Topics in
Analytic Number Theory

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A thesis submitted for the degree of

Doctor of Philosophy

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Abstract

In this thesis we prove several different results about the number of primes represented by linear functions.

The Brun-Titchmarsh theorem shows that the number of primes which are less than x and congruent to a modulo q is less than $(C + o(1))x/(\phi(q)\log x)$ for some value C depending on $\log x/\log q$. Different authors have provided different estimates for C in different ranges for $\log x/\log q$, all of which give $C > 2$ when $\log x/\log q$ is bounded. We show in Chapter 2 that one can take $C = 2$ provided that $\log x/\log q \geq 8$ and q is sufficiently large. Moreover, we also produce a lower bound of size $x/(q^{1/2}\phi(q))$ when $\log x/\log q \geq 8$ and is bounded. Both of these bounds are essentially best-possible without any improvement on the Siegel zero problem.

Let $k \geq 2$ and $\Pi(n) = \prod_{i=1}^{k}(a_i n + b_i)$ for some integers a_i, b_i $(1 \leq i \leq k)$. Suppose that $\Pi(n)$ has no fixed prime divisors. Weighted sieves have shown for infinitely many integers n that the number of prime factors $\Omega(\Pi(n))$ of $\Pi(n)$ is at most r_k, for some integer r_k depending only on k. In Chapter 3 and Chapter 4 we introduce two new weighted sieves to improve the possible values of r_k when $k \geq 3$. In Chapter 5 we demonstrate a limitation of the current weighted sieves which prevents us proving a bound better than $r_k = (1 + o(1))k\log k$ for large k.

Zhang has shown that there are infinitely many intervals of bounded length containing two primes, but the problem of bounded length intervals containing three primes appears out of reach. In Chapter 6 we show that there are infinitely many intervals of bounded length containing two primes and a number with at most 31 prime factors. Moreover, if numbers with up to 4 prime factors have 'level of distribution' 0.99, there are infinitely many integers n such that the interval $[n, n + 90]$ contains 2 primes and an almost-prime with at most 4 prime factors.

Contents

Chapter 1

Introduction

The central problem of prime number theory is to understand the distribution of the prime numbers. Specifically, given a set $\mathcal{S}$ of integers (or integer vectors), we would like to estimate the number of primes in $\mathcal{S}$ (or vectors in $\mathcal{S}$ with all components prime).

Unfortunately, we are unable to obtain a satisfactory answer for most sets $\mathcal{S}$ which we would like to consider. In this thesis we will restrict ourselves to the case when elements of $\mathcal{S}$ are defined by linear equations.

Problem. *Let $L_1, \dots, L_k$ be distinct linear functions with integer coefficients. How many integers n are there, such that $L_1(n), \dots, L_k(n)$ are all primes which are less than x?*

By 'linear function' we mean $L_i(n) = a_i n + b_i$ with $a_i \neq 0$ for each i. We note that if the product function $\Pi(n) = \prod_{i=1}^{k} L_i(n)$ has a fixed prime divisor, then there can be at most k values of n for which $L_1(n), \dots, L_k(n)$ are simultaneously prime. We therefore concentrate on the case when $\Pi(n)$ has no fixed prime divisor.

If $\Pi(n)$ has no fixed prime divisor, then we call the k-tuple $\{L_1, \dots, L_k\}$ of distinct linear functions *admissible*. It is conjectured (see [25]) that there should be infinitely many integers n for which $L_1(n), \dots, L_k(n)$ are simultaneously prime if $\{L_1, \dots, L_k\}$ is an admissible k-tuple.

Conjecture (Hardy-Littlewood prime k-tuple conjecture)**.** *Let $\mathcal{L} = \{L_1, \dots, L_k\}$ be an admissible k-tuple. Then, as $x \to \infty$,*

$$\#\{n \le x : L_1(n), \dots, L_k(n) \text{ all prime } \} \sim \mathfrak{S}(\mathcal{L}) \frac{x}{(\log x)^k}, \tag{1.0.1}$$

where

$$\mathfrak{S}(\mathcal{L}) = \prod_p \left(1 - \frac{\nu_{\mathcal{L}}(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}, \quad \nu_{\mathcal{L}}(p) = \#\{1 \le n \le p, \Pi(n) \equiv 0 \pmod p\}. \tag{1.0.2}$$

The prime k-tuples conjecture has far-ranging consequences, as it almost completely describes the 'small-scale' structure of the primes. We note that the famous twin prime conjecture is the special case $\mathcal{L} = \{n, n+2\}$ of the prime k-tuples conjecture.

The current state of our knowledge of the conjecture is very different in the case $k = 1$ compared to the case when $k > 1$.

1.1 The case $k = 1$

In the case $k = 1$ we are simply counting primes in an arithmetic progression $a \pmod q$ with $(a, q) = 1$. It is a classical theorem of Dirichlet [11] that there are infinitely many integers n such that $L_1(n)$ is prime (provided that L_1 has no fixed prime divisor), and (1.0.1) is established by the Siegel-Walfisz theorem [49]. The reason we are able to obtain an asymptotic estimate in this case is because Dirichlet L-functions can be used to count primes in an arithmetic progression, and this allows us to use powerful tools from complex analysis to study the problem.

We would like to be able to obtain the asymptotic (1.0.1) with some uniformity in the coefficients a_1, b_1 of $L_1(n) = a_1 n + b_1$. The Siegel-Walfisz theorem establishes this in the range $a_1 \leq (\log x)^C$ for any fixed $C > 0$, and it would be a major breakthrough if this uniformity could be extended to a wider range. Unfortunately, if we wish to establish such a result we would need to rule out the possible existence of so-called 'Siegel zeros'; these are real zeros of the Dirichlet L-function $L(s, \chi)$ to modulus a_1 which are very close to 1. This seems to be beyond our current techniques, and so we can only hope to prove upper or lower bounds when we consider values of a_1 outside of the range $a_1 \leq (\log x)^C$.

In Chapter 2 we consider this problem further, and improve the known upper and lower bounds for the number of primes less than x and congruent to $a \pmod q$ in the range $x > q^8$. In this range these results are essentially the best we can hope to prove without excluding the possible existence of Siegel zeros.

1.2 The case $k > 1$

When $k > 1$ the problem becomes much harder. The L-function method which gave an asymptotic in the case $k = 1$ fails in the case $k > 1$. There is no known k-tuple with $k > 1$ which is known to represent infinitely many primes.

The most promising techniques to the problem come from sieve methods. One can obtain upper bounds for the number of $n \leq x$ such that $L_1(n), \dots, L_k(n)$ are all prime, with the upper bound being of the correct order of magnitude. For example, by [24, Theorem 5.7] one has

$$\#\{n \leq x : L_1(n), \dots, L_k(n) \text{ all prime}\} \leq 2^k k!(1 + o(1))\mathfrak{S}(\mathcal{L})\frac{x}{(\log x)^k}. \tag{1.2.1}$$

Although we cannot prove that there are infinitely many integers n such that the $L_i(n)$ are simultaneously prime, using sieve methods we are also able to show that there are infinitely many integers n such that $L_1(n), \dots, L_k(n)$ are *almost prime* infinitely often, in the sense that their product has only a few prime factors.

For example, [24, Theorem 10.5] shows that there are infinitely many integers n such that the number of prime factors $\Omega(\Pi(n))$ of $\Pi(n)$ satisfies $\Omega(\Pi(n)) \leq r_k$, for some constant $r_k = k \log k + O(k)$ depending only on k.

In Chapter 3 and Chapter 4 we improve on the best known values of r_k when $3 \leq k \leq 10$, and in principle our method should produce the best bounds for $k > 10$. This improves on every case where we don't already have the best results we expect current methods produce.

1.3 L-function methods

We briefly summarise the role of L-functions in counting primes in arithmetic progressions. We let χ be a Dirichlet character to modulus q. By orthogonality of Dirichlet characters, we have

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \sum_{n \leq x} \chi(n)\Lambda(n), \tag{1.3.1}$$

where $\Lambda(n)$ is the Von-Mangoldt function (defined to be $\log p$ when $n = p^m$ and 0 otherwise). Using Perron's formula we obtain

$$\sum_{n \leq x} \chi(n)\Lambda(n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s} \right) \frac{x^s}{s} ds. \tag{1.3.2}$$

In the region $\Re(s) > 1$, we define

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \tag{1.3.3}$$

where the product expression follows from the fundamental theorem of arithmetic. We can then extend $L(s,\chi)$ to a meromorphic function defined on the entire complex plane by analytic continuation.

It is the fact that L-functions have this 'Euler product' expansion which means that they encode multiplicative properties of their coefficients. However, it is also this multiplicative structure of L-functions which means that they cannot easily be used to answer problems when we introduce non-multiplicative conditions, such as in the case of $k > 1$ of the prime k-tuples problem.

We now observe that

$$\frac{L'}{L}(s,\chi) = -\sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s}, \tag{1.3.4}$$

and so, substituting this into (1.3.2) we obtain

$$\sum_{n\leq x}\chi(n)\Lambda(n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s,\chi)\frac{x^s}{s}ds. \tag{1.3.5}$$

It is the fact that $\mathbf{1}_{n\equiv a \pmod q}(n)\Lambda(n)$ (where $\mathbf{1}_{n\equiv a \pmod q}(n)$ is the characteristic function of integers which are congruent to $a \pmod q$) can be written as a linear combination of coefficients of an L-function in this way which makes estimating the number of primes in an arithmetic progression amenable to the L-function technique.

By truncating the line of integration in (1.3.5) and moving it to $[-U - iT, -U + iT]$ we pick up residues at zeros of $L(s,\chi)$, at $s = 0$, and, if χ is a principal character, at $s = 1$. Estimating the size of these terms (and the errors introduced) gives the 'explicit formula'

$$\sum_{\substack{n\leq x\\ n\equiv a \pmod q}} \Lambda(n) = \frac{x}{\phi(q)} + \frac{1}{\phi(q)}\sum_{\chi}\overline{\chi(a)}\sum_{\rho}\frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right), \tag{1.3.6}$$

where the sum $\sum_\rho$ on the right hand side is over all zeros ρ of $L(s,\chi)$ with $0 < \Re(\rho) < 1$ and $|\Im(\rho)| < T$.

The explicit formula allows us to gain an asymptotic expression for the number of primes in an arithmetic progression if q is small. Since each zero ρ contributes a term of size approximately $x^{\Re(\rho)}$, we find that to obtain an asymptotic which is uniform in q we need to exclude the possibility of some zeros ρ having real part very close to 1.

The Generalised Riemann Hypothesis asserts that all non-trivial zeros of $L(s,\chi)$ have real part equal to $1/2$. If this were true, then we would obtain

$$\sum_{\substack{n\leq x\\ n\equiv a \pmod q}} \Lambda(n) = \frac{x}{\phi(q)} + O_\epsilon\left(x^{1/2+\epsilon}\right), \tag{1.3.7}$$

for any $\epsilon > 0$. This would give an asymptotic formula whenever $q \leq x^{1/2-\epsilon}$.

1.4 Sieve methods

Sieve methods are a set of techniques which have been developed to provide a very versatile means of attacking certain problems about the distribution of primes. In particular, they aim to be able to provide estimates in problems where the L-function method does not work, such as in the case $k > 1$ of the prime k-tuples problem where we have additive restrictions on the primes we wish to consider.

The basic aim of sieve methods is to estimate $\mathcal{S}(\mathcal{A}, z) = \#\{a \in \mathcal{A} : p|a \Rightarrow p \geq z\}$, which counts the number of elements in the set $\mathcal{A}$ with no prime factors less than a bound z. This is done by using a smoothed version of the inclusion-exclusion formula together with estimates for the number of elements of $\mathcal{A}$ in arithmetic progressions. It is typically not possible to obtain an asymptotic for $\mathcal{S}(\mathcal{A}, z)$, but instead we obtain an asymptotic estimate for some weighted sum which is an upper or lower bound for $\mathcal{S}(\mathcal{A}, z)$.

We assume a set $\mathcal{A}$ satisfies

$$\mathcal{A}_d = \#\{a \in \mathcal{A} : a \equiv 0 \pmod{d}\} = g(d)\#\mathcal{A} + r_d, \tag{1.4.1}$$

for some multiplicative function $g(d)$ and some error term r_d. (We note that the error term r_d used here is different to the constant r_k used earlier. Which quantity we refer to should be clear from the context.)

Then, for any choice of the coefficients $\lambda_d \in \mathbb{R}$ satisfying $\lambda_1 = 1$, we have

$$\begin{aligned}\mathcal{S}(\mathcal{A}, z) &\leq \sum_{a \in \mathcal{A}} \Bigg(\sum_{\substack{d|a \\ p|d \Rightarrow p<z}} \lambda_d \Bigg)^2 = \sum_{\substack{d,e \\ p|de \Rightarrow p<z}} \lambda_d \lambda_e \sum_{\substack{a \in \mathcal{A} \\ [d,e]|a}} 1 \\ &= \#\mathcal{A} \sum_{\substack{d,e \\ p|de \Rightarrow p<z}} \lambda_d \lambda_e g([d,e]) + O\Bigg(\sum_{\substack{d,e \\ p|de \Rightarrow p<z}} |\lambda_d||\lambda_e||r_{[d,e]}| \Bigg). \end{aligned} \tag{1.4.2}$$

If we have good estimates for $\mathcal{A}_d$, then r_d will typically be small, and so the final term will be small. We can often obtain an asymptotic estimate for the first sum for a suitable choice of λ_d, which then gives an upper bound for $\mathcal{S}(\mathcal{A}, z)$ which is of the correct order of magnitude.

More complex sieve methods then combine various different weighted upper and lower bounds for $\mathcal{S}(\mathcal{A}, z)$ to count primes or almost-primes in the set $\mathcal{A}$.

The most appealing feature of the above estimate is that it only required that $\mathcal{A}_d$ be reasonably well approximated. This is true of many of the sets we would like to investigate, and in particular the set of values of $\Pi(n)$.

Unfortunately, this flexibility means that estimates of this type alone are unable to prove directly that the set $\mathcal{A}$ contains primes - this is an obstruction known as the 'parity phenomenon'. This is a reason why sieve methods typically have to be combined with other techniques or only prove results about the existence of 'almost-primes' instead of primes in a set.

1.5 Small gaps between primes

One important approach to the prime k-tuples question is based on the sieve method of Goldston, Pintz and Yıldırım. This has been most spectacularly applied by Zhang [53] who has shown that for any admissible k-tuple with $k \geq 3500000$, at least two of the functions in the k-tuple are simultaneously prime infinitely often. In particular, the k-tuple $L_i(n) = n + p_{\pi^{-1}(k)+i} - p_{\pi^{-1}(k)+1}$ with $k = 3500000$ is admissible, since at $p_{\pi^{-1}(k)+1}$ each function is coprime to all primes $\leq k$, and the k-tuple clearly cannot have a fixed prime divisor larger than k. Therefore we have that

$$\liminf (p_{n+1} - p_n) \leq 7 \cdot 10^7. \qquad (1.5.1)$$

The method of Goldston, Pintz and Yıldırım considers the sum

$$S = \sum_{N \leq n < 2N} \left(\sum_{i=1}^{k} \chi_{\leq 1}(L_i(n)) - 1 \right) \left(\sum_{d | \Pi(n)} \lambda_d \right)^2, \qquad (1.5.2)$$

where $\chi_{\leq 1}$ is the characteristic function of the primes, and λ_d are real numbers which we are free to choose.

The idea is to show that $S > 0$ for a suitable choice of λ_d and for all large N. If we know that $S > 0$ then there must be at least one term in the sum over n which has a strictly positive contribution to S. However, since the λ_d are reals, we have $(\sum_{d|\Pi(n)} \lambda_d)^2 \geq 0$, and so the first expression in parentheses must be greater than 0. This is the case precisely when at least two of $L_i(n)$ are prime. Therefore if $S > 0$ for all large N, there must be infinitely many integers n such that at least two of the $L_i(n)$ are prime.

We note that the Goldston, Pintz Yıldırım method manages to avoid some problems associated with the parity phenomenon, since it doesn't directly attempt to produce a lower bound for the number of primes in the set of points represented by the k-tuple. Instead it uses the pigeonhole principle to deduce the existence of pairs of numbers which are simultaneously prime. The parity phenomenon in this context only indicates that the 'main term' contribution from each $\chi_{\leq 1}(L_i(n))$ term can be at most the contribution from the constant term 1/2.

In order to produce a suitable lower bound for S, the λ_d need to be chosen such that the weights $(\sum_{d|\Pi(n)} \lambda_d)^2$ are concentrated on n for which many of the $L_i(n)$ are prime. If we can only obtain accurate estimates (on average) for the number of primes in an arithmetic progression mod q with $q \leq N^\theta$, then in order to estimate S accurately we need to restrict the support of λ_d to $d \leq N^{\theta/2}$. This restricts the flexibility with which we can choose our λ_d, and correspondingly the weights $(\sum_{d|\Pi(n)} \lambda_d)^2$ are less concentrated on n for which many of the $L_i(n)$ are prime if θ is smaller.

The Bombieri-Vinogradov theorem shows that on average we have a good estimate for the number of primes in arithmetic progressions mod q for $q \leq N^{1/2-\epsilon}$. It turns out that as $k \to \infty$ this just fails to produce a positive lower bound for S. The breakthrough of Zhang was to extend earlier ideas of Bombieri, Friedlander and Iwaniec [1],[2],[3] to show that we can take q of size up to $N^{1/2+1/568}$ (provided q only has small prime factors). This allows us to obtain a positive lower bound for S when k is sufficiently large.

We consider this application further in Chapter 6, where we show that there are infinitely many intervals of bounded length containing two primes and a number with at most 31 prime factors.

Chapter 2

On the Brun-Titchmarsh theorem

2.1 Introduction

We let $\pi(x; q, a)$ denote the number of primes less than or equal to x which are congruent to $a \pmod q$. As mentioned in Chapter 1, it is a classical theorem of Walfisz [49] based on the work of Siegel that, for any fixed $N > 0$, uniformly for $q \leq (\log x)^N$ and $(a, q) = 1$, as $x \to \infty$ we have

$$\pi(x; q, a) \sim \frac{x}{\phi(q) \log x}. \tag{2.1.1}$$

It is generally believed that this asymptotic holds in a much wider range of q. If we assume the Generalised Riemann Hypothesis (GRH), then the asymptotic (2.1.1) holds uniformly in the much larger range $q \leq x^{1/2-\delta}$ for any fixed $\delta > 0$. Montgomery [35] has conjectured that (2.1.1) holds uniformly in the even larger range $q \leq x^{1-\delta}$. Friedlander *et al* [14] have shown for any A, the asymptotic formula (2.1.1) cannot hold for all $q \geq x/(\log x)^A$.

Any improvement in the range of q for which the asymptotic holds would exclude the possibility of the existence of zeros of Dirichlet L-functions in certain regions, but unfortunately such a result seems beyond our current techniques. Without this type of improvement, however, we cannot hope to prove results stronger than

$$o\left(\frac{x}{\phi(q) \log x}\right) \leq \pi(x; q, a) \leq \frac{2x}{\phi(q) \log x} \tag{2.1.2}$$

when $\log x / \log q$ is bounded.

Linnik [32], [33] gave a non-trivial lower bound for $\pi(x; q, a)$ for a wider range of q. He showed that there is a constant $L > 0$ such that, whenever $x > q^L$ and q is sufficiently large there is at least one prime in the arithmetic progression $\{n \leq x : n \equiv a \pmod q\}$ for any a

with $(a,q)=1$. Pan [40] showed that one can take $L \le 10,000$. This has subsequently been improved by many authors including (in chronological order) Chen [4], Jutila [30], Chen [6], Jutila [31], Chen [7], Graham [22], Wang [50], Chen and Liu [8], and Heath-Brown [26]. The best known result is due to Xylouris [52] (improving upon [51]), which shows that we can take $L = 5$.

Titchmarsh [48] used Brun's sieve to show that for $q < x$ we have the upper bound

$$\pi(x;q,a) \ll \frac{x}{\phi(q)\log(x/q)}. \tag{2.1.3}$$

The implied constant can be made explicit, and has been estimated by various authors. The strongest result of this type which holds for all ranges of q is due to Montgomery and Vaughan [36], who used the large sieve to obtain the following result.

Theorem 2.1 (Brun-Titchmarsh Theorem)**.** *For $x > q$ we have*

$$\pi(x;q,a) \le \left(\frac{2}{1-\log q/\log x}\right)\frac{x}{\phi(q)\log x}.$$

The constant $2/(1-\log q/\log x)$ of the Brun-Titchmarsh theorem should be compared with the constant $1+o(1)$ which Montgomery conjectures for $x > q^{1+\delta}$.

Since it appears unlikely that we can prove an upper bound with a constant less than 2 with the current techniques, any improvements are likely to reduce the factor $1/(1-\log q/\log x)$. Several authors including Motohashi [38], Goldfeld [16], Iwaniec [29] and Iwaniec and Friedlander [15] have made improvements of this type for different ranges of q. If we put

$$\theta = \frac{\log q}{\log x}, \tag{2.1.4}$$

then we have

$$\pi(x;q,a) \le \frac{(C+o(1))x}{\phi(q)\log x}, \tag{2.1.5}$$

where

$$C = \begin{cases} (2-((1-\theta)/4)^6)/(1-\theta), & 2/3 \le \theta, \\ 8/(6-7\theta), & 9/20 \le \theta \le 2/3, \\ 16/(8-3\theta), & \theta \le 9/20. \end{cases}$$

This improves the Brun-Titchmarsh bound of $C = 2/(1-\theta)$ slightly throughout the entire range of q. We note that in all cases we still have $C > 2$ for $\theta > 0$.

It has been known as a folklore amongst specialists that for θ less than some fixed constant we should be able to take $C = 2$. In this chapter we establish this, and give a quantitative bound for the range when this happens. We show that provided q is sufficiently large we can take

$$C = 2 \qquad \text{if } \theta \le 1/8.$$

2.2 Notation

We will let p represent a generic prime. We will consider the arithmetic progression where all terms are $\leq x$ and are congruent to $a \pmod{q}$. We will assume that q is larger than some fixed constant throughout, and so may not explicitly say that we are assuming q to be sufficiently large for a given statement to hold. We let χ denote a Dirichlet character $(\mathrm{mod}\ q)$ and χ_0 denote the principal character.

For the purposes of this chapter we shall define an 'η-Siegel zero' to be a real zero ρ of a Dirichlet L-function $L(s,\chi)$ which lies in the region

$$1 - \frac{\eta}{\log q} \leq \Re(\rho) \leq 1.$$

2.3 Main results

We improve on the Brun-Titchmarsh constant for some range of q. Instead of using sieve methods to count primes in arithmetic progressions we will use the analytic techniques developed in the estimation of Linnik's constant.

In Linnik's theorem one counts primes with a smooth weight, and estimating this requires estimating corresponding weighted sums over the zeros of Dirichlet L-functions. In the most successful work on Linnik's theorem only zeros of the form $\rho = 1 + O(1/\log q)$ make a significant contribution. In this problem we wish to count primes weighted by the characteristic function of the interval $[0, x]$, however, and this means we must consider all zeros $\rho = \beta + i\gamma$ with $\gamma \ll 1$ in the corresponding weighted sums over zeros. Thus the zero density estimates of Heath-Brown [26] are insufficient, and we need to extend them to this larger range.

Theorem 2.2. *There exists an effectively computable constant q_1, such that for $q \geq q_1$ and $x \geq q^8$ we have*

$$\pi(x;q,a) < \frac{2\,\mathrm{Li}(x)}{\phi(q)}.$$

We note that without excluding the possible existence of η-Siegel zeros for some $\eta > 0$ this is the strongest possible bound which we can hope to prove for $\log x/\log q$ bounded.

We also obtain lower bounds which are essentially the strongest possible for $\log x/\log q$ bounded without excluding the existence of an η-Siegel zero.

Theorem 2.3. *There exists an effectively computable constant q_2 such that for $q \geq q_2$ and $x \geq q^8$ we have*

$$\frac{\log q}{q^{1/2}}\left(\frac{x}{\phi(q)\log x}\right) \ll \pi(x;q,a).$$

Theorem 2.4. *Let $\epsilon > 0$. There exists an (ineffective) constant $q_3(\epsilon)$ such that for $q \geq q_3(\epsilon)$ and $x \geq q^8$ we have*

$$\frac{q^{-\epsilon}x}{\phi(q)\log x} \ll \pi(x;q,a).$$

Theorem 2.5. *Assume that there exists a constant $\eta > 0$ such that there are no η-Siegel zeros. Then there exists an effectively computable constant q_4 such that for $q \geq q_4$ and $x \geq q^8$ we have*

$$\frac{x}{\phi(q)\log x} \ll \pi(x;q,a) < \frac{2x}{\phi(q)\log x}.$$

Thus the number of primes in an arithmetic progression is close to the order predicted by GRH, provided $\log x/\log q \geq 8$ and q is sufficiently large. If there are no zeros exceptionally close to 1 then the number of primes has the same order as the asymptotic predicted by GRH.

In order to establish Theorems 2.2, 2.3, 2.4 and 2.5 we prove the following proposition.

Proposition 2.6. *There are fixed constants $\epsilon > 0$ and $\eta > 0$ such that:*

There exists an effectively computable constant q_5, such that if there is an η-Siegel zero $\rho_1 = 1 - \lambda_1/\log q$ to modulus $q \geq q_5$ then for $x \geq q^7$ we have

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{(1-\lambda_1)x}{\phi(q)}.$$

There exists an effectively computable constant q_6 such that if there are no η-Siegel zeros to modulus $q \geq q_6$ then for $x \geq q^{7.999}$ we have

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| < \frac{(1-\epsilon)x}{\phi(q)}.$$

We now establish Theorems 2.2, 2.3, 2.4 and 2.5 assuming Proposition 2.6.

By partial summation we have, for any constant $7 \leq A < 8$, we have

$$\begin{aligned}\pi(x;q,a) &= \frac{\theta(x;q,a)}{\log x} + \int_2^x \frac{\theta(t;q,a)}{t\log^2 t}dt \\ &= \frac{\theta(x;q,a)}{\log x} + \int_{q^A}^x \frac{\theta(t;q,a)}{t\log^2 t}dt + \int_{q^2}^{q^A} \frac{\theta(t;q,a)}{t\log^2 t}dt + \int_2^{q^2} \frac{\theta(t;q,a)}{t\log^2 t}dt. \qquad (2.3.1)\end{aligned}$$

Trivially for $t \le q^2$ we have $\theta(t;q,a) \le t\log t$. By the Brun-Titchmarsh Theorem, in the range $q^2 \le t \le q^A$ we have

$$\theta(t;q,a) \le (\log t)\pi(t;q,a) \ll \frac{t}{\phi(q)}. \tag{2.3.2}$$

We also note that $\theta(x;q,a) = \psi(x;q,a) + O(x^{1/2})$. Thus, uniformly for $x \ge q^8$ and $7 \le A \le 8$, we obtain

$$\pi(x;q,a) = \frac{\psi(x;q,a)}{\log x} + \int_{q^A}^{x} \frac{\psi(t;q,a)}{t\log^2 t}dt + O\left(x^{1/2} + \frac{q^A}{\phi(q)}\right). \tag{2.3.3}$$

This gives

$$\begin{aligned}\left|\pi(x;q,a) - \frac{\mathrm{Li}(x)}{\phi(q)}\right| \le \frac{1}{\log x}\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| + \int_{q^A}^{x} \frac{|\psi(t;q,a) - t/\phi(q)|}{t\log^2 t}dt \\ + O\left(x^{1/2} + \frac{q^A}{\phi(q)}\right).\end{aligned} \tag{2.3.4}$$

If there is an η-Siegel zero (where η is the constant from Proposition 2.6) then we choose $A = 7$ and by Proposition 2.6, uniformly for $q \ge q_5$ and $x \ge q^8$ we have

$$\begin{aligned}\left|\pi(x;q,a) - \frac{\mathrm{Li}(x)}{\phi(q)}\right| &\le \frac{(1-\lambda_1)x}{\phi(q)\log x} + \int_{q^7}^{x} \frac{1-\lambda_1}{\phi(q)\log^2 t}dt + O\left(x^{1/2} + \frac{q^7}{\phi(q)}\right) \\ &\le \frac{(1-\lambda_1)\,\mathrm{Li}(x)}{\phi(q)} + O\left(\frac{x^{7/8}}{\phi(q)}\right).\end{aligned} \tag{2.3.5}$$

By Pintz [42, Theorem 3] we have that $\lambda_1 \gg \log q/q^{1/2}$ (with the implied constant effectively computable). Thus for q sufficiently large the error term in (2.3.5) is at most

$$\frac{\lambda_1\,\mathrm{Li}(x)}{2\phi(q)}. \tag{2.3.6}$$

Thus for q sufficiently large and $x \ge q^8$ we have

$$\frac{x\log q}{q^{1/2}\phi(q)\log x} \ll \frac{\lambda_1\,\mathrm{Li}(x)}{2\phi(q)} \le \pi(x;q,a) \le \frac{(2-\lambda_1/2)\,\mathrm{Li}(x)}{\phi(q)} < \frac{2\,\mathrm{Li}(x)}{\phi(q)}, \tag{2.3.7}$$

with all constants effectively computable.

By Siegel's theorem [46], given any $\epsilon > 0$ there is a constant $C(\epsilon)$ such that if $q \ge C(\epsilon)$ we have $\lambda_1 \ge 2q^{-\epsilon}$. Here the constant $C(\epsilon)$ is not effectively computable. In this case, we have

$$\frac{xq^{-\epsilon}}{\phi(q)\log x} \le \frac{\lambda_1\,\mathrm{Li}(x)}{2\phi(q)} < \pi(x;q,a). \tag{2.3.8}$$

If there is no η-Siegel zero then we instead choose $A = 7.999$. By Proposition 2.6 and (2.3.4) there exists an $\epsilon > 0$ and q_6 such that uniformly for $x \geq q^8$ and for $q \geq q_6$ we have

$$\left|\pi(x; q, a) - \frac{\mathrm{Li}(x)}{\phi(q)}\right| \leq \frac{(1-\epsilon)x}{\phi(q)\log x} + \int_{q^{7.999}}^{x} \frac{1-\epsilon}{\phi(q)\log^2 t} dt + O\left(x^{1/2} + \frac{q^{7.999}}{\phi(q)\log x}\right)$$
$$= \frac{(1-\epsilon)\,\mathrm{Li}(x)}{\phi(q)} + O\left(\frac{x^{1-1/10{,}000}}{\phi(q)\log x}\right). \tag{2.3.9}$$

Thus for q sufficiently large and $x \geq q^8$ we have

$$\frac{x}{\phi(q)\log x} \ll \pi(x; q, a) < \frac{2x}{\phi(q)\log x}. \tag{2.3.10}$$

Theorems 2.2, 2.3, 2.4 and 2.5 now follow immediately from (2.3.7), (2.3.8) and (2.3.10).

2.4 Case 1: Siegel zeroes

We first consider the case when there are zeros very close to 1. For this section we assume that η-Siegel zeros exist for some small constant $\eta > 0$.

In order to establish Proposition 2.6 we will make use of the analytic techniques developed in the estimation of Linnik's constant. In particular, there are three main results which we use:

Proposition 2.7 (Zero-free region)**.** *There is a constant $c_1 > 0$ such that, for q sufficiently large, the function*

$$\prod_{\chi \pmod q} L(\sigma + it, \chi)$$

has at most one zero in the region

$$1 - \frac{c_1}{\log q(2 + |t|)} \leq \sigma.$$

Such a zero, if it exists, is real and simple, and the corresponding character must be a non-principal real character.

Proposition 2.8 (Deuring-Heilbronn phenomenon)**.** *There is a constant $c_2 > 0$ such that, if the exceptional zero $\rho_1 = 1 - \lambda_1/(\log q)$ from Proposition 2.7 exists, then for q sufficiently large, the function*

$$\prod_{\chi \pmod q} L(\sigma + it, \chi)$$

has no other zeros in the region

$$1 - \frac{c_2 \log(\lambda_1^{-1})}{\log q(2 + |t|)} \leq \sigma \leq 1.$$

Proposition 2.9 (Log-free zero-density estimate)**.** *There are constants $c_3 > 0$ and $C_3 > 0$ such that, for $T \geq 1$, one has*

$$\sum_{\chi \pmod q} N(\sigma, T, \chi) \leq C_3 (qT)^{c_3(1-\sigma)}.$$

Here

$$N(\sigma, T, \chi) = \#\{\rho : L(\rho, \chi) = 0, \quad \Re(\rho) \geq \sigma, \quad |\Im(\rho)| \leq T\}.$$

We recall that for the purposes of this article we are defining an η-Siegel zero to be a real zero ρ of some Dirichlet L-function in the region

$$1 - \frac{\eta}{\log q} \leq \rho \leq 1, \tag{2.4.1}$$

for a fixed small positive constant η.

We will choose $\eta \leq c_1/2$, so by Proposition 2.7 an η-Siegel zero, if it exists, must be simple, and the corresponding character must be a real character. Moreover, there can be at most one such zero. We label this exceptional zero $\rho_1 = 1 - \lambda_1/(\log q)$ with corresponding character χ_1. Thus we have that $\lambda_1 \leq \eta$. We will also make use of the fact that $\lambda_1 \gg_\epsilon q^{-1/2-\epsilon}$ (with the implied constant effectively computable), which follows from Dirichlet's class number formula.

By [23] and [26, Equation 1.4] we can take the constants c_2 and c_3 appearing in Proposition 2.8 and Proposition 2.9 as

$$c_2 = 2/3 - 1/1000, \qquad c_3 = 12/5 + 1/1000, \tag{2.4.2}$$

provided $\eta \leq c_4$, some suitably small absolute constant.

We wish to prove

$$\left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| \leq \frac{(1 - \lambda_1)x}{\phi(q)}. \tag{2.4.3}$$

We have

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \left(\sum_{n \leq x} \Lambda(n)\chi(n) \right). \tag{2.4.4}$$

We use the explicit formula (see, for example, [9]).

$$\sum_{n \leq x} \Lambda(n)\chi(n) = \varepsilon_1(\chi)x - \varepsilon_2(\chi)\frac{x^{\rho_1}}{\rho_1} - \sum_{\rho} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T} \right), \tag{2.4.5}$$

where $\varepsilon_1(\chi)$ and $\varepsilon_2(\chi)$ are defined by

$$\varepsilon_1(\chi) = \begin{cases} 1, & \chi = \chi_0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4.6}$$

$$\varepsilon_2(\chi) = \begin{cases} 1, & \chi \text{ is a character corresponding to the possible} \\ & \qquad\qquad \text{exceptional zero } \rho_1 \text{ of } \prod_\chi L(s,\chi), \\ 0, & \text{otherwise}, \end{cases} \tag{2.4.7}$$

and the sum $\sum_\rho$ is over all non-exceptional zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ in the region $\{0 < \beta < 1, |\gamma| < T\}$, counted with multiplicity.

We choose $T = q(\log x)^3/\lambda_1$ so that the last term is $o(\lambda_1 x/\phi(q))$.

Recalling that $\rho_1 = 1 - \lambda_1/\log q$, we have

$$\frac{x^{\rho_1}}{\rho_1} = x \exp\left(-\lambda_1 \frac{\log x}{\log q}\right) + o(\lambda_1 x). \tag{2.4.8}$$

Substituting (2.4.5) and (2.4.8) into (2.4.4) we obtain

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| \le \frac{x}{\phi(q)} \exp\left(-\lambda_1 \frac{\log x}{\log q}\right) + \frac{1}{\phi(q)} \sum_{\chi \pmod q} \sum_\rho \left|\frac{x^\rho}{\rho}\right| + o\left(\frac{\lambda_1 x}{\phi(q)}\right). \tag{2.4.9}$$

We now bound the sum

$$\sum_{\chi \pmod q} \sum_\rho \left|\frac{x^\rho}{\rho}\right|. \tag{2.4.10}$$

We first consider the case when $\log x > q^{1/3000}$.

Since $\lambda_1 \gg q^{-1/2-1/100}$ we have $T \ll q^{3/2+1/100}(\log x)^3 \ll (\log x)^{4600}$. By Proposition 2.7 (and recalling $\Re(\rho) \gg \lambda_1/\log q$ for all ρ) each zero in the sum (2.4.10) contributes at most

$$\left|\frac{x^\rho}{\rho}\right| \le x \frac{\exp((\Re(\rho) - 1)\log x)}{\Re(\rho)} \le x \exp\left(-c\frac{\log x}{\log\log x}\right) \tag{2.4.11}$$

for some constant $c > 0$. By Proposition 2.9 the total number of zeros in the sum is

$$\ll (qT)^{12/5+1/1000} \ll (\log x)^{20000}. \tag{2.4.12}$$

Therefore

$$\sum_{\chi \pmod q} \sum_\rho \left|\frac{x^\rho}{\rho}\right| \ll x(\log x)^{20000} \exp\left(-c\frac{\log x}{\log\log x}\right) = o(\lambda_1 x). \tag{2.4.13}$$

Thus we see that for x sufficiently large and $\log x > q^{1/3000}$, the right hand side of (2.4.9) is

$$\frac{x}{\phi(q)}\left(\exp\left(-\lambda_1\frac{\log x}{\log q}\right) + o(\lambda_1)\right) \le \frac{(1-\lambda_1)x}{\phi(q)}, \tag{2.4.14}$$

as required.

We now consider the case when $\log x \le q^{1/3000}$. In this case, since $\lambda_1 \gg q^{-1/2-1/1000}$, we have $T \ll q^{3/2+2/1000}$.

We begin by considering the contribution to the sum (2.4.10) from zeros in the rectangle

$$1 - \frac{m+1}{\log q} \le \Re(\rho) \le 1 - \frac{m}{\log q}, \qquad n \le |\Im(\rho)| \le 2n, \tag{2.4.15}$$

where $1 \le n \le T$ and $m \le 0.4 \log q$. Recalling that $m \le 0.4 \log q$, by Proposition 2.9 with $c_3 = 12/5 + 1/1000$ there are

$$\ll n^{2.41m/\log q} q^{2.41m/\log q} \ll n^{0.97} \exp(2.41m) \tag{2.4.16}$$

zeros in the rectangle. By Proposition 2.8 with $c_2 = 2/3 - 1/1000$ there are no zeros in the rectangle unless

$$\begin{aligned} m &\ge \left(\frac{2}{3} - \frac{1}{1000}\right)\left(\frac{\log q}{\log q(2+T)}\right) \log \lambda_1^{-1} \\ &\ge 0.266 \log \lambda_1^{-1}. \end{aligned} \tag{2.4.17}$$

If (2.4.17) holds then we see that each zero contributes

$$\begin{aligned} \left|\frac{x^\rho}{\rho}\right| &\le \frac{x}{n} \exp\left(-m \frac{\log x}{\log q}\right) = \frac{x}{n} \exp\left(-m\left(\frac{\log x}{\log q} - \frac{1}{0.266}\right)\right) \exp\left(-\frac{m}{0.266}\right) \\ &\le \frac{\lambda_1 x}{n} \exp\left(-m\left(\frac{\log x}{\log q} - 3.76\right)\right). \end{aligned} \tag{2.4.18}$$

Thus zeros in the rectangle give a total contribution of

$$\ll \frac{\lambda_1 x n^{0.97}}{n} \exp\left(-m\left(\frac{\log x}{\log q} - 3.76 - 2.41\right)\right) \ll \frac{\lambda_1 x}{n^{0.03}} \exp\left(-m\left(\frac{\log x}{\log q} - 6.17\right)\right). \tag{2.4.19}$$

We now sum this bound over $n = 2^j$ with $j \in \mathbb{N}$, and over $m \ge 0.266 \log \lambda_1^{-1}$. We see that provided $q^{6.18} \le x$, the contribution to the sum (2.4.10) from all non-exceptional zeros in the region

$$0.6 \le \Re(\rho) \le 1, \qquad 1 \le |\Im(\rho)| \le T \tag{2.4.20}$$

is

$$\ll \lambda_1 x \exp(-0.0266 \log \lambda_1{}^{-1}) \le \lambda_1 x \exp(-0.0266 \log \eta). \tag{2.4.21}$$

Therefore, provided η is sufficiently small, this is at most $\lambda_1 x$.

Similarly we consider the contribution to the sum (2.4.10) from zeros in the region

$$1 - \frac{m+1}{\log q} \le \Re(\rho) \le 1 - \frac{m}{\log q}, \qquad |\Im(\rho)| \le 1, \tag{2.4.22}$$

with $m \leq 0.4 \log q$. As above, each zero contributes

$$\ll \lambda_1 x \exp\left(-m\left(\frac{\log x}{\log q} - 3.76\right)\right). \tag{2.4.23}$$

The number of zeros in the rectangle is $\ll \exp(2.41m)$. Therefore the contribution of all such zeros is $\ll \lambda_1 x \exp(-0.0266 \log \eta) \leq \lambda_1 x$ for η sufficiently small.

Finally we consider zeros in the rectangles

$$0 \leq \Re(\rho) \leq 0.6, \qquad 0 \leq |\Im(\rho)| \leq \sqrt{T} \tag{2.4.24}$$

$$0 \leq \Re(\rho) \leq 0.6, \qquad \sqrt{T} \leq |\Im(\rho)| \leq T. \tag{2.4.25}$$

By symmetry of zeros around the line $\Re(s) = 1/2$ we have that $\Re(\rho) \gg \lambda_1/\log q$ for all such ρ. Thus, since $\lambda_1 \gg q^{-1/2-1/100}$ and $x > q$, each zero satisfying (2.4.24) contributes

$$\left|\frac{x^\rho}{\rho}\right| \ll \frac{x^{\Re(\rho)}}{\Re(\rho)} \ll x^{0.6}, \tag{2.4.26}$$

and similarly every zero satisfying (2.4.25) contributes

$$\left|\frac{x^\rho}{\rho}\right| \ll \frac{x^{0.6}}{\sqrt{T}}. \tag{2.4.27}$$

By Proposition 2.9 there are $\ll (q\sqrt{T})^{1+1/1000} \leq q^{1.76}$ zeros satisfying (2.4.24), and there are $\ll (qT)^{1+1/1000} \leq q^{1.76}\sqrt{T}$ zeros satisfying (2.4.25). Thus the combined contribution is

$$\ll x^{0.6} q^{1.76} \ll \lambda_1 x \left(\frac{q^{2.27}}{x^{0.4}}\right). \tag{2.4.28}$$

We see this is at most $\lambda_1 x$ for $x \geq q^6$ and q sufficiently large.

Since we have now covered all possible zeros in our sum, we see that for η sufficiently small and $x \geq q^{6.18}$ we have

$$\sum_{\chi \pmod q} \sum_\rho \left|\frac{x^\rho}{\rho}\right| \leq 3\lambda_1 x. \tag{2.4.29}$$

Substituting this into (2.4.9) we obtain

$$\left|\psi(x;q,a) - \frac{x}{\phi(q)}\right| \leq \frac{x}{\phi(q)}\left(\exp\left(-\lambda_1\frac{\log x}{\log q}\right) + 4\lambda_1\right). \tag{2.4.30}$$

We note that if $x \geq q^7$ and $\eta < 1/10$ then

$$\exp\left(-\lambda_1\frac{\log x}{\log q}\right) + 4\lambda_1 < 1 - \lambda_1, \tag{2.4.31}$$

since the function $1 - e^{-7t} - 5t$ is zero and increasing at 0, has a unique turning point and is positive at 1/10.

Thus we have shown that for η sufficiently small, $x \geq q^7$ and $\log x \leq q^{1/3000}$ we have

$$\left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| < \frac{(1 - \lambda_1)x}{\phi(q)}, \tag{2.4.32}$$

as required.

2.5 Case 2: No Siegel zeroes

We now consider the case where there are no η-Siegel zeros for some small fixed constant $\eta > 0$. In this case we have $\lambda_\rho \geq \eta$ for all zeros ρ with $|\Im(\rho)| \leq q^2$. Following the method in the previous section and using this zero free region, we can establish Proposition 2.6 if $\log x/ \log q$ is sufficiently large. To obtain an explicit lower bound for the range of $\log x/ \log q$ in which this holds, however, would require us to estimate the constant C_3 in Proposition 2.9, and would likely produce a very large bound if done directly.

We will follow the work done on the estimation of Linnik's constant to obtain an explicit lower bound for $\log x/ \log q$ for which the result holds. This section follows closely the method of Heath-Brown in [26, Section 13].

