

ISSN 1471-0498



**DEPARTMENT OF ECONOMICS
DISCUSSION PAPER SERIES**

AMBIGUITY AND THE HISTORICAL EQUITY PREMIUM

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**Number 550
April 2011
Revised January 2015**

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Ambiguity and the historical equity premium¹

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This version: January 2015

¹We are grateful to the Editor and the referees for their helpful comments and suggestions. We thank R. Bansal, P. Beaudry, H. Bhamra, J. Borovicka, T. Cogley, H. Chen, H. d'Albis, V. Gala, C. Gollier, L. Hansen, P. Klibanoff, H. Liu, T. Ramadorai and R. Uppal for helpful discussions. We also thank seminar and conference participants at Adam Smith Asset Pricing conference, AEA, RUD (Heidelberg), Northwestern (MEDS), Warwick, Leicester, Transatlantic Theory Workshop, EUI (Florence), UBC, Workshop on Ambiguity and Robustness in Macroeconomics and Finance (Becker-Friedman Inst.). Tallon thanks support from the Investissement d'Avenir Program (ANR-10-LABX-93).

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Abstract

This paper assesses the quantitative impact of ambiguity on the historically observed financial asset returns and prices. The single agent, in a dynamic exchange economy, treats the conditional uncertainty about the consumption and dividends next period as ambiguous. We calibrate the agent's ambiguity aversion to match *only* the first moment of the risk-free rate in data and condition the uncertainty each period on the actual, observed history of (U.S.) macroeconomic growth outcomes. Ambiguity aversion accentuates the conditional uncertainty endogenously in a dynamic way, depending on the history; e.g., it increases during recessions. We show the model implied time series of asset returns match observed return dynamics of first and second conditional moments, very substantially. In particular, we find the time-series properties of our model generated equity premium, which may be regarded as an index measure of revealed uncertainty, relates very closely to those of the macroeconomic uncertainty index recently developed in Jurado, Ludvigson, and Ng (2013).

J.E.L. Codes: G12, E21, D81, C63

Keywords: Ambiguity Aversion, Asset pricing, Equity premium puzzle, uncertainty shocks, time-varying uncertainty.

1 Introduction

This paper seeks to assess the *quantitative* impact of ambiguity on financial asset returns and prices, in particular, their dynamic paths as observed in history. Ambiguity refers to uncertainty about the “true” probability distribution governing future consumption and dividend outcomes. The decision maker’s ambiguity attitude determines how and to what extent such uncertainty affects his choices (e.g., whether he is averse to such uncertainty and if so, the level of aversion). Our goals are two fold: to connect the macroeconomic uncertainty *as it obtained on the path of history* to the movements in asset returns and prices along that path and to assess, quantitatively, the role of ambiguity sensitivity in that connection. To serve these goals we incorporate two components in our analysis. One, we only consider conditional uncertainty at information sets adapted to the path of observations of historical macroeconomic growth rates, as opposed to counterfactual, simulated sample paths. Two, our model of agent’s preferences departs from standard expected utility by simply allowing for ambiguity sensitivity; take that away, you get back to expected utility. These two components, together with the demonstration that they alone are sufficient to explain asset return dynamics very substantially, distinguish the contribution in this paper.

Ambiguity averse agents are inclined to choose actions whose consequences are more robust to the perceived ambiguity, e.g., a portfolio position whose (ex ante) value is relatively less affected by the uncertainty about probability distribution governing the future payoffs.¹ An important reason why ambiguity may be pervasive in economic and financial decision making is model uncertainty. For example, a typical professional investor may have different forecasting models for the same variable, or different parameter estimates for the same model, all of which are plausible on the basis of historical data. If the models make distinct (probabilistic) forecasts about key variables of interest, it is natural to seek a portfolio that is robust across the range of forecasts rather than optimizing exclusively to the forecast from a single model as argued, e.g., in Hansen (2007).

This paper considers a standard single agent, Lucas-tree, pure-exchange economy with two less standard assumptions. First, the agent’s belief about the consumption and dividend process is ambiguous, i.e., in each period he is uncertain about the exact probability distribution governing the realization of consumption and dividends in

¹See Dow and Werlang (1992), Epstein and Wang (1994), Mukerji and Tallon (2001), Caballero and Krishnamurthy (2008), Chen, Ju, and Miao (2009), Gollier (2011), Boyle, Garlappi, Uppal, and Wang (2010), Hansen and Sargent (2010), Maccheroni, Marinacci, and Ruffino (2013) and Uhlig (2010), *inter alia*.

the following period. Furthermore, this belief is dynamic, evolving as the agent learns from history. Second, the agent's preferences are ambiguity sensitive, modeled using the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005, 2009) (henceforth KMM2005, KMM2009).

The assumed source of the ambiguity in the agent's beliefs is the occurrence of periodic, temporary changes in the probability distribution governing next period's growth outcome due to the effect of the business cycle. These transient deviations are assumed to be governed by an auto-regressive ($AR(1)$) latent variable. The agent is, however, unsure about the value of the persistence parameter of the $AR(1)$ process since, even with a long run of growth data, it is difficult to distinguish the case where the latent variable is highly volatile but moderately persistent, from the case where the variable is less volatile but highly persistent. Uncertainty about persistence, in turn, makes it harder to estimate the evolving location of the latent variable precisely. Furthermore, depending on the observed history, the imprecision of the estimate of the location will vary over time, making the uncertainty about the probability distribution governing next period's growth vary over time.

The ambiguity averse agent's robustness concerns generate, endogenously, *doubt* and *pessimism*, to use the language of Abel (2002). The portfolio choice of the ambiguity averse agent in the model may be understood as that of an expected utility agent with an "as if" (probabilistic) belief that is more uncertain and pessimistic than the one obtained by objective inference, in the standard fashion, from data. The endogenous accentuation of doubt depends on the observed history and the level of ambiguity aversion. For instance, after a negative shock that follows a series of "normal" ones, the agent behaves as if the uncertainty is more severe and more persistent than what is implied by pure Bayesian inference (and the opposite, if it were a positive shock that broke the normal sequence). The level of ambiguity aversion is calibrated to match the average risk free rate (no other moment is used); all other parameters are either inferred/estimated from the history or fixed at values widely used in the literature.

We present two kinds of results on model implied conditional moments of rates of return and price-dividend ratio: (time-) averages of the moments over the sample period (1978-2011) and time series (and time series properties) of the moments over the same sample period, all based on conditional uncertainty at information sets reconcilable with historical growth data. We compare the level, volatility and dynamics of the model implied rates of return and price-dividend ratio to their counterparts in U.S. data.

The time-averaged conditional moments match the data moments at least as well as the best matches in the literature. However, in the literature, unlike here, the (model predicted) moments are typically unconditional moments, or conditional moments arising from simulated, not actual, histories. Our more distinctive results are those on the predicted time series of conditional moments statistics. Two key “stylized facts” our model matches are the counter-cyclicality of conditional equity premium and the procyclicality of conditional (excess) return volatility. Models in the literature have found it hard to explain these facts, especially the latter.

The model generated (conditional) equity premium is a measure of conditional (macroeconomic) uncertainty as revealed by the behavior of the agent in the model. We show its time-series properties match that of the purely statistical index of macroeconomic uncertainty, recently developed in Jurado, Ludvigson, and Ng (2013). Our model gives a theory of why an agent makes decisions following a positive shock that underplays the uncertainty and its persistence, while following a negative shock, takes decisions which imply a more severe and a more persistent uncertainty, thus explaining the key property of the index and the findings of the recent literature on uncertainty shocks. Such a theory (of endogenous asymmetric reaction to shocks) is needed to explain the dynamics of equity premium and uncertainty shocks because, as Campbell (2000) points out, “the evidence for heteroskedasticity in aggregate consumption is fairly weak.”

The rest of the paper is organized as follows. Section 2 introduces the relevant details of smooth ambiguity preferences, describes and analyzes the amended Lucas tree economy, assuming a general form of beliefs. In a subsection we describe and motivate the specific model of ambiguous beliefs we adopt. Section 3 first identifies the key mechanisms at work in our model and then presents and explains the quantitative implications of our model for asset prices and returns in the light of the mechanisms identified. In Section 4, using a thought experiment, we show that a decision maker with preferences and beliefs calibrated as our agent in the Lucas economy will demand a total uncertainty premium (for the Lucas tree) that is well within the bounds of the amounts widely considered as plausible. Section 5 discusses the more closely related literature. A final section concludes. The Appendix gathers several items, including, details of parameter values used in the model, details of the model including the specification of beliefs, how they are updated, and the formulae for rates of return.

2 The Model

2.1 Agent's preferences: recursive smooth ambiguity

We follow KMM2009, which develops a dynamic, recursive version of the smooth ambiguity model in KMM2005. In KMM2009 the basis of the dynamic model is the state space, the set of all observation paths generated by an event tree, a graph of decision/observation nodes. The root node of the tree, s^0 , branches out into a set of immediate successor nodes, $s^1 \equiv (s^0, s_1)$ where $s_1 \in \mathcal{S}_1$, the set of possible observations at time $t = 1$; and, so on. The decision maker (DM) chooses between consumption plans f , each of which associates a payoff to a node s^t in the event tree. The DM is uncertain about which stochastic process governs the probabilities on the event tree. The domain of this uncertainty is given by a parameter space $\Theta \ni \theta$, the set of unobservable parameters, over which the DM makes inference at each s^t . We denote by $\pi_\theta(s_{t+1} | s^t)$ the probability under *likelihood* distribution π_θ that the next observation will be s_{t+1} , given that node s^t is reached. The decision maker's prior on Θ is denoted by μ . KMM2009 give assumptions such that *recursive smooth ambiguity preferences* over plans f at a node s^t are updated and represented as:

$$V_{s^t}(f) = u(f(s^t)) + \beta \phi^{-1} \left[\int_{\Theta} \phi \left(\int_{\mathcal{S}_{t+1}} V_{(s^t, s_{t+1})}(f) d\pi_\theta(s_{t+1} | s^t) \right) d\mu(\theta | s^t) \right], \quad (1)$$

where $V_{s^t}(f)$ is a recursively defined (*direct*) value function, u characterizes attitude to risk, β is a discount factor, ϕ is a function characterizing the decision maker's ambiguity attitude, while $\mu(\cdot | s^t)$ denotes the Bayesian posterior. A concave ϕ characterizes ambiguity aversion, which is defined to be an aversion to mean preserving spreads in the distribution over expected utility values. In general, the model does not impose reduction between the second-order belief μ and the first-order probabilities π_θ 's; reduction only applies when ϕ is linear, representing an ambiguity neutral Bayesian expected utility maximizer.

Ambiguity aversion in this model is equivalent to the DM behaving as more risk averse when choosing between bets on θ than when choosing between objective lotteries. That is, the DM strictly prefers a lottery which yields a unit payoff with objective probability m (and 0 with probability $1 - m$) to a (same stakes) bet on an event $T \subset \Theta$, where $\mu(T) = m$ (and also strictly prefers the complementary lottery to the bet on the complementary event).² (The behavior is exactly analogous to the modal behavior in the

²See section D in the Appendix for details.

Ellsberg two-urn example: preference for betting on a draw from the urn with a known 50:50 mix over betting on a draw from the urn with unknown mix.) Hence, the second-order measure μ cannot be calibrated with a lottery; behaviorally, μ is not treated as an objective probability. The standard interpretation is that the DM views his belief about events such as T to be less reliable than an objective probability.

2.2 A Lucas-tree economy and Euler equations with general beliefs

There is an infinitely-lived agent, with recursive smooth ambiguity preferences, consuming a single good. He can trade in a short lived risk-free asset, whose holding and price at time t are denoted b_t and P_t^f respectively. There is also an asset (whose quantity is normalized to 1 unit) that yields a stochastic dividend at each period, D_t . The asset with uncertain dividend (the “risky” asset) has a price P_t at time t , and its holding is denoted e_t . Consumption at time t is denoted C_t . As in Bansal and Yaron (2004) and Campbell (1996) we will assume that dividend and consumption follow different stochastic processes, thus departing from the original Lucas tree economy. The gap between consumption and dividend is due to some (exogenously given) labor income l_t .³ Equilibrium will require that at each time $C_t = l_t + D_t$.

Next, we derive Euler equations that define equilibrium prices in this economy. At a node $\{C_\tau, D_\tau\}_{\tau=1}^t$, let μ_t denote the second-order belief, on parameters in Θ defining first-order probability distributions on immediate successors (C_{t+1}, D_{t+1}) . Beliefs are updated as a function of the observed realizations of the consumption and dividend signals according to Bayes law. Wealth at time $t + 1$ is $W_{t+1} = (P_{t+1} + D_{t+1})e_t + b_t + l_{t+1}$, and the budget constraint in period t is given by $C_t = W_t - P_t e_t - P_t^f b_t$. The agent’s maximization problem may be described in terms of a recursive Bellman equation given by:

$$J(W_t, \mu_t) = \max_{C_t, b_t, e_t} u(C_t) + \beta \phi^{-1}[E_{\mu_t}(\phi(E_{\pi_\theta}(J(W_{t+1}, \mu_{t+1}))))], \quad (2)$$

subject to the budget constraint and the law of motion of the two “state” variables (wealth and beliefs), where $J(W_t, \mu_t)$ denotes a recursively defined *indirect* value function (as opposed to the direct value function in eq. (1)). An equilibrium of this economy is given by $\{(P_\tau, P_\tau^f, e_\tau, b_\tau, C_\tau)\}_{\tau=1}^\infty$ such that the consumption and asset holding processes solve the maximization program and the market clears, i.e., $e_t = 1$, $b_t = 0$, $C_t = D_t + l_t$ at each

³It is thus equivalent to derive the stochastic process followed by C_t from the assumed processes for D_t and l_t as we do in this section or to assume directly a stochastic process for C_t and D_t , leaving the process for l_t implicit.

t . First order conditions are given by:

$$\beta \Upsilon_t E_{\mu_t} [\xi_t(\theta) E_{\pi_\theta} (u'(C_{t+1}))] = P_t^f u'(C_t) \quad (3)$$

$$\beta \Upsilon_t E_{\mu_t} [\xi_t(\theta) E_{\pi_\theta} ((P_{t+1} + D_{t+1}) u'(C_{t+1}))] = P_t u'(C_t) \quad (4)$$

where $\Upsilon_t = E_{\mu_t} [\phi'(E_{\pi_\theta}(J(W_{t+1}, \mu_{t+1}))) \times (\phi^{-1})' [E_{\mu_t} (\phi(E_{\pi_\theta}(J(W_{t+1}, \mu_{t+1}))))]]$ and

$$\xi_t(\theta) = \frac{\phi'(E_{\pi_\theta}(J(W_{t+1}, \mu_{t+1})))}{E_{\mu_t} [\phi'(E_{\pi_\theta}(J(W_{t+1}, \mu_{t+1})))]}. \quad (5)$$

The function ξ_t is a Radon–Nikodym derivative effecting a *node specific* change of measure, or “distortion”, on the posterior μ_t , akin to martingale distortions arising in robust control problems considered by Hansen and Sargent. The distortion is a function of the continuation expected values obtained at successor nodes. In this paper we assume $\phi(x) = -\exp(-\alpha x)/\alpha$, where the parameter α represents ambiguity attitude. This specification simplifies the expressions significantly, since we now have $\Upsilon_t = 1$. It is also assumed that $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. With these specifications, the Euler equations are as follows:

$$\beta R_t^f E_{\mu_t} [\xi_t(\theta) E_{\pi_\theta} [\exp(-\gamma g_{t+1})]] = 1 \quad (6)$$

$$\beta E_{\mu_t} [\xi_t(\theta) E_{\pi_\theta} [R_{t+1} \exp(-\gamma g_{t+1})]] = 1 \quad (7)$$

$$\Leftrightarrow \beta E_{\mu_t} \left[\xi_t(\theta) E_{\pi_\theta} \left[\left(\frac{\exp(z_{t+1}) + 1}{\exp(z_t)} \right) \exp(d_{t+1} - \gamma g_{t+1}) \right] \right] = 1 \quad (8)$$

where $z_t = \ln\left(\frac{P_t}{D_t}\right)$, $g_{t+1} = \ln\left(\frac{C_{t+1}}{C_t}\right)$, $d_{t+1} = \ln\left(\frac{D_{t+1}}{D_t}\right)$, the logarithm of price-dividend ratio, rates of growth of consumption and dividend, respectively, while $R_t^f = \frac{1}{P_t^f}$, $R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$ denote the risk-free and risky rates of return.

Remark 1 *These Euler equations look identical to ones obtained in a standard Bayesian model except for the inclusion of the distortion function, ξ_t . The distortion, in the case of ambiguity aversion, increases the (posterior) weight on likelihoods π_θ with lower expected continuation values, $E_{\pi_\theta}(J(W_{t+1}, \mu_{t+1}))$. One could splice together the one-period ahead predictive distributions, $[\xi_t(\theta) \times \mu_t(\theta)] \otimes \pi_\theta(g_{t+1}, d_{t+1})$, and construct an overall “as if” unconditional probability distribution over the event tree which could be reinterpreted as coming from a Bayesian model. However, seen by itself, the constructed as if distribution cannot be linked to the given set of likelihoods $\{\pi_\theta\}_{\theta \in \Theta}$; indeed, typically, it is not possible to obtain the constructed distribution by starting at the initial node with a different prior $\mu'_1 \neq \mu_1$ on Θ with μ'_t , $t > 1$, obtained by updating in the usual way. Hence,*

an understanding of the role of ambiguity aversion in the modeling exercise is that it provides a link between the subjective as if distribution and a specification of beliefs about possible data generating the processes $(\{\mu_t\}_t, \{\pi_\theta\}_{\theta \in \Theta})$; beliefs which, in principle, can be objectively reconciled with data.

