and $R(\mathcal{C}')$ we denote the $\mathcal{C}'$ points of D and R respectively; i.e. the subsets of $\mathcal{C}'$ which are defined by the formulas defining D and R. Similarly $D(\mathcal{C})$ and $R(\mathcal{C})$ denote their $\mathcal{C}$ points.

Lemma 2.1.54. *Given a Zariski morphism $f : D \to R$, any specialisation $\pi_D : D(\mathcal{C}') \to D(\mathcal{C})$ with $\mathrm{Dom}(\pi_D) \subseteq D(\mathcal{C}')$, there is a unique extension to $\pi_R : R(\mathcal{C}') \to R(\mathcal{C})$, with $\mathrm{Dom}(\pi_R) = f(\mathrm{Dom}(\pi_D))$, given by $\pi_R(f(x)) := f(\pi_D(x))$.*

Proof. In order to prove that the extension is a specialisation we need to show that for any positive quantifier free formula $\psi(y, z)$, for any $x_0, c \subset \mathrm{Dom}(\pi_D)$, $x_0 \in D$ and $y_0 = f(x_0)$,

$$\models \psi(y_0, c) \Rightarrow \models \psi(y_0^{\pi_R}, c^{\pi_D})$$

where $y_0^{\pi_R} = f(x_0^{\pi_D})$. Equivalently,

$$\models \psi(f(x_0), c) \Rightarrow \models \psi(f(x_0^{\pi_D}), c^{\pi_D})$$

which readily follows from the fact that $\psi(f(x), z)$ is equivalent to a positive quantifier free formula (see Remark 2.1.13). $\square$

Remark 2.1.55. The statement and the proof of Lemma 2.1.54 will be the same when we consider a multi-sorted Zariski structure $\mathcal{C}$. In fact, in Chapter 5, we will apply this lemma in the multi-sorted setting. Actually, we would like to be able consider topological sorts instead of definable sets. This motivates the following mild generalisation.

Lemma 2.1.56. *Let D and R be topological sorts in a Zariski structure $\mathcal{C}$. Let $f : D \to R$ be a morphism, and let $\pi_D : D(\mathcal{C}') \to D(\mathcal{C})$ be a specialisation with $\mathrm{Dom}(\pi_D) \subseteq D(\mathcal{C}')$. Then π_D induces a unique extension to $\pi_R : R(\mathcal{C}') \to R(\mathcal{C})$, with $\mathrm{Dom}(\pi_R) = f(\mathrm{Dom}(\pi_D))$, given by $\pi_R(f(x)) := f(\pi_D(x))$. Moreover, $\pi_R \times \pi_D : R(\mathcal{C}') \times D(\mathcal{C}') \to R(\mathcal{C}) \times D(\mathcal{C})$ is also a specialisation.*

Proof. This is similar to Lemma 2.1.54. Let $\psi(y)$ define a closed subset of $R(\mathcal{C}')$ (over $\emptyset$). We need to show that for any $x_0 \subset \mathrm{Dom}(\pi_D)$, and $y_0 = f(x_0)$,

$$\models \psi(y_0) \Rightarrow \models \psi(y_0^{\pi_R})$$

where $y_0^{\pi_R} = f(x_0^{\pi_D})$. Equivalently,

$$\models \psi(f(x_0)) \Rightarrow \models \psi(f(x_0^{\pi_D}))$$

which follows from the fact that $\psi(f(x)) \equiv \exists y\, f(x) = y \wedge \psi(y)$ is the pre-image of $\psi(y)$ under f, hence closed in $D(\mathcal{C}')$.

For the moreover part observe that $f \times \mathrm{id} : D \times D \to R \times D$ is a morphism. Then one can repeat a similar argument with $f \times \mathrm{id}$ in place of f. $\square$

Remark 2.1.57. Let (M, π) be a κ-universal specialisation, then M is κ-saturated. Indeed, let $p(x) \in S_1(B)$, for some $B \subseteq M$ with $|B| < \kappa$. Suppose a realises $p(x)$ in some elementary extension M' of M, and put $A = B \cup \{a\}$. Hence by κ-universality of π, there is an embedding $\sigma : A \to M$ over B such that $\sigma(a)$ realises $p(x)$ in M.

The following facts are Proposition 2.2.7 and 2.2.15 in [44]. The first one is due to more general results of Weglorz [40].

Fact 2.1.58. *Let M be a quasi-compact, and complete Zariski structure, and $M' \succeq M$. Then there is a total specialisation $\pi : M' \to M$. Moreover, any (partial) specialisation of M can be extended to a total specialisation.*

Fact 2.1.59. *Let M be a Zariski structure, and $M' \succeq M$. Then there is a κ-universal specialisation (M', π). Moreover, if M is quasi-compact, then π is a total specialisation.*

The following is one of the natural examples of a specialisation.

Example 2.1.60. Let K be an algebraically closed valued field, and k the residue field. Suppose $k \succeq K$ as fields. Consider K with the language $\mathcal{L}_{\text{Zar}}(k)$ for algebraic varieties defined over k. Then the residue map $res : K \to k$ where k is a specialisation.

However, note that res is a partial specialisation in this example. One can easily give an example of a total specialisation.

Example 2.1.61 (Example 2.2.4 in [44].)**.** Let ${}^*\mathbb{R}$ be the hyperreal numbers. Then the standard part map from ${}^*\mathbb{R}$ to $\mathbb{R}$ (which is a partial map) induces a total specialisation $\pi : \mathbb{P}^1{}^*\mathbb{R} \to \mathbb{P}^1\mathbb{R}$. Note that ${}^*\mathbb{R}$ interprets an elementary extension ${}^*\mathbb{C}$ of $\mathbb{C}$, therefore one can consider $\pi : \mathbb{P}^1{}^*\mathbb{C} \to \mathbb{P}^1\mathbb{C}$ in the same way, and show that it is a total specialisation.

Specialisations can be thought as replacements to the topology given by the positive quantifier free formulas. This remark can be made precise in the following:

Definition 2.1.62. Let $\pi : M \to M_0$ be a specialisation, a definable relation $S \subseteq M_0^n$ is said to be π-closed whenever $\pi({}^*S) \subseteq S$ where ${}^*S$ is the interpretation of S in M.

Fact 2.1.63 (Exercise 2.2.9 in [44])**.** *The family of π-closed sets satisfies the topological axioms (1)-(8).*

Fact 2.1.64 (Proposition 2.2.24 in [44])**.** *If $\pi : M \to M_0$ is a universal specialisation then any definable relation $S \subset M_0$ is closed if and only if it is π-closed.*

Important tools related to specialisations are the notions of infinitesimal neighbourhood and neighbourhood a point.

Definition 2.1.65. An *infinitesimal neighbourhood with respect to* π *of* $a \in M^n$ is the set $\mathcal{V}_a^\pi = \pi^{-1}(a)$. When π is clear from the context, I will drop it and write $\mathcal{V}_a$, and say *infinitesimal neighbourhood of* a.

Definition 2.1.66. A *neighbourhood with respect to* π *of* $a \in M^n$ is the type $Nbd_a^\pi(y) = \{\neg Q(C, y) : Q \text{ is a closed set}, M_0 \models \neg Q(c_0, a) \text{ where } c \in \mathcal{V}(c_0)\}$. When there is no ambiguity about π, I will write $Nbd_a(y)$.

The following fact from [44] explains the relation between these two notions:

Fact 2.1.67 (Lemma 2.2.21 in [44])**.** *Let* $\pi : M \to M_0$ *be a specialisation,*

1. $\mathcal{V}_a = Nbd_a(y) \cap \mathrm{Dom}(\pi)$.
2. *Assume* π *is* κ*-universal. If* $b \in M$ *and* $p(y)$ *is a type over* b *then there is an* $a \in \mathcal{V}_{a_0}$ *satisfying* $p(y)$ *if* $Nbd_{a_0}(y) \cup p(y)$ *is consistent.*

As specialisations are partial maps, an important notion is "maximality".

Definition 2.1.68. Let M_0 be a Zariski structure, $M \succeq M_0$ and $\pi : M \to M_0$ a specialisation. We say that $\mathrm{Dom}(\pi)$ is *maximal* or π *is maximally defined* (in short π is *maximal*) if there are no specialisations $\pi' : M \to M_0$ extending π non-trivially.

Proposition 2.1.69. *An* $\aleph_0$*-universal specialisation is maximally defined.*

Proof. Let M_0 be a Zariski structure and $M \succeq M_0$ and $\pi : M \to M_0$ be an $\aleph_0$-universal specialisation. Assume π is not maximally defined. Then there is an element $m \in M \setminus \mathrm{Dom}(\pi)$ such that there is a specialisation $\pi_{\{m\}} : \{m\} \cup \mathrm{Dom}(\pi) \to M_0$ extending π. Then, since π is universal there is an embedding $\sigma : \{m\} \to M$ over $\{m\} \cap M$ such that $\pi_{\{m\}_{|\{m\}}} = \pi \circ \sigma$. Since $m \in M$, we have $\sigma(m) = m$. But then this implies $\pi(m) = \pi_{\{m\}}(m)$. In particular it means π is defined on m. A contradiction to the choice of m. □

The converse direction is not true! Examples of maximal but not κ-universal specialisations (even if one assumes saturation) are constructed later in section 4.2.

One of the aims of this thesis is to find conditions under which the notions of maximality and universality coincide. One can see in Section 4.2 that this is not always the case.

However, the main aim as it is already pointed in the Introduction, is to answer *when is a specialisation is* κ*-universal?*

2.2 Ladder Theorem

In this section I will give a brief introduction of Zilber's Ladder Theorem. Ladder Theorem states that an uncountably categorical structure can be built up starting with a strongly minimal structure and taking covers. In its full generality the Ladder Theorem is not about Zariski structures. Therefore in this section I will work with an uncountably categorical structure which is not necessarily a Zariski structure. Towards the end of this section I will consider Ladder Theorem in the setting of Zariski structures.

At this point this section may seem rather irrelevant to specialisations. However at the end of Chapter 5, it will enable us to generalize the results from the previous parts to some uncountably categorical Zariski structures in the non-locally modular case. It will also point towards a Ladder Theorem for specialisations.

2.2.1 Definitions and the Main Statement

Let M be an uncountably categorical structure throughout this section unless stated otherwise.

Definition 2.2.1. Let $X \subseteq M$. An X-atom of M is an X-definable subset A of M^n for some n, such that A does not contain any proper X-definable sets.[2]

Definition 2.2.2. Let A and N be $\emptyset$-definable sets in M^{eq}, and A be an N-atom. The *Galois group of A over N*, denoted $\mathrm{Gal}(A/N)$ is the group of all N-elementary automorphisms of A. We also call $\mathrm{Gal}(A/N)$ the *binding group (of A to N)*.

In its full generality the Ladder Theorem is as follows:

Theorem 2.2.3 (Ladder Theorem, Theorem 5.0.1 in [47]). *Let M be an uncountably categorical structure and M_0 a strongly minimal set definable in M^{eq}. Then there are $\emptyset$-definable (in M^{eq}) sets $M_1, \dots, M_k$ such that $M_k = M$ and for each M_i-atom $A \subseteq M_{i+1}$ for any $i = 0, \dots, k$, the binding group $\mathrm{Gal}(A/M_i)$ is definable together with its action on A; and there exists an M_{i+1}-definable group G with $G \subseteq \mathrm{dcl}(M_i)$ and an M_{i+1}-definable isomorphism between G and $\mathrm{Gal}(A/M_i)$.*

Remark 2.2.4. In fact one can replace M with a set definable in M^{eq} in the Ladder Theorem, and the statement would still be true. Actually, in the original statement of the Ladder Theorem in [47], M and M_0 are taken to be interpretable sets in an uncountably categorical structure.

[2] in other words, if $F_n(X)$ is the boolean algebra of X-definable subsets of M^n, then atoms of $F_n(X)$ are called X-atoms.

2.2.2 Key Steps of the Proof

The proof of the Ladder Theorem is based on a detailed study of sets and groups of automorphisms interpretable in M. The proof constitutes most of the Chapter 5 in [47]. I will not repeat the full proof here. I will only repeat the key steps in the proof, which will also be relevant in this thesis.

2.2.2.1 Rank-Degree

Definition 2.2.5. Let A be a definable (in M^{eq}) set. By $\mathrm{RM}(A)$ I will denote the *Morley Rank* and by $\mathrm{dM}(A)$ the *Morley Degree* of A as usual. Let $X \subset M^{eq}$ and $\overline{a}$ a tuple in M^{eq}, then we define as usual $\mathrm{RM}(\overline{a}/X) := \mathrm{RM}(tp(\overline{a}/X))$ and $\mathrm{dM}(\overline{a}/X) := \mathrm{dM}(tp(\overline{a}/X))$. If $X = \emptyset$, I write $\mathrm{RM}(\overline{a})$ and $\mathrm{dM}(\overline{a})$ respectively.

The pair $(\mathrm{RM}(A), \mathrm{dM}(A))$ is called the *Rank-Degree* of A, and will be denoted by $\mathrm{RD}(A)$. Similarly, $\mathrm{RD}(\overline{a}/X)$ is defined to be $\mathrm{RD}(tp(\overline{a}/X))$.

Definition 2.2.6. Let A and N be $\emptyset$-definable sets and Z be a finite subset define

$$\mathrm{rd}_Z(A/N) := \{\min\{\mathrm{RD}(x/YNZ)\} : x \in A, Y \subseteq A, x \text{ is independent from } Y \text{ over } Z\}$$

If $Z = \emptyset$ write $\mathrm{rd}(A/N)$ instead of $\mathrm{rd}_\emptyset(A/N)$.

Fact 2.2.7 (Lemma 5.1.2 in [47])**.** *If $Z \subseteq N$, then* $\mathrm{rd}_Z(A/N) = \mathrm{rd}(A/N)$.

Fact 2.2.8 (Lemma 5.1.3 in [47])**.** *There is a tuple $\overline{a} \subseteq A$ such that for all $x \in A$ independent from $\overline{a}$, we have* $\mathrm{RD}(x/\overline{a}N) = \mathrm{rd}(A/N)$.

Definition 2.2.9. Let $\overline{a} \subset A$ be tuple satisfying Fact 2.2.8, Y a finite set and $x \in A$. The binary relation $E_{A,N}$ is defined by: $xE_{A,N}z$ if and only if there is a is finite tuple $\overline{y} \models tp(\overline{a})$ such that x and z are independent from y and

$$\{t : tp(t/\overline{y}N) = tp(x/\overline{y}N)\} = \{t : tp(t/\overline{y}N) = tp(z/\overline{y}N)\}$$

Following the notation in [47], I will denote $\{t : tp(t/\overline{y}N) = tp(x/\overline{y}N)\}$ by $x^{\overline{y}}$.

Fact 2.2.10 (Lemma 5.1.6 in [47])**.** *$E_{A,N}$ is a an equivalence relation on A, and it is $\emptyset$-definable.*

Fact 2.2.11 (Lemma 5.1.7 in [47])**.** *Let $x \in A$ be independent from $\overline{y}$ and $tp(\overline{y}) = tp(\overline{a})$. Then* $\mathrm{RM}(xE_{A,N} \setminus x^{\overline{y}}) < \mathrm{RM}(xE_{A,N})$ *and* $\mathrm{RM}(x^{\overline{y}} \setminus xE_{A,N}) < \mathrm{RM}(x^{\overline{y}})$. *Moreover, for all $x \in A$,* $\mathrm{RD}(xE_{A,N}) = \mathrm{rd}(A/N)$.

2.2.2.2 Binding Groups

For an N-atom A the binding group $\mathrm{Gal}(A/N)$ acts transitively on A. Zilber showed that if $\mathrm{rd}(A/N) = (0, 1)$, the binding group $\mathrm{Gal}(A/N)$ is definable in M^{eq}. Here I will recall this proof and some more properties about definable groups of automorphisms.

Let A^* be an N-atom defining the type $tp(\overline{a}/N)$. A^* is definable over some finite subset. However, since by Fact 2.2.7 it will not change the $\mathrm{rd}(A/N)$, we will add this finite set as a set of constants to the language and consider A^* to be $\emptyset$-definable. Moreover, clearly $\mathrm{Gal}(A/N)$ acts transitively on A^*.

Fact 2.2.12 (Lemma 5.2.3 in [47])**.** *There is a $\emptyset$-definable group G and a $\emptyset$-definable action $\diamondsuit : A^* \times G \to A^*$ of G on A^* such that $G \subseteq \mathrm{dcl}(N)$ and the action of G on A^* is sharply transitive.*

The following fact explains the relation between the groups $\mathrm{Gal}(A/N)$ and G. In particular it shows that $\mathrm{Gal}(A/N)$ is $\emptyset$-definable.

Fact 2.2.13 (Lemma 5.2.4 in [47])**.** *Let $\overline{x} \in A^*$. Then there is a unique group anti-isomorphism $p_{\overline{x}} : G \to \mathrm{Gal}(A/N)$ such that for all $g \in G$, $p_{\overline{x}}(g) \cdot \overline{x} = \overline{x} \diamondsuit g$*

Moreover, if $\overline{x}' = \gamma \cdot \overline{x}$ for some $\gamma \in \mathrm{Gal}(A/N)$, then $p_{\overline{x}'}(g) = \gamma p_{\overline{x}}(g)\gamma^{-1}$.

However, note that the definability of the action of $\mathrm{Gal}(A/N)$ does not follow from the above fact. For the definability of the action, one needs the following facts.

Fact 2.2.14 (Proposition 5.2.5 in [47])**.** *Let $\overline{x} \in A^*$. Then the group* $\mathrm{Gal}(A/N)$ *and its action on A is $\overline{x}$-definable in the following way:* $\mathrm{Gal}(A/N)$ *is interpreted as $(G, \times)$ where $\times$ is the anti-multiplication given in the Fact 2.2.13 (i.e. for any $g, h \in G$, $g \times h := hg$), and the action of* $\mathrm{Gal}(A/N)$ *on A is interpreted as the $\overline{x}$-definable map $(g, y) \mapsto p_{\overline{x}}(g) \cdot y$.*

Moreover, if $\mathrm{Gal}(A/N)$ *is abelian it is isomorphic to G, and it is $\emptyset$-definable with its action.*

Fact 2.2.15 (Proposition 5.2.6 in [47])**.** *Let B be a -definable N-atom. Suppose that* $\mathrm{Gal}(B/N)$ *is definable with its action and* $\mathrm{Gal}(B/N) \subseteq \mathrm{dcl}(N)$*. Then there is a tuple $\overline{b} \subseteq b$ such that $B \subseteq \mathrm{dcl}(\overline{b}N)$ and* $\mathrm{Gal}(B/N)$ *and its action are $\overline{b}$-definable.*

2.2.2.3 Proof of the Ladder Theorem

Fact 2.2.16 (Lemmas 5.3.1 and 5.3.2 in [47])**.** *Let* $\mathrm{RM}(a) > 1$*. Let N be a $\emptyset$-definable set, and assume that if* $\mathrm{RM}(N) < \mathrm{RM}(A/\emptyset)$*, then $a \notin \mathrm{acl}(N)$. Then there is a $\emptyset$-definable set N such that:*

1. $0 < \mathrm{RM}(N) < \mathrm{RM}(a)$

2. *There is an element* $b \in \mathrm{dcl}(a) \cap N$ *and a* b*-definable set* A *which is an* N*-atom such that* $a \in A$.

3. *For any* b*-definable equivalence relation* E_b *on* A*, either* aE_b *is finite or* $\mathrm{RM}(aE_b) = \mathrm{RM}(A)$.

Moreover, assuming conditions 2 and 3 above, one can find $\hat{a}$ *and an* N*-atom* $\hat{A}$ *such that* $\hat{a} \in \hat{A}$ *with* $\mathrm{acl}(\hat{a}) = \mathrm{acl}(a)$ *and*

4. $\mathrm{rd}(\hat{A}/N) = (0, 1)$.

Fact 2.2.17 (Lemma 5.3.3 in [47])**.** *Let* M *be a* $\emptyset$*-set with* $\mathrm{RM}(M) > 1$*. Then there are* $\emptyset$*-definable sets* $\hat{M}$ *and* N *such that*

1. $M \subseteq \mathrm{acl}(\hat{M})$.

2. $\mathrm{RM}(M) > \mathrm{RM}(N)$.

3. *If* $\hat{A} \subseteq \hat{M}$ *is an* N*-atom, then* $\mathrm{Gal}(\hat{A}/N)$ *is definable, its action is definable, and* $\mathrm{Gal}(\hat{A}/N) \subseteq \mathrm{dcl}(N)$.

Proof of the Ladder Theorem. First of all assume that $M \subseteq \mathrm{acl}(M')$ for some definable M'. In this case any M'-atom is finite. Then $\mathrm{Gal}(A/M')$ is definable with its action and $\mathrm{Gal}(A/M') \subseteq \mathrm{dcl}(M')$. Hence, for M almost strongly minimal the theorem is proved.

We proceed by induction on $\mathrm{RM}(M)$. Suppose $\mathrm{RM}(M) \leq 1$ and $M \not\subseteq \mathrm{acl}(M_0)$. Then there is an M_0-atom A with $\mathrm{RM}(A) = 1$ (otherwise, $M \subseteq \mathrm{acl}(M_0)$). Since $\mathrm{RM}(M) \leq 1$ there are only finitely many M_0-atoms of Morley rank 1. Therefore their union is $\emptyset$-definable. We may also assume their union is equal to M. This would imply there is a $\emptyset$-definable equivalence relation E on M with finitely many equivalence classes. Hence M/E is a finite $\emptyset$-definable set. Put $M_1 := M_0 \cup M/E$. Then, observe that if A is an M_1-atom it is $\bar{b}$-definable for some $\bar{b} \subset M_1$.

Use the last part of Fact 2.2.16 to construct

$$\widehat{M} = \bigcup \{\widehat{A} : A \subseteq M \text{ is an } M\text{-atom}\}$$

Clearly $\widehat{M}$ is $\emptyset$-definable, $M \subseteq \mathrm{acl}(\widehat{M})$ and $\mathrm{RM}(M) = \mathrm{RM}(\widehat{M}) = 1$. Moreover, all M_1 atoms in $\widehat{M}$ are of the for $\widehat{A}$ as described above. The group $\mathrm{Gal}(\widehat{A}/M_1)$ and its action are definable and $\mathrm{Gal}(\widehat{A}/M_1) \subseteq \mathrm{dcl}(M_1)$.

Therefore $M_0, M_1, M_2 := \widehat{M}, M_3 := M$ gives us the desired ladder.

Next, assume $\mathrm{RM}(M) > 1$ and the Ladder Theorem is valid for $\emptyset$-definable sets of lesser rank. Use Fact 2.2.17 to construct

$$M_n = N, \quad M_{n+1} := \widehat{M}, \quad M_{n+2} = M$$

By inductive hypothesis for N where N is as in Fact 2.2.17, there is a ladder $M_0, M_1, \ldots, M_n$. Therefore $M_0, M_1, \ldots, M_n, M_{n+1}, M_{n+2}$ is the required ladder. □

2.3 Ladder Theorem in the Setting of Zariski Structures

The Ladder Theorem, as recalled above is given in the general context of uncountably categorical structures. In this general setting, there is no need for the additional topological structure which is provided by Zariski structures.

However, the Zariski structures related to Quantum Harmonic Oscillator [35], and Quantum Zariski geometries [46] are the main examples of the Ladder construction. From the explicit description of these structures one can see they satisfy certain geometric assumptions.

In addition, an attempt to study the specialisations of a general two sorted Zariski structure (D, N, pr) where pr is a surjective morphism has shown us without such assumptions, behaviour of the specialisations can be wild.

When considering the Ladder Theorem for Zariski structures we need to restrict to the case where the rungs $M_0, \ldots, M_k = M$ satisfy certain geometric assumptions. These assumptions are of the kind structures in [35, 16, 46] satisfy. In addition, a further restriction is necessary for our considerations in Chapter 5. Namely, we restrict our consideration to the case where there is only one binding group acting on the fibres, instead of a family of groups as it were the case in the general setting of the Ladder Theorem.

The assumptions we will make in this thesis are formulated as Zariski cover structures, and defined in Chapter 5.

Definition 5.1.1. Let $\mathcal{C} := (C, M, \mathrm{pr})$ be a two-sorted Zariski structure such that

(i) There is a Zariski topological group G in $\mathcal{C}$ acting morphically and freely on C with Zariski continuous bijections.

(ii) M is interpretable in C as a topological sort and $\mathrm{pr} : C \to M$ denotes the canonical quotient-map. It is a $\emptyset$-definable surjection.

(iii) For each $m \in M$, the fibre $\mathrm{pr}^{-1}(m)$ is an orbit of an element in C, i.e. $\mathrm{pr}^{-1}(m) = G \cdot x$ for some $x \in C$.

(iv) The group G is a Zariski topological group in M (in particular G is interpretable as a topological sort in M).

Then $\mathcal{C} := (C, M, \mathrm{pr})$ is said to be a *(two sorted) Zariski cover structure*.

From now on, in the remaining of the thesis, by cover we will mean Zariski cover structure unless otherwise is stated.

In the considerations of Chapter 5 we will work under an additional assumption. Which is named Continuous Connections (see paragraph 5.1.18). As already noted in the introduction, unfortunately at this point we are unable to prove whether Quantum Harmonic Oscillator and Quantum Zariski Structures satisfy Continuous Connections assumption or not.

Chapter 3

Specialisations and Algebraically Closed Fields

3.1 Specializations and Algebraically Closed Valued Fields

In this chapter we will consider the structure (K, k, res) where K is an algebraically closed valued field, k its residue field and $res : K \to k$ is the residue map as a specialisation. For this consideration we are forced to make some restrictions. As we need to have $k \prec K$, we restrict ourselves to the case $\mathrm{Char}(K) = \mathrm{Char}(k)$. Given that we are in the equi-characteristic, both K and k are algebraically closed fields over the same prime subfield. Then one can (uncanonically) embed k into K, also one can choose this embedding in such a way that res is identity on k (by replacing the isomorphic copy of k in K with k if necessary). But then by quantifier elimination (Fact 2.1.3) this embedding is elementary, and we have $k \prec K$.

We will work with the relational language $\mathcal{L}_{\text{Zar}}$, which only has predicates for varieties in k^n (all $n \in \mathbb{N}$), whose relations are Zariski closed subsets of k^n (all $n \in \mathbb{N}$), and with the expanded language $\mathcal{L}^{res} := \mathcal{L}_{\text{Zar}} \cup \{res\}$ where res will be interpreted as the residue map. Note that k is $\mathcal{L}^{res}$ definable in K by the formula $\exists x(res(x) = y)$. From now on K will always be an algebraically closed valued field, k the (definable) residue field, $res : K \to k$ the residue map, $\mathcal{O}$ the corresponding valuation ring, and $\mathcal{M}$ the valuation ideal.

Let $\mathcal{L}_{\text{div}} = \{+, -, \cdot, 0, 1, |\}$ be the ring language augmented with a binary relation $|$ which is interpreted as $x|y$ if and only if $v(x) \leq v(y)$. Recall that ACVF is the complete theory given by the following sentences:

1. K is algebraically closed.

2. Valuation axioms.

3. There are $x, y \in K^\times$ such that $v(x) < v(y)$, where v is the valuation on K.

4. Characteristic of k and K.

in the language $\mathcal{L}_{\text{div}}$ and also in various other languages due to Robinson's analysis of algebraically closed valued fields [22, 31].

It is easy to see that one can write sentences 1, 2 and 4 in the language $\mathcal{L}^{res}$, one can also write 3 but it takes a bit more effort. First observe that $\mathcal{O}$ is $\mathcal{L}^{res}$ definable, by the formula $\exists y(res(x) = y)$. Then define $Res : K \times K \to k$ as follows:

$$Res(x, y) = \begin{cases} res(xy^{-1}) & \text{if } v(x) \geq v(y) \\ 0 & \text{otherwise} \end{cases}$$

So Res is $\mathcal{L}^{res}$ definable by the formula

$$\psi(x, y, z) : (xy^{-1} \in \mathcal{O} \wedge z = res(xy^{-1})) \vee (xy^{-1} \notin \mathcal{O} \wedge z = 0)$$

Now sentence 3 can can be written as $\exists x \exists y(Res(x, y) = 0)$. Therefore, one can indeed consider the theory ACVF to be living inside the theory specialisations (with some restrictions).

We start with a standard fact.

Lemma 3.1.1. *Let K be an algebraically closed field in the language $\mathcal{L}_{Zar}$. If $tr.d(K) = \kappa$ then K is κ-saturated. Conversely, if K is κ-saturated, then K has infinite transcendence degree.*

In the next theorem I will show any specialisation can be extended in such a way that its domain is a valuation ring. This is basically due to Chevalley's Place Extension Theorem. However the essential arguments are already present in Weil's Foundations of Algebraic Geometry [41].

Theorem 3.1.2. *Let k be an algebraically closed field and $K \succeq k$. Given a specialisation $\pi : K \to k$, it defines a residue map, $res : K \to k$ of a valuation ring $\mathcal{O}$.*

Proof. Let $R = \text{Dom}(\pi)$. Clearly we can assume R is a ring containing k (if not, take R to be the ring generated by $\text{Dom}(\pi)$. Then π naturally extends to a ring homomorphism from R to k which is a specialisation extending π to R). Then π is a surjective ring homomorphism with $\text{Ker}(\pi) = \mathcal{V}_0$. Hence $\mathcal{V}_0 \triangleleft R$ is maximal, and in particular prime. Then by Chevalley's Place Extension Theorem, there is a valuation ring $\mathcal{O}$ to which π extends.

Let us denote the extension of π to $\mathcal{O}$ by res and write $res : \mathcal{O} \to k$. Then $\mathrm{Ker}(res)$ is the unique maximal ideal (i.e. the valuation ideal) of $\mathcal{O}$ and $\mathcal{O}/\mathrm{Ker}(res) \simeq k$. □

Hence the specialisation denoted by res with the domain $\mathcal{O}$ above is a residue map coming from a valuation of the field K with residue field k.

Remark 3.1.3. If $\pi : K \to k$ is a specialisation of algebraically closed fields, by the above theorem we can consider its extension to a residue map $res : K \to k$. Therefore whenever we have a specialisation π we will consider the residue map res it defines instead of it. In other words one can always consider the structure (K, k, res) where res is a residue map as a specialisation on an algebraically closed field.

Remark 3.1.4. Recall that the infinitesimal neighbourhood of any element b in K is $\mathcal{V}_b = res^{-1}(b)$. Note that in this setting $\mathcal{V}_0$ is just the valuation ideal $\mathcal{M}$. Hence for any $b \in k$, the infinitesimal neighbourhood $\mathcal{V}_b$ is just the coset $b + \mathcal{M}$. Moreover, since ∞ is in k in the usual way, we will also consider the infinitesimal neighbourhood of ∞ and write $\mathcal{V}_\infty$.

3.1.1 Some Facts on Extensions of Valued Fields

I will recall some well known facts on extensions of valued fields. They are widely available in the literature on valued fields, e.g. in [38]. In this subsection, and also in the next one (subsection 3.1.2) we will consider a valued field as a tuple $(F, \mathcal{O}_F)$ where F is a field and $\mathcal{O}_F$ is a valuation subring of F. If F is algebraically closed we call $(F, \mathcal{O}_F)$ an *algebraically closed valued field*.

Definition 3.1.5. Let $(F, \mathcal{O}_F)$ and $(L, \mathcal{O}_L)$ be two valued fields. A valued field embedding $i : (F, \mathcal{O}_F) \to (L, \mathcal{O}_L)$ is a field embedding $i : F \to L$ such that $i(\mathcal{O}_F) = \mathcal{O}_L$.

