Holonomy of Cartan Connections

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THESIS

Abstract

This thesis looks into the holonomy algebras of Tractor/Cartan connections for both projective and conformal structures. Using a splitting formula and a cone construction in the Einstein case, it classifies all reductive, non-irreducible holonomy groups for conformal structures (thus fully solving the question in the definite signature case). The thesis then analyses the geometric consequences of of holonomy reduction for the projective Tractor connection. A general, Ricci-flat, cone construction pertains in the projective case, and this thesis fully classifies the irreducibly acting holonomy algebras by analysing which holonomy families admit a torsion-free Ricci-flat affine connection, and constructing cones with these properties.

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Contents

0.1 Introduction 1

1 The Cartan Connection **8**

1.1 Cartan and Tractor Connections 8

1.1.1 The Flat Models 8

1.1.2 The Cartan Connection 9

1.1.3 The Tractor Connection 10

1.1.4 Parabolic subalgebras 11

1.2 Preferred connections 14

1.2.1 Normal Cartan connections 17

1.3 Projective and conformal Cartan connections 18

1.3.1 Bracket formulas 20

1.3.2 Curvature formulas 22

1.3.3 Tractor bundles 26

2 The Conformal case **29**

2.0.1 Flat model and associated algebras 29

2.0.2 Preferred connections 29

2.0.3 Tractor connection 31

2.1 Einstein spaces 33
2.1.1 Reduced holonomy 33
2.1.2 Metric cones 38
2.1.3 Ricci-flat spaces 42
2.2 Decomposition theorem 45
2.2.1 Preliminaries 45
2.2.2 Preserved subbundles 46
2.2.3 Split bundle 49
2.2.4 Definite signature 58

3 The Projective case **62**

3.0.1 Flat model and associated algebras 62
3.0.2 Preferred connections 62
3.0.3 Tractor connection 65
3.1 Reducible holonomy: Ricci-flatness 66
3.1.1 Examples 71
3.2 Symplectic holonomy: Contact spaces 75
3.3 Complex holonomy: CR-spaces 80
3.3.1 Complex holonomy 80
3.3.2 Quaternionic holonomy 84
3.4 Orthogonal holonomy: Einstein spaces 85
3.5 Cone construction 88
3.5.1 Complex projective structures 94

4 Ricci-flat Holonomy **102**

4.1 Formal curvature modules 102

4.1.1 Spencer cohomology 102

4.1.2 The formal curvature 103

4.1.3 Complex algebras 106

4.2 Holonomy and Ricci-flatness 109

4.2.1 Volume forms and the Ricci tensor 109

4.3 Symplectic sub-groups 110

4.4 Split spaces: General case 113

4.5 Minimal Segre algebras 120

4.6 The case of E_6 126

5 Realisation of holonomy groups **133**

5.0.1 The list so far 133

5.1 Orthogonal holonomy 135

5.1.1 Full orthogonal holonomy 135

5.1.2 $\mathfrak{su}$ holonomies 139

5.1.3 $\mathfrak{sp}$ holonomies 141

5.1.4 Exceptional holonomies 141

5.2 Full holonomy 142

5.3 Complex holonomy 145

5.4 Quaternionic holonomy 148

5.5 Symplectic holonomies 149

5.5.1 Real symplectic 149

5.5.2 Complex symplectic 156

5.6 Low-dimension cases 157

A Invariant Linear Operators 164

B Twistor representations 166

C Wedge products for Complex spaces 169

D Two-dimensional projective holonomy 173

E Ricci-Flat Holonomies 177

0.1 Introduction

What is a geometry? It has always been easier to give examples than to answer this question precisely. Riemannian geometries, Klein geometies, symplectic geometries, projective geometries – what features unite them?

This thesis is a very small part of the ongoing effort, at least as far back as Cartan and Weyl, to attempt to put all geometries under one unifying roof – and, just as rapidly, to cut up that roof into seperate results for specific geometric structures.

This thesis looks at conformal and projective geometries; both members of the class of parabolic geometries, a group that includes, amongst others, almost Grassmanian, almost quaternionic, and co-dimension one CR structures. The central concepts emerged from E. Cartan's work [Car1], [Car2], (refined with discussions and arguments with H. Weyl [Wey1]), whose technique of 'moving frames' would ultimately develop into the concepts of principal bundles and Cartan connections – invariants that cover a vaste amount of geometric structures and furthermore allow for explicit calculations.

Despite the work of T.Y. Thomas [Tho1], [Tho2] who developed key ideas for Tractor calculus in the nineteen twenties and thirties, and S. Sasaki in 1943 [Sas], [SaYa], the subject fell into abeyance until the work of N. Tanaka [Tan] in 1979. Meanwhile, however, independent progress was being made in developing tools that would prove invaluble at a later stage, by Schouten [ScDa] (after whom the crucial tensor P is named), Kuiper who was to build the conformal development map [Kui] building on earlier work of Brinkmann's [Bri2] [Bri1], and, more recently, Kerbrat [Ker], and Eisenhart [Eis] who proved that the Cotton-York tensor [Cot] was the only obstruction to conformal flatness in three dimensions. That Cotton-York tensor reappears in this thesis in the curvature of the Cartan connection.

The whole subject was rediscovered and further developed by T.N. Bailey, M.G. Eastwood and R. Gover in 1994 [BEG]. Since then, there have been a series of papers by A. Čap and R. Gover [CaGo3], [CaGo2], [Gov], [CaGo1], developing a lot of the techniques that will be used in the present thesis – though those papers looked mainly at conformal geometry. Papers [CSS1], [CSS2] and [CSS3], by A. Čap, J. Slovák and V. Souček, develop similar methods in a more general setting. A recent paper by F. Leitner [Lei] then provided a classification of reducibly acting conformal holonomies, similarly to the results in this thesis – though using different methods involving con-

formal Killing forms. In effect, while this thesis looks at the standard representation of the Tractor connection, Leitner's looks at the exterior products of this standard representation; similiar results are proved in very different ways.

Early papers had focused on seeing the Cartan connection as a property of a principal bundle $\mathcal{P}$. But in the more recent ones, the principal bundle is replaced by an associated vector bundle, the *Tractor bundle* $\mathcal{T}$, and the Cartan connection by an equivalent connection form for $\mathcal{T}$, the *Tractor connection* $\overrightarrow{\nabla}$. With these tools, calculations are considerably simplified.

The purpose of this thesis is to analyse one of the invariants of the Tractor connection, the holonomy algebra, for both the conformal and projective geometries. This thesis acheives a limited sucess in acomplishing this goal; it classifies all the algebras acting reducibly but not irreducibly in the conformal case. In this way, it provides a complete classification for definite signature. It also classifies all the irreducible projective Tractor holonomies – and only those, though it provides some insight into the reduced holonomy situation.

In both cases, the situations that are not classified arise from the same problem: the difficulty of controlling the geometry in transverse directions, in the abscence of a metric. The conditions upon these constructions involve the Ricci tensor, making them into second-order, non-linear differential equations that are very hard to solve. Some advanced analysis would undoutably allow further insights; however, that is beyond the scope of this thesis.

Chapter 1 introduces Cartan and Tractor connections in the general case, showing how the Cartan connection – not a connection in the standard sense – is effectively a 'curved' version of flat homogeneous spaces. By 'flat', we mean a space G/P for groups $P \subset G$; by curved, we mean a space that infinitesimally approximates this structure. The analogy is with the flat Euclidean space and the curved Riemannian metrics that approximate it.

We then show how the Tractor connection – this is a standard connection – is equivalent to the Cartan connection. We shall then relate these constructions to more standard geometric invariants – the classes of 'preferred connections' on the tangent bundle. Though demonstrated in general, the various formulas will be given explicitly in the conformal and the projective cases. A few Lie algebra properties – especially algebra cohomology – and curvature formulas will be needed to show how the Tractor connection is built up from the preferred connections.

The detailed analysis of the conformal Tractor construction occupies chapter 2. The conformal Tractor bundle $\mathcal{T}$ is of rank $n+2$, where n is the dimension of the manifold. The Tractor connection preserves a metric of signature $(p+1, q+1)$ on $\mathcal{T}$, where (p, q) is the signature of the conformal structure; consequently, we shall be looking at subalgebras of $\mathfrak{so}(p+1, q+1)$.

There, we shall build on the well known fact that a parallel section of the Tractor bundle corresponds to the local existence of an Einstein metric, and use a cone construction to relate the Tractor holonomy in this case to standard affine holonomy. Then some standard results in cone holonomy allow us to fully classify these entirely, as long as the holonomy doesn't reduce any further. Well known examples of manifolds underlying specific cones – such as Sasaki-Einstein and 3-Sasaki manifolds – thus emerge as examples of manifolds with reduced Tractor holonomy. Conformally Ricci-flat manifolds will be dealt with from direct calculation; in their case, the Tractor holonomy is directly related to the holonomy of the Ricci-flat Levi-Civita connection.

It is not obvious that there should be any decomposition theorem for conformal holonomy, as the Tractor bundle of $M^m \times N^n$ is of rank $m+n+2$ whereas the combined rank of the Tractor bundles of the two manifolds is $m+n+4$. However, if M^m and N^n are conformally Einstein manifolds, then due to the existence of parallel sections, their combined ranks effectively reduce to $m+n+2$ and a decomposition theorem becomes conceivable. In fact, if the algebra is reductive and acts reducibly, one indeed has the following theorem:

Theorem 0.1.1. *Assume there is a bundle $K \subset \mathcal{T}$ of rank $l+1$ preserved by $\overrightarrow{\nabla}$ generating a foliation $U \subset T$, such that $T = U \oplus U^{\perp}$. Then there exists a metric g in the conformal class of M such that the manifold (M, g) splits locally as the direct product*

$$(M, g) = (N_1, h_1) \times (N_2, h_2)$$

where h_1 and h_2 are Einstein metrics with Einstein coefficients λ_1, λ_2, possibly zero, related by

$$(n-l-1)\lambda_1 = (1-l)\lambda_2.$$

The converse is also true. And in this situation the holonomy $\overrightarrow{\mathfrak{hol}}$ of $\overrightarrow{\nabla}$ is the direct sum of Lie

algebras

$$\overrightarrow{\mathfrak{hol}} = \overrightarrow{\mathfrak{hol}}_{N_1} \oplus \overrightarrow{\mathfrak{hol}}_{N_2}$$

where $\overrightarrow{\mathfrak{hol}}_{N_1}$ *is the holonomy of* $\overrightarrow{\nabla}_{N_1}$ *and* $\overrightarrow{\mathfrak{hol}}_{N_2}$ *that of* $\overrightarrow{\nabla}_{N_2}$.

The method used in [Lei] is quite different: there a preserved bundle corresponds to a single decomposable Tractor 'form' in the exterior product representation of the Tractor connection. Despite this, the same results are established.

In the definite signature case, a result by A.J. Di Scala and C. Olmos, [DiOl], shows that the only subalgebra of $\mathfrak{so}(n+1,1)$ acting irreducibly on $\mathbb{R}^{(n+1,1)}$ is itself. After demonstrating that this full algebra does in fact exist as a holonomy algebra for $n \geq 3$, we shall further show that all holonomy algebras must be reductive and acting reducibly in definite signature, and consequently we have a full classification of conformal holonomy algebras in this case.

Chapter 3 deals with the projective geometry case. The projective Tractor bundle $\mathcal{T}$ is of rank $n+1$, where n is the dimension of the manifold. The Tractor connection preserves a volume form on $\mathcal{T}$, so we are looking at holonomy algebras contained in $\mathfrak{sl}(n+1)$. We shall first analyse the consequences of reducibility (see Section 3.1), then show that the existence of symplectic, complex, hyper-complex and orthogonal structures on the Tractor bundle imply that the underlying manifold is projectively contact, CR, HR and Einstein, respectively. Holonomies of type $\mathfrak{su}$, for instance, correspond to projectively Sasaki-Einstein manifolds.

Unlike the conformal case, where cone constructions only existed in the conformally Einstein case, *all* projective manifolds have a cone construction whose affine holonomy is equal to the Tractor holonomy of the underlying manifold. Consequently projective Tractor holonomy is reduced to affine holonomy issues on specific – Ricci-flat, torsion-free – cone manifolds.

The Tractor bundle for the projective structure is of rank $n+1$; that of the conformal structure is of rank $n+2$. Once the Einstein condition reduces this rank to $n+1$ by preserving a Tractor, an isomorphism between the two becomes conceivable. And, indeed, a further result demonstrates that the projective and conformal holonomies of an Einstein manifold are isomorphic, as are the two cone constructions. In fact, an Einstein Levi-Civita connection has identical projective and conformal

cones. This allows us to state that

Proposition 0.1.2. *A subalgebra of $\mathfrak{so}(p+1,q+1)$ preserving a Tractor is a conformal holonomy algebra if and only if it is a projective holonomy algebra. Thus every projective holonomy algebra that preserves a metric on $\mathcal{T}$ is also a conformal holonomy algebra.*

Table 1 gives the list of these algebras. In the conformal case, they are acting reducibly but not irreducibly; in the projective case, they are acting irreducibly. Table 2 gives the remaining irreducibly acting projective holonomy algebras. These lists are arrived at in the last two chapters,

algebra $\mathfrak{g}$	representation V	restrictions	algebra $\mathfrak{g}$	representation V
$\mathfrak{so}(p,q)$	$\mathbb{R}^{(p,q)}$	$p+q \geq 5$	$\widetilde{\mathfrak{g}}_2$	$\mathbb{R}^{(4,3)}$
$\mathfrak{so}(n,\mathbb{C})$	$\mathbb{R}^n$	$n \geq 5$	$\mathfrak{g}_2(\mathbb{C})$	$\mathbb{C}^7$
$\mathfrak{su}(p,q)$	$\mathbb{C}^{(p,q)}$	$p+q \geq 3$	$\mathfrak{spin}(7)$	$\mathbb{R}^8$
$\mathfrak{sp}(p,q)$	$\mathbb{H}^{(p,q)}$	$p+q \geq 2$	$\mathfrak{spin}(4,3)$	$\mathbb{R}^{(4,4)}$
$\mathfrak{g}_2$	$\mathbb{R}^7$		$\mathfrak{spin}(7,\mathbb{C})$	$\mathbb{C}^8$

Table 1: Conformally and projectively Einstein Holonomy algebras

algebra $\mathfrak{g}$	representation V	restrictions	manifold properties
$\mathfrak{sl}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 3$	Generic
$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$	CR manifold
$\mathfrak{sl}(n,\mathbb{H})$	$\mathbb{C}^n$	$n \geq 2$	HR manifold
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$	Contact manifold
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathbb{C}^{2n}$	$n \geq 2$	CR-Contact manifold

Table 2: Projectively non-Einstein Holonomy algebras

by classifying the possible holonomy algebras of Ricci-flat cones.

Chapter 4 is essentially a stand-alone chapter, though vital for the rest of the work. In this

we move away from Cartan connections to look at ordinary torsion-free affine connections, and the possible holonomy algebras that can be generated from one that is Ricci-flat. This issue has not been looked at before in the literature; chapter 4 essentially answers this question in the negative sense, going through the list holonomies of irreducible, torsion-free affine connections established by S. Merkulov and L. Schwachhöfer in [MeSc1] and [MeSc2], to identify those families that do not allow Ricci-flat connections. Methods used vary with each type of holonomy family, and in a few cases (minimal Segre algebras, E_6 representations), very specific tools were constructed to deal with the issue.

The motivation for this, of course, is the cone construction discovered previously, and the consequent realisation that any projective Tractor holonomy must also be the affine holonomy of a Ricci-flat torsion-free affine connection.

Chapter 5 is the constructive counterpart to Chapter 4. In this we aim to build tractor manifolds or cone constructions with the remaining holonomy families. These constructions are long and technical, and generate no new mathematics; however they are needed to complete the lists, and a few are interesting in their own right; the fact that the existence of $\mathfrak{sl}(\mathbb{C})$ type holonomies is much easier to establish than $\mathfrak{sl}(\mathbb{R})$ types is intriguing. Some low-dimensional cases resist the general treatments; these are dealt with individually at the end of Chapter 5.

The general Ricci-flatness issue is not yet completely settled, however; some holonomy algebras have not been excluded by Chapter 4, though Chapter 5 demonstrated that they may not be the holonomies of Ricci-flat *cones*. Appendix E completes the list of Ricci-flat torsion-free affine holonomies, which is interesting in its own right. The short list is given in Table 3, and this, added to the previous two tables, establishes which holonomy algebras and representations can correspond to Ricci-flat, torsion-free affine connections.

algebra $\mathfrak{g}$	representation V
$\mathfrak{so}(p,q)$	$\mathbb{R}^{(p,q)}$, $p+q=4$
$\mathfrak{so}(4,\mathbb{C})$	$\mathbb{C}^4$
$\mathfrak{sl}(1,\mathbb{H})$	$\mathbb{H}$
$\mathfrak{sl}(2,\mathbb{C})$	$\mathbb{C}^2$

Table 3: Ricci-flat, non-normal Tractor holonomies

Chapter 1

The Cartan Connection

1.1 Cartan and Tractor Connections

1.1.1 The Flat Models

Traditionally, since Klein, geometries were defined by a manifold M and a Lie group G acting transitively and effectively on M. The stabilizer group of any point $x \in M$ is a sub-group $P \subset G$, which changes by conjugation as x varies.

From a more modern perspective, the focus has shifted to the groups G and P, with the underlying space M seen as the quotient

$$M = G/P.$$

A particular class of these spaces (the parabolic $|1|$-graded ones; see Section 1.1.4 for a definition) were classified in [KoNa]. A few standard examples, mainly drawn from [CSS1], are:

- Grassmannian spaces.

  Let $G = SL(p+q, \mathbb{R})$, and $P = (SL(p, \mathbb{R}) \times SL(q, \mathbb{R}) \times \mathbb{R}^*) \rtimes \mathbb{R}^{pq}$. Then $M = G/P$ are the real Grassmannian spaces. The 'curved' version of projective spaces $q = 1$ are one of the two main cases dealt with in this thesis; these are the manifolds modelled on $\mathbb{RP}^p$.

- Complex Grassmannian space.

  This is just the complex analogue of the previous spaces, $G = SL(p+q, \mathbb{C})$, and $P = (SL(p, \mathbb{C}) \times SL(q, \mathbb{C}) \times \mathbb{C}^*) \ltimes \mathbb{C}^{pq}$. Manifolds modelled on $\mathbb{CP}^p$ will also be used in this thesis.

- Conformal Spaces.

  Let $G = SO(m+1, n+1, \mathbb{R})$, and $P = CO(m, n) \ltimes \mathbb{R}^{(m,n)*}$. Then $M = G/P$ is the set of null lines in $\mathbb{R}^{(m+1,n+1)}$ – a quadric, the sphere in the definite signature case – and G is the group of Möbius transformations on it. This will be the other case dealt with in this thesis; the definite signature case $n = 0$ will be fully classified.

- Lagrange Grassmannian space.

  Let $G = Sp(2n, \mathbb{R})$, and $P = GL(n, \mathbb{R}) \ltimes S^2\mathbb{R}^{n*}$. The curved analogues of these spaces are called almost Lagrangian in [CSS1].

- Spinor spaces.

  Let $G = SO(n, n, \mathbb{R})$, and $P = GL(n, \mathbb{R}) \ltimes \wedge^2\mathbb{R}^{n*}$. These are the isotropic Grassmannian manifolds, which can be identified with the spaces of pure spinors. Their curved analogues are called almost spinorial.

We now aim to replace these homogeneous constructions with an inhomogenous infinitesimal analogue.

1.1.2 The Cartan Connection

The Cartan connection is a curved version of the flat geometries. Given *any* manifold M, it maps the tangent space T_M locally to the Lie algebra quotient,

$$(T_M)_x \cong \mathfrak{g}/\mathfrak{p},$$

for all x in M.

We will follow the exposition used in [CaGo3]. In all of the following, we assume that M is an n-dimensional manifold, with $\mathfrak{g}$ a semisimple Lie algebra and a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with $\mathfrak{p}$ of codimension

n in $\mathfrak{g}$. There are corresponding groups $P \subset G$; different choices of such groups may change the global properties of Cartan connections, but not the local ones.

Definition 1.1.1 (Cartan Connection). *On M, given a principal P-bundle $\mathcal{P} \to M$, a normal Cartan connection ω is a section of $T^*_{\mathcal{P}} \otimes \mathfrak{g}$, with the following properties:*

1. *ω is invariant under the P-action (P acting by Ad on $\mathfrak{g}$),*
2. *$\omega(\sigma_A) = A$, where σ_A is the fundamental vector field of $A \in \mathfrak{p}$,*
3. *$\omega_u : T\mathcal{P}_u \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.*

If $\mathfrak{p}$ is a parabolic algebra (see Section 1.1.4), we may make the further requirement that the connection be *normal*; this is a uniqueness condition for the Cartan connection of a particular geometry, similar to the torsion-free condition for a Levi-Civita connection. See [CaGo3] for a proof of the existence of a normal Cartan connection in all parabolic geometries.

Definition 1.1.2 (Normal Cartan Connection). *A Cartan connection for a parabolic geometry is normal if it has the following additional condition:*

4. *The 'curvature' $\kappa(\eta, \xi) = d\omega(\eta, \xi) + [\omega(\eta), \omega(\xi)]$ is such that $\partial^* \kappa = 0$ where ∂^* is the dual homology operator defined in Equation (1.2).*

The bundle $\mathcal{P}$ and the form ω together define the geometry. The first two conditions on ω are analogous to those of a standard connection. The third condition is very different, however, giving a pointwise isomorphism $T\mathcal{P}_u \to \mathfrak{g}$ rather than a map with kernel.

However the Cartan connection does give rise to a connection in the usual sense, the so-called Tractor connection.

1.1.3 The Tractor Connection

The inclusion $P \hookrightarrow G$ generates a principal bundle inclusion $i : \mathcal{P} \hookrightarrow \mathcal{G}$, with $\mathcal{G}$ a G-bundle, and generates a standard connection form:

Proposition 1.1.3. *There is a unique $\omega' \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that ω' is a standard connection form on $\mathcal{G}$ and $i^*\omega' = \omega$.*

Proof. At any point of $\mathcal{P} \hookrightarrow \mathcal{G}$, define $\omega'(X) = \omega(X)$ for $X \in \Gamma(T\mathcal{P})$, and $\omega'(\sigma_A) = A$ for σ_A the fundamental vector field of $A \in \mathfrak{g}$. These two formulas correspond whenever they are both defined (Property 2 from Definition 1.1.1), and completely define ω' on $\mathcal{P}$. Then define $\omega'_u = g^*(\omega'_{g(u)})$ in the general case, for $g(u) \in \mathcal{P}$. Property 1 for ω ensures this is well defined.

To see that ω' is indeed a connection, notice that for $v \in \mathcal{P}$, $\omega' : T\mathcal{G}_v \to \mathfrak{g}$ has maximal rank, since $\omega = \omega'|_{T\mathcal{P}} : T\mathcal{P}_v \to \mathfrak{g}$ is surjective. G-invariance of ω' generalises this property to all of $\mathcal{G}$. ■

This ω' is the Tractor connection; when we see it as a connection on an associated vector bundle, we shall designate it by $\overrightarrow{\nabla}$. The Tractor connection obviously generates a Cartan connection by pull-back to $T\mathcal{P}$. From now on, we shall use Cartan and Tractor connections interchangeably.

Remark. It is not the case that any G connection η will correspond to a Cartan connection via pull-back to $\mathcal{P}$, as the isomorphism condition $T\mathcal{P}_v \to \mathfrak{g}$ could be violated. In the language of Section 1.3.3, η must have a maximal second fundamental form on the canonical sub-bundles in the splitting of the Tractor bundle. This form is sometimes known as the soldering form [BuCa]. If so, then η comes from a Cartan connection.

1.1.4 Parabolic subalgebras

This definition of Cartan and Tractor connections is very broad, and little is known about the properties of these structures in general. There is one situation where a lot is known, however: when $\mathfrak{p}$ is a *parabolic* subalgebra of $\mathfrak{g}$. We shall use the definition of [CDS].

Definition 1.1.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra. For any subspace $\mathfrak{u}$ of $\mathfrak{g}$, let $\mathfrak{u}^\perp$ be the orthogonal complement with respect to the Killing form $(-,-)$ on $\mathfrak{g}$. A subalgebra $\mathfrak{p}$ is parabolic iff $\mathfrak{p}^\perp$ is the* nilradical *of $\mathfrak{p}$, i.e. its maximal nilpotent ideal.*

This definition implies that $\mathfrak{p}/\mathfrak{p}^\perp$ is a reductive Lie algebra, called the *Levi factor* and designated as $\mathfrak{g}_0$. This definition also gives rise to a filtration of $\mathfrak{g}$; first, one has a filtration of $\mathfrak{p}^\perp$ by defining $\mathfrak{g}_{(-1)} = \mathfrak{p}^\perp$ and $\mathfrak{g}_{(-j)} = [\mathfrak{p}^\perp, \mathfrak{g}_{(-j+1)}]$. Since $\mathfrak{p}^\perp$ is nilpotent, there exists an l such that $\mathfrak{g}_{(l+1)} = 0$ but $\mathfrak{g}_{(l)} \neq 0$. This gives $\mathfrak{p}^\perp$ a l-step filtration ($l = 0$ in the trivial case $\mathfrak{p} = \mathfrak{g}$ and $\mathfrak{p}^\perp = 0$). We may

extend this filtration to all of $\mathfrak{g}$ by defining $\mathfrak{g}_{(j)} = (\mathfrak{g}_{(-j-1)})^{\perp}$ for $j \geq 0$ so that

$$\mathfrak{g} = \mathfrak{g}_{(l)} \supset \mathfrak{g}_{(l-1)} \supset \ldots \supset \mathfrak{g}_{(0)} = \mathfrak{p} \supset \mathfrak{g}_{(-1)} = \mathfrak{p}^{\perp} \supset \ldots \supset \mathfrak{g}_{(-l)}.$$

It is easliy verified tht $[\mathfrak{g}_{(j)}, (\mathfrak{k})] \subseteq \mathfrak{g}_{(j+k)}$ making $\mathfrak{g}$ into a filtered Lie algebra. The corresponding graded Lie algebra is $gr\ \mathfrak{g} = \bigoplus_{-k}^{k} \mathfrak{g}_j$, where $\mathfrak{g}_j = \mathfrak{g}_{(j)}/\mathfrak{g}_{(j-1)}$, and $gr\mathfrak{g}$ is said to be $|l|$-graded. By a slight abuse of terminology, we say that $\mathfrak{g}$ itself is $|l|$-graded. This is not a major abuse, as

Lemma 1.1.5. *There are (non-canonical) splittings of the exact sequences*

$$0 \to \mathfrak{g}_{(j-1)} \to \mathfrak{g}_{(j)} \to \mathfrak{g}_j \to 0, \tag{1.1}$$

which induce Lie algebra isomorphisms between $\mathfrak{g}$ *and* $gr\ \mathfrak{g}$.

Proof of Lemma. Any semisimple Lie algebra admits a Cartan involution, an automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\sigma^2 = Id$ and $h(\eta, \xi) = (\sigma(\eta), \xi)$ is positive definite. We split each exact sequence (1.1) by identifying $\mathfrak{g}_j$ with the h-orthogonal complement to $\mathfrak{g}_{(j-1)}$ in $\mathfrak{g}_{(j)}$. Assume that $\eta \in \mathfrak{g}_{(j)}$ is h-orthogonal to $\mathfrak{g}_{(j-1)}$, i.e. $\sigma(\eta) \in \mathfrak{g}_{(j-1)}^{\perp} = \mathfrak{g}_{(-j)}$ and $\xi \in \mathfrak{g}_{(k)}$ is h-orthogonal to $\mathfrak{g}_{(k-1)}$. Then $[\eta, \xi] \in \mathfrak{g}_{(j+k)}$ and $\sigma[\eta, \xi] = [\sigma(\eta), \sigma(\xi)] \in \mathfrak{g}_{(-j-k)}$ so $[\eta, \xi]$ is h-orthogonal to $\mathfrak{g}_{(j+k+1)}$. Hence the splittings defined by σ induce a Lie algebra isomorphism. □

This splitting is non unique. But we may achieve good control of the possible splittings by using the *grading element*:

Lemma 1.1.6. *There is a unique element* ϵ_0 *in the centre of* $\mathfrak{g}_0$ *such that* $[\epsilon_0, \eta] = j\eta$ *for all* $\eta \in \mathfrak{g}_j$ *and all* j.

Proof of Lemma. Since $gr\ \mathfrak{g}$ is semisimple, the derivation defined by $\eta \to j\eta$ must be inner, in other words equal to $ad\ \epsilon_0$ for some $\epsilon_0 \in gr\ \mathfrak{g}$ (unique since $Z(\mathfrak{g}) = 0$). Since $[\epsilon_0, \epsilon_0] = 0$ and $[\epsilon_0, \eta] = 0$ for all $\eta \in \mathfrak{g}_0$, ϵ_0 must be in the centre of $\mathfrak{g}_0$. □

Proposition 1.1.7. *The splitting of* $\mathfrak{g}$ *into* $gr\ \mathfrak{g}$ *is defined entirely by the lift* ϵ *of* ϵ_0 *with respect to the exact sequence*

$$0 \to \mathfrak{p}^{\perp} \to \mathfrak{p} \to \mathfrak{g}_0 \to 0.$$

Proof. Decompose $\mathfrak{g}$ in terms of the eigenspaces of $ad\ \epsilon$. ■

This view of the splitting as a lift of the grading element will be important to define Weyl structures on a manifold.

Remark. Both the conformal $(\mathfrak{so}(p+1,q+1)/\mathfrak{co}(p,q) \ltimes \mathbb{R}^{(p,q)})$ and projective $(\mathfrak{sl}(n+1)/\mathfrak{gl}(n) \ltimes \mathbb{R}^n)$ algebras are parabolic, and $|1|$-graded.

Weyl structures

Define the group $G_0 = P/\exp \mathfrak{g}_{(-1)}$; the Lie algebra of G_0 is evidently $\mathfrak{g}_0$. Given any parabolic P-bundle $\mathcal{P}$, we may similarly construct the G_0-bundle $\mathcal{G}_0 = \mathcal{P}/\exp \mathfrak{g}_{(-1)}$.

We may define the Lie algebra bundle $\mathcal{P} \times_P \mathfrak{g}_0$ with the quotient action of P. This algebra bundle contains a canonical grading section E_0, defined as the constant function from $\mathcal{P}$ to ϵ_0; since ϵ_0 is P-invariant under the quotient action, this function is also P-invariant, and defines a section of $\mathcal{P} \times_P \mathfrak{g}_0$.

Definition 1.1.8. *Let $\mathcal{A} = \mathcal{G} \times_G \mathfrak{g} = \mathcal{P} \times_P \mathfrak{g}$ be a Lie algebra bundle. It contains the subundle $\mathcal{P} \times_P \mathfrak{g}_{(0)}$, which has a quotient map to $\mathcal{P} \times_P \mathfrak{g}_0$. A* Weyl *structure $\mathfrak{E}^W$ on M is a section of $\mathcal{A}$ that is a lift of the grading section E_0.*

These Weyl structures do have an immediate use, allowing one to split many canonical objects:

Proposition 1.1.9. *A Weyl structure gives a splitting of both the algebra bundle $\mathcal{A}$ and the Cartan connection ω.*

Proof. A Weyl structure $\mathfrak{E}^W$ is a map from $\mathcal{P}$ to $\mathfrak{p}$, whose image is always a lift of ϵ_0. This allows us to split $\mathfrak{g}$ as $gr\ \mathfrak{g}$ at every point of $\mathcal{P}$. As $\mathfrak{E}^W$ is P-invariant, so is this splitting, and we have resultant splittings:

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_l \oplus \mathcal{A}_{l-1} \oplus \ldots \oplus \mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_{-l} \\ \omega &= \omega_l + \omega_{l-1} + \ldots + \omega_0 + \ldots + \omega_{-l}. \end{aligned}$$

■

This is in fact a reduction of the structure group of $\mathcal{A}$ from P to G_0. And as an immediate corollary:

Corollary 1.1.10. *A Weyl structure gives a splitting of any vector bundle associated to the Tractor connection.*

The main motivation behind these Weyl structures will be explored in the next Section, where we look at equivalent, and more farmiliar geometric structures: preferred connections.

1.2 Preferred connections

There are other ways of characterising these geometries, quite apart from the Tractor connection. For instance, conformal structures are equivalently specified by a class of related metrics, and projective structures by a class of connections with identical geodesics. In this section, we shall see how all these structures are related to the Tractor connection. Moreover this will also give a very useful representation of the Tractor connection in terms of more usual affine connections.

In this section we shall produce various bundle isomorphisms, the equivalence of the Cartan connection with more conventional geometric structures, and various useful properties of these structures.

Recall the G_0-bundle $\mathcal{G}_0 = \mathcal{P}/\exp \mathfrak{g}_{(-1)}$. This bundle $\mathcal{G}_0$ is our first tie-in with more conventional geometries; in fact, for most parabolic geometries (though not the projective kind) this bundle ties down the geometry entirely.

Recall from Proposition 1.1.9 that $\mathfrak{E}^W$ defines a splitting of the Cartan connection

$$\omega = \omega_l + \ldots + \omega_0 + \ldots + \omega_l.$$

The central component ω_0 is a one-form on $\mathcal{P}$ with values in $\mathfrak{g}_{(0)}/\mathfrak{g}_{(-1)}$. It is P invariant under the quotient action of P. We may divide out by the action of $G_{(-1)} = \exp \mathfrak{g}_{(-1)}$ to get a one-form on $\mathcal{G}_0$ with values in $\mathfrak{g}_0$. It is easy to see, from the properties of ω, that this one-form (which we shall also denote ω_0) is $(G_0 = P/G_{(-1)})$-invariant and that $\omega_0(\sigma_A) = A$, where σ_A is the fundamental vector field of $A \in \mathfrak{g}_0$.

Proposition 1.2.1. *ω_0 is a standard connection form on the principal bundle $\mathcal{G}_0$ and on the tangent bundle.*

Proof. The fact that ω_0 is a connection form on $\mathcal{G}_0$ is an immediate consequence of the properties noted above. That $\mathcal{G}_0$ is a principal bundle for the tangent bundle is a consequence of the following important Lemma:

Lemma 1.2.2. *The algebra bundle $\mathcal{A}$ has a natural inclusion*

$$\mathcal{A} \supset T^*,$$

a natural projection

$$\mathcal{A} \to T,$$

and, given a choice of $\mathfrak{E}^W$, $\mathcal{A}$ splits as

$$\mathcal{A} = T \oplus \mathfrak{g}_0(T) \oplus T^*.$$

Proof of Lemma. This proof is from [CaGo3]. First notice that $\mathcal{A}$ has a natural filtration

$$\mathcal{A}_{(l)} \supset \mathcal{A}_{(l-1)} \supset \ldots \supset \mathcal{A}_{(-l)},$$

where $\mathcal{A}_{(j)} = \mathcal{P} \times_P \mathfrak{g}_{(j)}$; this last bundle is well defined since $\mathfrak{p} = \mathfrak{g}_{(0)}$ must preserve the filtration of $\mathfrak{g}$.

Let p be the projection $\mathcal{P} \to M$. Consider the map $\mathcal{P} \times \mathfrak{g} \to T$ given by $(u, A) \to T_p.(\omega(u)^{-1}(A))$. If A lies in $\mathfrak{p}$, then $\omega(u)^{-1}(A) = \sigma_A$ which is a vertical field, so this map factors smoothly to a map $\mathcal{P} \times \mathfrak{g}/\mathfrak{p} \to T$. The properties of ω immediately imply that this map factors further to a homomorphism $\mathcal{P} \times_P \mathfrak{g}/\mathfrak{p} \to T$ that covers the identity and is an isomorphism on each fiber, so is a bundle isomorphism. Consequently

$$\mathcal{A}/\mathcal{A}_{(0)} = \mathcal{P} \times_P \mathfrak{g}/\mathfrak{p} \cong T$$

The Killing form on $\mathfrak{g}$ identifies $(\mathfrak{g}/\mathfrak{p})^*$ with $\mathfrak{p}^\perp = \mathfrak{g}_{(-1)}$. Consequently, since the Killing form is $\mathfrak{g}$ invariant,

$$\mathcal{A}_{(-1)} = \mathcal{P} \times_P \mathfrak{g}_{(-1)} \cong T^*.$$

Now, given a choice of $\mathfrak{E}^W$, by Proposition 1.1.9 one has a splitting of $\mathcal{A}$ and consequently an identification

$$T = \mathcal{A}/\mathcal{A}_{(0)} = \mathcal{A}_l \oplus \ldots \oplus \mathcal{A}_1,$$

meaning that

$$\mathcal{A} = T \oplus \mathcal{A}_0 \oplus T^*.$$

Notice that $\mathcal{A}_0 = \mathcal{P} \times_P \mathfrak{g}_0$. Since the (quotient) action of $\mathfrak{g}_{(-1)}$ on $\mathfrak{g}_0$ is trivial, this is also $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_0$. Similarly, $T = \mathcal{G}_0 \times_{G_0} (\mathfrak{g}_l \oplus \ldots \oplus \mathfrak{g}_1)$. Then one merely has to note that this last vector space is a representation of $\mathfrak{g}_0$ via the Lie bracket of $gr\ \mathfrak{g}$. Consequently

$$\mathcal{A} = T \oplus \mathfrak{g}_0(T) \oplus T^*.$$

□

The previous lemma demonstrated that $\mathcal{G}_0$ is a principal bundle for the tangent bundle, making ω_0 into an affine connection on T. ■

Definition 1.2.3. *We call the various ω_0's the* preferred connections *of the Cartan connection. As they depend on a choice of Weyl structure $\mathfrak{E}^W$, they are also often known as* Weyl connections.

From now on we shall focus on the preferred connections, rather than on the equivalent Weyl structures.

Proposition 1.2.4. *All preferred connections for a given Cartan connection have the same torsion.*

Proof. The projection π^2 from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{p}$ is well defined (the use of the expression π^2 is connected to

the fact that Tractor bundles may be seen as second order jet-bundles, see Section 1.3.3). $\pi^2 \circ \omega$ is a one-form on $\mathcal{P}$ with values in $\mathfrak{g}/\mathfrak{p}$. This descends, dividing out by the action of $G_{(-1)}$, to a one-form on $\mathcal{G}_0$. From the definition of the isomorphism $\mathcal{A}/\mathcal{A}_{(0)} \cong T$, we can see that

$$\mathcal{G}_0 \times_{G_0} (\pi^2 \circ \omega) \quad = \quad Id_T,$$

the identity on T, implying that $\pi^2 \circ \omega$ is the canonical one-form of $\mathcal{G}_0$.

Now the 'curvature' of a Cartan connection is defined as

$$\kappa(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$

where ξ and η are lifts of vector fields on M; this expression is independent of the choice of such lifts, exactly as in the case of a usual curvature expression. And it is obviously P-invariant.

We now choose a Weyl structure $\mathfrak{E}^W$ and equivalent preferred connection, splitting the Cartan connection as $\omega^+ + \omega_0 + \omega^-$, where $\omega^1 = \sum_1^l \omega_j$ and $\omega^- = \sum_{-l}^{-1} \omega_j$. In this splitting we may calculate $\pi^2(\kappa)$ as

$$\begin{aligned} \pi^2(\kappa) &= \left(\pi^2(d\omega) + \pi^2([\omega, \omega])\right) \\ &= d\omega^+ + \left([\omega_0, \omega^+] + [\omega+, \omega_0]\right) \\ &= d\omega^+ + \omega_0 \wedge \omega^+. \end{aligned}$$

Consequently [KoNo] $\pi^2(\kappa)$ is the torsion of the preferred connection corresponding to ω_0. However, $\pi^2(\kappa)$ is defined invariantly; hence all preferred connections have the same torsion. ■

1.2.1 Normal Cartan connections

In this section we will understand a bit more about the 'normality' condition on the Cartan connection. First we will recall some Lie algebra cohomology [Bas]. For $\mathfrak{h}$ any Lie algebra and V a representation of $\mathfrak{h}$, define the space $D^k(V) = V \otimes \wedge^k \mathfrak{h}^*$ and the operator.

$$\partial : D^k(V) \to D^{k+1}(V)$$

to be

$$\partial D(X_0, \dots X_k) = \sum_{j=0}^{k} (-1)^j X_j . D(X_0, \dots, \widehat{X_j}, \dots, X_k).$$

This can easily be seen to be a complex, i.e. $\partial^2 = 0$. Define the dual spaces $C^k(V) = V^* \otimes \wedge^k \mathfrak{h}$. We may then define the dual operator

$$\partial^* : C^k(V) \to C^{k-1}(V)$$

as

$$\partial^* C(\nu_1, \dots \nu_{k-1}) = \sum_{j=1}^{m} \lambda^j . C(\lambda_j, \nu_1, \dots, \nu_{k-1}). \tag{1.2}$$

where $m = \text{rank}(V)$, (λ^j) are a basis for $\mathfrak{h}$ and (λ_j) a dual basis for $\mathfrak{h}^*$.

The 'curvature' κ is a section of $\Omega^2(M, \mathfrak{g})$. Using the equivalence $T^* = \mathcal{P} \times_P \mathfrak{g}_{(-1)}$ we may see κ as a P-invariant function from $\mathcal{P}$ to $\wedge^2 \mathfrak{g}_{(-1)} \otimes \mathfrak{g}$.

Definition 1.2.5. *Setting $V = \mathfrak{g}$ ($= \mathfrak{g}^*$ via the Killing form) and $\mathfrak{h} = \mathfrak{g}_{(-1)}$, the normality condition on the Cartan connection is that $\partial^* \kappa = 0$.*

1.3 Projective and conformal Cartan connections

In this section, we will restrict attention to the two most commonly studied parabolic geometries:

- conformal geometry, with $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$, $\mathfrak{p} = \mathfrak{co}(p,q) \rtimes \mathbb{R}^{(p,q)}$ and $\mathfrak{g}_0 = \mathfrak{co}(p,q)$,
- projective geometry, with $\mathfrak{g} = \mathfrak{sl}(n+1)$, $\mathfrak{p} = \mathfrak{gl}(n) \rtimes \mathbb{R}^n$ and $\mathfrak{g}_0 = \mathfrak{gl}(n)$.

In both geometries, the grading element ϵ_0 is $Id \in \mathfrak{g}_0$. For the confromal geometry, the structure is given entirely by the bundle $\mathcal{G}_0$; whereas $\mathcal{G}_0$ is the full frame bundle for the tangent bundle in the projective case, so provides no information. We will seek to explicitly describe the Tractor and preferred connections for these two geometries.

Recall from Lemma 1.2.2 that given a choice of preferred connection, $\mathcal{A}$ splits and a local section is of the form

$$\begin{pmatrix} X \\ \Psi \\ \nu \end{pmatrix},$$

with $X \in \Gamma(T)$, $\nu \in \Gamma(T^*)$ and $\Psi \in \Gamma(\mathfrak{g}_0(T))$.

