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Ancient Egyptian mathematics in the early 20th century: a mathematical view from Kiel, 1926

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In December 1926, the historian of mathematics Otto Neugebauer, then at the very beginning of his career, gave a seminar to mathematicians at the University of Kiel on the subject of ancient Egyptian fraction-reckoning. The content of Neugebauer's lecture was drawn from his doctoral dissertation, completed in Göttingen earlier that year, which had been inspired by the appearance in 1923 of a new edition of the most complete surviving source on ancient Egyptian mathematics, the so-called Rhind Mathematical Papyrus. This new edition, published by the Egyptologist Thomas Eric Peet, had served much more generally to revive interest in the subject of Egyptian mathematics, which was further explored by a small number of authors, including Peet and Neugebauer, throughout the rest of the 1920s. In their works, we see a range of approaches to the subject, from the cautiously context-sensitive style usually adopted by Egyptologists to the much more speculative mathematically-led methods of those scholars, such as Neugebauer, who came from strongly mathematical backgrounds. In this article, we present an annotated translation of Neugebauer's 1926 lecture and use it to paint a picture of differing attitudes towards the study of ancient mathematics.

Keywords: Ancient Egyptian mathematics; Otto Neugebauer; Thomas Eric Peet

Part 1. Introductory discussion

Background

Prior to the late 1870s, the academic world knew little about ancient Egyptian mathematics.[1] A long-standing tradition, stemming from Herodotus (fifth century BCE), held that the ancient Egyptians had been forced to invent

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[1]A longer and more detailed version of this background section, with fuller references, may be found in Hollings and Parkinson (2022).

geometry in order to meet the needs of land-surveying, and that this science had afterwards passed out of Egypt and into Greece.[2] But this is as far as the tradition went: it said nothing about the extent or nature of ancient Egyptian mathematical knowledge, and had no evidential basis either in archaeology or in direct textual sources. The initial inroads that were made into the reconstruction of ancient Egyptian languages and the decipherment of Egyptian modes of writing during the early nineteenth century included the identification of the numerical signs of the various scripts,[3] but still nothing could be said about the details of Egyptian mathematics, owing to a dearth of relevant original sources.

The situation changed, however, in the early 1860s, when the Scottish archaeologist Alexander Henry Rhind (1833–1863) purchased the two halves of an extensive papyrus from an antiquities dealer in Luxor. Following Rhind's early death very soon afterwards, the papyrus passed to the British Museum, where it remains to this day.[4] Its inclusion of what appear to be simple diagrams led to its being dubbed a 'geometrical papyrus' in its early mentions in print,[5] but it is generally known nowadays as the Rhind Mathematical Papyrus (hereafter, RMP). A small number of further sources on ancient Egyptian mathematics have subsequently come to light, but the RMP remains the largest and the most detailed document that has survived.[6]

The first detailed account of the content of the RMP was published by the Egyptologist August Eisenlohr (1832–1902) in the late 1870s (Eisenlohr 1877). Writing at a time when the understanding of the Egyptian language was still evolving, Eisenlohr looked to the mathematical content of the papyrus in order to reconstruct meaning where knowledge of the language was lacking. His work, which is still regarded as a remarkable achievement,[7] revealed the content of the papyrus to consist of an extensive reference table to aid in calculations with fractions (taking up most of the recto), and a sequence of over 80 arithmetical and geometrical problems, covering such topics as the distribution of rations in specified proportions and the calculation of certain areas. More fundamentally, the examination of the methods used to solve these problems provided the first detailed view of the processes of ancient Egyptian calculation, as well as an insight into the extent of the mathematical knowledge of ancient Egyptian scribes.[8] Eisenlohr's edition of the RMP caused excitement among mathematical readers in particular, since it pushed the (detailed) story of mathematics back several centuries further than it had previously been known – an important consideration for a discipline whose practitioners take great pride in its antiquity.[9] Thus, although no substantially new writings on ancient Egyptian mathematics appeared in print in the final years of the nineteenth century, there were a great many derivative works, based on Eisenlohr's treatment of the RMP but stripped of much of the philological commentary

[2] *Histories*, II.109.
[3] See, for example, Parkinson (1999, 32).
[4] Under museum numbers EA10057 (https://www.britishmuseum.org/collection/object/Y_EA10057) and EA10058 (https://www.britishmuseum.org/collection/object/Y_EA10058).
[5] See, for example, Birch (1868).
[6] For a current list of extant (hieratic) mathematical sources, see Imhausen (2016, § 9.1). Extracts from some of these sources may be found in Claggett (1999) and Katz (2007).
[7] See, for example, the recent comments of Imhausen (2021, 44).
[8] For an up-to-date account of the subject, see Imhausen (2016).
[9] On this point, see Hollings and Parkinson (2022, § 8).

and aimed more squarely at mathematical readers.[10] Gradually, the initial flurry of interest in ancient Egyptian mathematics died away as the subject was fully incorporated into the standard story of the history of mathematics.

Something of which mathematical readers were probably unaware, however, was the dynamic nature of the study of Egyptian philology at this time. Given the rapid evolution of the subject, it was an acknowledged fact that any edition of an ancient text would quickly go out of date,[11] as indeed happened to Eisenlohr's edition of the RMP. Some possible new readings of parts of the papyrus were suggested in the 1890s (Griffith 1891/1894), but it was clear that an entirely new edition was needed. This need would eventually be supplied by the Egyptologist Thomas Eric Peet (1882–1934). Peet had studied mathematics and classics in Oxford during the early years of the twentieth century, before embarking upon an archaeological career. Later, he turned to philology, as an editor and translator of ancient texts, and it is in this capacity that he published his edition of the RMP in 1923, by which time he was professor of Egyptology at the University of Liverpool.[12]

From the end of the nineteenth century until the appearance of Peet's edition of the RMP, the subject of ancient Egyptian mathematics had lain largely dormant, aside from the occasional further digest of Eisenlohr's writings. Peet's edition, however, immediately sparked a new interest, largely again from mathematicians, but also from at least a few Egyptologists, who were able and willing to incorporate an up-to-date interpretation of ancient Egyptian mathematics into their image of the wider civilization.[13] Peet's edition not only brought the latest knowledge of Egyptian language and culture to the study of ancient mathematics, but was also able to present a richer picture of the subject by drawing upon those additional sources that had come to light since Eisenlohr's writing in order to give a broader overview of the character of ancient Egyptian mathematics.[14] Crucially for the present discussion, Peet's edition also attracted a new type of reader, whom we shall refer to as the 'mathematician-philologists'.

In dealing with written historical materials, there exists a dividing line between those readers who have the languages to enable them to engage with the original sources and those who do not. While it would not have been unusual for a late-nineteenth-century mathematician or historian of mathematics to have possessed a knowledge of Latin or ancient Greek, with which they would have been able to read particular ancient sources, a knowledge of Ancient Egyptian among such readers was virtually unheard of – a circumstance that is unsurprising, given that philologists were still investigating the basic details of the language and lexicography. Thus, in regard to Egyptian mathematics, this linguistic dividing line was particularly sharp. Indeed, it was initially only through Eisenlohr's German translation of the RMP that mathematical readers were able to access the text at all. The subsequent derivative

[10]First among these was the treatment given by Cantor (1880), who had assisted in the preparation of Eisenlohr's edition. For a list of other such works, see Archibald (1927/1929).

[11]As the Egyptologist Francis Llewellyn Griffith (1862–1932) put it: 'time treats Egyptological commentaries *hardly*' (Griffith 1891/1894 26, n †, his emphasis).

[12]On Peet's career trajectory and his relationship to mathematics, see Hollings and Parkinson (2023).

[13]See, for example, the sections on numeration and metrology in Gardiner's *Egyptian grammar*, whose first edition appeared in 1927 (Gardiner 1957, §§ 259–266). More generally, on this reawakening of interest in Egyptian mathematics, see Hollings and Parkinson (2022).

[14]See, for instance, Peet (1923, 10).

accounts alluded to above further moulded the ancient mathematics into a form that was suited to modern mathematical readers, but at the cost of a greater separation from the original source material – a separation that was responsible (and continues to be responsible) for the more egregious errors and misinterpretations in the study of ancient Egyptian mathematics.[15] It became possible for the mathematical reader, by whom we mean any reader whose principal educational background was in mathematics, to focus exclusively on the details of the ancient mathematics without any wider considerations of context or language. They were free to begin to interpret that mathematics in light of their modern knowledge of the subject, with little concern for whether their interpretations were truly consistent with the original sources.[16] For Eisenlohr, the mathematically-led approach to the subject had been a sound, but temporary, means of opening a door onto an otherwise impenetrable subject. For the mathematical readers who followed him, it became the only way of dealing with material supplied by the Egyptologists. Driven by an implicitly idealist attitude, whereby ancient and modern mathematics differ only in the marks used to represent them, mathematicians did not perceive this to be a problem. Mathematical readers of ancient mathematics thus came to be characterized not only by their background but by their 'mathematical outlook': their readiness to interpret ancient sources in light of modern mathematical knowledge. The numerical superiority of such readers over those with a knowledge of the original languages led to the dominance of their point of view throughout much of the twentieth century (and arguably to this day).[17] We observe that the idea of 'outlook' is at least as important here as 'background': those few Egyptologists who had any substantial mathematics in their educational backgrounds – principally Peet, but also his sometime co-author Battiscombe Gunn (1883–1950), who had briefly studied engineering before turning to Egyptology – nevertheless maintained a 'philological outlook', whereby they placed greater emphasis on the language of original sources than on the details of the mathematics, in connection with which they took great pains not to impose modern views on ancient material. In a lecture of early 1934, for example, Peet commented on 'the necessity of tearing ourselves away from all modern conceptions and trying to start with blank minds, if we are to understand the process by which the Egyptians reached their results' (Peet 1934, 19–20).

Whereas Eisenlohr's readers were largely separated into the mutually exclusive camps of Egyptologists and mathematicians (among whom we also include those historians of mathematics who lacked knowledge of Ancient Egyptian), further scholars whom we shall dub the 'mathematician-philologists' emerged as a third (very small) category among Peet's readers who combined a mathematical background and outlook with a knowledge of ancient languages. Foremost among these readers was the young Otto Neugebauer (1899–1990), whose burgeoning interest in ancient

[15]For example, the spurious 'hieroglyphic equation' presented by Matthiessen (1878, 269). On this point, see also Hollings and Parkinson (2022, § 8). The need for readers' direct engagement with original materials has recently been re-emphasized by Imhausen (2021).

[16]We can cite here the widely circulated 'Egyptian value of π' (3.16), which arises when we equate the Egyptian method of calculating the area of a circle (namely, taking the square of eight-ninths of the diameter) to the formula πr^2. However, there is no evidence for any knowledge of the latter formula in ancient Egypt: see Imhausen (2009).

[17]See Hollings and Parkinson (2022) for a discussion of how the study of ancient Egyptian mathematics passed out of the hands of Egyptologists and into those of mathematicians.

science had been solidified by his being invited to review Peet's edition for a Danish journal.[18] His Göttingen doctoral dissertation on the principles of ancient Egyptian fraction-reckoning followed in 1926, marking the beginning of Neugebauer's rise to prominence as a historian of ancient mathematics and astronomy.[19] Throughout his career, Neugebauer stressed the need for scholars to learn the ancient languages necessary to read primary sources,[20] while at the same time promoting a strongly mathematical approach to the subject matter which is nowadays very often criticized for its failure to take account of broader cultural contexts.[21] The continued prevalence of the 'mathematical outlook' among readers who could not investigate the original texts was therefore legitimized, at least implicitly, by Neugebauer's work. A systematizing attitude towards ancient Egyptian mathematics is present in Neugebauer's dissertation work at the very beginning of his career, and this provides us with a prime example of this particular approach to the subject.

In his dissertation, Neugebauer put across two broad themes. The first was the assertion of the purely additive ('rein additiv') nature of ancient Egyptian mathematics: he denied Egyptian scribes a general notion of multiplication, arguing that the operations of doubling and decupling, for example, that appear in surviving sources should be interpreted as separate processes, rather than as instances of a general technique. The second part of Neugebauer's dissertation proposed a systematization of certain of the problems in the RMP, asserting an interdependence (not explicitly indicated in the original) between these problems and the construction of the table on the recto of the papyrus. In order to achieve this overarching scheme, Neugebauer was not averse to reordering the problems in the RMP or indeed interpolating whole new problems. What mattered to him was the identification of the fundamental principles upon which, from the point of view of modern mathematical sensibilities, the composition of the papyrus ought to have been based.[22]

In writing his dissertation, Neugebauer's main source for the RMP had been Peet's edition, and at some point in the mid-1920s he had begun corresponding with Peet on the subject of ancient Egyptian mathematics.[23] Peet's wider surviving correspondence, principally with his mentor Alan H Gardiner (1879–1963), provides a glimpse of his private opinions of Neugebauer's work. For the most part, these were reasonably positive: Peet seems to have been convinced by Neugebauer's 'additive' claims, if not by the certainty with which Neugebauer put his ideas across.[24] The systematization of problems within the RMP drew considerably fewer direct comments from Peet, however, and may indeed be covered by a later exasperated remark about the 'abstruse' nature of the correspondence that he was receiving from mathematical

[18]Neugebauer (1925). On Neugebauer's early career, see Rowe (2016b).

[19]Neugebauer (1926). On the content of Neugebauer's dissertation, see Ritter (2016) and Hollings and Parkinson (2020).

[20]See, for example, Neugebauer (1962, 49).

[21]See, for example, Robson (2007a, 60) and Imhausen (2009, 781).

[22]On the influence of the Göttingen mathematical school on Neugebauer's approach to the history of mathematics, see Rowe (2016a).

[23]The only two known surviving letters in this correspondence are discussed in Hollings and Parkinson (2020).

[24]Griffith Institute Archive, Oxford: AHG/42.230.92(2), Peet to Gardiner, 16th October 1926. See the discussion in Hollings and Parkinson (2020, § 4).

writers.[25] Peet's own approach to ancient mathematics was altogether more cautious, and displayed a reluctance to speculate beyond what was clearly and unequivocally written in the original sources. Indeed, his commitment to the text as written sometimes even led him to retain interpretations that were consistent with the written language but that did not in fact make *mathematical* sense.[26]

Peet's very small number of publications on this subject clearly display his cautious, language-led approach, whereas the mathematicians' approach that we have outlined above remained visible throughout Neugebauer's later works, not only those on Egyptian mathematics, but also his much more extensive writings on its Mesopotamian counterpart. We take this mathematical approach as the main focus of the present paper. We explore this through a case-study: we present in the second half of the present paper a translation of the typescript of a lecture, 'Über die Mathematik im alten Ägypten', that Neugebauer delivered at the University of Kiel in December 1926; this remained unpublished, although the tidiness of the typescript and the referencing that it contains suggest that it was being prepared for publication or circulation (Figure 1). The typescript is now in the Shelby White and Leon Levy Archives Center (IAS, Princeton), and is freely available online.[27] We provide a commentary which situates the lecture both within the scholarship of Neugebauer's time, and also within the subsequent literature; our lettered notes serve to interlink the appearances of related ideas in the translation and in this preliminary discussion. As an introduction, we provide a summary of the lecture, and also a short discussion of its style and attitudes, drawing a brief comparison with another lecture on ancient Egyptian mathematics, this one given by Peet in 1931.

Summary

Neugebauer's lecture was written in the language of a mathematician (see below), and it contained a summary of some of the ideas of his doctoral dissertation on Egyptian fraction-reckoning, packaged appropriately for an audience of mathematicians. The lecture was delivered in a mathematical didactics colloquium, and Neugebauer did seem to hint at points of comparison between the development of Egyptian numeration and the modern process of arithmetical education, but he never developed this theme explicitly,[28] and left the impression that he was addressing research mathematicians first and foremost.

Neugebauer introduced some important themes very early in his typescript. The first concerned what he termed 'genuinely scientific historical research' ('wirklich wissenschaftlichen Geschichtsforschung') [1],[29] in which phrase he united the systematizing approach to historical mathematics which we have mentioned above with serious attention to Egyptian philological scholarship. In doing so, Neugebauer displayed the

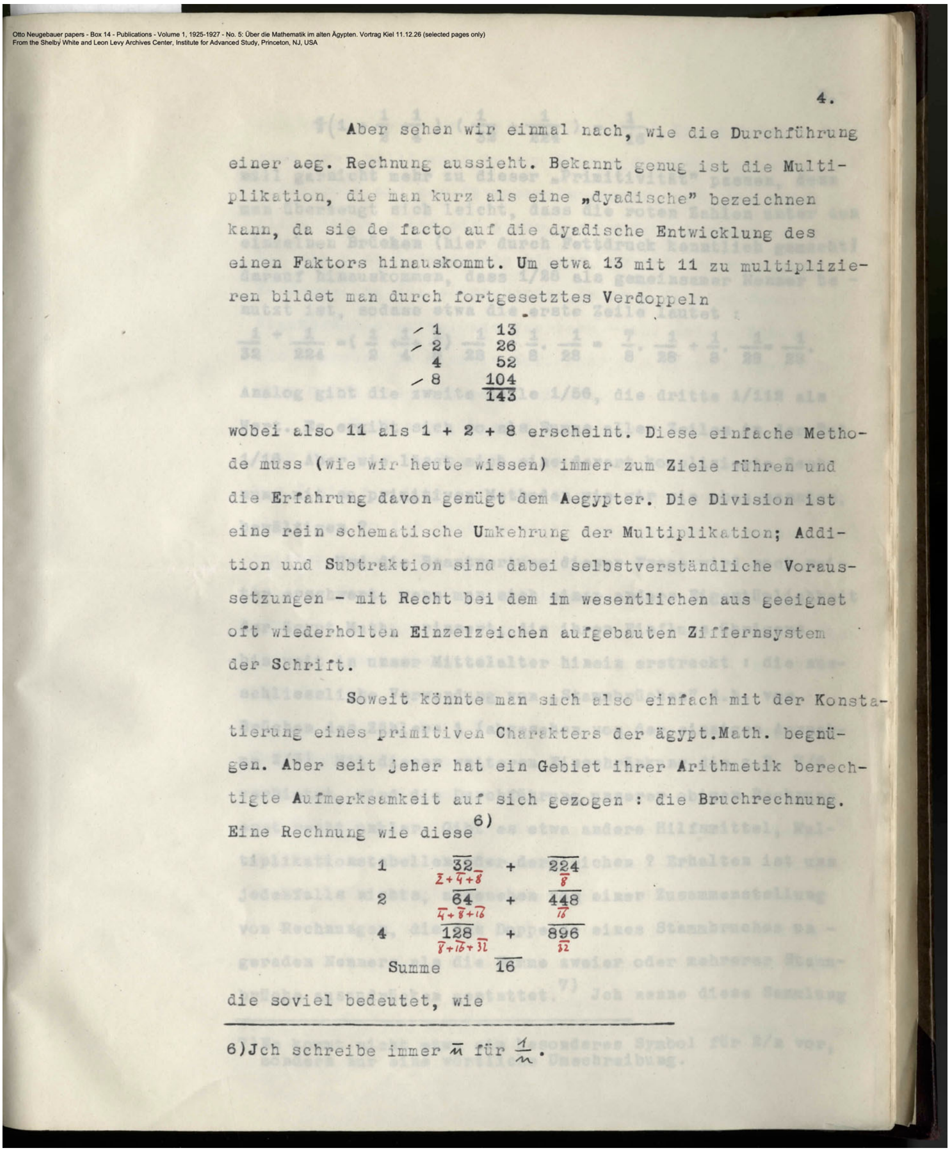
[25] Griffith Institute Archive, Oxford: AHG/42.230.39, Peet to Gardiner, 19th May 1930.