Fact 3.1.6 (Corollary 3.16 in [38])**.** *Let $(F, \mathcal{O}_F)$ be a valued field. Let F^{alg} be its algebraic closure and $\mathcal{O}_{F^{alg}} \subseteq F^{alg}$ be a valuation ring which dominates $\mathcal{O}_F$. Then any valued field embedding $i : (F, \mathcal{O}_F) \to (L, \mathcal{O}_L)$ to an algebraically closed L can be extended to a valued field embedding $(F^{alg}, \mathcal{O}_{F^{alg}}) \to (L, \mathcal{O}_L)$.*

Fact 3.1.7 (Lemma 3.22 in [38])**.** *Let $(F, \mathcal{O}_F)$ be a valued field, let x be transcendental over F and let $L = F(x)$. Then there is a unique valuation ring $\mathcal{O}_L \subseteq L$ with $\mathcal{O}_F \subseteq \mathcal{O}_L$ such that $x \in \mathcal{O}_L$ and $res(x)$ is transcendental over the residue field L_0.*

Furthermore, $L_0 = M_0(res(x))$ and $\Gamma_F = \Gamma_L$ where Γ_F and Γ_L are the corresponding value groups.

Fact 3.1.8 (Lemma 2.23 in [38]). *Let $(F, \mathcal{O}_F)$ be a valued field with valuation $v : F^\times \to \Gamma_F$, let x be transcendental over F and let $L = F(x)$. If h is an element of an ordered abelian group extending Γ_F such that $nh \notin \Gamma_F$ for all $n \geq 1$, then v extends uniquely to a valuation $w : L^\times \to \Gamma_F + h\mathbb{Z}$. Also $F_0 = L_0$.*

Fact 3.1.9 (Lemma 3.30 in [38]). *Let $(F, \mathcal{O}_F) \subseteq (L, \mathcal{O}_L)$ be a valued field extension with $M_0 \subseteq L_0$, also denote the valuation of L by v. Let $m_1, \ldots, m_n \in M$ and $x \in L \setminus M$. Suppose $v(x - m_i) \in vM^\times$ for all $i = 1, \ldots, n$. There there is an element $m \in M$ such that for all $i = 1, \ldots, n$, $v(m - m_i) = v(x - m_i)$.*

3.1.2 Quantifier Elimination for $Th(K)^{res}$

Let k be an algebraically closed field considered as a Zariski structure in its natural Zariski language $\mathcal{L}_{\text{Zar}}$. Let $K \succeq k$, we will consider the structure (K, k, π) in the language $\mathcal{L}_{\text{Zar}}^{\pi,\infty}(P) := \mathcal{L}_{\text{Zar}} \cup \{P, \pi, \infty\}$ where P is a unary predicate interpreted as k, π is a function symbol which will be interpreted as a specialisation $\pi : K \to k$, and ∞ is a new constant to be interpreted as a new element added to k. For dealing with substructures in a natural way we will also assume that the language $\mathcal{L}_{\text{Zar}}^{\pi,\infty}(P)$ contains function symbols $+$ and $\cdot$ to be interpreted as addition and multiplication.

Let $Th(K)^{res}$ be the theory given by the following:

1. $Th(K)$, the $\mathcal{L}_{\text{Zar}}$ theory of K.
2. ∞ will be a new element added to the residue field:
   $P(\infty) \wedge \forall a(P(a) \to (\infty + a = a + \infty = \infty \wedge a \cdot \infty = \infty \cdot a = \infty))$.
3. res is a non-trivial residue map:
   $\forall x(res(x) \neq \infty \vee res(x^{-1}) \neq \infty) \wedge \exists z(\neg P(z))$
4. The axiom scheme saying res is a specialization:
   $\{\forall x_1, \ldots, x_n(S(x_1, \ldots, x_n) \to S(res(x_1, \ldots, x_n))) : n \in \mathbb{N}, S \in \mathcal{L}_{\text{Zar}}\}$
5. res is identity on the residue field:
   $\forall x(P(x) \to res(x) = x)$

Let (M, M_0, res) be a model of $Th(K)^{res}$. Then $res : M \to M_0 \cup \{\infty\}$ is a place. So res corresponds to a unique valuation ring, denote it by $\mathcal{O}_M$. Therefore $(M, \mathcal{O}_M)$ is a non-trivial valued field. For the sake of notation, when there is no risk of ambiguity, instead of $res(m)$ (given that res is defined on m), I will write $\overline{m}$. I.e. whenever res is defined on $m \in M$ we define $\overline{m} := res(m)$. This convention will only be valid in this chapter.

We will use the following fact from [11] for proving $Th(K)^{res}$ admits quantifier elimination, in the language $\mathcal{L}_{\text{Zar}}^{\pi,\infty}(P)$.

Fact 3.1.10 (Theorem 5.7 in [11])**.** *For a theory T in a language $\mathcal{L}$ with $|\mathcal{L}| = \kappa$, the following are equivalent:*

1. *T admits quantifier elimination.*
2. *For any two models $\mathcal{M}$ and $\mathcal{N}$ of T where $|M| \leq \kappa$ and $\mathcal{N}$ is κ^+-saturated, and for any proper substructure S of $\mathcal{M}$ whit an embedding $i : S \to \mathcal{N}$, one can extend i to an embedding $j : S' \to \mathcal{N}$ for some proper substructure S' of $\mathcal{M}$ strictly containing S.*

Remark 3.1.11. Quantifier elimination for various theories of pairs of algebraically closed valued fields are already well known in model theoretic literature (see [2, 9, 22, 39] for example).

The main arguments we use in our proof of quantifier elimination (Lemmas 3.1.12, 3.1.13 and 3.1.14) are very close to Hrushovksi and Kazhdan's proof of quantifier elimination for a pair of algebraically closed valued fields in [15, Lemma 6.3]. And also close to Leloup [21].

Lemma 3.1.12. *Let (S, S_0, res) and (L, L_0, res) be two valued fields where L is an algebraically closed field. Consider the extension $(S^{alg}, S_0^{alg}, res)$ of (S, S_0, res) where S^{alg} is the algebraic closure of S. Then any $\mathcal{L}_{Zar}^{\pi,\infty}(P)$-embedding $i : (S, S_0, res) \to (L, L_0, res)$ extends to an $\mathcal{L}_{Zar}^{\pi,\infty}(P)$-embedding $(S^{alg}, S_0^{alg}, res) \to (L, L_0, res)$.*

Proof. First we will prove that there is an $\mathcal{L}_{\text{Zar}}^{\pi,\infty}(P)$-embedding

$$j_1 : (SS_0^{\text{alg}}, S_0^{\text{alg}}, res) \to (L, L_0, res)$$

extending i.

Observe that S_0 is relatively algebraically closed in S. So S is a purely transcendental extension of S_0. Therefore S and S_0^{alg} are linearly disjoint over S_0. Moreover, $S \otimes_{S_0} S_0^{\text{alg}}$ is a field since S_0^{alg} is an algebraic extension of S_0. Hence the compositum SS_0^{alg} is isomorphic (as fields) to the tensor product of fields $S \otimes_{S_0} S_0^{\text{alg}}$.

We know that i embeds S into L and S_0 into the algebraically closed field L_0. Extend i to a field embedding $j_0 : S_0^{\mathrm{alg}} \to L_0$. Then, $j_0(S_0^{\mathrm{alg}}) = i(S_0)^{\mathrm{alg}}$. As above, $i(S_0)$ is relatively algebraically closed in $i(S)$. Therefore $i(S)$ and $j_0(S_0^{\mathrm{alg}})$ are linearly disjoint over $i(S_0)$.

Then there is an isomorphism $j_1 : SS_0^{\mathrm{alg}} \to i(S)j_0(S_0^{\mathrm{alg}})$ such that $j_1(x) = i(x)$ for all $x \in S$ and $j_1(y) = j_0(y)$ for all $y \in S_0^{\mathrm{alg}}$. It is clear that j_1 preserves P as $j_1(S_0^{\mathrm{alg}}) = i(S_0)^{\mathrm{alg}} \subseteq L_0^{\mathrm{alg}} = P(L)$.

Clearly S^{alg} is an algebraic closure of SS_0^{alg}. Extend j_1 to a field embedding $j : S^{\mathrm{alg}} \to L$. Since j extends j_1, it maps S_0^{alg} into $L_0 = P(L)$. Hence j preserves P. Moreover, $j^{-1}(\mathcal{O}_L)$ is a valuation ring of S^{alg} lying above $\mathcal{O}_S$. Since S^{alg} is normal over S, there is a $\sigma \in \mathrm{Aut}(S^{\mathrm{alg}}|S)$ such that $j^{-1}(\mathcal{O}_L) = \sigma(\mathcal{O}_{S^{\mathrm{alg}}})$.

Since SS_0^{alg} is a normal extension of S, σ fixes SS_0^{alg} setwise. Moreover σ fixes S_0^{alg} setwise. Indeed, since S_0^{alg} is algebraically closed in SS_0^{alg}, any automorphism of SS_0^{alg} over S fixes S_0^{alg} setwise. Hence σ fixes S_0^{alg} setwise.

Then $j\sigma : (S^{\mathrm{alg}}, \mathcal{O}_{S^{\mathrm{alg}}}) \to (L, \mathcal{O}_L)$ is a valued field embedding which maps S_0^{alg} into L_0, and hence preserves P.

Next we will show that $j\sigma$ preserves the map res. Let $b \in \mathcal{O}_{S^{\mathrm{alg}}}$. If $b \in \mathcal{M}_{S^{\mathrm{alg}}}$, then $res(b) = 0$, and also $j\sigma(b) \in \mathcal{M}_L$. Therefore $j\sigma(res(b)) = j\sigma(0) = 0 = res(j\sigma(b))$. Now assume that $b \in \mathcal{O}_{S^{\mathrm{alg}}} \setminus \mathcal{M}_{S^{\mathrm{alg}}}$, i.e. $v(b) = 0$. Then $b = \frac{f(x)}{g(x)} = \frac{f_0+f_1x+\ldots+f_nx^n}{g_0+g_1x+\ldots+g_mx^m}$ for some $f(x), g(x) \in S^{\mathrm{alg}}[x] \setminus \{0\}$ with $v(f) = v(g)$. Furthermore we may assume that $v(f) = v(g) = 0$. Indeed, if this is not the case, divide all coefficients of f and g by a coefficient of g which has minimal valuation. Then, in particular, all coefficients of $f(x)$ and $g(x)$ are in $\mathcal{O}_{S^{\mathrm{alg}}}$.

Then

$$\begin{aligned}
j\left(res\left(\tfrac{f}{g}\right)\right) & = j(res f(x)(g(x))^{-1})) = \\
= \quad & j(res((f_0 + f_1x + \ldots + f_nx^n)(g_0 + g_1x + \ldots + g_mx^m)^{-1})) \\
= \quad & j(res(f_0 + f_1x + \ldots + f_nx^n)(res(g_0 + g_1x + \ldots + g_mx^m))^{-1}) \\
= \quad & j((\overline{f_0} + \overline{f_1}x + \ldots + \overline{f_n}x^n)(\overline{g_0} + \overline{g_1}x + \ldots + \overline{g_m}x^m)^{-1})) \\
= \quad & j((\overline{f_0} + \overline{f_1}x + \ldots + \overline{f_n}x^n))j(\overline{g_0} + \overline{g_1}x + \ldots + \overline{g_m}x^m)^{-1})) \\
= \quad & (i(\overline{f_0}) + i(\overline{f_1})y + \ldots + i(\overline{f_n})y^n)(i(\overline{g_0}) + i(\overline{g_1})y + \ldots + i(\overline{g_m})y^m))^{-1} \\
= \quad & \overline{(i(f_0) + i(f_1)y + \ldots + i(f_n))y^n)(i(g_0) + i(g_1)y + \ldots + i(g_m)y^m)^{-1})} \\
= \quad & res\left(j\left(\frac{f_0 + f_1x + \ldots + f_nx^n}{g_0 + g_1x + \ldots + g_mx^m}\right)\right)
\end{aligned}$$

Now assume $b \in S^{\text{alg}} \setminus \mathcal{O}_{S^{\text{alg}}}$. Then $res(b) = \infty$ and $j\sigma(res(b)) = j\sigma(\infty) = \infty$. On the other hand $j(b) \in i(S)(y) \setminus \mathcal{O}_{i(S)(y)}$. Therefore $res(j(b)) = \infty$. Hence $j(res(b)) = j(\infty) = \infty = res(j(b))$. $\square$

Lemma 3.1.13. *Let (M, M_0, res) and (L, L_0, res) be two models of $Th(K)^{res}$ with (L, L_0, res) is $|M|^+$-saturated. Let (S, S_0, res) be a substructure of (M, M_0, res) and $i : (S, S_0, res) \to (L, L_0, res)$ be an $\mathcal{L}_{Zar}^{\pi,\infty}(P)$-embedding. Then there is an extension $j : (F^{alg}, F_0^{alg}, res) \to (L, L_0, res)$ of i where F is the field generated by S.*

Lemma 3.1.14. *The theory $Th(K)^{res}$ admits quantifier elimination in the language $\mathcal{L}_{Zar}^{\pi}(P)$. Moreover the theory $Th(K)^{res}$ is complete.*

Proof. Let (M, M_0, res) and (N, N_0, res) be two models of $Th(K)^{res}$ such that (N, N_0, res) is $|M|^+$-saturated. Let $(S, S_0, res) \subseteq (M, M_0, res)$ be a proper substructure and suppose $i : (S, S_0, res) \to \mathcal{N}$ is an embedding, we will show that we can extend i to an embedding $j : (S', S_0', res) \to \mathcal{N}$ of some substructure $S' \subseteq M$ properly containing S.

By the above arguments it is enough to consider the case where S is a proper algebraically closed subfield of M, and $i : (S, S_0, res) \to (N, N_0, res)$ is an embedding.

Case1. $S_0 \neq M_0$. Let $x \in M_0 \setminus S_0$. Then $res(x) = x$ is transcendental over S_0 as S_0 is algebraically closed. Moreover $x \notin S$, and hence x is transcendental over S. Since (N, N_0, res) is saturated there is a $y \in L_0$ with $res(y) = y$ and $res(y) \notin i(S_0)$.

Therefore $res(y)$ is transcendental over $i(S_0)$ and y is transcendental over $i(S)$. The isomorphism $i : S \to i(S)$ extends to a field isomorphism $j : S(x) \to i(S)(y)$, where $j(x) = y$. Moreover, by Fact 3.1.7, $j : (S(x), \mathcal{O}_{S(x)}) \to (i(S)(y), \mathcal{O}_{i(S)(y)})$ is a valued field isomorphism and $j(res(x)) = j(x) = y = res(y) = res(j(x))$.

Also, observe that j maps $S_0(x)$ to $i(S_0)(y)$. Therefore it preserves the predicate P.

Next, for any $b \in S(x)$, we will show that $j(res(b)) = res(j(b))$. First assume that $b \in \mathcal{O}_{S(x)}$. If $b \in \mathcal{M}_{S(x)}$, then $res(b) = 0$, and also $j(b) \in \mathcal{M}_{i(S)(y)}$. Therefore $j(res(b)) = j(0) = 0 = res(j(b))$. Now assume that $b \in \mathcal{O}_{S(x)} \setminus \mathcal{M}_{S(x)}$, i.e. $v(b) = 0$. Then $b = \frac{f(x)}{g(x)} = \frac{f_0+f_1x+...+f_nx^n}{g_0+g_1x+...+g_mx^m}$ for some $f(x), g(x) \in S[x] \setminus \{0\}$ with $v(f) = v(g)$. Furthermore we may assume that $v(f) = v(g) = 0$. Indeed, if this is not the case, divide all coefficients of f and g by a coefficient of g which has minimal valuation. Then, in particular, all coefficients of $f(x)$ and $g(x)$ are in $\mathcal{O}_S$.

Then

$$\begin{array}{rl}
j\left(res\left(\frac{f}{g}\right)\right) & = j(resf(x)(g(x))^{-1})) = \\
= & j(res((f_0+f_1x+\ldots+f_nx^n)(g_0+g_1x+\ldots+g_mx^m)^{-1})) \\
= & j(res(f_0+f_1x+\ldots+f_nx^n)(res(g_0+g_1x+\ldots+g_mx^m))^{-1}) \\
= & j((\overline{f_0}+\overline{f_1}x+\ldots+\overline{f_n}x^n)(\overline{g_0}+\overline{g_1}x+\ldots+\overline{g_m}x^m)^{-1})) \\
= & j((\overline{f_0}+\overline{f_1}x+\ldots+\overline{f_n}x^n))j(\overline{g_0}+\overline{g_1}x+\ldots+\overline{g_m}x^m)^{-1})) \\
= & (i(\overline{f_0})+i(\overline{f_1})y+\ldots+i(\overline{f_n})y^n)(i(\overline{g_0})+i(\overline{g_1})y+\ldots+i(\overline{g_m})y^m))^{-1} \\
= & \overline{(i(f_0)+i(f_1)y+\ldots+i(f_n))y^n)(i(g_0)+i(g_1)y+\ldots+i(g_m)y^m)^{-1})} \\
= & res\left(j\left(\dfrac{f_0+f_1x+\ldots+f_nx^n}{g_0+g_1x+\ldots+g_mx^m}\right)\right)
\end{array}$$

Now assume $b \in S(x) \setminus \mathcal{O}_{S(x)}$. Then $res(b) = \infty$ and $j(res(b)) = j(\infty) = \infty$. On the other hand $j(b) \in i(S)(y) \setminus \mathcal{O}_{i(S)(y)}$. Therefore $res(j(b)) = \infty$. Hence $j(res(b)) = j(\infty) = \infty = res(j(b))$.

Case2. $S \cap M_0 = M_0$ and $vS^\times \neq vM^\times$. Let $g \in vM^\times \setminus vS^\times$. Since (N, N_0, res) is $|M|^+$-saturated, $vN^\times$ is $|M|^+$-saturated as an ordered abelian group, moreover $vN^\times$ is non-trivial. Also, $vi(S^\times)$ and $vS^\times$ are divisible groups. Therefore, there is $h \in vN^\times \setminus viS^\times$ such that for all $s \in S$, $g < v(s)$ if and only if $h < v(i(s))$.

Hence the isomorphism $vS^\times \to vi(S^\times)$ extends to the following ordered abelian group isomorphism

$$\begin{array}{rcl}
vS^\times + g\mathbb{Z} & \to & vi(S^\times) + h\mathbb{Z} \\
v(s) + gn & \mapsto & v(i(s)) + hn
\end{array}$$

Now pick elements $x \in M^\times$ and $y \in L^\times$ with valuations g and h respectively (i.e. $v(x) = g$ and $v(y) = h$). Clearly, $x \notin S$ and $y \notin i(S)$. Therefore, x and y are transcendental over S and $i(S)$ respectively.

Let $S' := S(x)$, $R := i(S)$ and $R' := i(S)(y)$. We can extend i to a field isomorphism $j : S' \to R'$. Moreover, by Fact 3.1.8 j is a valued field embedding. Therefore j maps $\mathcal{O}_{S'}$ isomorphically to $\mathcal{O}_{R'}$, and $\mathcal{M}_{S'}$ to $\mathcal{M}_{R'}$. Hence we can extend j naturally to $j : \mathcal{O}_{S'}/\mathcal{M}_{S'} \to \mathcal{O}_{R'}/\mathcal{M}_{R'}$.

Let $S'_0 := P(S') = M_0 \cap S' = M_0$ and $R'_0 := P(R') = N_0 \cap S' \subset N_0$. Then j maps $P(S')$ to $P(R')$ since it extends i, and i maps M_0 into N_0. Hence j preserves the predicate P.

It follows that j preserves the map res from the same argument in Case 1 above. Therefore j is also an $\mathcal{L}^{\pi}_{\text{Zar}}(P)$-embedding.

Case 3. $S \cap M_0 = M_0$ and $vS^\times = vM^\times$. We may assume $vS^\times = vi(S^\times)$ by identifying $(S, S \cap M_0, res)$ with its image under i. Let $x \in M \setminus S$. Let $f(x) \in S[x]$ be a non-zero element. Then $f(x) = u \prod_i (x - s_i)$ for some $u, s_i \in S$. Therefore $v(f) = v(u) + \sum_i v((x - s_i))$. Hence the valuation $v_{|S(x)} : S(x) \to vS$ is determined by $v_{|S} : S \to vS$ and $S \to vS^\times$ given by $s \mapsto v(x - s)$.

By Fact 3.1.9 and saturation there exists an element $y \in L \setminus i(S)$ such that $v(y - s) = v(x - s)$ for all $s \in S$. Next, extend the isomorphism i to a field isomorphism $j : S(x) \to i(S)(y)$ by $j(x) = y$. Then j is a valued field embedding.

As in Case 2, $P(S(x)) = M_0 \cap S(x) = M_0$ and $P(i(S)(y)) = N_0 \cap i(S)(y) \subset N_0$. Then j maps $P(S(x))$ to $P(i(S)(y))$ since it extends i, and i maps M_0 into N_0. Hence j preserves the predicate P.

Again, it follows that j preserves the map res from the argument used in Case 1 above. Therefore j is also an $\mathcal{L}^{\pi}_{\text{Zar}}(P)$-embedding.

This establishes quantifier elimination for $Th(K)^{res}$.

For completeness, consider (k, k, id) where $id : k \cup \{\infty\} \to k \cup \{\infty\}$ is identity. Take an element t which is transcendental over k and adjoin it to k. This is a prime substructure of the theory $Th(K)^{res}$ (i.e. it embeds into any model of $Th(K)^{res}$). Hence this theory is complete, as a theory with quantifier elimination and a prime structure is complete. □

Lemma 3.1.15. *Let (N, N_0, res) be a κ-saturated model of $Th(K)^{res}$ (in the language $\mathcal{L}^{\pi,\infty}_{Zar}(P)$), then $res : N \to N_0$ is a κ-universal specialisation.*

Proof. Let $N' \succeq N$ as $\mathcal{L}_{\text{Zar}}$ structures, and $A \subseteq N'$ with $|A| < \kappa$. Let $res' : A \cup N \to N_0$ be a specialisation extending res. We may assume that $res'(a) \in A$ for each $a \in A$ (if not, we can add those elements from N_0 to A, the resulting set will still have cardinality less than κ).

We need to show that there is an elementary embedding $\sigma : A \to N$ over $A \cap N$ such that $res'_{|A} = res \circ \sigma$.

Enumerate $A \setminus N$ as $\{b_i : i < \lambda\}$. Let $A_0 = A \cap N$, and $\sigma_0 := id = A \cap N \to N$, and let $A_j = A_0 \cup \{b_i : j < i\}$. By induction we will build a chain of partial elementary embeddings $\sigma_i : A_i \to L$ such that $\sigma_i \subseteq \sigma_j$ for $i < j$ and $res'_{|A_i} = res \circ \sigma_i$.

Let $i = j+1$ and $\sigma_j : A_j \to N$ be a partial elementary embedding. Consider the $\mathcal{L}_{\text{Zar}}^{\pi,\infty}(P)$ type

$$p(x) = \{\varphi(x, \sigma_j(\overline{a})) : N' \models \varphi(b_{j+1}, \overline{a}), \overline{a} \in A_j\}$$

Since $|A_j| < \kappa$, by κ-saturation of (N, N_0, res), the type $p(x)$ is realised by some $c_i \in N$. Then define $\sigma_i(b_i) = c_i$. To see that $res'(b_i) = res(\sigma_i(b_i))$, observe that $res'(b_i) \in A_0$. Then $res'(b_i) = \beta$ for some $\beta \in A_0$. Then the formula $res(x) = \beta$ is in $p(x)$. Hence $res(\sigma_i(b_i)) = res(c_i) = \beta_i$. Since σ_j is a partial elementary embedding, it follows that σ_i is partial elementary.

If i is a limit ordinal, let $\sigma_i := \bigcup_{j<i} \sigma_j$. Then define $\sigma := \bigcup_{j<\lambda} \sigma_j : A \to N$. Clearly, σ is an $\mathcal{L}_{\text{Zar}}$ partial elementary map, and moreover $res'|_A = res \circ \sigma$. $\square$

The following theorem follows from the completeness of $Th(K)^{res}$ and Lemma 3.1.15.

Theorem 3.1.16. *Let (M, M_0, res) be a model of $Th(K)^{res}$. Then (M, M_0, res) is elementarily equivalent to a model (N, N_0, res) such that $res : N \to N_0$ is κ-universal.*

Proof. Let (N, N_0, res) be a κ-saturated model of $Th(K)^{res}$, then by Lemma 3.1.15 the specialisation $res : N \to N_0$ is κ-universal. Since the theory $Th(K)^{res}$ is complete, $(M, M_0, res) \equiv (N, N_0, res)$. $\square$

Remark 3.1.17. The symbol ∞ in the language $\mathcal{L}_{\text{Zar}}^{\pi,\infty}(P)$ is for convenience. Especially to consider res as a map defined on the whole structure; in terms of valuation theory, to consider it as a place in the proper way.

However, we can consider the language $\mathcal{L}_{\text{Zar}}^{\pi}(P)$ without the ∞ symbol, and consider res as a partial map. The theory $Th(K)^{res}$ with the exception of axiom 2 can be stated in $\mathcal{L}_{\text{Zar}}^{\pi}(P)$. This axiom only specifies how ∞ will be treated. Moreover, all the arguments in the proofs of Lemmas 3.1.14 and 3.1.15 are valid for $\mathcal{L}_{\text{Zar}}^{\pi}(P)$ (with keeping in mind that res is a partial map in this setting, and being cautious accordingly). Theorem 3.1.16 will follow in the same way.

In Theorem 3.1.2 I have shown that any specialisation of an algebraically closed field can be extended to have a valuation ring as its domain. In other words, it can be extended to a place. It is easy to see that valuation rings are maximal domains for specialisations of algebraically closed fields. In other words, for algebraically closed fields maximally defined specialisations are places. On the other hand places are naturally identified with the equivalence classes of valuations. Therefore maximally defined specialisations $K \to k$ are in one to one correspondence with the equivalence classes of valuations $v : K \to \Gamma$ with the residue field k.

In particular if we consider a specialisation $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$, then π can be extended to total specialisation. Which will again be a place $res : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$ such that $res^{-1}(k)$ is a valuation ring R and $res^{-1}(\infty) = \mathbb{P}^1(K) \setminus R$. In this case all total specialisations $\mathbb{P}^1(K) \to \mathbb{P}^1(k)$ are in a one to one correspondence with the equivalence classes of valuations $v : K \to \Gamma$ with residue field k as above. A longer, but more geometric and constructive proof of this fact is given by de Piro [3].

Remark 3.1.18. For complete Zariski structures any specialisation can be extended to a total specialisation. Hence the notion of maximally defined specialisation coincides with total specialisation in this case.

The notion of maximally defined specialisation will be relevant to prove similar results in the rest of this thesis.

3.2 Specialisations over Algebraic Varities

In this section we will consider the universality of specialisations of algebraic (quasi-projective) varieties.

In this section G will denote an algebraic (quasi-projective) variety defined over a field k which is a subfield of an algebraically closed field K. $G(K)$ -K points of G- can be thought as a Zariski structure in a natural way in the language $\mathcal{L}_{\text{Zar}}$, which only has predicates for k-definable subvarieties of $G(K)^n$ (all $n \in \mathbb{N}$).

It is a well known fact that $G(K)$ is bi-interpretable with K where K has the structure of an algebraically closed field together with named elements from k. This bi-interpretation induces an isomorphism between $Aut(K)$ and $Aut(G)$ [27, Chap. 3, Prop. 1.4. part (ii)][1], hence we may think that $Aut(K)$ acts on G in the same way $Aut(G)$ does.

Let $A \subseteq G(K)$. Since $G(K)$ is k-definably covered by (quasi) affine varieties we can consider the subfield generated by A and k, which, we will denote by $k(A)$. To be more precise $k(A) = k(\overline{a} : [\overline{a}] \in A \cap U_i)$ where $[\overline{a}]$ is the homogeneous coordinates of a point and $U_i = \{[\overline{a}] : a_i = 1\}$.

Lemma 3.2.1. *Let $A \subseteq G(K)$, then $acl_G(A) = G(k(A)^{alg})$.*

Proof. $g \in acl_G(A) \iff g$ has finitely many conjugates under $Aut(G/A)$
$\iff g$ has finitely many conjugates under $Aut(K/A) \iff g \in G(k(A)^{\text{alg}})$. ☐

[1]Note that in the proposition referenced here it is actually proved that two countable, ω-categorical structures are bi-interpretable if and only if they have homeomorphic automorphism groups. However the direction we required above is true without assuming countability and ω-categoricity, as it can be seen from the proof.

3.2.1 Specialisation Induced by the Residue Map

Let $res : K \to k$ be a specialisation as in Section 3.1, and let G be an algebraic variety defined over k. Then, res gives rise to a specialisation $Res : G(K) \to G(k)$. In the case $G = \mathbb{P}^1$ we have already seen this specialisation in the Example 2.1.61. Here I will recall this example in more detail and generalize this construction to quasi-projective varieties .

Recall that $\mathbb{P}^1 = U_0 \cup H_0$ where $U_0 = \{(x_0 : x_1) : x_0 = 1\}$ and $H_0 = \{(x_0 : x_1) : x_0 = 0\}$. Then H_0 consists of a single point, denoted by ∞, and U_0 is isomorphic to $\mathbb{A}^1$ via $(1 : x_1) \mapsto x_1$.

Therefore we can write $\mathbb{P}^1(K) = \mathbb{A}^1(K) \cup \{\infty\} = K \cup \{\infty\}$ and similarly $\mathbb{P}^1(k) = k \cup \{\infty\}$. Then one can define $Res : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$ for $(1 : x_1) \in U_0(K)$ as $Res(1 : x_1) = (1 : res(x_1))$ given $x_1 \in \mathrm{Dom}(res)$. One can extend Res to the whole of $\mathbb{P}^1(K)$: Define $Res(\infty) = \infty$, and for all $(1 : x_1) \in U_1$ with $x_1 \notin \mathrm{Dom}(res)$, define $Res(1 : x_1) = \infty$.

Next we will consider $\mathbb{P}^n$. Recall that $\mathbb{P}^n = U_0 \cup \ldots \cup U_n$ where $U_i = \{(x_0 : \ldots : x_n) : x_i = 1\}$ for $i = 0, \ldots, n$. Here, U_i is isomorphic to $\mathbb{A}^n$ via $(x_0 : \ldots : x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Therefore define $Res : \mathbb{P}^n(K) \to \mathbb{P}^n(k)$ as follows: For $(x_0 : \ldots : x_n) \in U_i(K)$ define $Res(x_0 : \ldots : x_n) = (res(x_0) : \ldots : res(x_{i-1}) : 1 : res(x_{i+1}) : \ldots : res(x_n))$.

Remark 3.2.2. For any $(x_0 : \ldots : x_n) \in \mathbb{P}^n(K)$ there is a $\lambda \in K$ such that $\lambda x_0, \ldots, \lambda x_n$ are in the valuation ring $\mathrm{Dom}(res)$ but not all of them are in the valuation ideal, i.e. there is some j such that $res(\lambda x_j) \neq 0$. Hence $(\lambda x_j)^{-1} \in \mathrm{Dom}(res)$. Then $(x_0 : \ldots : x_n) = (\lambda x_0 : \ldots : \lambda x_n) = (\frac{\lambda x_0}{\lambda x_j} : \ldots : \frac{\lambda x_{j-1}}{\lambda x_j} : 1 : \frac{\lambda x_{j+1}}{\lambda x_j} : \ldots : \frac{\lambda x_n}{\lambda x_j}) = (\frac{x_0}{x_j} : \ldots : \frac{x_{j-1}}{x_j} : 1 : \frac{x_{j+1}}{x_j} : \ldots : \frac{x_n}{x_j})$ and $\frac{x_0}{x_j}, \ldots : \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j} \in \mathrm{Dom}(res)$.

