Communication at the Zero Lower Bound: The Case for Forward Guidance

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Abstract

Forward guidance has become an integral part of the policy arsenal of central banks, yet its effectiveness was assumed to rely on the bank's commitment to high future inflation. In this paper I show a novel mechanism which generates disagreement in interest rate forecasts between the central bank (CB) and the agents and calls for forward guidance as a communication device without any policy deviations.

The zero lower bound (ZLB) acts as an informational curtain for adaptively learning agents as they cannot observe the path of the hypothetical shadow rate. I prove analytically that this biases agents' forecast upwards towards earlier lift-off, consistent with data from the FED, Swedish Riksbank and surveys. The disagreement coupled with the learning agents results in unanchored expectations and explosive dynamics. Forward guidance then restores stability at the ZLB by preventing spurious expectational drift. Thus, the paper calls for transparent communication by the CB in ZLB environments. Although such communication is welfare improving, no forward guidance puzzle is present.

Keywords: Delphic forward guidance; unanchored expectations; central bank communication; zero lower bound; adaptive learning;

JEL Classifications: E43; E52; E58; E61;

1 Introduction

In the aftermath of the Global Financial Crisis not only did interest rates in many countries reach their lower bound for the first time, but they also remained constrained for an unprecedented duration. The Federal Funds Rate in the US remained at zero between 2008 - 2015, only briefly recovering before being plunged down by the onset of the pandemic. Other developed countries (e.g. EU, UK, Japan, etc.) did not exhibit that brief recovery to positive interest rates and as a result experienced over a decade of ultra-low interest rates limited by their bound.

This experience spurred a large literature trying to understand the behaviour of the economy in such novel circumstances. This phenomenon was also accompanied by unconventional monetary policy instruments to which central banks (CBs) resorted once the interest rates were no longer flexible. Understanding the mechanisms and limitations of such policy tools will help us better navigate future recessions.

In this paper I focus on one of these instruments - forward guidance, that is, the then unconventional practice of central bank communication about the future path of the policy rate. I build a novel theory of why the need for such communication arises when interest rates are bounded below, what potential benefits it brings and what the dangers of not doing so are.

Generally, there are two reasons for doing forward guidance – policy commitment or transmission of useful information to the public. Both have equal empirical support (Campbell et al. 2017) but there is a distinct lack of models and understanding about why and how the central bank would communicate – what information is transmitted and how would the agents react or learn.

This paper tries to fill this gap and illustrates a mechanism through which the zero lower bound (ZLB) [1] biases the public's expectations and provides a role for the central bank to correct them through communication. In particular, the ZLB acts as an informational curtain for the agents – it truncates the interest rate data

1. Note that throughout I will be using the term "zero lower bound" despite some countries having experienced negative interest rates. This is because the effective lower bound in the US was indeed zero and it represents a convenient benchmark for the theoretical model. The arguments of the paper hold regardless of the exact level of the binding lower bound.

below zero, preventing them from updating their old optimistic beliefs about the recovery of the economy and their anticipated trajectory of the policy rate. This gives rise to a forecast disagreement between the agents and the central bank with agents expecting a higher path of future interest rates, giving reason for the central bank to communicate and inform them about its own projection for the policy rate, i.e. engage in forward guidance.

Section 2 provides empirical evidence for forecast disagreement between central banks and private agents at the ZLB. In particular, I show that in both the US and Sweden markets systematically expected earlier lift-off from the ZLB than the bank's internal projections. Yet, above the ZLB there is no evidence of forecast disagreement giving credence to the role of the ZLB in biasing private expectations.

Section 3 builds a New Keynesian model with learning agents who observe the realised shocks and try to econometrically estimate how these affect the endogenous aggregate variables - that is, they are trying to learn the underlying model of the economy. I augment the standard approach to learning by asking the agents to keep track of a notional shadow rate along with understanding the binding ZLB, thus allowing them to better cope with the new regime. Despite their higher sophistication, I prove analytically that the agents will always have an upward bias in their forecasts in the face of a binding ZLB. I also incorporate forward guidance messages about the future of the policy rate in the agents' expectations and learning, thus enabling the central bank to correct their misspecified beliefs.

The reason for the expectational bias stems from the fact that a binding ZLB means the central bank cannot accommodate fully the shock by lowering the interest rate enough. Thus, output gap and inflation will suffer proportionately more compared to what the agents had experienced in normal times (i.e. during the Great Moderation mid-1980s to 2007). Given lower output and inflation, a standard policy rule would imply a lower trajectory for the interest rate. Due to the truncation of the shadow rate by the ZLB, the learning agents have no way of observing the new trajectory. Communication about the future of the policy rate, as in the case of forward guidance, is the least informationally dense message which the central bank can use to correct the agents' misspecified beliefs.

Finally, Section 4 studies numerically the size of the forecast disagreement and the implications of forward guidance. A shock comparable to the Great Recession results in private expectations anticipating a lift-off from the ZLB 4 quarters earlier than the central bank, which has model-consistent expectations. Policy communication proves to be effective in correcting private expectations, but most importantly foregoing any communication at all runs the increased risk of expectations becoming unanchored - resulting in instability. This result calls for the regular use of forward guidance at the ZLB as a communication device even when abstracting from any policy commitment.

The literature on forward guidance (FG) largely agrees that there are two main classifications of FG depending on the underlying reasons for its use. The seminal work of Krugman et al. (1998) and Eggertsson and Woodford (2003) showed that promises of lower interest rates for longer than originally intended can largely mitigate the negative effects of a binding ZLB on interest rates. The stimulus comes through agents expecting low interest rates in the future (i.e. accommodative monetary policy) and higher inflation, hence cutting back less on present investment and consumption. Campbell et al. (2012) label this approach Odyssean Forward Guidance. Although certainly valid, this type of forward guidance suffers from time-inconsistency of the commitment to higher future inflation (i.e. "commitment to irresponsibility") and to potential credibility concerns if the period of the commitment spans more than one set of policy committee members with potentially different policy preferences.

Campbell et al. (2012) also acknowledge a more established form of FG, pursued by the Reserve Bank of New Zealand and the Riksbank in Sweden, for example. In essence, such CBs engage in regular forecasts of the path of their policy rate given their current economic outlook, hence it was dubbed Delphic forward guidance. This type of FG inherently lacks a commitment as it is understood that the forecast might change next quarter when more data becomes available or new shocks arrive. Such informational FG may be useful to the public if the CB has better information about the state of the economy - a view I take in this paper.

Moreover, in Marinkov (2020) I propose another function of forward guidance as a communication strategy for latent policy change. There, the learning agents and the central bank share their expectational facility and model of the economy, but the central bank possesses private information about its exact policy rule. Then, the ZLB again acts as an informational curtain for the learners who fail to estimate a potential policy change[2] as the interest rate is bound by the ZLB . In that case, forward guidance is a useful tool in helping them learn the new policy regime through announcing future lift-off dates [3].

Here, in the present paper I build on this previous work but pursue a more fundamental, structural reason for FG. Instead of considering a policy regime change, I show that the non-linearity introduced by the ZLB itself acts as a regime change for the agents and this creates disagreement between their policy rate forecasts and those of the central bank, who knows the precise structure of the economy. Importantly, this happens without any change in the policy parameters. Therefore, FG acts as a helping hand for learners to update their perceived law of motion of the economy under the ZLB regime. Such information revelation about the structure of the economy is akin to Delphic forward guidance. Although empirically supported by Campbell et al. (2017), they and others[4] only incorporate Odyssean FG through anticipated monetary policy shocks in their models and do not study theoretically or numerically the effects and nature of Delphic FG. The model here allows for Delphic FG by showing a channel which could explain the observed policy rate forecast disagreement in the data between central banks and the private sector.

The main insight is that the zero lower bound calls for a necessary increase in transparency and communication by the central bank at the ZLB because it acts both as a regime change and an information curtain preventing agents from correctly adjusting their expectations about the path of the interest rate. First, forward guidance is shown to have a welfare-improving effect by helping the agents update

2. E.g. as a result of introduction of macroprudential policies or changing Taylor policy coefficients.

3. Marinkov (2020) explores various communication and interpretation schemes for the FG signal. Wrong interpretation or small weights of the signal are shown to still be marginally better than no communication at all. The stimulative effects of a prolonged ZLB duration are modest and no forward guidance puzzle is present.

4. see Eggertsson and Woodford (2003), Del Negro et al. (2012), Campbell et al. (2012), Ben Zeev et al. (2017) among others

their expectations even in the absence of interest rate observations. The benefit is modest and no forward guidance puzzle is present. Second, forward guidance helps prevent an expectational drift due to agents expecting an earlier lift-off from the ZLB. This is numerically shown to improve the stability of the system by keeping it tighter within the basin of convergence to the rational expectations equilibrium. This is a novel result which complements prior work on the stability implications of monetary policy in learning models (see Evans and Honkapohja (2003)) . In the model this communication is achieved solely through forward guidance, yet in reality a combination of FG and asset purchases might be needed to achieve the necessary shift in expectations. For instance, Campbell et al. (2017) and Andrade et al. (2019) show that FG was successful at shifting short-term expectations but quantitative easing was more adept at affecting the longer end of expectations. Relatedly, Bhattarai et al. (2022) argue that quantitative easing might be useful as a commitment device for Odyssean forward guidance.

2 Evidence of forecast disagreement at the ZLB

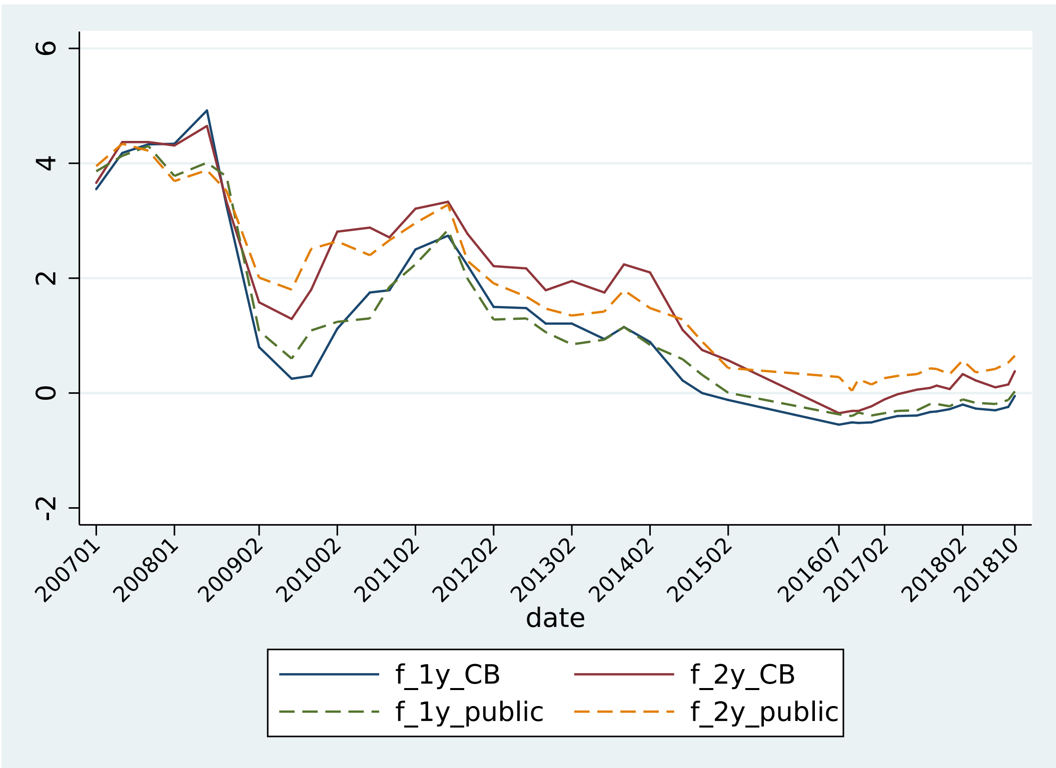
The Great Recession and the followed long spell of binding ZLB were unprecedented events that caught the public by surprise. This lead to high levels of disagreement by historical standards among private forecasters (Andrade et al. 2019) and to aggregate forecast errors compared to historically well-performing empirical policy rules. In particular, Engen et al. (2015) show that during 2008-2014 Blue Chip forecasts systematically overestimated the Treasury Bill compared to what standard Taylor rules would have suggested. The authors conclude that the forecasters' beliefs evidently reflected both a misunderstanding of the FED's reaction function and excessive optimism about the likely speed of the recovery.

Marinkov (2020) considered the case of an unobserved policy change at the ZLB as a reason for the use of communication-type forward guidance. Instead, here I explore the evidence of forecasts disagreement between central banks and the markets in order to uncover whether the markets' optimism suggested by Engen et al. (2015) also persists when compared to the CBs' best internal forecasts rather than stylised policy rules.

To disentangle the disagreement between the CB and private agents both their forecasts are needed. Among major central banks the Swedish Riksbank is one of the few who publish internal consensus interest rate forecasts along with private market forecasts. They began releasing their internal forecasts in the 2007 issue of their Monetary Policy Report.

Figure 1 plots the 1-year-ahead and 2-year-ahead repo rate forecasts for both the Riksbank (solid lines) and the public (dashed lines). As expected, they are not too disparate from one another, yet there are two important features of data. First, whenever interest rates are expected to be binding to some lower bound, the private forecasts are always supportive of an earlier lift-off than the Riksbank's. Second, Sweden is a special case among developed economies because it dipped twice to the zero lower bound (ZLB), thus it provides more comparable data above and below the ZLB and allows for testing the theory that the ZLB causes disagreement between the CB and the agents.

Figure 1: Forecasts of Swedish repo rate



Source: Riksbank's Monetary Policy Report 2007-2018

Table 1: 1-year-ahead disagreement on Swedish repo rate

Based on private agents' expected 1-year-ahead repo rate

	count	mean	se(mean)	min	max
Low	16	-.1135	.0085	-.18	-.05
High	24	.0107	.0755	-.79	.91

Based on Riksbank's expected 1-year-ahead repo rate

	count	mean	se(mean)	min	max
Low	18	-.1391	.0192	-.37	-.05
High	21	.0616	.0804	-.79	.91

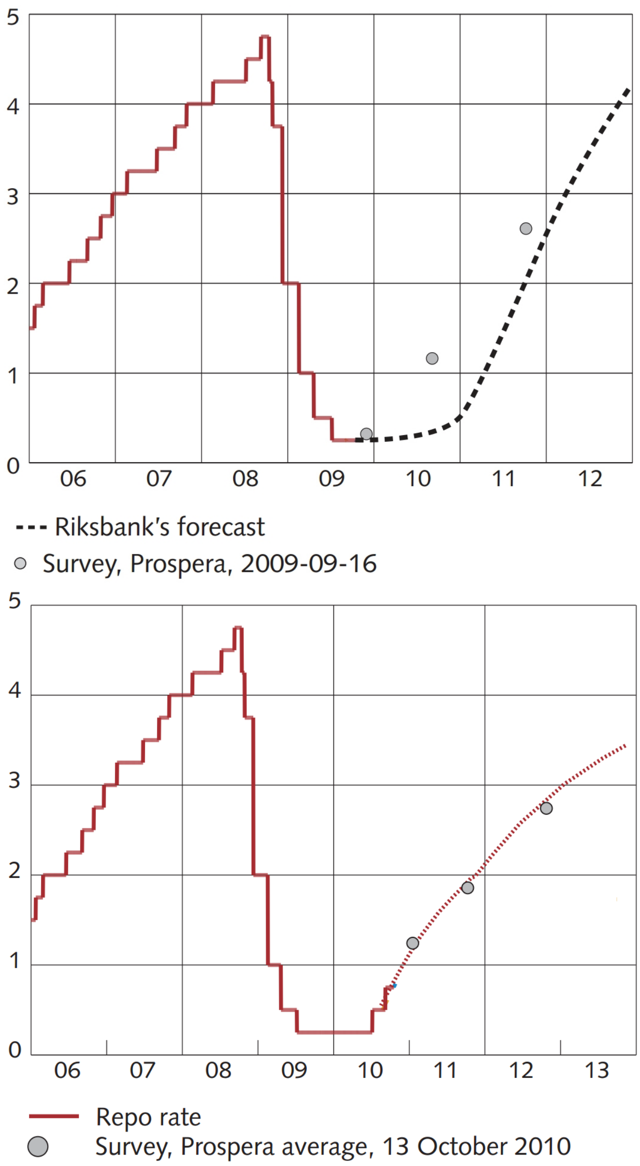
Note: 'High' and 'Low' states refer to 1-year-ahead expected repo rate above or below 0.25, respectively. The first block defines 'High' and 'Low' based on private agent's expectations and the second - on Riksbank's forecast.
Source: Riksbank Monetary Policy Report (2007-2018)

To quantify the disagreement between the agents and the Riksbank Table 1 computes the difference between the forecasts of the Bank and those of the market. The measure is set up such that a positive disagreement means that the Riksbank expects higher future repo rate than the market. The data is split in two regimes - Low and High, where Low is classified as expected 1-year-ahead repo rate to be smaller than 0.25, and High - to be larger than 0.25. The table shows the classification according to future expected repo rates by both Riksbank and the market. Further robustness classifications on horizons and cut-offs are performed in Table A.1 in appendix A.1.