We define the following quantities which we shall use for the rest of the chapter:

$$M = \frac{\log x}{\log q}, \qquad \mathcal{L} = \log q, \qquad \mathcal{Z}(\chi) = \{\rho : L(\rho, \chi) = 0, 0 \leq \Re(\rho) \leq 1\}, \tag{2.5.1}$$

$$\phi_\chi := \begin{cases} \frac{1}{4}, & q \text{ cube-free or } \operatorname{ord}(\chi) \leq \log q, \\ \frac{1}{3}, & \text{otherwise.} \end{cases} \tag{2.5.2}$$

2.5.1 Weighted sum over primes

We wish to investigate

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n). \tag{2.5.3}$$

We fix a small positive constant $\epsilon > 0$ and let

$$f(t) = \begin{cases} 0, & t \leq 1/2 \\ \frac{\log x}{\epsilon}(t - 1/2), & 1/2 \leq t \leq 1/2 + \epsilon/\log x \\ 1, & 1/2 + \epsilon/\log x \leq t \leq 1 \\ 1 - \frac{\log x}{\epsilon}(t - 1), & 1 \leq t \leq 1 + \epsilon/\log x \\ 0, & 1 + \epsilon/\log x \leq t. \end{cases} \tag{2.5.4}$$

The Brun-Titchmarsh theorem for primes in short intervals (see [36], for example) states that

$$\pi(x; q, a) - \pi(x - y; q, a) \leq \frac{2y}{\phi(q) \log y/q}. \tag{2.5.5}$$

We replace the sum

$$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) \tag{2.5.6}$$

with the weighted sum

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right). \tag{2.5.7}$$

By the Brun-Titchmarsh theorem for primes in short intervals and for ϵ sufficiently small, the error introduced by making this change is

$$\begin{aligned} &\leq \sum_{\substack{x \leq n \leq xe^{\epsilon} \\ n \equiv a \pmod q}} \Lambda(n) + \sum_{n \leq e^{\epsilon} x^{1/2}} \Lambda(n) \leq (\log xe^{\epsilon})(\pi(xe^{\epsilon}; q, a) - \pi(x; q, a)) + O(x^{1/2}) \\ &\leq \frac{4\epsilon x}{\phi(q)}. \end{aligned} \tag{2.5.8}$$

Thus in order to prove

$$\left| \psi(x; q, a) - \frac{x}{\phi(q)} \right| \leq \frac{(1 - \epsilon)x}{\phi(q)}, \tag{2.5.9}$$

it is sufficient to prove that

$$\left| \sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right) - \frac{x}{\phi(q)} \right| < \frac{(1 - 5\epsilon)x}{\phi(q)}. \tag{2.5.10}$$

We also note that

$$\sum_{\substack{n=1 \\ n \equiv a (\mathrm{mod}\ q)}}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right) = \frac{1}{\phi(q)} \sum_{\chi \pmod q} \overline{\chi(a)} \left(\sum_{n=1}^{\infty} \Lambda(n) f\left(\frac{\log n}{\log x}\right) \chi(n) \right). \tag{2.5.11}$$

We now replace χ in the inner sum with the primitive character χ^* which induces it. This introduces an error

$$\ll \frac{1}{\phi(q)} \sum_{\chi} \sum_{p|q} \sum_{x^{1/2} \leq p^e \leq x e^{\epsilon}} \log p \ll \sum_{p|q} \log x$$
$$\ll q^{\epsilon} \log x \leq \epsilon x \tag{2.5.12}$$

(recalling that $x > q$).

Therefore it is sufficient to prove that

$$\left| \sum_{\chi} \overline{\chi}(a) \sum_{n=1}^{\infty} \Lambda(n) \chi^*(n) f\left(\frac{\log n}{\log x}\right) - x \right| \leq (1 - 6\epsilon)x. \tag{2.5.13}$$

2.5.2 Sum over zeroes

We let F be the Laplace transform of f. Hence

$$F(s) = \int_0^{\infty} \exp(-st) f(t) dt$$
$$= e^{-s} \left(\frac{1 - \exp(s/2)}{-s}\right) \left(\frac{1 - \exp(\frac{\epsilon}{\log x} s)}{-\frac{\epsilon}{\log x} s}\right) \exp\left(-\frac{\epsilon}{\log x} s\right). \tag{2.5.14}$$

From the Laplace inversion formula we have

$$f\left(\frac{\log n}{\log x}\right) = \frac{\log x}{2\pi i} \int_{2-i\infty}^{2+i\infty} n^{-s} F(-s \log x) ds. \tag{2.5.15}$$

Therefore, for $\chi \neq \chi_0$ we obtain

$$\sum_{n=1}^{\infty} \Lambda(n) \chi^*(n) f\left(\frac{\log n}{\log x}\right) = \frac{\log x}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(-\frac{L'}{L}(s, \chi^*)\right) (F(-s \log x))\, ds$$
$$= \frac{\log x}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \left(-\frac{L'}{L}(s, \chi^*)\right) (F(-s \log x))\, ds$$
$$- \log x \sum_{\rho} F(-\rho \log x) \tag{2.5.16}$$

where $\sum_{\rho}$ indicates a sum over all non-trivial zeros of $L(s, \chi)$.

On $\Re s = -\frac{1}{2}$ we have (see, for example, [9, Chapter 19] for the estimate for $L(s, \chi*)$)

$$\frac{L'}{L}(s, \chi^*) \ll \log(q(1 + |s|)), \qquad F(-s \log x) \ll x^{-1/4} |s|^{-2} (\log x)^{-1}. \tag{2.5.17}$$

Hence, recalling that $q \leq x$,

$$\frac{\log x}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \left(-\frac{L'}{L}(s, \chi^*) \right) (F(-s \log x))\, ds = O(x^{-1/4} \log x). \tag{2.5.18}$$

Thus

$$\begin{aligned} \sum_{\chi \neq \chi_0} \left| \sum_{n=1}^{\infty} \Lambda(n) \chi^*(n) f\left(\frac{\log n}{\log x} \right) \right| &\leq \log x \sum_{\chi \neq \chi_0} \sum_{\rho} |F(-\rho \log x)| \\ &\quad + O(q x^{-1/4} \log x) \\ &\leq \log x \sum_{\chi \neq \chi_0} \sum_{\rho} |F(-\rho \log x)| + \epsilon x. \end{aligned} \tag{2.5.19}$$

We now consider the case $\chi = \chi_0$. We note that χ_0^* is identically 1. Hence by the prime number theorem we have

$$\left| \sum_{n=1}^{\infty} \Lambda(n) \chi_0^*(n) f\left(\frac{\log n}{\log x} \right) - x \right| \leq 3\epsilon x. \tag{2.5.20}$$

Thus, putting together (2.5.19) and (2.5.20), we have

$$\begin{aligned} &\left| \sum_{\chi} \overline{\chi}(a) \sum_{n=1}^{\infty} \Lambda(n) \chi^*(n) f\left(\frac{\log n}{\log x} \right) - x \right| \\ &\quad \leq \left| \sum_{n=1}^{\infty} \Lambda(n) \chi_0^*(n) f\left(\frac{\log n}{\log x} \right) - x \right| + \sum_{\chi \neq \chi_0} \left| \sum_{n=1}^{\infty} \Lambda(n) \chi^*(n) f\left(\frac{\log n}{\log x} \right) \right| \\ &\quad \leq 4\epsilon x + \log x \sum_{\chi \neq \chi_0} \sum_{\rho} \left| F(-\rho \log x) \right|. \end{aligned} \tag{2.5.21}$$

In particular it is sufficient to prove that

$$\log x \sum_{\chi \neq \chi_0} \sum_{\rho} \left| F(-\rho \log x) \right| \leq (1 - 10\epsilon) x. \tag{2.5.22}$$

We now consider the contribution from the other characters where $\chi \neq \chi_0$. We first consider all zeros $\rho = \beta + i\gamma$ of all L-functions $L(s, \chi)$ (with $\chi \neq \chi_0$) in the rectangle

$$1 - \frac{m+1}{\log q} \leq \beta \leq 1 - \frac{m}{\log q}, \qquad n \leq |\gamma| \leq 2n \tag{2.5.23}$$

for $n \geq 1$.

From Proposition 2.9 with $c_3 = 3$ (and using the fact that the zeros are symmetric about $\sigma = 1/2$) we have

$$\sum_{\chi} N(\sigma, \chi, T) \ll q^{3(1-\sigma)} \left(1 + T^{3/2} \right). \tag{2.5.24}$$

Thus there are $\ll e^{3m}\left(1+n^{3/2}\right)$ such zeros in the rectangle. Each zero contributes

$$\log x \left|F(-\rho \log x)\right| \ll x \frac{\exp(-m\frac{\log x}{\log q})}{\epsilon n^2} \tag{2.5.25}$$

to the right hand side of (2.5.21).

Thus, provided $M = \log x / \log q > 3$, there is a constant R (depending only on ϵ) such that the contribution of all zeros in the rectangles with $\max(m, n) \geq R$ is at most ϵx.

Similarly, we consider zeros in the rectangle

$$\max\left(\frac{1}{2}, 1 - \frac{m+1}{\log q}\right) \leq \beta \leq 1 - \frac{m}{\log q}, \qquad |\gamma| \leq 1. \tag{2.5.26}$$

There are $\ll e^{3m}$ such zeros, and each zero contributes

$$\ll x \frac{\exp(-m\frac{\log x}{\log q})}{\epsilon}. \tag{2.5.27}$$

Therefore, provided $M > 3$, the contribution from all zeros in rectangles with $m \geq R$ is $\leq \epsilon x$.

We now consider the final rectangle

$$0 \leq \beta \leq \frac{1}{2}, \qquad |\gamma| \leq 1. \tag{2.5.28}$$

There are $\ll q^{3/2}$ zeros in this rectangle. All zeros must have $\beta \geq q^{-1/2-1/100}$ for q sufficiently large (by symmetry of zeros about the critical line and the non-existence of Siegel zeros which are within $q^{-1/2-1/100}$ of 1). Therefore each zero contributes $\ll x^{1/2} q^{2/100}/\epsilon$. Thus the contribution from these zeros is

$$\ll \frac{x^{1/2} q^{3/2+1/50}}{\epsilon} \leq \epsilon x. \tag{2.5.29}$$

Thus at a cost of $3\epsilon x$ we only need to consider the contribution of zeros ρ satisfying

$$|1 - \Re(\rho)| \ll_\epsilon \frac{1}{\log q}, \qquad \Im(\rho) \ll_\epsilon 1. \tag{2.5.30}$$

For such ρ, and for ϵ sufficiently small and q sufficiently large, we have

$$\left|\left(\frac{1 - x^{-\rho/2}}{\rho}\right) e^{\epsilon \rho}\right| \leq 1 + 3\epsilon. \tag{2.5.31}$$

Also, for any $z \in \mathbb{C}$ with $\Re(z) \geq 0$ we have

$$\left|\frac{1 - e^{-z}}{z}\right| \leq 1. \tag{2.5.32}$$

Thus, putting $\Re(\rho) = 1 - \lambda_\rho / \log q$, we obtain

$$\begin{aligned}\log x \left| F(-\rho \log x) \right| &= x \exp(-(1-\rho)\log x) \left| \left(\frac{1 - x^{-\rho/2}}{\rho} \right) \left(\frac{1 - e^{-\epsilon\rho}}{\epsilon\rho} \right) e^{\epsilon\rho} \right| \\ &\le x \exp\left(-\lambda_\rho \frac{\log x}{\log q} \right) (1 + 3\epsilon) \\ &= x \exp\left(-M\lambda_\rho \right) (1 + 3\epsilon). \end{aligned} \tag{2.5.33}$$

Thus we have shown that

$$\left| \psi(x; q, a) - \frac{x}{\psi(q)} \right| \le 12\epsilon \frac{x}{\phi(q)} + (1 + 3\epsilon) \frac{x}{\phi(q)} \sum_{\chi \neq \chi_0} \sideset{}{^*}\sum_{\rho} \exp(-M\lambda_\rho), \tag{2.5.34}$$

where $\sideset{}{^*}\sum$ represents a sum over all zeros of $L(s, \chi)$ in

$$\mathcal{R} = \left\{ z : 1 - \frac{R}{\log q} \le \Re(z) \le 1, \Im(z) \le R \right\}, \tag{2.5.35}$$

with R a constant (independent of x and q).

2.6 Zero density estimates

We wish to estimate the sum

$$\sum_{\chi \neq \chi_0} \sum_{\rho \in \mathcal{R} \cap Z(\chi)} \exp(-M\lambda_\rho). \tag{2.6.1}$$

We do this by obtaining a zero density estimate for zeros in $\mathcal{R}$ by means of different weighted sums over zeros of $L(s, \chi)$. We note that by the log-free zero density estimate given in Proposition 2.9 this sum is finite for any $M \in \mathbb{R}$. We specifically wish to show that the sum is < 1 when $M = 7.999$.

Similar sums have been looked at in the estimation of Linnik's constant. We will broadly follow the approach of Heath-Brown in [26], but most of the estimates must be extended to cover a region where $\Im(\rho) \ll 1$ instead of $\Im(\rho) \ll \mathcal{L}^{-1}$.

We split $\mathcal{R}$ vertically into smaller rectangles each with height $1/\mathcal{L}$. We put

$$\mathcal{R}_m := \left\{ z : 1 - \frac{R}{\mathcal{L}} \le \Re(z) \le 1, \frac{m - 1/2}{\mathcal{L}} \le |\Im(z)| \le \frac{m + 1/2}{\mathcal{L}} \right\}. \tag{2.6.2}$$

We label our non-principal characters (mod q) as $\chi^{(1)}, \chi^{(2)}, \dots$ in some order with $\chi_1 = \chi^{(1)}$. For each character $\chi^{(j)}$, and for each rectangle $\mathcal{R}_m$ for which $L(s, \chi^{(j)})$ has a zero in $\mathcal{R}_m$ we pick a zero of $L(s, \chi^{(j)})$ with greatest real part, which we label $\rho^{(j,m)}$.

We introduce the notation

$$\rho^{(j,m)} = \beta^{(j,m)} + i\gamma^{(j,m)}, \qquad 1 - \beta^{(j,m)} = \frac{\lambda^{(j,m)}}{\log q}, \qquad \gamma^{(j,m)} = \frac{\nu^{(j,m)}}{\log q}. \tag{2.6.3}$$

We also specifically label special zeros ρ_1, ρ_1' and ρ_2. We let ρ_1 be a zero of $\prod_\chi L(s,\chi)$ which is in $\mathcal{R}$ and has largest real part. We let χ_1 be the corresponding character. We let ρ_2 be a zero of $\prod_{\chi,\chi\neq\chi_1,\overline{\chi}_1} L(s,\chi)$ which is in $\mathcal{R}$ and has largest real part. We let ρ_1' be a zero of $L(s,\chi_1)$ which is in $\mathcal{R}$ and is not ρ_1 or $\overline{\rho}_1$ but otherwise has largest real part. If ρ_1 is not a simple zero we simply have $\rho_1' = \rho_1$.

For simplicity we argue as if ρ_1, ρ_1', ρ_2 all exist. Our argument is simpler and stronger if any of these do not exist.

We now wish to estimate separately a weighted sum over rectangles and a weighted sum over zeros in any such rectangle. Specifically we wish to prove the following three lemmas:

Lemma 2.10. *For any $\delta > 0$ any $m \in \mathbb{Z}$ and any constant $K > 0$ we have for $q > q_0(\delta)$ that*

$$\sum_{\rho\in\mathcal{R}_m\cap Z(\chi^{(j)})} B_1(\lambda_\rho) \leq C_1(\lambda^{(j,m)})$$

where

$$B_1(\lambda) = \frac{(1-\exp(-K\lambda))^2}{\lambda^2 + 1/4},$$
$$C_1(\lambda) = \frac{\phi_\chi(1-\exp(-2K\lambda))}{2\lambda} + \frac{2K\lambda - 1 + \exp(-2K\lambda)}{2\lambda^2} + \delta.$$

Lemma 2.11. *We have*

$$\sum_{j,m}\left(\frac{e^{3.245\lambda^{(j,m)}} + e^{2.826\lambda^{(j,m)}}}{0.21} + \frac{e^{1.240\lambda^{(j,m)}} + e^{1.128\lambda^{(j,m)}}}{0.056}\right)^{-1} \leq 11.9288.$$

We actually prove a slightly more general result than Lemma 2.11. Let $(\chi^{(i)})_{i\in I}$ be a set of characters (mod q). Then for any $\delta > 0$ and $q > q_0(\delta)$ we will prove

$$\sum_{m\in\mathbb{Z}, i\in\mathbb{I}} B_2(\lambda^{(i,m)}) \leq C_2 \tag{2.6.4}$$

where

$$B_2(\lambda) = \left(\frac{e^{2\lambda x_1} + e^{2\lambda x_0}}{x_1 - x_0} + \frac{e^{2\lambda u_1} + e^{2\lambda u_0}}{u_1 - u_0}\right)^{-1}, \tag{2.6.5}$$

$$C_2 = \left(\frac{x_1 + x_0 - v - u_1}{2w(v - u_1)}\right)(1 + G_2) + \delta. \tag{2.6.6}$$

Here G_2 will be defined in (2.6.58), and x_1, x_0, v, u_1, u_0, w are all positive constants satisfying

$$x_1 > x_0, \qquad x_0 > v + w + 1/3, \qquad v > u_1, \qquad u_1 > u_0, \qquad u_0 > 2w + 1/3. \quad (2.6.7)$$

Lemma 2.11 is this with x_1, x_0, v, u_1, u_0, w given by (2.6.73) and (2.6.74).

Lemma 2.12. *Let $g : [0, \infty) \to \mathbb{R}$ be a non-negative continuous function which is supported on $[0, x_0)$ for some $x_0 > 0$, is twice differentiable on $(0, x_0)$ and has a bounded second derivative on $(0, x_0)$. Moreover, assume the Laplace transform G of g satisfies $\Re(G(z)) \geq 0$ for $\Re(z) \geq 0$.*

Let $0 \leq \lambda_{11} \leq \lambda_1$ and $0 \leq \lambda \leq 2$ be such that

$$G(\lambda - \lambda_{11}) > g(0)/6 \qquad \textit{and} \qquad (G(\lambda - \lambda_{11}) - g(0)/6)^2 > G(-\lambda_{11})g(0)/6.$$

Then for any $\delta > 0$ and $q > q_0(\delta, g)$ we have

$$\sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} 1 \leq \frac{G(-\lambda_{11})G_3}{(G(\lambda - \lambda_{11}) - g(0)/6)^2 - G(-\lambda_{11})g(0)/6} + \delta.$$

Here

$$G_3 = \sup_{j_1, m_1} \sum_{m_2} \left(\left| \Re \left(G(-\lambda_{11} + i(\gamma^{(j_1,m_1)} - \gamma^{(j_1,m_2)})) \right) \right| - g(0)/6 \right).$$

In particular, we have the zero density results given by Table 2.1.

We will now proceed to prove each of these Lemmas in turn.

We note here that we can easily ensure the L given in [26, Lemma 6.1] satisfies $R \leq L \leq \frac{1}{10}\mathcal{L}$ rather than just $L \leq \frac{1}{10}\mathcal{L}$ by following exactly the same argument but with this restriction. This means that all the results of Heath-Brown [26] and Xylouris [51] which consider zeros in the region

$$1 - \frac{\log \log \mathcal{L}}{3\mathcal{L}} \leq \sigma \leq 1, \qquad |t| \leq L \quad (2.6.8)$$

also apply to the zeros which we consider in $\mathcal{R}$.

2.6.1 First zero density estimate

We now consider zeros within one of the rectangles $\mathcal{R}_m$. We follow almost identically the argument of Heath-Brown in [26, Lemma 13.3].

We put

$$h_1(t) = \begin{cases} \sinh(K\lambda - t\lambda), & 0 \le t \le K \\ 0, & t \ge K, \end{cases} \tag{2.6.9}$$

$$H_1(z) = \int_0^\infty e^{-zt} h_1(t)dt = \frac{1}{2}\left(\frac{e^{K\lambda}}{\lambda + z} + \frac{e^{-K\lambda}}{\lambda - z} - \frac{2\lambda e^{-Kz}}{\lambda^2 - z^2}\right), \tag{2.6.10}$$

$$H_2(z) = \left(\frac{1 - e^{-Kz}}{z}\right)^2, \tag{2.6.11}$$

for some constants $K \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, which will be declared later.

We note that

$$\Re(H_1(it)) = \frac{\lambda e^{K\lambda}}{2}\left|\frac{1 - e^{-K(\lambda+it)}}{\lambda + it}\right|^2 = \frac{\lambda e^{K\lambda}}{2}\left|H_2(\lambda + it)\right|. \tag{2.6.12}$$

Since $H_1(z)$ and $H_2(\lambda + z)$ tend uniformly to zero in $\Re(z) \ge 0$ as $|z| \to \infty$, and $\Re(H_1(z)) = \lambda e^{K\lambda}|H_2(\lambda + z)|/2$ when $\Re(z) = 0$, by [26, Lemma 4.1] we have

$$\Re(H_1(z)) \ge \frac{\lambda e^{K\lambda}}{2}|H_2(\lambda + z)| \tag{2.6.13}$$

whenever $\Re(z) \ge 0$.

We fix a character $\chi = \chi^{(j)} \ne \chi_0$ and take $\lambda = \lambda^{(j,m)}$. Therefore $L(s,\chi)$ has no zeros in the region $\{\sigma > 1 - \lambda/\mathcal{L}\} \cap \mathcal{R}_m$.

Thus

$$\sum_{\rho \in \mathcal{R}_m \cap Z(\chi)} |H_2((1 - \rho + im/\mathcal{L})\mathcal{L})| \le \frac{2e^{-K\lambda}}{\lambda} \sum_{\rho \in \mathcal{R}_m \cap Z(\chi)} \Re(H_1((s - \rho)\mathcal{L})), \tag{2.6.14}$$

where $s = 1 - \lambda/\mathcal{L} + im/\mathcal{L}$.

By [26, Lemma 5.2] and [26, Lemma 5.3] we have (recalling that $|m| \ll \mathcal{L}$ so $|\Im(s)| \le \mathcal{L}$ for q sufficiently large), for any given $\delta > 0$ and $q > q(\delta)$ the upper bound

$$\begin{aligned} \sum_{\rho \in \mathcal{R}_m \cap Z(\chi)} \Re(H_1((s - \rho)\mathcal{L})) &\le \frac{h_1(0)\phi_\chi}{2} + \mathcal{L}^{-1}\left|\sum_{n=1}^\infty \Lambda(n)\Re\left(\frac{\chi(n)}{n^s}\right)h_1(\mathcal{L}^{-1}\log n)\right| + \delta \\ &\le \frac{h_1(0)\phi_\chi}{2} + \mathcal{L}^{-1}\sum_{n=1}^\infty \Lambda(n)\left(\frac{\chi_0(n)}{n^{\Re(s)}}\right)h_1(\mathcal{L}^{-1}\log n) + \delta \\ &\le \frac{h_1(0)\phi_\chi}{2} + |H_1((\Re(s) - 1)\mathcal{L})| + 2\delta. \end{aligned} \tag{2.6.15}$$

This gives

$$\sum_{\rho \in \mathcal{R}_m \cap Z(\chi)} \left|\frac{1 - e^{-K\lambda_\rho - iK(m - \gamma_\rho\mathcal{L})}}{\lambda_\rho + i(m - \gamma_\rho\mathcal{L})}\right|^2 \le \frac{\phi_\chi(1 - e^{-2K\lambda})}{2\lambda} + \frac{2K\lambda - 1 + e^{-2K\lambda}}{2\lambda^2} + 2\delta. \tag{2.6.16}$$

Since $\rho \in \mathcal{R}_m$, we have $|m - \gamma_\rho \mathcal{L}| \leq 1/2$. Thus, recalling that $\chi = \chi^{(j)}$ and $\lambda = \lambda^{(j,m)}$, we obtain

$$\sum_{\rho \in \mathcal{R}_m \cap Z(\chi^{(j)})} \frac{(1 - e^{-K\lambda_\rho})^2}{\lambda_\rho^2 + 1/4} \leq \frac{\phi_\chi (1 - e^{-2K\lambda^{(j,m)}})}{2\lambda^{(j,m)}} + \frac{2K\lambda^{(j,m)} - 1 + e^{-2K\lambda^{(j,m)}}}{2(\lambda^{(j,m)})^2} + 2\delta. \tag{2.6.17}$$

Hence Lemma 2.10 holds.

2.6.2 Second zero density estimate

We now prove Lemma 2.11. The proof uses ideas originally due to Graham [22]. We follow the method of [26, Section 11], but extend the result to a weighted sum over zeros rather than just characters. We do this by using integrated exponential weights instead of exponential weights, an idea originally due to Jutila [31].

We adopt similar notation to that of [26, Section 11]. We put

$$U_0 = q^{u_0}, U_1 = q^{u_1}, X_0 = q^{x_0}, X_1 = q^{x_1}, V = q^v, W = q^w \tag{2.6.18}$$

with constant exponents $0 < w < u_0 < u_1 < v < x_0 < x_1$ to be declared later. We put

$$U = q^u, X = q^x \tag{2.6.19}$$

with $u_0 \leq u \leq u_1$ and $x_0 \leq x \leq x_1$ parameters which we will integrate over.

We define

$$\psi_d = \begin{cases} \mu(d), & 1 \leq d \leq U_1 \\ \mu(d)\frac{\log V/d}{\log V/U_1}, & U_1 \leq d \leq V \\ 0, & d \geq V, \end{cases} \tag{2.6.20}$$

and

$$\theta_d = \begin{cases} \mu(d)\frac{\log W/d}{\log W}, & 1 \leq d \leq W \\ 0, & d \geq W. \end{cases} \tag{2.6.21}$$

We wish to study the sum

$$J(\rho^{(j,m)}, \chi) := w_{j,m} \sum_{n=1}^{\infty} \left(\sum_{d|n} \psi_d \right) \left(\sum_{d|n} \theta_d \right) \chi(n) n^{-\rho^{(j,m)}} j(n), \tag{2.6.22}$$

where

$$j(n) = \left(\frac{\int_{x_0}^{x_1} \int_{u_0}^{u_1} (e^{-n/X} - e^{-n\mathcal{L}^2/U}) du dx}{(u_1 - u_0)(x_1 - x_0)} \right) \tag{2.6.23}$$

and $w_{j,m}$ are some non-negative weights.

We start with the following weighted-sum result.

Lemma 2.13. *For $x_0 > w + v + \phi_{\chi^{(j)}}$ we have:*

$$w_{j,m}^2 \leq (1 + O(\mathcal{L}^{-1})) \left| J(\rho^{(j,m)}, \chi^{(j)}) \right|^2 .$$

Proof. The argument of [26, Pages 317-318] shows that for $x_0 > w + v + \phi_{\chi^{(j)}}$

$$1 + O(\mathcal{L}^{-1}) = \sum_{n=1}^{\infty} \left(\sum_{d|n} \psi_d \right) \left(\sum_{d|n} \theta_d \right) \chi^{(j)}(n) n^{-\rho^{(j,m)}} \left(e^{-n/X} - e^{-n\mathcal{L}^2/U} \right). \tag{2.6.24}$$

We note that in [26] the definition of ψ_d is slightly different (it is defined with constants labelled U and V rather than U_1 and V as in our case), but this does not affect the argument in any way since $U_1 \geq U$.

Multiplying the above expression by weights $w_{j,m}$ and integrating over $x \in [x_0, x_1]$ and $u \in [u_0, u_1]$ gives

$$w_{j,m} = (1 + O(\mathcal{L}^{-1})) J(\rho^{(j,m)}, \chi^{(j)}). \tag{2.6.25}$$

Squaring both sides of the above expression gives the result. □

We sum the expression of Lemma 2.13 over all zeros $\rho^{(j,m)}$. We let $\sum_{j,m}$ denote this sum. Thus

$$\sum_{j,m} w_{j,m}^2 \leq (1 + O(\mathcal{L}^{-1})) \sum_{j,m} \left| J(\rho^{(j,m)}, \chi^{(j)}) \right|^2 . \tag{2.6.26}$$

We now use the duality principle (see, for example, [9, Chapter 27]).

Lemma 2.14 (Duality Principle)**.** *If*

$$\sum_n \left| \sum_{j,m} a_{n,j,m} C_{j,m} \right|^2 \leq B \sum_{j,m} \left| C_{j,m} \right|^2$$

for all choices of the coefficients $C_{j,m}$, then

$$\sum_{j,m} \left| \sum_n a_{n,j,m} b_n \right|^2 \leq B \sum_n |b_n|^2$$

for any choice of b_n.

We wish to use Lemma 2.14 with

$$a_{n,j,m} = w_{j,m} \chi^{(j)}(n) n^{1/2-\rho^{(j,m)}} \left(\sum_{d|n} \theta_d \right) j(n)^{1/2}, \tag{2.6.27}$$

$$b_n = \left(\sum_{d|n} \psi_d \right) n^{-1/2} j(n)^{1/2} \tag{2.6.28}$$

to bound this sum. We note that

$$\sum_{n=1}^{\infty} a_{n,j,m} b_n = J(\rho^{(j,m)}, \chi^{(j)}). \tag{2.6.29}$$

First we evaluate $\sum b_n^2$.

Lemma 2.15. *For $x_0 > v$ we have:*

$$\sum_{n=1}^{\infty} |b_n|^2 = (1 + O(\mathcal{L}^{-1} \log \mathcal{L})) \frac{x_1 + x_0 - u_1 - v}{2(v - u_1)}.$$

Proof. The argument leading to equation (11.14) of [26, Page 319] shows that provided $x > v$ we have

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} \psi_d \right)^2 n^{-1} (e^{-n/X} - e^{-n\mathcal{L}^2/U}) = (1 + O(\mathcal{L}^{-1} \log \mathcal{L})) \frac{2x - u_1 - v}{2(v - u_1)}. \tag{2.6.30}$$

(We recall that our definition of ψ_d used parameters U_1 and V rather than U and V.) Since we have $x \geq x_0 > v$ this holds in our case.

Therefore, integrating with respect to $x \in [x_0, x_1]$ and $u \in [u_0, u_1]$ and dividing through by $(x_1 - x_0)(u_1 - u_0)$ gives

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} \psi_d \right)^2 n^{-1} \frac{\int_{x_0}^{x_1} \int_{u_0}^{u_1} (e^{-n/X} - e^{-n\mathcal{L}^2/U}) du dx}{(x_1 - x_0)(u_1 - u_0)}$$
$$= (1 + O(\mathcal{L}^{-1} \log \mathcal{L})) \frac{x_1 + x_0 - u_1 - v}{2(v - u_1)}. \tag{2.6.31}$$

Hence the result holds. ◻

Therefore in order to use Lemma 2.14 we want to find a bound B such that

$$\sum_{n=1}^{\infty} \left| \sum_{j,m} a_{n,j,m} C_{j,m} \right|^2 \leq B \sum_{j,m} \left| C_{j,m} \right|^2 \tag{2.6.32}$$

for any possible choice of $C_{j,m}$.

Expanding the left hand side, terms are of the form

$$\sum_{n=1}^{\infty} a_{n,j_1,m_1} a_{n,j_2,m_2} C_{j_1,m_1} \overline{C}_{j_2,m_2}$$
$$= C_{j_1,m_1} \overline{C}_{j_2,m_2} w_{j_1,m_1} w_{j_2,m_2} \sum_{n=1}^{\infty} \left(\sum_{d|n} \theta_d \right)^2 \chi^{(j_1)}(n) \overline{\chi}^{(j_2)}(n) n^{1-\rho^{(j_1,m_1)} - \overline{\rho}^{(j_2,m_2)}} j(n). \tag{2.6.33}$$

To ease notation we let

$$\rho_{(1)} = \rho^{(j_1,m_1)}, \qquad \rho_{(2)} = \rho^{(j_2,m_2)}, \tag{2.6.34}$$

and correspondingly define $\chi_{(1)}, \chi_{(2)}, \beta_{(1)}, \beta_{(2)}, \lambda_{(1)}, \lambda_{(2)}, \gamma_{(1)}, \gamma_{(2)}$.

We first deal with the terms when $\chi_{(1)} \neq \chi_{(2)}$. We let

$$J_2(s,\chi) = \sum_{w_1,w_2\le W} \theta_{w_1}\theta_{w_2}\chi([w_1,w_2])[w_1,w_2]^{-s}. \tag{2.6.35}$$

(Here $[a,b]$ denotes the least common multiple of a and b).

By the inverse Laplace transform of the exponential function we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2 \chi_{(1)}(n)\overline{\chi}_{(2)}(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}(e^{-n/X} - e^{-n\mathcal{L}^2/U}) \\
&\quad = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})(X^s - (U\mathcal{L}^{-2})^s) \\
&\qquad\qquad\qquad \times \Gamma(s)J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})ds \\
&\quad = \frac{1}{2\pi i}\int_{2-\beta_{(1)}-\beta_{(2)}-1/k-i\infty}^{2-\beta_{(1)}-\beta_{(2)}-1/k+i\infty} L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})(X^s - (U\mathcal{L}^{-2})^s) \\
&\qquad\qquad\qquad \times \Gamma(s)J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi_{(1)}\overline{\chi}_{(2)})ds.
\end{aligned} \tag{2.6.36}$$

where $k > 1$ is a fixed constant (to be declared later).

We now bound each of the terms in the integration along the line $\Re(s) = 2-\beta_{(1)}-\beta_{(2)}-1/k$ and for $\chi \neq \chi_0$. By [26, Lemma 2.5] we have

$$L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi) \ll_k q^{\phi_\chi/k+1/k^2}(1+|t|), \tag{2.6.37}$$

where we recall that $\phi_\chi = 1/4$ or $1/3$ for each χ. Moreover, we also have the bounds

$$\Gamma(s) \ll e^{-|t|}, \tag{2.6.38}$$

$$\begin{aligned}
J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi) &\ll \sum [w_1,w_2]^{-1+1/k} \ll \sum_{n\le W^2} n^{-1+1/k}d(n)^2 \\
&\ll W^{2/k}\mathcal{L}^3.
\end{aligned} \tag{2.6.39}$$

Thus, letting $\chi = \chi_{(1)}\overline{\chi}_{(2)}$, we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{1-\beta_{(1)}-\beta_{(2)}-1/k-i\infty}^{1-\beta_{(1)}-\beta_{(2)}-1/k+i\infty} L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi)\Gamma(s)(X^s - (U\mathcal{L}^{-2})^s)J_2(s+\rho_{(1)}+\overline{\rho}_{(2)}-1, \chi)ds \\
&\qquad \ll (q^{\phi_\chi}W^2U^{-1}\mathcal{L}^3)^{1/k}q^{1/k^2}\mathcal{L}^2(U\mathcal{L}^{-2})^{2-\beta_{(1)}-\beta_{(2)}} \\
&\qquad \ll (q^{\phi_\chi}W^2U^{-1})^{1/k}q^{2/k^2}.
\end{aligned} \tag{2.6.40}$$

(We recall that $1-\beta_{(1)}$ and $1-\beta_{(2)}$ are $o(1)$.)

This is $O(\mathcal{L}^{-1})$ provided that k is chosen sufficiently large and (remembering that $\phi_\chi \le 1/3$ for all χ) provided that

$$u_0 > 2w + 1/3. \tag{2.6.41}$$

The terms with $\chi_{(1)} \neq \chi_{(2)}$ therefore contribute

$$\ll \mathcal{L}^{-1}\left(\sum_{j,m}|C_{j,m}|w_{j,m}\right)^2 \ll \mathcal{L}^{-1}\left(\sum_{j,m}w_{j,m}^2\right)\sum_{j,m}|C_{j,m}|^2. \tag{2.6.42}$$

We now consider terms with $\chi_{(1)} = \chi_{(2)}$. Such terms are of the form

$$C_{(1)}\overline{C}_{(2)}(w_{(1)}w_{(2)})\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n). \tag{2.6.43}$$

Lemma 2.16. *For $x > v$ we have:*

$$\left|\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n)\right| \le \left|\frac{(1+O(\mathcal{L}^{-1}\log\mathcal{L}))}{w\mathcal{L}^2(2-\rho_{(1)}-\overline{\rho}_{(2)})^2}\right|$$
$$\times\left|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho_{(2)}}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho_{(2)}}}}{u_1-u_0}\right|+O(\mathcal{L}^{-1}).$$

Proof. We observe that

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}\left(e^{-n/X}-e^{-n\mathcal{L}^2/U}\right)$$
$$=\sum_{d_1,d_2}\theta_{d_1}\theta_{d_2}\chi_0([d_1,d_2])[d_1,d_2]^{1-\rho_{(1)}-\overline{\rho}_{(2)}}$$
$$\times\sum_{k=1}^{\infty}k^{1-\rho_{(1)}-\overline{\rho}_{(2)}}\chi_0(k)\left(e^{-k[d_1,d_2]/X}-e^{-k[d_1,d_2]\mathcal{L}^2/U}\right). \tag{2.6.44}$$

Again, by the inverse Laplace transform of the exponential function we have

$$\sum_{k=1}^{\infty}\chi_0(k)k^{1-\rho_{(1)}-\overline{\rho}_{(2)}}(e^{-k[d_1,d_2]/X}-e^{-k[d_1,d_2]\mathcal{L}^2/U})$$
$$=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}L(s+\rho_{(1)}+\overline{\rho}_{(2)}-1,\chi_0)\Gamma(s)\left(\left(\frac{X}{[d_1,d_2]}\right)^s-\left(\frac{U}{\mathcal{L}^2[d_1,d_2]}\right)^s\right)ds. \tag{2.6.45}$$

We again move the line of integration to $\Re(s) = 2-\beta_{(1)}-\beta_{(2)}-1/k$, and by exactly the same reasoning, we have that the integral over this contour is negligible when $u_0 > 2w$. We encounter a simple pole at $s = 2-\rho_{(1)}-\overline{\rho}_{(2)}$, however, which contributes

$$\frac{\phi(q)}{q}\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})\left(\left(\frac{X}{[d_1,d_2]}\right)^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-\left(\frac{U}{\mathcal{L}^2[d_1,d_2]}\right)^{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right). \tag{2.6.46}$$

Thus

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}\left(e^{-n/X}-e^{-n\mathcal{L}^2/U}\right)$$
$$=\frac{\phi(q)}{q}\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})\left(X^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-(U\mathcal{L}^{-2})^{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right)\sum_{d_1,d_2}\frac{\theta_{d_1}\theta_{d_2}\chi_0([d_1,d_2])}{[d_1,d_2]}+O(\mathcal{L}^{-1}). \tag{2.6.47}$$

We now perform the integrations with respect to x and u. We have

$$\frac{1}{(x_1-x_0)(u_1-u_0)}\int_{x_0}^{x_1}\int_{u_0}^{u_1}\left(X^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-(U\mathcal{L}^{-2})^{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right)du\,dx \tag{2.6.48}$$
$$=\left(\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\right)\frac{1}{\mathcal{L}(2-\rho_{(1)}-\overline{\rho}_{(2)})}.$$

Thus

$$\left|\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n)\right|$$
$$\leq\frac{\phi(q)}{\mathcal{L}q}\left|\frac{\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})}{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right|\times\left|\sum_{d_1,d_2}\frac{\theta_{d_1}\theta_{d_2}\chi_0([d_1,d_2])}{[d_1,d_2]}\right|$$
$$\times\left|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\right|+O(\mathcal{L}^{-1}). \tag{2.6.49}$$

We now estimate the sum over d_1, d_2. This gives

$$\left|\sum_{d_1,d_2\leq W}\theta_{d_1}\theta_{d_2}[d_1,d_2]^{-1}\chi_0([d_1,d_2])\right|=\frac{1}{N}\left|\sum_{d_1,d_2\leq W}\theta_{d_1}\theta_{d_2}\left(\frac{q}{\phi(q)}\sum_{\substack{[d_1,d_2]|n\\ n\leq N\\ (n,q)=1}}1+O(q)\right)\right|$$
$$=\frac{q}{\phi(q)N}\sum_{\substack{n\leq N\\ (n,q)=1}}\left(\sum_{d|n}\theta_d\right)^2+O(qW^2N^{-1})$$
$$\leq\frac{q}{\phi(q)N}\sum_{n\leq N}\left(\sum_{d|n}\theta_d\right)^2+O(qW^2N^{-1}). \tag{2.6.50}$$

Graham [21] has shown that for $N > q^2W^2$

$$N^{-1}\sum_{n\leq N}\left(\sum_{d|n}\theta_d\right)^2=\frac{1+O(\mathcal{L}^{-1})}{\log W}. \tag{2.6.51}$$

Hence, for $N > q^3W^2$, we have

$$\left|\sum_{d_1,d_2}\theta_{d_1}\theta_{d_2}[d_1,d_2]^{-1}\chi_0([d_1,d_2])\right| \le \frac{q}{\phi(q)N}\sum_{n\le N}\left(\sum_{d|n}\theta_d\right)^2 + O\left(q^{-2}\right)$$
$$= \frac{(1+O(\mathcal{L}^{-1}))q}{\phi(q)\log W}$$
$$= (1+O(\mathcal{L}^{-1}))\frac{q}{\phi(q)w\mathcal{L}}. \qquad (2.6.52)$$

Thus

$$\left|\sum_{n=1}^{\infty}\left(\sum_{d|n}\theta_d\right)^2\chi_0(n)n^{1-\rho_{(1)}-\overline{\rho}_{(2)}}j(n)\right| \le \frac{(1+O(\mathcal{L}^{-1}))}{\mathcal{L}^2w}\left|\frac{\Gamma(2-\rho_{(1)}-\overline{\rho}_{(2)})}{2-\rho_{(1)}-\overline{\rho}_{(2)}}\right|$$
$$\times\left|\frac{X_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-X_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{x_1-x_0}-\frac{U_1^{2-\rho_{(1)}-\overline{\rho}_{(2)}}-U_0^{2-\rho_{(1)}-\overline{\rho}_{(2)}}}{u_1-u_0}\right| + O(\mathcal{L}^{-1}). \qquad (2.6.53)$$

We recall the Weierstrass product expansion of $\Gamma(s)$:

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)^{-1}e^{s/n}. \qquad (2.6.54)$$

Since $2-\beta_{(1)}-\beta_{(2)} = O(\mathcal{L}^{-1}\log\mathcal{L})$, we see that when $s = 2-\rho_{(1)}-\overline{\rho}_{(2)}$ we have

$$|\Gamma(s)| \le \frac{e^{-\gamma\Re(s)}}{|s|}\prod_{n=1}^{\infty}\left|1+\frac{s}{n}\right|^{-1}e^{\Re(s)/n}$$
$$\le \frac{1+O(\mathcal{L}^{-1}\log\mathcal{L})}{|s|}\prod_{n=1}^{\infty}\left(1+\frac{\Re(s)}{n}\right)^{-1}e^{\Re(s)/n}$$
$$\le \frac{1+O(\mathcal{L}^{-1}\log\mathcal{L})}{|2-\rho_{(1)}-\overline{\rho}_{(2)}|}\prod_{n=1}^{\infty}\left(1+O\left(\frac{\Re(s)}{n^2}\right)\right)$$
$$\le \frac{1+O(\mathcal{L}^{-1}\log\mathcal{L})}{|2-\rho_{(1)}-\overline{\rho}_{(2)}|}. \qquad (2.6.55)$$

This completes the proof. □

To simplify notation we put

$$j_2(a,b) := \frac{1}{\mathcal{L}^2(2-a-\overline{b})^2}\left|\frac{X_1^{2-a-\overline{b}}-X_0^{2-a-\overline{b}}}{x_1-x_0}-\frac{U_1^{2-a-\overline{b}}-U_0^{2-a-\overline{b}}}{u_1-u_0}\right|. \qquad (2.6.56)$$

Thus the sum over all the terms of the form (2.6.43) with $\chi_{(1)} = \chi_{(2)}$ is

$$\le \frac{(1+O(\mathcal{L}^{-1}\log\mathcal{L}))}{w}\sum_{j,\,m_1,\,m_2}\left|C_{j,m_1}C_{j,m_2}w_{j,m_1}w_{j,m_2}j_2(\rho^{(j,m_1)},\rho^{(j,m_2)})\right|$$
$$+ O\left(\mathcal{L}^{-1}\sum_{j,\,m_1,\,m_2}\left|C_{j,m_1}C_{j,m_2}w_{j,m_1}w_{j,m_2})\right|\right). \qquad (2.6.57)$$

We put

$$G_2 = \sup_{j,m_1} \sum_{m_2} \left| w_{j,m_1} w_{j,m_2} j_2(\rho^{(j,m_1)}, \rho^{(j,m_2)}) \right|. \tag{2.6.58}$$

Since $2|C_{(1)}C_{(2)}| \leq |C_{(1)}|^2 + |C_{(2)}|^2$, we have

$$\sum_{j,m_1,m_2} \left| C_{j,m_1} C_{j,m_2} w_{j,m_1} w_{j,m_2} j_2(\rho^{(j,m_1)}, \rho^{(j,m_2)}) \right| \leq G_2 \sum_{j,m} |C_{j,m}|^2. \tag{2.6.59}$$

Combining (2.6.42) and (2.6.59) we have

$$\sum_{n=1}^{\infty} \left| \sum_{j,m} a_{n,j,m} C_{j,m} \right|^2 \leq \left(\frac{G_2}{w} \left(1 + O(\mathcal{L}^{-1} \log \mathcal{L})\right) + O\left((\mathcal{L}^{-1} \sum_{j,m} w_{j,m}^2) \right) \right) \sum_{j,m} \left| C_{j,m} \right|^2 \tag{2.6.60}$$

for any choice of the coefficients $C_{j,m}$.

Therefore, by (2.6.26), Lemma 2.14 and Lemma 2.15 we have

$$\sum_{j,m} w_{j,m}^2 \leq \left(1 + O(\mathcal{L}^{-1} \log \mathcal{L})\right) \left(\frac{G_2}{w} + O\left(\mathcal{L}^{-1} \sum_{j,m} w_{j,m}^2 \right) \right) \left(\frac{x_1 + x_0 - u_1 - v}{2(v - u_1)} \right), \tag{2.6.61}$$

which gives

$$\sum_{j,m} w_{j,m}^2 \leq \left(1 + O(\mathcal{L}^{-1} \log \mathcal{L})\right) \left(\frac{x_1 + x_0 - u_1 - v}{2w(v - u_1)} \right) G_2. \tag{2.6.62}$$

We are therefore left to choose suitable weights $w_{j,m}$, bound G_2 and choose suitable constants w, u_0, u_1, v, x_0, and x_1.