Therefore, any point $x \in \mathbb{P}^n(K)$ has can be written in homogeneous coordinates as $x = (x_0 : \ldots : x_{j-1} : 1 : x_{j+1} : \ldots : x_n) \in U_j$ for some j with $x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n \in \mathrm{Dom}(res)$. It immediately follows that Res is defined on every point of $\mathbb{P}^n(k)$, i.e. $\mathrm{Dom}(Res) = \mathbb{P}^n(K)$.

Next, as we define Res separately on each of the sets U_i, we need to check that it is well defined on the intersection of such sets. This will be immediate from the following remark:

Remark 3.2.3. One can alternatively define $Res : \mathbb{P}^n(K) \to \mathbb{P}^n(k)$ as follows. Let $(x_0 : \ldots, x_n) \in \mathbb{P}^n(K)$. There is a $\lambda \in K$ such that $\lambda x_0, \ldots, \lambda x_n \in \mathrm{Dom}(res)$ such that $res(\lambda x_i) \neq 0$ for some $i = 1, \ldots, n$. Then define $Res(x_0 : \ldots, x_n) = (res(\lambda x_0) : \ldots : res(\lambda x_n))$.

The definition of Res as given above does not depend on λ. For if $\mu \in K$ is an other element such that $\mu x_0, \ldots, \mu x_n \in \mathrm{Dom}(res)$ with $res(\mu x_j) \neq 0$, it follows that $(res(\lambda x_0) : \ldots : res(\lambda x_n)) = (res(\mu x_0) : \ldots : res(\mu x_n))$.

Clearly, Res defined in the remark above coincides with the map Res defined separately on each U_i. Moreover, the map Res, is well defined on the intersections $U_i \cap U_j$ for all i, j by the same remark.

Next, we need to check that $Res : \mathbb{P}^n(K) \to \mathbb{P}^n(k)$ is a specialisation. Since $res : k \to k$ is identity, $Res : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ is also identity. Let, $S \subseteq (\mathbb{P}^n(K))^m$ be a closed subset and let $\overline{x} \in \mathrm{Dom}(Res) \cap S$. Write $\overline{x} = (z_1, \ldots, z_m)$ where $z_i = (z_{i0} : \ldots : z_{in})$ for all $i = 1 \ldots, m$. There is a finite set of homogeneous polynomials $\{f_j(Z_{01}, \ldots, Z_{0n}, \ldots, Z_{m0}, \ldots, Z_{mn})\} \subseteq k[Z_{01}, \ldots, Z_{0n}, \ldots, Z_{m0}, \ldots, Z_{mn}]$ such that $f_i(z_{01}, \ldots, z_{0n}, \ldots, z_{m0}, \ldots, z_{mn}) = 0$ for all $i = 1, \ldots, m$.

Since $res : K \to k$ is a specialisation, $f_i(res(\overline{z})) = 0$ for all i, where $\overline{z} = (z_{01}, \ldots, z_{0n}, \ldots, z_{m0}, \ldots, z_{mn})$. Therefore, $Res(\overline{x}) \in S$.

3.2.2 Specialisation Induced on the Field from a Variety

In this subsection I will show that a maximal specialisation on G comes from a maximal specialisation $\pi_K : K \to k$. The first main statement of this subsection is the following:

Proposition 3.2.10. *Let G be an affine variety defined over k. Let $\pi_G : G(K) \to G(k)$ be a maximal specialisation. Then there is a maximal specialisation $\pi_K : K \to k$ such that* $\mathrm{Dom}(\pi_G) = G(K) \cap (\mathrm{Dom}(\pi_K))^n$ *and for any $\overline{x} \in \mathrm{Dom}(\pi_G)$, its image $\pi_G(\overline{x}) = (x_1^{\pi_K}, \ldots, x_n^{\pi_K})$.*

To prove Proposition 3.2.10, one first needs some preparation. Let k be an algebraically closed field (seen as a Zariski structure) and $K \succeq k$. Let G an affine variety over k and $\pi_G : G(K) \to G(k)$ be a specialisation. We can interpret k in $G(k)$ (and similarly K in $G(K)$) via the projections $\mathrm{pr}_i : G(k) \to k$ for $i = 1, \ldots, n$. Clearly $\mathrm{pr}_i(G(k)) = k \setminus X_i$ for some finite $X_i \subseteq k$. Moreover, the equivalence relation $\sim_i$ on G for each i, defined by $(x_1, x_2, \ldots, x_n) \sim_i (x'_1, x'_2, \ldots, x'_n)$ if and only if $\mathrm{pr}_i(x_1, x_2, \ldots, x_n) = \mathrm{pr}_i(x'_1, x'_2, \ldots, x'_n)$ is closed.

For each $i = 1, \ldots, n$, we define $\pi_i : K \setminus X_i \to k \setminus X_i$ as follows: first, define $\mathrm{Dom}(\pi_i) := \mathrm{pr}_i(\mathrm{Dom}(\pi_G))$. Second, for any $x \in \mathrm{pr}_i(\mathrm{Dom}(\pi_G))$ define $\pi_i(x)$

as follows: Pick an element $(x_1, x_2, \ldots, x_n) \in \mathrm{pr}_i^{-1}(x) \cap \mathrm{Dom}(\pi_G)$. Then define $\pi_i(x) = \mathrm{pr}_i(\pi_G(x_1, x_2, \ldots, x_n))$. It is clear that π_i is well defined for each i, since $\sim_i$ is a closed equivalence relation. Next we will show that π_i is a specialisation.

Note that, $K \setminus X_i$ and $k \setminus X_i$ are quasi-affine varieties, and we will consider them with the natural Zariski topologies on their Cartesian powers as Zariski geometries.

$$\begin{tikzcd} G(K) \arrow[r, "\pi_G"] \arrow[d, "\mathrm{pr}_i"] & G(k) \arrow[d, "\mathrm{pr}_i"] \\ K_i := K \setminus X_i \arrow[r, "\pi_i"] & k_i := k \setminus X_i \end{tikzcd}$$

Proposition 3.2.4. *For each $i \in \{1, \ldots, n\}$, $\pi_i : K_i \to k_i$ is a specialisation.*

Proof. Let S be a closed set of $K_i := K \setminus X_i$ and $\overline{a} \in S \cap \mathrm{Dom}(\pi_i)$. We want to show that $\pi_i(\overline{a}) \in S$. Since pr_i is continuous, $T := \mathrm{pr}_i^{-1}(S)$ is closed, and since $\overline{a} \in \mathrm{Dom}(\pi_i)$, there is $\overline{\alpha} \in \mathrm{Dom}(\pi_G)$ such that $\mathrm{pr}_i(\overline{\alpha}) = \overline{a}$. Then $G(K_i) \models T(\alpha)$, and since π_G is a specialisation, $G(k_i) \models T(\pi_G(\overline{\alpha}))$.

Then, by definition of pr_i, we have $\mathrm{pr}_i(\pi_G(\alpha)) \in \mathrm{pr}_i(T) = S$. Therefore, by definition, $\pi_i(\overline{a}) \in S$. Hence π_i is a specialisation. □

Therefore we can think of $\pi_G : G(K) \to G(k)$ in terms of the specialisations $\pi_1, \ldots, \pi_n$. Let $\overline{x} \in \mathrm{Dom}(\pi_G)$, and say $\pi_G(\overline{x}) = \pi_G(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$. By construction $\pi_1(x_1) = \mathrm{pr}_1(\pi_G(\overline{x})) = y_1, \ldots, \pi_n(x_n) = \mathrm{pr}_1(\pi_G(\overline{x})) = y_n$. Then we can write $\pi_G(x_1, \ldots, x_n) = (x_1^{\pi_1}, \ldots, x_n^{\pi_n})$.

Proposition 3.2.5. *For any $i, j \in \{1, \ldots, n\}$, the specialisations π_i and π_j are the same on the intersection $\mathrm{Dom}(\pi_i) \cap \mathrm{Dom}(\pi_j)$ of their domains; i.e $\pi_i = \pi_j$ on $\mathrm{Dom}(\pi_i) \cap \mathrm{Dom}(\pi_j)$.*

Proof. First of all observe that since $\mathrm{Dom}(\pi_i) \cap \mathrm{Dom}(\pi_j) \supseteq k \setminus X_i \cup X_j$, it is not empty.

Let $z \in \mathrm{Dom}(\pi_i) \cap \mathrm{Dom}(\pi_j)$. Then there are tuples $\overline{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathrm{Dom}(\pi_G)$ and $\overline{y} = (y_1, \ldots, y_j, \ldots, y_n) \in \mathrm{Dom}(\pi_G)$ such that $x_i = y_j = z$.

Define

$$C_{ij} = \{(\overline{x}, \overline{y}) \in (G(K))^2 : x_i = y_j\}$$

Then $\overline{x}, \overline{y} \in C \cap \mathrm{Dom}(\pi_G)$. Clearly C is a closed subset of $(G(K))^2$. Then, since π_G is a specialisation, $\pi_G(\overline{x}), \pi_G(\overline{y}) \in C$. Then $\pi_i(x_i) = \pi_j(y_j)$. Hence $\pi_i(z) = \pi_j(z)$. □

Define $D := \mathrm{Dom}(\pi_1) \cup \ldots \cup \mathrm{Dom}(\pi_n)$. Next we will define a common extension of the specialisations $\pi_1, \ldots, \pi_n$ to D. Define $\pi_K^0 : D \to k \setminus X$ where $X = X_1 \cap \ldots \cap X_n$ as follows: let $x \in D$, then by definition there is an $i = 1, \ldots, n$ such that $x \in \mathrm{Dom}(\pi_i)$. Define $\pi_K^0(x) = \pi_i(x)$. Since $\pi_1, \ldots, \pi_n$ are the same on the intersection $\mathrm{Dom}(\pi_1) \cap \ldots \cap \mathrm{Dom}(\pi_n)$, this is well defined. We now need to show that π_K^0 is a specialisation.

Proposition 3.2.6. *$\pi_K^0 : D \to k \setminus X$ is a specialisation.*

Proof. Let S be a closed set of $K \setminus X$ and let $\overline{z} = (z_1, \ldots, z_n) \in D^n$ be such that $\models S(z_1, \ldots, z_n)$. We will show that $\models S(z_1^{\pi_K^0}, \ldots, z_n^{\pi_K^0})$.

For each $i = 1, \ldots, n$ there is a $j_i \in \{1, \ldots, n\}$ such that $z_i \in \mathrm{Dom}(\pi_{j_i})$, since $\overline{z} \in D^n$. Then for all i there is a tuple $\overline{x}_i = (x_{1i}, \ldots, x_{ji}, \ldots, x_{ni}) \in \mathrm{Dom}(\pi_G)$ such that $z_i = x_{ji}$.

Next, look at the set

$$C = \{(\overline{x}_1, \ldots, \overline{x}_n) \in G(K)^n : S(x_{j1}, \ldots, x_{jn})\}$$

Then C is a closed subset of $G(K)^n$. Therefore, since π_G is a specialisation, $\models C(\overline{x}_1^{\pi_G}, \ldots, \overline{x}_n^{\pi_G})$. Recall that $\overline{x}_i^{\pi_G} = (x_{1i}^{\pi_1}, \ldots, x_{ji}^{\pi_{ji}}, \ldots, x_{ni}^{\pi_n})$. Then in particular, $S(x_{j1}^{\pi_{j1}}, \ldots, x_{jn}^{\pi_{jn}})$. But, $S(x_{j1}^{\pi_{j1}}, \ldots, x_{jn}^{\pi_{jn}}) = S(x_{j1}^{\pi_K^0}, \ldots, x_{jn}^{\pi_K^0})$, since π_K^0 is a common extension of all $\pi_1, \ldots, \pi_n$. Then $\models S(z_1^{\pi_K^0}, \ldots, z_n^{\pi_K^0})$ as $z_i = x_{ji}$. Hence π_K^0 is a specialisation. $\square$

The specialisation $\pi_K^0 : D \to k \setminus X$ is not defined on the finite subset $X \subset k$. However, as $X \subset k$ and since any specialisation $K \to k$ must be identity on k we can canonically extend π_K^0 to a specialisation $\pi_K^0 : K \to k$ simply by defining it to be identity on X.

From now on, whenever we write $\pi_K^0 : K \to k$, it will always refer to the specialisation of K with domain $D \cup X$, and range k (so that $\pi_K^0 : D \cup X \to k$ is onto). Further, the specialisation $\pi_K^0 : K \to k$ induces a specialisation $\pi_G^0 : G(K) \to G(k)$ in the natural way: Define $\mathrm{Dom}(\pi_G^0) := G(K) \cap (\mathrm{Dom}(\pi_K^0))^n$, and for all $\overline{x} = (x_1, \ldots, x_n) \in \mathrm{Dom}(\pi_G^0)$, define $\pi_G^0(\overline{x}) := (x_1^{\pi_K^0}, \ldots, x_n^{\pi_K^0})$.

Proposition 3.2.7. *The specialisation $\pi_G^0 : G(K) \to G(k)$ induced by $\pi_K^0 : K \to k$ is an extension of the specialisation $\pi_G : G(K) \to G(k)$, and $\pi_G^0 = \pi_G$ on $\mathrm{Dom}(\pi_G)$.*

Proof. Let $\pi_G^0 : G(K) \to G(k)$ be the specialisation induced by π_K^0. Then by construction, $\mathrm{Dom}(\pi_G^0) = G(K) \cap (\mathrm{Dom}(\pi_K^0))^n$, and for all $\overline{x} = (x_1, \ldots, x_n) \in \mathrm{Dom}(\pi_G^0)$, $\pi_G^0(\overline{x}) = (x_1^{\pi_K^0}, \ldots, x_n^{\pi_K^0})$.

Next, we will show that $\mathrm{Dom}(\pi_G) \subseteq \mathrm{Dom}(\pi_G^0)$. Let $\overline{x} = (x_1, \ldots, x_n) \in \mathrm{Dom}(\pi_G)$. Then $x_i \in \mathrm{Dom}(\pi_i)$ for all $i = 1, \ldots, n$. Then $x_1, \ldots, x_n \in \mathrm{Dom}(\pi_K^0)$, since π_K^0 is a common extension of all of the specialisations $\pi_1, \ldots, \pi_n$. Hence $(x_1, \ldots, x_n) \in \mathrm{Dom}(\pi_G^0)$.

Last, we will show that $\pi_G^0 = \pi_G$ on $\mathrm{Dom}(\pi_G)$. Let $\overline{x} = (x_1, \ldots, x_n) \in \mathrm{Dom}(\pi_G)$. Then $x_i \in \mathrm{Dom}(\pi_i)$ for all $i = 1, \ldots, n$. Therefore $\pi_G(\overline{x}) = (x_1^{\pi_1}, \ldots, x_n^{\pi_n}) = (x_1^{\pi_K^0}, \ldots, x_n^{\pi_K^0}) = \pi_G^0(\overline{x})$. Hence π_G^0 is an extension of π_G. □

Note here that if $\pi_G : G(K) \to G(k)$ is a maximal specialisation we will have $\mathrm{Dom}(\pi_G) = \mathrm{Dom}(\pi_G^0)$ and $\pi_G^0 = \pi_G$.

At this point we are ready to prove Proposition 3.2.10. However, before proceeding to the proof of Proposition 3.2.10, we will prove another result.

Proposition 3.2.8. *Let $\pi_G : G(K) \to G(k)$ be a maximal specialisation. If the specialisation $\pi_K^0 : K \to k$ induced by π_G (as defined before Proposition 3.2.6) is κ-universal, then π_G is κ-universal.*

Proof. Suppose that π_K^0 is κ-universal. We will show that π_G is also κ-universal. Let $G(K') \succeq G(K)$, and $A \subseteq G(K')$ with $|A| < \kappa$. Let $\pi_{G'} : G(K) \cup A \to G(k)$ be a specialisation extending π_G.

By Propositions 3.2.4, 3.2.5, 3.2.6, $\pi_{G'}$ induces a specialisation $\pi_{K'}^0 : K \to k$ such that $\pi_{G'}(x_1, \ldots, x_n) = (x_1^{\pi_{K'}^0}, \ldots, x_n^{\pi_{K'}^0})$.

Next, we claim that $\pi_{K'}^0$ extends π_K^0. Indeed, let $a \in \mathrm{Dom}(\pi_K^0)$. Then $a \in \mathrm{Dom}(\pi_i)$ for some i. Therefore there is a tuple $(a_1, \ldots, a, \ldots, a_n) \in \mathrm{Dom}(\pi_G)$.

By construction of π_K^0, we have

$$\pi_G(a_1, \ldots, a, \ldots, a_n) = (a_1^{\pi_K^0}, \ldots, a^{\pi_K^0}, \ldots, a_n^{\pi_K^0})$$

Similarly

$$\pi_{G'}(a_1, \ldots, a, \ldots, a_n) = (a_1^{\pi_{K'}^0}, \ldots, a^{\pi_{K'}^0}, \ldots, a_n^{\pi_{K'}^0})$$

But $\pi_{G'}(a_1, \ldots, a, \ldots, a_n) = \pi_G(a_1, \ldots, a, \ldots, a_n)$, since $\pi_{G'}$ extends π_G. Therefore

$$(a_1^{\pi_K^0}, \ldots, a^{\pi_K^0}, \ldots, a_n^{\pi_K^0}) = (a_1^{\pi_{K'}^0}, \ldots, a^{\pi_{K'}^0}, \ldots, a_n^{\pi_{K'}^0})$$

Hence $\pi_{K'}^0(a) = \pi_K^0(a)$ as desired.

Since π_K^0 is κ-universal, there is an embedding $\sigma_K : B \to K$ such that $\pi_K^0 = \pi_{K'}^0 \circ \sigma_K$ on B where $B = \bigcup_{i=1}^{n} \mathrm{pr}_i(A)$. Define $\sigma : A \to G(K)$ as $\sigma(x_1, \ldots, x_n) = (\sigma_K(x_1), \ldots, \sigma_K(x_n))$.

Next, we will show that $\pi_{G'} = \pi_G \circ \sigma$ on A. Let $(a_1, \ldots, a_n) \in A$. Then

$$\begin{aligned} \pi_{G'}(a_1, \ldots, a_n) &= (a_1^{\pi^0_{K'}}, \ldots, a_n^{\pi^0_{K'}}) = ((\sigma_K(a_1))^{\pi^0_K}, \ldots, (\sigma_K(a_n))^{\pi^0_K}) \\ &= \pi_G(\sigma_K(a_1), \ldots, \sigma_K(a_n)) = \pi_G \circ \sigma(a_1, \ldots, a_n) \end{aligned}$$

□

Remark 3.2.9. Let G be an affine variety defined over k. Let $res : K \to k$ be a κ-universal specialisation. The specialisation $Res : G(K) \to G(k)$ induced by res is κ-universal: $\mathrm{Dom}(Res) = (\mathrm{Dom}(res))^n \cap G(k)$, and for any $\overline{a} \in \mathrm{Dom}(Res)$, we define $Res(\overline{a}) = (res(a_1), \ldots, res(a_n))$. Clearly, the specialisation $Res^0_K : K \to k$ induced by Res is a restriction of res. Since res is maximal, $Res^0_K = res$. Then Res is κ-universal.

Proposition 3.2.10. *Let G be an affine variety defined over k. Let $\pi_G : G(K) \to G(k)$ be a maximal specialization. Then there is a maximal specialisation $\pi_K : K \to k$ such that $\mathrm{Dom}(\pi_G) = G(K) \cap (\mathrm{Dom}(\pi_K))^n$ and for any $\overline{x} \in \mathrm{Dom}(\pi_G)$, its image $\pi_G(\overline{x}) = (x_1^{\pi_K}, \ldots, x_n^{\pi_K})$.*

Proof of Proposition 3.2.10. Let $\pi_G : G(K) \to G(k)$ be a maximal specialisation. Look at the coordinate projections $\mathrm{pr}_i : G(K) \to K \setminus X_i$ as before and define $\pi_i : K \setminus X_i \to k \setminus X_i$ as follows: Domain of π_i is $\mathrm{pr}_i(\mathrm{Dom}(\pi_G))$ and for any $x \in \mathrm{pr}_i(\mathrm{Dom}(\pi_G))$ there is an element $(x_1, \ldots, x_n) \in \mathrm{pr}_i^{-1}(x) \cap \mathrm{Dom}(\pi_G)$. Define $\pi_i(x) = \mathrm{pr}_i(\pi_G(x_1, \ldots, x_n))$. By Proposition 3.2.4, π_i is a specialisation for all i.

Let $D := \mathrm{Dom}(\pi_1) \cup \ldots \cup \mathrm{Dom}(\pi_n)$. We define $\pi^0_K : D \to k \setminus X$ as follows: for any $x \in D$, there is an $i = 1, \ldots, n$ such that $x \in \mathrm{Dom}(\pi_i)$. Then define $\pi^0_K(x) = \pi_i(x)$. By Proposition 3.2.5, π^0_K is well defined and by Proposition 3.2.6 it is a specialisation.

Extend π^0_K to X by defining $\pi^0_K(x) = x$ for all $x \in X$. Then we get an extension of π^0_K to a specialisation $\pi^0_K : K \to k$.

If π^0_K is a maximal specialisation, take π_K to be π^0_K. Then, by Proposition 3.2.7 and the observation following it, we are done.

So suppose π^0_K is not maximal. First, by extending π^0_K to the ring generated by $\mathrm{Dom}(\pi^0_K)$ we may assume that $\mathrm{Dom}(\pi^0_K)$ is a ring. In this case π^0_K is a ring homomorphism. Then by Chevalley's Extension Theorem, we may extend π^0_K to a residue map, call it π_K. It is clear that π_K is a maximal specialisation on K.

Next, let $\pi^+_G : G(K) \to G(k)$ be the specialisation induced on $G(K)$ by π_K. I.e. $\mathrm{Dom}(\pi^+_G) = G(K) \cap (\mathrm{Dom}(\pi_K))^n$, and if $\overline{x} = (x_1, \ldots, x_n) \in \mathrm{Dom}(\pi^+_G)$, then $\pi^+_G(\overline{x}) := (x_1^{\pi_K}, \ldots, x_n^{\pi_K})$.

Then again by Proposition 3.2.7 and the observation following it, $\mathrm{Dom}(\pi_G) = \mathrm{Dom}(\pi_G^+) = G(K) \cap (\mathrm{Dom}(\pi_K))^n$ and $\pi_G = \pi_G^+$. Therefore $\pi_G(\overline{x}) = \pi_G^+(\overline{x}) = (x_1^{\pi_K}, \ldots, x_n^{\pi_K})$ for any $\overline{x} \in \mathrm{Dom}(\pi_G)$. $\square$

Next, we will consider the case where G is a quasi-projective variety defined over k. Therefore G is an open subset of some closed projective set $X \subseteq \mathbb{P}^n$. Let $\mathbb{A}_i^n := \{(z_0, \ldots, z_n) : z_i \neq 0\}$. We know that $\mathbb{P}^n = \bigcup_{i=0}^{n} \mathbb{A}_i^n$. Also $\mathbb{A}_i^n$ is isomorphic to $\mathbb{A}^n$ for all i via

$$\begin{aligned} h_i : \mathbb{A}_i^n &\rightarrow \mathbb{A}^n \\ (z_0, \ldots, z_n) &\mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right) \end{aligned}$$

where the tuple $\left(\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}\right)$ does not contain the i^{th} coordinate.

It follows that $G = \bigcup_{i=0}^{n} U_i$ where $U_i = \mathbb{A}_i^n \cap G$. Clearly each $\mathbb{A}_i^n$ is an open subset of $\mathbb{P}^n$. Therefore each U_i is an open subset of G. Moreover one can see that each U_i is a locally closed subset of $\mathbb{A}_i^n$. Since $\mathbb{A}_i^n$ is isomorphic to $\mathbb{A}^n$, each U_i, via restriction of h_i, is isomorphic to a locally closed subset of $\mathbb{A}_n$, call it V_i. So each U_i is a quasi-affine variety. The sets U_i are called *affine pieces of* G, and a particular U_i will be called the i^{th} affine piece.

Proposition 3.2.11. *Let G be a quasi-projective variety, and $\pi_G : G(K) \rightarrow G(k)$ be a specialisation. Then π_G induces a specialisation $\pi_{U_i} : U_i(K) \rightarrow U_i(k)$ for each i where U_i is the i^{th} affine piece of G.*

Proof. Define $\pi_{U_i} : U_i(K) \rightarrow U_i(k)$ as follows: π_{U_i} is not defined on $U_i \setminus \mathrm{Dom}(\pi_G)$. Let $x \in U_i(K) \cap \mathrm{Dom}(\pi_G)$. If $\pi_G(x) \in U_i(k)$, define $\pi_{U_i}(x) := \pi_G(x)$. If $\pi_G(x) \notin U_i(k)$, then π_{U_i} will not be defined on x. Hence $\mathrm{Dom}(\pi_{U_i}) = \{x \in \mathrm{Dom}(\pi_G) : \pi_G(x) \in U_i(k)\}$.

Next, we will show that $\pi_{U_i} : U_i(K) \rightarrow U_i(k)$ is a specialisation. Since π_G is identity on $G(k)$, it is clear that π_{U_i} is identity on $U(k)$. Now, let S be a closed subset of $(U_i(K))^m$, and suppose $U_i(K) \models S(\overline{a})$ for some $\overline{a} \in \mathrm{Dom}(\pi_{U_i})$.

Let $\overline{S}$ be the closure of S in $(G(K))^n$. Then $G(K) \models \overline{S}(\overline{a})$. Since π_G is a specialisation, $\pi_G(\overline{a}) \in \overline{S}$ (i.e. $G(k) \models \overline{S}(\pi_G(\overline{a}))$). Since $\overline{a} \in U_i(K)$, we know that $\pi_G(\overline{a}) = \pi_{U_i}(\overline{a})$. Therefore $\pi_{U_i}(\overline{a}) \in U_i(k) \cap \overline{S}(k) = S(k)$. $\square$

Remark 3.2.12. Let $\alpha \in U_i(K) \cap \mathrm{Dom}(\pi_G)$ with $a := \pi_G(\alpha) \notin U_i(k)$. So, $\pi_G(\alpha) \in U_j(k)$ for some $j \neq i$. Then $\alpha \in U_j(K)$. Suppose not, i.e. suppose $\alpha \notin U_j(K)$. Then $\alpha_j = 0$. But this is a Zariski closed relation, and since π_G is a specialisation, it must be preserved by π_G. Therefore, $a_i = 0$. However, $a \in U_j(k)$ means exactly that $a_i \neq 0$. Which is a contradiction. Hence $\alpha \in U_j(K)$.

Moreover, as $\alpha \in \mathrm{Dom}(\pi_G)$, it will be in the domain of $\pi_{U_j} : U_j(K) \to U_j(k)$, which is constructed as in the above proof.

As already remarked above, the affine pieces U_i of G are locally closed in $\mathbb{A}^n_i$, and $\mathbb{A}^n_i$ is isomorphic to the affine space $\mathbb{A}^n$, via h_i. Then U_i is isomorphic to a locally closed subset $V_i \subseteq \mathbb{A}^n$ via h_i.

Proposition 3.2.13. *Let U_i be an affine piece of G. Then the specialisation π_{U_i} : $U_i(K) \to U_i(k)$ induces a specialisation $\pi_{V_i} : V_i(K) \to V_i(k)$.*

Proof. Define $\pi_{V_i} := h_i \circ \pi_{U_i} \circ h_i^{-1}$, and let $\mathrm{Dom}(\pi_{V_i}) = h_i^{-1}(\mathrm{Dom}(\pi_{U_i}))$. First, let us verify that π_{V_i} is identity on $V_i(k)$. Let $a \in V_i(k)$, then $h^{-1}(a) \in U_i(k)$. Since π_{U_i} is identity on $U_i(k)$, we get $\pi_{U_i}(h_i^{-1}(a)) = h_i^{-1}(a)$. Then

$$\pi_{V_i}(a) = h_i \circ \pi_{U_i} \circ h_i^{-1}(a) = h_i(\pi_{U_i}(h_i^{-1}(a))) = h_i(h_i^{-1}(a)) = a$$

Next, let S be a closed subset of $(V_i(K))^n$, and suppose $V_i(K) \models S(\overline{a})$ for some $\overline{a} \in \mathrm{Dom}(\pi_{V_i})$. Let $T := h_i^{-1}(S)$. Since h_i^{-1} is a homeomorphism, T is closed in $(U_i(K))^n$. Then, since π_{U_i} is a specialisation, $\pi_{U_i}(h_i^{-1}(\overline{a})) \in T$. Then $\pi_{V_i}(\overline{a}) = h_i(\pi_{U_i}(h_i^{-1}(\overline{a}))) \in h_i(T) = S$. Hence π_{V_i} is a specialisation. $\square$

We can repeat the argument we used to construct the specialisations π_i in Proposition 3.2.4. Consider the coordinate projections $\mathrm{pr}_j : V_i(k) \to k$ for each $j = 1, \ldots, n$. Then $\mathrm{pr}_j(V_i(k)) = k \setminus X_{ij}$ for some finite $X_{ij} \subset k$. Define $\pi_{V_{ij}} : K \setminus X_{ij} \to k \setminus X_{ij}$ as before: First we define $\mathrm{Dom}(\pi_{V_{ij}}) := \mathrm{pr}_j(\mathrm{Dom}(\pi_{V_i}))$. Then for any $x \in \mathrm{Dom}(\pi_{V_{ij}})$, to define $\pi_{V_{ij}}(x)$ pick an element $(x_1, \ldots, x_n) \in \mathrm{pr}_j^{-1}(x) \cap \mathrm{Dom}(\pi_{V_i})$, and define $\pi_{V_{ij}}(x) := \mathrm{pr}_j(\pi_{V_i}(x_1, \ldots, x_n))$.

As proved in Proposition 3.2.4, $\pi_{V_{ij}} : K \setminus X_{ij} \to k \setminus X_{ij}$ is a specialisation for each $j = 1, \ldots, n$. Also, for each $j, k = 1, \ldots, n$, the specialisations $\pi_{V_{ij}}$ and $\pi_{V_{ik}}$ are equal on the intersection $\mathrm{Dom}(\pi_{V_{ij}}) \cap \mathrm{Dom}(\pi_{V_{ik}})$ of their domains (Proposition 3.2.5). Moreover there is a common extension $\pi^0_{iK} : D_i \to k \setminus X_i$ of the specialisations $\pi_{V_{i1}}, \ldots, \pi_{V_{in}}$ where $D_i = \bigcup_{j=1}^{n} \mathrm{Dom}(\pi_{V_{ij}})$ and $X_i = \bigcap_{j=1}^{n} X_{ij}$ (Proposition 3.2.6). Since $X_i \subseteq k$, and since any specialisation $K \to k$ must be identity on k, we can extend π^0_{iK} to a specialisation $\pi^0_{iK} : K \to k$ with $\mathrm{Dom}(\pi^0_{iK}) = D_i \cup X_i$. Therefore, for each i we have a specialisation $\pi^0_{iK} : K \to k$.

The proposition below immediately follows from the Proposition 3.2.7.

Proposition 3.2.14. *The specialisation $\pi^0_{V_i} : V_i(K) \to V_i(k)$ induced by $\pi^0_{iK} : K \to k$ is an extension of the specialisation $\pi_{V_i} : V_i(K) \to V_i(k)$ for all i, and $\pi^0_{V_i} = \pi_{V_i}$ on* $\mathrm{Dom}(\pi_{V_i})$.