Lemma 1.3.1. *Under a change of preferred connection, this section changes as*

$$\begin{pmatrix} X \\ \Psi \\ \nu \end{pmatrix} \to \begin{pmatrix} X \\ \Psi + [\Upsilon, X] \\ \nu + [\Upsilon, \Psi] + \frac{1}{2}[\Upsilon, [\Upsilon, X]] \end{pmatrix}, \tag{1.3}$$

for a one-form Υ.

Proof of Lemma. Since a Weyl structure is equivalently a lift of the grading section of $\mathcal{P} \times_P \mathfrak{g}_0 = \mathcal{A}_0$ in the exact sequence $0 \to \mathcal{A}_{(-1)} \to \mathcal{A}_{(0)} \to \mathcal{A}_0 \to 0$, the difference between two Weyl structures is a section of $\mathcal{A}_{(-1)} \cong T^*$, in other words a one-form. Call it Υ. Then the preceding formula gives the action of $\exp \Upsilon$ upon the algebras bundle $\mathcal{A}$. □

Remark. Note that this proof is valid for all $|1|$-graded parabolic geometries, not just the projective and conformal ones.

We know that every preferred connection has the same torsion τ. However, the following theorem gives a better understanding of what makes a connection preferred:

Theorem 1.3.2. *Every connection that preserves the conformal / projective structure and has the torsion* τ *is a preferred connection.*

The previous lemma shows that two preferred connections ∇ and $\widehat{\nabla}$ differ by

$$\nabla_X Y = \widehat{\nabla}_X Y + [\Upsilon, X].Y,$$

for some one-form Υ. The fact that connections of this type span all the connections preserving the structure with required torsion will be proved seperately in the conformal and projective chapters, as the two proofs are technical and quite distinct one from the other.

Proposition 1.3.3. *In the conformal and projective cases, the preferred connections of a normal Cartan connection are torsion-free.*

Proof. I shall not reproduce this proof, given in [CaGo3]. It demonstrates this result using Kostant's version of the Bott-Borel-Weil theorem from [Och] and [Kos] to compute various cohomology spaces, and thus demonstrate that if the Cartan curvature $\kappa : \mathcal{P} \to \wedge^2\mathfrak{g}_{-1} \otimes \mathfrak{g}$ is ∂^*-closed, then for projective and conformal geometries, κ is in fact a map to $\wedge^2\mathfrak{g}_{-1} \otimes \mathfrak{p}$; in other words, $\pi^2(\kappa)$, the torsion of the preferred connections, must vanish. ■

Remark. In the conformal case, the fact that one can always choose torsion-free connections preserving the structure is a direct consequence of the existence of Levi-Civita connections for the various metrics in the conformal class.

Remark. A more direct proof demonstrates the existence of torsion-free connections preserving a projective structure. Let ∇' be any affine connection, with torsion τ. Then $\nabla = \nabla' - \frac{1}{2}\tau$ is a torsion-free connection, and if X is the tangent vector of a geodesic of ∇',

$$\nabla_X X = \nabla'_X X - \frac{1}{2}\tau(X, X) = \nabla'_X X,$$

so any geodesic of ∇' is a geodesic of ∇.

From now on, we will take our Cartan connections normal and our preferred connections torsion-free. All these constructions can be reversed; so to every projective and conformal structure, there corresponds a (unique) normal Cartan connection.

1.3.1 Bracket formulas

At this point it becomes usefull to compute the Lie bracket on the Algebra bundle $\mathcal{A}$. This will be usefull for the change of splitting and the change of connection formulas, as well as providing an

explicit formula for the Lie algebra action of T and T^* on $\mathcal{A}$ and associated bundles. Fix a preferred connection and a splitting of $\mathcal{A}$.

In the projective case, $\mathcal{A}$ is an $\mathfrak{sl}(n+1)$-bundle and decomposes as

$$\left(\begin{array}{c|c} A & X \\ \hline \nu & a \end{array}\right),$$

where A is an element of $\mathfrak{gl}(T)$, X an element of T, ν an element of T^* and $a = -$ trace A. One identifies $B \in \mathfrak{g}_0$ with (A, a) where

$$\begin{aligned} A &= B - \frac{\text{trace } B}{n+1} Id, \\ a &= -\frac{\text{trace } B}{n+1}. \end{aligned}$$

Then one can easily calculate the Lie bracket,

$$\begin{aligned} [B, C] &= BC - CB, \\ [B, X] &= B(X), \\ [B, \nu] &= -B(\nu), \\ [X, \nu] &= X \otimes \nu + \nu(X) Id. \end{aligned}$$

In the conformal cases now, choose a preferred connection that preserves a metric g, and in that splitting, define

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(for a general preferred connection, replace g with $\mathbf{g}$, the conformal metric; see the beginning of

Chapter 2.0.3 for more details). Then the algebra bundle $\mathcal{A}$ preserves h by definition, and splits as

$$\left(\begin{array}{c|c|c} -a & \nu & 0 \\ \hline X & A & -g\nu^t \\ \hline 0 & -X^t g & a \end{array}\right),$$

where $A \in \mathfrak{so}(T) \subset \mathfrak{co}(T) = \mathfrak{g}_0(T)$, $X \in T$, $\nu \in T^*$, and $a \in \mathbb{R}$. Then identify $B = A + aId \in \mathfrak{g}_0$ with (A, a). Thus the Lie bracket is

$$\begin{aligned} [B, C] &= BC - CB, \\ [B, X] &= B(X), \\ [B, \nu] &= -B(\nu), \\ [X, \nu] &= X(\nu)Id + X \otimes \nu - g\nu^t \otimes X^t g. \end{aligned}$$

1.3.2 Curvature formulas

In order to proceed, we need some of the properties of the preferred connections, as well as a local formula for the Tractor connection.

Given the curvature R of a preferred connection, the trace is the Ricci tensor Ric. Since all the $\mathfrak{g}_0$ algebras come from a reductive group G_0 when acting on T and associated bundles, one also has the totally trace-free part of R, the *Weyl* tensor W. Note that the Weyl tensor depends both on the preferred connection, and on the algebra $\mathfrak{g}_0$. For instance a given Levi-Civita connection has different Weyl tensor depending on whether one looks at the conformal or projective structures that it generates.

The tensor W is projectively/conformally invariant; this fact comes from the curvature formula for the Tractor connection, Equation (1.8): just as we proved previously that $\pi^2(\kappa)$ (the torsion) is independent of the choice of ∇, if the torsion vanishes, the next component $\pi^1(\kappa)$ will be invariant: but that is just the Weyl curvature.

The rest of the curvature information is contained in the Ricci tensor Ric. From this we shall construct an equivalent tensor, the rho-tensor P, constructed differently for the different geometries.

Conformal examples

In the conformal case, P is defined by

$$\begin{aligned} \mathsf{P}_{hj} &= -\frac{1}{n-2}\left(\frac{1}{n}\mathsf{Ric}_{hj} + \frac{n-1}{n}\mathsf{Ric}_{jh} - \frac{1}{2n-2}R\mathbf{g}_{hj}\right), \\ \mathsf{Ric}_{hj} &= \mathsf{P}_{jh} - (n-1)\mathsf{P}_{hj} - \left(\mathsf{P}_{kl}\mathbf{g}^{kl}\right)\mathbf{g}_{hj} \end{aligned}$$

where R is the scalar curvature (the trace taken via $\mathbf{g}$) and $\mathbf{g}$ itself is the section of $\odot^2\big(T^* \otimes (\wedge^n T^*)^{-\frac{1}{n}}\big)$ defined by

$$\mathbf{g} = (\det(g))^{-\frac{2}{n}} g$$

for any metric g in the conformal class (see Section 2.0.3). This definition does not depend on which g we use.

In terms of $\mathsf{Ric}_{(hj)}$ and $\mathsf{Ric}_{[hj]}$, the symmetric and anti-symmetric components of Ric_{hj},

$$\mathsf{P}_{hj} = \frac{1}{n}\mathsf{Ric}_{[hj]} - \frac{1}{n-2}\left(\mathsf{Ric}_{(hj)} - \frac{1}{2n-2}R\mathbf{g}_{hj}\right).$$

The full curvature then becomes:

$$R_{hjkl} = W_{hjkl} + 2\mathbf{g}_{k[h}\mathsf{P}_{j]l} - 2\mathbf{g}_{l[h}\mathsf{P}_{j]k} - 2\mathsf{P}_{[hj]}\mathbf{g}_{kl} \tag{1.4}$$

Proof. This formula is easily checked by taking traces and by the symmetries of the curvature tensor R. ∎

Under a change of connection given by a one-form Υ, this tensor changes as

$$\widehat{\mathsf{P}}_{hj} = \mathsf{P}_{hj} - \nabla_j \Upsilon_h + \frac{1}{2}\Upsilon_{hj}^2.$$

We can define another useful tensor, the Cotton-York tensor:

$$CY_{hjk} = \nabla_h \mathsf{P}_{jk} - \nabla_j \mathsf{P}_{hk}.$$

One- and two-dimensions

Though any two-manifold is conformally flat, with an infinite-dimensional local conformal tranformation group, paper [Cal] and other unpublished papers by the same author extend the concept of conformal Cartan connections to one and two dimensions, by constructing Möbius structures. As in higher dimensions, a choice of Weyl structure determines a splitting of the associated Tractor bundle. There is an ambiguity, however, in the trace-free symmetric part of the P-tensor; this may be chosen freely.

Definition 1.3.4. *For our purposes, we shall take*

$$\mathsf{P}_{hj} = -\frac{1}{2}\mathsf{Ric}_{jh}.$$

This is not a conformally invariant definition. However, we shall be using it in a specific metric (Einstein, with constant scalar curvature), where it makes sense and allows one to extend the reach of the decomposition theorem down to lower dimensions.

In one dimension, we may easily require

$$\mathsf{P}_{hj} = 0,$$

which is conformally invariant. This also fits our definitions.

Projective examples

In the projective case, P is defined by

$$\mathsf{P}_{hj} = -\frac{n}{n^2-1}\mathsf{Ric}_{hj} - \frac{1}{n^2-1}\mathsf{Ric}_{jh}, \tag{1.5}$$

$$\mathsf{Ric}_{hj} = -n\mathsf{P}_{hj} + \mathsf{P}_{jh}.$$

In terms of $\mathsf{Ric}_{(hj)}$ and $\mathsf{Ric}_{[hj]}$, the symmetric and anti-symmetric components of Ric_{hj},

$$\mathsf{P}_{hj} = -\frac{1}{n-1}\mathsf{Ric}_{(hj)} - \frac{1}{n+1}\mathsf{Ric}_{[hj]}.$$

Then the full curvature becomes:

$$R_{hj}{}^{k}{}_{l} = W_{hj}{}^{k}{}_{l} + \mathsf{P}_{hl}\delta^{k}_{j} + \mathsf{P}_{hj}\delta^{k}_{l} - \mathsf{P}_{jl}\delta^{k}_{h} - \mathsf{P}_{jh}\delta^{k}_{l}. \tag{1.6}$$

Proof. This formula is easily checked by taking traces and by the symmetries of the curvature tensor R. ∎

Under a change of connection given by a one-form Υ, this tensor changes as

$$\widehat{\mathsf{P}}_{hj} = \mathsf{P}_{hj} - \nabla_j\Upsilon_h + \frac{1}{2}\Upsilon^2_{hj}, \tag{1.7}$$

as in the conformal case. We can also define the Cotton-York tensor as before:

$$CY_{hjk} = \nabla_h\mathsf{P}_{jk} - \nabla_j\mathsf{P}_{hk}.$$

General formulas

A choice of preferred connection ∇ splits the Lie algebra bundle $\mathcal{A}$ as $T^* \oplus \mathfrak{g}_0(T) \oplus T$. In this splitting, the Tractor connection becomes:

$$\overrightarrow{\nabla} = \nabla + \rho(X) + \rho\mathsf{P}(X),$$

where ρ denotes the action of $\mathcal{A}$.

Proof. Using the change of splitting formula (1.3) and the formula for the change of P, one can check that this expression is independent of the choice of preferred connection. It is easy to see that it corresponds to a Cartan connection as its second fundamental form on $\mathcal{A}_{-1} \subset \mathcal{A}$ is maximal;

in other words, for any non-zero section s of $\mathcal{A}_{-1}$, the map $T \to \mathcal{A}_0$ given by $X \to \overrightarrow{\nabla}_X s/\mathcal{A}_{-1}$ is injective. This second fundamental form is often called a soldering form. It is then easy to check that we are in the presence of a Cartan connection, pull back this connection form to the bundle $\mathcal{P}$. Then this obeys all the conditions of a Cartan connection; the fact that $\omega_u : T\mathcal{P}_u \to \mathfrak{g}$ is an isomorphism is a direct consequence of this property of the soldering form.

Furthermore the curvature can be calculated:

$$R^{\overrightarrow{\nabla}}_{X,Y} = \begin{pmatrix} T(X,Y) \\ W(X,Y) \\ CY(X,Y) \end{pmatrix}. \tag{1.8}$$

which shows that $\overrightarrow{\nabla}$ is normal, as the torsion vanishes – thus $\pi^2 \circ R^{\overrightarrow{\nabla}} = 0$ – and since κ is the pull-back of $R^{\overrightarrow{\nabla}}$ to $\mathcal{P}$.

As normal Cartan connections are unique (up to isomorphism) for a particular geometry on the manifold, this expression is the normal Cartan connection for this geometric structure. ■

We shall designate our main focus of investigation, the local holonomy algebra of $\overrightarrow{\nabla}$ as

$$\overrightarrow{\mathfrak{hol}}.$$

Remark. We now have everything we need for the Tractor connection, except a good bundle for it to operate on.

1.3.3 Tractor bundles

In the conformal case, as we shall see later in Corollary 2.0.1, a choice of a section s of the weight bundle $L^1 \cong (\wedge^n T^*)^{\frac{-1}{n}}$ implies a choice of preferred connection ∇. This must preserve the volume form s^{-n}, and thus has a symmetric Ricci tensor. Then one can construct the trace free part of that tensor:

$$\Delta : s \to s(\mathsf{Ric}^s - \frac{1}{n} g^s R^s),$$

where g^s is the metric corresponding to s, and R^s the scalar curvature. It turns out that this operator is linear, second order, and conformally invariant.

Proposition 1.3.5. *The kernel of*

$$\Delta : J^2(L^1) \to \odot^2 T_0^* \otimes L^1$$

is a rank $(n+2)$ *bundle that has an action of the Lie algebra bundle* $\mathcal{A}$ *and therefore admits a Tractor connection.*

We shall not prove this result, which is done in [CaGo3] and [CaGo2]. Instead, we shall *define* this bundle as

$$\mathcal{T}^* = \mathcal{G} \times_G \mathbb{R}^{(p+1,q+1)},$$

via the standard action of $G = SO(p+1, q+1)$ on $\mathbb{R}^{(p+1,q+1)}$. This bundle we shall call the dual Tractor Bundle; its dual $\mathcal{T} = (\mathcal{T}^*)^*$ will be the Tractor bundle, where we shall be performing most of our calculations in the conformal case.

In the projective case, similarly a section s of $L^{\frac{n}{n+1}}$ defines a preferred connection ∇, then it turns out that the operator

$$\Delta : s \to s.\big(sym(\mathsf{Ric}^s)\big)$$

is second order, linear and projectively invariant. In this case, since Δ is bijective on the included subbundle of $\odot^2 T^* \otimes L^{\frac{n}{n+1}}$ in $J^2(L^{\frac{n}{n+1}})$, one can identify

$$\text{Ker } \Delta \cong J^1(L^{\frac{n}{n+1}}).$$

It turns out that this kernel admits a $\mathcal{A}$ action. We thus identify the projective Tractor bundle with the dual $\left(J^1\left(L^{\frac{n}{n+1}}\right)\right)^*$. See [CaGo3] for details; from our perspective,

$$\mathcal{T}^* = \mathcal{G} \times_G \mathbb{R}^{n+1},$$

with $G = SL(n+1)$ acting on $\mathbb{R}^{n+1}$ in the usual fashion.

These two bundles we shall call the Tractor bundles for the conformal and projective geometries. There are other 'Tractor' bundles corresponding to different representations of G (most notably the adjoint representation, the exterior powers of the standard representations [Lei], and the twistor representation - see Appendix B; see also [CaGo2]), but we shall not need to use them in this thesis.

Chapter 2

The Conformal case

2.0.1 Flat model and associated algebras

In the conformal case, for signature (p, q), the flat model is the collection of null-lines in $\mathbb{R}^{(p+1,q+1)}$, and the Möbius transformations are all the isometries of $\mathbb{R}^{(p+1,q+1)}$. In other words

$$\begin{aligned} G = &\quad SO(p+1, q+1), \\ P = &\quad CO(p, q) \rtimes \mathbb{R}^{(p,q)}, \\ G_0 = &\quad\quad CO(p, q). \end{aligned}$$

2.0.2 Preferred connections

We must now prove the result left unproved in the last chapter, namely that every conformal connection with required torsion is a preferred connection. Remember that two preferred connections are related by a one-form Υ:

$$\nabla_X Y = \nabla'_X Y + [\Upsilon, X].Y.$$

Let ∇ be a preferred connection, and $\widehat{\nabla}$ a conformal connection with the same torsion. Then

$$\nabla = \widehat{\nabla} + \Psi$$

with Ψ a one-form with values in $\mathfrak{co}(p,q)$. Since ∇ and $\widehat{\nabla}$ have the same torsion, $\Psi \in \Gamma(\mathcal{H})$ where

$$\mathcal{H} = \left(\odot^2 T^* \otimes T\right) \cap \left(T^* \otimes \mathfrak{co}(p,q)\right).$$

However the uniqueness result for the Levi-Civita connection (in every signature) implies that

$$\left(\odot^2 T^* \otimes T\right) \cap \left(T^* \otimes \mathfrak{so}(p,q)\right) = 0.$$

Hence the rank of $\mathcal{H}$ is at most n. It suffices then to prove that the map

$$\begin{aligned} \phi : \Gamma(T^*) &\rightarrow \Gamma(\mathcal{H}) \\ \phi(\Upsilon)(X) &= [\Upsilon, X] \end{aligned}$$

is injective. But

$$[\Upsilon, X] = -Id\Upsilon(X) - \Upsilon \otimes X + g(X) \otimes g(\Upsilon),$$

using any metric g in the conformal class. Then we can see that

$$\text{trace } \phi(\Upsilon) = -n\Upsilon,$$

proving the injectivity of ϕ.

In fact this proves another result, namely that

Corollary 2.0.1. *For a given class of preferred connections, a choice of connection in that class is equivalent to a connection on any weighted bundle $L^a = (\wedge^n T)^{a/n}$, $a \neq 0$.*

Proof. Any preferred connection ∇ determines a connection on L^a. If l is a section of L^a and ∇'

another preferred connection differing from ∇ via Υ,

$$\nabla l = \nabla' l - a\Upsilon l.$$

So different preferred connections determine different connections on L^a. Conversely, any connection on L^a differs from ∇ by such an Υ, and so corresponds to the action of the preferred connection $\nabla + \phi(\Upsilon)$. ■

The next little lemma is an absolutely crucial one, though well known:

Lemma 2.0.2. *A torsion-free affine connection will preserve a volume form if and only if it has symmetric Ricci tensor.*

The action of the curvature of a connection on the volume bundle is known by taking the trace of the last terms. Then the proof comes directly from the first Bianchi identity:

$$\begin{aligned} R_{hj\ k}^{\ \ k} &= R_{kj\ h}^{\ \ k} + R_{hk\ j}^{\ \ k} \\ &= \mathsf{Ric}_{hj} - \mathsf{Ric}_{jh}. \end{aligned}$$

2.0.3 Tractor connection

The algebra bundle $\mathcal{A}$ has invariant inclusions and projections, and hence so does the tractor bundle $\mathcal{T}$. In details,

$$L^{-1} \subset T[-1] \oplus L^{-1} \subset \mathcal{T},$$

where $T[-1] = T \otimes L^{-1}$, and

$$\mathcal{T} \to L^1 \oplus T[-1] \to L^1.$$

Call π^1 the projection onto $L^1 \oplus T[-1]$, and π^2 that onto L^1. Now, given a choice of preferred connection ∇, there is a splitting of the algebra bundle $\mathcal{A}$ and hence of the Tractor bundle

$$\mathcal{T} = L^1 \oplus T[-1] \oplus L^{-1}.$$

Then the Tractor connection is $\overrightarrow{\nabla}_X = \nabla_X + X + \mathsf{P}(X)$, or, more explicitly,

$$\overrightarrow{\nabla}_X \begin{pmatrix} x \\ Y \\ z \end{pmatrix} = \begin{pmatrix} \nabla_X x - \mathbf{g}(X,Y) \\ \nabla_X Y + zX - x\mathsf{P}(X) \\ \nabla_X z + \mathsf{P}(X,Y) \end{pmatrix},$$

where $\mathbf{g}$ is the bilinear map $\odot^2 T[-1] \to \mathbb{R}$ determined by any metric g in the conformal class. In fact this map is the same whatever metric g we choose; thus we call $\mathbf{g}$ the weightless metric (though it is not a metric in the standard sense, being a bilinear form on $T[-1]$, not on T).

Given a section s of L^1, we can form the connection ∇ on L^1 by requiring

$$\nabla s = 0,$$

and consequently get a preferred connection ∇. Since ∇ preserves the volume form s^{-n}, it must be a metric Levi-Civita connection. The metric it corresponds to is

$$g^s = s^{-2}\mathbf{g}.$$

In future, when talking about preferred $\mathfrak{sl}(n)$ connections, we will just define them by s itself. Notice that by using s we may get an extra isomorphism

$$\mathcal{T} \cong \mathbb{R} \oplus T \oplus \mathbb{R}.$$

The formula for changing a splitting by a one-form Υ (see Theorem 1.3) is given explicitly by

$$\begin{pmatrix} x \\ Y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ Y + \Upsilon^* x \\ z - \Upsilon(Y) - \frac{1}{2}\mathbf{g}(\Upsilon,\Upsilon)x \end{pmatrix},$$

where $\Upsilon^* \in T[-2]$ is the dual to Υ using $\mathbf{g}$.

2.1 Einstein spaces

2.1.1 Reduced holonomy

A major and venerable result [Sas] in the case of conformal Tractor connections, is the fact that a preserved vector $v \in \Gamma(\mathcal{T})$ corresponds to an Einstein metric in the conformal class. We shall be requiring that $\pi^2(v) \neq 0$ to define this metric; the next lemma shows that we can expect this to be the case on 'most' of the manifold. But before, we shall define:

Definition 2.1.1. *There is an invariant metric h on $\mathcal{T}$, since $\overrightarrow{\nabla}$ has structure group $SO(p+1,q+1)$. Explicitly it is given by*

$$h < \begin{pmatrix} a \\ B \\ c \end{pmatrix}, \begin{pmatrix} x \\ Y \\ z \end{pmatrix} >= az + xc + \mathbf{g}(B,Y).$$

Definition 2.1.2. *From now on, we shall designate the canonical $L^{-1} \subset \mathcal{T}$ as E, and consequently the other canonical bundle bundle $T[-1] \oplus L^{-1}$ as $E^{\perp}$, using the Tractor metric h.*

Lemma 2.1.3. *If $K \subset \mathcal{T}$ is a bundle preserved by $\overrightarrow{\nabla}$, then both*

$$E = L^{-1} \quad \text{and} \quad E^{\perp} = T[-1] \oplus L^{-1}$$

are transverse to K off a set of strictly smaller dimension. As a consequence of this, $\pi^2(K) \neq 0$ on an open dense subset of M.

Proof of Lemma. First notice that the rank of E is one, whereas $E^{\perp}$ is of rank $n+1$ in a rank $n+2$ bundle. Hence being transverse to the first means that $K \cap E = 0$ (except where $K = \mathcal{T}$) and being transverse to the second implies that K is not contained in $E^{\perp}$.

To prove this lemma, it suffices to show that for any preserved $K \neq \mathcal{T}$,

$$K \cap E = 0$$

on an open, dense set. For then the bundle $K^{\perp}$ is also a preserved bundle, also with $K^{\perp} \cap E = 0$ on an open, dense set; consequently K is not contained in $E^{\perp}$, or equivalently is transverse to it.

Define the second fundamental form of $\overrightarrow{\nabla}$ on E:

$$\begin{aligned} S : E &\rightarrow T^* \otimes \left(\mathcal{T}/E \cong \pi^1(\mathcal{T})\right) \\ S(z) &= \pi^1(\overrightarrow{\nabla} z). \end{aligned}$$

Remembering that $\pi^1(\mathcal{T}) = E^{\perp} = T[-1] \oplus E$, $S(z)$ is just the map $X \rightarrow zX$. The fact that S is maximal in this way is a consequence of $\overrightarrow{\nabla}$ coming from a Cartan connection.

Hence the local parallel transport of the bundle E around any open set spans $E^{\perp}$. However, the second fundamental form S' of $E^{\perp}$ is given by

$$S'_X(Y, z) = -\mathbf{g}(X, Y).$$

Consequently, the local parallel transport of the bundle $E^{\perp}$ (and hence that of E) around any open set is all of $\mathcal{T}$. In other words, if K is preserved by local parallel transport, then $K \cap E$ cannot be non-zero on any open set. □

Before proving the general statement for Einstein metrics, it is as well to exclude the possibility of a preserved line subbundle without a preserved section:

Lemma 2.1.4. *Let $\mathcal{L}$ be a line bundle inside $\mathcal{T}$, preserved by $\overrightarrow{\nabla}$. Then $\overrightarrow{\nabla}$ preserves a section l of $\mathcal{L}$.*

Proof of Lemma. This result is trivially true if the metric h, when restricted to $\mathcal{L}$, is non-

degenerate; then one picks any section of constant norm. So now assume that $\mathcal{L}$ is a null bundle.

The projection π^2 maps $\mathcal{L}$ non-degenerately to L^1 (on an open, dense set of M), and is hence an isomorphism between them. Let σ be the inverse map $L^1 \to \mathcal{L}$. Then we may define a connection ∇ on L^1 by

$$\nabla s = \pi^2\big(\overrightarrow{\nabla}\sigma(s)\big).$$

It is easy to see that this is indeed a connection. Like any connections on L^1, it extends to a preferred connection on the tractor bundle. In the splitting determined by this connection, a section l of $\mathcal{L}$ is of the form (x, Y, z). However,

$$\begin{aligned} \nabla_X s &= \pi^2\big(\overrightarrow{\nabla}_X\sigma(s)\big) \\ &= \nabla_X s - \mathbf{g}(X, Y). \end{aligned}$$

Consequently, $Y = 0$, and since l is null, $z = 0$ too. Then since ∇ preserves $\mathcal{L} = \mathbb{R}.(x, 0, 0)$, one must have $\mathsf{P}^\nabla = 0$. In other words, ∇ is Ricci-flat, hence must preserve a volume form v. Setting

$$l = \begin{pmatrix} v^{\frac{-1}{n}} \\ 0 \\ 0 \end{pmatrix}$$

implies

$$\overrightarrow{\nabla} l = 0.$$

□

We may now turn back to the main theorem of this section.

Theorem 2.1.5. *If v is a section of the Tractor bundle with*

$$\overrightarrow{\nabla} v = 0$$

then the metric $(\pi^2(v))^{-2}\mathbf{g}$ *is an Einstein metric on the open, dense set where* $\pi^2(v) \neq 0$.

Conversely, if g *is an Einstein metric in the conformal class,*

$$g = \lambda \mathsf{Ric}^g.$$

Then in the splitting defined by the section $\det(g)^{\frac{1}{n}}$, *the vector* $v = (1, 0, \frac{-\lambda}{2n-2})$ *is preserved:*

$$\vec{\nabla} v = 0.$$

Proof. The converse is easy to see. Since g is Einstein,

$$\mathsf{P}_{hj} = -\frac{\lambda}{2n-2} g_{hj},$$

so

$$\vec{\nabla}_X v = \begin{pmatrix} 0 \\ \frac{\lambda}{2n-2} X - \frac{\lambda}{2n-2} X \\ 0 \end{pmatrix} = 0.$$

From now on, we shall restrict attention to the open dense set where $\pi^2(v)$ is non-zero.

Then the metric $g = (\pi^2(v))^{-2}\mathbf{g}$ defines a splitting of $\mathcal{T}$. In that splitting,

$$v = \begin{pmatrix} 1 \\ Y \\ z \end{pmatrix}$$

for some $Y \in \Gamma(T)$ and $z \in C^\infty(M)$. However, for all X,

$$0 = \vec{\nabla}_X v = \begin{pmatrix} -g(X, Y) \\ \nabla_X Y - \mathsf{P}(X) + zX \\ \nabla_X z + \mathsf{P}(X, Y) \end{pmatrix}.$$

Directly from this, $Y = 0$, and consequently z is a constant, and

$$\mathsf{P}(X) = zX$$

Hence $\mathsf{P} = zg$, and g is an Einstein metric with Einstein coefficient

$$\lambda = -(2n-2)z$$

■

Remark. Notice that this includes the Ricci-flat case $\lambda = z = 0$. Notice also that the sign of λ is the opposite of the square-norm of v.

Example 1. The classic examples of this are the various conformally Einstein metrics on the sphere S^n. The sphere is conformally flat, so there are many holonomy preserved sections of its Tractor bundle.

A preserved section u of negative norm corresponds to the Spherical metric $g = \pi^2(u)^{-2}\mathbf{g}$ on the whole space. In this case, $\pi^2(u)$ is nowhere zero.

A preserved section u of zero norm corresponds to the Euclidean metric $g = \pi^2(u)^{-2}\mathbf{g}$ on $\mathbb{R}^n \cong S^n \backslash \{\infty\}$. In this case, $\pi^2(u)(b) \neq 0$ for $b \neq \infty$.

A preserved section u of positive norm corresponds to the Hyperbolic metric $g = \pi^2(u)^{-2}\mathbf{g}$ on two half spheres of S^n. In this case $\pi^2(u)$ is zero only on the equator S^{n-1} cutting S^n into two.

One- and Two-dimensions

Any bundle must be flat in one-dimension, so this is the case with our 'Tractor' bundle. We further have

Proposition 2.1.6. *Any Tractor connection in two dimensions with a preserved Tractor u is flat.*

Proof. The preceding proof readily adapts, giving us a section $\pi^2(u)$ of L^1, a preferred connection ∇ and a splitting of the Tractor bundle corresponding to this section, and a tensor P such that

$$\mathsf{P} = Id.$$

However, [Cal], the only curvature element of a Tractor/Möbius connection in two dimensions is the Cotton-York tensor – which must vanish entirely, making the connection flat. ■

Remark. The ambiguity in the choice of P, mentioned in Section 1.3.2, is not a factor here, as the condition that v be preserved drastically reduces our freedom in choosing P.

2.1.2 Metric cones

The metric cone is the prime way of classifying the holonomy groups of $\overrightarrow{\nabla}$. It relates the holonomy of a conformally Einstein (not Ricci-flat) connection to that of a torsion-free affine connection on a cone one dimension higher than the manifold. This construction is related to the Ambient Metric construction [FeHi], and is a special case of the double cone construction *with torsion* of [ArLe].

Definition 2.1.7 (Cone). *A cone is an affine manifold $(\mathcal{C}, \nabla)$ with a special vector field Q such that:*

- ∇ *is torsion-free,*
- ∇ *is Q-invariant,*
- $\nabla Q = Id$.

By being Q invariant, we mean the definition of [KoNo]: that if $[X, Q] = [Y, Q] = 0$, then $[\nabla_X Y, Q] = 0$. Equivalently, ∇ is preserved by the action of the one parameter subgroup generated by Q.

Lemma 2.1.8. *The property of being Q-invariant may be replaced with the condition that all curvature terms involving Q vanish; these are equivalent, given the other two conditions.*

Proof of Lemma. $R_{-,-}Q = 0$ by definition. Now let X and Y be vector fields commuting with Q. Then

$$\begin{aligned} R_{Q,X}Y &= (\nabla_Q \nabla_X - \nabla_X \nabla_Q)Y \\ &= [Q, \nabla_X Y]. \end{aligned}$$

And that expression being zero is precisely what it means for ∇ to be Q invariant. □

Lemma 2.1.9. *If the connection ∇ is a Levi-Civita connection, then the condition $\nabla Q = Id$ is enough to ensure that $(\mathcal{C}, \nabla)$ is a cone.*

Proof of Lemma. A Levi-Civita connection is automatically torsion free. Moreover, since $R_{-,-}Q = 0$, the first Bianchi identity implies that

$$R_{Q,X}Y = R_{Q,Y}X.$$

So, since ∇ is metric,

$$R_{Q,-} \in \Gamma\big((T^* \odot T^* \otimes T^*) \cap (T^* \otimes T^* \wedge T^*)\big).$$

However that last bundle is zero. One can see this either by noticing that this bundle is $\mathfrak{so}(T^*)^{(1)}$, which must be zero by Table 4.1, or by direct calculation. Let Ψ be a section of this bundle, then

$$\begin{aligned}\Psi(X,Y,Z) &= -\Psi(X,Z,Y) = -\Psi(Z,X,Y) = +\Psi(Z,Y,X)\\ &= +\Psi(Y,Z,X) = -\Psi(Y,X,Z) = -\Psi(X,Y,Z).\end{aligned}$$

So $\Psi = 0$, and all curvature terms involving Q vanish, which, by Lemma 2.1.8, implies that $(\mathcal{C}, \nabla)$ is a cone. □

We can now construct the metric cone.

Theorem 2.1.10. *If (M, g) is an Einstein manifold with Einstein constant $\lambda \neq 0$, then the tractor holonomy of $\overrightarrow{\nabla}$ is equal to the holonomy of the Levi-Civita connection of $\widehat{h}$ on the (pseudo-Riemannian) cone:*

$$\begin{aligned}\mathcal{C}(M) &= \mathbb{R} \times M\\ \widehat{h} &= \exp(2q).(dq^2 - 2\mathsf{P}),\end{aligned}$$

where q is the coordinate along $\mathbb{R}$.

Proof. Let Q be the vector field along $\mathbb{R}$, and ∇ the Levi-Civita connection of g, $\widehat{\nabla}$ that of $\widehat{h}$. Then

if X and Y are sections of TM, $\widehat{\nabla}$ is given by:

$$\begin{aligned} \widehat{\nabla}Q &= Id \\ \widehat{\nabla}_Q X &= X \\ \widehat{\nabla}_X Y &= \nabla_X Y + 2\mathsf{P}(X,Y), \end{aligned} \tag{2.1}$$

since $\mathsf{P} = -\frac{\lambda}{2n-2}g$, implying that $\nabla\mathsf{P} = 0$. This is obviously a cone. Let ϕ be a path in $\{0\} \times M$ and $Y + aQ$ a parallel transported vector along ϕ; so

$$\widehat{\nabla}_{\phi'}(Y + aQ) = 0$$

Let μ be another path in $\mathcal{C}(M)$, such that μ and ϕ have same endpoints and same projection to M. Then

Lemma 2.1.11. *The parallel transport of vectors using $\widehat{\nabla}$ along the paths μ and ϕ are the same.*

Proof of Lemma. Extend $Y + aQ$ into the $\mathbb{R}$ direction as $\exp(-q)(Y + aQ)$. Then

$$\widehat{\nabla}_Q \exp(-q)(Y + aQ) = 0,$$

and

$$\widehat{\nabla}_X \exp(-q)(Y + aQ) = \exp(-q)\widehat{\nabla}_X(Y + aQ),$$

Consequently,

$$\widehat{\nabla}_{\mu'} \exp(-q)(Y + aQ) = 0,$$

as μ' and ϕ' have the same TM component. So the values of this at the end points of μ (where q must be zero) are the same as for those at the endpoints of ϕ. □

We've shown that the holonomy of $\widehat{\nabla}$ is generated by curves in $\{0\} \times M$. we now need to show that its holonomy is the same as $\overrightarrow{\nabla}$ on $\mathcal{T}$. Since g is Einstein, non-Ricci-flat, $\mathcal{T}$ has a preserved

non-null section v. So the holonomy of $\overrightarrow{\nabla}$ restricts to $v^{\perp}$. We aim to find an isomorphism

$$v^{\perp} \cong T\mathcal{C}(M).$$

In the splitting given by ∇,

$$v = \begin{pmatrix} 1 \\ 0 \\ \frac{-\lambda}{2n-2} \end{pmatrix}$$

so we make the identification

$$\begin{pmatrix} \frac{n-1}{\lambda} \\ 0 \\ \frac{1}{2} \end{pmatrix} \cong Q$$

and

$$\begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix} \cong X$$

Then one can see that $\overrightarrow{\nabla}$ and $\widehat{\nabla}$ are isomorphic connections along $M \cong \{0\} \times M$ given this equivalence of sections of $T\mathcal{C}(M)$ and $\mathcal{T}$:

$$\begin{aligned} \overrightarrow{\nabla}_Y \begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ \nabla_Y X \\ 0 \end{pmatrix} + 2\mathsf{P}(Y,X) \begin{pmatrix} \frac{n-1}{\lambda} \\ 0 \\ \frac{1}{2} \end{pmatrix} \\ &\cong \widehat{\nabla}_Y X, \end{aligned}$$

since

$$\mathsf{P} = \frac{-\lambda}{2n-2} g,$$

as ∇ is Einstein. Also

$$\overrightarrow{\nabla}_Y \begin{pmatrix} \frac{n-1}{\lambda} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ Y \\ 0 \end{pmatrix} \cong Y.$$

As a direct result of this and Lemma 2.1.11, $\overrightarrow{\nabla}$ and $\widehat{\nabla}$ must have the same holonomy. ■

We shall not however make use of this construction in this chapter. For it turns out that this construction is the same as the projective cone construction of Section 3.5 – for a projectively Einstein structure. Existence issues for various holonomies will be dealt with more fully in that context.

2.1.3 Ricci-flat spaces

The conformally Ricci-flat spaces have their own, simpler theory. Let g be the Ricci-flat metric, with (preferred) Levi-Civita connection ∇; then in the splitting it defines, the vector $(1, 0, 0)$ is covariantly constant. This implies that there is no T^* component in the Tractor holonomy bundle $\overrightarrow{\mathfrak{hol}}$ of $\overrightarrow{\nabla}$. More than that, if $\nabla_{\phi'(t)} Y = 0$ for some path ϕ,

$$\overrightarrow{\nabla}_{\phi'(t)} \begin{pmatrix} \int g(\phi(t), Y)dt \\ Y \\ 0 \end{pmatrix} = \begin{pmatrix} g(\phi(t), Y) - g(\phi(t), Y) \\ \nabla_{\phi'(t)} Y \\ 0 \end{pmatrix} = 0.$$

Recall that given ∇, the algebra bundle of $\overrightarrow{\nabla}$ splits as

$$\mathcal{A} = T^* \oplus \mathfrak{co}(T) \oplus T.$$

Hence if $\mathfrak{hol}^\nabla$ is the affine holonomy bundle of ∇, it is the projection π^1 of $(\overrightarrow{\mathfrak{hol}})$ onto the $\mathfrak{co}(T)$ component (since there is no T^* component). Now, without T^* component, $\pi^1 : \overrightarrow{\mathfrak{hol}} \to \mathfrak{co}(T)$ is an algebra homomorphism. Consequently, there is an inclusion ι of $\mathfrak{hol}^\nabla$ into $\overrightarrow{\mathfrak{hol}}$ as a subalgebra

bundle. In other words

$$\iota(\mathfrak{hol}^\nabla) \subset \overrightarrow{\mathfrak{hol}} \subset \mathfrak{hol}^\nabla \oplus T.$$

In the last case, the Lie bracket is given by being standard on $\mathfrak{hol}^\nabla$, trivial on T, and the natural action of $\mathfrak{hol}^\nabla$ on T in cross terms (see Section 1.3.1).

We shall assume that $\mathfrak{hol}^\nabla$ acts irreducibly on T (if not, then we are in the case of the decomposition theorem, see Section 2.2). Then $\mathfrak{hol}^\nabla \oplus T$ decomposes into two irreducible components under the action of $\mathfrak{hol}^\nabla$, namely $\iota(\mathfrak{hol}^\nabla)$ and T (since the action of $(0,h,X)$ on T is equal to that of $(0,h,0)$, for any $h \in \mathfrak{hol}^\nabla$). Consequently

$$\overrightarrow{\mathfrak{hol}} = \iota(\mathfrak{hol}^\nabla) \text{ or } \overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^\nabla \oplus T.$$

Lemma 2.1.12. *$\overrightarrow{\mathfrak{hol}} = \iota(\mathfrak{hol}^\nabla)$ if and only if (M,∇) is a cone in the sense of Definition 2.1.7.*

Proof of Lemma. If (M,∇) is a cone, there exists a vector field Q such that $\nabla Q = Id$. Now $g(Q)$ is a one-form such that $\nabla g(Q) = g$. Since ∇ is torsion-free and g is symmetric, this means that $g(Q)$ is a closed one-form. So, locally, there exists a function α with $d\alpha = g(Q) = \nabla\alpha$. This means that the section

$$\begin{pmatrix} \alpha \\ Q \\ -1 \end{pmatrix}$$

is preserved by $\overrightarrow{\nabla}$. So $\overrightarrow{\mathfrak{hol}}$ cannot contain T.

Conversely, imagine that $\overrightarrow{\mathfrak{hol}} = \iota(\mathfrak{hol}^\nabla)$. Then $\iota(\mathfrak{hol}^\nabla)$ preserves the bundle $T[-1] \oplus L^{-1} \subset \mathcal{T}$. The action of $\iota(\mathfrak{hol}^\nabla)$ commutes with the natural projection

$$T[-1] \oplus L^{-1} \to T[-1].$$

Consequently, since $\iota(\mathfrak{hol}^\nabla)$ is reductive and acts reducibly, $\mathcal{T}$ must contain one summand equal to $T[-1]$, as well as a preserved section; thus $\iota(\mathfrak{hol}^\nabla)$ must preserve a second section, say v. Since the

other bundles span $T[-1] \oplus L^{-1}$, one must have $\pi^2(v) \neq 0$. Since v is preserved, $\pi^2(v)$ is a constant. Scale it to be minus one, so that v is of the form

$$v = \begin{pmatrix} \alpha \\ Q \\ -1 \end{pmatrix}.$$

The condition that v be preserved by $\overrightarrow{\nabla}$ translates to requiring that $\nabla Q = Id$. Then since ∇ is a Levi-Civita connection, Lemma 2.1.9 implies that (M, ∇) is a cone. □

We have consequently proved the theorem:

Theorem 2.1.13. *If (M, g) is conformally equivalent to a Ricci-flat manifold whose affine holonomy $\mathfrak{hol}^{\nabla}$ acts irreducibly on T, and is not a cone in the sense of Definition 2.1.7, then its Tractor holonomy is*

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^{\nabla} \oplus T$$

where the Lie bracket is standard on $\mathfrak{hol}^{\nabla}$, trivial on T, and the action of $\mathfrak{hol}^{\nabla}$ on T for cross terms. If (M, g) is conformally equivalent to a Ricci-flat cone, then

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^{\nabla}.$$

We have already seen, in Section 2.1.2, that an Einstein cone has the same affine holonomy as the Tractor holonomy of its underlying manifold. But this theorem shows that the Einstein cone has Tractor holonomy also equal to these two holonomies.