2.3 Beliefs and how they are applied in the evaluation of the Lucas tree

2.3.1 Description

We now describe the specific belief about the Lucas tree economy that we apply in our analysis. It is assumed the agent believes the growth rate of consumption (g_t) and dividends (d_t) are partly driven by a common latent state, x_t , which evolves according to an $AR(1)$ process with persistence ρ . While it is assumed there is a single persistence parameter operating through history, the agent is unsure what it is, believing there are two possible values of the parameter, high (ρ_h) or low (ρ_l). At time t the agent puts probability η_t on persistence being low and $(1 - \eta_t)$ on persistence being high. Each possible process is⁴:

$$\begin{aligned} x_{k,t+1} &= \rho_k x_{k,t} + \sigma_{x_k} \varepsilon_{x_k,t+1} \\ d_{k,t+1} &= \bar{d} + \psi x_{k,t+1} + \sigma_{d_k} \varepsilon_{d_k,t+1} = \bar{d} + \psi (\rho_k x_{k,t} + \sigma_{x_k} \varepsilon_{x_k,t+1}) + \sigma_{d_k} \varepsilon_{d_k,t+1} \\ g_{k,t+1} &= \bar{g} + x_{k,t+1} + \sigma_{g_k} \varepsilon_{g_k,t+1} = \bar{g} + \rho_k x_{k,t} + \sigma_{x_k} \varepsilon_{x_k,t+1} + \sigma_{g_k} \varepsilon_{g_k,t+1} \end{aligned} \quad (9)$$

where $(\varepsilon_{g_k,t+1}, \varepsilon_{d_k,t+1}, \varepsilon_{x_k,t+1})' \sim N(0, I)$, for $k = l, h$. We denote using, \bar{g} , \bar{d} the long-run growth rate of consumption and dividend, respectively. The shock $x_{k,t}$ is the temporary deviation from the trend (identified by the long-run growth rate). The interpretation is that the mean of the distribution on growth is partly fixed by the long-run trend and partly by a temporary shock to productivity due to the business cycle. The business cycle effect on the productivity across the economy is not observed directly. Though an innovation in each period, today's business cycle shock is, naturally, related to previous period's shock, and, so, is modeled by a auto-regressive latent variable. The factor ψ accounts for the empirically observed greater volatility of dividend relative to that of consumption.⁵ Note, there is a *different* tuple of volatility parameters $(\sigma_{g_k}, \sigma_{d_k}, \sigma_{x_k})$ associated with each possible value of persistence, ρ_k .

The agent is assumed to know the values of parameters $(\bar{g}, \bar{d}, \sigma_{g_k}, \sigma_{d_k}, \sigma_{x_k}, \psi)$. The agent observes, contemporaneously, the consumption and dividend growths. Given

⁴When $\eta_0 = 0$, the model reduces to the CASE I in Bansal and Yaron (2004).

⁵This modeling device was introduced in Abel (1999) and is followed widely in the finance literature and may be interpreted as the "leverage ratio" on (expected) consumption growth.

$x_{k,t}$, ρ_k and the current node $\{(C_\tau, D_\tau)\}_{\tau=0}^t$ the probability distribution over the immediate successor nodes, identified by (g_{t+1}, d_{t+1}) , is the product of two conditionally independent, given $x_{k,t}$ and ρ_k , Normal distributions,

$$g_{k,t+1} \sim N\left(\bar{g} + \rho_k x_{k,t}, \sigma_{g_k}^2 + \sigma_{x_k}^2\right) \text{ and } d_{k,t+1} \sim N\left(\bar{d} + \psi \rho_k x_{k,t}, \sigma_{d_k}^2 + \sigma_{x_k}^2\right).$$

This product distribution is the typical first-order distribution, the object $\pi_\theta(\cdot | s^t)$ in the abstract KMM formulation, with $(\rho_k, x_{k,t})$ playing the role of the unobserved parameter “ θ ”. (Note, since the volatilities $\sigma_{g_k}, \sigma_{d_k}, \sigma_{x_k}$ may vary with k , the parameter fixes *both* mean and variance.)

Thus, the domain (i.e., the support) of the second-order uncertainty at time t is an union of two component sets, $\{\rho_l x_{l,t} | x_{l,t} \in \mathbb{R}\} \cup \{\rho_h x_{h,t} | x_{h,t} \in \mathbb{R}\}$. The agent’s prior belief ascribes a measure to each component set: the measure on the first component is given by $\eta_0 \otimes N(0, \sigma_0^2)$ and that on the second by $(1 - \eta_0) \otimes N(0, \sigma_0^2)$. The agent updates beliefs using Bayes rule, based on the history of growth realizations and the presumption that the economy conforms to one of the two processes described in (9). Let $\hat{x}_{k,t} \equiv E[x_{k,t} | g_{k,1}, \dots, g_{k,t}, d_{k,1}, \dots, d_{k,t}]$ denote the expectation of $x_{k,t}$ conditional on the history of growth rates up to t if the beliefs were updated assuming $\rho = \rho_k$ is the data generating process. The filtered latent state corresponding to process k , $\hat{x}_{k,t}$, is obtained by applying the (steady state) Kalman filter that takes the process with $\rho = \rho_k$ as the “true” data generating process. The agent’s posterior belief then ascribes a measure on the first component set given by $\eta_t \otimes N(\hat{x}_{l,t}, \Omega_l)$ and that on the second by $(1 - \eta_t) \otimes N(\hat{x}_{h,t}, \Omega_h)$, where $\Omega_k, k = l, h$, denotes the steady state variance associated with the Kalman filter based on the process with $\rho = \rho_k$ and η_t shows the posterior belief on ρ_l . Hence, the agent’s posterior may be summarized by the tuple, $(\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$.⁶

We now turn to the evaluation of the Lucas tree with the specified beliefs. Denote by $\hat{x}_{k,t+1}^{(i)}$, $i = l, h$, $k = l, h$, the agent’s forecast for the (one period ahead) update using a Kalman filter which takes the model with $\rho = \rho_k$ as the data generating process, when the data is actually generated by the $\rho = \rho_k$ model. Correspondingly, $\eta_{t+1}^{(l)}$ (respectively $\eta_{t+1}^{(h)}$) is the posterior probability that the low persistence process is the correct model when the low (high) persistence model is the data generating process. The direct continuation value is a function of the current node but does not distinguish between two histories which have the same current consumption and same current belief, summarized by $(\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$. The function is defined by the following recursion:

⁶See section B in the Appendix for further details about the updating.

$$V(C_t; \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = u(C_t) + \beta \phi^{-1}(\mathcal{V}_{t+1},) \quad (10)$$

where

$$\begin{aligned} \mathcal{V}_{t+1} \equiv & \eta_t E_{\hat{x}_{l,t}} \left[\phi \left(E_{x_{l,t}} \left[V \left(C_t \exp(g_{l,t+1}), \hat{x}_{h,t+1}^{(l)}, \hat{x}_{l,t+1}^{(l)}, \eta_{t+1}^{(l)} \right) \right] \right) \right] \\ & + (1 - \eta_t) E_{\hat{x}_{h,t}} \left[\phi \left(E_{x_{h,t}} \left[V \left(C_t \exp(g_{h,t+1}), \hat{x}_{h,t+1}^{(h)}, \hat{x}_{l,t+1}^{(h)}, \eta_{t+1}^{(h)} \right) \right] \right) \right]. \end{aligned}$$

To see how the KMM representation is being implemented, we note the following. The argument of a $\phi(\cdot)$ is an expectation of the continuation value/utility at successor nodes, where the expectation $E_{x_{k,t}}$ is taken with respect to the typical first-order distribution described earlier, defined by fixing the “parameter pair” $(\rho_k, x_{k,t})$. The measure on $(\rho_k, x_{k,t})$ is given by $\eta_{t,k} \otimes N(\hat{x}_{k,t}, \Omega_k)$ and we calculate the expectation of the functions $\phi(\cdot)$ by applying this measure, which corresponds to the second-order measure μ_t in the KMM representation.

2.3.2 Motivation for the beliefs model and parameter choice

Hamilton (1989) pioneered the idea of modeling consumption growth as an auto-regressive process, with parametric shifts occurring through Markovian transitions on latent states. That paper also showed that the idea was a particularly good fit for the U.S. growth experience through the improved facility of capturing the effect of business cycles. Hence, the basic functional form of (9) with a given ρ_k , is a plausible starting point for describing the beliefs of an investor for whom the key source of uncertainty is the business cycle.

Adding uncertainty about ρ_k to (9) is empirically justified and improves it as a framework for understanding and quantifying ambiguity about macroeconomic growth; this enables (9) to encapsulate a theory of why it is difficult to precisely estimate the probability distribution of growth, and of why and how that imprecision will vary with history. The key is that the two uncertainties, about persistence ρ_k , and about $x_{k,t}$, which controls the mean of the distribution, go hand in hand: they interact and reinforce each other to make the belief about the “true” growth distribution unreliable and inference about it imprecise. Shephard and Harvey (1990) explains that it is very difficult (in that it would take an inordinately long series of observations) to determine whether the true growth process is a very persistent process where the persistent component has a small volatility or whether it is a moderately persistent process with a persistent component

that has a large volatility. Thus, uncertainty about the volatility of the latent variable makes the persistence parameter difficult to estimate. Indeed, even after almost a century of data the learning, far from settling down on one value of ρ_k , produces posteriors η_t that have varied continually between 0.3 and 0.7. In turn, the uncertainty about ρ_k degrades the inference on the evolving latent variable $x_{k,t}$. The expectation of this variable is tracked by the Kalman filter, but the specification of the Kalman filter is determined by the value of the persistence parameter. Since that is not reliably known, the Kalman forecast is imprecise.

This understanding of the uncertainty described by (9) motivates how it is represented in the different parts of the KMM preference functional. Given $(\rho_k, x_{k,t})$, the uncertainty about the parameters of the distribution on growth is almost objective since the other parameters fixing the distribution may be reliably estimated given this knowledge and the run of data. On the other hand, the uncertainty about $(\rho_k, x_{k,t})$, though probabilistically represented, may be viewed as a deeper uncertainty, far less reliably estimated and more variable. Thus, the former uncertainty appears as a first-order belief in the KMM functional (i.e., “inside” the ϕ) whereas the latter uncertainty is treated as a second-order uncertainty (i.e., “outside” the ϕ).

There are two reasons for choosing a two point support for the uncertainty about persistence. One is computational limitation (with more than two points the number of “state variables” in the dynamic problem that we have to solve goes beyond the state of art capabilities). The second is that a two-point support is an efficient way of capturing Shephard and Harvey (1990)’s key insight that the crucial empirical confound underlying the uncertainty is the confound between a high persistence combined with low volatility parameters on one hand and low persistence combined with larger volatility parameters, on the other. We were guided in part by findings in the literature, and in part by our own empirical investigations, in choosing the values of ρ_k . One substantial strand of literature (the long run risk literature, pioneered in Bansal and Yaron (2004)) argues there is strong justification, based on asset pricing moments, for assuming a high value of ρ (we set $\rho_h = 0.85$ as the standard case and 0.90 for robustness checks, which corresponds to the endpoints of the interval of values suggested by this literature). Another strand points out that pure consumption growth data suggests a more moderate value; we set $\rho_l = 0.30$, motivated by studies in Beeler and Campbell (2009) and Constantinides and Ghosh (2010).⁷ It is generally agreed the estimates are quite fragile. Our

⁷Constantinides and Ghosh (2010) provide a GMM estimate (based on the years 1931-2006) of $\rho = 0.32$ (see their Table 4). Though we set $\rho_l = 0.30$, (we found) values between 0.25 and 0.40 have virtually

own investigations found, setting $\rho_l = 0.2$ and $\rho_h = 0.85$, η_t is approximately 50% in 1977, the beginning of the model evaluation period, and is consistently in the interval $[0.3, 0.7]$ throughout the period 1978-2011, demonstrating how difficult it is to separate the two persistence models on the basis of growth data.

The time-series parameters of the model (except for the persistence parameters ρ_k , and the leverage-ratio parameter ψ) were estimated using maximum likelihood on annual U.S. data from 1930 to 1977 (see section A in the Appendix for details about the data set and the parameter values.) The remaining years in the data set, 1978-2011, were used in the evaluation of the model. Our aim was to have the longest run of data for the evaluation of the model. Going back beyond 1977 causes problems in that the parameter estimates change significantly through the 70s because of the macroeconomic events. By starting the evaluation at 1978, the maintained assumption that the agent behaves as if he knows the parameter values of the model becomes more credible. Turning to preference parameters, in all cases the ambiguity aversion parameter α was calibrated to produce a real risk-free rate of 1.5%, averaged over $t = 1978, \dots, 2011$, which is the average observed rate in that period. Section 4 discusses whether the calibrated level is plausible for an individual agent. No other moments were used in the choice of α . Choice of the other preference parameters follows the standard practice in the literature.

3 Implications of the model for asset returns and prices

3.1 Understanding the mechanism of ambiguity aversion

A good way to understand the key channels through which ambiguity aversion affects asset returns in our model is by understanding how the distortion function, ξ , shown in eqn (5) shapes the “as if” belief of the agent, i.e., the (probabilistic) belief which supports the action chosen by the agent in equilibrium. We identify two main mechanisms. The first works through the endogenous pessimism and added doubt that the as if belief embodies, at any one point in time, compared to the belief of an agent with rational expectations based on the processes underlying the specified belief model. The second mechanism is an endogenous accentuation of the cyclical variation in uncertainty.

identical posteriors (and implications for rates of returns).

3.1.1 Endogenous pessimism and doubt

The intuition behind the first channel can be more transparently understood in the special case of the model of beliefs where there is no uncertainty about the persistence (e.g., $\eta_0 = 0$). Under this assumption the argument $(x_{t,t}, \eta_t)$ drops out of the value function described in (10), and the distortion is given as (suppressing “ k ” subscripts);⁸

$$\xi_t(x_t | C_t, \hat{x}_t; \alpha) \equiv \frac{\exp(-\alpha(E_{x_t}(V(C_{t+1}; \hat{x}_{t+1}))))}{E_{\hat{x}_t}[\exp(-\alpha(E_{x_t}(V(C_{t+1}; \hat{x}_{t+1}))))]}. \quad (11)$$

The effect of ξ_t is to create an “as if” posterior, i.e., a distorted posterior, $\tilde{\mu}_t \equiv \xi_t(x_t) \otimes N(\hat{x}_t, \Omega)$. In the case of ambiguity aversion, i.e., $\alpha > 0$, it is evident from eq. (11) that $\tilde{\mu}_t$ puts relatively greater probability mass (compared to μ_t) on x_t ’s that generate probability distributions associated with lower expected continuation values, $E_{x_t}(V(C_{t+1}; \hat{x}_{t+1}))$. The distorted posterior gives rise to an “as if” conditional one-step-ahead distribution on growth which we call the *twisted (predictive) distribution*

$$g_{t+1} \sim \xi_t(x_t) \otimes N(\hat{x}_t, \Omega) \otimes N(\rho x_t + \bar{g}, \sigma_x^2 + \sigma_g^2). \quad (12)$$

When $\xi_t(x_t) = 1$ the formula (12) describes the belief of a Savage-Bayes rational (or, equivalently, ambiguity neutral) agent, a useful benchmark. Such an agent, whom we dub “Bayesian,” is uncertain about x_t with belief about growth described by a mixture of normals. The twisted distribution, on the other hand, describes the predictive “as if” belief of an ambiguity sensitive agent.

Another useful benchmark is the predictive belief of an agent with “rational expectations”, narrowly defined. This distribution, $N(\rho \hat{x}_t + \bar{g}, \sigma_x^2 + \sigma_g^2)$, arises from a posterior that is *degenerate* on \hat{x}_t . As Figure 1 shows, compared to the rational expectations distribution, the twisted distribution has a lower mean *and* a larger spread. Abel (2002) argues that one can account for the observed equity premium and the risk-free rate by invoking pessimism and doubt in an otherwise standard asset pricing model. Pessimism is deemed, by Abel, as a subjective distribution on growth that is first order stochastically dominated by the “objective” distribution; doubt, corresponds to a subjective distribution that is a mean preserving spread of the objective distribution. Evidently, an ambiguity averse agent’s conditional (“as if”) beliefs, in effect, incorporate *endogenously both*

⁸Henceforth, we shall write ξ_t as a function of direct continuation value $V(\cdot)$ instead of the indirect value, $J(W_{t+1}, \mu_{t+1})$. In a single agent economy consumption is exogenously determined, and so it is possible to solve for the continuation value at any node on the event tree without solving for the equilibrium prices first.

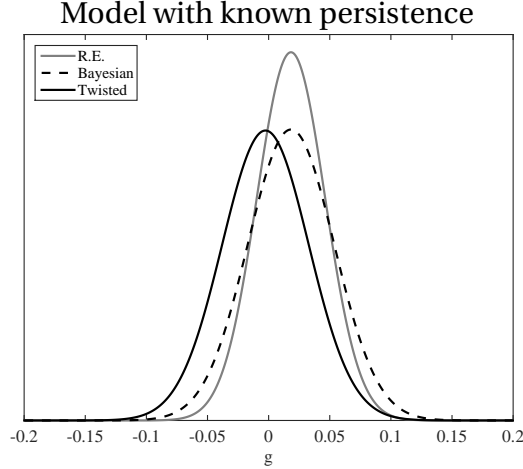


Figure 1: **Beliefs and “as-if” beliefs:** The agent’s “as-if” belief about the conditional distribution of consumption growth with no uncertainty about the latent state (R.E.), with uncertainty about the latent state but without ambiguity aversion (Bayesian) and with ambiguity aversion about the uncertainty of the latent state (Twisted). The distributions were computed using $\rho = 0.85$, and the level of consumption and latent state as the average over 1978–2011.

these elements while the Bayesian agent only incorporates the doubt. These observations will be the key to understanding our results on time averages of conditional returns moments.