We note that, using Cauchy's inequality, we have

$$\begin{aligned} &\left| \frac{X_1^{2-\rho_{(1)}-\bar{\rho}_{(2)}} - X_0^{2-\rho_{(1)}-\bar{\rho}_{(2)}}}{x_1 - x_0} - \frac{U_1^{2-\rho_{(1)}-\bar{\rho}_{(2)}} - U_0^{2-\rho_{(1)}-\bar{\rho}_{(2)}}}{u_1 - u_0} \right| \\ &\quad \leq \left(\frac{e^{(\lambda_{(1)}+\lambda_{(2)})x_1} + e^{(\lambda_{(1)}+\lambda_{(2)})x_0}}{x_1 - x_0} + \frac{e^{(\lambda_{(1)}+\lambda_{(2)})u_1} + e^{(\lambda_{(1)}+\lambda_{(2)})u_0}}{u_1 - u_0} \right) \\ &\quad \leq \left(\frac{e^{2\lambda_{(1)}x_1} + e^{2\lambda_{(1)}x_0}}{x_1 - x_0} + \frac{e^{2\lambda_{(1)}u_1} + e^{2\lambda_{(1)}u_0}}{u_1 - u_0} \right)^{1/2} \left(\frac{e^{2\lambda_{(2)}x_1} + e^{2\lambda_{(2)}x_0}}{x_1 - x_0} + \frac{e^{2\lambda_{(2)}u_1} + e^{2\lambda_{(2)}u_0}}{u_1 - u_0} \right)^{1/2}. \end{aligned} \tag{2.6.63}$$

Also

$$\begin{aligned} \sum_{\rho_{(2)}} \left| \mathcal{L}^{-2} (2 - \rho_{(1)} - \rho_{(2)})^{-2} \right| &= \sum_{\rho_{(2)}} \frac{1}{(\lambda_{(1)} + \lambda_{(2)})^2 + (v_{(1)} - v_{(2)})^2} \\ &\leq 2 \sum_{m=0}^{\infty} \frac{1}{(\lambda_{(1)} + \lambda_{(2)})^2 + m^2}, \end{aligned} \tag{2.6.64}$$

since $|\Im(\rho^{(j,m_1)}) - \Im(\rho^{(j,m_2)})| \geq (|m_1 - m_2| - 1)/\mathcal{L}$ by our choice of the rectangles $\mathcal{R}_m$.

Motivated by these observations we choose

$$w_{j,m} = \left(\frac{e^{2\lambda^{(j,m)}x_1} + e^{2\lambda^{(j,m)}x_0}}{x_1 - x_0} + \frac{e^{2\lambda^{(j,m)}u_1} + e^{2\lambda^{(j,m)}u_0}}{u_1 - u_0} \right)^{-1/2}. \tag{2.6.65}$$

We assume from here on that we are only considering zeros $\rho^{(j,m)}$ with $\lambda^{(j,m)} \geq \lambda_{min}$, for some fixed value of λ_{min}.

We now wish to estimate G_2, and so bound $\sum_{\rho_{(2)}} |w_{(1)}w_{(2)}j_2(\rho_{(1)}, \rho_{(2)})|$. We assume $\rho_{(1)}$ is in a rectangle $\mathcal{R}_{m_1}$ and then consider the contributions $G_{2,c}$ from zeros in rectangles $\mathcal{R}_{m_2}$ where $|m_1 - m_2| = c \in \mathbb{Z}$ (since we have picked a fixed zero in each rectangle, there are at most 2 zeros corresponding to each choice of c).

We first consider $c = 0$. In this case $\rho_{(2)} = \rho_{(1)}$ (and there is only one zero). This contributes at most

$$\begin{aligned} G_{2,0} &\leq \sup_{\rho_{(1)}} \left| j(\rho_{(1)}, \rho_{(1)}) w_{(1)}^2 \right| \\ &= \sup_{\rho_{(1)}} \left(\frac{X_1^{2-2\beta_{(1)}} - X_0^{2-2\beta_{(1)}}}{x_1 - x_0} - \frac{U_1^{2-2\beta_{(1)}} - U_0^{2-2\beta_{(1)}}}{u_1 - u_0} \right) \\ &\qquad \times \left(\frac{X_1^{2-2\beta_{(1)}} + X_0^{2-2\beta_{(1)}}}{x_1 - x_0} + \frac{U_1^{2-2\beta_{(1)}} + U_0^{2-2\beta_{(1)}}}{u_1 - u_0} \right)^{-1} (2\lambda_{(1)})^{-2} \\ &= \sup_{\lambda_{(1)} \geq \lambda_{min}} \left(\frac{e^{2x_1\lambda_{(1)}} - e^{2x_0\lambda_{(1)}}}{x_1 - x_0} - \frac{e^{2u_1\lambda_{(1)}} - e^{2u_0\lambda_{(1)}}}{u_1 - u_0} \right) \\ &\qquad \times \left(\frac{e^{2x_1\lambda_{(1)}} + e^{2x_0\lambda_{(1)}}}{x_1 - x_0} + \frac{e^{2u_1\lambda_{(1)}} + e^{2u_0\lambda_{(1)}}}{u_1 - u_0} \right)^{-1} (2\lambda_{(1)})^{-2}. \end{aligned} \tag{2.6.66}$$

We now deal with the case $1 \leq c \leq 6$. This means that $c - 1 \leq |\Im(\rho_{(1)}) - \Im(\rho_{(2)})| \leq c + 1$, and there are at most two zeros $\rho_{(2)}$. These zeros contribute at most

$$\begin{aligned} 2 &\sup_{\substack{\lambda_{(1)}, \lambda_{(2)} \geq \lambda_{min} \\ c-1 \leq t \leq c+1}} \left(\frac{e^{2x_1\lambda_{(1)}} + e^{2x_0\lambda_{(1)}}}{x_1 - x_0} + \frac{e^{2u_1\lambda_{(1)}} + e^{2u_0\lambda_{(1)}}}{u_1 - u_0} \right)^{-1/2} \\ &\times \left(\frac{e^{2x_1\lambda_{(2)}} + e^{2x_0\lambda_{(2)}}}{x_1 - x_0} + \frac{e^{2u_1\lambda_{(2)}} + e^{2u_0\lambda_{(2)}}}{u_1 - u_0} \right)^{-1/2} \left((\lambda_{(1)} + \lambda_{(2)})^2 + t^2 \right)^{-1}. \\ &\times \left| \frac{e^{x_1(\lambda_{(1)} + \lambda_{(2)} + it)} - e^{x_0(\lambda_{(1)} + \lambda_{(2)} + it)}}{x_1 - x_0} - \frac{e^{u_1(\lambda_{(1)} + \lambda_{(2)} + it)} - e^{u_0(\lambda_{(1)} + \lambda_{(2)} + it)}}{u_1 - u_0} \right|. \end{aligned} \tag{2.6.67}$$

As in (2.6.63), it follows from Cauchy's inequality that

$$\left(\frac{e^{(\lambda_{(1)}+\lambda_{(2)})x_1}+e^{(\lambda_{(1)}+\lambda_{(2)})x_0}}{x_1-x_0}+\frac{e^{(\lambda_{(1)}+\lambda_{(2)})u_1}+e^{(\lambda_{(1)}+\lambda_{(2)})u_0}}{u_1-u_0}\right)^2$$

$$\leq\left(\frac{e^{2x_1\lambda_{(2)}}+e^{2x_0\lambda_{(2)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(2)}}+e^{2u_0\lambda_{(2)}}}{u_1-u_0}\right)\left(\frac{e^{2x_1\lambda_{(1)}}+e^{2x_0\lambda_{(1)}}}{x_1-x_0}+\frac{e^{2u_1\lambda_{(1)}}+e^{2u_0\lambda_{(1)}}}{u_1-u_0}\right). \quad (2.6.68)$$

Hence

$$G_{2,c}\leq 2\sup_{\substack{\lambda\geq\lambda_{min}\\ c-1\leq t\leq c+1}}\left|\frac{e^{x_1(2\lambda+it)}-e^{x_0(2\lambda+it)}}{x_1-x_0}-\frac{e^{u_1(2\lambda+it)}-e^{u_0(2\lambda+it)}}{u_1-u_0}\right|$$

$$\times\left(\frac{e^{2x_1\lambda}+e^{2x_0\lambda}}{x_1-x_0}+\frac{e^{2u_1\lambda}+e^{2u_0\lambda}}{u_1-u_0}\right)^{-1}\left(4\lambda^2+t^2\right)^{-1}. \quad (2.6.69)$$

When $c\geq 7$ we use the simple estimate:

$$G_{2,c}\leq 2\sup_{\substack{\lambda_{(1)},\lambda_{(2)}\geq\lambda_{min}\\ c-1\leq t\leq c+1}}\left((\lambda_{(1)}+\lambda_{(2)})^2+t^2\right)^{-1}\leq\frac{2}{4\lambda_{min}^2+(c-1)^2}. \quad (2.6.70)$$

For given constants x_1, x_0, u_1, u_0, w, v and λ_{min} we use *Mathematica*®*'s*[1] NMaximize function to calculate the bounds above for $G_{2,0}$ and $G_{2,c}$ for $1\leq c\leq 6$. We can estimate the bound given for $G_{2,c}$ when $7\leq c\leq 101$ exactly, and then for $c\geq 102$ we use an integral comparison to see that

$$\sum_{c\geq 102}G_{2,c}\leq\sum_{m\geq 101}\frac{2}{4\lambda_{min}^2+m^2}\leq\int_{100}^{\infty}\frac{2}{4\lambda_{min}^2+t^2}dt$$

$$\leq\frac{\tan^{-1}(\lambda_{min}/50)}{\lambda_{min}}. \quad (2.6.71)$$

We can then use this information to estimate G_2.

$$G_2\leq G_{2,0}+\sum_{1\leq c\leq 6}G_{2,c}+\sum_{6\leq m\leq 100}\frac{2}{4\lambda_{min}^2+m^2}+\frac{\tan^{-1}(\lambda_{min}/50)}{\lambda_{min}}. \quad (2.6.72)$$

As is the case in [26], it is optimal to choose $u_0=2w+1/3+\delta$ and $x_0=w+v+1/3+\delta$ with δ small. We will take $\delta=2/3000$ for our purposes. We are then left to choose suitable positive constants w, $u_1\geq u_0$, $v\geq u_1$ and $x_1\geq x_0$. We fix these now as

$$w=0.115,\quad u_0=0.564,\quad u_1=0.620, \quad (2.6.73)$$

$$v=0.964,\quad x_0=1.413,\quad x_1=1.623. \quad (2.6.74)$$

[1] *Mathematica* is a registered trademark of Wolfram Research, Inc.

We consider $\lambda_{min} = 0.35$. For this value we calculate that

$$G_2 \leq 0.650. \tag{2.6.75}$$

Putting everything together we obtain

$$\sum_{\substack{j,m \\ \lambda^{(j,m)} \geq 0.35}} \left(\frac{e^{3.246\lambda^{(j,m)}} + e^{2.826\lambda^{(j,m)}}}{0.210} + \frac{e^{1.240\lambda^{(j,m)}} + e^{1.128\lambda^{(j,m)}}}{0.056} \right)^{-1} \leq 11.9288. \tag{2.6.76}$$

2.6.3 Third zero density estimate

We now prove Lemma 2.12. The proof uses the ideas from [26, Section 12] to obtain a stronger zero density estimate close to 1, but again we need to extend this to our larger region with $\Im(\rho) \ll 1$. Specifically we wish to estimate

$$N^*(\lambda) := \#\left\{\rho^{(j,m)} \in \mathcal{R} : \lambda^{(j,m)} \leq \lambda\right\} \tag{2.6.77}$$

in the range $0 \leq \lambda \leq 2$. We note that from the log-free zero density bound, that for $0 \leq \lambda \leq 2$ we have that $N^*(\lambda)$ is uniformly bounded in q and λ.

Throughout this section we assume that we have a fixed non-negative constant λ_{11} such that $\lambda_{11} \leq \lambda_1$. We put $\beta_{11} = 1 - \lambda_{11}/\mathcal{L}$.

We adopt the notation of [26]. We put

$$K(s,\chi) := \sum_{n=1}^{\infty} \Lambda(n) \Re\left(\frac{\chi(n)}{n^s}\right) g\left(\mathcal{L}^{-1} \log n\right) \tag{2.6.78}$$

for some function g which satisfies:

Condition 1: $g : [0,\infty) \to \mathbb{R}$ is continuous, g is supported on $[0, x_0)$ for some $x_0 > 0$, g is twice differentiable on $(0, x_0)$ and g'' is bounded on $(0, x_0)$.

Condition 2: g is non-negative and its Laplace transform G satisfies $\Re(G(z)) \geq 0$ for $\Re(z) \geq 0$. We start with the following estimate

Lemma 2.17. *Let g be a function satisfying Conditions 1 and 2 and let $\delta > 0$. Let $q > q_0(\delta, g)$ and $\lambda_1 \geq \lambda_{11}$.*

If

$$G(\lambda - \lambda_{11}) > g(0)/6 \qquad \textit{and} \qquad (G(\lambda - \lambda_{11}) - g(0)/6)^2 > G(-\lambda_{11})g(0)/6 \tag{2.6.79}$$

then we have

$$N^*(\lambda) \leq \frac{G(-\lambda_{11})G_3}{(G(\lambda - \lambda_{11}) - g(0)/6)^2 - G(-\lambda_{11})g(0)/6} + \delta,$$

where G_3 will be defined in equation (2.6.93).

Proof. The first inequality of [26, Section 12] shows that for $q > q_0(g, \delta_1)$ we have

$$\mathcal{L}^{-1} K(\beta_{11} + i\gamma^{(j,m)}, \chi^{(j)}) \leq g(0)\phi_{\chi^{(j)}}/2 + \delta_1 - G(\lambda^{(j,m)} - \lambda_{11}). \tag{2.6.80}$$

Therefore, for any zero $\rho^{(j,m)}$ with $G(\lambda^{(j,m)} - \lambda_{11}) > g(0)\phi_{\chi^{(j)}}/2$ we obtain

$$0 < G(\lambda^{(j,m)} - \lambda_{11}) - g(0)\phi_{\chi^{(j)}}/2 \leq -\mathcal{L}^{-1} K(\beta_{11} + i\gamma^{(j,m)}, \chi^{(j)}) + \delta_1. \tag{2.6.81}$$

We note that $G(\lambda^{(j,m)} - \lambda_{11})$ is a decreasing function in $\lambda^{(j,m)}$ and recall that $\phi_\chi \leq 1/3$ for all characters χ. Therefore, if

$$G(\lambda - \lambda_{11}) > g(0)/6, \tag{2.6.82}$$

then for any $\lambda^{(j,m)} \leq \lambda$ we have that

$$\begin{aligned} 0 \leq G(\lambda - \lambda_{11}) - g(0)/6 &\leq G(\lambda^{(j,m)} - \lambda_{11}) - g(0)\phi_{\chi^{(j)}}/2 \\ &\leq -\mathcal{L}^{-1} K(\beta_{11} + i\gamma^{(j,m)}, \chi^{(j)}) + \delta_1. \end{aligned} \tag{2.6.83}$$

We sum over all j, m for which $\lambda^{(j,m)} \leq \lambda$. Thus for $q > q_0(g, \delta_1)$ we have

$$\begin{aligned} N^*(\lambda)(G(\lambda - \lambda_{11}) - g(0)/6) &\leq \sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} G(\lambda^{(j,m)} - \lambda_{11}) - g(0)/6 \\ &\leq -\mathcal{L}^{-1} \sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} K(\beta_{11} + i\gamma^{(j,m)}, \chi^{(j)}) + \sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} \delta_1 \\ &= -\mathcal{L}^{-1} \sum_{n=1}^{\infty} \Lambda(n) n^{-\beta_{11}} g(\mathcal{L}^{-1} \log n) \Re\Bigg(\sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} \chi^{(j)}(n) n^{-i\gamma^{(j,m)}} \Bigg) + \delta_2 \\ &\leq \mathcal{L}^{-1} \sum_{n=1}^{\infty} \Lambda(n) n^{-\beta_{11}} \chi_0(n) g(\mathcal{L}^{-1} \log n) \Bigg| \sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} \chi^{(j)}(n) n^{-i\gamma^{(j,m)}} \Bigg| + \delta_2 \\ &\leq \Sigma_1^{1/2} \Sigma_2^{1/2} + \delta_2 \end{aligned} \tag{2.6.84}$$

where

$$\delta_2 = \sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} \delta_1, \tag{2.6.85}$$

$$\Sigma_1 = \mathcal{L}^{-1} \sum_{n=1}^{\infty} \Lambda(n) n^{-\beta_{11}} \chi_0(n) g(\mathcal{L}^{-1} \log n), \tag{2.6.86}$$

$$\Sigma_2 = \mathcal{L}^{-1} \sum_{n=1}^{\infty} \Lambda(n) n^{-\beta_{11}} g(\mathcal{L}^{-1} \log n) \Bigg| \sum_{\substack{j,m \\ \lambda^{(j,m)} \leq \lambda}} \chi^{(j)}(n) n^{-i\gamma^{(j,m)}} \Bigg|^2. \tag{2.6.87}$$

By [26, Lemma 5.3] for $q > q_0(g, \delta_1)$ we have

$$\Sigma_1 = \mathcal{L}^{-1} K(\beta_{11}, \chi_0) \leq G(-\lambda_{11}) + \delta_1. \tag{2.6.88}$$

We expand the square in Σ_2 and see that

$$\Sigma_2 = \Re(\Sigma_2) = \mathcal{L}^{-1} \sum_{\substack{j_1, j_2, m_1, m_2 \\ \lambda^{(j_1, m_1)}, \lambda^{(j_2, m_2)} \leq \lambda}} K(\beta_{11} + i(\gamma^{(j_1, m_1)} - \gamma^{(j_2, m_2)}), \chi^{(j_1)} \overline{\chi}^{(j_2)}). \tag{2.6.89}$$

By [26, Lemma 5.3] the terms with $j_1 = j_2$ contribute a total

$$\begin{aligned} &\mathcal{L}^{-1} \sum_{j_1, m_1, m_2} K(\beta_{11} + i(\gamma^{(j_1, m_1)} - \gamma^{(j_1, m_2)}), \chi_0) \\ &\qquad \leq \sum_{j_1, m_1, m_2} \left(\left| \Re \left(G(-\lambda_{11} + i(\nu^{(j_1, m_1)} - \nu^{(j_1, m_2)})) \right) \right| + \delta_1 \right). \end{aligned} \tag{2.6.90}$$

By [26, Lemma 5.2], the terms with $j_1 \neq j_2$ contribute

$$\mathcal{L}^{-1} \sum_{j_1 \neq j_2, m_1, m_2} K(\beta_{11} + i(\gamma^{(j_1, m_1)} - \gamma^{(j_2, m_2)}), \chi^{(j_1)} \overline{\chi}^{(j_2)}) \leq \sum_{j_1 \neq j_2, m_1, m_2} (g(0)/6 + \delta_1). \tag{2.6.91}$$

Putting these together we get

$$\Sigma_2 \leq \sum_{\substack{j_1, m_1, m_2 \\ \lambda^{(j_1, m_1)}, \lambda^{(j_1, m_2)} \leq \lambda}} \left(\left| \Re \left(G(-\lambda_{11} + i(\nu^{(j_1, m_1)} - \nu^{(j_1, m_2)})) \right) \right| - g(0)/6 \right) + N^*(\lambda)^2 g(0)/6 + \delta_3. \tag{2.6.92}$$

We put

$$G_3 := \sup_{j_1, m_1} \sum_{m_2} \left(\left| \Re \left(G(-\lambda_{11} + i(\gamma^{(j_1, m_1)} - \gamma^{(j_1, m_2)})) \right) \right| - g(0)/6 \right), \tag{2.6.93}$$

so

$$\Sigma_2 \leq N^*(\lambda)^2 g(0)/6 + N^*(\lambda) G_3 + \delta_3. \tag{2.6.94}$$

Putting together (2.6.84), (2.6.88) and (2.6.94) we obtain

$$\begin{aligned} N^*(\lambda)^2 (G(\lambda - \lambda_{11}) - g(0)/6)^2 &\leq \Sigma_1 \Sigma_2 + \delta_2 \\ &\leq (G(-\lambda_{11}) + \delta_1)(N^*(\lambda)^2 g(0)/6 + N^*(\lambda) G_3 + \delta_3) + \delta_2. \end{aligned} \tag{2.6.95}$$

Since $N^*(\lambda)$ is bounded uniformly for $0 \leq \lambda \leq 2$ by the log-free zero density estimate, all the sums and terms are finite. Therefore, by a suitable choice of δ_1 we have for given $\delta > 0$ and $q > q_0(g, \delta)$ that

$$N^*(\lambda) \left((G(\lambda - \lambda_{11}) - g(0)/6)^2 - G(-\lambda_{11}) g(0)/6 \right)^2 \leq G(-\lambda_{11}) G_3 + \delta \tag{2.6.96}$$

Therefore the lemma holds. □

We are now left to choose a suitable function g and evaluate this expression. As in the work of Heath-Brown [26] and Xylouris [51] we choose

$$g(t) := \begin{cases} \int_{t-\gamma}^{\gamma}(\gamma^2 - x^2)(\gamma^2 - (t-x)^2)dx \\ \quad = -\frac{1}{30}t^5 + \frac{2\gamma^2}{3}t^3 - \frac{4\gamma^3}{3}t^2 + \frac{16\gamma^5}{15}, & t \in [0, 2\gamma), \\ 0, & t \geq 2\gamma, \end{cases} \tag{2.6.97}$$

for some constant $\gamma > 0$.

We see that g is the convolution of $g_2(x) = \max(0, \gamma^2 - x^2)$ with itself. Therefore, since $g_2(-t) = g_2(t)$ we have

$$\begin{aligned} \Re(G(it)) &= \int_0^{\infty} g(t)\cos(ty)dt \\ &= \int_0^{\infty}\int_{-\infty}^{\infty} g_2(\tau)g_2(t-\tau)\cos(ty)d\tau dt \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} g_2(a)g_2(b)\cos(ya+yb)dadb \\ &= 2\left(\int_0^{\infty} g_2(a)\cos(ay)\right)^2 \geq 0. \end{aligned} \tag{2.6.98}$$

Therefore, by [26, Lemma 4.1], g satisfies Condition 2 that $\Re(z) \geq 0 \Rightarrow \Re(G(z)) \geq 0$. We also see that g is twice differentiable on $(0, 2\gamma)$ and its second derivative is continuous and bounded, and so also fulfills Condition 1.

We see the Laplace transform G is

$$\begin{aligned} G(z) &= \int_0^{\infty} e^{-zt}g(t)dt \\ &= \begin{cases} \frac{16\gamma^5}{15}z^{-1} - \frac{8\gamma^3}{3}z^{-3} + 4\gamma^2(1 + e^{-2\gamma z})z^{-4} \\ \qquad\qquad + 4(-1 + e^{-2\gamma z} + 2\gamma z e^{-2\gamma z})z^{-6}, & z \neq 0, \\ \frac{8\gamma^6}{9}, & z = 0. \end{cases} \end{aligned} \tag{2.6.99}$$

We bound G_3 in the same manner as we did in proving Lemma 2.11. We recall

$$G_3(\lambda) = \sup_{m_1, j_1} \sum_{m_2} \left(\left|\Re(G(-\lambda_{11} + i(\nu^{(j_1,m_1)} - \nu^{(j_2,m_2)})))\right| - g(0)/6\right). \tag{2.6.100}$$

As in the proof of Lemma 2.11, we consider the contribution $G_{3,c}$ of zeros from rectangles $\mathcal{R}_{m_2}$ with $|m_1 - m_2| = c \in \mathbb{Z}$.

We first consider $G_{3,0}$. There is only one zero $\rho^{(j_1,m_2)} = \rho^{(j_1,m_1)}$, if it exists. Thus

$$G_{3,0} \leq G(-\lambda_{11}) - g(0)/6. \tag{2.6.101}$$

For $G_{3,c}$ with $1 \leq c \leq 5$ we see that there are at most 2 zeros both with $c-1 \leq |v^{(j_1,m_1)} - v^{(j_1,m_2)}| \leq c+1$. These contribute

$$G_{3,c} \leq 2\max\left(\sup_{c-1\leq t\leq c+1} |\Re(G(-\lambda_{11}+it))| - g(0)/6, 0\right). \tag{2.6.102}$$

We estimate these using *Mathematica's* NMaximize function.

We use a simpler bound to estimate $G_{3,c}$ with $c \geq 6$. Letting $z = x+iy$ we have

$$\begin{aligned}|\Re(G(z))| &\leq \left|\frac{16\gamma^5}{15}\Re(z^{-1})\right| + \left|8\frac{\gamma^3}{3}\Re(z^{-3})\right| + 4\gamma^2\left|\Re\left((1+e^{-2\gamma z})z^{-4}\right)\right| \\ &\quad + 4\left|\Re\left((-1+e^{-2\gamma z}+2\gamma z e^{-2\gamma z})z^{-6}\right)\right| \\ &\leq \frac{16\gamma^5 x}{15(x^2+y^2)} + \frac{8\gamma^3(|x|^3+3|x|y^2)}{3(x^2+y^2)^3} + \frac{4\gamma^2(1+e^{-2\gamma x})}{(x^2+y^2)^2} \\ &\quad + 4(1+e^{-2\gamma x}+2\gamma(x^2+y^2)^{1/2}e^{-2\gamma x})(x^2+y^2)^{-3}.\end{aligned} \tag{2.6.103}$$

We let $G_4(x,y)$ denote the final quantity on the right hand side of (2.6.103). We see that $G_4(x,y)$ is decreasing in y, and so

$$\begin{aligned}G_{3,c} &\leq 2\max\left(\sup_{c-1\leq|t|\leq c+1}\left|\Re(G(-\lambda_{11}+it))\right| - g(0)/6, 0\right) \\ &\leq 2\max\left(G_4(-\lambda_{11}, c-1) - g(0)/6, 0\right).\end{aligned} \tag{2.6.104}$$

We estimate this directly. We note that if $G_4(-\lambda_{11}, c_1 - 1) \leq g(0)/6$ then $G_{3,c} \leq 0$ for all $c \geq c_1$.

Using these estimates we can then bound G_3 for any given value of our parameter γ and a given lower bound λ_{11} for λ_1.

We consider separately the cases $\lambda_1 \geq 0.35$, $\lambda_1 \geq 0.40$, $\lambda_1 \geq 0.44$, $\lambda_1 \geq 0.52$, $\lambda_1 \geq 0.60$, $\lambda_1 \geq 0.66$ and $\lambda_1 \geq 6/7$. In each case we choose the value of $\gamma \in \{1.00, 1.01, \ldots, 1.60\}$ which gives the best bound whilst ensuring that conditions (2.6.79) still hold.

We give the results in the following table. We note that in comparison with [26, Table 13] these are worse by a factor of approximately 4, but are counting the number of rectangles containing a zero rather than just the number of characters.

Table 2.1: Third Zero Density Estimate

λ	Bound for $N^*(\lambda)$						
	$\lambda_1 \geq 0.35$	$\lambda_1 \geq 0.40$	$\lambda_1 \geq 0.44$	$\lambda_1 \geq 0.52$	$\lambda_1 \geq 0.60$	$\lambda_1 \geq 0.66$	$\lambda_1 \geq 6/7$
0.74	30	29	28	27	26	26	-
0.75	31	30	29	28	27	26	-
0.76	32	31	30	29	28	27	-
0.77	33	32	31	30	29	28	-
0.78	34	33	32	31	29	29	-
0.79	35	34	33	32	30	29	-
0.80	36	35	34	32	31	30	-
0.81	37	36	35	33	32	31	-
0.82	38	37	36	34	33	32	-
0.83	40	38	37	35	34	33	-
0.84	41	39	38	37	35	34	-
0.85	42	41	40	38	36	35	-
0.86	44	42	41	39	37	36	-
0.87	45	44	42	40	38	37	34
0.88	47	45	44	41	39	38	35
0.89	49	47	45	43	41	39	36
0.90	51	49	47	44	42	40	37
0.91	53	50	49	46	43	42	38
0.92	55	52	51	47	45	43	39
0.93	57	54	52	49	46	44	40
0.94	59	57	55	51	48	46	41
0.95	62	59	57	53	49	47	43
0.96	65	61	59	55	51	49	44
0.97	68	64	61	57	53	51	45
0.98	71	67	64	59	55	52	47
0.99	74	70	67	61	57	54	48
1.00	78	73	70	64	59	56	50
1.01	82	77	73	67	62	58	51
1.02	86	80	76	70	64	61	53
1.03	91	84	80	73	67	63	55
1.04	96	89	84	76	70	66	57
1.05	101	94	88	80	73	68	59
1.06	108	99	93	83	76	71	61
1.07	114	105	98	88	79	74	63
1.08	122	111	104	92	83	78	65
1.09	131	118	110	97	87	81	68
1.10	141	127	117	103	91	85	71
1.11	152	136	125	109	96	89	73
1.12	164	146	134	115	101	94	76
1.13	179	157	143	122	107	98	80
1.14	197	171	155	130	113	104	83
1.15	218	186	167	139	120	110	87

Continued on next page...

λ	Bound for $N^*(\lambda)$ - *continued from previous page*						
	$\lambda_1 \geq 0.35$	$\lambda_1 \geq 0.4$	$\lambda_1 \geq 0.44$	$\lambda_1 \geq 0.52$	$\lambda_1 \geq 0.60$	$\lambda_1 \geq 0.66$	$\lambda_1 \geq 6/7$
1.16	243	205	182	150	128	116	91
1.17	274	226	199	161	136	123	95
1.18	313	253	220	175	146	131	100
1.19	365	286	244	190	156	140	105
1.20	435	328	274	208	169	149	110
1.21	536	383	312	229	183	160	116
1.22	695	458	361	255	199	173	123
1.23	981	568	426	286	218	187	130
1.24	1642	742	518	326	241	203	138
1.25	4835	1063	658	377	268	222	146
1.26	∞	1844	895	446	301	245	156
1.27		6602	1382	543	343	272	167
1.28		∞	2967	690	397	305	179
1.29			∞	940	470	347	193
1.30				1457	573	400	208
1.31				3156	729	471	226
1.32				∞	995	569	247
1.33					1549	716	272
1.34					3398	958	302
1.35					∞	1433	338
1.36						2782	382
1.37						35205	438
1.38						∞	513
1.39							614
1.40							763
1.41							998
1.42							1430
1.43							2480
1.44							8791
1.45							∞

2.7 Proof of Proposition 2.6

We wish to estimate

$$\sum_{\chi\neq\chi_0}\sum_{m\in\mathbb{Z}}\sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi)}\exp(-M\lambda_\rho).$$

We do this by Lemmas 2.10, 2.11 and 2.12.

We split the argument into 2 sections, when there is a zero close to one (in which case it must be a real zero from a real character) and when there are no zeros close to one (and so

ρ_1 or χ_1 might be complex).

The work in this section follows along the same lines as that of [26, Sections 14 and 15].

2.7.1 A zero close to 1

We consider the case when $\eta \le \lambda_1 \le 0.35$. By [51, Table 11] we see that such a zero cannot exist if χ_1 or ρ_1 is complex, and hence ρ_1 must be a real zero corresponding to a real character. Moreover, ρ_1 is simple. Since χ_1 is real we have that $\phi_{\chi_1} = 1/4$.

We first consider the contribution from characters $\chi^{(j)} \neq \chi_1$. We note that

$$\frac{\exp(-M\lambda)}{B_1(\lambda)} = \left(\frac{\lambda}{\sinh(K\lambda/2)}\right)^2 \left(1 + \frac{1}{4\lambda^2}\right) e^{-(M-K)\lambda}. \tag{2.7.1}$$

The first two terms in the product are decreasing in λ, and so for $M \ge K$ this is a decreasing function of λ. Therefore for all $\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})$, if $M \ge K$, we have

$$\exp(-M\lambda_\rho) \le \frac{\exp(-M\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} B_1(\lambda_\rho). \tag{2.7.2}$$

Thus by Lemma 2.10 we have

$$\sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} \exp(-M\lambda_\rho) \le \frac{\exp(-M\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} B_1(\lambda_\rho) \le \frac{\exp(-M\lambda^{(j,m)}) C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}. \tag{2.7.3}$$

We note that

$$\frac{\exp(-2x_1\lambda)}{B_2(\lambda)} \qquad \text{and} \qquad C_1(\lambda)$$

are decreasing functions in λ. Thus for $M \ge 2x_1 + K$ we have that

$$\frac{\exp(-M\lambda) C_1(\lambda)}{B_1(\lambda) B_2(\lambda)} \tag{2.7.4}$$

is a decreasing function in λ. Since for $\chi^{(j)} \neq \chi_1$ we have $\lambda^{(j,m)} \ge \lambda_2$, this gives us

$$\sum_{\substack{j,m \\ \chi^{(j)} \neq \chi_1, \chi_0}} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} \exp(-M\lambda_\rho) \le \frac{\exp(-M\lambda_2) C_1(\lambda_2)}{B_2(\lambda_2) B_1(\lambda_2)} \sum_{\substack{j,m \\ \chi^{(j)} \neq \chi_1, \chi_0}} B_2(\lambda^{(j,m)}). \tag{2.7.5}$$

We now consider the contribution from the character χ_1. We give the zero ρ_1 close to 1 special treatment, and so treat the rectangle $\mathcal{R}_0$ which contains ρ_1 differently ($\rho_1 \in \mathcal{R}_0$ since ρ_1 is real).

We first consider the contribution from rectangles $\mathcal{R}_m$ with $m \neq 0$. Using the same ideas as above we have

$$\sum_{m\neq 0}\sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \leq \frac{\exp(-M\lambda_1')C_1(\lambda_1')}{B_2(\lambda_1')B_1(\lambda_1')} \sum_{\substack{m\neq 0\\ \chi^{(j)}=\chi_1}} B_2(\lambda^{(j,m)}). \tag{2.7.6}$$

We now consider the rectangle $\mathcal{R}_0$. We have

$$\begin{aligned}\sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) &\leq \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \sum_{\substack{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi)\\ \rho\neq\rho_1}} B_1(\lambda_\rho)\\ &\leq \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi)} B_1(\lambda_\rho)\\ &\leq \exp(-M\lambda_1) + \frac{\exp(-M\lambda_1')C_1(\lambda_1)}{B_1(\lambda_1')}. \end{aligned}\tag{2.7.7}$$

We note that $B_2(\lambda)$ and $C_1(\lambda)$ are both decreasing in λ. Therefore

$$\sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \leq \exp(-M\lambda_1) + \left(\frac{\exp(-M\lambda_1')C_1(0)}{B_1(\lambda_1')B_2(\lambda_1')}\right) B_2(\lambda_1). \tag{2.7.8}$$

Combining this with (2.7.6) and using the fact the C_1 is decreasing we obtain

$$\begin{aligned}\sum_{\substack{j,m\\ \chi^{(j)}=\chi_1}}\sum_{\rho\in\mathcal{R}_m\cap\mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \leq{}& \frac{\exp(-M\lambda_1')C_1(0)}{B_1(\lambda_1')B_2(\lambda_1')} \sum_{\substack{j,m\\ \chi^{(j)}=\chi_1}} B_2(\lambda^{(j,m)})\\ &+ \exp(-M\lambda_1).\end{aligned}\tag{2.7.9}$$

Now combining (2.7.9) and (2.7.5) we get

$$\begin{aligned}\sum_{\chi\neq\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)} \exp(-M\lambda_\rho) &\leq \exp(-M\lambda_1) + C_4(\lambda_1',\lambda_2)\sum_{j,m} B_2(\lambda^{(j,m)})\\ &\leq \exp(-M\lambda_1) + C_4(\lambda_1',\lambda_2)C_2,\end{aligned}\tag{2.7.10}$$

where

$$C_4(\lambda_1',\lambda_2) = \max\left(\frac{\exp(-M\lambda_2)C_1(\lambda_2)}{B_1(\lambda_2)B_2(\lambda_2)}, \frac{\exp(-M\lambda_1')C_1(0)}{B_1(\lambda_1')B_2(\lambda_1')}\right). \tag{2.7.11}$$

By [26, Lemmas 8.4 and 8.8] for any $\delta > 0$ and for all $q \geq q_0(\delta)$ we have

$$\lambda_1', \lambda_2 \geq \left(\frac{12}{11} - \delta\right)\log(\lambda_1^{-1}). \tag{2.7.12}$$

Also by [26, Tables 4 and 7] for $\lambda_1 \leq 0.35$ we have that

$$\lambda_1' \geq 2.19, \qquad \lambda_2 \geq 1.42. \tag{2.7.13}$$

Thus, since $C_4(\lambda_1', \lambda_2)$ is decreasing in λ_1' and λ_2, we have for any constant B with $0 \leq B \leq M - K - 2x_1$ the upper bound

$$C_4(\lambda_1',\lambda_2) \leq \exp\left(-\left(\frac{12}{11} - \delta\right) B \log(\lambda_1^{-1})\right)$$
$$\times \max\left(\frac{\exp(-1.42(M-B))C_1(1.42)}{B_1(1.42)B_2(1.42)}, \frac{\exp(-2.19(M-B))C_1(0)}{B_1(2.19)B_2(2.19)}\right). \tag{2.7.14}$$

We choose

$$B = 1, \delta = 0.01, K = 0.66 \tag{2.7.15}$$

and as before

$$w = 0.115, \quad u_0 = 0.564, \quad u_1 = 0.620, \tag{2.7.16}$$
$$v = 0.964, \quad x_0 = 1.413, \quad x_1 = 1.623. \tag{2.7.17}$$

Given M we can now explicitly calculate the above quantities. For $M = 7.5$ we obtain

$$\sum_{\chi \neq \chi_0} \sum_{\rho \in \mathcal{R} \cap \mathcal{Z}(\chi)} \exp(-7.5\lambda_\rho) \leq \exp(-7.5\lambda_1) + 2.38 \times \lambda_1^{1.08}. \tag{2.7.18}$$

We see that the right hand side is a function which is 1 when $\lambda_1 = 0$, and is decreasing at 0. Moreover, it is convex (has positive second derivative) on $(0, \infty)$ and so can have at most one turning point, which would be a minimum should it exist. Therefore the right hand side is always < 1 for $\lambda_1 \in [\eta, 0.35]$ if it is < 1 at 0.35.

Calculating this at 0.35 with $M = 7.5$ gives 0.8628.., and so this is < 1 for $\lambda_1 \in [\eta, 0.35]$ provided $M \geq 7.5$.

2.7.2 No zeroes close to 1

We now consider the case when $\lambda_1 \geq 0.35$.

As above, for characters $\chi^{(j)} \neq \chi_1, \overline{\chi}_1$, we have

$$\sum_{\rho \in \mathcal{R} \cap \mathcal{Z}(\chi^{(j)})} \exp(-M\lambda_\rho) \leq \sum_m \frac{\exp(-M\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi^{(j)})} B_1(\lambda_\rho)$$
$$\leq \sum_m \frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}. \tag{2.7.19}$$

We now consider the contributions for the character χ_1 (and $\overline{\chi_1}$ if χ_1 complex). We separate out the contribution of ρ_1 (and $\overline{\rho}_1$ if it exists). To do this we put

$$n_1(\chi_1) = \begin{cases} 2, & \chi_1 \text{ complex} \\ 1, & \text{otherwise} \end{cases} \tag{2.7.20}$$

$$n_2(\chi_1) = \begin{cases} 2, & \chi_1 \text{ real and } \rho_1 \text{ complex} \\ 1, & \text{otherwise} \end{cases} \tag{2.7.21}$$

$$n_3(\chi_1) = \begin{cases} 2, & \chi_1 \text{ real and } \rho_1 \text{ complex and } \rho_1 \notin \mathcal{R}_0 \\ 1, & \text{otherwise.} \end{cases} \tag{2.7.22}$$

We then have

$$\sum_{\rho \in \mathcal{R} \cap \mathcal{Z}(\chi_1)} \exp(-M\lambda_1) = n_2(\chi_1) \exp(-M\lambda_\rho) + \sum_m \sum_{\substack{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi_1) \\ \rho \neq \rho_1, \overline{\rho}_1}} \exp(-M\lambda_\rho). \tag{2.7.23}$$

We separate out the contribution from the rectangle $\mathcal{R}_{m_1}$ which contains ρ_1. If $\chi_1 = \chi^{(1)}$ is real and ρ_1 is complex then we also separate the rectangle $\mathcal{R}_{m_2}$ which contains $\overline{\rho}_1$ if this is different to $\mathcal{R}_{m_1}$. We note that all zeros in either of these rectangles have either $\lambda_\rho = \lambda_1$ or $\lambda_\rho \geq \lambda_1'$. The zeros in any other rectangle $\mathcal{R}_m$ have $\lambda_\rho \geq \lambda^{(1,m)}$. We then use Lemma 2.10 again. This gives

$$\begin{aligned}
\sum_{\rho \in \mathcal{R} \cap \mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) &= n_2(\chi_1) \exp(-M\lambda_1) + \sum_{\substack{\rho \in (\mathcal{R}_{m_1} \cup \mathcal{R}_{m_2}) \cap \mathcal{Z}(\chi_1) \\ \rho \neq \rho_1, \overline{\rho}_1}} \exp(-M\lambda_\rho) \\
&\quad + \sum_{m \neq m_1, m_2} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi_1)} \exp(-M\lambda_\rho) \\
&\leq \sum_{m \neq m_1, m_2} \frac{\exp(-M\lambda^{(1,m)})}{B_1(\lambda^{(1,m)})} \sum_{\rho \in \mathcal{R}_m \cap \mathcal{Z}(\chi_1)} B_1(\lambda_\rho) + \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \sum_{\rho \in (\mathcal{R}_{m_1} \cup \mathcal{R}_{m_2}) \cap \mathcal{Z}(\chi_1)} B_1(\lambda_\rho) \\
&\quad + n_2(\chi_1) \left(\exp(-M\lambda_1) - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} B_1(\lambda_1) \right) \\
&\leq \sum_{m \neq m_1, m_2} \frac{\exp(-M\lambda^{(1,m)}) C_1(\lambda^{(1,m)})}{B_1(\lambda^{(1,m)})} + n_3(\chi_1) \frac{\exp(-M\lambda_1') C_1(\lambda_1)}{B_1(\lambda_1')} \\
&\quad + n_2(\chi_1) \left(\exp(-M\lambda_1) - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} B_1(\lambda_1) \right) \\
&= (n_2(\chi_1) B_1(\lambda_1) - n_3(\chi_1) C_1(\lambda_1)) \left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)} - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')} \right) \\
&\quad + \sum_m \frac{\exp(-M\lambda^{(1,m)}) C_1(\lambda^{(1,m)})}{B_1(\lambda^{(1,m)})}.
\end{aligned} \tag{2.7.24}$$

If χ_1 is complex we follow the same argument and obtain the same result for $\overline{\chi}_1$.

Putting together (2.7.19) and (2.7.24) we obtain

$$\sum_{\chi\neq\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)}\exp(-M\lambda_\rho)\leq\sum_{m,j}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}+A_1, \tag{2.7.25}$$

where

$$A_1=n_1(\chi_1)\Big(n_2(\chi_1)B_1(\lambda_1)-n_3(\chi_1)C_1(\lambda_1)\Big)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)}-\frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right). \tag{2.7.26}$$

We now use Lemmas 2.11 and 2.12 to estimate the sum on the right hand side of (2.7.25). We fix a constant Λ (to be declared later) and consider separately the terms with $\lambda^{(j,m)}>\Lambda$ and $\lambda^{(j,m)}\leq\Lambda$. We use Lemma 2.11 to estimate the first set of terms, and Lemma 2.12 to estimate the second set.