Table 1 shows that regardless of the classification private agents expect an earlier lift-off than the CB (negative and significant average disagreement) when the economy is a Low regime of near zero interest rates. Moreover, the High regime of normal times exhibits no systematic forecast bias for either party. As a case in point, Figure 2 shows that during the first ZLB spell in 2009 agents expected a higher interest rate path than the Bank, but already a year later when the interest rate left the ZLB expectations aligned perfectly.

It is worth noting that disagreement between the Riksbank and the market continued throughout the ZLB spell. This is an unexpected fact because of Riksbank's open and explicit interest rate forecasts which one would expect are one of the most transparent and informative means of CB communication. Perhaps, the market did not put a high enough weight on their routine announcements while the unprece-

Figure 2: Forecasts of Swedish repo rate 2009-2010



Source: Riksbank's Monetary Policy Report 2009-2010

Table 2: Private FFR forecasts observation count by horizon

	1q	2q	3q	4q	5q	6q	7q	8q
count	47	48	47	31	18	16	12	13

Source: Private economists Federal funds rate forecasts from the Wall Street Journal Forecasting Survey (March 2003 - December 2014)

dented forward guidance by the Federal Reserve and the Bank of England among others had a notable effect on market expectations as shown by Engen et al. (2015), Andrade et al. (2019) and Campbell et al. (2017). See Marinkov (2020) on the implications of imprecise or unconvincing forward guidance in a model with learning agents.

The Federal Reserve did not use to publish routine interest rate forecasts. However, the Federal Reserve Tealbook (formerly Greenbook) publishes internal consensus projections by the research staff at the Board of Governors. These are prepared before each FOMC meeting and are released to the public with a five year lag. At the time of writing, projections for the federal funds rate are available for the period January 1981 - December 2014[5] at 8 projections per year, corresponding to the respective FOMC meetings. The forecast horizons differ for each meeting and could go up to 8 quarters ahead.

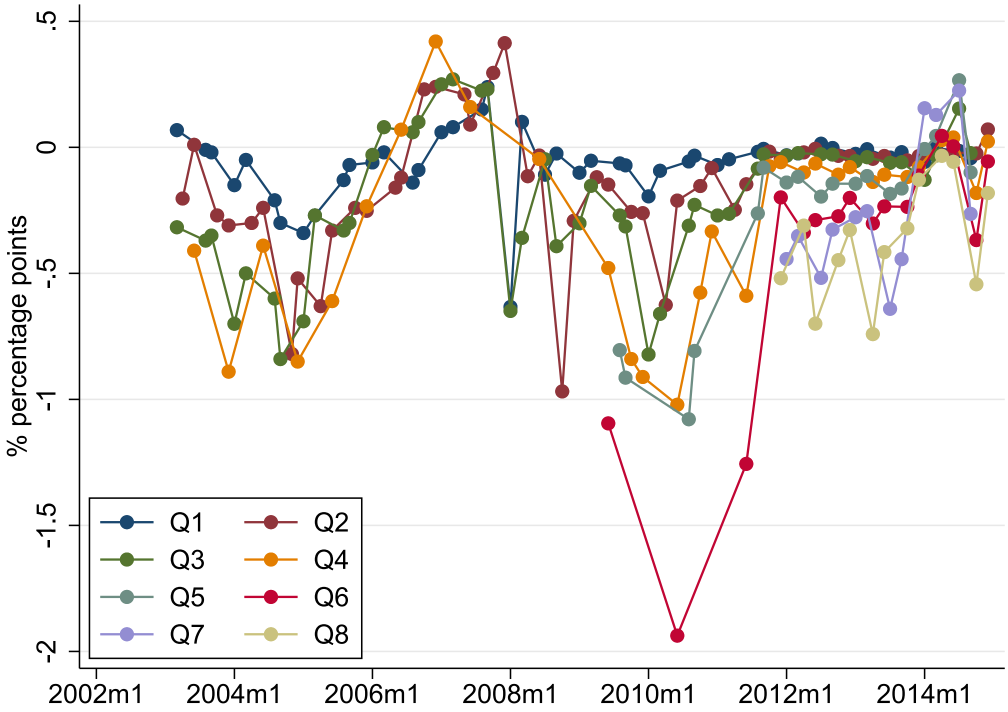
For the market's FFR forecasts I use the Wall Street Journal (WSJ) Economic Forecasting Survey which is available monthly since January 2003. It provides forecasts from a panel of academic, business and financial economists, and the forecasts are usually elicited for the end of next June and December. Since 2009-2011 the forecast horizon has been increased to sometimes 3 years ahead but the forecasts are always for the months of June and December. After matching the forecast dates with the FOMC meetings from the Tealbook, as well as the implied forecast horizons of the two datasets, we get the following number of corresponding forecasts:

Figure 3 shows the FFR forecast disagreement between the Tealbook and the WSJ survey at any given date for up to 8 quarters ahead. The series are computed by subtracting the average WSJ survey FFR forecast for a given horizon of x quarters

5. The data could be obtained under the "Output Gap and Financial Assumptions from the Board of Governors" section in the Federal Reserve Bank of Philadelphia's Real-Time Data. Upon enquiry, it was confirmed that currently no future expansions of the federal funds rate (FFR) projection releases are planned.

from the FED's forecast for the same horizon at a given time t. Thus, negative values imply that at time t the market expects higher FFR at $t + x$ than the FED, that is $E_t^{WSJ} i_{t+x} > E_t^{FED} i_{t+x}$.

Figure 3: Federal funds rate forecast disagreement



Source: Federal Reserve Tealbook and Wall Street Journal Economic Forecasting Survey

Notably, periods of rapid monetary policy loosening and reaching an effective lower bound, such as the dot-com bubble or the financial crisis, are seen by the forecasters as temporary. They expect the FFR to be more quickly reverted to the previous "normal" levels than what the FED projects. We see that this disagreement quickly vanishes with the hiking of the rate in 2005-2007, and the beginning of calendar- and threshold-based forward guidance messages from early 2012[6]. The disagreement is larger for longer horizons, indicating the forecasters expect an overall higher trajectory for return to normal. Interestingly, even after a sustained period at the ZLB and a slew of unconventional monetary policies, until mid-2013 the markets still expect higher FFR 1.5-2 years ahead. These dynamics of disagreement are in

6. Quantitative easing certainly played an important role for this, yet its analysis is beyond the scope of this paper, instead, please refer to Bhattarai et al. 2022 who argue that quantitative easing could act as a commitment device for forward guidance.

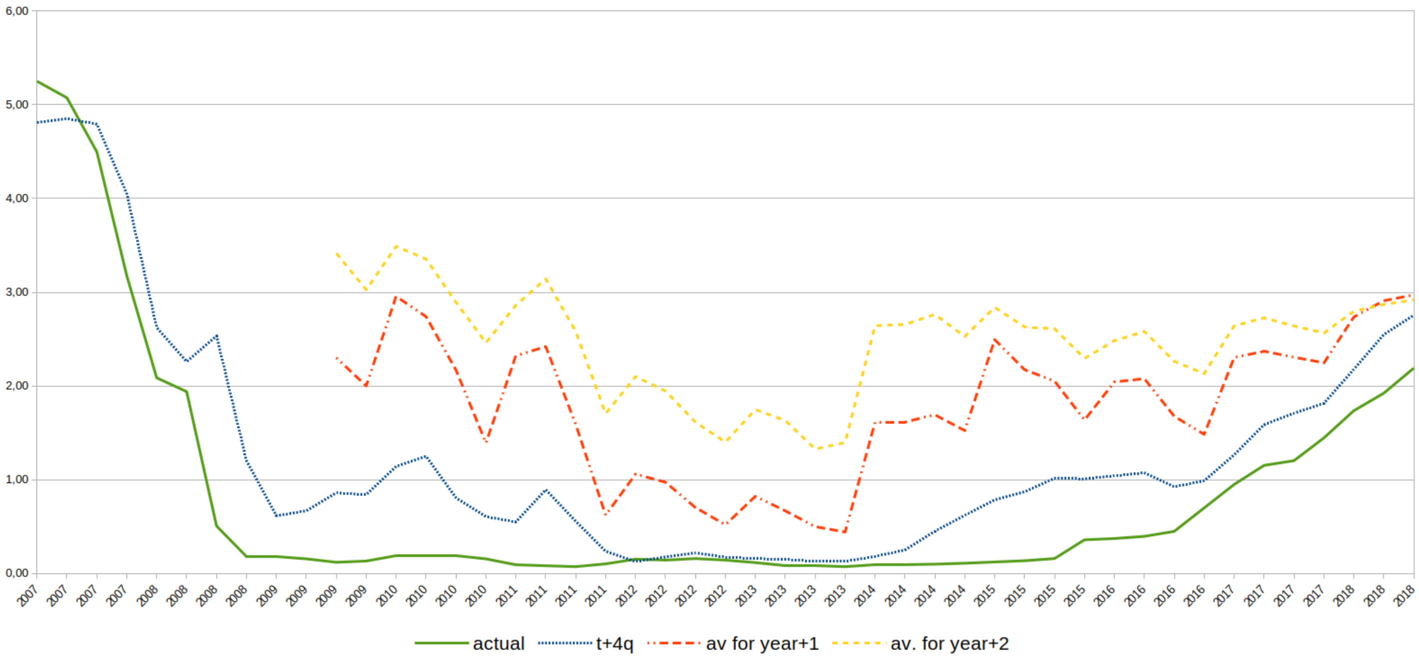
accordance with the case of the Riksbank above - both suggesting that in low-interest rate environments private agents tend to expect higher policy rate trajectories than the central bank. This policy forecast disagreement will be the main object of central bank communication when we later model forward guidance as working though an information transmission channel.

Using the Blue Chip Financial Forecasts (BCFF) survey, Cieslak (2018) finds that private FFR forecast errors are large and negative during monetary policy easing (i.e. ecoonmic downturns) and are more predictable than the actual future FFR. This is particularly evident during the beginning of the ZLB period (2007-2009). The author finds that Tealbook FED staff forecasts are more precise than the market's but still feature predictable forecast errors. She concludes that real-time forecastin is too demanding in reality for the assumption of full-information rational expectations to hold. Although it does not study forecast disagreement between the market and the FED, that paper provides another piece of evidence that markets have a hard time predicting the FFR in recessions when interest rates are quickly approaching their lower bound.

Similarly, Figure 4 shows the average expectations of professional forecasters in the US for the Treasury bill rate. The period of explicit date- and state-contingent forward guidance (2011-2013) saw market expectations converging closely to what ended up being the actual rate, closing their forecast errors. Yet, before that period and even after it around 2015, market expectations were higher than what the T-bill rate ended up being, resulting in significant negative forecast errors. This is indirect evidence of a similar pattern as observed in Sweden and the US FFR above.

Finally, abstracting from disagreement, Campbell et al. (2012) find that future monetary policy tightening lowers unemployment expectations and increases inflation expectations in the US, contrary to the predictions of New Keynesian models. They interpret this finding as evidence for successfully communicated Delphic forward guidance by the FOMC. Campbell et al. (2017) study empirically the hypothesis that FOMC's meeting announcements carried Delphic forward guidance. They classify private information of the FED by the difference between the Tealbook forecasts on inflation, GDP growth and unemployment rate and Bluechip survey of private forecasters' expectations. They find that the four-quarter ahead futures contract rate is statistically positively correlated with policy makers' forecast of future GDP growth being higher than the market expects (and lower for unemployment). This is evidence that the committee's private information about the future of the economy was transmitted through the FOMC's announcements - supporting a Delphic forward guidance interpretation.

Figure 4: Private Forecasts of US T-bill rate



Source: Survey of Professional Forecasters

3 Model

To explain why disagreement between the central bank and the private agents arises at the zero lower bound I build a simple New Keynesian model featuring adaptive learning as an expectation formation framework for the agents. The model environment is the canonical New Keynesian model with a representative consumer and monopolistically competitive firms subject to Calvo pricing.

This section begins by outlining the model under rational expectations before introducing adaptive learning as the expectations formation framework of the agents. Marinkov (2020) describes in detail the difference between rational and learning agents and how each model interacts with the non-linearity of an occasionally binding ZLB. Notable differences in the current paper are the reduced form knowledge of the Taylor rule by the agents and the presence of the lagged interest rate as a state variable. As will later become clear the first assumption eliminates the simultaneity in determining the output gap, inflation and the policy rate, while the second makes the learners' forecasts of output gap and inflation more responsive to the ZLB - a necessity pointed out in Marinkov (2020). As a result, the ZLB biases the interest rate forecasts of the agents towards earlier lift-off.

Lastly, the section describes how forward guidance can be incorporated in the learning framework.

3.1 Rational Expectations

As extensively discussed in Woodford (2003), under rational expectations (RE) the linearised aggregate economy of the canonical New Keynesian model can be summarized by the following two equations:

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t\right) \tag{1}$$

$$\pi_t = \kappa x_t + \beta \mathbb{E}_t \pi_{t+1} + u_t \tag{2}$$

with shock processes

$$r_t = \rho_r r_{t-1} + \varepsilon_t^r, \quad \varepsilon_t^r \sim N(0, \sigma_r^2) \tag{3}$$

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u, \quad \varepsilon_t^u \sim N(0, \sigma_u^2) \tag{4}$$

where x_t is the current output gap, defined as the difference between output and its natural rate in an economy with fully flexible prices; π_t denotes the inflation rate; i_t the nominal interest rate; β is the discount factor; σ is the coefficient of relative risk aversion; and κ is a convolution of structural parameters. All endogenous variables are expressed as log-deviations from their steady state values. Thus, in steady state $x = \pi = i = 0$. Finally, r_t and u_t stand for exogenous natural rate and cost-push shocks, respectively, and follow $AR(1)$ processes.