[26] We see this, for example, in his initial treatment of Problem 51 of the RMP (Peet 1923, 91–94). We hope to deal with this point at greater length elsewhere. See also note 53 below.

[27] Shelby White and Leon Levy Archives Center (IAS, Princeton): Otto Neugebauer papers, Box 14 – Publications – Volume 1, 1925–1927 – No 5: Über die Mathematik im alten Ägypten. Vortrag Kiel 11.12.26. https://albert.ias.edu/handle/20.500.12111/2923.

[28] See note *z* in Part 2.

[29] Numbers in square brackets in this summary and the following discussions correspond to the page numbering of Neugebauer's typescript, as indicated in Part 2 of the present paper.



Otto Neugebauer papers - Box 14 - Publications - Volume 1, 1925-1927 - No. 5: Über die Mathematik im alten Ägypten. Vortrag Kiel 11.12.26 (selected pages only)
From the Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA

4.

Aber sehen wir einmal nach, wie die Durchführung einer aeg. Rechnung aussieht. Bekannt genug ist die Multiplikation, die man kurz als eine „dyadische" bezeichnen kann, da sie de facto auf die dyadische Entwicklung des einen Faktors hinauskommt. Um etwa 13 mit 11 zu multiplizieren bildet man durch fortgesetztes Verdoppeln

/ 1	13
/ 2	26
4	52
/ 8	104
	143

wobei also 11 als 1 + 2 + 8 erscheint. Diese einfache Methode muss (wie wir heute wissen) immer zum Ziele führen und die Erfahrung davon genügt dem Aegypter. Die Division ist eine rein schematische Umkehrung der Multiplikation; Addition und Subtraktion sind dabei selbstverständliche Voraussetzungen - mit Recht bei dem im wesentlichen aus geeignet oft wiederholten Einzelzeichen aufgebauten Ziffernsystem der Schrift.

Soweit könnte man sich also einfach mit der Konstatierung eines primitiven Charakters der ägypt.Math. begnügen. Aber seit jeher hat ein Gebiet ihrer Arithmetik berechtigte Aufmerksamkeit auf sich gezogen : die Bruchrechnung. Eine Rechnung wie diese[6)]

1	$\overline{32}$ ($\bar{2}+\bar{4}+\bar{8}$)	+	$\overline{224}$ ($\bar{8}$)
2	$\overline{64}$ ($\bar{4}+\bar{8}+\overline{16}$)	+	$\overline{448}$ ($\overline{16}$)
4	$\overline{128}$ ($\bar{8}+\overline{16}+\overline{32}$)	+	$\overline{896}$ ($\overline{32}$)
Summe	$\overline{16}$		

die soviel bedeutet, wie

6) Jch schreibe immer $\bar{n}$ für $\frac{1}{n}$.

Figure 1. Page from Neugebauer's lecture typescript, showing ancient Egyptian methods of calculation. Image courtesy of the Shelby White and Leon Levy Archives Center (IAS, Princeton).

twin foundations of his academic identity: his education in the Göttingen mathematical school, with its focus on the search for general principles and rigorous foundations in mathematics, and his grounding in German comparative philology, with its strong textual focus.[30] Indeed, the mathematical, philosophical, and philological sources that Neugebauer cited throughout the typescript confirm the impression of his straddling several disciplines; he comfortably combined mathematical

[30]See notes *cc* and *ee* in Part 2.

explanations with philological evidence.[31] There was also a hint of criticism of the anecdotal forms of historical writing which are often found in the history of mathematics, and of which Neugebauer would become even more critical in later years.[32] A further point made in passing in Neugebauer's opening paragraphs, and which he revisited at the end of the lecture, was the relative neglect of 'pre-Greek' mathematics by historians – the lecture as a whole might be taken as a call for attention to such subjects.[33]

Neugebauer began his treatment of ancient Egyptian mathematics by discussing the main sources available for study [2], placing the emphasis on the RMP, and mentioning other sources only in passing. Eisenlohr's and Peet's editions of the RMP were noted, with the latter described as 'meeting all scientific requirements' ('allen wissenschaftlichen Anforderungen gerecht werdende'). In a slight contradiction to the usual modern view of Neugebauer and his work (as indicated above), he also stressed the importance of drawing upon wider Egyptological scholarship, and quoted from a satirical literary letter of the Ramessid Period which gives an indication of the place of mathematics in Egyptian culture [3].[34]

The detailed mathematics of the lecture began with a brief outline of the 'dyadic' Egyptian scheme for multiplication [4], and moved quickly to a sample calculation with fractions, which also served to illustrate a more sophisticated Egyptian calculational technique, the so-called 'method of red auxiliaries', to which Neugebauer returned later in the lecture.[35] The observation that Egyptian fractional arithmetic was confined almost exclusively to (in modern terms) unit fractions led into a discussion of the recto table of the RMP [6], and the methods that modern authors have suggested for its construction.[36] As prelude to a brief description of his own ideas in this direction, Neugebauer next took a step back and considered the formation of the Egyptian system of numeration more generally [10], beginning with integers, but then quickly returning to issues relating to fractions, and the reason for the Egyptian use of unit fractions in particular. The discussion was gradually brought back to the dyadic scheme of multiplication, the RMP recto table, and the method of red auxiliaries, all of which were united with ideas about the formation of the system of fractions to provide an apparently very coherent view of why ancient Egyptian arithmetic worked the way it did.

In a later part of the lecture [21], Neugebauer moved away from the Egyptian setting, and sought a comparison with the mathematics of ancient Mesopotamia. Again taking a systematic approach to the development of early systems of numeration, he examined and sought to explain the greater range of subdivisions that existed in Sumerian metrological systems than in the Egyptian setting. This led Neugebauer into a very brief discussion of the Babylonian sexagesimal positional number system (whose development went on to be the subject of his *Habilitationschrift*, completed the following year) [26],[37] together with some musings on how the

[31]See, for example, the discussions on his page 15.
[32]See note *b* in Part 2.
[33]See note *dddd* in Part 2.
[34]See note *e* in Part 2.
[35]See note *ccc* in Part 2.
[36]See note *o* in Part 2.
[37]See note *kkk* in Part 2.

circumstances surrounding its emergence might be compared with those connected with later developments in Egypt.

In concluding his lecture, Neugebauer moved into territory that would probably have been more familiar to his mathematical audience than any of the foregoing, namely ancient Greek mathematics [28]. He posed the question of what mathematics the Greeks might have taken from Egypt, but left it largely open as a problem in need of further research.

Discussion

The fact that Neugebauer's lecture was aimed at a mathematical audience is immediately clear in the language that he used. Although he focused on research in historical topics, Neugebauer was fully embedded within and accepted by a broader mathematical community.[38] This meant, among other things, that he knew how to write in a style that would appeal to mathematicians. For instance, he betrayed his Göttingen mathematical background in his Kiel lecture when he wrote of making these historical studies 'well founded' ('gut fundiert'), and indicated that his lecture should be taken as a 'programme' for doing so [2]. His explanations of the details of the mathematics were certainly briefer than those that we find, for example, in Peet's edition of the RMP, suggesting that Neugebauer did not expect these parts to present any difficulty to his audience.[39] In parallel with other, more judgemental, writers on this subject,[40] Neugebauer referred to the apparently 'primitive' nature of ancient Egyptian mathematics [4], although it soon became clear that this was not how he regarded the calculational techniques that he was discussing – we might speculate that in applying the word 'primitive', he was trying to pre-empt the dismissive attitude that he might have expected from members of the audience. The remark in his opening paragraph [1] regarding the open-mindedness of his listeners may indeed have been a tactical move in this respect. The exactness inherent in his proposed 'scientific' approach to history was also likely to appeal to the mathematical mind.[41]

Other aspects of the typescript are also not unlike a mathematical paper: the way in which Neugebauer carefully set up and used specialized terminology, for example.[42] In addition, he deployed established mathematical terminology, such as in a passing analogy involving piecewise-linear approximation [8]. Elsewhere, he even used set-theoretic language to label the processes that he was describing: in the systematic foundational approach that he took to the development of Egyptian numeration, for instance, with his use of a notion of 'basis' [16], and in his discussion of 'extensions' to the number 'domain' [20].[43] This was very much a language that would have been understood and appreciated by mathematicians, adding to the sense of precision that Neugebauer was trying to convey. Moreover, at the heart of the research programme that Neugebauer described there was a thread not unlike one that we might find in a

[38]See, for example, the discussion in Hollings and Parkinson (2022, § 9).
[39]Or the briefness could be due to the fact that this is a lecture typescript that was not intended for formal publication.
[40]See, for example, the views of the authors cited in note *i* in Part 2.
[41]Indeed, this was an approach to historical writing that would earn Neugebauer acclaim from mathematicians elsewhere: see note *b* in Part 2.
[42]For instance, his notions of 'natural fraction' [9] and 'complement fraction' [13].
[43]See note *eee* in Part 2.

line of contemporaneous mathematical research: the quest to uncover hidden rules and general principles. He did not ask *whether* there was a general rule underpinning the construction of the recto table of the RMP, but *what* it was. Having given an outline of the preliminary work on ancient mathematics that had been done by philologists, Neugebauer pointed towards the need for further research by mathematical specialists [28]. In doing so, he created an impression of a dynamic field of research that was open to mathematicians, rather than a settled collection of facts, which may have helped to lend the discussion a greater air of legitimacy. And, like in any good mathematics talk, Neugebauer ended on an open problem.[44]

Comparison

By way of context for Neugebauer's Kiel lecture, we make a few brief comments on another lecture on ancient Egyptian mathematics: one given by Peet at the Rylands Library in Manchester in February 1931. In several respects, a comparison of these two lectures is not entirely fair. To begin with, Neugebauer's survives only as an unpublished typescript, while we have Peet's in the form of a polished and published article (Peet 1931b). Moreover, Peet's lecture seems to have been aimed at a generally interested public, in contrast to Neugebauer's more specialized mathematical audience. The relative professional statuses of the two lecturers differed significantly as well: Neugebauer had only recently completed his doctoral dissertation, whereas Peet was the well-established Egyptologist who was about to move from one British Chair in Egyptology to another (he moved from Liverpool to Oxford in 1933). Neugebauer's typescript includes a number of references to the wider literature, but nothing on the scale of the detailed referencing included by Peet, whose article, moreover, is considerably longer, being a general overview of ancient Egyptian mathematics, in contrast to Neugebauer's narrower focus. With regard to referencing, Peet had access to the much richer literature dealing specifically with Egyptian mathematics that had developed in the five years that had passed – this of course included several of Neugebauer's own subsequent works.[45] The most significant additional source to which Peet had access in 1931 was the Moscow Mathematical Papyrus (MMP). He had obtained photographs of it prior to the publication of his edition of the RMP, and while, as we can see from Neugebauer's lecture, the MMP was widely known to exist in 1926, the detailed study of its content took place only gradually during the 1920s, culminating in the edition published by the Russian Orientalist V V Struve (1889–1965) in 1930.[46] Thus, Peet was in the comfortable position of being able to look back over a decade of publications in this area, and was able to express different opinions from those he had included in his own earlier writings (see below). Nevertheless, despite these differences, a comparison of the two lectures illustrates the contrasting approaches of these two authors to the subject of ancient Egyptian mathematics.

Peet opened the published version of his lecture by acknowledging the two possible approaches to ancient mathematics that we have mentioned above: the

[44]Indeed, insofar as it is possible to judge it from a typescript, Neugebauer's lecture matches favourably with the later advice given by Paul Halmos on how to deliver a mathematical talk (Halmos 1974).
[45]On the growth in the literature on ancient Egyptian mathematics during the 1920s, see Hollings and Parkinson (2022).
[46]Struve (1930). On the MMP, see notes *c* and *aaaa* in Part 2.

mathematically internalist attitude, and the broader, more contextualizing approach. In Peet's view, however, these represent two successive stages in the study of ancient mathematics: first we must understand the details of the mathematics (as Eisenlohr had done), but we must then seek to place that mathematics within its proper setting (as Peet aimed to do in his edition of the RMP). We might speculate that Peet was quietly speaking out here against the more narrowly mathematical writings on the subject that had appeared during the second half of the 1920s (perhaps the 'abstruse' ideas to which we alluded earlier).

The next sections of Peet's lecture summarized the available primary sources, and introduced Egyptian numerical notation, along with the basic operations of arithmetic: in particular, the scheme for calculation that Neugebauer had labelled 'dyadic'. In what was presumably a simplification for his non-specialist audience, Peet only mentioned the hieroglyphic numerals and made little mention of those of the cursive hieratic script in which the surviving mathematical texts are actually written; Neugebauer had similarly side-stepped this point by not introducing any of the original numerical notation at all.[47] In Peet's lecture, detailed sections follow on techniques for fractional arithmetic, including the method of red auxiliaries (which he called the 'common denominator method'). He addressed the construction of the RMP recto table directly and firmly with the assertion that 'no general formula was used, […] the results were purely empirical and obtained by gradual collection'.[48] Peet did not employ Neugebauer's simplified notation for unit fractions;[49] this had yet to be widely adopted by writers on this subject.[50]

Peet gave several sample problems in Egyptian arithmetic (more than Neugebauer had had space for), some drawn from the RMP and others constructed to illustrate particular techniques. He also addressed the controversial issue of whether ancient Egyptian scribes had any notion akin to our algebraic equations – he thought not, preferring a graphical justification of certain solutions in the RMP.[51] In this connection, he found that he had to disagree, however politely, with some recent suggestions by Neugebauer (1930) as to how certain apparently 'algebraic' problems in the RMP and MMP had been solved; indeed, it was not just the technique suggested by Neugebauer (tantamount to the manipulation of algebraic equations) with which Peet found fault, but also his assertion that this was a uniform method that had been applied to several problems: in Peet's view 'the outstanding characteristic of Egyptian mathematics is precisely the lack of any such uniformity' (Peet 1931b, 422).

This treatment of arithmetical problems in Peet's lecture was followed by something that Neugebauer had not included in his own (very possibly for reasons of time), namely a discussion of Egyptian geometry. Here, Peet was able to draw upon his own recent investigations, in particular of the geometrical problems in the MMP.[52] He worked systematically through what was known about Egyptian knowledge of the areas and volumes of certain shapes. The area of a triangle received particular attention, as this had been an especially contentious point. Peet carefully

[47]See note *h* in Part 2.
[48]Peet (1931b, 414). See notes *g* and *o* in Part 2.
[49]See notes 6 and *j* in Part 2.
[50]For example, Struve, despite his personal and professional links to Neugebauer (see note *c* in Part 2), did not employ this notation in his edition of the MMP.
[51]Peet (1931b, 421). On the question of Egyptian algebra, see the discussion in note *s* in Part 2.
[52]Gunn and Peet (1929) and Peet (1931a).

summarized the philological and mathematical evidence for the various possible interpretations of the relevant problem in the RMP (Problem 51), before arriving at the conclusion that Egyptian scribes did indeed have a general technique for finding the area of an arbitrary triangle, a point upon which he had previously cast doubt, on philological grounds.[53]

The final five pages of Peet's lecture took up the question of the 'general character' of Egyptian mathematics, and here he again had cause to revise a prior opinion. In his edition of the RMP, he had asserted that 'everything [in Egyptian mathematics] is expressed in concrete terms' (Peet 1923, 10), but by 1931, he was willing to grant an element of abstraction to ancient Egyptian mathematical thinking (Peet 1931b, 437). In a manner that is entirely consistent with all that we have noted about Peet so far, this change of view stemmed not from a re-examination of any mathematics in the RMP and other sources, but from philological detail: namely the realization that the 'classifier'-sign that is used in the Egyptian word (*ʿḥʿ*) that indicates a 'quantity' in an arithmetical problem is the sign which denotes an abstract notion.[54] Peet's further discussion of the 'general character' of Egyptian mathematics focused largely on whether it had a 'scientific nature', an active topic of discussion in the literature of the time, and one that Neugebauer had touched upon in his own lecture several years earlier.[55] Peet's treatment of this issue was careful and thoughtful, and considered both sides of the argument, but came to few firm conclusions, other than to deny (via a quotation from Neugebauer, who was of the same mind as Peet on this point) the legitimacy of the question 'was ancient Egyptian mathematics scientific?'. In his final remarks, Peet retained the same non-judgemental tone that he had employed throughout the lecture in connection with Egyptian mathematics, and warned against the use of words such as 'unscientific' which might be seen to carry reproach. He reserved his most critical comments for those other contemporary authors whom he believed to have been incautious in their assumptions,[56] but even here his remarks were considered and carefully worded, firm without being absolute: in contrast to the bolder style often adopted by Neugebauer, Peet would say that something was 'not very convincing' rather than assert that it was 'wrong'.[57]

Two final points of contrast between the styles of Peet and Neugebauer lie in aspects that one included in his lecture but the other did not. We have seen that a comparison of Egyptian and Mesopotamian mathematics appeared towards the end of Neugebauer's lecture, but no such material was included in Peet's. Such a comparison did appear briefly in Peet's edition of the RMP,[58] but the fact that he did not include anything of this kind in his lecture is arguably due to more than simply a lack of time

[53]Eight years earlier, Peet had arguably been *overly* cautious in his approach to this issue, and had backed away from giving any firm conclusion on this point in his edition of the RMP, namely that it was impossible to say whether a general technique for finding the area of a triangle was known (Peet 1923, 91–94). In his review of the edition, however, Gunn cut across Peet's indecision and asserted that the evidence for the calculation was clear (Gunn 1926, 133). Peet had evidently also become convinced of this by 1931.