We first consider the terms with $\lambda^{(j,m)}>\Lambda$.

$$\sum_{\substack{j,m\\ \lambda^{(j,m)}>\Lambda}}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}=\sum_{j,m}\left(\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})B_2(\lambda^{(j,m)})}\right)B_2(\lambda^{(j,m)}). \tag{2.7.27}$$

Again we note that

$$\frac{\exp(-K\lambda)}{B_1(\lambda)},\qquad\frac{\exp(-2x_1\lambda)}{B_2(\lambda)},\qquad\text{and}\qquad C_1(\lambda)$$

are all decreasing functions of λ. Therefore, provided $M\geq K+2x_1$, we have

$$\begin{aligned}\sum_{\substack{j,m\\ \lambda^{(j,m)}>\Lambda}}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}&\leq\left(\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\right)\sum_{\substack{j,m\\ \lambda^{(j,m)}>\Lambda}}B_2(\lambda^{(j,m)})\\&=\left(\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\right)\sum_{j,m}B_2(\lambda^{(j,m)})-\left(\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\right)\sum_{\substack{j,m\\ \lambda^{(j,m)}\leq\Lambda}}B_2(\lambda^{(j,m)})\\&\leq\frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)}-\left(\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\right)\sum_{\substack{j,m\\ \lambda^{(j,m)}\leq\Lambda}}B_2(\lambda^{(j,m)}).\end{aligned} \tag{2.7.28}$$

Hence

$$\begin{aligned}\sum_{m,j}\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})}&\leq\frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)}\\&\quad+\sum_{\substack{j,m\\ \lambda^{(j,m)}\leq\Lambda}}\left(\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})B_2(\lambda^{(j,m)})}-\frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}\right)B_2(\lambda^{(j,m)}).\end{aligned} \tag{2.7.29}$$

We therefore are left to evaluate

$$\sum_{\substack{j,m \\ \lambda^{(j,m)} \le \Lambda}} \left(\frac{\exp(-M\lambda^{(j,m)})C_1(\lambda^{(j,m)})}{B_1(\lambda^{(j,m)})B_2(\lambda^{(j,m)})} - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)} \right) B_2(\lambda^{(j,m)}). \tag{2.7.30}$$

To ease notation we put

$$D(\lambda) = \left(\frac{\exp(-M\lambda)C_1(\lambda)}{B_1(\lambda)B_2(\lambda)} - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)} \right) B_2(\lambda). \tag{2.7.31}$$

We note that $D(\lambda)$ is a decreasing function of λ (and is non-negative for $\lambda \le \Lambda$).

We separate the terms for λ_1 and put $\lambda^* = \min(\lambda_1', \lambda_2)$. This gives

$$\sum_{\substack{j,m \\ \lambda^{(j,m)} \le \Lambda}} D(\lambda^{(j,m)}) = n_3(\chi_1)n_1(\chi_1)D(\lambda_1) + \sum_{\substack{j,m \\ \lambda^* \le \lambda^{(j,m)} \le \Lambda}} D(\lambda^{(j,m)}). \tag{2.7.32}$$

We put $\Lambda_r = \Lambda - (0.01)r$ and define s such that $\Lambda_{s+1} \le \lambda^* < \Lambda_s$. We then split the sum into a sum over the different ranges $\Lambda_{r+1} \le \lambda^{(j,m)} < \Lambda_r$. This gives

$$\begin{aligned} \sum_{\substack{j,m \\ \lambda^* \le \lambda^{(j,m)} \le \Lambda}} D(\lambda^{(j,m)}) &\le \sum_{r=0}^{s-1} \sum_{\substack{j,m \\ \Lambda_{r+1} \le \lambda^{(j,m)} \le \Lambda_r}} D(\lambda^{(j,m)}) + \sum_{\substack{j,m \\ \lambda^* \le \lambda^{(j,m)} \le \Lambda_s}} D(\lambda^{(j,m)}) \\ &\le (N^*(\Lambda_s) - n_1(\chi_1)n_3(\chi_1))D(\lambda^*) + \sum_{r=0}^{s-1} (N^*(\Lambda_r) - N^*(\Lambda_{r+1}))D(\Lambda_{r+1}). \end{aligned} \tag{2.7.33}$$

Note that we have used the fact that $D(\lambda)$ is decreasing in λ.

Rearranging the sum using Abel's identity, we have

$$\begin{aligned} \sum_{\substack{j,m \\ \lambda^* \le \lambda^{(j,m)} \le \Lambda}} D(\lambda^{(j,m)}) \le -n_1(\chi_1)n_3(\chi_1)D(\lambda^*) + N^*(\Lambda_s)(D(\lambda^*) - D(\Lambda_s)) \\ + \sum_{r=0}^{s-1} N^*(\Lambda_r)(D(\Lambda_{r+1}) - D(\Lambda_r)), \end{aligned} \tag{2.7.34}$$

since $D(\Lambda) = 0$.

Since $D(\Lambda_{r+1}) \ge D(\Lambda_r)$ and $D(\lambda^*) \ge D(\Lambda_s)$ we may replace $N^*(\lambda)$ with an upper bound, say $N_0^*(\lambda)$. This gives

$$\sum_{\substack{j,m \\ \lambda^* \le \lambda^{(j,m)} \le \Lambda}} D(\lambda^{(j,m)}) \le -n_1(\chi_1)n_3(\chi_1)D(\lambda^*) + N_0^*(\Lambda_s)D(\lambda^*) + \sum_{r=0}^{s-1} (N_0^*(\Lambda_r) - N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}). \tag{2.7.35}$$

Hence

$$\sum_{\substack{j,m \\ \lambda^{(j,m)}\le\Lambda}} D(\lambda^{(j,m)}) \le n_1(\chi_1)n_3(\chi_1)(D(\lambda_1) - D(\lambda^*)) + N_0^*(\Lambda_s)D(\lambda^*) + \sum_{r=0}^{s-1}(N_0^*(\Lambda_r) - N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}). \tag{2.7.36}$$

Putting (2.7.25), (2.7.29) and (2.7.36) together we obtain

$$\sum_{\chi\neq\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)} \exp(-M\lambda_\rho) \le \frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)} + N_0^*(\Lambda_s)D(\lambda^*) + A_1' + \sum_{r=0}^{s-1}(N_0^*(\Lambda_r) - N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}), \tag{2.7.37}$$

where

$$A_1' = n_1(\chi_1)n_2(\chi_1)B_1(\lambda_1)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)} - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right) + n_1(\chi_1)n_3(\chi_1)\left(D(\lambda_1) - D(\lambda^*)\right) - n_1(\chi_1)n_3(\chi_1)C_1(\lambda_1)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)} - \frac{\exp(-M\lambda_1')}{B_1(\lambda_1')}\right). \tag{2.7.38}$$

We now wish to bound this when we consider λ_1, λ_1' and λ_2 constrained in size. Specifically, we consider $\lambda_1 \in [\lambda_{11}, \lambda_{12}]$, $\lambda_2 \ge \lambda_{21}$ and $\lambda_1' \ge \lambda_{11}'$.

By definition $N_0^*(\Lambda_s) \ge n_1(\chi_1)n_3(\chi_1)$, and so the coefficient of $D(\lambda^*)$ is non-negative. Since D is a decreasing function, the right hand side of (2.7.37) is decreasing as a function of λ_2. The term $B_1(\lambda_1)$ occurs $n_2(\chi_1)/n_3(\chi_1)$ times in the sum

$$\sum_{\rho\in\mathcal{R}_0\cap\mathcal{Z}(\chi_1)} B_1(\lambda_\rho).$$

Since the sum is $\le C_1(\lambda_1)$, and all terms in the sum are positive we have that

$$n_2(\chi_1)B_1(\lambda_1) \le n_3(\chi_1)C_1(\chi_1). \tag{2.7.39}$$

Therefore, by expanding out A' we see that the right hand side of (2.7.37) is also decreasing as a function of λ_1'.

Therefore we may replace λ_1' and λ_2 with their lower bounds λ_{11}' and λ_{21} respectively.

Considering this bound as a function of λ_1 we find that the right hand side is

$$n_1(\chi_1)n_3(\chi_1)\left(\frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}C_1(\lambda_1) - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}B_2(\lambda_1)\right) + n_1(\chi_1)n_2(\chi_1)B_1(\lambda_1)\left(\frac{\exp(-M\lambda_1)}{B_1(\lambda_1)} - \frac{\exp(-M\lambda_{11}')}{B_1(\lambda_{11}')}\right) + C, \tag{2.7.40}$$

where C is independent of λ_1. We see this is

$$\leq n_1(\chi_1)n_3(\chi_1)\left(\frac{\exp(-M\lambda'_{11})}{B_1(\lambda'_{11})}C_1(\lambda_{11}) - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}B_2(\lambda_{12})\right)$$
$$+ 2B_1(\lambda_{11})\left(\frac{\exp(-M\lambda_{11})}{B_1(\lambda_{11})} - \frac{\exp(-M\lambda'_{11})}{B_1(\lambda'_{11})}\right) + C. \quad (2.7.41)$$

Therefore we obtain

$$\sum_{\chi\neq\chi_0}\sum_{\rho\in\mathcal{R}\cap\mathcal{Z}(\chi)} \exp(-M\lambda_\rho) \leq \frac{\exp(-M\Lambda)C_1(\Lambda)C_2}{B_1(\Lambda)B_2(\Lambda)} + N_0^*(\Lambda_s)D(\lambda^*) + A''_1$$
$$+ \sum_{r=0}^{s-1}(N_0^*(\Lambda_r) - N_0^*(\Lambda_{r+1}))D(\Lambda_{r+1}), \quad (2.7.42)$$

where

$$A''_1 = 2B_1(\lambda_{11})\left(\frac{\exp(-M\lambda_{11})}{B_1(\lambda_{11})} - \frac{\exp(-M\lambda'_{11})}{B_1(\lambda'_{11})}\right)$$
$$+ n_4\left(\frac{\exp(-M\lambda'_{11})}{B_1(\lambda'_{11})}C_1(\lambda_{11}) - \frac{\exp(-M\Lambda)C_1(\Lambda)}{B_1(\Lambda)B_2(\Lambda)}B_2(\lambda_{12}) - D(\lambda^*)\right), \quad (2.7.43)$$

and n_4 is chosen to be 1 or 2 so as to give the largest value for A''_1.

We now proceed to estimate (2.7.42) for various ranges of λ_1 which cover the region $\lambda_1 \geq 0.35$. We consider

$$M = 7.8. \quad (2.7.44)$$

For each range of λ_1 we use the lower bounds for λ'_1 and λ_2 as given by [51, Table 2, 3, 7] and [26, Table 4 and 7]. We use the upper bounds for N_0^* as calculated in Table 2.1.

We give these bounds on λ'_1 and λ_2, our choices of Λ and the calculation of the right hand side of (2.7.42) in Table 2.7.2.

We see that for each range of λ_1 we obtain an upper bound for (2.7.42) which is < 0.99. Since the expression is decreasing in M, this holds for all $M \geq 7.8$. We have therefore established Proposition 2.6 by taking $\epsilon = 10^{-3}$.

Table 2.2: Calculation of the RHS of (2.7.42) for different ranges of λ_1.

λ_{11}	λ_{12}	λ_{21}	λ'_{11}	Λ	Total RHS of (2.7.42)
0.35	0.40	1.29	2.10	1.29	0.8579...
0.40	0.44	1.18	2.03	1.27	0.9821...
0.44	0.46	1.08	1.66	1.28	0.9213...
0.46	0.48	1.08	1.53	1.28	0.9120...
0.48	0.50	1.08	1.47	1.28	0.9041...
0.50	0.52	1.00	1.40	1.28	0.9304...
0.52	0.54	1.00	1.34	1.31	0.8049...
0.54	0.56	0.92	1.28	1.31	0.8427...
0.56	0.58	0.92	1.23	1.31	0.8385...
0.58	0.60	0.92	1.18	1.31	0.8349...
0.60	0.62	0.85	1.13	1.34	0.7782...
0.62	0.64	0.85	1.09	1.34	0.7756...
0.64	0.66	0.79	1.04	1.34	0.8363...
0.66	0.68	0.79	1.00	1.36	0.7652...
0.68	0.70	0.79	0.96	1.36	0.7636...
0.70	0.72	0.745	0.93	1.36	0.8241...
0.72	0.74	0.745	0.91	1.36	0.8229...
0.74	0.76	0.745	0.89	1.36	0.8219...
0.76	0.78	0.76	0.86	1.36	0.7988...
0.78	0.80	0.78	0.84	1.36	0.7708..
0.80	0.82	0.80	0.83	1.36	0.7463...
0.82	0.86	0.82	0.827	1.36	0.7243...
0.86	∞	0.86	0.86	1.44	0.5110...

Chapter 3

Almost-prime k-tuples

3.1 Introduction

We consider a set of distinct integer linear functions

$$L_i(x) = a_i x + b_i, \qquad i \in \{1, \dots, k\}. \tag{3.1.1}$$

We say such a set of functions is *admissible* if their product has no fixed prime divisor. That is, for every prime p there is an integer n_p such that none of $L_i(n_p)$ are a multiple of p. We are interested in the following conjecture, which we recall from Chapter 1.

Conjecture (Prime k-tuples Conjecture)**.** *Given an admissible set of distinct integer linear functions $L_i(x)$ ($i \in \{1, \dots, k\}$), there are infinitely many integers n for which all the $L_i(n)$ are prime.*

With the current technology it appears impossible to prove any case of the prime k-tuples conjecture for $k \geq 2$.

Although we cannot prove that the functions are simultaneously prime infinitely often, we are able to show that they are *almost prime* infinitely often, in the sense that their product has only a few prime factors. This was most notably achieved by Chen [5] who showed that there are infinitely many primes p for which $p + 2$ has at most 2 prime factors. His method naturally generalises to show that for a pair of admissible functions the product $L_1(n)L_2(n)$ has at most 3 prime factors infinitely often.

Similarly sieve methods can prove analogous results for any k. We can show that the product of k admissible functions $\Pi(n) = L_1(n) \dots L_k(n)$ has at most r_k prime factors infinitely often, for some explicitly given value of r_k. The prime k-tuples conjecture is equivalent to showing we can have $r_k = k$ for all k. The current best values of r_k grow asymptotically like $k \log k$ and explicitly for small k we can take $r_2 = 3$ (Chen, [5]), $r_3 = 8$ (Porter, [43]), $r_4 = 12$, $r_5 = 16$, $r_6 = 20$ (Diamond and Halberstam [10]), $r_7 = 24$, $r_8 = 28$, $r_9 = 33$, $r_{10} = 38$ (Ho and Tsang, [28]).

The problem of determining the bounds r_k has attracted much interest over the past 50 years due to the fact it gives a typical numerical indication of the strength of a sieve of dimension k.

3.2 Main result

Our main result is the following theorem.

Theorem 3.1. *Given an admissible set of k distinct linear functions, for infinitely many $n \in \mathbb{N}$ the product $\Pi(n)$ has at most r_k prime factors, where r_k is given in Table 3.1 below.*

Table 3.1: Bounds for $\Omega(\Pi(n))$

k	3	4	5	6	7	8	9	10
r_k	8	11	15	18	22	26	30	34

Theorem 3.1 improves the previous best known bounds for $k \geq 4$, which were obtained by Diamond and Halberstam [10] for $4 \leq k \leq 6$ and by Ho and Tsang [28] for $7 \leq k \leq 10$. We fall just short of proving $r_k \leq 7$ for $k = 3$, and so fail to improve upon a result of Porter [43]. This comparison is shown in Table 3.2. We prove these results using a sieve which

Table 3.2: Bounds for $\Omega(\Pi(n))$

k	3	4	5	6	7	8	9	10
Previous best bound	8	12	16	20	24	28	33	38
New bound	8	11	15	18	22	26	30	34

is a combination of a weighted sieve similar to Selberg's $\Lambda^2\Lambda^-$ sieve (see [45]), and the Graham-Goldston-Pintz-Yıldırım sieve (see [17]) used to count numbers with a specific number of prime factors. We comment that the assumption that the functions are distinct is not strictly necessary. Essentially the same argument can give the bounds which are at least as strong when the functions are not distinct.

In Chapter 4, we will also improve the bound when $k = 3$, using an argument based on the Diamond-Halberstam-Richert sieve rather than Selberg's sieve.

We note that for k large our method only improves lower order terms, and so we do not improve the asymptotic bound $r_k \sim k \log k$. Indeed, we will consider the case of large k in Chapter 5.

3.3 Key ideas

We wish to show that for any sufficiently large N we have

$$\sum_{N \leq n \leq 2N} (c - \Omega(\Pi(n))) \left(\sum_{d|\Pi(n)} \lambda_d \right)^2 > 0 \tag{3.3.1}$$

for some real numbers λ_d and some constant integer $c > 0$. From this it is clear that there must be some $n \in [N, 2N]$ such that $\Omega(\Pi(n)) \leq c$. Since this is true for all sufficiently large N, it follows that there are infinitely many integers n such that $\Omega(\Pi(n)) \leq c$.

The work of Heath-Brown [27] and Ho and Tsang [28] considered a similar sum, but used the divisor function $d(\Pi(n))$ instead of the number-of-prime-factors function Ω. Using the divisor function has the advantage that there are stronger level-of-distribution results available, but we find that this is outweighed by the fact that the Ω function is relatively much smaller than the divisor function on numbers with many prime factors.

The Ω function has Bombieri-Vinogradov style equidistribution results (as shown by Motohashi [39]), and so we would expect we should be able to estimate the above sum directly, in a method similar to Heath-Brown [27] or Selberg [45] when they considered the divisor function instead. We encounter some technical difficulties when attempting to translate this argument, however.

Instead we express $\Omega(n)$ as a weighted sum over small prime factors (as in the weighted sieve method of Diamond and Halberstam [10]) and a remaining positive contribution which we split up depending on the number of prime factors of each of the $L_j(n)$.

Diamond and Halberstam used a weighted sieve. The method relied on the fact that for n square-free we have the inequality

$$\Omega(n) \leq \sum_{\substack{p|n \\ p \leq y}} \left(1 - \frac{\log p}{\log y}\right) + \frac{\log n}{\log y}. \tag{3.3.2}$$

We note that this inequality is strict if n has a prime factor which is larger than y. This results in a loss in the argument which has a noticeable effect when we apply this to k-tuples when k is small. Assuming that $y \geq n^{1/2}$ and n is square-free, we can write instead an equality

$$\Omega(n) = \sum_{\substack{p|n \\ p\leq y}} \left(1 - \frac{\log p}{\log y}\right) + \frac{\log n}{\log y} + \sum_{r=1}^{\infty} \chi_r(n), \tag{3.3.3}$$

where

$$\begin{aligned} \chi_r(n) &= \begin{cases} 1 - \frac{\log p_r}{\log y}, & n = p_1 \dots p_r \text{ with } p_1 \leq \dots \leq p_{r-1} < y < p_r, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} -\left(\frac{\log n}{\log y} - 1 - \sum_{i=1}^{r-1} \frac{\log p_i}{\log y}\right), & n = p_1 \dots p_r \text{ with } p_1 \leq \dots \leq p_{r-1} < y < p_r \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3.4}$$

For fixed r we can evaluate Selberg-type weighted sums over $\chi_r(L_i(n))$ using the method of Graham *et al* in [17] as an extension of the original 'GPY method'. We note that the contribution from $\chi_r(n)$ is always negative, so we can obtain a lower bound by simply omitting terms when $r > h$ for some constant h. The contribution of the χ_r terms decreases quickly with r, and so we in practice only need to calculate the contribution when r is small (we only consider the contributions of χ_r when $r \leq 4$ in this work). This is the key difference in our approach to previous methods, and allows us to obtain the improvements given by Theorem 3.1.

3.4 Initial considerations

We adopt similar notation to that of Graham *et al* in [17].

Let $\mathcal{L} = \{L_1, L_2, \dots, L_k\}$ be an admissible k-tuple of linear functions. We define

$$\Pi(n) = \prod_{i=1}^{k} L_i(n) = (a_1 n + b_1) \dots (a_k n + b_k), \tag{3.4.1}$$

$$\nu_p(\mathcal{L}) = \#\{1 \leq n \leq p : \Pi(n) \equiv 0 \pmod{p}\}. \tag{3.4.2}$$

We note that admissibility is equivalent to the condition

$$\nu_p(\mathcal{L}) < p \qquad \text{for all primes } p. \tag{3.4.3}$$

We also see that $\nu_p(\mathcal{L}) \leq k$ for all primes p, and so the above condition holds automatically for $p > k$.

For technical reasons we adopt a normalisation of our linear functions, as done originally by Heath-Brown in [27]. Since we are only interested in showing any admissible k-tuple has at most r_k prime factors infinitely often (for some explicit r_k), by considering the functions $L_i(An+B)$ for suitably chosen constants A and B, we may assume without loss of generality that our functions satisfy the following hypothesis.

Hypothesis 3.2. *$\mathcal{L} = \{L_1, \dots, L_k\}$ is an admissible k-tuple of linear functions. The functions $L_i(n) = a_i n + b_i$ $(1 \le i \le k)$ are distinct with $a_i > 0$. Each of the coefficients a_i is composed of the same primes, none of which divides the b_j. If $i \ne j$, then any prime factor of $a_i b_j - a_j b_i$ divides each of the a_l.*

For a set of linear functions satisfying Hypothesis 3.2 we define

$$A = \prod_{i=1}^{k} a_i. \tag{3.4.4}$$

We note that in this case

$$\nu_p(\mathcal{L}) = \begin{cases} 0, & p|A, \\ k, & p \nmid A. \end{cases} \tag{3.4.5}$$

We also define the *singular series* $\mathfrak{S}(\mathcal{L})$ of $\mathcal{L}$ when $\mathcal{L}$ satisfies Hypothesis 3.2.

$$\mathfrak{S}(\mathcal{L}) = \prod_{p|A}\left(1 - \frac{1}{p}\right)^{-k} \prod_{p\nmid A}\left(1 - \frac{k}{p}\right)\left(1 - \frac{1}{p}\right)^{-k}. \tag{3.4.6}$$

We note that $\mathfrak{S}(\mathcal{L})$ is positive.

As is common with the Selberg sieve, for some parameter R_2 we impose the condition

$$\lambda_d = 0 \qquad \text{if } d \ge R_2 \text{ or } d \text{ not square-free or } (d, A) \ne 1. \tag{3.4.7}$$

We wish to choose the λ_d to maximize the sum (3.3.1), but this will be difficult to do optimally. We proceed by reparameterising the form in λ_d into new variables y_r and y_r^* which will almost diagonalise it. We define

$$y_r = \mu(r) f_1(r) \sum_d{}' \frac{\lambda_{dr}}{f(dr)}, \tag{3.4.8}$$

$$y_r^* = \mu(r) f_1^*(r) \sum_d{}' \frac{\lambda_{dr}}{f^*(dr)}, \tag{3.4.9}$$

where here and from now on, the $'$ by the summation indicates that the sum is over all values of the indices which are square-free and coprime to A. For square-free d coprime to

A, the functions f, f_1, f^* and f_1^* are defined by

$$f(d) = \prod_{p|d} \frac{p}{k}, \tag{3.4.10}$$

$$f_1(d) = (f * \mu)(d) = \prod_{p|d} \frac{p-k}{k}, \tag{3.4.11}$$

$$f^*(d) = \prod_{p|d} \frac{p-1}{k-1}, \tag{3.4.12}$$

$$f_1^*(d) = (f^* * \mu)(d) = \prod_{p|d} \frac{p-k}{k-1}. \tag{3.4.13}$$

We note that by Möbius inversion we have

$$\lambda_d = \mu(d) f(d) \sum_r{}' \frac{y_{rd}}{f_1(rd)}. \tag{3.4.14}$$

Thus the λ_d (and hence also the y_r^*) are defined uniquely by a choice of the y_r. The conditions (3.4.7) will be satisfied if the same conditions apply to the y_r.

For some polynomial P (to be determined later), we choose

$$y_r = \begin{cases} \mu^2(r)\mathfrak{S}(\mathcal{L})P\left(\frac{\log R_2/r}{\log R_2}\right), & \text{if } r \le R_2 \text{ and } (r, A) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4.15}$$

The exact polynomial P we choose is given later in Table 3.3. We now turn our attention to the proof of the theorem.

3.5 Proof of Theorem 3.1

We consider the sum

$$S = S(\nu; N, R_1, R_2, \mathcal{L}) = \sum_{N \le n \le 2N} w(n)\Lambda^2(n), \tag{3.5.1}$$

where

$$w(n) = \nu - \sum_{p|\Pi(n)} \left(1 - \frac{\log p}{\log R_1}\right), \tag{3.5.2}$$

$$\Lambda^2(n) = \Big(\sum_{\substack{d|\Pi(n) \\ d \le R_2}} \lambda_d\Big)^2. \tag{3.5.3}$$

We note that if $\Pi(n)$ is square-free then

$$w(n) = \nu - \Omega(\Pi(n)) + \frac{\log \Pi(n)}{\log R_1}. \tag{3.5.4}$$

We see that for $n \in [N, 2N]$ and some fixed $h \in \mathbb{Z}_{>0}$ we have

$$\begin{aligned} w(n) &= \nu - \sum_{j=1}^{k} \sum_{p|L_j(n)} \left(1 - \frac{\log p}{\log R_1}\right) \\ &\geq \nu - \sum_{j=1}^{k} \sum_{\substack{p|L_j(n) \\ p \leq R_1 \text{ or } \Omega(L_j(n)) \leq h}} \left(1 - \frac{\log p}{\log R_1}\right) \\ &\geq \nu - \sum_{j=1}^{k} \sum_{\substack{p|L_j(n) \\ p \leq R_1}} \left(1 - \frac{\log p}{\log R_1}\right) + \sum_{j=1}^{k} \sum_{r=1}^{h} \chi_r(L_j(n)), \end{aligned} \tag{3.5.5}$$

where

$$\chi_r(n) = \begin{cases} \frac{\log N}{\log R_1} - 1 - \sum_{i=1}^{r-1} \frac{\log p_i}{\log R_1}, & \text{if } n = p_1 \dots p_r \text{ with} \\ & \quad n^{\epsilon} < p_1 < \dots < p_{r-1} \leq n^{\log R_1 / \log N} < p_r \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.6}$$

Thus

$$\begin{aligned} \sum_{\substack{N \leq n \leq 2N \\ \Pi(n) \text{ square-free}}} \left(\nu - \Omega(\Pi(n)) + \frac{\log \Pi(n)}{\log R_1}\right) \Lambda^2(n) &= S - S' \\ &\geq \nu S_0 - S' - T_0 + \sum_{j=1}^{k} \sum_{r=1}^{h} T_{r,j}, \end{aligned} \tag{3.5.7}$$

where

$$S_0 = \sum_{N \leq n \leq 2N} \Lambda^2(n), \tag{3.5.8}$$

$$S' = \sum_{\substack{N \leq n \leq 2N \\ \Pi(n) \text{ not square-free}}} w(n) \Lambda^2(n), \tag{3.5.9}$$

$$T_0 = \sum_{N \leq n \leq 2N} \sum_{\substack{p|\Pi(n) \\ p \leq R_1}} \left(1 - \frac{\log p}{\log R_1}\right) \Lambda^2(n), \tag{3.5.10}$$

$$T_{r,j} = \sum_{N \leq n \leq 2N} \chi_r(L_j(n)) \Lambda^2(n). \tag{3.5.11}$$

We can evaluate S_0, S', T_0 and T_i using weighted forms of the Selberg sieve. We state the results here and prove them in the following sections. To ease notation we now fix as constants

$$r_1 = \frac{\log R_1}{\log N}, \qquad r_2 = \frac{\log R_2}{\log N}. \tag{3.5.12}$$

We view r_1, r_2, k, A and our polynomial P as fixed, and so any constants implied by the use of O or $\ll$ notation may depend on these quantities without explicit reference.

Proposition 3.3. *Let $\mathcal{L}$ satisfy Hypothesis 3.2. Let $W_0 : [0, r_1/r_2] :\to \mathbb{R}_{\geq 0}$ be a piecewise smooth non-negative function. Let λ_d, y_d be defined in terms of a polynomial P as given in* (3.4.14) *and* (3.4.15). *Assume that $r_1 \geq r_2$. Then there exists a constant C such that if $R_1 R_2^2 \leq N(\log N)^{-C}$ then we have*

$$\sum_{N \leq n \leq 2N} \left(\sum_{\substack{p | \Pi(n) \\ p \leq R_1}} W_0\left(\frac{\log p}{\log R_2}\right) \right) \left(\sum_{\substack{d | \Pi(n) \\ d \leq R_2}} \lambda_d \right)^2 = \frac{\mathfrak{S}(\mathcal{L}) N (\log R_2)^k}{(k-1)!} J_0 + O_{W_0}\left(N (\log N)^{k-1} (\log\log N)^2 \right),$$

where

$$\begin{aligned} J_0 &= J_{01} + J_{02} + J_{03}, \\ J_{01} &= k \int_0^1 \frac{W_0(y)}{y} \int_0^{1-y} (P(1-x) - P(1-x-y))^2 x^{k-1} dx dy, \\ J_{02} &= k \int_0^1 \frac{W_0(y)}{y} \int_{1-y}^1 P(1-x)^2 x^{k-1} dx dy, \\ J_{03} &= k \int_1^{r_1/r_2} \frac{W_0(y)}{y} \int_0^1 P(1-x)^2 x^{k-1} dx dy. \end{aligned}$$

Proposition 3.4. *Let $\mathcal{L}$ satisfy Hypothesis 3.2. Given $\epsilon > 0$ and $r \in \mathbb{Z}_{>0}$, let*

$$\mathcal{A}_r := \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(1 - r_2, 1 - x_{r-1}) \right\}.$$

Let $W_r : [0,1]^{r-1} \to \mathbb{R}_{\geq 0}$ be a piecewise smooth function supported on $\mathcal{A}_r$ such that

$$\frac{\partial}{\partial x_j} W_r(x) \ll W_r(x) \qquad \text{uniformly in} \left\{ x \in \mathcal{A}_r : \frac{\partial}{\partial x_j} W_r(x) \text{ exists} \right\}.$$

Let

$$\beta_r(n) = \begin{cases} W_r\left(\frac{\log p_1}{\log n}, \ldots, \frac{\log p_{r-1}}{\log n}\right), & n = p_1 p_2 \ldots p_r \text{ with } p_1 < \cdots < p_r, \\ 0, & \text{otherwise,} \end{cases}$$

Let λ_d, y_d be defined in terms of a polynomial P as given in (3.4.14) *and* (3.4.15). *Then there is a constant C such that if $R_2^2 \leq N^{1/2} (\log N)^{-C}$, we have*

$$\sum_{N \leq n \leq 2N} \beta_r(L_j(n)) \left(\sum_{\substack{d | \Pi(n) \\ d \leq R_2}} \lambda_d \right)^2 = \frac{\mathfrak{S}(\mathcal{L}) N (\log R_2)^{k+1}}{(k-2)! (\log N)} J_r + O_{W_r}\left(N (\log\log N)^r (\log N)^{k-1} \right),$$

where

$$J_r = \int_{(x_1,\dots,x_{r-1})\in\mathcal{A}_r} \frac{W_r(x_1,\dots,x_{r-1}) I_1(r_2^{-1}x_1,\dots,r_2^{-1}x_{r-1})}{\left(\prod_{i=1}^{r-1} x_i\right)\left(1-\sum_{i=1}^{r-1} x_i\right)} dx_1\dots dx_{r-1},$$

$$I_1 = \int_0^1 \left(\sum_{J\subseteq\{1,\dots,r-1\}} (-1)^{|J|}\tilde{P}^+\left(1-t-\sum_{i\in J} x_i\right)\right)^2 t^{k-2}dt,$$

$$\tilde{P}^+(x) = \begin{cases} \int_0^x P(t)dt, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3.5. *Let $\mathcal{L}$ satisfy Hypothesis 3.2. Let λ_d, y_d be defined in terms of a polynomial P as given in* (3.4.14) *and* (3.4.15). *Then there exists a constant C such that if $R_2^2 \leq N^{1/2}(\log N)^{-C}$ then*

$$\sum_{\substack{N\leq n\leq 2N \\ \Pi(n) \text{ not square-free}}} \left(\sum_{\substack{d|\Pi(n) \\ d\leq R_2}} \lambda_d\right)^2 \ll N(\log N)^{k-1}\log\log N.$$

We also quote a result [17, Theorem 7] which is based on the original result of Goldston *et al* in [20].

Proposition 3.6. *Let $\mathcal{L}$ satisfy Hypothesis 3.2. Let λ_d, y_d be defined in terms of a polynomial P as given in* (3.4.14) *and* (3.4.15). *Then there is a constant C such that if $R_2^2 \leq N(\log N)^{-C}$, we have*

$$\sum_{N\leq n\leq 2N} \left(\sum_{\substack{d|\Pi(n) \\ d\leq R_2}} \lambda_d\right)^2 = \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{(k-1)!} J + O\left(N(\log N)^{k-1}\right)$$

where

$$J = \int_0^1 P(1-t)^2 t^{k-1} dt.$$

Using Propositions 3.3, 3.4, 3.6 and 3.5 we can now bound our sum S in terms of the integers k and h and the polynomial P. For some $\epsilon > 0$ we choose

$$r_1 = \frac{1}{2} + \epsilon, \qquad r_2 = \frac{1}{4} - \epsilon, \tag{3.5.13}$$

so that the conditions of all the propositions are satisfied.

Proposition 3.6 gives the size of S_0 immediately.

Using Proposition 3.5 we have

$$
\begin{aligned}
S' &= \sum_{\substack{N\le n\le 2N\\ \Pi(n)\text{ not square-free}}} w(n)\Lambda^2(n)\\
&\le \sum_{\substack{N\le n\le 2N\\ \Pi(n)\text{ not square-free}}} \left(\nu + \frac{\log \Pi(n)}{\log R_1}\right)\Lambda^2(n)\\
&\le \sum_{\substack{N\le n\le 2N\\ \Pi(n)\text{ not square-free}}} \left(\nu + \frac{k+\epsilon}{r_1}\right)\Lambda^2(n)\\
&\ll N(\log N)^{k-1}\log\log N. \qquad (3.5.14)
\end{aligned}
$$

To estimate T_0 and the $T_{r,j}$ we choose

$$W_0(x) = 1 - \frac{r_2}{r_1}x, \qquad (3.5.15)$$

$$W_j(x_1,\dots,x_{j-1}) = \begin{cases} \frac{1}{r_1} - 1 - \frac{1}{r_1}\sum_{i=1}^{j-1} x_i, & \epsilon < x_1 < \cdots < x_{j-1} \\ & \text{and } \sum_{i=1}^{r-1} x_i < 1 - r_1 \\ 0, & \text{otherwise,} \end{cases} \qquad (3.5.16)$$

which satisfy the conditions of Propositions 3.3 and 3.4 respectively.

By Proposition 3.3 we have

$$
\begin{aligned}
T_0 &= \sum_{N\le n\le 2N}\left(\sum_{\substack{p\mid\Pi(n)\\ p\le R_1}} W_0\left(\frac{\log p}{\log R_2}\right)\right)\left(\sum_{\substack{d\mid\Pi(n)\\ d\le R_2}}\lambda_d\right)^2\\
&= \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{(k-1)!}J_0 + O\left(N(\log N)^{k-1}\log\log N\right), \qquad (3.5.17)
\end{aligned}
$$

where

$$
\begin{aligned}
J_0 &= J_{01} + J_{02} + J_{03}, &(3.5.18)\\
J_{01} &= k\int_0^1 \frac{r_1 - r_2 y}{r_1 y}\int_0^{1-y} (P(1-x) - P(1-x-y))^2 x^{k-1}dxdy, &(3.5.19)\\
J_{02} &= k\int_0^1 \frac{r_1 - r_2 y}{r_1 y}\int_{1-y}^1 P(1-x)^2 x^{k-1}dxdy, &(3.5.20)\\
J_{03} &= k\int_1^{r_1/r_2} \frac{r_1 - r_2 y}{r_1 y}\int_0^1 P(1-x)^2 x^{k-1}dxdy. &(3.5.21)
\end{aligned}
$$

By Proposition 3.4 we have

$$\begin{aligned}
T_{r,j} &= \sum_{N \le n \le 2N} \chi_r(L_j(n))\Lambda^2(n) \\
&= \sum_{N \le n \le 2N} \beta_r(L_j(n))\Lambda^2(n) \\
&= \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^{k+1}}{(k-2)!(\log N)} J_r + O_r\left(N(\log\log N)^{r+1}(\log N)^{k-1}\right), \qquad (3.5.22)
\end{aligned}$$

where

$$\beta_r(n) = \begin{cases} W_r\left(\frac{\log p_1}{\log n}, \dots, \frac{\log p_{r-1}}{\log n}\right), & n = p_1 p_2 \dots p_r \text{ with } p_1 < \dots < p_r, \\ 0, & \text{otherwise,} \end{cases} \qquad (3.5.23)$$

$$J_r = \int_{(x_1,\dots,x_{r-1})\in\mathcal{A}_r} \frac{W_r(x_1,\dots,x_{r-1})I_1(r_2^{-1}x_1,\dots,r_2^{-1}x_{r-1})}{\left(\prod_{i=1}^{r-1} x_i\right)\left(1-\sum_{i=1}^{r-1} x_i\right)} dx_1 \dots dx_{r-1}. \qquad (3.5.24)$$

Therefore we see that

$$\nu S_0 - S' + T_0 + \sum_{j=1}^{k}\sum_{r=1}^{h} T_{r,j} = \frac{N\mathfrak{S}(\mathcal{L})(\log R_2)^k}{(k-1)!}\left(\nu J - J_0 + r_2 k(k-1)\left(\sum_{r=1}^{h} J_r\right)\right) \qquad (3.5.25)$$
$$+ O\left(\frac{N(\log N)^k}{\log\log N}\right).$$

Therefore we put

$$\nu = \frac{J_0 - r_2 k(k-1)(\sum_{r=1}^{h} J_r)}{J} + \epsilon. \qquad (3.5.26)$$

We then see that for any N sufficiently large we have

$$\nu S_0 - S' - T_0 + \sum_{j=1}^{k}\sum_{r=1}^{h} T_{r,j} > 0. \qquad (3.5.27)$$

Thus we have

$$\Omega(\Pi(n)) \le \left\lfloor \frac{J_0 - r_2 k(k-1)(\sum_{r=1}^{h} J_r)}{J} + \frac{k}{r_1} + 2\epsilon \right\rfloor \qquad (3.5.28)$$

infinitely often.

With these fixed, given k, h and a polynomial P we obtain a bound on $\Omega(\Pi(n))$. To make calculations feasible we choose $h = 3$ (except we take $h = 4$ when $k = 10$). Numerical experiments indicate that the bounds of Theorem 3.1 cannot be improved by increasing h except possibly when $k = 5$.

We can now explicitly write down the integrals J_1, J_2 and J_3, splitting the integral up depending on whether $\tilde{P}^+$ (defined in Proposition 3.4) is positive or not. We put

$$\tilde{P}(x) = \int_0^x P(t)dt. \qquad (3.5.29)$$

Then we have that

$$J_1 = \left(\frac{1-r_1}{r_1}\right)\int_0^1 \tilde{P}(1-x)^2 x^{k-2}dx + O(\epsilon). \tag{3.5.30}$$

Similarly

$$J_2 = J_{21} + J_{22} + J_{23} + O(\epsilon), \tag{3.5.31}$$

where

$$J_{21} = \int_0^1 \frac{1-r_1-r_2y}{r_1y(1-r_2y)}\int_0^{1-y}\left(\tilde{P}(1-x)-\tilde{P}(1-x-y)\right)^2 x^{k-2}dxdy, \tag{3.5.32}$$

$$J_{22} = \int_0^1 \frac{1-r_1-r_2y}{r_1y(1-r_2y)}\int_{1-y}^1 \tilde{P}(1-x)^2 x^{k-2}dxdy, \tag{3.5.33}$$

$$J_{23} = \int_1^{(1-r_1)/r_2} \frac{1-r_1-r_2y}{r_1y(1-r_2y)}\int_0^1 \tilde{P}(1-x)^2 x^{k-2}dxdy. \tag{3.5.34}$$

Finally

$$J_3 = J_{31} + J_{32} + J_{33} + J_{34} + J_{35} + J_{36} + J_{37} + J_{38} + O(\epsilon), \tag{3.5.35}$$

where

$$J_{31} = \int_1^{(1-r_1)/2r_2}\int_y^{(1-r_1)/r_2-y} \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))}\int_0^1 \tilde{P}(1-x)^2 x^{k-2}dxdzdy, \tag{3.5.36}$$

$$J_{32} = \int_0^1\int_y^{(1-r_1)/r_2-y} \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))}\int_{1-y}^1 \tilde{P}(1-x)^2 x^{k-2}dxdzdy, \tag{3.5.37}$$

$$J_{33} = \int_0^1\int_1^{(1-r_1)/r_2-y} \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))} \times \int_0^{1-y}\left(\tilde{P}(1-x)-\tilde{P}(1-x-y)\right)^2 x^{k-2}dxdzdy, \tag{3.5.38}$$

$$J_{34} = \int_0^1\int_y^1 \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))} \times \int_{1-z}^{1-y}\left(\tilde{P}(1-x)-\tilde{P}(1-x-y)\right)^2 x^{k-2}dxdzdy, \tag{3.5.39}$$

$$J_{35} = \int_{1/2}^1\int_y^1 \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))} \times \int_0^{1-z}\left(\tilde{P}(1-x)-\tilde{P}(1-x-y)-\tilde{P}(1-x-z)\right)^2 x^{k-2}dxdzdy, \tag{3.5.40}$$

$$J_{36} = \int_0^{1/2}\int_{1-y}^1 \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))} \times \int_0^{1-z}\left(\tilde{P}(1-x)-\tilde{P}(1-x-y)-\tilde{P}(1-x-z)\right)^2 x^{k-2}dxdzdy, \tag{3.5.41}$$

$$J_{37} = \int_0^{1/2}\int_y^{1-y} \frac{1-r_1-r_2(y+z)}{r_1yz(1-r_2(y+z))}$$

$$\times \int_{1-y-z}^{1-z} \left(\tilde{P}(1-x) - \tilde{P}(1-x-y) - \tilde{P}(1-x-z)\right)^2 x^{k-2} dx dz dy, \quad (3.5.42)$$

$$J_{38} = \int_0^{1/2} \int_y^{1-y} \frac{1 - r_1 - r_2(y+z)}{r_1 yz(1 - r_2(y+z))} \int_0^{1-y-z} \Big(\tilde{P}(1-x) - \tilde{P}(1-x-y)$$

$$- \tilde{P}(1-x-z) + \tilde{P}(1-x-y-z)\Big)^2 x^{k-2} dx dz dy. \quad (3.5.43)$$

We now have explicit representations of J, J_0, J_1, J_2 and J_3. We can calculate these by numerical integration given k and a polynomial P.

Table 3.3 gives close to optimal polynomials for $3 \le k \le 10$ and the corresponding bounds obtained if we take ϵ sufficiently small. These give the results claimed in Theorem 3.1 except for $k = 10$.

Table 3.3: Bounds for $\Omega(\Pi(n))$

k	Bound on $\Omega(\Pi(n))$	Polynomial $P(x)$
3	8.220...	$1 + 14x$
4	11.653...	$1 + 22x$
5	15.306...	$1 + 33x$
6	18.936...	$1 + 10x + 40x^2$
7	22.834...	$1 + 10x + 60x^2$
8	26.860...	$1 + 10x + 80x^2$
9	30.942...	$1 + 30x + 300x^3$
10	35.158...	$1 + 35x - 10x^2 + 400x^3$

For $k = 10$ we find an improvement if we also include the contribution when one of the $L_i(n)$ has 4 prime factors (we omit the explicit integrals here). In this case we choose the polynomial

$$P(x) = 1 + 10x + 150x^2. \quad (3.5.44)$$

This gives us the bound 34.77... and so 10-tuples infinitely often have at most 34 prime factors, verifying Theorem 3.1.

3.6 The quantities T_δ and T_δ^*

Before proving the propositions, we first establish some results about the quantities

$$T_\delta = \sum_{d,e}{}' \frac{\lambda_d \lambda_e}{f([d,e,\delta]/\delta)}, \tag{3.6.1}$$

$$T_\delta^* = \sum_{d,e}{}' \frac{\lambda_d \lambda_e}{f^*([d,e,\delta]/\delta)}. \tag{3.6.2}$$

Here we use $[a_1, \ldots, a_n]$ to denote the least common multiple of $a_1, \ldots, a_n$, and we recall the definition of $\sum'$ from (3.4.8) as being restricted to square-free values coprime to A.

Most of these results already exist in some form in the literature. These results will underlie the proof of the propositions. We note that in [17] Graham, Goldston, Pintz and Yıldırım used slightly different notation (our quantity T_δ^* is labelled T_δ).

We first put T_δ and T_δ^* into an almost-diagonalised form.

Lemma 3.7. *We have*

$$T_\delta = \sum_{\substack{a \\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1(a)} \left(\sum_{s|\delta} \mu(s) y_{as} \right)^2,$$

$$T_\delta^* = \sum_{\substack{a \\ (a,\delta)=1}}{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) y_{as}^* \right)^2,$$

where

$$y_a^* = \frac{\mu^2(a) a}{\phi(a)} \sum_m{}' \frac{y_{ma}}{\phi(m)}.$$

Proof. The result for T_δ is shown, for example, in [45, Page 85]. The result for T_δ^* is proven in [17, Lemma 6]. □

We now again quote a Lemma from [17], which expresses the y_a^* in terms of the polynomial P which we used to define the variables y_a.

Lemma 3.8. *Let*

$$y_a = \begin{cases} \mu^2(a) \mathfrak{S}(\mathcal{L}) P\left(\frac{\log R_2/a}{\log R_2}\right), & \text{if } 0 \le a < R_2 \text{ and } (a, A) = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Then we have for $(a, A) = 1$ *and* $a < R_2$ *that*

$$y_a^* = \mu^2(a) \frac{\phi(A)}{A} \mathfrak{S}(\mathcal{L}) (\log R_2) \tilde{P}\left(\frac{\log R_2/a}{\log R_2}\right) + O(\log\log R_2),$$

where

$$\tilde{P}(x) = \int_0^x P(t)dt.$$

If $(a, A) \neq 1$ *or* $a \geq R_2$ *then we have*

$$y_a^* = 0.$$

Proof. This is proven in [17, Lemma 7]. □

We will repeatedly use the following results.