The model is closed with a monetary policy (MP) rule subject to the zero lower bound (ZLB):

$$\begin{aligned} s_t =& \rho_i s_{t-1} + (1-\rho_i)(\chi_\pi \pi_t + \chi_x x_t) \\ i_t =& \max\{i^*, s_t\} \end{aligned} \tag{5}$$

where s_t is the unconstrained shadow rate and i_t is the realised policy rate subject to the lower bound i^*. Note that above the ZLB this monetary policy rule implies that the policy and shadow interest rates coincide - that is $s_t = i_t$ if $s_t \geq i^*$, while i_t is constrained by the ZLB where s_t is not - $i_t = i^*$ if $s_t < i^*$.

The reaction parameters satisfy $\chi_\pi > 1$ and $\chi_x > 0$, and the interest rate smoothing - $\rho_i \in (0,1)$. The constant $i^* = 1 - 1/\beta < 0$ represents the effective lower bound on interest rates since, otherwise, as explained in Eggertsson and Woodford (2003) agents would choose to hold all their assets in cash. I will refer to it as the ZLB to be consistent with the arguments in the Introduction and with real world analogies.[7]

3.2 Expectations formation

The specification of expectations employed is adaptive learning (ADL). In particular, agents do not know the true structure of the economy and make forecasts as econometricians using simple regression models[8]. Namely, they make forecasts according to the aggregate policy functions from the minimum state-variable RE solution to the model:

$$\underset{3\times1}{Y_t} \equiv \begin{bmatrix} x \\ \pi \\ i \end{bmatrix}_t = \underset{3\times3}{\Gamma} \begin{bmatrix} u_t \\ r_t \\ i_{t-1} \end{bmatrix} \equiv \Gamma Z_t \tag{6}$$

where due to the smoothing in the Taylor rule, the lagged interest rate becomes a state variable[9].

7. Appendix A.2 provides estimates for the policy coefficients in the monetary policy rule.
8. Following the 'consistency principle' of Evans and Honkapohja (2001)
9. Note that (6) represents the solutions of the model under RE without a ZLB. If the ZLB is

Adaptive learning is a linear updating procedure, yet the ZLB creates a non-linearity in the path of the interest rate as in (5), so agents must understand it cannot be realised below i^*. To get around this issue I model the agents as forming expectations about the shadow interest rate and then applying the ZLB to their expectations just like the policy is being set. Hence, importantly, they use realised rather than shadow prices when forming expectations of x_t and π_t. During a period of binding ZLB the agents use the shadow rate to form lift-off expectations consistent with the known policy prior to the ZLB. When $s_t < i^*$ if they were to use $i_{-1} = i^*$ as a basis for expectations for $t = 0, 1...$ in (6), they would have an upward bias in their projected paths for the interest rate because the ZLB i^* is higher than the shadow rate at $t = -1$. Hence, as outlined below, I assume that above the ZLB agents rely solely on realised prices. When the ZLB binds, on the other hand, due to a lack of exact observable data on the policy rate, they rely on their shadow rate projections for keeping track of the full path the policy instrument below the ZLB. Thus, even though the use of a shadow rate complicates the notation, this dichotomy is necessary for more realistic and sophisticated expectations. In this sense the imperfect knowledge of the adaptively learning agents here is conservative.

As shown next, each period as additional data become available the agents update their forecasting model by updating the coefficients to their perceived transition matrix Φ_t using a recursive constant gain algorithm. I assume that they observe the disturbances r_t and u_t and know their autoregressive coefficients [10].

Adaptive Learning

Denote by $S_t \equiv \begin{bmatrix} u_t & r_t & s_{t-1} \end{bmatrix}'$ the state variables vector using the shadow interest rate. Then, just like in the rational expectations solution in (6) the learning agents use the state variables vector and a transition matrix to forecast the endogenous variables vector $\begin{bmatrix} x_t & \pi_t & s_t \end{bmatrix}'$. Unlike RE, however, they do not know the correct transition matrix Γ from (6) and instead use their perceived 3-by-3 transition matrix

respected, when binding the solution of the model will be piece-wise linear featuring a sequence of different policy transformations Γ^i for every period i when the ZLB is binding.

10. Eusepi and Preston (2010) show that this assumption can be dispensed with and instead agents would estimate those coefficients. For simplicity, it is maintained.

Φ_{t-1} from the end of period $t-1$. Remember that the RE state variables vector with actual prices is $Z_t \equiv \begin{bmatrix} u_t & r_t & i_{t-1} \end{bmatrix}'$. Given the discussion above I assume agents use the vector of realised prices Z_t to form expectations of the output gap and inflation, but they use S_t to forecast the interest rate and respect the ZLB.

$$\begin{aligned} \hat{\mathbb{E}}_t \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{t-1} Z_t \\ \hat{\mathbb{E}}_t s_t &= \phi_{3,t-1} S_t \\ \hat{\mathbb{E}}_t i_{t+j} &= \max\left\{ i^*,\ \hat{\mathbb{E}}_t s_{t+j} \right\}, \qquad j \geq 0 \end{aligned} \tag{7}$$

where $\hat{\mathbb{E}}$ is the expectations operator for the learners and $\phi_{n,t}$ is the n^{th} row of their perceived 3-by-3 transition matrix Φ_t. Agents update this perceived law of motion (PLM) by a recursive constant gain algorithm using the discrepancies between their expectations of endogenous variables $\hat{\mathbb{E}}Y_t$ and the actual realizations Y_t. They weigh this discrepancy by the historical variance-covariance matrix R_{t-1} of the endogenous variables and use the weighted forecast discrepancy for error correction. Each error correction term is given a constant gain weight τ against their prior beliefs from $t-1$ [11]. Finally, they update the variance-covariance matrix R_t in a similar fashion.

$$\begin{aligned} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_t &= \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{t-1} + \tau R_{t-1}^{-1} Z_t \left(\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} - \hat{\mathbb{E}}_t \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} \right)' \\ \phi_{3,t} &= \phi_{3,t-1} + \tau R_t^{-1} S_t (i_t - \hat{\mathbb{E}}_t i_t) \\ R_t &= R_{t-1} + \tau (Z_t Z_t' - R_{t-1}) \end{aligned} \tag{8}$$

3.3 Bounded Rationality and the Actual Law of Motion

Replacing RE with ADL means that the structural equations of the economy (1)-(2) need to be modified accordingly. For a related class of models Preston (2005) and Eusepi and Preston (2018) argue that under ADL aggregate expectations $\hat{\mathbb{E}}_t$ are an

11. Note that here I assume constant gain learning instead of the decreasing gain learning used in Evans and Honkapohja (2001). The reason is that the former is more useful for tracking regime changes, while the latter is useful for studying asymptotic convergence properties of learning models to their RE counterparts. Given the current emphasis on the ZLB, tracking is a necessary feature of the model.

average of the expectations of heterogeneous households and firms who know only their own objectives, constraints and beliefs and cannot compute aggregate probability laws, i.e. cannot obtain model-consistent expectations like RE. Thus, agents act rationally when it comes to their own objective functions but unlike rational agents fail to anticipate the aggregate laws of motion and resort to econometric learning as in section 3.2. A representative agent occurs when a symmetric equilibrium is assumed in which although everyone's problem is identical, no individual is aware of that and as a result the representative agent cannot compute aggregate probability laws. This breaks the law of iterated expectations (LIE) for the operator $\hat{\mathbb{E}}$, and hence the recursion from which the aggregate demand (1) and Phillips curve (2) equations are derived. These two equations under ADL and $\hat{\mathbb{E}}$ then depend on a long horizon expectations reading:

$$x_t = \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \frac{1}{\sigma} (i_T - \pi_{T+1} - r_T) \right] \tag{9}$$

$$\pi_t = \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa (x_T + u_T) + (1-\alpha)\beta \pi_{T+1} \right] \tag{10}$$

where $\hat{\mathbb{E}}_t$ again stands for the expectations of the adaptive learners and α is the Calvo probability of not being able to reset prices. I will refer to these two equations as the actual law of motion (ALM) of the economy.

Yet, Honkapohja et al. (2012) point out that assuming a continuum of symmetrical agents as is the case in the used NK model, one could still apply the LIE and resort to one period ahead Euler equation learning. I keep the infinite horizon learning for two reasons.

First, it allows for incorporation of FG as information about future policy rates into the law of motion for the aggregate variables, contrary to the Euler equation learning approach. Second, because agents do not know the structure of the economy, they cannot foresee how the ZLB will change the actual law of motion of the system. Consider equation (1) and apply $\hat{\mathbb{E}}$ instead of the RE operation like the Euler equation learning approach advocated by Honkapohja et al. (2012) would prescribe. Now, if the ZLB is expected to be binding for a few periods ahead, its effect should come through expectations of the output gap ($\hat{\mathbb{E}}_t x_{t+1}$) and inflation ($\hat{\mathbb{E}}_t \pi_{t+1}$). But these are only gradually updated (as described in section 3.2), implying that although agents respect the ZLB in their forecasts for the interest rate, they are completely oblivious of its future effects on inflation and the output gap when only the one-period-ahead Euler equation (1) is used. In contrast, suppose that agents expect $t = T^{ZLB}$ as the last period of binding ZLB. Then in the long horizon approach (9) they could set $\hat{\mathbb{E}}_t i_{t+s} = i^*$ for all $t + s \leq T^{ZLB}$. Then, the expected duration of the ZLB has an effect on the realisations of the output gap both through the current and future binding periods, which in turn is reflected on future inflation as in (10). Thus, the economy driven by the learners features minimal deviations from the rational expectations economy which are reflected only in the recursively updated $\hat{\mathbb{E}}_{t+s} x_{t+s+j}$ and $\hat{\mathbb{E}}_{t+s} \pi_{t+s+j}$ for $s \geq 1$, $j \geq 0$.

These arguments are related to the discussion of the use of long horizon expectations for anticipated structural changes in Evans et al. (2008).

Disagreement between the CB and the learning agents

Suppose the economy exists for a long enough period with no extreme shocks that bring it to the ZLB. Then, following the forecast and updating procedures from section 3.2 the learning agents converged to the RE solution of the model in (6) This implies that at some period $t-s$ the perceived transition matrix has converged to the actual one - $\Phi_{t-s} = \Gamma$. Therefore, the agents have fully learned the model with no binding ZLB. The period of the Great Moderation is a useful analogy for this scenario[12].

Now, suppose the economy is hit by a demand shock ε_r at period t which brings the interest rate to the ZLB for at least 2 periods. Since the agents respect the ZLB in their expectations they know that today the interest rate will be at the ZLB - $\hat{\mathbb{E}} i_t = i^*$. Hence, from (8) the error correction term for the interest rate's law of motion is zero and no updating occurs - $\phi_{3,t} = \phi_{3,t-1}$. On the other hand, their

12. Note that due to the constant gain learning the PLM Φ_t converges to the rational expectations equilibrium transition matrix Γ in distribution rather than point-wise. See Evans and Honkapohja (2001) for a proof. For the sake of argument here I consider the Great moderation example as having $\Phi_{t-s} = \Gamma$.

perceived LOMs for output gap and inflation ($\phi_{1,t-1}$ and $\phi_{2,t-1}$) are the first and second rows of the transition matrix for a world with no binding ZLB ($\Phi_{t-1} = \Gamma$). A model prescribed by Γ is characterised by an active monetary policy rule which accommodates demand shocks. This, however, is no longer true with a binding ZLB which locks the interest rate at an inefficiently high level i^*. Therefore the agents' forecasts for time t will be based on $t-1$ beliefs of the Great Moderation and will be too optimistic. At the end of period t they will observe the realisations and update their expectations as in (8). Overall, during the expected period of the ZLB the agents will not update their perceived law of motion for the shadow rate but will update their beliefs for the laws of motion of output gap and inflation.

I assume the central bank knows the ALM of the model (9) and (10) and observes agents' expectations $\hat{\mathbb{E}}Y_t$ for all endogenous variables[13]. Upon observing agents' expectations the CB plugs them into the ALM equations (9) and (10) and obtains model-consistent forecasts. Given its projections for output gap and inflation it uses the MP rule (5) and forms projections for the shadow rate. Because the CB's shadow rate forecasts are based on constantly updated $\hat{\mathbb{E}}_t$ expectations through the ALM, it is better able to anticipate the trajectory of the interest rate than the agents, who due to their fulfilled expectations of a binding ZLB in the immediate future fail to adjust the law of motion for the interest rate ($\phi_{3,t+s} = \phi_{3,t-1}$ if $\hat{\mathbb{E}}_{t+s-1} i_{t+s} = i^*$). The binding ZLB means a given shock hits harder output gap and inflation, necessitating a lower shadow rate. Due to the informational curtain of the ZLB, the agents do not update their shadow rate forecasts and hence expect an earlier lift-off than the model-consistent forecasts of the CB.

The agents gradually update their output gap and inflation expectations, but the binding ZLB prevents them from understanding how the new regime changes the dynamics of the MP rule even in the absence of an explicit policy change. The only source of change in the system is the ZLB which affects the propagation of the state variables Z_t to the endogenous variables Y_t.

Proposition 1. *Suppose the economy is brought to the zero lower bound after a*

13. Considering the vast amounts of information collected and processed by central banks as well as their sophisticated forecast models this does seem like a realistic assumption.

period of convergence to a rational expectations model with no binding ZLB. Then, the mechanics described above result in a disagreement between the agents and the central bank about the future path of the interest rate even in the absence of any policy change. Namely, the agents expect an earlier lift-off from the ZLB than the central bank.

Proof: *see Appendix B.1*

The assumption that the central bank knows the true model of the economy implies that the agents are also too optimistic relative to the actual realisations. As discussed in section 2, this is supported by the empirical evidence for policy rate forecasts in the US and Sweden, as well as, in financial markets as shown by Cieslak (2018).

3.4 Forward guidance

Henceforth, I assume that in order to correct the disagreement about the future path of interest rates the CB uses forward guidance by truthfully revealing its expected lift-off date during every period of a binding ZLB. Next I describe how forecasts are made and outline the different forward guidance experiments. Section 4 presents simulations for each experiment and discusses their implications and effectiveness.

3.4.1 Forecasting

Every period the agents form long-run expectations $\hat{\mathbb{E}}_t \left\{x_j, \pi_j, i_j, s_j\right\}_{j=t}^{\infty}$ as outlined in section 3.2. This allows them to estimate the last period of binding ZLB defined as:

$$T^{ag} \text{ such that } \begin{cases} \hat{\mathbb{E}}_t i_{T^{ag}} = i^* \\ \hat{\mathbb{E}}_t i_{T^{ag}+1} > i^* \end{cases} \tag{11}$$

The central bank is assumed to have rational model-consistent expectations, but no choice variable and to truthfully reveal its expectations, thus abstracting from strategic behaviour. It observes agents' expectations ($\hat{\mathbb{E}}_t Y_{t+j},\ j > 0$ and T^{ag}) and uses them to form expectations according to the structural equations of the model

(9)-(10). Then it sets its instrument i_t according to the policy rule (5) and in a similar fashion to (11) obtains its expectation of the last period of binding ZLB - T^{cb}. As per Proposition 1, we would have $T^{cb} > T^{ag}$, because agents' expectations adjust to reflect the new regime[14] brought by the ZLB only gradually through observations. This disagreement about the path of the interest rate is the rationale for FG.

3.4.2 Experiments

At period $t = 1$ the economy is in its RE equilibrium above the ZLB. Then a large persistent natural rate shock (ε_2^r - for calibration see Table B.1), pushes it to the ZLB. Both the agents and the CB anticipate a lift-off date according to the described procedures above. Whenever forecasts disagree, there is scope for forward guidance. Three cases of such CB communication are considered. In all cases where communication occurs, the CB is assumed to release its beliefs truthfully, abstracting from strategic behaviour.