Remark 3.2.15. As a result of the proposition above, $\pi_{V_i}(x_1, \ldots, x_n) = (x_1^{\pi^0_{iK}}, \ldots, x_n^{\pi^0_{iK}})$ for any $(x_1, \ldots, x_m) \in \mathrm{Dom}(\pi_{V_i})$, for each i. By definition $\pi_{V_i}(x_1, \ldots, x_n) = h_i \circ \pi_{U_i} \circ h_i^{-1}(x_1, \ldots, x_n)$. Write $h_i^{-1}(x_1, \ldots, x_n)$ in homogeneous coordinates as $(z_0 : \ldots : 1 : \ldots : z_n)$ where $z_0 = x_1, \ldots, z_{i-1} = x_i, z_{i+1} = x_{i+1}, \ldots z_n = x_n$ and $z_i = 1$.

Therefore

$$(x_1^{\pi^0_{iK}}, \ldots, x_n^{\pi^0_{iK}}) = h_i \circ \pi_{U_i} \circ h_i^{-1}(x_1, \ldots, x_n) = h_i \circ \pi_{U_i}(z_0 : \ldots : 1 : \ldots : z_n)$$

Then $\pi_{U_i}(z_0 : \ldots : 1 : \ldots : z_n) = h_i^{-1}(x_1^{\pi^0_{iK}}, \ldots, x_n^{\pi^0_{iK}})$. Write $h_i^{-1}(x_1^{\pi^0_{iK}}, \ldots, x_n^{\pi^0_{iK}})$ in homogeneous coordinates as $(x_1^{\pi^0_{iK}} : \ldots : 1 : \cdots : x_n^{\pi^0_{iK}})$. Moreover $(x_1^{\pi^0_{iK}} : \ldots : 1 : \cdots : x_n^{\pi^0_{iK}}) = (z_0^{\pi^0_{iK}} : \ldots : 1 : \ldots : z_n^{\pi^0_{iK}})$. Therefore

$$\pi_{U_i}(z_0 : \ldots : 1 : \ldots : z_n) = (z_0^{\pi^0_{iK}} : \ldots : 1 : \ldots : z_n^{\pi^0_{iK}})$$

If $\pi^0_{iK} : K \to k$ is not maximal, we can extend it to a maximal specialisation $\pi_{iK} : K \to k$ as before. Therefore, we can say that on each affine piece U_i of G, the specialisation π_{U_i} induced by the specialisation $\pi_G : G(K) \to G(k)$, can be given in terms of a maximal specialisation $\pi_{iK} : K \to k$.

By construction, $\pi_G = \bigcup_{i=0}^{n} \pi_{U_i}$. Hence, the specialisation π_G can be given in terms of maximal specialisations π_{iK} for $i = 0, \ldots, n$ on the corresponding affine pieces.

3.2.3 Zariski Groups in Non-Locally Modular Zariski Geometries

Let G be a Zariski group interpretable in a non-locally modular Zariski geometry. Here I will make an analysis of the specialisations of such Z-groups of the following types:

1. G is a definably simple non-abelian group.

2. G is an abelian group with no definable infinite subgroups.

This analysis comes as a consequence of the Classification Theorem and the results of the previous sections of this chapter.

It is reasonable to restrict to these cases as the binding groups in the Zilber's Ladder Theorem are exactly of these two types. If one considers an uncountably categorical Zariski geometry, the binding groups in the Ladder Theorem becomes Zariski groups.

As a result of the Classification Theorem, Zilber proves that Z-groups satisfy the algebraicity conjecture. More precisely the following theorem:

Fact 3.2.16 (Theorem 4.4.6 in [44])**.** *A definably simple Z-group G with (EU) such that some one dimensional irreducible Z-subset C in G is pre-smooth is Zariski isomorphic to an algebraic group $\hat{G}(K)$ for some algebraically closed field K.*

Therefore, as G can be taken to be an algebraic group, the analysis carried out in previous section (in particular Remark 3.2.15) gives the following result:

Proposition 3.2.17. *Let G be a definably simple, Z-group. Then G is (isomorphic to) an algebraic group $G(K)$. Let $Res : G(K) \to G(k)$ be the specialisation induced by $res : K \to k$. Let $U_0, \ldots, U_n$ be the affine pieces of G. Then there are maximal specialisations $\pi_{iK} : K \to k$ for each of the affine pieces such that $Res(z_0 : \ldots : z_n) = (z_0^{\pi_{iK}} : \ldots : z_n^{\pi_{iK}})$ for $(z_0 : \ldots : z_n) \in \mathrm{Dom}(Res) \cap U_i(K)$. In particular $\pi_{iK} = res$ for all i.*

Chapter 4

Specialisations of Finite Fibrations of Zariski Structures

In this chapter we start with considering finite fibrations of Zariski structures. In particular, we will construct a fibration C_0 of $\mathbb{P}^1(k)$ such that a given κ-universal specialisation $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$ can be lifted to a specialisation $\pi_C : C \to C_0$ which is not κ-universal. This will also show that maximality of a specialisation does not need to imply universality.

The main motivation for considering finite fibrations is the Classification Theorem of one dimensional pre-smooth Zariski structures given by Hrushovski and Zilber in [16] (see Theorem 4.0.3 below). Where they show that any pre-smooth, ample Zariski structure is a finite fibration of $\mathbb{P}^1(K)$ for some algebraically closed K.

In the last section, we turn our attention to a specific case. The Zariski structures we will consider in this section were constructed in the processes of producing the first example of a non-algebraic Zariski geometry in [16, sect. 10]. We show that universality of the specialisations of these structures are controlled by the universality of the specialisations on the base Zariski structure. This section can be seen as a prelude to Chapter 5. Where we aim to extend the discussions of this section, and particularly Theorem 4.4.7 to Zariski cover structures. Which is achieved under certain additional assumptions.

Definition 4.0.1. Let M_0 be a Zariski structure, and C_0 a topological structure. Then a map $\mathrm{pr} : C_0 \to M_0$ is called a finite fibration of M_0 when the following conditions are satisfied:

(i) $\mathrm{pr} : C_0 \to M_0$ is surjective and continuous, and $|\mathrm{pr}^{-1}(x)|$ is finite for all $x \in M_0$.

(ii) There is a $\emptyset$-definable equivalence relation $\sim$ on C_0 such that the fibres of pr are $\sim$-equivalence classes.

(iii) The two sorted structure (C_0, M_0, pr) does not induce new structure on M_0. I.e. the $\emptyset$-definable subsets of M_0^t for all t in the two sorted structure are $\emptyset$-definable in the structure M_0.

When it is clear from the context we will often omit pr and say that C_0 is (finite) fibration of M_0.

Definition 4.0.2. Let C be an one dimensional Zariski Geometry. A *plane curve* over C is an irreducible one dimensional subset of C^2. A *family of plane curves* is a closed irreducible subset $X \subseteq T \times C^2$, where T is a closed irreducible subset of C^n, such that $X(t)$ is plane curve for $t \in T$. A one dimensional Zariski geometry is said to be *ample* if there is a family $X \subseteq T \times C^2$ of plane curves such that for generic $a, b \in C^2$ there is a $t \in T$ with $a, b \in X(t)$ (i.e. $X(t)$ passes through a, b).

The following theorem is one of the main results in [16], and it is the main motivation to consider specialisations of finite fibrations of Zariski structures. A more general version of this result in a broader context is Theorem 4.4.1. in [44].

Theorem 4.0.3. *(Theorem B in [16]) Let C be a pre-smooth, irreducible and ample Zariski geometry. Then there is a surjective, finite (i.e. with finite fibres) Zariski map* $\mathrm{pr} : C \to \mathbb{P}^1(K)$ *for an algebraically closed field K; further* pr *maps constructible sets to constructible sets. Moreover, there is a finite subset $F \subseteq C$ such that* pr *is closed on $C \setminus F$.*

Then, one sees that C interprets $\mathbb{P}^1(K)$ and hence the algebraically closed field K. So, $\mathbb{P}^1(K) \subseteq C^{eq}$. In the following results, the notions "algebraic over", and "acl" are to be considered in C^{eq}. C is prime over $\mathbb{P}^1(K)$ has a slightly different meaning than the usual notion; it means that whenever $C' \models Th(C)$ such that C' interprets, (and hence C'^{eq} contains) an isomorphic copy of $\mathbb{P}^1(K)$, then C embeds elementarily into C'.

Lemma 4.0.4. *Let C be an ample Zariski geometry, and* $\mathrm{pr} : C \to \mathbb{P}^1(K)$ *be the finite Zariski map in Theorem 4.0.3. Then C is algebraic over $\mathbb{P}^1(K)$; hence it is also prime over $\mathbb{P}^1(K)$.*

Proof. Let E_{pr} denote the inverse image of the graph of $=$ on $\mathbb{P}^1(K)$. Then E_{pr} is a definable (in fact closed) equivalence relation on C. Moreover each equivalence class is finite. Hence, each equivalence class of E_{pr} is algebraic over $\mathbb{P}^1(K)$. Therefore C, being the union of E_{pr}-equivalence classes, is algebraic over $\mathbb{P}^1(K)$.

Suppose $B \models Th(C)$, and $\mathbb{P}^1(K)$ embeds elementarily into B/E_{pr}. Then $acl(\mathbb{P}^1(K))$ embeds elementarily into $acl(B/E_{\mathrm{pr}})$. Hence, C embeds elementarily into B.

$\square$

Let C_0 be an ample Zariski Geometry, $C \succ C_0$, and $\pi : C \to C_0$ a specialisation. By Theorem 4.0.3 there are finite Zariski maps $\mathrm{pr} : C \to \mathbb{P}^1(K)$ and $\mathrm{pr}_0 : C_0 \to \mathbb{P}^1(k)$. There is also the specialisation $Res : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$ induced by the residue map from K to k. Moreover we know that there is a κ-universal specialisation $\pi' : C' \to C_0$. Similarly there are finite Zariski maps $\mathrm{pr}' : C' \to \mathbb{P}^1(K')$ and $\mathrm{pr}_0 : C_0 \to \mathbb{P}^1(k)$; and $Res' : \mathbb{P}^1(K') \to \mathbb{P}^1(k)$ which is induced by the residue map between K' and k. In this case we have the following pictures:

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[d, "\mathrm{pr}"] & C_0 \arrow[d, "\mathrm{pr}_0"] \\ \mathbb{P}^1(K) \arrow[r, "Res"] & \mathbb{P}^1(k) \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "\pi'"] \arrow[d, "\mathrm{pr}'"] & C_0 \arrow[d, "\mathrm{pr}_0"] \\ \mathbb{P}^1(K') \arrow[r, "Res'"] & \mathbb{P}^1(k) \end{tikzcd}$$

One may aim to show that essentially two pictures can be identified if $Res : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$ is κ-universal; proving $\pi : C \to C_0$ is κ-universal. By Lemma 4.0.4 there is an embedding σ of C into C'. Moreover, we know that σ extends to an embedding of C^{eq} into C'^{eq}, we will again denote this extension by σ. One can see that for any two elements of C which are in the same fibre of pr, their images under σ are in the same fibre of C'. Furthermore, for any $a \in \mathbb{P}^1(K)$, the fibre over a in C maps to the fibre over σa in C'.

Consider the special case $K = K'$ and $Res = Res'$. Under this assumption, by Lemma 4.0.4, C and C' are isomorphic as Zariski geometries (Hence are C^{eq} and C'^{eq}). Again, call this isomorphism σ. Remark that σ is an isomorphism of Zariski geometries. However, it does not necessarily preserve specialisations.

In the next section, I will consider finite fibrations of Zariski structures and in section 4.2 I will give an example of a specialisation $\pi_C : C \to C_0$ where C and C_0 are fibrations of $\mathbb{P}^1(K)$ and $\mathbb{P}^1(k)$ respectively such that π_C is not κ-universal but it is a lifting of a κ-universal specialisation $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$.

4.1 Finite Fibrations of Zariski Structures

Let M_0 be a Zariski structure, C_0 a non-empty set and $\mathrm{pr} : C_0 \to M_0$ be an n to one function (i.e. for each $x \in M_0$ the fibre $\mathrm{pr}^{-1}(x)$ has n elements). The topology of M_0 can be lifted to C_0 as follows:

Say R is a closed relation of M_0, we will lift R to a relation R_{pr} of C_0 by declaring $C_0 \models R_{\mathrm{pr}}(\overline{\alpha})$ if and only if $M_0 \models R(\mathrm{pr}(\overline{\alpha}))$. In other words, for any closed relation R of M_0, the relation R_{pr} is its pull-back via pr. Hence, C_0 becomes a structure of the language whose basic relations are R_{pr} for each closed relation R of M_0; denote this language by $\mathcal{L}_{\mathrm{pr}}$. Next, declare the sets definable by positive quantifier free formulas with parameters to be closed. Closed sets can be described more explicitly.

Lemma 4.1.1. *Let $\mathcal{F}_t = \{\mathrm{pr}^{-1}(S) : S \subseteq M_0^t$ is closed$\}$. Then $\mathcal{F}_t$ is closed under finite unions and finite intersections and has the descending chain property.*

Proof. It is clear that $\mathcal{F}_t$ is closed under finite unions and finite intersections. Let $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_l \supseteq \ldots$ be a descending chain of elements of $\mathcal{F}_t$. Then $\mathrm{pr}(F_1) \supseteq \mathrm{pr}(F_2) \supseteq \ldots \supseteq \mathrm{pr}(F_l) \supseteq \ldots$ is a descending chain of closed subsets of M_0^t. Since M_0^t is Noetherian, it stabilizes. Then the chain $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_l \supseteq \ldots$ must stabilize as well since they are pre-images.

□

In particular the topology given by $\mathcal{F}_1$ on C_0 is known as the *weak topology with respect to* pr. However, note that the collection $\mathcal{F}_t$ in the above lemma does not include many closed sets; most importantly the graph of $=$ (i.e. diagonals). Let $\Delta(x, y)$ be the graph of $=$.

Lemma 4.1.2. *Let $\mathcal{F}_t^{\Delta}$ be the collection of subsets of C_0^t defined by the formulas*

$$F_i(x_1, \ldots, x_t) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j)$$

for some $F_i \in \mathcal{F}_t$ and some $k \leq t$ (note that k can be 0). Also let $\mathcal{F}_t^{\Delta}$ include the formulas obtained by permuting the coordinates of the formulas described above. Then $\mathcal{F}_t^{\Delta}$ has the descending chain condition. Moreover $\mathcal{F}_t^{\Delta}$ is closed under arbitrary intersections.

Proof. Let $T_1 \supseteq T_2 \supseteq \ldots$ be a descending chain of elements of $\mathcal{F}_t^{\Delta}$. Then each T_i can be written as

$$T_i(x_1, \ldots, x_t) = F_i(x_1, \ldots, x_t) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j)$$

We may assume that if $T_i \supseteq T_j$, then $F_i \supseteq F_j$. Clearly, since $T_j \subseteq F_i$ we have $F_i \wedge T_j = T_j$. Therefore by replacing F_i with $F_i \wedge F_j$ in T_j we may assume $F_i \supseteq F_j$. Therefore we have a descending chain $F_1 \supseteq F_2 \supseteq \ldots$ of elements of $\mathcal{F}_t$. But we know that $\mathcal{F}_t$ has the descending chain condition. So there is some m such that for any $\nu > m$, we have $F_\nu = F_m$. Hence for any $\nu > m$ the set T_ν is of the following form

$$T_\nu(x_1, \ldots, x_t) = F_\nu(x_1, \ldots, x_t) \wedge \bigwedge_{i<j\leq k_\nu} \Delta(x_i, x_j)$$

Next, observe that one can have finitely many non-empty sets of this form. Hence the descending chain $T_1 \supseteq T_2 \supseteq \ldots$ must stabilize.

For the second part, it is clear by construction that $\mathcal{F}_t^\Delta$ is closed under finite intersections. Then, since $\mathcal{F}_t^\Delta$ has the descending chain condition, any infinite intersection reduces to a finite one. $\square$

For any t, a closed subset of C_0^t can be written as a finite disjunction of the form

$$S_1(\overline{a}_1, x_1, \ldots, x_t) \vee \ldots \vee S_l(\overline{a}_l, x_1, \ldots, x_t)$$

where $\overline{a}_i$ are tuples (possibly of different lengths) from C_0 and $S_i \in \mathcal{F}_{t+k_i}^\Delta$ for all i and k_i is the length of the tuple $\overline{a}_i$.

By Lemma 4.1.2 this is a Noetherian topology on each C_0^t. Moreover pr is continuous, open and closed. Also observe that M_0 is (homeomorphic to) the quotient topology with respect the map pr.

Proposition 4.1.3. *Any $\emptyset$-definable subset of C_0^t projects to an $\emptyset$-definable subset of M_0^t.*

Lemma 4.1.4. *Let $S \subseteq C_0^t$ be a closed irreducible set. Then*

$$S(x_1, \ldots, x_t) = T_{\mathrm{pr}}(x_{\sigma(1)}, \ldots, x_{\sigma(t)}) \wedge \bigwedge_{i<j\leq k} \Delta(x_{\sigma(i)}, x_{\sigma(j)})$$

where $\sigma \in Sym(t)$ a permutation of the coordinates, $T \subseteq M_0^t$ is a closed irreducible subset, and T_{pr} is its pull back under pr*.*

Proof. Let S be a closed irreducible subset of C_0^t. Then S is a finite union of some elements of $\mathcal{F}_t^\Delta$. In fact, since S is irreducible it must be of the form

$$S(x_1, \ldots, x_t) = T_{\mathrm{pr}}(x_{\sigma(1)}, \ldots, x_{\sigma(t)}) \wedge \bigwedge_{i<j\leq k} \Delta(x_{\sigma(i)}, x_{\sigma(j)})$$

for some closed $T \subseteq M_0^t$ with the inverse image T_{pr} under pr.

Next, we will show that T is irreducible. Suppose not. Then $T = F \cup Q$ for some closed subsets $F, Q \subset M_0^t$ such that $F \not\subseteq Q$ and $Q \not\subseteq F$.

Therefore

$$\begin{aligned}
S(x_1,\ldots,x_t) &= [F(x_{\sigma(1)},\ldots,x_{\sigma(t)}) \vee Q(x_{\sigma(1)},\ldots,x_{\sigma(t)})] \wedge \bigwedge_{i<j\leq k_1} \Delta(x_{\sigma(i)},x_{\sigma(j)}) \\
&= [F(x_{\sigma(1)},\ldots,x_{\sigma(t)}) \wedge \bigwedge_{i<j\leq k_1} \Delta(x_{\sigma(i)},x_{\sigma(j)})] \vee [Q(x_{\sigma(1)},\ldots,x_{\sigma(t)}) \wedge \\
&\quad \bigwedge_{i<j\leq k_1} \Delta(x_{\sigma(i)},x_{\sigma(j)})]
\end{aligned}$$

Which contradicts with the assumption that S irreducible. Therefore T must be irreducible. $\square$

Lemma 4.1.5. *Theory of C_0 admits quantifier elimination in the language $\mathcal{L}_{\mathrm{pr}}$.*

Proof. Consider a formula $\varphi(\overline{y}) : \exists x\Theta(x,\overline{y})$ where $\Theta(x,\overline{y})$ is a conjunction of atomic and negated formulas. We may assume that $\Theta(x,\overline{y})$ is a conjunction of formulas from $\mathcal{F}_t^{\Delta} \cup \neg\mathcal{F}_t^{\Delta}$ (where $\neg\mathcal{F}_t^{\Delta} = \{\neg\varphi : \varphi \in \mathcal{F}_t^{\Delta}\}$).

Since $\mathcal{F}_t^{\Delta}$ is closed under finite conjunctions, we may assume that only one of the conjuncts of $\varphi(\overline{y})$ is from $\mathcal{F}_t^{\Delta}$ and all of the other conjuncts are elements of $\neg\mathcal{F}_t^{\Delta}$.

It is enough to prove that we can eliminate the existential quantifiers in the formulas of the form

$$\varphi(\overline{x}) : \exists x_l(F(x_1,\ldots,x_t) \wedge \bigwedge_{i<j\leq k} \Delta(x_i,x_j))$$

and

$$\psi(\overline{x}) : \exists x_l\neg(F(x_1,\ldots,x_t) \wedge \bigwedge_{i<j\leq k} \Delta(x_i,x_j))$$

where $\overline{x}$ is the tuple $(x_1,\ldots,\hat{x}_l,\ldots,x_t)$ (where $\hat{x}_l$ means that the coordinate x_l is omitted from the tuple).

In the first case, let us first assume that $\bigwedge_{i<j\leq k} \Delta(x_i,x_j)$ does not involve the quantified variable x_l. Then it is enough to show that the formula $\exists x_l(F(x_1,\ldots,x_t))$ is equivalent to a quantifier free formula. Let R denote the set defined by the formula $\exists x_l(F(x_1,\ldots,x_t))$. Observe that $\mathrm{pr}(R)$ is defined by the formula $\exists z_l(S(z_1,\ldots,x_k)$ where S is the closed relation on M_0^n which lifts to F (i.e. $F = S_{\mathrm{pr}}$ as defined at the beginning of this subsection). Since M_0 eliminates quantifiers it is clear that $\exists z_l S(z_1,\ldots,z_t)$ is equivalent to some quantifier free $S'(z_1,\ldots,z_t)$. Therefore, $\exists x_l(F(x_1,\ldots,x_t))$ is equivalent to $S'_{\mathrm{pr}}(x_1,\ldots,\hat{x}_l,\ldots,x_n)$. Now it follows that φ is equivalent to $S'_{\mathrm{pr}}(x_1,\ldots,\hat{x}_l,\ldots,x_n) \wedge \bigwedge_{i<j\leq k} \Delta(x_i,x_j)$.

Now, suppose that the part $\bigwedge_{i<j\leq k} \Delta(x_i, x_j)$ involves x_l. In this case the formula $x_l = x_j$ for some $j > k$ appears in the formula φ. Therefore φ is equivalent to $(F(x_1, \ldots, x_j \ldots, x_j, \ldots, x_n) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j))$.

For the second case, let us rewrite $\psi(\overline{x})$ as $\neg\forall x_l(F(x_1, \ldots, x_t) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j))$. We may assume that the part $\bigwedge_{i<j\leq k} \Delta(x_i, x_j)$ does not involve the quantified variable x_l. Therefore, it is enough to show that the formula $\forall x_l(F(x_1, \ldots, x_t))$ is equivalent to a quantifier free formula.

Let T denote the set defined by $\forall x_l(F(x_1, \ldots, x_t))$. Observe that the set $\mathrm{pr}(T)$ is defined by the formula $\forall z_l(Q(z_1, \ldots, z_t))$ where Q is the closed relation on M_0^t which lifts to F. I.e. $F = Q_{\mathrm{pr}}$ as defined in the beginning of this section. Since we can eliminate quantifiers in M_0, the formula $\forall z_l(Q(z_1, \ldots, z_t))$ is equivalent to some quantifier free formula $Q'(z_1, \ldots, z_t)$. It follows that $\forall x_l(F(x_1, \ldots, x_t)$ is equivalent to $Q'_{\mathrm{pr}}(x_1, \ldots, x_t)$. Therefore the formula $\psi(\overline{x})$ is equivalent to the quantifier free formula $Q'_{\mathrm{pr}}(x_1, \ldots, x_t) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j))$. □

Let $\mathcal{F}^{\Delta}$ be the collection of sets which are definable by positive quantifier free $\mathcal{L}_{\mathrm{pr}}$-formulas.

Lemma 4.1.6. $(C_0, \mathcal{F}^{\Delta})$ *is a Noetherian topological structure.*

Proof. The topological axioms 1-5 and 7 follow immediately. To show Axiom 6 holds: Let $T \subseteq C_0^k$ and $R \subseteq C_0^l$ be basic closed sets (i.e. $T \in \mathcal{F}_k^{\Delta}$ and $R \in \mathcal{F}_l^{\Delta}$), then there are closed $F_1 \in \mathcal{F}_k$ and, $F_2 \in \mathcal{F}_l$ such that $T(x_1, \ldots, x_k) = F_1(x_1, \ldots, x_k) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j)$ and $R(y_1, \ldots, y_l) = F_2(y_1, \ldots, y_l) \wedge \bigwedge_{i<j\leq l} \Delta(y_i, y_j)$. Then, clearly

$$(T \times R)(\overline{x}, \overline{y}) = (F_1(\overline{x}) \times F_2(\overline{y})) \wedge \bigwedge_{i<j\leq k} \Delta(x_i, x_j) \wedge \bigwedge_{i<j\leq l} \Delta(y_i, y_j)$$

Since $F_1(\overline{x}) \times F_2(\overline{y}) \in \mathcal{F}_{k+l}$, clearly $T \times R \in \mathcal{F}_{k+l}^{\Delta}$

Next, let $T \subseteq C_0^k$ and $R \subseteq C_0^l$ be arbitrary closed sets. So, $T = T_1 \cup \ldots \cup T_t$ and $R = R_1 \cup \ldots \cup R_r$ for some $T_1, \ldots, T_t \in \mathcal{F}_k^{\Delta}$ and $R_1, \ldots, R_r \in \mathcal{F}_l^{\Delta}$. Then, $T \times R = \bigcup_{i,j=1}^{t,r} (T_i \times R_j)$, where by the previous case, $T_i \times R_j \in \mathcal{F}_{k+l}^{\Delta}$. Hence $T \times R \subset C_0^{k+l}$ is closed.

Axiom 8 is also straightforward: Let $T \subseteq C_0^{k+l}$ be closed and $c \in C_0^k$. Then $S := \mathrm{pr}(T) \subseteq M_0^{k+l}$ is closed, and $S(\mathrm{pr}(c), M_0^l)$ is closed. Then by the construction of the topology on C_0^k, the set $T(c', C_0^l)$ is closed for any $c' \in \mathrm{pr}^{-1}(\mathrm{pr}(c))$. Hence $T(c, C_0^l)$ is closed.

The descending chain property follows from Lemma 4.1.2.

□

By Lemma 4.1.2 and Lemma 4.1.6, C_0 together with the collection of sets which are definable by positive quantifier free $\mathcal{L}_{\mathrm{pr}}$-formulas with parameters as closed sets is a Noetherian topological structure.

One can consider M_0 as the quotient C_0/E_{pr} where E_{pr} is the inverse image of $=$ under pr; hence a closed relation. One can consider the quotient topology on each Cartesian power $M_0^k = (C_0/E_{\mathrm{pr}})^k$. So M_0 is a topological sort in C_0. Moreover, by the construction of C_0, the quotient topologies on Cartesian powers of M_0 are exactly the topologies coming from the Zariski structure on M_0.

Also observe that the structure $(C_0, \mathcal{F}^{\Delta})$ is definable in (M_0, τ) where τ is the union of Noetherian topologies on all Cartesian powers of M_0: Let $X = M_0 \times \{m_0, \dots m_{n-1}\}$ for some $m_0, \dots m_{n-1} \in M_0$. Then X is definable in M_0. For any relation $R_{\mathrm{pr}} \subseteq C_0^K$ of C_0 consider the definable relation $R_X := \{(\overline{x_1}, \dots, \overline{x_k}) \in X^k : (x_{11}, x_{21}, \dots, x_{k1}) \in R\}$ where x_{i1} is the first coordinate of the tuple $\overline{x_i}$ for $i = 1, \dots, k$. Clearly R_X is a definable (in M_0) relation of X. Moreover the structures $(C_0, \mathcal{F}^{\Delta})$ and (X, τ_X) are isomorphic where τ_X is the collection of set o the form R_X as defined above.

4.2 A Family of Counterexamples

Let $M_0 = \mathbb{P}^1(k)$ for some algebraically closed field k. Let C_0 be a non-empty set and $\mathrm{pr} : C_0 \to M_0$ be a finite to one map such that C_0 is in n to one correspondence with $\mathbb{P}^1(k)$ via pr. The topological structure on $\mathbb{P}^1(k)$ can be lifted to C_0 as discussed in the previous section. So $\mathrm{pr} : C_0 \to M_0$ becomes a finite fibration (with fibres of cardinality n).

Let C be an elementary extension of C_0. Then C can also be made a topological structure. Let E_{pr} denote the inverse image of $=$ on $\mathbb{P}^1(k)$. It is a closed equivalence relation on C_0, hence it is also closed on C. It is easy to see that each E_{pr} class in M has n points. Hence the quotient M/E_{pr} will have the structure of $\mathbb{P}^1(K)$ with its Zariski topology for some $K \succeq k$, via an extension of pr to C, which I will call pr again abusing the notation. I.e. there is an algebraically closed field $K \succeq k$ such that C is in n to one correspondence with $\mathbb{P}^1(K)$ via (an extension) of the Zariski map pr.

Consider a specialisation $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$. By definition π is identity on $\mathbb{P}^1(k)$. Let $\pi_C : C \to C_0$ be a specialisation which is a lifting of π, then π_C must be identity on C_0 again by the definition of specialisation.

Consider a fibre $\overline{c}$ in C, i.e. $\overline{c} = \mathrm{pr}^{-1}(x)$ for some $x \in \mathbb{P}^1(K)$. Assume $x \notin \mathbb{P}^1(k)$ and $x \in \mathrm{Dom}(\pi)$. Since π_C is lifting π, it must map this fibre to the fibre over $\pi(x)$, say $\mathrm{pr}^{-1}(\pi(x)) = \overline{c_0}$. This could be possible in the following two ways:

1. $\pi_C(\overline{c}) \subsetneq \overline{c_0}$. I.e π_C is not a bijection. In this case I will say π_C *degenerates* the fibre $\overline{c}$.

2. $\pi_C(\overline{c}) = \overline{c_0}$. I.e. π_C is a bijection on the fibre $\overline{c}$.

Proposition 4.2.1. *Let $\pi_C : C \to C_0$ be a specialisation. Assume that on any fibre in $C \setminus C_0$, π_C is not bijective (i.e. π_C satisfies first condition on all fibres in $C \setminus C_0$). Then π_C cannot be κ-universal.*

Proof. Let $\mathbb{P}^1(K') \succeq \mathbb{P}^1(K)$ and $\mathrm{pr} : C' \to \mathbb{P}^1(K')$ be a fibration with fibres of size n of $\mathbb{P}^1(K')$. Then C' can be seen as an $\mathcal{L}_{\mathrm{pr}}$-structure. Moreover it is a Noetherian topological structure, and $C' \succeq C$.

Let $\pi' : \mathbb{P}^1(K') \to \mathbb{P}^1(k)$ be a specialisation extending $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$. Lift π' to a specialisation $\pi_{C'} : C' \to C_0$ as follows: On C, define $\pi_{C'} = \pi_C$ so that it extends π_C. On $C' \setminus C$ define $\pi_{C'}$, as explained above, by lifting the specialisation $\pi' : \mathbb{P}^1(K') \to \mathbb{P}^1(k)$ to the fibres in C' such that for a fibre $\overline{c} \subset C' \setminus C$, we have $\pi_{C'}(c_i) \neq \pi_{C'}(c_j)$ for any distinct $c_i, c_j \in \overline{c}$ (I.e. $\pi_{C'}$ is a bijection on every fibre in $C' \setminus C$).

Now take any subset A of C' of cardinality smaller than κ. Then it is clear that there is no embedding of $A \setminus (A \cap C)$ into C which respects π_C as any embedding also needs to preserve fibres. $\square$

One can follow the second possibility described above to construct a similar counter example:

Proposition 4.2.2. *Let $\pi_C : C \to C_0$ be a specialization lifting $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$. Suppose that on any fibre over any element of* $\mathrm{Dom}(\pi)$*, the specialisation π_C is a bijection. Then π_C is not κ-universal.*

It is important to note that the counterexamples considered here shows that the condition $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$ is maximally defined is not enough to achieve universality. In the examples here maximally defined will coincide with $\pi : \mathbb{P}^1(K) \to \mathbb{P}^1(k)$

is a total specialisation. It is clear from above that even under this assumption one still gets the counterexamples as constructed above.