3.1.2 Endogenous accentuation of cyclical variation in uncertainty

To understand the second mechanism we return to the beliefs model without the restriction of $\eta_0 = 0$. Learning about persistence leads to time-varying mixing of the two processes through η_t . This produces a posterior *predictive* belief about consumption growth which is heteroskedastic across time, even though in each process (with a given persistence) the growth distribution is homoskedastic. The mean and variance of the mixture distribution on the latent state are,

$$\hat{x}_t = \eta_t \hat{x}_{l,t} + (1 - \eta_t) \hat{x}_{h,t}, \quad (13)$$

$$Var_r(x_t) = \eta_t \Omega_l + (1 - \eta_t) \Omega_h + \eta_t (1 - \eta_t) (\hat{x}_{h,t} - \hat{x}_{l,t})^2. \quad (14)$$

It is as if the agent has two forecasting models. When the history is such that both models explain that history just as well, i.e., η_t is close to 0.5 and yet their core forecasts markedly disagree, i.e., $(\hat{x}_{h,t} - \hat{x}_{l,t})^2$ is large, the uncertainty, as shown by the variance, rises. In contrast in the case with $\eta_0 = 0$, what happens over time to the posterior is that its mean

\hat{x}_t may change but not its variance, ensuring a homoskedastic predictive.⁹

The endogenously time varying uncertainty in our model, due to learning about the persistence, creates a potential for *uncertainty shocks*, sudden sharp increases in uncertainty about consumption growth. One way an uncertainty shock can come about is as follows. A sequence of moderately positive growth realizations, being quite consistent with high and low persistent processes, brings η_t close to 1/2. If one or more negative realizations arise after such a sequence, $(\hat{x}_{h,t} - \hat{x}_{l,t})^2$ increases, thus increasing $Var_t(x_t)$. Ambiguity aversion exacerbates the time-variation of the Savage-Bayes uncertainty by endogenously accentuating that uncertainty asymmetrically between positive and negative shocks, creating “as if” uncertainty shocks that are far sharper than what is reflected by dynamics of $Var_t(x_t)$.

To see how, consider the following. The distorted posterior is a mixture of two component distorted posteriors, $\xi_t^k \otimes \eta_t \otimes N(\hat{x}_{k,t}, \Omega_k)$ for $k = h, l$, where ξ_t^k is as in eq. (23) in Section B.1.2 in the Appendix. Let $\tilde{x}_{k,t}$ denote the mean of a distorted component posterior, $\xi_t^k \otimes N(\hat{x}_{k,t}, \Omega_k)$. Due to the greater persistence, the aggregate uncertainty around $\hat{x}_{h,t} - \hat{x}_{l,t}$ – captured by Ω_h – is larger than that around $\hat{x}_{l,t}$. Since the distortion function is proportional to a negative exponential, it has more bite on a distribution which has more probability mass on the left tail by whipping up that mass even more; hence, we have $\hat{x}_{h,t} - \tilde{x}_{h,t} > \hat{x}_{l,t} - \tilde{x}_{l,t}$. Which means that $(\hat{x}_{h,t} - \hat{x}_{l,t})^2 > (\tilde{x}_{h,t} - \tilde{x}_{l,t})^2$ when $\hat{x}_{h,t} > \hat{x}_{l,t}$ (as would be, following a positive shock) and $(\hat{x}_{h,t} - \hat{x}_{l,t})^2 < (\tilde{x}_{h,t} - \tilde{x}_{l,t})^2$ when $\hat{x}_{h,t} < \hat{x}_{l,t}$ (following a negative shock). Hence, when $\hat{x}_{h,t} < \hat{x}_{l,t}$, the components of the mixture yielding the as if posterior are further apart compared to the components of the Bayesian posterior (and, conversely, when $\hat{x}_{h,t} > \hat{x}_{l,t}$). This has two implications. One, $\overline{Var}_t(x_t)$, the variance of the distorted posterior¹⁰ understates that of the Bayesian posterior following a positive shock, and exaggerates it following a negative shock, making it more pronouncedly counter-cyclical than $Var_t(x_t)$. Two, the distorted posterior demonstrates a significant negative skewness compared to the Bayesian posterior in recessionary periods, but not in good times.

⁹The case for introducing time-varying volatility of macroeconomic variables has been argued strongly in the recent literature, e.g. Fernandez-Villaverde and Rubio-Ramírez (2010). Much of this literature, including Bansal and Yaron (2004) (see their main, CASE II, model) models this by positing an exogenously specified stochastic volatility. Beeler and Campbell (2009) and Constantinides and Ghosh (2010) argue that the assumption of highly persistent stochastic volatility of innovations to consumption (key factor underpinning the exogenous specification of stochastic volatility) is not well supported empirically. In contrast, the time-varying heteroskedasticity generated endogenously in our model is a *forecast* uncertainty, of beliefs, empirically driven by the history of growth outcomes and consistent with a stationary volatility of consumption shocks.

¹⁰See section B.1.1 in the Appendix for an analytical expression.

The left panel in Figure 2 shows how $\hat{x}_{h,t}$ and $\hat{x}_{l,t}$ have moved with the business cycle. The right panel compares the variance of the posterior and the variance of the distorted posterior showing that the latter greatly amplifies movements in the former, especially at downturns. Figure 2 also shows that in 1992 $\hat{x}_{h,t} < \hat{x}_{l,t}$ while in 1999 $\hat{x}_{h,t} > \hat{x}_{l,t}$, though $|\hat{x}_{h,t} - \hat{x}_{l,t}|$ were similar in these two years. Figure 3 demonstrates how much more significant the effect of the distortion was on the posterior in the latter year.

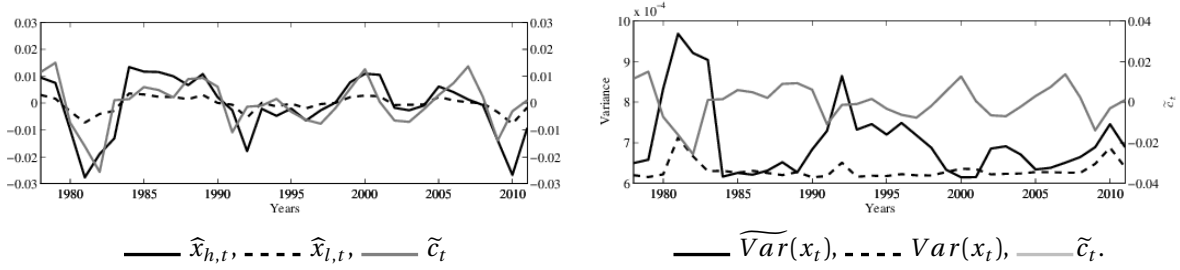


Figure 2: **Explaining time-varying ambiguity:** The left panel shows the filtered latent variables assuming that the high ($\hat{x}_{h,t}$) and low ($\hat{x}_{l,t}$) persistence as the DGP. The right panel graphs the conditional variance of the latent state variable ($Var_t(x_t)$) and the “as if” conditional variance ($\widetilde{Var}_t(x_t)$). In both panels the gray line shows the HP-filtered consumption growth, indicating the business cycle.

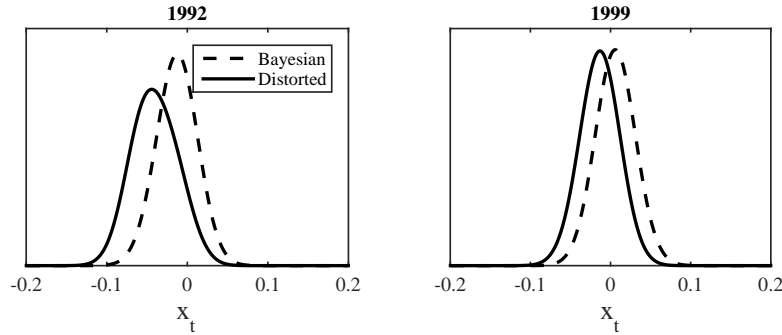


Figure 3: **Time-varying distortion:** The two panels plot beliefs about the latent state without ambiguity aversion (Bayesian) and with ambiguity aversion. The left panel shows a “bad” year where $\hat{x}_{h,t} < \hat{x}_{l,t}$, and the right panel shows a “good” year where $\hat{x}_{h,t} > \hat{x}_{l,t}$.

The following argument focused on the uncertainty about ρ_k offers another, and perhaps pithier, intuition.¹¹ The ambiguity averse agent behaves as if he forecasts consumption growth putting more weight (compared to the Bayesian posterior) on the “worst case” persistence, i.e., the ρ_k that minimizes the expected continuation utility. When

¹¹We thank a referee for suggesting this.

consumption growth is below the mean, the worst case persistence parameter is ρ_h , suggesting that we will remain below the mean for a long time. In contrast, when consumption growth is above the mean, the worse case is that the persistence is ρ_l , so we revert quickly to the mean. Thus, the ambiguity averse agent, endogenously behaves as if the uncertainty is more persistent and severe following negative shocks than in normal times (*even though* $\eta_t \simeq 1/2$). These insights about the asymmetric reaction to good and bad news will be key to understanding how ambiguity aversion affects conditional returns and their variation over time, in particular, over the business cycle.

3.2 Comparing model implications with data

We use annual data on real per-capita consumption C_t and estimates of $\hat{x}_{k,t}$ corresponding to the filtration imposed by the observed history of growth of real consumption and of real dividends to obtain a time series of model implied *conditional moments* of the annual rates of return using our numerical solution technique (see section E in the Appendix.) We compute the model implied price-dividend ratio applying the relationship

$$R_{t+1} = \frac{\exp(p_{t+1} - d_{t+1}) + 1}{\exp(p_t - d_t)} \exp(d_{t+1}) \quad (15)$$

where d_t is taken from the historical data, R_{t+1} and p_{t+1} are computed from the model, and the recursion is started from the actual price-dividend ratio in 1977 ($t = 0$). Throughout the exercise, the level of ambiguity aversion was calibrated so that the average risk-free rate was 1.5%.

We present and discuss two kinds of results on model implied conditional moments of rates of return and price-dividend ratio: averages of the moments over the sample period, 1978-2011 in section 3.2.1, and time series (and time series properties) of the moments over the same sample period in section 3.2.2. In section 3.2.3 we compare the time-series of our model implied equity premium with the leading macroeconomic uncertainty index in the literature.

3.2.1 Time averages of moments

Table 1 reports the model implied conditional moments of returns and price-dividend ratio, time averaged over the sample period. The panels in Figure 4 show the comparative statics of ambiguity aversion and risk aversion on the conditional rates of return. The model's match of the first moments of returns is quite perfect and second moments are predicted to a large extent.

Returns and Volatility								
	γ	α	$E(r)$	$E(r - r^f)$	$\sigma(r^f)$	$\sigma(r)$	$\sigma(r - r^f)$	
Data			8.08	6.68	2.20	16.5	16.1	
	1.0	31.5	6.61	5.08	1.20	22.2	22.2	
	2.0	17.8	7.36	5.85	2.58	23.0	23.0	
	2.5	11.3	7.97	6.46	3.29	23.55	23.6	
	3.0	6.65	8.66	7.14	3.96	24.17	24.2	
Robustness Checks								
$\rho_h = 0.90$	2.5	7.30	7.88	6.36	3.83	23.5	23.6	
$\rho_l = 0.25$	2.5	11.1	7.98	6.48	3.05	23.7	23.7	
$\psi = 2.50$	2.5	11.3	7.58	6.07	3.15	23.6	23.5	
$\beta = 0.965$	2.5	13.0	9.15	7.62	3.44	23.8	23.8	
$\beta = 0.97$	2.5	12.2	8.56	7.05	3.36	23.7	23.7	
Bayesian	2.5	≈ 0	7.62	0.62	1.70	23.1	23.2	
Price-Dividend Ratio								
	γ	α	$\mathbb{E}(P/D)$	$\sigma(P/D)$	$\mathbb{E}(p - d)$	$\sigma(p - d)$	AC1	AC2
Data			45.513	19.954	3.724	0.445	0.803	0.759
	1.0	31.5	29.3	4.34	3.37	0.15	0.51	0.48
	2.0	17.8	32.3	5.92	3.46	0.19	0.65	0.60
	2.5	11.3	44.0	14.5	3.73	0.34	0.85	0.78
	3.0	6.65	52.9	22.2	3.88	0.43	0.88	0.81
Robustness Checks								
$\rho_h = 0.90$	2.5	7.30	42.9	13.7	3.71	0.33	0.84	0.78
$\rho_l = 0.25$	2.5	11.1	44.3	14.8	3.74	0.35	0.85	0.78
$\psi = 2.5$	2.5	11.3	39.6	11.1	3.64	0.29	0.82	0.75
$\beta = 0.965$	2.5	13.0	59.9	28.1	3.98	0.49	0.89	0.82
$\beta = 0.97$	2.5	12.2	51.3	20.6	3.86	0.42	0.88	0.81
Bayesian	2.5	≈ 0	40.0	11.5	3.65	0.30	0.82	0.75
Bayesian, $\beta = .97$	2.5	≈ 0	46.3	16.5	3.77	0.37	0.86	0.79

Table 1: The top panel contains the average of the predicted *conditional* moments of rates of return (on dividend claim) for different values of γ and calibrated α . Immediately below is a series of robustness checks where the parameter in the left-most column was changed from the basic specification ($\rho_h = 0.85$, $\rho_l = 0.3$, $\psi = 3$, $\beta = 0.975$), taking $\gamma = 2.5$ as part of the baseline specification. The bottom panel contains the time-averaged model implied price/dividend ratio statistics over the period 1978–2011. AC1 and AC2 denote the first and second order autocorrelation of $p - d$.

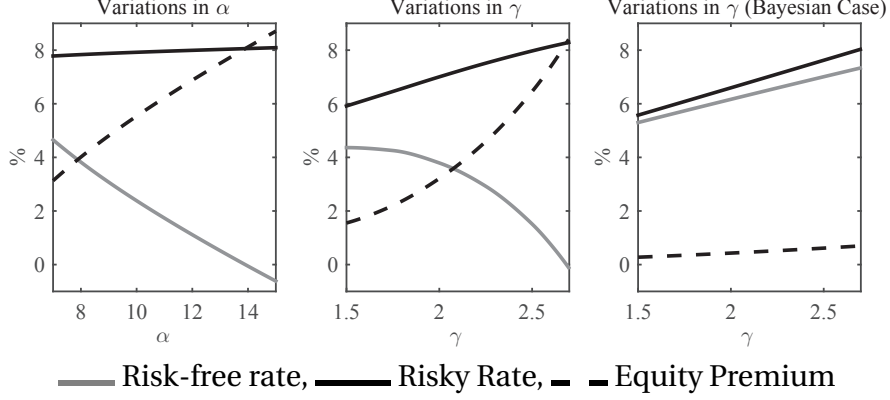


Figure 4: **Comparative statics** : In the left panel, α varies with γ fixed at 2.5. In the middle panel, α was fixed at 11.3 and γ varies. The average comparative statics are constructed by first computing the comparative statics for each year using the filtered values \hat{x}_t and then averaging across $t = 1978, \dots, 2011$. The right panel depicts the Bayesian case, i.e., with $\alpha \approx 0$. (The graphs correspond to our model with unknown persistence.)

To help us understand these results (which were obtained numerically) we consider analytical approximations¹² for the rates of return for the case where persistence is known, e.g., with $\eta_t = 0$. The risk-free rate is approximated as:

$$r_t^f = -\ln \beta + \gamma \bar{g} + \gamma \rho \tilde{x}_t - \frac{\gamma^2}{2} \left(\sigma_x^2 + \sigma_g^2 + \rho^2 \widetilde{Var}_t(x_t) \right). \quad (16)$$

where \tilde{x}_t is the mean of the distorted posterior at time t .

An increase in ambiguity aversion, α , decreases \tilde{x}_t making the agent behave as if he were expecting a lower endowment income in future states. Implying, a rise in demand for the risk-free asset (a “flight to quality”, as termed by Caballero and Krishnamurthy (2008)) driving up its equilibrium price and lowering the risk-free rate. The accentuation of doubt, working through $\widetilde{Var}_t(x_t)$ reinforces the effect. This is a key effect of ambiguity aversion. Note, when $\alpha > 0$ the term $\gamma \rho \tilde{x}_t$ acts to dampen the effect of $\gamma \bar{g}$, making the comparative static of γ on the risk free rate very different, qualitatively and quantitatively, depending on whether $\alpha > 0$ or $\alpha = 0$, as a comparison of the middle and right panels of Figure 4 shows. Hence, it is not possible to replicate the effect of ambiguity aversion by turning it off and simply varying γ .