1. **Baseline no FG** - the agents expect a lift-off at T^{ag} and are surprised by the continuing ZLB. They gradually update their beliefs by comparing s_t and i_t.

2. **FG as the length of the ZLB spell** - the CB releases T^{cb} and if different from T^{ag}, the agents adopt it outright in their expectations. This is reflected in the aggregate demand equation (9). Note that in this case the law of motion for the interest rate is not updated, so even at lift-off date (T^{cb}) there might be some disagreement between the CB and the agents.

3. **FG interpreted by adjusting** $\hat{\mathbb{E}}_t s_t = \boldsymbol{\phi_{3,t-1}} \begin{bmatrix} u_t \\ r_t \\ s_{t-1} \end{bmatrix}$

$$\phi'_{3,t-1} = \phi_{3,t-1} + \lambda R_{t-1}^{-1} S_{T^{cb}} \left(i^* - \hat{\mathbb{E}}_t i_{T^{cb}} \right) \tag{12}$$

- Equation (12) shows that now when the CB announces T^{cb} the agents try to adjust their perceived LOM for the shadow rate such that as of today

14. as manifested through the transition matrix Φ_t.

their expectations for date T^{cb} are for $\hat{\mathbb{E}}_t i_t = i^*$.

- here $\lambda = \tau$ gives weight to FG announcements as 1 quarter worth of data. Variation in λ can proxy how credible or well understood FG is.

4 Experiments

This section presents the conducted experiments and results. Throughout, I use the parameter and shock values from Table B.1 in appendix B.2 and the estimated monetary policy rule from Table A.2b in appendix A.2 . Impulse responses are calculated as point-wise median from 5000 simulations of random iid ε_r and ε_u shocks. This is done in order to provide enough variability for the learners to update their perceived transition matrix. The zero lower bound is respected throughout and the only commonality between simulations is the negative natural rate shock at period $t = 2$.

4.1 No forward guidance

Figure 5 below shows the impulse response functions (IRFs) of x_t, π_t and i_t (both expected and realised) to a large negative natural rate shock in period 2, which results in a prolonged period of binding ZLB. The bold black line shows the actual end-of-period realisations of the endogenous variables, while the dashed red line is the beginning-of-period expectations of the agents. Both expectations and realisations are, as expected - below the schedules which would have occurred was there not ZLB constraint. Moreover, as explained in previous sections, agents' expectations of future output gap and inflation only change with observations even if they understand what the ZLB means for the path of the interest rate. Thus, initially they expect a faster recovery, yet since the ZLB changes the economy's response to shocks, the actual output gap and inflation turn out to be lower. The constant gain learning results in a quick updating of beliefs and convergence of the dotted and solid lines.

Note that without FG the agents' projected shadow rate will be identical to the hypothetical one if no lower bound constraint existed (green thin solid line).

Figure 5: no FG - IRFs

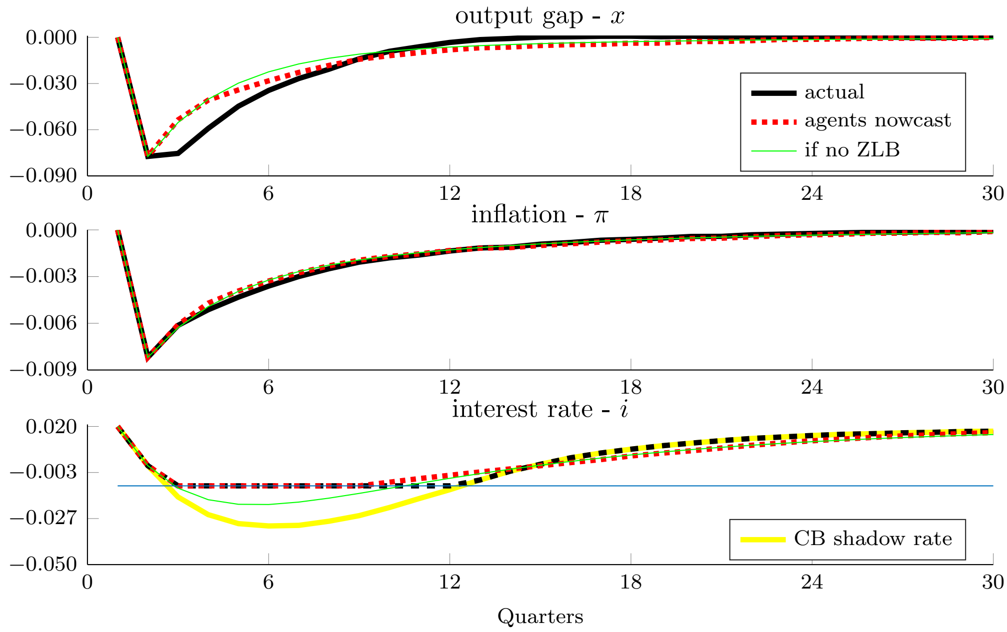


Figure 6 below zooms in on the end of the ZLB spell to highlight the disagreement about the lift off date between the agents and the CB. Even in this parsimonious model disagreement does occur and it is around 150 basis points at period 9 when the agents expect lift-off next period. A richer model featuring more persistence (e.g. habit formation or price indexation) as used by central banks today is likely to produce even larger disagreements. Finally, Figure 7 plots the expected duration of the ZLB of both agents and the CB when asked at every period. Disagreement persists with agents consistently expecting a 3-4 quarters shorter ZLB duration than the more informed CB.

4.2 "Period" forward guidance

Now suppose whenever disagreement occurs at the beginning of a period (as in Figure 7), the CB announces T^{cb} and the agents outright adopt it without changing their perception of the law of motion for the interest rate. Naturally, now the expected durations of the ZLB coincide throughout (Figure B.1 in appendix B.3). This situation is akin to the framework of forward guidance as anticipated shocks by Del Negro et al. (2012). Agents understand the length of the ZLB spell will be different but do not update their perceived LOM of the interest rate. Notably the

Figure 6: no FG - interest rate paths

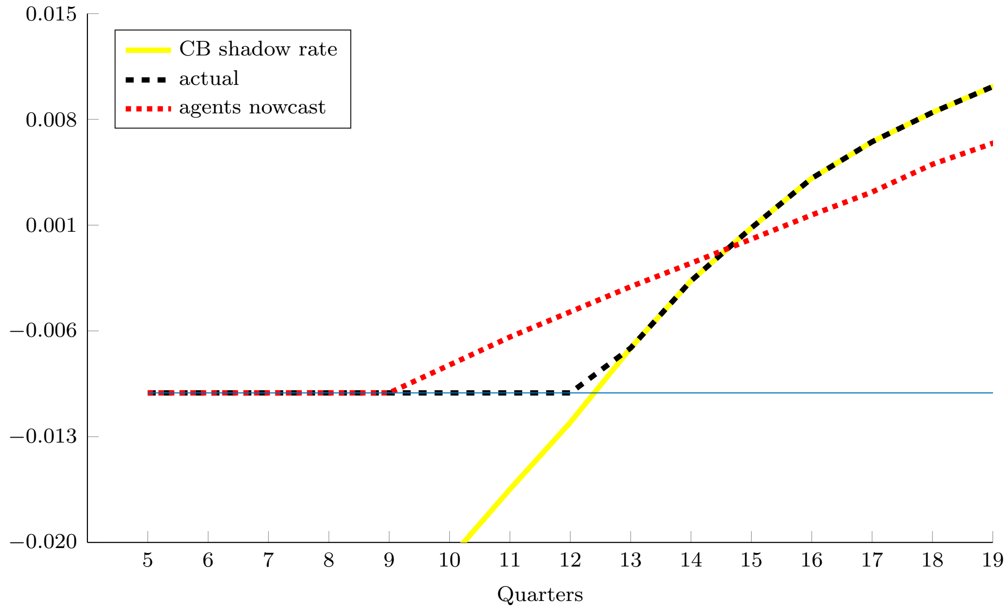
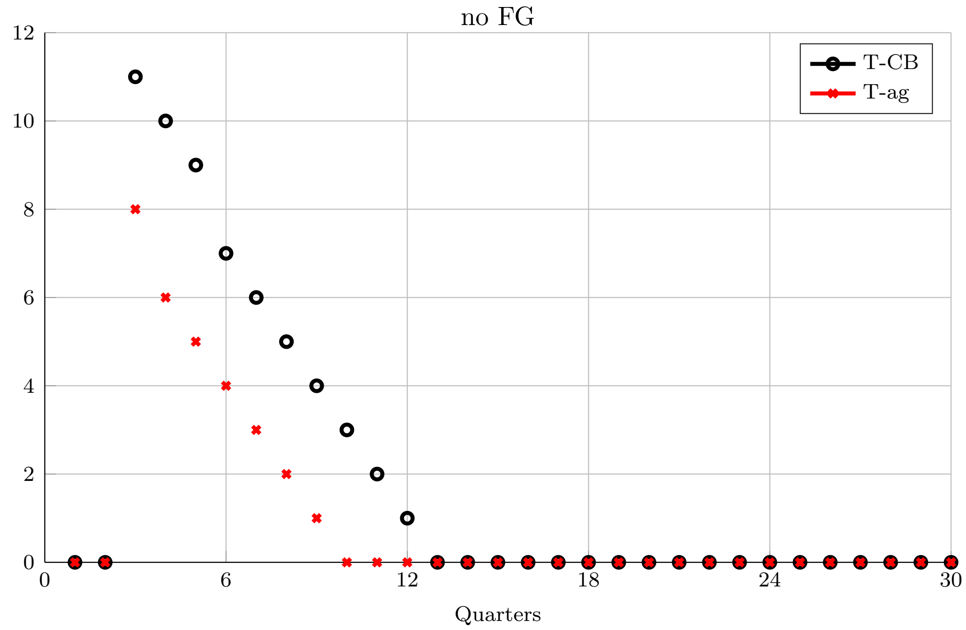


Figure 7: no FG - Anticipated duration of ZLB



agents' perceived LOM during the ZLB is misspecified but since no updating has occurred, it is in fact very close to the correct one upon exit from the ZLB[15].

4.3 Update from forward guidance

Such smooth transfer of information as above is not very likely in practice. Rather, suppose that the CB again announces T^{cb} but instead of directly adopting it, the agents use their usual learning procedure aiming to adjust their expectations for the interest rate at time T^{cb} ($\hat{\mathbb{E}}_t i_{T^{cb}}$) to equal i^*. Note that this communication scheme resembles the conditional FG that CBs have implemented in practice.

There are two differences with this learning step compared to their usual updating. First, it regards further than 1 period ahead forecasts. Second, the learning gain (λ) here can be varied to emulate the credibility of the message released by the CB. Here it is assumed that $\lambda = \tau = 0.02$, or the agents view CB's FG announcements as just another data point. See Marinkov (2020) for comparisons of the effects of different λ's. The learning for FG announcements (12) is restated below.

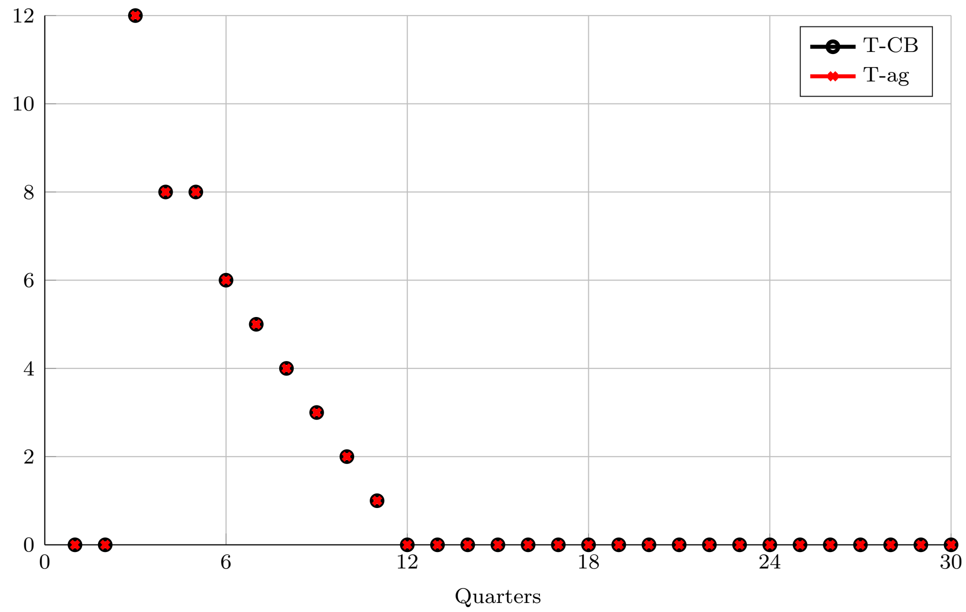
$$\phi_3' = \phi_{3,t-1} + \lambda R_{t-1}^{-1} S_{T^{cb}} \left(i^* - \hat{\mathbb{E}}_t i_{T^{cb}} \right)$$

A benefit of the "learning FG" scenario is that it could be beneficial in cases of earlier or delayed lift-offs than announced due to future shocks. Agents could better anticipate those if they have updated their perceived LOM for the interest rate. A potential downside compared to the "period" FG above is that this communication causes an expectational change in the perceived law of motion of the agents which might threaten the stability of the system.

Figure 8 shows the corresponding anticipated ZLB durations. Given that the agents solve a linear problem in order to match the announced lift-off date (12), it is no surprise that their perceived duration of the ZLB coincides with the CB's announcement. Notice that in period 4 the common perceived duration drops below the value of the no communication case in Figure 7. This happens because of the feedback of the updated long-run agents' expectations from (12) into the ALM (9)

15. Note that the agent's perceived pre-crisis LOM differs to the one immediately after lift-off due to the smoothing in the policy function, and hence the path-dependency of the ZLB period.

Figure 8: "Learning FG" - Anticipated duration of ZLB



and (10).

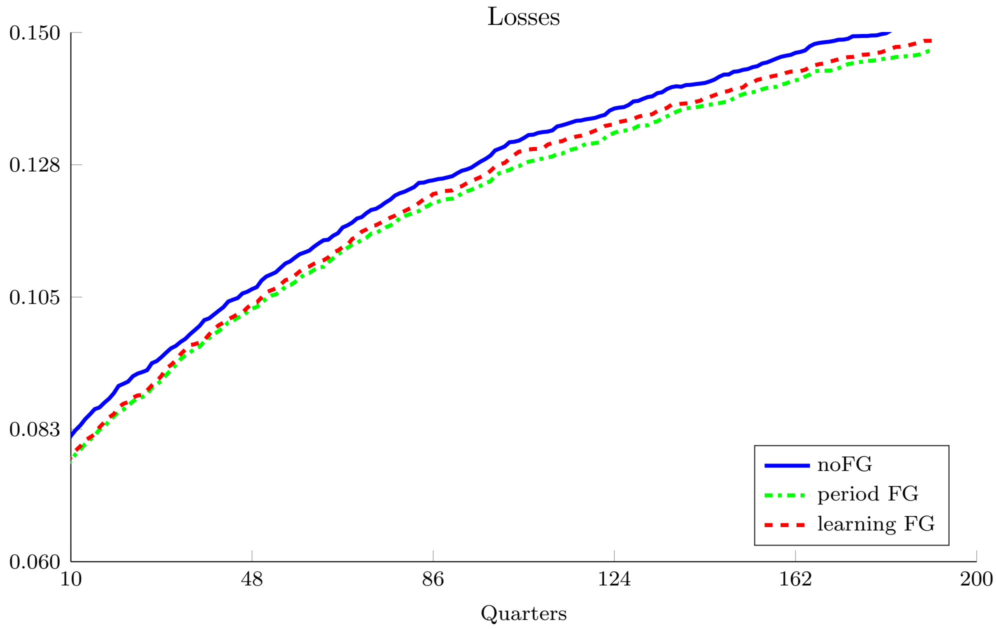
4.4 Welfare comparisons

Figure 9 plots the cumulative welfare loss associated with the cases for forward guidance described above. It is computed through a standard central bank welfare loss function: $L_t = L_{t-1} + \beta^{t-1}\left(\pi_t^2 + 0.5x_t^2\right)$. Naturally, period forward guidance has the best welfare outcome since it results in full agreement and in contrast to the learning forward guidance it does not create any expectational drift from the announcements. Thus, after lift-off agents still hold their pre-crisis beliefs about the law of motion of the interest rate, which are in fact the correct ones for the case of above the ZLB. Although this is welfare improving, the gains are marginal and no forward guidance puzzle is present.