Also note that as remarked in the previous section the structure C_0 is definable in $\mathbb{P}^1(k)$; hence it is interpretable in ACF. On the other hand C_0 is not bi-interpretable with $\mathbb{P}^1(k)$: Let $\mathrm{Aut}(\mathbb{P}^1(k))$ be the automorphism group of $\mathbb{P}^1(k)$ (as a topological structure). Then one can compute $\mathrm{Aut}(C_0)$ to be $\mathbb{Z}/n\mathbb{Z}\,\mathrm{Wr}_W\,\mathrm{Aut}(\mathbb{P}^1(k))$. Clearly these two groups are not homeomorphic, which implies C_0 and M_0 are not bi-interpretable.

4.3 Additional Topological Structure

Remark that in the construction above there are no relations between the points a given fibre. But in many natural examples of Zariski geometries there is a structure on the fibres. Particularly in all examples of Zariski geometries constructed in [16, 46, 35] there is a finite group H acting transitively on each fibre of C_0. I will show that for the examples given in [16] counterexample above cannot be constructed, but before I will consider a simpler situation.

4.3.1 Restricted Inequality Predicate

4.3.1.1 One Sorted Construction

To the language $\mathcal{L}_{\mathrm{pr}}$ we add the definable predicate

$$\mathfrak{I}(x, y) : x \neq y \wedge x E_{\mathrm{pr}} y$$

This predicate will be called the restricted inequality, Let $\mathcal{L}_{\mathrm{ineq}} = \mathcal{L}_{\mathrm{pr}} \cup \{\mathfrak{I}(x, y)\}$. Then one defines a richer topology on each Cartesian power of C_0: Declare the sets defined by positive quantifier free formulas of $\mathcal{L}_{\mathrm{ineq}}$ to be $\mathcal{L}_{\mathrm{ineq}}$-closed.

Lemma 4.3.1. *C_0 is a Noetherian $\mathcal{L}_{ineq}$-topological structure.*

Proof. For an $\mathcal{L}_{\mathrm{pr}}$-closed subset S of C_0^n consider

$$S(x_1, \cdots, x_n) \wedge \bigwedge_{i<j\leq k} \mathfrak{I}(x_i, x_j) \wedge \bigwedge_{t<l} \mathfrak{I}(x_t, a_t)$$

for some $k, l \in \mathbb{N}$. Since we are keeping the $\mathcal{L}_{\mathrm{pr}}$-closed sets, k and l can be zero.

Denote the collection of sets of this form by $\mathfrak{F}_n$ for each n.

Claim 1. The collection $\mathfrak{F}_n$ has the descending chain condition for all n.

Proof. Let $F_1 \supseteq F_2 \supseteq \cdots$ be a descending chain. Each F_i is of the form

$$F_i(x_1, ...x_n) = S_i(x_1, ..., x_n) \wedge \bigwedge_{i<j\leq k} \mathfrak{I}(x_i, x_j) \wedge \bigwedge_{t<l} \mathfrak{I}(x_t, a_t)$$

for some $a_i \in C_0$.

Observe that, if $F_i \supseteq F_j$ we can assume $S_i \supseteq S_j$. Indeed, since $F_j \subseteq S_i$ we have $S_i \wedge F_j = F_j$. Hence we may assume that if $F_i \supseteq F_j$ we have $S_i \supseteq S_j$ by replacing F_j with $S_i \wedge F_j$ if necessary.

Therefore $S_1 \supseteq S_2 \supseteq ...$ is a descending chain. Since $\mathcal{L}_{\text{pr}}$-topology is Noetherian, there is some m such that for any $\nu > m$, we have $S_\nu = S_m$. So, for $\nu > m$ the sets F_ν are of the form

$$S_m(x_1, ..., x_n) \wedge \bigwedge_{i<j\leq k_\nu} \mathfrak{I}(x_i, x_j) \wedge \bigwedge_{t<l_\nu} \mathfrak{I}(x_i, a_i)$$

Observe that one can only have finitely many non-empty sets of this form. Hence, the chain $F_1 \supseteq F_2 \supseteq \cdots$ stabilizes. □

It is clear that, by construction, the family $\mathfrak{F}_n$ is closed under finite intersections; and since $\mathfrak{F}_n$ has (DCC), it follows it is closed under intersections. Moreover, as the closed subsets of C_0^n are the sets defined (with parameters) by positive boolean combinations of predicates in $\mathcal{L}_{\text{ineq}}$, they can be written as finite unions of sets form $\mathfrak{F}_n$ and singletons. Hence the $\mathcal{L}_{\text{ineq}}$-closed sets give a Noetherian topology on C_0^n.

The topological axioms easily follow from the characterization of closed sets given above. Remark that the predicate $\mathfrak{I}$ is definable in the language $\mathcal{L}_{\text{pr}}$, and we have quantifier elimination in $\mathcal{L}_{\text{pr}}$, hence we also have quantifier elimination in the expanded language $\mathcal{L}_{\text{ineq}}$.

□

Lemma 4.3.1 shows that when we enrich the structure of C_0 with the restricted inequality predicate, it still will be a Noetherian topological structure (with a richer topology). With the restricted inequality predicate the specialisations of C_0 have the following property:

Proposition 4.3.2. *Let $C \succeq C_0$ and $\pi : C \to C_0$ a specialisation. Then π is a bijection on* $\text{pr}^{-1}(m)$*.*

Proof. Let $c, c' \in \text{pr}^{-1}(m)$. Suppose $\pi(c) = \pi(c')$ but $c \neq c'$. Then $\mathfrak{I}(c, c')$. Since $\mathfrak{I}$ is closed, $\mathfrak{I}(\pi(c), \pi(c'))$. Which implies $\pi(c) \neq \pi(c')$. A contradiction. □

Even if we consider C_0 as an $\mathcal{L}_{\mathrm{pr}}$-topological structure, a maximal specialisation $\pi_C : C \to C_0$ must satisfy the following property: Let $\pi : M \to M_0$ be the projection of π_C, then for an $m \in \mathrm{Dom}(\pi)$ we have $\mathrm{pr}^{-1}(m) \subseteq \mathrm{Dom}(\pi_C)$.

When C_0 is enriched with the restricted inequality predicate, and considered as an $\mathcal{L}_{\mathrm{ineq}}$-topological structure, the observation above with Proposition 4.3.2 gives the following: For a maximal specialisation $\pi_C : C \to C_0$ and its projection $\pi : M \to M_0$, if $\pi(m) = m_0$, then π_C maps the fibre $\mathrm{pr}^{-1}(m)$ to the fibre $\mathrm{pr}^{-1}(m_0)$ bijectively.

Theorem 4.3.3. *Consider C_0 as an $\mathcal{L}_{ineq}$-topological structure. If $\pi_C : C \to C_0$ is a maximal specialisation which projects to a specialisation $\pi : M \to M_0$ then π_C is κ-universal if π is.*

Proof. Let $C' \succeq C$, and $A \subseteq C$ with $|A| < \kappa$ and $\pi_A : C \cup A \to C_0$ be a specialisation extending π_C. Then its projection $\pi' : \mathrm{pr}(A) \cup M \to M_0$ is a specialisation extending π. Since π is κ-universal, there is an embedding $\sigma : \mathrm{pr}(A) \to M$ over $\mathrm{pr}(A) \cap M$ such that $\pi \circ \sigma = \pi'$ on A.

One can lift σ to an embedding $\sigma_A : A \to C$ as follows: First observe that one can assume A contains full fibres. I.e. one can assume that if $\mathrm{pr}^{-1}(m) \cap A \neq \emptyset$ then $\mathrm{pr}^{-1}(m) \subseteq A$. Therefore, one can also observe that $\mathrm{pr}(A) \setminus (\mathrm{pr}(A) \cap M) = \mathrm{pr}(A \setminus (A \cap C))$; call this set Y. Enumerate its image under π' as $\pi'(Y) = \{y_i : i < \kappa\}$. Then, $\{Y \cap \mathcal{V}_{y_i} : y_i \in \pi'(Y)\}$ is a partition of Y.

Let $D = A \setminus (A \cap C)$, and $D_0 = A \cap C$. On D_0 define $\sigma_0 : D_0 \to C$ as *id*. Let $D_i = D \cap \mathrm{pr}^{-1}(\mathcal{V}^{\pi'}_{y_i})$. Let $x \in \mathrm{pr}(D) \cap \mathcal{V}_{y_i} = Y \cap \mathcal{V}_{y_i}$. Since π is κ-universal, $\pi'(x) = \pi(\sigma(x))$. Moreover the fibre $\mathrm{pr}^{-1}(\pi(\sigma(x)))$ is the bijective image of $\mathrm{pr}^{-1}(x)$ under π_A by Proposition 4.3.2, and of $\mathrm{pr}^{-1}(\sigma(x))$ under π_C. So $\mathrm{pr}^{-1}(\sigma(x))$ needs to be the bijective image of $\mathrm{pr}^{-1}(x)$ under σ_i.

Consider $\mathrm{pr}^{-1}(\pi(\sigma(x)))$ as an ordered tuple, denoted by $\overline{c_0}$. Since π_A maps the fibre $\mathrm{pr}^{-1}(x)$ to $\mathrm{pr}^{-1}(\pi(\sigma(x)))$, one can consider $\mathrm{pr}^{-1}(x)$ as an ordered tuple $\overline{c'} \in \mathcal{V}^{\pi_A}_{\overline{c_0}}$. Similarly $\mathrm{pr}^{-1}(\sigma(x))$ can be considered as an ordered tuple $\overline{c} \in \mathcal{V}^{\pi_C}_{\overline{c_0}}$.

Define $\sigma_i(\overline{c'}) = \overline{c}$. Repeating this process for all $x \in \mathrm{pr}(D) \cap \mathcal{V}^{\pi'}_{y_i}$ one can define $\sigma_i : D_i \to C$. Hence $\sigma_A := \bigcup_i \sigma_i : A \to C$ is an embedding over $A \cap C$ such that $\pi_A = \pi_C \circ \sigma_A$. $\square$

One can summarize the result achieved in this subsection in the following corollary of Lemma 4.3.1 and Theorem 4.3.3.

Corollary 4.3.4. *Let M_0 be an $\mathcal{L}$-Zariski structure, and C_0 be a finite fibration of M_0 with the $\mathcal{L}_{\mathrm{pr}}$-topological structure. Then if $\mathcal{L}_{ineq}$ is the definitional expansion of $\mathcal{L}_{\mathrm{pr}}$*

with the (definable) restricted inequality predicate, C_0 can be seen as a $\mathcal{L}_{ineq}$-topological structure. Moreover, if $C \succeq C_0$ are $\mathcal{L}_{\mathrm{pr}}$-topological structures and $\pi_C : C \to C_0$ is a maximal specialisation such that its projection $\pi : M \to M_0$ is κ-universal, then so is π_C.

Remark 4.3.5. Let $\mathrm{pr} : C_0 \to M_0$ be an n to one function where $C_0 \neq \emptyset$. Consider C_0 with the $\mathcal{L}_{\mathrm{pr}}$ topological structure on it. Hence $\mathrm{pr} : C_0 \to M_0$ becomes a finite fibration. One may naturally wish to consider C_0 as an $\mathcal{L}_{\mathrm{pr}}$-Zariski structure. A natural candidate for dimension is the following: Let $S \subseteq C_0^m$ be closed, define the dimension of S as $\dim S := \dim \mathrm{pr}(S)$. Then we extend dim to constructible sets in the usual way: For a constructible $Q \subseteq C_0^n$ define $\dim Q := \dim \overline{Q}$. By the quantifier elimination dim is defined on all definable sets.

Note that the inverse image of the diagonal in M_0 is an irreducible and closed subset of C_0^2, and the diagonal in C_0 is a closed and irreducible proper subset of the inverse image of the diagonal in M_0. Both of these sets have dimension one. Hence Strong Irreducibility fails for this notion of dimension.

In fact this argument shows us that we need the restricted inequality predicate added to the language $\mathcal{L}_{\mathrm{pr}}$ as a part of the topological structure on C_0 if we wish the consider it as a Zariski structure with the above notion of dimension.

4.3.1.2 Two Sorted Construction

Alternatively one can consider the same construction in the setting of multi sorted structures: Consider the $\mathcal{L}$-topological structure M_0 and the $\mathcal{L}_{\mathrm{pr}}$-topological structure C_0 where $\mathcal{L}_{\mathrm{pr}}$ is the language whose predicates are pull-backs via pr of closed set of M_0.

Let $\tilde{\mathcal{L}} := \mathcal{L} \cup \mathcal{L}_{\mathrm{pr}} \cup \{\mathrm{pr}\}$. The aim is to prove that (M_0, C_0) is a 2-sorted $\tilde{\mathcal{L}}$-structure. I will do this in several steps.

Let F be the following closed subset of $(M_0 \times C_0)^n$:

$$F := S(x_1, \ldots, x_n) \wedge T(u_1, \ldots, u_n) \wedge \chi(\overline{x}, \overline{u})$$

Where

- $\chi(\overline{x}, \overline{u}) : \bigwedge\limits_{i,j} (\mathrm{pr}(u_i) = x_j) \wedge \bigwedge\limits_{k} (\mathrm{pr}(c) = x_k) \wedge \bigwedge\limits_{l} (\mathrm{pr}(u_l) = a)$ with $a \in M_0$, $c \in C_0$ and $0 \leq j, k, i, l \leq n$.

- $S \subseteq M_0^n$ and $T \subseteq C_0^n$ are closed.

Let $\mathfrak{C}_n$ denote the collection of the sets of the form given above.

Lemma 4.3.6. *The family $\mathfrak{C}_n$ has descending chain condition (DCC).*

Proof of the lemma above follows from an argument similar to the one used to prove $\mathfrak{F}_n$ has DCC in Lemma 4.3.1. Next, I will show that (M_0, C_0) admits quantifier elimination in the language $\tilde{\mathcal{L}}$.

Lemma 4.3.7. *The $\tilde{\mathcal{L}}$ structure (M_0, C_0) admits quantifier elimination.*

Proof. The proof is similar to the proof of Lemma 4.1.5. For that reason I shall be brief. Let φ be a formula with a single existential quantifier where the quantifier free part is a formula from the collection $\mathfrak{C}_n$. There are three cases to consider:

- If φ is of the form $\exists x_\nu(S(x_1, \ldots, x_n) \wedge T(u_1, \ldots, u_m) \wedge \chi(\overline{x}, \overline{u}))$. Since the parts $T(u_1, \ldots, u_m)$ and $\bigwedge_l(\mathrm{pr}(u_l) = a)$ does not concern the variable x_ν it is enough to consider $\exists x_\nu(S(x_1, \ldots, x_n) \wedge \chi'(\overline{x}, \overline{u}))$ where $\chi'(\overline{x}, \overline{u}) : \bigwedge_{i,j}(\mathrm{pr}(u_i) = x_j) \wedge \bigwedge_k(\mathrm{pr}(c) = x_k)$. If x_ν does not appear among the variables in $\chi'(\overline{x}, \overline{u})$ then this follows from Lemma 4.1.5. Otherwise either $\mathrm{pr}(u_\mu) = x_\nu$ or $\mathrm{pr}(c) = x_\nu$ appears. Then φ is equivalent to the formula $S(x_1, \ldots, x_n) \wedge T(u_1, \ldots, u_m) \wedge \chi(\overline{x}, \overline{u})$ with x_ν either replaced by $\mathrm{pr}(u_\mu)$ or by $\mathrm{pr}(c)$.

- If φ is of the form $\exists u_\mu(S(x_1, \ldots, x_n) \wedge T(u_1, \ldots, u_n) \wedge \chi(\overline{x}, \overline{u}))$. In this case φ will be equivalent to the formula

$$S(x_1, \ldots, x_n) \wedge \mathrm{pr}(T)(\mathrm{pr}(u_1), \ldots, \mathrm{pr}(u_n)) \wedge \chi(\overline{x}, \overline{u}))$$

  where $\mathrm{pr}(u_\mu)$ is either replaced by x_ν for some ν or for some $a \in M_0$ depending on which of the expressions $\mathrm{pr}(u_\nu) = x_\nu$ or $\mathrm{pr}(u_\nu) = a$ appears in the formula φ.

$\square$

Lemma 4.3.8. *$(M_0 \times C_0)^n$ is an $\tilde{\mathcal{L}}$-topological structure.*

Proof. Topological axioms 1-8 hold by definition. Moreover closed subsets of $(M_0 \sqcup C_0)^n$ can be written as a finite union of elements of $\mathfrak{C}_n$ for any n. By construction $\mathfrak{C}_n$ is closed under finite intersections, since it has (DCC), it is also closed under arbitrary intersections. Therefore $\mathfrak{C}_n$ gives a Noetherian topology on $(M_0 \times C_0)^n$. $\square$

To this language we add the following (definable) predicate

$$I(x, y, z) : x, y \in \mathrm{pr}^{-1}(z) \ \& \ x \neq y$$

We will call this predicate the restricted inequality. Let $\mathcal{L}_{\mathrm{ineq}} = \tilde{\mathcal{L}} \cup \{I(x, y, z)\}$. Then one can define a topology on $(C_0 \times M_0)^n$: Declare the sets defined by positive formulas of $\mathcal{L}_{\mathrm{ineq}}$ as $\mathcal{L}_{\mathrm{ineq}}$-closed.

Lemma 4.3.9. (C_0, M_0) *is a multi-sorted structure with respect to* $\mathcal{L}_{ineq}$*. Moreover, the new topology induced on the Cartesian powers of* C_0 *from the* $\mathcal{L}_{ineq}$ *-topology makes it a topological structure.*

Proof. For an $\mathcal{L}_{\mathrm{pr}}$-closed subset S of C_0^n define

$$F(S, Z) = S(x_1, \cdots, x_n) \wedge \bigwedge_{i<j\leq k} I(x_i, x_j, z_{ij}) \wedge \bigwedge_{i<l} I(x_i, a_i, z_i)$$

for some $k, l \leq n$.

Observe that, if $F(S_1, Z_1) \supseteq F(S_2, Z_2)$ we can assume $S_1 \supseteq S_2$. Otherwise replace S_2 with $S_1 \cap S_2$, and we get $F(S_2, Z_2) = F(S_1 \cap S_2, Z_2)$.

Claim 1. Every descending chain of the sets $F(S, Z)$ stabilizes.

Proof. Let $F(S_1, Z_1) \supseteq F(S_2, Z_2) \supseteq \cdots$ be a descending chain. By the observation above we can assume $S_1 \supseteq S_2 \supseteq \cdots$. Since $\mathcal{L}_{\mathrm{pr}}$-topology is Noetherian, there is some m such that for any $t > m$, we have $S_t = S_m$.

Therefore, for $t > m$ the sets $F(S_t, Z_t)$ are of the form

$$S_m(x_1, \cdots, x_n) \wedge \bigwedge_{i<j\leq k} I(x_i, x_j, z_{tij}) \wedge \bigwedge_{i<l} I(x_i, a_i, z_{ti})$$

Observe that one can only have finitely many non-empty sets of this form. Hence, the chain $F(S_1, Z_1) \supseteq F(S_2, Z_2) \supseteq \cdots$ stabilizes. □

It is clear that, by construction, the sets $F(S, Z)$ are closed under intersections. Moreover, as the closed subsets of C_0^n are the sets defined (with parameters) by positive boolean combinations predicates of $\mathcal{L}_{\mathrm{ineq}}$, they can be written as a finite union of sets of the form $F(S, Z)$ and singletons. Hence the $\mathcal{L}_{\mathrm{ineq}}$ closed sets give a Noetherian topology on C_0^n.

The topological axioms easily follow by the characterization of closed sets as finite unions of $F(S, Z)$ and singletons. Remark that the predicate I is already definable in the language $\mathcal{L}_{\mathrm{pr}}$, and we have quantifier elimination in $\mathcal{L}_{\mathrm{pr}}$, we also have quantifier elimination in the expanded language $\mathcal{L}_{\mathrm{ineq}}$.

□

The topology on C_0 described here is the same topology that is considered in Lemma 4.3.1. Therefore one can prove Theorem 4.3.3 again in this setting. I will state the Theorem for the sake of completeness, but without proof since the proof is similar to the proof of Theorem 4.3.3.

Theorem 4.3.10. *If $\pi_C : C \to C_0$ is a maximal specialisation which projects to a specialisation $\pi : M \to M_0$ then π_C is κ-universal if π is.*

The reason for repeating the same constructions in a slightly different setting is because it became apparent that there is value in studying multi-sorted Zariski structures. Especially to deal with the more sophisticated Zariski structures such as the ones presented in [35, 46]. Also in the next chapter, we will mostly work in the multi-sorted setting.

4.4 First Example of Non-algebraic Zariski Geometry

Next, as mentioned at the beginning of this chapter, we will consider a case where there is a finite group H acting on C_0 and the action of H on fibres in C_0 is transitive. Also the additional topological structure on C_0 will come from this action.

For this subsection, let M_0 be a one dimensional Zariski geometry and G be a group acting on M_0 such that for any $g \in G$ the graph of g -as a binary relation- is a closed and irreducible set. Further, we assume that the action of G is semi-free (i.e. if $g \neq 1$ and $gx = x$ then $g'x = x$ for all $g' \in G$). Recall the following fact from [16].

Fact 4.4.1. *(Proposition 10.1 in [16]) Let M_0 be a one dimensional Zariski geometry, G a group acting on M_0 with a semi-free action (i.e. if $g \neq 1$ and $gx = x$ then $g'x = x$ for all $g' \in G$). Suppose $\mathrm{p} : G^* \to G$ is a homomorphism with a finite kernel H, then there is a Zariski geometry C_0 on which G^* acts semi-freely and there is a surjective Zariski map $\mathrm{pr} : C_0 \to M_0$ such that $\mathrm{pr}(g^* m) = \mathrm{p}(g^*)\,\mathrm{pr}(m)$ for all $g^* \in G^*$ and all $m \in C_0$. Moreover if M_0 is complete then so is C_0.*

I will sketch the construction of C_0 in the proof given in [16] as it will be useful for some future arguments. First I will fix some notation: Let $\mathrm{Fix}_{M_0}(G)$ denote the set $\{x \in M_0 : gx = x \text{ for all } g \in G\}$ and for $g \in G$ define $\mathrm{Fix}_{M_0}(g) := \{x \in M_0 : gx = x\}$. Note that, by the definition of semi-free action, if x is a fix point for some $g \in G \setminus \{1\}$, then it is a fix point for all elements of G. Hence for any $g \in G$, $\mathrm{Fix}_{M_0}(g) = \mathrm{Fix}_{M_0}(G)$. Which, in turn implies that $\mathrm{Fix}_{M_0}(G)$ is a closed subset of M_0.

Let C_0 be a set such that G^* acts semi-freely on it in the same way G acts on M_0. I.e. $|\operatorname{Fix}_{M_0}(G)| = |\operatorname{Fix}_{C_0}(G^*)|$, and the number of non degenerate G^* orbits is the same as number of non degenerate G orbits. Furthermore, we may also assume that $\operatorname{Fix}_{M_0}(G)$ is finite. Indeed, if it is infinite then $\operatorname{Fix}_{M_0}(G) = M_0$. Then we can take $C_0 = M_0$ and G^* with a trivial action on C_0.

For $m, n \in C_0$ define $m \sim n$ if and only if there is an $h \in H$ such that $hx = y$. Clearly $\sim$ is an equivalence relation invariant under the action of G^*. Moreover $\sim$ is closed (hence definable). So H acts trivially on the quotient $C_0/{\sim}$, and hence the action of G^* will give an action of G on this quotient. By construction $C_0/{\sim}$ is isomorphic to M_0 as G-sets, say via a map pr. This map pr can be extended naturally to $\mathrm{pr} : C_0 \to M_0$ so that $\mathrm{pr}(\operatorname{Fix}_{C_0}(G^*)) = \operatorname{Fix}_{M_0}(G)$ and $\mathrm{pr}(g^*m) = \mathrm{p}(g^*)\,\mathrm{pr}(m)$ for all $g^* \in G^*$.

Then C_0 is made into a Zariski geometry by defining pull-backs of the closed sets of M_0^n via pr and graphs of elements of G^* as basic relations (of an associated language) and declaring the Boolean combinations of these predicates as closed sets.

Remark 4.4.2. As C_0 is stable, M_0 is stably embedded in (C_0, M_0). By construction $\mathrm{pr} : C_0 \to M_0$ is a continuous surjection. And lastly, the finite group H (interpretable in M_0) is acting regularly on each fibre.

Next, let C be a κ-saturated elementary extension of C_0. Then C is a one dimensional Zariski geometry, and G^* acts semi-freely C:

- $\operatorname{Fix}_{C_0}(G^*)$ is algebraic over $\emptyset$. Hence it does not change in elementary extensions. So $\operatorname{Fix}_{C_0}(G^*) = \operatorname{Fix}_C(G^*)$.

- Let $x \in C \setminus \operatorname{Fix}_{C_0}(G^*)$. Then x cannot be fixed by any $g^* \neq 1$. Hence G^* acts freely on $C \setminus C_0$, and hence semi-freely on C.

- Let $x \in C \setminus C_0$, then $H \cdot x \cap C_0 = \emptyset$ (where $H \cdot x$ is the orbit of x under the action of H).

Therefore one can consider $\sim$ as an equivalence relation on C in the same way. One can make $C/{\sim}$ a Zariski structure. Topology on $(C/{\sim})^n$ for all n is defined via the natural projection map: Let T be a subset of $(C/{\sim})^n$, it is closed if its pre-image under the natural projection $\mathrm{pr} : C \to (C/{\sim})^n$ is closed in C^n. Clearly these topologies are Noetherian. Dimension of a closed irreducible set is defined as follows: First remark that the natural projection $C \to C/{\sim}$ is continuous, open and closed with respect to the topology on $C/{\sim}$ defined as above (Lemma 3.7.3 in [44]). Now, let $T \subseteq (C/\sim)^n$ be closed and irreducible, then $T = \mathrm{pr}(S)$ for some closed irreducible $S \subseteq M^n$. Then the dimension of T is defined as

$$\dim(T/S) := \dim(S) - \min\{\dim(\mathrm{pr}^{-1}(a) \cap S) : a \in T\}$$

Note that at first sight the definition of $\dim(T/S)$ depends on S. It is proved by Zilber (Lemma 3.7.5 in [44]) that if one chose another closed irreducible $S' \subseteq C^n$ with $\mathrm{pr}(S) = T$, then $\dim(T/S) = \dim(T/S')$. Hence it is independent from the choice of S, so I will denote $\dim(T/S)$ by $\dim(T)$.

Since the topologies on each $(C/\sim)^n$ are Noetherian, any closed set T has a unique irreducible decomposition. Then for an arbitrary closed set T, its dimension $\dim(T)$ is defined as the maximum of dimensions of its irreducible components.

Fact 4.4.3. *$C/\sim$ with the topologies on its Cartesian powers and dimension as defined above is a pre-smooth Zariski structure. Moreover, since C is essentially uncountable, so is $C/\sim$; hence it is a Zariski geometry.*

With the obvious interpretations $C/\sim$ is a structure of the language associated to M_0 (this is the language which has a predicate for every closed set of M_0). Moreover the G-set isomorphism $\mathrm{pr}^{-1} : M_0 \to C_0/\sim$ gives an elementary embedding of M_0 into $C/\sim$. Hence; (by replacing $C_0/\sim$ with M_0) $C/\sim$ can be thought as an elementary extension of M_0. Therefore talking about specialisations of $C/\sim$ into M_0 makes sense.

Remark that if $\pi : C/\sim \to M_0$ is a specialisation and $a \in C/\sim$ is fixed under the action of G (i.e. if $a \in \mathrm{Fix}_{C/\sim}(G)$), then its image πa is in $\mathrm{Fix}_{M_0}(G)$. Although, for an element $a \in C/\sim \cap \mathrm{Dom}(\pi)$ which is not fixed under the action of G, its image $\pi(a)$ can be a fixed point of the action of G in M_0.

This observation motivates the following definition.

Definition 4.4.4. Let M_0 be a Zariski geometry and G a group acting semi-freely on M_0 and $M \succeq M_0$ so G acts on M semi-freely. Let $\pi : M \to M_0$ be a specialisation and $a \in M \setminus \mathrm{Fix}_M(G)$. Then π is said to *degenerate a (with respect to the action G)* if $\pi(a) \in \mathrm{Fix}_{M_0}(G)$.

Proposition 4.4.5. *Let $\pi_C : C \to C_0$ be a specialisation lifting π and $a \in \mathrm{Dom}(\pi)$.*

1. *If $m \in \mathrm{pr}^{-1}(a) \cap \mathrm{Dom}(\pi_C)$, then $\pi_C(m) \in \mathrm{pr}^{-1}(\pi(a))$. Further, if $m, n \in \mathrm{pr}^{-1}(a) \cap \mathrm{Dom}(\pi_C)$ then $\pi_C(m), \pi_C(n) \in \mathrm{pr}^{-1}(\pi(a))$.*

2. *If $\mathrm{pr}^{-1}(a) \subset \mathrm{Dom}(\pi_C)$ and if π does not degenerate a, then π_C must be a bijection on $\mathrm{pr}^{-1}(a)$.*

3. *Let $\pi_C : C \to C_0$ be a specialisation. Then it is a lifting of a specialisation $C/\sim \to M_0$.*

Proof. 1. Let $m \in \mathrm{pr}^{-1}(a)$ and say $\pi_C(m) = m_0$. By the definition of lifting, $\pi_C(m)/\sim = \pi(m/\sim) = \pi(a)$. So, $m_0 \in \mathrm{pr}^{-1}(\pi(a))$. Since $\sim$ is a closed equivalence relation, if $n \sim m$ then $\pi_C(m) \sim \pi_C(n)$. Hence (together with the previous argument) $\pi_C(m), \pi_C(n) \in \mathrm{pr}^{-1}(\pi(a))$.

2. Suppose $\mathrm{pr}^{-1}(a) \subset \mathrm{Dom}(\pi_C)$, but π_C is not one to one. So, $\pi_C(m) = \pi_C(n)$ for some $m, n \in \mathrm{pr}^{-1}(a)$. But there is an $h \in H$ such that $hm = n$. Recall that the graph h is a closed set. Hence $h\pi_C(m) = \pi_C(n) = \pi_C(m)$. So $\pi_C(m) \in \mathrm{Fix}_{M_0}(G^*)$. By the previous part $\pi_C(m) \in \mathrm{pr}^{-1}(\pi(a))$. But then $\pi(a)$ must be fixed point under the action of G by the construction of C_0 and since $\pi_C(m) \in \mathrm{Fix}_{C_0}(G^*)$. Which contradicts with the assumption that π does not degenerate a.

3. Consider the set $\mathrm{Dom}(\pi_C)/\sim$. Clearly $\mathrm{Dom}(\pi_C)$ contains C_0, hence $\mathrm{Dom}(\pi_C)/\sim$ contains M_0. Now define $\pi : \mathrm{Dom}(\pi_C)/\sim \to M_0$ as follows: For $a \in \mathrm{Dom}(\pi_C)/\sim$, there is $\overline{m} \in \mathrm{Dom}(\pi_C)$ such that $\mathrm{pr}(m) = a$. Say $\pi_C(m) = m_0$, and $\mathrm{pr}(m_0) = a_0$. Define $\pi(a) := a_0$. It is clear that π is a specialisation and π_C is lifting π.

□

Proposition 4.4.6. *Let $\pi : C/\sim \to M_0$ be a specialisation. Then π can be lifted to a specialisation $\pi_C : C \to C_0$ such that if $m \in \mathrm{Dom}(\pi_C)$, then the orbit of m under the action of the finite group H is contained in $\mathrm{Dom}(\pi_C)$. Moreover, if π is a maximal specialisation then so is π_C.*

Proof. By Proposition 4.4.5, if $a \in \mathrm{Dom}(\pi)$ then any specialisation lifting π must map the fibre over a to the fibre over $\pi(a)$.