2.2 Decomposition theorem

2.2.1 Preliminaries

Definition 2.2.1. *Given a (pseudo-)Riemannian metric g on M with Levi-Civita connection ∇, a subbundle $U \subset T$ is* umbilical *for the connection ∇, if for any sections X and Y of U,*

$$\nabla_X Y = \widetilde{\nabla}_X Y + g(X,Y)H,$$

for $\widetilde{\nabla}$ some connection on U, and H a vector field.

Remark. Note that an umbilical subbundle is automatically integrable, as

$$\begin{aligned}[X,Y] = \nabla_X Y - \nabla_Y X &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X + \big(g(X,Y) - g(Y,X)\big)H \\ &= \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X,\end{aligned}$$

a section of U.

Lemma 2.2.2. *U being umbilical is equivalent to*

$$\nabla_X Y \in \Gamma(U), \tag{2.2}$$

whenever X and Y are orthogonal sections of U.

Proof of Lemma. If U is umbilical, then Equation (2.2) is true by definition

$$\begin{aligned}\nabla_X Y &= \widetilde{\nabla}_X Y + g(X,Y)H \\ &= \widetilde{\nabla}_X Y \in \Gamma(U).\end{aligned}$$

So we now assume Equation (2.2) and aim to prove umbilicity.

Fix a section σ of $T \to T/U$, and consequently a connection $\widetilde{\nabla}$ on U, by projecting along the image of σ. Then the map $\Phi = \nabla - \widetilde{\nabla}$ is bilinear, $U^* \otimes U^* \to \sigma(T/U)$, and symmetric since $\nabla_X Y + \nabla_Y X$ is a section of U. By assumption, $\Phi(X,Y) = 0$ whenever $g(X,Y) = 0$. This implies

that $\Phi(X,-)=0$ whenever$X \in \Gamma(U \cap U^{\perp})$. Consequently Φ is a section of $\odot^2 V^* \otimes \sigma(T/U)$, where $V = U/(U \cap U^{\perp})$, and g descends to a non-degenerate metric on V.

Now let (X_j) be a frame of V, chosen so that the $g(X_j, X_k)$ are nowhere zero (one can do this, for instance, by choosing a standard orthonormal frame (X_j) and mapping $X_j \to X_j + \frac{1}{2n}\sum_{l=1}^{j} X_l$). Pick H in $\sigma(T/U)$ such that $\Phi(X_1, X_1) = g(X_1, X_1)H$. Then since X_1 is orthogonal to $\tau_{1,1,j} = g(X_1, X_1)X_j - g(X_j, X_1)X_1$, one has $\Phi(X_1, \tau) = 0$ and hence

$$\begin{aligned}\Phi(X_1, X_j) &= \frac{1}{g(X_1, X_1)} g(X_j, X_1)\big(g(X_1, X_1)H\big) \\ &= g(X_j, X_1)H.\end{aligned}$$

The same argument with the orthogonal sections $\tau_{j,1,k}$ and X_j demonstrates

$$\Phi(X_j, X_k) = g(X_j, X_k)H.$$

This extends trivially to the whole of U. Thus $\nabla_X Y = \widetilde{\nabla}_X Y + \Phi(X,Y) = \widetilde{\nabla}_X Y + g(X,Y)H$ in general, proving the result. □

Note that if we change ∇ by $\Upsilon^* = H$, we can make U into a totally geodesic foliation. In other words, there is a class of preferred connections for which U is totally geodesic, differing by Υ's such that Υ^* is a section of U.

Definition 2.2.3 (Induced conformal structure). *If $U \subset TM$ is an umbilical foliation, then there is an induced conformal structure on any leaf N of U (i.e. $U|_N = TN$). It is given by the collection of preferred connections ∇ for which U is totally geodesic. Call these the U-preferred connections; they restrict to connections on N, as N is totally geodesic. If U is non-degenerate with respect to the conformal structure, this is equivalent to restricting the metrics of TM to N.*

2.2.2 Preserved subbundles

Let K be a subbundle of $\mathcal{T}$ of rank k, preserved by $\overrightarrow{\nabla}$. Then K defines a sub-bundle U of T as follows. We assume, from Lemma 2.1.3, that K is locally transverse to both E and $E^{\perp}$. Recall that $E \cong L^{-1} \subset \mathcal{T}$ is the canonical line bundle, and that $E^{\perp} \cong TM[-1] \oplus L^{-1}$ is of rank $n+1$ in $\mathcal{T}$.

Hence, $K \cap E^{\perp}$ is a bundle of rank $k-1$, and π^1 is injective on $(K \cap E^{\perp})$ (since $K \cap E = 0$, so π^1 is injective on K). Moreover $\pi^1(E^{\perp}) = T[-1]$, so

$$U = \pi^1(K \cap E^{\perp}) \subset T[-1]$$

is a well defined, rank $k-1$ bundle. Use any section of L^1 to get the isomorphism $T \cong T[-1]$. Since changing the section simply results in scaling any element of $T[-1]$, we may see U as a well-defined subbundle of T.

Theorem 2.2.4. *U is an integrable, umbilical foliation of T. Moreover, U is either Einstein (i.e all leaves N of U are conformally Einstein under the restricted conformal structure), lightlike or has an extra preserved lightlike subfoliation $V \subset U$ with the same properties as U.*

Proof. Let X and Y be orthogonal sections of U. Fix any metric in the conformal class. Then

$$\begin{pmatrix} 0 \\ Y \\ z \end{pmatrix},$$

is a section of $K \cap E^{\perp}$ for some z. Then

$$\vec{\nabla}_X \begin{pmatrix} 0 \\ Y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \nabla_X Y + zX \\ z' \end{pmatrix},$$

for some z'. Since K is preserved by $\vec{\nabla}$, this is a section of K; it is clearly a section of $E^{\perp}$. As a consequence, we know that

$$\nabla_X Y + zX \quad \in \quad \Gamma(U).$$

Thus $\nabla_X Y$ is also a section of $\Gamma(U)$, making U umbilical, and hence integrable.

Going back to the initial bundle K for a second, note that there must be an orthogonal preserved

bundle $K^\perp$. Then we may as before construct the subbundle of T:

$$\pi^1(E^\perp \cap K^\perp).$$

Since the metric h restricts to $\mathbf{g}$ on $E^\perp$, one can see that this bundle is simply $U^\perp$. There are three cases to consider:

- $U = U^\perp$.

  Then U is lightlike.

- $U \cap U^\perp = 0$.

  Then U splits T, as defined in the next section, and, by Theorem 2.2.6, must be conformally Einstein.

- $U \cap U^\perp = V$, with $V \neq 0$ and $V \neq U$.

  In this case, $V = \pi^1\left((K \cap K^\perp) \cap E^\perp\right)$. And $K \cap K^\perp$ is a $\overrightarrow{\nabla}$-preserved bundle, so V has the same properties as U. On top of this, V is lightlike from its definition.

■

Proposition 2.2.5. *There is a Tractor bundle $\mathcal{T}_U$ on the leaves N of the foliation defined by U, and a well-defined inclusion $\mathcal{T}_U \subset \mathcal{T}$.*

Proof. If ∇ is a U-preferred connection – one that makes U, and its foliation, totally geodesic – in the splitting of $\mathcal{T}$ that it defines,

$$\mathcal{T} = L^1 \oplus T[-1] \oplus L^{-1}.$$

Define $\mathcal{T}_U$ as the subbundle

$$\mathcal{T}_U = L^1 \oplus U[-1] \oplus L^{-1}.$$

To check this is well defined, we change ∇ to ∇', another U-preferred connection. This is equivalent to changing ∇ by an $\Upsilon \in \Gamma(g(U) \subset T^*)$ for any metric g in the conformal class. Then the splitting

changes as:

$$\begin{pmatrix} x \\ Y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ Y + \Upsilon^* x \\ z - \Upsilon(Y) - \frac{1}{2}\mathbf{g}(\Upsilon, \Upsilon)x \end{pmatrix},$$

which, since Υ^* is a section of U, does not change the definition of $\mathcal{T}_U$ nor its inclusion into $\mathcal{T}$. ■

Remark. The bundle $\mathcal{T}_U$ is preserved by $\overrightarrow{\nabla}$ along directions in U – since the Ricci-tensor, hence the rho-tensor, restricts on totally geodesic foliations like U – but there is no reason to suppose that the action of $\overrightarrow{\nabla}$ is the same as that of the Tractor connection $\overrightarrow{\nabla}_U$ of the leaves of U themselves.

2.2.3 Split bundle

A bundle U splits T if

$$U \oplus U^{\perp} = T,$$

equivalently, if $U \cap U^{\perp} = 0$. *We shall not consider non-split bundles any further*, as the methods used in this thesis are insufficient to deal with the non-split case. In the split case, we aim to prove the following theorem:

Theorem 2.2.6. *Assume there is a bundle K of rank k preserved by $\overrightarrow{\nabla}$, and the foliation U that it generates splits T. Let $l = k - 1$ be the rank of U. Then there exists a metric g in the conformal class of M such that the manifold (M, g) splits locally as the direct product*

$$(M, g) = (N_1, h_1) \times (N_2, h_2)$$

where h_1 and h_2 are Einstein metrics with Einstein coefficients λ_1, λ_2, possibly zero, related by

$$(n - l - 1)\lambda_1 = (1 - l)\lambda_2.$$

The converse is also true. And in this situation the holonomy $\overrightarrow{\mathfrak{hol}}$ of $\overrightarrow{\nabla}$ is the direct sum of Lie

algebras

$$\overrightarrow{\mathfrak{hol}} = \overrightarrow{\mathfrak{hol}}_{N_1} \oplus \overrightarrow{\mathfrak{hol}}_{N_2}$$

where $\overrightarrow{\mathfrak{hol}}_{N_1}$ *is the holonomy of* $\overrightarrow{\nabla}_{N_1}$ *and* $\overrightarrow{\mathfrak{hol}}_{N_2}$ *that of* $\overrightarrow{\nabla}_{N_1}$.

There are really two situations here: the case when $K \cap K^\perp$ is of rank one, and that where it is of rank zero (in all other cases, $U \cap U^\perp \neq 0$).

K degenerate

If $K \cap K^\perp = \mathcal{L}$, a line bundle, necessarily null, then by Lemma 2.1.4 there must be a preserved section l of $\mathcal{L}$ and hence a Ricci-flat metric g on M, with Levi-Civita connection ∇.

Now we have the bundles U and $U^\perp$ as before, both integrable and umbilical.

Lemma 2.2.7. *Let X be a section of U. Then for any $A \in \Gamma(T)$, $\nabla_A X$ is a section of U.*

Proof of Lemma. In the splitting defined by g, one section of K is the Einstein vector

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Since v is also a section of $K^\perp$, K must lie in $v^\perp$. In other words, K is of the form

$$\begin{pmatrix} \mathbb{R} \\ U \\ 0 \end{pmatrix}.$$

Now consider

$$\overrightarrow{\nabla}_A \begin{pmatrix} 0 \\ X \\ 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{g}(A, X) \\ \nabla_A X \\ 0 \end{pmatrix}.$$

Since $\overrightarrow{\nabla}$ preserves K, $\nabla_A X$ must be a section of U. □

This shows that U (and $U^\perp$) are totally geodesic foliations. Moreover, they are preserved by ∇ in every direction.

Remark. As a consequence of that, if X and B are commuting sections of U and $U^\perp$ respectively,

$$\nabla_X B = \nabla_B X = 0.$$

Let $h_1 = g|_U$, Y and X be sections of U, A any section of T. Then

$$\begin{aligned}(\nabla_A h_1)(X,Y) &= A.h_1(X,Y) - h_1(\nabla_A X, Y) - h_1(X, \nabla_A Y)\\ &= A.g(X,Y) - g(\nabla_A X, Y) - g(X, \nabla_A Y)\\ &= (\nabla_A g)(X,Y)\\ &= 0,\end{aligned}$$

as $\nabla_A X$ and $\nabla_A Y$ are sections of U, and $h_1 = g$ on sections of U. Consequently we have demonstrated, for h_1 and for $h_2 = g|_{U^\perp}$:

Lemma 2.2.8. *∇h_1 and ∇h_2 are both zero.*

Now pick sections X and Y of U commuting with a section B of $U^\perp$. By the previous lemma

$$B.h_1(X,Y) = 0,$$

so the Lie derivative of h_1 in the direction of B is

$$\big(\mathcal{L}_B h_1\big)(X,Y) = B.h_1(X,Y) - h_1([B,X],Y) - h_1(X,[B,Y]) = 0.$$

We may choose local coordinates that respect the foliations U and $U^\perp$ to get frames (X^j) of U and (B^k) of $U^\perp$, commuting with one-another. Consequently, if N_1 is a leaf of U and N_2 a leaf of $U^\perp$, h_1 is preserved by translation along N_2 and vice-versa. This demonstrates that

Proposition 2.2.9. *Locally, $(M,g) = (N_1,h_1) \times (N_2,h_2)$.*

This implies that $\nabla|_U$ is the Levi-Civita connection of h_1, and $\nabla|_{U^\perp}$ that of h_2. To finish this exploration, we require:

Lemma 2.2.10 (Restricted Ricci curvature). *Given a foliation U preserved by ∇, the Ricci tensor of $\nabla|_U$ is the Ricci tensor of ∇, restricted to U.*

Proof of Lemma. Notice that this condition makes U integrable and totally geodesic. Let $(X_j), (B_j)$ be a coordinate frame for T, with $X_j \in \Gamma(U)$ and the (B_j) complementary. Then

$$\mathsf{Ric}(X_j, X_k) = \left(\sum_l X_l^* \lrcorner R_{X_l, X_j} X_k\right) + \left(\sum_l B_l^* \lrcorner R_{B_l, X_j} X_k\right).$$

But the second term on the right is zero, as $R_{-,-}X_j$ must be a section of U, and the first term is just the Ricci curvature of ∇_U. □

Consequently, one can see that ∇ is Ricci-flat on U and on $U^\perp$ (hence on N_1 and N_2).

In this case the relation

$$(n-l-1)\lambda_1 = (1-l)\lambda_2.$$

is trivially satisfied, as both λ_j are zero. The converse to this construction is trivial: a direct product of Ricci-flat spaces is Ricci-flat. Then K may be reconstructed as

$$K = \begin{pmatrix} \mathbb{R} \\ TN_1 \\ 0 \end{pmatrix}$$

in the global Ricci-flat metric's splitting. Since TN_1 must be totally geodesic, $\overrightarrow{\nabla}$ preserves K and

$$K^\perp = \begin{pmatrix} \mathbb{R} \\ TN_2 \\ 0 \end{pmatrix}.$$

Now notice that since all P are zero, $\overrightarrow{\nabla}$ acts on $\mathcal{T}_{N_1}$ along N_1 exactly as the Tractor connection $\overrightarrow{\nabla}_{N_1}$ does. Moreover, $\overrightarrow{\nabla}$ acts trivially on $\mathcal{T}_{N_1}$ along N_2. Since the opposite result holds for $\mathcal{T}_{N_2}$, and since

these two tractor bundles span all of $\mathcal{T}$, one has

$$\overrightarrow{\mathfrak{hol}} = \overrightarrow{\mathfrak{hol}}_{N_1} \oplus \overrightarrow{\mathfrak{hol}}_{N_2}.$$

K non-degenerate

We seek to imitate the proofs of the previous section in the case where $K \cap K^\perp = 0$. First of all, we seek to find an imitation of the Ricci-flat metric g. We shall use a preferred connection rather than a metric – though it will turn out to be a metric connection in the end.

Starting off, pick ∇' such that U is totally geodesic. In the rest of these proofs, X and Y will be sections of U, B and C sections of $U^\perp$.

Since $U^\perp$ is umbilical,

$$\nabla'_B C = \widetilde{\nabla}'_B C + H\tilde{g}(B, C),$$

for some $H \in \Gamma(U)$ and any metric $\tilde{g}$ in the conformal class. Then replace ∇' with ∇, by adding the one-form $\Upsilon = \tilde{g}(H)$. This connection makes $U^\perp$ totally geodesic, but since

$$\nabla_X Y = \nabla'_X Y + \Upsilon(X)Y + \Upsilon(Y)X - H\tilde{g}(X, Y)$$

is a section of U, then the bundle U remains totally geodesic under ∇. In fact ∇ is the sole preferred connection that makes U and $U^\perp$ totally geodesic – as adding any $\Upsilon \neq 0$ would destroy this property on at least one of these bundles.

Now we try and calculate K and $K^\perp$ in the splitting given by ∇. We know that elements of $K \cap E^\perp$ are of the form

$$\begin{pmatrix} 0 \\ X \\ z \end{pmatrix},$$

for some $z \in \Gamma(L^{-1})$ depending on X, and elements of $K \cap E^{\perp}$ are of the form

$$\begin{pmatrix} 0 \\ B \\ z' \end{pmatrix}.$$

Hence, choosing Y such that Y and X are not orthogonal,

$$\overrightarrow{\nabla}_Y \begin{pmatrix} 0 \\ X \\ z \end{pmatrix} = \begin{pmatrix} -\mathrm{g}(Y,X) \\ \nabla_Y X - zY \\ z'' \end{pmatrix}$$

now the middle piece is a section of U as well, so there exists a section

$$v_1 = \begin{pmatrix} a \\ 0 \\ z'' \end{pmatrix}$$

in K, with $a \neq 0$. Since $K^{\perp}$ must be orthogonal to this vector, $K^{\perp} \cap E^{\perp}$ must be of the form

$$\begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix},$$

and the similar result goes for $K \cap E^{\perp}$. Consequently, as before, we have

Lemma 2.2.11. *For any $A \in \Gamma(T)$, $\nabla_A X$ is a section of U.*

We may, as before, choose frames (X^j) and (B^k) for these bundles such that the frames commute. Then

$$\nabla_{X^j} B^k = \nabla_{B^k} X^j = 0.$$

This implies that the curvature tensor of ∇ splits into two components, its curvature on U and its curvature on $U^{\perp}$. The Ricci-tensor does the same, (see Lemma 2.2.10), as does the rho-tensor, since

U and $U^\perp$ are orthogonal. So

$$\mathsf{P} = \mathsf{P}_1 + \mathsf{P}_2.$$

We now aim to prove:

Lemma 2.2.12. *The connection ∇ is metric.*

Proof of Lemma. Consider the section v in K, and

$$\vec{\nabla}_B v = \begin{pmatrix} \nabla_B a \\ z''B + a\mathsf{P}(B) \\ \nabla_B z'' \end{pmatrix}.$$

The middle term $z''B + a\mathsf{P}(B) = z''B + a\mathsf{P}_2(B)$ must be zero, showing that $\mathbf{g}^{jk}(\mathsf{P}_2)_{ij}$ is some multiple of the identity – hence that P_2 is a symmetric tensor. As the same is true of P_1, ∇ has symmetric rho-tensor, hence symmetric Ricci-tensor, hence preserves a volume form, hence preserves a metric g in the conformal class. □

Defining $h_1 = g|_U$, $h_2 = g|_{U^\perp}$, one can, exactly as in Proposition 2.2.9, get the proof of the decomposition:

Proposition 2.2.13. *Locally, $(M, g) = (N_1, h_1) \times (N_2, h_2)$, where N_1 is a leaf of U and N_2 is a leaf of $U^\perp$.*

Moreover, we've shown that P_1 and P_2 are multiples of h_1 and h_2 respectively; consequently Ric_1 and Ric_2 are as well, so both N_1 and N_2 are Einstein manifolds, with coefficients λ_1 and λ_2. We now aim to show the relation between these coefficients.

The scalar curvature R of ∇ is $l\lambda_1 + (n-l)\lambda_2$. Hence the rho-tensor, by Equation (1.4), is:

$$\begin{aligned} \mathsf{P}_1 &= -\frac{1}{n-2}\left(\mathsf{Ric}_1 - \frac{1}{2n-2}Rh_1\right) \\ &= -\frac{(2n-2-l)\lambda_1 + (l-n)\lambda_2}{(n-2)(2n-2)}h_1. \end{aligned}$$

$$\begin{aligned} \mathsf{P}_2 &= -\frac{1}{n-2}\left(\mathsf{Ric}_2 - \frac{1}{2n-2}Rh_2\right) \\ &= -\frac{(-l)\lambda_1 + (n-2+l)\lambda_2}{(n-2)(2n-2)}h_2. \end{aligned}$$

Now there is a section

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ f \end{pmatrix}$$

of K (we may freely use 1, as we have established that ∇ is metric, hence got an isomorphism $L^1 \cong \mathbb{R} \times M$), and a corresponding section

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ f' \end{pmatrix}$$

of $K^{\perp}$. Since v_2 is orthogonal to v_1, $f' = -f$. Then

$$\overrightarrow{\nabla}_B v_1 = \begin{pmatrix} 0 \\ fB - \mathsf{P}_2(B) \\ \nabla_B f \end{pmatrix}$$

as a consequence of this, we see that f is a constant and

$$f = -\frac{(2n-2-l)\lambda_1 + (l-n)\lambda_2}{(n-2)(2n-2)}.$$

carrying out a similar operation on v_2 yields the following formula

$$f = \frac{(-l)\lambda_1 + (n-2+l)\lambda_2}{(n-2)(2n-2)}.$$

Equating these terms and re-arranging gives us the required

$$(n-l-1)\lambda_1 = (1-l)\lambda_2.$$

There is, however, a rather more fundamental reason for this seemingly arbitrary equality. For:

Proposition 2.2.14. *The condition*

$$(n-l-1)\lambda_1 = (1-l)\lambda_2.$$

is equivalent to the rho-tensor P_{N_1} *of* $\nabla|_{N_1}$ *being equal to the restriction of the rho-tensor on* M*,*

$$\mathsf{P}_{N_1} = \mathsf{P}|_U = \mathsf{P}_1.$$

Proof.

$$\begin{aligned}\mathsf{P}_1 - \mathsf{P}_{N_1} &= \left(-\frac{(2n-2-l)\lambda_1 + (l-n)\lambda_2}{(n-2)(2n-2)} - \frac{-\lambda_1}{2(l-1)}\right) h_1 \\ &= ((n-l-1)\lambda_1 - (1-l)\lambda_2)\left(\frac{(n-l)}{(l-1)(n-2)(2n-2)}\right) h_1\end{aligned}$$

Similarly

$$\begin{aligned}\mathsf{P}_2 - \mathsf{P}_{N_2} &= \left(-\frac{(-l)\lambda_1 + (n-2+l)\lambda_2}{(n-2)(2n-2)} - \frac{-\lambda_2}{2(n-l-1)}\right) h_2 \\ &= ((n-l-1)\lambda_1 - (1-l)\lambda_2)\left(\frac{l}{(l-1)(n-2)(2n-2)}\right) h_2\end{aligned}$$

Consequently, $\mathsf{P}_1 = \mathsf{P}_{N_1}$ if and only if $\mathsf{P}_2 = \mathsf{P}_{N_2}$, and if and only if $(n-l-1)\lambda_1 = (1-l)\lambda_2$. ■

This is the essence of the decomposition: because of this result, $\overrightarrow{\nabla}$ operates on $\mathcal{T}_{N_1}$ along $TN_1 = U$ just as the reduced Tractor connection $\overrightarrow{\nabla}_{N_1}$ does. Now let v_2 be the Einstein vector in $\mathcal{T}_{N_1}$; then $\overrightarrow{\nabla}$ along $TN_2 = U^\perp$ will operate trivially on

$$K = v_2^\perp \cap \mathcal{T}_{N_1},$$

since K is the sum of elements of $(0, X, 0)$ and v_1. Consequently the holonomy algebra of $\overrightarrow{\nabla}$ restricted to K is $\overrightarrow{\mathfrak{hol}}_{N_1}$.

The similar result holds for $K^\perp$. Thus, since $K \oplus K^\perp = \mathcal{T}$,

$$\overrightarrow{\mathfrak{hol}} = \overrightarrow{\mathfrak{hol}}_{N_1} \oplus \overrightarrow{\mathfrak{hol}}_{N_2}.$$

To reverse this decomposition, define (M, g) as $(N_1, h_1) \times (N_2, h_2)$ with N_1 and N_2 Einstein with Einstein coefficients related as above. Then the overall Tractor connection $\overrightarrow{\nabla}$ will be generated by $\overrightarrow{\nabla}_{N_1}$ and $\overrightarrow{\nabla}_{N_2}$ as above. Then let v_2 be the Einstein vector of $\mathcal{T}_{N_1}$. Then the bundle

$$K = v_2^\perp \cap \mathcal{T}_{N_1},$$

is preserved by $\overrightarrow{\nabla}$ as is its orthogonal complement

$$K = v_1^\perp \cap \mathcal{T}_{N_2},$$

where v_1 is the Einstein vector of $\mathcal{T}_{N_2}$. Note that $v_1 \in \Gamma(K)$ and $v_2 \in \Gamma(K^\perp)$, which explains the somewhat odd numbering of them.

2.2.4 Definite signature

In conclusion from the preceding, once we have the classification of conformally Einstein holonomy of chapter 5, we shall have a full classification for all those holonomy algebras $\overrightarrow{\mathfrak{hol}}$ preserving a bundle K such that $K \cap K^\perp$ is of rank zero or one. In the definite signature case, we are looking at subalgebras of

$$\mathfrak{so}(n+1, 1),$$

so $K \cap K^\perp$ has rank at most one, giving a full classification of all reducible holonomies in this case. But paper [DiOl] shows that there are no proper subalgebras of $\mathfrak{so}(m, 1)$ acting irreducibly on $\mathbb{R}^{(m,1)}$. In other words, once we have shown the following proposition, we have a full classification in the

definite signature case.

Proposition 2.2.15. *For $p+q>3$, there exist conformal manifolds with Tractor holonomy*

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{so}(p+1,q+1).$$

Proof. We assume that $p>q$ as $\mathfrak{so}(p+1,q+1) = \mathfrak{so}(q+1,p+1)$. Given an inner-product space $\mathbb{R}^{(s,t)}$, we define the quadric

$$S^{(s,t)}(a) = \{x \in \mathbb{R}^{(s,t)} | g(x,x) = a\}.$$

The standard spheres are included in this picture as $S^n = S^{(n+1,0)}(1)$. We may assume $a>0$, as $S^{(s,t)}(a) = S^{(t,s)}(-a)$.

Now $S^{(s,t)}(a)$ is an Einstein manifold with a metric of signature $(s-1,t)$ and positive Einstein coefficient. The $S^{(s,t)}(a)$ are also conformally flat (pick a point on $S^{(s,t)}(a)$ and use it to do a conformally invariant stereographic projection onto a flat manifold).

Now consider the product

$$M \quad = \quad S^{(p-1,q)}(1) \times S^{(3,0)}(2).$$

This manifold is not Einstein (as the 'radii' of the spheres are different) and is of signature (p,q).

Definition 2.2.16 (Indecomposable). *A manifold is said to be* indecomposable *if there are no $\overrightarrow{\nabla}$-preserved bundles K of rank $k>1$.*

A symmetric space (S,g) is a manifold such that $\nabla^g R^g = 0$ for R^g the full curvature tensor. It is quite easy to show, using the infinitesimal holonomy developed by S. Kobayashi and K. Nomizu [KoNo], that any indecomposable conformal manifold that is conformal to a symmetric space has maximal Tractor holonomy in its category i.e. $\mathfrak{so}(p+1,q+1)$ if it is not Einstein, $\mathfrak{so}(p,q+1)$ or $\mathfrak{so}(p+1,q)$ if it is.

Now M is a symmetric space, indecomposable as the Einstein coefficients of $S^{(p-1,q)}(1)$ and

$S^{(3,0)}(2)$ are both positive non-zero, hence

$$0 < (n-l-1))\lambda_1 \neq (1-l)\lambda_2 < 0,$$

as $l = p+q-2$. So since M is non-Einstein, it has maximal holonomy $\mathfrak{so}(p+1, q+1)$. ■

Notice that this argument does not work when $p+q=3$, as M is then decomposable as a conformal manifold – and any symmetric 3-space is conformally flat. The existence of a conformal 3-fold with maximal Tractor holonomy is the subject of the next proposition.

Proposition 2.2.17. *There exist conformal manifolds with Tractor holonomy*

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{so}(p+1, q+1),$$

where $p+q=3$.

Proof. In three dimensions, the conformal Weyl tensor vanishes, [Wey2] (this is easy to see by comparing the amount of independent components of the Riemann and Ricci curvatures), and the full obstruction to integrability is carried by the last piece of the Tractor curvature, the Cotton-York tensor, which is consequently conformally invariant.

Lemma 2.2.18. *If a definite signature three-manifold has non-vanishing Tractor holonomy, it has full* $\mathfrak{so}(4,1)$ *Tractor holonomy.*

Proof of Lemma. $\mathfrak{so}(4,1)$ has no subalgebra acting irreducibly on $\mathbb{R}^{(4,1)}$, see [DiOl]. If the algebra reduces, the manifold is either Einstein (which means its Cotton-York tensor vanishes, as it is a derivative of the Ricci tensor, so the manifold is conformally flat) or decomposes. But any two- or one-dimensional manifolds are conformally flat, so its holonomy must vanish if it decomposes. □

So to complete this proof, we just need the following lemma:

Lemma 2.2.19. *There exist three-manifolds that are not conformally flat – hence with non-vanishing Cotton-York tensors.*

Proof of Lemma. It is well know that these exist (see paper [Suy], which demonstrates this result, by looking at three-dimensional hypersurfaces in $\mathbb{R}^4$). But the following easy argument confirms it: By paper [Bry1], a metric on M^n is defined by $n(n+1)/2$ local functions, whereas the space of local diffeormorphisms is given by n local functions. Thus the 'moduli space' of local metrics is of rank $n(n-1)/2$, which is 3 when $n = 3$. To change from one metric to another using a conformal transformation, we must scale by a single function (equivalently, with a closed one-form Υ). Consequently there must exist non-conformally flat metrics on three-manifolds. □

This demonstrates the proposition. ■

Remark. The same proof works for $\mathfrak{so}(3,2)$, though there we must exclude the possible $\mathfrak{so}(2,1) \subset \mathfrak{so}(3,2)$ and possible reduced holonomy algebras – as the decomposition theorem might not work, depending on whether preserved subbundle are degenerate.

To get around this, we use the 'patching' idea of Proposition 5.6.6, except here we are deforming the metric to a flat metric, using bump functions (rather than deforming the connection). But then we may conjugate any eventual $\mathfrak{h} \subset \mathfrak{so}(3,2)$ to get the full $\mathfrak{so}(3,2)$ (patching several times, if needed). In order for patching to work, we must avoid Ricci-flat manifolds, so that $L^1 \oplus T[-1] \subset \mathcal{T}$ is not preserved – but those manifolds are conformally flat in three dimensions anyway.

Chapter 3

The Projective case

3.0.1 Flat model and associated algebras

The flat model in the projective case is simply the n-dimensional projective space $\mathbb{PR}^n$, and the set of projective transformations is just $\mathbb{P}SL(n+1)$.

$$\begin{aligned} G &= \mathbb{P}SL(n+1), \\ P &= GL(n) \rtimes \mathbb{R}^n, \\ G_0 &= GL(n). \end{aligned}$$

Notice that the Lie algebra of $\mathbb{P}SL(n+1)$ is simply $\mathfrak{sl}(n+1)$. As our concern in this paper is about the algebra rather than the group, we shall generally fail to distinguish between $\mathbb{P}SL(n+1)$ and $SL(n+1)$.

3.0.2 Preferred connections

We must now prove the result left unproved in the first chapter, namely that every affine connection that preserves the projective structure and has the required torsion is a preferred connection.

Remember that two preferred connections are related by a one-form Υ:

$$\nabla_X Y = \nabla'_X Y + [\Upsilon, X].Y.$$

Let ∇ be a preferred connection, and $\widehat{\nabla}$ an affine connection with same projective structure and same torsion. Then

$$\nabla = \widehat{\nabla} + \Psi$$

with $\Psi \in \Gamma(\mathcal{H})$ a one-form with values in $\mathfrak{gl}(n)$. Since ∇ and $\widehat{\nabla}$ have same torsion,

$$\mathcal{H} \subset \left(\odot^2 T^* \otimes T\right) \cap \left(T^* \otimes \mathfrak{gl}(n)\right).$$

Since they also have the same projective structure, for all $X \in \Gamma(T)$ we have a function f_X so that

$$\Psi(X, X) = f_X X$$

However the symmetry of Ψ implies that

$$\Psi(X, Y) = \frac{1}{2}\left(\Psi(X+Y, X+Y) - \Psi(X, X) - \Psi(Y, Y)\right).$$

Hence Ψ is entirely determined by the value of $\Psi(X, X)$ for different X. Choosing a local frame (X^h), define the one-form Υ by $\Upsilon(X^h) = -\frac{1}{2} f_{X^h}$. Then

Lemma 3.0.1. *For all Z, $\Upsilon(Z) = -\frac{1}{2} f_Z$, and $\Psi(X, Y) = \frac{1}{2}\left(f_Y X + f_X Y\right)$.*

Proof of Lemma. Let $\Psi(X, Y) = aX + bY$. Define $Z = X + \mu Y$. Then

$$\Psi(Z, Z) \quad = \quad f_X X + \mu^2 f_Y Y + 2\mu a X + 2\mu b Y.$$

However $\Psi(Z, Z)$ is a multiple of Z, so

$$\mu(f_X + 2\mu a) \quad = \quad (\mu^2 f_Y + 2\mu b).$$

Since this equality is valid for all μ, we must have $f_X = 2b$ and $f_Y = 2a$. Hence $\Psi(X,Y) = \frac{1}{2}(f_Y X + f_X Y)$.

Moreover, $\Psi(Z,Z) = (f_X + \mu f_Y) Z$, so $f_Z = f_X + \mu f_Y$. This shows that f_Z depends linearly on Z; in other words, there exists a one-form ν, such that

$$\nu(Z) = f_Z.$$

Then since $\Upsilon(X^i) = -\frac{1}{2}\nu(X^h)$ on the frame, $\Upsilon(Z) = -\frac{1}{2}\nu(Z) = -\frac{1}{2}f_Z$ for all Z. □

We may put these results together to show that

$$\begin{aligned} \Psi(X,Y) = &\quad -\Upsilon(X)Y - \Upsilon(Y)X \\ = &\quad [\Upsilon, X].Y, \end{aligned}$$

which implies that the map

$$\begin{aligned} \phi : \Gamma(T^*) &\rightarrow \Gamma(\mathcal{H}) \\ \phi(\Upsilon)(X) &= [\Upsilon, X], \end{aligned}$$

is bijective.

In fact this proves another result, namely that

Corollary 3.0.2. *For a given class of preferred connections, a choice of connection in that class is equivalent to a connection on any weight bundle $L^a = (\wedge^n T)^{a/n}$, $a \neq 0$.*

Proof. Notice first that

$$\text{trace } \phi(\Upsilon) = \quad -(n+1)\Upsilon,$$

Any preferred connection ∇ determines a connection on L^a. If l is a section of L^a and ∇' another preferred connection differing from ∇ via Υ,

$$\nabla l = \nabla' l - a\frac{n+1}{n}\Upsilon l.$$

So different preferred connections determine different connections on L^a. Conversely, any connection on L^a differs from ∇ by such an Υ, and so corresponds to the action of the preferred connection $\nabla + \phi(\Upsilon)$. ■

Corollary 3.0.3. *For every projective structure, there exist volume-form preserving – $\mathfrak{sl}(n)$ – preferred connections, for every volume form ν.*

Proof. The connection ∇ on L^{-n} defined by

$$\nabla\nu = 0$$

defines, by the previous corollary, a preferred connection preserving ν. In future, when talking about preferred $\mathfrak{sl}(n)$ connections, we will often just define them by ν itself. ■

3.0.3 Tractor connection

Given a choice of preferred connection ∇, there is a decomposition of the algebra bundle $\mathcal{A}$ and hence of the Tractor bundle

$$\mathcal{T} = T[\mu] \oplus L^{\mu}$$

with $\mu = -\frac{n}{n+1}$. The Tractor connection is given by $\overrightarrow{\nabla}_X = \nabla_X + X + \mathsf{P}(X)$, or, more explicitly,

$$\overrightarrow{\nabla}_X \begin{pmatrix} Y \\ a \end{pmatrix} = \begin{pmatrix} \nabla_X Y + Xa \\ \nabla_X a + \mathsf{P}(X, Y) \end{pmatrix}.$$

The formula for changing a splitting by a one-form Υ (see Theorem 1.3) is given explicitly by

$$\begin{pmatrix} Y \\ a \end{pmatrix} \to \begin{pmatrix} Y \\ a - \Upsilon(Y) \end{pmatrix}. \tag{3.1}$$

3.1 Reducible holonomy: Ricci-flatness

This section will provide a description of the geometric meanings of reducible Tractor holonomy. We will not, however, fully classify this case, similar to the fact that reducible holonomy is not fully classified in the affine case. In this section, by co-volume forms, we mean elements such as

$$X^1 \wedge X^2 \wedge \ldots \wedge X^k$$

where (X^j) is a frame for a bundle of rank k.

Let $\widetilde{K} \subset \mathcal{T}$ be a rank $k \leq n$ subbundle preserved by $\overrightarrow{\nabla}$.

Lemma 3.1.1. *On an open dense subset of the manifold, L^μ is not a subbundle of $\widetilde{K}$.*

Proof of Lemma. This fact (the equivalent to Lemma 2.1.3 in the conformal case), is a consequence of the fact that the second fundamental form of L^μ is maximal, since $\overrightarrow{\nabla}$ comes from a Cartan connection.

In more details, let $\pi^1 : \mathcal{T} \to \mathcal{T}/L^\mu = T[\mu]$ be the quotient projection. Then the second fundamental form of L^μ,

$$S : L^\mu \longrightarrow T^* \otimes T[\mu]$$

is defined by

$$S(s)(X) \quad = \quad \pi^1\left(\overrightarrow{\nabla}_X s\right) = sX.$$

In consequence the image of sections of L^μ under $\overrightarrow{\nabla}$ span all of $\mathcal{T}$. □

From now on we shall assume, by restricting to open, dense subsets of M, that $L^\mu \cap \widetilde{K} = 0$. Hence the projection π^1 is injective on $\widetilde{K}$. Given any nowhere-zero section s of L^μ, define $K \subset T$ as $s^{-1}\pi(\widetilde{K})$. This bundle does not depend on a choice of s, as changing s changes the scaling but not the bundle.

Theorem 3.1.2. *K is an integrable, totally geodesic foliation, and there are preferred connections ∇ such that ∇ preserves K and $\mathsf{P}^\nabla(-, Y) = 0$ for any section Y of K. We may furthermore choose*

∇ so that it preserves a co-volume form on K. If $\widetilde{\nabla}$ preserves a co-volume form on $\widetilde{K}$, then there exists $\mathfrak{sl}(n)$-preferred connections ∇ with these properties.

Most of this section will be devoted to proving this. We choose a splitting of $\mathcal{T} = T[\mu] \oplus L^{\mu}$ such that $\widetilde{K} \subset T[\mu]$, and the preferred ∇ corresponding to this splitting.

Let X and Y be sections of K, then

$$\begin{pmatrix} Ys \\ 0 \end{pmatrix}$$

is a section of $\widetilde{K}$, for any $s \in \Gamma(L^{\mu})$. Then

$$\widetilde{\nabla}_X \begin{pmatrix} Ys \\ 0 \end{pmatrix} = \begin{pmatrix} (\nabla_X Y)s + Y(\nabla_X s) \\ s\mathsf{P}(X,Y) \end{pmatrix}. \tag{3.2}$$

Since this must also be a section of $\widetilde{K}$, one must have $\nabla_X Y$ as a section of K, and consequently $[X,Y] = \nabla_X Y - \nabla_Y X$ is a section of K. Hence

Proposition 3.1.3. *K is integrable and totally geodesic.*

If one were to view X as any section of T rather than K in Equation (3.2), one sees that ∇ preserves K and

$$\mathsf{P}(-,Y) = 0,$$

since K has no L^{μ} component.

Remark. Note that as a consequence of this, P is zero on $K \otimes K$, hence Ric is zero on this foliation as well. Since K is preserved by ∇ Lemma 2.2.10 implies that, $\mathsf{Ric}^K = \mathsf{Ric}^M|_{K\otimes K}$; in other words, the leaves of the foliations K are Ricci-flat under the connection ∇ restricted to these leaves.

Lemma 3.1.4. *We may choose ∇ so that it preserves a co-volume form on K.*

Proof of Lemma. Since $\nabla|_K$ is Ricci-flat, it must preserve a co-volume form τ *along* K. Thus

$$\nabla\tau = \omega \otimes \tau,$$

where ω is a one-form with $\omega(K) = 0$. Now $[\Upsilon, X]$ acts on τ by taking the trace of the first k components; or, in other words,

$$\begin{aligned} [\Upsilon, X].\tau &= -(\Upsilon \otimes X + \Upsilon(X)Id)\,\tau \\ &= -\Upsilon(\tau) \wedge X - k(\Upsilon(X))\tau. \end{aligned}$$

In other words, if we change preferred connections from ∇ to ∇' by the choice of

$$\Upsilon = -\frac{1}{k}\omega,$$

then

$$\nabla'\tau = \omega \otimes \tau - \frac{k}{k}\omega \otimes \tau = 0.$$

Since $\Upsilon(K) = 0$, then by Equation (3.1), ∇' still determines a splitting with $\widetilde{K} \subset T[\mu] \subset \mathcal{T}$. □

Proposition 3.1.5. *$\overrightarrow{\nabla}$ preserves a co-volume form on $\widetilde{K}$ if and only if ∇' is an $\mathfrak{sl}(n)$ connection.*

Proof. Since $\mathsf{P}^{\nabla'}(-, Y) = 0$ for a section Y of T, $\overrightarrow{\nabla}$ acts on $\widetilde{K}$ in the same way that ∇' acts on $K[\mu]$.