[54]This sign shows a papyrus roll, indicating words to do with writing, written things, and thus abstract concepts (Gardiner's sign list Y1; here and below, references to Gardiner's sign list are to the version found in Gardiner 1957).

[55]See notes *g* and *fff* in Part 2. For a discussion of various views on this point, see Hollings and Parkinson (2022, § 8).

[56]See, for example, his comments on Kurt Vogel's opinions on 'Egyptian algebra': Peet (1931b, 423, n 2).

[57]Again, see, for example, Peet's comments concerning claims for 'Egyptian algebra': Peet (1931b, 423).

[58]See notes *jjj* and *lll* in Part 2.

and space. For Neugebauer, this cross-cultural comparison was a central part of his approach to ancient mathematics – he stressed this in his lecture [27], and would have justified it in much the same terms as the legitimacy of considering such material through a modern mathematical lens. For Peet, on the other hand, such a comparison may still have been insightful, but his focus was placed more firmly on Egyptian mathematics in itself as part of Egypt's wider culture, rather than as part of a larger cross-cultural study in the history of mathematics. While Peet, like Neugebauer, drew comparisons with ancient *Greek* mathematics (Peet 1931b, 441), this was probably because the compulsion to use Greece as a benchmark in the study of any ancient culture was an ingrained aspect of contemporaneous (classical) education.

One last point that reinforces Neugebauer's focus on mathematical detail in contrast to Peet's view of ancient Egyptian mathematics as a cultural activity is the fact that the individual was much more visible in Peet's lecture than in Neugebauer's. The mathematics that Peet described was the creation and tool of ancient scribes whose motivations and thought processes he attempted, in his cautious way, to capture, just as he had transcribed and edited their papyri. It was therefore quite natural for Peet to propose empirical, even accidental, modes of discovery for the mathematics that appears in ancient sources, and for him to take the long view that these ideas were collected by gradual accretion over centuries.[59] The many mistakes that are preserved in these sources evoke an image of imperfect and varied individuals, who were, moreover, members of that tiny minority of ancient Egyptian society that was both literate and numerate. And even for these people, mathematics was just a small part of their world. This contrasts strongly with Neugebauer's veritable rejoicing in 'the disappearance of the individual in Egyptian cultural history', as he put it in an article published the year after his Kiel lecture.[60] In Neugebauer's view, the search for the individual scribe was a distraction from the quest to understand the nature of 'exact thought' in the ancient world.[61] By his downplaying of the specific human factor, Neugebauer's scribes became implicitly idealized ancient mathematicians who strove after truth in much the same way as their modern counterparts. There was little room in this view for 'untidy' ideas about empirical discoveries – hence the search for the general rules upon which ancient Egyptian mathematics must surely (in his view) have been founded.

We conclude our discussion of these lectures of Neugebauer and Peet, and of the approaches to the study of ancient mathematics that they represent, by noting that although we have presented a picture of largely opposing methodologies, the relationship between the two was not at all one of antagonism. Peet's opinion of Neugebauer's work was in general quite positive: for example, he described Neugebauer's dissertation as both 'valuable' and 'enlightening if difficult' (Peet 1930a, 270). Although Peet was always ready to condemn inadequate scholarship and to disagree with conclusions,[62] he never

[59]See Peet's remarks above on the RMP recto table.

[60]'Das Verschwinden der Einzelpersönlichkeit in der ägyptischen Kulturgeschichte ist nicht immer als ein Vorzug angesehen worden.' (Neugebauer 1927c, 40).

[61]See the comments in Siegmund-Schultze (2016, 103).

[62]He was particularly critical, for instance, of the quality of the scholarship of Arnold Buffum Chace (1845–1932), who published his own edition of the RMP in the late 1920s. Having seen some of Chace's draft material earlier in the decade, Peet declined a collaboration that Chace had proposed, and commented in a private letter to Gardiner: 'Frankly, I don't think his work is, or ever will be of the quality with which I want to be associated' (Griffith Institute Archive, Oxford: AHG/42.230.176(1), Peet to Gardiner,

questioned the value of the mathematically-led works produced by Neugebauer and others; there was a place for these contributions to the literature, beside more restrained investigations, but such mathematical speculations were of limited interest to Peet if they did not add to an overview of ancient Egyptian culture.[63] Conversely, Neugebauer always maintained a deep respect for Peet and his work, which had of course provided the foundation for his own investigations, and many years later, he was still referring to Peet's Rylands lecture as an 'excellent brief summary of Egyptian mathematics' (Neugebauer 1962, 92). We suggest that Peet, as one of the few Egyptologists to whom Neugebauer could easily communicate his thoughts, provided an anchor for Neugebauer within an Egyptological community that did not always take his ideas seriously.[64] Neugebauer's 1926 lecture shows that he recognized the value of historians of mathematics engaging with the wider Egyptological literature and the important role to be played by intermediary figures and resources,[65] and it was only in later decades that greater animosity emerged between the arguably more extreme and more entrenched 'mathematical' and 'cultural' camps in the study of ancient mathematics.[66] It was in precisely this positive light that in 1931 Neugebauer had reviewed Peet's Rylands lecture for *Zentralblatt für Mathematik und ihre Grenzgebiete* as:

> Summary report on the current state of our knowledge of Egyptian mathematics, critically considering the latest literature. In particular, those who are not directly connected with the field are given an extremely carefully and cautiously drawn picture of the entire problem situation, which they can really rely on.[67]

Part 2. Translation and commentary

English translation of Neugebauer's lecture

Neugebauer's discussion here of ancient mathematics is written entirely in the historical present, but since this occasionally creates ambiguities (what does 'now' mean?), we have adopted the past tense as appropriate throughout the translation. Numbered footnotes, as well as the initial footnote marked by an asterisk, appear in Neugebauer's typescript; notes denoted by letters, gathered at the end of the translation, form our commentary. Since Neugebauer's citations are often abbreviated, we add to his footnotes references to the items as given in our bibliography.

7th December 1921). Chace's edition of the RMP (Chace 1927/1929), with its greater emphasis on mathematics than on philology, went on to be the preferred edition of mathematicians, while Peet's was used by Egyptologists; see the recent comments of Imhausen (2021), and also those in Hollings and Parkinson (2022, §§ 6, 8).

[63]In some cases, he noted particular works as being of greater interest to mathematicians than to Egyptologists: see, for example, Peet (1930b).

[64]See, for example, the comments of Ritter (2016, 158) on the reception of Neugebauer's ideas by Egyptologists.

[65]See, for example, the remarks on his page 3 about the resources that 'are already made accessible to non-Egyptologists in a very convenient and reliable way'.

[66]See Schneider (2016) as an introduction to the relevant literature. On changing methodologies in the study of ancient Greek mathematics in particular, see Saito (1998).

[67]'Zusammenfassender Bericht über den gegenwärtigen Stand unserer Kenntnisse der ägyptischen Mathematik unter kritischer Berücksichtigung der neuesten Literatur. Insbesondere dem Fernerstehenden wird ein überaus sorgfältig und vorsichtig gezeichnetes Bild der ganzen Problemlage gegeben, auf das er sich wirklich verlassen darf.' (Zbl 0002.37801)

On mathematics in ancient Egypt

by

O. Neugebauer, Göttingen*

I do not want to begin my lecture, which, as you know, deals with the mathematics of Egypt, without a word of thanks. First of all, my thanks to Herr Prof. Toeplitz,[a] who gave me the opportunity to speak here before you. But then it is for me a particular satisfaction to see that you are interested in a topic that is generally so little up-to-date as Egyptian mathematics. I can see in this a sign that you recognize the justification of a working method that regards a detailed study of individual questions of historical development, with the greatest possible reference to all of our knowledge of the cultural epoch in question, as the necessary prerequisite for genuinely scientific historical research. This line of work, which has been gaining more and more importance over the last few years, hopefully means the end of a state of affairs in the history of mathematics which – at least as far as the history of pre-Greek mathematics is concerned – can sometimes only be compared with the rule of force, where it is more about boldness of assertion than thoroughness of investigation.[b]

I turn to my subject: Egyptian mathematics. But even within this, there will be only one sub-area around which my report is organized: fraction-reckoning. However, since this can only be a report and not an exhaustive presentation, I will allow myself a few digressions [2] that are intended to illuminate the context with further questions. Such a consideration is then necessarily at the same time a criticism of what has been achieved so far, and indeed a negative one, because it shows what remains to be done and, worse, what should actually have been done before the actual investigation can be considered really well founded. But that is perhaps too much to ask in such an under-worked area as pre-Greek mathematics, at least for the strength of one individual; and so I ask you to regard the second part of my lecture in particular as just a programme.

Now to Egyptian mathematics itself. I must start with a few remarks about our sources. You all know the name of the most important of them: it is the mathematical 'Papyrus Rhind' of the British Museum, which was published in 1877 by Eisenlohr as 'a mathematical handbook of the ancient Egyptians',[1] and recently (1923) it was revised, meeting all scientific requirements, by the English Egyptologist T. E. Peet.[2] The documents of Egyptian mathematics that are otherwise available to us are too scanty for me to tire you by listing them.[3] However, there is a second well-preserved mathematical papyrus in the Moscow collection, of which only a small fragment has been published so far;[4,c] the rest of its content is kept strictly [3] confidential. So if accordingly our knowledge of Egyptian mathematics seems to rest on very weak legs, one must still emphasize one kind of source that is usually not only kept quiet but actually ignored: our general knowledge of the history and culture of Egypt. Our knowledge in this field is so rich, based on innumerable papyri of religious and

*Lecture held on 11 Dec 1926 in the mathematical didactics colloquium of the University of Kiel.

[1] A. Eisenlohr, Ein math. Handbuch d. alten Aegypter, Leipzig 1877 [Eisenlohr 1877].

[2] T. E. Peet, The Rhind Math. Papyrus, Liverpool 1922 [*sic*] [Peet 1923].

[3] See T. E. Peet, l.c. 6ff.

[4] B. Tourajeff [Turaev], The volume of the truncated pyramid in Egyptian mathematics, Ancient Egypt 1917 [Touraeff 1917].

literary content, on innumerable inscriptions in tombs and temples of all epochs over 3000 years: history that one cannot ignore any more than one can ignore other expressions of Greek intellectual life when dealing with Greek mathematics. And in Egypt (very much in contrast to other parts of oriental history) we are also in the fortunate position that all these things are already made accessible to non-Egyptologists in a very convenient and reliable way. I will immediately make use of it by reading to you a passage from a famous 'literary pamphlet' which relates directly to the questions in which we are interested here.[5,d] An administrator writes to a colleague:

[*quotation absent from typescript; rest of line left blank*][e]

Here you have a characteristic Egyptian text in front of you; Egyptian in every respect, in its entire habitus and especially instructive for us due to its attitude to mathematics – any given problem could be taken directly from Pap. Rhind.

[4] But let's see what an Egyptian calculation looks like. The multiplication, which can be briefly described as 'dyadic', is well known, since it actually amounts to the dyadic development of one factor.[f] To multiply 13 by 11, one forms by continued doubling

$$\begin{array}{ccr} / & 1 & 13 \\ / & 2 & 26 \\ & 4 & 52 \\ / & 8 & \underline{104} \\ & & 143 \end{array}$$

where 11 appears as 1 + 2 + 8. This simple method (as we know today) must always lead to the goal and the experience of it was sufficient for the Egyptian.[g] Division is a purely schematic reversal of multiplication; addition and subtraction are self-evident prerequisites – and rightly so given the script's numerical system, which is essentially composed of individual characters that are repeated appropriately often.[h] So far, one could be content with the simple statement that Egyptian mathematics was primitive.[i] But one area of their arithmetic has always drawn legitimate attention: fraction-reckoning. A calculation like this[6,j]

$$\begin{array}{cccc} 1 & \underset{\bar{\mathbf{2}}+\bar{\mathbf{4}}+\bar{\mathbf{8}}}{\overline{32}} & + & \underset{\bar{\mathbf{8}}}{\overline{224}} \\ 2 & \underset{\bar{\mathbf{4}}+\bar{\mathbf{8}}+\overline{\mathbf{16}}}{\overline{64}} & + & \underset{\overline{\mathbf{16}}}{\overline{448}} \\ 4 & \underset{\bar{\mathbf{8}}+\overline{\mathbf{16}}+\overline{\mathbf{32}}}{\overline{128}} & + & \underset{\overline{\mathbf{32}}}{\overline{896}} \\ & \text{Sum} & \overline{16} & \end{array}$$

which means much the same as[k] [5]

$$\left(1+\frac{1}{2}+\frac{1}{4}\right)\cdot\left(\frac{1}{32}+\frac{1}{224}\right)=\frac{1}{16},$$

[5] From Erman, Die Literatur der alten Aegypter, Leipzig 1923, p. 281ff [Erman 1927, 223ff in the English translation].

[6] I always write $\overline{n}$ for $\frac{1}{n}$.

no longer fits in with this 'primitiveness', because it is easy to convince oneself that the red numbers under the individual fractions (indicated here by boldface)[l] mean that 1/28 is used as a common denominator, so that the first line reads:

$$\frac{1}{32}+\frac{1}{224}=\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\frac{1}{28}+\frac{1}{8}\cdot\frac{1}{28}=\frac{7}{8}\cdot\frac{1}{28}+\frac{1}{8}\cdot\frac{1}{28}=\frac{1}{28}.$$

Similarly, the second line gives 1/56, the third 1/112, as the value. The sum of all rows is actually 1/16. But how can such a complicated calculation be done with the primitive methods we mentioned above?

And the answer to this question is made even more difficult if one remembers another peculiarity of Egyptian mathematics, which incidentally extended its influence well into our Middle Ages: the exclusive use of 'unit fractions',[m] i.e. fractions of numerator 1 (apart from the one exception 2/3).[n] With this further restriction that, e.g. 7/8 is prohibited, the execution of our calculation above becomes even more unclear. Are there any other aids, multiplication tables or the like? In any case, nothing has been preserved for us, apart from a compilation of calculations that allow the duplicate of a unit fraction with odd denominator to be expressed as the sum of two or more unit fractions.[7] I call this collection [6] of calculations the '2/*n*-table' for short. This table fits perfectly into the framework of dyadic multiplication and the monopoly of the unit fractions, because the multiplication of 1/*n* by an integer *m* never in fact goes beyond unit fractions as long as we understand how to resolve the resulting fraction $\frac{2}{v}$ into a sum of unit fractions with every doubling – and numerators other than 2 never appear. But for our problem above that doesn't help.

But this 2/*n*-table, the most extensive copy of which is preserved in Pap. Rhind (it ranges from $n = 5$ to $n = 101$), has its pitfalls. Even if its purpose is clear, it is not at all obvious according to which law, when a fraction 2/*n* is broken down into a sum of unit fractions, these latter are chosen from the large set that is in principle admissible. In particular, the simplest decomposition $2/n = 1/n + 1/n$ never occurs.[o]

It is no wonder that all of this has provoked so many attempts at explanation. I mention here only a very extensive work of almost 200 pages by Hultsch in the Abhandlungen of the Leipzig Academy.[8,p] The result is the establishment of such a cumbersome system of rules (even the concept of prime numbers plays a role)[q] that it certainly cannot be memorized by any modern European, and therefore also probably not by an ancient Egyptian. [7] Mathematically, too, Hultsch's methods are not clear to me; he says, e.g.:[9] 'The axiom to start with is that all division is based on a delimitation by multiplication', an axiom whose necessity does not make sense to me at all.[r] And Rodet solves our first problem simply by claiming:[s] 'Instead of fractions, one substitutes whole numbers or almost whole numbers, on which the operation can be carried out more easily.'[10,t] But I do not see why it should result in a more convenient calculation in our example above, if one uses 1/28 instead of the least common multiple 896

[7] There was no special symbol for 2/*n*, only a verbal description.
[8] F. Hultsch, Die Elemente der aeg. Teilungsrechnung I, Abh. d. sächs. Ges. d. Wiss. 1895 [Hultsch 1895].
[9] l.c. p. 91.
[10] Journal Asiatique XVIII, p. 219 [Rodet 1881]. The French original reads: 'on substitue aux fractions des nombres entieres ou presque entieres [*sic*] sur lesquels l'operation s'effectue plus aisément.'

of the denominator.[u] So one must think a little more carefully what the operation with fractional numbers is based on in a people who first had to create their own scientific aids.