The first moment of the risky rate is approximated as

$$E_t r_t = \text{Const}_1 + \rho (\gamma - \psi) \tilde{x}_t + \psi \rho \hat{x}_t - \frac{\rho^2}{2} [(\gamma - \psi)^2 \text{Const}_2] \widetilde{Var}_t(x_t) \quad (17)$$

where $E_t \equiv E_{\hat{x}_t} E_{x_t}$ describes the *conditional* expectation of a Savage-Bayes rational observer/analyst who observes these prices and uses the same information as the agent

¹²See Appendix C for details of the derivation.

to predict dividend at $t + 1$. Const_1 and Const_2 collect terms which are constant across time and not affected by ambiguity aversion. An increase in α has two countervailing effects. The first effect, given by $\rho\gamma\tilde{x}_t$, was also present in the expression for the risk-free rate; the intuition here is analogous. The second effect is in the term $-\rho\psi\tilde{x}_t$. As α increases \tilde{x}_t decreases, hence decreasing the (“as if”) expected future dividend payoff from the asset causing the agent to want to pay less for the asset. With $\gamma \leq 3$ and $\psi = 3$, as we have here, the second effect dominates (very slightly) and equilibrium risky rate varies positively (but quite minimally) with ambiguity aversion.

The approximation for the equity premium may be written as

$$E_t r_t - r_t^f = \text{Const}_3 + \psi\rho(\hat{x}_t - \tilde{x}_t) + \frac{\rho^2}{2}[\gamma^2 - (\psi - \gamma)^2 \text{Const}_2] \widetilde{\text{Var}}_t(x_t). \quad (18)$$

where we have explicitly left the two terms which are affected by ambiguity aversion, $(\hat{x}_t - \tilde{x}_t)$ and $\widetilde{\text{Var}}_t(x_t)$. The first term shows that the premium increases with ambiguity aversion (the difference $(\hat{x}_t - \tilde{x}_t)$ increases when α is increased) and the magnitude of this effect is accentuated by persistence and leverage. A doubt factor also comes into play (principally) through its effect on the risk free rate, discussed earlier. Since, the risk free rate is conditionally non-stochastic, the conditional volatility of equity premium coincides with that of the risky rate. The overwhelming factor fixing the (average) conditional volatility of risky return is the volatility of the dividend claim, in turn determined by the volatility of the latent state multiplied by ψ and ρ .

The (log of) price dividend ratio, z , is approximated as showing that it follows the expectation of the distorted posterior on the latent variable.¹³

$$z_{t+1} = A_0 + A_1\tilde{x}_{t+1} \quad (19)$$

To summarize, ambiguity aversion gets the first moment of equity premium right by holding the risk free rate down while affecting the risky rate only very marginally. The volatility comes from two sources, the uncertainty about the latent state accentuated by the uncertainty about the persistence and the leverage factor. The two models which have produced matches as close as we have here on average moments are those in Ju and Miao (2012) and Bansal and Yaron (2004). In the former, matching the second moment statistics requires the inclusion of an extra effect in preferences à la Epstein-Zin with an intertemporal elasticity of substitution (EIS) greater than 1, quite apart from ambiguity

¹³The formulas for A_0, A_1 are given by eqs. (32) and (31) in section C in the Appendix. Notably, given our parameter values, A_1 is positive and proportional to $\rho(\psi - \gamma)$.

aversion. The crucial ingredient required to match second moments in the latter paper is the exogenously set stochastic volatility. Furthermore, in these models the moments arise from simulated histories, not actual histories, whereas the results here are averages of moments conditioned on the actual history.

3.2.2 Time series profiles of conditional rates of return and price-dividend ratio

Perhaps the most distinctive results of the analysis in this paper concerns the time series of conditional moments. These are largely driven by dynamics of the as if belief explained in section 3.1.2. Figure 5 demonstrates this quite vividly in the case of the equity premium. Studies have estimated conditional moments of equity premium on historical data, notably Kandel and Stambaugh (1990) and Whitelaw (1994). The latter summarizes a key finding as follows (pp 526; in the quote “expected return” is the conditional first moment of equity returns in excess of risk free rate) :

The expected return seems to reach a maximum at the trough of the business cycle and reach a minimum before, or at, the peak of the business cycle. Expected returns appear to decrease during economic expansions and increase during economic contractions. In contrast, the conditional volatility appears to reach a maximum earlier in the business cycle, at or slightly after the peak in the cycle, and to reach a minimum just after the business cycle trough.

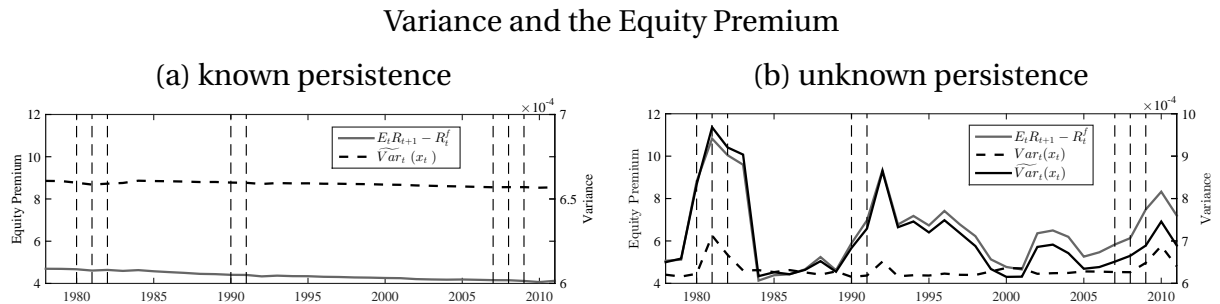


Figure 5: **Movements in variance and model implied equity premium:** Panel (a) shows the conditional equity premium and the conditional variance of the “as-if” posterior from the model with known persistence, $\rho = 0.85$. Panel (b) shows the same as well as the variance of the undistorted posterior for the model with unknown persistence. Vertical dashed lines indicate years featuring a recession.

Figures 5(b) and 6(a) show how well the series predicted by our model match the above quote. Equity premium, as predicted by the model, is counter-cyclical; it’s cor-

relation with H-P filtered consumption growth is -0.59. Whitelaw (1994) estimates the contemporaneous correlation between the first and second (conditional) moments (of equity premium) to be -0.34; based on the data considered for this paper, which pertains to a different time period and frequency, the correlation of the same two statistics in our model is -0.86.

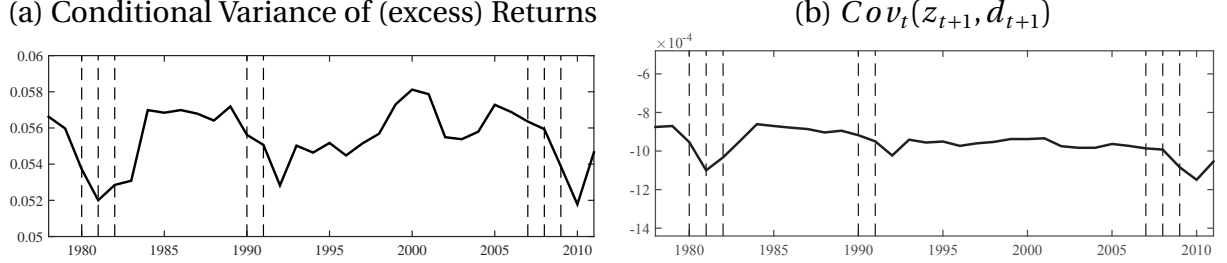


Figure 6: The two panels depict the conditional variance of (excess) returns and $Cov_t(z_{t+1}, d_{t+1})$, implied by the model, demonstrating the close link between the dynamics of the two. Vertical dashed lines indicate years featuring a recession.

What accounts for the pro-cyclical volatility of returns in our model? Starting from the standard approximation for the risky rate (eqn. 28 in section C in the Appendix), its variance may be seen to be composed as:

$$Var_t(r_t) \simeq \kappa_1^2 Var_t(z_{t+1}) + Var_t(d_{t+1}) + 2\kappa_1 Cov_t(z_{t+1}, d_{t+1}).$$

(In our data, $\kappa_1 = 0.98$.) It turns out the time averaged variance is completely swamped by the term $Var_t(d_{t+1})$ ($\overline{Var_t(r_t)} = 0.0555$, $\overline{Var_t(d_{t+1})} = 0.0541$ and $\overline{Var_t(z_{t+1})} = 1.17e - 4$). However, as seen from Figure 6, the dynamics of $Var_t(r_t)$ are very largely determined by $Cov_t(z_{t+1}, d_{t+1})$. To see an intuition why this covariance is negative and even more so in recessionary times, note, (belief about) d_{t+1} is determined by the Bayesian posterior, with mean \hat{x}_t , while z_{t+1} is guided by \tilde{x}_t , the mean of the distorted posterior. As explained in section 3.1.2, \tilde{x}_t is below \hat{x}_t and in recessions, even more so and less mean reverting (i.e., more persistent) than \hat{x}_t . Hence, in recessions there is a bigger measure of events where d_{t+1} realizes above its mean while z_{t+1} stays below its mean. The price-dividend ratio is function of the agent's view of the longer term prospects while the dividend is just the outcome in the next period; the former may remain relatively downbeat and sluggish, especially in recessionary times, despite a positive outcome of the latter.

Models in the literature have found it hard to explain counter-cyclical of conditional equity premium and pro-cyclical of conditional (excess) return volatility, especially the latter. A device used to obtain stochastic volatility, quite common in the

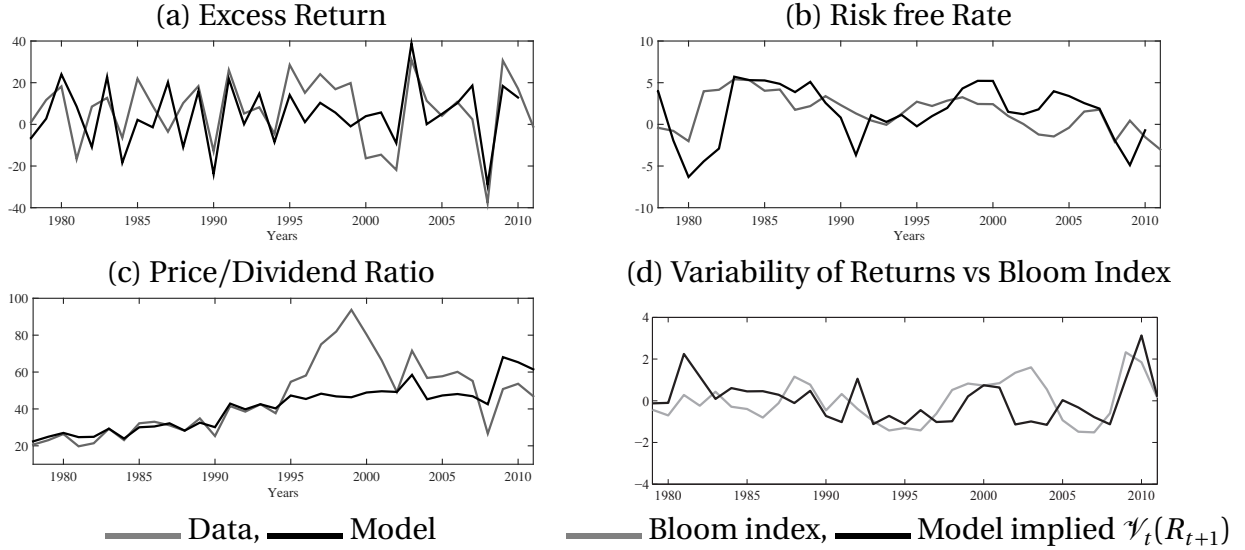


Figure 7: **Returns and Price–Dividend Ratio:** Panel (a) contains a plot of the model-implied excess return along with the actual excess return. Panel (b) shows the model-implied risk-free rate along with the actual real risk-free rate. Panel (c) contains the actual and model implied price-dividend ratios. Panel (d) shows the time-series of $\mathcal{V}_t(R_{t+1}) \equiv \sqrt{E_t(R_{t+1} - ER_{t+1})^2}$ and stock market volatility index constructed in Bloom (2009). For comparison purposes, both the Bloom index and $\mathcal{V}_t(R_{t+1})$ are normalized by their respective mean levels. So, on the vertical axis, we measure the (signed) percentage deviations from the respective means.

asset pricing/equity premium literature, is to have a Markovian model of consumption growth involving up to four latent states, with each state implying a one-period-ahead Gaussian growth distribution with a mean that is high or low and variance that is high or low. Learning varies the posterior weight on the high variance state, creating stochastic volatility. Kandel and Stambaugh (1990) calibrates such a model to match consumption growth and unconditional returns moment statistics. The calibration however, does not deliver a stochastic volatility that ensures a counter cyclical conditional equity premium (nor pro-cyclical excess returns volatility). Whitelaw (2000)’s model is in the same genre, and while it manages to produce the negative correlation between expectation and volatility of excess returns, it fails to match other features of the equity return data (e.g., equity premium is much too low). Ju and Miao (2012) and Bansal and Yaron (2004) obtain a counter-cyclical equity premium but *not* pro-cyclical volatility.

While equity premium is not directly observed, we do observe the realized risky rate, risk free rate, the *realized* excess return (the difference between the two) and the price-dividend ratio. Figure 7 plots these and the corresponding series implied by the model

(each point shows the value of the variable forecast by the model at a date given the information set at *that* date).¹⁴ This sets out a stark, stiff test for the model. The predictions are evidently good, especially for returns but reasonably good too for the price-dividend ratio. The correlation of the realized risky rate and excess return with \hat{x}_t is -0.08 and -0.1 in data compared to -0.07 and -0.21, respectively, in the model prediction. The instantaneous correlation between R and (p-d) is positive in the data (0.54) and in the model 0.66. The correlation of the linearly detrended (in logs), HP-filtered (in logs) and “raw” (i.e., unfiltered) predicted ratio and the correspondingly treated price-dividend observed in data are 0.67, 0.77 and 0.83, respectively. However, we cannot match the period between 1995 and 2000 which corresponds to the dot-com bubble (see, e.g., Kraay and Ventura (2007)). This is only to be expected in our model, where prices are determined in general equilibrium entirely based on the stochastic evolution of *real* output. In this respect, it is significant that the actual price-dividend returns to the predicted path following the collapse of the bubble.

Panel (d) of Figure 7 plots the model implied times series of the (square root of) conditional expectation of the deviation of the rate of return from its unconditional¹⁵ mean. One may consider this a measure of variability of risky returns and as shown, it is a good match with Bloom (2009)’s measure of stock market volatility (the correlation between the two is 0.38).

Excess returns tend to mean revert over long horizons. Applying a statistic used in the literature (see, e.g. Guvenen (2009)) that aggregates consecutive autocorrelation coefficients of excess returns from the U.S. data in our 1978-2011 sample, we find a strong pattern of mean reversion, shown in the second row in Table 2. The third row displays the model counterparts of this measure of mean reversion, which are consistent with the signs and rough magnitudes of these statistics in the data. Such mean reversion is a clear departure from the martingale hypothesis of returns and is sometimes linked to the predictability of returns. Table 3 allows a comparison, between the data and the model implications, of coefficients from predictive regressions of annual returns on lagged price-dividend ratio. The estimated coefficients match sign and while the model implied coefficients are smaller they are within the 95% confidence interval of the corresponding

¹⁴The fact that we use annual data inevitably makes the time alignment across variables rather imperfect, which needs to be taken into account when reading the graphs.

¹⁵More precisely, the unconditional mean $ER_{t+1} \equiv T^{-1} \sum_{t=1}^T R_t$, where R_t is as implied by the model given the observed history growth outcomes up to t .

estimates in the data.¹⁶

Cumulative Autocorrelation						
Lag, in years	1	2	3	4	5	
Data	-0.16	-0.30	-0.32	-0.79	-0.33	
Model implied returns	-0.54	-0.35	-0.58	-0.76	-0.52	

Table 2: **Mean reversion of returns:**Autocorrelation structure of excess returns in the data and as implied by the model (baseline specification). The cumulative autocorrelation is defined as $(\sum_{i=1}^j Corrl((R_t - R_t^f), (R_{t-i} - R_{t-i}^f)))$.

$\sum_{n=1}^N r_{t+n} = \theta_0 + \theta_p(p_t - d_t) + \varepsilon_{t+n}$			
Data		Data	Model
N	θ_p	95% C.I.	θ_p
3	-0.56	[-1.30;0.18]	-0.07
5	-1.03	[-2.03;-0.02]	-0.14

Note: standard errors are robust to heteroskedasticity and autocorrelation.

Table 3: **Predictability Regression Coefficients (1978–2011):** The table reports coefficients from predictive regressions of annual returns on lagged price-dividend ratios over the sample period, 1978-2011, in the data and in the time-series implied by the model. The third column shows the 95% confidence interval on the estimated regression coefficient.

3.2.3 Equity premium and macro-uncertainty measures

The model implied equity premium is the conditional expectation of the model implied return of a share of the macroeconomic equity in excess of the (model implied) risk free return. The risk free return may be understood as the return under the assumption the asset delivers the conditionally expected (or, the forecasted) payoff *for sure*. The premium is the compensation for the uncertainty that the equity may deliver a payoff different from what is forecast, hence a compensation for possible forecast error. Since the taste parameters (e.g., attitudes toward time and uncertainty) have been held fixed across time in the model, we may interpret the movements in equity premium to be

¹⁶The estimates of coefficients from model implied values are fragile since the nature of the exercise limits us to historical sample points and hence very few observations (for the longer 5-year horizon, there are only about 30 observations). In the literature, predictability regressions are typically run on data obtained from model simulations; Beeler and Campbell (2009), e.g., use a million such data points.