If ∇' preserves a nowhere zero section s of L^{μ}, then $\overrightarrow{\nabla}$ preserves $s^k\tau$ on $\widetilde{K}$. Conversely, if $\overrightarrow{\nabla}$ preserves a co-volume form $\widetilde{\tau}$ on $\widetilde{K}$, then

$$\widetilde{\tau} = t\tau$$

for t some nowhere-zero section of $L^{k\mu}$. Then

$$\begin{aligned} 0 &= \nabla'\widetilde{\tau} \\ &= \nabla' t\tau \\ &= (\nabla' t)\tau + t(\nabla'\tau) \\ &= (\nabla' t)\tau. \end{aligned}$$

Hence $\nabla' t = 0$, so ∇' is an $\mathfrak{sl}(n)$ connection. ■

Corollary 3.1.6. *Theorem 3.1.2 clearly has a converse: let ∇ be a preferred connection with a preserved totally geodesic integrable foliation K such that $\mathsf{P}^{\nabla}(Y,-) = 0$ for a section Y of K. Then $\overrightarrow{\nabla}$ preserves a subbundle $\widetilde{K}$ of $\mathcal{T}$. If there exists $\mathfrak{sl}(n)$-preferred connections with these properties which preserve co-volume forms on K, then $\overrightarrow{\nabla}$ preserves a co-volume form on $\widetilde{K}$.*

As a consequence of this, if $\widetilde{K}$ is a rank n bundle, then $K = T$, and there exists a Ricci-flat preferred connection ∇ on M. Since it is Ricci-flat, it must preserve a volume form, hence:

Corollary 3.1.7. *If $\overrightarrow{\nabla}$ preserves a rank $k = n$ bundle $\widetilde{K}$, it always preserves a volume form on $\widetilde{K}$.*

Remark. In the general case, we have a wide variety of splittings – hence preferred connections – with the required inclusion $\widetilde{K} \subset T[\mu] \subset \mathcal{T}$. When $k = n$, however, $\widetilde{K}$ and $T[\mu]$ have same rank, so there is a single splitting of $\mathcal{T}$ with this property. So we may talk about *the* Ricci-flat preferred connection ∇ on $K = T$.

Notice that since the rho-tensor of ∇ is zero on K, as is the rho-tensor of $\nabla|_K$, the tractor connection of K is a restriction of that of M:

$$\overrightarrow{\nabla}^K_X \begin{pmatrix} Ys^\nu \\ t^\nu \end{pmatrix} = \overrightarrow{\nabla}_X \begin{pmatrix} Ys \\ t \end{pmatrix}$$

whenever X and Y are sections of K, and $\nu = \frac{\mu_K}{\mu_M} = \frac{n(k+1)}{(n+1)k}$. This provides another useful tool for calculating the Tractor holonomy: namely, that the holonomy of $\overrightarrow{\nabla}$ contains the holonomies of $\overrightarrow{\nabla}^K$ along every leaf of the foliation K.

There is another useful characterisation in the 'nearly irreducible' case, where $n = k$:

Theorem 3.1.8. *If $\overrightarrow{\nabla}$ preserves a bundle K of rank n and acts irreducibly on K then the holonomy algebra of $\overrightarrow{\nabla}$ is*

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^{\nabla} \oplus T \quad \text{or} \quad \overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^{\nabla},$$

where $\mathfrak{hol}^{\nabla}$ is the affine holonomy algebra of the Ricci-flat preferred connection ∇ on M. The Lie

bracket is given by the standard one on $\mathfrak{hol}^\nabla$*, the trivial one on* T*, and action of* $\mathfrak{hol}^\nabla$ *on* T *in cross terms.*

The second equality holds if ∇ *is a cone in the sense of Definition 2.1.7, the first if it is not.*

Proof. Remember the algebra bundle splitting,

$$\mathcal{A} = T^* \oplus \mathfrak{gl}(n, \mathbb{R}) \oplus T.$$

In the splitting given by ∇, $T[\mu] = \widetilde{K}$ is preserved by $\overrightarrow{\nabla}$, thus there can be no T^* component to the holonomy of $\overrightarrow{\nabla}$. As $\overrightarrow{\nabla}$ and ∇ act identically on $T[\mu]$, the $T \otimes T^*$ component of the holonomy of $\overrightarrow{\nabla}$ must be the affine holonomy of ∇. Then given the conditions on $\overrightarrow{\nabla}$, $\mathfrak{hol}^\nabla$ must act irreducibly on $T[\mu]$.

Then the algebra $\mathfrak{hol}^\nabla \oplus T$ decomposes into two pieces, $\mathfrak{hol}^\nabla$ and T, under the action of $\mathfrak{hol}^\nabla$. In other words, if the holonomy of $\overrightarrow{\nabla}$ has any T component, it has the full T.

Lemma 3.1.9. $\overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^\nabla$ *if and only if* (K, ∇) *is a cone in the sense of Definition 2.1.7.*

Proof of Lemma. Since (K, ∇) is a cone, there exists a vector field Q such that $\nabla Q = Id$. Thus the section

$$\begin{pmatrix} Q \\ -1 \end{pmatrix}$$

is preserved by $\overrightarrow{\nabla}$. So $\overrightarrow{\mathfrak{hol}}$ cannot contain T.

Conversely, imagine that $\overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^\nabla$. The action of $\mathfrak{hol}^\nabla$ commutes with the natural projection

$$\mathcal{T} \to T[\mu].$$

Consequently, since $\mathfrak{hol}^\nabla$ is reductive and acts reducibly, $\mathcal{T}$ must contain one summand equal to $T[\mu]$, as well as a preserved section, say v. Obviously $\pi^1(v) \neq 0$. Since v is preserved, $\pi^1(v)$ is a

constant. Scale it to be minus one, so that v is of the form

$$v = \begin{pmatrix} Q \\ -1 \end{pmatrix}.$$

The condition that v be preserved by $\overrightarrow{\nabla}$ translates to requiring that $\nabla Q = Id$. Proving that (K, ∇) is a cone is harder. Define a cone over K, $\mathcal{C} = K \times \mathbb{R}$ with Q' the $\mathbb{R}$-factor. Extend the connection by requiring $\nabla Q' = Id$ and torsion-freeness. Since this is a projective cone (see latter Section 3.5), it has the same holonomy as $(K, \overrightarrow{\nabla})$. Now

$$\nabla(Q - Q') = 0,$$

and since $\overrightarrow{\mathfrak{hol}} = \mathfrak{hol}^{\nabla}$ is reductive and acts reducibly, there exists a foliation of $\mathcal{C}$ transverse to $S = Q - Q'$, and by the torsion-free affine version of the de Rham decomposition theorem – closely following the Riemannian argument– ∇ is S-invariant. Since it is also Q' invariant, it must be Q invariant.

In other words, (K, ∇) must be a cone. □

■

We shall see, in Section 3.5, that a projective cone has the same affine holonomy as the Tractor holonomy of its underlying manifold. But this theorem shows that the projective cone has Tractor holonomy also equal to these two holonomies.

3.1.1 Examples

There is no complementary foliation to K (unlike the definite signature conformal case) and the condition $\mathsf{P}(-, Y) = 0$ is a second order non-linear differential one; consequently it is hard to understand exactly what restrictions they impose on the projective structure. A pair of examples, however, suffice to show that these restrictions are geometrically not that strong, even when the various dimensions or co-dimensions are low.

Proposition 3.1.10. *The condition* $\mathsf{P}(-,Y)=0$ *is truly a restriction on the rho-tensor; one may have connections* ∇ *with this property where* $\mathsf{Ric}^{\nabla}(-,Y)\neq 0$*, even when* K *is of co-dimension one.*

Proof. Obviously, ∇ cannot be an $\mathfrak{sl}(n)$ connection. Define ∇ on $\mathbb{R}^{n-1}\times\mathbb{R}$ as being any Ricci-flat connection on $\mathbb{R}^{n-1}$. Then let X,Y be commuting sections of $T\mathbb{R}^{n-1}$, and Z the $\mathbb{R}$-factor. Let x be a local coordinate on $\mathbb{R}^{n-1}$, $X(x)=1$ and $Z(x)=Y(x)=0$. Then define the remaining non-zero terms of ∇ as

$$\begin{aligned}\nabla_X Z=\nabla_Z X &= -cxX\\ \nabla_Y Z=\nabla_Z Y &= cxY\\ \nabla_Z Z &= axZ,\end{aligned}$$

for constants a and c. Then the only non-zero terms of Ric of ∇ are

$$\begin{aligned}\mathsf{Ric}(X,Z) &= -c-a\\ \mathsf{Ric}(Z,X) &= c\end{aligned}$$

Then it suffices to set $c=\frac{a}{1-n}$ to get the required

$$\mathsf{P}(Z,X) = 0.$$

The generalisations of this construction are obvious. ∎

More importantly, one has:

Proposition 3.1.11. *Assume* $k\geq 3$*. Then* ∇ *restricted to one leaf of* K *may be flat, even if it is non-flat and with maximal holonomy* $\mathfrak{sl}(k)$ *when restricted to a different leaf of* K*. This result remains valid if* ∇ *is an* $\mathfrak{sl}(n)$ *connection or not, Ricci-flat or not, and whatever the codimension of* K *is.*

Proof. Let $M=\mathbb{R}^4$, with coordinates x_1,x_2,x_3 and z, and corresponding vector fields X^1,X^2,X^3

and Z. Let f be the smooth function

$$f(z) = \begin{cases} \exp(-\frac{1}{z^2}), & z > 0 \\ 0, & z \leq 0 \end{cases}$$

Then define the torsion-free connection ∇ by:

$$\nabla_Z Z = x_2 a X^2, \tag{3.3}$$

for some constant a, and

$$\nabla_{X^j} X^j = f(z) x_{j+1} X^{j+2},$$

cycling j modulo 3. All the other expressions for ∇ in terms of these vector fields are zero. The curvature of ∇ is given by

$$\begin{aligned} R(X^j, X^{j+1})X^j &= -fX^{j+2}, \\ R(X^j, X^{j+2})X^j &= -f x_{j+1} x_j X^{j+1}, \\ R(Z, X^j)X^j &= Z(f) x_{j+1} X^{j+2}, \\ R(Z, X^2)Z &= -aX^2 - x^2 a f x^3 X^1, \end{aligned}$$

and all the other curvature terms are zero. The only non-zero Ricci curvature term is

$$\mathsf{Ric}(Z, Z) = a,$$

so we may make ∇ Ricci-flat or not by choosing a zero or non-zero. Since the co-volume form

$$X^1 \wedge X^2 \wedge X^3 \wedge Z$$

is preserved,

$$\mathsf{P}(-, X^j) = \frac{1}{1-n} \mathsf{Ric}(-, X^j) = 0.$$

We can see that the leaves $z = c$ are totally geodesic, and that ∇ restricted to them is Ricci-flat. Moreover ∇ is flat on $c \leq 0$ and non-flat on $c > 0$. The holonomy algebra of ∇ along a leaf $c > 0$ is spanned, by the Ambrose-Singer Theorem [KoNo], by elements of the form

$$\begin{aligned} X^j &\to X^{j+1}, \\ X^j &\to X^{j+2}. \end{aligned}$$

And the closure under the Lie bracket of elements of this form is the full $\mathfrak{sl}(3)$ holonomy.

If we want ∇ not to be a $\mathfrak{sl}(n)$ connection, replace Equation (3.3) with

$$\begin{aligned} \nabla_Z Z &= x_2 a X^2 + b x_1 Z, \\ \nabla_Z X^1 = \nabla_{X^1} Z &= -x_1 c X^1, \\ \nabla_Z X^2 = \nabla_{X^2} Z &= x_1 c X^2, \end{aligned}$$

for some constants b and $c = \frac{b}{1-n}$. Then the extra curvature will disappear upon taking the Ricci trace, apart from

$$\begin{aligned} \mathsf{Ric}(Z, Z) &= a, \\ \mathsf{Ric}(Z, X^1) &= -c, \\ \mathsf{Ric}(X^1, Z) &= b + c. \end{aligned}$$

So, as before, we have the required $\mathsf{P}(-, X^j) = 0$, and a non-symmetric Ric, hence a non $\mathfrak{sl}(n)$ connection ∇. The properties of ∇ restricted to leaves of K have not changed, so the preceding results still apply. ■

Remark. These results can then be generalised to a wide variety of varying holonomy groups. So it seems that the condition $\mathsf{P}(-, Y) = 0$ is not enough to pin down the geometry in any significant way.

3.2 Symplectic holonomy: Contact spaces

It turns out that a symplectic structure on the Tractor bundle corresponds to a canonical contact structure on the manifold, though an actual contact form depends on a choice of preferred connection. A projectively invariant understanding of what is happening is given by the contact distribution $U \subset T$, where any geodesic that starts tangential to U will remain tangential to U.

But before proceeding, we must define what we understand by a contact structure.

A contact structure on a manifold of dimension $n = 2m + 1$ is a maximally non-integrable distribution $U \subset T$ of rank $2m$. Calling $\mathcal{L}$ the quotient bundle, we may dualise the quotient map and get the exact sequence

$$0 \longrightarrow \mathcal{L}^* \longrightarrow T^* \longrightarrow U^* \longrightarrow 0.$$

A section θ of $\mathcal{L}^*$ is thus a one-form such that

$$\theta(U) = 0,$$

the maximal non-integrability condition translating to the fact that the volume form

$$\Theta = \wedge^m(d\theta) \wedge \theta$$

is nowhere zero.

Notice that though there is no canonical isomorphism between sections of $T[a]$ and T, there is an isomorphism between subbundles of these two bundles, since scaling does not change a subbundle.

Now assume that we have a projective structure with a preserved nondegenerate alternating form ω on $\mathcal{T}$. We shall call this a symplectic form, for as we shall see in the cone construction of Section 3.5, ω is just a standard symplectic form on the cone. Given a preserved symplectic form ω on $\mathcal{T}$, this allows us to define two bundles; $U = \pi\left((L^\mu)^\perp\right)$, where the $\perp$ is taken with respect to the symplectic structure. Since $L^\mu \subset (L^\mu)^\perp$, U is a subbundle of T of rank $n - 1 = 2m$.

Conversely, we may define the bundle $\mathcal{L}^* = \omega(L^\mu) \subset \mathcal{T}^*$. Since $\mathcal{L}^*(L^\mu) = 0$ by definition, $\mathcal{L}^*$ is

in fact a subbundle of $T^*[-\mu] \subset \mathcal{T}^*$. Again, we may consider $\mathcal{L}^*$ as a subbundle of T^*.

Note that since $\mathcal{L}^*$ is zero on any lift of U into $\mathcal{T}$, $\mathcal{L}^*(U) = 0$.

Theorem 3.2.1. *A symplectic form ω on the Tractor bundle $\mathcal{T}$ corresponds to a canonical contact structure on M.*

Proof. A choice of section s of L^μ – equivalently, a choice of $\mathfrak{sl}(n)$ preferred connection ∇ – defines a section $\theta = s.\omega(s)$ of $\mathcal{L}^*$. Furthermore, since ∇ defines a splitting of $\mathcal{T}$, we may define a two-form ω' as

$$\omega' = s^2\left(\omega|_{T[\mu]}\right).$$

We aim to show that

Lemma 3.2.2. $d\theta = 2\omega'$.

Proof of Lemma. We know that $\overrightarrow{\nabla}\omega = 0$. From this, we may deduce the properties of ∇ itself. Let X, Y and Z be sections of U. Let R be the *Reeb* vector field of θ, ω'; i.e. $\omega'(R, -) = 0$ and $\theta(R) = 1$. Notice that this implies $\omega(s, R) = 1$, and $\omega(R, T[\mu]) = 0$. Using s, we identify $T[\mu]$ and T. Then

Lemma 3.2.3. ∇ *has the following properties:*

1. $\nabla_A R \in \Gamma(U)$ *for A any section of T.*
2. $\nabla_R X \in \Gamma(U)$.
3. $[R, X] \in \Gamma(U)$.
4. *The R component of $\nabla_X Y$ is $\omega'(Y, X)$.*
5. $\nabla_A \omega'$ *is zero on $U \otimes U$.*

Proof of Lemma.

$$0 = A.\omega(R, s) \quad = \quad \omega(\overrightarrow{\nabla}_A R, s) + \omega(R, \overrightarrow{\nabla}_A s)$$

$$\begin{aligned} &= \omega(\nabla_A R, s) + \mathsf{P}(A,R)\omega(s,s) + \omega(R,A) \\ &= \omega(\nabla_A R, s) + 0 + 0. \end{aligned}$$

so $\nabla_A R$ is a section of U, proving 1. Similarly

$$\begin{aligned} 0 = R.\omega(X,s) &= \omega(\overrightarrow{\nabla}_R X, s) + \omega(X, \overrightarrow{\nabla}_R s) \\ &= \omega(\nabla_R X, s) + \mathsf{P}(R,X)\omega(s,s) + \omega(X,R) \\ &= \omega(\nabla_R X, s) + 0 + 0. \end{aligned}$$

so $\nabla_R X$ is also a section of U, proving 2. Then 3 is a direct consequence of 1 and 2.

To prove 4 consider

$$\begin{aligned} 0 = X.\omega(Y,s) &= \omega(\overrightarrow{\nabla}_X Y, s) + \omega(Y, \overrightarrow{\nabla}_X s) \\ &= \omega(\nabla_X Y, s) + 0 + \omega'(Y,X). \end{aligned}$$

and the R component of $\nabla_X Y$ is just $-\omega(\nabla_X Y, s)$. For the final statement, again let A be any section of T, and

$$\begin{aligned} A.\omega'(X,Y) &= \\ A.\omega(X,Y) &= \omega(\overrightarrow{\nabla}_A X, Y) + \omega(X, \overrightarrow{\nabla}_A Y) \\ &= \omega(\nabla_A X, Y) + \omega(X, \nabla_A Y) \\ &\quad + \omega(s\mathsf{P}(A,X), Y) + \omega(X, s\mathsf{P}(A,Y)) \\ &= \omega'(\nabla_A X, Y) + \omega'(X, \nabla_A Y) + 0. \end{aligned}$$

demonstrating 5, since

$$(\nabla_A \omega')(X,Y) = A.\omega'(X,Y) - \left(\omega'(\nabla_A X, Y) + \omega'(X, \nabla_A Y)\right).$$

□

Now we may calculate $d\theta$.

$$\begin{aligned} d\theta(R,X) &= R.\theta(X) - X.\theta(R) - \theta([X,R]) \\ &= 0 \end{aligned}$$

since $\theta(R)$ and $\theta(X)$ are constants, and $[X,R]$ is a section of U.

$$\begin{aligned} d\theta(X,Y) &= X.\theta(Y) - Y.\theta(X) - \theta([X,Y]) \\ &= 0 - \theta(\nabla_X Y - \nabla_Y X) \\ &= -(2\omega'(Y,X)) \\ &= 2\omega'(X,Y). \end{aligned}$$

Hence $d\theta = 2\omega'$. □

To show that U defines a contact structure, it suffices to show that

$$\wedge^m \omega' \wedge \theta$$

is non-degenerate. But this is immediate as ω' is non-degenerate on U and zero on $\mathbb{R}.R$, whereas θ is zero on U and non-zero $\mathbb{R}.R$. ■

Note that although U and $\mathcal{L}^*$ are invariantly defined (and hence so is the contact structure), we need a choice of volume-preserving preferred connection to get an explicit θ or ω'.

Projectively, these structures imply

Proposition 3.2.4. *If $\phi : [0,1] \to M$ is a local geodesic that is tangent to U at some point, then it is tangent to U at every point (i.e. it is a Legendrian curve).*

Proof. Reparameterise ϕ so that the geodesic is parameterized by the affine parameter of ∇. Now $\phi' = X + aR$, and the geodesic equation becomes

$$0 = \nabla_{\phi'}\phi' = \phi'(a)R + \omega'(X,X)R + Y,$$

for some section Y of U. Then since $\omega'(X,X)=0$, we must have a constant along $\phi(t)$. So if $a=0$ at any point in the image of ϕ, $a=0$ at every point. ■

To invert this construction – start from some projective torsion-free connection ∇ which preserves a contact structure as above and generate a Tractor connection which preserves a symplectic form ω – we must add an additional integrability condition to those of Lemma 3.2.3. Given the Reeb vector field, we may split T^* as $\mathcal{L}^* \oplus U^*$. Let Ξ be the projection onto U^*. Since ω' is non-degenerate as a map $U \to U^*$, $\omega'^{-1} : U^* \to U$ is well defined. Then

Lemma 3.2.5. *If ∇ is a preferred $\mathfrak{sl}(n)$ connection of a Tractor connection preserving a symplectic structure, the P tensor of ∇ must obey the following formula:*

$$\nabla_A R \;=\; \omega'^{-1} \circ \Xi \circ \mathsf{P}(A).$$

Proof of Lemma. For any section A of T,

$$\begin{aligned} 0 &= A.\omega(R,Y) \\ &= \omega(\nabla_A R, Y) + \omega(R, s\mathsf{P}(A,Y)) \\ &= \omega(\nabla_A R, Y) - \mathsf{P}(A,Y) \\ &= \omega'(\nabla_A R, Y) - \mathsf{P}(A,Y). \end{aligned}$$

Since this formula is valid for all Y – though not upon replacing Y with R – we get the required result. □

And then it is quite easy to see that any ∇ that obeys all these conditions will generate a symplectic Tractor connection.

3.3 Complex holonomy: CR-spaces

3.3.1 Complex holonomy

It turns out that a complex structure J on the Tractor bundle $\mathcal{T}$ corresponds to the existence of CR-structures on M. The projective interpretation of this is hard to see: for though the Reeb vector field is well defined, the actual distributions and CR-structure vary depending on the choice of preferred connections. Notice that for any section s of the canonical bundle $L^{\mu} \subset \mathcal{T}$, one has a well defined vector field $R = s^{-1}\pi(Js) \subset \Gamma(T)$. Dividing out by the action of R gives an infinitesimal covering of a $\mathbb{C}$-projective structure on a manifold one dimension lower. If the the Tractor connection is moreover R-invariant, then this is a proper covering of this structure. See Section 3.5.1 for more details on this.

But first we must define what we mean by a CR-structure.

A CR-space is a manifold of odd dimension n with a distribution $H \subset T$ of rank $n-1$ and an endomorphism $J : H \to H$ such that $J^2 = -Id$, and that obeys two integrability conditions:

1. If $X, Y \in \Gamma(H)$ then $[JX, Y] + [X, JY] \in \Gamma(H)$,

2. The Nijenhuis tensor

$$N_J(A, B) = J([JA, B] + [A, JB]) - [JA, JB] + [A, B].$$

vanishes identically.

Now, given a projective structure with a complex structure J on the Tractor bundle $\mathcal{T}$, we have a canonical *Reeb* vector field, defined by

$$R = s^{-1}\pi\left(Js\right),$$

for any nowhere zero section s of L^{μ}.

There is a special class of connections within the preferred connections of this projective structure; namely those whose splitting has the property $J(L^{\mu}) \subset T[\mu]$. Since the class of preferred connections

corresponds to the class of *all* affine connections on L^μ, it corresponds to the class of all splittings of the sequence

$$0 \to L^\mu \longrightarrow \quad \mathcal{T} \cong \left(J^1(L^\mu)\right)^* \quad \longrightarrow T[\mu] \longrightarrow 0,$$

hence we can definitely find ones with the property $J(L^\mu) \subset T[\mu]$. Call these the $\mathbb{C}$-preferred connections. By choosing sections of $L^\mu \to M$ tangent to $J(L^\mu)$, we get $\mathbb{C}$-preferred connections that are volume preserving.

Then we have a subbundle in this splitting

$$H = T[\mu] \cap J(T[\mu]).$$

However, different choices of connection result in different bundles H, as they result in a different choice of bundle $T[\mu] \subset \mathcal{T}$; in this way, the reverse of the contact case, the Reeb vector is canonical but the distribution is not. J descends to a complex structure on H, hence to a complex structure on $H[-\mu]T$, since $H[-\mu] \otimes (H[-\mu])^* = H \otimes H^*$.

We now fix a $\mathbb{C}$-preferred connection ∇, and seek to deduce its properties from those of $\overrightarrow{\nabla}$.

Theorem 3.3.1. *If $\mathcal{T}$ has a preserved complex structure J, then a choice of $\mathbb{C}$-preferred connection ∇ gives a CR structure on M.*

The proof of this is detailed in the rest of this section.

Lemma 3.3.2. *If X, Y are sections of H, ∇ has the following properties:*

1. $\mathsf{P}(R, R) = -1$ *and* $\nabla_R R = 0$,
2. $\mathsf{P}(R, X) = 0$ *and* $\nabla_X R = JX$ *is a section of* H,
3. $\nabla_R X$ *is a section of* H *thus,*
4. $[R, X]$ *is also a section of* H,
5. $\widetilde{\nabla} J = 0$, *where* $\widetilde{\nabla}$ *is* ∇ *projected onto* H *along* R,
6. $\nabla_X Y = -\mathsf{P}(X, JY)R + Z$, *for* Z *a section of* H.

Proof of Lemma. Let s be any nowhere zero section of L^{μ}. As ∇ is $\mathbb{C}$-preferred, $R = s^{-1}Js$.

$$\begin{aligned} s\mathsf{P}(R,R) + s\nabla_R R &= \overrightarrow{\nabla}_R R - (\nabla_R s)R \\ &= J\overrightarrow{\nabla}_R s - (\nabla_R s)R \\ &= J(sR) + J(\nabla_R s) - (\nabla_R s)R \\ &= -s, \end{aligned}$$

since $J(\nabla_R s) = (\nabla_R s)R$ (this is true as both are zero whenever $\nabla_R s$ is zero, and otherwise $R = (\nabla_R s)^{-1}J(\nabla_R s)$). This proves 1. Similarly

$$\begin{aligned} s\mathsf{P}(X,R) + \nabla_X R &= \overrightarrow{\nabla}_X R - (\nabla_X s)R \\ &= J\overrightarrow{\nabla}_X s - (\nabla_X s)R \\ &= J(X), \end{aligned}$$

which is also a section of H by the definition of H, proving 2. For 3

$$\begin{aligned} J((\nabla_R X)s) &= J(s\mathsf{P}(R,X) + (\nabla_R X)s) \\ &= J\overrightarrow{\nabla}_R Xs - J(\nabla_R s)X \\ &= (s\mathsf{P}(R,JX) + (\nabla_R JX)s) + (\nabla_R s)JX - J((\nabla_R s)X) \\ &= (\nabla_R JX)s, \end{aligned}$$

implying that $\nabla_R X$ has no R component, as this would require an s component in $J\nabla_R X$. Then 4 is a direct consequence of 2 and 3.

The previous proof implies that $\widetilde{\nabla}_R J = 0$; in order to prove 5 one merely needs to show that $J\nabla_X Y$ and $\nabla_X JY$ differ only by multiples of s and R.

$$\begin{aligned} J(\nabla_X Y)s &= J(\overrightarrow{\nabla}_X Ys) - J(s\mathsf{P}(X,Y) - (\nabla_X s)Y) \\ &= \overrightarrow{\nabla}_X JY - Rs\mathsf{P}(X,Y) - (\nabla_X s)JY \\ &= (\nabla_X JY)s + s\mathsf{P}(X,JY) - Rs\mathsf{P}(X,Y). \end{aligned}$$

We get the further result that

$$\nabla_X Y = J^{-1}\left(\nabla_X JY + s\mathsf{P}(X, JY) - R\mathsf{P}(X, Y)\right).$$

implying that the R component of $\nabla_X JY$ is $\mathsf{P}(X, Y)$, proving 6.

And, inverting all these steps, one can see that any affine connection with these properties will generate a complex Tractor connection. This is all related to the complex projective structure, see Section 3.5.1. □

Proposition 3.3.3. *This ∇ generates a CR structure on M.*

Proof. The first condition of integrability, that $[JX, Y] + [X, JY] \in \Gamma(H)$ for $X, Y \in \Gamma(H)$ is easily checked:

$$\begin{aligned}
[JX, Y] + [X, JY] &= \nabla_{JX} Y - \nabla_Y JX + \nabla_X JY - \nabla_{JY} X \\
&= -\mathsf{P}(JX, JY) - \mathsf{P}(Y, X) + \mathsf{P}(X, Y) + \mathsf{P}(JY, JX) + Z \\
&= Z,
\end{aligned}$$

for Z a section of H.

Now

$$\begin{aligned}
N_J(X, Y) &= J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \\
&= J\left(\nabla_{JX} Y - \nabla_Y JX + \nabla_X JY - \nabla_{JY} X\right) \\
&\quad -\nabla_{JX} JY + \nabla_{JY} JX + \nabla_X Y - \nabla_Y X \\
&= J\left(\widetilde{\nabla}_{JX} Y - \widetilde{\nabla}_Y JX + \widetilde{\nabla}_X JY - \widetilde{\nabla}_{JY} X\right) \\
&\quad -\widetilde{\nabla}_{JX} JY + \widetilde{\nabla}_{JY} JX + \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X,
\end{aligned}$$

by the first integrability condition. But that last expression is zero since $\widetilde{\nabla} J = 0$. ■

3.3.2 Quaternionic holonomy

Assume now that $\overrightarrow{\nabla}$ preserves three complex structures J_1, J_2, J_3, with the usual quaternionic relations:

$$J_\alpha J_\beta = -\delta_{\alpha\beta} Id + \epsilon_{\alpha\beta\gamma} J_\gamma.$$

As before we may choose splittings of $\mathcal{T}$ such that $T[\mu]$ is tangential to $J_1(L^\mu)$, $J_2(L^\mu)$ and $J_3(L^\mu)$. Call these the $\mathbb{H}$-preferred connections. Since

$$\overrightarrow{R} J_\alpha(L^\mu) = J_\alpha \overrightarrow{R}(L^\mu) = 0,$$

we may choose a *section* of $L^\mu \to M$ tangential to these bundles, giving us a volume preserving $\mathbb{H}$-preferred connection ∇.

As before, we have the well defined Reeb vectors R_1, R_2 and R_3, invariants of the projective structure, and, via the choice of ∇, three distributions H_1, H_2 and H_3.

We get the further relations

$$\begin{aligned} J_1 R_2 = \quad & -J_2 R_1 = \quad R_3 \\ J_2 R_3 = \quad & -J_3 R_2 = \quad R_1 \\ J_3 R_1 = \quad & -J_1 R_3 = \quad R_2 \end{aligned}$$

and the distribution

$$\begin{aligned} \widehat{H} &= H_1 \cap H_2 \cap H_3 \\ &= T \cap J_1 T \cap J_2 T \cap J_3 T, \end{aligned}$$

is stable under the actions of all the automorphisms J_1, J_2 and J_3, and these obey the quaternionic relations on $\widehat{H}$.

Call any manifold with this sort of structure an HR-manifold. Paper [Biq] deals with similar structures.

Theorem 3.3.4. *Any Tractor connection that preserves hyper-complex structures J_1, J_2, J_3 has a class of $\mathbb{H}$-preferred connections. These connections define an HR-structure on the manifold, with canonical Reeb vectors and non-canonical distributions.*

It might be worth enquiring what happens when $\overrightarrow{\nabla}$ preserves not a hyper-complex structure, but a quaternionic one; i.e. preserves the span of J_1, J_2 and J_3 without preserving any one individually. This however, is not possible, as a consequence of Theorem 4.5.1.

3.4 Orthogonal holonomy: Einstein spaces

In this section we aim to show that $\overrightarrow{\nabla}$ preserving a metric on $\mathcal{T}$ is equivalent to the existence of an Einstein, non-Ricci-flat, preferred connection ∇.

Some explanations as to what we mean by an Einstein connection in this case:

Definition 3.4.1. *∇ is Einstein if Ric^{∇} is non-degenerate and*

$$\nabla \mathsf{Ric}^{\nabla} = 0.$$

Notice this also implies that $\nabla \det(\mathsf{Ric}^{\nabla}) = 0$, so ∇ is an $\mathfrak{sl}(n)$ connection. Thus Ric^{∇} is symmetric, and ∇ is the Levi-Civita connection of the 'metric' Ric^{∇}, meaning that ∇ is an Einstein connection in the standard sense, with Einstein coefficient 1.

Proposition 3.4.2. *If ∇ is an Einstein connection, then $\overrightarrow{\nabla}$ preserves a metric h on $\mathcal{T}$.*

Proof. Let $s \in L^{\mu}$ be a section corresponding to ∇. Then in the splitting defined by ∇, consider the metric

$$h(\begin{pmatrix} X \\ a \end{pmatrix}, \begin{pmatrix} Y \\ b \end{pmatrix}) = s^{-2}\left(-\mathsf{P}(X,Y) + ab\right).$$

Note that where Ric is of signature (p,q), h is of signature $(p+1,q)$. In a more general setting, if $\mathsf{Ric} = \lambda g$ for some metric g of signature (p,q), then h is of signature $(p+1,q)$ when $\lambda > 0$ and $(q+1,p)$ when $\lambda < 0$.

Remembering the formulas for the Tractor connection, and using s implicitly:

$$\begin{aligned}
Z.h(\begin{pmatrix} X \\ 0 \end{pmatrix}, \begin{pmatrix} Y \\ 0 \end{pmatrix}) &= -Z.\mathsf{P}(X,Y) \\
&= -\mathsf{P}(\nabla_Z X, Y) - \mathsf{P}(X, \nabla_Z Y) \\
&= h(\overrightarrow{\nabla}_Z \begin{pmatrix} X \\ 0 \end{pmatrix}, \begin{pmatrix} Y \\ 0 \end{pmatrix}) + h(\begin{pmatrix} X \\ 0 \end{pmatrix}, \overrightarrow{\nabla}_Z \begin{pmatrix} Y \\ 0 \end{pmatrix}) \\
Z.h(\begin{pmatrix} X \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix}) &= 0 \\
&= \mathsf{P}(Z,X)a - \mathsf{P}(Z,X)a \\
&= h(\overrightarrow{\nabla}_Z \begin{pmatrix} X \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ a \end{pmatrix}) + h(\begin{pmatrix} X \\ 0 \end{pmatrix}, \overrightarrow{\nabla}_Z \begin{pmatrix} 0 \\ a \end{pmatrix}) \\
Z.h(\begin{pmatrix} 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}) &= (\nabla_Z a)b + a\nabla_Z b \\
&= h(\overrightarrow{\nabla}_Z \begin{pmatrix} 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix}) + h(\begin{pmatrix} 0 \\ a \end{pmatrix}, \overrightarrow{\nabla}_Z \begin{pmatrix} 0 \\ b \end{pmatrix}),
\end{aligned}$$

hence

$$\overrightarrow{\nabla} h = 0.$$

■

Conversely:

Proposition 3.4.3. *If $\overrightarrow{\nabla}$ preserves a metric h on $\mathcal{T}$, then there exists an Einstein preferred connection ∇ on an open dense submanifold of M.*

Proof. We need first to show that $L^\mu \subset \mathcal{T}$ cannot degenerate for h, at least on an open dense subset.

Assume $h(s,s) = 0$ at $x \in M$ for some nowhere zero $s \in \Gamma(L^\mu)$. Then

$$X.h(s,s) = 2h(\nabla_X s, s)$$

$$\begin{aligned} &= 2h(\begin{pmatrix} Xs \\ \nabla_X s \end{pmatrix}, s) \\ &= 2h(\begin{pmatrix} Xs \end{pmatrix}, s), \end{aligned}$$

and since $\begin{pmatrix} Xs \\ 0 \end{pmatrix}$ spans an n-dimensional subset of $\mathcal{T}$, this quantity must be non-zero for most X, bar a $(n-1)$-dimensional subset of T_x.

Now on most points of M, we may define a special section $s \in \Gamma(L^\mu)$ by requiring

$$h(s,s) = 1.$$

and the associated preferred connection ∇ with $\nabla s = 0$. Consequently

$$\begin{aligned} 0 &= X.h(s,s) \\ &= 2h(\begin{pmatrix} X \\ 0 \end{pmatrix}, s) \end{aligned}$$

so $L^\mu \perp T[\mu]$. Moreover

$$\begin{aligned} 0 &= X.h(Ys,s) \\ &= h(\begin{pmatrix} \nabla_X Y \\ \mathsf{P}(X,Y) \end{pmatrix}, s) + h(Ys, Xs) \\ &= \mathsf{P}(X,Y) + h(Ys, Xs). \end{aligned}$$

Hence

$$h(\begin{pmatrix} X \\ a \end{pmatrix}, \begin{pmatrix} Y \\ b \end{pmatrix}) = s^{-2}\left(-\mathsf{P}(X,Y) + ab\right),$$

as before. As well as this,

$$\begin{aligned} X.\mathsf{P}(Y,Z) &= X.h(Ys, Zs) \\ &= h(\begin{pmatrix} \nabla_X Y \\ \mathsf{P}(X,Y) \end{pmatrix}, sZ) + h(sY, \begin{pmatrix} \nabla_X Z \\ \mathsf{P}(X,Z) \end{pmatrix}) \\ &= \mathsf{P}(\nabla_X Y, Z) + \mathsf{P}(Y, \nabla_X Z), \end{aligned}$$

so

$$\nabla_X \mathsf{P} = 0.$$

■

3.5 Cone construction

So far we have seen the properties of various holonomy groups, but no tool that would enable one to classify them. The main tool for that is the cone construction, defined as follows:

Definition 3.5.1. *The vector line-bundle $L^{-\mu}$ has a principal $\mathbb{R}^+$-bundle – the quotient of the full frame bundle of T by the action of the simple piece $SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$. Call this bundle $\mathcal{C}(M)$, the cone over M. Let π be the projection $\pi : \mathcal{C}(M) \to M$.*

Theorem 3.5.2. *If $(M, \overrightarrow{\nabla})$ is a projective manifold, then there exists a torsion-free Ricci-flat affine connection $\widehat{\nabla}$ on $\mathcal{C}(M)$, which has the same holonomy as $\overrightarrow{\nabla}$.*

This construction bears similarities to the conformal ambient metric construction presented in [FeHi] and [CaGo1]; however, instead of using a metric, we shall use the P-tensor, and will not be extending the cone into a second dimension. The rest of this section will be dedicated to proving this.

Fix a preferred connection ∇; this defines not only a splitting of $\mathcal{T}$, but also, because it is a connection on L^μ, an $\mathbb{R}^+$-invariant splitting of the projection sequence

$$0 \longrightarrow \mathbb{R}^+ \longrightarrow T\mathcal{C}(M) \xrightarrow{d\pi} TM \longrightarrow 0.$$

Define Q to be the vector field on $C(M)$ generated by the action of $\mathbb{R}^+$. For the rest of this section, let X, Y and Z be sections of $TM \subset TC(M)$. Then define the connection $\widehat{\nabla}$ by

$$\begin{aligned} \widehat{\nabla} Q &= Id, \\ \widehat{\nabla}_Q Y &= Y, \\ \widehat{\nabla}_X Y &= \nabla_X Y + \mathsf{P}(X,Y)Q. \end{aligned}$$

Notice that $C(M)$ is a cone in the sense of Definition 2.1.7.

Lemma 3.5.3. *$\widehat{\nabla}$ is torsion-free.*

Proof of Lemma. Since the splitting of $TC(M)$ is $\mathbb{R}^+$ invariant,

$$[Q,Y] = 0 = Y - Y = \widehat{\nabla}_Q Y - \widehat{\nabla}_Y Q.$$

Now consider $[X,Y]$. It is clear that $\pi_*[X,Y] = [\pi_* X, \pi_* Y]$, giving the horizontal element of $[X,Y]$. Now let ω be the connection one-form, associated with ∇.

$$\begin{aligned} d\omega(X,Y) &= X.\omega(Y) - Y.\omega(X) - \omega([X,Y]) \\ &= -\omega([X,Y]), \end{aligned}$$

the vertical component we are looking for. Since X and Y are horizontal,

$$\begin{aligned} d\omega(X,Y) &= d\omega(X,Y) + \omega \wedge \omega(X,Y) \\ &= \Theta(X,Y), \end{aligned}$$

with Θ the curvature of ω. In other words, the vertical component of $[X,Y]$ is minus the curvature of ∇. Now the curvature of ∇ on L^n is

$$\begin{aligned} R_{hj\ k}^{\ \ k} &= R_{kj\ h}^{\ \ k} + R_{hk\ j}^{\ \ k} \\ &= \mathsf{Ric}_{jh} - \mathsf{Ric}_{hj}, \end{aligned}$$

twice the anti-symmetric part of the Ricci tensor. Thus on L^μ,

$$\begin{aligned}[X,Y] &= -\frac{\mu}{n}\left(-\mathsf{Ric}(X,Y)+\mathsf{Ric}(Y,X)\right)Q+\pi_*[X,Y]\\ &= \left(\mathsf{P}(X,Y)-\mathsf{P}(Y,X)\right)Q+\nabla_X Y-\nabla_Y X\\ &= \widehat{\nabla}_X Y-\widehat{\nabla}_Y X,\end{aligned}$$

since $-\frac{\mu}{n}=\frac{1}{n+1}$ and by Equation (1.5),

$$\mathsf{P}_{hj}-\mathsf{P}_{jh} = \frac{1}{n+1}\left(\mathsf{Ric}_{jh}-\mathsf{Ric}_{hj}\right).$$

□

Lemma 3.5.4. *$\widehat{\nabla}$ is Ricci-flat.*

Proof of Lemma. Let R be the curvature of ∇. Then the curvature of $\widehat{\nabla}$ is:

$$\begin{aligned}R_{X,Q}Y &= \nabla_X\nabla_Q Y-\nabla_Q\nabla_X Y-\nabla_{[X,Q]}Y\\ &= \widehat{\nabla}_X Y-\widehat{\nabla}_X Y+0\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\widehat{R}_{X,Y}Q &= \widehat{\nabla}_X\widehat{\nabla}_Y Q-\widehat{\nabla}_Y\widehat{\nabla}_X Q-\widehat{\nabla}_{[X,Y]}Q\\ &= \widehat{\nabla}_X Y-\widehat{\nabla}_Y X-[X,Y]\\ &= 0,\end{aligned}$$

while

$$\begin{aligned}\widehat{R}_{X,Q}Q &= \widehat{\nabla}_X\widehat{\nabla}_Q Q-\widehat{\nabla}_Q\widehat{\nabla}_X Q-\widehat{\nabla}_{[X,Q]}Q\\ &= X-X-0\\ &= 0.\end{aligned}$$

The only non-zero component is:

$$\begin{aligned}
\widehat{R}_{X,Y}Z &= \widehat{\nabla}_X\widehat{\nabla}_Y Z - \widehat{\nabla}_Y\widehat{\nabla}_X Z - \widehat{\nabla}_{[X,Y]}Z \\
&= R_{X,Y}Z + \mathsf{P}(Y,Z)X - \mathsf{P}(X,Z)Y + (\mathsf{P}(Y,X) - \mathsf{P}(X,Y))Z \\
&\quad +(\nabla_X\mathsf{P})(Y,Z)Q - (\nabla_Y\mathsf{P})(X,Z)Q \\
&= W_{X,Y}Z + CY_{X,Y,Z}Q,
\end{aligned}$$

by Equation (1.6), with W the Weyl tensor and CY the Cotton-York tensor of ∇. Hence the curvatures of $\widehat{\nabla}$ and $\overline{\nabla}$ are the same. Then the trace of $\widehat{R}$ is

$$\begin{aligned}
\widehat{\mathsf{Ric}}_{X,Y} &= \sum_j X_j^* \lrcorner \left(\widehat{R}_{X^j,X}Y\right) + Q^*(\widehat{R}_{Q,X}) \\
&= \sum_j X_j^* \lrcorner \left(W_{X^j,X}Y\right) + 0 \\
&= 0,
\end{aligned}$$

as the Weyl tensor is trace-free, where (X^j, Q) is a local frame for $T\mathcal{C}(M)$ and (X_j^*, Q^*) a dual frame. □

Proposition 3.5.5. *$\widehat{\nabla}$ is projectively invariant.*

Proof. Choose another connection ∇' and a corresponding splitting. Then in this splitting $\widehat{\nabla}$ must be

$$\begin{aligned}
\widehat{\nabla}Q &= Id, \\
\widehat{\nabla}_Q Y &= Y, \\
\widehat{\nabla}_X Y &= \nabla_X Y + S(X,Y)Q.
\end{aligned}$$

for some section S of $T^* \otimes T^*$. By the previous arguments, the requirement of torsion-freeness implies that the anti-symmetric part of S is the anti-symmetric part of P'. Then the Ricci-flatness requirement implies that $S = \mathsf{P}'$. ■

Remark. One can get the preferred connections from the cone connection via the following method:

given a Q-invariant splitting of $T\mathcal{C}(M)$, one has a connection ∇ on $T \subset T\mathcal{C}(M)$ by projecting $\widehat{\nabla}$ along Q. In other words

$$\nabla_X Y = \pi_*(\widehat{\nabla}_X Y).$$

And, of course, ∇ is the preferred connection corresponding to our chosen splitting of $T\mathcal{C}(M)$.