Lemma 3.9. *For* $u \geq 1$ *we have*

$$\sum_{a<u}{}' \frac{\mu^2(a)}{f_1(a)} = \frac{(\log u)^k}{\mathfrak{S}(\mathcal{L})k!} + O((\log 2u)^{k-1}),$$
$$\sum_{a<u}{}' \frac{\mu^2(a)}{f_1^*(a)} = \frac{A}{\phi(A)} \frac{(\log u)^{k-1}}{\mathfrak{S}(\mathcal{L})(k-1)!} + O((\log 2u)^{k-2}).$$

Proof. This is follows, for example, from [17, Lemma 3]. □

Lemma 3.10. *Let* $\kappa, A_1, A_2, L > 0$*. Suppose that* γ *is a multiplicative function such that*

$$0 \leq \frac{\gamma(p)}{p} \leq 1 - A_1, \qquad -L \leq \sum_{w \leq p < z} \frac{\gamma(p)\log p}{p} - \kappa \log \frac{z}{w} \leq A_2,$$

for any $2 \leq w \leq z$*. Let* $F : [0, 1] \to \mathbb{R}$ *be a piecewise differentiable function, and let* g *be the multiplicative function defined by*

$$g(d) = \prod_{p|d} \frac{\gamma(p)}{k - \gamma(p)}.$$

Then

$$\sum_{d<z} \mu^2(d)g(d)F\left(\frac{\log z/d}{\log z}\right) = \frac{c_\gamma (\log z)^\kappa}{\Gamma(\kappa)} \int_0^1 F(1-x)x^{\kappa-1}dx + O_{A_1,A_2,\kappa}(c_\gamma LM(F)(\log z)^{k-1}),$$

where

$$c_\gamma = \prod_p \left(1 - \frac{\gamma(p)}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^\kappa,$$
$$M(F) = \sup_{0 \leq 1 \leq x} (|F(x)| + |F'(x)|).$$

Proof. This is [17, Lemma 4]. □

In order to estimate the terms T_δ^* we wish to remove the condition $(a,\delta)=1$ in the summation over a, and remove the constraint caused by y_a and y_a^* only being supported on square-free a. We let

$$P_a = \begin{cases} \mathfrak{S}(\mathcal{L})P\left(\frac{\log R_2/a}{\log R_2}\right), & \text{if } 0 \le a < R_2 \\ 0, & \text{otherwise,} \end{cases} \tag{3.6.3}$$

$$P_a^* = \begin{cases} \frac{\phi(A)}{A}\mathfrak{S}(\mathcal{L})(\log R_2)\tilde{P}\left(\frac{\log R_2/a}{\log R_2}\right), & \text{if } 0 \le a < R_2 \\ 0, & \text{otherwise,} \end{cases} \tag{3.6.4}$$

so that these are equal to y_a and $y_a^* + O(\log\log R_2)$ respectively when a is square-free and coprime to A.

Lemma 3.11. *Let* $(\delta, A) = 1$. *Then we have*

$$T_\delta = \sideset{}{'}\sum_{a} \frac{\mu^2(a)}{f_1(a)}\left(\sum_{s|\delta}\mu(s)P_{as}\right)^2 + O\left(d(\delta)^2(\log R_2)^{k-1}\log\log R_2\right),$$
$$T_\delta^* = \sideset{}{'}\sum_{a} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P_{as}^*\right)^2 + O\left(d(\delta)^2(\log R_2)^{k}\log\log R_2\right).$$

Proof. We only prove the result for T_δ^* here, the result for T_δ follows from a completely analogous argument. We see that since $P_a^* \ll \log R_2$ we have

$$\begin{aligned} T_\delta^* &= \sideset{}{'}\sum_{\substack{a \\ (a,\delta)=1}} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P_{as}^* + O(d(\delta)\log\log R_2)\right)^2 \\ &= \sideset{}{'}\sum_{\substack{a \\ (a,\delta)=1}} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P_{as}^*\right)^2 + O\left(d(\delta)^2(\log R_2)(\log\log R_2)\sum_{a<R_2}\frac{\mu^2(a)}{f_1^*(a)}\right). \end{aligned} \tag{3.6.5}$$

By Lemma 3.9 the error term above is $O(d(\delta)^2(\log R_2)^k\log\log R_2)$.

We see that to prove the result it is sufficient to prove

$$\sideset{}{'}\sum_{\substack{a \\ (a,\delta)\neq 1}} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P_{as}^*\right)^2 \ll (\log R_2)^k d(\delta)^2(\log\log R_2). \tag{3.6.6}$$

Since all terms in the sum are non-negative, we have

$$\sideset{}{'}\sum_{\substack{a \\ (a,\delta)\neq 1}} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P_{as}^*\right)^2 \le \sum_{p|\delta}\sideset{}{'}\sum_{\substack{a \\ p|a}} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P_{as}^*\right)^2. \tag{3.6.7}$$

We consider the inner sum. By the Cauchy-Schwarz inequality we have

$$\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 = \sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta/p}\mu(s)(P^*_{as}-P^*_{asp})\right)^2$$
$$\ll d(\delta)\sum_{s|\delta/p}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(P^*_{as}-P^*_{asp}\right)^2. \qquad (3.6.8)$$

We split the summation over a depending on whether the P^*_{as} and P^*_{asp} terms vanish (since $P^*_b = 0$ for $b \geq R_2$).

$$\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 \ll d(\delta)\sum_{s|\delta/p}\sum_{a'<R_2/sp^2}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}\left(P^*_{a'ps}-P^*_{a'sp^2}\right)^2$$
$$+ d(\delta)\sum_{s|\delta/p}\sum_{R_2/sp^2\leq a'<R_2/sp}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}(P^*_{a'ps})^2. \qquad (3.6.9)$$

We substitute in the value of P^*.

$$\frac{1}{d(\delta)}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|q}\mu(s)P^*_{as}\right)^2$$
$$\ll (\log R_2)^2\sum_{s|\delta/p}\sum_{a'<R_2/sp^2}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}\left(\tilde{P}\left(1-\frac{\log a'ps}{\log R_2}\right)-\tilde{P}\left(1-\frac{\log a'sp^2}{\log R_2}\right)\right)^2$$
$$+ (\log R_2)^2\sum_{s|\delta/p}\sum_{R_2/sp^2\leq a'<R_2/sp}{}' \frac{\mu^2(a'p)}{f_1^*(a'p)}\tilde{P}\left(1-\frac{\log a'ps}{\log R_2}\right)^2. \qquad (3.6.10)$$

In the first sum above both the arguments of the polynomials differ by $\log p/\log R_2$. Since they are fixed polynomials on a bounded range, the derivative of the polynomial is $\ll 1$ and so the difference is $\ll \log p/\log R_2$. In the second sum we just use the trivial bound $\tilde{P}(x) \ll 1$.

This gives

$$\frac{1}{d(\delta)}\sum_{\substack{a\\p|a}}{}' \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|q}\mu(s)P^*_{as}\right)^2 \ll \frac{(\log p)^2}{f_1^*(p)}\sum_{s|\delta/p}\sum_{a<R_2/sp^2}{}' \frac{\mu^2(a)}{f^*(a)}$$
$$+ \frac{(\log R_2)^2}{f(p)}\sum_{s|\delta/p}\sum_{R_2/sp^2\leq a<R_2/sp}{}' \frac{\mu^2(a)}{f^*(a)}. \qquad (3.6.11)$$

Using Lemma 3.9 we see that the first sum is $\ll d(\delta)(\log p)^2(\log R_2)^{k-1}/f_1^*(p)$ and the second sum is $\ll d(\delta)(\log p)(\log R_2)^k/f_1^*(p)$ because of the range of summation over a.

Thus

$$\sideset{}{'}\sum_{\substack{a\\ p|a}} \frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 \ll d(\delta)^2\frac{\log p}{f_1^*(p)}(\log R_2)^k \ll d(\delta)^2\frac{\log p}{p}(\log R_2)^k. \tag{3.6.12}$$

Summing over all $p|\delta$ gives the bound

$$d(\delta)^2(\log R_2)^k\sum_{p|\delta}\frac{\log p}{p}. \tag{3.6.13}$$

Splitting the sum into a sum over $p \leq \log R_2$ and a sum over $p > \log R_2$ we get the bound

$$d(\delta)^2(\log R_2)^k(\log\log R_2). \tag{3.6.14}$$

This gives (3.6.6), and hence the Lemma. □

Essentially the same argument as above also yields a useful bound on the size of T_δ and T^*_δ.

Lemma 3.12. *Let* $(\delta, A) = 1$. *Then we have*

$$\begin{aligned} T_\delta &\ll \min_{p|\delta}(\log p)\, d(\delta)^2(\log R_2)^{k-1},\\ T^*_\delta &\ll \min_{p|\delta}(\log p)\, d(\delta)^2(\log R_2)^k + d(\delta)^2(\log R_2)^k\log\log R_2. \end{aligned}$$

Proof. For $p|\delta$ we have (using the fact all terms are non-negative)

$$\begin{aligned} T^*_\delta &= \sideset{}{'}\sum_{\substack{a\\ (a,\delta)=1}}\frac{\mu^2(a)}{f_1^*(a)}\left(\sum_{s|\delta}\mu(s)P^*_{as}\right)^2 + O(d(\delta)^2(\log R_2)^k\log\log R_2)\\ &\ll d(\delta)\sideset{}{'}\sum_{a}\frac{\mu^2(a)}{f_1^*(a)}\sum_{s|\delta/p}\left(P^*_{as}-P^*_{asp}\right)^2 + d(\delta)^2(\log R_2)^k\log\log R_2\\ &\ll d(\delta)(\log R_2)^2\sum_{s|\delta/p}\ \sideset{}{'}\sum_{a<R_2/sp}\frac{\mu^2(a)}{f_1^*(a)}\left(\tilde{P}\left(1-\frac{\log as}{\log R_2}\right)-\tilde{P}\left(1-\frac{\log asp}{\log R_2}\right)\right)^2\\ &\quad + d(\delta)(\log R_2)^2\sum_{s|\delta/p}\ \sideset{}{'}\sum_{R_2/sp\leq a<R_2/s}\frac{\mu^2(a)}{f_1^*(a)}\tilde{P}\left(1-\frac{\log as}{\log R_2}\right)^2\\ &\quad + d(\delta)^2(\log R_2)^k\log\log R_2. \end{aligned} \tag{3.6.15}$$

Noting the difference of the polynomials in the first sum is $\ll \log p/\log R_2$, and the polynomial in the second sum is $\ll 1$, we have

$$\begin{aligned} T^*_\delta &\ll d(\delta)(\log p)^2\sum_{s|\delta/p}\ \sideset{}{'}\sum_{a<R_2/sp}\frac{\mu^2(a)}{f_1^*(a)} + d(\delta)(\log R_2)^2\sum_{s|\delta/p}\ \sideset{}{'}\sum_{R_2/sp\leq a<R_2/s}\frac{\mu^2(a)}{f_1^*(a)}\\ &\quad + d(\delta)^2(\log R_2)^k\log\log R_2. \end{aligned} \tag{3.6.16}$$

Appealing to Lemma 3.9 as in the previous lemma we obtain

$$T^*_\delta \ll d(\delta)^2(\log p)(\log R_2)^k + d(\delta)^2(\log R_2)^k \log\log R_2. \tag{3.6.17}$$

The result for T_δ follows by a completely analogous argument. In this case the first line holds without the $O(d(\delta)^2(\log R_2)^k \log\log R_2)$ term, and so the final expression also holds without this term. □

With these results we are able to get an integral expression for T_δ and T^*_δ when δ has a bounded number of prime factors.

Lemma 3.13. *Let $p_1, \dots, p_{r-1} \nmid A$ for some primes $p_1, \dots, p_{r-1}$. Then we have*

$$\begin{aligned} T_{p_1\dots p_{r-1}} &= (\log R_2)^k \frac{\mathfrak{S}(\mathcal{L})}{(k-1)!} I_0\left(\frac{\log p_1}{\log R_2}, \dots, \frac{\log p_{r-1}}{\log R_2}\right) \\ &\quad + O_r((\log R_2)^{k-1} \log\log R_2), \\ T^*_{p_1\dots p_{r-1}} &= (\log R_2)^{k+1} \frac{\phi(A)\mathfrak{S}(\mathcal{L})}{A(k-2)!} I_1\left(\frac{\log p_1}{\log R_2}, \dots, \frac{\log p_{r-1}}{\log R_2}\right) \\ &\quad + O_r((\log R_2)^k \log\log R_2). \end{aligned}$$

Here

$$\begin{aligned} I_0(x_1, \dots, x_{r-1}) &= \int_0^1 \left(\sum_{J \subseteq \{1,\dots,r-1\}} P^+\left(1 - t - \sum_{j\in J} x_j\right)(-1)^{|J|}\right)^2 t^{k-1} dt, \\ I_1(x_1, \dots, x_{r-1}) &= \int_0^1 \left(\sum_{J \subseteq \{1,\dots,r-1\}} \tilde{P}^+\left(1 - t - \sum_{j\in J} x_j\right)(-1)^{|J|}\right)^2 t^{k-2} dt, \\ P^+(x) &= \begin{cases} P(x), & x \geq 0, \\ 0, & \textit{otherwise}, \end{cases} \\ \tilde{P}^+(x) &= \begin{cases} \int_0^x P(t)dt, & x \geq 0, \\ 0, & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. Let $\delta = p_1 \dots p_{r-1}$. By Lemmas 3.7 and 3.11 we have that

$$T^*_\delta = \sum_a{}' \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) P^*_{as} \right)^2 + O_r\left((\log R_2)^k \log\log R_2\right). \tag{3.6.18}$$

Substituting in the value P^*_a (and noting that $(\delta, A) = 1$) we obtain

$$\begin{aligned} T^*_\delta = \frac{\phi(A)^2}{A^2}(\log R_2)^2 \mathfrak{S}(\mathcal{L})^2 \sum_{(a,A)=1} \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|\delta} \mu(s) \tilde{P}^+\left(\frac{\log R_2/as}{\log R_2}\right)\right)^2 \\ + O_r\left((\log R_2)^k \log\log R_2\right). \end{aligned} \tag{3.6.19}$$

We also have

$$T_{p_1\dots p_{r-1}} = \mathfrak{S}(\mathcal{L})^2 \sum_{(a,A)=1} \frac{\mu^2(a)}{f_1(a)} \left(\sum_{s|p_1\dots p_{r-1}} \mu(s) P^+ \left(\frac{\log R_2/as}{\log R_2} \right) \right)^2 + O_r\left((\log R_2)^{k-1} \right). \quad (3.6.20)$$

We can now estimate the main term using Lemma 3.10. First we put

$$\begin{aligned}
\gamma(p) &= \begin{cases} \frac{p(k-1)}{p-1}, & p \nmid A, \\ 0, & \text{otherwise}, \end{cases} \\
g(d) &= \prod_{p|d} \frac{\gamma(p)}{p-\gamma(p)}, \qquad (3.6.21) \\
F(t) &= F_{x_1,\dots,x_{r-1}}(t) = \sum_{J\subseteq\{1,\dots,r-1\}} (-1)^{|J|} \tilde{P}^+ \left(t - \sum_{j\in J} x_j \right).
\end{aligned}$$

We note that by our normalisation, one has $\gamma(p) < p$ for all p, so $g(d)$ is always positive and well-defined.

If we put $x_i = \log p_i / \log R_2$ for each $i \in \{1, \dots, r-1\}$ then we see that

$$\sum_{(a,A)=1} \frac{\mu^2(a)}{f_1^*(a)} \left(\sum_{s|p_1\dots p_{r-1}} \mu(s) \tilde{P}^+ \left(\frac{\log R_2/as}{\log R_2} \right) \right)^2 = \sum_{d\le R_2} \mu^2(d) g(d) F\left(\frac{\log R_2/d}{\log R_2} \right). \quad (3.6.22)$$

Since F is a continuous piecewise differentiable function we can apply Lemma 3.10 which gives

$$\begin{aligned}
\sum_{d\le R_2} \mu^2(d) g(d) F\left(\frac{\log R_2/d}{\log R_2} \right) &= \frac{A(\log R_2)^{k-1}}{\phi(A)\mathfrak{S}(\mathcal{L})(k-2)!} \int_0^1 F(1-t) t^{k-2} dt \\
&\quad + O\left((\log R_2)^{k-2} \log\log R_2 \right). \qquad (3.6.23)
\end{aligned}$$

Similarly we follow the same procedure instead with

$$\begin{aligned}
\gamma(p) &= \begin{cases} k, & p \nmid A \\ 0, & \text{otherwise}, \end{cases} \\
G(t) &= \sum_{J\subseteq\{1,\dots,r-1\}} (-1)^{|J|} P^+ \left(t - \sum_{j\in J} x_j \right). \qquad (3.6.24)
\end{aligned}$$

This yields

$$\begin{aligned}
&\sum_{(a,A)=1} \frac{\mu^2(a)}{f_1(a)} \left(\sum_{s|p_1\dots p_{r-1}} \mu(s) P^+ \left(\frac{\log R_2/as}{\log R_2} \right) \right)^2 \\
&\qquad = \frac{(\log R_2)^k}{\mathfrak{S}(\mathcal{L})(k-1)!} \int_0^1 G(1-t)^2 t^{k-1} dt + O\left((\log R_2)^{k-1} \log\log R_2 \right). \qquad \square
\end{aligned}$$

We also require a bound on the size of the sieve coefficients λ_d.

Lemma 3.14. *We have that*

$$\lambda_d \ll (\log R_2)^k.$$

Proof. This is proven in [17, Proof of Theorem 7]. □

We finish this section with a partial summation lemma, which will be useful later on.

Lemma 3.15. *Let $0 \leq a < b$ be fixed constants. Let $V : [a,b] \to \mathbb{R}_{\geq 0}$ be a continuous piecewise smooth function. If V satisfies $V(x) \ll x$ uniformly for $x \in [a,b]$ then we have*

$$\sum_{R^a \leq p \leq R^b} \frac{1}{p} V\left(\frac{\log p}{\log R}\right) = \int_a^b \frac{V(u)}{u} du + O\left(\frac{M(V) \log\log R}{\log R}\right),$$

where

$$M(V) = \sup_{t \in [a,b]} \left(1 + |V'(t)|\right).$$

Proof. The result follows straightforwardly by partial summation and the prime number theorem.

If $a = 0$ then we replace a with $\log 2/\log R$. This leaves the left hand side of the result unchanged, and introduces an error

$$\int_a^{\log 2/\log R} \frac{V(u)}{u} du \ll \frac{1}{\log R} \tag{3.6.25}$$

to the right hand side, which can be absorbed into the error term.

By the prime number theorem

$$\pi(y) = \frac{y}{\log y}\left(1 + O\left(\frac{1}{\log y}\right)\right). \tag{3.6.26}$$

Therefore, by partial summation we have

$$\begin{aligned}\sum_{R^a \leq p_j \leq R^b} \frac{1}{p} V\left(\frac{\log p}{\log R}\right) &= O\left(\frac{1}{\log R}\right) + \int_{R^a}^{R^b} \frac{t}{t^2 \log t} V\left(\frac{\log t}{\log R}\right)\left(1 + O\left(\frac{1}{\log t}\right)\right) dt \\ &\quad - \int_{R^a}^{R^b} \frac{t}{t^2 (\log t)(\log R)} V'\left(\frac{\log t}{\log R}\right)\left(1 + O\left(\frac{1}{\log t}\right)\right) dt \\ &= \int_a^b \frac{V(u)}{u} du + O\left(\int_a^b \frac{1 + |V'(u)|}{u \log R} du\right) + O\left(\frac{1}{\log R}\right) \\ &= \int_a^b \frac{V(u)}{u} du + O\left(\frac{M(V) \log\log R}{\log R}\right).\end{aligned}$$

□

3.7 Proof of Propositions 3.3-3.5

3.7.1 Proof of Proposition 3.3

We consider the weighted sum of Proposition 3.3 in a similar way to previous work on Selberg's $\Lambda^2\Lambda^-$ sieve, which in its basic form considers the weight $W_0(x) = -1$. We have

$$\sum_{N\le n\le 2N}\left(\sum_{\substack{p|\Pi(n)\\ p\le R_1}} W_0\left(\frac{\log p}{\log R_2}\right)\right)\left(\sum_{\substack{d|\Pi(n)\\ d\le R_2}}\lambda_d\right)^2 = \sum_{p\le R_1} W_0\left(\frac{\log p}{\log R_2}\right)\sum_{d,e\le R_2}\lambda_d\lambda_e\sum_{\substack{N\le n\le 2N\\ [p,d,e]|\Pi(n)}} 1$$

$$= N\sideset{}{'}\sum_{p\le R_1} W_0\left(\frac{\log p}{\log R_2}\right)\sideset{}{'}\sum_{d,e\le R_2}\frac{\lambda_d\lambda_e}{f([d,e,p])} + O_{W_0}(E_1)$$

$$= N\sideset{}{'}\sum_{p\le R_1} W_0\left(\frac{\log p}{\log R_2}\right)\frac{T_p}{f(p)} + O_{W_0}(E_1), \tag{3.7.1}$$

where

$$E_1 = \sum_{p\le R_1}\sum_{d,e\le R_2}\left|\lambda_d\lambda_e r_{[d,e,p]}\right|, \qquad r_d = \sum_{\substack{N<n\le 2N\\ d|\Pi(n)}} 1 - \frac{N}{f(d)}. \tag{3.7.2}$$

By Lemma 3.14 we have $\lambda_d \ll (\log N)^k$, and we note that $r_d \le k^{\omega(d)}$. Therefore, we have

$$E_1 \ll (\log N)^{2k}\sum_{\substack{p\le R_1\\ d,e\le R_2}}\mu^2([d,e,p])k^{\omega([d,e,p])}$$

$$\ll (\log N)^{2k}\sum_{r\le R_2^2R_1}\mu^2(r)(7k)^{\omega(r)}$$

$$\ll (\log N)^{2k}R_2^2R_1\sum_{r\le R_2^2R_1}\frac{\mu^2(r)(7k)^{\omega(r)}}{r}$$

$$\ll (\log N)^{2k}R_2^2R_1\prod_{p\le R_2^2R_1}\left(1+\frac{7k}{p}\right)$$

$$\ll (\log N)^{9k}R_2^2R_1. \tag{3.7.3}$$

Thus for $R_2^2R_1 \le N(\log N)^{-9k}$ we have $E_1 \ll N$.

By Lemma 3.13 we have

$$T_p = (\log R_2)^k\frac{\mathfrak{S}(\mathcal{L})}{(k-1)!}I_0\left(\frac{\log p}{\log R_2}\right) + O\Big((\log N)^{k-1}\log\log N\Big), \tag{3.7.4}$$

where

$$I_0(x) = \int_0^1\left(P_1^+(1-t) - P_1^+(1-t-x)\right)^2 t^{k-1}dt. \tag{3.7.5}$$

Recalling that $f(p) = p/k$ for $p \nmid A$, we see that the error terms from T_p contribute

$$\ll_{W_0} (\log N)^{k-1} \log\log N \sum_{p \leq R_1} \frac{1}{p} \ll (\log N)^{k-1} (\log\log N)^2. \tag{3.7.6}$$

Therefore we are left to estimate the sum

$$\sum_{p \leq R_1}{}' \frac{1}{p} W_0\left(\frac{\log p}{\log R_2}\right) I_0\left(\frac{\log p}{\log R_2}\right). \tag{3.7.7}$$

We note that if $t \leq 1 - x$ then $P^+(1-t) - P^+(1-t-x) \ll x$, and so

$$I_0(x) \ll x. \tag{3.7.8}$$

If $1 - x \leq t \leq 1$ then since the interval has length x we also have

$$I_0(x) \ll x. \tag{3.7.9}$$

By the piecewise smoothness of $I_0(x)$ and $W_0(x)$ we have uniformly for $x \in [0, r_1/r_2]$

$$I_0'(x) \ll 1, \qquad W_0'(x) \ll_{W_0} 1. \tag{3.7.10}$$

Therefore by Lemma 3.15, we have

$$\sum_{p \leq R_1} \frac{1}{p} W_0\left(\frac{\log p}{\log R_2}\right) I_0\left(\frac{\log p}{\log R_2}\right) = \int_0^{r_1/r_2} \frac{W_0(u)}{u} I_0(u) du + O_{W_0}\left(\frac{\log\log N}{\log N}\right). \tag{3.7.11}$$

By (3.7.8) we see that the contribution to the above sum for primes which divide A is

$$\ll \frac{1}{\log N}. \tag{3.7.12}$$

This gives the result.

3.7.2 Proof of Proposition 3.4

We will follow a similar argument to that of Graham *et al* [17] where the result was obtained with $r = 2$ and $W_2(x_1, x_2) = 1$. Thorne [47] extended this in the natural way to consider $r > 2$, again without the weighting W_r. In order to introduce the weighting by W_r, it is necessary to establish a Bombieri-Vinogradov style result for numbers with r prime factors weighted by W_r.

Lemma 3.16. *Let*

$$\beta_r(n) = \begin{cases} W_r\left(\frac{\log p_1}{\log n}, \dots, \frac{\log p_{r-1}}{\log n}\right), & n = p_1 p_2 \dots p_r \text{ with } p_1 \le \dots \le p_r, \\ 0, & \text{otherwise,} \end{cases}$$

for some piecewise smooth function $W_r : [0,1]^{r-1} \to \mathbb{R}$.

Let

$$\Delta_{\beta,r}(x;q) = \max_{y \le x} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{y<n\le 2y \\ n\equiv a \pmod q}} \beta_r(n) - \frac{1}{\phi(q)} \sum_{\substack{y<n\le 2y \\ (n,q)=1}} \beta_r(n) \right|. \tag{3.7.13}$$

For every fixed integer $h > 0$, *and for every* $C > 0$ *there exists a constant* $C' = C'(C,h)$ *such that if* $Q \le x^{1/2}(\log x)^{-C'}$ *then we have*

$$\sum_{q \le Q} \mu^2(q) h^{\omega(q)} \Delta_{\beta,r}(x;q) \ll_{C,h,W_r} x(\log x)^{-C}. \tag{3.7.14}$$

Proof. This result follows from the Bombieri-Vinogradov theorem for numbers with exactly r prime factors, as proven by Motohashi [39], and the continuity of W_r.

We assume that W_r is smooth. The result can be extended to piecewise smooth functions by taking smooth approximations.

We fix a constant $C > 0$, an integer h, and a function W_r.

We let

$$\chi_{\delta,\eta}(n) = \begin{cases} 1, & n = p_1 p_2 \dots p_r \text{ with } n^{\eta_i} \le p_i \le n^{\delta_i} \ \forall i \\ & \quad \text{and } p_1 < p_2 < \dots < p_r. \\ 0, & \text{otherwise.} \end{cases} \tag{3.7.15}$$

By Motohashi's result [39, Theorem 2], we have that uniformly for any choice of constants δ_i and η_i ($i = 1, \dots, r$) there is a constant $C' = C'(C,h)$ such that if $Q \le x^{1/2}(\log x)^{-C'}$ then we have

$$\sum_{q \le Q} \mu^2(q) h^{\omega(q)} \max_{\substack{y,a \\ y \le x \\ (a,q)=1}} \left| \sum_{\substack{y\le n\le 2y \\ n\equiv a \pmod q}} \chi_{\delta,\eta}(n) - \frac{1}{\phi(q)} \sum_{\substack{y\le n\le 2y \\ (n,q)=1}} \chi_{\delta,\eta}(n) \right| \ll_{C,h} x(\log x)^{-(C+h)(r+1)}. \tag{3.7.16}$$

We choose $\delta_i \in \{(\log x)^{-C-h}, 2(\log x)^{-C-h}, \dots, \lceil(\log x)^{C+h}\rceil(\log x)^{-C-h}\}$ separately for each $i \in \{1, \dots, r\}$, subject to the constraint $\delta_i \le \delta_{i+1}$ $(1 \le i \le r-1)$. For each choice of the δ_i we take $\eta_i = \delta_i - (\log x)^{-C-h}$ for $1 \le i \le r$. We put

$$W_r(\delta) = W_r(\delta_1, \delta_2, \dots, \delta_{r-1}). \tag{3.7.17}$$

We notice that by the smoothness of W_r we have that

$$\begin{aligned}\beta_r(n) &= \sum_{\delta} \chi_{\delta,\eta}(n)\left(W_r(\delta) + O((\log x)^{-C-h})\right) \\ &= \sum_{\delta} \chi_{\delta,\eta}(n) W_r(\delta) + O\left((\log x)^{-C-h}\right). \end{aligned} \tag{3.7.18}$$

Here $\sum_\delta$ indicates a sum over all the $O((\log x)^{r(C+h)})$ possible choices of the δ_i.

Therefore, we have that

$$\begin{aligned}\sum_{\substack{y\le n\le 2y \\ n\equiv a \pmod q}} \beta_r(n) - \frac{1}{\phi(q)} \sum_{\substack{y\le n\le 2y \\ (n,q)=1}} \beta_r(n) &= \sum_{\delta} W_r(\delta)\left(\sum_{\substack{y\le n\le 2y \\ n\equiv a \pmod q}} \chi_{\delta,\eta}(n) - \frac{1}{\phi(q)} \sum_{\substack{y\le n\le 2y \\ (n,q)=1}} \chi_{\delta,\eta}(n)\right) \\ &\quad + O\left((\log y)^{-C-h}\frac{y}{\phi(q)}\right). \end{aligned} \tag{3.7.19}$$

Thus, for $Q \le x(\log x)^{-C'}$ we have

$$\begin{aligned}&\sum_{q\le Q} \mu^2(r) h^{\omega(q)} \Delta_{\beta,r}(x;q) \\ &\qquad \le \sum_{\delta} W_r(\delta) \sum_{q\le Q} \mu^2(r) h^{\omega(q)} \max_{\substack{a,y \\ y\le x \\ (a,q)=1}} \left| \sum_{\substack{y\le n\le 2y \\ n\equiv a \pmod q}} \chi_{\delta,\eta}(n) - \frac{1}{\phi(q)} \sum_{\substack{y\le n\le 2y \\ (n,q)=1}} \chi_{\delta,\eta}(n) \right| \\ &\qquad\quad + O\left((\log x)^{-(C+h)} \sum_{q\le Q} \mu^2(q) h^{\omega(q)} \frac{x}{\phi(q)}\right) \\ &\qquad \ll \sum_{\delta} W_r(\delta) x(\log x)^{-(C+h)(r+1)} + x(\log x)^{-(C+h)} \prod_{p\le Q}\left(1 + \frac{h}{p-1}\right) \\ &\qquad \ll x(\log x)^{-C}. \end{aligned}$$

□

With this, we can adapt the argument of Thorne [47] slightly to rewrite the main term in terms of the quantities T_q^*.

Lemma 3.17. *We have*

$$\begin{aligned}\sum_{N\le n\le 2N} \beta_r(L_j(n))\left(\sum_{d|\Pi(n)} \lambda_d\right)^2 &= \frac{AN}{\phi(A)(\log N)} \sum_{\substack{p_1,\dots,p_{r-1} \\ N^\epsilon < p_1 < p_2 < \dots < p_{r-1} \\ q < \min(N/R_2, N/p_{r-1})}} \frac{T_q^*}{q} \alpha\left(\frac{\log p_1}{\log R_2}, \dots, \frac{\log p_{r-1}}{\log R_2}\right) \\ &\quad + O_{W_r}\left(N(\log N)^{k-1}(\log\log N)^{r-1}\right), \end{aligned}$$

where

$$q = \prod_{i=1}^{r-1} p_i,$$

$$T_\delta^* = \sum_{\substack{d,e \\ (d,A)=(e,A)=1}} \frac{\lambda_d \lambda_e}{f^*([d,e,\delta]/\delta)},$$

$$\alpha(x_1, \dots, x_{r-1}) = \frac{W_r(r_2 x_1, \dots, r_2 x_{r-1})}{1 - r_2 \sum_{i-1}^{r-1} x_i}.$$

Proof. Thorne [47] considers essentially the same sum but without the weighting by W_r. In his argument up until equation (4.14) on Page 23, this difference only affects the argument when he appeals to the Bombieri-Vinogradov theorem for E_h numbers (where $h \le r$). Lemma 3.16 gives the equivalent Bombieri-Vinogradov style result when weighting by W_r, and so exactly the same argument follows through. The only additional assumption of Thorne is that he restricts the consideration to numbers $n = p_1 \dots p_r$ satisfying

$$\exp(\sqrt{\log N}) < p_1 < \cdots < p_r \qquad \text{and} \qquad R_2 < p_r. \tag{3.7.20}$$

This is satisfied if for a fixed $\epsilon > 0$ we require W_r to be supported on

$$\mathcal{A}_r = \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < \min(1 - r_2, 1 - x_{r-1}) \right\}. \tag{3.7.21}$$

This gives us in our case (the equivalent of Thorne's equation (4.14), but with the explicit error term he calculates)

$$\begin{aligned} \sum_{N \le n \le 2N} \beta_r(L_j(n)) \left(\sum_{d | \Pi(n)} \lambda_d \right)^2 = {} & \sum_{d,e}{}' \lambda_d \lambda_e \sum_{p_1, \dots, p_{r-1}}^{*} \frac{d_{k-1}([d,e,q]/q)}{\phi(a_j [d,e,q]/q)} \\ & \times \sum_{\substack{a_j N/q \le m \le 2 a_j N/q \\ m \text{ prime}}} W_r \left(\frac{\log p_1}{\log mq}, \dots, \frac{\log p_{r-1}}{\log mq} \right) \\ & + O(N). \end{aligned} \tag{3.7.22}$$

Here and from now on we use the symbol $\sum^*$ to indicate that we are summing over primes $p_1, \dots, p_{r-1}$ with

$$\left(\frac{\log p_1}{\log N}, \dots, \frac{\log p_{r-1}}{\log N} \right) \in \mathcal{A}_r, \tag{3.7.23}$$

and to ease notation we have put

$$q = \prod_{i=1}^{r-1} p_i. \tag{3.7.24}$$

Again we assume for simplicity that W_r is smooth. By taking smooth approximations one can establish the result for piecewise-smooth W_r.

Estimating the inner sum gives

$$\begin{aligned}
&\sum_{\substack{a_jN/q\le m\le 2a_jN/q\\ m\text{ prime}}} W_r\left(\frac{\log p_1}{\log mq},\dots,\frac{\log p_{r-1}}{\log mq}\right)\\
&\quad=\left(W_r\left(\frac{\log p_1}{\log N},\dots,\frac{\log p_{r-1}}{\log N}\right)+O\left(\frac{1}{\log N}\right)\right)\left(\pi\left(\frac{2a_jN}{q}\right)-\pi\left(\frac{a_jN}{q}\right)\right)\\
&\quad=W_r\left(\frac{\log p_1}{\log N},\dots,\frac{\log p_{r-1}}{\log N}\right)\frac{a_jN}{\log N}\left(\frac{\log N}{\log N-\log q}\right)\left(1+O\left(\frac{1}{\log N}\right)\right).
\end{aligned}\tag{3.7.25}$$

We note that (by Hypothesis 3.2) if $d|\Pi(n)$ then $(d,A)=1$. Therefore $(a_j,[d,e,q]/q)=1$, so $\phi(a_j[d,e,q]/q)=\phi(a_j)\phi([d,e,q]/q)$. Together these give

$$\begin{aligned}
&\sum_{N\le n\le 2N}\beta_r(L_j(n))\left(\sum_{d|\Pi(n)}\lambda_d\right)^2\\
&\quad=\frac{a_jN}{\phi(a_j)\log N}\sideset{}{^*}\sum_{p_1,\dots,p_{r-1}}\frac{T_q^*W_r\left(\frac{\log p_1}{\log N},\dots,\frac{\log p_{r-1}}{\log N}\right)\log N}{q(\log N-\log q)}(1+O((\log N)^{-1}))\\
&\qquad+O(N)\\
&\quad=\frac{a_jN}{\phi(a_j)\log N}\sideset{}{^*}\sum_{p_1,\dots,p_{r-1}}\frac{T_q^*}{q}\alpha\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_{r-1}}{\log R_2}\right)\left(1+O\left(\frac{1}{\log N}\right)\right)\\
&\qquad+O(N),
\end{aligned}\tag{3.7.26}$$

where

$$\alpha(x_1,\dots,x_{r-1})=\frac{W_r(r_2x_1,\dots,r_2x_{r-1})}{1-r_2\sum_{i-1}^{r-1}x_i}.\tag{3.7.27}$$

We note that a_j and A are composed of the same prime factors, so $a_j/\phi(a_j)=A/\phi(A)$. Therefore the main term is that of the Lemma.

By Lemma 3.12 we have

$$T_q^*\ll_r(\log N)^k\log p_1+(\log N)^k\log\log N.\tag{3.7.28}$$

We also have

$$\alpha(x_1,\dots,x_{r-1})\ll_{W_r}1.\tag{3.7.29}$$

Thus the $O(1/\log N)$ term contributes

$$\ll_{W_r,r}N(\log N)^{k-2}\sideset{}{^*}\sum_{p_1,\dots,p_{r-1}}\frac{\log p_1+\log\log N}{p_1\dots p_{r-1}}\ll_{W_r,r}N(\log N)^{k-1}(\log\log N)^{r-2}.\tag{3.7.30}$$

This gives the result. □

Lemma 3.18. *We have*

$$\sum_{p_1,\ldots,p_{r-1}}^{*} \frac{T_q^*}{q}\alpha\left(\frac{\log p_1}{\log R_2},\ldots,\frac{\log p_{r-1}}{\log R_2}\right)$$
$$= (\log R_2)^{k+1}\frac{\phi(A)\mathfrak{S}(\mathcal{L})}{A(k-2)!}\int\cdots\int \frac{I_1(u_1,\ldots,u_{r-1})\alpha(u_1,\ldots,u_{r-1})}{u_1u_2\ldots u_{r-1}}du_1\ldots du_{r-1}$$
$$+ O\Big((\log\log N)^r(\log N)^k\Big), \tag{3.7.31}$$

where the integration is subject to the constraints

$$\epsilon < u_1 < \cdots < u_{r-1}, \qquad \text{and} \qquad \sum_{i=1}^{r-1} u_i \le \min(r_2^{-1}-1, r_2^{-1}-u_{r-1}). \tag{3.7.32}$$

Proof. By Lemma 3.13, for $q = p_1p_2\ldots p_{r-1}$ we have

$$T_q^* = (\log R_2)^{k+1}\frac{\phi(A)\mathfrak{S}(\mathcal{L})}{A(k-2)!}I_1\left(\frac{\log p_1}{\log R_2},\ldots,\frac{\log p_{r-1}}{\log R_2}\right) + O_r((\log N)^k\log\log N). \tag{3.7.33}$$

Thus summing the error term over $p_1,\ldots p_{r-1}$ gives a contribution

$$\sum_{p_1,\ldots,p_{r-1}}^{*} \frac{1}{q}\alpha\left(\frac{\log p_1}{\log R_2},\ldots,\frac{\log p_{r-1}}{\log R_2}\right)(\log N)^k\log\log N$$
$$\ll_{W_r} (\log N)^k\log\log N\left(\sum_{p\le N}\frac{1}{p}\right)^{r-1}$$
$$\ll_{W_r} (\log N)^k(\log\log N)^r. \tag{3.7.34}$$

We are therefore left to evaluate the main term

$$\sum_{p_1,\ldots,p_{r-1}}^{*} \frac{1}{q}\alpha\left(\frac{\log p_1}{\log R_2},\ldots,\frac{\log p_{r-1}}{\log R_2}\right)I_1\left(\frac{\log p_1}{\log R_2},\ldots,\frac{\log p_{r-1}}{\log R_2}\right). \tag{3.7.35}$$

We will now apply Lemma 3.15 to $p_{r-1},\ldots,p_1$ in turn to estimate the sum (3.7.35).

For $u_1,\ldots,u_j \in [0, r_2^{-1}]$ we put

$$V_j(u_1,\ldots,u_j) = \int\cdots\int \frac{1}{\prod_{i=j+1}^{r-1}u_i}\alpha(u_1,\ldots,u_{r-1})\,I_1(u_1,\ldots,u_{r-1})\,du_{j+1}\ldots du_{r-1}, \tag{3.7.36}$$

where the integration is subject to $u_j < u_{j+1} < \cdots < u_{r-1}$ and $\sum_1^{r-1}u_i \le \min(r_2^{-1}-1, r_2^{-1}-u_{r-1})$.

As in the proof of Lemma 3.12, since $\tilde{P}$ is continuous and its derivative is uniformly bounded on $[0,1]$, we have that

$$
\begin{aligned}
I_1(u_1,\dots,u_{r-1}) &= \int_0^1 \left(\sum_{J\subseteq\{1,\dots,r-1\}} \tilde{P}^+\left(1-t-\sum_{i\in J}u_i\right)(-1)^{|J|}\right)^2 t^{k-2}dt\\
&\ll_r \int_0^1 \sum_{J\subseteq\{1,\dots,r-1\}\setminus\{j\}} \left(\tilde{P}^+\left(1-t-\sum_{i\in J}u_i\right)-\tilde{P}^+\left(1-t-u_j-\sum_{i\in J}u_i\right)\right)^2 t^{k-2}dt\\
&\ll_r u_j^2. \qquad (3.7.37)
\end{aligned}
$$

Thus, since $\alpha(u_1,\dots,u_{r-1})\ll 1$, we have uniformly for $u_1,\dots,u_j\in[0,r_2^{-1}]$

$$
\begin{aligned}
V_j(u_1,\dots,u_j) &\ll u_j^2\int\dots\int \frac{1}{\prod_{i=j+1}^{r-1}u_i}du_{j+1}\dots du_{r-1}\\
&\ll u_j^2(1+|\log 1/u_j|^r)\\
&\ll u_j. \qquad (3.7.38)
\end{aligned}
$$

Moreover, essentially the same argument shows that uniformly for $u_1,\dots,u_j\in[0,r_2^{-1}]$ we have

$$\frac{\partial}{\partial u_j}I_1(u_1,\dots,u_{r-1})\ll_r u_j. \qquad (3.7.39)$$

Thus since

$$\frac{\partial}{\partial u_j}\alpha(u_1,\dots,u_{r-1})\ll_r 1 \qquad (3.7.40)$$

we have that

$$
\begin{aligned}
\frac{\partial}{\partial u_j}V_j(u_1,\dots,u_j) &\ll u_j\int\cdots\int \frac{1}{\prod_{i=j+1}^{r-1}u_i}du_{j+1}\dots du_{r-1}\\
&\ll 1. \qquad (3.7.41)
\end{aligned}
$$

Thus the condition of Lemma 3.15 applies for the function V_j. Applying Lemma 3.15 in turn to $V_{r-1},V_{r-2},\dots,V_1$ gives the result. We note that the error terms contribute a total which is $\ll(\log N)^k(\log\log N)^{r-1}$. □

3.7.3 Proof of Proposition 3.5

By Lemma 3.14, we have $\lambda_d \ll (\log N)^k$. Therefore we have

$$\sum_{p\le AN^{1/2}} \sum_{\substack{N\le n\le 2N \\ p^2|\Pi(n)}} \left(\sum_{\substack{d|\Pi(n) \\ d\le R_2}} \lambda_d \right)^2 = N \sum_{p\le AN^{1/2}} \sum_{d,e\le R_2}{}' \frac{\lambda_d \lambda_e}{f([d,e,p^2])} + O\left(\sum_{p\le AN^{1/2}} \sum_{d,e\le R_2} |\lambda_d \lambda_e r_{[d,e,p^2]}| \right)$$

$$\ll N \sum_{p\le N} \frac{T_p}{p^2} + O\left((\log N)^{2k} \sum_{r\le R_2^2 AN^{1/2}} \mu^2(r)(7k)^{\omega(r)} \right). \qquad (3.7.42)$$

We first bound the error term

$$\sum_{r\le R_2^2 AN^{1/2}} \mu^2(r)(7k)^{\omega(r)} \ll R_2^2 N^{1/2} \sum_{r\le AR_2^2 N^{1/2}} \frac{\mu^2(r)(7k)^{\omega(r)}}{r}$$

$$\ll R_2^2 N^{1/2} \prod_{p\le AR_2^2 N^{1/2}} \left(1 + \frac{7k}{p}\right)$$

$$\ll R_2^2 N^{1/2} (\log N)^{7k}. \qquad (3.7.43)$$

Thus for $R_2 \le N^{1/4}(\log N)^{-5k}$ the error term is $O(N)$.

By Lemma 3.12, we have that

$$T_p \ll (\log N)^{k-1} \log p + (\log N)^{k-1} \log\log N. \qquad (3.7.44)$$

Thus

$$\sum_{p\le AN^{1/2}} \sum_{\substack{N\le n\le 2N \\ p^2|\Pi(n)}} \left(\sum_{\substack{d|\Pi(n) \\ d\le R_2}} \lambda_d \right)^2 \ll N(\log N)^{k-1} \sum_{p\le N} \frac{\log p + \log\log N}{p^2} + O(N)$$

$$\ll N(\log N)^{k-1} \log\log N. \qquad (3.7.45)$$

Chapter 4

Almost-prime 3-tuples

4.1 Introduction

In this chapter we focus on the $k = 3$ case of the prime k-tuples conjecture. We consider a set of distinct admissible integer linear functions

$$L_i(x) = a_i x + b_i, \qquad i \in \{1, 2, 3\}, \tag{4.1.1}$$

and recall that the condition that $\{L_1, L_2, L_3\}$ is admissible means that for every prime p there is an integer n_p such that $L_1(n_p)L_2(n_p)L_3(n_p)$ is coprime to p. We are interested in the following conjecture.