4.5 Beliefs' drift and Stability

This section discusses the underlying updating of beliefs in the three experiments. Figure 10 shows the drifts in the elements of the transition matrix Φ_t mapping states into expectations of endogenous variables. Although in the long-run these converge

Figure 9: Welfare losses

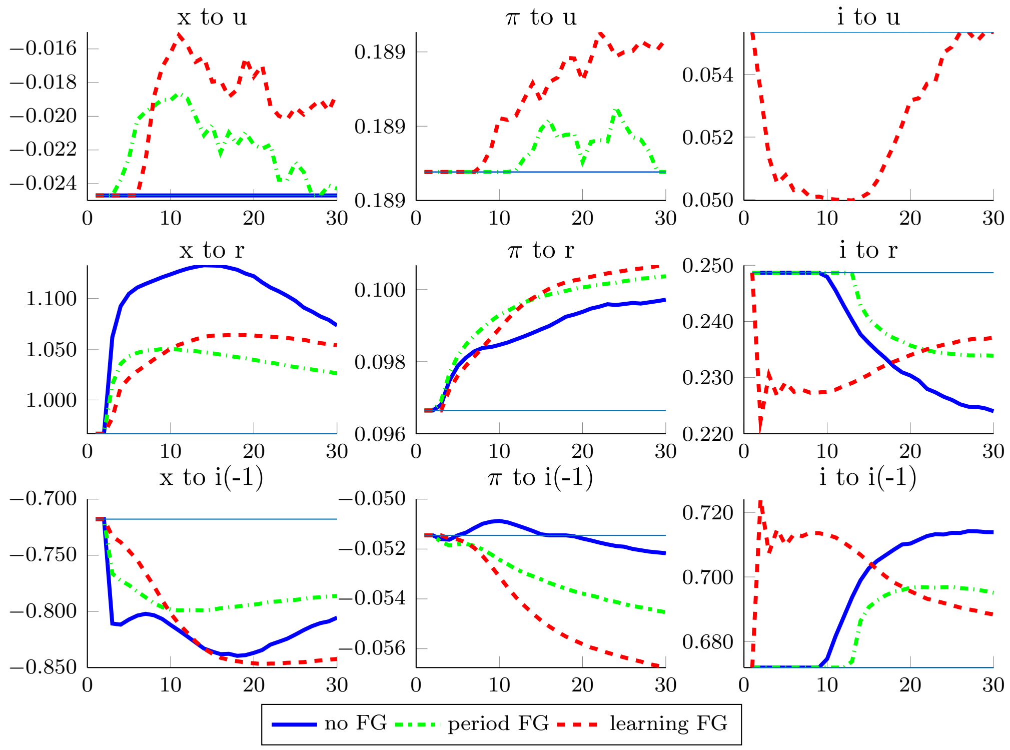


back to their equilibrium values under RE[16], they exhibit a prolonged drift away that lasts much longer after the ZLB is no longer binding. Regarding the welfare of the economy in the presence of future volatility this may be important in richer models or if shocks had larger variances such that future binding ZLB periods were more likely. Note that it is also consistent with the findings in section 2 where private expectations in Sweden remained higher than the CB's even in the end of the second ZLB spell.

Importantly, the drift is especially dangerous in the case of no communication. As already established, the agents expect an earlier lift-off than the CB. After their last anticipated period of ZLB - T^{ag}, they expect an interest rate above the ZLB but observe it still remains at the ZLB. This causes them to increase their perceived persistence of the policy rate through their learning algorithm (8). All figures above are median outcomes from Monte Carlo simulations. Nonetheless, during these simulations I find that over 10,000 draws and 500 periods over half of the draws end up in instability due to perceived unit root in the law of motion of the interest rate. Figure 11 plots the impulse responses of an identical economy as in the no

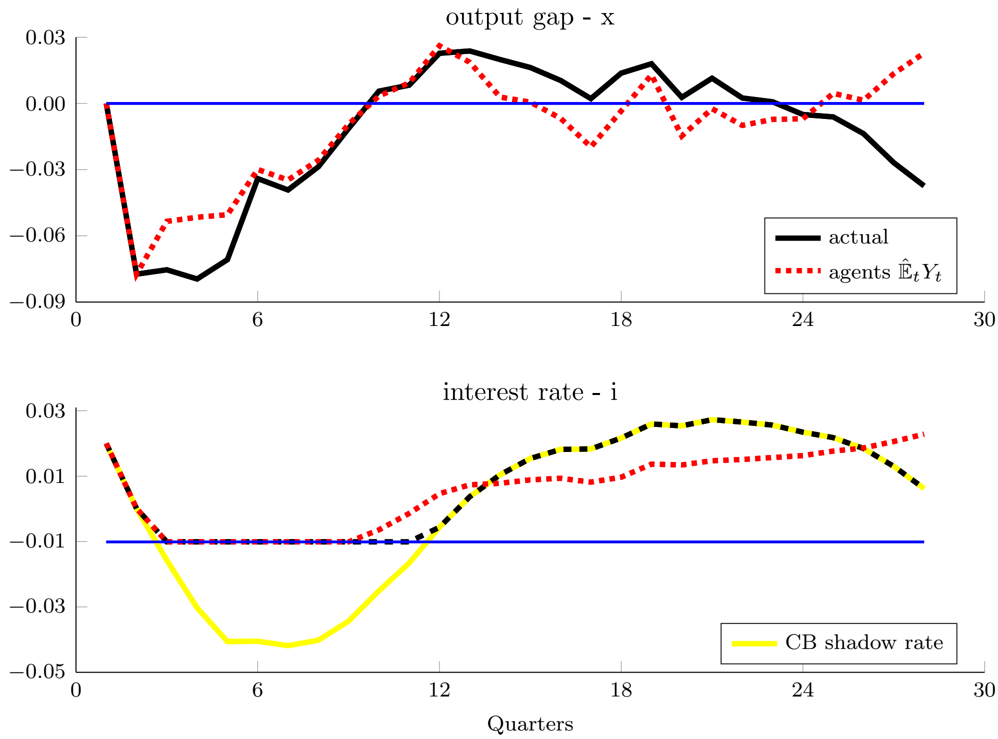
16. Due to constant gain learning instead of decreasing gain learning they converge to a distribution centered around their RE values (Evans and Honkapohja 2001)

Figure 10: Drift in perceived transition matrix Φ



communication case but it allows for moderate future shocks after the initial period. The familiar disagreement about the lift-off date and the severity of the crisis are still present. This time, however a sequence of very small negative demand shocks in period 12 push the economy beyond the bounds of stability. As established above, after their expected lift-off date (period 9) the agents observe a still binding ZLB which causes them to increase their perceived persistence of the interest rate. Iterated in their medium-run expectations in the ALM (9), this creates a boom in the economy around period 12. The central bank increases the interest rate in order to tame the boom, but this creases more disagreement in the interest rate forecasts with the agents. Given the small negative shocks at period 12 and the increasing policy rate, the agents again are lead to belief that the interest rate depends more on its past value rather than shocks. This again affects the medium to long-run expectations of the agents who now (around period 18) expect very high interest rates in the future, thus causing the economy to experience a recession. The CB, following its Taylor rule, quickly lowers interest rates, thus creating yet another big disagreement between with the agents. This causes even higher perceived persistence

Figure 11: no FG - single simulation IRFs



of the interest rate until around period 30 it surpasses 1 (unit root) and renders the economy explosive. The trajectory of the policy rate disagreement and the continual drift towards a unit root of the perceived persistence of the interest rate are depicted in Figure 12.

The learning literature has long established that the stability of the economy is greatly improved by a CB which reacts not to actual data as assumed here, but to the expectations of the agents (see Evans and Honkapohja (2003)). This is a remarkable analytical result in environments with no regime changes such as a zero lower bound. Here, I numerically make the case that FG can greatly improve the stability of a system with occasionally binding ZLB even when the CB reacts to contemporaneous data. The reason for this is that the communication provided by the CB circumvents the problem of the unobservable shadow rate to the agents who adjust their expectations. This helps minimize the initial expectational drift caused by the ZLB period and keeps the economy tighter within the basin of convergence, which greatly improves its stability.

Forward guidance in both of its iterations considered above has a stabilizing effect on the economy by keeping expectational drift at bay, thus preserving stability. Figure 12 shows on the first row the disagreement between the agents and the CB for the interest rate nowcast and on the second the AR(1) persistence in the perceived law of motion of the interest rate in the same Monte Carlo draw as in Figure 11 [17]. Although the two FG schemes exhibit some disagreement after the lift-off date, it is very contained and does not cause big drifts in the perceived persistence of the interest rate. The case of no communication, however, shows that disagreement keeps growing even after the lift-off and this is fuelled by an upward drifting perceived interest rate persistence. Once the perceived interest rate reaches 1, the system becomes explosive due to the long-horizon expectations in (9).

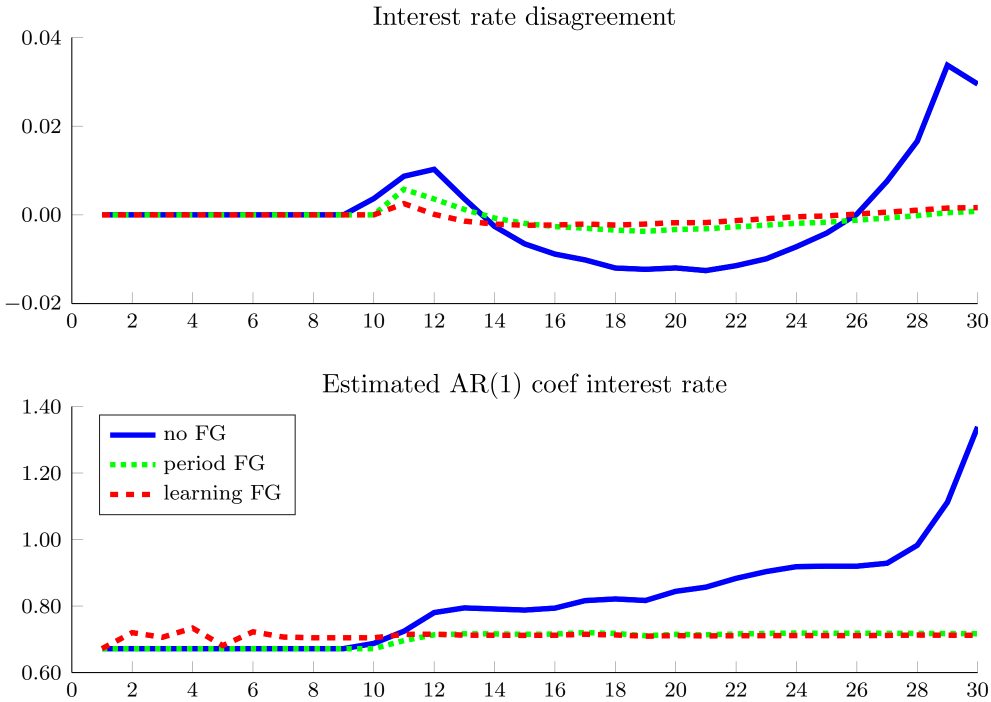
This analysis shows that forward guidance can be used at the ZLB to restore stability to the system. This is so because if no communication is issued, the learners will wrongly think the prolonged ZLB reflects higher persistence in the interest rate. In the presence of a "perfect storm" of shocks, their updating quickly leads them to believe there is a unit root in the interest rate's law of motion since it does not react to shocks (the shadow rate does, but it is unobserved). When this happens, the economy becomes unstable. Forward guidance prevents this spurious drift in expectations and preserves the stability of the economy. An upcoming extension of the paper will focus on numerically studying the limits of the basin of attraction by mapping the magnitudes of the "perfect storm" shocks which result in unstable economies in the MC draws.

5 Conclusion

This paper shows that the zero lower bound calls for a necessary increase in transparency and communication by the central bank because the ZLB non-linearity distorts private agents' expectations of the trajectory of the policy rate. The private agents' and central bank's expectations diverge because the bank is better able to understand the effects of the new ZLB regime on the aggregate law of motion

17. Figure B.2 in the Appendix shows how an economy with the same sequence of shocks as in Figure 11 but with period FG preserves its stability has suffers less volatility.

Figure 12: Interest rate disagreement ($\hat{\mathbb{E}}_{ag} - \mathbb{E}_{cb}$) and perceived AR(1) persistence of interest rate



of the economy. In particular, a binding ZLB causes private agents to expect and earlier lift-off than the CB does. The median welfare effects of forward guidance are not negligible, but neither are they huge, so no forward guidance puzzle is present. Importantly, however, forward guidance can be used as a stabilizing tool to ensure stability at the ZLB by preventing spurious expectational drift. In the baseline model communication is achieved through forward guidance, yet in reality a combination of FG and asset purchases might be needed to achieve the necessary shift in expectations.

Avenues for future work include allowing for central bank learning and considering optimal policy. A different expectation formation in the form of rational inattention also has the potential to explain disagreement and the effectiveness of forward guidance [18].

18. Note that this is similar to varying the weight on CB announcements λ studied in Marinkov (2020)

Appendices

A Data

A.1 Robustness policy rate forecasts disagreement - Swedish Riksbank

Table A.1 performs robustness checks on disagreement between the Riksbank and private agents. As in Table 1 in Low regimes agents expect on average higher interest rates than the CB (disagreement is negative and significant). In High states there is no significant disagreement between the CB and the agents at 3-months and 1-year forecast horizons. Interestingly, there is some evidence that at 2-year forecast horizons the Riksbank expect higher interest rates than the agents (disagreement is positive and significant). This might be due to better long-run forecasting abilities of the central bank or it might reflect a private agents' perception of more past-dependent policy compared to what the CB claims.