Let $B = \mathrm{pr}^{-1}(\mathrm{Dom}(\pi))$. Enumerate B as $\{b_i : i < \lambda\}$ where λ is the cardinality of B. Let $B_0 = C_0$ and $\pi_0 = \mathrm{id} : B_0 \to C_0$. Let $B_j = B_0 \cup \{b_i : i < j\}$. We will extend this to a specialisation $\pi_C : B \to C_0$.

Suppose $j = i+1$ and $\pi_i : B_i \to C_0$ is a specialisation. To extend π_i to B_j look at

$$p(x) = \{\varphi(x, \pi_i(a_1), ..., \pi_i(a_n)) : \varphi(b_j, a_1, ..., a_n) \in \mathrm{Diag}^+(b_j B_i)\}$$

Let $m_j = \mathrm{pr}(b_j)$, we will show that there is a realization of $p(x)$ in $\mathrm{pr}^{-1}(\pi(m_j))$. Let $S_{\mathrm{pr}}(x, \pi_i(a_1), ..., \pi_i(a_n)) \in p(x)$ be the pull-back of a closed set S of M_0. Since $C \models S_{\mathrm{pr}}(b_i, a_1, ..., a_n)$, by construction $C/\sim \models S(m_i, \mathrm{pr}(a_1), ..., \mathrm{pr}(a_n))$. Then, as π

is a specialisation, $M_0 \models S(\pi(m_i), \pi(\mathrm{pr}(a_1)), ..., \pi(\mathrm{pr}(a_n)))$. Since π_i is a partial lifting of the specialisation π, we observe $\pi(\mathrm{pr}(a_i)) = \mathrm{pr}(\pi_i(a_i))$ for $i = 1, ..., n$. Hence $S(\pi(m_i), \mathrm{pr}(\pi_i(a_1)), ..., \mathrm{pr}(\pi_i(a_n)))$. Therefore, by construction, $C_0 \models S_{\mathrm{pr}}(c, \pi_i(a_1), ..., \pi_i(a_n))$ for any $c \in \mathrm{pr}^{-1}(\pi(m_i))$.

The remaining formulas in $p(x)$ are of the type $g^*(x) = \pi_i(a_l)$ for some $g^* \in G^*$ such that $g^*(b_i) = a_l$. Next, one needs to show that the partial type $\{g_k^*(x) = \pi_i(a_l) : g_k^* \in G^* \text{ and } g_k^*(b_i) = a_l\}$ has a realization in $\mathrm{pr}^{-1}(\pi(m_i))$.

Note that this is a finite type, and in particular we have

$$C \models (g_1^*)^{-1}(a_{l_1}) = \ldots = (g_t^*)^{-1}(a_{l_t})$$

Since π is a specialisation

$$C_0 \models (g_1^*)^{-1}(\pi_i(a_{l_1})) = \ldots = (g_t^*)^{-1}(\pi_i(a_{l_t}))$$

Therefore $(g_1^*)^{-1}(\pi_i(a_{l_1}))$ realizes $p(x)$.

Next, we will show that $(g_1^*)^{-1}(\pi_i(a_{l_1})) \in \mathrm{pr}^{-1}(\pi(m_i))$. By construction

$$\mathrm{pr}((g_1^*)^{-1}(\pi_i(a_{l_1}))) = p((g_1^*)^{-1}) \cdot (\mathrm{pr}(\pi_i(a_{l_1}))) \tag{4.1}$$

Also by the same argument,

$$m_i = \mathrm{pr}((g_1^*)^{-1}(a_{l_1})) = p((g_1^*)^{-1}) \cdot (\mathrm{pr}(a_{l_1})) \tag{4.2}$$

Recall that, since the graph of $(g_k^*)^{-1}$ is a closed set, its projection is also closed. Also observe that projection of the graph of $((g_k^*)^{-1})$ is the graph of $p((g_1^*)^{-1}) \in G$. Hence the graph of $p((g_1^*)^{-1})$ is a closed set of $C/\sim$. Therefore by (4.2) one sees $\pi(m_i) = p((g_1^*)^{-1}) \cdot (\mathrm{pr}(\pi_i(a_{l_1})))$. Hence by (4.1), $(g_1^*)^{-1}(\pi_i(a_{l_1})) \in \mathrm{pr}^{-1}(\pi(m_i))$.

Together with the previous argument we observe that $(g_1^*)^{-1}(\pi_i(a_{l_1}))$ is a realization of $p(x)$ in $\mathrm{pr}^{-1}(\pi(m_i))$. Now define $\pi_j : B_j \to C_0$ as $\pi_j = \pi_i$ on B_i and $\pi_j(b_i) = (g_1^*)^{-1}(\pi_i(a_{l_1}))$.

If j is a limit ordinal then take $\pi_j = \bigcup_{i<j} \pi_i$. Clearly, each $\pi_j : B_j \to M_0$ for $j < \lambda$ is a specialization, and partially lifts π. Finally, define $\pi_C := \bigcup_{j<\lambda} \pi_j : B \to C_0$. It is clear that $\pi_C : B \to C_0$ is a specialisation lifting π to C. $\square$

Theorem 4.4.7. *Let $\pi_C : C \to C_0$ be a maximal specialisation. If the specialisation $\pi : C/\sim \to M_0$ induced by π_C is κ-universal, then π_C is κ-universal.*

Proof. Let $C' \succeq C$ and $A \subseteq C'$ with $|A| < \kappa$; let $\pi_{C'} : A \cup C \to C_0$ be a specialisation extending π_C. Moreover, we can assume that if $a \in A$, then $H \cdot a \subseteq A$ (in other words we can assume that if $a \in A$, then $b \in A$ for all b such that $\mathrm{pr}(a) = \mathrm{pr}(b)$). By part (3) of Proposition 4.4.5 there is a specialisation $\pi' : \mathrm{Dom}(\pi_{C'})/{\sim} \to M_0$ which lifts to $\pi_{C'}$. Clearly π' extends π. Since π is κ-universal, there is an elementary embedding $\sigma : A/{\sim} \to C/{\sim}$ over $A/{\sim} \cap C/{\sim}$ such that $\pi' = \pi \circ \sigma$ on $A/{\sim}$.

We will lift σ to an embedding $\sigma_A : A \to C$ such that $\pi_{C'} = \pi_C \circ \sigma_A$ on A. Enumerate $A \setminus C = \{a_i : i < \lambda\}$ where $\lambda < \kappa$. Also assume that the enumeration is done in a way so that all elements of any given fibre lying inside $A \setminus C$ is listed consecutively before listing an element from a different fibre. Let $A_0 = A \cap C$ and define $\sigma_0 := \mathrm{id} : A_0 \to C$. Let $A_j := A_0 \cup \{a_i : i < j\}$.

Next, let $j = i + 1$ and assume that there is an embedding $\sigma_i : A_i \to C$ such that $\pi_{C'} = \pi_C \circ \sigma_i$ on A_i. We will extend this embedding to an embedding $\sigma_j : A_j \to C$.

Look at the type

$$p(x) = \{\varphi(x, \sigma_i(b_1), \ldots, \sigma_i(b_n)) : \varphi(a_i, b_1, \ldots, b_n) \in \mathrm{Diag}(a_i A_i)\}$$

Let $m_i := \mathrm{pr}(a_i)$, we will show that $p(x)$ has a realisation in $\mathrm{pr}^{-1}(\sigma(m_i))$ in several steps. There are four types of formulae in $p(x)$:

1. $S_{\mathrm{pr}}(x, \sigma_i(b_1), \ldots, \sigma_i(b_n))$ where S_{pr} is the pull-back of a closed set S of $C/{\sim}$
2. $\neg T_{\mathrm{pr}}(x, \sigma_i(b_1), \ldots, \sigma_i(b_n))$ where T_{pr} is the pull-back of a closed set T of $C/{\sim}$
3. $g^*(x) = \sigma_i(b_l)$ where $g^* \in G$
4. $g^*(x) \neq \sigma_i(b_l)$ where $g^* \in G$

We will show that there is an element in $\mathrm{pr}^{-1}(\sigma(m_i))$ that realises all types of the formulas in $p(x)$ as described above.

Let $S_{\mathrm{pr}}(x, \sigma_i(b_1), \ldots, \sigma_i(b_n)) \in p(x)$ be the pull-back of a closed set S of $C/{\sim}$. Since $C' \models S_{pr}(a_i, b_1, \ldots, b_n)$, by construction $C'/{\sim} \models S(m_i, \mathrm{pr}(b_1), \ldots, \mathrm{pr}(b_n))$. Since $\sigma : A/{\sim} \to C/{\sim}$ is an embedding we have $C/{\sim} \models S(\sigma m_i, \sigma\, \mathrm{pr}(b_1), \ldots, \sigma\, \mathrm{pr}(b_n))$. Since $\sigma_i : A_i \to C$ is a partial lifting of the embedding σ, and since $b_1, \ldots, b_n \in A_i$, we observe that $\sigma(\mathrm{pr}(b_i)) = \mathrm{pr}(\sigma_i(b_i))$ for all $i = 1, \ldots, n$. Hence one can write $C/{\sim} \models S(\sigma m_i, \mathrm{pr}(\sigma_i(b_1)), \ldots, \mathrm{pr}(\sigma_i(b_n)))$. So, by construction $C \models S_{\mathrm{pr}}(c, \sigma_i(b_1), \ldots, \sigma_i(b_n))$ for any $c \in \mathrm{pr}^{-1}(\sigma m_i)$.

Next, let $\neg T_{\mathrm{pr}}(x, \sigma_i(b_1), \ldots, \sigma_i(b_n)) \in p(x)$, where T_{pr} is the pull-back of some closed set T of $C/{\sim}$. Then $C' \models \neg T_{\mathrm{pr}}(a_i, b_1, \ldots, b_n)$. So, by construction $C'/{\sim} \models$

$\neg T(m_i, \mathrm{pr}(b_1), \dots, \mathrm{pr}(b_n))$. Otherwise, if $C'/{\sim} \models T(m_i, \mathrm{pr}(b_1), \dots, \mathrm{pr}(b_n))$, then by definition $C' \models T_{\mathrm{pr}}(t, b_1, \dots, b_n)$ for all $t \in \mathrm{pr}^{-1}(m_i)$. In particular, this would mean $C' \models T_{\mathrm{pr}}(a_i, b_1, \dots, b_n)$.

Since $\sigma : A/{\sim} \to C/{\sim}$ is an embedding, it preserves negated atomic formulas. Hence we have $C/{\sim} \models \neg T(\sigma m_i, \sigma\, \mathrm{pr}(b_1), \dots, \sigma\, \mathrm{pr}(b_n))$. Since σ_i partially lifts σ, we have $\sigma(\mathrm{pr}(b_i)) = \mathrm{pr}(\sigma_i(b_i))$. Therefore $C/{\sim} \models \neg T(\sigma m_i, \mathrm{pr}(\sigma_i(b_1)), \dots, \mathrm{pr}(\sigma_i(b_n)))$. Hence $C \models \neg T_{\mathrm{pr}}(c, \sigma_i(b_1), \dots, \sigma_i(b_n))$ for any $c \in \mathrm{pr}^{-1}(\sigma m_i)$.

The remaining formulas in $p(x)$ are either of the type $g^*(x) = \sigma_i(b_l)$ or of type $g^*(x) \neq \sigma_i(b_l)$. First, we will deal with the formulas of the type $g^*(x) = \sigma_i(b_l)$ in $p(x)$.

We will show that the partial type

$$q(x) := \{g_k^*(x) = \sigma_i(b_l) : g_k^* \in G^* \text{ and } g_k^*(a_i) = b_l\}$$

has a realisation in $\mathrm{pr}^{-1}(\sigma m_i)$. First, let us assume that this partial type is finite. Then, in particular we have

$$C' \models (g_1^*)^{-1}(b_{l_1}) = \ldots = (g_t^*)^{-1}(b_{l_t})$$

Since $\sigma_i : A_i \to C$ is an elementary embedding

$$C \models (g_1^*)^{-1}(\sigma_i(b_{l_1})) = \ldots = (g_t^*)^{-1}(\sigma_i(b_{l_t}))$$

Therefore $(g_1^*)^{-1}(\sigma_i(b_{l_1}))$ satisfies the partial type $q(x)$ assuming that it is finite.

Hence, we showed that any finite subset of $q(x)$ has a realisation. Hence, by compactness, this type itself has a realisation, call it d. By saturation $d \in C$. Also, there are $g_r^* \in G^*$ and $b_{l_r} \in A_i$ such that, $g_r^*(d) = \sigma_i(b_{l_r})$. So we can write $d = (g_r^*)^{-1}(\sigma_i(b_{l_r}))$. Therefore $(g_r^*)^{-1}(\sigma_i(b_{l_r}))$ realises the partial type $q(x)$.

Next, we will show that $(g_r^*)^{-1}(\sigma_i(b_{l_r}))$ realises the partial type

$$r(x) := \{f_k^*(x) \neq \sigma_i(b_l) : f_k^* \in G^* \text{ and } f_s^*(a_i) \neq b_l\}$$

Suppose that $(g_r^*)^{-1}(\sigma_i(b_{l_r}))$ does not realise this partial type. Then there is $f_s^* \in G^*$ and $b_{q_s} \in A_i$ such that $f_s^*(x) \neq \sigma_i(b_{q_s}) \in r(x)$ and $f_s^*((g_r^*)^{-1}(\sigma_i(b_{l_r}))) = \sigma_i(b_{q_s})$.

Rewrite this equality as $(g_r^*)^{-1}(\sigma_i(b_{l_r})) = (f_s^*)^{-1}\sigma_i(b_{q_s})$. Since σ_i in an elementary embedding, $(g_r^*)^{-1}(b_{l_r}) = (f_s^*)^{-1}(b_{q_s})$. On the other hand, we know that $(g_r^*)^{-1}(b_{l_r}) = a_i$ and $(f_s^*)^{-1}(b_{q_s}) \neq a_i$. Which leads to $a_i \neq a_i$, hence a contradiction. Therefore $(g_r^*)^{-1}(\sigma_i(b_{l_r}))$ realises the partial type $r(x)$ as required.

Now, we will show that $(g_r^*)^{-1}(\sigma_i(b_{l_r}))$ is in $\mathrm{pr}^{-1}(\sigma m_i)$. By construction

$$\mathrm{pr}((g_r^*)^{-1}(\sigma_i(b_{l_r}))) = p((g_r^*)^{-1}) \cdot \mathrm{pr}(\sigma_i(b_{l_r})) \tag{4.3}$$

Similarly,

$$m_i = \mathrm{pr}((g_r^*)^{-1}(b_{l_r})) = p((g_r^*)^{-1}) \cdot \mathrm{pr}(b_{l_r}) \tag{4.4}$$

From the proof of Proposition 4.4.6, we know that the graph of $p((g_r^*)^{-1})$ is a closed set. By (4.4), since σ is an embedding, $\sigma(m_i) = p((g_r^*)^{-1}) \cdot \mathrm{pr}(\sigma_i(b_{l_r}))$. Then by (4.3), we have $(g_r^*)^{-1}(\sigma_i(b_{l_r})) \in \mathrm{pr}^{-1}(\sigma(m_i))$. Then $(g_r^*)^{-1}(\sigma_i(b_{l_r}))$ is a realisation of the type $p(x)$ in $\mathrm{pr}^{-1}(\sigma(m_i))$.

Define $\sigma_j : A_j \to C$ as $\sigma_j = \sigma_i$ on A_i and $\sigma_j(a_i) = (g_r^*)^{-1}(\sigma_i(b_{l_r}))$. By construction $\sigma_j : A_j \to M$ is an elementary embedding. It remains to show that $\pi_{C'} = \pi_C \circ \sigma_j$ on A_j. By the induction hypothesis, $\pi_{C'} = \pi_C \circ \sigma_i$ on A_i. So, we only need to show that $\pi_{C'}(a_i) = \pi_C \circ \sigma_j(a_i)$.

If there is a $g^* \in G^*$ and $b_l \in A_i$ such that $g^* \cdot a_i = b_l$, then since $\pi_{C'}$ is a specialisation,

$$g^* \cdot \pi_{C'}(a_i) = \pi_{C'}(b_l) = \pi_C(\sigma_i(b_l)) \tag{4.5}$$

On the other hand, since $\sigma_j : A_j \to C$ is an embedding, $g \cdot \sigma_j(a_i) = \sigma_i(b_l)$. Since π_C is a specialisation and the graph of g^* is a closed set,

$$g^* \cdot \pi_C(\sigma_j(a_i)) = \pi_C(\sigma_i(b_l)) \tag{4.6}$$

Therefore, by equations (4.5) and (4.6), $\pi_{C'}(a_i) = \pi_C(\sigma_j(a_i))$.

Now, suppose that such g^* and $b_l \in A_i$ does not exist. Then a_i must be the first element from the fibre $\mathrm{pr}^{-1}(m_i)$ we embedded, i.e. there is no $b_l \in \mathrm{pr}^{-1}(m_i) \cap A_i$ with $b_l \neq a_i$. Otherwise, if there is an element $b_l \in \mathrm{pr}^{-1}(m_i) \cap A_i$ with $b_l \neq a_i$, then there is an element $h \in H \subseteq G^*$ such that $h \cdot a_i = b_l$. Hence we fall into the case considered in the previous paragraph.

Let $a \in \mathrm{pr}^{-1}(m_i)$ with $a_i \neq a$. Then there are no $g^* \in G$ and $b_l \in A_i$ such that $g^* \cdot a = b_l$. In other words, if a_i is not sent into A_i under the action of G^*, then any element of the fibre $\mathrm{pr}^{-1}(m_i)$ cannot be sent into A_i under the action of G^*. Then m_i cannot be sent into $\mathrm{pr}(A_i)$ under the action of G. Therefore, σm_i cannot be sent into $\sigma(\mathrm{pr}(A_i))$ under the action of G. Which implies that the fibre $\mathrm{pr}^{-1}(\sigma m_i)$ cannot be mapped into $\sigma_i(A_i)$ under the action of G^*.

Note that in this case the type $p(x)$ does not contain any formulas of the type $g^*(x) = \sigma_i(b_l)$. And as we showed in the above paragraph, the partial type $\{f_k^*(x) \neq \sigma_i(b_l) : f_k^* \in G^* \text{ and } f_s^*(a_i) \neq b_l\} \subseteq p(x)$ is realised by all elements in the fibre $\mathrm{pr}^{-1}(\sigma m_i)$. We have also shown that other formulas in $p(x)$ are realised by all elements in $\mathrm{pr}^{-1}(\sigma m_i)$. Moreover, we claim that there exist an element $a \in \mathrm{pr}^{-1}(\sigma m_i)$ such that $\pi_C(a) = \pi_{C'}(a_i)$.

First, suppose that $\pi_{C'}(a_i) \in \mathrm{Fix}_{C_0}(G^*)$, then $\pi'(m_i) \in \mathrm{Fix}_{M_0}(G)$. Then, since σ is an elementary embedding with $\pi' = \pi \circ \sigma$ on $\mathrm{pr}(A)$, we have $\pi(\sigma m_i) = \pi'(m_i) \in \mathrm{Fix}_{M_0}(G)$. Then, π_C maps $\mathrm{pr}^{-1}(\sigma m_i)$ onto $\pi_{C'}(a_i)$, as it lifts π. Therefore, in this case any element of $\mathrm{pr}^{-1}(\sigma m_i)$ realises the type $p(x)$, and furthermore we have $\pi_C(a) = \pi_{C'}(a_i)$ for any $a \in \mathrm{pr}^{-1}(\sigma m_i)$. Thus, we pick an element $a \in \mathrm{pr}^{-1}(\sigma m_i)$ and define $\sigma_j(a_i) = a$.

If $\pi_{C'}(a_i) \notin \mathrm{Fix}_{C_0}(G^*)$, then $\pi_{C'}$ does not degenerate the fibre $\mathrm{pr}^{-1}(m_i)$. Moreover, $\pi(\sigma m_i) = \pi'(m_i) \notin \mathrm{Fix}_{M_0}(G)$. Hence, π_C does not degenerate the fibre $\mathrm{pr}^{-1}(\sigma m_i)$. Then by Proposition 4.4.5 (2), π_C is a bijection on the fibre $\mathrm{pr}^{-1}(\sigma m_i)$. Hence, there exists an $a \in \mathrm{pr}^{-1}(\sigma m_i)$ such that $\pi_C(a) = \pi_{C'}(a_i)$. Then, in this case define $\sigma_j(a_i) = a$. It is clear that $\sigma_j : A_i \to C$ is an embedding with $\pi_M(a_i) = \pi_C(\sigma_j(a_i))$.

If j is a limit ordinal then define $\sigma_j := \bigcup_{i<j} \sigma_j : A_j \to C$. Clearly $\sigma_i : A_i \to C$ is an elementary embedding such that $\pi_{C'} = \pi_C \circ \sigma_j$.

To finish the proof, define $\sigma_A := \bigcup_{j<\lambda} \sigma_j : A_j \to C$. By construction σ_A is an elementary embedding satisfying $\pi_{C'} = \pi_C \circ \sigma_A$ on A. $\square$

Corollary 4.4.8. *Let $M_0 := M_0(k)$ be an elliptic or a rational curve over an algebraically closed field k of characteristic zero, and C_0 is constructed as above. Let $C \succeq C_0$, then $M := C/\sim = M_0(K)$ for some $K \succeq k$ (i.e. $C/\sim$ is the set of K-points of the same curve M_0). More importantly the specialisation $Res : M \to M_0$ where $Res(b) = (res(b_1), ..., res(b_n))$ for the residue map $res : K \to k$ lifts to a maximal specialisation $\pi_C : C \to C_0$. Also, if $Res : M \to M_0$ is κ-universal, then so is π.*

Proof. By Theorem 4.4.7, it lifts to a κ-universal specialisation $\pi_C : C \to C_0$. $\square$

In the examples given in [16, 44], M_0 is again a rational or an elliptic curve with $G \leq \mathrm{ZAut}(M_0)$, and G^* and H satisfies:

- G^* is nilpotent,
- There is an integer μ such that G^* has a non-abelian subgroup with index at least μ.
- H does not split, i.e. there are no subgroups $T \leq G^*$ such that $H \cdot T = G^*$.

and C_0 is constructed as described above. In this case it is shown (by Theorem C in [16] and Lemma 5.1.15 in [44]) that C_0 constructed in this way is a non-algebraic Zariski geometry (i.e. it is not interpretable in an algebraically closed field).

Chapter 5

Specializations of Zariski Cover Structures

In this chapter we will consider finite and infinite covers of Zariski structures in a more general setting. This should be seen as a natural next step to Chapter 4.

In the first section we analyse Zariski cover structures. They are a certain type of a cover of Zariski structures. The natural examples of covers constructed in [35, 16, 46] are all Zariski cover structures.

Second section studies specialisations, and particularly universal specialisations of Zariski cover structures. The main result is a characterisation of $\aleph_0$-universal specialisations of Zariski cover structures under a certain assumption (Theorem 5.2.14).

Remark 5.0.1. Before we start let us make the following convention. Let C be a Zariski structure together with its natural Zariski language $\mathcal{L}_{\text{Zar}}$. We will assume that $\mathcal{L}_{\text{Zar}}$ contains names for all elements of C.

5.1 Zariski Cover Structures

Definition 5.1.1. Let $\mathcal{C} := (C, M, \text{pr})$ be a two-sorted Zariski structure such that

(i) There is a Zariski topological group G in $\mathcal{C}$ acting morphically and freely on C with Zariski continuous bijections.

(ii) M is interpretable in C as a topological sort and $\text{pr} : C \to M$ denotes the canonical quotient-map. It is a $\emptyset$-definable surjection.

(iii) For each $m \in M$, the fibre $\text{pr}^{-1}(m)$ is an orbit of an element in G, i.e. $\text{pr}^{-1}(m) = G \cdot x$ for some $x \in C$.

(iv) The group G is a Zariski topological group in M (in particular G is interpretable as a topological sort in M).

Then $\mathcal{C} := (C, M, \mathrm{pr})$ is said to be a *(two sorted) Zariski cover structure*.

Remark 5.1.2. Since C is stable, M is stably embedded in $\mathcal{C}$.

Proposition 5.1.3. *The action of G on each fibre of* pr *is transitive and free (i.e. regular).*

Proof. Immediate from the definition. □

Lemma 5.1.4. *Let $\mathcal{C} = (C, M, \mathrm{pr})$ be a Zariski cover structure. Then*

(i) the map $\mathrm{pr} : C \to M$ is a morphism;

(ii) M is isomorphic to C/G; i.e. there is a bijective morphism $C/G \to M$.

Proof. (i) The map pr is the natural quotient map. Hence it is a morphism (see Fact 2.1.15).

(ii) Since the action of the group G is free, it is also proper by definition (see Definition 2.1.44, (ii) and (iii)). Therefore the equivalence relation E_G defined by

$$cE_Gd \text{ if and only if } \exists g \in G\,(g \cdot c = d)$$

is closed. Hence C/E_G is a topological sort. Moreover, pr is constant on the classes of E_G. By Definition 5.1.1 (iii), E_G is the pre-image of $=$ under pr. Then $\mathrm{pr}/E_G : C/G \to M$ is an isomorphism.

□

Remark 5.1.5. Clearly pr induces a natural projection $\mathrm{pr} : C^n \to M^n$ for any n, for which we use the same notation. Since it is a product of morphisms, it is also a morphism.

Remark 5.1.6. The notion of Zariski cover structure is compatible with the more general notion of cover given by Hrushovski in [14] and repeated by Ahlbrandt and Ziegler in [1]. The notion of Zariski cover structure has some additional topological properties coming from Zariski structures, and as the main perspective is to establish a theory of specialisations; which requires the topological structure. One main difference is that, in Zariski cover structures the same group is acting on fibres. For general covers, there is more flexibility, different groups are allowed to act on different fibres.

Example 5.1.7. One can consider a special case of the construction we discussed in Chapter 4, of covers related to the first example of a non-algebraic Zariski geometry. This case is considered in [44] and it is also shown there that a similar example of a non-algebraic Zariski geometry can be produced following this case. It is easy to see that these covers are Zariski cover structures.

Also from the detailed description of their closed sets, it is easy to see that the covers related to the Quantum Harmonic Oscillator, and Quantum Zariski geometries are Zariski cover structures. All the necessary ingredients are already present in papers these structures are first presented ([35, 46]).

Next, we proceed to analyse the relations between the fibres. This analysis will be important in the next section, where we consider specialisations of Zariski cover structures. The following definition is essential for our considerations.

Definition 5.1.8. Let $\bar{b} = (b_1, \ldots, b_n) \in C^n$, and $\mathrm{pr}(\bar{b}) = (m_1, \ldots, m_n) = \overline{m}$. Let $A \subset C$. We say $\bar{b}$ is *strongly independent in fibres over A* if

$$loc(\bar{b}/M \cup A) = \{\bar{c} \in C^n : \mathrm{pr}\ \bar{c} = \overline{m}\}$$

where $loc(\bar{b}/M \cup A)$ is the locus of $\bar{b}$ over $M \cup A$.

Lemma 5.1.9. *Let $\mathcal{C} = (C, M, \mathrm{pr})$ be a Zariski cover structure, and $m \subset M$. Suppose there is $b' \in \mathrm{pr}^{-1}(m)$ strongly independent in fibres over A for some $A \subseteq \mathcal{C}$. Then every $b \in \mathrm{pr}^{-1}(m)$ is independent in fibres over A and is generic over $A \cup M$.*

Proof. If b is dependent, it satisfies for some $a \in A$, and $m' \in M$ a positive quantifier free formula $Q(m', a, b)$, with $\dim Q(m', a, y) < \dim \mathrm{pr}^{-1}(m)$. Then $b' \in g \cdot Q(m', a, C)$ for some $g \in G(M)$, which gives a similar formula for b'. □

Lemma 5.1.10. *Suppose $\mathcal{C} \preceq \mathcal{C}'$, where $\bar{b}' \subset C' \setminus C$, and $|\bar{b}'| = n$; also let $\bar{b}'$ be strongly independent in fibres over $A \subseteq C$. Let $m' := \mathrm{pr}(b')$. Then*

(i) the locus of b' over $A \cup M'$ is of the form $S(y, m')$ where

$$S(y, x) \equiv x = \mathrm{pr}(y)$$

(ii) the locus of $b'm'$ over $A \cup M$ is of the form $S(y, x)$ & $R(x, a)$ for some $a \subset M$, and $R(y, z)$ a positive quantifier free formula over $\emptyset$.

Proof. (i) Immediate from the definition.

(ii) Let $Q(y, x, z)$ be a positive quantifier free formula such that $Q(y, x, c)$ is the locus of $b'm'$ over $A \cup M$, where $c \subset A \cup M$.

Let $R(x, a)$ be the locus of m' over $A \cup M$, where $a \subset A \cup M$. Since M is totally transcendental and stably embedded in $\mathcal{C}$, we may choose $a \subset M$. Since M is a submodel of M', the locus $R(x, a)$ is irreducible. We may assume $a \subset c$ and $Q(y, x, z) \equiv Q(y, x, z)$ & $R(x, z)$.

For $m \in M^n$, let $S_m(y)$ be the formula $S(y, m)$; which is equivalent to $\mathrm{pr}(y) = m$. Let S_m be the fibre in the respective model. Set

$$R^0(M, c) = \{m \in M^n : S_m \cap Q(C, m, c) \neq \emptyset\}$$

as the projection of $Q(C, m, c) \subset C^n \times M^n$ on M^n.

Note that $R^0(M, c)$ is a dense subset of $R(M, c)$, since m' is a generic point in the M' versions of both.

We now consider the action of $g \in G^n$ on the set $C^n \times M^n$

$$(b, m) \mapsto (g \cdot b, m)$$

By our assumptions it is continuous, and thus $g \cdot Q(C, M, c)$ is closed.

By (i), $Q(C', m', c) = S_{m'}$, and this is a generic fibre. Hence G^n acts transitively on the generic fibre $Q(C', m', c)$. So for any $g \in G(M)$,

$$g \cdot Q(C', m', c) = Q(C', m', c)$$

Hence, in $\mathcal{C}$, for any $g \in G$ and any $m \in R^0(M, c)$, if m is generic over g, c, then

$$g \cdot Q(C, m, c) = Q(C, m, c)$$

By the addition formula, for a given $g \in G(M)$,

$$\dim g \cdot Q(C, M, c) \cap Q(C, M, c) \geq$$
$$\geq \dim G^n + \dim\{m \in R^0(M, c) : g \cdot Q(C, m, c) = Q(C, m, c)\} \geq \dim Q(C, M, c)$$

and hence, since $Q(C, M, c)$ is irreducible,

$$g \cdot Q(C, M, c) = Q(C, M, c)$$

for all $g \in G(M)$. This proves that every fibre $Q(C, m, c)$ is stable under the action of G, hence

$$Q(C, m, c) = S_m(C) \text{ for all } m \in R^0(M, c)$$

Then $\mathrm{pr}^{-1} R^0(M, c) = Q(C, M, c)$ and $Q(C, M, c)$ is a closed G-invariant set. By definition, the image $R^0(M, c)$ is closed in the sort M^n. Clearly then $R^0(M, c) = R(M, a)$ and thus $S(y, x)$ & $R(x, a) \equiv Q(y, x, c)$.