Conjecture (Prime k-tuples Conjecture, $k = 3$)**.** *Let $\{L_1, L_2, L_3\}$ be admissible. Then there are infinitely many integers n for which each of $L_1(n)$, $L_2(n)$, and $L_3(n)$ are prime.*

Although we cannot hope to prove that the functions are simultaneously prime infinitely often with the current technology, we can use sieve methods are able to show that L_1, L_2, and L_3 are simultaneously *almost prime* infinitely often.

Chen's result [5] that any pair of admissible functions have at most 3 prime factors infinitely often is the best bound we can hope to prove with the current methods (since an improvement from 3 to 2 in the bound would imply the prime k-tuples conjecture for $k = 2$). When we only consider a triple of functions, however, we do not necessarily have the best bounds that the current techniques can produce. We would not expect to be able to prove that $\Omega(L_1(n)L_2(n)L_3(n)) \leq 4$ infinitely often (since this would imply that two of the linear functions are simultaneously prime infinitely often), but *a priori* there is no reason why we should think it impossible to prove a bound of 5 instead of 4.

The previous best result on triples of functions was a result of Porter [43] from 1972 that showed for any admissible triple $\{L_1, L_2, L_3\}$ there are infinitely many integers n such that the product $L_1(n)L_2(n)L_3(n)$ has at most 8 prime factors. In this chapter we improve this result, improving the bound on the number of prime factors from 8 to 7.

4.2 Main result

Theorem 4.1. *Let $\mathcal{L} = \{L_1, L_2, L_3\}$ be an admissible 3-tuple of distinct integer linear functions. Then there are infinitely many n for which the product $L_1(n)L_2(n)L_3(n)$ has at most 7 prime factors.*

Results such as Theorem 4.1 which show that k-tuples take few prime factors infinitely often tend to use weighted sieves. When $k = 2$ the best results fix one of the functions to attain a prime value, and use a $(k-1)$-dimensional sieve to show that the remaining functions have few prime factors infinitely often. When $k \geq 4$ better bounds are obtained by instead using a k-dimensional sieve on all the factors. When $k = 3$ both approaches can show that a 3-tuple has at most 8 prime factors infinitely often. In this chapter we mix these two approaches, using k-dimensional sieves to estimate some terms and $(k-1)$-dimensional sieves to estimate other terms. This is the key innovation which allows us to reduce the bound r_3 from 8 to 7.

A similar mixed approach can be used to gain an improvement over the approach of Diamond and Halberstam when $k \geq 4$, but in this case we find it is superior to use a different argument based on the Selberg sieve, given in Chapter 3.

Theorem 4.1 follows almost immediately from the following proposition, which may be viewed as a sharpening of Diamond and Halberstam's Theorem 11.1 in [10], tailored to our specific application.

Proposition 4.2. *Let $\mathcal{L} = \{L_1, \dots, L_k\}$ be an admissible k-tuple of distinct integer linear functions. Let u, v be fixed real numbers satisfying $1 < v/(v-1) < u < v$ and $v > \beta_k$. Let r_k be a natural number satisfying*

$$r_k > \max(N(u, v; k), uk - 1)$$

where

$$N(u,v;k) = uk - 1 + \frac{k}{f_k(v)}\left(I_1 - I_2 - \frac{e^\gamma(u-1)}{v} f_{k-1}\left(\frac{v}{2}\right)\right),$$
$$I_1 = \int_{1/v}^{1/u} \min\left(F_k(v-vs), e^\gamma F_{k-1}\left(\frac{v}{2}\right) w(v-vs)\right)\frac{1-us}{s}ds,$$
$$I_2 = \int_{1/u}^{1-1/v} \max\left(f_k(v-vs), e^\gamma f_{k-1}\left(\frac{v}{2}\right) w(v-vs)\right)\frac{us-1}{s}ds.$$

Then there are infinitely many n for which the product $L_1(n)\dots L_k(n)$ has at most r_k prime factors.

Here F_k and f_k are the upper and lower sieve functions from the k-dimensional Diamond-Halberstam-Richert sieve[1]*, and β_k is the sifting limit of f_k. The function $w(u)$ is the Buchstab function defined by the delay differential equation given by* (4.4.39).

In [10, Theorem 11.1] one has a similar result but instead of $N(u,v;k)$ one has the expression

$$uk - 1 + \frac{k}{f_k(v)}\int_{1/v}^{1/u} F_k(v-vs)\frac{1-us}{s}ds. \tag{4.2.1}$$

We can see that certainly whenever $f_{k-1}(v/2) > 0$ Proposition 4.2 gives a superior bound. The optimal choice of u and v when using [10, Theorem 11.1] with $k = 3$ is approximately $u = 1.5$, $v = 12$, and since with these values $f_2(v/2) > 0$ we expect a small improvement.

Using numerical integration we can establish Theorem 4.1 from Proposition 4.2. Given an admissible 3-tuple of integer linear functions, we apply Proposition 4.2 with $u = 2$ and $v = 12$.

To compute the bound $N(2,12;3)$ we used the *Mathematica* ®[2] technical computing software. We used the package written by William Galway (this package is available at `http://www.math.uiuc.edu/SieveTheoryBook`) to calculate the sieve functions f_k and F_k,

[1]The functions f_k and F_k are described in detail in [10, Chapter 6]. They are the solutions to the delay-differential equations

$$\begin{aligned}
F_k(u) &= 1/j_k(u/2), && 0 \le u \le \alpha_k,\\
f_k(u) &= 0, && 0 \le u \le \beta_k,\\
j_k(u) &= e^{-\gamma k}u^k/\Gamma(k+1), && 0 \le u \le 1,\\
(u^k F_k(u))' &= ku^{k-1} f_k(u-1), && \alpha_k \le u,\\
(u^k f_k(u))' &= ku^{k-1} F(u-1), && \beta_k \le u,\\
u j_k'(u) &= k j_k(u) - k j_k(u-1), && u \ge 1.
\end{aligned}$$

[2]*Mathematica* is a registered trademark of Wolfram Research, Inc.

and the code written by Michael Trott (available at `http://mathworld.wolfram.com/notebooks/Combinatorics/BuchstabFunction.nb`) to calculate the Buchstab function $w(u)$. We used the following code to calculate the integrals I_1 and I_2 and the bound $N(u, v; k)$:

```
I1[u_, v_, k_] := NIntegrate[Min[UpperSieveFunc[k, v − v∗s],Exp[EulerGamma]∗UpperSieveFunc[k − 1, v
    /2]∗BuchstabFunction[v − v∗s]]∗(1 − u∗s)/s, {s, 1/v, 1/u}];

I2[u_, v_, k_] := NIntegrate[Max[LowerSieveFunc[k, v − v∗s],Exp[EulerGamma]∗LowerSieveFunc[k − 1, v
    /2]∗BuchstabFunction[v − v∗s]]∗(u∗s −1)/s, {s, 1/u, 1 − 1/v}];

NBound[u_, v_, k_] := u∗k − 1 + k/LowerSieveFunc[k, v]∗(I1[u, v, k] − I2[u,v, k] −Exp[EulerGamma]∗(u
    − 1)/v∗LowerSieveFunc[k − 1, v/2]);
```

We can then calculate that

$$N(2, 12; 3) = 6.943\ldots \tag{4.2.2}$$

and so we may take $r_k = 7$ in Proposition 4.2.

We note that the proof of Proposition 4.2 makes much use of the fact we are dealing with integer linear functions for which much more is known about the distribution of the values they take - Diamond and Halberstam's Theorem 11.1 holds in much more general circumstances. This also gives no change to the asymptotic bound of $k \log k + O(k)$ for the number of prime factors of a k-tuple when k is large.

4.3 Proof of Proposition 4.2

To simplify the argument we adopt a normalisation of our functions, as we did in Chapter 3. Since we are only interested in the showing any admissible k-tuple has at most r_k prime factors infinitely often (for some explicit r_k), by considering the functions $L_i(An+B)$ instead of $L_i(n)$ for suitably chosen constants A and B, we may assume without loss of generality that our functions satisfy the following hypothesis.

Hypothesis 4.3. *$\mathcal{L} = \{L_1, \ldots, L_k\}$ is an admissible k-tuple of linear functions. The functions $L_i(n) = a_i n + b_i$ ($1 \le i \le k$) are distinct with $a_i > 0$. Each of the coefficients a_i is composed of the same primes none of which divides the b_j. If $i \ne j$, then any prime factor of $a_i b_j - a_j b_i$ divides each of the a_l.*

Let $\mathcal{L} = \{L_1, \ldots, L_k\}$ be an admissible k-tuple satisfying Hypothesis 4.3. We view this k-tuple as fixed, and so any implied constants from $\ll$ or O-notation may depend on the

k-tuple without explicit reference. We define

$$\Pi(n) = \prod_{i=1}^{k} L_i(n), \tag{4.3.1}$$

$$A = \prod_{i=1}^{k} a_i, \tag{4.3.2}$$

$$v_p(\mathcal{L}) = \#\{1 \le n < p : \Pi(n) \equiv 0 \pmod{p}\}. \tag{4.3.3}$$

By Hypothesis 4.3 we have

$$v_p(\mathcal{L}) = \begin{cases} k, & p \nmid A, \\ 0, & p|A. \end{cases} \tag{4.3.4}$$

Finally, we define the quantity

$$P(z) = \prod_{p<z} p. \tag{4.3.5}$$

We consider the sum

$$S = S(\tau; N, z) = \sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z))=1}} (\tau - \Omega(\Pi(n))). \tag{4.3.6}$$

If for some fixed constant τ we can show $S(\tau; N, z) > 0$ (for suitable z, N), then there must be an $n \in [N, 2N)$ such that $\Omega(\Pi(n)) < \tau$. Thus if we can show $S(\tau; z, N) > 0$ for all sufficiently large N (with suitable z depending on N) then there are infinitely many n such that $\Omega(\Pi(n)) < \tau$.

We first split the sum S up as a weighted sum over the prime factors of each of the functions $L_j(n)$, based on a new parameter $y \ge z$. This gives

$$\begin{aligned} S &= \sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z))=1}} (\tau - \Omega(\Pi(n))) \\ &= \sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z))=1}} \left(\tau - \frac{\log \Pi(n)}{\log y} - \sum_{p|\Pi(n)} \left(1 - \frac{\log p}{\log y}\right)\right) + O\Bigg(\sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z))=1 \\ \Pi(n) \text{ not square-free}}} \log N \Bigg) \\ &= \sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z))=1}} \left(\tau - k\frac{\log N}{\log y} + O\left(\frac{1}{\log y}\right) - \sum_{j=1}^{k} \sum_{p|L_j(n)} \left(1 - \frac{\log p}{\log y}\right)\right) + O(S'). \end{aligned} \tag{4.3.7}$$

Here

$$S' = \sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z))=1 \\ \Pi(n) \text{ not square-free}}} \log N \ll \log N \sum_{z \le p \ll N^{1/2}} \sum_{\substack{N \le n < 2N \\ p^2 | \Pi(n)}} 1. \tag{4.3.8}$$

We note that by Hypothesis 4.3 if $p^2|\Pi(n)$ then $p^2|L_j(n)$ for some $1 \leq j \leq k$. Thus if $z = N^{1/v}$ for some fixed constant $v > 1$ we have

$$\begin{aligned} S' &\ll \log N \sum_{z \leq p \ll N^{1/2}} \left(\frac{kN}{p^2} + O(1)\right) \\ &\ll \frac{N \log N}{z} + N^{1/2} \log N \\ &\ll N^{1-1/2v}. \end{aligned} \tag{4.3.9}$$

We return to (4.3.7). We reverse the order of summation over p and n, and split the contribution up depending on whether the terms are positive or negative. This gives

$$\begin{aligned} S =& \left(\tau - k\frac{\log N}{\log y} + O\left(\frac{1}{\log y}\right)\right) \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1}} 1 - \sum_{j=1}^{k} \sum_{z \leq p < y} \left(1 - \frac{\log p}{\log y}\right) \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1 \\ p|L_j(n)}} 1 \\ &+ \sum_{j=1}^{k} \sum_{y \leq p < 2a_jN + b_j} \left(\frac{\log p}{\log y} - 1\right) \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1 \\ p|L_j(n)}} 1 + O(N^{1-1/2v}). \end{aligned} \tag{4.3.10}$$

We notice that the inner sum in the third term makes a contribution only if $L_j(n)$ is a multiple of p and has all prime factors of size at least z. Therefore either $p \ll N/z$ and $L_j(n)$ has prime factors other than p, or $L_j(n) = p$. We split the term depending on which of these two is the case.

Thus

$$\begin{aligned} \sum_{y \leq p < 2a_jN + b_j} \left(\frac{\log p}{\log y} - 1\right) \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1 \\ p|L_j(n)}} 1 =& \sum_{y \leq p \ll N/z} \left(\frac{\log p}{\log y} - 1\right) \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1 \\ p|L_j(n)}} 1 \\ &+ \left(\frac{\log N}{\log y} - 1 + O\left(\frac{1}{\log y}\right)\right) \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1 \\ L_j(n) \text{ prime}}} 1. \end{aligned} \tag{4.3.11}$$

Substituting this into (4.3.10) gives

$$S = S_1 - S_2 + S_3 + S_4 + O(N^{1-1/2v}), \tag{4.3.12}$$

where

$$S_1 = \left(\tau - k\frac{\log N}{\log y} + O\left(\frac{1}{\log y}\right)\right) \sum_{\substack{N\le n<2N \\ (\Pi(n),P(z))=1}} 1, \tag{4.3.13}$$

$$S_2 = \sum_{j=1}^{k} \sum_{z\le p<y} \left(1 - \frac{\log p}{\log y}\right) \sum_{\substack{N\le n<2N \\ (\Pi(n),P(z))=1 \\ p|L_j(n)}} 1, \tag{4.3.14}$$

$$S_3 = \sum_{j=1}^{k} \sum_{y\le p\ll N/z} \left(\frac{\log p}{\log y} - 1\right) \sum_{\substack{N\le n<2N \\ (\Pi(n),P(z))=1 \\ p|L_j(n)}} 1, \tag{4.3.15}$$

$$S_4 = \left(\frac{\log N}{\log y} - 1 + O\left(\frac{1}{\log y}\right)\right) \sum_{j=1}^{k} \sum_{\substack{N\le n<2N \\ (\Pi(n),P(z))=1 \\ L_j(n)\text{ prime}}} 1. \tag{4.3.16}$$

We wish to use the Diamond-Halberstam-Richert sieve to get lower bounds for S_1, S_3 and S_4 and an upper bound for S_2. This will then give a lower bound for our sum S. We summarise these results in the following proposition.

Proposition 4.4. *Let $N > N_0$ and u, v be fixed constants satisfying $1 < v/(v-1) < u < v$ and let $\tau > uk$. Let*

$$V(z) = \prod_{\substack{p<z \\ p\nmid A}} \left(1 - \frac{k}{p}\right), \qquad y = N^{1/u}, \qquad z = N^{1/v}.$$

Then we have

$$S_1 \ge (\tau - ku)NV(z)f_k(v) + O\left(\frac{N}{(\log N)^k \log\log N}\right), \tag{4.3.17}$$

$$S_2 \le kNV(z)I_1 + O\left(\frac{N}{(\log N)^k \log\log N}\right), \tag{4.3.18}$$

$$S_3 \ge kNV(z)I_2 + O\left(\frac{N}{(\log N)^k \log\log N}\right), \tag{4.3.19}$$

$$S_4 \ge kNV(z)\frac{(u-1)e^\gamma f_{k-1}\left(\frac{v}{2}\right)}{v} + O\left(\frac{N}{(\log N)^k \log\log N}\right), \tag{4.3.20}$$

where, as in Proposition 4.2,

$$I_1 = \int_{1/v}^{1/u} \min\left(F_k(v - vs),\ e^\gamma F_{k-1}\left(\frac{v}{2}\right)w(v - vs)\right)\frac{1 - us}{s}ds, \tag{4.3.21}$$

$$I_2 = \int_{1/u}^{1-1/v} \max\left(f_k(v - vs),\ e^\gamma f_{k-1}\left(\frac{v}{2}\right)w(v - vs)\right)\frac{us - 1}{s}ds. \tag{4.3.22}$$

Here f_k and F_k are the lower and upper sieve functions of the Diamond-Halberstam-Richert sieve of dimension k, and $w(u)$ is the Buchstab function.

We now establish Proposition 4.2 from Proposition 4.4.

Given u, v satisfying the conditions of Proposition 4.4 we then have that for N sufficiently large

$$S \geq NV(z)\left((\tau - ku)f_k(v) - kI_1 + kI_2 + \frac{(u-1)ke^\gamma}{v} f_{k-1}\left(\frac{v}{2}\right)\right) + o\left(\frac{N}{(\log N)^k}\right), \quad (4.3.23)$$

where

$$I_1 = \int_{1/v}^{1/u} \min\left(F_k\left(v\left(1-s\right)\right),\ e^\gamma F_{k-1}\left(\frac{v}{2}\right) w\left(v\left(1-s\right)\right)\right) \frac{1-us}{s} ds, \quad (4.3.24)$$

$$I_2 = \int_{1/u}^{1-1/v} \max\left(f_k\left(v\left(1-s\right)\right),\ e^\gamma f_{k-1}\left(\frac{v}{2}\right) w\left(v\left(1-s\right)\right)\right) \frac{us-1}{s} ds. \quad (4.3.25)$$

We have that

$$V(z) \gg \frac{N}{(\log N)^k}, \quad (4.3.26)$$

and so

$$S \geq NV(z)\left((\tau - ku)f_k(v) - kI_1 + kI_2 + \frac{(u-1)ke^\gamma}{v} f_{k-1}\left(\frac{v}{2}\right) + o(1)\right). \quad (4.3.27)$$

Since $v > \beta_k$ we have that $f_k(v) > 0$. Thus we have that $S > 0$ provided N is sufficiently large and provided

$$\tau > uk + \frac{k}{f_k(v)}\left(I_1 - I_2 - \frac{(u-1)e^\gamma}{v} f_{k-1}\left(\frac{v}{2}\right)\right). \quad (4.3.28)$$

Let $r_k \geq uk - 1$ be a natural number with

$$r_k > N(u, v; k) = uk - 1 + \frac{k}{f_k(v)}\left(I_1 - I_2 - \frac{(u-1)e^\gamma}{v} f_{k-1}\left(\frac{v}{2}\right)\right). \quad (4.3.29)$$

Then by the argument above we have

$$S\left(r_k + 1; N, N^{1/v}\right) = \sum_{\substack{N \leq n < N \\ (\Pi(n), P(N^{1/v}))=1}} \left(r_k + 1 - \Omega(\Pi(n))\right) > 0 \quad (4.3.30)$$

for all sufficiently large N. Thus $\Omega(\Pi(n)) \leq r_k$ for at least one $n \in [N, 2N)$ for all N sufficiently large, so the k-tuple has at most r_k prime factors infinitely often. Thus Proposition 4.2 holds.

4.4 Proof of Proposition 4.4

We first let N be sufficiently large so that $L_i(n)$ is strictly increasing for $n \geq N$ for all $1 \leq i \leq k$ (this happens because $a_i > 0$ for all i by Hypothesis 4.3). In particular $\Pi(n_1) \neq \Pi(n_2)$ for any $n_1 \neq n_2$ with $n_1, n_2 \geq N$. This assumption is not strictly necessary, but simplifies notation since we do not have to address set/sequence issues.

4.4.1 Estimation of S_1

The sum in S_1 is already of the correct form to be estimated. We let

$$\mathcal{A} = \{\Pi(n) : N \leq n < 2N\}, \tag{4.4.1}$$

$$\mathcal{A}_d = \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}, \tag{4.4.2}$$

and $\mathcal{P}$ be the set of primes.

We use the standard sieve notation $S(\mathcal{B}, \mathcal{Q}, z)$ to denote the number of elements of the set $\mathcal{B}$ which are coprime to all the primes in the set $\mathcal{Q}$ that are less than z. We see that

$$\sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z))=1}} 1 = S(\mathcal{A}, \mathcal{P}, z). \tag{4.4.3}$$

By virtue of our normalisation from Hypothesis 4.3 we have that

$$\#\mathcal{A}_d = g_1(d)N + O(k^{\omega(d)}), \tag{4.4.4}$$

where $g_1(d)$ is the multiplicative function defined by

$$g_1(p) = \begin{cases} k/p, & p \nmid A, \\ 0, & p | A. \end{cases} \tag{4.4.5}$$

Thus applying [10, Theorem 9.1] (with the 'y' from their notation taken to be $N(\log N)^{-6k}$) and recalling that $z = N^{1/v}$ we obtain

$$\begin{aligned} S(\mathcal{A}, \mathcal{P}, z) \geq N \prod_{\substack{p<z \\ p \nmid A}} \left(1 - \frac{k}{p}\right) \left(f_k\left(v - 6k\frac{\log\log N}{\log N}\right) + O\left(\frac{(\log\log N)^2}{(\log N)^{1/(2k+2)}}\right)\right) \\ + O\left(\sum_{m<N(\log N)^{-6k}} \mu^2(m)(4k)^{\omega(m)}\right), \end{aligned} \tag{4.4.6}$$

where f_k is the Diamond-Halberstam-Richert lower sieve function of dimension k.

We see that

$$\sum_{m<N(\log N)^{-6k}} \mu^2(m)(4k)^{\omega(m)} \le N(\log N)^{-6k} \sum_{m<N} \frac{\mu^2(m)(4k)^{\omega(m)}}{m}$$
$$\le N(\log N)^{-6k} \prod_{p<N} \left(1 + \frac{4k}{p}\right)$$
$$\ll N(\log N)^{-2k}. \tag{4.4.7}$$

By our construction of A we have that

$$V(z) = \prod_{\substack{p<z \\ p\nmid A}} \left(1 - \frac{k}{p}\right) \asymp (\log z)^{-k} \asymp (\log N)^{-k}, \tag{4.4.8}$$

and so the error therm contributes a negligible amount.

By [10, Lemma 6.2] $f_k(x)$ satisfies a Lipschitz condition for $x \ge 1$. Therefore

$$S(\mathcal{A}, \mathcal{P}, z) \ge NV(z)\left(f_k(v) + O\left(\frac{1}{\log\log N}\right)\right). \tag{4.4.9}$$

We recall that $y = N^{1/u}$ for some fixed constant u. Thus provided $\tau \ge ku$ we have

$$S_1 = \left(\tau - ku + O\left(\frac{1}{\log N}\right)\right) S(\mathcal{A}, \mathcal{P}, z)$$
$$\ge (\tau - ku)NV(z)f_k(v) + O\left(\frac{N}{(\log N)^k \log\log N}\right). \tag{4.4.10}$$

4.4.2 Estimation of S_2

We will obtain two different upper bounds for the inner sum in S_2, one of which will give stronger results when p is small, and the other will cover the case when p is large. We first note that

$$S_2 = \sum_{z\le p<y} \left(1 - \frac{\log p}{\log y}\right) \sum_{j=1}^{k} \sum_{\substack{N\le n<2N \\ (\Pi(n),P(z))=1 \\ p|L_j(n)}} 1$$
$$= \sum_{z\le p<y} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, \mathcal{P}, z). \tag{4.4.11}$$

We can use [10, Theorem 9.1] to give an upper bound in the same manner as our bound for S_1. We note that since $u > v/(v-1)$ we have $N(\log N)^{-2k-5}/p \ge z$ for N sufficiently large.

Thus [10, Theorem 9.1] gives (using the 'y' from their notation as $N(\log N)^{-2k-5}/p$)

$$S(\mathcal{A}_p, \mathcal{P}, z) \leq \frac{k}{p} NV(z)\left(F_k\left(v - v\frac{\log p}{\log N}\right) + O\left(\frac{1}{\log\log N}\right)\right)$$
$$+ O\left(\sum_{d \leq N(\log N)^{-2k-5}/p} \mu^2(d) 4^{\omega(d)}\right). \tag{4.4.12}$$

Thus summing over $p \in [P, 2P)$ and treating the error term as before, we obtain

$$\sum_{P \leq p < 2P} \left(1 - \frac{\log p}{\log y}\right) S(\mathcal{A}_p, \mathcal{P}, z) \leq \frac{k \log 2}{\log P}\left(1 - \frac{\log P}{\log y}\right) NV(z) F_k\left(v - v\frac{\log P}{\log N}\right)$$
$$+ O\left(\frac{N}{(\log P)(\log N)^k \log\log N}\right). \tag{4.4.13}$$

When P is small the above bound provides a good estimate, but when P is large we can do better. We define

$$\Pi^{(j)}(n) = \prod_{i \neq j} L_i(n), \tag{4.4.14}$$

$$\mathcal{A}^{(j)} = \{\Pi^{(j)}(n) : N \leq n < 2N, (L_j(n), P(z)) = 1\}, \tag{4.4.15}$$

$$\mathcal{A}^{(j,d)} = \{\Pi^{(j)}(n) : N \leq n < 2N, (L_j(n), P(z)) = 1, L_j(n) \equiv 0 \pmod{d}\}. \tag{4.4.16}$$

We then see that, since our forms are coprime by Hypothesis 4.3, we have

$$S(\mathcal{A}_p, \mathcal{P}, z) = \sum_{j=1}^{k} S(\mathcal{A}^{(j,p)}, \mathcal{P}, z). \tag{4.4.17}$$

The terms $S(\mathcal{A}^{(j,p)}, \mathcal{P}, z)$ correspond to $(k-1)$-dimensional sieves rather than k-dimensional sieves. Reducing the sieve dimension in this way can improve estimates, but the set of $n \in [N, 2N)$ such that $(L_j(n), P(z)) = 1$ does not have as strong level of distribution results when p is small, which means that this step is only useful when p is relatively large.

Since any $L_j(n)$ can have at most $\lfloor v \rfloor$ prime factors if $(L_j(n), P(z)) = 1$ and N is sufficiently large, we have

$$\sum_{P \leq p < 2P} S(\mathcal{A}^{(j,p)}, \mathcal{P}, z) = \sum_{r=1}^{\lfloor v \rfloor} S(\mathcal{B}^{(j)}_{P,r}, \mathcal{P}, z) + O(S'_P), \tag{4.4.18}$$

where

$$\mathcal{B}^{(j)}_{P,r} = \Big\{\Pi^{(j)}(n) : N \leq n < 2N, (L_j(n), P(z)) = 1,$$
$$L_j(n) \text{ has at least } r \text{ prime factors in } [P, 2P)\Big\}, \tag{4.4.19}$$

$$S'_P = \sum_{\substack{N \leq n < 2N \\ (\Pi(n), P(z)) = 1 \\ p^2 | \Pi(n) \text{ for some } p \in [P, 2P)}} 1. \tag{4.4.20}$$

We define

$$f_{P,r}(n) = \begin{cases} 1, & n \text{ has at least } r \text{ prime factors in } [P, 2P) \text{ and } (n, P(z)) = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{4.4.21}$$

To estimate $S(\mathcal{B}_{P,r}^{(j)}, \mathcal{P}, z)$ we wish to estimate the number of elements $\#(\mathcal{B}_{P,r}^{(j)})_d$ of $\mathcal{B}_{P,r}^{(j)}$ which are a multiple of some integer d.

We see that $(\mathcal{B}_{P,r}^{(j)})_d$ is empty unless $(d, A) = 1$ by Hypothesis 4.3. If $(d, A) = 1$ then we have

$$\begin{aligned} \#(\mathcal{B}_{P,r}^{(j)})_d &= \sum_{\substack{N \le n < 2N \\ d | \Pi^{(j)}(n)}} f_{P,r}(L_j(n)) \\ &= \sum_{\substack{d_1 \dots d_k = d \\ d_j = 1}} \sum_{\substack{N \le n < 2N \\ d_i | L_i(n)\ \forall i}} f_{P,r}(a_j n + b_j). \end{aligned} \tag{4.4.22}$$

We let $m = a_j n + b_j$ so $a_j N + b_j \le m < 2a_j N + b_j$ and $m \equiv b_j \pmod{a_j}$. The condition $d_i | L_i(n)$ introduces the condition $a_i m \equiv a_i b_j - a_j b_i \pmod{d_i}$, since $(a_j, d_i) = 1$ (because $(d, A) = 1$). We can combine all these congruence conditions via the Chinese remainder theorem to give $m \equiv m_0 \pmod{a_j d}$ for some m_0. By Hypothesis 4.3 we see that m_0 is coprime to $a_j d$. Thus

$$\begin{aligned} \#(\mathcal{B}_{P,r}^{(j)})_d &= \sum_{\substack{d_1 \dots d_k = d \\ d_j = 1}} \sum_{\substack{a_j N + b_j \le m < 2a_j N + b_j \\ m \equiv m_0 \pmod{a_j d}}} f_{P,r}(m) \\ &= \sum_{\substack{d_1 \dots d_k = d \\ d_j = 1}} \left(\frac{X_{P,r}^{(j)}}{\phi(d)} + O(E_{P,r}(d)) \right). \end{aligned} \tag{4.4.23}$$

Here we have defined

$$X_{P,r}^{(j)} = \frac{1}{\phi(a_j)} \sum_{a_j N + b_j \le m < 2a_j N + b_j} f_{P,r}(m), \tag{4.4.24}$$

$$E_{P,r}(q) = \max_{(a,q)=1} \left| \sum_{\substack{a_j N + b_j \le m < 2a_j N + b_j \\ m \equiv a \pmod{q}}} f_{P,r}(m) - \frac{\phi(a_j)}{\phi(q)} X_{P,r}^{(j)} \right|. \tag{4.4.25}$$

The function $f_{P,r}$ is a sum of $O(1)$ characteristic functions of numbers with a fixed number of prime factors (at most $\lfloor \nu \rfloor$) where the prime factors lie in specific intervals (each factor is prescribed to be in one of $[z, P)$, $[P, 2P)$ or $[2P, N)$). By Motohashi [39, Theorem 2] for any fixed $C > 0$ there is a $C'(C) > 0$ such that any such characteristic function $\mathbf{1}$ satisfies

$$\sum_{q \le N^{1/2} (\log N)^{-C'(C)}} \mu^2(q) (4k)^{\omega(q)} E_{\mathbf{1}}(q) \ll N (\log N)^{-C}, \tag{4.4.26}$$

where

$$E_{\mathbf{1}}(q) = \max_{(a,q)=1} \Bigg| \sum_{\substack{a_jN+b_j \le m < 2a_jN+b_j \\ m \equiv a \pmod q}} \mathbf{1}(m) - \frac{1}{\phi(q)} \sum_{\substack{a_jN+b_j \le m < 2a_jN+b_j \\ (m,q)=1}} \mathbf{1}(m) \Bigg|. \tag{4.4.27}$$

Since $f_{P,r}(n) = 0$ unless $(n, P(z)) = 1$, for any $d|P(z)$ we have that

$$\sum_{\substack{a_jN+b_j \le m < 2a_jN+b_j \\ (m,a_jd)=1}} f_{P,r}(m) = \phi(a_j) X_{P,r}^{(j)}. \tag{4.4.28}$$

Thus, since $f_{P,r}$ is a sum of $O(1)$ functions satisfying (4.4.26), we have

$$\sum_{\substack{q \le N^{1/2}(\log N)^{-C'(C)} \\ q|P(z)}} \mu^2(q)(4k)^{\omega(q)} E_{P,r}(q) \ll N(\log N)^{-C}. \tag{4.4.29}$$

Thus if

$$(\mathcal{B}_{P,r}^{(j)})_d = \{b \in \mathcal{B}_{P,r}^{(j)} : b \equiv 0 \pmod d\}, \tag{4.4.30}$$

then from Hypothesis 4.3 for square-free d we have

$$\#(\mathcal{B}_{P,r}^{(j)})_d = g_2(d) X_{P,r}^{(j)} + O((k-1)^{\omega(d)} E_{P,r}(a_j d)). \tag{4.4.31}$$

Here $g_2(d)$ is defined on primes by

$$g_2(p) = \begin{cases} \frac{k-1}{p-1}, & p \nmid A, \\ 0, & \text{otherwise}, \end{cases} \tag{4.4.32}$$

and then extended multiplicatively to all square-free d.

With this we can now apply [10, Theorem 9.1]. We obtain (taking the 'y' from their notation to be $N^{1/2}(\log N)^{-C'(2k)}$)

$$\begin{aligned} S(\mathcal{B}_{P,r}^{(j)}, \mathcal{P}, z) \le X_{P,r}^{(j)} \prod_{\substack{p<z \\ p \nmid A}} \left(1 - \frac{k-1}{p-1}\right) \left(F_{k-1}\left(\frac{v}{2}\right) + O\left(\frac{\log\log N}{\log N}\right)\right) \\ + O\Big(\sum_{\substack{q \le N^{1/2}(\log N)^{-C'(2k)} \\ q|P(z)}} \mu^2(q)(4k)^{\omega(q)} E_{P,r}(q) \Big). \end{aligned} \tag{4.4.33}$$

We see from (4.4.29) that last term is $O(N(\log N)^{-2k})$.

We notice that by Mertens' theorem

$$\begin{aligned} \prod_{\substack{p<z \\ p \nmid A}} \left(1 - \frac{k-1}{p-1}\right) &= \prod_{\substack{p<z \\ p \nmid A}} \left(1 - \frac{k}{p}\right)\left(1 - \frac{1}{p}\right)^{-1} \\ &= V(z)\left(\frac{e^{\gamma}\phi(A)\log z}{A} + O(1)\right). \end{aligned} \tag{4.4.34}$$

We are therefore left to evaluate the terms $X_{P,r}^{(j)}$. From the definition of $X_{P,r}^{(j)}$

$$\sum_{r=1}^{\lfloor v \rfloor} X_{P,r}^{(j)} = \frac{1}{\phi(a_j)} \sum_{\substack{a_jN+b_j \le m < 2a_jN+b_j \\ (m,P(z))=1}} \sum_{r=1}^{\lfloor v \rfloor} f_{P,r}(m). \tag{4.4.35}$$

If m is square-free and $(m, P(z)) = 1$, then since $f_{P,r}$ counts integers which have at least r prime factors in $[P, 2P)$, we have

$$\sum_{r=1}^{\lfloor v \rfloor} f_{P,r}(m) = \sum_{\substack{P \le p < 2P \\ p|m}} 1. \tag{4.4.36}$$

Thus

$$\begin{aligned}
\sum_{r=1}^{\lfloor v \rfloor} X_{P,r}^{(j)} &= \frac{1}{\phi(a_j)} \sum_{\substack{a_jN+b_j \le m < 2a_jN+b_j \\ (m,P(z))=1}} \sum_{\substack{P \le p < 2P \\ p|m}} 1 + O\Bigg(\sum_{\substack{a_jN+b_j \le m < 2a_jN+b_j \\ (m,P(z))=1 \\ m \text{ not square-free}}} 1 \Bigg) \\
&= \frac{1}{\phi(a_j)} \sum_{\substack{a_jN \le m < 2a_jN \\ (m,P(z))=1}} \sum_{\substack{P \le p < 2P \\ p|m}} 1 + O(1) + O\Bigg(\sum_{z \le p} \sum_{\substack{m \ll N \\ p^2 | m}} 1 \Bigg) \\
&= \frac{1}{\phi(a_j)} \sum_{P \le p < 2P} \sum_{\substack{a_jN/p \le n < 2a_jN/p \\ (n,P(z))=1}} 1 + O\left(\frac{N}{z} \right).
\end{aligned} \tag{4.4.37}$$

We can evaluate the inner sum asymptotically. By [37, Theorem 7.11] we have

$$\sum_{\substack{a_jN/p \le n < 2a_jN/p \\ (n,P(z))=1}} 1 = w\left(v - v\frac{\log p}{\log N} \right) \frac{a_jN}{p \log z} + O\left(\frac{N}{p(\log N)^2} \right). \tag{4.4.38}$$

Here we have used that $N/p > z$ since $u > v/(v-1)$. The function $w(u)$ is the Buchstab function defined by the delay differential equation

$$\begin{aligned}
w(u) &= u^{-1}, && \text{for } 1 \le u \le 2, \\
(uw(u))' &= w(u-1), && \text{for } u > 2.
\end{aligned} \tag{4.4.39}$$

Since $w'(u)$ exists and is uniformly bounded for all $u \in [1, +\infty) \backslash \{2\}$, we have that w satisfies $w(x+y) - w(x) \ll y$ for $x \ge 1, y \ge 0$. Therefore, putting together (4.4.37) and (4.4.38), we obtain

$$\begin{aligned}
\sum_{r=1}^{\lfloor v \rfloor} X_{P,r}^{(j)} &= \frac{a_jN}{\phi(a_j)\log z} \sum_{P \le p < 2P} \frac{1}{p} w\left(v - v\frac{\log p}{\log N} \right) + O\left(\frac{N}{(\log P)(\log N)^2} \right) \\
&= \frac{a_jN \log 2}{\phi(a_j)(\log P)(\log z)} w\left(v - v\frac{\log P}{\log N} \right) + O\left(\frac{N}{(\log P)(\log N)^2} \right).
\end{aligned} \tag{4.4.40}$$

Substituting (4.4.34), (4.4.40) and (4.4.33) into (4.4.18), we obtain

$$\sum_{P\le p<2P} S(\mathcal{A}_p^{(j)},\mathcal{P},z) \le \frac{e^{\gamma}\phi(A)a_j\log 2}{A\phi(a_j)\log P} NV(z)F_{k-1}\left(\frac{v}{2}\right)w\left(v-v\frac{\log P}{\log N}\right) + O\left(\frac{N}{(\log P)(\log N)^k(\log\log N)}\right) + O(S'_P). \tag{4.4.41}$$

By Hypothesis 4.3, the integers a_j and A have the same prime factors (ignoring multiplicity). Thus $\phi(A)a_j = A\phi(a_j)$. Using this, and combining (4.4.41) with (4.4.17) we obtain

$$\sum_{P\le p<2P} S(\mathcal{A}_p,\mathcal{P},z) \le \frac{ke^{\gamma}\log 2}{\log P} NV(z)F_{k-1}\left(\frac{v}{2}\right)w\left(v-v\frac{\log P}{\log N}\right) + O\left(\frac{N}{(\log P)(\log N)^k(\log\log N)}\right) + O(S'_P). \tag{4.4.42}$$

To ease notation we let $s = \log P/\log N$ and recall that $y = N^{1/u}$. Combining (4.4.42) with (4.4.13) gives

$$\begin{aligned}\sum_{P\le p<2P}\left(1-\frac{\log p}{\log y}\right)S(\mathcal{A}_p,\mathcal{P},z) &\le \frac{k(1-us)NV(z)\log 2}{s\log N}\min\left(F_k(v-vs),\ e^{\gamma}F_{k-1}\left(\frac{v}{2}\right)w(v-vs)\right)\\ &\quad + O\left(\frac{N}{(\log P)(\log N)^k\log\log N}\right) + O(S'_P).\end{aligned} \tag{4.4.43}$$

An application of partial summation now gives

$$\begin{aligned} S_2 &= \sum_{z\le p<y}\left(1-\frac{\log p}{\log y}\right)S(\mathcal{A}_p,\mathcal{P},z)\\ &\le kNV(z)\int_{1/v}^{1/u}\min\left(F_k(v-vs),\ e^{\gamma}F_{k-1}\left(\frac{v}{2}\right)w(v-vs)\right)\frac{1-us}{s}ds\\ &\quad + O\left(\frac{N}{(\log N)^k\log\log N}\right) + O\left(\sum_{P=2^n\in[z,y)} S'_P\right).\end{aligned} \tag{4.4.44}$$

We note that $S'_P \le S' \ll N^{1-1/2v}$ by (4.3.9), so the final term is negligible.

4.4.3 Estimation of S_3

Exactly the same argument as the one we used to bound S_2 allows us to bound S_3 from below. Using the lower bounds in place of the upper bounds, we obtain

$$\begin{aligned} S_3 &= \sum_{y \le p < AN/z} \left(\frac{\log p}{\log y} - 1 \right) S(\mathcal{A}_p, \mathcal{P}, z) \\ &\ge kNV(z) \int_{1/u}^{1-1/v} \max\left(f_k(v - vs),\ e^{\gamma} f_{k-1}\left(\frac{v}{2}\right) w(v - vs) \right) \frac{us-1}{s} ds \\ &\quad + O\left(\frac{N}{(\log N)^k \log\log N} \right). \end{aligned} \tag{4.4.45}$$

4.4.4 Estimation of S_4

We see that

$$\begin{aligned} S_4 &= \left(\frac{\log N}{\log y} - 1 + O\left(\frac{1}{\log N} \right) \right) \sum_{j=1}^{k} \sum_{\substack{N \le n < 2N \\ (\Pi(n), P(z)) = 1 \\ L_j(n) \text{ prime}}} 1 \\ &= \left(\frac{\log N}{\log y} - 1 + O\left(\frac{1}{\log N} \right) \right) \sum_{j=1}^{k} S(C^{(j)}, \mathcal{P}, z), \end{aligned} \tag{4.4.46}$$

where

$$C^{(j)} = \{\Pi^{(j)}(n) : N \le n < 2N, L_j(n) \text{ prime}\}. \tag{4.4.47}$$

By the prime number theorem for primes in arithmetic progressions we have

$$\begin{aligned} \#C^{(j)} &= \#\{a_j N + b_j \le p < 2a_j N + b_j : p \equiv b_j \pmod{a_j}\} \\ &= \frac{a_j}{\phi(a_j)} \frac{N}{\log N} + O\left(\frac{N}{\log^2 N} \right). \end{aligned} \tag{4.4.48}$$

We let

$$\begin{aligned} \#(C^{(j)})_d &= \#\{c \in C^{(j)} : c \equiv 0 \pmod{d}\} \\ &= g_2(d) \#C^{(j)} + O((k-1)^{\omega(a_j d)} E(a_j d)), \end{aligned} \tag{4.4.49}$$

where $g_2(d)$ is the function described by (4.4.32) and

$$E(d) = \max_{(a,d)=1} \left| \sum_{\substack{a_j N + b_j \le p < 2a_j N + b_j \\ p \equiv a \pmod{d}}} 1 - \frac{1}{\phi(d)} \sum_{a_j N + b_j \le p < 2a_j N + b_j} 1 \right|. \tag{4.4.50}$$

By the Bombieri-Vingradov theorem, for any constant C there is a $C' = C'(C)$ such that

$$\sum_{q \leq N^{1/2}(\log N)^{-C'}} \mu^2(a_j q)(4k)^{\omega(a_j q)}|E(a_j q)| \ll N(\log N)^{-C}. \tag{4.4.51}$$

Thus we can apply [10, Theorem 9.1] (with 'y' from their notation as $N^{1/2}(\log N)^{-C'}$) to give

$$S(C^{(j)}, \mathcal{P}, z) \geq \#C^{(j)} \prod_{\substack{p<z \\ p \nmid A}} \left(1 - \frac{k-1}{p-1}\right) f_{k-1}\left(\frac{v}{2}\right) + O\left(\frac{N}{(\log N)^k \log\log N}\right) \tag{4.4.52}$$

We note that, as before,

$$\prod_{\substack{p<z \\ p \nmid A}} \left(1 - \frac{k-1}{p-1}\right) = \frac{\phi(A)}{A} V(z)(e^{\gamma} \log z + O(1)). \tag{4.4.53}$$

Thus, since $\phi(a_j)/a_j = \phi(A)/A$, we have

$$\begin{aligned} S(C^{(j)}, \mathcal{P}, z) &\geq \left(\frac{N}{\log N} + O\left(\frac{N}{(\log N)^2}\right)\right) V(z)\left(e^{\gamma} \log z + O(1)\right) f_{k-1}\left(\frac{v}{2}\right) \\ &\quad + O\left(\frac{N}{(\log N)^k \log\log N}\right) \\ &= \frac{NV(z)e^{\gamma} f_{k-1}\left(\frac{v}{2}\right)}{v} + O\left(\frac{N}{(\log N)^k \log\log N}\right). \end{aligned} \tag{4.4.54}$$

This gives

$$S_4 \geq kNV(z)\frac{(u-1)e^{\gamma} f_{k-1}\left(\frac{v}{2}\right)}{v} + O\left(\frac{N}{(\log N)^k \log\log N}\right). \tag{4.4.55}$$

This completes the proof of Proposition 4.4.

Chapter 5

Long k-tuples

5.1 Introduction

We consider an admissible set of distinct integer linear functions $L_i(x) = a_i x + b_i$. In this chapter we consider the problem when k is large.

It is a classical result of Miech [34] that there are infinitely many integers n such that the product $\Pi(n)$ has at most $k \log k + O(k)$ prime factors. Heath-Brown [27] extended this by showing that for infinitely many integers n each function $L_i(n)$ has individually at most $C \log k + O(1)$ prime factors, with $C = 2/\log 2$.