	Persistence in years		p-value Test (1)-(2)	Level of Significance
	w/ recession (1)	w/o recession (2)		
Model Cond. Eqty. Prm.	0.90 (0.16;5.12)	0.64 (0.13;5.12)	0.007	99%
JLN Uncertainty index	0.95 (0.20;4.77)	0.56 (0.16;3.46)	0.15	85%
Bayes case Eq Prm ($\alpha \simeq 0$)	0.66 (0.14;4.82)	0.67 (0.14;4.80)	0.83	–

Table 4: **Counter-cyclical Persistence:** Columns 2 and 3 show estimates, corresponding to the time-series indicated in column 1, of the AR(1) parameter and between parenthesis its standard deviation and the associated Student-t statistic in years with and without recessionary episodes, respectively; the final columns show the p-value of the test for statistical significance of the difference in estimates in columns 2 and 3 and the associated level of significance. The final row of the table shows these numbers for the series obtained from the model with ambiguity neutrality (i.e., $\alpha \simeq 0$).

the Bayesian case (i.e., by setting $\alpha \simeq 0$) yields correlations of -0.02 and 0.11 , for levels and differences, respectively. We have already noted the pronounced counter-cyclicity of the model implied conditional equity premium (a correlation of -0.61 with the Kalman filtered latent variable). The JLN index is similarly counter-cyclical, with a correlation of -0.60 with the filtered value of our latent variable. Another salient feature is persistence: both series are persistent but the *persistence is significantly greater in years with recessionary episodes*, as the numbers reported in the first two rows of Table 4 show. The final row of the table shows, in contrast, the model generated conditional equity premium in the Bayesian case demonstrates no significant difference in persistence across the business cycle. JLN emphasize in their concluding remarks that the key features of macroeconomic uncertainty are its counter-cyclicity and its persistence during recessions. These two features speak directly to the mechanism at work in our model and the latter feature, in particular, provides a very discriminating test of the mechanism. As we showed, the Bayesian uncertainty does increase, if minimally, following a shock; but *that increase is symmetric* with respect to the sign of shock. It is ambiguity aversion that is responsible for the asymmetric behavioral response to good and bad news and

annual measure. We compare the equity premium to the trailing 12 month average of the monthly JLN measure. This is done to facilitate a more realistic alignment of when data is made available to market participants. However, the adjustment is far from perfect. Our equity premium variable, by construction, is based on the annual GDP growth report, and hence effectively shows the uncertainty lagged by about a year. This is worth bearing in mind when looking at the graphs.

for increasing the (as if) belief on high persistence in recessionary periods.

Could these features obtain in a model with stochastic volatility but no ambiguity aversion? Clearly in a Markovian style model described in the previous section, the transition probability kernel may be set to imply a relatively high persistence of the state with low growth and high variance. However, when such a model is calibrated to data, as e.g. in Kandel and Stambaugh (1990), it fails to generate conditional returns moments with time series properties (including persistence) that match the data. In Bansal and Yaron (2004) even though the equity premium is persistent, the persistence is not shown to vary across the business cycle. A more fundamental difference between these stochastic volatility based asset pricing models and ours is that in the former there is no explanation, as such, of the variation in volatility: it is as if agents become more uncertain because they believe they happen to be in a state where future economic shocks are assumed to be more volatile. Furthermore, as Campbell (2000), p. 1545, remarks, “A difficulty with these efforts is that the evidence for heteroskedasticity in aggregate consumption is fairly weak.” In contrast, our model gives a theory *why* an agent makes decisions following a positive shock that underplays the inferred uncertainty and, after a negative shock that follows a series of “normal” ones, behaves as if the uncertainty is more severe and persistent than pure Bayesian inference would suggest.

Recently, Orlik and Veldkamp (2014) have constructed a measure of macroeconomic uncertainty which also comes with a theory why such uncertainty is more countercyclical than stochastic volatility alone. In their model the agent does not know the true distribution of macroeconomic outcomes, but estimates its parameters in the way of a Bayesian econometrician using real time (GDP) data. They measure uncertainty as the conditional standard deviation of GDP growth, which captures uncertainty about the distributions’ estimated parameters. When the forecasting model admits only normally-distributed outcomes, they find small, acyclical changes in uncertainty. But when the forecasting model is enlarged in a specific way, so that agents also estimate parameters that regulate skewness, uncertainty fluctuations become more pronouncedly countercyclical. However, they find the uncertainty diminishes secularly and significantly due to the learning of the parameters. To rectify this they add an exogenously specified stochastic volatility component which, like in Bansal and Yaron (2004), has a persistence that is *independent* of the business cycle. They report that their measure has a correlation of 0.31 with the JLN uncertainty index (recall, for our model this correlation is 0.58).

4 Assessing the calibrated value of ambiguity aversion

Here we discuss a way of assessing the plausibility of the calibrated levels of ambiguity aversion in terms of implied individual (as opposed to market) behavior.

In standard analysis of the equity premium question, the value of (relative) risk aversion parameter is motivated by using a thought experiment; the typical question being how much an agent would pay to avoid a given risk. Arguably, neither the question nor the intuitive answer refers to the expected utility model, or any formal model of decision making for that matter. We now consider as a thought experiment the implied uncertainty premium of an individual investor with preferences and dynamic belief evaluating a Lucas tree prospect, precisely like the agent in our model. We find the investor is willing to pay an overall uncertainty premium (a sum of the risk premium and the ambiguity premium) that is well within the bounds of what is regarded as intuitively plausible per the *standard* intuition and analysis.

Our thought experiment consists of an offer at time t to our Lucas economy agent, with preference parameters (γ, α, β) , to replace the uncertain consumption prospect he faces with a fixed consumption in each period, now and for ever. Define the *consumption certainty equivalent*, $c^*(\gamma, \alpha, \beta; c_t)$, to be the c^* that makes the agent indifferent, given information at t , between the plan (c^*, c^*, c^*, \dots) and his endowed stochastic consumption plan (c_t, c_{t+1}, \dots) . Hence, $c^*(\gamma, \alpha = 0, \beta; c_t)$ is the certainty equivalent for the Bayesian agent and $c^*(\gamma = 0, \alpha = 0, \beta; c_t)$ that for the risk neutral agent. The *risk premium* is $\mathcal{R}(\gamma, \alpha, \beta; c_t) \equiv c^*(0, 0, \beta; c_t) - c^*(\gamma, 0, \beta; c_t)$, and the *ambiguity premium* is $\mathcal{A}(\gamma, \alpha, \beta; c_t) \equiv c^*(\gamma, 0, \beta; c_t) - c^*(\gamma, \alpha, \beta; c_t)$. Finally, define $\gamma^*(\gamma, \alpha, \beta; c_t)$, to be the value

	$\beta = 0.975$			$\beta = 0.965$
γ	2.0	2.5	3.0	2.5
α	17.75	11.35	6.65	13.00
$\gamma^*(\gamma, \alpha, \beta)$	3.48706	3.51019	3.52078	3.76367

Table 5: **Uncertainty premia in the thought experiment:** We report the time-average of $\gamma^*(\gamma, \alpha, \beta; c_t)$ computed at each t on the sample path.

of the relative risk aversion parameter which solves the following equation:

$$\mathcal{R}(\gamma^*, \alpha = 0, \beta; c_t) = \mathcal{R}(\gamma, \alpha, \beta; c_t) + \mathcal{A}(\gamma, \alpha, \beta; c_t) \Leftrightarrow c^*(\gamma^*, 0, \beta; c_t) = c^*(\gamma, \alpha, \beta; c_t). \quad (20)$$

On the right hand side of the first equality in (20) we have the *total uncertainty premium* paid by our agent with preference parameters (γ, α, β) . On the left hand side, it is the

uncertainty premium of an ambiguity neutral agent facing the same uncertain prospect as the one on the right. Table 5 reports calculations with $\gamma = 2, 2.5, 3$ and α set to the corresponding calibrated values used in our model. Hence, our agent is calibrated to pay as much uncertainty premium (in *total*) as a standard expected utility agent with relative risk aversion around 3.5. Almost every equity premium study in the literature considers this amount of uncertainty premium very much within the range of plausibility in the context of a financial economy (Mehra and Prescott (1985), e.g. , had argued on this basis that $\gamma \leq 10$ was plausible). In this sense, the calibrated uncertainty attitude parameters, *taken together*, make a plausible preference configuration for an individual DM in a financial economy.

5 Related literature

We describe next how the analysis here relates to other explanations in the literature (of the observed behavior of equity premium) based on *aggregate* uncertainty in representative agent frameworks. Three papers closest to ours are Bansal and Yaron (2004), Hansen and Sargent (2010) and Ju and Miao (2012).

Bansal and Yaron (2004) pioneered the use of the (basic) model of beliefs we apply to show how long run risk (LRR) and aversion to such risk (while allowing a Kreps and Porteus (1978)/Epstein and Zin (1989)/Weil (1989) like separation of EIS from risk aversion) could explain aspects of the observed equity premium. The changes we introduce are: (1) letting the belief about the latent state be the full Bayes posterior, instead of degenerate, probability-one-belief on the filtered state; (2) letting the agent be uncertain about the value of the persistence parameter; (3) letting the agent treat (1) and (2) as ambiguity. We show these changes are sufficient to yield a model of beliefs where the (endogenously accentuated) uncertainty varies enough over time, without resorting to an exogenously specified stochastic volatility.

Hansen and Sargent (2010) study the effect of model uncertainty and robustness concerns on the price of macroeconomic risk. The single agent believes the economy evolves according to a model like we have here. However, their agent processes belief differently, by applying *two* “risk-sensitivity operators”. The first operator, which may be interpreted in terms of an enhanced risk aversion obtained via a Kreps and Porteus (1978) style preference for earlier resolution of risk, applies to the evaluation (of the consumption plan) *conditional* on each of the two values of ρ . The other operator may be interpreted as a KMM2005 style smooth ambiguity aversion transformation where the

agent's second order uncertainty is a two point (Bernoulli) belief, where each point in the support is the conditional evaluation given a ρ . Hence, while uncertainty about the two values of ρ is treated as ambiguity, the uncertainty about the latent state, given ρ , is *not* processed as ambiguity, unlike in our model. Their paper gives results on what they call, "uncertainty price" but does not derive results and (statistics on) model implied rates of return and price-dividend ratio. However, for the sake of comparison, we implemented an amended version their preference model, with simply the second (KMM style) risk sensitivity operator on the two point belief but not including the other operator, on our data set. The predicted time-averaged equity premium (conditional on actual history) is about 0.6%. Neither do the time-series properties of the conditional equity premium produce a better match; e.g., it has a negative correlation with the JLN index.

Ju and Miao (2012) use a modified smooth ambiguity framework to assess the effect of ambiguity on dynamics of asset prices. In the model of beliefs there the latent state variable driving the (mean) growth rate in the economy may take only two possible values. The preference model incorporates an EIS effect, in addition to ambiguity aversion, with the EIS parameter set at 1.5. They produce statistics on *unconditional* moments of returns and prices, by averaging across simulated, counter-factual paths, which match data well. They also report, using graphs, model implied conditional returns and prices along the observed, historical sample path; here, their model is evidently less successful. As panel B in their Figure 3 shows, throughout the post-war period the (second-order) belief has been almost completely stuck (almost Dirac) on the same latent (high-growth) state. Hence, the results we obtain about predicted time series of moments of conditional returns (even counter-cyclical equity premium) would not obtain in their model if actual history were applied.¹⁸

Veronesi (1999) constructs and theoretically analyzes a dynamic, rational expectations, expected utility representative agent model of asset pricing where beliefs are based on two hidden states (each specifying a mean growth rate) and shows that it implies time-varying expected returns and prices. However, it is a theoretical exercise and does not show what actual values and magnitudes are implied along information paths based on observed history. Gollier (2011) shows analytically, using a (static) smooth ambiguity model, that an increase in ambiguity aversion may not, in general, increase the eq-

¹⁸Recently, Strzalecki (2013) has shown that it is theoretically possible that recursive ambiguity frameworks have some preference for early resolution inseparably mixed in with ambiguity aversion. Compared to the model in the present paper what is different about the preferences in Ju and Miao (2012) and Hansen and Sargent (2010) is that those include separate components explicitly adding preference for early resolution above and beyond what may be already mixed in with ambiguity aversion.

uity premium, thereby making a good case for empirical investigation of the question. Abel (2002), Cecchetti, Lam, and Nelson (2000), Giordani and Soderlind (2006), Jouini and Napp (2006), show that exogenously introducing pessimism and doubt in beliefs can generate a realistic equity premium and risk-free rate. Our results are driven by similar elements of pessimism and doubt, but in our framework these arise endogenously. Barro (2006), and Weitzman (2007) show that rare risks and/or heavy tails may contribute to the large equity premium and low risk-free rate observed in the data. Our contribution focuses on “common” uncertainty near the current growth rate rather than on “rare” uncertainty, and so is easier to relate to observed consumption data. Constantinides (1990) and Campbell and Cochrane (1999) study models with habits in consumption which can match the level, variation and counter-cyclicity of the equity premia. Habits effectively allow the risk aversion to vary endogenously over the business cycle. The crucial difference to our paper is that we have constant aversion (to ambiguity and risk) but our agent faces time-varying uncertainty and it is variation in that uncertainty, rather than variation in the aversion to it, which causes the returns and premia to vary.

6 Concluding remarks

Our model applied three elements: Uncertainty and learning about persistent hidden states describing temporary shocks to fundamentals; uncertainty and learning about the level of persistence; treating both these uncertainties as ambiguous and incorporating a level of ambiguity aversion calibrated to match the average risk-free rate. The first two elements, compatible with a Bayesian agent (but not with rational expectations), explain quite substantially the average volatility of returns and prices, and also the level of risky rate. Ambiguity aversion was important in explaining the levels of risk free rate and equity premium, and for shaping the dynamics of all the variables, especially the first and second moments (conditional) equity premium through the channel of an endogenously accentuated “as if” uncertainty.

Our results show that observed levels and movements of moments of asset returns can be very substantively explained on the basis of aggregate macroeconomic risk, conditional on the *actual history* of aggregate output growth reports. That both first and second moments of conditional excess returns have the cyclical properties that match the data is a novel contribution. As was the finding that the model implied conditional equity premium matches the time series properties of the JLN macroeconomic uncertainty index, thereby giving a theory of uncertainty shocks and the counter cyclical na-

ture of their severity and persistence. Thus, consistent with JLN's conjecture, we do find that Knightian uncertainty can provide a good explanation of dynamics of macroeconomic uncertainty. Finally, it is worth appreciating the minimality of the departure from expected utility that was sufficient to capture so many aspects of returns data. These observations are very suggestive of the potential for this approach in domains of macro-finance research where effects of endogenously time-varying uncertainty are of interest.

A Data and estimation of parameters of the stochastic models

Equity returns are computed using the CRSP value-weighted index. Dividend growth is imputed using the difference in the returns on the value-weighted index with and without dividends multiplied by the market value. The risk-free rate was taken from Ken French's data library. Consumption is defined as the sum of services and non-durable consumption and was taken from BEA Table 1.1. Population was taken from BEA Table 2.2. Both per-capita consumption growth and dividend growth were converted to real terms using the average CPI for the year taken from the BLS. Annual data was available from 1930 until 2011, a total of 82 observations.

Turning to preference parameters, in all cases the ambiguity aversion parameter α was calibrated to produce a real risk-free rate of 1.5%, averaged over $t = 1978, \dots, 2011$, which is the average observed rate in that period. No other moments were used in the choice of α . The relative risk aversion parameter γ was allowed to range between 1 (log utility) and 3, regarded as plausible in macroeconomic models (Ljungqvist and Sargent, 2004, pg. 426); the “baseline” calibration set $\gamma = 2.5$.¹⁹ The discount factor β was set to 0.975, which corresponds to the discount rate used in BY. To check for robustness we varied a number of the key non-estimated parameters, including $\rho = 0.9$, $\beta \in \{.965, .97, .98\}$ and $\psi = 2.5$.

The long-run risk model was fit to annual data using maximum likelihood. Parameter estimates are shown in Table 6. All parameters, except ρ and ψ were estimated using data 1930–1977. The mean of consumption and dividends, \bar{g} and \bar{d} , respectively were set to their values in the period 1930 – 1977. The variances of the latent state process, consumption growth and dividend growth were estimated using the Kalman Filter. The dividend leverage parameter, ψ , was set to 3 as in BY, although Constantinides and Ghosh (2010) estimated it to be slightly lower, close to the value we use for robustness checks ($\psi = 2.5$).

¹⁹If the two smooth ambiguity preferences do not share the same risk attitude it is not necessarily true that a more concave ϕ means more ambiguity aversion. Hence α is meaningfully calibrated *given* a value of γ ; not independent of γ .

Parameter Estimates

Parameter	$\psi = 3$				$\psi = 2.5$	
	$\rho = .25$	$\rho = .3$	$\rho = .85$	$\rho = .9$	$\rho = .3$	$\rho = .85$
\bar{g}	1.92 (0.302)	1.92 (0.302)	1.92 (0.302)	1.92 (0.302)	1.92 (0.302)	1.92 (0.302)
\bar{d}	2.31 (2.21)	2.31 (2.21)	2.31 (2.21)	2.31 (2.21)	2.02 (2.21)	2.02 (2.21)
σ_g^2	0.048 (0.016)	0.046 (0.016)	0.025 (0.010)	0.020 (0.007)	0.047 (0.017)	0.026 (0.008)
σ_d^2	4.49 (0.893)	4.51 (0.892)	4.75 (0.909)	4.73 (0.902)	4.64 (0.914)	4.81 (0.918)
σ_x^2	0.054 (0.013)	0.054 (0.013)	0.051 (0.019)	0.059 (0.021)	0.054 (0.013)	0.050 (0.021)

Table 6: **Parameter estimates** (standard errors below in parentheses) using annual data and the long-run risk model, shown above, using data from 1930 until 1977. All variance estimates and their standard errors have been multiplied by 100.