I give an example that is taken from Pap. Rhind and with which you are already familiar in its character.[v] Provisions are to be distributed among 10 people; first 1 loaf of bread: everyone gets 1/10. Now 7 loaves of bread; we answer: per man 7/10. But how is that supposed to be done in reality? Divide each loaf of bread into 10 pieces and then give each man 7 pieces? The Egyptian result is much more convenient: $2/3 + 1/30$, thus in practice simply 2/3. – The question of by [8] which algorithm one arrives at such a result should not concern us now, but anyone who is presented with the problem in this way will probably immediately suspect: well, the Egyptian computer will proceed in such a way that the most convenient unit fraction (i.e. here the largest) is pulled out,[w] so that only the smallest possible error remains.[x] In fact, this conclusion has been drawn often enough; but on closer inspection it turns out that one cannot get through like this without putting the term 'convenient' into a true bed of Procrustes. And there is good reason for that, because the essential prerequisite for this conclusion is the existence and mastery of certain, albeit relatively simple, mathematical methods from which one can choose a suitable one – just as we can choose between a piecewise-linear approximation or an approximation using polynomials or periodic functions. But with the Egyptians we only find one method and this is of course not always the most convenient mathematically and how it came about we certainly do not know. So with a choice between means that are in themselves equal, it is nothing. And yet a vague feeling, which is certainly alive even in a person who is not very keen on pure theory, tells us that the solution $2/3 + 1/30$ is something more reasonable, something more natural than the solution 7/10, which one can only see clearly when one knows the tenth and counts 7 of this. And so [9] we are touching on a concept that I consider fundamental for the understanding of all practical arithmetic in antiquity: the concept of the 'natural fraction': 1/3, 2/3, a half, a quarter,[y] which are actually terms of quantity that are vividly familiar to everyone from childhood, which one knows without ever having to know anything about 'mixed fractions', 'displayed division', 'numerator', and 'denominator'. And here there is indeed also a choice for practical arithmetic. If at all possible, these natural fractions must come to the fore, they have the first claim to be called upon to help when one can no longer cope with the familiar whole numbers. Of course, they too are not entirely sufficient if one has ever stepped down the steep path of arithmetic with non-whole numbers. But everything that is new here is just a necessary evil if one knows how to control it over time. It is just an inevitable consequence of algorithmic calculation, whose real support and object are still the natural numbers, i.e. whole numbers and natural fractions build the new fractional quantities that must now be introduced – I call them 'algorithmic fractions' for short.[z]

I have hereby mentioned the crucial point of further investigations. I have to confine myself only to alluding to the proof for the justification of [10] such a division of all fractions into two categories and I must refer to my work 'Ueber die Grundlagen der ägypt. Bruchrechnung'[11,aa] for the full details. But it is a series of partly general, partly specific Egyptian peculiarities of thinking that finally merge into a large complex and only produce what can be called a mathematical method as the last

[11]Berlin, Springer, 1926 [Neugebauer 1926].

product, so that I have to try to make these individual features stand out as clearly as possible.

First, the foundation of all mathematics, numbers. The numerals were probably originally written by joining individual marks together:[bb] the classic example of Hilbert's 'chalk marks'.[cc] And this is an exact analogy to the origin of every script, to the 'picture script', in which the numbers are represented by counting the individual marks. This alone should lead to the priority of the cardinal numbers over the ordinal numbers; but beyond that, the philological proof of this priority has long been provided.[dd] In his brilliant study of numbers and number words,[12,ee] to which even mathematicians cannot be pointed emphatically enough, Sethe has shown how the concept of ordinal numbers proves linguistically to denote that which makes the row 'full'; [11] that the 'fourth' means something like 'that which concludes the series of four'.[ff] I shall come back to this point once more and only want to mention that in ignorance of such detailed investigations into the actual historical process, the dogma has become established among mathematicians that the ordinal number is the original concept. This view is taken recently by Weyl[13] and, as far as I know, comes from Kronecker.[14,gg] But the linguistic investigation goes much further, it shows how the higher units that are necessary in the formation of a number system, such as tens or hundreds, are now in their turn the object of counting, e.g. they also attract possessive suffixes, and so on.[hh] It turns out that in the early days of language there was no question of a multiplicative conception of numerals, that the foundation of all mathematics is pure counting, is additive.[ii]

I turn to the fractions. Here again the language clearly shows the distinction between 'natural' and 'algorithmic'. The former are independent individuals with independent names: 1/2, 1/3, 2/3 have designations that fall completely out of the uniform scheme of fractional notation for $1/n$ with large n, [12] e.g. the Egyptian 1/2 means 'side', etc.[jj] And much more, one can see how what 'natural fraction' means is a function of time: one can peel out the oldest part of the fraction signs, such as 1/2, 1/4, 3/4, 1/3, 2/3, which is then gradually expanded with new fraction signs, very similar also to how the increasing natural whole numbers end at different times with the simple designation 'many'. Thus, for example, in Egypt in historical times the numeral for million disappears, later also that for 100,000.[15,kk]

So far we have dealt only with peculiarities which mutatis mutandis probably apply to the beginnings of all cultures, even if they are most easily recognizable in Egypt. But now a special Egyptian momentum comes into play: the iron adherence to a tradition. Anyone who has dealt only a little with Egyptian culture will notice this again and again: in religion, architecture, literature.[ll] Not that everything remains unchanged – that is an impossibility everywhere – but, as Erman so aptly put it: 'The Egyptians loved to carry their eggshells around with them.'[mm] For mathematics, this is expressed in the dominant importance of a fixed scheme. Just as the funerary formulae, which were originally intended only for the king or his [13] grandees but which eventually appeared on every private grave, are actually absurd, misleading and purely formulaic but were still taken literally,[nn] in exactly the same way,

[12] K. Sethe, Von Zahlen und Zahlworten bei den alten Aegyptern, Strassburg 1916 [Sethe 1916a].
[13] H. Weyl, Handb. d. Philos. Abt. II A p. 28 [Weyl 1927].
[14] L. Kronecker, Ueber den Zahlbegriff, Crelle 101, p. 339 [Kronecker 1887].
[15] Sethe, l.c. p. 11ff.

an attempt was made to solve problems that require a more complicated set of figures with a method of calculation that went back to the oldest times. Egyptian mathematics did not strive to invent new methods for new problems, and thus perhaps also to cope with old problems more easily, but the basic tendency of Egyptian mathematics was: to overcome further problems by using the old methods with a minimum of modification.

But with that I have actually anticipated something and have to return to the Egyptian fraction concept, especially to the 'unit fraction'. As you can see, this concept has for our current understanding taken a back seat to the concept of natural fractions in general. And in fact before we can understand the meaning of the other, we must first learn more about the fate of this term. The natural fractions are individuals; they are counted accordingly like things, like the units. Take some $1/n$; if we count such $n$ths, a caesura arises automatically: after n steps we run into the full old unit of n $n$ths equal to 1. The fraction $\frac{n-1}{n}$, the so-called 'complement fraction', plays [14] a special role in this series: add another nth and the whole row is full. And with that we again find the connection to the philological investigation and at the same time build the bridge to the ordinal number: the complement fraction is expressed by a cardinal number: *τά τρία μερη* 'the three parts' for 3/4.[oo] However, the last 'completing' unit fraction must be denoted by an ordinal number: *τό τεταρτον μέρος* for 1/4.[16] (I mention by the way that it is precisely this fact that provides the starting point for the treatment of ordinal numbers in Sethe's investigation.) You see: next to the 'unit fraction' $1/n$, the complement fraction occurs in an entirely singular manner. The set of fractions is thus attacked from two sides, so to speak; from below through the new unit of counting, the unit fraction, and from above through the last step of counting, the complement fraction – whereby the meaning of the latter as clearly easy to grasp is also understandable psychologically.

And what happens to the fractions between unit fraction and complement fraction? They are nothing but uninteresting trivial intermediate stages of counting: if I have k $n$ths and count 1 nth further, then I simply have $k + 1 = m$ such things. The 'mixed fraction' $\frac{m}{n}$ which appears to us in our multiplicative way of thinking as the result of multiplying an integer m by the fraction $\frac{1}{n}$ or as a 'displayed division' or as a 'ratio' can [15] only be defined by $\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$ for additive thinking. Let us now think back to our '$2/n$-table' for a moment: it now goes without saying that the decomposition of the double of $1/n$ into $\overline{n} + \overline{n}$ does not occur, because this decomposition is 'trivial', it is the definition of $2/n$ but never a result.[pp] Any further doubling would never lead beyond the 'definition equation' of m/n, namely $\overline{n} + \overline{n} + \cdots + \overline{n}$.

But let's stay with our fractions themselves and see how the influence of counting, the influence of additive mathematics, now asserts itself. What does the domain of natural fractions look like now? 1/2 is at the same time a unit fraction and a complement fraction; 1/3 and 2/3 are indeed conceived in this sequence (as the language shows); then 1/4 and 3/4 in some way. Both complement fractions, 2/3 and 3/4, had their own symbol, but 3/4 only in the earliest period, and this was later lost.[qq] But why is this so? This is where the Egyptian tendency towards a fixed scheme intervenes. How is it with whole numbers? The most important step in counting is the step from 1 to 2, 'taking it again', which also finds its linguistic counterpart in the special

[16]Sethe, l.c. p. 106.

grammatical form of the dual.[17,rr] We already know this process of taking again as the dominant method of Egyptian multiplication [16] – not 'multiplication' in our sense of course, but just a repeated addition. In doing so, however, a selection suddenly occurs from the set of all integers: the numbers 1, 2, 4, 8, 16, ... form, so to speak, the 'basis' of the representation of all other numbers. The result of the dyadic multiplication process is the distinction of a single special series of numbers, the 'dyadic series' as I will call it in a moment. From then on, there is a fixed, 'dyadic', scheme for calculating with whole numbers.

And now the problem gradually arises of calculating with fractions, of getting serious about their relationship to the whole numbers and their relationships to one another. How this had to be done in Egypt has already been shown above: the usual scheme was simply extended to the new domain. This means that the fractions (at first only the limited range of natural fractions is to be considered) are given a new arrangement that conforms to the dyadic scheme; the 'dyadic series' is continued on the other side through the fractions 2, 4, 8, ...[ss] I call this sequence the '1/2-series'. You see that 3/4 no longer plays a role in this scheme, it becomes superfluous in fraction-reckoning and its symbol disappears from the script. But the old range of natural fractions still contains a complement fraction, which fits into a dyadic scheme: the fraction 'two parts', as [17] 2/3 is called in Egyptian.[tt] Next to the '1/2-series', there is the '1/3-series' 2/3 = 3, 3, 6, ...[uu] This sequence 3 3 [*sic*] became the generally prevailing one in Egyptian mathematics,[vv] to the extent that *a*/3 was calculated by forming 2*a*/3![ww] Thus, the domain of natural fractions was given an entirely new structure; but at the same time one also sees the onset of a process of dissolution, because the dyadic scheme, consistently applied, naturally leads far beyond the 'natural fractions'.

But the bond that links the natural fractions with one another and distinguishes them from the now developed domain of algorithmic fractions is strong enough to allow them once again, and crucially for everything else, to come to the fore: in the problem of decomposing 2/*n* into a sum of unit fractions. That this is still the case is of course due to the fact that the problem of the 2/*n*-decomposition arises precisely at the moment when one tries to calculate with fractions at all, and it is naturally only at this moment that the process of algorithmic resolution can begin to be used.

This brings us to the beginning of proper Egyptian mathematics, or rather to fraction-reckoning. The path that we have had to take to get back to this point was not a short one, but it was certainly not a short and comfortable one in reality, and I am convinced that one must first fully recognize one's difficulties before one can arrive at a correct insight into [18] proper Egyptian mathematics. Now that we have it behind us, the direction in which we have to go on is clear ahead of us. What now had to follow no longer provides many principally new insights, and I do not want to tire you with the details of an investigation that requires the consideration of the numerical material available to us and is reproduced in full detail in the second part of my work on Egyptian fraction-reckoning; I will only give the basic idea of this investigation here and refrain from discussing all the details.

I tie this in again with the dyadic multiplication of whole numbers. This is based on the dyadic development of one factor or, as we can also say, on its composition from members of the dyadic series. The result then appears as the sum of terms of the form

[17]See, e.g. J. Wackernagel, Vorlesungen über Syntax I, Basel 1920, p. 73ff [Wackernagel 1920–1924].

1.n, 2.n, 4.n, ... In the 2/n-table, there is no longer a whole factor n, but a unit fraction $\overline{n}$; it should be doubled and the result of this operation, namely $\overline{n}+\overline{n}$, has to be expressed somehow with the help of other unit fractions in order in general to get out of the domain of trivial representation. Does the involvement of the first term $\overline{2}.\overline{n}$ of the dyadic series extended into the fractional domain help to build up $2.\overline{n}$? If this attempt is to succeed, the extension from half an nth to 2 $n$ths must [19] be a unit fraction, i.e. it must be expressed in modern terms:

$$\frac{2}{n}-\frac{1}{2n}=\frac{3}{2n}=\frac{1}{m}.$$

From this it follows that the decomposition of 2/n into a sum of two unit fractions of this type is only possible if n is divisible by 3.[xx] How all these things look from the standpoint of Egyptian mathematics, I cannot now deal with here; only the general scheme is already obvious: instead of the elements 2n, 4n, 8n there now appear $n$ths with the natural fractions, first of the 1/2-series, then also of the 1/3-series, as coefficients – just like the natural numbers of the dyadic series earlier.[yy] The problem then is to determine the supplementary last unit fraction, the 1/m, or the 'completion term', as I call it.[zz] The calculation necessary for this, 'completion calculation' is what the Egyptians called it in a characteristic way,[aaa] then leads in turn to an actual technique of fraction-reckoning, which, however, is still based on the two pillars of Egyptian mathematics: counting and the dyadic scheme. Counting in particular manifests itself in the fact that the fractions were provided with 'auxiliary numbers' (in red script),[bbb] which allow a fraction of the smaller denominator to be counted directly via a larger denominator derived from it, e.g.

$$\begin{array}{ccc}\overline{4} & + & \overline{28}\\ \mathbf{7} & & \mathbf{1}\end{array}$$

[20] – in other words, a circumstance that comes down conceptually to our 'least common denominator', but is quite different from it in practice.[ccc] This is because the dyadic procedure is now applied entirely schematically to such a simple calculation obtained from elsewhere (how should not interest us now), i.e. to fractions and auxiliary numbers, via which one can effortlessly arrive at such complicated results as the one given at the very beginning, which can be derived from our current example (which, by the way, represents the decomposition of 2/7) by means of continued halving.[ddd] So if one wants to summarize our result very briefly in modern terminology, one can say: the 'integral domain' of the natural whole numbers is extended by the 'natural fractions' and this new domain is subjected to the old scheme of operations.[eee] Thus, the whole complexity of Egyptian fraction-reckoning does not dissolve into simplicity but into sheer elaboration, which, for its part, offered the advantage of great certainty without having to be in the least clear about the conceptual roots of this 'method'.[fff] And with this we have closed the first circle of our consideration and have come back to the Egyptian as a human being, as he describes himself, perhaps drastically but certainly true to life: 'I am one who loves food and hates chatter.'[18, ggg]

[18] I owe this quotation from an unpublished New York passage from the 11th Dynasty to Mr. J. Polotzky [*sic*], Göttingen.

[21] But at the end of our considerations we want to drop the exclusive restriction to Egypt and ask how what has been reported so far can be classified in the somewhat more general framework of an investigation of 'pre-scientific mathematics' (to use an expression of Hankel).[hhh]

It is evident from the outset that an examination of Egyptian mathematics must be accompanied by a comparison with the primitive mathematics of other peoples in order to gain clarity about what is common property and what is specifically Egyptian.[iii] And it goes without saying that there is one cultural group to which one has to pay special attention: that of Mesopotamia.[jjj] So I will now try to give a broad outline of the course of development in this area, but I have to emphasize that I must reserve the detailed presentation of the reasons for my opinions for a separate work.[kkk]

I expressed myself so carefully with 'culture of Mesopotamia' because this culture is in fact not a uniform one. You are sure to know that the supremacy of the Semitic inhabitants of Babylonia and Assyria was preceded by an era in which a people of non-Semitic race were in control: the Sumerians.[lll] We must first ask about their calculation methods if we want to understand the following development, which is known to be characterized by the catchphrase 'sexagesimal system'.

[22] But the picture of Sumerian mathematics as we encounter it in the research of Thureau-Dangin and, more recently, of Deimel – to give just these two names[19,mmm] – is far from being as clear as in Egypt. Even in the oldest texts known to us, a strange mixed system can be seen in every respect: although the basis of the numerical notation is undoubtedly an additive one here too, one can already feel a clearly multiplicative-algorithmic tendency. And then there is the fact that the number system is at least a muddle of a 10-system and a 6-system – not a pure 60-system! – which is also written with characters that, because of their small number, can have different meanings – not only with regard to their absolute value, but also with regard to their use in various domains of practical systems of measurement, such as measures of area or capacity. So one is not faced with very encouraging factual material. The only thing that is immediately apparent is that the so-called sexagesimal system can only be the end-product of a long development. And from this point of view, the position taken by, for example, Thureau-Dangin is self-evident, namely that it is wrong to present this system from the outset as original and as the result of conscious, so-called 'scholarly' considerations: the question of the origin of the Babylonian 'sexagesimal system' can only be asked when one has clarity about the development of Sumerian mathematics.[nnn]

And here again it is a long-established fact that it is neither sixty nor six that plays a special role, but rather the sixth.[20] The problem is to clarify the role of this fraction.[ooo]

It is to be expected that here, too, it cannot be a case of an absolute dominance of a single fraction over all others from the outset. And in fact, closer inspection shows that other 'natural fractions' were also in use and play their role:[ppp] the complement

[19]For all: Thureau-Dangin, Recherches sur l'origine de l'ecriture cunéiforme I, Paris 1898/99 [Thureau-Dangin 1898–1899] – Numération et Métrologie Sumériennes, Rev. Ass. Bd. 18 (1921) [Thureau-Dangin 1921]. A. Deimel, Sumerische Grammatik, Rom 1924 [Deimel 1924].