It appears difficult to improve on the leading term $k \log k$ of Miech's bound. Roughly this appears to be because any sieve applied to a k-tuple has an effect similar to removing all terms for which $\Pi(n)$ has prime factors less than $n^{c/k}$ (for some constant c). Integers n with all prime factors of size at least $n^{c/k}$ have $\log k + O(1)$ prime factors on average, and all the most successful sieve results rely on weighted sieves which essentially estimate the average number of prime factors of $\Pi(n)$ after the sieve has been applied.

This limitation was also demonstrated explicitly by Ramaré [44], who showed that there are infinitely many integers n for which the product $L_1(n) \dots L_k(n)$ has *exactly* $(1 + o(1))k \log k$ distinct prime factors, and each individual function $L_i(n)$ at most $k^{1/2}$ distinct prime factors. In both cases the number of prime factors were not counted with multiplicity.

We extend this result to show there are infinitely many n for which the product $\Pi(n)$ has exactly $(1+o(1))k \log k$ prime factors (counted with multiplicity), and each individual function has $O_\epsilon(k^\epsilon)$ prime factors (again counted with multiplicity).

This also points at the limitations of the type of sieve method employed; it seems unlikely that any sieve of the form employed in Chapter 3 would be able to prove a result better than $\Omega(\Pi(n)) \leq k \log k + O(k)$ for large k.

5.2 Main result

Theorem 5.1. *Let $\epsilon > 0$ and let $\mathcal{L} = \{L_1 \dots, L_k\}$ be an admissible k-tuple of integer linear functions. Then there are infinitely many n for which both of the following holds:*

1. $\Omega(\Pi(n)) = k \log k + O_\epsilon(k(\log k)^{1-\epsilon})$.
2. $\Omega(L_i(n)) = O_\epsilon(k^\epsilon)$ *for all* $i \in \{1, \dots, k\}$.

We comment that our method can be extended to show that any admissible k-tuple has $\Omega(\Pi(n)) = (\alpha + o(1))k \log k$ infinitely often (with $\Omega(L_i(n)) = O_\epsilon(k^\epsilon)$ as above) for any choice of fixed $\alpha \geq 1$.

We establish Theorem 5.1 using a sieve method similar to the Goldston-Pintz-Yıldırım sieve. Our result holds for a very wide choice of the sieve coefficients λ_d. If we define the λ_d in terms of a polynomial P as in equations (3.4.15) and (3.4.14), then our result holds for any fixed choice of P with $P(0) \neq 0$. One can easily extend this result to include some polynomials P which depend also on k, and in particular the choice $P(x) = x^l$ with $l = o(k)$, which appears to be essentially optimal in the case of small gaps between primes. Theorem 5.1 does not hold for any choice of a continuous function P, however, since it fails to hold for the choice

$$P(y) = \begin{cases} 0, & y \leq 1 - 1/k, \\ 1 - k(y-1), & y \geq 1 - 1/k. \end{cases} \tag{5.2.1}$$

The upper bound $\Omega(\Pi(n)) \leq (1 + o(1))k \log k$ appears to hold for all choices of P.

5.3 Proof of Theorem 5.1

We adopt the notation of Chapter 3. In particular, without loss of generality we assume Hypothesis 3.2 holds, and the sieve coefficients λ_d are defined by (3.4.15) and (3.4.14) in terms of some fixed polynomial P (which is independent of k). We let $\sum'$ denote a sum over values which are square-free and coprime to $A = \prod_{i=1}^k a_i$. We recall that

$$\Lambda^2(n) = \Big(\sum_{\substack{d|\Pi(n) \\ d \leq R_2}} \lambda_d \Big)^2. \tag{5.3.1}$$

We will prove the theorem by showing for every $m > 0$, for any fixed choice of the polynomial P with $P(0) \neq 0$ and for $N \gg_k 1$ we have that

$$S_m = \sum_{\substack{N \leq n < 2N \\ \Pi(n) \text{ square-free}}} \sum_{i=1}^{k} \left(\sum_{p | L_i(n)} 1 - \log k \right)^{2m} \Lambda^2(n) \ll_m \sum_{\substack{N \leq n < 2N \\ \Pi(n) \text{ square-free}}} k(\log k)^{2m-1} \Lambda^2(n). \quad (5.3.2)$$

If (5.3.2) holds, then we see there must be some $n \in [N, 2N)$ such that

$$\sum_{i=1}^{k} (\Omega(L_i(n)) - \log k)^{2m} = O_m(k(\log k)^{2m-1}). \quad (5.3.3)$$

From this it follows immediately that $\Omega(L_i(n)) \ll_{m,\epsilon} k^{1/(2m)+\epsilon}$ for each i, and that

$$\Omega(\Pi(n)) - k \log k \leq k \left(\frac{\sum_{i=1}^{k} (\Omega(L_i(n)) - \log k)^{2m}}{k} \right)^{1/(2m)} \ll_m k(\log k)^{1-1/(2m)}. \quad (5.3.4)$$

To establish (5.3.2) we split the sum over primes depending on whether the primes are smaller than some bound R_1 (in which case we will be able to obtain an asymptotic estimate) or larger than R_1 (these terms will have a negligible contribution for large k).

$$S_m = \sum_{\substack{N \leq n < 2N \\ \Pi(n) \text{ square-free}}} \sum_{i=1}^{k} \left(\sum_{\substack{p | L_i(n) \\ p \leq R_1}} 1 + O\left(\frac{\log N}{\log R_1} \right) - \log k \right)^{2m} \Lambda^2(n). \quad (5.3.5)$$

We fix $R_1 = N^{r_1}$ for some constant $r_1 > 0$, which will depend only on m. Therefore the $O(\log N / \log R_1)$ term is $O_m(1)$. We first remove the condition that $\Pi(n)$ be square-free. Thus

$$S_m = \sum_{N \leq n < 2N} \sum_{i=1}^{k} \left(\sum_{\substack{p | L_i(n) \\ p \leq R_1}} 1 + O_m(1) - \log k \right)^{2m} \Lambda^2(n) + O_m(S'_m), \quad (5.3.6)$$

where

$$S'_m = k \sup_i \sum_{\substack{N \leq n < 2N \\ \Pi(n) \text{ not square-free}}} \left(\sum_{\substack{p | L_i(n) \\ p \leq R_1}} 1 \right)^{2m} \Lambda^2(n) + k(\log k)^{2m} \sum_{\substack{N \leq n < 2N \\ \Pi(n) \text{ not square-free}}} \Lambda^2(n). \quad (5.3.7)$$

We now expand the inner expression using the binomial theorem. This gives

$$S_m = \sum_{j=0}^{2m} \binom{2m}{j} (-\log k)^{2m-j} \sum_{i=1}^{k} S_m^{(i)}(j) + O_m\left(k \sup_{j \leq 2m-1, i} (\log k)^{2m-1-j} S_m^{(i)}(j) \right) + O_m(S'_m), \quad (5.3.8)$$

where

$$S_m^{(i)}(j) = \sum_{N\le n<2N}\left(\sum_{\substack{p_1,\dots,p_j|L_i(n)\\ p_1,\dots,p_j\le R_1}} 1\right)\Lambda^2(n). \tag{5.3.9}$$

The following proposition gives estimates for $S_m^{(i)}(j)$ and S'_m which allow us to deduce the theorem.

Proposition 5.2. *Let $R_1^{2m}R_2^2 \le N^{1/2}(\log N)^{-2^{m+3}k}$ and $N^\epsilon \le R_1 \le R_2$ for some fixed $\epsilon > 0$. Then we have*

$$\begin{aligned} S_m^{(i)}(j) &= P(0)^2(\log k)^j\frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{k!} + O_{m,\epsilon}\left((\log k)^{j-1}\frac{N(\log R_2)^k}{k!}\right)\\ &\quad + O_{k,m}(N(\log\log N)^{2m}(\log N)^{k-1}),\\ S'_m &\ll_{k,m} (\log\log N)^{2m}(\log N)^{k-1}. \end{aligned}$$

Using Proposition 5.2 we can now prove (5.3.2), and hence the theorem. We set $R_2 = N^{1/8}$ and $R_1 = N^{1/(9m)}$. By Proposition 3.6 and Proposition 3.5 we have

$$\sum_{\substack{N\le n<2n\\ \Pi(n)\text{ square-free}}}\Lambda^2(n) = \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{(k-1)!}\int_0^1 P(1-t)^2t^{k-1}dt + O_k((\log N)^{k-1}\log\log N). \tag{5.3.10}$$

We note that $P(1-t) = P(0) + O(1-t)$. This gives

$$\sum_{\substack{N\le n<2n\\ \Pi(n)\text{ square-free}}}\Lambda^2(n) = \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{k!}\left(P(0)^2 + O(k^{-1})\right) + O_k((\log N)^{k-1}\log\log N). \tag{5.3.11}$$

We substitute the expressions for $S_m^{(i)}(j)$ and S'_m from Proposition 5.2 into (5.3.8). This gives

$$\begin{aligned} S_m &= \frac{\mathfrak{S}(\mathcal{L})N(\log R_2)^k}{k!}\sum_{j=1}^{2m}\binom{2m}{j}(-1)^jk(\log k)^{2m}P(0)^2 + O_m\left(k(\log k)^{2m-1}\frac{N(\log R_2)^k}{k!}\right)\\ &\quad + O_{k,m}(N(\log\log N)^{2m}(\log N)^{k-1})\\ &= O_m\left((\log k)^{2m-1}\frac{N(\log N)^k}{(k-1)!}\right) + O_{k,m}(N(\log\log N)^{2m}(\log N)^{k-1}). \end{aligned} \tag{5.3.12}$$

Thus for $N > N_0(k)$, provided $P(0)\neq 0$, we have

$$S_m \ll_m (\log k)^{2m-1}\frac{N(\log N)^k}{(k-1)!} \ll k(\log k)^{2m-1}\sum_{\substack{N\le n<2n\\ \Pi(n)\text{ square-free}}}\Lambda^2(n). \tag{5.3.13}$$

This gives the theorem for any fixed polynomial P with $P(0)\neq 0$. Therefore it just remains to establish Proposition 5.2.

5.4 Proof of Proposition 5.2

We begin by establishing some preparatory lemmas.

Lemma 5.3. *Given integers $r \geq 0$, $k \geq 2$ and a real number $c \in (0, 1]$ we have*

$$\sum_{j=1}^{k} \binom{k}{j} \frac{(-1)^{j+1} c^j}{j^r} = \frac{(\log k)^r}{r!} + O_{r,c}((\log k)^{r-1}).$$

Proof. We define $H_k^{(m)}$ recursively by

$$H_k^{(0)} = 1 - (1-c)^k, \tag{5.4.1}$$

$$H_k^{(m)} = \sum_{j=1}^{k} \frac{H_j^{(m-1)}}{j} \qquad \text{for } m \geq 1. \tag{5.4.2}$$

We now show by induction that for all integers $k \geq 1$ and $m \geq 0$ we have

$$H_k^{(m)} = \sum_{j=1}^{k} \binom{k}{j} \frac{(-1)^{j+1} c^j}{j^m}. \tag{5.4.3}$$

We notice that for any $k \geq 1$

$$H_k^{(0)} = 1 - (1-c)^k = \sum_{j=1}^{k} \binom{k}{j} (-1)^{j+1} c^j. \tag{5.4.4}$$

Similarly, for any $m \geq 0$ we have

$$H_1^{(m)} = c = \sum_{j=1}^{1} \binom{1}{j} \frac{(-1)^{j+1} c^j}{j^m}. \tag{5.4.5}$$

We now assume that (5.4.3) holds for all pairs (m, k) with $m + k \leq B$. Using the fact that $\binom{k}{j} = \binom{k-1}{j-1} + \binom{k-1}{j}$ and $\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1}$ we see that if $m + k = B + 1$ (and $m, k \geq 1$) then we have

$$\begin{aligned}
\sum_{j=1}^{k} \binom{k}{j} \frac{(-1)^{j+1} c^j}{j^m} &= \sum_{j=1}^{k} \binom{k-1}{j-1} \frac{(-1)^{j+1} c^j}{j^m} + \sum_{j=1}^{k-1} \binom{k-1}{j} \frac{(-1)^{j+1} c^j}{j^m} \\
&= \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} \frac{(-1)^{j+1} c^j}{j^{m-1}} + H_{k-1}^{(m)} \\
&= \frac{H_k^{(m-1)}}{k} + \sum_{j=1}^{k-1} \frac{H_j^{(m-1)}}{j} \\
&= H_k^{(m)}.
\end{aligned} \tag{5.4.6}$$

Thus (5.4.3) holds for all $m \geq 0$ and $k \geq 1$ by induction.

We now estimate the size of the $H_k^{(m)}$ using the property (5.4.2). We see that

$$\begin{aligned} H_k^{(1)} &= \sum_{j=1}^{k} \frac{1}{j} - \sum_{j=1}^{k} \frac{(1-c)^j}{j} \\ &= \log k + O(1) + \log c + \sum_{j=k+1}^{\infty} \frac{(1-c)^j}{j} \\ &= \log k + O_c(1). \end{aligned} \tag{5.4.7}$$

We now see by induction that

$$H_k^{(m)} = \sum_{j=1}^{k} \frac{(\log j)^{m-1}}{j(m-1)!} + O_{c,m}\left(\sum_{j=1}^{k} \frac{(\log k)^{m-2}}{j}\right) = \frac{(\log k)^m}{m!} + O_{c,m}\left((\log k)^{m-1}\right). \tag{5.4.8}$$

□

Lemma 5.4. *Let $k > 1$ and $r_1 \leq r_2$. Let*

$$I_m = \int_{0<z_1<\dots<z_m<r_1/r_2} \frac{1}{z_1 \dots z_m} \int_0^1 \left(\sum_{\mathcal{J}\subseteq\{1,\dots,m\}} (-1)^{|\mathcal{J}|} P^+\left(1 - x - \sum_{i\in\mathcal{J}} z_i\right)\right)^2 x^{k-1} dx dz_1 \dots dz_m,$$

$$P_{max} = \sup_{x\in[0,r_1/r_2]} |P'(x)| + |P(x)|.$$

Then we have

$$I_m = \frac{P(0)^2(\log k)^m}{m!k} + O_{m,r_1,r_2}\left(\frac{P_{max}^2(\log k)^{m-1}}{k}\right).$$

Proof. To ease notation we write $\int'$ for the integration over $z_1, \dots z_m$ in the region $0 < z_1 < \dots < z_m < r_1/r_2$, and $P_{\mathcal{J}}^+$ for $P^+(1-x-\sum_{i\in\mathcal{J}} z_i)$.We separate out the $(P(1-x)-P(1-x-z_1))^2$ term. This gives

$$\begin{aligned} I_m &= \int' \frac{1}{z_1 \dots z_m} \int_0^1 \left(P^+(1-x) - P^+(1-x-z_1)\right)^2 x^{k-1} dx dz_1 \dots dz_m \\ &+ O_m\left(\sup_{\substack{\emptyset\neq\mathcal{J}_1\subseteq\{2,\dots m\} \\ \mathcal{J}_2\subseteq\{1,\dots,m\}}} \int' \frac{1}{z_1 \dots z_m} \int_0^1 \left|(P_{\mathcal{J}_1}^+ - P_{\mathcal{J}_1\cup\{1\}}^+)P_{\mathcal{J}_2}^+\right| x^{k-1} dx dz_1 \dots dz_m \right). \end{aligned} \tag{5.4.9}$$

For the first integral, we note that in the range $1 - z_1 \leq x \leq 1$ we have $P^+(1-x) = P(1-x)$ and $P^+(1-x-z) = 0$. Moreover, $P(1-x)^2 = P(0)^2 + O(z_1 P_{max}^2)$. In the range $0 \leq x \leq 1 - z_1$ we have that $P^+(1-x) - P^+(1-x-z_1) = P(1-x) - P(1-x-z_1) = O(z_1 P_{max})$. Thus the inner integral is

$$\int_{1-z_1}^1 P(0)^2 x^{k-1} dx + O_m\left(\int_0^1 z_1 P_{max}^2 x^{k-1} dx\right) = \frac{P(0)^2(1-(1-z_1)^k)}{k} + O_m\left(\frac{z_1 P_{max}^2}{k}\right). \tag{5.4.10}$$

We now see that the error term $O_m(z_1 P_{max}^2/k)$ contributes a total

$$\ll \frac{P_{max}^2}{k}\int' \frac{z_1}{z_1\dots z_m} dz_1\dots dz_m = \frac{r_1 P_{max}^2}{r_2 k}. \tag{5.4.11}$$

To estimate the contribution from the first term we expand out the numerator using the binomial formula, and then can perform integrations with respect to $z_1\dots z_m$. This gives a contribution

$$\begin{aligned}&\frac{P(0)^2}{k}\sum_{j=1}^{k}\binom{k}{j}(-1)^{j+1}\int_{0\le z_1\le\dots\le z_m\le r_1/r_2}\frac{z_1^{j-1}}{z_2\dots z_m}dz_1\dots dz_m\\ &=\frac{P(0)^2}{k}\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^{j+1}(r_1/r_2)^j}{j^m}\\ &=\frac{P(0)^2(\log k)^m}{m!k}+O_{m,r_1,r_2}\left(\frac{P(0)^2(\log k)^{m-1}}{k}\right),\end{aligned} \tag{5.4.12}$$

where we have used Lemma 5.3 in the last line.

We now consider the second integral given in (5.4.9). In the range $x > 1-\sum_{i\in\mathcal{J}_1} z_i$ we have $P^+_{\mathcal{J}_1} = P^+_{\mathcal{J}_1\cup\{1\}} = 0$, since the arguments are negative. In the range $x < 1-\sum_{i\in\mathcal{J}_1\cup\{1\}} z_i$ we have $|P^+_{\mathcal{J}_1} - P^+_{\mathcal{J}_1\cup\{1\}}| \ll z_1 P_{max}$ uniformly in z_1 since neither term vanishes and their arguments differ by z_1. In the remaining range $1-\sum_{i\in\mathcal{J}_1\cup\{1\}} z_i \le x \le 1-\sum_{i\in\mathcal{J}_1} z_i$ we simply bound the integrand by P_{max}^2. This shows that the inner integral in the error term is

$$\ll \int_0^1 P_{max}^2 z_1 x^{k-1}dx + \int_{1-\sum_{i\in\mathcal{J}_1\cup\{1\}} z_i}^{1-\sum_{i\in\mathcal{J}_1} z_i} P_{max}^2 x^{k-1}dx \ll \frac{P_{max}^2 z_1}{k} + z_1(1-z_2)^{k-1}P_{max}^2. \tag{5.4.13}$$

We see the first term contributes a total $O(P_{max}^2/k)$ by (5.4.11). By expanding using the binomial formula and following the same calculation as (5.4.12) we find the second term contributes a total $O(P_{max}^2(\log k)^{m-1}/k)$ to (5.4.9). This completes the proof. ◻

Lemma 5.5. *Let $R_1, R_2 \gg N^\epsilon$, $R_1 \le R_2$ and $R_1^{2m}R_2^2 \ll N(\log N)^{-2^{r+3}k}$. Then we have*

$$\sideset{}{'}\sum_{p_1<\dots<p_j\le R_1}\frac{T_{p_1\dots p_j}}{p_1\dots p_j} = (\log k)^j\frac{(\log R_2)^k\mathfrak{S}(\mathcal{L})}{k!j!}\left(P(0)^2+O_{j,\epsilon}\left(\frac{1}{\log k}\right)\right) + O_{k,j,\epsilon}((\log N)^{k-1}(\log\log N)^{j+1})$$

Proof. By Lemma 3.13 we have

$$T_{p_1\dots p_j} = (\log R_2)^k\frac{\mathfrak{S}(\mathcal{L})}{(k-1)!}I_0\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_j}{\log R_2}\right) + O_{k,j}((\log R_2)^{k-1}\log\log R_2). \tag{5.4.14}$$

We see that the final term contributes

$$\begin{aligned}
&\ll_{k,j} (\log R_2)^{k-1} \log\log R_2 \sideset{}{'}\sum_{p_1<\cdots<p_j\le R_1} \frac{1}{p_1 \dots p_j} \\
&\ll (\log N)^{k-1}(\log\log N)\left(\sum_{p\le R_1}\frac{1}{p}\right)^j \\
&\ll (\log N)^{k-1}(\log\log N)^{j+1}. \qquad (5.4.15)
\end{aligned}$$

We now concentrate on the contribution from I_0. By an analogous argument to the one used in the proof of Proposition 3.3 we have that

$$\begin{aligned}
&\sideset{}{'}\sum_{p_1<\cdots<p_j\le R_1} \frac{I_0\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_j}{\log R_2}\right)}{p_1\dots p_j} \\
&= \int_{0<z_1<\cdots<z_j<r_1/r_2} \frac{1}{z_1\dots z_j}\int_0^1 \left(\sum_{\mathcal{J}\subseteq\{1,\dots,j\}} (-1)^{|\mathcal{J}|} P^+\left(1-x-\sum_{i\in\mathcal{J}} z_i\right)\right)^2 x^{k-1} dx dz_1 \dots dz_j \\
&\quad + O_{j,\epsilon}\left(\frac{(\log\log N)^j}{\log N}\right) \qquad (5.4.16)
\end{aligned}$$

Thus by Lemma 5.4 we have

$$\sideset{}{'}\sum_{p_1<\cdots<p_j\le R_1} \frac{I_0\left(\frac{\log p_1}{\log R_2},\dots,\frac{\log p_j}{\log R_2}\right)}{p_1\dots p_j} = \frac{(\log k)^j}{j!k}\left(P(0)^2 + O_{j,\epsilon}\left(\frac{1}{\log k}\right)\right) + O_{j,\epsilon}\left(\frac{(\log\log N)^j}{\log N}\right). \qquad (5.4.17)$$

Combining (5.4.15) and (5.4.17) then gives the result. □

We are now in a position to prove Proposition 5.2. We first consider $S_m^{(i)}(j)$. We have

$$S_m^{(i)}(r) = \sideset{}{'}\sum_{p_1,\dots p_r\le R_1} \sum_{d,e\le R_2} \lambda_d\lambda_e \sideset{}{'}\sum_{\substack{N\le n<2N \\ [d,e]|\Pi(n) \\ p_1,\dots,p_r|L_i(n)}} 1. \qquad (5.4.18)$$

As with previous estimations from Chapter 3, we have that for a, b square-free

$$\sum_{\substack{N\le n<2N \\ a|\Pi(n) \\ b|L_i(n)}} 1 = \frac{N}{f([a,b])k^{\omega(b)}} + O(k^{\omega([a,b])}). \qquad (5.4.19)$$

Thus

$$
\begin{aligned}
S_m^{(i)}(r) = N &\sideset{}{'}\sum_{\substack{p_1,\dots,p_r\le R_1\\ p_i\neq p_j \text{ for } i\neq j}} \sideset{}{'}\sum_{d,e\le R_2} \frac{\lambda_d\lambda_e}{k^r f([d,e,p_1,\dots,p_r])} + O\Bigg(\sum_{p_1,\dots,p_r\le R_1}\sum_{d,e\le R_2}|\lambda_d\lambda_e| k^{\omega([d,e,p_1,\dots,p_r])}\Bigg)\\
&+ O_r\Bigg(N\sup_{s<r}\sideset{}{'}\sum_{\substack{p_1,\dots,p_s\\ p_i\neq p_j \text{ for } i\neq j}}\sideset{}{'}\sum_{d,e\le R_2}\frac{\lambda_d\lambda_e}{k^s f([d,e,p_1,\dots,p_s])}\Bigg)\\
= r!N &\sideset{}{'}\sum_{p_1<\dots<p_r\le R_1}\frac{T_{p_1\dots p_r}}{k^r p_1\dots p_r} + O_r\Bigg(N\sup_{s<r}\sideset{}{'}\sum_{p_1<\dots<p_s}\frac{T_{p_1\dots p_s}}{k^s p_1\dots p_s}\Bigg) + O(E_r). \qquad (5.4.20)
\end{aligned}
$$

As in the proof of Proposition 3.6, we have

$$
\begin{aligned}
E_r &\ll (\log R_2)^{2k}\sum_{p_1,\dots,p_r\le R_1}\sum_{d,e\le R_2}\mu^2([d,e,p_1,\dots,p_r])k^{\omega([d,e,p_1,\dots,p_r])}\\
&\ll (\log R_2)^{2k}\sum_{a\le R_1^r R_2^2}\mu^2(a)(2^{r+2}k)^{\omega(a)}\\
&\ll (\log R_2)^{2^{r+3}k}R_1^r R_2^2. \qquad (5.4.21)
\end{aligned}
$$

Therefore, by Lemma 5.5 we have

$$
\begin{aligned}
S_m^{(i)}(r) = P(0)^2(\log k)^r\frac{N(\log R_2)^k\mathfrak{S}(\mathcal{L})}{k!} &+ O_{r,\epsilon}\left((\log k)^{r-1}\frac{N(\log R_2)^k}{k!}\right)\\
&+ O_{r,k,\epsilon}\left(N(\log N)^{k-1}(\log\log N)^r\right) + O\left((\log R_2)^{2^{r+3}k}R_1^r R_2^2\right). \qquad (5.4.22)
\end{aligned}
$$

Therefore if $R_1^r R_2^2 \le N(\log N)^{-2^{r+3}k}$ the final error term can be absorbed into the penultimate one, which gives the first result of Proposition 5.2.

Finally, we consider S_m'. By Proposition 3.5 we have for $R_2^2 \le N^{1/2}$

$$
k(\log k)^{2m}\sum_{\substack{N\le n<2N\\ \Pi(n) \text{ not square-free}}}\Lambda^2(n) \ll_k (\log N)^{k-1}\log\log N. \qquad (5.4.23)
$$

As with our estimation of $S_m^{(i)}(r)$ above, we have

$$
\begin{aligned}
\sum_{\substack{N\le n<2N\\ \Pi(n) \text{ not square-free}}}\Bigg(\sum_{\substack{p|L_i(n)\\ p_i\le R_1}}1\Bigg)^{2m}\Lambda^2(n) &\ll \sideset{}{'}\sum_{p\ll N^{1/2}}\sideset{}{'}\sum_{p_1,\dots,p_{2m}\le R_1}\sideset{}{'}\sum_{d,e\le R_2}\lambda_d\lambda_e\sum_{\substack{N\le n<2N\\ [d,e,p^2,p_1,\dots,p_{2m}]|\Pi(n)}}1\\
&\ll N\sideset{}{'}\sum_{p\ll N^{1/2}}\sideset{}{'}\sum_{p_1,\dots,p_{2m}\le R_1}\frac{T_{[p,p_1,\dots,p_{2m}]}}{f([p^2,p_1,\dots,p_{2m}])} + E'. \qquad (5.4.24)
\end{aligned}
$$

Here, the error term E' contributes

$$E' \ll (\log R_2)^{2k} \sum_{p \ll N^{1/2}} \sum_{p_1 \dots p_{2m} \le R_1} \sum_{d,e \le R_2} \mu^2([d, e, p^2, p_1, \dots, p_{2m}]) k^{\omega([d,e,p,p_1,\dots,p_{2m}])}$$
$$\ll (\log R_2)^{2^{2m+5}} N^{1/2} R_1^{2m} R_2^2. \tag{5.4.25}$$

By Lemma 3.12 we have that

$$T_{[p,p_1,\dots,p_{2m}]} \ll_m (\log p)(\log R_2)^{k-1}.$$

Thus

$$\sum_{p \ll N^{1/2}}{}' \sum_{p_1,\dots,p_{2m} \le R_1}{}' \frac{T_{[p,p_1,\dots,p_{2m}]}}{f([p^2, p_1, \dots, p_{2m}])} \ll_k (\log R_2)^{k-1} \left(\sum_{p \ll N^{1/2}} \frac{\log p}{p^2} \right) \left(\sum_{p' \le R_1} \frac{1}{p_1} \right)^{2m}$$
$$\ll (\log R_2)^{k-1} (\log \log R_2)^{2m}. \tag{5.4.26}$$

Putting together (5.4.23), (5.4.24), (5.4.25) and (5.4.26) then gives the result.

Chapter 6

Bounded length intervals containing two primes and an almost-prime

6.1 Introduction

We are interested in trying to understand how small gaps between primes can be. If we let p_n denote the n^{th} prime, it is conjectured that

$$\liminf_n (p_{n+1} - p_n) = 2. \tag{6.1.1}$$

This is the famous twin prime conjecture. More generally, we can look at the difference $p_{n+k} - p_n$. It would follow from the Hardy-Littlewood prime k-tuples conjecture that

$$\liminf_n (p_{n+k} - p_n) \ll k \log k. \tag{6.1.2}$$

In particular, we expect that $\liminf_n (p_{n+1} - p_n)$ is finite for each k.

For $k = 1$, the recent breakthrough of Zhang [53] has shown unconditionally that

$$\liminf_n (p_{n+1} - p_n) < 7 \cdot 10^7. \tag{6.1.3}$$

For $k > 1$, we have much less precise knowledge. The best results are due to Goldston, Pintz and Yıldırım [18], who have shown

$$\liminf_n \frac{p_{n+k} - p_n}{\log p_n} < e^{-\gamma}(\sqrt{k} - 1)^2. \tag{6.1.4}$$

In particular, we do not know whether $\liminf (p_{n+k} - p_n)$ is finite when $k > 1$.

Both unconditional results are based on the 'GPY method' for showing the existence of small gaps between primes. This method relies heavily on results about primes in arithmetic progressions. We say that the primes have 'level of distribution' θ if, for any constant A, there is a constant $C = C(A)$ such that

$$\sum_{q \le x^{\theta}(\log x)^{-C}} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{p \equiv a \pmod q \\ p \le x}} 1 - \frac{\mathrm{Li}(x)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A}. \tag{6.1.5}$$

The Bombieri-Vinogradov theorem states that the primes have level of distribution $1/2$, and the major ingredient in Zhang's proof that $\liminf(p_{n+1} - p_n)$ is finite is a slightly weakened version of the statement that the primes have level of distribution $1/2 + 1/584$.

It is believed that further improvements in the level of distribution of the primes are possible, and Elliott and Halberstam [12] conjectured the following much stronger result.

Conjecture (Elliott-Halberstam Conjecture)**.** *For any fixed $\epsilon > 0$, the primes have level of distribution $1 - \epsilon$.*

Friedlander and Granville [13] have shown that the primes do not have level of distribution 1, and so the Elliott-Halberstam conjecture represents the strongest possible result of this type.

Under the Elliott-Halberstam conjecture the GPY method gives [19] that

$$\liminf_n (p_{n+1} - p_n) \le 16. \tag{6.1.6}$$

If we consider $k > 1$, however, we are unable to prove such strong results, even under the full strength of the Elliott-Halberstam conjecture. In particular we are unable to prove that there are infinitely many intervals of bounded length that contain at least 3 primes. The GPY methods can still be used, but even with the Elliott-Halberstam conjecture we are only able to prove that

$$\liminf_n \frac{p_{n+2} - p_n}{\log p_n} = 0. \tag{6.1.7}$$

From the prime k-tuples conjecture, we expect that $\liminf_n (p_{n+2} - p_n) = 6$.

Therefore it appears that we are unable to show that $\liminf(p_{n+2} - p_n)$ is finite with the current methods. As an approximation to the conjecture, it is common to look for *almost-prime* numbers instead of primes, where almost-prime indicates that the number has only a 'few' prime factors.

Pintz [41] has shown that Zhang's result can be extended to show that there are infinitely many intervals of bounded length which contain two primes and a number with at most $O(1)$ prime factors. Pintz doesn't give an explicit bound on the number of prime factors for the almost-prime.

We extend this work to show that there are infinitely many intervals of bounded length which contain two primes and a number with at most 31 prime factors. Moreover, if we assume a generalized version of the Elliott-Halberstam conjecture, then the constant 31 can be reduced to 4.

Our method for showing this relies on a modified GPY method which also counts almost-primes, which might have wider interest and applications.

6.2 A generalized Elliott-Halberstam hypothesis

We introduce an assumption on numbers with exactly r prime factors in arithmetic progressions of level θ.

Given constants $0 \le \eta_i \le \delta_i \le 1$ for $1 \le i \le r$ we define

$$\beta_{r,\eta,\delta}(n) = \begin{cases} 1, & n = p_1 p_2 \dots p_r \text{ with } n^{\eta_i} \le p_i \le n^{\delta_i} \text{ for } 1 \le i \le r, \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.1}$$

We put

$$\Delta_{r,\eta,\delta}(x; q, a) = \sum_{\substack{x < p \le 2x \\ p \equiv a \pmod q}} \beta_{r,\eta,\delta}(n) - \frac{1}{\phi(q)} \sum_{x < p \le 2x} \beta_{r,\eta,\delta}(n), \tag{6.2.2}$$

$$\Delta^*_{r,\eta,\delta}(x; q) = \max_{y \le x} \max_{\substack{a \\ (a,q)=1}} \left| \Delta_{r,\eta,\delta}(y; q, a) \right|. \tag{6.2.3}$$

We can now state the hypothesis that we consider, the generalized Bombieri-Vinogradov hypothesis of level θ for E_r numbers, GBV(θ,r).

Hypothesis GBV(θ,r). *For every constant $A > 0$ and integer $h > 0$ there is a constant $C = C(A, h)$ such that if $Q \le x^\theta (\log x)^{-C}$ then uniformly for $0 \le \eta_i \le \delta_i \le 1$ ($1 \le i \le r$) we have*

$$\sum_{q \le Q} \mu^2(q) h^{\omega(q)} \Delta^*_{r,\eta,\delta}(x; q) \ll_A x (\log x)^{-A}.$$

We note that Motohashi [39] has shown that Hypothesis GBV(θ,r) holds for all $\theta \le 1/2$ and all r. Hypothesis GBV(θ,r) is purely conjectural for all (θ, r) with $\theta > 1/2$.

6.3 Main results

Theorem 6.1. *There are infinitely many integers n such that the interval* $[n, n+10^8]$ *contains two primes and a number with at most 31 prime factors.*

Theorem 6.2. *Let* $\theta \geq 0.99$*, and assume Hypothesis GBV(θ,r) holds for* $1 \leq r \leq 4$*. Then there exist infinitely many integers n such that the interval* $[n, n+90]$ *contains two primes and one other integer with at most 4 prime factors.*

We comment that we can extend Theorem 6.1 to show that there are infinitely many intervals of bounded length containing two primes and k_2 almost-primes, with each of the k_2 almost-primes having at most $\lfloor 3\log_2 5000k_2 \rfloor$ prime factors.

6.4 A modified GPY method

We consider a set of distinct integer linear functions $\mathcal{L} = \{L_1, \dots, L_k\}$ which is admissible. (We recall that a such set is admissible if for every prime p there is an integer n_p such that every function evaluated at n_p is coprime to p).

We now consider the sum

$$S = S(r; N, \mathcal{L}) = \sum_{N \leq n \leq 2N} \left(\sum_{i=1}^{k-1} \chi_{\leq 1}(L_i(n)) + \chi_{\leq r}(L_k(n)) - 2 \right) \left(\sum_{\substack{d \mid \Pi(n) \\ d \leq R}} \lambda_d \right)^2, \tag{6.4.1}$$

where

$$\chi_{\leq r}(n) = \begin{cases} 1, & n \text{ has at most } r \text{ prime factors} \\ 0, & \text{otherwise,} \end{cases} \tag{6.4.2}$$

$$\Pi(n) = \prod_{i=1}^{k} L_i(n), \tag{6.4.3}$$

and the λ_d and R are real numbers which we declare later.

If we can show that $S > 0$ then we know there must be at least one $n \in [N, 2N]$ for which the terms in parentheses give a positive contribution to S. The second term in our expression for S is a square, and so is always non-negative. We see that the first term in parentheses is positive only when, amongst the $L_i(n)$ $(1 \leq i \leq k)$, there are either two primes and one number with at most r prime factors, or at least three primes. If we choose all our original functions to be of the form $L_i(n) = n + h_i$ (with $h_i \geq 0$) then all these integers then lie in an interval $[m, m+H]$, where $H = \max_i h_i$.

To simplify notation we put

$$\Lambda^2(n) = \Bigg(\sum_{\substack{d|\Pi(n)\\ d\leq R}} \lambda_d\Bigg)^2. \tag{6.4.4}$$

To avoid confusion we mention that $\Lambda^2(n)$ is unrelated to the von-Mangoldt function.

We expect to be able to show that $S > 0$ for suitably large k and r if the primes have level of distribution $\theta > 1/2$. This is because the original GPY method shows that for sufficiently large size of k (depending on ϵ) we can choose the λ_d to give

$$\sum_{N\leq n\leq 2N}\sum_{i=1}^{k-1}\chi_{\leq 1}(L_i(n))\Lambda^2(n) \geq (2\theta-\epsilon)\sum_{N\leq n\leq 2N}\Lambda^2(n). \tag{6.4.5}$$

Moreover, since $\Lambda^2(n)$ is small when $\Pi(n)$ has many prime factors, we expect for sufficiently large r (depending on k and ϵ) that

$$\sum_{N\leq n\leq 2N}(1-\chi_{\leq r}(L_k(n)))\Lambda^2(n) \leq \epsilon\sum_{N\leq n\leq 2N}\Lambda^2(n). \tag{6.4.6}$$

And so provided that $\theta > 1/2+\epsilon$ we expect that

$$S \gg_\epsilon \sum_{N\leq n\leq 2N}\Lambda^2(n) > 0. \tag{6.4.7}$$

If we wish, we can repeat this to show for k, r sufficiently large any k-tuple contains 2 primes and m almost-primes, whenever we can show that it contains 2 primes by the GPY method.

6.5 Proof of Theorem 6.1

We broadly follow the above approach to establish Theorem 6.1. To simply the argument, however, we first make a couple of simplifications. We will use the inequality

$$\chi_{\leq r}(n) \geq 1 - \frac{\tau(n)}{2^{r+1}} \qquad \text{if } n \text{ is square-free}, \tag{6.5.1}$$

to simplify the analysis of the $\chi_{\leq r}$ term (here τ denotes the divisor function). Since this is only valid for n square-free, we will also restrict the sum to n for which $\Pi(n)$ is square-free. As in Chapter 3, this has a negligible effect on the final asymptotics.

We let $L_i^{(1)}(n) = n + h_i$ ($1 \le i \le k_0$) be our admissible k_0-tuple. As in previous chapters, we adopt a normalization of our functions. We let $L_i(n) = L_i^{(1)}(An + a_0) = An + b_i$ where the constants $A, a_0 > 0$ are chosen such that for all primes p we have

$$\#\{1 \le a \le p : \prod_{i=1}^{k_0} L_i(n) \equiv 0 \pmod p\} = \begin{cases} k_0, & p \nmid A, \\ 0, & p | A. \end{cases} \tag{6.5.2}$$

We now set $\Pi(n) = \prod_{i=1}^{k_0} L_i(n)$. We consider the sum

$$S = S(B) = \sum_{\substack{N \le n < 2N \\ \Pi(n) \text{ square-free}}} \Big(\sum_{i=1}^{k_0-1} \chi_{\le 1}(L_i(n)) - 1 - \frac{\tau(L_{k_0}(n))}{B} \Big) \Big(\sum_{d | \Pi(n)} \lambda_d \Big)^2, \tag{6.5.3}$$

where the λ_d are real constants (to be chosen later).

We wish to show, for a suitable choice of positive constants B and k_0, that $S > 0$ for any large N. If this holds we see there must be infinitely many integers n such that two of the $L_i^{(1)}(n)$ are prime and one other of the $L_i^{(1)}(n)$ has at most $\lfloor \log_2 B \rfloor$ prime factors.

We first remove the condition that $\Pi(n)$ be square-free in the sum over n, and then we split S up into separate terms which we will estimate individually. This gives

$$\begin{aligned} S &\ge \sum_{N \le n < 2N} \Big(\sum_{i=1}^{k_0-1} \chi_{\le 1}(L_i(n)) - 1 - \frac{\tau(L_{k_0}(n))}{B} \Big) \Big(\sum_{d | \Pi(n)} \lambda_d \Big)^2 - kS' \\ &= -S_1 + \sum_{i=1}^{k_0-1} S_2(L_i) - \frac{1}{B} S_3 - kS', \end{aligned} \tag{6.5.4}$$

where

$$S' = \sum_{\substack{N \le n < 2N \\ \Pi(n) \text{ not square-free}}} \Big(\sum_{d | \Pi(n)} \lambda_d \Big)^2, \tag{6.5.5}$$

$$S_1 = \sum_{N \le n < 2N} \Big(\sum_{d | \Pi(n)} \lambda_d \Big)^2, \tag{6.5.6}$$

$$S_2(L_i) = \sum_{N \le n < 2N} \chi_{\le 1}(L_i(n)) \Big(\sum_{d | \Pi(n)} \lambda_d \Big)^2, \tag{6.5.7}$$

$$S_3 = \sum_{N \le n < 2N} \tau(L_{k_0}(n)) \Big(\sum_{d | \Pi(n)} \lambda_d \Big)^2. \tag{6.5.8}$$

We will use the following proposition to estimate the terms above.

Proposition 6.3. *Let $\varpi = 1/1168$ and $D = N^{1/4+\varpi}/A$. Let $D_1 = N^{\varpi}/A$ and $\mathcal{P} = \prod_{p\le D_1} p$. For $d < D$ with $d|\mathcal{P}$ we let*

$$\lambda_d = \frac{\mu(d)}{(k_0+l_0)!}\left(\log\frac{D}{d}\right)^{k_0+l_0},$$

and let $\lambda_d = 0$ otherwise. Then we have

$$\begin{aligned}
S' &= o(N(\log N)^{k_0+2l_0}),\\
S_1 &\le \frac{\mathfrak{S}N(\log D)^{k_0+2l_0}}{(k_0+2l_0)!}\binom{2l_0}{l_0}(1+\kappa_1+o(1))),\\
S_2(L_i) &\ge \frac{\mathfrak{S}N(\log D)^{k_0+2l_0+1}}{(k_0+2l_0+1)!\log N}\binom{2l_0+2}{l_0+1}(1-\kappa_2)\\
&\quad + O(N(\log N)^{k_0+2l_0-1}\log\log N),\\
S_3 &\le \frac{\mathfrak{S}N(\log D)^{k_0+2l_0}}{(k_0+2l_0-1)!}\binom{2l_0-2}{l_0-1}\left(\frac{6l_0-4}{l_0(k_0+2l_0)}+\frac{\log N}{\log D}+\kappa_3\left(6\varpi+\frac{\log N}{\log D}\right)+o(1)\right)\\
&\quad + O(N(\log N)^{k_0+2l_0-1}\log\log N).
\end{aligned}$$

where

$$\begin{aligned}
\kappa_1 &= \delta_1(1+\delta_2^2+(\log 293)k_0)\binom{k_0+2l_0}{k_0},\\
\kappa_2 &= \delta_1(1+\delta_2^2+(\log 293)k_0)\binom{k_0+2l_0+1}{k_0-1},\\
\kappa_3 &= \delta_1(1+\delta_2^2+(\log 293)(k_0+1))\binom{k_0+2l_0-1}{k_0+1},\\
\delta_1 &= (1+4\varpi)^{-k_0+1},\\
\delta_2 &= 1+\sum_{\nu=1}^{293}\frac{(\log 293)^\nu(k_0+1)^\nu}{\nu!},\\
\mathfrak{S} &= \prod_{p|A}\left(1-\frac{1}{p}\right)^{-k_0}\prod_{p\nmid A}\left(1-\frac{k_0}{p}\right)\left(1-\frac{1}{p}\right)^{-k_0}.
\end{aligned}$$

We can now establish our main theorem using Proposition 6.3. Substituting the bounds into (6.5.4) we obtain

$$S \ge \frac{N\mathfrak{S}(\log D)^{k_0+2l_0}}{(k_0+2l_0)!}\binom{2l_0}{l_0}\left(\frac{(k_0-1)(2l_0+1)(1+4\varpi)(1-\kappa_2)}{(k_0+2l_0+1)(2l_0+2)}-1-\kappa_1-\frac{c_0}{B}\right), \tag{6.5.9}$$

where

$$c_0 = \frac{l_0(k_0+2l_0)}{4l_0-2}\left(\frac{6l_0-4}{l_0(k_0+2l_0)}+\frac{\log N}{\log D}+\kappa_3\left(6\varpi+\frac{\log N}{\log D}\right)\right). \tag{6.5.10}$$

We now choose $k_0 = 4.5\times 10^6$, $l_0 = 300$. By a simple computation analogous to that giving [53, inequality (4.21)] we certainly have

$$\kappa_1, \kappa_2, \kappa_3 \le \exp(-1000). \tag{6.5.11}$$

Thus, by computation, we see that for N sufficiently large we have

$$\begin{aligned}\frac{(k_0-1)(2l_0+1)(1+4\varpi)(1-\kappa_2)}{(k_0+2l_0+1)(2l_0+2)} - 1 - \kappa_1 &\geq \left(1 - \frac{1}{600} - \frac{602}{4500000}\right)\frac{1172}{1168} - 1 - 3e^{-1000}\\ &\geq 0.0016, \end{aligned}\tag{6.5.12}$$

and

$$c_0 \leq k_0 + 2l_0 + 2 \leq 460000. \tag{6.5.13}$$

We now choose $B = 2^{32} - 1 \geq 4000000000$, and we see that

$$\begin{aligned} S &\geq \frac{N\mathfrak{S}(\log D)^{k_0+2l_0}}{(k_0+2l_0)!}\binom{2l_0}{l_0}\left(0.0016 - \frac{4600000}{4000000000}\right)\\ &\geq 0.00045\frac{N\mathfrak{S}(\log D)^{k_0+2l_0}}{(k_0+2l_0)!}\binom{2l_0}{l_0}. \end{aligned}\tag{6.5.14}$$

Thus, for any admissible k_0-tuple of linear functions of the type we have considered, there are infinitely many integers n for which two of the functions are prime at n, and another function has at most 31 prime factors. A computation now reveals that

$$\pi(10^8) - \pi(4.5 \times 10^6) \geq 4.5 \times 10^6. \tag{6.5.15}$$

Therefore we can form an admissible k_0-tuple of linear functions of the form $L_i(n) = n + h_i$ with $0 \leq h_i \leq 10^8$, by letting $h_i = p_{m+i} - p_{m+1}$ where p_m is the largest prime smaller than 4.5×10^6. This shows that there are infinitely many intervals of length at most 10^8 which contain two primes and a number with at most 31 prime factors.