B Details of the model

B.1 Beliefs and the direct value function

The agent believes that the stochastic evolution of the economy follows a persistent latent state process given by a BY type specification with either a low persistence (ρ_l) or a high persistence (ρ_h), but does not know for sure which. That is, he believes either of the models described in equation (9) represent the true data generating process. Define $\hat{x}_{k,t} \equiv E[x_{k,t} | g_{k,1}, \dots, g_{k,t}, d_{k,1}, \dots, d_{k,t}]$, $k = l, h$, to denote the filtered x at time t conditional on the observed history of growth rates (of consumption and dividend), if the history were interpreted and beliefs updated using a Kalman filter which takes the model with $\rho = \rho_k$ as the data generating process. At any node on the growth path, at a time t , the agent's beliefs may be summarized by the tuple $(\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$, where the first two elements show the beliefs about the latent state variable conditional on alternative assumptions about the true data generating process (low or high persistence, respectively) while the last element shows the posterior belief that the true data generating process is the low persistence model. We denote by $\hat{x}_{k,t+1}^{(i)}$, $i = l, h$, $k = l, h$, the agent's forecast for the (one period ahead) update to his belief about the filtered x if the growth outcome next period (along with the previous history) were interpreted using a Kalman filter which takes the model with $\rho = \rho_k$ as the data generating process, when the data is actually generated by the i persistence model. The direct value function obtains as

follows:

$$\begin{aligned}
V(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) &= (1 - \beta) \frac{C_t^{1-\gamma}}{1-\gamma} \\
&\quad - \frac{\beta}{\alpha} \ln \left[\eta_t \left\{ \int_{-\infty}^{\infty} \exp \left(-\alpha \iiint_{-\infty}^{\infty} V(C_t \exp(g_{l,t+1}), \hat{x}_{l,t+1}^{(l)}(\vec{\epsilon}_{l,t+1}), \right. \right. \right. \\
&\quad \left. \left. \left. \hat{x}_{h,t+1}^{(l)}(\vec{\epsilon}_{l,t+1}), \eta_{t+1}^{(l)}(\vec{\epsilon}_{l,t+1})) dF(\vec{\epsilon}_{l,t+1}) \right) dF(x_{l,t}) \right\} \right. \\
&\quad \left. + (1 - \eta_t) \left\{ \int_{-\infty}^{\infty} \exp \left(-\alpha \iiint_{-\infty}^{\infty} V(C_t \exp(g_{h,t+1}), \hat{x}_{l,t+1}^{(h)}(\vec{\epsilon}_{h,t+1}), \right. \right. \right. \\
&\quad \left. \left. \left. \hat{x}_{h,t+1}^{(h)}(\vec{\epsilon}_{h,t+1}), \eta_{t+1}^{(h)}(\vec{\epsilon}_{h,t+1})) dF(\vec{\epsilon}_{h,t+1}) \right) dF(x_{h,t}) \right\} \right]
\end{aligned} \tag{21}$$

where $\vec{\epsilon}_{l,t+1} = [\epsilon_{x_{l,t+1}} \epsilon_{d_{l,t+1}} \epsilon_{g_{l,t+1}}]$ is a 3 by 1 vector of standard normal shocks (and so is $\vec{\epsilon}_{h,t+1}$) and η_t is the posterior probability at time t that the model with ρ_l is the data generating process. $F(\vec{\epsilon}_{l,t+1})$ and $F(\vec{\epsilon}_{h,t+1})$ are both trivariate independent standard normal distributions. $F(x_{k,t})$, $k = l, h$, is a normal distribution with mean $\hat{x}_{k,t}$ and variance Ω_k , where Ω_k is defined below. The updates for $\hat{x}_{k,t+1}^{(i)}$ are obtained as follows:

$$\begin{aligned}
\hat{x}_{l,t+1}^{(l)}(\vec{\epsilon}_{l,t+1}) &= \rho_l \hat{x}_{l,t} + K_l \nu_{l,t+1}^{(l)} \\
\hat{x}_{h,t+1}^{(l)}(\vec{\epsilon}_{l,t+1}) &= \rho_h \hat{x}_{h,t} + K_h \nu_{h,t+1}^{(l)} \\
\hat{x}_{l,t+1}^{(h)}(\vec{\epsilon}_{h,t+1}) &= \rho_l \hat{x}_{l,t} + K_l \nu_{l,t+1}^{(h)} \\
\hat{x}_{h,t+1}^{(h)}(\vec{\epsilon}_{h,t+1}) &= \rho_h \hat{x}_{h,t} + K_h \nu_{h,t+1}^{(h)}
\end{aligned}$$

where $\nu_{k,t+1}^{(i)}$, $(i) = (l)$ or $(i) = (h)$ and $k = l, h$, denote the ‘‘surprises’’. For example, when the DGP is $(i) = (l)$ and the filter uses ρ_k , $k = h$, the surprise is defined

$$\nu_{h,t+1}^{(l)} = \begin{bmatrix} g_{l,t+1} - \bar{g} - \rho_h \hat{x}_{h,t} \\ d_{l,t+1} - \bar{d} - \psi \rho_h \hat{x}_{h,t} \end{bmatrix} = \begin{bmatrix} \bar{g} - \bar{g} + \rho_l x_{l,t} - \rho_h \hat{x}_{h,t} + \sigma_{x_l} \epsilon_{x_{l,t+1}} + \sigma_{g_l} \epsilon_{g_{l,t+1}} \\ \bar{d} - \bar{d} + \psi \rho_l x_{l,t} - \psi \rho_h \hat{x}_{h,t} + \psi \sigma_{x_l} \epsilon_{x_{l,t+1}} + \sigma_{d_l} \epsilon_{d_{l,t+1}} \end{bmatrix}.$$

The Kalman gain parameters, K_k , $k = l, h$, depending on whether low or high persistence model is assumed to be the true model, respectively, are

$$K_k = \rho_k \Omega_k [1 \ \psi] \hat{F}_k^{-1}, \quad \text{where} \quad \hat{F}_k = \begin{bmatrix} \Omega_k + \sigma_{g_k}^2 & \psi \Omega_k \\ \psi \Omega_k & \psi \Omega_k + \sigma_{d_k}^2 \end{bmatrix}$$

Finally, Ω_k , $k = l, h$, is defined as the solution to

$$\Omega_k = \rho_k^2 \Omega_k - \rho_k^2 \Omega_k^2 [1 \ \psi] \hat{F}_k^{-1} [1 \ \psi]' + \sigma_{x_k}^2$$

The Bayes update of η_t is obtained as follows :

$$\eta_{t+1}^{(l)}(\vec{\epsilon}_{l,t+1}) = \frac{\eta_t L(\nu_{l,t+1}^{(l)}, \hat{F}_l)}{\eta_t L(\nu_{l,t+1}^{(l)}, \hat{F}_l) + (1-\eta_t) L(\nu_{h,t+1}^{(l)}, \hat{F}_h)}$$

$$\eta_{t+1}^{(h)}(\vec{\epsilon}_{h,t+1}) = \frac{\eta_t L(\nu_{l,t+1}^{(h)}, \hat{F}_l)}{\eta_t L(\nu_{l,t+1}^{(h)}, \hat{F}_l) + (1-\eta_t) L(\nu_{h,t+1}^{(h)}, \hat{F}_h)}$$

where the likelihood is

$$L(\nu_{j,t+1}^{(i)}, \hat{F}_j) = \frac{1}{2\pi|\hat{F}_j|} \exp\left(-\frac{(\nu_{j,t+1}^{(i)})' \hat{F}_j^{-1} \nu_{j,t+1}^{(i)}}{2}\right) \quad \text{where } i = l, h \text{ and } j = l, h.$$

B.1.1 Mean and variance of the distorted posterior

The mean of the distorted (or, “as if”) posterior is given by:

$$\tilde{x}_t = \eta_t \int_{-\infty}^{\infty} (x_{l,t}) \xi_t^{(l)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) dF(x_{l,t}) + (1-\eta_t) \int_{-\infty}^{\infty} (x_{h,t}) \xi_t^{(h)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) dF(x_{h,t}) \quad (22)$$

and the variance, by:

$$\begin{aligned} \widetilde{Var}_t(x_t) \equiv & \eta_t \int_{-\infty}^{\infty} (x_{l,t}^2) \xi_t^{(l)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) dF(x_{l,t}) \\ & + (1-\eta_t) \int_{-\infty}^{\infty} (x_{h,t}^2) \xi_t^{(h)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) dF(x_{h,t}) - \tilde{x}_t^2 \end{aligned}$$

B.1.2 The rates of return

The risky rate of return is a function of four state variables, $C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t$, just like V and ξ_t . In the sequel, it should be clear that variables in $t+1$ are evaluated using the relevant stochastic components. Let $C_{k,t+1} = C_t \exp(g_{k,t+1})$, $k = l, h$. The risk rate, R_t , will satisfy:

$$\begin{aligned} & \beta \eta_t \int_{-\infty}^{\infty} \xi_t^{(l)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \left(\iiint_{-\infty}^{\infty} R_t(C_{l,t+1}, \hat{x}_{l,t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \eta_{t+1}^{(l)}) \times \right. \\ & \quad \left. (u'(\exp(g_{l,t+1}))) dF(\vec{\epsilon}_{l,t+1}) \right) dF(x_{l,t}) \\ & + \beta (1-\eta_t) \int_{-\infty}^{\infty} \xi_t^{(h)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \left(\iiint_{-\infty}^{\infty} R_t(C_{h,t+1}, \hat{x}_{l,t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \eta_{t+1}^{(h)}) \times \right. \\ & \quad \left. (u'(\exp(g_{h,t+1}))) dF(\vec{\epsilon}_{h,t+1}) \right) dF(x_{h,t}) = 1 \end{aligned}$$

where,

$$\xi_t^{(l)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \frac{\phi' \left(\iiint_{-\infty}^{\infty} V(C_{l,t+1}, \hat{x}_{l,t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \eta_{t+1}^{(l)}) dF(\vec{\epsilon}_{l,t+1}) \right)}{\Psi} \quad (23)$$

and

$$\xi_t^{(h)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \frac{\phi' \left(\iiint_{-\infty}^{\infty} V(C_{h,t+1}, \hat{x}_{l,t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \eta_{t+1}^{(h)}) dF(\vec{\epsilon}_{h,t+1}) \right)}{\Psi} \quad (24)$$

with

$$\begin{aligned} \Psi = & \eta_t \int_{-\infty}^{\infty} \phi' \left(\iiint_{-\infty}^{\infty} V(C_{l,t+1}, \hat{x}_{l,t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \eta_{t+1}^{(l)}) dF(\vec{\epsilon}_{l,t+1}) \right) dF(x_{l,t}) \\ & + (1 - \eta_t) \int_{-\infty}^{\infty} \phi' \left(\iiint_{-\infty}^{\infty} V(C_{h,t+1}, \hat{x}_{l,t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \eta_{t+1}^{(h)}) dF(\vec{\epsilon}_{h,t+1}) \right) dF(x_{h,t}) \end{aligned}$$

Then, we have

$$\begin{aligned} E_t R_t = & \eta_t \iiint_{-\infty}^{\infty} R_t(C_{l,t+1}, \hat{x}_{l,t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \eta_{t+1}^{(l)}) dF(\vec{\epsilon}_{l,t+1}) dF(x_{l,t}) \\ & + (1 - \eta_t) \iiint_{-\infty}^{\infty} R_t(C_{h,t+1}, \hat{x}_{l,t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \eta_{t+1}^{(h)}) dF(\vec{\epsilon}_{h,t+1}) dF(x_{h,t}) \end{aligned}$$

and the risk-free rate is

$$\begin{aligned} R_t^f = & \left[\beta \eta_t \int_{-\infty}^{\infty} \xi_t^{(l)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \left(\iiint_{-\infty}^{\infty} (u'(\exp(g_{l,t+1}))) dF(\vec{\epsilon}_{l,t+1}) \right) dF(x_{l,t}) \right. \\ & \left. + \beta (1 - \eta_t) \int_{-\infty}^{\infty} \xi_t^{(h)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \left(\iiint_{-\infty}^{\infty} (u'(\exp(g_{h,t+1}))) dF(\vec{\epsilon}_{h,t+1}) \right) dF(x_{h,t}) \right]^{-1} \end{aligned}$$

and so the equity premium is $E_t R_t^p = E_t R_t - R_t^f$. The variance of equity premium is computed as

$$\sigma^2(R_t^p) = E_t R_t^2 - (E_t R_t)^2$$

where

$$\begin{aligned} E_t R_t^2 = & \eta_t \iiint_{-\infty}^{\infty} (R_t(C_{l,t+1}, \hat{x}_{l,t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \eta_{t+1}^{(l)}))^2 dF(\vec{\epsilon}_{l,t+1}) dF(x_{l,t}) \\ & + (1 - \eta_t) \iiint_{-\infty}^{\infty} (R_t(C_{h,t+1}, \hat{x}_{l,t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \eta_{t+1}^{(h)}))^2 dF(\vec{\epsilon}_{h,t+1}) dF(x_{h,t}) \end{aligned}$$

C An analytical approximation for rates of return in the case of known persistence model

This section develops an analytical approximation to the equilibrium rates of return in the model with known persistence. The crucial assumption on which the following sec-

ond order approximation analysis depends is that $E_{\tilde{\mu}_t}$ operates with respect to some normal distribution $N(\tilde{x}_t, \tilde{\Omega})$. As the numbers (reporting skewness and excess kurtosis) in Table 7 generated using the accurate numerical approximation demonstrate, Normality is a fairly accurate description.

Model with known persistence				
	x_t		$g_{c,t}$	
	E	σ	E	σ
Rat. Exp.	–	–	0.018	0.028
Bayesian	-0.002	0.023	0.018	0.032
Twisted	-0.023	0.024	-0.003	0.032
	sk	κ	sk	κ
Rat. Exp.	–	–	0.000	0.000
Bayesian	0.000	-0.000	0.000	-0.000
Twisted	0.000	-0.000	0.000	0.000

Model with unknown persistence				
	x_t		$g_{c,t}$	
	E	σ	E	σ
Bayesian	-0.001	0.024	0.019	0.034
Twisted	-0.022	0.028	-0.002	0.037
	sk	κ	sk	κ
Bayesian	-0.003	0.013	-0.003	0.017
Twisted	-0.005	-0.053	-0.038	-0.029

Table 7: **Conditional moments of distributions.** In each case, $\gamma = 2.5$ and α was set such that the model generates an average risk-free rate of 1.5%. C_t , $\hat{x}_{\ell,t}$, $\hat{x}_{h,t}$ and η_t are set equal to their mean in the data. sk and κ denote skewness and excess kurtosis (relative to a Gaussian distribution), respectively. The latent state variable is known to a rational expectations agent and so the conditional distribution is degenerate.

Assumption 1 [Approximating assumption 1] $\tilde{\mu}_t = N(\tilde{x}_t, \Omega)$.

Recall that $\tilde{\mu}_t \equiv \xi_t(x_t) \otimes N(\hat{x}_t, \Omega)$ and thus has density given by

$$\tilde{f}(x_t) = \xi_t(x_t | C_t, \hat{x}_t; \alpha) \frac{1}{\sqrt{2\pi\Omega}} \exp\left(-\frac{(x_t - \hat{x}_t)^2}{2\Omega}\right). \quad (25)$$

This assumption is thus equivalent to assuming that eq. (25) is exactly a normal density with the same variance as the Bayesian posterior Ω but with a different mean (\tilde{x}_t

instead of \hat{x}_t). Let $E_t \equiv E_{\hat{x}_t} E_{x_t}$; $\tilde{E}_t \equiv E_{\tilde{\mu}_t} E_{x_t} \equiv E_{\tilde{x}_t} E_{x_t}$. It is useful to recall, if x_t is normally distributed, then for any $k \in \mathbb{R}$,

$$E_t [\exp(k x_t)] = \exp\left(k E_t x_t + \frac{k^2}{2} \text{Var}_t(x_t)\right)$$

Also, $\widetilde{\text{Var}}_t(x_t) \equiv \text{Var}_{\tilde{\mu}_t}(x_t) = \Omega$ and $\text{Var}_t(x_t) = \text{Var}_{\mu_t}(x_t) = \Omega$ and all ε terms have expectation zero under both \tilde{E}_t and E_t since the terms have expectation zero conditional on x_t .

The first Euler equation relating to the risk-free asset may be rewritten as follows:

$$\begin{aligned} 1 &= \beta R_t^f \tilde{E}_t [\exp(-\gamma \bar{g} - \gamma \rho x_t - \gamma \sigma_x \varepsilon_{x,t+1} - \gamma \sigma_g \varepsilon_{g,t+1})] \\ &= \beta R_t^f \exp\left(-\gamma \bar{g} - \gamma \rho \tilde{x}_t + \frac{\gamma^2}{2} (\sigma_x^2 + \sigma_g^2) + \frac{\gamma^2 \rho^2}{2} \widetilde{\text{Var}}_t(x_t)\right). \end{aligned}$$

Taking logs and rearranging terms we obtain an approximate solution for the risk-free rate of return:

$$r_t^f = -\ln \beta + \gamma \bar{g} + \gamma \rho \tilde{x}_t - \frac{\gamma^2}{2} (\sigma_x^2 + \sigma_g^2 + \rho^2 \widetilde{\text{Var}}_t(x_t)). \quad (26)$$

The second Euler equation relating to the risky asset may then be written as:

$$\tilde{E}_t \exp\left[\ln \beta + \ln\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right) - \gamma \ln\left(\frac{C_{t+1}}{C_t}\right)\right] = 1 \quad (27)$$

We adopt the following approximation (to the risky rate of return), proposed in Campbell and Shiller (1988).