Remark. Two such splittings will differ via

$$X \to X' = X + \Upsilon(X)Q$$

for some one-form Υ on M. This is the origin of the fact that two preferred connections differ by the action of a one-form Υ.

Lemma 3.5.6. *Let ϕ and ϕ' be two paths in $\mathcal{C}(M)$ with identical endpoints such that*

$$\pi(\phi) = \pi(\phi').$$

Then the holonomy transforms of $\overrightarrow{\nabla}$ along ϕ and ϕ' are the same.

Proof of Lemma. Let $X + aQ$ be a vector field, parallel transported along ϕ,

$$\widehat{\nabla}_{\dot{\phi}}(X + aQ) = 0.$$

Now there is a (local) invariant extension of $X + aQ$ in the direction of the cone, $e^{-q}(X + aQ)$ where q is a local coordinate, $q = 0$ (locally) along ϕ and $Q(q) = 1$. Consequently,

$$\widehat{\nabla}_Q(e^{-q}(X + aQ)) = 0,$$

and

$$\widehat{\nabla}_Y(e^{-q}(X + aQ)) = e^{-q}\widehat{\nabla}_Y(X + aQ),$$

so since $\dot{\phi}' = \dot{\phi} + bQ$ for some function b,

$$\begin{aligned}\widehat{\nabla}_{\dot{\phi}'}(e^{-q}(X+aQ)) &= e^{-q}\widehat{\nabla}_{\dot{\phi}}(X+aQ) + b\widehat{\nabla}_Q(e^{-q}(X+aQ)) \\ &= 0.\end{aligned}$$

Then since $q = 0$ locally at both endpoints of ϕ and ϕ', the result is proved. □

To complete this section and give a point to it all, one has to show the final result:

Theorem 3.5.7. *Differentiating $\mathcal{T}$ along T via $\overrightarrow{\nabla}$ or differentiating $TC(M)$ along $T \subset TC(M)$ is an isomorphic operation.*

Proof. A section s of L^μ is isomorphic with a $\mathbb{R}^+$-invariant function $C(M) \to \mathbb{R}$. In our case, we require that

$$Q(s) = \mu s.$$

Then we may identify $(sY, s) \in \Gamma(\mathcal{T})$ with $(sY, sQ) \in \Gamma(TC(M))$. Under this identification it is clear that

$$\overrightarrow{\nabla}_X(sY, s) \cong \widehat{\nabla}_X(sY, sQ).$$

■

As a simple consequence of this and Lemma 3.5.6,

Corollary 3.5.8. *$\overrightarrow{\nabla}$ and $\widehat{\nabla}$ have same holonomy.*

So in order to classify holonomy groups of $\overrightarrow{\nabla}$, one has to look at those groups that can arise as the affine holonomy groups of Ricci-flat cones. By an abuse of notation, so as not to clutter up with too many connection symbols, we will also designate $\widehat{\nabla}$ with the symbol $\overrightarrow{\nabla}$.

Proposition 3.5.9. *This construction is the same as the conformal Einstein cone construction of Section 2.1.2. For in that case, the conformal P^{co} is given by*

$$\mathsf{P}^{co} = \frac{-\lambda}{2n-2}g$$

$$= \frac{1}{2} \cdot \frac{-\mathsf{Ric}}{n-1}$$
$$= \frac{1}{2}\mathsf{P}^{pr}$$

with P^{pr} *the projective rho-tensor. Then replacing the* $2\mathsf{P}^{co}$ *term in the conformal Einstein cone Equation (2.1) with the equal term* P^{pr} *yields the standard projective cone.*

In this way, by classifying projective Tractor holonomy groups, we shall also classify conformal Tractor holonomy groups for conformally Einstein structures.

3.5.1 Complex projective structures

Let M^{2n+1} be a projective manifold with a complex structure J on $\mathcal{T}$ – hence on the cone $\mathcal{C}(M)$. Assume that $\overrightarrow{\nabla}$ is R-invariant, where $R = JQ$. Similarly to the proof of Lemma 2.1.8, being R-invariant is equivalent to the disappearance of all curvature terms involving R.

Then we may divide out $\mathcal{C}(M)$ by the action of Q and R to get a manifold N. Call this projection $\Pi : \mathcal{C}(M) \to N$. Notice that Π factors through M:

$$\mathcal{C}(M) \longrightarrow M \longrightarrow N.$$

Lemma 3.5.10. *N has a canonical complex structure.*

Proof of Lemma. The complex structure J is invariant along Q and R. Let $\eta : N \to \mathcal{C}(M)$ be a section of Π. Then we may define the complex structure on N via

$$J^N(X) = \Pi_* \left(J(\eta_* X) \right).$$

Notice that this definition is independent of η, since if η' is another section of Π, $\eta'_*(X) = \eta_*(X) + aQ + bJQ$, so

$$\begin{aligned} \Pi_* \left(J(\eta'_* X) \right) &= \Pi_* \left(J(\eta_* X + aQ + bJQ) \right) \\ &= \Pi_* \left(J(\eta_* X) \right) + \Pi_* \left(aJQ - bQ \right) \\ &= \Pi_* \left(J(\eta_* X) \right). \end{aligned}$$

□

Now remember what happens when $\mathcal{C}(M)$ has a complex structure; for a choice of $\mathbb{C}$-preferred connection ∇, one has a CR distribution $H \subset TM$, which will project bijectively onto TN when dividing out by the action of $R = JQ$.

Lemma 3.5.11. *Since $\overrightarrow{\nabla}$ is Q- and R-invariant, one may produce a torsion-free connection $\widetilde{\nabla}$ on N by projecting $\overrightarrow{\nabla}$ along Q and R – equivalently, by projecting ∇ along R.*

The set of all $\widetilde{\nabla}$ preserve some structure on N; call this the complex projective structure. But what exactly is it?

Lemma 3.5.12. *All $\widetilde{\nabla}$ preserve the complex structure J^N.*

Proof of Lemma. This is a direct consequence of the properties of ∇, as given in Lemma 3.3.2.

□

Definition 3.5.13 (Generalised complex geodesics). *A generalised complex geodesic is a map $\psi : \mathbb{R} \to N$ such that*

$$\overrightarrow{\nabla}_{\dot{\psi}} \dot{\psi} \in \Gamma(B),$$

where B is the bundle spanned by $\dot{\psi}$ and $J\dot{\psi}$. Since a real geodesic is a fortiori a generalised complex geodesic, these exist at all points, in every direction. However they are non-unique; for instance the image of any geodesic in $\mathcal{C}(M)$ is a generalised complex geodesic in N.

Definition 3.5.14 (Complex geodesics). *A complex geodesic on a complex manifold $(N, J, \widetilde{\nabla})$ is a map μ from a domain $U \subset \mathbb{C}$ to N such that $\mu(U)$ is totally geodesic [MoMo], [Leb]. They exist if the connection ∇ is holomorphic – paper [MoMo] erroneously claims their existence in the general case.*

Obviously any curve inside a complex geodesic is a generalised complex geodesic. Note that a complex geodesic is a function $\mathbb{C} \to N$, whereas generalised complex geodesic are functions $\mathbb{R} \to N$.

Lemma 3.5.15. *All connections $\widetilde{\nabla}$ have the same generalised complex geodesics, and, if and when they exist, the same complex geodesics.*

Proof of Lemma. All $\widetilde{\nabla}$ correspond to $\overrightarrow{\nabla}$ projected onto various foliations H. So if X is a generalised complex geodesic,

$$\overrightarrow{\nabla}_X X = fX + gJX + \aleph,$$

for functions f and g, where $\aleph$ denotes terms in Q and R. Changing the foliation to that corresponding to $\widetilde{\nabla}'$ involves replacing X with $X' = X + (\Upsilon^{\mathbb{C}} \llcorner_{\mathbb{C}} X)Q$ for some complex $(1,0)$-form $\Upsilon^{\mathbb{C}}$ (in other words, $X' = X + (\Upsilon^{\mathbb{C}} \llcorner_{\mathbb{R}} X)Q - (\Upsilon^{\mathbb{C}} \llcorner_{\mathbb{R}} JX)JQ$ under the identification $TN'_{\mathbb{C}} \cong TN$).

Consequently

$$\begin{aligned} \overrightarrow{\nabla}_{X'} X' &= \overrightarrow{\nabla}_X X + 2(\Upsilon^{\mathbb{C}} \llcorner_{\mathbb{C}} X)X + \aleph' \\ &\left(= \overrightarrow{\nabla}_X X + (\Upsilon^{\mathbb{C}} \llcorner_{\mathbb{R}} X)X - (\Upsilon^{\mathbb{C}} \llcorner_{\mathbb{R}} JX)JX + \aleph'\right). \end{aligned}$$

So $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ have the same generalised complex geodesics. Given a complex geodesic μ, let ψ be any curve in it – hence a generalised complex geodesic. Then, as μ is a complex map,

$$\widetilde{\nabla}'_{\dot{\psi}} \dot{\psi} \quad \text{and} \quad \widetilde{\nabla}'_{\dot{\psi}} J\dot{\psi}$$

are sections of $\mu_*(TU)$. Consequently $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ have same complex geodesics as well. □

Definition 3.5.16 (Complex projective structure). *A complex projective structure on N is given by the complex structure J^N and the generalised complex geodesics. The connections $\widetilde{\nabla}$ are the preferred connections for this structure.*

By an analogous argument to that given for the real case in Section 3.0.2, if $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ are two connections in this class

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}'_X Y + \Upsilon^{\mathbb{C}}(X)Y + \Upsilon^{\mathbb{C}}(Y)X.$$

Again, as in Corollary 3.0.2, the preferred connection $\widetilde{\nabla}$ is bijectively determined by its effect on

powers of the complex weight bundle

$$L_{\mathbb{C}}^{-n} \cong \wedge^{(n,0)} T_{\mathbb{C}}^{*}.$$

We then define the complex Tractor bundle $\mathcal{T}_N^{\mathbb{C}}$ of N to be $T\mathcal{C}(M)$ projected onto N, and the complex cone connection $\mathcal{C}^{\mathbb{C}}(N)$ to be $\mathcal{C}(M)$. The point of these constructions is:

Theorem 3.5.17. *By looking at all possible $(\mathcal{C}(M), \overrightarrow{\nabla}, J)$ that are R-invariant, one generates all possible complex projective manifolds N. Moreover, M can be reconstructed from N.*

Proof. To prove this, we shall construct a complex cone $\mathcal{C}^{\mathbb{C}}(N)$ for any complex projective manifold N. Then M comes directly from dividing $\mathcal{C}^{\mathbb{C}}(N)$ by the action of Q.

Given a N with a complex projective structure, choose a preferred connection $\widetilde{\nabla}$. Then the Ricci tensor of $\widetilde{\nabla}$ splits into four pieces:

$$\widetilde{\mathsf{Ric}} = l_s + l_a + h_s + h_a,$$

where

$$\begin{aligned}
l_s(X,Y) &= \frac{1}{4}\left(\widetilde{\mathsf{Ric}}(X,Y) + \widetilde{\mathsf{Ric}}(Y,X) - \widetilde{\mathsf{Ric}}(JX,JY) - \widetilde{\mathsf{Ric}}(JY,JX)\right), \\
l_a(X,Y) &= \frac{1}{4}\left(\widetilde{\mathsf{Ric}}(X,Y) - \widetilde{\mathsf{Ric}}(Y,X) - \widetilde{\mathsf{Ric}}(JX,JY) + \widetilde{\mathsf{Ric}}(JY,JX)\right), \\
h_s(X,Y) &= \frac{1}{4}\left(\widetilde{\mathsf{Ric}}(X,Y) + \widetilde{\mathsf{Ric}}(Y,X) + \widetilde{\mathsf{Ric}}(JX,JY) + \widetilde{\mathsf{Ric}}(JY,JX)\right), \\
h_a(X,Y) &= \frac{1}{4}\left(\widetilde{\mathsf{Ric}}(X,Y) - \widetilde{\mathsf{Ric}}(Y,X) + \widetilde{\mathsf{Ric}}(JX,JY) - \widetilde{\mathsf{Ric}}(JY,JX)\right).
\end{aligned}$$

Here, l_s is the J-linear symmetric component of the tensor $\widetilde{\mathsf{Ric}}$, l_a the J-linear anti-symmetric component, h_s the J-hermitian symmetric component and h_a the J-hermitian anti-symmetric component. Then define the complex projective rho-tensor $\mathsf{P}^{\mathbb{C}}$ as

$$\mathsf{P}^{\mathbb{C}} = -\frac{l_s}{2n-2} - \frac{l_a + h_s + h_a}{2n+2}.$$

If $\widetilde{\nabla}$ preserves a complex volume form up to real multiplication, then

$$\mathsf{P}^{\mathbb{C}} = -\frac{l_s}{2n-2} - \frac{h_a}{2n+2}.$$

If $\widetilde{\nabla}$ preserves a complex volume form up to imaginary multiplication – equivalently, preserves a real volume form – then

$$\mathsf{P}^{\mathbb{C}} = -\frac{l_s}{2n-2} - \frac{h_s}{2n+2}.$$

Finally, if $\widetilde{\nabla}$ preserves a complex volume form, then

$$\mathsf{P}^{\mathbb{C}} = -\frac{l_s}{2n-2},$$

and, in this case, $l_s = \widetilde{\mathsf{Ric}}$.

There is also a complex projective Weyl tensor, $W^{\mathbb{C}}$. In details, this is given by

$$\begin{aligned} R_{hj\ \ l}^{\ \ k} &= (W^{\mathbb{C}})_{hj\ \ l}^{\ \ k} + \big((\mathsf{P}^{\mathbb{C}})_{hl}\delta^k_j - (\mathsf{P}^{\mathbb{C}}J)_{hl}J^k_j - (\mathsf{P}^{\mathbb{C}})_{jl}\delta^k_h + (\mathsf{P}^{\mathbb{C}}J)_{jl}J^k_h\big) \\ &\quad + \big((\mathsf{P}^{\mathbb{C}})_{hj}\delta^k_l - (\mathsf{P}^{\mathbb{C}})_{jh}\delta^k_l - (\mathsf{P}^{\mathbb{C}}J)_{hj}J^k_l + (\mathsf{P}^{\mathbb{C}}J)_{jh}J^k_l\big) \end{aligned} \tag{3.4}$$

where $(\mathsf{P}^{\mathbb{C}}J)_{hj} = (\mathsf{P}^{\mathbb{C}})_{hm}J^m_j$. If we take the tensor products to be complex, this expression becomes

$$\begin{aligned} R_{hj\ \ l}^{\ \ k} &= (W^{\mathbb{C}})_{hj\ \ l}^{\ \ k} + 2\big((\mathsf{P}^{\mathbb{C}})_{hl} \otimes \delta^k_j - (\mathsf{P}^{\mathbb{C}})_{jl} \otimes \delta^k_h\big) \\ &\quad + 2\big((\mathsf{P}^{\mathbb{C}})_{hj} \otimes \delta^k_l - (\mathsf{P}^{\mathbb{C}})_{jh} \otimes \delta^k_l\big). \end{aligned}$$

The complex Cotton-York tensor is also defined,

$$\begin{aligned} CY^{\mathbb{C}}(X,Y;Z) &= +(\widetilde{\nabla}_X\mathsf{P}^{\mathbb{C}})(Y,Z) - (\widetilde{\nabla}_Y\mathsf{P}^{\mathbb{C}})(X,Z) \\ &\quad - i(\widetilde{\nabla}_X\mathsf{P}^{\mathbb{C}})(Y,JZ) + i(\widetilde{\nabla}_Y\mathsf{P}^{\mathbb{C}})(X,JZ). \end{aligned}$$

For simplicity's sake, let $\widetilde{\nabla}$ be a preferred connection that preserves a complex volume form. The formulas work for all $\widetilde{\nabla}$, but we won't need that level of generality. Then let $\mathcal{C}^{\mathbb{C}}(N) = \mathbb{R}^2 \times N$, and

let Q and R be the vectors in the direction of $\mathbb{R}^2$. Extend J^N by defining $JQ = R$, and define the connection $\overrightarrow{\nabla}$ as

$$\begin{aligned}
\overrightarrow{\nabla} Q &= Id, \\
\overrightarrow{\nabla} R &= J, \\
\overrightarrow{\nabla}_X Y &= \widetilde{\nabla}_X Y + \mathsf{P}^{\mathbb{C}}(X,Y)Q - \mathsf{P}^{\mathbb{C}}(X,JY)R,
\end{aligned}$$

and defining the rest of the terms by torsion-freeness. $\overrightarrow{\nabla}$ obviously preserves J, and, as in the real projective case,

Lemma 3.5.18. *$\overrightarrow{\nabla}$ is Ricci-flat.*

Proof of Lemma. Let X, Y and Z be sections of H. The only non-zero components of the curvature of $\overrightarrow{\nabla}$ is

$$\begin{aligned}
\overrightarrow{R}_{X,Y}Z &= \overrightarrow{\nabla}_X \overrightarrow{\nabla}_Y Z - \overrightarrow{\nabla}_Y \overrightarrow{\nabla}_X Z - \overrightarrow{\nabla}_{[X,Y]} Z \\
&= \widetilde{R}_{X,Y}Z \\
&\quad + \mathsf{P}^{\mathbb{C}}(Y,Z)X - \mathsf{P}^{\mathbb{C}}(X,Z)Y \\
&\quad - \mathsf{P}^{\mathbb{C}}(Y,JZ)JX + \mathsf{P}^{\mathbb{C}}(X,JZ)JY \\
&\quad + (\nabla_X \mathsf{P}^{\mathbb{C}})(Y,Z)Q - (\nabla_Y \mathsf{P}^{\mathbb{C}})(X,Z)Q \\
&\quad - (\nabla_X \mathsf{P}^{\mathbb{C}})(Y,JZ)R + (\nabla_Y \mathsf{P}^{\mathbb{C}})(X,JZ)R.
\end{aligned}$$

Most of these terms will disappear upon taking the Ricci trace. In fact

$$\begin{aligned}
\overrightarrow{\mathsf{Ric}}(X,Y) &= \widetilde{\mathsf{Ric}}(X,Y) - \mathsf{P}^{\mathbb{C}}(X,Z) + 2n\mathsf{P}^{\mathbb{C}}(X,Z) + \mathsf{P}^{\mathbb{C}}(JX,JZ) \\
&= \widetilde{\mathsf{Ric}}(X,Z) + (2n-2)\mathsf{P}^{\mathbb{C}}(X,Z) \\
&= 0.
\end{aligned}$$

and all other Ricci terms are evidently zero. □

Let $\nu : N \to \mathcal{C}^{\mathbb{C}}(N)$ be a section of the projection $\mathcal{C}^{\mathbb{C}}(N) \to N$. Then projecting $\overrightarrow{\nabla}$ along Q and R gives us a preferred connection $\widetilde{\nabla}^{\nu}$ on N.

Proposition 3.5.19. *The construction of the complex cone connection $\overrightarrow{\nabla}$ is independent of ν.*

Proof. Using ν to include $TN \subset T\mathcal{C}^{\mathbb{C}}(N)$, the connection $\overrightarrow{\nabla}$ is given by

$$\begin{aligned} \overrightarrow{\nabla} Q &= Id, \\ \overrightarrow{\nabla} R &= J, \\ \overrightarrow{\nabla}_X Y &= \widetilde{\nabla}_X Y + S(X,Y)Q + S'(X,Y)R, \end{aligned}$$

for some sections S, S' of $TN^* \otimes TN^*$. The torsion-freeness of $\overrightarrow{\nabla}$ implies that S and T must be symmetric. Then the J invariance of $\overrightarrow{\nabla}$ implies

$$\begin{aligned} J\widetilde{\nabla}_X Y + S(X,Y)R - S'(X,Y)Q &= \overrightarrow{\nabla}_X JY \\ &= \widetilde{\nabla}_X Y + S(X,JY)Q + S'(X,JY)R, \end{aligned}$$

so $S'(X,Y) = -S(X,JY)$. This also means that S must be J-linear. Then the Ricci-flatness of $\overrightarrow{\nabla}$ forces

$$S(X,Z) = -\frac{\widetilde{\mathsf{Ric}^{\nu}}(X,Z)}{(2n-2)} = (\mathsf{P}^{\mathbb{C}})^{\nu}(X,Z).$$

■

One may then define the manifold M by dividing out $\mathcal{C}^{\mathbb{C}}(N)$ out by the action of Q. Since $\mathcal{C}^{\mathbb{C}}(N)$ is a real cone – as $\overrightarrow{\nabla} Q = Id$ – this generates a real projective structure on M, independently of the choice of preferred, complex volume-form preserving, connection $\widetilde{\nabla}$ on N. As stated before, one does not need the complex volume-form preserving condition – but it makes the calculations much simpler. ■

Notice that if a preferred connection $\widetilde{\nabla}$ is holomorphic, then the whole construction is just the complexification of the real case.

Remark. In fact, one may say that a general $\widetilde{\nabla}$ connection gives splittings of $T\mathcal{C}(M)$ and TM. If $\widetilde{\nabla}$ preserves a complex volume form up to real multiplication, then the second splitting comes in fact from a section $N \to M$. If $\widetilde{\nabla}$ preserves a complex volume form up to complex multiplication, then the first splitting comes from a section $M \to \mathcal{C}(M)$. And if, as in the example we've dealt with, $\widetilde{\nabla}$ preserves both, then everything is generated by an overall section $N \to \mathcal{C}(M)$.

Remark. There is a close connection between a change of real $\mathbb{C}$-preferred connection on M, $\nabla \to \nabla'$ and the corresponding change of complex preferred connections $\widetilde{\nabla} \to \widetilde{\nabla}'$ on N. The first two differ by a one-form Υ that is zero on R. Then Υ can be made R invariant by a suitable choice of isomorphisms $H \cong TN$. This makes Υ equivalent to a one-form $\Upsilon^{\mathbb{C}}$ on N, which is the one-form giving the difference between $\widetilde{\nabla}$ and $\widetilde{\nabla}'$. The converse of this is true as well.

Remark. In terms of splittings of $T\mathcal{C}(M) \cong T\mathcal{C}^{\mathbb{C}}(N)$, a splitting $TN \subset T\mathcal{C}^{\mathbb{C}}(N)$given by a preferred connection $\widetilde{\nabla}$ on N extends to a splitting $TM \subset T\mathcal{C}(M)$ by simply defining

$$TM = TN \oplus \mathbb{R}(R).$$

The connection corresponding to this splitting is ∇, the $\mathbb{C}$-preferred connection that generated $\widetilde{\nabla}$ in the first place (Lemma 3.5.11).

In this setting, the complex Tractor connection becomes, as in the real case,

$$\overrightarrow{\nabla}^{\mathbb{C}}_X = \widetilde{\nabla}_X + X + \mathsf{P}^{\mathbb{C}}(X),$$

with a complex action of X and $\mathsf{P}^{\mathbb{C}}$. The co-Tractor bundle is then $\mathcal{T}^* = J^1(K)$, where

$$K = (L_{\mathbb{C}})^{\frac{n}{n+1}},$$

is a complex weighted line bundle, with $(L_{\mathbb{C}})^n \cong \wedge^{(n,0)} T^*_{\mathbb{C}}$, analogously to the real case.

See papers [MoMo] and [KoOc] for more information. The twistor results of [Hit] are also related.

Paper [PPS] details what is actually a quaternionic projective structure, with a hypercomplex cone construction. See the later Section 5.4 for more details.

Chapter 4

Ricci-flat Holonomy

In the previous chapter, we established that all projective Tractor connections correspond to a torsion-free, Ricci-flat affine connection on a cone, and that the two connections have the same holonomy algebras. When looking at Merkulov's and Schwachhöfer's full list of torsion-free irreducible holonomies [MeSc1], it is the purpose of this chapter to classify them as to when and whether they may be Ricci-flat.

In the metric case, it is well known that the Levi-Civita connections with holonomies $\mathfrak{su}(p,q)$ and $\mathfrak{sp}(p,q)$ must be Ricci-flat [Yau], whereas those with holonomy $\mathfrak{u}(p,q)$ and $\mathfrak{sp}(p,q)\oplus\mathfrak{sp}(1)$ cannot be Ricci-flat. Those with holonomy $\mathfrak{so}(n)$ may be Ricci-flat or not.

It is also well known that Ricci-flat symmetric spaces must have reduced holonomy [KaOl] (in the definite signature case, they must be flat, since the Ricci tensor must be a non-zero multiple of the Killing form on the Lie algebra restricted to a non-degenerate subspace, [Ebe]). Thus we need only look at those irreducible holonomy groups which are non-symmetric.

4.1 Formal curvature modules

4.1.1 Spencer cohomology

See [MeSc1] for an introduction to Spencer cohomology.

Let $\mathfrak{h}$ be a Lie algebra and V an $\mathfrak{h}$-module. Since $\mathfrak{h} \subset \mathfrak{gl}(V) = V \otimes V^*$, we can inductively define the modules:

$$\begin{aligned} \mathfrak{h}^{(-1)} &= V \\ \mathfrak{h}^{(0)} &= \mathfrak{h} \\ \mathfrak{h}^{(k)} &= [\mathfrak{h}^{(k-1)} \otimes V^*] \cap [V \otimes \odot^{k+1} V^*]. \end{aligned}$$

Furthermore, if $C^{k,l}(\mathfrak{h}) = \mathfrak{h}^{(k)} \otimes \wedge^{l-1} V^*$ we may define the map

$$\partial : C^{k,l}(\mathfrak{h}) \to C^{k-1,l+1}(\mathfrak{h}),$$

via anti-symmetrisation on the last l indices. Since $\partial^2 = 0$, there is a complex

$$C^{k+1,l-1}(\mathfrak{h}) \xrightarrow{\partial} C^{k,l}(\mathfrak{h}) \xrightarrow{\partial} C^{k-1,l+1}(\mathfrak{h})$$

whose cohomology at the centre term is defined to be $H^{k,l}(\mathfrak{h})$. This is called the (k,l) Spencer cohomology group for $(\mathfrak{h}, V)$.

4.1.2 The formal curvature

Given an algebra $\mathfrak{g}$ and a faithful representation V, there is a naturally defined operator

$$\partial \left(\wedge^2 V^* \otimes \mathfrak{g} \right) \to \wedge^3 V^* \otimes V,$$

considering $\mathfrak{g}$ as a subset of $V^* \otimes V$, ∂ is just antisymmetrisation over the three components. Then we define

$$K(\mathfrak{g}) = \ker \partial.$$

In other words $K(\mathfrak{g})$ obeys the first Bianchi identity. The point of this construction is clear; if there is a torsion-free connection ν on a principal frame bundle $\mathcal{G}$ of the tangent bundle, then the curvature

of ν is a section of

$$\mathcal{G} \times_G K(\mathfrak{g}).$$

Hence we can deduce algebraic facts about the curvature of a G-connection from the module $K(\mathfrak{g})$.

By our results on Spencer cohomology from Section 4.1.1, we know that

$$0 \longrightarrow \partial\left(V^* \otimes \mathfrak{g}^{(1)}\right) \longrightarrow K(\mathfrak{g}) \longrightarrow H^{1,2}(\mathfrak{g}) \longrightarrow 0.$$

Since we will be dealing with reductive $\mathfrak{g}$'s acting reducibly, there actually is a splitting

$$K(\mathfrak{g}) = \partial\left(V^* \otimes \mathfrak{g}^{(1)}\right) \oplus H^{1,2}(\mathfrak{g}).$$

Both of these components have a geometric interpretation; the obstruction for the given G-structure being flat, given that it is 1-flat – equivalently, M admitting a flat connection with principal bundle $\mathcal{G}$, given that it admits a torsion-free one – lies in

$$\mathcal{G} \times_G H^{1,2}(\mathfrak{g})$$

whereas different torsion-free connections preserving the G-structure differ by sections of

$$\mathcal{G} \times_G \mathfrak{g}^{(1)}.$$

Remark. It is rare for an algebra to have both a $\mathfrak{g}^{(1)}$ and an $H^{1,2}(\mathfrak{g})$ component – both an obstruction to integrability and a wide class of associated connections – though a few do, such as the conformal $\mathbb{R}.\mathfrak{so}(p,q)$ and the almost Grassmannian $\mathbb{F}.\mathfrak{sl}(n,\mathbb{F}).\mathfrak{sl}(2,\mathbb{F})$.

The full list of complex groups with non-zero $\mathfrak{g}^{(1)}$ is as follows [MeSc1]:

Group G	representation V	$\mathfrak{g}^{(1)}$
$SL(n,\mathbb{C})$	$\mathbb{C}^n, n \geq 2$	$\left(V \otimes \odot^2 V^*\right)_0$
$GL(n,\mathbb{C})$	$\mathbb{C}^n, n \geq 1$	$V \otimes \odot^2 V^*$
$GL(n,\mathbb{C})$	$\odot^2\mathbb{C}^n, n \geq 2$	V^*
$GL(n,\mathbb{C})$	$\wedge^2\mathbb{C}^n, n \geq 5$	V^*
$GL(m,\mathbb{C}).GL(n,\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^n, m,n \geq 2$	V^*
$Sp(2n,\mathbb{C})$	$\mathbb{C}^{2n}, n \geq 2$	$\odot^3 V^*$
$\mathbb{C}^*.Sp(2n,\mathbb{C})$	$\mathbb{C}^{2n}, n \geq 2$	$\odot^3 V^*$
$CO(n,\mathbb{C})$	$\mathbb{C}^n, n \geq 5$	V^*
$\mathbb{C}^*.Spin(10,\mathbb{C})$	$\mathbb{C}^{16}$	V^*
$\mathbb{C}^*.E_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	V^*

Since $K(\mathfrak{g})$ is a formal curvature module, we may define the formal Ricci-curvature module $R(\mathfrak{g})$ by taking the trace of $K(\mathfrak{g})$. Then possible Ricci-flat curvatures will lie inside the kernel of

$$K(\mathfrak{g}) \to R(\mathfrak{g}).$$

If, on the other hand, this map has no kernel; in other words, if

$$K(\mathfrak{g}) \cong R(\mathfrak{g}),$$

then we say that $\mathfrak{g}$ has curvature of Ricci-type. Obviously a connection whose holonomy algebra is of Ricci-type cannot be Ricci-flat without being flat.

4.1.3 Complex algebras

In this section, any tensor product is complex unless stated otherwise. Let $\mathfrak{g}_{\mathbb{R}}$ be a real Lie algebra, with a corresponding complex Lie algebra $\mathfrak{g}$. Let $V_{\mathbb{R}}$ be a real representation of $\mathfrak{g}_{\mathbb{R}}$, and $V = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ the corresponding representation of $\mathfrak{g}$. For any two complex spaces W and U,

$$W \otimes_{\mathbb{R}} U = (W \otimes U) \oplus (\overline{W} \otimes U),$$

the $+1$ and -1 eigen-spaces of the operator $J \otimes J$ (remember that $(\overline{W} \otimes U)$ is $(W \otimes \overline{U})$ with opposite complex structure). Similarly, the module $\wedge^2_{\mathbb{R}} V^* \otimes_{\mathbb{R}} \mathfrak{g}$ splits into three sub-modules:

$$\begin{aligned} \wedge^2_{\mathbb{R}} V^* \otimes_{\mathbb{R}} \mathfrak{g} &= \left(\wedge^{(2,0)} V^* \otimes \mathfrak{g}\right) \oplus \left(\wedge^{(1,1)} V^* \otimes_{\mathbb{R}} \mathfrak{g}\right) \oplus \left(\wedge^{(0,2)} V^* \otimes \mathfrak{g}\right) \\ &= P_1 \oplus P_2 \oplus P_3, \end{aligned}$$

where $\wedge^{(n,m)} V^*$ are as defined in Appendix A. Note that $\wedge^{(1,1)} V^*$ is just the space of skew-hermitian forms; this space does *not* have a complex structure itself, hence the real tensor product in the central term. Denote by p_1, p_2, p_3 the projections onto these sub-modules. These modules are disjoint from the point of view of the ∂ map:

Lemma 4.1.1. *If $\partial(a) = 0$, then $\partial p_j(a) = 0$ for all j.*

Proof of Lemma. Assume $\partial(a) = 0$. The module P_1 is contained in the module $\otimes^3 V^* \otimes V$, so $\partial(P_1) \subset \wedge^{(3,0)} V^* \otimes V$. By the results of Appendix A, $\partial(P_2)$ and $\partial(P_3)$ are both contained in $\wedge^{2,1} V^* \otimes_{\mathbb{R}} V$. Consequently, $\partial p_1(a)$ must be zero. From now on, by replacing a with $a - p_1(a)$, we may assume that $p_1(a) = 0$.

The operator $\Theta = J \otimes J \otimes J \otimes J$ operates naturally on $\wedge^2_{\mathbb{R}} V^* \otimes_{\mathbb{R}} V^* \otimes_{\mathbb{R}} V$, and, since ∂ is an antisymmetrisation of this space and Θ is entirely symmetric,

$$\Theta \circ \partial = \partial \circ \Theta.$$

However, $\Theta(p_2(a)) = -p_2(a)$ and $\Theta(p_3(a)) = p_3(a)$, so

$$\partial p_2(a) = \tfrac{1}{2}\left(\partial(a) - \Theta\partial(a)\right) = 0$$

and

$$\partial p_3(a) = \tfrac{1}{2}\left(\partial(a) + \Theta\partial(a)\right) = 0.$$

□

On the other hand, $\wedge^{(2,0)} V^* \otimes \mathfrak{g}$ is just the complexification of the real module $\wedge^2_{\mathbb{R}} V^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$. So we can directly classify this piece of the complex module in terms of the real one:

Proposition 4.1.2. $p_1\big(K(\mathfrak{g})\big) = K(\mathfrak{g}_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$.

The next lemma deals with the P_3 component:

Lemma 4.1.3. *∂ is injective on P_3.*

Proof of Lemma. Let b_1 be an element of P_3. Then $\partial(b_1)$ equals $\frac{1}{3}(b_1 + b_2 + b_3)$ where b_2 and b_3 are the two cyclic permutations of b_1. However, if we apply $\theta = J \otimes J$ to the first two components of these elements, we see that:

$$\begin{aligned} \theta b_1 &= -b_1 \\ \theta b_2 &= b_2 \\ \theta b_3 &= b_3. \end{aligned}$$

Accordingly, $b_1 = \frac{3}{2}\big(\partial(b_1) - \theta\partial(b_1)\big)$, directly displaying the injectivity of ∂ on P_3. □

Putting this together with Lemma 4.1.1 implies that $p_3(a)$ must be zero if $\partial(a) = 0$. In other

words,

$$p_3(K(\mathfrak{g})) = 0.$$

Thus:

Theorem 4.1.4. *The formal curvature module $K(\mathfrak{g})$ splits as*

$$K(\mathfrak{g}) = K_1(\mathfrak{g}) \oplus K_2(\mathfrak{g}),$$

where $K_1(\mathfrak{g})$ is the complexification of $K(\mathfrak{g}_{\mathbb{R}})$ and $K_2(\mathfrak{g}) \subset \wedge^{(1,1)}V^* \otimes_{\mathbb{R}} \mathfrak{g}$.

Furthermore, the formal Ricci module splits into the sum of the traces of these two modules:

$$R(\mathfrak{g}) = R_1(\mathfrak{g}) \oplus R_2(\mathfrak{g}),$$

with $R_1(\mathfrak{g})$ a J-symmetric space, and $R_2(\mathfrak{g})$ a J-hermitian space.

Note that since this splitting result is true for $\mathfrak{gl}(n,\mathbb{C})$, it is also true for any $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$, even if $\mathfrak{g}$ is not itself a complex algebra (such as $\mathfrak{u}(n)$).

Example 2. To illustrate these two bundles, we can use two metric examples; first of all, let $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$. The complex metric gives an isomorphism $\mathfrak{g} \cong \wedge^{(2,0)}V^*$, and the extra metric condition that $R_{hjkl} = R_{klhj}$ gives us

$$\begin{aligned} K_1(\mathfrak{g}) &\subset \mathfrak{g} \otimes \mathfrak{g}, \\ K_2(\mathfrak{g}) &= 0. \end{aligned}$$

And, of course, the Ricci tensor of such a connection must be J-symmetric.

Example 3. Conversely, let $\mathfrak{g} = \mathfrak{u}(n)$. The hermitian metric gives an isomorphism $\mathfrak{g} = \wedge^{(1,1)}V^*$, and with the condition $R_{hjkl} = R_{klhj}$ as before,

$$\begin{aligned} K_1(\mathfrak{g}) &= 0, \\ K_2(\mathfrak{g}) &\subset \mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}. \end{aligned}$$

And, of course, these Kähler manifolds must have J-hermitian Ricci tensor.

4.2 Holonomy and Ricci-flatness

4.2.1 Volume forms and the Ricci tensor

Let $\mathcal{E}_n = \wedge^n T^*$ be the volume bundle on a manifold M^n, and ∇ a torsion free-connection on M. Then the curvature $R_{hj}{}^k{}_l$ of ∇ acts on $\mathcal{E}_n$ via its trace $R_{hj}{}^k{}_k$. However, since ∇ is torsion-free, the first Bianchi identity gives, as in Lemma 2.0.2,

$$\begin{aligned} R_{hj}{}^k{}_k &= R_{kj}{}^k{}_h + R_{hk}{}^k{}_j \\ &= \mathsf{Ric}_{hj} - \mathsf{Ric}_{jh}. \end{aligned}$$

This demonstrates the next Lemma (which is just Lemma 2.0.2 again, presented here for completeness):

Lemma 4.2.1. *A torsion-free connection ∇ preserves a volume form if and only if its Ricci tensor is symmetric.*

Similarly, if $n = 2m$ and ∇ preserves a complex structure, let $\mathcal{E}_m^{\mathbb{C}} = \wedge^{m,0} T_{\mathbb{C}}^*$ be the complex volume bundle. Then the curvature of ∇ acts on $\mathcal{E}_m^{\mathbb{C}}$ via the complex trace

$$\operatorname{trace}_{\mathbb{C}} R = \frac{1}{2}\left(\operatorname{trace}_{\mathbb{R}} R + i \operatorname{trace}_{\mathbb{R}} JR\right).$$

The first term is just the skew-symmetric part of the Ricci tensor, as before. The second term is given by

$$R_{hj}{}^k{}_l J^l_k = \left(R_{lj}{}^k{}_h + R_{hl}{}^k{}_j\right) J^l_k.$$

Since ∇ preserves the complex structure, $R_{hj}{}^k{}_l J^n_k = R_{hj}{}^n{}_k J^k_l$, implying that the previous formula becomes:

$$R_{hj}{}^k{}_l J^l_k = R_{lj}{}^l{}_k J^k_h + R_{hl}{}^l{}_k J^k_j$$

$$= \quad -\mathsf{Ric}_{jk}J^k_h + \mathsf{Ric}_{hk}J^k_j,$$

the skew-symmetric part of Ric$J$. This gives us the result:

Lemma 4.2.2. *A torsion-free connection ∇ preserves a complex volume form if and only if the tensors* Ric *and* Ric$J$ *are both symmetric.*

And this gives us our first tool for classifying Ricci-Flat spaces, notably that

Proposition 4.2.3. *A Ricci-flat space (M, ∇) with ∇ torsion-free, has a preserved real volume form, and, if ∇ preserves a complex structure, it also has a preserved complex volume form.*

Example 4. Looking back at Example 2, $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, we see that its Ricci tensor is J-symmetric. Being a metric connection, its Ricci tensor must also be symmetric, so we come to the unsurprising conclusion that a connection with holonomy $\mathfrak{so}(n, \mathbb{C})$ must preserve a complex volume form.

Example 5. On the other hand, Example 3 shows that $\mathfrak{g} = \mathfrak{u}(n)$ has a Ricci tensor that is J-hermitian, in other words J-skew. This gives us the slightly more interesting conclusion that a Kähler manifold has a preserved complex volume form (i.e. has $\mathfrak{su}(n)$ holonomy) if and only if it is Ricci-flat.

4.3 Symplectic sub-groups

These are the various sub-groups of the symplectic and complex symplectic groups, $\mathfrak{sl}(2n, \mathbb{R})$ and $\mathfrak{sl}(2n, \mathbb{C})$. The list of such groups that can appear as irreducible holonomy groups is as follows

[MeSc1]:

Group G	Representation V	Group G	Representation V
$Sp(2n,\mathbb{R})$	$\mathbb{R}^{2n}$	E_7^5	$\mathbb{R}^{56}$
$Sp(2n,\mathbb{C})$	$\mathbb{C}^{2n}$	E_7^7	$\mathbb{R}^{56}$
$SL(2,\mathbb{R})$	$\mathbb{R}^4 = \odot^3\mathbb{R}^2$	$E_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
$SL(2,\mathbb{C})$	$\mathbb{C}^4 = \odot^3\mathbb{C}^2$	$Spin(2,10)$	$\mathbb{R}^{32}$
$SL(2,\mathbb{R}).SO(p,q)$	$\mathbb{R}^{2(p+q)}, p+q \geq 3$	$Spin(6,6)$	$\mathbb{R}^{32}$
$SL(2,\mathbb{C}).SO(n,\mathbb{C})$	$\mathbb{C}^{2n}, n \geq 3$	$Spin(12,\mathbb{C})$	$\mathbb{C}^{32}$
$Sp(1).SO(n,\mathbb{H})$	$\mathbb{H}^n, n \geq 2$	$Sp(6,\mathbb{R})$	$\mathbb{R}^{14} \subset \wedge^3\mathbb{R}^6$
$SL(6,\mathbb{R})$	$\mathbb{R}^{20} \cong \wedge^3\mathbb{R}^6$	$Sp(6,\mathbb{C})$	$\mathbb{C}^{14} \subset \wedge^3\mathbb{C}^6$
$SL(6,\mathbb{C})$	$\mathbb{C}^{20} \cong \wedge^3\mathbb{C}^6$		
$SU(1,5)$	$\mathbb{R}^{20}$		
$SU(3,3)$	$\mathbb{R}^{20}$		

This section aims to prove the following theorem:

Theorem 4.3.1. *All the groups in that list, apart from $Sp(2n,\mathbb{R})$ and $Sp(2n,\mathbb{C})$ themselves, have curvature of Ricci-type:*

$$K(\mathfrak{g}) \cong R(\mathfrak{g}).$$

In other words, connections with these holonomies cannot be Ricci-flat without being flat.