Table A.1: Disagreement on Swedish repo rate

Based on private agents' expected 1-year-ahead repo rate				
	$\mathbb{E}_t i_{t+1y} \lesseqgtr 0.25$		$\mathbb{E}_t i_{t+1y} \lesseqgtr 0.75$	
	mean	se(mean)	mean	se(mean)
dis_3m_low	-.011	.0051	-.0426	.018
dis_1y_low	-.1135	.0085	-.1502	.0213
dis_2y_low	-.3111	.0414	-.306	.038
dis_3m_high	-.0376	.0423	-.0128	.0458
dis_1y_high	.0107	.0755	.0616	.0804
dis_2y_high	.1539	.0807	.2157	.0826

Based on Riksbank's expected 1-year-ahead repo rate				
	$\mathbb{E}_t i_{t+1y} \lesseqgtr 0.25$		$\mathbb{E}_t i_{t+1y} \lesseqgtr 0.75$	
	mean	se(mean)	mean	se(mean)
dis_3m_low	-.0322	.0155	-.041	.0172
dis_1y_low	-.1391	.0192	-.1822	.0378
dis_2y_low	-.2947	.0384	-.3262	.0414
dis_3m_high	-.0128	.0458	-.013	.0482
dis_1y_high	.0616	.0804	.1042	.0717
dis_2y_high	.2157	.0826	.262	.072

A.2 Estimating the Taylor rule

Abstracting from the ZLB, the theory in (5) implies that the following relationship holds for the nominal interest rate:

$$i_t - \bar{r} = \rho_i(i_{t-1} - \bar{r}) + (1 - \rho_i)\left(\chi_\pi(\pi_t - \bar{\pi}) + \chi_x x_t\right)$$

or equivalently

$$i_t = \bar{r}(1 - \rho_i) + \rho_i i_{t-1} + (1 - \rho_i)\left(\chi_\pi(\pi_t - \bar{\pi}) + \chi_x x_t\right) \tag{13}$$

where $\bar{r}$ stands for the natural rate of interest and $\bar{\pi}$ - for the inflation target of the central bank, both of which are netted out in the theoretical model in (5) since all variables in the model are in deviations from steady state. Thus, I estimate the following empirical interest rate model from which I then back out the implied the Taylor parameters in (13) and, respectively, (5):

$$i_t = a_0 + a_i i_{t-1} + a_\pi \pi_t + a_x x_t + \varepsilon_i \tag{14}$$

The data used is from the Federal Reserve Economic Data and includes the official output gap, the Personal Consumption Expenditures (PCE) Excluding Food and Energy (chain-type price index) and the FED funds rate (FFR). The frequency is quarterly and the sample is chosen for the period January 1st 1987 - October 1st 2007 in order to coincide with the end of the Volcker administration at the FED and the onset of the financial crisis. The same sample period has also been chosen by Taylor (1993) and Kahn (2012).

Table A.2 shows the estimates for the regression in (14) as well as the implied Taylor coefficients in (13) and (5).

Note that the a monetary rule with persistence implies stronger reaction to deviations from target for inflation and output gap. This is because a higher persistence means that inflation or output gap shocks have a longer lasting effect on the interest rate and hence the CB would choose to react more strongly to such shocks as they happen.

Lastly, Figure A.1 plots the actual FED funds rate versus the implied monetary

Table A.2: Monetary policy rules

(a) empirical interest rate models

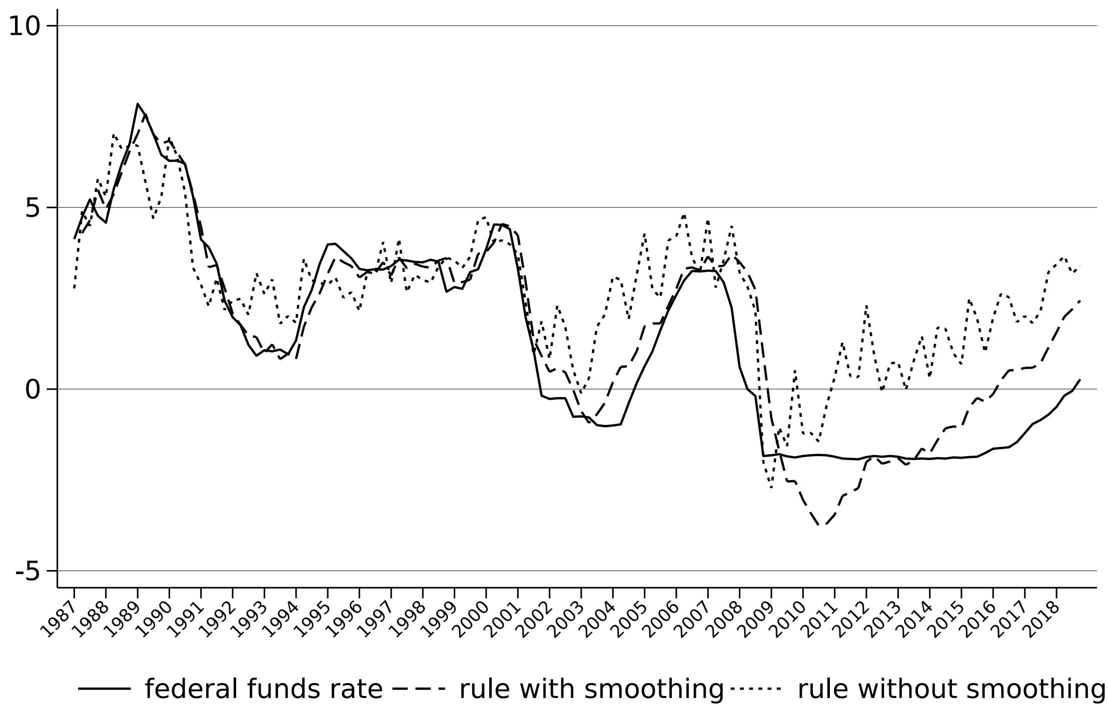
	no smoothing	smoothing
ffr_{-1}		0.79***
		(0.04)
inflation	1.41***	0.43***
	(0.13)	(0.08)
output gap	0.70***	0.27***
	(0.09)	(0.04)
Constant	3.33***	0.64***
	(0.13)	(0.15)
Observations	60	59
Adjusted R^2	0.655	0.941

(b) implied Taylor coefficients

ρ_i	0.791
χ_π	2.073
χ_x	1.302
$\bar{r}$	3.060

rules in Table A.2. Since the sample for the estimates in Table A.2 ends in October 1st 2007, all values this date show the out of sample fit of the two empirical models. Notably, both visually and evidenced by the higher R^2 in Table A.2a the model with smoothing fits the data much better not only during the sample period including the Great Moderation but also out of sample suggesting a long period of binding ZLB.

Figure A.1: Estimated US Taylor rules and the FFR



B Model

B.1 Proof of disagreement at the ZLB

Suppose the economy has always been above the ZLB until period t when a big negative shock is realised. Thus, at period $t-1$ the agents have converged to the REE, that is $\hat{\mathbb{E}}_{t-1}x_{t-1} = x_{t-1}$ and $\Phi_{t-2} = \Gamma$, so in normal times no adjustment is expectations should occur between period $t-1$ and t. In more detail:

$$\hat{\mathbb{E}}_{t-1}x_{t-1} \equiv \phi_{x,t-2}Z_{t-1} = x_{t-1} \tag{15}$$

$$= \hat{\mathbb{E}}_{t-1}\sum_{T=t-1}^{\infty}\beta^{T-t}\left[(1-\beta)x_{T+1} - \frac{1}{\sigma}\left(i_T - \pi_{T+1} - r_T\right)\right] \tag{16}$$

$$= \hat{\mathbb{E}}_{t-1}\sum_{T=t-1}^{\infty}\beta^{T-t}\left[(1-\beta)x_{T+1} - \frac{1}{\sigma}\left(s_T - \pi_{T+1} - r_T\right)\right] \tag{17}$$

where the last equation uses the pre-crisis expectations of the agents that no ZLB is coming - $\hat{\mathbb{E}}_{t-1}i_{t+j} = \hat{\mathbb{E}}_{t-1}s_{t+j},\ \forall j$.

It will be argued below that the realisation of a strong negative shock and the following anticipation of the ZLB by the agents cause a wedge between the period t versions of the equations (16) and (17) where the former will be lower. This means that the agents will underestimate the negative effect of the ZLB on output. In the presence of an anticipated binding ZLB it will be shown that

$$x_t = \hat{\mathbb{E}}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[(1-\beta)x_{T+1} - \frac{1}{\sigma}\left(i_T - \pi_{T+1} - r_T\right)\right] < \tag{18}$$

$$\hat{\mathbb{E}}_t\sum_{T=t}^{\infty}\beta^{T-t}\left[(1-\beta)x_{T+1} - \frac{1}{\sigma}\left(s_T - \pi_{T+1} - r_T\right)\right] = \hat{\mathbb{E}}_t x_t \tag{19}$$

The same argument applies to $\pi_t < \hat{\mathbb{E}}_t\pi_t$, yet as will be seen to a smaller degree due to the stronger interest rate channel for the output gap. These two results together will ensure that $i_t < \hat{\mathbb{E}}_t i_t$, thus showing the disagreement between CB and the agents who don't anticipate the structural effects of the ZLB on output gap and inflation but only its effects on the interest rate and hence have biased expectations towards a higher shadow rate and an earlier lift-off.

B.1.1 ALM at the ZLB - output gap

The learning mechanism ensures $PLM \to ALM$ above the ZLB, that is - the nowcast of endogenous variables equals their realisations:

$$\hat{\mathbb{E}}_t x_t = x_t = \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \frac{1}{\sigma} \left(i_T - \pi_{T+1} - r_T \right) \right] \tag{20}$$

Now suppose that between some periods $t+m$ and $t+n$, where $n \geq m \geq 0$, the ZLB is expected to hold, thus the anticipated duration of binding ZLB is $n-m+1$. The agents understand that the interest rate cannot be lower than the ZLB i^*, so they form expectations of the shadow rate and censor it at the ZLB to obtain their actual expected future interest rate between $t+m$ and $t+n$. Even though the agents correctly understand the direct effect of the ZLB on the interest rate, they do not anticipate the changes in the LOM for x and π stemming from the ZLB. That is, they understand the interest rate cannot go below the ZLB and will use such censored interest rate for forecasting x and π but they do not understand that the censoring itself causes the LOMs of x and π (i.e. the coefficients in Γ_t) to be different as well. The CB keeps its reaction function unchanged but because of these changed LOM the LOM in terms of state variables of the policy rate itself changes. Without communication at the ZLB the agents have no way of observing this change and amending their forecasts. This creates a forecast disagreement between the CB and the agents with the former expecting a longer ZLB spell. In what follows we prove the existence and direction of this disagreement and discuss its reliance on the anticipated ZLB spell duration.

In the Technical Appendix we have established that the ALM of x_t depends on u_t, r_t and $\mathtt{Y_b(3,t)} \equiv \hat{\mathbb{E}}_t \sum\limits_{T=t}^{\infty} \beta^{T-t} i_T$. I have assumed that the agents know the autoregressive coefficients of u_t and r_t and they can only update the shocks' impact on endogenous variables through observations. Therefore, it is sufficient to establish how $\mathtt{Y_b(3,t)}$ changes in the presence of the ZLB to understand how the ALM of x_t

changes too.

During the binding ZLB period[19] from $t+m$ to $t+n$ including the expectation of the discounted sum of interest rates is:

$$\sum_{T=t+m}^{t+n} \beta^{T-t} i^* = i^* \beta^m \sum_{T=0}^{n-m} \beta^T = i^* \beta^m \frac{1-\beta^{n-m+1}}{1-\beta} \tag{21}$$

By definition, if no binding ZLB is anticipated, this sum is not defined and may be treated as zero. Thus, we may write:

$$\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} i_T = \begin{cases} \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} s_T - \hat{\mathbb{E}}_t \sum_{T=t+m}^{t+n} \beta^{T-t} s_T + \sum_{T=t+m}^{t+n} \beta^{T-t} i^*, & \text{if } n \geq m \geq 0 \\ \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} s_T, & \text{otherwise} \end{cases} \tag{22}$$

where s_t is the shadow rate such that $\hat{\mathbb{E}}_t i_T = \max(i^*, \hat{\mathbb{E}}_t s_t)$. In words, the first case corrects the infinite sum of discounted expected interest rates by censoring the region where the ZLB is expected to bind. While in the second, when no ZLB is expected, the expected interest rate coincides with the shadow rate. Importantly, notice that increasing the censored region, either by longer duration of the ZLB $(n-m)$ or a deeper recession (shadow rate much lower than i^*), deviates the sum of discounted expected interest rates further from the sum of expected discounted shadow rates. This will prove a crucial element in understanding the source of disagreement between the agents and the CB.

Since we have already computed the infinite sum of discounted interest rates without a binding ZLB, we know the first term on the right, while the third was

19. Note that due to the linear forecasting rules agents might anticipate maximum 1 uninterrupted duration of binding ZLB. This is in accordance with typical real-word forecasting.

computed in (21). Thus, we only need to compute the corrective second term:

$$\begin{aligned}
\hat{\mathbb{E}}_t \sum_{T=t+m}^{t+n} \beta^{T-t} s_T &= \beta^m \hat{\mathbb{E}}_t i_{t+m} + \beta^{m+1} \hat{\mathbb{E}}_t i_{t+m+1} + \dots = \\
&= \beta^m \left[\hat{\mathbb{E}}_t i_{t+m} + \beta \hat{\mathbb{E}}_t i_{t+m+1} + \dots \right] = \\
&= \beta^m \left[\phi_{it-1} \hat{\mathbb{E}}_t Z_{t+m} + \phi_{it-1} \hat{\mathbb{E}}_t Z_{t+m+1} + \dots \right] = \\
&= \beta^m \phi_{it-1} \left[\Psi_{t-1}^m Z_t + \Psi_{t-1}^{m+1} Z_t + \dots + \Psi_{t-1}^n Z_t \right] = \qquad (23) \\
&= \beta^m \phi_{it-1} \left[V_{t-1} D_{t-1}^m V_{t-1}^{-1} + \dots + V_{t-1} D_{t-1}^n V_{t-1}^{-1} \right] Z - t = \qquad (24) \\
&= \beta^m \phi_{it-1} V_{t-1} D_{t-1}^m \left[I + D_{t-1} + \dots + D_{t-1}^{n-m} \right] V_{t-1}^{-1} Z_t \qquad (25)
\end{aligned}$$

where we define Ψ_{t-1} as the perceived law of motion of the state variables:

$$\hat{\mathbb{E}}_t Z_{t+1} = \Psi_{t-1} Z_t =$$

$$= \underbrace{\begin{bmatrix} \rho_u & 0 & 0 & 1 & 0 & 0 \\ 0 & \rho_r & 0 & 0 & 1 & 0 \\ \phi_{iu,t-1} & \phi_{ir,t-1} & \phi_{ii,t-1} & \phi_{i\varepsilon_u,t-1} & \phi_{i\varepsilon_r,t-1} & \phi_{ic,t-1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\Psi_{t-1}} \underbrace{\begin{bmatrix} u_{t-1} \\ r_{t-1} \\ i_{t-1} \\ \varepsilon_{u,t} \\ \varepsilon_{r,t} \\ c_{t-1} \end{bmatrix}}_{Z_t} \qquad (26)$$

The transformation between (23) and (24) is an eigendecomposition of the matrix $\Psi_{t-1} = V_{t-1} D_{t-1} V_{t-1}^{-1}$, where D is a diagonal matrix of Ψ's eigenvalues and V is a matrix featuring its eigenvectors as columns. It can be shown that:

$$D_{t-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{ii,t-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_r & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_u \end{bmatrix} \tag{27}$$

$$V_{t-1} = \begin{bmatrix} -\frac{1}{\rho_u} & 0 & 0 & 0 & 0 & \frac{\rho_u - \phi_{ii,t-1}}{\phi_{iu,t-1}} \\ 0 & -\frac{1}{\rho_r} & 0 & 0 & \frac{\rho_r - \phi_{ii,t-1}}{\phi_{ir,t-1}} & 0 \\ \frac{\phi_{iu,t-1}}{\phi_{ii,t-1}\rho_u} - \frac{\phi_{i\varepsilon_u,t-1}}{\phi_{ii,t-1}} & \frac{\phi_{ir,t-1}}{\phi_{ii,t-1}\rho_r} - \frac{\phi_{i\varepsilon_r,t-1}}{\phi_{ii,t-1}} & -\frac{\phi_{ic,t-1}}{\phi_{ii,t-1}} & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Note that in the case where $\phi_{ii,t-1} = \rho_u$ or $\phi_{ii,t-1} = \rho_r$ at least two of the eigenvectors become colinear and the decomposition is not possible. In what follows for the sake of brevity I will omit the subscript $t-1$ on the elements of the law of motion for i - ϕ_i.