[20]Thureau-Dangin, R.A. l.c. p. 124. Sethe, Die Zeitrechnung d. alten Aegypter, Göttinger Nachr. phil. hist. 1920 p. 101 [Sethe 1919–1920]. Deimel, l.c. passim.

fraction 5/6 occurs, of course 1/2, then 1/3, 2/3. So the beginnings are no different than in Egypt. But one thing stands out particularly clearly in the tangle of Sumerian numeral and fractional notation: that all of these quantities must primarily be considered in their relationship to measured quantities. [24] It is precisely in the countless 'economic texts' in which these signs appear again and again, only in connection with measures, with the subdivision of the larger units of measure, that the fractions gain their meaning. And only this history of development allows us to comprehend that we encounter such a variety in the type of subdivisions: sometimes decadic, sometimes dyadic, or by thirds, sixths, etc.[qqq] If one takes this as a starting point, the meaning of a fraction like 1/6 is no longer anything wonderful. The fact that 1/6 comes very much to the fore can of course be partly due to the convenient divisibility properties of 6, which mean that 1/2, 1/3, 2/3 can also be counted in sixths. But basically this exceptional position is certainly an accidental one that does not have to be 'explained', but can only be justified by the convenience of the smaller units of measurement that come about with it, just as in Egypt 1/7 of the 'cubit' as a 'palm' and 1/4 of the palm as a 'finger' gave a practically usable and descriptive division of length measurement. Incidentally, among the Sumerians the division of the cubit was not a six-division, but a decimal one; on the other hand, the six-division occurs, for example, essentially in the case of a capacity measure for grain, according to which the area dimensions are then measured.[21, rrr]

[25] But now there is a decisive difference compared to Egypt. Here as there, the decimal system prevails among the integers, and here as there the natural fractions play their independent role. But the dyadic schematization in Egypt suppressed any emergence of fractions other than those that fit into the dyadic scheme – I only remind you of what I said above about 3/4 – and so, for example, 1/6 is also part of the '1/3-series' as a completely meaningless element. Of course, this does not mean that these fractions could also be made to disappear from popular consciousness, any more than our purely decimal coin system was able to displace the 'Groschen'.[sss] Among the Sumerians, however, no such uniform system developed. Rather, the originally decadic range of the integers was influenced by the fractions, which led naturally to a jumble of six-, ten-, thirty-, sixty-, even eighteen-quantities, which also made necessary those tabulations that are characteristic of the Babylonian culture. This chaos was only put to an end later, probably by the Semites, through a kind of compromise between ten-division and six-division, the so-called 'sexagesimal system', which, as a somewhat mathematically conceived system, perhaps eventually only remained in the learned schools, while the arithmetic of the [26] people became once again purely decimal.[22, ttt]

And in all probability, at the same time as the sexagesimal standardization of what is traditional, another regulation takes place that has been given more mathematical importance than it actually deserves: the introduction of some kind of 'positional system'. The Sumerian script was originally a picture script, as the oldest texts show with all the clarity one could wish for. But the inconvenience of the writing material, of the tone, forced the signs to be extremely schematized, broken down into individual strokes, which ultimately resulted in what is called 'cuneiform'. Accordingly, the number of numerals available was also very limited, which led to

[21] Cf. Deimel, l.c. p. 194f.
[22] Thureau-Dangin, R.A. l.c. p. 124f.

different values being assigned to one character, starting at sixty with that of the old unit – at most, in careful texts, differentiated by its size.[23, *uuu*] Thus one arrived at a sign change, which I want to indicate briefly with the schema '*a b a b*', a schema that was then characteristic of the developed sexagesimal system, but which required that any missing members of such a series be marked in some way, first with an empty space, [27] then more clearly by a special mark. There is no question of a sign for 'zero', it is not a sign for a mathematical concept but rather a 'diacritical' sign of the script. The mathematical idea of position was first grasped by the Indians (probably around the turn of our era[24]) – certainly not in Babylon.[*vvv*]

Once again the comparison with Egypt comes to mind, where, starting out from a picture script, a writing system emerged which could almost immediately have made an alphabetic script available, yet needed a foreign people uninhibited by a tradition of creating things themselves, and by means of a genuine 'discovery', to be given its unique form as one of the most important instruments of human culture.[25, *www*]

You see how much the comparison of different cultures can teach and how necessary it is. Consequently, this leads to the task of pursuing the origin of the number concept itself and its development. If one takes a work such as that recently published by W. Schmidt on 'Sprachformen [*sic*] und Sprachkreise der Erde'[26] and looks at, for example, the map showing [28] the distribution of number systems, one notices how much has already been accomplished by comparative linguistics in these matters and how much there is still to be done for the historical foundations of mathematics.[*xxx*]

But let us leave the early history of mathematics and return finally to Egypt, to a question that is of greater importance for the further development of mathematics: What is the material that the Greeks took from here? It seems to me that most of the work still needs to be done to answer this question. One has to examine how the tendencies in Egyptian mathematics described in the first part of my lecture continued to develop and transform, and how they gradually prepared a multiplicative method of calculation.[*yyy*]

With the appearance of the Greeks, a new phase in the history of mathematics began.[*zzz*] And yet, I believe, one should not overlook the continuity of the trend here either. This is least of all true of geometry, whose purely mathematical significance was only now being discovered – the calculation of a storage volume was no doubt regarded by the Egyptians just as much as a mere application of numerical calculation as the distribution of provisions, because the first geometrical propositions, for example, about the areas of simple figures, have their psychological roots in the same empirical observations which also led to the fact that twelve [29] loaves of bread can be divided between three people in groups of four and between four people in groups of three.[*aaaa*] But if we consider arithmetic, it was no doubt much more familiar to every common man than abstract geometric propositions, and therefore in much closer connection with the preceding development. In his 'Vorlesungen

[23] Deimel, l.c. p. 184.
[24] Bühler, Grundriss der indo-arischen Philologie, vol. I, 11 [Bühler 1896].
[25] See Sethe, Der Ursprung des Alphabets, Göttinger Nachr. 1916. Gesch. Mitteilungen p. 88ff [Sethe 1916b]. (Recently published as a special edition (Sethe 1926).)
[26] Heidelberg 1926, Kulturgesch. Bibl. 1. Reihe, vol. 5 [Schmidt 1926].

über Zahlentheorie',[27] Kronecker expresses his admiration for the fact that 'given the inadequacy of the Greek numerals' it was possible to prove that a number $2^{n-1}p$ is a 'perfect' number, i.e. is equal to the sum of its divisors (including 1, but of course not itself) if

$$1 + 2 + 2^2 + \ldots + 2^{n-1} = p$$

is a prime number.[bbbb] Of course, the whole type of this problem is absolutely Greek and above all requires a full understanding of the concept of divisibility. But once you have found this concept, it seems obvious to me to apply it to the traditional number formations. And the sum of the powers of 2 perhaps belongs to these in the first place – this is exactly our 'dyadic series', the main tool of Egyptian mathematics. And the fact that one gains the freedom to approach the old with new ideas is perhaps precisely due to the Greek number system, which had broken the bond that connects all old number systems with the beginning of mathematics, [30] additive numbers.[cccc]

Well – all of these things still require closer examination and justification. Nevertheless, I did not want to restrict myself anxiously to what is reasonably well-secured, partly to shed light on the debit and credit of our knowledge of pre-Greek mathematics, but partly to show you how every historical consideration points beyond itself and thus bears the stamp of the restlessly changing – like the object itself.[dddd]

Notes on the translation

a. Otto Toeplitz (1881–1940) was a German mathematician with a strong interest in the history of mathematics (Robinson 2008). He was the originator of the so-called 'genetic method' in the teaching of mathematics, whereby an idealized historical narrative provides motivation (Fried and Jahnke 2015). Toeplitz was subsequently Neugebauer's co-editor for the Springer series *Quellen und Studien zur Geschichte der Mathematik* (Dauben 1999, 13–15).

b. This is probably an early barb aimed at anecdotal forms of writing on the history of mathematics. In a short methodological article (Neugebauer 1927b) published the year after his Kiel lecture, Neugebauer implicitly criticized David Eugene Smith's *History of mathematics* (Smith 1923/1925) in this regard. Later on, when he was a firmly established academic, Neugebauer's criticisms of such writings became more explicit: he complained, for example, of the 'moralizing and anecdotal attitude' displayed by Moritz Cantor in his *Vorlesungen über Geschichte der Mathematik* (Neugebauer 1956), and had little enthusiasm for the work of the historian of science George Sarton (Rowe 2016b, 54). The 'scientific' approach to history that Neugebauer adopted in his dissertation won praise from some of his mathematical readers: 'It is gratifying to see how the history of mathematics, in the hands of some researchers, is gradually being raised from the stage of reportage or chronicle to science.' (Bieberbach 1929: 'Es ist erfreulich zu sehen, wie die Geschichte der Mathematik unter den Händen einiger

[27][Kronecker 1901], p. 8.

Forscher allmählich aus dem Stadium der Reportage oder Chronik zur Wissenschaft erhoben wird.')

c. Neugebauer evidently did not yet know about the paper of Tsinzerling (1925), which made use of previously unpublished material from B V Turaev, who had died in 1920. Neugebauer would later play an editorial role in the publication of an edition of the Moscow Mathematical Papyrus (Struve 1930) as the first volume of the 'A' strand ('Quellen') of *Quellen und Studien* (see note *a*).

d. Adolf Erman (1854–1937), cited here by Neugebauer, was a German Egyptologist who exerted a significant influence on the study of Egyptian philology, a major work being his co-edited *Wörterbuch der Aegyptischen Sprache* (Erman and Grapow 1926–1963). Erman spent the majority of his career in Berlin, from where he retired as Professor of Egyptology in 1923. There is no specific evidence of a personal link between Neugebauer and Erman, the generational difference probably being too great (Neugebauer does not appear in the name index of Erman's autobiography, for example: Erman 1929), though Neugebauer's early Egyptological work was heavily influenced by Erman's student Kurt Sethe (see note *ee*). On Erman, see Bierbrier (2012, 180–181).

e. The citation in Neugebauer's note refers to Erman's German translation of a satirical literary letter composed in the Ramessid Period, and known principally from the late 19th Dynasty Papyrus Anastasi I (BM EA 10247, roughly 1200 BCE). On the pages indicated, one scribe, Hori, ridicules the abilities of another, Amenemope, by impugning his ability to solve certain mathematical problems relating to the digging of a lake, the construction of a brick ramp, and the transport of an obelisk. The text was edited by Gardiner (1911), and a full modern edition and commentary is provided by Fischer-Elfert (1986, 1992); a convenient English translation is Wente (1990, 98–110). For its mathematical content, see Imhausen (2016, § 14.3).

f. Neugebauer was evidently simplifying matters here for the purposes of his lecture, since he would have known that operations other than doubling were permissible – these are well attested in the RMP: see Ritter (2000).

g. The theme of empirical arithmetical discoveries by Egyptian scribes, as opposed to the discoveries of results that are 'proved' in a more modern sense, is one that runs through much of the period's literature on ancient Egyptian mathematics (see, for example, Peet 1923). Indeed, mathematical authors (for example, Archibald 1949) have tended to use the presence or absence of a notion of proof (necessarily in a modern Western sense) as a test of whether a culture possesses a 'scientific' form of mathematics, a test that ancient Egyptian mathematics fails. See the discussion of this point in Hollings and Parkinson (2022, § 8).

h. The label often applied to such a system of numerals is 'cumulative-additive' (Chrisomalis 2010, 12). Although hieroglyphic numerals follow the pattern that Neugebauer describes here, his insertion of the word 'essentially' presumably serves to gloss over a slight complication connected with the cursive hieratic script in which the RMP and other mathematical papyri are written: although many of the higher numerical signs have their origins in the repetition of simpler signs, this is not always obvious from the cursive and ligatured forms subsequently taken in hieratic. See the tables of comparison in Gillings (1972, Appendix 12); see also note *bb*.

i. This is a statement that had frequently been made by both mathematical and Egyptological authors; see, for example, Sloley (1922).

j. The notation $\overline{n}$ was introduced by Neugebauer in his doctoral dissertation, and is now standard within the study of ancient Egyptian mathematics; its convenience lies in the fact that it provides a good representation of Egyptian (unit) fractional notation, which, in the hieratic script, simply added a dot above the sign for the corresponding integer. In another parallel with the original script, addition of fractions is indicated by juxtaposition; e.g. $1\,\overline{2}\,\overline{4} = 1 + \frac{1}{2} + \frac{1}{4}$. In this lecture typescript, Neugebauer was a little inconsistent in his typesetting of fractions: $\frac{1}{n}$, $1/n$ and $\overline{n}$ all appear (in the translation, we follow his particular usage in each case).

k. The 2 and the 4 in the leftmost column should read $\overline{2}$ and $\overline{4}$, respectively – Neugebauer omitted the fractional bars, accidentally replicating a common scribal error in the RMP: the omission of fractional dots (see note *j*). The calculation given here is a corrected version of one found in Problem 15 of the RMP. The papyrus gives the total as $\overline{16}$, and the red numbers appear as given here, but the other values in the right-hand column were evidently copied down wrong: first $\overline{228}$ for $\overline{224}$, and the error was then carried forward with $\overline{456}$ in place of $\overline{448}$ and $\overline{912}$ in place of $\overline{896}$. The scribe also omitted the fractional dots from all three instances of $\overline{32}$, that of $\overline{64}$, and the final appearance of $\overline{16}$, but realised that something had gone awry with the calculation and so added an abbreviated form of the verb *thì* ('to go wrong') at the end. Problem 15 is one of the collection of problems within the RMP that Peet dubbed the 'first group of completions', and which were key to a systematization of problems that Neugebauer proposed in his dissertation: see note *ddd* and Hollings and Parkinson (2020).

l. The use of boldface to indicate red ink in the original papyrus is another modern typesetting convention that was apparently introduced by Neugebauer in his dissertation (Peet had used italics, for example). Although we use boldface here, the relevant numbers actually appear in (handwritten) red ink in the typescript, despite Neugebauer's comment about boldface. Neugebauer may have illustrated his lecture at a blackboard, perhaps using blackboard bold, or else red chalk, to distinguish the red numerals.

m. This almost exclusive use of unit fractions has often been presented as a 'restriction' under which Egyptian scribes laboured, thereby feeding into the view of Egyptian mathematics as 'primitive' (see note *i*). Even the term 'unit fraction' is problematic, since it implicitly carries with it the notions of numerator and denominator which are alien to the Egyptian context. Since fractional arithmetic accounts for the vast majority of the calculations that appear in surviving sources, this has also been the central focus of modern writings on Egyptian mathematics. An early question within the study of these ideas, but not one that was addressed here by Neugebauer, was whether there existed a *notion* of non-unit fractions, even if in most cases there was no *notation* (see the discussion in Høyrup 2004). More recent treatments have tended to place Egyptian fractional arithmetic into a natural metrological setting, which in particular sets the mathematics within a wider cultural context, but Neugebauer confined himself mostly to discussions of abstract numbers (consistent with his 'high-level' and largely context-free approach to the subject). For general discussions of the concept and arithmetic of fractions in ancient Egypt, see the chapters of Ritter, Caveing, and Guillemot in Benoit et al. (1992).

n. The fraction $\frac{2}{3}$ had a special status, and its own sign, as the only non-unit fraction in regular use in Egyptian arithmetic (Imhausen 2016, 53). In Neugebauer's fractional bar notation (see notes 6 and *j*), it is written $\bar{\bar{3}}$. Moreover, taking two thirds of a given value, as a single process, was one of the permissible operations alluded to in note *f*; a common means for a scribe to find one third of a quantity was first to find two thirds and then halve this (as in RMP Problem 19, for example). But such a process, which seems unusual to modern mathematical eyes, may admit a simple explanation, as offered by Peet (1931b, 415): 'This is not quite so paradoxical as might appear, for we may regard $\frac{2}{3}$ of x as that number which when its half is added to it gives x. In this way we get at $\frac{2}{3}$ by the use only of the fraction $\frac{1}{2}$, which was more fundamental than $\frac{1}{3}$, since it only involved division by 2.'

o. Other examples of the table appear within the Middle Kingdom Lahun Mathematical Fragments (Griffith 1898, vol 1, 15–16; Imhausen and Ritter 2004, 93) and the Mathematical Leather Roll, though the latter source would have been unknown to Neugebauer at the time of his lecture, since it was only unrolled in 1927 (Glanville 1927). As Neugebauer hints here and below, the quest to find a uniform method for the construction of these tables has been a major theme in the modern study of Egyptian mathematics, with his dissertation being one of the first works in this direction. However, these investigations have tended to employ anachronistic mathematical attitudes (see, for example, the comments in Imhausen 2007, 19); Peet's view was that the results entered into the table were 'purely empirical and obtained by gradual collection' (Peet 1931b, 414). Reference to the table as concerning the duplication of unit fractions may be a modern imposition, since the original is expressed solely in terms of division: *nỉs 2 ḫnt* … (literally 'call 2 out of …', usually taken to be a technical term for 'divide 2 by …': see Peet 1923, 14; Gunn 1926, 124–125; Imhausen 2016, 93). However, Peet (1923, 16) argued from the internal evidence of certain calculations in the RMP that both conceptions of this process were present in the mind of the scribe.