Fix a given $\mathfrak{g}$ on the list, a proper subset of $\mathfrak{sp}(V,\mathbb{F})$, $V \cong \mathbb{R}^{2n}$. There are canonical manifolds with full holonomy $\mathfrak{g}$; they are constructed in [CMS2] using perturbed Poisson structures, and locally any manifold with $\mathfrak{g}$-holonomy is constructed in this way. However, we shall not need this explicit construction, as we shall demonstrate this theorem algebraically.

Fix a given symplectic form $\eta \in \wedge^2 V^*$. Given η, and since $\mathfrak{g}$ is semisimple, we have a $\mathfrak{g}$-invariant projection

$$\odot^2 V \to \mathfrak{g}.$$

Call $u \circ v$ the projection of $u \odot v$. Then, by [CMS2], [CaSc] and [CMS1], the following equalities hold for all $\mathfrak{g}$ in the list:

$$\eta(Au, v) = (A, u \circ v) \tag{4.1}$$

$$\begin{aligned}(u \circ v, s \circ t) - (u \circ t, s \circ v) = \big(&2\eta(u,s)\eta(v,t) \\ &+\eta(u,t)\eta(v,s) + \eta(u,v)\eta(s,t)\big),\end{aligned} \tag{4.2}$$

for all $A \in \mathfrak{g}$ and all $u, v, s, t \in V$, with

$$(-,-) = -\frac{1}{4n+4}B$$

where B is the Killing form on $\mathfrak{g}$ (which is the restriction of the Killing form on $\mathfrak{sp}(V,\mathbb{F})$). There is an injection of $Ad(\mathfrak{g})$ into $K(\mathfrak{g})$ given by $A \to \rho_A$,

$$\begin{aligned}\rho_A : \quad \wedge^2 V &\longrightarrow \mathfrak{g} \\ u \wedge v &\longrightarrow 2\eta(u,v)A - u \circ (Av) + v \circ (Au).\end{aligned}$$

The fact that $\rho_A \in K(\mathfrak{g})$ is guaranteed by Equations 4.1 and 4.2. Paper [MeSc1] demonstrates that the whole of $K(\mathfrak{g})$ is constructed in this manner. Then we have the following Proposition, coming from [CaSc]:

Proposition 4.3.2. $\mathsf{Ric}(\rho_A) = 0$ *iff* $A = 0$.

Proof. We shall use the following lemma:

Lemma 4.3.3. *For any element $k \in K(\mathfrak{sp}(V,\mathbb{F}))$,*

$$\mathsf{Ric}(k)(x,y) = \eta(k(\eta^{-1})x, y).$$

Proof of Lemma. Let (e_j, f_j) be a basis for V such that, when using the summation convention, $\eta^{-1} = e_j \wedge f_j$. Thus, continuing with the summation convention,

$$\begin{aligned}\mathsf{Ric}(k)(x,y) &= tr(k(x,-)y) = \eta(R_{(e_j,x)}y, f_j) - \eta(R_{(f_j,x)}y, e_j)\\ &= -\eta(R_{(x,e_j)}f_j, y) - \eta(R_{(f_j,x)}e_j, y)\\ &= \eta(R_{(e_j,f_j)}x, y),\end{aligned}$$

as η maps k to an element of $\wedge^2 V^* \otimes \odot^2 V^*$. □

Now suppose $\mathsf{Ric}(\rho_A) = 0$. This is the case iff $\rho_A(\eta^{-1}) = 0$. But then [CaSc] demonstrates $\rho_A(\eta^{-1}) = 0$ only when $A = 0$. ■

We have consequently demonstrated that $K(\mathfrak{g}) \cong R(\mathfrak{g})$, or in other words that $\mathfrak{g}$ is of Ricci-type.

4.4 Split spaces: General case

Let (M, ∇), be a manifold with affine connection, whose holonomy algebra bundle acts irreducibly on T. Let $\mathfrak{g}$ be the fiber of the holonomy algebra at a point, and V the fiber of T at the same point. By our assumptions, V is an irreducible representation of $\mathfrak{g}$.

Then we call M a split space if V is in some way the tensor product of smaller representations of $\mathfrak{g}$. In details, we say that $\mathfrak{g}$ is a *maximal* algebra if there does not exist a non-symmetric holonomy algebra $\mathfrak{h}$ such that $\mathfrak{g}$ is a strict subalgebra of $\mathfrak{h}$ and

$$[\mathfrak{h}, \mathfrak{h}] = [\mathfrak{g}, \mathfrak{g}].$$

More intuitively, $\mathfrak{g}$ is maximal if it has the maximal allowed centre. For instance, $\mathfrak{gl}(n)$, $\mathfrak{co}(n)$ and

$\mathfrak{u}(n)$ are maximal, whereas $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$ are not. The algebra $\mathfrak{spin}(7)$ is also maximal, since $\mathbb{R} \oplus \mathfrak{spin}(7)$ is not a possible holonomy algebra.

Then the following table gives the maximal split algebras:

Algebra $\mathfrak{g}$	Representation V	Restrictions
$\mathfrak{gl}(n,\mathbb{R})$	$\odot^2_{\mathbb{R}}\mathbb{R}^n$	$n \geq 3$
$\mathfrak{gl}(n,\mathbb{R})$	$\wedge^2_{\mathbb{R}}\mathbb{R}^n$	$n \geq 5$
$\mathfrak{gl}(n,\mathbb{C})$	$\odot^2_{\mathbb{C}}\mathbb{C}^n$	$n \geq 3$
$\mathfrak{gl}(n,\mathbb{C})$	$\wedge^2_{\mathbb{C}}\mathbb{C}^n$	$n \geq 5$
$\mathfrak{gl}(n,\mathbb{C})$	$H^+_n(\mathbb{C}) \cong \wedge^{1,1}\mathbb{C}^n$	$n \geq 3$
$\mathfrak{gl}(n,\mathbb{H})$	$H^+_n(\mathbb{H})$	$n \geq 3$
$\mathfrak{gl}(n,\mathbb{H})$	$H^-_n(\mathbb{H})$	$n \geq 2$
$\mathbb{C} \oplus \mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(r,\mathbb{C})$	$\mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{C}^r$	$m > r \geq 2$ or $m \geq r > 2$
$\mathbb{R} \oplus \mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(r,\mathbb{R})$	$\mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^r$	$m > r \geq 2$ or $m \geq r > 2$
$\mathbb{R} \oplus \mathfrak{sl}(m,\mathbb{H}) \oplus \mathfrak{sl}(r,\mathbb{H})$	$\mathbb{H}^m \otimes_{\mathbb{H}} \mathbb{H}^r \cong \mathbb{R}^{4mr}$	$m > r \geq 1$ or $m \geq r > 1$

Here $H^+_n(\mathbb{F})$ is the space of self-adjoint n by n matrices with entries in $\mathbb{F}$, whereas $H^-_n(\mathbb{F})$ is the

complementary space of skew adjoint ones. Notice that under multiplication by i,

$$H_n^+(\mathbb{F}) \cong H_n^-(\mathbb{C}) = \wedge^{1,1}\mathbb{C}^n$$

where $\wedge^{1,1}\mathbb{C}^n$ is defined in Appendix C.

All the algebras on this table share the property that

$$\mathfrak{g}^{(1)} = V^*,$$

see Table 4.1 and [MeSc1]. Then we aim to prove the following theorem:

Theorem 4.4.1.

$$\partial\left(V^* \otimes_{\mathbb{R}} \mathfrak{g}^{(1)}\right) \cong R(\mathfrak{g}).$$

This is enough to specify all of the algebras on this table except for the minimal Segre ones:

$$\mathbb{C} \oplus \mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}),$$

$$\mathbb{R} \oplus \mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}),$$

$$\mathbb{R} \oplus \mathfrak{sl}(m,\mathbb{H}) \oplus \mathfrak{sl}(1,\mathbb{H}).$$

For in all other cases the obstruction tensor $H^{1,2}(\mathfrak{g}) = 0$, [Bry1], so

Theorem 4.4.2. *All algebras on the table except for the minimal Segre are of Ricci-type. Consequently, neither they nor any of their subalgebras may be holonomy algebras of Ricci-flat connections.*

That theorem will remove most of what's left of possible cone holonomies.

For the rest of this section, any unspecified tensor product $\otimes$ is taken to be a real tensor product. Let $W \cong \mathbb{R}^m$ and $U \cong \mathbb{R}^r$, and let $E = \mathbb{R}^m \otimes \mathbb{R}^r$. Choose (X_k) and (Y_j), basis of $\mathbb{R}^m$ and $\mathbb{R}^r$, with dual basis (x^k) and (y^j). Then define $\mu : E^* \to E^* \odot E^* \otimes E$,

$$\mu(ab) = ay^j \otimes x^k b \otimes X_k Y_j + x^k b \otimes ay^j \otimes X_k Y_j,$$

summing over repeated indexes.

Lemma 4.4.3. *The function μ is independent of the choice of basis (X_k) and (Y_j), and is injective.*

Proof of Lemma. μ is the sum of two elements, each a reordering of the tensor product

$$ab \otimes x^j X_j \otimes y^k Y_k = ab \otimes Id_{\mathbb{R}^m} \otimes Id_{\mathbb{R}^r}$$

and that element is obviously independent of the basis. For injectivity, note that the trace of $\mu(ab)$ over the last two elements is

$$\text{trace } \mu(ab) \quad = \quad (m+r)ab.$$

□

Given complex structures J^U and J^W on U and W, we can define the inclusion of the complex tensor product into the real one, $U \otimes_J W \subset E$, with $J = (J^U, J^W)$. This is the subbundle spanned by elements of the form

$$a \otimes b - J^W a \otimes J^U b.$$

Similarly, if U is a right quaternionic structure, $J_1^U J_2^U = -J_3^U$, and W a left quaternionic structure, $J_1^W J_2^W = J_3^W$, we may define the quaternionic tensor product bundle $U \otimes_{\mathbb{H}} W$ inside E as the intersection

$$\big(U \otimes_{J_1} W\big) \cap \big(U \otimes_{J_3} W\big) \cap \big(U \otimes_{J_3} W\big),$$

$J_k = (J_k^U, J_k^W)$ as before. In fact, we need only take the intersections of the first two bundles.

Similarly, in the case when $m = r$, $W = U$, we may define the alternating $W \wedge W$ and symmetric spaces $W \odot W$ in the usual way. Then all of our representation spaces V are intersections of these various bundles; for instance in the real Segre case

$$V = E,$$

whereas in the complex symmetric case,

$$V = \left(W \otimes_{(+J,+J)} W\right) \cap \left(W \odot W\right)$$

while the complex self-adjoint bundle is given by

$$V = \left(W \otimes_{(+J,-J)} W\right) \cap \left(W \odot W\right),$$

and so on. As any complex structure J has a dual action on the dual bundle, and the transpose operation applies naturally to a space and its dual, for any of our spaces $V \subset E$, we have a well defined $V^* \subset E^*$. Consequently, we have a well defined projection $p : E^* \to V^*$.

Proposition 4.4.4. *For the representation space $V^* \subset E^*$ of a split algebra $\mathfrak{g}$,*

$$p \circ \mu(V^*) = \mathfrak{g}^{(1)},$$

where p is operating on the first element of $\mu(V^*)$.

Proof. Since we know that $\mathfrak{g}^{(1)} \cong V^*$ and that μ is injective, it suffices to show that $p \circ \mu(V^*) \subset \mathfrak{g}^{(1)}$. First we shall use the lemma:

Lemma 4.4.5. $\mu(V^*)\llcorner(V) \subset \mathfrak{g}$.

Proof of Lemma. In the Segre case,

$$\mu(ab)\llcorner CD = a(C)x^k b \otimes X_k D + b(D)ay^j \otimes CY^j.$$

This corresponds to an element of $\mathfrak{gl}(m,\mathbb{R}) \oplus \mathfrak{gl}(r,\mathbb{R})$.

In the skew case,

$$\begin{aligned} \mu(ab - ba)\llcorner(CD - DC) \quad = \quad & +a(C)x^k b \otimes X_k D + b(D)ay^j \otimes CY^j \\ & -b(C)x^k a \otimes X_k D - a(D)by^j \otimes CY^j \\ & -a(D)x^k b \otimes X_k C - b(C)ay^j \otimes DY^j \end{aligned}$$

$$+b(D)x^k a \otimes X_k C + a(C)by^j \otimes DY^j,$$

which corresponds to

$$a(C)b \otimes D - b(C)a \otimes D - a(D)b \otimes C + b(D)a \otimes C \in \mathfrak{g},$$

acting diagonally inside $\mathfrak{gl}(m,\mathbb{R}) \oplus \mathfrak{gl}(m,\mathbb{R})$. The proof in the symmetric case is the same, modulo a few sign differences.

In the complex case:

$$A = \mu(ab - JaJb)\llcorner(CD - JCJD)$$

$$=$$

$$+a(C)x^k b \otimes X_k D + b(D)ay^j \otimes CY^j$$

$$-Ja(C)x^k Jb \otimes X_k D - Jb(D)Jay^j \otimes CY^j$$

$$-a(JC)x^k b \otimes X_k JD - b(JD)ay^j \otimes JCY^j$$

$$+Ja(JC)x^k Jb \otimes X_k JD + Jb(JD)aby^j \otimes JCY^j \quad ,$$

then, using the fact that $Ja(C) = a(JC)$ and $Id = X^k \otimes x_k = -JX^k \otimes Jx_k$, one has

$$A(+ef) = A(-JeJf),$$

implying that A is contained in

$$\mathfrak{gl}(m,J) \oplus \mathfrak{gl}(r,J).$$

The lemma in the general case then follows from intersections of these various constructions. □

This lemma establishes that $p \circ \mu(V^*) \subset V^* \otimes \mathfrak{g}$. Moreover, if $v \in V$, $w \in V^*$,

$$p \circ \mu(w)(v) = \mu(w)(v)$$

by definition of what p is. Consequently $p \circ \mu(w)$ remains symmetric in the first two elements;

consequently

$$p \circ \mu(V^*) \subset \left(V^* \odot V^* \otimes V\right) \cap \left(V^* \otimes \mathfrak{g}\right) = \mathfrak{g}^{(1)}.$$

■

Before continuing, we shall see what properties $\mu(V^*)$ and $p \circ \mu(V^*)$ share; for the first is easier to work with. First of all, we know that for $v \in V$, $w \in V^*$,

$$p \circ \mu(w)(v) = \mu(w)(v).$$

However, we shall also need:

Lemma 4.4.6. *Both $\mu(V^*)$ and $p \circ \mu(V^*)$ have the same trace over the last two elements – equivalently over the first and last element.*

Proof of Lemma. The projection p commutes with the operation of taking traces. However, the trace formula is

$$\text{trace } \mu(ab) \quad = \quad (m+r)ab.$$

So trace $\mu(V^*) \subset V^*$. Therefore, as p is the identity on V^*,

$$\text{trace } \mu(V^*) = p \circ \text{trace } \mu(V^*) = \text{trace } p \circ \mu(V^*).$$

□

We are now in a position to prove the main theorem. Let $\mathbf{R}$ be the operator taking the Ricci-trace. Recall that:

$$\begin{aligned} \partial(cd \otimes \mu(ab)) \quad = \quad & (cd \otimes ay^j) \otimes x^k b \otimes X_k Y_j + (cd \otimes x^k b) \otimes ay^j \otimes X_k Y_j \\ & -(ay^j \otimes cd) \otimes x^k b \otimes X_k Y_j - (x^k b \otimes cd) \otimes ay^j \otimes X_k Y_j \end{aligned}$$

Lemma 4.4.7. *The linear maps $\mathbf{R}\ \partial(Id_{E^*} \otimes p \circ \mu)$ and $\mathbf{R}\ \partial(Id_{E^*} \otimes \mu)$, both mapping $E^* \otimes E^*$ to*

itself, are equal on $V^* \otimes V^* \subset E^* \otimes E^*$.

Proof of Lemma. This is a direct consequence of the two identities for $p \circ \mu$ and μ that we have just seen. The first two terms of $\mathbf{R}\ \partial(V^* \otimes \mu(V^*))$ involve evaluating an element of $\mu(V^*)$ on an element of V^*; the second two terms involve the tensor product of an element of V^* with the trace of an element of $\mu(V^*)$. And one can replace μ with $p \circ \mu$ in all these cases. □

Then the final statement is a consequence of:

Proposition 4.4.8. *The linear map* $\mathbf{P} = \mathbf{R}\ \partial(Id_{E^*} \otimes \mu)$ *is an isomorphism from* $E^* \otimes E^*$ *to itself.*

Proof.

$$\mathbf{P}(cd \otimes ab) = ad \otimes cb + cb \otimes ad - (m+r)cd \otimes ab.$$

Therefore

$$\frac{-1}{m+r}\mathbf{P}(cd \wedge ab) = cd \wedge ab,$$

and since $m + r > 2$,

$$\frac{2}{4-(m+r)^2}\left(\mathbf{P}(ad \odot cb) + \frac{m+r}{2}\mathbf{P}(cd \odot ab)\right) = cd \odot ab.$$

showing that $\mathbf{P}$ is surjective, and, equivalently, bijective. ■

All this implies that the $\partial(V^* \otimes \mathfrak{g}^{(1)})$ component of the curvature is of Ricci-type. Then the whole curvature must be of Ricci-type, except for the minimal Segre algebras. We will deal with those in the next chapter.

4.5 Minimal Segre algebras

There is no uniform terminology for algebras of this type. As the general algebras $\mathbb{C} \oplus \mathfrak{sl}(m, \mathbb{C}) \oplus \mathfrak{sl}(r, \mathbb{C})$ are sometimes called Segre structures, I have elected to call them 'minimal Segre' when r

is minimal – though they are sometimes referred to as 'paraconformal'. Recall that these are the algebras

$$\mathbb{C} \oplus \mathfrak{sl}(m_1, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}),$$

$$\mathbb{R} \oplus \mathfrak{sl}(m_2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}),$$

$$\mathbb{R} \oplus \mathfrak{sl}(m_3, \mathbb{H}) \oplus \mathfrak{sl}(1, \mathbb{H}).$$

Notice that for $m_1 = m_2$ and $m_1 = 2m_3$, the second two algebras are real forms of the first. Furthemore, Ricci-flatness forces the preservation of a complex volume-form by Lemma 4.2.2; we shall consequently only have to use the complex algebra $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ in this section.

Theorem 4.5.1. *Let ∇ be a Ricci-flat affine connection whose holonomy is contained in $\mathfrak{sl}(m, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Then its holonomy is contained in $\mathfrak{sl}(m, \mathbb{C})$.*

As a direct consequence of this theorem, we can say that any subalgebra or real form of $\mathfrak{g}$ acting irreducibly, cannot be a Ricci-flat holonomy algebra. This concerns the following algebras:

Algebra	Representation
$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$	$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^n \cong \mathbb{C}^{2n}, \quad n \geq 3$
$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R})$	$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^n \cong \mathbb{R}^{2n}, \quad n \geq 3$
$\mathfrak{sl}(1, \mathbb{H}) \oplus \mathfrak{sl}(n, \mathbb{H})$	$\mathbb{H}^1 \otimes_{\mathbb{H}} \mathbb{H}^n \cong \mathbb{R}^{4n}, \quad n \geq 2$
$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(2n, \mathbb{C})$	$\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^{2n} \cong \mathbb{C}^{4n}, \quad n \geq 2$
$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sp}(2n, \mathbb{R})$	$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^{2n} \cong \mathbb{R}^{4n}, \quad n \geq 2$
$\mathfrak{sp}(1) \oplus \mathfrak{sp}(p, q)$	$\mathbb{H} \otimes_{\mathbb{H}} \mathbb{H}^{(p,q)} \cong \mathbb{R}^{(4q,4q)}, \quad p + q \geq 2$

Though of course in that last case the result – that a Ricci-flat quaternionic-Kähler manifold is hyper-Kähler – is well known, [Sal2] and [Ale].

In order to prove this theorem, we shall use the quaternionic approach from paper [AlMa], modified to cover the full complex case.

Let $\mathcal{G}$ be the frame bundle for the $\mathfrak{g}$-structure, and let J_1, J_2 and J_3 be sections of the bundle $Q = \mathcal{G} \times_G \mathfrak{sl}(2,\mathbb{C})$, choosen so they obey the quaternionic identities $J_\alpha J_\beta = -\delta_{\alpha\beta} Id + \epsilon_{\alpha\beta\gamma} J_\gamma$. Thus the complex span of these elements cover all of Q.

Let ∇ be any connection associated to this $\mathfrak{g}$-structure. The curvature R^∇ of ∇ decomposes as

$$R^{\nabla'} + \Omega^1 J_1 + \Omega^2 J_2 + \Omega^3 J_3,$$

where $R^{\nabla'}$ is a curvature terms with values in $\mathcal{G} \times_G \mathfrak{sl}(m,\mathbb{C})$, and the Ω_α are sections of $\wedge^2 T^* \otimes \mathbb{C}$.

Note the commutator relation

$$\left[R^\nabla, J_\alpha\right] = 2\left(\Omega^\gamma J_\beta - \Omega^\beta J_\gamma\right)$$

where (α,β,γ) is a cyclic permutation of $(1,2,3)$. Let $\Omega^{\alpha'}$ and $\Omega^{\alpha''}$ be the real and imaginary parts of Ω^α. Since all elements of $\mathfrak{g}$ are trace-free, we may calculate the Ω^α using the formula

$$\Omega^{\alpha'}_{(X,Y)} = -\frac{1}{4m} Tr(R^\nabla_{(X,Y)} \circ J_\alpha)$$

and

$$\Omega^{\alpha''}_{(X,Y)} = \frac{1}{4m} Tr(R^\nabla_{(X,Y)} \circ iJ_\alpha).$$

Note that these traces are real traces.

There are two other operators we shall be needing: the i-linearity operator $\tilde{}$ and the operator $\hat{}$, the hermitian operator with respect to the complex structure. In details, for any section F of

$\wedge^2 T^* \otimes \mathbb{C}$,

$$\widetilde{F}(X,Y) = \frac{1}{2}(F(X,Y) - iF(X,iY))$$

while

$$\widehat{F}(X,Y) = \frac{1}{4}\big(F(X,Y) - \sum_{k=1}^{3} F(J_k X, J_k Y)\big).$$

It is easy to see that both these operators are projections, i.e. square to themselves. The i-linearity operator has certain interesting properties; indeed

Lemma 4.5.2. *If F is a section of $\wedge^{(2,0)}T^* \otimes \mathbb{C}$ – the tensor product in this expression is real – then $\widetilde{F}$ is skew-symmetric. If F is a section of $\wedge^{(1,1)}T^* \otimes \mathbb{C}$, then*

$$F(X,Y) = \widetilde{F}(X,Y) - \widetilde{F}(Y,X).$$

Proof of Lemma.

$$\begin{aligned}\widetilde{F}(X,Y) + \widetilde{F}(Y,X) &= \frac{1}{2}\big(F(X,Y) + F(Y,X) - iF(X,iY) - iF(Y,iX)\big)\\ &= 0 + \frac{-i}{2}\big(F(X,iY) + F(iY,X)\big)\\ &= 0,\end{aligned}$$

if $F \in \Gamma(\wedge^{(2,0)}T^* \otimes \mathbb{C})$. On the other hand,

$$\begin{aligned}\widetilde{F}(X,Y) - \widetilde{F}(Y,X) &= \frac{1}{2}\big(F(X,Y) - F(Y,X) - iF(X,iY) + iF(Y,iX)\big)\\ &= F(X,Y) + \frac{-i}{2}\big(F(X,iY) + F(iY,X)\big)\\ &= F(X,Y),\end{aligned}$$

if $F \in \Gamma(\wedge^{(1,1)}T^* \otimes \mathbb{C})$. □

We now aim to show that

Proposition 4.5.3. *If ∇ is Ricci-flat, then $\Omega^\alpha = 0$ for all α.*

Proof. We shall prove this statement purely algebraically. Since we may, as in section 4.1.3, split the curvature module into two components, the i-symmetric and i-hermitian components, both obeying the Bianchi identity and whose Ricci tensors are respectively i-symmetric and i-hermitian, it suffices to prove this result in the two cases where ∇ is assumed to have purely i-symmetric and purely i-hermitian curvature.

We shall deal with the first case first. Notice that this implies that Ω^α is a section of $\wedge^{(2,0)}T^* \otimes \mathbb{C}$.

Let (E^k) be a local frame on the manifold, with dual frame (e_k). Then using the Bianchi identity, the function $-4m\Omega^{\alpha'}_{(J_\alpha X, J_\alpha Y)}$ is equal to

$$\begin{aligned}
Tr(R^\nabla_{(J_\alpha X, J_\alpha Y)} \circ J_\alpha) &= \sum_k \left(R^\nabla_{(J_\alpha X, J_\alpha Y)} J_\alpha E^k\right) \llcorner e_k \\
&= -\sum_k \left(R^\nabla_{(J_\alpha Y, J_\alpha E^k)} J_\alpha X\right) \llcorner e_k - \sum_k \left(R^\nabla_{(J_\alpha E^k, J_\alpha X)} J_\alpha Y\right) \llcorner e_k \\
&= -\sum_k \left(R^\nabla_{(J_\alpha Y, J_\alpha E^k)} X\right) \llcorner J_\alpha e_k - \sum_k \left(R^\nabla_{(J_\alpha E^k, J_\alpha X)} Y\right) \llcorner J_\alpha e_k \\
&\quad -\sum_k \left(2\Omega^\gamma_{(J_\alpha Y, J_\alpha E^k)} J_\beta X - 2\Omega^\beta_{(J_\alpha Y, J_\alpha E^k)} J_\gamma X\right) \llcorner e_k \\
&\quad -\sum_k \left(2\Omega^\gamma_{(J_\alpha E^k, J_\alpha X)} J_\beta Y - 2\Omega^\beta_{(J_\alpha E^k, J_\alpha X)} J_\gamma Y\right) \llcorner e_k \\
&= -\mathsf{Ric}(J_\alpha Y, X) + \mathsf{Ric}(J_\alpha X, Y) \\
&\quad -2\big(\Omega^{\gamma'}_{(J_\alpha Y, J_\gamma X)} + \Omega^{\beta'}_{(J_\alpha Y, J_\beta X)} + \Omega^{\gamma'}_{(J_\gamma Y, J_\alpha X)} + \Omega^{\beta'}_{(J_\beta Y, J_\alpha X)}\big) \\
&\quad -2\big(\Omega^{\gamma''}_{(J_\alpha Y, iJ_\gamma X)} + \Omega^{\beta''}_{(J_\alpha Y, iJ_\beta X)} + \Omega^{\gamma''}_{(iJ_\gamma Y, J_\alpha X)} + \Omega^{\beta''}_{(iJ_\beta Y, J_\alpha X)}\big)
\end{aligned}$$

The Ric terms disappear, of course, and using the corresponding expression for $-4m\Omega^{\alpha''}_{(J_\alpha X, J_\alpha Y)}$, one gets the equation

$$-4m\Omega^\alpha_{(J_\alpha X, J_\alpha Y)} = -4\big(\widetilde{\Omega}^\gamma_{(J_\alpha Y, J_\gamma X)} + \widetilde{\Omega}^\beta_{(J_\alpha Y, J_\beta X)} + \widetilde{\Omega}^\gamma_{(J_\gamma Y, J_\alpha X)} + \widetilde{\Omega}^\beta_{(J_\beta Y, J_\alpha X)}\big), \tag{4.3}$$

since the Ω^α are i-symmetric. Notice that this equation implies that Ω^α is completely i-symmetric, i.e. that $\Omega^\alpha = \widetilde{\Omega}^\alpha$. By replacing Y with $J_\alpha Y$ and defining $\Omega = \sum_k \Omega^k_{(\cdot, J_k \cdot)}$, we may rewrite this equation as

$$(m-2)\Omega^\alpha_{Y,(J_\alpha X)} + \Omega_{(J_\alpha X, J_\alpha Y)} + \Omega_{(Y,X)} = 0. \tag{4.4}$$

By summing over $\alpha = 1, 2, 3$, we get the identity

$$(m+1)\Omega_{(Y,X)} + \sum_{\alpha} \Omega_{(J_\alpha X, J_\alpha Y)} = 0,$$

from which it follows that, if Ω^s and Ω^a are the symmetric and anti-symmetric parts of Ω,

$$-m\Omega^s = 4\widehat{\Omega}^s \quad \text{and} \quad (m+2)\Omega^a = 4\widehat{\Omega}^a.$$

which, since $m+2 \neq 4$ and $-m \neq 4$, implies that $\Omega = 0$ and hence, by Equation (4.4), that $\Omega^\alpha = 0$.

We now turn to the i-hermitian piece, for which the proof starts in the same manner, except that Equation (4.3) becomes

$$-4m\Omega^\alpha_{(J_\alpha X, J_\alpha Y)} = -4\big(\widetilde{\Omega}^\gamma_{(J_\alpha Y, J_\gamma X)} + \widetilde{\Omega}^\beta_{(J_\alpha Y, J_\beta X)} - \widetilde{\Omega}^\gamma_{(J_\alpha X, J_\gamma Y)} - \widetilde{\Omega}^\beta_{(J_\alpha X, J_\beta Y)}\big).$$

Notice the exchange of indices and signs in the last two terms. Since the section $\widetilde{F}$ is defined by the relation $\widetilde{F}(X, iY) = i\widetilde{F}(X, Y)$ and $\widetilde{F}$ remains i-hermitian if F is, we may deduce that

$$m\widetilde{\Omega}^\alpha_{(J_\alpha X, J_\alpha Y)} = -\widetilde{\Omega}^\gamma_{(J_\alpha X, J_\gamma Y)} - \widetilde{\Omega}^\beta_{(J_\alpha X, J_\beta Y)},$$

or, equivalently, after replacing X with $J_\alpha X$,

$$(m-1)\widetilde{\Omega}^\alpha_{(X, J_\alpha Y)} + \widetilde{\Omega}_{(X,Y)} = 0.$$

Summing over α gives us $(m+2)\widetilde{\Omega} = 0$ and, consequently,

$$\widetilde{\Omega}^\alpha = 0.$$

And then the relation $\Omega^\alpha(X, Y) = \widetilde{\Omega}^\alpha(X, Y) - \widetilde{\Omega}^\alpha(Y, X)$ from Lemma 4.5.2 gives us the required vanishing of Ω^α. ■

And this is all we need to prove Theorem 4.5.1.

4.6 The case of E_6

Also present in the table of possible irreducible torsion-free affine holonomy groups are various subalgebras and real forms of

$$\mathfrak{g} = \mathbb{C}^* \cdot \mathfrak{e}_6^{\mathbb{C}}.$$

We aim to prove that $\mathfrak{g}$ is of Ricci-type, and that consequently all subalgebras and real forms of it are. The representation space of $\mathfrak{g}$ is

$$V \cong \mathbb{C}^{27}.$$

This is the standard representation space of $\mathfrak{g}$. The algebra $\mathfrak{e}_6^{\mathbb{C}}$ is in fact defined as the maximal algebra preserving a certain non-degenerate cubic Ψ on V [Ada]. Non-degeneracy means that the Ψ-induced maps, $V \to \odot^2 V^*$ and $\odot^2 V \to V^*$ are of maximum rank. The full algebra $\mathfrak{g}$ must preserve Ψ up to scaling.

The Dynkin diagram of $\mathfrak{e}_6^{\mathbb{C}}$ has six nodes, and the maximal weights are given by sextuplets of non-negative integers. In this optic,

$$\begin{aligned} V &= (1,0,0,0,0,0) \\ V^* &= (0,0,0,0,1,0). \end{aligned}$$

The dual representation of $\mathfrak{e}_6^{\mathbb{C}}$ on V^* must preserve a non-degenerate cubic $\Theta \in \odot^3 V$. We choose the scale of Θ by requiring, in abstract index notation,

$$\Psi_{jkl}\Theta^{jkl} = 27.$$

We will use [MPR] in order to calculate various tensor products of representations of $\mathfrak{e}_6^{\mathbb{C}}$. Using Θ, there is a decomposition of

$$\odot^2 V^* = V \oplus U.$$

Using [MPR], one has that U is irreducible and

$$U = (0,0,0,0,2,0).$$

This decomposition implies that,

$$\Psi_{jkm}\Theta^{jkl} = Id^l_m,$$

as a map $V \to V$ or $V^* \to V^*$. Similarly,

$$\Psi_{jpm}\Theta^{jkl} = \Pi^{kl}_{pm},$$

where Π is the projection of $\odot^2 V^*$ onto its submodule V, along U. Using [MPR], we can decompose $\odot^2 V^* \otimes V$,

$$\begin{aligned}\odot^2 V^* \otimes V = \quad & (1,0,0,0,2,0) \oplus (0,0,0,0,1,1) \oplus (0,0,0,0,2,0) \\ \oplus \quad & (0,1,0,0,0,0) \oplus U^* \oplus 2V^*.\end{aligned}$$

Now, we know by [Joy] and [MeSc1] that

$$\begin{aligned}\mathfrak{g}^{(1)} &= V^* \\ H^{1,2}(\mathfrak{g}) &= 0.\end{aligned}$$

Therefore the module $\mathfrak{g}^{(1)}$ is contained inside the two V^* components of the previous decomposition. Then define two maps $V^* \to \odot^2 V^* \otimes V$ by

$$\begin{aligned}\mu_1(v_j) &= \Psi_{kmr}\Theta^{jkl}v_j \\ \mu_2(v_j) &= v_j Id^l_k + v_k Id^l_j\end{aligned}$$

where (jk) denotes symmetrisation of the indices.

Lemma 4.6.1. *μ_1 and μ_2 are injective and non-isomorphic.*

Proof of Lemma. The traces of μ_1 and μ_2 are

$$\begin{aligned} \text{trace } \mu_1(v_j) &= v_j \\ \text{trace } \mu_2(v_j) &= (28)v_j, \end{aligned}$$

proving that both are non-zero, hence (as V^* is irreducible) injective. Now contract them with an element w_l of V^*:

$$\begin{aligned} \mu_1(v_j) \llcorner w_l &= \Psi_{kmr}\Theta^{jkl}v_j w_l = \Pi^{jk}_{mr}(v_j w_l) \\ \mu_2(v_j) \llcorner w_l &= v_j w_l + w_j v_l, \end{aligned}$$

and these two elements cannot be isomorphic for general v_j and w_l. □

Consequently any map $\nu : V^* \to \odot^2 V^* \otimes V$ is given as

$$\nu = \lambda_1\mu_1 + \lambda_2\mu_2,$$

for complex constants λ_1 and λ_2.

Proposition 4.6.2. *Let $\nu : V^* \to \mathfrak{g}^{(1)}$ be an invariant isomorphic map. Then ν has $\lambda_1 = -\lambda_2$.*

Proof. We shall leave abstract index notation to the side for the moment, and we will need to provide a more explicit description of Ψ and Θ. There is an inclusion $\mathfrak{sp}(8,\mathbb{C}) \subset \mathfrak{e}_6^{\mathbb{C}}$ described as follows. Let ω be the symplectic form for $\mathfrak{sp}(8,\mathbb{C})$. Then there is a map

$$\wedge^2\mathbb{C}^{8*} \to \wedge^8\mathbb{C}^{8*},$$

by wedging with ω^3. The kernel of this map is 27-dimensional; we shall call it V^*, as it is the dual natural representation space of $\mathfrak{e}_6^{\mathbb{C}}$. To confirm this, consider the non-degenerate cubic Θ defined on it by

$$\Theta(a,b,c)\omega^4 = a \wedge b \wedge c \wedge \omega.$$

This cubic is obviously preserved by $\mathfrak{sp}(8,\mathbb{C})$, giving us the required inclusion. Let (X^j) be a basis for V, with dual basis (η_j). We may express ω as

$$\eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4 + \eta_5 \wedge \eta_6 + \eta_7 \wedge \eta_8.$$

Consequently a basis for V is given by

$$\eta_1 \wedge \eta_2 - \eta_3 \wedge \eta_4,$$

$$\eta_1 \wedge \eta_2 - \eta_5 \wedge \eta_6,$$

$$\eta_1 \wedge \eta_2 - \eta_7 \wedge \eta_8,$$

$$\eta_\alpha \wedge \eta_\beta,$$

where α and β are numbers chosen from distinct sets in the collection $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$.

We will work on an explicit example to find the (unique) relation between λ_1 and λ_2. So let

$$\begin{aligned} a &= \eta_4 \wedge \eta_6 \\ b &= \eta_1 \wedge \eta_4 \\ c &= \eta_3 \wedge \eta_6 \\ d &= \eta_2 \wedge \eta_5. \end{aligned}$$

As a consequence, $\Theta(ab) = \Theta(bc) = 0$ and $\Theta(bc) \neq 0$.

Now

$$a \wedge d \wedge \omega = -\eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6 \wedge \eta_7 \wedge \eta_8.$$

The only basis element this wedges with in a non-trivial way is $\eta_1 \wedge \eta_3$, to give 1. Consequently,

$$\Theta(ad) = X^1 \wedge X^3 = Z.$$

We now aim to calculate $\Pi(ad) = \Psi(Z)$. If (Y^σ) is a basis for $\odot^2 V^*$ with dual basis (y_σ),

$$\Psi(Z) = \sum_\sigma \Psi(ZY^\sigma) y_\sigma.$$

If (Y^σ) is the tensor product of the basis elements of V^*, the only Y^σ such that $\Psi(ZY^\sigma) \neq 0$ are

$$\begin{aligned}
X^2 \wedge X^5 &\odot X^4 \wedge X^6 \\
X^2 \wedge X^6 &\odot X^4 \wedge X^5 \\
X^2 \wedge X^7 &\odot X^4 \wedge X^8 \\
X^2 \wedge X^8 &\odot X^4 \wedge X^7 \\
X^2 \wedge X^4 &\odot \frac{1}{2}(X^1 \wedge X^2 - X^5 \wedge X^6) \\
X^2 \wedge X^4 &\odot \frac{1}{2}(X^1 \wedge X^2 - X^7 \wedge X^8).
\end{aligned}$$

Consequently there is an $\eta_2 \wedge \eta_5 \odot \eta_4 \wedge \eta_6$ summand in $\Pi(ad)$. In other words, if $W = X^4 \wedge X^6$,

$$\Pi(ad) \llcorner W = \eta_2 \wedge \eta_5 = d.$$

Note that $\Psi(d, b, c) = -1$.

An element e of $\mathfrak{g}$ preserves Θ up to scale. In our case $\Theta(abc) = 0$, so there is no issue of scale. Explicitly,

$$0 = \Theta((e \cdot a), b, c) + \Theta(a, (e \cdot b), c) + \Theta(a, b, (e \cdot c))$$

since Θ is zero on abc. Because of the choices of a, b, and c that we made, this ensured that

$$0 = \Theta((e \cdot a), b, c).$$

The above formula must hold replacing e with the element of $\mathfrak{e}_6^{\mathbb{C}}$ that is $\nu(d) \llcorner W$. This implies

$$\begin{aligned}
0 &= \lambda_1 \Theta(d, b, c) + \lambda_2 \Theta(d, b, c) \\
&= -(\lambda_1 + \lambda_2),
\end{aligned}$$

since

$$\begin{aligned} \mu_1(d) \llcorner a \llcorner W = \Pi(ad) \llcorner W &= d \\ \mu_2(d) \llcorner a \llcorner W = W(d)a + W(a)d &= d. \end{aligned}$$

■

Theorem 4.6.3. *The algebra*

$$\mathfrak{g} = \mathbb{C}^* \cdot \mathfrak{e}_6^{\mathbb{C}}.$$

acting on $V \cong \mathbb{C}^{27}$*, is of Ricci-type.*

Proof. The curvature bundle of $\mathfrak{g}$ is

$$K(\mathfrak{g}) = \partial(V^* \otimes m(V^*))$$

As before, define $\mathbf{R}$ as the Ricci-trace map $V^* \otimes V^*$ to itself. Then by the properties of μ_1 and μ_2 expounded in Lemma 4.6.1,

$$\mathbf{R}(w \otimes v) = \lambda_1 \Pi(w \odot v) - 2\lambda_1 w \odot v + 27\lambda_1 w \otimes v.$$

Now the image of $\mathbf{R}$ is not symmetric, and $\wedge^2 V^*$ is an irreducible representation of $\mathfrak{e}_6^{\mathbb{C}}$; consequently the entire $\wedge^2 V^*$ is in the image of $\mathbf{R}$. Now looking at the symmetric part:

$$\mathbf{R}(w \odot v) = \lambda_1 \big(\Pi(w \odot v) + 25 w \odot v\big).$$

Consequently

$$\mathbf{R}(\Pi(w \odot v)) = 26\lambda_1 \Pi(w \odot v),$$

and

$$\mathbf{R}((1-\Pi)(w\odot v)) = 25\lambda_1(1-\Pi)(w\odot v).$$

Consequently, as $\lambda_1 \neq 0$ since ν is non-trivial, $\mathbf{R}$ is an isomorphism, and

$$\mathfrak{g} = \mathbb{C}^* \cdot \mathfrak{e}_6^{\mathbb{C}}$$

is of Ricci-type. ■

Chapter 5

Realisation of holonomy groups

5.0.1 The list so far

In the previous chapter, we found holonomy algebras that could not correspond to Ricci-flat torsion-free (RFTF) affine connections. We have not proved the converse, notably that those not treated in the last chapter do correspond to RFTF connections. We will do this in this chapter by constructing projective cones with each remaining holonomy algebra, apart from a few low dimensional exceptions, and in doing so demonstrate:

Proposition 5.0.1. *Let* $\mathfrak{g}$ *be a holonomy algebra acting irreducibly on the tangent space. If there exists a RFTF connection with holonomy* $\mathfrak{g}$*, then apart from a few low dimensional exceptions, there exists a projective cone with holonomy* $\mathfrak{g}$*.*

It is very fortunate for our classification result that this is the case, that the holonomy algebra is not an invariant restrictive enough to rule out the cone construction in general.