Plugging in the equation for D_{t-1} into (25) and computing the series in the parentheses we obtain:

$$\hat{\mathbb{E}}_t \sum_{T=t+m}^{t+n} \beta^{T-t} s_T = \beta^m \phi_{it-1} V_{t-1} D_{t-1}^m \left[I + D_{t-1} + \cdots + D_{t-1}^{n-m}\right] V_{t-1}^{-1} Z_t =$$

$$= \beta^m \phi_{it-1} V_{t-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\phi_{ii,t-1}^m - \phi_{ii,t-1}^n}{1-\phi_{ii,t-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\rho_r^m - \rho_r^n}{1-\rho_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\rho_u^m - \rho_u^n}{1-\rho_u} \end{bmatrix} V_{t-1}^{-1} Z_t = \tag{29}$$

$$= \begin{bmatrix} \frac{\beta^m}{\phi_{ii}-\rho_u} \left[\frac{\phi_{ii}\phi_{iu}(\phi_{ii}^m - \phi_{ii}^n)}{1-\phi_{ii}} - \frac{(\phi_{ii}\phi_{iu} - \phi_{ii}\phi_{ir} + \phi_{ir}\rho_u)(\rho_u^m - \rho_u^n)}{1-\rho_u}\right] \\ \frac{\beta^m}{\phi_{ii}-\rho_r} \left[\frac{\phi_{ii}\phi_{ir}(\phi_{ii}^m - \phi_{ii}^n)}{1-\phi_{ii}} - \frac{(\phi_{ii}\phi_{ir} - \phi_{ii}\phi_{iu} + \phi_{iu}\rho_r)(\rho_r^m - \rho_r^n)}{1-\rho_r}\right] \\ \frac{\beta^m \phi_{ii}(\phi_{ii}^m - \phi_{ii}^n)}{1-\phi_{ii}} \\ \frac{\beta^m}{\phi_{ii}-\rho_u} \left[\frac{(\phi_{ii}^m - \phi_{ii}^n)(\phi_{iu} + \phi_{i\varepsilon_u}(\phi_{ii}-\rho_u))}{1-\phi_{ii}} - \frac{(\rho_u^m - \rho_u^n)(\phi_{ii}\phi_{iu} - \phi_{ir}(\phi_{ii}-\rho_u))}{\rho_u(1-\rho_u)}\right] \\ \frac{\beta^m}{\phi_{ii}-\rho_r} \left[\frac{(\phi_{ii}^m - \phi_{ii}^n)(\phi_{ir} + \phi_{i\varepsilon_r}(\phi_{ii}-\rho_r))}{1-\phi_{ii}} - \frac{(\rho_r^m - \rho_r^n)(\phi_{ii}\phi_{ir} - \phi_{iu}(\phi_{ii}-\rho_r))}{\rho_r(1-\rho_r)}\right] \\ \frac{\beta^m \phi_{ic}(\phi_{ii}^m - \phi_{ii}^n)}{1-\phi_{ii}} \end{bmatrix}' \begin{bmatrix} u_{t-1} \\ r_{t-1} \\ i_{t-1} \\ \varepsilon_{u,t} \\ \varepsilon_{r,t} \\ c_{t-1} \end{bmatrix} \tag{30}$$

Since the shadow rate reacts positively to all state variables, that is - $\phi_i > 0$, it means that $\hat{\mathbb{E}}_t \sum_{T=t+m}^{t+n} \beta^{T-t} s_T$ will also be a positive function of each state variable in Z_t. This can be verified by considering each term of the correction vector in (30). It can be shown that for either case $\phi_{ii} \lesseqgtr \rho_u$ or $\phi_{ii} \lesseqgtr \rho_r$ all terms are always positive. Therefore, the component containing the sum of discounted expected interest rates in the output gap ALM ($\mathtt{Y_b(3,t)}$) is corrected negatively for each state variable in the presence of a binding ZLB. Remember that:

$$x = (1-\beta)\mathtt{Y_b(1,t)} + \frac{1}{\sigma}\mathtt{Y_b(2,t)} - \frac{1}{\sigma}\mathtt{Y_b(3,t)} = \tag{31}$$

$$= (1-\beta)\left[\frac{\phi_{xu,t-1}}{1-\beta\rho_u}u_t + \frac{\phi_{xr,t-1}}{1-\beta\rho_r}r_t + \phi_{xi,t-1}\mathtt{Y_b(3,t)}\right] + \ldots$$
$$\frac{1}{\sigma}\left[\frac{\phi_{\pi u,t-1}}{1-\beta\rho_u}u_t + \frac{\phi_{\pi r,t-1}}{1-\beta\rho_r}r_t + \phi_{\pi i,t-1}\mathtt{Y_b(3,t)}\right] - \frac{1}{\sigma}\mathtt{Y_b(3,t)} \tag{32}$$

where $\phi_{x,i} < 0$ and $\phi_{\pi,i} < 0$, thus $\mathtt{Y_b(3,t)}$ negatively affects x. Because the ZLB correction term (30) negatively affects $\mathtt{Y_b(3,t)}$, it means that the presence of a binding ZLB results in unambiguously larger responses of x to all state variables in Z_t, confirm the intuition that once the policy rate cannot fully accommodate shocks, the latter end up hitting harder.

Notice also that the correction vector in (30) is increasing in the ZLB duration $m - n$ [20].

B.1.2 ALM at the ZLB - inflation

From Section **??** we know that

$$\begin{aligned}\pi_t =& \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa(x_T + u_T) + (1-\alpha)\beta\pi_{T+1}\right] \\ =& \kappa \mathtt{Y_{ab}(1,t)} + (1-\alpha)\beta \mathtt{Y_{ab}(2,t)} + \frac{\kappa}{(1-\alpha\beta\rho_u)} u_t \end{aligned} \tag{33}$$

where

$$\begin{aligned}\mathtt{Y_{ab}(1,t)} &\equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} x_T = \\ &= \phi_{xt-1} Z_t + \frac{\alpha\beta\phi_{xu,t-1}}{1-\alpha\beta\rho_u} u_t + \frac{\alpha\beta\phi_{xr,t-1}}{1-\alpha\beta\rho_r} r_t + \alpha\beta\phi_{xi,t-1} \mathtt{Y_{ab}(3,t)} \qquad (34) \\ \mathtt{Y_{ab}(2,t)} &\equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \pi_{T+1} = \\ &= \frac{\phi_{\pi u,t-1}}{1-\alpha\beta\rho_u} u_t + \frac{\phi_{\pi r,t-1}}{1-\alpha\beta\rho_r} r_t + \phi_{\pi i,t-1} \mathtt{Y_{ab}(3,:)} \qquad (35) \\ \mathtt{Y_{ab}(3,t)} &\equiv \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} i_T = \\ &= \frac{1}{(1-\alpha\beta\phi_{ii,t-1})} \left(\frac{\alpha\beta\phi_{iu,t-1}}{1-\alpha\beta\rho_u} u_t + \frac{\alpha\beta\phi_{ir,t-1}}{1-\alpha\beta\rho_r} r_t + \phi_{it-1} Z_t \right) \qquad (36)\end{aligned}$$

The equivalent sum of discounted interest rates here is:

$$\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} i_T = \begin{cases} \hat{\mathbb{E}}_t \sum\limits_{T=t}^{\infty} (\alpha\beta)^{T-t} s_T - \hat{\mathbb{E}}_t \sum\limits_{T=t+m}^{t+n} (\alpha\beta)^{T-t} s_T + \sum\limits_{T=t+m}^{t+n} (\alpha\beta)^{T-t} i^*, & \text{if } n \geq m \geq 0 \\ \hat{\mathbb{E}}_t \sum\limits_{T=t}^{\infty} (\alpha\beta)^{T-t} s_T, & \text{otherwise} \end{cases} \tag{37}$$

The eigen decomposition of the ZLB correction term is analogous to (25) and

20. This again can be shown to hold for either case $\phi_{ii} \lesseqgtr \rho_u$ or $\phi_{ii} \lesseqgtr \rho_r$.

similarly the presence of a binding ZLB results in lower reaction of $\mathtt{Y_{ab}(3,t)}$ to the state variables vector Z_t. Since $\alpha, \beta, \kappa \in (0,1)$ and $\phi_{x,i} < 0$ and $\phi_{\pi,i} < 0$, then $\mathtt{Y_{ab}(3,t)}$ enters negatively in π and hence the ZLB correction means that π_t is more reactive to the state variable vector Z_t. Note, however, that due to $\kappa \& \alpha < 1$ the correction in π_t is much smaller than that in x_t meaning that the agents are making larger forecast errors on output gap forecasts than inflation forecasts.

B.1.3 ALM at the ZLB - policy rate

The CB observes the realisations of output gap and inflation and sets its shadow policy rate according to:

$$s_t = \rho_i s_{t-1} + (1-\rho_i)(\chi_\pi \pi_t + \chi_x x_t) \tag{38}$$

Naturally, $\chi_x, \chi_\pi > 0$, thus the ZLB corrections in x_t and π_t from the previous sections imply that CB shadow rate reacts more strongly to the state variable vector Z_t. In particular, if a negative demand or supply shock drives the interest rate to the ZLB, then the CB would act to adjust its shadow rate and skew its trajectory towards lower values, thus prolonging the a priori expected ZLB duration. In essence, the CB fully understands the indirect impacts of the ZLB on output gap and inflation and chooses lower policy rates when the ZLB is binding despite its policy coefficients χ_x and χ_π remaining unchanged. This creates disagreement between the CB and the agents who expect the policy rate to still follow its pre-ZLB trajectory. That is, the agents expect an earlier lift-off due to the non-linear correction to the ALM as a result of the ZLB.

B.2 Calibration

Table B.1: Calibration

parameter	value	source
α	0.75	sticky prices last for 3 quarters
β	0.99	implying 4.1 % annual rate of return
κ	0.024	Woodford(2003)
σ	3	implying IES of $\frac{1}{3}$
ρ_r	0.9	arbitrary
ρ_u	0.4	irrelevant
ρ_i	0.85	consistent with staff estimates
σ_r, σ_u	0.015	only for welfare loss calculations
ε_2^r	-0.07	a "Great Recession" shock
τ	0.02	standard in learning lit; robust to changes

B.3 Figures

Figure B.1: "Period FG" - Anticipated duration of ZLB

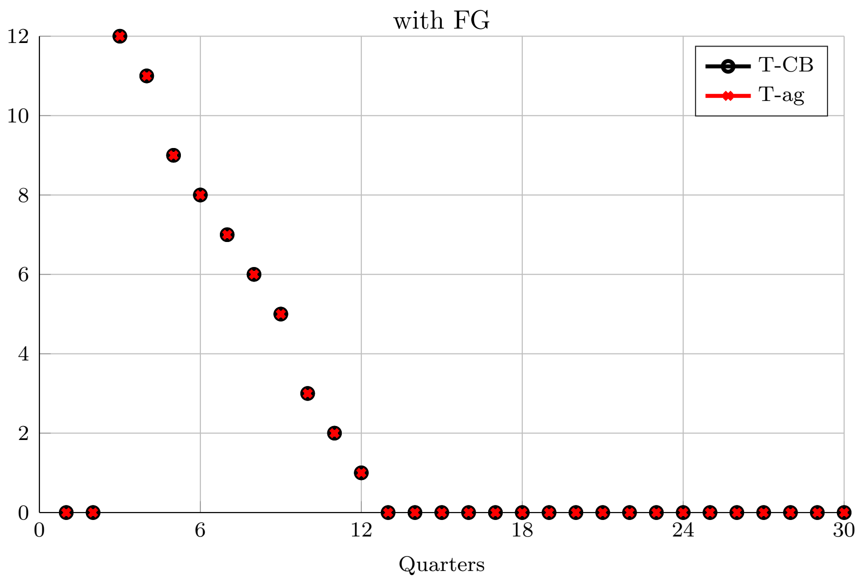
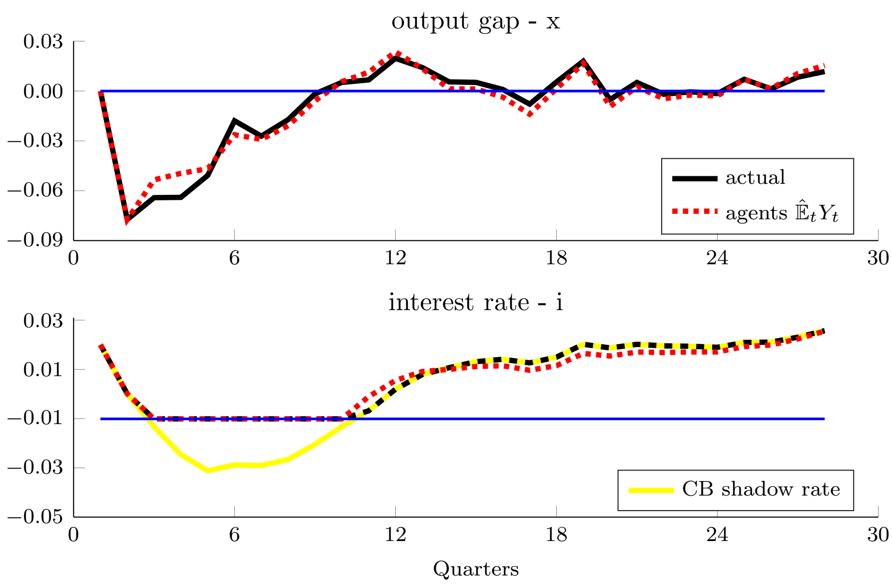


Figure B.2: period FG - single simulation IRFs



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Technical Appendix

Communication at the Zero Lower Bound:

The Case for Forward Guidance

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1 Calculations for the simulation algorithm

The Matlab program `2020_A0_TRAINING_noepsi_VAR3_smooth_LRlearning.m` calls the rational expectations solution to the model from the Dynare file `./nk_baseline_RE/NK_VAR3.mod`. It uses the implied decision rules and moments to initiate the learning economy's transition matrix `Phi` and variance-covariance matrix `R_T`. It then proceeds to simulate the economy with a sequence of shocks for $\{\varepsilon_u, \varepsilon_r\}$ and checks whether the learning algorithm converges to the RE values in the long run. There is some minor volatility in the transition matrix and VCV which is due to the approximation limits of Dynare and the difference between the simulated and theoretical variances of the shocks in Matlab.

Since this section explains the derivations for the computations in Matlab, I will often use Matlab notation for denoting variables.

1.1 Calculation of the Actual Law of Motion (ALM)

The ALM is determined by:

$$x_t = \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \frac{1}{\sigma} \left(i_T - \pi_{T+1} - r_T \right) \right] \tag{1}$$

$$\pi_t = \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa (x_T + u_T) + (1-\alpha)\beta \pi_{T+1} \right] \tag{2}$$

To compute the values of x and π at any given time we need expressions for the infinite sums of each component in (1) and (2) which can be computed in Matlab.