We comment here that with slightly more care one can take κ_1, κ_2, and κ_3 to be rather smaller than the expressions given in Proposition 6.3. This allows us to show that $S > 0$ for smaller values of k, which in turn allows us to reduce the number of prime factors required for the almost-prime from 31 to 29. Moreover, any improvement in the constant ϖ occurring in Zhang's paper would give a corresponding improvement here. By choosing k and l optimally, we would have that there are infinitely intervals of bounded length containing two primes and one almost-prime with $\approx 3\log_2 \frac{3}{4\varpi}$ prime factors.

6.6 Lemmas

The proof of the bounds for the sums S', S_1 and S_2 essentially already exists in the literature. Ho and Tsang [28] evaluate a sum very similar to S_3, but in their case the λ_d are non-zero on some square-free $d < D$ for which $d \nmid \mathcal{P}$. We therefore require some estimates to show that the error in replacing our sieve weights by the ones used by Ho and

Tsang is small, analogously to [53, Sections §4 and §5]. Our work naturally relies heavily on the papers [53] and [28], and is far from self-contained. We recall the definitions of $D, D_1, \mathcal{P}, \varpi, \lambda_d$ and $\mathfrak{S}$ from Proposition 6.3. As in [53], we also define the quantity $D_0 = (\log D)^{1/k_0}$.

Lemma 6.4. *Let ϱ_3 be the multiplicative function supported on square-free integers coprime to A satisfying $\varrho_3(p) = k_0 + 1 - k_0^2/p$ for $p \nmid A$. Then*

$$\sum_{N \le n < 2N} \tau(L_{k_0}(n)) \left(\sum_{d | \Pi(n)} \lambda_d \right)^2 = \frac{N\phi(A)}{A}((\log N + O(1))M_1 - 2M_2 + M_3) + o(N(\log N)^{k_0+2l_0}),$$

where

$$M_1 = \sum_{d,e|\mathcal{P}} \frac{\lambda_d \lambda_e \varrho_3([d,e])}{[d,e]},$$
$$M_2 = \sum_{p \nmid A} \frac{2(p-k_0)\log p}{(k_0+1)p - k_0} \sum_{\substack{d,e|\mathcal{P} \\ p|d}} \frac{\lambda_d \lambda_e \varrho_3([d,e])}{[d,e]},$$
$$M_3 = \sum_{p \nmid A} \frac{2(p-k_0)\log p}{(k_0+1)p - k_0} \sum_{\substack{d,e|\mathcal{P} \\ p|d,e}} \frac{\lambda_d \lambda_e \varrho_3([d,e])}{[d,e]}.$$

Proof. This follows from the argument of [27, Pages 254-255], with changes only to the notation. We note that the λ_d are supported on $d < D < N^{1/3-\epsilon}$, as required for the argument. □

Lemma 6.5. *Let ϱ_3 be as defined in Lemma 6.4, and let*

$$g(y) = \begin{cases} \frac{1}{(k_0+l_0)!}\left(\log \frac{D}{y}\right)^{k_0+l_0}, & y < D, \\ 0, & \text{otherwise,} \end{cases}$$
$$\mathcal{A}_3(d) = \sum_{(r,d)=1} \frac{\mu(r)\varrho_3(r)}{r} g(dr),$$
$$\theta_3(d) = \prod_{p|d} \left(1 - \frac{\varrho_3(p)}{p}\right)^{-1}.$$

Then if $d < D$ is square-free we have

$$\mathcal{A}_3(d) = \frac{\theta_3(d)}{(l_0-1)!} \mathfrak{S} \frac{A}{\phi(A)} \left(\log \frac{D}{d}\right)^{l_0-1} + O\left((\log D)^{l_0-2+\epsilon}\right),$$
$$\sum_{d \le x^{1/4}} \frac{\varrho_3(d)\theta_3(d)}{d} = \frac{(1+4\varpi)^{-k_0-1}}{(k_0+1)!} \mathfrak{S}^{-1} \frac{\phi(A)}{A} (\log D)^{k_0+1} + O((\log D)^{k_0-1}).$$

Proof. The proof is entirely analogous to that of [53, Lemmas 3 and 4], the only difference being we have ϱ_3, $\mathfrak{S}A/\phi(A)$, k_0+1 and l_0-1 in place of ϱ_1, $\mathfrak{S}$, k_0 and l_0 in the argument. □

Lemma 6.6. *Let*

$$M_1^* = \sum_{d,e} \frac{\lambda_d \lambda_e \varrho_3([d,e])}{[d,e]}.$$

Then we have that

$$|M_1 - M_1^*| \leq \kappa_3 \frac{A}{\phi(A)} \binom{2l_0-2}{l_0-1} \frac{\mathfrak{S}(\log D)^{k_0+2l_0-1}}{(k_0+2l_0-1)!}(1+o(1)),$$

where

$$\kappa_3 = \delta_1(1+(\delta_2)^2 + (\log 293)(k_0+1))\binom{k_0+2l_0-1}{k_0+1},$$
$$\delta_1 = (1+4\varpi)^{-k_0+1},$$
$$\delta_2 = 1 + \sum_{\nu=1}^{293} \frac{(\log 293)^\nu (k_0+1)^\nu}{\nu!}.$$

Proof. The proof is entirely analogous to §4 of [53], using Lemma 6.5 in place of [53, Lemma 2 and Lemma 3] and replacing ϱ_1, $\mathfrak{S}$, k_0 and l_0 with ϱ_3, $\mathfrak{S}A/\phi(A)$, k_0+1 and l_0-1 in the relevant places. □

Lemma 6.7. *Let*

$$M_3^* = \sum_{\substack{p \nmid A \\ p \leq D_1}} \frac{2(p-k_0)\log p}{(k_0+1)p - k_0} \sum_{p|d,e} \frac{\lambda_d \lambda_e \varrho_3([d,e])}{[d,e]}.$$

Then we have

$$|M_3 - M_3^*| \leq 2\varpi\kappa_3 \frac{A}{\phi(A)} \binom{2l_0-2}{l_0-1} \frac{\mathfrak{S}(\log D)^{k_0+2l_0}}{(k_0+2l_0-1)!}(1+o(1)),$$

where κ_3 is defined in Lemma 6.6.

Proof. We first fix p and consider the difference in the inner sums over d and e. This inner sum can be evaluated by essentially the same argument as section §4 of [53]. The condition $p|d,e$ corresponds to $p|(d,e)$, which in the notation of [53, section §4] introduces the condition $p|d_0$. Writing d_0 in place of d_0/p then gives in place of the sums Σ_1, Σ_2 and

Σ_3 the sums

$$\Sigma_{1,p} = \sum_{d_0 \le x^{1/4}/p} \sum_{d_1} \sum_{d_2} \frac{\mu(d_1 d_2)\varrho_3(pd_0d_1d_2)}{pd_0d_1d_2} g(pd_0d_1)g(pd_0d_1),$$

$$\Sigma_{2,p} = \sum_{\substack{d_0 \le x^{1/4}/p \\ d_0|\mathcal{P}}} \sum_{d_1|\mathcal{P}} \sum_{d_2|\mathcal{P}} \frac{\mu(d_1 d_2)\varrho_3(pd_0d_1d_2)}{pd_0d_1d_2} g(pd_0d_1)g(pd_0d_1),$$

$$\Sigma_{3,p} = \sum_{\substack{x^{1/4}/p < d_0 \le D/p \\ d_0\nmid\mathcal{P}}} \sum_{d_1} \sum_{d_2} \frac{\mu(d_1 d_2)\varrho_3(pd_0d_1d_2)}{pd_0d_1d_2} g(pd_0d_1)g(pd_0d_1).$$

The analysis now follows essentially as before. When [53, Lemma 3] is used to estimate the terms $\mathcal{A}_1(d)$ we can instead use the inequality

$$\mathcal{A}_3(dp) \le \theta_3(p)\mathcal{A}_3(d) + O((\log D)^{l_0-2+\epsilon}). \tag{6.6.1}$$

The only other additional constraint is that $(d, p) = 1$, which can be dropped for an upper bound in the final estimations. This argument then gives

$$|\Sigma_{1,p}| + |\Sigma_{2,p}| + |\Sigma_{3,p}| \le \frac{(1+o(1))\varrho_3(p)\theta_3(p)^2}{p}\kappa_3 \frac{A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{\mathfrak{S}(\log D)^{k_0+2l_0-1}}{(k_0+2l_0-1)!}. \tag{6.6.2}$$

We now sum this bound over p to obtain a total error of

$$\begin{aligned}
&\kappa_3 \frac{A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{\mathfrak{S}(\log D)^{k_0+2l_0-1}}{(k_0+2l_0-1)!} \sum_{p \le D_1} \frac{\varrho_3(p)\theta_3(p)^2}{p}\frac{2(p-k_0)\log p}{(k_0+1)p-k_0} \\
&= \kappa_3 \frac{A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{\mathfrak{S}(\log D)^{k_0+2l_0-1}}{(k_0+2l_0-1)!} \sum_{p \le D_1} \left(\frac{2\log p}{p} + O\left(\frac{\log p}{p^2}\right)\right) \\
&= (2+o(1))(\log D_1)\kappa_3 \frac{A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{\mathfrak{S}(\log D)^{k_0+2l_0-1}}{(k_0+2l_0-1)!}.
\end{aligned} \tag{6.6.3}$$

□

Lemma 6.8. *Let*

$$M_2^* = \sum_{\substack{p\nmid A \\ p \le D_1}} \frac{2(p-k_0)\log p}{(k_0+1)p-k_0} \sum_{p|d} \frac{\lambda_d\lambda_e\varrho_3([d,e])}{[d,e]}.$$

Then we have that

$$|M_2 - M_2^*| \le 2\varpi\kappa_3 \frac{A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{\mathfrak{S}(\log D)^{k_0+2l_0}}{(k_0+2l_0-1)!}(1+o(1)),$$

where κ_3 is defined in Lemma 6.6.

Proof. Analogously to §4 of [53], we first bound the difference $|M_2 - M_2^*|$ by

$$\sum_{p \nmid A} \frac{2(p-k_0)\log p}{(k_0+1)p - k_0}(|\Sigma_{1,p}^*| + |\Sigma_{2,p}^*| + |\Sigma_{3,p}^*|), \tag{6.6.4}$$

where

$$\Sigma_{1,p}^* = \sum_{\substack{d_0 \le x^{1/4} \\ (d_0,p)=1}} \sum_{(d_1,p)=1} \sum_{(d_2,p)=1} \frac{\mu(d_1 d_2 p)\varrho_3(p d_0 d_1 d_2)}{p d_0 d_1 d_2} g(p d_0 d_1) g(d_0 d_2), \tag{6.6.5}$$

$$\Sigma_{2,p}^* = \sum_{\substack{d_0 \le x^{1/4} \\ (d_0,p)=1 \\ d_0 | \mathcal{P}}} \sum_{\substack{(d_1,p)=1 \\ d_1 | \mathcal{P}}} \sum_{\substack{(d_2,p)=1 \\ d_2 | \mathcal{P}}} \frac{\mu(d_1 d_2 p)\varrho_3(p d_0 d_1 d_2)}{p d_0 d_1 d_2} g(p d_0 d_1) g(d_0 d_2), \tag{6.6.6}$$

$$\Sigma_{3,p}^* = \sum_{\substack{x^{1/4} < d_0 \le D \\ d_0 \nmid \mathcal{P}}} \sum_{(d_1,p)=1} \sum_{(d_2,p)=1} \frac{\mu(d_1 d_2 p)\varrho_3(p d_0 d_1 d_2)}{p d_0 d_1 d_2} g(p d_0 d_1) g(d_0 d_2). \tag{6.6.7}$$

We first consider $\Sigma_{1,p}$. we wish to put this into a simpler form. Since ϱ_3 is supported only on square-free integers, we can insert the conditions $(d_0, d_1) = (d_0, d_2) = (d_1, d_2) = 1$. With these conditions we may split up the arguments of μ and ρ_3 due to their multiplicativity. We then rewrite the condition $(d_1, d_2) = 1$ using Möbius inversion. This gives

$$\begin{aligned}
\Sigma_{1,p}^* &= \sum_{\substack{d_0 \le x^{1/4} \\ (d_0,p)=1}} \sum_{(d_1,d_0 p)=1} \sum_{(d_2,d_0 p)=1} \frac{\mu(d_1)\mu(d_2)\mu(p)\varrho_3(p)\varrho_3(d_0)\varrho_3(d_1)\varrho_3(d_2)}{p d_0 d_1 d_2} \\
&\qquad \times g(p d_0 d_1) g(d_0 d_2) \sum_{q_1 | d_1, d_2} \mu(q_1) \\
&= \frac{-\varrho_3(p)}{p} \sum_q \frac{\mu(q)\varrho(q)^2}{q^2} \sum_{d_0 \le x^{1/4}} \frac{\varrho_3(d_0)}{d_0} \sum_{(d_1, d_0 p q_1)=1} \frac{\varrho_3(d_1)\mu(d_1)}{d_1} g(p d_0 d_1 q_1) \\
&\qquad \times \sum_{(d_2, d_0 p q_1)=1} \frac{\mu(d_2)\varrho_3(d_2)}{d_2} g(d_0 d_2 q_1) \\
&= \frac{-\varrho_3(p)}{p} \sum_q \frac{\mu(q)\varrho(q)^2}{q^2} \sum_{d_0 \le x^{1/4}} \frac{\varrho_3(d_0)}{d_0} \mathcal{A}_3(d_0 p q) \sum_{(d_2, d_0 p q_1)=1} \frac{\mu(d_2)\varrho_3(d_2)}{d_2} g(d_0 d_2 q_1).
\end{aligned} \tag{6.6.8}$$

We rewrite the condition $(d_2, p) = 1$ in the inner sum by Möbius inversion. This gives

$$\begin{aligned}
\sum_{(d_2, d_0 p q_1)=1} \frac{\mu(d_2)\varrho_3(d_2)}{d_2} g(d_0 d_2 q_1) &= \sum_{(d_2, d_0 q_1)=1} \frac{\mu(d_2)\varrho_3(d_2)}{d_2} g(d_0 d_2 q_1) \sum_{q_2 | p, d_2} \mu(q_2) \\
&= \sum_{q_2 | p} \frac{\varrho_3(q_2)}{q_2} \sum_{(d_2, d_0 q_1 q_2)=1} \frac{\mu(d_2)\varrho_3(d_2)}{d_2} g(d_0 d_2 q_1 q_2) \\
&= \mathcal{A}_3(d_0 q_1) + \frac{\varrho_3(p)}{p} \mathcal{A}_3(d_0 q_1 p).
\end{aligned} \tag{6.6.9}$$

Thus we obtain

$$\Sigma_{1,p}^{*}=\frac{-\varrho_3(p)}{p}\sum_{(d_0,p)=1}\frac{\varrho_3(d_0)}{d_0}\sum_{(q,pd_0)=1}\frac{\mu(q)\varrho_3(q)^2}{q^2}$$
$$\times\left(\mathcal{A}_3(d_0pq)\mathcal{A}_3(d_0q)+\frac{\varrho_3(p)}{p}\mathcal{A}_3(d_0pq)^2\right). \tag{6.6.10}$$

Analogously to the argument in [53], we can restrict the sum over q to $q\le D_0$ at a cost of an error $O(D_0^{-1}p^{-1}(\log D)^B)$ for some constant B. Letting $d=d_0q$ then gives

$$\Sigma_{1,p}^{*}=\frac{-\varrho_3(p)}{p}\sum_{\substack{d\le x^{1/4}D_0\\(d,p)=1}}\frac{\varrho_3(d)\theta_3^*(d)}{d}\left(\mathcal{A}_3(dp)\mathcal{A}_3(d)+\frac{\varrho_3(p)}{p}\mathcal{A}_3(dp)^2\right)+O\left(\frac{(\log D)^B}{pD_0}\right), \tag{6.6.11}$$

where

$$\theta_3^*(d)=\sum_{\substack{d_0q=d\\d_0<x^{1/4}\\q<D_0}}\frac{\mu(q)\varrho_3(q)}{q}. \tag{6.6.12}$$

An analogous argument can be applied to $\Sigma_{2,p}^*$ and $\Sigma_{3,p}^*$ which gives

$$\Sigma_{2,p}^{*}=\frac{-\varrho_3(p)}{p}\sum_{\substack{d\le x^{1/4}D_0\\d|\mathcal{P}\\(d,p)=1}}\frac{\varrho_3(d)\theta_3^*(d)}{d}\left(\mathcal{A}_3^*(dp)\mathcal{A}_3^*(d)+\frac{\varrho_3(p)}{p}\mathcal{A}_3^*(dp)^2\right)+O\left(\frac{(\log D)^B}{pD_0}\right),$$

$$\Sigma_{3,p}^{*}=\frac{-\varrho_3(p)}{p}\sum_{\substack{d\le x^{1/4}D_0\\d|\mathcal{P}\\(d,p)=1}}\frac{\varrho_3(d)\tilde{\theta}_3^*(d)}{d}\left(\mathcal{A}_3(dp)\mathcal{A}_3(d)+\frac{\varrho_3(p)}{p}\mathcal{A}_3(dp)^2\right)+O\left(\frac{(\log D)^B}{pD_0}\right),$$

where

$$\mathcal{A}_3^*(d)=\sum_{\substack{(r,d)=1\\r|\mathcal{P}}}\frac{\mu(r)\varrho_3(r)g(dr)}{r}, \tag{6.6.13}$$

$$\tilde{\theta}^*(d)=\sum_{\substack{d_0q=d\\x^{1/4}<d_0}}\frac{\mu(q)\varrho_3(q)}{q}. \tag{6.6.14}$$

The rest of Zhang's argument now essentially follows as before. The differences are, as in Lemma 6.7, when Zhang uses the asymptotic expression for $\mathcal{A}_1(d)$ we instead use the upper bound from the inequality (6.6.1), and in the final estimations from the sums over d we drop the condition $(d,p)=1$ to obtain an upper bound. This gives us

$$|\Sigma_{1,p}|+|\Sigma_{2,p}|+|\Sigma_{3,p}|\le\kappa_3\frac{A}{\phi(A)}\left(\frac{\varrho_3(p)\theta_3(p)}{p}+\frac{\varrho_3(p)\theta_3(p)^2}{p}\right)\binom{2l_0-2}{l_0-1}$$
$$\times\frac{\mathfrak{S}(\log D)^{k_0+2l_0-1}}{(k_0+2l_0-1)!}(1+o(1)). \tag{6.6.15}$$

We now perform the summation over p. We see that

$$\sum_{p<D_1}\frac{2(k_0-p)\log p}{(k_0+1)p-k_0}\left(\frac{\varrho_3(p)\theta_3(p)}{p}+\frac{\varrho_3(p)\theta_3(p)^2}{p^2}\right)=\sum_{p\le D_1}\left(\frac{2\log p}{p}+O\left(\frac{\log p}{p^2}\right)\right)$$
$$\le (2+o(1))(\log D). \qquad (6.6.16)$$

This gives us the bound stated in the Lemma. □

Lemma 6.9. *Let M_1^*, M_2^* and M_3^* be defined as in Lemmas 6.6, 6.7 and 6.8. We have that*

$$M_1^*=\frac{\mathfrak{S}A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{1}{(k_0+2l_0-1)!}(\log D)^{k_0+2l_0-1}+O\left(\frac{(\log D)^{k_0+2l_0-1}}{\log\log D}\right),$$
$$|M_2^*|\le\frac{\mathfrak{S}A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{2(l_0-1)}{l_0(k_0+2l_0)!}(\log D)^{k_0+2l_0}+O\left(\frac{(\log D)^{k_0+2l_0}}{\log\log D}\right),$$
$$|M_3^*|\le\frac{\mathfrak{S}A}{\phi(A)}\binom{2l_0-2}{l_0-1}\frac{2}{(k_0+2l_0)!}(\log D)^{k_0+2l_0}+O\left(\frac{(\log D)^{k_0+2l_0}}{\log\log D}\right).$$

Proof. This follows from the estimation of the equivalent terms '$M_{2,1}, M_{2,2}, M_{2,3}$', adapted to our notation, which is performed in [28, Pages 40-44]. We note that our sieve weights differ from those used in [28] only by a constant factor of $(\log D)^{k_0+l_0}/(k_0+l_0)!$. The only difference in the argument is that we have the additional restriction that $p\le D_1$ in the terms M_2^* and M_3^*. However, at the point in the argument when the sum over p is evaluated, we may drop this requirement to obtain a bound rather than an asymptotic estimate. Since all further estimations are over terms of the same signs, these bounds correspondingly produce a upper bounds for $|M_2^*|$ and $|M_3^*|$. With further effort one can asymptotically evaluate the terms M_2^* and M_3^*, but the loss in our argument here is comparable to the size of κ_3, which will be small. □

6.7 Proof of Proposition 6.3

We can now complete the proof of Proposition 6.3. The first statement which bounds S' follows from the argument of [28, Page 45]. The result is larger by a factor $(\log D)^{k_0+2l_0}$ since each of our λ_d are larger by a constant factor of $(\log D)^{k_0+l_0}/(k_0+l_0)!$. The second and third statements which bound S_1 and $S_2(L)$ are the equivalent statements to the bounds [53, Inequalities (4.20) and (5.6)]. We note that in Zhang's work the linear equations are of the form $L_i(n)=n+h_i$ rather than $An+a_0+h_h$. This essentially leaves the proof of the result for S_1 unchanged, but causes a very minor change in the proof of the bound S_2. We have

$$\begin{aligned} S_2(L_j) &= \sum_{d,e} \lambda_d \lambda_e \sum_{\substack{N \le n < 2N \\ [d,e] \| \Pi(n)}} \chi_{\le 1}(L_j(n)) \\ &= \frac{(\pi(2AN) - \pi(AN))}{\phi(A)} \sum_{d,e} \frac{\lambda_d \lambda_e \varrho_2([d,e])}{\phi([d,e])} + O(E_j) + O(N^\epsilon) \\ &= \frac{AN(1+o(1))}{\phi(A)\log N} \sum_{d,e} \frac{\lambda_d \lambda_e \varrho_2([d,e])}{\phi([d,e])} + O(E_j) + O(N^\epsilon). \end{aligned} \tag{6.7.1}$$

where ϱ_2 is the multiplicative function defined on square-free integers with

$$\varrho_2(p) = \begin{cases} k_0 - 1, & p \nmid A, \\ 0, & \text{otherwise}, \end{cases} \tag{6.7.2}$$

$$E_j = \sum_{\substack{d < D^2 \\ d | \mathcal{P}, (d,A)=1}} \tau_3(d) \varrho_2(d) \sum_{c \in C_j^*(d)} |\Delta(\chi_{\le 1}; Ad, c)|, \tag{6.7.3}$$

$$\Delta(\chi_{\le 1}; d, c) = \sum_{\substack{AN \le n < 2AN \\ n \equiv c \pmod d}} \chi_{\le 1}(n) - \frac{1}{\phi(d)} \sum_{AN \le n < 2AN} \chi_{\le 1}(n), \tag{6.7.4}$$

$$C_j^*(d) = \Big\{ c : 1 \le c \le Ad, (c,d) = 1, c \equiv h_j + a_0 \pmod A, \prod_{i=1}^{k_0} (c - h_j + h_i) \equiv 0 \pmod d \Big\}. \tag{6.7.5}$$

Since $A = O(1)$, $D_1 = N^{\varpi}/A$ and $D = N^{1/4+\varpi}/A$, Zhang's Theorem 2 now bounds E_j by essentially the same argument. We see that, by the Chinese remainder theorem, there is a bijection $C_i^*(qr) \to C_i^*(q) \times C_i^*(r)$ when $|\mu(qrA)| = 1$, which gives the relevant equivalent of [53, Lemma 5]. The only other change required is a trivial adjustment to the terms in the argument following Zhang's inequality (10.6) to take into account the additional congruence restriction $c \equiv h_j + a_o \pmod A$. The rest of the main analysis of S_2 goes through correspondingly. The only change is that in Lemmas 2 and 3 we have $\phi(A)\mathfrak{S}/A$ in place of $\mathfrak{S}$. This causes us to gain a factor $\phi(A)/A$, which cancels with the factor $A/\phi(A)$ which we have in (6.7.1). The final statement bounding S_3 is a consequence of simply combining the results of Lemmas 6.4, 6.6, 6.7, 6.8 and 6.9.

6.8 Proof of Theorem 6.2

We now establish Theorem 6.2 using our modified GPY method, assuming Hypothesis $GBV(0.99, r)$ for $1 \le r \le 4$. We consider the sum S, defined by

$$S = \sum_{N \le n \le 2N} \left(\sum_{i=1}^{k-1} \chi_{\le 1}(L_i(n)) + \chi_{\le 4}(L_k(n)) - 2 \right) \Lambda^2(n)$$
$$= \sum_{i=1}^{k-1} Q_1(L_i) + Q_2(L_k) - 2Q_3, \tag{6.8.1}$$

where

$$Q_1(L) = \sum_{N \le n \le 2N} \chi_{\le 1}(L(n)) \Lambda^2(n), \tag{6.8.2}$$
$$Q_2(L) = \sum_{N \le n \le 2N} \chi_{\le 4}(L(n)) \Lambda^2(n), \tag{6.8.3}$$
$$Q_3 = \sum_{N \le n \le 2N} \Lambda^2(n). \tag{6.8.4}$$

We recall that $\Lambda^2(n)$ is defined by (6.4.4) in terms of λ_d and R. We now set

$$R = N^{0.99/2} (\log N)^{-C}, \tag{6.8.5}$$

where $C > 0$ is a constant chosen sufficiently large so we can use the estimates of the hypotheses $GBV(0.99, r)$, for $1 \le r \le 4$.

In our situation, the choice of good values for λ_d and the corresponding evaluation of Q_1, Q_2, Q_3 already exists in the literature. As in Chapter 3, we define λ_d in terms of a polynomial P via equations (3.4.14) and (3.4.15). We quote from [17, Theorem 7 and Theorem 9]. These results give for $L \in \mathcal{L}$ and for sufficiently large C that we can choose the λ_d such that

$$Q_1(L) \sim \frac{\mathfrak{S}(\mathcal{L}) N (\log R)^{k+1}}{(k-2)! \log N} \int_0^1 \tilde{P}(1-t)^2 t^{k-2} dt, \tag{6.8.6}$$
$$Q_3 \sim \frac{\mathfrak{S}(\mathcal{L}) N (\log R)^{k}}{(k-1)!} \int_0^1 P(1-t)^2 t^{k-1} dt. \tag{6.8.7}$$

where

$$\mathfrak{S}(\mathcal{L}) \text{ is a positive constant depending only on } \mathcal{L}, \tag{6.8.8}$$
$$\tilde{P}(x) = \int_0^x P(t) dt. \tag{6.8.9}$$

We now concentrate on the remaining term $Q_2(L_k)$. We split the contribution depending on whether $L_k(n)$ has exactly 1, 2, 3 or 4 prime factors. Thus

$$Q_2(L_k) = Q_1(L_k) + Q'_{2,2} + Q'_{2,3} + Q'_{2,4}, \tag{6.8.10}$$

where

$$Q'_{2,j} = \sum_{N \le n \le 2N} \beta_j(L_k(n))\Lambda^2(n), \tag{6.8.11}$$

and

$$\beta_j(n) = \begin{cases} 1, & n \text{ has exactly } j \text{ prime factors} \\ 0, & \text{otherwise.} \end{cases} \tag{6.8.12}$$

For technical reasons we find it harder to deal with terms arising when $L(n)$ has a prime factor less than N^ϵ or no prime factor greater than R. Thus we obtain a lower bound for Q'_{1j} by replacing $\beta_j(L_0(n))$ with $\beta'_j(L_0(n))$, where

$$\beta'_j(n) = \begin{cases} 1, & n = p_1 p_2 \dots p_j \text{ with } n^\epsilon < p_1 < \cdots < p_j \text{ and } n^{0.505} < p_j \\ 0, & \text{otherwise.} \end{cases} \tag{6.8.13}$$

We can then obtain these asymptotic lower bounds. By following an equivalent argument to prove Proposition 3.4 but using Hypothesis GBV(0.99,j) to bound the error terms we have

$$Q'_{2,j} \ge (1 + o(1)) \frac{\mathfrak{S}(\mathcal{L})(\log R)^{k+1}}{(k-2)! \log N} J_j, \tag{6.8.14}$$

where

$$J_r = \int_{(x_1,\dots,x_{r-1}) \in \mathcal{A}_r} \frac{I_1(Bx_1, \dots, Bx_{r-1})}{\left(\prod_{i=1}^{r-1} x_i\right)\left(1 - \sum_{i=1}^{r-1} x_i\right)} dx_1 \dots dx_{r-1}, \tag{6.8.15}$$

$$I_1 = \int_0^1 \left(\sum_{J \subseteq \{1,\dots,r-1\}} (-1)^{|J|} \tilde{P}^+(1 - t - \sum_{i \in J} x_i) \right)^2 t^{k-2} dt, \tag{6.8.16}$$

$$\tilde{P}^+(x) = \begin{cases} \int_0^x P(t)dt, & x \ge 0 \\ 0, & \text{otherwise,} \end{cases} \tag{6.8.17}$$

$$B = \frac{2}{0.99}, \tag{6.8.18}$$

$$\mathcal{A}_r = \left\{ x \in [0,1]^{r-1} : \epsilon < x_1 < \cdots < x_{r-1}, \sum_{i=1}^{r-1} x_i < B^{-1} \right\}. \tag{6.8.19}$$

Substituting (6.8.6), (6.8.7) and (6.8.14) into our expression (6.8.1) for S, we obtain

$$S \ge \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!} \left(\frac{0.99(k-1)}{2} (kJ_1 + J_2 + J_3 + J_4) - 2I_0 + o(1) \right), \tag{6.8.20}$$

where J_r is given above and

$$I_0 = \int_0^1 P(1-t)^2 t^{k-1} dt. \tag{6.8.21}$$

Therefore given a polynomial P we can get an asymptotic lower bound for S by explicitly calculating the integrals I_0, J_1, J_2, J_3 and J_4.

Explicitly we have for $r = 1$

$$J_1 = \int_0^1 \tilde{P}(1-t)^2 t^{k-2} dt. \tag{6.8.22}$$

Similarly, for $r = 2$ we have

$$J_2 = J_{21} + J_{22} + O(\epsilon), \tag{6.8.23}$$

where

$$J_{21} = \int_0^1 \frac{B}{x(B-x)} \int_0^{1-x} \left(\tilde{P}(1-t) - \tilde{P}(1-t-x)\right)^2 t^{k-2} dt dx, \tag{6.8.24}$$

$$J_{22} = \int_0^1 \frac{B}{x(B-x)} \int_{1-x}^1 \tilde{P}(1-t)^2 t^{k-2} dt dx. \tag{6.8.25}$$

For $r = 3$ we have

$$J_3 = J_{31} + J_{32} + J_{33} + J_{34} + O(\epsilon), \tag{6.8.26}$$

where

$$J_{31} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_{1-x}^1 \left(\tilde{P}(1-t)\right)^2 t^{k-2} dt dy dx, \tag{6.8.27}$$

$$J_{32} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_{1-y}^{1-x} \left(\tilde{P}(1-t) - \tilde{P}(1-t-x)\right)^2 t^{k-2} dt dy dx, \tag{6.8.28}$$

$$J_{33} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_{1-x-y}^{1-y} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y)\right)^2 t^{k-2} dt dy dx, \tag{6.8.29}$$

$$J_{34} = \int_0^{1/2} \int_x^{1-x} \frac{B}{xy(B-x-y)} \int_0^{1-x-y} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) + \tilde{P}(1-t-x-y)\right)^2 t^{k-2} dt dy dx. \tag{6.8.30}$$

Finally, for $r = 4$ we have

$$J_4 = J_{41} + J_{42} + J_{43} + J_{44} + J_{45} + J_{46} + J_{47} + J_{48} + J_{49} + J_{410} + J_{411} + O(\epsilon), \tag{6.8.31}$$

where

$$J_{41} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x}^1 \left(\tilde{P}(1-t)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.32}$$

$$J_{42} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-y}^{1-x} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.33}$$

$$J_{43} = \int_0^{1/4} \int_x^{1/2-x} \int_{x+y}^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-y}^{1-y} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.34}$$

$$J_{44} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{x+y} \frac{B}{xyz(B-x-y-z)} \int_{1-z}^{1-y} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.35}$$

$$J_{45} = \int_0^{1/4} \int_x^{1/2-x} \int_{x+y}^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-z}^{1-x-y} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) + \tilde{P}(1-t-x-y)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.36}$$

$$J_{46} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{x+y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-y}^{1-z} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.37}$$

$$J_{47} = \int_0^{1/4} \int_x^{1/2-x} \int_{x+y}^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-z}^{1-z} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.38}$$

$$J_{48} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{x+y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-z}^{1-x-y} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.39}$$

$$J_{49} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-y-z}^{1-x-z} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) + \tilde{P}(1-t-x-z)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.40}$$

$$J_{410} = \int_0^{1/3} \int_x^{(1-x)/2} \int_y^{1-x-y} \frac{B}{xyz(B-x-y-z)} \int_{1-x-y-z}^{1-y-z} \times \left(\tilde{P}(1-t) - \tilde{P}(1-t-x) - \tilde{P}(1-t-y) - \tilde{P}(1-t-z) + \tilde{P}(1-t-x-y) + \tilde{P}(1-t-x-z) + \tilde{P}(1-t-y-z)\right)^2 t^{k-2} dt dz dy dz, \tag{6.8.41}$$

$$
\begin{aligned}
J_{411} = &\int_0^{1/3}\int_x^{(1-x)/2}\int_y^{1-x-y}\frac{B}{xyz(B-x-y-z)}\int_0^{1-x-y-z}\\
&\times\Big(\tilde{P}(1-t)-\tilde{P}(1-t-x)-\tilde{P}(1-t-y)-\tilde{P}(1-t-z)+\tilde{P}(1-t-x-y)\\
&+\tilde{P}(1-t-x-z)+\tilde{P}(1-t-y-z)-\tilde{P}(1-t-x-y-z)\Big)^2 t^{k-2}dtdzdydz. \quad (6.8.42)
\end{aligned}
$$

We choose $k = 22$ and $P(t) = 1 + 60t - 300t^2 + 3500t^3$. We find that numerical integration gives

$$I_0 = \frac{121351}{59202} = 2.04978\ldots, \quad (6.8.43)$$

$$J_1 = \frac{228380}{18027009} = 0.01266\ldots, \quad (6.8.44)$$

$$J_2 \geq 0.041 + O(\epsilon), \quad (6.8.45)$$

$$J_3 \geq 0.048 + O(\epsilon), \quad (6.8.46)$$

$$J_4 \geq 0.028 + O(\epsilon). \quad (6.8.47)$$

Thus we have that

$$
\begin{aligned}
S &\geq \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!}\left(\frac{0.99(k-1)}{2}(kJ_1 + J_2 + J_3 + J_4) - 2I_0 + O(\epsilon) + o(1)\right)\\
&\geq \frac{\mathfrak{S}(\mathcal{L})(\log R)^k}{(k-1)!}\left(0.013 + O(\epsilon) + o(1)\right). \quad (6.8.48)
\end{aligned}
$$

In particular, for N sufficiently large and ϵ sufficiently small we have $S > 0$, and so there are infinitely many n for which an admissible 22-tuple attains at least two prime values and one value with at most 4 prime factors.

The set $\{0, 6, 8, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 60, 66, 74, 78, 80, 84, 86, 90\}$ is an admissible 22-tuple, and so the interval $[n, n + 90]$ infinitely often contains at least two primes and an integer with at most 4 prime factors.

We remark here that if we can take the level of distribution $\theta = 1 - \delta$ for every $\delta > 0$ then we can take $k = 19$ instead of 22, which reduces the length of the interval to 80.

References

[1] Bombieri, E., Friedlander, J. B., and Iwaniec, H. Primes in arithmetic progressions to large moduli. *Acta Math. 156*, 3-4 (1986), 203–251.

[2] Bombieri, E., Friedlander, J. B., and Iwaniec, H. Primes in arithmetic progressions to large moduli. II. *Math. Ann. 277*, 3 (1987), 361–393.

[3] Bombieri, E., Friedlander, J. B., and Iwaniec, H. Primes in arithmetic progressions to large moduli. III. *J. Amer. Math. Soc. 2*, 2 (1989), 215–224.

[4] Chen, J.-R. On the least prime in an arithmetical progression. *Sci. Sinica 14* (1965), 1868–1871.

[5] Chen, J. R. On the representation of a larger even integer as the sum of a prime and the product of at most two primes. *Sci. Sinica 16* (1973), 157–176.

[6] Chen, J. R. On the least prime in an arithmetical progression and two theorems concerning the zeros of Dirichlet's L-functions. *Sci. Sinica 20*, 5 (1977), 529–562.

[7] Chen, J. R. On the least prime in an arithmetical progression and theorems concerning the zeros of Dirichlet's L-functions. II. *Sci. Sinica 22*, 8 (1979), 859–889.

[8] Chen, J. R., and Liu, J. On the least prime in an arithmetical progression and theorems concerning the zeros of Dirichlet's L-functions. V. In *International Symposium in Memory of Hua Loo Keng, Vol. I (Beijing, 1988)*. Springer, Berlin, 1991, pp. 19–42.

[9] Davenport, H. *Multiplicative number theory*, third ed., vol. 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. Revised and with a preface by Hugh L. Montgomery.

[10] Diamond, H., and Halberstam, H. Some applications of sieves of dimension exceeding 1. In *Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995)*, vol. 237 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1997, pp. 101–107.

[11] Dirichlet, P. *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält*. Abhandlungen der Preussischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse. 1839.

[12] Elliott, P. D. T. A., and Halberstam, H. A conjecture in prime number theory. In *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*. Academic Press, London, 1970, pp. 59–72.

[13] Friedlander, J., and Granville, A. Limitations to the equi-distribution of primes. I. *Ann. of Math. (2) 129*, 2 (1989), 363–382.

[14] Friedlander, J., Granville, A., Hildebrand, A., and Maier, H. Oscillation theorems for primes in arithmetic progressions and for sifting functions. *J. Amer. Math. Soc. 4*, 1 (1991), 25–86.

[15] Friedlander, J., and Iwaniec, H. The Brun-Titchmarsh theorem. In *Analytic number theory (Kyoto, 1996)*, vol. 247 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge, 1997, pp. 85–93.

[16] Goldfeld, D. M. A further improvement of the Brun-Titchmarsh theorem. *J. London Math. Soc. (2) 11*, 4 (1975), 434–444.

[17] Goldston, D. A., Graham, S. W., Pintz, J., and Yildirim, C. Y. Small gaps between products of two primes. *Proc. Lond. Math. Soc. (3) 98*, 3 (2009), 741–774.

[18] Goldston, D. A., Pintz, J., and Yildirim, C. Y. Primes in tuples. III. On the difference $p_{n+\nu} - p_n$. *Funct. Approx. Comment. Math. 35* (2006), 79–89.

[19] Goldston, D. A., Pintz, J., and Yildirim, C. Y. Primes in tuples. I. *Ann. of Math. (2) 170*, 2 (2009), 819–862.

[20] Goldston, D. A., Pintz, J., and Yildirim, C. Y. Primes in tuples. II. *Acta Math. 204*, 1 (2010), 1–47.

[21] Graham, S. An asymptotic estimate related to Selberg's sieve. *J. Number Theory 10*, 1 (1978), 83–94.

[22] Graham, S. On Linnik's constant. *Acta Arith. 39*, 2 (1981), 163–179.

[23] Graham, S. W. *Applications of Sieve Methods*. ProQuest LLC, Ann Arbor, MI, 1977. Thesis (Ph.D.)–University of Michigan.

[24] Halberstam, H., and Richert, H.-E. *Sieve methods*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London-New York, 1974. London Mathematical Society Monographs, No. 4.

[25] Hardy, G., and Littlewood, J. Some problems of partitio numerorum; iii: On the expression of a number as a sum of primes. *Acta Mathematica 44*, 1 (1923), 1–70.

[26] Heath-Brown, D. R. Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression. *Proc. London Math. Soc. (3) 64*, 2 (1992), 265–338.

[27] Heath-Brown, D. R. Almost-prime k-tuples. *Mathematika 44*, 2 (1997), 245–266.

[28] Ho, K.-H., and Tsang, K.-M. On almost prime k-tuples. *J. Number Theory 120*, 1 (2006), 33–46.

[29] Iwaniec, H. On the Brun-Titchmarsh theorem. *J. Math. Soc. Japan 34*, 1 (1982), 95–123.

[30] Jutila, M. A new estimate for Linnik's constant. *Ann. Acad. Sci. Fenn. Ser. A I No. 471* (1970), 8.

[31] Jutila, M. On Linnik's constant. *Math. Scand. 41*, 1 (1977), 45–62.

[32] Linnik, U. V. On the least prime in an arithmetic progression. I. The basic theorem. *Rec. Math. [Mat. Sbornik] N.S. 15(57)* (1944), 139–178.

[33] Linnik, U. V. On the least prime in an arithmetic progression. II. The Deuring-Heilbronn phenomenon. *Rec. Math. [Mat. Sbornik] N.S. 15(57)* (1944), 347–368.

[34] Miech, R. J. Almost primes generated by a polynomial. *Acta Arith. 10* (1964/1965), 9–30.

[35] Montgomery, H. L. Problems concerning prime numbers. In *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Northern Illinois Univ., De Kalb, Ill., 1974)*. Amer. Math. Soc., Providence, R. I., 1976, pp. 307–310. Proc. Sympos. Pure Math., Vol. XXVIII.

[36] Montgomery, H. L., and Vaughan, R. C. The large sieve. *Mathematika 20* (1973), 119–134.

[37] Montgomery, H. L., and Vaughan, R. C. *Multiplicative number theory. I. Classical theory*, vol. 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[38] Motohashi, Y. On some improvements of the Brun-Titchmarsh theorem. *J. Math. Soc. Japan 26* (1974), 306–323.

[39] Motohashi, Y. An induction principle for the generalization of Bombieri's prime number theorem. *Proc. Japan Acad. 52*, 6 (1976), 273–275.

[40] Pan, C. D. On the least prime in an arithmetical progression. *Sci. Record (N.S.) 1* (1957), 311–313.

[41] Pintz, J. Polignac numbers, conjectures of Erdős on gaps between primes, arithmetic progressions in primes, and the bounded gap conjecture. *Available at* `http://arxiv.org/abs/1305.6289`.

[42] Pintz, J. Elementary methods in the theory of L-functions. II. On the greatest real zero of a real L-function. *Acta Arith. 31*, 3 (1976), 273–289.

[43] Porter, J. W. Some numerical results in the Selberg sieve method. *Acta Arith. 20* (1972), 417–421.

[44] Ramaré, O. On long κ-tuples with few prime factors. *Proc. Lond. Math. Soc. (3) 104*, 1 (2012), 158–196.

[45] Selberg, A. *Collected papers. Vol. II*. Springer-Verlag, Berlin, 1991. With a foreword by K. Chandrasekharan.

[46] Siegel, C.-L. Über die Classenzahl quadratischer Zahlkörper. *Acta Arith*, 1 (1936), 83–86.

[47] Thorne, F. Bounded gaps between products of primes with applications to ideal class groups and elliptic curves. *Int. Math. Res. Not. IMRN*, 5 (2008), Art. ID rnm 156, 41.

[48] Titchmarsh, E. C. A divisor problem. *Rend. Circ. Math. Palermo 54* (1930), 414–429.

[49] Walfisz, A. Zur additiven Zahlentheorie. II. *Math. Z. 40*, 1 (1936), 592–607.

[50] Wang, W. On the least prime in an arithmetic progression. *Acta Math. Sinica (N.S.) 7*, 3 (1991), 279–289. A Chinese summary appears in Acta Math. Sinica **3**5 (1992), no. 4, 575.

[51] Xylouris, T. On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L-functions. *Acta Arith. 150*, 1 (2011), 65–91.

[52] Xylouris, T. *Über die Nullstellen der Dirichletschen L-Funktionen und die kleinste Primzahl in einer arithmetischen Progression.* 2011. Bonn, Univ., Diss., URN : urn: nbn: de: hbz: 5N-27156.

[53] Zhang, Y. Bounded gaps between primes. *Ann. of Math.(2), to appear*.