The list of holonomy algebras not ruled out from the last chapter is:

algebra $\mathfrak{g}$	representation V	algebra $\mathfrak{g}$	representation V
$\mathfrak{so}(p,q)$	$\mathbb{R}^{(p,q)}$, $p+q \geq 3$	$\mathfrak{spin}(3,4)^*$	$\mathbb{R}^{(4,4)}$
$\mathfrak{so}(n,\mathbb{C})$	$\mathbb{C}^n$, $n \geq 3$	$\mathfrak{spin}(7,\mathbb{C})^*$	$\mathbb{C}^7$
$\mathfrak{su}(p,q)^*$	$\mathbb{C}^{(p,q)}$, $p+q \geq 3$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathbb{R}^n$, $n \geq 2$
$\mathfrak{sp}(p,q)^*$	$\mathbb{H}^{(p,q)}$, $p+q \geq 2$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n$, $n \geq 1$
$\mathfrak{g}_2^*$	$\mathbb{R}^7$	$\mathfrak{sl}(n,\mathbb{H})^*$	$\mathbb{H}^n$, $n \geq 1$
$\widetilde{\mathfrak{g}}_2^*$	$\mathbb{R}^{(3,4)}$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathbb{R}^{2n}$, $n \geq 2$
$\mathfrak{g}_2(\mathbb{C})^*$	$\mathbb{C}^7$	$\mathfrak{sp}(2n,\mathbb{C})$	$\mathbb{C}^{2n}$, $n \geq 2$
$\mathfrak{spin}(7)^*$	$\mathbb{R}^8$		

Algebras whose associated connections *must* be Ricci-flat have been marked with a star.

Remember the properties of the projective cone of Section 3.5 :

Definition 5.0.2. *A manifold $(C, \overrightarrow{\nabla})$ is a projective cone if:*

- $\overrightarrow{\nabla}$ *is Ricci-flat.*
- $\overrightarrow{\nabla}$ *is torsion-free.*
- *There exists $Q \in \Gamma(T_C)$ with $\overrightarrow{\nabla} Q = Id_{T_C}$.*
- $\overrightarrow{\nabla}$ *is Q-invariant.*

Recall from Lemma 2.1.8, that the property of being Q-invariant may be replaced with the equivalent condition that all curvature terms involving Q vanish.

Remark. Most constructions in this chapter will be done by taking the direct product of projective manifolds with known properties. The crux of these ideas is to exploit the fact that projective structures do not respect the taking of direct products: we shall construct examples with maximal Tractor holonomy from the direct product of projectively flat, non-flat manifolds.

5.1 Orthogonal holonomy

The bulk of the work, like the bulk of the possible holonomy groups, lie in this section. We shall construct projective cones for the first ten holonomy algebras. By Proposition 3.5.9 this will also deal with all remaining reducible conformal holonomy algebras.

We will use two approaches: either constructing a projective manifold whose Tractor connection has the holonomy we need, or directly building a projective cone with the required holonomy (and the underlying projective manifold would then emerge by projecting along the cone direction).

5.1.1 Full orthogonal holonomy

Here we aim to show that there exist projective manifolds with full $\mathfrak{so}(p,q)$ holonomy algebras. The main theorem is:

Theorem 5.1.1. *Let (M^m, ∇^M) and (N^n, ∇^N) be projectively-flat Einstein manifolds, with non-zero Ricci-curvature. Then $(C, \nabla) = (M \times N, \nabla^M \times \nabla^N)$ has full orthogonal holonomy.*

Proof. Since M is projectively flat, it has vanishing projective Weyl tensor; since it is Einstein, it has symmetric Ricci and rho tensors. Consequently the full curvature of ∇^M is given by Equation (1.6):

$$(R^M)_{hj\ \ l}^{\ \ k} = \frac{1}{1-m}\left(\mathsf{Ric}^M_{hl}(\delta^M)^k_j - \mathsf{Ric}^M_{jl}(\delta^M)^k_h\right),$$

with a similar result for ∇^N. Consequently the full curvature of ∇ is

$$R_{hj\ \ l}^{\ \ k} = (R^M)_{hj\ \ l}^{\ \ k} + (R^N)_{hj\ \ l}^{\ \ k},$$

and its Ricci curvature is

$$\mathsf{Ric}_{jl} = \mathsf{Ric}^M_{jl} + \mathsf{Ric}^N_{jl}.$$

Thus the rho tensor of ∇ is

$$\mathsf{P}_{jl} = \frac{1}{1-m-n}\big((1-m)\mathsf{P}^M_{jl} + (1-n)\mathsf{P}^N_{jl}\big).$$

In other words, the projective Weyl tensor of (C, ∇) is

$$\begin{aligned} &(R^M)_{hj\ \ l}^{\ \ k} + (R^N)_{hj\ \ l}^{\ \ k} - \big(\mathsf{P}_{hl}\delta^k_j - \mathsf{P}_{jl}\delta^k_h\big) \\ &= \\ &\tfrac{1}{1-m}\mathsf{Ric}^M_{hl}(\delta^M)^k_j + \tfrac{1}{1-n}\mathsf{Ric}^N_{hl}(\delta^N)^k_j - \tfrac{1}{(1-m-n)}\big(\mathsf{Ric}^M_{hl} + \mathsf{Ric}^N_{hl}\big)\delta^k_j \end{aligned} \tag{5.1}$$

minus the corresponding term with h and j commuted. The Cotton-York tensor vanishes, as $\nabla\mathsf{Ric}^M = \nabla\mathsf{Ric}^N = 0$. This expression therefore contains the full curvature of the Tractor connection $\overrightarrow{\nabla}$. Given the splitting defined by ∇,

$$\mathcal{A} = T^* \oplus \mathfrak{gl}(T) \oplus T,$$

we may start computing the central $(0, \overrightarrow{\mathfrak{hol}}_0, 0) \subset \overrightarrow{\mathfrak{hol}}$ term, by the use of the Ambrose-Singer Theorem [KoNo] on the Weyl tensor W. Because ∇ itself is Einstein (metric $\mathsf{Ric}^M + \mathsf{Ric}^N$, Einstein coefficient one), we know that $\overrightarrow{\mathfrak{hol}}_0 \subset \mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N)$. Then let

$$\begin{aligned} \mu_1 &= \frac{1}{1-m} - \frac{1}{1-m-n} = \frac{-n}{(1-m)(1-m-n)} \\ \mu_2 &= \frac{1}{1-n} - \frac{1}{1-m-n} = \frac{-m}{(1-n)(1-m-n)}. \end{aligned}$$

Then if X, Y are sections of T_M,

$$W(X,Y) = \mu_1 \mathsf{Ric}^M(X,-)Y - \mu_1 \mathsf{Ric}^M(Y,-)X.$$

Thus $\overrightarrow{\mathfrak{hol}}_0$ must contain $\mathfrak{so}(\mathsf{Ric}^M)$. Similarly for $\mathfrak{so}(\mathsf{Ric}^N)$. These terms lie diagonally inside the maximal bundle:

$$\begin{pmatrix} \mathfrak{so}(\mathsf{Ric}^M) & 0 \\ 0 & \mathfrak{so}(\mathsf{Ric}^M) \end{pmatrix}.$$

The upper-right and lower-left components are isomorphic, as representations of $\mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{so}(\mathsf{Ric}^N)$, to $\mathbb{R}^n \otimes \mathbb{R}^m$ and $\mathbb{R}^m \otimes \mathbb{R}^n$, respectively. They are both irreducible as representations, being tensor products of irreducible representations of distinct algebras. Consequently, decomposing $\mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N)$ as a representation of $\mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{so}(\mathsf{Ric}^N)$, one sees that

$$\overrightarrow{\mathfrak{hol}}_0 = \mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{so}(\mathsf{Ric}^N) \quad \text{or} \quad \overrightarrow{\mathfrak{hol}}_0 = \mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N),$$

To show that we are in the second case, one merely needs to consider, for $X \in \Gamma(T_M), A \in \Gamma(T_N)$,

$$W(X,Y) = \frac{-1}{1-m-n}\left(\mathsf{Ric}^M(X,-)A \; - \; \mathsf{Ric}^N(A,-)X\right),$$

evidently not an element of $\mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{so}(\mathsf{Ric}^N)$.

Since ∇ is Einstein, it must preserve a volume form ν, and we know that $\overrightarrow{\nabla}$ preserves a metric $h = \mathsf{Ric}^M + \mathsf{Ric}^N - \nu^2$ on $\mathcal{T}$. The algebra $\mathfrak{so}(h)$ decomposes as $\mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N) \oplus T$ in terms of the action of $\mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N)$; the Lie bracket on $\mathfrak{so}(h)$ is given by the natural action of the first component on the latter. Consequently, as before,

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N) \quad \text{or} \quad \overrightarrow{\mathfrak{hol}} = \mathfrak{so}(h).$$

To show the latter, we turn to infinitesimal holonomy. Since ∇ annihilates both Ricci tensors, we

have the expression, for X, Y, Z now sections of T:

$$\overrightarrow{\nabla}(0, W, 0)(X; Y, Z) = (W(Y, Z)\mathsf{P}(X),\ 0,\ W(Y, Z)X).$$

And one may evidently choose X, Y, Z to make that last expression non-zero. ■

Now we need to find projectively flat manifolds with the required properties. The ideal candidates spring to mind: the quadrics of Proposition 2.2.15. Using them, we may construct manifolds of dimension ≥ 4 with orthogonal holonomy of signature $(a, b+1)$ for any non-negative integers a and b. However, since orthogonal holonomy with signature $(a, b+1)$ is equivalent to signature $(b+1, a)$, we actually have all the orthogonal holonomy algebras in dimension ≥ 4. For dimension three, one must remember that all orthogonal projective holonomy corresponds to the conformal holonomy of a conformally Einstein structure. However all conformally Einstein 3-folds are conformally flat; hence there cannot exist non-trivial orthogonal projective holonomy in dimension three, and, a fortiori, in dimensions two and one.

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{so}(p, q),\ \mathbb{R}^{(p,q)},\ p+q \geq 5\big),$$

are possible projective (and conformal) holonomy algebras.

Theorem 5.1.2. *Let (M^{2m}, ∇^M) and (N^{2n}, ∇^N) be $\mathbb{C}$-projectively-flat complex Einstein manifolds, with non-zero Ricci-curvature, which is moreover symmetric under the complex structure. Then $(H, \nabla) = (M \times N, \nabla^M \times \nabla^N)$ has full orthogonal $\mathbb{C}$-projective holonomy.*

Proof. In this case,

$$\begin{aligned} \mathsf{P}^{\mathbb{C}}_M &= \frac{1}{2(1-m)}\mathsf{Ric}^M \\ \mathsf{P}^{\mathbb{C}}_N &= \frac{1}{2(1-n)}\mathsf{Ric}^N. \end{aligned}$$

Now $\nabla = \nabla^N \times \nabla^M$ has Ricci tensor $\mathsf{Ric}^N + \mathsf{Ric}^M$, a symmetric and $\mathbb{C}$-linear tensor; thus ∇ must preserve a complex volume form ν. Then the $\mathbb{C}$-projective holonomy of $M \times N$ must preserve the

complex metric $\mathsf{P}^{\mathbb{C}}_M + \mathsf{P}^{\mathbb{C}}_N - \nu^2$. Moreover,

$$(R^M)_{hj\ l}^{\ k} = \frac{1}{(1-m)}\big(\mathsf{Ric}^M_{hl} \otimes_{\mathbb{C}} (\delta^M)^k_j - \mathsf{Ric}^M_{jl} \otimes_{\mathbb{C}} (\delta^M)^k_h\big),$$

and similarly for N. With these observations, the proof then proceeds in exactly the same way as in the real case. ■

To construct such manifolds, one takes the complex versions of the quadrics in the previous argument, and their direct product as before.

By the previous results of Theorem 3.5.17 any $\mathbb{C}$-projective manifold $M \times N$ with $\mathbb{C}$-projective holonomy algebra $\overrightarrow{\mathfrak{hol}}$ corresponds to a real projective manifold one dimension higher, with $\overrightarrow{\mathfrak{hol}}$ as (real) projective holonomy algebra.

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{so}(n,\mathbb{C}),\ \mathbb{C}^n,\ n \geq 5\big),$$

are possible projective (and conformal) holonomy algebras.

5.1.2 $\mathfrak{su}$ holonomies

When we talk of a manifold with Tractor holonomy $\mathfrak{su}(p,q)$, we are talking about, by definition, a conformally/projectively Einstein manifold whose metric cone is Ricci-flat and has holonomy $\mathfrak{su}(p,q)$. In other words this is a Sasaki-Einstein manifold. The existence of Sasaki-Einstein manifolds has been addressed in [BFGK] and [Boh] as well as [BGN]; I have adapted their proof, changing the terminology slightly.

Let (N^{2m}, g, ω, J) be a Kähler-Einstein manifold, of complex signature (p,q). Normalise the metric (inversing the signature if needed) so that the Einstein coefficient is 1. Normalise the Kähler form ω so $g = \omega(-, J-)$. Then let M be an S^1 bundle over N, with a connection form η such that

$$d\eta = 2\mu\omega,$$

for $\mu = \frac{1}{1-2m}$. Since we are working locally, there are no existence issues, this is just the $\frac{2\mu}{n}$ complex weight bundle. Let S be the vector field generated by the S^1 action. This whole construction implies that if X and Y are sections of TN and $\overline{X}$, $\overline{Y}$ their S-invariant lifts into TM,

$$[\overline{X},\overline{Y}] = \overline{[X,Y]} + 2\mu\omega(X,Y).$$

Then for $X, Y \in \Gamma(TN)$, the connection $\widetilde{\nabla}$ is

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \mu\omega(X,Y)S \\ \widetilde{\nabla}_S Y = \widetilde{\nabla}_Y S &= JY \\ \widetilde{\nabla}_S S &= 0, \end{aligned}$$

Notice that $\widetilde{\nabla}$ is torsion-free and preserves the metric $h = \mu g - (ds)^2$.

Proposition 5.1.3. *$(M, \widetilde{\nabla})$ is a Sasaki-Einstein manifold for a suitable choice of μ.*

Proof. The curvature of ∇ is

$$\begin{aligned} \widetilde{R}_{SY}X = \widetilde{R}_{XY}S &= 0 \\ \widetilde{R}_{SY}S &= -Y \\ \widetilde{R}_{XY}Z &= R_{XY}Z + \mu\big(\omega(Y,Z)JX - \omega(X,Z)JY\big). \end{aligned}$$

Implying that

$$\begin{aligned} \widetilde{\mathsf{Ric}}(S,Y) &= 0 \\ \widetilde{\mathsf{Ric}}(S,S) &= 2m \\ \widetilde{\mathsf{Ric}}(X,Y) &= g(X,Y) - \mu g(X,Y). \end{aligned}$$

By definition of h and μ, $\widetilde{\mathsf{Ric}} = -2mh$. Hence $(M, \widetilde{\nabla})$ is Einstein. Moreover S is a Killing vector field for h, $\nabla S = J$ is a complex structure on the lift of TN into TM.

Consequently $(M, \widetilde{\nabla})$ is Sasaki-Einstein. ∎

Notice that the real signature of $(M, \widetilde{\nabla})$ is $(2p, 2q+1)$ or $(q, p+1)$, depending on the sign of the Einstein coefficient for N, so we get all signatures of Sasaki-Einstein manifolds.

Remark. Because of the sign conventions used, the real signature of the cone over these Sasaki-Einstein manifolds is $(2p, 2q+2)$ or $(2q, 2p+2)$, as one expects.

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{su}(m,n),\ \mathbb{C}^{(m,n)},\ m+n \geq 3\big),$$

are possible projective (and conformal) holonomy algebras.

5.1.3 $\mathfrak{sp}$ holonomies

When we talk of a manifold with Tractor holonomy $\mathfrak{sp}(p,q)$, we are talking about, by definition, a conformally/projectively Einstein manifold whose metric cone is Ricci-flat and has holonomy $\mathfrak{sp}(p,q)$. In other words this is a 3-Sasaki manifold.

The proof of this is exactly as above, except that one uses N, an Einstein Quaternionic-Kähler, as the base manifold, and M is a principal $SU(2) \cong Sp(1)$ bundle.

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{sp}(m,n),\ \mathbb{H}^{(m,n)},\ m+n \geq 3\big),$$

are possible projective (and conformal) holonomy algebras.

5.1.4 Exceptional holonomies

Bryant [Bry2] constructs manifolds with exceptional holonomy as cones on other manifolds. All manifolds with exceptional holonomy are Ricci-flat, so these are Ricci-flat cones by definition.

In [Bry2], Bryant shows that the real cone on $SU(3)/T^2$ has holonomy G_2 and the real cone on $SU(2,1)/T^2$ has holonomy $\widetilde{G}_2$. Moreover the *complex* cone on $SL(3,\mathbb{C})/T^2_{\mathbb{C}}$ has holonomy $G_2^{\mathbb{C}}$; this corresponds to $SL(3,\mathbb{C})/T^2_{\mathbb{C}}$ having $\mathbb{C}$-projective holonomy $G_2^{\mathbb{C}}$. And, of course, this implies

that there exists a manifold one dimension higher – hence of dimension 15 – with real projective holonomy $G_2^{\mathbb{C}}$.

Similarly the cone on $SO(5)/SO(3)$ has holonomy $Spin(7)$. The other $Spin(7)$ cases weren't dealt with in the paper, but one can extend the arguments there to show that the real cone on $SO(3,2)/SO(2,1)$ has holonomy $Spin(3,4)$ and that the complex cone on $SO(5,\mathbb{C})/SO(3,\mathbb{C})$ has holonomy $Spin(7,\mathbb{C})$.

5.2 Full holonomy

Here we aim to show that there exist projective manifolds with full $\mathfrak{sl}(n,\mathbb{R})$ holonomy algebras. The main theorem is:

Theorem 5.2.1. *Let (M^m,∇^M) and (N^n,∇^N) be projectively-flat manifolds, non-Einstein but with non-degenerate symmetric Ricci tensors. Then $(C=M\times N,\nabla=\nabla^M\times\nabla^N)$ has full holonomy $\mathfrak{sl}(n+m,\mathbb{R})$.*

Proof. This proof is initially modelled on that of the existence of full orthogonal holonomy in Theorem 5.1.1. But first we need:

Lemma 5.2.2. *The Cotton-York tensor of ∇ vanishes.*

Proof of Lemma. Both manifolds are projectively flat, so have no Tractor curvature. Since the Tractor curvature includes their Cotton-York tensor (Equation (1.8)), this last must vanish. So if X and Y are sections of TM, X' and Y' sections of TN,

$$\begin{aligned}(\nabla_X \mathsf{Ric}^M)(Y,-) &= (\nabla_Y \mathsf{Ric}^M)(X,-)\\ (\nabla_{X'} \mathsf{Ric}^N)(Y',-) &= (\nabla_{Y'} \mathsf{Ric}^N)(X',-).\end{aligned}$$

Then since Ric^M is covariantly constant in the N direction (and vice versa),

$$\begin{aligned}0=(\nabla_{X'} \mathsf{Ric}^M)(Y,-) &= (\nabla_Y \mathsf{Ric}^M)(X',-)\\ 0=(\nabla_{X'} \mathsf{Ric}^N)(Y,-) &= (\nabla_Y \mathsf{Ric}^N)(X',-).\end{aligned}$$

Consequently the Cotton-York tensor of (C, ∇) vanishes. □

Exactly as in Theorem 5.1.1, there exists a summand $\mathfrak{h} = \mathfrak{so}(\mathsf{Ric}^M + \mathsf{Ric}^N) \subset \overrightarrow{\mathfrak{hol}}_0$. Under the action of $\mathfrak{h}$, the bundle $\mathfrak{sl}(\mathcal{T})$ splits as

$$\mathfrak{sl}(\mathcal{T}) = \mathfrak{h} \oplus \odot^2_0 TC \oplus TC \oplus TC^* \oplus \mathbb{R}.$$

Here the bundles TC and TC^* are isomorphic as representations of $\mathfrak{h}$.

Now using infinitesimal holonomy, we consider the first derivative:

$$\overrightarrow{\nabla} \begin{pmatrix} 0 \\ W \\ 0 \end{pmatrix} (X; Y, Z) = \begin{pmatrix} W(Y,Z)\mathsf{P}(X) \\ (\nabla_X W)(Y,Z) \\ W(Y,Z)X \end{pmatrix}. \tag{5.2}$$

Let X and Y be sections of TM, Z a section of TN. Then Equation (5.1) implies that the central term is

$$(\nabla_X W)(Y,Z) = \frac{-1}{1-m-n}(\nabla_X \mathsf{Ric})(Y,-)Z.$$

Since M is non-Einstein, there must exist X and Y such that this term in non-zero. This term is evidently not a section of $\mathfrak{h}$, so

$$\mathfrak{h} \oplus \odot^2_0 TC = \mathfrak{sl}(TC) \subset \overrightarrow{\mathfrak{hol}}.$$

Now $\mathfrak{sl}(TC)$ does distinguish between TC and TC^*; thus looking at Equation (5.2), we can see that $T \oplus T^* \subset \overrightarrow{\mathfrak{hol}}_0$. Then the last $\mathbb{R}$ term is generated by the Lie bracket between TC and TC^*, so

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{sl}(\mathcal{T}).$$

■

We now need to show the existence of such manifolds; in order to do that, we have

Proposition 5.2.3. *There exist manifolds with the conditions of Theorem 5.2.1.*

Proof. Consider $\mathbb{R}^n$, with standard coordinates x^l and frame $X^l = \frac{\partial}{\partial x^l}$ and let ∇' be the standard flat connection on N. Using a one form Υ, the connection changes to

$$\nabla_X Y = \nabla'_X Y + \Upsilon(X)Y + \Upsilon(Y)X.$$

Similarly, since ∇' is Ricci-flat, the rho-tensor of ∇ is, by Equation (1.7),

$$\mathsf{P}_{hj} = -\nabla'_j \Upsilon_h + \frac{1}{2}\Upsilon_{hj}^2.$$

Now if we choose $\Upsilon = dx_1 + \sum_l x_l dx_l$, the tensor P is given by

$$\mathsf{P} = \sum_l (x_l dx_1 \odot dx_l - dx_l \odot dx_l) + O(2)$$

This is non-degenerate at the origin. Since $\Upsilon = dx_1 + O(1)$,

$$\nabla_{X^1} X^2 = X^2 + O(1)$$

and

$$\begin{aligned}(\nabla_{X^1}\mathsf{P})(X^2, X^2) &= X^1 \cdot \mathsf{P}(X^2, X^2) - \mathsf{P}(\nabla_{X^1} X^2, X^2) - \mathsf{P}(X^2, \nabla_{X^1} X^2) \\ &= 0 - 2\mathsf{P}(X^2, X^2) \\ &= -2 + O(1).\end{aligned}$$

So ∇ is non-Einstein at the origin. Since being non-degenerate and non-Einstein are open conditions, there exists a neighbourhood of the origin with both these properties. Define this to be N. One needs lastly to see that P (and thus Ric) is symmetric – equivalently, that ∇ preserves a volume form. One can either see it directly by the formula for P, or one can observe that since ∇' preserves a volume form, the preferred connection ∇ preserves one if and only if Υ is closed. But this is immediate since

$$\Upsilon = d\left(x_1 + \sum_l \frac{x_l^2}{2}\right).$$

■

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{sl}(n,\mathbb{R}),\ \mathbb{R}^n,\ n \geq 5\big),$$

are possible projective holonomy algebras.

5.3 Complex holonomy

To show that one has full complex holonomy is actually simpler than in the real case. The crucial theorem is:

Theorem 5.3.1. *Let (M^{2m}, ∇^M) and (N^{2n}, ∇^N) be $\mathbb{C}$-projectively-flat manifolds, both Einstein, with non-degenerate Ricci tensors. Assume further that Ric^M is $\mathbb{C}$-linear while Ric^N is $\mathbb{C}$-hermitian. Then $(C = M \times N, \nabla = \nabla^M \times \nabla^N)$ has full complex holonomy $\mathfrak{sl}(n+m,\mathbb{C})$.*

Proof. In this case,

$$\begin{aligned} \mathsf{P}^{\mathbb{C}}_M &= \frac{1}{2(1-m)}\mathsf{Ric}^M \\ \mathsf{P}^{\mathbb{C}}_N &= \frac{1}{2(-1-n)}\mathsf{Ric}^N. \end{aligned}$$

Consequently the curvature tensors of ∇^M and ∇^N are given, according to Equation (3.4), by

$$\begin{aligned} (R^M)_{hj\ \ l}^{\ \ k} &= \tfrac{-1}{m-1}\ \big(\mathsf{Ric}^M_{hl} \otimes_{\mathbb{C}} (\delta^M)^k_j - \mathsf{Ric}^M_{jl} \otimes_{\mathbb{C}} (\delta^M)^k_h\big) \\ (R^N)_{hj\ \ l}^{\ \ k} &= \tfrac{-1}{n+1}\ \big(\mathsf{Ric}^N_{hl} \otimes_{\mathbb{C}} (\delta^M)^k_j - \mathsf{Ric}^N_{jl} \otimes_{\mathbb{C}} (\delta^M)^k_h \\ &\qquad + \mathsf{Ric}^N_{hj} \otimes_{\mathbb{C}} \delta^k_l - \mathsf{Ric}^N_{jh} \otimes_{\mathbb{C}} \delta^k_l\big). \end{aligned}$$

As usual, the complex Cotton-York tensor is zero, meaning the full curvature of the Tractor connection is contained in the Weyl tensor. We aim to calculate the $\mathbb{C}$-projective holonomy of C. From now on, any implicit tensor product is taken to be complex. Then as in the proof of Theorem 5.1.1,

it is easy to see that if $X, Y \in \Gamma(TM')$,

$$W(X,Y) = \mu_1 \mathsf{Ric}^M(X,-)Y - \mu_1 \mathsf{Ric}^M(Y,-)X.$$

Alternatively, if $X, Y \in \Gamma(TN')$,

$$\begin{aligned} W(X,Y) &= \mu_3 \mathsf{Ric}^N(X,-)Y - \mu_3 \mathsf{Ric}^N(Y,-)X \\ &\quad + \mu_3\big(\mathsf{Ric}^N(X,Y) - \mathsf{Ric}^N(Y,X)\big), \end{aligned}$$

where

$$\mu_3 = \frac{1}{m+n+1} - \frac{1}{n+1}.$$

Consequently, we can see that

$$\mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{u}(\mathsf{Ric}^N) \subset \overrightarrow{\mathfrak{hol}}_0 \subset \overrightarrow{\mathfrak{hol}}.$$

Where $\overrightarrow{\mathfrak{hol}}_0$ is the $\mathfrak{gl}(TC')$ component of $\overrightarrow{\mathfrak{hol}}$, the $\mathbb{C}$-projective holonomy algebra of C. Now under the action of $\mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{u}(\mathsf{Ric}^N)$, $\mathfrak{gl}(TC')$ splits as

$$\mathfrak{gl}(TC_{\mathbb{C}}) = \left(\begin{array}{c|c} \mathfrak{so}(\mathsf{Ric}^M) \oplus \odot^2_0 TM_{\mathbb{C}} \oplus \mathbb{C} & A \\ \hline B & \mathfrak{u}(\mathsf{Ric}^N) \oplus i\mathfrak{u}(\mathsf{Ric}^N) \end{array}\right).$$

Here $A = TM' \otimes_{\mathbb{C}} (TN')^*$ and $B = (TM')^* \otimes_{\mathbb{C}} TN'$. These are irreducible, but *not* isomorphic representations of $\mathfrak{so}(\mathsf{Ric}^M) \oplus \mathfrak{u}(\mathsf{Ric}^N)$, because of the action $\mathfrak{u}(\mathsf{Ric}^N)$. Now if X is a section of TM' and Y is a section of TN',

$$W(X,Y) = \frac{\mathsf{Ric}^M(X,-)Y}{m+n-1} - \frac{\mathsf{Ric}^N(Y,-)X}{m+n+1},$$

an element of $A \oplus B$ that is neither in A nor in B. Consequently $A \oplus B \subset \overrightarrow{\mathfrak{hol}}_0$. But the span of

$A \oplus B$ under the Lie bracket is the full algebra $\mathfrak{sl}(TC')$. So

$$\mathfrak{sl}(TC') \oplus i\mathbb{R} \subset \overrightarrow{\mathfrak{hol}}_0.$$

Let $\mathfrak{h} = \mathfrak{sl}(TC') \oplus i\mathbb{R}$. Under the action of $\mathfrak{h}$, the full algebra $\mathfrak{sl}(\mathcal{T}, \mathbb{C})$ splits as

$$\mathfrak{sl}(\mathcal{T}, \mathbb{C}) = \mathfrak{h} \oplus TC' \oplus (TC')^* \oplus \mathbb{R}.$$

Lemma 5.3.2. $\mathfrak{h} \oplus TC' \oplus (TC')^* \subset \overrightarrow{\mathfrak{hol}}$

Proof of Lemma. This is the standard argument, involving infinitesimal holonomy. TC' and $(TC')^*$ are irreducible non-isomorphic representations of $\mathfrak{h}$. Then

$$\overrightarrow{\nabla}(0, W, 0)(X; Y, Z) = (W(Y, Z)\mathsf{P}^{\mathbb{C}}(X),\ 0,\ W(Y, Z)X).$$

Consequently $TC' \oplus (TC')^* \subset \overrightarrow{\mathfrak{hol}}$. □

To end the proof, notice that you can generate the final $\mathbb{R}$ term by taking the Lie bracket on $TC' \oplus (TC')^*$. So

$$\overrightarrow{\mathfrak{hol}} = \mathfrak{sl}(\mathcal{T}, \mathbb{C}).$$

■

To construct an explicit example of the previous, it suffices to take M as a complex version of the quadrics of Proposition 2.2.15, and N to be the (Einstein-Kähler) projective plane. As a consequence of this, we have manifolds with full $\mathbb{C}$-projective holonomy, which corresponds, by Theorem 3.5.17, to a real projective manifold one dimension higher, with same real projective holonomy algebra.

Consequently,

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{sl}(n, \mathbb{C}),\ \mathbb{C}^n,\ n \geq 4\big),$$

are possible projective holonomy algebras.

5.4 Quaternionic holonomy

The holonomy algebra $\mathfrak{sl}(n,\mathbb{H})$ forces the manifold to be Ricci-flat by definition [Bar] and [AlMa], so we shall focus on the cone conditions.

Paper [PPS], building on ideas from [Sal1] and [Joy], demonstrates that when one has a hypercomplex cone construction $(\mathcal{C}(M),\overrightarrow{\nabla},I,J,K)$, such that $\overrightarrow{\nabla}$ is invariant under the actions of IQ, JQ, KQ and – trivially – Q, one may divide out by these actions to get a Quaternionic manifold N. Furthermore, a choice of compatible splitting of $T\mathcal{C}(M)$ is equivalent to a choice of torsion-free connection preserving the quaternionic structure. Thus we have the following natural definitions:

Definition 5.4.1 (Quaternionic Projective Structure). *A quaternionic projective structure is simply a reduction of the structure group of the tangent bundle to* $\mathfrak{gl}(n,\mathbb{H})\oplus\mathfrak{sl}(1,\mathbb{H})$. *The preferred connections are just the torsion-free connections preserving this structure. The total space of the cone construction is the bundle*

$$L^{-\frac{n}{n+1}}\otimes H,$$

where H *is the natural rank* 4 *bundle associated to* $\mathfrak{sl}(1,\mathbb{H})$.

The definition of [AlMa] for the change of quaternionic connection by a choice of one-form is exactly analogous to our formulas for the change of real or complex preferred connections. See paper [ADM] for the definition of the quaternionic Weyl tensor (recalling that any quaternionic-Kähler manifold is Einstein, so any expression involving the metric can be replaced with one involving the Ricci tensor, for the general case).

In fact, our results are somewhat stronger than in the complex case: since $\overrightarrow{\nabla}$ is hypercomplex,

$$\overrightarrow{R}_{X,Y}=\overrightarrow{R}_{IX,IY}$$

by [PPS] and [Sal1]. Consequently all curvature terms involving IQ, JQ and KQ vanish and, as in the proof of Lemma 2.1.8,

Proposition 5.4.2. *Every hypercomplex cone is* IQ*-,* JQ*- and* KQ*-invariant, and thus every hypercomplex cone corresponds to a quaternionic projective structure.*

Given this definition, one may construct examples similarly to the real and complex cases; indeed:

Theorem 5.4.3. *Let (M^{4m}, ∇^M) and (N^{4n}, ∇^N) be quaternionic projectively-flat manifolds, non-Einstein but with non-degenerate symmetric Ricci tensors. Then $(C = M \times N, \nabla = \nabla^M \times \nabla^N)$ has full quaternionic Tractor holonomy $\mathfrak{sl}(n+m, \mathbb{H})$.*

The proof is analogous to the real case, and one can choose M and N to be quaternionic projective spaces, with a suitable non Einstein connection, again as in the real case.

Then one may construct the quaternionic cone and divide out by the action of Q to get a real projective manifold with the same Tractor holonomy.

Consequently,

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{sl}(n, \mathbb{H}),\ \mathbb{H}^n,\ n \geq 3\big),$$

are possible projective holonomy algebras.

5.5 Symplectic holonomies

The constructions used here were originally discovered in a different context by Simone Gutt, to whom I am very grateful. Paper [BCGRS] also contains the construction of what is effectively a 'symplectic projective structure', with its own Weyl and rho tensors. Though we will not use or detail this explicitly, it is implicitly underlying some aspects of the present proof.

5.5.1 Real symplectic

Let M^{2n+1} be a contact manifold, with a choice of contact form $\alpha \in \Gamma(T^*)$. We may then define the Reeb vector field $E \in \Gamma(T)$ on M by

$$\begin{aligned} \alpha(E) &= 1, \\ d\alpha(E, -) &= 0. \end{aligned}$$

Since α is a contact form, this suffices to determine E entirely. Let $H \subset T$ be the contact distribution defined by $\alpha(H) = 0$.

Lemma 5.5.1. *If $X \in \Gamma(H)$ then $[E, X] \in \Gamma(H)$.*

Proof of Lemma. By definition,

$$\begin{aligned} 0 = d\alpha(E, X) &= E \cdot \alpha(X) - X \cdot \alpha(E) - \frac{1}{2}\alpha([E, X]) \\ &= -\frac{1}{2}\alpha([E, X]). \end{aligned}$$

Hence $[E, X] \in \Gamma(H)$. □

Lemma 5.5.2. $\mathcal{L}_E \alpha = 0$.

Proof of Lemma. For X a section of H,

$$\begin{aligned} (\mathcal{L}_E\alpha)(X) &= \mathcal{L}_E(\alpha(X)) - \alpha(\mathcal{L}_E X) \\ &= 0 - \alpha([E, X]) = 0. \end{aligned}$$

Similarly

$$\begin{aligned} (\mathcal{L}_E\alpha)(E) &= \mathcal{L}_E(\alpha(E)) - \alpha(\mathcal{L}_E E) \\ &= E \cdot 1 - 0 = 0. \end{aligned}$$

□

Lemma 5.5.3. $\mathcal{L}_E(d\alpha) = 0$

Proof of Lemma. Immediate since $[\mathcal{L}, d] = 0$. □

This gives us the following proposition:

Proposition 5.5.4. *Dividing out by the action of the one-parameter sub-group generated by E gives a map $\pi : M \to (N, \nu)$ with (N, ν) a symplectic manifold and $d\alpha = \pi^*\nu$.*

If X and Y are now sections of T_N, they have unique lifts $\overline{X}$ and $\overline{Y}$. Then since $d\alpha(\overline{X},\overline{Y}) = -\frac{1}{2}\alpha([\overline{X},\overline{Y}])$, we have

$$[\overline{X},\overline{Y}] = \overline{[X,Y]} - 2\nu(X,Y)E. \tag{5.3}$$

The point of all these constructions in the following theorem:

Theorem 5.5.5. *Given $\pi : M \to (N,\nu,\nabla)$ such that M is a contact manifold with contact form α with $d\alpha = \pi^*\nu$, and ∇ a connection preserving ν, there exists a Ricci-flat, torsion-free cone connection $\overrightarrow{\nabla}$ on $\mathfrak{M} = \mathbb{R}\times M$ that preserves the symplectic form $e^{2q}(d\alpha + dq\wedge\alpha)$, where q is the coordinate along $\mathbb{R}$.*

Proof. Let s be a section of $\odot^2 TN^*$, U a section of TN and f a function on N. Then define the following connection on $\mathfrak{M}$:

$$\begin{aligned}
\overrightarrow{\nabla}_{\overline{X}}\overline{Y} &= \overline{\nabla_X Y} - \nu(X,Y)E - s(X,Y)Q\\
\overrightarrow{\nabla}_E\overline{X} &= \overrightarrow{\nabla}_{\overline{X}}E = \overline{\sigma X} + \nu(X,U)Q\\
\overrightarrow{\nabla}_E E &= \pi^* fQ - \overline{U}\\
\overrightarrow{\nabla}_Q X &= X\\
\overrightarrow{\nabla}_Q E &= E\\
\overrightarrow{\nabla}Q &= Id.
\end{aligned} \tag{5.4}$$

Where $s(X,Y) = \nu(X,\sigma Y)$, or, in other words, $\sigma^k_j = s_{hj}\nu^{hk}$. One can see immediately from Equation (5.3) that $\overrightarrow{\nabla}$ is torsion-free. It is obviously a cone connection. On top of that:

Proposition 5.5.6. *$\overrightarrow{\nabla}$ is a symplectic connection, for the non-degenerate symplectic form $\omega = e^{2q}(d\alpha + dq\wedge\alpha)$.*

Proof. By direct calculation. ∎

We may now calculate the curvature of $\overrightarrow{\nabla}$; it is, for R the curvature of ∇,

$$\overrightarrow{R}_{Q,--} = \overrightarrow{R}_{-,-}Q = 0$$

$$
\begin{aligned}
\overrightarrow{R}_{X,Y}Z &= R_{X,Y}Z + 2\nu(X,Y)\sigma Z + \\
&\quad ((\sigma Y)\nu(X,Z) - (\sigma X)\nu(Y,Z) + Ys(X,Z) - Xs(Y,Z)) \\
&\quad + (\nu(X, D(Y,Z)) - \nu(Y, D(X,Z)))\, Q \\
\overrightarrow{R}_{X,Y}E &= D(X,Y) - D(Y,X) \\
&\quad + (\nu(Y, \nabla_X U) - \nu(X, \nabla_Y U) + 2f\nu(X,Y))Q \\
\overrightarrow{R}_{X,E}Y &= D(X,Y) + (\nu(Y, \sigma^2 X) + f\nu(X,Y) + \nu(Y, \nabla_X U))Q \\
\overrightarrow{R}_{X,E}E &= (X.f + s(X,U) - \nu(\sigma X, U))Q \\
&\quad + fX - \nabla_X U - \sigma^2 X.
\end{aligned}
\tag{5.5}
$$

Where $D(X,Y) = (\nabla_X \sigma)(Y) + \nu(Y,U)X - \nu(X,Y)U$. Taking traces, with Ric the Ricci curvature of ∇,

$$
\begin{aligned}
\overrightarrow{\mathsf{Ric}}_{-,Q} &= \overrightarrow{\mathsf{Ric}}_{Q,-} = 0 \\
\overrightarrow{\mathsf{Ric}}_{X,Z} &= \mathsf{Ric}_{X,Z} + \\
&\quad (\text{trace } \sigma)\nu(X,Z) + 3\nu(\sigma X, Z) + (1-2n)s(X,Z) \\
\overrightarrow{\mathsf{Ric}}_{X,E} &= -\text{trace } (\nabla_X \sigma) + i(X).\text{trace}[Y \to \nabla_Y \sigma] + (2n+1)\nu(X,U) \\
\overrightarrow{\mathsf{Ric}}_{E,X} &= i(X).\text{trace}[Y \to \nabla_Y \sigma] + (2n+1)\nu(X,U) \\
\overrightarrow{\mathsf{Ric}}_{E,E} &= f2n - \text{trace } (\nabla U) - \text{trace } (\sigma^2).
\end{aligned}
$$

Now $\nu(\sigma X, Y) = \nu_{km}\sigma^k_j X^j Y^m = \nu_{km} s_{ij} \nu^{ik} X^j Y^m = -s(X,Y)$, and trace σ = trace $\nabla_X \sigma = 0$, so $\overrightarrow{\mathsf{Ric}}$ is symmetric, as expected. So the full expression is:

$$
\begin{aligned}
\overrightarrow{\mathsf{Ric}}_{-,Q} &= \overrightarrow{\mathsf{Ric}}_{Q,-} = 0 \\
\overrightarrow{\mathsf{Ric}}_{X,Z} &= \mathsf{Ric}_{X,Z} - (2n+2)s(X,Z) \\
\overrightarrow{\mathsf{Ric}}_{X,E} &= +i(X).\text{trace}[Y \to \nabla_Y \sigma] + (2n+1)\nu(X,U) \\
\overrightarrow{\mathsf{Ric}}_{E,X} &= +i(X).\text{trace}[Y \to \nabla_Y \sigma] + (2n+1)\nu(X,U) \\
\overrightarrow{\mathsf{Ric}}_{E,E} &= f2n - \text{trace } (\nabla U) - \text{trace } (\sigma^2).
\end{aligned}
$$

Choose $s = \frac{1}{2n+2}\mathrm{Ric}$, and, for $\eta = \mathrm{trace}[Y \to \nabla_Y \sigma]$, define $U = -\frac{1}{2n+1}\nu(\eta, -)$. Finally, let $f = \frac{1}{2n}(\mathrm{trace}\,(\nabla U) + \mathrm{trace}\,(\sigma^2))$. Then $\overrightarrow{\nabla}$ is Ricci-flat, as theorised. ■

Remark. It is nearly certainly not the case, however, that *every* Ricci-flat symplectic cone connection can be generated in the above manner; for the $\overrightarrow{\nabla}$ so generated is E invariant, which is not a general condition for a symplectic connection.

We now aim to construct an explicit connection ∇ such that the $\overrightarrow{\nabla}$ it generates has maximal holonomy.