Assumption 2 [Approximating assumption 2]:

$$r_t \equiv \ln\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right) \simeq \kappa_0 + \kappa_1 z_{t+1} - z_t + d_{t+1} \quad (28)$$

where $z_t = \ln\left(\frac{P_t}{D_t}\right)$ and κ_0 and κ_1 are approximating constants.

Next, we conjecture that the log price-dividend ratio is given by

$$z_t = A_0 + A_1 \tilde{x}_t. \quad (29)$$

Our final assumption is that the mean of the distorted conditional distribution is an affine function of the mean of the (contemporaneous) undistorted, Bayesian conditional distribution, which holds well in our data, see Figure 9.

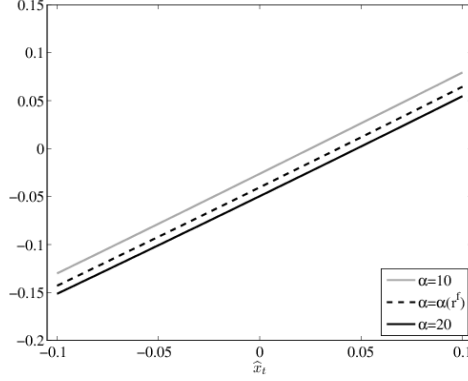


Figure 9: $\tilde{x}_t = E_{\tilde{\mu}_t}(x_t)$ plotted against \hat{x}_t . The level of consumption is set to the average value between 1978 and 2011. In each case, $\gamma = 2.50$.

Assumption 3 [Approximating assumption 3] $\tilde{x}_t = \delta_0 + \delta_1 \hat{x}_t$ for $t = 1, 2, \dots$, $\delta_1 > 0$.

Note this assumption implies trivially that $\hat{x}_t = (\tilde{x}_t - \delta_0)/\delta_1$. Hence, we obtain a second order approximation of the second Euler equation as follows:

$$1 = \tilde{E}_t \exp \left[\ln(\beta) + \kappa_0 + \kappa_1 z_{t+1} - z_t + d_{t+1} - \gamma g_{t+1} \right]$$

Plugging the guess for z_t and using the processes of growth rates, and using Assumptions 1 and 3, we obtain

$$1 = \tilde{E}_t \exp \left[\ln(\beta) + \bar{d} - \gamma \bar{g} + \kappa_0 + (\kappa_1 - 1)A_0 + \kappa_1 A_1 (\delta_0 + \delta_1 \hat{x}_{t+1}) - A_1 \tilde{x}_t + (\psi - \gamma)\rho x_t + (\psi - \gamma)\sigma_x \varepsilon_{x,t+1} + \sigma_d \varepsilon_{d,t+1} - \gamma \sigma_g \varepsilon_{g,t+1} \right]. \quad (30)$$

In the expression for \hat{x}_{t+1} from the Kalman filter, let $K = [K_g, K_d]$. Then, we have now an expression for \hat{x}_{t+1} which is equal to (substituting d_{t+1} and g_{t+1} using their dynamics in the model):

$$\hat{x}_{t+1} = \rho \hat{x}_t (1 - K_g - \psi K_d) + (K_g + \psi K_d)\rho x_t + (K_g + \psi K_d)\sigma_x \varepsilon_{x,t+1} + K_g \sigma_g \varepsilon_{g,t+1} + K_d \sigma_d \varepsilon_{d,t+1}$$

Taking the log of eq. (30) and using $\hat{x}_t = \frac{\tilde{x}_t - \delta_0}{\delta_1}$. Hence,

$$\begin{aligned} 0 = & \ln(\beta) + \bar{d} - \gamma \bar{g} + \kappa_0 + (\kappa_1 - 1)A_0 + \kappa_1 A_1 \delta_0 - \delta_0 (\kappa_1 A_1 \rho (1 - K_g - \psi K_d)) \\ & + [\kappa_1 A_1 \rho (1 - K_g - \psi K_d) + \rho \kappa_1 A_1 \delta_1 (K_g + \psi K_d) + (\psi - \gamma)\rho - A_1] \tilde{x}_t \\ & + \rho^2 (\kappa_1 A_1 \delta_1 (K_g + \psi K_d) + \psi - \gamma)^2 \widetilde{Var}_t(x_t)/2 \\ & + (\psi - \gamma + \kappa_1 A_1 \delta_1 (K_g + \psi K_d))^2 \sigma_x^2 / 2 \\ & + (\kappa_1 A_1 \delta_1 K_d + 1)^2 \sigma_d^2 / 2 + (\kappa_1 A_1 \delta_1 K_g - \gamma)^2 \sigma_g^2 / 2 \end{aligned}$$

Since this approximation must be valid for any \tilde{x}_t , we collect the \tilde{x}_t terms, set the expression equal to zero and we have

$$\kappa_1 A_1 \rho (1 - K_g - \psi K_d) + \rho \kappa_1 A_1 \delta_1 (K_g + \psi K_d) + (\psi - \gamma) \rho - A_1 = 0$$

which must hold for all \tilde{x}_t . Hence,

$$A_1 = \frac{\rho(\psi - \gamma)}{1 - \rho \kappa_1 (1 - (1 - \delta_1)(K_g + \psi K_d))} \quad (31)$$

Doing the same for the constant terms, we have

$$\begin{aligned} (1 - \kappa_1) A_0 = & \ln(\beta) + \bar{d} - \gamma \bar{g} + \kappa_0 + \kappa_1 A_1 \delta_0 - \delta_0 (\kappa_1 A_1 \rho (1 - K_g - \psi K_d)) \\ & + \rho^2 (\kappa_1 A_1 \delta_1 (K_g + \psi K_d) + \psi - \gamma)^2 \widetilde{Var}_t(x_t) / 2 \\ & + (\psi - \gamma + \kappa_1 A_1 \delta_1 (K_g + \psi K_d))^2 \sigma_x^2 / 2 \\ & + (\kappa_1 A_1 \delta_1 K_d + 1)^2 \sigma_d^2 / 2 + (\kappa_1 A_1 \delta_1 K_g - \gamma)^2 \sigma_g^2 / 2 \end{aligned} \quad (32)$$

Using eq. (29) and that $E_t \tilde{x}_{t+1} = \delta_0 + \delta_1 E_t \hat{x}_{t+1}$ where $E_t \hat{x}_{t+1} = \rho \hat{x}_t (1 - K_g - \psi K_d) + (K_g + \psi K_d) \rho E_t x_t = \rho \hat{x}_t$, we obtain

$$E_t r_t = \kappa_0 + A_0 (\kappa_1 - 1) + \kappa_1 A_1 \delta_0 (1 - \rho) + \bar{d} + A_1 (\kappa_1 \rho - 1) \tilde{x}_t + \psi \rho \hat{x}_t \quad (33)$$

and so the Equity premium is then

$$\begin{aligned} E_t r_t - r_t^f = & \kappa_0 + A_0 (\kappa_1 - 1) + \kappa_1 A_1 \delta_0 (1 - \rho) + \bar{d} + A_1 (\kappa_1 \rho - 1) \tilde{x}_t + \psi \rho \hat{x}_t \\ & + \ln(\beta) - \gamma \bar{g} - \gamma \rho \tilde{x}_t + \frac{\gamma^2}{2} (\sigma_x^2 + \sigma_g^2 + \rho^2 \widetilde{Var}_t(x_t)) \end{aligned} \quad (34)$$

Note that when $\delta_1 = 1$, as is true in our data (see Figure 9), A_1 simplifies to $-\rho(\psi - \gamma) / (\kappa_1 \rho - 1)$.

We need values of the approximating constants, κ_0 and κ_1 , to compute the log price-dividend ratio. Beeler and Campbell (2009) obtain the constants as follows

$$\begin{aligned} \bar{z} &= \frac{\sum z_t}{N} \\ \kappa_1 &= \frac{\exp \bar{z}}{1 + \exp \bar{z}} \\ \kappa_0 &= \ln(1 + \exp \bar{z}) - \kappa_1 \bar{z}. \end{aligned}$$

D Ambiguity of second-order beliefs

Let T be a second-order event, i.e., $T \subset \Theta$, with $\mu(T) = m$. Consider two prospects. One, a bet on this event, which pays x on the event and y off it, with $x > y$. Two, a lottery, ℓ_m which pays x with probability m and y with probability $1 - m$. Notice, when ϕ is concave, by Jensen's inequality,

$$m\phi(u(x)) + (1-m)\phi(u(y)) < \phi(m(u(x)) + (1-m)(u(y))) \quad (35)$$

The LHS of (35) is the evaluation of the bet on T while the RHS is the evaluation of the lottery, per the smooth ambiguity model. Similarly, the bet on the complementary event T^c is dispreferred to ℓ_{1-m} given a concave ϕ . Indeed, ambiguity aversion implies we cannot find a calibrated lottery event such that betting on that lottery event is same as betting on T ; there is no lottery probability that is same as μ . Hence, when ϕ is concave, the second-order measure μ cannot be calibrated with a lottery; behaviorally, μ is not treated as an objective probability.

As shown formally in section 2.4 in Klibanoff, Marinacci, and Mukerji (2012), this is the heart of the argument that establishes that ambiguity of a first-order event E implies that non-null and non-universal second-order events concerning the probability of E are treated as ambiguous. Hence, the smooth ambiguity model property of expected utility evaluation of second-order acts (e.g., bets on events in Θ) does not mean that the DM treats these acts as based on unambiguous events.

E Online Appendix – Details of the numerical solution procedure

E.1 Solution Method

This section describes the minimum weighted residuals method we use to obtain an approximate solution for the value function and the risky rate. We then explain how we assess the accuracy of the method.

Both the value function and the risky rate are approximated by a parametric function of the form

$$\Phi_y(X_t) = \exp\left(\sum_{i_c, i_h, i_\ell, i_\eta \in \mathcal{I}} \theta_{i_c, i_h, i_\ell, i_\eta}^y H_{i_c}(\varphi_c(C_t)) H_{i_h}(\varphi_h(\hat{x}_{h,t})) H_{i_\ell}(\varphi_\ell(\hat{x}_{\ell,t})) H_{i_\eta}(\varphi_\eta(\eta_t))\right)$$

where $X_t \equiv (C_t, \hat{x}_{h,t}, \hat{x}_{\ell,t}, \eta_t)$ denotes the vector of state variables²⁰ and $y \in \{V, R\}$. The set of indices \mathcal{I} is defined by

$$\mathcal{I} = \{i_z = 1, \dots, n_z; z \in \{C, h, \ell, \eta\} | i_c + i_h + i_\ell + i_\eta \leq \max(n_c, n_h, n_\ell, n_\eta)\}$$

Implicit in the definition of this set is that we are considering a complete basis of polynomials.²¹ $H_i(\cdot)$ is a Hermite polynomial of order i and $\varphi_z(\cdot)$ is a strictly increasing function that maps \mathbb{R} into \mathbb{R} . This function is used to map Hermitian nodes into values for the vector of state variables, $X_t \equiv (C_t, \hat{x}_{h,t}, \hat{x}_{\ell,t}, \eta_t)$.²² The parameters θ^y , $y \in \{V, R\}$, are then determined by a minimum weighted residuals method. More precisely, we define the residuals associated to both the direct Value function equation, $\mathcal{R}_V(\theta^V; X_t)$, and the Euler equations for risky assets (consumption claims and dividend claims), $\mathcal{R}_R(\theta^V; X_t)$, as

$$\mathcal{R}_V(\theta^V; X_t) \equiv \Phi_V(C_t, \hat{x}_t^h, \hat{x}_t^\ell, \eta_t) - (1 - \beta)u(C_t) - \frac{\beta}{\alpha} \ln(\mathcal{V}_{t+1})$$

where

$$\begin{aligned} \mathcal{V}_{t+1} \equiv & \eta_t \int_{-\infty}^{\infty} \exp\left(-\alpha \iiint_{-\infty}^{\infty} \Phi_V(C_{t+1}^{(\ell)}, \hat{x}_{h,t+1}^{(\ell)}, \hat{x}_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}) dF(\vec{\epsilon}_{\ell,t+1})\right) dF(x_{\ell,t}) + \\ & (1 - \eta_t) \int_{-\infty}^{\infty} \exp\left(-\alpha \iiint_{-\infty}^{\infty} \Phi_V(C_{t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \hat{x}_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}) dF(\vec{\epsilon}_{h,t+1})\right) dF(x_{h,t}) \end{aligned}$$

²⁰When persistence is known, the vector of state variables reduces to $X_t = (C_t, x_t)$ and the approximant takes the simpler form $\Phi_y(X_t) = \exp\left(\sum_{i_c, i_x \in \mathcal{I}} \theta_{i_c, i_x}^y H_{i_c}(\varphi_c(C_t)) H_{i_x}(\varphi_x(\hat{x}_t))\right)$.

²¹See Judd (1998), Chapter 7.

²²We use this function in order to be able to narrow down the range of values taken by the state variables, such that the approximation performs better when evaluated on the data.

and

$$\mathcal{R}_R(\theta^R, \theta^V; X_t) \equiv u'(C_t) - \beta \mathcal{E}_{t+1}$$

where

$$\begin{aligned} \mathcal{E}_{t+1} \equiv & \eta_t \int_{-\infty}^{\infty} \left(\xi_{\ell,t} \iiint_{-\infty}^{\infty} u'(C_{t+1}^{(\ell)}) \Phi_R(C_{t+1}^{(\ell)}, \widehat{x}_{h,t+1}^{(\ell)}, \widehat{x}_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}) \underbrace{\frac{D_{t+1}^{(\ell)}}{D_t}}_{(i)} dF(\vec{\mathcal{E}}_{\ell,t+1}) \right) dF(x_{\ell,t}) \\ & + (1 - \eta_t) \int_{-\infty}^{\infty} \left(\xi_{h,t} \iiint_{-\infty}^{\infty} u'(C_{t+1}^{(h)}) \Phi_R(C_{t+1}^{(h)}, \widehat{x}_{h,t+1}^{(h)}, \widehat{x}_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}) \underbrace{\frac{D_{t+1}^{(h)}}{D_t}}_{(ii)} dF(\vec{\mathcal{E}}_{h,t+1}) \right) dF(x_{h,t}) \end{aligned}$$

where $\vec{\mathcal{E}}_{\nu,t+1} = \{\varepsilon_{x,\nu,t+1}, \varepsilon_{d,\nu,t+1}, \varepsilon_{g,\nu,t+1}\}$, with $\nu \in \{h, \ell\}$ is a vector of standard normal shocks with distribution $F(\vec{\mathcal{E}}_{\nu,t+1})$. (i) and (ii) are only present in the dividend claim case. We also define

with

$$\begin{aligned} \Psi_t \equiv & \eta_t \int_{-\infty}^{\infty} \phi' \left(\iiint_{-\infty}^{\infty} \Phi_V(C_{t+1}^{(\ell)}, \widehat{x}_{h,t+1}^{(\ell)}, \widehat{x}_{\ell,t+1}^{(\ell)}, \eta_{t+1}^{(\ell)}) dF(\vec{\mathcal{E}}_{\ell,t+1}) \right) dF(x_{\ell,t}) \\ & + (1 - \eta_t) \int_{-\infty}^{\infty} \phi' \left(\iiint_{-\infty}^{\infty} \Phi_V(C_{t+1}^{(h)}, \widehat{x}_{h,t+1}^{(h)}, \widehat{x}_{\ell,t+1}^{(h)}, \eta_{t+1}^{(h)}) dF(\vec{\mathcal{E}}_{h,t+1}) \right) dF(x_{h,t}) \end{aligned}$$

In both cases, $C_{t+1}^{(\nu)}$, $\widehat{x}_{h,t+1}^{(\nu)}$, $\widehat{x}_{\ell,t+1}^{(\nu)}$, $\eta_{t+1}^{(\nu)}$, $\nu \in \{h, \ell\}$, are obtained using the dynamic equations described in subsection B.1. These expressions are simplified when the agent is certain about the persistence. This case amounts to setting $\eta_t = 0$ for all t in the preceding expressions and consider only one process for \widehat{x}_t .

The vector of parameters θ^V and θ^R are then determined by projecting the residuals on Hermite polynomials. This then defines a system of orthogonality conditions which is solved for θ^V and θ^R . More precisely, we solve²³

$$\begin{aligned} \langle \mathcal{R}_V(\theta^V; X_t) | \mathcal{H}(X_t) \rangle &= \int \mathcal{R}_V(\theta^V; X_t) \mathcal{H}(X_t) \Omega(X_t) dX_t = 0 \\ \langle \mathcal{R}_R(\theta^R, \theta^V; X_t) | \mathcal{H}(X_t) \rangle &= \int \mathcal{R}_R(\theta^R, \theta^V; X_t) \mathcal{H}(X_t) \Omega(X_t) dX_t = 0 \end{aligned}$$

where

$$\mathcal{H}(X_t) \equiv H_{i_c}(\varphi_h(C_t)) H_{i_h}(\varphi_h(\widehat{x}_t^h)) H_{i_\ell}(\varphi_\ell(\widehat{x}_t^\ell)) H_{i_\eta}(\varphi_\eta(\eta_t)) \text{ with } i_c + i_h + i_\ell + i_\eta \leq \max(n_c, n_h, n_\ell, n_\eta)$$

²³It should be clear to the reader that the integral refers to a multidimensional integration problem, as we integrate over C , x^h , x^ℓ and η .

and

$$\Omega(X_t) \equiv \omega(\varphi_h(C_t))\omega(\varphi_h(x_t^h))\omega(\varphi_\ell(x_t^\ell))\omega(\varphi_\eta(\eta_t))$$

where $\omega(x) = \exp(-x^2)$ is the appropriate weighting function for Hermite polynomials. Note that since the knowledge of the risky interest rate is not needed to evaluate the direct value function in equilibrium, the system can be solved recursively. We therefore first solve the value function approximation problem, and use the result vector of parameters θ^V to solve for the risky rate problem.