p. Friedrich Hultsch (1833–1906) was a German classical philologist whose interest in ancient metrology led him into the study of ancient mathematics more generally. Although Hultsch did publish a short supplement to his 1895 work (Hultsch 1901), in which he compared calculations in the RMP to those in a later Demotic papyrus, the projected second part of that work never appeared. On Hultsch, see Baader (1974).

q. Eisenlohr (1877, vol 1, 33) had also brought considerations of prime numbers to his study of the recto table in the RMP, and was duly criticized by Neugebauer alongside Hultsch (Neugebauer 1926, 9); the relevant reference for Hultsch, not given explicitly by Neugebauer, is Hultsch (1895). See the comments in note *xx*.

r. Hultsch's original German (with his emphasis, omitted by Neugebauer, restored) reads: 'Es ist auszugehen von dem Axiom, dass alle D i v i s i o n auf einer U m g r e n z u n g d u r c h M u l t i p l i c a t i o n beruht.' 'Delimitation' ('Umgrenzung') here refers to the approximation of a quantity via successively tighter upper and lower bounds. Hultsch's inspiration for suggesting this method for division came from his own proposed reconstruction of the lost method of Theodorus of Cyrene (fifth century BCE) for approximating square roots, which he had outlined in detail in an earlier publication (Hultsch 1893). Judicious use of upper and lower bounds for the parts of a quantity, combined with a form of ‘multiplying up’ reminiscent of long division (hence ‘delimitation by multiplication’), enabled Hultsch to provide a method for the *calculation* rather than the *approximation* of a quotient, but the resulting process is rather more complicated than anything preserved in the RMP. Peet (1923, 16) referred to Hultsch’s work as ‘a long study of considerable complexity’ and disputed many of his assertions.

s. Léon Rodet (1832?–1895) was a mathematician and oriental scholar with a particular interest in the mathematics of ancient India and the mediaeval Middle East. The investigation of mediaeval Islamic algebra may be what led Rodet into ancient Egyptian studies, where his work focused on what Eisenlohr had claimed to be algebraic problems in the RMP. Rodet drew parallels with techniques employed in certain mediaeval sources (Rodet’s parallels are summarized by Peet 1923, 18), but denied the ancient Egyptians a notion of algebra. On Rodet, see: https://data.bnf.fr/en/12459787/leon_rodet/ (last accessed 7 January 2024). On so-called ‘Egyptian algebra’, see Imhausen (2001).

t. In fact, the original French appears with much emphasis from Rodet: ‘on «substitue aux fractions des nombres *entiers* ou *presque entiers*» sur lesquels l’opération s’effectue plus aisément.’ Neugebauer perhaps omitted the quotation marks because it is not clear what purpose they serve in this passage. Judging by the surrounding discussion in Rodet (1881), the words between the guillemets may have been Rodet’s own French translation of a phrase from Eisenlohr (1877), but he does not give a specific reference.

u. What Neugebauer probably means here is that we would only displace the difficulty. Rather than scaling up each row by a factor of 28, as is done in the example as given (though Neugebauer phrases his discussion in terms of $\overline{28}$), if we instead scale up by a factor of 896, then all of the red (bold) numbers become integers: **28** and **4** in the first row, **14** and **2** in the second, and **7** and **1** in the third. These are easily added to make **56**, but then we still have the problem of how to divide **56** by 896.

v. The division of 1 loaf of bread among 10 people is RMP Problem 1; division of 7 loaves is Problem 4.

w. Neugebauer is being a little loose in his terminology, since the largest fraction that has been pulled out here is clearly not a unit fraction. But, as we have already noted (see note *n*), the status of $\overline{\overline{3}}$ within Egyptian arithmetic was on a par with the unit fractions, so it is a valid candidate for ‘most convenient’ fraction within Neugebauer’s argument. In place of ‘unit fraction’, we might substitute the term ‘natural fraction’ that Neugebauer introduces later.

x. In common with most problems in the RMP, the scribe does not in fact give any indication of how the answer $\overline{\overline{3}}\ \overline{30}$ was obtained; it is simply stated, and then verified via the standard two-column method: 1 and $\overline{\overline{3}}\ \overline{30}$ are set down as the first row, which is then doubled three times (making use of results from the recto table and possibly other memorized arithmetical facts), and the resulting rows with 2 and 8 in the left-hand column are marked off for addition. The corresponding sum in the right-hand column is

$1\,\overline{3}\,\overline{15} + 5\,\overline{2}\,\overline{10}$, which the scribe concludes to be the required 7. Peet (1923, 51) posits the existence of a now-lost table for addition of fractions as the means by which the scribe calculated this last sum.

y. It is not clear whether Neugebauer meant to stop at $\frac{1}{4}$ or whether he intended his notion of 'natural fractions' to encompass all so-called dimidiated fractions: reciprocals of powers of 2. The latter certainly had an important role to play in Egyptian calculation: see, for example, Ritter (2000, 131).

z. Here and below, Neugebauer seems to hint at the relevance to this discussion of the development of the understanding of fractions within a (modern) mathematical education – a reasonable direction for him to go in, considering that this talk was delivered within a mathematical didactics seminar. However, he does not develop this line of thought any further, preferring a philological approach (see below). For a recent discussion of these ideas in a pedagogical context, see Sidney et al. (2019).

aa. For a survey of the content of Neugebauer's dissertation (whose title translates into English as *On the foundations of Egyptian fraction-reckoning*), see Ritter (2016); see also Hollings and Parkinson (2020).

bb. On the history of numerical notation, see Chrisomalis (2010); Egyptian numerals appear in chapter 2. See also note *h*.

cc. Neugebauer refers here to the foundational programme of David Hilbert (1863–1943), which called for a formalization of mathematics in an axiomatic form. Fundamental to Hilbert's programme were signs or strokes on the page or blackboard which served as primitive objects that could be manipulated formally according to prescribed rules, but which nevertheless carried intuitive information; moreover, the collection of available signs could always be augmented by the introduction of additional strokes (see, for example, Zach 2019). The strong influence of the attitudes of the Göttingen mathematical school (with Hilbert at its centre) on Neugebauer's approach to the history of mathematics has been discussed by Rowe (2016a). This influence is made explicit in a short article that Neugebauer published the year after his Kiel lecture (Neugebauer 1927a), in which he drew parallels between Hilbert's proof theory and the early historical development of number systems (see the remarks in Rowe 2016b, 35–36).

dd. For a more recent discussion of this point, see Wiese (2003, ch 4).

ee. Kurt Sethe (1869–1934) was a German Egyptologist who, alongside his teacher Erman (see note *d*) and the British Egyptologist Alan H Gardiner (see note *www*), is regarded as one of the greatest Egyptian philologists of the early twentieth century. In writing the work cited here, Sethe was one of the few Egyptologists to take an interest in numerical sources, and the text remains a standard reference on this topic. As Professor of Egyptology in Göttingen, Sethe provided Neugebauer with his early Egyptological training before departing for Berlin to succeed Erman in 1923, though he and Neugebauer remained in contact thereafter, and Sethe wrote a glowing report on the latter's dissertation (Ritter 2016, 146). Neugebauer later acknowledged the influence of Sethe's 1916 text on his own thinking: that it is possible to approach historical research in a precise manner (Neugebauer 1930, 303). Indeed, the rigorous philological training that Neugebauer received from Sethe is the second pillar upon which all of his subsequent work was founded, the first being his education in the Göttingen mathematical school (on which, see note *cc*). On Sethe, see Bierbrier (2012, 502–503).

ff. Neugebauer is paraphrasing Sethe (1916a, 109).

gg. In stressing the fundamental importance of the ordering of numbers, Weyl quotes Schopenhauer: 'Every number presupposes the preceding as the basis of its being: I can only get to ten through all of the preceding ones …' (Schopenhauer 1813, § 38: 'Jede Zahl setzt die vorhergehenden als Grunde ihres Seyns voraus: zur Zehn kann ich nur gelangen durch alle vorhergehenden …'). Thus, for Weyl, numbers are most often characterized by their position in a sequence. Made more formal, the rationale for giving priority to the concept of ordinal numbers stems from the need to ensure that the assignment of a cardinal to a given set is independent of the ordering of the elements of that set – hence the asserted primacy of ordinals. Weyl cites the prior discussions of this point by Helmholtz (1887) and Kronecker (1887). On Kronecker's views on the foundations of mathematics, see Boniface (2005).

hh. This point is treated by Sethe in his section II.6. For a discussion in a more general setting, see Hurford (1987, § 5.1).

ii. As we described in our 'Background' section, the assertion of the purely additive nature of ancient Egyptian arithmetic was one of the central themes of Neugebauer's dissertation, and the subject of his first chapter. See below and the references in note *aa*.

jj. The word in question is *gs* (Sethe 1916a, 74–75; Erman and Grapow 1926–1963, vol V, 191). 'One third' is *rꜣ* (Sethe 1916a, 81–82; Erman and Grapow 1926–1963, vol II, 392), which can also stand more generally for 'part', while 'two thirds' is *rꜣwỉ* (or other variants), literally 'the two parts' (Sethe 1916a, 91–97; Erman and Grapow 1926–1963, vol II, 392; Gardiner 1957, § 265). These were not the only fractions to have had special names; for example, several dimidiated fractions (defined in note *y*) had particular names (Sethe 1916a, 75–80), which gives us reason to believe that Neugebauer did indeed intend to include these among his 'natural fractions' (see Neugebauer's pp. 12–13). However most other fractions usually appeared as the numerals described in note *j*, rather than as number words. Indeed, numeral signs were favoured over number words more generally: see Chrisomalis (2020, 152–154).

kk. Neugebauer is alluding here to the fact that some cultures' counting systems apparently allow for only a finite set of number words, even in compounds. The existence of such spoken number systems in pre-literate societies has been widely reported in writings on the history of mathematics (for an example roughly contemporary to Neugebauer's Kiel lecture, see Smith 1923/1925, vol I, 6), and has often been taken as evidence of the inability of supposedly 'primitive' cultures to count beyond low numbers (see the discussion in Barany 2014). The reality, however, is otherwise: as several authors have demonstrated (see, for example, Harris 1987), a lack of dedicated words for higher numbers does not preclude the handling of these quantities. For a recent discussion of this point in the context of Papua New Guinea, see Owens and Lean (2018, 282). Ancient Egypt is somewhat different from the cultures usually discussed in this context because it was not confined to an oral tradition (here instead see the discussions in Chrisomalis 2020). The word for million is *ḥḥ* (Erman and Grapow 1926–1963, vol IV, 152–153), which is written in hieroglyphs with the sign of a Heh-god with upraised arms to support the sky, a personification of infinity, 𓁨 (Gardiner's sign list C11). This word was also used for large (unspecified) quantities, and, according to Sethe, in later periods 'millions' were expressed through multiples of the word for 100,000, *ḥfn* (Erman and Grapow 1926–1963, vol IV, 152–153; Sethe 1916a, 12; Gardiner 1957, §§ 99, 259; Chrisomalis 2010, 37). Later on, this latter word also came to denote an innumerable multitude; perhaps relevantly the word was related to the word for 'tadpole' (Erman and Grapow 1926–1963, vol III, 74–75), which is seen in the sign, 𓆏 (Gardiner's sign I8), that is consistently used to write this word in hieroglyphs (Sethe 1916a, 13–14); Sethe noted that the associated word stem *ḥfl* survives in modern Arabic in words relating to large quantities (Sethe 1916a, 14).

ll. This supposed cultural conservatism of Egypt is an orientalist idea that derives in part from the fact that continuity with the past was a major theme of Egyptian state ideology: see, for example, Kemp (1991, 20). However, as Neugebauer points out in his following sentence, this did not result in a static culture; rather, innovations were often legitimized through associations with the past; as Kemp (1991, 59) puts it: 'Inventing traditions was something that the Egyptians were very good at.' Writing in a medical context and having demonstrated that Egyptian medical practitioners adopted foreign ideas, Ritner (2000, 110) asserts that 'Egyptian conservatism should not be misconstrued as resistance to new or different ideas but recognized instead as a reluctance to discard completely society's valued, older conceptions'. Conservatism versus innovation in ancient Egyptian science and technology is a theme considered by Shaw (2012). In a mathematical setting, Peet (1923, 16) cites 'the amazing conservatism of the Egyptian mind in every branch of life' when dismissing as a *non sequitur* the suggestion that the lack of notation for non-unit fractions in ancient Egypt entailed the lack of the notion (see note *m*) – Peet argues that even if Egyptian scribes had a conception of non-unit fractions, they would still have been compelled to retain the traditional notation which only accommodated unit fractions. See the comments on this point in Chrisomalis (2020, 154), which draws upon the notion of 'decorum' in ancient Egyptian textual culture (Baines 2007).

mm. This quotation is attributed elsewhere to Sethe (Schott 1964, VIII). The earliest printed instance of it that we have been able to find was published four years after Neugebauer's lecture (Sethe 1930, 1), where it nevertheless gives the impression of already being a well-worn phrase: 'It has been rightly said of the ancient Egyptians that, unlike other peoples, they always carried their eggshells with them' ('Man hat von dem Volk der alten Ägypter mit Recht gesagt, daß es im Unterschied zu anderen Völkern seine Eierschalen immer mit sich herumgetragen habe.'). This is an idiomatic German phrase that refers to the way in which a newborn chick will take its first steps with pieces of eggshell still stuck to its body.

nn. The word 'formulae' suggests that Neugebauer was thinking of the traditional long-attested funerary invocation 'A boon which the king gives' (see, for example, Gardiner 1957, 170–173 (Excursus B); Franke 2003). However, he might instead be referring to the extensive corpus of funerary rituals and incantations which are first found in royal pyramids in the Old Kingdom (the 'Pyramid Texts'), then on non-royal coffins in the First Intermediate

Period and Middle Kingdom (the 'Coffin Texts'), and then on shrouds and papyri in the New Kingdom (the 'Book of the Dead'; for an overview, see Allen 1988).

oo. Here and below, the Greek text has been added into the typescript by hand. Neugebauer's insertion of Greek phrases stems from Sethe's treatment of this topic, wherein he demonstrates that the construction of the relevant number phrases in Egyptian parallels that in Greek: see Sethe (1916a, 105).

pp. As several examples in the RMP attest (see, for example, Peet 1923, 16), it was a known fact among Egyptian scribes that any fraction of the form $\overline{2m}$ could be doubled simply as $\overline{m}$ (which is why the recto table of the RMP only deals with $2 \div n$ for odd n), so the temptation to write $2 \div n$ as $\overline{n} + \overline{n}$ would presumably not have arisen for even n. The scribe's evident aversion to writing $\overline{n} + \overline{n}$ in any instance has been addressed by other authors in a similar manner to Neugebauer, though without the emphasis on the additive nature of Egyptian arithmetic. For example, in his *Egyptian grammar* (first published in 1927), into whose numerical and metrological parts Peet had some input (see the references in Hollings and Parkinson 2020, 70), Gardiner (1957, § 265) argued from the ordinal sense of phrases like 'seventh part' that any such quantity must necessarily be unique: 'in any series of seven, only one part could be the seventh, namely that which occupied the seventh place in the row of seven equal parts laid out for inspection'. Elsewhere, systematizing mathematical authors have taken it as an unexplained fundamental principle of Egyptian arithmetic that in the decomposition of an arbitrary fraction as a sum of unit fractions, the same term cannot appear twice: see, for example, the 'precepts' of Gillings (1972, 49). Their absence from surviving Egyptian texts did not prevent Neugebauer from later employing such expressions as $\overline{n} + \overline{n}$ within his theoretical apparatus for explaining the structure of the recto table of the RMP (Neugebauer 1930).

qq. A specific hieroglyphic sign for $\frac{3}{4}$ appears, for example, on the royal annals preserved on the Palermo Stone (dating from the late Old Kingdom) in a measurement of length as $\frac{3}{4}$ of a finger (see note *rrr*); a modified version of the sign also appears in a metrological context in a New Kingdom temple inscription, but the use of either form was otherwise rare (Sethe 1916a, 98; Gardiner's sign list D23; Ritter 1992). A more commonly used hieroglyphic fraction sign was that for $\frac{2}{3}$ (Gardiner's sign list D22). Unlike the other simple fractions (Imhausen 2016, 53), there does not seem to have been a special hieratic sign for $\frac{3}{4}$, which suggests that it indeed did not have a common role to play in Egyptian arithmetic, as Neugebauer argues below.

rr. The reference given here is suggestive of Neugebauer's general philological grounding: Jacob Wackernagel (1853–1938) was a Swiss linguist whose major work was a comprehensive grammar of Sanskrit. He taught for a time in Göttingen, but did not overlap there with Neugebauer. See Wachter (2013).

ss. Neugebauer has presumably omitted fractional bars here, and should have written 'the fractions $\overline{2}$, $\overline{4}$, $\overline{8}$, …'.

tt. See notes *n* and *jj*.

uu. Notationally, the use of an equals sign here is a little odd, but should perhaps be interpreted simply as an informal shorthand for the series. Once again, however, fractional bars are missing, and the series should presumably read $\overline{\overline{3}}$, $\overline{3}$, $\overline{6}$, …: more repeated halving, but this time starting from $\overline{\overline{3}}$.

vv. There does not appear to be any deliberate reason for the large space here, but again fractional bars are missing: $\overline{\overline{3}}$ $\quad\overline{3}$.