Let V be the standard representation of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$. Then $\mathfrak{g}$ is isomorphic, via the alternating form ν, with $V^* \odot V^*$. The Lie bracket is given, in terms of this isomorphism, as

$$[ab, cd] = \nu(b, c)ad + \nu(b, d)ac + \nu(a, c)bd + \nu(a, d)bc.$$

We know that $H^{(1,2)}(\mathfrak{g}) = 0$ and that all symplectic structures are flat. Moreover $\mathfrak{g}^{(1)} = \odot^3 V^*$, means that any symplectic connection is locally isomorphic with a section $U \to \odot^3 TN^*$, for $U \subset N$ open. Choosing local symplectic coordinates (x_j) such that

$$\nu = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \ldots + dx_{2n-1} \wedge dx_{2n},$$

we may define the symplectic connection ∇ as

$$\nabla = d + \sum_{j \neq 1} x_1 dx_1 (dx_j)^2 + \sum_{k \neq 1,2} x_2 dx_2 (dx_k)^2.$$

Notice that $\nabla = d + O(1)$. We may calculate the curvature of ∇ as

$$\begin{aligned} R &= \nabla \wedge \nabla \\ &= 2\sum_{j \neq 1} (dx_1 \wedge dx_j) \otimes (dx_1 dx_j) + 2 \sum_{k \neq 1,2} (dx_2 \wedge dx_k) \otimes (dx_2 dx_k) + O(2). \end{aligned}$$

When taking the Ricci trace using the symplectic form ν, all terms apart from $(dx^1 \wedge dx^2) \otimes (dx^1 dx^2)$

vanish. Consequently the Ricci tensor is

$$\mathsf{Ric} = 2(dx_1 dx_2) + O(2).$$

And, of course,

$$\nabla \mathsf{Ric} = O(1).$$

This allows us to simplify the curvature equations. By definition $U = O(1)$, so

$$\begin{aligned} \overrightarrow{R}_{X,Y}Z &= R_{X,Y}Z + 2\nu(X,Y)\sigma Z + O(1) \\ &\quad ((\sigma Y)\nu(X,Z) - (\sigma X)\nu(Y,Z) + Ys(X,Z) - Xs(Y,Z)) \end{aligned}$$

$s = \frac{1}{2n+2}\mathsf{Ric}$ as before.

Proposition 5.5.7. $\overrightarrow{\nabla}$ *has full symplectic holonomy.*

Proof. Still working in our chosen basis, we notice that because of our conditions on the Ricci tensor, for one of j and k in the set $(1,2)$ but $(j,k) \neq (1,2)$,

$$\overrightarrow{R}_{X^j,X^k}Z = R_{X^j,X^k}Z + O(1)$$

where $X^j = \frac{\partial}{\partial x^j}$. This means, by the Ambrose-Singer Theorem [KoNo], that elements of the form $dx_1 dx_j|_0$, $j \neq 2$ and $dx_2 dx_k|_0$, $k \neq 1$, are contained in $\overrightarrow{\mathfrak{hol}}$, the infinitesimal holonomy algebra of $\overrightarrow{\nabla}$ at 0. Now we may take a few Lie brackets:

$$[dx_1 dx_k, dx_2 dx_j] = dx_k dx_j + \nu(dx_k, dx_j)dx_1 dx_2 + O(1) \tag{5.6}$$

implying

$$[dx_1 dx_k, dx_2 dx_j] - [dx_1 dx_j, dx_2 dx_k] = 2dx_k dx_j + O(1).$$

Consequently $dx_k dx_j|_0 \in \overrightarrow{\mathfrak{hol}}$. By (5.6), we also have $dx_1 dx_2|_0$ in this bundle. To show that we have

all of $\mathfrak{sp}(\nu,\mathbb{R})$, we need only to add the elements $dx_2dx_2|_0$ and $dx_1dx_1|_0$. These are generated, for j odd, by

$$\begin{aligned}\left[dx_1dx_j, dx_1dx_{j+1}\right] &= dx_1dx_1 + O(1)\\ \left[dx_2dx_j, dx_2dx_{j+1}\right] &= dx_2dx_2 + O(1).\end{aligned}$$

Under the action of $\mathfrak{sp}(\nu,\mathbb{R})$, the full algebra $\mathfrak{sp}(\omega,\mathbb{R})$ splits as

$$\mathfrak{sp}(\omega,\mathbb{R}) = \mathfrak{sp}(\nu,\mathbb{R}) \oplus 2V \oplus \mathfrak{sp}(\omega/\nu,\mathbb{R}),$$

where the last module is a trivial representation for $\mathfrak{sp}(\nu,\mathbb{R})$.

Lemma 5.5.8. *If $\overrightarrow{\mathfrak{hol}}$ acts irreducibly on $T\mathfrak{M}|_0$, then $\overrightarrow{\mathfrak{hol}} = \mathfrak{sp}(\omega,\mathbb{R})$.*

Proof of Lemma. If $\overrightarrow{\mathfrak{hol}}$ acts irreducibly on $T\mathfrak{M}|_0$, then

$$\mathfrak{sp}(\nu,\mathbb{R}) \oplus 2V \subset \overrightarrow{\mathfrak{hol}},$$

and the $2V$ generate the remaining piece $\mathfrak{sp}(\omega/\nu,\mathbb{R})$ through the Lie bracket. □

So in order to finish this proof, we need to show that $\overrightarrow{\mathfrak{hol}}$ acts irreducibly on $T\mathfrak{M}_0$, or equivalently,

Lemma 5.5.9. *If there exists $\mathcal{K} \subset T\mathfrak{M}$ with $TN_0 \subset \mathcal{K}_0$ and such that $\mathcal{K}$ is preserved by $\vec{\nabla}$, then $\mathcal{K} = T\mathfrak{M}$.*

Proof of Lemma. First of all $\mathcal{K}$ has a non-trivial intersection with TN away from 0. So let $s \in \Gamma(K) \cap TN$ such that $s(0) = X^1$. Then by Equation (5.4)

$$(\vec{\nabla}_{X^1} s)(0) = \frac{1}{2n+1}Q + t,$$

whereas

$$(\vec{\nabla}_{X^2} s)(0) = E + t',$$

where $t, t' \in TN_0 \subset \mathcal{K}_0$. Consequently $\mathcal{K} = T\mathfrak{M}$, and the holonomy algebra of $\overrightarrow{\nabla}$ acts irreducibly on $T\mathfrak{M}$. □

■

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{sp}(2n, \mathbb{R}),\ \mathbb{R}^{2n},\ n \geq 3\big),$$

are possible projective holonomy algebras.

5.5.2 Complex symplectic

The previous proof works exactly the same in the holomorphic category.

Consequently

$$\big(\mathfrak{g}, V\big) \cong \big(\mathfrak{sp}(2n, \mathbb{C}),\ \mathbb{C}^{2n},\ n \geq 3\big),$$

are possible projective holonomy algebras.

5.6 Low-dimension cases

Some low-dimensional algebras are possible affine holonomy algebras, but have not yet been either constructed or ruled out as normal Tractor holonomy algebras. They are:

algebra $\mathfrak{g}$	representation V	Dimensions
$\mathfrak{so}(p,q)$	$\mathbb{R}^{(p,q)}$	$p+q=3,4$
$\mathfrak{so}(n,\mathbb{C})$	$\mathbb{C}^n$	$n=3,4$
$\mathfrak{sp}(p,q)$	$\mathbb{H}^{(p,q)}$	$p+q=2^*$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathbb{R}^n$	$n=2,3^*,4^*$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n$	$n=1,2,3^*$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathbb{H}^n$	$n=1,2^*$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n=2^*$
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathbb{C}^{2n}$	$n=2^*$

Those marked with stars are those algebras that *can* appear as projective normal Tractor holonomy algebras.

Proposition 5.6.1. *The low-dimensional* $\mathfrak{so}$ *algebras cannot appear as projective holonomy algebras.*

Proof. From proposition 2.1.6, *any* two-dimensional Cartan connection, which preserves a Tractor, is flat.

Dimensional considerations imply that the conformal Weyl tensor vanishes in 3 dimensions, [Wey2]. The obstruction to conformal flatness is carried entirely by the Cotton-York tensor, which of course vanishes for an Einstein space.

So any 3-dimensional Einstein space is conformally – hence projectively – flat. This eliminates the real $\mathfrak{so}$ and the $\mathfrak{su}$, as the underlying manifold must be such a manifold. The complex $\mathfrak{so}$ has $\mathbb{C}$-linear curvature (see Example 2), so is automatically holomorphic – so disappears just as in the real case, as the holomorphic Weyl tensor must also vanish in three complex dimensions. ■

Since every one-dimensional manifold is projectively flat, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(1,\mathbb{C})$ are not possible Tractor holonomy algebras – they are not even possible Ricci-flat algebras, in fact.

Lemma 5.6.2. $\mathfrak{sl}(2,\mathbb{C})$ *is not a possible Tractor holonomy algebra.*

Proof of Lemma. Assume that $\overrightarrow{\nabla}$ is a cone connection with this holonomy, and let $R = JQ$. From an adaptation of Lemma 2.1.8, we know that a cone connection is R-invariant if and only if all curvature terms involving R vanish. For a connection with holonomy $\mathfrak{sl}(2,\mathbb{C})$, being Ricci-flat is equivalent to having J-hermitian curvature. Consequently

$$\overrightarrow{R}_{R,X} = \overrightarrow{R}_{JR,JX} = -\overrightarrow{R}_{Q,JX} = 0.$$

So $\overrightarrow{\nabla}$ is R-invariant, and, as in Section 3.5.1, there is a complex projective manifold N of complex dimension one, for which $\overrightarrow{\nabla}$ is the complex cone connection.

Any two torsion-free complex connections $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ on N differ by a one-form $\Xi \in \Omega^{(1,0)}(N)$

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}'_X Y + \Xi(X)Y$$

Since $\Xi(X)Y = \Xi(Y)X$, we can set $\Upsilon^{\mathbb{C}} = \frac{1}{2}\Xi$ to see that $\widetilde{\nabla}$ and $\widetilde{\nabla}'$ define the same complex projective structure. So every complex projective structure on N is flat, implying that $\overrightarrow{\nabla}$ itself must be flat. □

Lemma 5.6.3. $\mathfrak{sl}(1,\mathbb{H})$ *is not a possible Tractor holonomy algebra.*

Proof of Lemma. We know what $\overrightarrow{\nabla}$ must be, explicitly; it is given by one vector field Q with $\overrightarrow{\nabla}Q = Id$, and three (non-commuting) vector fields $J_\alpha Q$ such that

$$\overrightarrow{\nabla} J_\alpha Q = J_\alpha.$$

and

$$[J_\alpha Q, J_\beta Q] = -2J_\alpha J_\beta Q$$

whenever $\alpha \neq \beta$. But in this case all the curvature of $\overrightarrow{\nabla}$ vanishes. □

Proposition 5.6.4. *The algebras* $\mathfrak{sl}(2,\mathbb{H})$ *and* $\mathfrak{sp}(p,q)$, $p+q=2$, *do exist as Tractor holonomy algebras.*

Proof. As seen in Section 5.4, any hypercomplex Tractor connection corresponds to a quaternionic projective structure on a manifold N, in other words a $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(1,\mathbb{H}) \oplus \mathfrak{sl}(1,\mathbb{H})$ structure. However this last algebra is equal to $\mathfrak{co}(4)$ – if one takes $\mathbb{H}$ as a model space, a definite-signature metric g is given by $g(a,b) = Re(a\bar{b})$, and it is easy to see that $\mathfrak{g}$ preserves g up to scaling.

As usual, a subgroup of $\mathfrak{sl}(2,\mathbb{H})$ acting reducibly on $\mathbb{H}^2$ corresponds to a conformally Ricci-flat 4-fold. But, from our efforts on conformal holonomy of Chapter 2, we know there exist non conformally Ricci-flat manifolds in four dimensions. The subgroups acting irreducibly on $\mathbb{H}^2$ are

$$\mathfrak{sp}(2,0) \cong \mathfrak{sp}(0,2),\ \mathfrak{sp}(1,1),\ \mathfrak{sl}(2,\mathbb{H}),$$

corresponding respectively to conformally Einstein with $\lambda < 0$, conformally Einstein with $\lambda > 0$, and not conformally Einstein at all. Examples of all these constructions, without further holonomy reductions, exist in four dimensions, see Theorem 5.1.1 and Proposition 2.2.15. ∎

Remark. The argument for the rest of this section can be paraphrased as 'if we have a manifold with non-trivial Tractor holonomy, we can conjugate the holonomy algebra by gluing the manifold to a copy of itself with a twist, to generate the full algebra'. The subtleties will be in making the manifold flat around the gluing point. This argument only works if the flattening respects whatever structures – complex or symplectic – we are attempting to preserve. We must also avoid using Ricci-flat connections, as then conjugation will not give us the full algebras; but it is simple to pick a preferred connection that is not Ricci-flat.

Proposition 5.6.5. *The algebras* $\mathfrak{sl}(3,\mathbb{R})$ *and* $\mathfrak{sl}(3,\mathbb{C})$, *do exist as Tractor holonomy algebras.*

Proof. The projective Weyl tensor vanishes in two real dimensions, and consequently the full obstruction to projective flatness is carried by the Cotton-York tensor (see Equation (1.8)). Cartan [Car2] proved propositions about two dimensional projective structures that are equivalent to stating that the only possible tractor holonomy algebras are $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^{2*}$; see Appendix D for a direct proof of this.

In order to prove the existence of a manifold with full tractor holonomy, we shall use the following proposition:

Proposition 5.6.6. *Assume there exists a manifold M^2 with non-trivial Tractor holonomy. Then there exists a manifold N^2 with full Tractor holonomy.*

Proof. As we've seen, the Tractor holonomy of M is $\mathfrak{sl}(3,\mathbb{R})$ or $\mathfrak{h} = \mathfrak{sl}(2,\mathbb{R}) \rtimes \mathbb{R}^{2*}$. Since the former gives us our result directly, assume the latter; since a non-trivial holonomy algebra must be non-trivial on some set, we have a set $U \subset M$ such that the local holonomy at any point of U is $\mathfrak{h}$. Let ∇ be any preferred connection of this projective structure. Choose local coordinates (x_j) on $U \subset M$, and let $\widehat{\nabla}$ be the flat connection according to these local coordinates. Let f be a bump function, and define $\nabla' = f\nabla + (1-f)\widehat{\nabla}$. This is a torsion free connection, and since $\nabla' = \nabla$ where $f = 1$, has Tractor holonomy containing $\mathfrak{h}$.

Take two copies U_1 and U_2 of (U, ∇', x_j) and identify two small flat patches of them – patches where ∇'_1 and ∇'_2 are flat – using the rule $x_j - a_j \to s(x_j - b_j)$ for s some element of $SL(2,\mathbb{R})$, a a point in the flat part of U_1, b a point in the flat part of U_2, and the x_j local, flat coordinates. This identifies flat sections with flat sections, so does not affect the local holonomy around these patches. The local derivative of s is $Ds(X^j) = s(X^j)$.

Restrict U_1 and U_2 so that the construction we get is a manifold. Since s maps flat sections to flat sections, $\nabla'_1 = \nabla'_2$ whenever they are both defined. So we have a globally defined ∇'.

Changing s changes the inclusion of the holonomy-preserved vector from U_1 into U_2, thus changes the inclusion $\mathfrak{h} \subset \overrightarrow{\mathfrak{hol}}_x$ by conjugation on the $\mathfrak{sl}(2,\mathbb{R})$ factor of $\mathcal{A}_x$ defined by ∇'. But any two conjugate non-identical copies of $\mathfrak{h}$ generate all of $\mathfrak{sl}(3,\mathbb{R})$, so we are done. ∎

Then we may conclude with the following lemma:

Lemma 5.6.7. *There exists manifolds M^2 with non trivial Tractor holonomy.*

Proof of Lemma. To do so, it suffices to find a manifold with non-trivial Cotton-York tensor. But if we have local coordinates x and y such that $\nabla_X Y = \nabla_Y X = \nabla_Y Y = 0$ and $\nabla_X X = y^2 Y$,

then ∇ is torsion-free and

$$\mathsf{Ric}^{\nabla} = 2y\ dx \otimes dx,$$

thus

$$CY^{\nabla} = 4\ dx \wedge dy \otimes dx.$$

□

We may use these same ideas to construct a manifold with complex projective Tractor holonomy $\mathfrak{sl}(3,\mathbb{C})$ – and hence a real projective manifold with same holonomy, one dimension higher. The existence proof Lemma 5.6.7 works in the holomorphic category, and in then has a tractor holonomy algebra containing $h \otimes \mathbb{C}$. Then given a holomorphic M with these properties, we can use the trick of Proposition 5.6.6, with (x_j) holomorphic coordinates, to get $\nabla' = f\nabla + (1-f)\widehat{\nabla}$. This obviously preserves the complex structure (though it is not holomorphic), and we can then patch $U_1^{\mathbb{C}}$ and $U_2^{\mathbb{C}}$ together using $s \in SL(2,\mathbb{C})$, which also preserves the complex structure. ■

Corollary 5.6.8. *Tractor holonomy* $\mathfrak{sl}(4,\mathbb{C})$ *also exists.*

Proof. The cone over any manifold with Tractor holonomy $\mathfrak{sl}(3,\mathbb{R})$ has same Tractor holonomy, see Theorem, 3.1.8. Then we construct the cone over the manifold of the previous proposition, choose a preferred connection that does *not* make it Ricci flat (so that the tangent bundle $T[\mu]$ is not holonomy preserved), and then use the same patching process to conjugate $\mathfrak{sl}(3,\mathbb{R})$ and get full Tractor holonomy. ■

Proposition 5.6.9. *The algebras* $\mathfrak{sp}(4,\mathbb{R})$ *and* $\mathfrak{sp}(4,\mathbb{C})$*, do exist as Tractor holonomy algebras. They even exist for the 'symplectic projective' construction of Section 5.5.1.*

Proof. This is a sketch of a proof, without going into too many details. The Lie algebra $\mathfrak{sp}(2n+2,\mathbb{R})$ splits into

$$\mathfrak{sp}(2n+2,\mathbb{R}) = V^* \oplus \big(\mathfrak{sp}(2n,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R})\big) \oplus V,$$

where $V \cong \mathbb{R}^{2n}$, $[V,V] \subset \mathfrak{sp}(2,\mathbb{R})$ and $[V^*,V^*] \subset \mathfrak{sp}(2,\mathbb{R})$. The Lie bracket between V and V^* is given by

$$[X,\xi] = X(\xi)\cdot\nu|_{\mathfrak{sp}(2,\mathbb{R})} + X\otimes\xi + \nu(\xi)\otimes\nu(X),$$

ν the symplectic structure. Note here that $\nu|_{\mathfrak{sp}(2,\mathbb{R})}$ is a map $V \to V^*$, equal to the identity under the isomorphism $V \cong V^*$ given by $\nu|_{\mathfrak{sp}(2n,\mathbb{R})}$, the other piece of ν. We may then interpret the construction of Section 5.5.1 as a 'symplectic projective structure' whose preferred connections change via

$$\nabla_X Y \to \nabla'_X Y = \nabla_X Y + [X,\Upsilon]\cdot Y$$

for some one-form Υ. This implies that there exist non-flat symplectic projective manifolds in two dimensions (as $\odot^3\mathbb{R}^2 = (\mathfrak{sp}(2,\mathbb{R}))^{(1)}$ is of dimension four, while $\mathbb{R}^{2*} = T^*_x$ is of dimension two).

Then since the tangent space of the underlying manifold N^2 cannot be preserved by $\overrightarrow{\nabla}$ (since N cannot be Ricci-flat without being flat) we may construct a patching argument as in Proposition 5.6.6 to get the full tractor holonomy, using three copies patched together if need be. The process still works, as given any symplectic connection ∇ with symplectic form ν, and $\widehat{\nabla}$ a flat connection preserving ν, then $\nabla' = f\nabla + (1-f)\widehat{\nabla}$ also preserves ν.

To generalise this argument to the complex case is slightly subtle, as we are no longer in the case of a manifold that can be made holomorphic, and the complex symplectic curvature expressions (the complex equivalent of Equations 5.5) become considerably more complicated – though Equations 5.5 remain valid if we look at the holomorphic (J-commuting) part of the curvature only.

Therefore we may start with a holomorphic symplectic connection, not $\mathbb{C}$-symplectically flat. These exist by the same argument as in the real case. Then we use partition of unity 'patching' arguments on this manifold, to conjugate whatever holonomy algebra it has locally, and thus to create a manifold with full Tractor holonomy. This manifold is no longer holomorphic, but the terms from the anti-holomorphic part of the curvature cannot reduce the holonomy algebra; and since they must be contained in $\mathfrak{sp}(2n+2,\mathbb{C})$, they can't increase it either. ■

Remark. Note that these last proofs can construct $\mathfrak{sl}(n,\mathbb{F})$ and $\mathfrak{sl}(2n,\mathbb{F})$ for all the larger n as well; however the constructions throughout the preceding chapter have the advantage of providing

explicit examples, and to have their local holonomy equal to their general holonomy.

In conclusion, the possible metric, irreducible projective Tractor holonomy algebras are given by

algebra $\mathfrak{g}$	representation V	restrictions	algebra $\mathfrak{g}$	representation V
$\mathfrak{so}(p,q)$	$\mathbb{R}^{(p,q)}$	$p+q \geq 5$	$\widetilde{\mathfrak{g}}_2$	$\mathbb{R}^{(4,3)}$
$\mathfrak{so}(n,\mathbb{C})$	$\mathbb{R}^n$	$n \geq 5$	$\mathfrak{g}_2(\mathbb{C})$	$\mathbb{C}^7$
$\mathfrak{su}(p,q)$	$\mathbb{C}^{(p,q)}$	$p+q \geq 3$	$\mathfrak{spin}(7)$	$\mathbb{R}^8$
$\mathfrak{sp}(p,q)$	$\mathbb{H}^{(p,q)}$	$p+q \geq 2$	$\mathfrak{spin}(4,3)$	$\mathbb{R}^{(4,4)}$
$\mathfrak{g}_2$	$\mathbb{R}^7$		$\mathfrak{spin}(7,\mathbb{C})$	$\mathbb{C}^8$

these are also the possible maximal conformally Einstein conformal Tractor holonomy algebras. The possible non-metric, irreducible projective Tractor holonomy algebras are:

algebra $\mathfrak{g}$	representation V	restrictions	manifold properties
$\mathfrak{sl}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 3$	Generic
$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$	CR manifold
$\mathfrak{sl}(n,\mathbb{H})$	$\mathbb{C}^n$	$n \geq 2$	HR manifold
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$	Contact manifold
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathbb{C}^{2n}$	$n \geq 2$	CR-Contact manifold

Appendix A

Invariant Linear Operators

We will assume familiarity with jet-bundles and their uses; paper [Swe] provides a good overview. The constructions presented here are not unrelated to those of [BCEG].

For a vector bundle B, we shall use the standard formula

$$0 \longrightarrow \odot^k T^* \otimes B \longrightarrow J^k(B) \longrightarrow J^{k-1}(B) \longrightarrow 0$$

and the equality $J^0(B) = B$, without comment.

Remark. Notice that if B is a vector bundle associated to the frame bundle of the tangent bundle, and if one has a choice of connection ∇, the k-th order differential operator

$$Sym(\nabla \circ \nabla \ldots \nabla) : \Gamma(B) \to \Gamma(\odot^k T^* \otimes B)$$

yields a splitting of this sequence.

Remembering the inclusions

$$T^* \subset \mathcal{B} \subset \mathcal{A}$$

where $\mathcal{B} = \mathcal{P} \times_P \mathfrak{p}$, we say that a bundle is associated to $\mathcal{A}$ (or $\mathcal{B}$) if it admits an action of $\mathcal{A}$ (or $\mathcal{B}$) that respects the Lie bracket. This is the same as saying they have structure bundle $\mathcal{G}$ or $\mathcal{P}$.

Now let B, C be vector bundles associated to $\mathcal{B}$ (note that any vector bundle with structure group $\mathcal{G}_0$, for instance the tangent bundle, can be made into a $\mathcal{B}$ bundle by assuming the action of $T^* \subset \mathcal{B}$ to be trivial. This choice respects any inclusion $\mathcal{G}_0 \subset \mathcal{P}$). Papers [CSS1] and especially [Slo] show that *most* invariant operators arise from the so-called *semi-holonomic* jet bundles $\overline{J}^k(B)$. These bundles, which are associated to $\mathcal{B}$, are located 'between' the standard jet bundles and the non-commutative jet bundles $J^1(J^1(\ldots J^1)(B)$:

$$J^k(B) \subset \overline{J}^k(B) \subset J^1(J^1(\ldots J^1)(B).$$

Then an invariant operator is identified as a $\mathcal{B}$-invariant linear map

$$\phi : \overline{J}^k(B) \to C.$$

The restriction ϕ to $J^k(B) \subset \overline{J}^k(B)$ yields a linear differential operator $\Gamma(B) \to \Gamma(C)$ in the usual way; this is the invariant operator.

Appendix B

Twistor representations

In paper [Bas], Baston presents another space on which the Tractor connection can naturally act; this is in fact the spin representation of $\mathfrak{so}(p+1,q+1)$. Let T_0 and T_1 be the reduced spin representations of $\mathfrak{co}(p,q)$; in other words if $p+q$ is odd, $T_0 = T_1$ and if $p+q$ is even, these are the two irreducible spin representations.

The tangent space acts on these spaces by Clifford multiplication, interchanging the T_0 and T_1 in the even dimensional case. Consequently we have a well defined operator

$$C : \Omega^1 \otimes T_0 \to T_1.$$

The Dirac operator D is given by $C \circ \nabla$. However, $\Omega^1 \otimes T_0$ has a second irreducible piece – equivalently, we have an invariant lifting of T_1 into $\Omega^1 \otimes T_0$. If we project ∇ onto this second bundle, we have a second operator, that is also conformally invariant by [Feg]. The roots of this operator are the *twistor-spinors.*

In details, these are the spinors $\psi \in \Gamma(T_0)$ such that for all $X \in \Gamma(T)$,

$$\nabla_X \psi + \frac{1}{n} X.D\psi = 0.$$

Alternatively, there exists a section π of T_1 such that

$$\nabla_X \psi + X \cdot \pi = 0.$$

We thus have a well defined bundle $\mathcal{T}$ and a projection of the exact sequence for the jet bundle $J^1(T_0)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1 \otimes T_0 & \longrightarrow & J^1(T_0) & \longrightarrow & T_0 & \longrightarrow & 0 \\ & & \downarrow \mathrm{tr} & & \downarrow & & \| & & \\ 0 & \longrightarrow & T_1 & \longrightarrow & \mathcal{T} & \longrightarrow & T_0 & \longrightarrow & 0 \end{array}$$

The invariant lift of T_1 into $\Omega^1 \otimes T_0$ gives a lift of $\mathcal{T}$ into $J^1(T_0)$. This bundle corresponds to the spin representation of $\mathfrak{so}(p,q)$, and is the Twistor-Tractor bundle. Paper [Bas] then demonstrates that this bundle has a natural connection on it, isomorphic with the Tractor connection. A preserved Twistor-Tractor thus implies the existence of a twistor-spinor.

Paper [Hab] by Katharina Habermann analyses solutions to this twistor equation; she shows that these imply that the manifold is conformally Einstein, of non-negative scalar curvature in the definite signature case. The Tractor holonomy groups G_2 and $Spin(7)$ actually correspond to the existence of twistor-spinors on the manifold.

We can get this result directly from this thesis, by noting first of all that $\mathfrak{so}(n+1,1)$ does not preserve a spinor; hence any holonomy algebra that preserves a Twistor-Tractor must reduce, thus be conformally Einstein. And the only holonomy algebra with negative Einstein coefficient (see Tables 1 and 2) is $\mathfrak{so}(n,1)$, which doesn't preserve a spinor either: hence we have the non-negative condition.

Twistor-spinors are conformally invariant generalisations of the concept of a Killing spinor, a spinor ψ such that

$$\nabla_X \psi = \lambda X \cdot \psi,$$

for some constant λ. Historically, Killing spinors have also been used for holonomy reduction. In [Bär], C. Bär showed that having a Killing spinor is equivalent to having a parallel spinor on the

metric cone. So weak holonomy SU(3) and nearly Kählerian structures are also detected by the Tractor connection.

Appendix C

Wedge products for Complex spaces

Let (V, J) be a vector space with complex structure J. By $\overline{V}$ we mean $(V, -J)$, the same vector space with opposite complex structure. All wedge and tensor products we use will be real unless stated otherwise. Let $\alpha : \otimes^j V \to \wedge^j V$ be the natural antisymmetrisation map. Let $\otimes^{(n,m)} V = \otimes^n_{\mathbb{C}} V \otimes_{\mathbb{C}} \otimes^m_{\mathbb{C}} \overline{V}$. Then we shall define the space $\wedge^{(n,m)}$ as:

Definition C.0.10.

$$\wedge^{(n,m)} V = \alpha \left(\otimes^{(n,m)} V \right).$$

Obviously, $\wedge^{(n,m)} V = \overline{\wedge^{(m,n)} V}$, implying that $\wedge^{(n,m)} V$ and $\wedge^{(m,n)} V$ are the same spaces. And, of course, $\wedge^{(n,0)} V = \wedge^n_{\mathbb{C}} V$. The point of all this is to get the following decomposition result:

Proposition C.0.11.

$$\wedge^k V = \oplus \wedge^{(n,m)} V,$$

where $n + m = k$, *and* $n \geq m$.

Proof. Given ω, an element of $\wedge^{(n,m)}V$, we can state that

Lemma C.0.12.

$$\alpha(J \otimes J\omega) = -\theta(n,m)\omega$$

where $J \otimes J$ is taken to act (for instance) on the first two entries of ω, and

$$\theta(n,m) = \frac{n^2 - n + m^2 - m - 2mn}{k(k-1)}$$

Proof of Lemma. ω is the sum of terms of the form $\frac{1}{k!}\sum_\sigma -1^{sg(\sigma)}\sigma(a_1, a_2, \ldots, a_k)$ with σ a permutation of k elements, $sg(\sigma)$ the sign of the permutation, and $a_1 a_2 \ldots a_k$ an element of $\left(\otimes^{(n,m)}V\right)$.

If the first two elements of $\sigma(a_1, a_2, \ldots, a_k)$ are a_i and a_j, with i, j both $\leq n$ or both $> n$, then $J \otimes J\sigma(a_1, a_2, \ldots, a_k) = -\sigma(a_1, a_2, \ldots, a_k)$. There are

$$(n^2 - n + m^2 - m)(k-2)!$$

such terms.

On the other hand, if $i \leq n$ and $j > n$ or vice-versa, then

$$J \otimes J\sigma(a_1, a_2, \ldots, a_k) = \sigma(a_1, a_2, \ldots, a_k).$$

There are $2nm(k-2)!$ such terms.

On the other hand, it is obvious that

$$\alpha(a_1 a_2 \ldots a_k) = -1^{sg(\sigma)}\alpha(\sigma(a_1, a_2, \ldots a_k)),$$

hence

$$\begin{aligned} \alpha(J \otimes J\omega) &= \frac{1}{k!}(k-2)!\left(2mn - n^2 + n - m^2 + m\right)\omega \\ &= -\theta(n,m)\omega. \end{aligned}$$

□

To prove the main proposition, merely note that $\theta(n,m)$ can be re-written as

$$\theta(n,m) \quad = \quad \frac{(n-m)^2-k}{k(k-1)},$$

making θ an obviously injective function in $|n-m|$, and allowing us to decompose $\wedge^k V$ in terms of the eigenspaces of $\alpha \circ J \otimes J$, i.e. in terms of $\wedge^{(n,m)}V$. ■

Proposition C.0.13. *$\partial(\wedge^{(n,m)}V \otimes V) \subset \wedge^{(n+1,m)}V \oplus \wedge^{(n,m+1)}V$ if $n \neq m$, and $\partial(\wedge^{(n,n)}V \otimes V) \subset \wedge^{(n+1,1)}V$.*

Proof. If $\alpha_k(q) \otimes v$ is an element of $\wedge^{(n,m)}V \otimes V$, for q an element of $\otimes^{(n,m)}V$, then

$$\partial(\alpha_k(q) \otimes v) = \alpha_{k+1}(q \otimes v).$$

But $q \otimes v$ decomposes as $q \otimes_{\mathbb{C}} v + q \otimes_{\mathbb{C}} \overline{v}$, elements of $\otimes^{(n+1,m)}V$ and $\otimes^{(n,m+1)}V$ respectively. ■

Given V and J, one may define V' and $V'' = \overline{V'}$ as the $+i$ and $-i$ eigenspaces of J in the complexification of V, $V_{\mathbb{C}}$. Then given the sum

$$V_{\mathbb{C}} = V' \oplus V''$$

one may define the standard complex wedge product

$$\wedge^{(n,m)}V_{\mathbb{C}} = \wedge^n V' \otimes \wedge^m V''.$$

Furthermore, there is a projection $p : V_{\mathbb{C}} \to V$,

$$p(v) = \frac{1}{2}\left(v + \overline{v}\right).$$

Proposition C.0.14.

$$p\big(\wedge^{(n,m)} V_{\mathbb{C}}\big) = \wedge^{(n,m)}V.$$

Proof. A basis element of $\wedge^{(n,m)}V_{\mathbb{C}}$ is composed of all permutations of elements of the form

$$\alpha = a_1 a_2 \dots a_n \overline{a}_{n+1} \dots \overline{a}_{n+m}$$

Hence a basis element for $p\big(\wedge^{(n,m)} V_{\mathbb{C}}\big)$ is composed of all permutations of elements of the form

$$p(\alpha) = b_1 \dots b_{n+m}$$

where $b_j = p(a_j)$, so $p\big(\wedge^{(n,m)} V_{\mathbb{C}}\big)$ is indeed inside $\wedge^{n+m}V$. Now

$$p(ia_j) = Jp(a_j),$$

whereas

$$p(i\overline{a}_j) = -Jp(a_j)$$

so $p(\alpha)$ is an element of $\otimes^n_{\mathbb{C}}V \otimes_{\mathbb{C}} \otimes^m_{\mathbb{C}}\overline{V} = \otimes^{(n,m)}V$. This demonstrates the result. ■

Appendix D

Two-dimensional projective holonomy

This appendix, as a minor illustration of some of the techinques used here, is to fully classify all possible projective Tractor holonomy algebras in two dimensions. This completes the idea of Cartan [Car2] in this case. We aim to prove the following theorem:

Theorem D.0.15. *The only possible projective Tractor holonomy algebras on a two-manifold are* $\mathfrak{sl}(3,\mathbb{R})$ *and* $\mathfrak{sl}(2) \rtimes \mathbb{R}^{2*}$.

The full holonomy $\mathfrak{sl}(3,\mathbb{R})$ is a possible holonomy algebra, see Proposition 5.6.5. Apart from that, we know that if the Tractor holonomy preserves a bundle of rank two, the manifold is projectively Ricci-flat, see Theorem 3.1.2. But any two-manifold that is Ricci-flat is flat, so these algebras are excluded.

The only subalgebras of $\mathfrak{sl}(3,\mathbb{R})$ that fit this criteria are $\mathfrak{so}(p,q)$ with $p+q=3$ and $\mathfrak{g} \rtimes \mathbb{R}^{2*}$, where $\mathfrak{g}$ is one of $\mathfrak{gl}(2,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{co}(p',q')$ and $\mathfrak{so}(p',q')$, with $p'+q'=2$. The algebra $\mathfrak{so}(p,q)$ implies that our manifold is Einstein. But any Einstein manifold in two dimensions is conformally flat, and the conformal and projective Tractor connections are isomorphic for Einstein manifolds, see Proposition 3.5.9. Thus this algebra is excluded.

Proposition D.0.16. *Any non-trivial projective Tractor holonomy algebra for a two-manifold must have dimension at least five.*

Proof. The projective Weyl tensor vanishes in two dimensions, so the full obstruction to projective flatness is carried by the Cotton-York tensor. Consequently, the is an element of the form $(\xi, 0, 0)$, $\xi \in T^*$ within the holonomy algebra. Let X and Y form a local frame for T, chosen so that $\xi(X) = 0$, $\xi(Y) = 1$. Since the holonomy algebra is preserved by the tractor connection, one must have elements of the form

$$(\xi', [X, \xi], 0) \quad \text{and} \quad (\xi'', [Y, \xi], 0)$$

within the holonomy algebra. These elements are obviously linearly independent. Iterating this, one gets algebra elements of the form

$$(\xi''', \tau, [X, \xi].Y) \quad = \quad (\xi''', \tau, X)$$

and

$$(\xi'''', \tau', [Y, \xi].Y) \quad = \quad (\xi'''', \tau', 2Y),$$

and these five elements are clearly linearly independent. ∎

Thus, as $\mathfrak{co}(p', q') \rtimes \mathbb{R}^{2*}$ and $\mathfrak{so}(p', q') \rtimes \mathbb{R}^{2*}$ have dimensions four and three respectively, these are not possible projective Tractor holonomy algebras. For the two remaining cases, we shall construct examples with holonomy $\mathfrak{sl}(2) \rtimes \mathbb{R}^{2*}$ and then prove that there exists no holonomy $\mathfrak{gl}(2) \rtimes \mathbb{R}^{2*}$ manifolds.

Holonomy $\mathfrak{gl}(2) \rtimes \mathbb{R}^{2*}$ implies that there is a connection ∇ that does not preserve a volume form, and a vector field Y preserved up to scaling by ∇ such that $\mathsf{P}^{\nabla}(-, Y) = 0$, see Theorem 3.1.2. Holonomy $\mathfrak{sl}(2) \rtimes \mathbb{R}^{2*}$ implies the same thing, except that ∇ must preserve a volume form, and consequently the condition $\mathsf{P}^{\nabla}(-, Y) = 0$ is equivalent to $\mathsf{Ric}^{\nabla}(-, Y) = 0$. Since we have excluded all other reduced holonomy algebras, it suffices to find such ∇'s with non-vanishing Cotton-York tensors to get these two algebras.

Let us deal with the $\mathfrak{sl}(2) \rtimes \mathbb{R}^{2*}$ case first. Let x and y be local coordinates, $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$

and define the connection ∇ by

$$\begin{aligned} \nabla_Y Y &= 0, \\ \nabla_Y X = \nabla_X Y &= 0, \\ \nabla_X X &= \frac{y^2}{2} Y. \end{aligned}$$

Then ∇ preserves Y and has curvature

$$R^\nabla = 2y \; dx \wedge dy \otimes Y \otimes dx,$$

consequently ∇ preserves a volume-form and has Ricci tensor

$$\mathsf{Ric}^\nabla = y \; dx \otimes dx,$$

and Cotton-York tensor

$$CY = -2 \; dx \wedge dy \otimes dx,$$

implying that we have the required holonomy.

Consider now the algebra $\mathfrak{gl}(2) \ltimes \mathbb{R}^{2*}$, and assume that we have a projective manifold with this holonomy. We may choose a connection ∇ that has the required properties for P^∇ and preserves a volume-form on the preserved bundle; as it is a line-bundle, this is equivalent to preserving a vector field Y. Choose X, a vector field commuting with Y, and relevant local coordinates. Then the connection ∇ must be of the type

$$\begin{aligned} \nabla_Y Y &= 0, \\ \nabla_Y X = \nabla_X Y &= 0, \\ \nabla_X X &= Z, \end{aligned}$$

for some vector field Z. Since $R^\nabla_{-,-} Y = 0$, the only curvature terms involve $dx \wedge dy \otimes (Y dx)$ and $dx \wedge dy \otimes (X dx)$. Consequently the only Ricci curvature terms are $dx \otimes dx$ and $dy \otimes dx$. However,

since

$$\mathsf{P}^{\nabla}(X,Y) \;=\; -\frac{2}{3}\mathsf{Ric}^{\nabla}(X,Y) - \frac{1}{3}\mathsf{Ric}^{\nabla}(Y,X),$$

by Equation 1.5, we can see that $\mathsf{P}^{\nabla}(X,Y) = 0$ if and only if the $dy \otimes dx$ term in the Ricci curvature vanishes. But this is equivalent to ∇ preserving a volume form, so the holonomy algebra reduces to $\mathfrak{sl}(2) \rtimes \mathbb{R}^{2*}$.

Appendix E

Ricci-Flat Holonomies

In this Appendix, we aim to finish the classification of which holonomy algebras acting irreducibly can correspond to a Ricci-flat connection. For though we have excluded many holonomy algebras from being Ricci-flat, and we have constructed Ricci-flat cones for most of the others, we have not settle the existence of general Ricci-flat connections in some cases. These are the algebras concerned (those that can be Ricci-flat have been marked with a $*$):

algebra $\mathfrak{g}$	representation V	Dimensions
$\mathfrak{so}(p,q)$	$\mathbb{R}^{(p,q)}$	$p+q=3,4^*$
$\mathfrak{so}(n,\mathbb{C})$	$\mathbb{C}^n$	$n=3,4^*$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathbb{R}^n$	$n=2$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathbb{C}^n$	$n=1,2^*$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathbb{H}^n$	$n=1^*$

Of these, we can immediately exclude $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(1,\mathbb{C})$, as any Ricci-flat two-manifold is flat. In contrast, any $\mathfrak{sl}(n,\mathbb{H})$ connection must be Ricci-flat by definition. Manifolds with holonomy $\mathfrak{so}(p,q)$, $p+q=3$, have vanishing Weyl tensor as all three-manifolds do. However, a Ricci-flat manifold has full curvature contained in the Weyl tensor. Thus Ricci-flat manifolds with these holonomies must

be flat. The result holds, similarly, in the holomorphic category of $\mathfrak{so}(3,\mathbb{C})$.

For the case of $\mathfrak{sl}(2,\mathbb{C})$, let x and y be complex coordinates with corresponding holomorphic vector fields $X, Y \in \Gamma(T_{\mathbb{C}})$. Then define the connection ∇ as

$$\nabla_{\overline{X}}\overline{X} = \overline{\nabla_{\overline{X}}X} = \overline{\nabla_X\overline{X}} = \nabla_X X \quad = \quad fY$$

and all other terms involving X, Y and their conjugates are zero. Here f is a complex-valued function that is independent of y (i.e. $Yf = \overline{Y}f = 0$). This ∇ is a torsion-free connection respecting the real structure on $T_{\mathbb{C}}$ – consequently equivalent to a real connection representing the corresponding complex structure on T. The curvature of ∇ is given by

$$R_{\overline{X}X}X \quad = \quad (\overline{X}f)Y - (Xf)\overline{Y},$$

the corresponding $R_{\overline{X}X}\overline{X}$ term, and all other curvature terms are zero. This makes ∇ Ricci-flat. We now use f as a bump function to smoothly move ∇ to a flat connection (moving along the x direction, of course), while remaining Ricci-flat and complex along the way. We may then use the patching argument from Proposition 5.6.6 and suitable local choices of f to construct a complex, Ricci-flat connection ∇ such that, directly from the curvature, we have the holonomy elements

$$X \to Y,$$

and

$$Y \to iX.$$

And these two elements generate the full $\mathfrak{sl}(T,\mathbb{C})$ holonomy.

In order to generate the remaining holonomy algebras, we turn to the Schwarzschild metric [EGH]. In this (Lorentzian) case, the metric is

$$g = -Cdt^2 + \frac{1}{C}dr^2 + r^2(d\theta^2 + \sin\theta d\psi^2).$$

Where $C = 1 - \frac{2M}{r}$ for some mass M. There is also an Euclidean Schwarzschild metric (given by replacing $-dt^2$ with dt^2) and a split Schwarzschild metric (given by replacing $d\psi^2$ with $-d\psi^2$). Of course, since the metric is real-analytic in the coordinates, there is also a complex Schwarzschild metric, my considering t, r, θ and ψ as complex coordinates.

Then elementary but laborious calculations establish that all these metrics are Ricci-flat, and all have maximal holonomy – it turns out that the curvature tensor is enough to generate the full holonomy algebra.

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