□

Corollary 5.1.11. *If $Q(y, x, z)$ is the locus of (b, m, c) over $\emptyset$ and $Q(C, m, c) = S_m$ then $Q(C, m', c') \neq \emptyset$ implies $Q(C, m', c') = S_{m'}$ for any m', c'.*

Proof. By assumption G^n acts transitively on the generic fibres $Q(C, m', c')$. So for any $g \in G(M)$ we have

$$g \cdot Q(C, m', c') = Q(C, m', c').$$

Hence, in $\mathcal{C}$, for any $g \in G$ and for any generic m', c' satisfying $\exists y\, Q(y, x, z)$ we have

$$g \cdot Q(C, m', c') = Q(C, m', c').$$

As all elements of $\mathcal{C}$ are named by convention (see Remark 5.0.1) it is the prime model. In particular that elements of M are named. Therefore, m', c' is generic over any $g \in G(M)$. Then, $g \cdot Q(C, M, C) \cap Q(C, M, C)$ contains the original generic element bmc. Hence, since $Q(C, M, C)$ is irreducible,

$$g \cdot Q(C, M, C) = Q(C, M, C)$$

for all $g \in G(M)$. Which shows that every fibre $Q(C, m', c')$ is invariant under the action of $G(M)$. Hence

$$Q(C, m', c') = S_{m'} \text{ when } \exists y\, Q(y, m', c').$$

□

Corollary 5.1.12. *Under the assumptions of Corollary 5.1.11, $\exists y\; Q(y, x, z)$ defines a closed set.*

Proof. The topology on the sort $M \times M \times C$ can be defined from the sort $C \times M \times C$ by the equivalence relation $(y, x, z) \sim (g \cdot y, x, z)$ and the corresponding action of G,

$$C/G \times M \times C \simeq M \times M \times C.$$

Corollary 5.1.11 together with Lemma 5.1.10 proves that Q defines a G-invariant closed subset of $C \times M \times C$. Hence Q/G is closed. This is homeomorphic to the set defined by $\exists y\; Q(y, x, z)$. □

Lemma 5.1.13. *Under the assumptions of Lemma 5.1.10, let b' be strongly independent over $A \subseteq C$ and $m' = \mathrm{pr}(b')$. Then $S_{m'}(y)$ defines a complete type (an atom of the Boolean algebra) over $M' \cup A$.*

Proof. Suppose that the formula $S(y, m')$ does not define a complete type over $A \cup M'$.

Then there is a positive quantifier-free $Q(y, x, z)$ over $\emptyset$ such that for some $a \subset M' \cup A$,

$$\vDash \exists y\, Q(y, m', a) \,\&\, S(y, m') \;\; \& \neg Q(b', m', a).$$

Let b'' satisfy $Q(b'', m', a)$. We may assume that Q and a are such that $Q(y, x, a)$ is the locus of $b''m'$ over $M' \cup A$. So $Q(C', m', a) \subset S_{m'}$.

Since $G(M')$ acts transitively on the fibres, $b' = g \cdot b''$ for some $g \in G(M')$. Hence $g \cdot Q(C', m', a) \subset S_{m'}$, and $g \cdot Q(C', m', a)$ is Zariski closed set defined over $M' \cup A$. Since b' is strongly independent in fibres over A, it follows that $g \cdot Q(C', m', a) = S_{m'}$. Hence $Q(C', m', a) = S_{m'}$. A contradiction. □

Lemma 5.1.14. *Let $b \in C^n$ be strongly independent in fibres over $\emptyset$. Let $m' \subset M'^k$ and $Q(y, x', x, z)$ is a positive quantifier free formula and $mw \subset M$, $m = \mathrm{pr}(b)$, such that $Q(y, x', m, w)$ is the locus of bm' over mw. Then there is a positive quantifier free formula $R(x', x, z)$ over $\emptyset$ such that*

$$Q(y, x', x, z) \equiv R(x', x, z) \;\&\; x = \mathrm{pr}(y)$$

Proof. Let $R(x', x, z)$ be the formula $\exists y\, Q(y, x', x, z)$.

Claim 1. $Q(y, x', x, z) \equiv R(x', x, z) \,\&\, x = \mathrm{pr}(y)$.

Indeed, the implication from left to right is obvious. To see the inverse we need to prove that for any (a', b, a, c)

$$\vDash \forall y\, (Q(b, a', a, c) \;\&\; a = \mathrm{pr}\ y) \to Q(y, a', a, c)$$

But this formula immediately follows from Lemma 5.1.13. This proves the claim.

The claim implies that the closed subset Q of $C^n \times M^l$, some l, is saturated with respect to the equivalence relation on C^n given by the action of group G^n. Moreover, the subset R defined by $R(x', x, z)$ in the topological sort M^l is closed by the definition of the topology (see Definition 5.1.1). Hence $R(x', x, z)$ is positive quantifier free. □

For the rest of this chapter, especially for the next section, where we will consider the specialisations of Zariski cover structures, we are making further restrictions. We are making the following assumptions, which are to be valid for the rest of the chapter:

- G is definably almost simple (i.e. proper definable normal subgroups are finite); and any definable normal subgroup $H \lhd G^k$ (for any $k \in \mathbb{N}$) is definable without parameters.

- The group G is definable in M, so that $G \subset M^l$ is a definable subset for some l.

- The fibres $\mathrm{pr}^{-1}(m)$, for all $m \in M$, are atoms over M.

It is important here to note that generally the definition of definably almost simple requires the group to be non-abelian, and (definably) connected. In our version we are not making this restriction. In other words, we are allowing abelian groups, and groups which are not connected to be definably almost simple (e.g. finite groups are automatically definably almost simple).

Lemma 5.1.15. *Let $b = b_1 b_2$, where $b_1 = (b_{11}, \ldots, b_{1n}) \in C^n$ is a tuple and b_2 is a singleton such that $b_{11}, \ldots, b_{1n}, b_2$, is not strongly independent in fibres over $\emptyset$. Assume also that the locus $\varphi(b_1, C)$ of b_2 over Mb_1 is a proper subset of the fibre containing b_2.*

Then there is a finite, $\emptyset$-definable $H \lhd G$ such that

$$H \cdot b_2 = \varphi(b_1, C)$$

and $\varphi(b_1, C)$ is an atom over $M \cup \{b_1\}$. Moreover, for any $b_1' \equiv_M b_1$ there is b_2' such that

$$H \cdot b_2' = \varphi(b_1', C).$$

Proof. Note first that under the assumptions, $\mathrm{pr}^{-1}(m_2)$ is an atom over M.

Let $\varphi(b_1, C)$ be the locus of b_2 over Mb_1. Observe that $\varphi(b_1, C)$ is an atom over Mb_1. If not we may assume that there is a proper Mb_1-closed subset $\psi(b_1, C) \subset \varphi(b_1, C)$. There is a $g \in G$ such that $b_2 \in g \cdot \psi(b_1, C)$. But $g \cdot \psi(b_1, C)$ is Mb_1-closed.

Define a binary relation E_{b_1} on $\mathrm{pr}^{-1}(m_2)$ as follows:

$$E_{b_1}(x, y) :\Leftrightarrow \exists g \in G \ (x \in g \cdot \varphi(b_1, C) \ \& \ y \in g \cdot \varphi(b_1, C)).$$

E_{b_1} is an Mb_1-definable equivalence relation on $\mathrm{pr}^{-1}(m_2)$. All equivalence classes are shifts of $\varphi(b_1, C)$ by elements of G.

Reflexivity and symmetry of E_{b_1} is obvious. We need the following claim to prove transitivity:

Claim 1. Let $g_1, g_2 \in G$, and assume $g_1 \cdot \varphi(b_1, C) \cap g_2 \cdot \varphi(b_1, C) \neq \emptyset$. Then $g_1 \cdot \varphi(b_1, C) = g_2 \cdot \varphi(b_1, C)$.

Proof. Suppose $y \in g_1 \cdot \varphi(b_1, C) \cap g_2 \cdot \varphi(b_1, C)$. Then there is an $h \in G$ such that $h \cdot y = b_2$. Then, $b_2 \in hg_1 \cdot \varphi(b_1, C) \cap hg_2 \cdot \varphi(b_1, C)$. Since $\varphi(b_1, C)$ is the locus of b_2 over Mb_1

$$hg_1 \cdot \varphi(b_1, C) = hg_2 \cdot \varphi(b_1, C) = \varphi(b_1, C) = loc(b_2/Mb_1).$$

Then it follows that $g_1 \cdot \varphi(b_1, C) = g_2 \cdot \varphi(b_1, C)$. This proves the claim. □

Next we will show that $E_{b_1}(y, z)$ is transitive. Let $E_{b_1}(x, y)$ and $E_{b_1}(y, z)$. Then there are $g_1, g_2 \in G$ such that

$$x \in g_1 \cdot \varphi(b_1, C) \ \&\ y \in g_1 \cdot \varphi(b_1, C)$$
$$y \in g_2 \cdot \varphi(b_1, C) \ \&\ z \in g_2 \cdot \varphi(b_1, C).$$

Observe that $g_1 \cdot \varphi(b_1, C) \cap g_2 \cdot \varphi(b_1, C) \neq \emptyset$, namely y is in the intersection. Then by the previous claim

$$g_1 \cdot \varphi(b_1, C) = g_2 \cdot \varphi(b_1, C).$$

Therefore $x \in g_2 \cdot \varphi(b_1, C)$. Hence $E_{b_1}(x, z)$.

Claim 2. E_{b_1} is M-definable.

Proof. Let $b_1' \equiv_M b_1$. Then $\varphi(b_1', C)$ is the locus of some b_2' over Mb_1'. Then by repeating the argument above one sees that $E_{b_1'}$, defined in the same way, is an equivalence relation whose classes are shifts of $\varphi(b_1', C)$; which is an atom over Mb_1'. In particular, $b_2 \in g \cdot \varphi(b_1', C)$ for some $g \in G$. Since $b_1' = f \cdot b_1$ for some $f \in G^n$, we see that $g \cdot \varphi(b_1', C)$ is Mb_1-definable and hence, $g \cdot \varphi(b_1', C) = \varphi(b_1, C)$. It follows, E_{b_1} and $E_{b_1'}$ have the same classes, and thus are equal. The claim follows. □

Let $E := E_{b_1}$. Define the subset $H \subset G$ as follows:

$$g \in H :\Leftrightarrow g \cdot b_2 \in \varphi(b_1, C)$$

Then H is actually a subgroup: First note that $g \cdot \varphi(b_1, C) = \varphi(b_1, C)$. Then it is immediate that product of two elements of H is again in H. It also follows from this observation that H is closed under inversion; let $h \in H$ and $b_2' := h^{-1} \cdot b_2$. Then $b_2' \in h^{-1} \cdot \varphi(b_1, C)$. By definition of H we also know that $b_2 \in h^{-1} \cdot \varphi(b_1, C)$. Since $h^{-1} \cdot \varphi(b_1, C) \cap \varphi(b_1, C) \neq \emptyset$, we know that $h^{-1} \cdot \varphi(b_1, C) = \varphi(b_1, C)$. Hence $b_2' = h^{-1} \cdot b_2 \in \varphi(b_1, C)$. Moreover, $H \cdot b_2 = \varphi(b_1, C)$.

Claim 3. For any $c \in \mathrm{pr}^{-1}(m_2)$ we have $E(c, C) = H \cdot c$.

Proof. The set

$$\{c \in \mathrm{pr}^{-1}(m_2) : \ E(c, C) = H \cdot c\}$$

is M-definable (as M is stably embedded in C) and contains b_2. Hence, it must be equal to $\mathrm{pr}^{-1}(m_2)$ since the latter is an atom. This proves the claim. $\square$

It follows that $E(x, y) \equiv \exists h \in H \ y = h \cdot x$, that is E is induced by the action of the definable subgroup H. By Fact 2.1.43, H is a closed subgroup. Then by the proof of 2.1.47, E is a Zariski closed subset of $\mathrm{pr}^{-1}(m_2) \times \mathrm{pr}^{-1}(m_2)$.

Also, H is normal. Indeed, any E-class has the form $H \cdot c$, and for any $g \in G$ we have a class $H \cdot (g \cdot c)$. On the other hand, by definition of E, we know that the action of an element of G takes an equivalence class to another equivalence class. Therefore $g \cdot (H \cdot c) = g \cdot (c/E)$ is an equivalence class. Moreover, we have $g \cdot (H \cdot c) = H \cdot (g \cdot c)$ which implies

$$gHg^{-1} = H$$

Since G is assumed to be definably almost simple, H is finite. Also, H is $\emptyset$-definable by assumption on G. $\square$

Theorem 5.1.16. *Let $b = b_1 b_2$, where $b_1 = (b_{11}, \ldots, b_{1n}) \in C^n$ is strongly independent in fibres over U and $b_2 = (b_{21}, \ldots, b_{2k})$ is such that $b_{11}, \ldots, b_{1n}, b_{2i}$, for each $i = 1, \ldots, k$ is not strongly independent. Let $m_1 = \mathrm{pr}(b_1)$, and $m_2 = \mathrm{pr}(b_2)$.*

Then there are $\emptyset$-definable normal finite subgroups $H_i \triangleleft G$ for $i = 1, \ldots, k$, a $\emptyset$-definable subgroup $H \leq H_1 \times \cdots \times H_k$, and a positive quantifier free formula $\varphi(y_1, y_2)$ over UM such that

$$\models \varphi(b_1, b_2) \text{ and } H \cdot b_2 = \varphi(b_1, C^k)$$

and $\varphi(b_1, C^k)$ is an atom over $M \cup \{b_1\} \cup U$.

Proof. First, let us consider the case $U = \emptyset$. For any b_{2i}, its locus $\varphi_i(b_1, C)$ over Mb_1 is a proper subset of $\mathrm{pr}^{-1}(m_{2i})$. For $i = 1, \ldots, k$, let H_i be the finite, $\emptyset$-definable normal subgroup of G given by Lemma 5.1.15. Let $\varphi(b_1, C^k)$ be the locus of b_2 over Mb_1. Then

$$\varphi(b_1, C^k) \subset (H_1 \times \cdots \times H_k) \cdot b_2$$

Observe that, similarly to the proof of Lemma 5.1.15, $\varphi(b_1, C^k)$ is an atom over Mb_1. From a similar argument it follows that there is an Mb_1-definable subgroup $H < G^k$ such that $\varphi(b_1, C^k) = H \cdot b_2$. In fact, $H < H_1 \times \cdots \times H_k$. Therefore H

is finite. In particular, it is a finite subset of the prime model; hence $\emptyset$-definable as elements of the prime model are named. This proves the theorem in case $U = \emptyset$.

In the general case we may assume that $U = u \subset C$ is a finite subset, and thus $b_1' := ub_1$ would satisfy the assumptions we used for $b_1 b_2$ in Lemma 5.1.15 and its application. $\square$

The parameters used from M in the formula φ in Lemma 5.1.15 and Theorem 5.1.16 are hidden for the sake of brevity. Let $b = b_1 b_2$ be a tuple split into its maximal strongly independent and dependent parts as above, with $\mathrm{pr}(b_1) = m_1$ and $\mathrm{pr}(b_2) = m_2$. The parameters involved in φ from M are m_1, m_2 and some $s_b \subset M$.

Corollary 5.1.17. *Given $b = b_1 b_2$ is as in Theorem 5.1.16, there is a U-definable function $f_u : D_u \to C^k/H$, where $\{s_b m_1 m_2\} \times \mathrm{pr}^{-1}(m_1) \subseteq D_u$, and $m_1 = \mathrm{pr}(b_1)$, such that*

$$f_u(s_b, m_1, m_2, b_1) = H \cdot b_2 = \hat{b}_2 \in \mathrm{pr}^{-1}(m_2)/H$$

Proof. $f_u(x, x_1, x_2, y_1) = z$ is given by the formula

$$\exists y_2 (\varphi(x, x_1, x_2, y_1, y_2) \;\&\; z = H \cdot y_2)$$

D_u is defined by

$$(x, x_1, x_2, y_1) \in D_u \Leftrightarrow \exists y_2 \left\{\varphi_u(x, x_1, x_2, y_1, y_2) \;\&\; \forall y_2' \left(\varphi_u(x, x_1, x_2, y_1, y_2') \to y_2' \in H \cdot y_2\right)\right\}$$

$\square$

5.1.18. Continuous Connections (CC).

We assume that for every $b = b_1 b_2$, and U as in Theorem 5.1.16, there is a finite $s_b \subset M$ and a function f_u as in Corollary 5.1.17 such that, in notations of Corollary 5.1.17:

- There is a closed u-definable subset D_u^0 of $M^l \times C^n$ (where $l = |s_b| + |m_1| + |m_2|$) containing (s_b, m_1, m_2, b_1) such that $f_u(x, x_1, x_2, y_1)$ is defined and $f : D_u^0 \to C^{|m_2|}/H$ is a morphism.

- There is a closed u-definable subset $D_u^\dagger$ of $M^t \times C^{k_1} \times C^{k_2}/H$ (where $t = |m_1| + |m_2|$, $k_1 = |b_1|$ and $k_2 = |b_2|$) containing $(m_1, m_2, b_1, \hat{b}_2)$ and a morphism $f_u^\dagger : D_u^\dagger \to M$.

- If $(x, x_1, x_2, y_1) \in D_u^0$ and $f_u(x, x_1, x_2, y_1) = \hat{y}_2$, then $(x_1, x_2, y_1, \hat{y}_2) \in D_u^\dagger$ and $f^\dagger(x_1, x_2, y_1, \hat{y}_2) = x$.

Lemma 5.1.19. *Let $\mathcal{C} = (C, M, \mathrm{pr})$ be a Zariski cover structure satisfying the Continuous Connections assumption, then every model of $\mathrm{Th}(\mathcal{C})$ is a Zariski cover structure and satisfies (CC).*

Proof. Let $\mathcal{D} \models \mathrm{Th}(\mathcal{C})$. First let us check that $\mathcal{D}$ is a Zariski cover structure. It is clear that $\mathcal{D}$ is a two sorted Zariski structure. We will write $\mathcal{D} = (D, N)$. The interpretation of the topological Zariski group G gives a Zariski topological group B interpretable in N. Also B will act morphically and freely with Zariski automorphisms as all of these are first order properties. This is enough to see that $\mathcal{D}$ is a Zariski cover structure. By construction, $\mathcal{D}$ satisfies (CC). $\square$

Remark 5.1.20. In general, showing a certain Zariski cover structure satisfies (CC) seems to require "efficient" quantifier elimination. One, in particular, which will enable us to describe closed relations between fibres as easily and explicitly possible.

We conclude this section with a preliminary analysis of (CC) for a special case of the covers related to the first example of a non-algebraic Zariski structure. We will consider the following special case. Let us briefly recall the construction.

Let M be a one dimensional Zariski geometry, and $B \leq ZAut(M)$ be a group acting freely on M by Zariski automorphisms. Let B^* be a group extension of B with a finite kernel G:

$$1 \to G \to B^* \to B \to 1$$

Let C be a set such that B^* acts freely on it in the same way B acts on M. I.e. the number of B^* orbits is the same as number of G orbits.

For $c, d \in C$ define $c \equiv d$ if and only if there is an $h \in G$ such that $h \cdot c = d$. Clearly $\equiv$ is an equivalence relation invariant under the action of B^*. Moreover $\equiv$ is closed (hence definable). So G acts trivially on the quotient $C/\equiv$, and hence the action of B^* will give an action of B on this quotient. By construction $C/\equiv$ is isomorphic to M as B-sets, say via a map pr. This map pr can be extended naturally to $\mathrm{pr} : C_0 \to M_0$ so that $\mathrm{pr}(g^* c) = p(g^*)\,\mathrm{pr}(c)$ for all $g^* \in B^*$, where $p : B^* \to B$ is the group homomorphism.

Then C_0 is made into a Zariski geometry by defining pull-backs of the closed sets of M_0^n via pr and graphs of elements of G^* as basic relations (of an associated language) and declaring the Boolean combinations of these predicates as closed sets.

Let $\overline{b} = (b_1, \ldots, b_n) \in C^n$. Then $\overline{b}$ is strongly independent in fibres over $\emptyset$ if and only if b_i is not in the orbit of b_j for any i, j.

Now let $\overline{b} \in C^n$, write $\overline{b} = b_1 b_2$ where b_1 is maximal strongly independent in fibres over $\emptyset$ and b_2 is the rest. Let $\varphi(b_1, C^k)$ be the locus of b_2 over Mb_1. By

Corollary 5.1.17, there is a $\emptyset$-definable function $f : D \to C^k/H$ where $H < G^k$, $\{s_b m_1 m_2\} \times \mathrm{pr}^{-1}(m_1) \subseteq D$, $m_1 = \mathrm{pr}(b_1), m_2 = \mathrm{pr}(b_2)$ and $s_b \subset M$ is a parameter. The function f is defined as

$$\exists y_2 (\varphi(x, x_1, x_2, y_1, y_2) \,\&\, z = H \cdot y_2)$$

and D is defined as

$$(x, x_1, x_2, y_1) \in D \Leftrightarrow \exists y_2 \,\{\varphi(x, x_1, x_2, y_1, y_2) \,\&\, \forall y_2' \,(\varphi(x, x_1, x_2, y_1, y_2') \to y_2' \in H \cdot y_2)\}$$

In the next paragraph we will see that the parameter s_b is actually unnecessary. Later in the analysis, we will even conclude that m_1, m_2 are also not necessary as parameters.

We claim that the locus $\varphi(b_1, C^k)$ is the singleton $\{b_2\}$. Write $b_1 = (b_{11}, \ldots, b_{1n})$ and $b_2 = (b_{21}, \ldots, b_{2k})$. Since $b_1 b_{2i}$, for all $i = 1, \ldots, k$, are dependent in fibres, there is a $b_{1j} \in b_1$ and a $g_{ij} \in B^*$ such that $g_{ij} \cdot b_{1j} = b_{2i}$. The pair g_{ij} and b_{1j} uniquely determine b_{2i}. Therefore, b_2 is in the definable closure of b_1. Moreover, as the graphs of elements of B^* are closed sets, we see that $\{b_2\}$ is a b_1-closed set (i.e. it is a closed set only using b_1 as a parameter). Hence the locus $\varphi(b_1, C^k)$ of b_2 is the singleton $\{b_2\}$. Hence the only subgroup H of $(B^*)^k$ such that $H \cdot b_2 = \varphi(b_1, C^k)$ is the trivial subgroup.

From the analysis above, we see that

$$\models \forall x, x_1, x_2, y_1, y_2 \,(\varphi(x, x_1, x_2, y_1, y_2) \to \bigwedge_{i,j} g_{ij}(y_{1j}) = y_{2i}).$$

Which implies

$$\models \forall x, x_1, x_2, y_1 \,\exists! y_2 \,(\varphi(x, x_1, x_2, y_1, y_2)).$$

With this observation, Corollary 5.1.17 yields that there is a $\emptyset$-definable $f : D \to C^k$ given by $\varphi(x, x_1, x_2, y_1, y_2)$ and $D = M^t \times C^n$ where $t = n + k + |x|$. Since D and C^k are pre-smooth, and f has a closed graph, by the Closed Graph Theorem (Lemma 2.1.11) f is continuous. Hence a morphism. For further reduction, one can take $D^0 := \{s_b m_1 m_2\} \times \mathrm{pr}^{-1}(m_1)$, which is a finite, and hence a closed subset of $M^t \times C^n$. And directly show that $f \times \mathrm{id} : D^0 \times C^m \to C^n \times C^m$ is continuous for all n. Hence, again a morphism.

Furthermore, as it is evident from the above analysis, the parameters s_b, m_1, m_2 are not essential. As we have shown Mb_1-locus of b_2 is the same as the locus of b_2

over b_1. But the b_1-locus of b_2 is given by $\bigwedge_{i,j} g_{ij}(y_{1j}) = y_{2i}$. Which defines a function

$$C^n \to C^k$$
$$b_1 \mapsto b_2$$

where b_{2i} are obtained from the corresponding b_{1j} and g_{ij}. As the parameter s_b is not needed in this case, it seems that the function $f^\dagger$ will not play a role. Indeed, as it is done in the proof of Theorem 4.4.7, the analysis of the relevant relations of the type $g_{ij}(y_{1j}) = y_{2i}$ is enough to obtain the result without using $f^\dagger$.

5.2 Specialisations of Zariski Cover Structures

In this section we analyse the specialisations of Zariski cover structures. The first two results (Lemma 5.2.1 and Theorem 5.2.2) are however, not restricted to Zariski cover structures and are valid for the more general setting of (multi-sorted) Zariski structures. It was certainly possible to include them to the general discussion about specialisations in Chapter 2. However, it is more appropriate to present them here, as they are directly related to the analysis of specialisations of Zariski cover structures, and will be used in many of the results in this chapter. For this reason, although they are stated for the general case, it is helpful to keep in mind they will be applied in the setting of Zariski cover structures in the rest of this chapter.

Lemma 5.2.1. *Let $\mathcal{C}$ be a Zariski structure which is prime and minimal over an $\emptyset$-definable subset M. Then $\mathcal{C}$ is atomic over $M \cup A$ for any $A \subseteq C$.*

Proof. The theory is $Th(\mathcal{C})$ is ω-stable. Then there exists $\mathcal{C}^\flat \succeq \mathcal{C}$ which contains $M \cup A$ and is atomic over $M \cup A$.

Since $\mathcal{C}$ is prime and minimal over M, we get $\mathcal{C}^\flat = \mathcal{C}$. Hence $\mathcal{C}$ is atomic over $M \cup A$. □

Following Lemma 5.2.1, a useful characterisation of $\aleph_0$-universal specialisations can be given. The following equivalence will certainly be useful for the analysis of $\aleph_0$-universal specialisations of Zariski cover structures (in particular see Theorem 5.2.14). Note that the second condition of the Theorem 5.2.2 below implies "π is an $\aleph_0$-universal specialisation".

Theorem 5.2.2. *Let $\mathcal{C}_0 \prec \mathcal{C}$ be a Zariski structure and its $\aleph_0$-saturated extension, $\pi : \mathcal{C} \to \mathcal{C}_0$ a specialisation. Suppose also that there is a $\emptyset$-definable subset M such that for every $A \subset C$, $\mathcal{C}$ is atomic over $M \cup A$.*

Then the following are equivalent:

(i) For any finite $c \subset M \cup \mathrm{Dom}(\pi)$*, any finite tuple* $b' \in \mathcal{C}' \succeq \mathcal{C}$*, and for any specialisation*

$$\pi' : Cb' \to C_0$$

extending π*, there is* $b \subset \mathrm{Dom}(\pi)$ *such that* $b \equiv_c b'$ *and* $\pi(b) = \pi'(b')$*.*

(ii) For any finite $a \subset \mathcal{C}$*, any finite tuple* $b' \in \mathcal{C}' \succeq \mathcal{C}$*, for any specialisation*

$$\pi' : Cb' \to C_0$$

extending π*, there is* $b \subset \mathrm{Dom}(\pi)$ *such that* $b \equiv_a b'$ *and* $\pi(b) = \pi'(b')$*.*

Proof. We only need to prove (i) implies (ii). Suppose for a contradiction that (ii) fails for some π', a and b'. Then, a cannot be a subset of $\mathrm{Dom}(\pi)$, as we are assuming (i).

Let $p = \mathrm{tp}(a/M \cup \mathrm{Dom}(\pi))$. By assumption, $\mathcal{C}$ is atomic over $M \cup \mathrm{Dom}(\pi)$. Therefore p is principal; so it is equivalent to a formula $P(z)$ over some $c \subset M \cup \mathrm{Dom}(\pi)$.

Let $q(z, y) = \mathrm{tp}(a, b'/\emptyset)$ and

$$t(y) = \bigwedge_{Q \in q} \exists z\ P(z)\ \&\ Q(z, y)$$

By construction, t is a type over c. Clearly, b' realises t. By (i) there is $b \in \mathrm{Dom}(\pi)$ realising t, and $\pi(b) = \pi'(b')$. Then $\{P(z)\} \cup q(z, b)$ is consistent, and by saturation must have a realisation in $\mathcal{C}$. Since P is complete (is an atom) over $M \cup \mathrm{Dom}(\pi)$, we have $P(z) \vdash q(z, b)$. It follows $\models q(a, b)$, and $b \equiv_a b'$. A contradiction to our assumptions. $\square$

At this point, it is reasonable to repeat the assumptions made towards the end of the previous section. The following are standing assumptions for the rest of this chapter:

- G is definably almost simple (i.e. definable normal subgroups are finite); and any definable normal subgroup $H \triangleleft G^k$ (for any $k \in \mathbb{N}$) is definable without parameters.

- The group G is definable in M, so that $G \subset M^l$ is a definable subset, for some l.

- The fibres $\mathrm{pr}^{-1}(m)$, for all $m \in M$, are atoms over M.

Proposition 5.2.3. *Let $\mathcal{C}_0 \prec \mathcal{C}$ be a Zariski structure and its extension, let $\pi : \mathcal{C} \to \mathcal{C}_0$ be an $\aleph_0$-universal specialisation. Assume that M is a sort in $\mathcal{C}$ and $\mathrm{pr} : C \to M$ is a Zariski cover structure satisfying (CC).*

Let $\pi_M : M \to M_0$ be the restriction of π to the substructure. Then π_M is $\aleph_0$-universal.

Proof. Let $M' \succ M$, and $n' \subset M'$ be a finite tuple. Let $\pi'_M : M' \to M_0$ be a specialisation extending $\pi_M : M \to M_0$ with $n' \subset \mathrm{Dom}(\pi'_M)$, and $\pi'_M(n') = n_0$.

We claim that $\pi'_M \cup \pi$ is a specialisation extending π. This will imply that the type of n' over a finite subset of M is realised in $\mathcal{C}$, and so in M, by some n such that $\pi(n) = n_0$. In other words, that π_M is $\aleph_0$-universal.

Consider a positive quantifier free formula $Q(x', y)$ in $\mathcal{C}$ such that $Q(x', b)$, for some b in $\mathrm{Dom}(\pi)$, is the locus of n' over $\mathrm{Dom}(\pi)$. Let $n := \mathrm{pr}(b)$. Then it follows $n \in \mathrm{Dom}(\pi)$.

Let $b = b_1 b_2$ be the splitting of b into b_1, which is maximal strongly independent in fibres over $\emptyset$, and b_2 such that

$$\hat{b}_2 = f(s_b, m_1, m_2, b_1), \quad s_b = f^\dagger(m_1, m_2, b_1, \hat{b}_2)$$

where $m_1 = \mathrm{pr}(b_1)$ and $m_2 = \mathrm{pr}(b_2)$. Note that $m_1, m_2 \in \mathrm{Dom}(\pi)$, since $b_1, b_2 \in \mathrm{Dom}(\pi)$.

We have $\hat{b}_2 \in \mathrm{Dom}(\pi)$, since $b_2 \in \mathrm{Dom}(\pi)$. Then $s_b \in \mathrm{Dom}(\pi)$, since $f^\dagger$ is a morphism. Now we replace $Q(x', y)$ with the formula

$$Q^*(x', x, w, y_1) \equiv \exists y_2 \; Q(x', y_1 y_2) \;\&\; [\hat{y}_2 = f(w, x, y_1)]$$

where $[\hat{y}_2 = f(w, x, y_1)]$ is the positive quantifier free formula (see 5.1.18)

$$(w, x, y_1) \in D^0 \;\&\; (x, y_1 \hat{y}_2) \in D^\dagger \;\&\; \hat{y}_2 = f(w, x, y_1).$$

In particular $\hat{y}_2 = H \cdot y_2 \in C^{|b_2|}/H$, where H is the finite group given in relation to b_2 by Theorem 5.1.16.