ww. See note *n*. In saying that this scheme became 'generally prevailing', Neugebauer is probably alluding to the ubiquity in the surviving sources and apparently fundamental nature of calculations involving $\overline{\overline{3}}$. He may also have had Problems 61 and 61B of the RMP in mind: the former contains a short table of calculations with fractions, in many cases using $\overline{\overline{3}}$, while 61B is the only example in the papyrus of the statement of a general rule, namely to find $\overline{\overline{3}}$ of any (odd) aliquot part. Peet (1923, 103) argues that the placement of Problem 61, as the first problem on the verso of the papyrus, seemingly added into a margin that had previously been left blank, suggests its role as an accessible reference table. The appearance of this partial table in the papyrus led Gillings (1972, ch 4) to speculate that tables for calculation with $\overline{\overline{3}}$ may have been as widely used by Egyptian scribes as those for $2 \div n$ seem to have been, and he offers suggestions as to how such a table might have been constructed. Prompted by the survival of such tables from much later periods, Peet (1923, 8, 20) took their existence during the Middle Kingdom for granted, and further evidence of the fundamental nature of the concept of 'two parts' (see note *jj*).

xx. Neugebauer's assertion here is consistent with what we find in the recto table of the RMP: whenever *n* is divisible by 3, the expression of $2 \div n$ as a sum of unit fractions always contains the term $\overline{2n}$. Beginning with Eisenlohr (1877, 30–35), most authors who have sought to find systematic rules in the construction of the $2 \div n$ table have done so through an examination of the factors of *n*, which naturally leads the modern mathematical mind to consider *prime* factors. As we have already noted (see note *q*), both Eisenlohr and Hultsch took this approach, but were criticized by Neugebauer for doing so: extensive application of ideas of divisibility was alien to the 'purely additive' scheme that Neugebauer was establishing for Egyptian arithmetic. Neugebauer's 'dyadic' approach was designed precisely to sidestep the apparent difficulty of explaining the construction of fractions from a purely additive viewpoint within a system that (he claimed) lacked a general notion of multiplication, and therefore also of division (Neugebauer 1926, 20). The mechanism by which Neugebauer achieved this in a manner that is consistent with the content of the RMP is elaborated upon in note *zz*. To return to Neugebauer's formula for the case when *n* is divisible by 3, we note that the use of such modern formulae was widespread in discussions of Egyptian mathematics, at least as a convenient shorthand, despite concerns about the unintentional imposition of modern attitudes (Peet 1923, 60; Neugebauer 1926, 14). In his dissertation, Neugebauer (1926, 22) also proposed formulae for the cases when *n* is divisible by 5 or by 7, but these do not reproduce the results of the RMP in all instances. Looking at things in a different way that does not require an explicit formula (in the manner of Gillings 1972, 54, for example), we note that all the resolutions of $2 \div 3k$ (for odd $k \geq 3$) that appear in the RMP may be derived from that of $2 \div 9$ via successive division of the terms by 3 (assuming that we allow such an operation), but that analogous processes applied to $2 \div 5$ and to $2 \div 7$ do not produce the same results as those listed in the RMP for $2 \div 5k$ and $2 \div 7k$, respectively. Here we see one drawback to the attempted use of prime numbers in this context. It appears that Neugebauer did consider this point (Neugebauer 1926, 22); had it worked, it would have furnished him with a scheme for the systematization of the $2 \div n$ table that was as simple as the one that he proposed for the first group of completions (see notes *k* and *ddd*). Instead, he found that any such proposed systematization must necessarily be rather more involved, a feature common to all subsequent attempts to explain the table.

yy. In his dissertation, Neugebauer (1926, 22) applies the term 'Kennziffern' to the elements of the 1/2-series in their role here, which we might translate as 'indices' ('reference numbers' might be another possible translation, but this term is already employed elsewhere: see note *ccc*).

zz. To expand slightly on Neugebauer's explanation, a general principle that does genuinely appear in the RMP in the context of the recto table is that of breaking 2 down into two or more parts, each of which can then be operated with separately. The simplest examples appear precisely for those $2 \div n$ for which n is divisible by 3, wherein 2 is written as $1\,\overline{2} + \overline{2}$. The resolution of $2 \div n$ as a sum of unit fractions is then found as the sum of $1\,\overline{2} \times \overline{n}$ and $\overline{2} \times \overline{n}$. Both products, the first being Neugebauer's 'completion term' and the latter the so-called 'principal term' ('Hauptgleid': Neugebauer 1926, 21), are easily found via Neugebauer's dyadic considerations. The calculation of the completion term is reminiscent of those in the first group of completions ('(1 + simple fractions)× (sum of unit fractions)') – hence Neugebauer's desire to link these to the $2 \div n$ table (see notes *k* and *ddd*). A range of other decompositions of 2 is used throughout the rest of the table (as summarized by Peet 1923, 37): for example, $1\,\overline{2}\,\overline{4} + \overline{4}$ for $n = 7$, and $1\,\overline{\overline{3}}\,\overline{6} + \overline{6}$ for $n = 11$. For n divisible by 5 but not by 3, the decomposition $1\,\overline{\overline{3}} + \overline{3}$ is used frequently but not uniformly, namely for $n = 5, 25, 65, 85$, but not for $n = 35, 55, 95$; a different decomposition is used in each of the latter three cases, which are also the instances in which the multiplicative scheme mentioned in note *xx* breaks down. For $n = 35$, for example, 2 is broken down as $1\,\overline{6} + \overline{\overline{3}}\,\overline{6}$. It is in an additional line of explanation that appears in the determination of $2 \div 35$ but in no other entries in the table that Gillings sees the key to understanding the structure of the table as a whole: see Gillings (1972, 77–80).

aaa. The term 'completion' ('Ergänzung') translates the Egyptian term *skm* ('to cause to be complete'), and Neugebauer uses it here in the sense of what Peet called the 'second group of completions' (Peet 1923, 58–60): Problems 21–23 of the RMP. These are problems concerning the subtraction of fractions: Problem 21, for example, asks 'what completes $\overline{\overline{3}}\,\overline{15}$ into 1?' (Peet 1923, 58). The use of the word 'completion' in this connection is easily explained, but it is harder to justify for the first group of completions (Problems 7–20; see note *k*). The collective name for these latter problems was coined by Peet in light of the fact that Problem 7 is headed 'Example of completion' (*tp n skmt*), though it has been suggested elsewhere (Chace 1927/1929, vol I, 23) that this was a misplaced heading for the second group, a not unreasonable suggestion, in light of the many copying errors found in the RMP. On these two apparently different uses of the word *skm*, see Peet (1923, 53) and Gunn (1926, 130). Although Neugebauer's formula for the case where n is divisible by 3 expresses the completion term $\overline{m}$ as the difference of two fractions, it would be circular to suggest that this is how $\overline{m}$ was calculated, since the application of the formula requires knowledge of $2 \div n$, which is what the determination of $\overline{m}$ is intended to give. Thus, despite the fact that Neugebauer's formula presents the calculation of the completion term in the style of the second group of completions, i.e. as a difference of fractions, the calculation, as outlined in note *zz*, was actually more akin to the problems of the first group of completions.

bbb. Which appears here in bold: see note *l*.

ccc. This 'method of red auxiliaries', which we have already seen in Neugebauer's example of multiplying $\overline{32}\,\overline{224}$ by $1\,\overline{2}\,\overline{4}$, is, as he indicated earlier, one of the more sophisticated techniques in Egyptian arithmetic. It differs from the modern method of finding the least common denominator of a set of fractions in that the (red) auxiliary numbers that result from the scaling up by a given factor (often termed the 'reference number' in modern writings on this topic) need not be integers; see also the comments in note *u*. The reference number was generally chosen so that the resulting red auxiliaries were either integers or simple fractions such as dimidiated fractions or elements of Neugebauer's 1/3-series, which might then easily be added. The sum of the auxiliaries would finally need to be divided by the reference number to obtain the required result. This last calculation is often omitted from the surviving sources, such as in Neugebauer's earlier example of RMP Problem 15, but where it does appear (for example, in Problem 21 of the RMP) it is accomplished by what would probably have been (for an experienced scribe) an easy application of the standard two-column method for division, which in many respects is the reverse of that for multiplication: 1 and the divisor are written down as the first entries of the columns and the dividend is sought as a sum of entries in the second column, whereupon the quotient may be found as the sum of the corresponding entries in the first column. For Peet's comments on the method of red auxiliaries (which he called the 'common denominator method'), see Peet (1923, 17–18).

ddd. The linking of this decomposition with the so-called 'first group of completions' (see note *k*) provided Neugebauer with one of the main illustrations of his systematization of the early content of the RMP which he used in turn to explain the construction of the $2 \div n$ table. In the example given here, Neugebauer has extracted the first row of the calculation in RMP Problem 7 (and also in Problems 7B and 10, which replicate 7). In the papyrus, this row is then halved to give $\overline{8}\,\overline{56}$, with corresponding auxiliaries (for reference number 28, as above) $\mathbf{3\,\overline{2}}$ and $\mathbf{\overline{2}}$, then halved again to give $\overline{16}\,\overline{112}$ (auxiliaries $\mathbf{1\,\overline{2}\,\overline{4}}$ and $\mathbf{\overline{4}}$). The auxiliaries add to **14**, which is seen to be half of the reference number 28 (no explicit calculation is given). Hence the overall result of Problem 7: $1\,\overline{2}\,\overline{4} \times \overline{4}\,\overline{28} = \overline{2}$. Neugebauer's earlier example corresponds to RMP Problem 15, whose overall result (again obtained via red auxiliaries using reference number 28) is $1\,\overline{2}\,\overline{4} \times \overline{32}\,\overline{224} = \overline{16}$, which, as Neugebauer observes here, may be obtained by halving the quantities in Problem 7 three times. In his dissertation, Neugebauer illustrated this transition in the form of a table (Neugebauer 1926, Tafel III.1), in which one moves from Problem 7 to Problem 15 via the intermediate Problems 11B (Problem 7 halved: $1\,\overline{2}\,\overline{4} \times \overline{8}\,\overline{56} = \overline{4}$) and 13 (Problem 11B halved: $1\,\overline{2}\,\overline{4} \times \overline{16}\,\overline{112} = \overline{8}$). However, there is no Problem 11B in the papyrus: this was interpolated by Neugebauer for the sake of his schematic reorganization of the first group of completions. For a more detailed discussion of these ideas and their connection to the $2 \div n$ table, see Hollings and Parkinson (2020).

eee. The language of 'domains' and their 'extensions' that Neugebauer uses throughout this lecture is indicative of the set-theoretic way of expressing mathematics that he would have learnt during his university education, and was perhaps a deliberate choice here, given the mathematical nature of his audience; as a point of contrast, we note that this kind of language would have been much less prevalent in Peet's early-twentieth-century Oxford mathematical education than in Neugebauer's 1920s Göttingen education (see the comments in Hollings and Parkinson 2023, § 2). The use, moreover, of 'integral domain' ('Integritätsbereich'), though not in its modern sense, may point towards Neugebauer's exposure to such abstract structural ideas through his attendance of Emmy Noether's Göttingen lectures, where he certainly encountered Abelian groups (Rowe 2016b, 16, 32). Interestingly, in a letter to Helmut Hasse of 2nd May 1928, Noether included Neugebauer in a list of people to whom she planned to send offprints of a joint paper with Richard Brauer (Roquette and Lemmermeyer 2022, § 3.14). Doubt is cast on Neugebauer's familiarity with the concepts of modern algebra, however, by a mistake made by him a few years later as a commissioning editor at Springer: he commissioned a book on algebras alongside one on hypercomplex numbers, not realizing at first that the two notions coincide (Roquette and Lemmermeyer 2022, p 151).

fff. That is to say that Egyptian arithmetical methods *worked*, but that the scribes who used them did not worry about *why* they worked. In his dissertation, Neugebauer had been one of those authors who denied the ancient Egyptians a 'scientific' mathematics precisely because they lacked a modern notion of proof (Neugebauer 1926, 17); see note *g*. More generally, Neugebauer has fallen here into the judgemental language often applied by mathematical writers in condemnation of the 'cumbersome' processes of Egyptian arithmetic: see, for example, Miller (1905).

ggg. Hans Jacob Polotsky (1905–1991) was a Swiss-born Israeli philologist who specialized in Egyptology and Semitic languages. He was a contemporary of Neugebauer's in 1920s Göttingen, but spent most of his career (from 1934 onwards) at the Hebrew University of Jerusalem. See Bierbrier (2012, 439–440). The phrase quoted by Neugebauer eventually appeared in Polotsky's 1928 doctoral dissertation, an analysis of funerary inscriptions of the 11th Dynasty, which was subsequently published as Polotsky (1929). Neugebauer refers to a First Intermediate Period funerary stela of Megegi in the Metropolitan Museum of Art in New York (accession number 14.2.6, https://www.metmuseum.org/art/collection/search/544007; Hayes 1953, 152), which contains a standard autobiographical claim found in several inscriptions (Polotsky 1929, 23–24; see also Janssen 1946, 66 (Aw.70–71)): 'Ich bin ein {Er liebt das Gute und hasst das Böse}': 'I am one who loves good and hates evil'. However, Neugebauer's quotation is actually from another different contemporaneous stela, that of Idu in the British Museum (EA 1059; see Fischer 1981, 64–66), which Polotsky (23–24) gives as 'Ich bin ein {er liebt das Essen und hasst das Schwatzen} und ein Genosse des Festtages' ('I am one who loves food and hates chatter, and a comrade of the feast day';

Neugebauer has ‘einer’ in place of the first ‘ein’, and an umlaut in ‘Schwatzen’). A more recent English translation is ‘I am one who loves eating and hates discussion, a companion on the holiday’ (Fischer 1981, 65). The confusion between the two modern locations is an easy slip.

hhh. Hermann Hankel (1839–1873) was a German mathematician whose main work was in hypercomplex number systems and the theory of functions (Crowe 2008). He also took an interest in the history of mathematics, and in his 1869 inaugural lecture as professor in Tübingen he gave a broad survey of the development of mathematics during the preceding centuries. This lecture included a subsequently much-quoted (if over-simplistic) passage concerning the cumulative development of mathematics: ‘In most of the sciences, one generation tears down what another has built, and what one has established another destroys. In mathematics alone, each generation puts a new floor on the old substructure.’ (Hankel 1869, 34: ‘In den meisten Wissenschaften pflegt eine Generation das niederzureissen, was die andere gebaut, und was jene gesetzt, hebt diese auf. In der Mathematik allein setzt jede Generation ein neues Stockwerk auf den alten Unterbau.’) His posthumously published *Zur Geschichte der Mathematik in Alterthum und Mittelalter* (Hankel 1874) has been criticized for its inaccuracies (see Folkerts et al. 2002, 122–123), though a fair assessment should note that his writing came too early for Eisenlohr’s 1877 revelations concerning ancient Egyptian mathematics, and similarly pre-dated the detailed understanding of Mesopotamian mathematics (Høyrup 2016a, 62–64). It is to Hankel’s 1874 book that Neugebauer alludes in relation to so-called ‘prescientific’ (‘vorwissenschaftlich’) mathematics. On the question of ‘scientific’ mathematics in ancient Egypt, see also note *g*. The influence on Neugebauer of Hankel’s approach to the history of mathematics is discussed by Chaigneau (2019, § 2.3.2).

iii. Such a comparison is consistent with the philological tradition within which Neugebauer was working: contrasts with Babylonian, Hebrew, Arabic, Roman, and (most particularly) Greek sources appear throughout Sethe’s *Zahlen und Zahlworten*, for example (see note *oo*).

jjj. Peet had earlier introduced his own comparison of Egyptian and Babylonian mathematics by noting that ‘[a]s early as 3000 B.C. it would seem that the world was in possession of two separate and highly developed systems of mathematics, the Egyptian and the Babylonian’ (Peet 1923, 27), making their joint discussion entirely natural. Elsewhere, in a slightly different setting, he justified his restriction to hieratic sources over demotic ones in terms of avoiding ‘the possibility of contamination from Greek mathematics’ and thereby studying material that was more purely Egyptian (Peet 1923, 7).

kkk. Neugebauer is probably referring here to his *Habilitationsschrift*, which dealt with the origins of the Babylonian sexagesimal system (Neugebauer 1927c): see also note *nnn*. From around 1930 onwards, Neugebauer largely left the study of Egyptian mathematics (but not astronomy) behind, in favour of Mesopotamian topics, probably because of the greater availability of sources and their more challenging mathematical content (see, for example, the comments in Ritter 2016, 159–160).

lll. This distinction between ancient cultures in the same part of the world has often been somewhat flattened by writers on the history of mathematics, with 'Babylonian' used as a catch-all term: see, for example, Smith (1923/1925, vol I, 35–41). The inappropriateness of that term in the case of the famous cuneiform tablet Plimpton 322, for instance, has been noted by Robson (2001), and more recently the organization of the book Robson (2008) has served to re-emphasize the distinctions between successive cultures. Even the word 'Mesopotamian' is not without its problems in this context (Robson 2008, 274), but we retain it here as convenient terminology. Peet's use of 'Babylonian' in the quotation in note *jjj* is not as simplistic as it might at first appear: he followed this passage with some brief comments on the different cultures of ancient Mesopotamia ('The term Babylonian is however here a complex one …'). The term 'Semitic' has fallen out of use in the racial sense that was common in the early twentieth century, but is still employed in reference to languages and their associated cultures.

mmm. François Thureau-Dangin (1872–1944) was a French Assyriologist who spent most of his career in the department of oriental antiquities at the Louvre. He established a classification of cuneiform signs that is still used by Assyriologists, took the first steps towards the decipherment of the Sumerian language, and was one of the pioneers of the study of ancient Mesopotamian mathematics (André-Salvini 1997, 2016). A number of authors have contrasted the approaches of Neugebauer and Thureau-Dangin to the latter topic, observing that their works do not simply fall into the 'mathematician vs philologist' framework that we might naively expect to find (Proust 2016, 218–221; see also Høyrup 2016a, 2016b; Melville 2016). Their relationship does, however, appear to have been one of competition (Proust 2016, 229–231). For a comparative study of the works of Thureau-Dangin and Neugebauer, see Chaigneau (2019). Anton Deimel (1865–1954) was a German Assyriologist and theologian who was professor of Assyriology at the Pontifical Biblical Institute from 1909 and is regarded as the founder of Sumerian studies (Falkenstein 1957). The work of Deimel cited here by Neugebauer contains a large collection of source material on early number signs. In 1927, Neugebauer spent some time in Rome studying Sumerian with Deimel (Neugebauer 1927c, 5).