1.1.1 ALM - output gap

Taking first the output gap:

$$\begin{aligned}
x_t =&\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) x_{T+1} - \frac{1}{\sigma} \left(i_T - \pi_{T+1} - r_T \right) \right] \\
=&(1-\beta)\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} x_T - \frac{1}{\sigma}\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} i_T + \frac{1}{\sigma}\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \pi_{T+1} + \frac{1}{\sigma}\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} r_T \\
=&\underbrace{(1-\beta)\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} x_T}_{\texttt{Y}_\texttt{b}\texttt{(1,t)}} - \underbrace{\frac{1}{\sigma}\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} i_T}_{\texttt{Y}_\texttt{b}\texttt{(3,t)}} + \underbrace{\frac{1}{\sigma}\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \pi_{T+1}}_{\texttt{Y}_\texttt{b}\texttt{(2,t)}} + \frac{r_t}{\sigma(1-\beta\rho_r)}
\end{aligned} \tag{3}$$

Since the lagged interest rate is a state variable, the infinite expectational sums for x_T and π_T require infinite sums of forecasts for i_T. Thus, it is necessary that we first compute the sum for the interest rate - `Y_b(3,t)`, before obtaining the ones for `Y_b(1,t)` and `Y_b(2,t)`.

Abusing proper mathematical notation for the sake of brevity the following derivations shows how to obtain an expression for `Y_b(3,t)`=$f_3(u_t, r_t, Z_t)$

$$\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} i_T = \hat{\mathbb{E}}_t i_t + \beta \hat{\mathbb{E}}_t i_{t+1} + \beta^2 \hat{\mathbb{E}}_t i_{t+2} + \beta^3 \hat{\mathbb{E}}_t i_{t+3} + \beta^4 \hat{\mathbb{E}}_t i_{t+4} + \dots$$

$$= \phi_{it-1} Z_t + \beta \phi_{it-1} \begin{bmatrix} u_t \\ r_t \\ \phi_{it-1} Z_t \\ 0 \\ 0 \end{bmatrix} + \beta^2 \phi_{it-1} \begin{bmatrix} \rho_u u_t \\ \rho_r r_t \\ \phi_{it-1} Z_{t+1} = \phi_{it-1} \begin{bmatrix} u_t \\ r_t \\ \phi_{it-1} Z_t \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}$$

$$+ \beta^3 \phi_{it-1} \begin{bmatrix} \rho_u^2 u_t \\ \rho_r^2 r_t \\ \phi_{it-1} Z_{t+2} = \phi_{it-1} \begin{bmatrix} \rho_u u_t \\ \rho_r r_t \\ \phi_{it-1} Z_{t+1} = \phi_{it-1} \begin{bmatrix} u_t \\ r_t \\ \phi_{it-1} Z_t \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}$$

$$+\beta^4\phi_{it-1}\begin{bmatrix}\rho_u^3 u_t\\ \rho_r^3 r_t\\ \phi_{it-1}Z_{t+3}=\phi_{it-1}\begin{bmatrix}\rho_u^2 u_t\\ \rho_r^2 r_t\\ \phi_{it-1}Z_{t+2}=\phi_{it-1}\begin{bmatrix}\rho_u u_t\\ \rho_r r_t\\ \phi_{it-1}Z_{t+1}=\phi_{it-1}\begin{bmatrix}u_t\\ r_t\\ \phi_{it-1}Z_t\\ 0\\ 0\end{bmatrix}\\ 0\\ 0\end{bmatrix}\\ 0\\ 0\end{bmatrix}\\ 0\\ 0\end{bmatrix}$$

$$+\dots$$

$$=\underbrace{\phi_{it-1}Z_t}_{\hat{\mathbb{E}}_t i_t}+\underbrace{\beta\phi_{uit-1}u_t+\beta\phi_{rit-1}r_t+\beta\phi_{iit-1}\phi_{it-1}Z_t}_{\hat{\mathbb{E}}_t i_{t+1}}$$

$$+\underbrace{\beta^2\phi_{uit-1}\rho_u u_t+\beta^2\phi_{rit-1}\rho_r r_t+\beta^2\phi_{iit-1}\phi_{uit-1}u_t+\beta^2\phi_{iit-1}\phi_{rit-1}r_t+\beta^2\phi_{iit-1}^2\phi_{it-1}Z_t}_{\hat{\mathbb{E}}_t i_{t+2}}$$

$$+\beta^3\phi_{uit-1}\rho_u^2 u_t+\beta^3\phi_{rit-1}\rho_r^2 r_t+\beta^3\phi_{iit-1}\phi_{uit-1}\rho_u u_t$$

$$+\underbrace{\beta^3\phi_{iit-1}\phi_{rit-1}\rho_r r_t+\beta^3\phi_{iit-1}^2\phi_{uit-1}u_t+\beta^3\phi_{iit-1}^2\phi_{rit-1}r_t+\beta^3\phi_{iit-1}^3\phi_{it-1}Z_t}_{\hat{\mathbb{E}}_t i_{t+3}}$$

$$+\beta^4\phi_{uit-1}\rho_u^3 u_t+\beta^4\phi_{rit-1}\rho_r^3 r_t+\beta^4\phi_{iit-1}\phi_{uit-1}\rho_u^2 u_t+\beta^4\phi_{iit-1}\phi_{rit-1}\rho_r^2 r_t$$

$$+\underbrace{\beta^4\phi_{iit-1}^2\phi_{uit-1}\rho_u u_t+\beta^4\phi_{iit-1}^2\phi_{rit-1}\rho_r r_t+\beta^4\phi_{iit-1}^3\phi_{uit-1}u_t+\beta^4\phi_{iit-1}^3\phi_{rit-1}r_t+\beta^4\phi_{iit-1}^4\phi_{it-1}Z_t}_{\hat{\mathbb{E}}_t i_{t+4}}$$

$$+\dots$$

$$
\begin{aligned}
&= \frac{1}{1-\beta\phi_{iit-1}}\phi_{it-1}Z_t + \beta\phi_{uit-1}u_t \Bigg(\underbrace{1}_{\hat{\mathbb{E}}_t i_{t+1}} + \underbrace{\beta\rho_u + \beta\phi_{iit-1}}_{\hat{\mathbb{E}}_t i_{t+2}} + \underbrace{\beta^2\rho_u^2 + \beta^2\phi_{iit-1}\rho_u + \beta^2\phi_{iit-1}^2}_{\hat{\mathbb{E}}_t i_{t+3}} \\
&+ \underbrace{\beta^3\rho_u^3 + \beta^3\phi_{iit-1}\rho_u^2 + \beta^3\phi_{iit-1}^2\rho_u + \beta^3\phi_{iit-1}^3}_{\hat{\mathbb{E}}_t i_{t+4}} + \dots \Bigg) \\
&+ \beta\phi_{rit-1}r_t \Bigg(\underbrace{1}_{\hat{\mathbb{E}}_t i_{t+1}} + \underbrace{\beta\rho_r + \beta\phi_{iit-1}}_{\hat{\mathbb{E}}_t i_{t+2}} + \underbrace{\beta^2\rho_r^2 + \beta^2\phi_{iit-1}\rho_r + \beta^2\phi_{iit-1}^2}_{\hat{\mathbb{E}}_t i_{t+3}} \\
&+ \underbrace{\beta^3\rho_r^3 + \beta^3\phi_{iit-1}\rho_r^2 + \beta^3\phi_{iit-1}^2\rho_r + \beta^3\phi_{iit-1}^3}_{\hat{\mathbb{E}}_t i_{t+4}} + \dots \Bigg)
\end{aligned}
$$

Focusing on u_t, denote $\beta\rho_u = a$ and $\beta\phi_{iit-1} = b$. Then, the term relating to u_t above reads:

$$
\begin{aligned}
&\beta\phi_{uit-1}u_t \left(1 + a + b + a^2 + ab + b^2 + a^3 + a^2b + ab^2 + b^3 + \dots\right) \\
&= \beta\phi_{uit-1}u_t \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a^i b^j = \beta\phi_{uit-1}u_t \sum_{i=0}^{\infty} a^i \sum_{j=0}^{\infty} b^j = \frac{\beta\phi_{uit-1}u_t}{(1-a)(1-b)} \\
&= \frac{\beta\phi_{uit-1}u_t}{(1-\beta\rho_u)(1-\beta\phi_{iit-1})}
\end{aligned}
$$

Applying analogous logic for r_t and substituting above gives:

$$
\begin{aligned}
&\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} i_T = \frac{1}{1-\beta\phi_{iit-1}}\phi_{it-1}Z_t + \frac{\beta\phi_{uit-1}u_t}{(1-\beta\rho_u)(1-\beta\phi_{iit-1})} + \frac{\beta\phi_{rit-1}r_t}{(1-\beta\rho_r)(1-\beta\phi_{iit-1})} \\
&= \frac{1}{(1-\beta\phi_{iit-1})}\left(\frac{\beta\phi_{uit-1}}{1-\beta\rho_u}u_t + \frac{\beta\phi_{rit-1}}{1-\beta\rho_r}r_t + \phi_{it-1}Z_t\right) = \boxed{\texttt{Y}_{\texttt{b}}\texttt{(3,t)}}
\end{aligned}
$$

Next we compute the component for the sum of output gap expectations:

$$
\begin{aligned}
\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} x_{T+1} &= \hat{\mathbb{E}}_t x_{t+1} + \beta \hat{\mathbb{E}}_t x_{t+2} + \beta^2 \hat{\mathbb{E}}_t x_{t+3} + \beta^3 \hat{\mathbb{E}}_t x_{t+4} + \ldots \\
&= \phi_{xt-1} \underbrace{\begin{bmatrix} u_t \\ r_t \\ \hat{\mathbb{E}}_t i_t \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+1}} + \beta \phi_{xt-1} \underbrace{\begin{bmatrix} \rho_u u_t \\ \rho_r r_t \\ \hat{\mathbb{E}}_t i_{t+1} \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+2}} + \ldots \\
&= \frac{\phi_{uxt-1}}{1-\beta\rho_u} u_t + \frac{\phi_{rxt-1}}{1-\beta\rho_r} r_t + \phi_{ixt-1} \mathtt{Y_b(3,:)} = \boxed{\mathtt{Y_b(1,t)}}
\end{aligned}
$$

Analogously for π:

$$
\begin{aligned}
\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} \beta^{T-t} \pi_{T+1} &= \hat{\mathbb{E}}_t \pi_{t+1} + \beta \hat{\mathbb{E}}_t \pi_{t+2} + \beta^2 \hat{\mathbb{E}}_t \pi_{t+3} + \beta^3 \hat{\mathbb{E}}_t \pi_{t+4} + \ldots \\
&= \phi_{\pi t-1} \underbrace{\begin{bmatrix} u_t \\ r_t \\ \hat{\mathbb{E}}_t i_t \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+1}} + \beta \phi_{\pi t-1} \underbrace{\begin{bmatrix} \rho_u u_t \\ \rho_r r_t \\ \hat{\mathbb{E}}_t i_{t+1} \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+2}} + \ldots \\
&= \frac{\phi_{u\pi t-1}}{1-\beta\rho_u} u_t + \frac{\phi_{r\pi t-1}}{1-\beta\rho_r} r_t + \phi_{i\pi t-1} \mathtt{Y_b(3,:)} = \boxed{\mathtt{Y_b(2,t)}}
\end{aligned}
$$

Plugging everything back in the ALM for the output gap (3) gives an expression for the realisation of the output gap that can be directly computed in Matlab:

$$
x_t = \mathtt{Y(1,t)} = (1-\beta)\mathtt{Y_b(1,t)} - \frac{1}{\sigma}\mathtt{Y_b(3,t)} + \frac{1}{\sigma}\mathtt{Y_b(2,t)} + \frac{r_t}{\sigma(1-\beta\rho_r)} \tag{4}
$$

1.1.2 ALM - inflation

Next we repeat the procedure for inflation:

$$
\begin{aligned}
\pi_t =& \hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa(x_T + u_T) + (1-\alpha)\beta\pi_{T+1}\right] \\
=& \kappa\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} x_T + \kappa\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} u_T + (1-\alpha)\beta\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \pi_{T+1} \\
=& \kappa\underbrace{\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} x_T}_{\mathtt{Y_{ab}(1,t)}} + (1-\alpha)\beta\underbrace{\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \pi_{T+1}}_{\mathtt{Y_{ab}(2,t)}} + \frac{\kappa}{(1-\alpha\beta\rho_u)} u_t \qquad (5)
\end{aligned}
$$

$$
\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} x_T = \hat{\mathbb{E}}_t x_t + \alpha\beta\hat{\mathbb{E}}_t x_{t+1} + \alpha^2\beta^2\hat{\mathbb{E}}_t x_{t+2} + \dots
$$

$$
= \phi_{xt-1} Z_t + \alpha\beta\phi_{xt-1} \underbrace{\begin{bmatrix} u_t \\ r_t \\ \hat{\mathbb{E}}_t i_t = \phi_{it-1} Z_t \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+1}} + \alpha^2\beta^2\phi_{xt-1} \begin{bmatrix} \rho_u u_t \\ \rho_r r_t \\ \hat{\mathbb{E}}_t i_{t+1} = \phi_{it-1}\hat{\mathbb{E}} Z_{t+1} = \phi_{it-1} \begin{bmatrix} u_t \\ r_t \\ \phi_{it-1} Z_t \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}
$$

$$
+ \dots = \phi_{xt-1} Z_t + \frac{\alpha\beta\phi_{uxt-1}}{1-\alpha\beta\rho_u} u_t + \frac{\alpha\beta\phi_{rxt-1}}{1-\alpha\beta\rho_r} r_t + \alpha\beta\phi_{ixt-1} \underbrace{\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^T i_T}_{\mathtt{Y_{ab}(3,t)}} = \boxed{\mathtt{Y_{ab}(1,t)}}
$$

Again we see that the sum of interest rate expectations are necessary for computing the sums of output gap and inflation expectations. Analogously to the derivations in the previous section we obtain:

$$
\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} i_T = \ = \frac{1}{(1-\alpha\beta\phi_{iit-1})} \left(\frac{\alpha\beta\phi_{uit-1}}{1-\alpha\beta\rho_u} u_t + \frac{\alpha\beta\phi_{rit-1}}{1-\alpha\beta\rho_r} r_t + \phi_{it-1} Z_t \right) = \boxed{\mathtt{Y_{ab}(3,t)}}
$$

Next, we compute `Y_ab(2,t)`:

$$\begin{aligned}
&\hat{\mathbb{E}}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \pi_{T+1} = \hat{\mathbb{E}}_t \pi_{T+1} + \alpha\beta \hat{\mathbb{E}}_t \pi_{t+2} + \alpha^2\beta^2 \hat{\mathbb{E}}_t \pi_{t+3} + \dots \\
&= \phi_{\pi t-1} \underbrace{\begin{bmatrix} u_t \\ r_t \\ \hat{\mathbb{E}}_t i_t \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+1}} + \alpha\beta\phi_{\pi t-1} \underbrace{\begin{bmatrix} \rho_u u_t \\ \rho_r r_t \\ \hat{\mathbb{E}}_t i_{t+1} \\ 0 \\ 0 \end{bmatrix}}_{\hat{\mathbb{E}}_t Z_{t+2}} + \dots \\
&= \frac{\phi_{u\pi t-1}}{1-\alpha\beta\rho_u} u_t + \frac{\phi_{r\pi t-1}}{1-\alpha\beta\rho_r} r_t + \phi_{i\pi t-1} \mathtt{Y_{ab}(3,:)} = \boxed{\mathtt{Y_{ab}(2,t)}}
\end{aligned}$$

Finally, plugging these components in the ALM for inflation (5) we obtain a computable expression for the realisation of inflation:

$$\pi_t = \kappa \mathtt{Y_{ab}(1,t)} + (1-\alpha)\beta \mathtt{Y_{ab}(2,t)} + \frac{\kappa}{(1-\alpha\beta\rho_u)} u_t \tag{6}$$