We claim that Q^* defines a Zariski closed set. Let $k = |x' x w|$ and $t = |y| = |y_1 y_2|$. Define the function

$$\mathrm{id} \times \mathrm{p} : M^k \times C^t \to M^k \times C^{|y_1|} \times C^{|y_2|}/H$$

by $\mathrm{id} \times \mathrm{p}(x' x w y_1 y_2) = (x' x w y_1 \hat{y}_2)$; i.e. it is identity on $M^k \times C^{|y_1|}$ and $\mathrm{p} : C^{|y_2|} \to C^{|y_2|}/H$ is the canonical quotient map. The image $M^k \times C^{|y_1|} \times C^{|y_2|}/H$ can be identified with the sort $(M^k \times C^t)/H$ by taking the action of H on M^k to be trivial. Since H is finite, $(M^k \times C^t)/H$ is an orbifold; and $\mathrm{id} \times \mathrm{p}$ becomes the canonical quotient map.

It follows that $Q^*(x', x, w, y_1)$ is the image (in the orbifold $(M^k \times C^t)/H$) under $\mathrm{id} \times \mathrm{p}$ of the closed set defined by

$$Q(x', y_1 y_2) \ \& \ [\hat{y}_2 = f(w, x, y_1)]$$

Then by Lemma 2.1.48, $Q^*(x', x, w, y_1)$ is closed.

Next we claim that $Q^*(x', n, s_b, b_1)$ defines the locus of n' over $\mathrm{Dom}(\pi)$. It is enough to prove that

$$Q^*(M', n, s_b, b_1) \subseteq Q(M', b)$$

Note that by the construction of formula Q^*,

$$\vDash Q^*(n'', n, s_b, b_1) \Rightarrow \ \vDash Q(n'', b_1 b_2') \text{ for some } b_2' \in \hat{b}_2$$

where $\hat{b}_2 = H \cdot b_2$. By Theorem 5.1.16, $H \cdot b_2$ is an atom over $M' b_1$. Hence $\vDash Q(n'', b_1 b_2')$ if and only if $\vDash Q(n'', b_1 b_2)$. This completes the proof of the claim.

Clearly,

$$\vDash Q^*(n', n, s_b, b_1)$$

and $Q^*(x', x, w, y_1)$ satisfies assumptions of Lemma 5.1.14. Hence, splitting $x = x_1 x_2$, $|x_1| = |y_1|$ (in correspondence with $n = m_1 m_2$) we get by Lemma 5.1.14

$$Q^*(x', x_1 x_2, w, y_1) \equiv R(x', x_1, x_2, w) \ \& \ x_1 = \mathrm{pr}(y_1)$$

for some positive quantifier free formula R.

This implies that the locus of n' over $\mathrm{Dom}(\pi)$ is determined by the formula $R(x', m_1, m_2, s_b)$, where $m_1, m_2, s_b \subset \mathrm{Dom}(\pi_M)$. Since $\pi'_M(n') = n_0$ we get

$$\vDash R(n_0, \pi(m_1), \pi(m_2), \pi(s_b))$$

and hence

$$Q(n_0, \pi(b))$$

This proves that $\pi'_M \cup \pi$ preserves positive quantifier free formulae over $\mathrm{Dom}(\pi)$ and hence is an extension of specialisation π. $\square$

In the rest of this section we will study specialisations on Zariski cover structures. Let $\mathcal{C}_0 = (C_0, M_0, \mathrm{pr})$ be a two sorted Zariski cover structure $\mathrm{pr} : C_0 \to M_0$. Let $\mathcal{L}$ denote the Zariski language for the two sorted Zariski structure (C_0, M_0). Let $\mathcal{C} = (C, M, \mathrm{pr})$, such that $\mathcal{C} \succeq \mathcal{C}_0$ and let $\pi : \mathcal{C} \to \mathcal{C}_0$ be a maximal specialisation such that its restriction $\pi_M : M \to M_0$ is an $\aleph_0$-universal specialisation. This sets the context for the rest of this section. For all of the results, the assumptions made here on $(\mathcal{C}, \mathcal{C}_0, \pi)$ are valid; most importantly, that π is maximal and π_M is $\aleph_0$-universal.

By $\mathcal{L}^\pi$ we denote the language $\mathcal{L}$ expanded with a symbol π which will be interpreted as the specialisation. In fact, we will consider M_0 along with the definable Zariski topological group G as assumed previously. We will write $G(M_0)$ to indicate the realisation of G in the Zariski structure M_0. Therefore the elementary extension M will be considered with the corresponding definable Zariski topological group $G(M)$. Often we will also consider π_M as a specialisation $G(M) \to G(M_0)$ of the Zariski topological groups in the natural way.

Lemma 5.2.4. *Let $\mathcal{C}_0$ be a Zariski cover structure, and $\mathcal{C} \succeq \mathcal{C}_0$ be an extension. Let $T = D/H$ be an orbifold where $H < G$ is a finite $\emptyset$-definable subgroup acting freely on a constructible set $D \subset \mathcal{C}$. Let $\pi : \mathcal{C} \cup T(\mathcal{C}) \to \mathcal{C}_0 \cup T(\mathcal{C}_0)$ be a maximal specialisation, and let $t \in T(\mathcal{C}) \cap \mathrm{Dom}(\pi)$. Then $\mathrm{p}^{-1}(t) \subset \mathrm{Dom}(\pi)$.*

Proof. Consider an $a \in \mathrm{p}^{-1}(t)$ and a positive quantifier free formula $Q(y, z)$, such that for some $c \subset C$, the formula $Q(y, c)$ is the locus of a over $\mathrm{Dom}(\pi)$. Denote

$$Q(x, y, z) :\equiv Q(y, z) \ \& \ \mathrm{p}_T(y) = x$$

By Lemma 2.1.48 we see that the formula $\exists y\, Q(x, y, z)$ defines a closed subset which by construction contains (t, c) and so does contain $(\pi(t), \pi(c))$. The latter means that

$$\mathcal{C}_0 \vDash \exists y (Q(y, \pi(c)) \ \& \ \mathrm{p}_T(y) = t)$$

Let a_0 satisfy the formula $Q(y, \pi(c)) \ \& \ \mathrm{p}_T(y) = t$.

Now it is clear that setting $\pi(a) := a_0$ we will have extension of the specialisation π to a. Since π is maximal, a must be in $\mathrm{Dom}(\pi)$.

Since H is a substructure of the prime model, $H \subset \mathrm{Dom}(\pi)$. Recall that $\mathrm{p}^{-1}(t) = H \cdot a$. By Lemma 2.1.54, $H \cdot a \subset \mathrm{Dom}(\pi)$. □

Lemma 5.2.5. *Suppose that every $\aleph_0$-saturated model of the $\mathcal{L}^\pi$-theory* $\mathrm{Th}(\mathcal{C}, \mathcal{C}_0, \pi)$ *is $\aleph_0$-universal. Then every κ-saturated model of the theory is κ-universal.*

Proof. Let $(\mathcal{D}, \mathcal{D}_0, \pi)$ be a κ-saturated model of the theory $\mathrm{Th}(\mathcal{C}, \mathcal{C}_0, \pi)$. So in particular it is $\aleph_0$-saturated. Hence by assumption π is $\aleph_0$-universal. Let $\mathcal{D}' \succeq \mathcal{D}$, $A \subseteq \mathcal{D}'$ with $|A| < \kappa$ and $\pi' : A \cup \mathcal{D} \to \mathcal{D}_0$ a specialisation extending π. Let $A_0 := A \cap \mathcal{D}$, and $A' := A \setminus A_0$. Also, for any element of A, without loss of generality we may assume that its image under π' is in A_0.

Let $a' \subset A'$ be an arbitrary finite tuple, and $B \subset A_0$ be an arbitrary finite subset. Let $\pi'(a') =: a_0$. We need to show that the $\mathcal{L}^\pi$-type

$$p_{a'}(x/B) := \mathrm{tp}(a'/B) \cup \{\pi(x) = a_0\}$$

is satisfiable in $\mathcal{D}$, where $\mathrm{tp}(a'/B)$ is the $\mathcal{L}$-type.

Since a' is finite and π' is $\aleph_0$-universal, there is an embedding $\sigma : a'B \to \mathcal{D}$ over B such that $\pi'(a') = \pi(\sigma a')$. Set $a = \sigma(a')$. Then, $a \models \mathrm{tp}(a'/B)$, and $\pi(a) = a_0 = \pi'(a')$.

By compactness, the type

$$\bigcup_{\substack{a' \subset A' \\ B \subset A_0}} p_{a'}(x/B)$$

where a' ranges over all finite tuples in A' and B ranges over all finite subsets of A_0 is realisable in some elementary extension of $(\mathcal{D}, \mathcal{D}_0, \pi)$. But since $(\mathcal{D}, \mathcal{D}_0, \pi)$ is κ-saturated, in fact there is a realisation of this type in $(\mathcal{D}, \mathcal{D}_0, \pi)$. $\square$

We will study $\aleph_0$-universal specialisations of Zariski cover structures. We will consider $\mathcal{C}' \succeq \mathcal{C}$, a finite subset $A \subset M \cup \mathrm{Dom}(\pi)$, and a finite tuple b' in $\mathcal{C}'$, a specialisation $\pi' : \mathcal{C} \cup \{b'\} \to \mathcal{C}_0$ extending $\pi : \mathcal{C} \to \mathcal{C}_0$. We aim to show that there is a b in $\mathcal{C}$ such that

$$\pi(b) = \pi'(b') \text{ and } \mathrm{tp}(b'/A) = \mathrm{tp}(b/A) \tag{5.1}$$

Lemma 5.2.6. *Let $\mathcal{C}' \succeq \mathcal{C}$, $A \subset M \cup \mathrm{Dom}(\pi)$ be a finite subset, $b' \subseteq \mathcal{C}'$ be a finite tuple, and $\pi' : \mathcal{C}b' \to \mathcal{C}_0$ be a specialisation extending π. Suppose $b' \subset M'$. Then we can find $b \subset M$ such that $\pi(b) = \pi'(b')$ and $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$ (i.e. property (5.1) holds).*

Proof. Since M' is totally transcendental and stably embedded in $\mathcal{C}'$, there is a finite $A_{M'} \subset M'$, such that $\mathrm{tp}(b'/A_{M'}) \vdash \mathrm{tp}(b'/A)$. Since the restriction $\pi'_M : \{b'\} \cup M \to M_0$ of π' is a specialisation extending π_M, and since π_M is $\aleph_0$-universal, there is $b \subset M$ such that $\pi(b) = \pi'(b')$ and $\mathrm{tp}(b'/A_{M'}) = \mathrm{tp}(b/A_{M'})$. Property 5.1 follows. $\square$

From here on we will assume that M is $\aleph_0$ saturated.

Lemma 5.2.7. *Let $\mathcal{C}' \succeq \mathcal{C}$ and $b' \subseteq \mathcal{C}'$ be a finite tuple. Let $\pi' : \mathcal{C}b' \to \mathcal{C}_0$ be a specialisation extending π, and $A \subset M \cup \mathrm{Dom}(\pi)$ be a finite subset. Suppose that* $\mathrm{pr}(b') = \mathrm{pr}(a)$ *for some $a \subset C \cap \mathrm{Dom}(\pi)$. Then there is $b \subset \mathcal{C}$ such that $\pi(b) = \pi'(b')$ and* $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$ *(i.e. property 5.1 holds).*

Proof. Since b' and a are in the same fibre, $b' = g' \cdot a$ for some unique $g' \in G(M')$. Since the group action is free, we may assume g' is in the domain of π'_M. If not, using Lemma 2.1.54, one can extend π'_M to a specialisation $M' \to M_0$ which is defined on g'. With the abuse of notation we will again denote this extension by $\pi'_M : M' \to M_0$, and also write $g' \in \mathrm{Dom}(\pi'_M)$. Let $a_0 := \pi(a)$, $b_0 = \pi'(b')$, and $g_0 = \pi'_M(g')$.

It follows that π'_M is an extension of the specialisation π_M. Since π_M is $\aleph_0$-universal there is $g \in G(M)$ such that $\mathrm{tp}(g/Aa) = \mathrm{tp}(g'/Aa)$ and $\pi_M(g) = g_0$.

Let $b := g \cdot a$. By this definition, $\mathrm{tp}(b'/Aa)$ and $\mathrm{tp}(b/Aa)$ are determined by $\mathrm{tp}(g'/Aa)$ and $\mathrm{tp}(g/Aa)$ respectively. So we have $\mathrm{tp}(b'/Aa) = \mathrm{tp}(b/Aa)$. Finally, again by freeness of the group action, $\pi(b)$ is defined. Then $\pi(b) = \pi(g) \cdot \pi(a) = g_0 \cdot a_0$, that is $\pi(b) = b_0$. Hence our choice of b satisfies property 5.1.

$\square$

Remark 5.2.8. Below, in the proofs of Lemma 5.2.9, Lemma 5.2.10 and Proposition 5.2.12 we make a case distinction between G being infinite and finite. This may look strange at first. As already mentioned in Chapter 2 (see Remark 2.1.42) the main difference here is between G being connected and not connected. The only non-connected case in our setting is when G is finite. By definition, when G is infinite it is irreducible; as G is a group, that means it is connected.

In addition, we keep the case distinction between finite and infinite to emphasize that these cases also correspond to the situation where the fibres of the cover are infinite or finite.

Lemma 5.2.9. *Suppose G is infinite. Let $m \in M^n \cap \mathrm{Dom}(\pi)$, with $m_0 = \pi(m)$, and $b_0 \in \mathrm{pr}^{-1}(m_0) \cap C_0^n$. Let $\mathcal{C}' \succeq \mathcal{C}$. Then, for any $b' \in C'^n \cap \mathrm{pr}^{-1}(m)$ generic over* $\mathrm{Dom}(\pi)$ *there is a specialisation $\pi' : \mathcal{C}b' \to \mathcal{C}_0$ extending π such that $\pi'(b') = b_0$.*

Proof. Note that since G is connected, so is G^n; and hence $\mathrm{pr}^{-1}(m)$ is irreducible. Let $b' \in C'^n$ be an element of $\mathrm{pr}^{-1}(m)$ which is generic over C.

Consider a positive quantifier free formula $Q(y, x, z)$ over $\emptyset$ and a tuple c from $\mathrm{Dom}(\pi)$ such that $\vDash Q(b', m, c)$. We may assume without loss of generality that Q defines the locus of b', m, c over $\emptyset$. By the genericness assumption, $Q(y, m, c) \equiv \mathrm{pr}(y) = m$.

Hence the assumptions of Corollary 5.1.11 are satisfied. Then by Corollary 5.1.12 $\exists y\, Q(y,x,z)$ defines a closed set.

It follows

$$Q(\mathcal{C}_0, \pi(m), \pi(c)) \neq \emptyset \text{ and } Q(\mathcal{C}_0, \pi(m), \pi(c)) = S_{m_0}$$

Hence $\vDash Q(b_0, m_0, \pi(c))$ for any such $Q(y,x,c)$.

Set $\pi'(b') = b_0$. By construction π' is a specialisation extending π. $\square$

Lemma 5.2.10. *Let $\mathcal{C}_0 \preceq \mathcal{C}$ be a Zariski cover structure and its extension. Suppose $m \in M^n \cap \mathrm{Dom}(\pi)$. Then there is $b \in \mathrm{pr}^{-1}(m) \cap \mathrm{Dom}(\pi)$.*

Proof. It is enough to prove the statement for each coordinate of m, so we can assume $n = 1$. We write $\mathrm{pr}^{-1}(m)$ in C as C_m.

When G is finite the statement follows from Lemma 5.2.4. Assume G is infinite and let $\mathcal{C}' \succeq \mathcal{C}$ such that $\mathcal{C}'$ is $|\,\mathrm{Dom}(\pi)|^+$-saturated. We will consider the following two cases.

Case 1. There is $b' \in C'_m$ independent in fibres over $\mathrm{Dom}(\pi)$. Then, by Lemma 5.1.9, every $b \in C_m$ is independent in fibres and is generic over $\mathrm{Dom}(\pi)$. By Lemma 5.2.9 there is an extension π' such that $b \in \mathrm{Dom}(\pi')$. But π is maximal and so $\pi'(b) = \pi(b)$.

Case 2. Any $b' \in C'_m$ is dependent in fibres over $\mathrm{Dom}(\pi)$. Choose $b' \in C'_m$ generic over $\mathrm{Dom}(\pi)$. By Lemma 5.2.9 there is an extension π', and $b' \in \mathrm{Dom}(\pi')$.

Since b' is a singleton, according to notations of the paragraph 5.1.18, we have $b'_1 = \emptyset$ and $b' = b'_2$. Further, there is $s_{b'}$ in M' and $u \in \mathrm{Dom}(\pi)$ such that

$$f_u(s_{b'}, m) = \hat{b}', \quad f_u^\dagger(m, \hat{b}') = s_{b'}$$

The second equality together with the fact that $f^\dagger$ is a morphism implies that π' can be extended to $s_{b'}$. With an abuse of notation we will write $s_{b'} \in \mathrm{Dom}(\pi')$. Consider the restriction of π' to M' and denote it by $\pi'_M : M' \to M_0$.

By Lemma 5.2.6, there is a $s_b \in M$ such that $\mathrm{tp}(s_{b'}/mu) = \mathrm{tp}(s_b/mu)$ and $\pi'(s_{b'}) = \pi(s_b)$. Set $\hat{b} = f_u(s_b, m)$. Then $\hat{b}$ is an element of the topological sort C/H where $H < G$ is finite (see Theorem 5.1.16), and by continuity of f_u, we have $\hat{b} \in \mathrm{Dom}(\pi)$. Hence by Lemma 5.2.4 applied to the sort C/H, there is a $b \in \mathrm{Dom}(\pi)$ such that $\hat{b} = H \cdot b$.

$\square$

Corollary 5.2.11. *Suppose $m \in M^n \cap \mathrm{Dom}(\pi)$, and $A \subset C$ with $|A| < \aleph_0$. Suppose there is $b' \in \mathrm{pr}^{-1}(m)$ in C' and π' extends π so that $b' \in \mathrm{Dom}(\pi')$. Then there is $b \in \mathrm{pr}^{-1}(m) \cap \mathrm{Dom}(\pi)$ such that $\pi(b) = \pi'(b')$ and $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$. In other words property 5.1 holds.*

Proof. Follows from Lemma 5.2.10 and Lemma 5.2.7. □

Next we will consider property 5.1 in more detail. Let us recall that we are considering $\mathcal{C}' \succeq \mathcal{C}$, a finite subset

$$A \subset M \cup \mathrm{Dom}(\pi), \tag{5.2}$$

a finite tuple b', and a specialisation $\pi' : \mathcal{C} \cup \{b'\} \to \mathcal{C}_0$ extending $\pi : \mathcal{C} \to \mathcal{C}_0$. Suppose b' satisfies property 5.1. I.e. there exist a tuple $b \in \mathcal{C}$ such that

$$\pi(b) = \pi'(b') \text{ and } \mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$$

Split $b' = b'_1 b'_2 \in C'^{n+k}$ so that $b'_1 \in C'^n$ is maximal strongly independent in fibres over $\mathrm{Dom}(\pi)$ and b'_2 is the rest.

Then by Continuous Connections,

$$\hat{b}'_2 = f_u(s_{b'}, m'_1, m'_2, b'_1) \text{ and } s_{b'} = f_u^{\dagger}(m'_1, m'_2, b'_1, \hat{b}'_2) \tag{5.3}$$

for the morphism f_u over a finite $u \subset \mathrm{Dom}(\pi)$, for some tuple $s_{b'}$ in M', where $m'_1 = \mathrm{pr}(b'_1)$, and $m'_2 = \mathrm{pr}(b'_2)$. And $\hat{b}'_2 = H \cdot b'_2$ is an element of C'^k/H with H an $\emptyset$-definable finite subgroup of G^n (see Theorem 5.1.16 and Corollary 5.1.17).

Proposition 5.2.12. *Let $\mathcal{C}' \succeq \mathcal{C}$. Assume $b' \in C'^n$ is strongly independent in fibres over $\mathrm{Dom}(\pi)$, $m' = \mathrm{pr}(b')$, $s_{b'} \subset M'$ and $\pi' : \mathcal{C}' \to \mathcal{C}_0$ a specialisation, defined on $s_{b'} m' b'$, extending π. Let $A \subset M \cup \mathrm{Dom}(\pi)$ be of cardinality less than $\aleph_0$. Then there is $s_b m b \subset \mathrm{Dom}(\pi)$ in $\mathcal{C}$ such that*

$$\pi(s_b m b) = \pi'(s_{b'} m' b') \textit{ and } \mathrm{tp}(s_b m b/A) = \mathrm{tp}(s_{b'} m' b'/A)$$

Proof. Since π_M is $\aleph_0$-universal, there is $s_b m \subset M \cap \mathrm{Dom}(\pi)$ such that $\mathrm{tp}(s_{b'} m'/A) = \mathrm{tp}(s_b m/A)$ and $\pi'(s_{b'} m') = \pi(s_b m') =: s_0 m_0$.

Case 1 If G is finite. Then, by Lemma 5.2.4, we get $\mathrm{pr}^{-1}(m) \subset \mathrm{Dom}(\pi)$.

Since the sizes of fibres are equal and π preserves the discrete Zariski topology on the fibres,

$$\pi(\mathrm{pr}^{-1}(m)) = \mathrm{pr}^{-1}(m_0)$$

and π is a bijection on $\mathrm{pr}^{-1}(m)$. In particular, there is $b \in \mathrm{pr}^{-1}(m)$ such that $\pi(b) = b_0$.

We claim that $\mathrm{tp}(s_{b'}m'b'/A) = \mathrm{tp}(s_b mb/A)$. This follows from the fact that $\mathrm{tp}(s_{b'}m'/A)$ expresses that $S(y, m')$ is an atom over $s_{b'}A$, and hence, by equality of types, $S(y, m)$ is an atom over $s_b A$. (Here $S(y, x) \equiv x = \mathrm{pr}(y)$, see Lemma 5.1.10).

Case 2 If G is infinite. Then By Lemma 5.2.9, for any $b'' \in \mathrm{pr}^{-1}(m)$ that is generic over C there is an extension π'' of π such that $\pi''(b'') = b_0$. Note that $\mathrm{tp}(b'/Am')$ is generic.

By Corollary 5.2.11, there exists $b \in \mathrm{pr}^{-1}(m) \cap \mathrm{Dom}(\pi)$ satisfying the generic type over Am in $\mathrm{pr}^{-1}(m)$ with $\pi(b) = b_0$. The type $\mathrm{tp}(b'/Am')$ is generic in $\mathrm{pr}^{-1}(m')$ by assumption on b'. Also $\mathrm{tp}(m'/A) = \mathrm{tp}(m/A)$ by the above choice, hence $\mathrm{tp}(b'/A) = \mathrm{tp}(b/A)$.

$\square$

Corollary 5.2.13. *Let $A \subset M \cup \mathrm{Dom}(\pi)$ be of cardinality less than $\aleph_0$. Assume that $\mathcal{C}$ satisfies the Continuous Connections assumption. Then property (5.1) holds for any $b' \in C'^n \cap (\mathrm{Dom}(\pi'))^n$, where $\mathcal{C}' \succeq \mathcal{C}$ and $\pi' : \mathcal{C}' \to \mathcal{C}_0$ is a specialisation extending π.*

Proof. Write $b' = b'_1 b'_2$ where b'_1 is maximal strongly independent in fibres over A, and b'_2 is the rest. We need to find $b \in \mathcal{C}$ such that

$$\pi(b) = \pi'(b') \text{ and } \mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$$

Let $\mathrm{pr}(b'_1) =: m'_1, \mathrm{pr}(b'_2) =: m'_2$. By Continuous Connections,

$$\hat{b}'_2 = f_u(s_{b'}, m'_1, m'_2, b'_1)$$

for a morphism f_u over a finite u, and $s_{b'} \subset M'$ is some finite tuple. We may assume that $m'_1, m'_2, s_{b'} \in \mathrm{Dom}(\pi')$.

Now, taking Au instead of A in Proposition 5.2.12 and repeating the same argument for $s_{b'}m'_1m'_2b'_1$ gives

$$\mathrm{tp}(s_b, m_1, m_2, b_1/Au) = \mathrm{tp}(s_{b'}, m'_1, m'_2, b'_1/Au)$$

and

$$\pi(s_b, m_1, m_2, b_1) = \pi'(s_{b'}, m'_1, m'_2, b'_1) \tag{5.4}$$

Set $\hat{b}_2 := f_u(s_b, m_1, m_2, b_1)$. It follows from the equality between types, that

$$\mathrm{tp}(b_1\hat{b}_2/Au) = \mathrm{tp}(b'_1\hat{b}'_2/Au)$$

By applying Lemma 2.1.56 to $f_u^\dagger$, and (5.4) we get $\pi(\hat{b}_2) = \pi'(\hat{b}'_2)$.

The set $\hat{b}_2 = H \cdot b_2$ is an atom over AMb_1 by Theorem 5.1.16 . So any choice of such a b_2 satisfies

$$\mathrm{tp}(b_1 b_2/Au) = \mathrm{tp}(b'_1, b'_2/Au)$$

By Lemma 5.2.4 we can choose b_2 such that $\pi(b_2) = \pi'(b'_2)$. □

All of the analysis of specialisations of Zariski cover structures carried in this section yields the following theorem. Which gives a characterisation of $\aleph_0$-universal specialisations of Zariski cover structures satisfying the Continuous Connections assumption.

Theorem 5.2.14. *Let $\mathcal{C}_0 = (C_0, M_0, \mathrm{pr}) \preceq \mathcal{C} = (C, M, \mathrm{pr})$ be two Zariski cover structures satisfying the Continuous Connections assumption, and $\mathcal{C}$ is an $\aleph_0$-saturated extension. Let $\pi : \mathcal{C} \to \mathcal{C}_0$ be a specialisation. Then the following are equivalent:*

(i) π is $\aleph_0$-universal;

(ii) the restriction $\pi_M : M \to M_0$ is $\aleph_0$-universal and $\pi : \mathcal{C} \to \mathcal{C}_0$ is a maximal specialisation;

(iii) the restriction $\pi_M : M \to M_0$ is $\aleph_0$-universal, $\pi : \mathcal{C} \to \mathcal{C}_0$ is a maximal specialisation, and the following holds

$$\forall m \in \mathrm{Dom}(\pi)\ \exists c \in \mathrm{Dom}(\pi)\ \ \mathrm{pr}(c) = m \tag{5.5}$$

Proof. First we note that $\mathcal{C}$ and M satisfy the assumptions of Lemma 5.2.1. Hence we may use Theorem 5.2.2(ii) as the criterion for universality. So, in order to prove ((ii)⇒(i)) we need to satisfy property (5.1) for $A \subset M \cup \mathrm{Dom}(\pi)$. This is Corollary 5.2.13.

((i)⇒(ii)) follows from Proposition 5.2.3, since any $\aleph_0$-universal specialisation is maximal.

((ii)⇒(iii)) is proved in Lemma 5.2.10.

((iii)⇒(ii)) Immediate. □

Remark 5.2.15. Under the assumption $\pi : \mathcal{C} \to \mathcal{C}_0$ is maximal and its restriction $\pi_M : M \to M_0$ is $\aleph_0$-universal, (5.5) implies that $\pi(S_m(C)) = S_{\pi(m)}(C_0)$, since together with c we have $G(M_0) \cdot c \subset \mathrm{Dom}(\pi) \cap S_m(C)$.

Theorem 5.2.16. *Under assumptions of Theorem 5.2.14 consider the structure* $(\mathcal{C}, \mathcal{C}_0, \pi)$ *in the language of specialisations and its substructure* $\pi_M : M \to M_0$. *Suppose* $\pi : \mathcal{C} \to \mathcal{C}_0$ *is maximal, and any* κ*-saturated model of the theory of specialisations of* $\pi_M : M \to M_0$ *is* κ*-universal.*

Then any κ*-saturated model of the theory of specialisations of* $\pi : \mathcal{C} \to \mathcal{C}_0$ *is* κ*-universal.*

Proof. Consider the theory of specialisation $\pi : \mathcal{C} \to \mathcal{C}_0$. Pick a κ-saturated model of the theory. By Lemma 5.1.19 we may assume it is $\pi : \mathcal{C} \to \mathcal{C}_0$. Then, by our assumptions, $\pi_M : M \to M_0$ is κ-universal. Now by Theorem 5.2.14 we get that π is $\aleph_0$-universal. Lemma 5.2.5 completes the proof. □

5.3 An Easy Example

Let K be an algebraically closed field. Let $\mu = \mathbb{Z}/2\mathbb{Z}$ written multiplicatively as $\{-1, 1\}$. Set $C := K \times \mu$. Consider the natural action of μ on C given by

$$\begin{aligned} \Theta : C \times \mathbb{Z}/2\mathbb{Z} &\to C \\ ((m, \varepsilon), 1) &\mapsto (m, \varepsilon) \\ ((m, \varepsilon), -1) &\mapsto (m, -\varepsilon) \end{aligned}$$

where $\varepsilon = \pm 1$.

Let $\mathcal{L}$ be the language with names for elements for C, a predicate for $\mathrm{pr}^{-1}(S)$ for every closed $S \subseteq K^n$, and a function g which will be interpreted as the element $-1 \in \mu$, as a function through its action.

Proposition 5.3.1. *C is a Noetherian $\mathcal{L}$-topological structure. I.e. positive quantifier free $\mathcal{L}$-formulas give a family of Noetherian topologies on C^n all n. Moreover theory of C admits quantifier elimination.*

We will omit the proof as this is immediate and very similar to the discussion in Section 4.1.

Proposition 5.3.2. *The function $g : C \to C$ is a morphism with respect to the $\mathcal{L}$-topology.*

Let $\mathcal{L}^f$ be the language $\mathcal{L}$ expanded with another function symbol f to be interpreted as

$$\begin{aligned} f : C &\to C \\ (k, \varepsilon) &\mapsto (k + 1, \varepsilon) \end{aligned}$$

Proposition 5.3.3. *The collection of positive quantifier free $\mathcal{L}^f$-formulas gives a Noetherian topological structure on C. Moreover the $\mathcal{L}^f$-theory of C admits quantifier elimination.*

Proposition 5.3.4. *The functions $f : C \to C$ and $g : C \to C$ are morphisms with respect to the $\mathcal{L}^f$-topological structure.*

Proof. We will prove this for f. A similar argument will work for g. One needs to show that $f \times \mathrm{id} : C \times C^n \to C \times C^n$ is continuous. Let $Q \subseteq C \times C^n$ be a closed subset. Then

$$(f \times \mathrm{id})^{-1} Q = \{(c, \overline{d}) \in C \times C^n : Q(f(c), \overline{d})\}$$

Which is closed by the description of closed sets in $\mathcal{L}^f$-topology. $\square$

With the Krull dimension, C will be a $\mathcal{L}^f$-Zariski structure. Observe that the equivalence relation

$$cEd :\Leftrightarrow \exists\, h \in \mu \text{ such that } h \cdot c = d$$

defines a closed equivalence relation on C^2. Hence $M := C/\mu$ is a topological sort; and with the induced dimension it will be a Zariski structure. The natural quotient map $\mathrm{pr} : C \to M$ is a morphism, and the group μ is definable in M as a Zariski topological group. It is also immediate that the action of μ on C is morphic and free. Hence $\mathcal{C} = (C, M, \mathrm{pr})$ is a Zariski cover structure.

Proposition 5.3.5. *Quantifier free $\mathcal{L}^f$-formulas are boolean combination of formulas of the type $P(\mathrm{pr}(y_1), \ldots, \mathrm{pr}(y_n)) = 0$ and $y_i = t_{ij} y_j$ where t is a term built from f and elements of μ, and $P \in K[X_1, \ldots, X_n]$.*

From Proposition 5.3.5, it follows that all non-trivial closed relations between fibres can be expressed in terms of compositions of f and elements of μ. Which will be morphisms. Hence $\mathcal{C}$ satisfies Continuous Connections.

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