nnn. Neugebauer captures here something of the complex picture of the development of ancient Mesopotamian numeration, which indeed would only grow richer and more complicated as further sources emerged and were interpreted over the course of the twentieth century. The central hypothesis of Neugebauer's *Habilitationsschrift* (see also note *kkk*), which he outlines briefly here, and which drew heavily upon source material published by Deimel (note *mmm*), was that the Babylonian sexagesimal system was a simplification of an older Sumerian numeral system that had been used for metrological purposes within Sumerian economic life; prior attempts to explain the origins of the system had looked to astronomical applications, an explanation that Neugebauer refuted. Thureau-Dangin (1932, 1939) also asserted a Sumerian origin for Babylonian numerals, though in the form of an absolute system of numeration (see Høyrup 2016b, 182–184; for a recent account of the development of the sexagesimal system, see Friberg 2019). All of these speculations, however, long pre-dated the full decipherment of proto-cuneiform sources, which revealed the existence of several different prior systems of numerical signs, each used to count different commodities (Melville 2016;

Chrisomalis 2010, ch 7; Robson 2008, 75–83). Each of these systems had separate signs for 1 and 10, which were deployed in a cumulative-additive manner. Many also had a sign for 60, and in one instance a sign for 600 which was formed as a combination of those for 10 and 60; this feature was also present in the Sumerian numerals known to Neugebauer (see Neugebauer 1927c, Tabel I) – hence his comment about a 'multiplicative-algorithmic tendency'. Indeed, other early examples, unknown to Neugebauer in the 1920s, exhibit a bisexagesimal structure: two copies of a sign for 60 are combined to make one for 120, which is then further augmented with the sign for 10 to make one for 1200 (see Chrisomalis 2010, Table 7.1). Neugebauer alludes to the presence of a sign for 10 in the Sumerian numeral system when he notes that this is not a 'pure 60-system'. His remark about the small number of signs used probably concerns the fact that both in archaic Sumerian numerals and in their eventual cuneiform successors, the signs for 1 and 60 are essentially the same (Chrisomalis 2010, Tables 7.10 and 7.11) – in principle, the latter was a larger version of the former, but in practice the size difference was minimal (particularly in the cuneiform case), since both signs were written with the same stylus. The second part of Neugebauer's remark is less clear, but may refer to the early adoption of the cuneiform numerals for non-discrete applications alongside their original use in discrete counting (see, for example, Robson 2008, 76).

ooo. Thureau-Dangin (1921, 124), for example, presents $\frac{1}{6} = 10 \times \frac{1}{60}$ as a natural quantity to emerge in a sexagesimal system with sub-base 10; Peet (1923, 28) offers the same purely mathematical justification. Sethe (1916a, 67), on the other hand, cites a parallel with the Roman *uncia* to assert metrological reasons for the prominence of $\frac{1}{6}$, since it frequently appears as a conversion factor between different units of measure (see, for example, Chambon 2011); for Sethe, $\frac{1}{6}$ becomes a new unit of calculation which then easily gives rise to $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$. In the paper cited by Neugebauer, Sethe further cites the construction of a regular hexagon (consisting of six equilateral triangles) via the repeated marking off of a circle's radius on its circumference, and points to the later Babylonian use of a six-spoked wagon wheel. The factor $\frac{1}{6}$ also has a central role to play in the very theory of the origin of the sexagesimal system that Neugebauer disputed (see note *nnn*): that 60 is a convenient base because 360 is close to the number of days in the year.

ppp. The special signs used for these various fractions in the Old Babylonian period (early second millennium BCE) may be found in Benoit et al. (1992, 10). See also Neugebauer (1927c, Tabel II).

qqq. On the role played by metrology in the formation of systems of fractions in the ancient Near East, see the chapter of Ritter in Benoit et al. (1992); on Mesopotamian fractions specifically, see those of Bruins, Michel, Démare-Lafont, and Glassner. In particular, the chapter by Michel deals with the economic tablets that Neugebauer alludes to here. The various subdivisions mentioned by Neugebauer can be seen in the summary of Mesopotamian metrological systems given by Robson (2008, Appendix A) (see also Robson 2007a, 70–72). For example, in the Old Babylonian period (as defined in note *ppp*), the capacity measure of 1 *ban* was made up of 10 *sila* (1 *sila* $\approx$ 1 litre), and 6 *ban* made 1 *bariga*.

rrr. On ancient Egyptian metrology, see Imhausen (2016, ch 6). As Neugebauer indicates, the basic unit of length, the 'cubit' (*mḥ*, literally 'forearm': Erman and Grapow 1926–1963, vol II, 120), was divided into 7 'palms' (*šsp*: Erman and Grapow 1926–1963, vol IV, 535), and each of these into 4 'fingers' (*ḏbʿ*: Erman and Grapow 1926–1963, vol V, 565). Based on the average length of extant cubit rods, the Egyptian cubit is often quoted as having been approximately 52.5 cm, but there was no standardized unit (moreover, the sample here is skewed, since most surviving cubit rods date from the New Kingdom: Imhausen 2016, 168–170). The Mesopotamian cubit (Sumerian *kuš*$_3$, Akkadian *ammatu* = 'forearm') was equivalent to approximately 50 cm, and was divided into 30 fingers (*šu-si*, *ubānu*), which at an early period had been conceived as consisting of 3 'double-hands' (*šu-du*$_3$*-a*, *šīzu*), each made up of 10 fingers (Robson 2008, Table A.2) – hence Neugebauer's comment about a decimal subdivision. Neugebauer's final sentence here probably relates to a system of area measurement, the so-called 'seed measure', that emerged in the early first millennium BCE, whereby areas were considered in proportion to fixed measures of capacity; the most common instance of this system identified an area of 100 cubits ×100 cubits with a capacity of $33\frac{1}{3}$ *sila* (see note *qqq*; Powell 1990, §II B.2; Robson 2008, Table A.4). Moreover, in an earlier period, the same terms and scales of units had been used for both area and volume measures: the basic unit was the *sar*, which as an area measure was equal to 1 square rod (1 rod = 12 cubits), and as a volume measure to 1 area *sar* × 1 cubit (Robson 2008, Table A.3).

sss. 'Groschen' was originally the name for a 12 pfennig coin in the German states, but after decimalization in the late nineteenth century, it became a nickname for the new 10 pfennig coin.

ttt. Once again, the various factors mentioned here by Neugebauer, including 18, may be found in the summary of Mesopotamian metrological systems given by Robson (2008, Appendix A). As we have already observed (note *nnn*), a range of numeral systems were in use at different times and for different purposes in ancient Mesopotamia, each displaying its own combination of decimal and sexagesimal features. These systems differed across successive cultures, and were used to measure and quantify different commodities (Chrisomalis 2010, ch 7). The later sexagesimal *positional* system (see below in the main text) was in particular used only for very specialized purposes (Robson 2008, 75–83), in contrast to the emphasis that is now usually placed upon it in general writings on the history of mathematics. Neugebauer's distinction between the numeration of 'learned schools' and the 'arithmetic of the people' may be a false dichotomy in a culture where literacy and numeracy were confined almost exclusively to an educated elite (Robson 2007b). On the other hand, if 'arithmetic of the people' refers to the supposed day-to-day numeracy of the illiterate population at large, then we can make no substantial comment on something that by its very nature went unrecorded. In this context, however, Thureau-Dangin (1921, 125) invoked the decimal nature of Semitic lexical numerals (see also Chrisomalis 2010, 248).

uuu. See the remarks on the sizes of cuneiform numeral signs in note *nnn*. On the development of Mesopotamian cuneiform, see Daniels and Bright (1996, § 3).

vvv. Neugebauer's schema '*a b a b*' is presumably intended to convey the fact that in the Babylonian positional notation, a 'two-digit' sexagesimal number (i.e. one with two sexagesimal positions) is expressed via alternating clusters of signs for 1 (Neugebauer's '*b*', standing in the left-hand position for 60) and signs for 10 (Neugebauer's '*a*', standing in the left-hand position for 600). A 'missing member' of the extended series is then a missing '*a b*' pair. On the Babylonian zero-marker, see Chrisomalis (2010, 252–253). A more recent assessment than that of the German indologist Georg Bühler (1837–1898) whom Neugebauer cites here (on Bühler, see Kirfel 1955) places the development of the positional system in India (with the sign for zero that would eventually evolve into the number concept) in the seventh century CE (Chrisomalis 2010, ch 6). Neugebauer evidently considered a 'zero' necessary for a true understanding of 'position' in this context.

www. These comments reflect the same dismissive attitudes as the earlier ones about ancient Egyptian conservatism (see note *ll*). The final allusion is to the controversial theory that the modern alphabet had its origins in Egyptian hieroglyphs, via the intermediate Proto-Sinaitic script, but the development of the alphabet is not such a simple progression and 'discovery' as Neugebauer implies (for a summary, see Parkinson 1999, 181–184 or Lam 2010; for a recent development of the theory, see, for example, Goldwasser 2012). This issue was examined by Sethe in the 1916 article cited here by Neugebauer, although this work was completed without knowledge of Alan Gardiner's proposed decipherment of the Proto-Sinaitic script, published that same year (Gardiner 1916), and so Sethe followed up his original work with a further article (Sethe 1917); his two papers were published together as a single volume (Sethe 1926) shortly before Neugebauer delivered his lecture in Kiel. The Egyptologist Alan H Gardiner (1879–1963), who exerted a major influence on British Egyptology during the first half of the twentieth century, and who was particularly supportive of Peet's career (Bierbrier 2012, 205–207; Hollings and Parkinson 2020, § 1.3), had strong links to German Egyptology, which may be seen very clearly in an acknowledgement at the beginning of Sethe's 1917 article:

> It is a particular pleasure for me to be one of the first to be able to point out to the scientific world in Germany this highly significant work [Gardiner 1916] by an excellent English scholar who is a close friend of mine and who, despite all the storms that are currently raging through the world, has not allowed his calm view of things to become clouded towards his German friends. (Sethe 1917, 438: 'Es ist mir eine besondere Freude, als einer der ersten die wissenschaftliche Welt Deutschlands auf diese höchst bedeutsame Arbeit des mir nah befreundeten ausgezeichneten englischen Gelehrten hinweisen zu können, der sich trotz aller Stürme, die zur Zeit die Welt durchtosen, den ruhigen Blick für die Dinge seinen deutschen Freunden gegenüber nicht hat trüben lassen.')

xxx. Neugebauer cites here a work by the German-Austrian linguist, ethnologist, historian of religion, and Catholic priest Wilhelm Schmidt (1868–1954), who was particularly active in studying the languages of Southeast Asia and Oceania, though the cited work is a more general systematization of the languages of the wider world; on Schmidt, see Gusinde (1954). In more recent years, the languages of Papua New Guinea and Oceania have been the focus of studies relating to the development of numeration: see, for example, Owens et al. (2018).

yyy. Because of the central place that ancient Greek mathematics occupies in the standard story of the development of mathematics (see note *zzz*), much of the early work on ancient Egypt and Mesopotamia in this connection focused on the question of how ideas from these parts of the world related to Greek mathematics, rather than considering Egyptian or Mesopotamian mathematics in their own rights (see note *dddd*). There is a long-standing – if generalized – tradition, stemming from Herodotus, that knowledge of geometry first entered Greece from Egypt, but the precise nature of this influence is elusive (Imhausen 2016, 117–118, 180–181). For Peet's comments on the Greek view of Egyptian mathematics, see Peet (1923, 31–32); see also the discussion in Hollings and Parkinson (2022, § 2). For further detailed treatments of the influence of ancient Egypt on Greek mathematics, see Fowler (1987, § 8.1) and Cuomo (2001, chs 1 and 2). A major feature of Neugebauer's subsequent work was the assertion of a Babylonian influence on Greek thought, argued on the basis of mathematics rather than documentary evidence (Robson 2008, §§ 9.3–9.5; Rowe 2016a, 133).

zzz. Traditionally, the general tendency in the (Eurocentric) study of the history of mathematics has been to assert that 'true' mathematics began in ancient Greece. Not only is this where the modern notion of 'proof' is deemed to have first emerged, but it was also in this setting that mathematics seemingly became an intellectual pursuit in its own right, rather than simply a means to practical problem-solving. If we insist on taking these two points as criteria for 'true' mathematics, then ancient Egyptian mathematics certainly falls short (see notes *g* and *fff*), but a much broader view of what constitutes mathematics is now more usually taken: see, for example, Stedall (2012). On the views of nineteenth- and twentieth-century mathematicians on ancient mathematics and its relationship to its modern counterpart, see Hollings and Parkinson (2022, § 8). On different traditions of proof around the world, see Chemla (2012).

aaaa. Although some of the problems in the RMP are accompanied by diagrams, and were for that reason labelled 'geometrical' by Eisenlohr (1877, 118–133), the style of the mathematical content of these problems does not differ significantly from that in the problems deemed 'arithmetical': problems are set up verbally, and then the necessary calculations are performed via the standard two-column method. For example, Problem 51 presents the reader with a picture of a triangular piece of land and asks for its area (Imhausen 2016, 121–122). The base of the triangle (4 rods (*ḫt*); 1 rod = 100 cubits) and its height (10 rods) are given both as labels on the diagram and also next to this in the text. The remainder of the problem instructs the reader to halve the base and then multiply this by the height, giving an area of 20 aroura (*sṯꜣt*; 1 aroura = 1 square rod), with an accompanying two-column verification of the calculation (a minor complication, which this problem has in common with other geometrical ones in the RMP, is that all calculations are carried out in terms of cubits, so implicit unit conversions occur at the beginning and end). Thus, the geometrical problems in the papyrus are 'geometrical' only by dint of context and not necessarily by mathematical content. What geometry there is here is a long way from the formal deductive scheme of Euclid's *Elements.* In the 1920s, hints of further geometrical problems in the then little-studied Moscow Mathematical Papyrus (MMP) (Touraeff 1917; see also note *c*) excited speculation among mathematicians that the papyrus might contain material that was closer to the Euclidean ideal of 'pure' geometry (see, for example, Karpinski 1929), but subsequent investigations revealed that the geometry of the MMP was of a broadly similar empirical style to that of the RMP (Struve 1930).

bbbb. Kronecker's full sentence was: 'Given the inadequacy of the Greek numerals, this achievement of Greek mathematics is truly to be admired, no matter how easy the proof is to carry out with our present tools.' (Kronecker 1901, 8–9: 'Bei der Unzulänglichkeit der griechischen Zahlzeichen ist diese Leistung der griechischen Mathematik wahrhaft zu bewundern, so leicht auch der Beweis mit unseren jetzigen Hifsmitteln zu führen ist.') He then gave a modern proof in just a few lines. This result on perfect numbers appears as Proposition IX.36 of Euclid's *Elements*, where the proof is considerably longer and uses ideas about continuous proportion from earlier in the text which are expressed verbally and do not rely in any significant way on Greek numerical notation (Heath 1908, vol II, 421–425). Indeed, Kronecker may be guilty here of a common misconception, which is to assume that any historical numeral system must have been used for calculation. This is a perception that stems from the fact that the Hindu-Arabic numerals that are in daily use in the modern world may be used *both* for recording numbers *and* for performing computations. In fact, most historical numeral systems (including those of ancient Greece) had only a representational role, recording values that had been calculated elsewhere, on an abacus, counting board, or similar device; see the discussion of this point in Chrisomalis (2020, ch 2). On Greek numerals, see note *cccc*.

cccc. Early Greek numerical notation was cumulative-additive (note *h*) with base 10 and sub-base 5. By around 500 BCE, the so-called acrophonic numerals (so named because they derived from the initial letters of the corresponding number words) were in use in Athens; these were also cumulative-additive, but with some multiplicative features: the sign for 50 combined those for 10 and 5, for example. However, the numeral system to which Neugebauer is undoubtedly referring here is the much more common alphabetic (a.k.a. Ionic or Milesian) system, which is attested from the sixth century BCE. In this system, letters of the alphabet stand for numbers: $\overline{\alpha} = 1$, $\overline{\beta} = 2$, etc, where the horizontal bars indicate that the numeral is intended, rather than the letter. The 24 letters of the now-standard Greek alphabet, plus three letters that have since fallen out of use, provided 27 signs to stand for the units 1–9, the tens 10–90, and the hundreds 100–900; thus, for example, $\overline{\psi\kappa\delta} = 724$, since $\overline{\psi} = 700$, $\overline{\kappa} = 20$, and $\overline{\delta} = 4$. Further embellishments enabled the expression of numbers higher than 999. As Neugebauer indicates, these numerals have no additive features, though the significance of this observation in this context with regard to practical calculation may be overstated: see the comments in note *bbbb*. On ancient Greek numerical notation, see Heath (1921, 29–45) or Chrisomalis (2010, chs 4 and 5).

dddd. As we have already observed (see notes *yyy* and *zzz*), the respective mathematical developments of ancient Egypt and ancient Mesopotamia were studied through the lens of how they related to what was already known about ancient Greek mathematics, hence the widespread use of the term employed here by Neugebauer: ‘pre-Greek’ (‘vorgriechisch’). The more that Neugebauer worked on these topics, however, the greater the respect he gained for them, and the more determined he became that they ought to be studied for their own sake, rather than simply as precursors of Greek mathematics: see, for example, Neugebauer (1929), and also Neugebauer’s plenary lecture at the 1936 International Congress of Mathematicians (Neugebauer 1937; Hollings and Siegmund-Schultze 2020, § 9.9). Consequently, the label ‘pre-Greek’ gradually disappeared from Neugebauer’s work: see the comments in Swerdlow (1993, 147).

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