Integrals are approximated using a monomial approach whenever we face a multidimensional integration problem (inner integrals in the computation of expectations and projections) and a Gauss Hermitian quadrature approach when dealing with unidimensional integrals (outer integrals in the computation of expectations).²⁴

The algorithm imposes that several important choices be made for the algorithm parameters. The first one corresponds to the degree of polynomials we use for the approximation. The results are obtained with polynomials of order

- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (5, 2, 2, 2)$ for the value function when $\rho_h = 0.85$,
- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (4, 2, 2, 2)$ for the value function when $\rho_h = 0.90$
- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (3, 3, 3, 3)$ for the interest rate,
- $(n_c, n_{x_h}, n_{x_\ell}, n_\eta) = (2, 4, 4, 1)$ for the asset prices.

The second choice pertains to the number of nodes. We use 8 nodes in each dimension (4096 nodes). The transform functions $\varphi(\cdot)$ are assumed to be linear $\varphi_z(x) = \kappa_z x$ where κ_z , $z \in \{c, h, \ell, \eta\}$ is a constant chosen such that the focus of the approximation is put on values of state variables taken in the data. More precisely, we set $\kappa_c = 2.0817$, $\kappa_h = 40$, $\kappa_\ell = 350$ and $\kappa_\eta = 1$.

The number of nodes used in the uni-dimensional quadrature method used in the outer integral involved in the computation of expectations is set to 12. In the case of the multidimensional integrals, we use a degree 5 rule for an integrand on an unbounded

²⁴See Judd (1998), chapter 7.

range weighted by a standard normal.²⁵ Finally, the stopping criterion is set to 1e-6.

Given these parameters, the algorithm associated to each problem works as follows

1. Choose two candidate vectors of parameters θ^V and θ^R
2. Find the nodes, r_{j_z} , $j_z = 1, \dots, m_z$, at which the residuals are evaluated. These nodes corresponds to the roots of the different Hermite polynomials involved in the approximation, then compute the values of the state variables as

$$C_{j_c} = \varphi_c^{-1}(r_{j_c}), x_{j_h}^h = \varphi_h^{-1}(r_{j_h}), x_{j_\ell}^\ell = \varphi_\ell^{-1}(r_{j_\ell}), \eta_{j_\eta} = \varphi_\eta^{-1}(r_{j_\eta})$$

3. Evaluate the residuals $\mathcal{R}_V(\theta^V; X_t)$ and $\mathcal{R}_R(\theta^R, \theta^V; X_t)$ and compute the orthogonality conditions

$$\langle \mathcal{R}_V(\theta^V; X_t) | \mathcal{H}(X_t) \rangle \text{ and } \langle \mathcal{R}_R(\theta^R, \theta^V; X_t) | \mathcal{H}(X_t) \rangle.$$

4. If the orthogonality conditions are satisfied, in the sense the residuals are lower than the stopping criterion ε , then the vector of parameters are given by θ^V and θ^R . Else update θ^V and θ^R using a Gauss Newton algorithm and go back to step 1.

E.2 Computation of Returns

Given an approximate solution for the value function and the risky return, and given a sequence $\{X_t\}_{t=t_1}^{t=t_2} = \{C_t, \hat{x}_{h,t}, \hat{x}_{\ell,t}, \eta_t\}_{t=t_1}^{t=t_2}$ of annual observations of aggregate per-capita consumption, beliefs and prior probabilities in the time periods $t = t_1$ through $t = t_2$ we compute the conditional n^{th} order moment of the risky rate in period t as

$$E_t^n R_{t+1} = \iiint_{-\infty}^{\infty} \Phi(X_{t+1})^n dF(\vec{\varepsilon}_{t+1}) dF(x_t) \quad (36)$$

²⁵More precisely, we approximate

$$\begin{aligned} \int_{\mathbb{R}^k} F(x) \exp\left(\sum_{i=1}^k x_i^2\right) dx &\simeq a_0 F(0) + a_1 \sum_{i=1}^k (F(re_i) + F(-re_i)) + \\ &+ a_2 \sum_{i=1}^{k-1} \sum_{j=i+1}^k (F(se_i + se_j) + F(se_i - se_j) + F(-se_i + se_j) + F(-se_i - se_j)) \end{aligned}$$

where e_i denotes the i^{th} column vector of the identity matrix of order k . $r = \sqrt{1 + \frac{k}{2}}$, $s = \frac{\sqrt{2r}}{2}$, $a_0 = \frac{2\pi^{\frac{k}{2}}}{k+2}$, $a_1 = \frac{4-k}{4(k+2)} a_0$ and $a_2 = \frac{a_0}{2(k+2)}$. See Judd (1998) for greater details.

The model average n -th order moment is then computed as

$$ER^n = \frac{1}{t_2 - t_1} \left[\sum_{t=t_1}^{t=t_2} E_t^n R_{t+1} - (E_t^1 R_{t+1})^n \right] \quad (37)$$

Similarly, given a sequence $\{C_t, \hat{x}_{h,t}, \hat{x}_{l,t}, \eta_t\}_{t=t_1}^{t=t_N}$, the risk-free rate can be directly computed

$$R_t^f = \left[\beta \eta_t \int_{-\infty}^{\infty} \xi_t^{(l)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \left(\iiint_{-\infty}^{\infty} (U'(\exp(g_{l,t+1}))) dF(\vec{\epsilon}_{l,t+1}) \right) dF(x_{l,t}) \right. \\ \left. + \beta (1 - \eta_t) \int_{-\infty}^{\infty} \xi_t^{(h)}(C_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) \left(\iiint_{-\infty}^{\infty} (U'(\exp(g_{h,t+1}))) dF(\vec{\epsilon}_{h,t+1}) \right) dF(x_{h,t}) \right]^{-1}$$

Just as in the preceding section, integrals are approximated using a monomial approach whenever we face a multidimensional integration problem (inner integrals in the computation of expectations and projections) and a Gauss Hermitian quadrature approach when dealing with uni-dimensional integrals (outer integrals in the computation of expectations). The n -order moments are then obtained in a similar fashion as for the risky rate.

The (conditional) equity premium at time t , is the random variable denoted $R_t^p \equiv E_t^1 R_{t+1} - R_t^f$. Therefore, the n -order order moments of the equity premium can be computed as in eq. (37).

E.3 Accuracy

Our measure of accuracy of the risky rate builds heavily on previous work by Judd (1992). Since we are mostly interested in the empirical properties of the model, we mainly evaluate the accuracy of the solution for the data. Accuracy is assessed by considering the following rearrangement of the Euler equation error (both in the case of the consumption claim based approach and the dividend claim based approach)

$$\mathcal{E}(X_t) = \frac{u'^{-1}(\beta \mathcal{E}_{t+1})}{C_t} - 1$$

This measure then gives us the error an agent would make by using the approximate solution for the risky rate as a rule of thumb for deciding investing one additional dollar as asset holding. This quantity is computed for each value of the state variables in the data. Then three measures, formerly proposed by Judd (1992) are considered

$$E_1 = \log_{10}(E(|\mathcal{E}(X_t)|)), \quad E_2 = \log_{10}(E(\mathcal{E}(X_t)^2)), \quad \text{and} \quad E_\infty = \log_{10}(\sup |\mathcal{E}(X_t)|)$$

The first measure corresponds to the average absolute error, the second one corresponds to the quadratic average of the error, while the last one reports the maximal error an agent would make using the rule of thumb. All measures are expressed in \log_{10} terms, which furnishes a natural way of interpreting the accuracy measure. For instance, a value of E_1 equal to -4 indicates that an agent who uses the approximated decision rule would make –on average– a mistake of 1 dollar for each 10000 dollars invested in the risky asset. These measures are evaluated outside the grid points that are used to compute the approximation. Since our ultimate goal is to assess the quantitative relevance of the model, we need to make sure that our approximation performs well for the data we use. Hence, the measures are evaluated using the data. Results for both models are reported in Table 8 and show that the approximation is accurate.

γ	Known persistence				Unknown Persistence			
	α	E_1	E_2	E_∞	α	E_1	E_2	E_∞
2.0	11.51	-4.98	-8.18	-4.52	17.75	-3.63	-5.63	-3.34
2.5	7.24	-5.54	-9.29	-5.09	11.35	-4.07	-6.50	-3.77
3.0	4.21	-8.66	-15.59	-8.05	6.65	-5.78	-9.93	-5.48

Table 8: **Accuracy of the Numerical Solution:** This table reports the measure of accuracy for the Euler equation. In each case, α was set such that the model generates a risk-free rate of 1.5%.

For example, let us consider the case of known persistence with $\gamma = 2$, an agent who uses the approximate solution based on consumption claims would make, on average, a 1 dollar mistake for every 95,500 dollars invested in the assets, while the maximal error would be of the same order. Good performances are valid for the two values of persistence (ρ) we consider. In the model with unknown persistence, the performances of the approximation slightly deteriorate. This accuracy loss is essentially due to the structure of the problem. When persistence is known, the model is almost log-linear, such that our approximation performs remarkably well. In the full model, the quasi log-linearity is lost as we have to compose probabilities of each model. Increasing the degree of the polynomials yields some (marginal) improvements but (i) leave the results almost unchanged and (ii) comes at a substantial computational cost. We therefore kept the degrees of the polynomials as they are. The accuracy properties of the approximate solution are very similar for the parametrization we consider in the robustness check exercise.²⁶

²⁶Accuracy is actually improved by increasing persistence, lowering the leverage and the discount factor.

References

- ABEL, A. (1999): "Risk premia and term premia in general equilibrium," *Journal of Monetary Economics*, 43(1), 3–33.
- (2002): "An exploration of the effects of pessimism and doubt on asset returns," *Journal of Economic Dynamics and Control*, 26(7-8), 1075–1092.
- BANSAL, R., AND A. YARON (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4), 1481–1509.
- BARRO, R. (2006): "Rare disasters and asset markets in the twentieth century*," *The Quarterly Journal of Economics*, 121(3), 823–866.
- BEELER, J., AND J. CAMPBELL (2009): "The long-run risks model and aggregate asset prices: an empirical assessment," Discussion paper, NBER.
- BLOOM, N. (2009): "The impact of uncertainty shocks," *Econometrica*, 77(3), 623–685.
- BOYLE, P., L. GARLAPPI, R. UPPAL, AND T. WANG (2010): "Keynes Meets Markowitz: The Trade-off Between Familiarity and Diversification," Discussion paper, London Business School.
- CABALLERO, R., AND A. KRISHNAMURTHY (2008): "Collective risk management in a flight to quality episode," *The Journal of Finance*, 63(5), 2195–2230.
- CAMPBELL, J. (1996): "Understanding Risk and Return," *Journal of Political Economy*, 104, 298–345.
- CAMPBELL, J., AND J. COCHRANE (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107(2), 205.
- CAMPBELL, J., AND R. SHILLER (1988): "The dividend-price ratio and expectations of future dividends and discount factors," *Review of Financial Studies*, 1(3), 195–228.
- CAMPBELL, J. Y. (2000): "Asset pricing at the millennium," *The Journal of Finance*, 55(4), 1515–1567.

- CECCHETTI, S., P. S. LAM, AND C. M. NELSON (2000): "Asset Pricing with Distorted Beliefs: Are Equity Returns Too Good to Be True?," *The American Economic Review*, 90, 787–805.
- CHEN, H., N. JU, AND J. MIAO (2009): "Dynamic asset allocation with ambiguous return predictability," Discussion paper, Massachusetts Institute of Technology.
- CONSTANTINIDES, G. (1990): "Habit formation: A resolution of the equity premium puzzle," *The Journal of Political Economy*, 98(3), 519–543.
- CONSTANTINIDES, G., AND A. GHOSH (2010): "Asset pricing tests with long run risks in consumption growth," Working paper 16618, Chicago Booth GSB and NBER.
- DOW, J., AND S. D. C. WERLANG (1992): "Uncertainty aversion, risk aversion, and the optimal choice of portfolio," *Econometrica: Journal of the Econometric Society*, 60(1), 197–204.
- EPSTEIN, L., AND T. WANG (1994): "Intertemporal asset pricing under Knightian uncertainty," *Econometrica: Journal of the Econometric Society*, 62(2), 283–322.
- EPSTEIN, L., AND S. ZIN (1989): "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework," *Econometrica: Journal of the Econometric Society*, 57, 937–969.
- FERNANDEZ-VILLAYERDE, J., AND J. RUBIO-RAMÍREZ (2010): "Macroeconomics and Volatility: Data, Models, and Estimation," Working Paper 16618, University of Pennsylvania and NBER.
- GIORDANI, P., AND P. SODERLIND (2006): "Is there evidence of pessimism and doubt in subjective distributions? Implications for the equity premium puzzle," *Journal of Economic Dynamics and Control*, 30(6), 1027–1043.
- GOLLIER, C. (2011): "Portfolio choices and asset prices: The comparative statics of ambiguity aversion," *The Review of Economic Studies*, 78(4), 1329–1344.
- GUVENEN, F. (2009): "A Parsimonious Macroeconomic Model for Asset Pricing," *Econometrica*, 77(6), 1711–1750.
- HAMILTON, J. (1989): "A New Approach to Economic Analysis of Nonstationary Time Series," *Econometrica*, 57(2), 357–384.

- HANSEN, L. (2007): “Beliefs, Doubts and Learning: Valuing Macroeconomic Risk,” *American Economic Review*, 97(2), 1–30.
- HANSEN, L., AND T. SARGENT (2010): “Fragile beliefs and the price of uncertainty,” *Quantitative Economics*, 1(1), 129–162.
- JOUINI, E., AND C. NAPP (2006): “Heterogeneous beliefs and asset pricing in discrete time: An analysis of pessimism and doubt,” *Journal of Economic Dynamics and Control*, 30, 1233–1260.
- JU, N., AND J. MIAO (2012): “Ambiguity, Learning, and Asset Returns,” *Econometrica*, 80(2), 559–591.
- JUDD, K. (1992): “Projection methods for solving aggregate growth models* 1,” *Journal of Economic Theory*, 58(2), 410–452.
- (1998): *Numerical Methods in Economics*. MIT Press, Cambridge, MA.
- JURADO, K., S. C. LUDVIGSON, AND S. NG (2013): “Measuring uncertainty,” Discussion paper, National Bureau of Economic Research.
- KANDEL, S., AND R. F. STAMBAUGH (1990): “Expectations and volatility of consumption and asset returns,” *Review of Financial Studies*, 3(2), 207–232.
- KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): “A smooth model of decision making under ambiguity,” *Econometrica*, 73(6), 1849–1892.
- (2009): “Recursive smooth ambiguity preferences,” *Journal of Economic Theory*, 144(3), 930–976.
- (2012): “On the Smooth Ambiguity Model: A Reply,” *Econometrica*, 80(3), 1303–1321.
- KRAAY, A., AND J. VENTURA (2007): “The Dot-Com Bubble, the Bush Deficits, and the U.S. Current Account,” in *G7 Current Account Imbalances: Sustainability and Adjustment*, pp. 457–496. National Bureau of Economic Research Inc.
- KREPS, D., AND E. PORTEUS (1978): “Temporal Resolution of Uncertainty and Dynamic Choice Theory,” *Econometrica*, 46(1), 185–200.
- LJUNGQVIST, L., AND T. SARGENT (2004): *Recursive macroeconomic theory*. The MIT Press.

- MACCHERONI, F., M. MARINACCI, AND D. RUFFINO (2013): "Alpha as Ambiguity: Robust Mean-Variance Portfolio Analysis," *Econometrica*, 81, 1075–1113.
- MEHRA, R., AND E. PRESCOTT (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15(2), 145–161.
- MUKERJI, S., AND J. M. TALLON (2001): "Ambiguity aversion and incompleteness of financial markets," *Review of Economic Studies*, 68(4), 883–904.
- ORLIK, A., AND L. VELDKAMP (2014): "Understanding uncertainty shocks and the Role of the Black Swan," Working paper 20445, NBER working paper, <http://www.nber.org/papers/w20445>.
- SHEPHARD, N., AND A. HARVEY (1990): "On the probability of estimating a deterministic component in the local level model," *Journal of Time Series Analysis*, 11(4), 339–347.
- STRZALECKI, T. (2013): "Temporal resolution of uncertainty and recursive models of ambiguity aversion," *Econometrica*, 81(3), 1039–1074.
- UHLIG, H. (2010): "A model of a systemic bank run," *Journal of Monetary Economics*, 57(1), 78–96.
- VERONESI, P. (1999): "Stock market overreactions to bad news in good times: a rational expectations equilibrium model," *Review of Financial Studies*, 12(5), 975–1007.
- WEIL, P. (1989): "The equity premium puzzle and the risk-free rate puzzle," *Journal of Monetary Economics*, 24(3), 401–421.
- WEITZMAN, M. (2007): "Subjective expectations and asset-return puzzles," *The American Economic Review*, 97(4), 1102–1130.
- WHITELAW, R. F. (1994): "Time Variations and Covariations in the Expectation and Volatility of Stock Market Returns," *Journal of Finance*, 49(2), 515–541.
- WHITELAW, R. F. (2000): "Stock market risk and return: An equilibrium approach," *Review of Financial Studies*, 13(3), 521–547.