

Asymptotic solutions for a relativistic formulation of the generalized nonextensive Thomas-Fermi model

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Abstract

The semiclassical Thomas-Fermi model for heavy atoms was recently extended to include nonextensive statistical mechanics and relativistic effects. While this generalized model has potential application to neutron stars and in understanding the screening process in relativistic dense astrophysical plasmas, there has not been a study on the behavior of the Thomas-Fermi potential $\Phi(x)$ due to nonextensive statistical mechanics and relativistic effects. In the present paper, we extend this literature by obtaining asymptotic solutions through the application of the δ -expansion method, which was previously employed to study the standard Thomas-Fermi equation. Making use of this asymptotic solution, we approximate the value of the critical slope $\Phi'(0)$ for various values of the thermodynamic and relativistic corrections to the standard Thomas-Fermi model. These results allow us to understand how the addition of thermodynamic and relativistic corrections modify the behavior of the Thomas-Fermi potential.

Keywords: Thomas-Fermi equation; nonextensive statistical mechanics; relativistic effects; delta expansion method

1. Introduction

The Thomas-Fermi model [1, 2] is a statistical model of an atom which gives a semi-classical approach to understanding ground state potentials and charge densities in large atoms and metals [3], while it is also of relevance to astrophysical scale objects, such as neutron stars [4]. The Thomas-Fermi equation is given by the second-order nonlinear ordinary differential equation

$$\frac{d^2\Phi}{dx^2} = \frac{\Phi^{3/2}}{\sqrt{x}}, \quad (1)$$

subject to the boundary conditions

$$\Phi(0) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \Phi(x) = 0. \quad (2)$$

The Thomas-Fermi potential $\Phi(x)$ describes the charge density in atoms of high atomic number.

The Thomas-Fermi equation (1) is a nonlinear differential equation boundary-value problem that is difficult to solve numerically, owing to the nonlinearity, the singularity at $x = 0$, and the semi-infinite domain on which the boundary value problem is prescribed. Due to these complications, the equation has served as a good test problem for a variety of solution methods. This equation has been approximated via various numerical or asymptotic methods; see [5, 6, 7, 8, 9, 10] for a few examples of analytical or hybrid analytical-numerical approaches employed to obtain approximate solutions to (1). Often, one is also interested in calculating the value of $\Phi'(0)$, the initial decay rate at the origin. This value acts as a sort of eigenvalue [11] for the problem, and the calculation of this quantity is often used for benchmarking analytical or numerical methods. The value $\Phi'(0) = -1.588071022611375312718684508$ appears to be the most accurate approximation available in the recent literature [12]. Physically, the value $\Phi'(0)$ tells one how rapidly the solutions decay. Due to the importance of the value $\Phi'(0)$, it is sometimes referred to as the “critical slope” of the Thomas-Fermi potential.

The Thomas-Fermi equation (1) was extended by Martinenko and Shivamoggi [13] to the nonextensive statistical mechanics regime, which resulted in the equation

$$\frac{d^2\Phi}{dx^2} = \frac{\Phi^{3/2}}{\sqrt{x}} \left\{ 1 + \tilde{v} \frac{x}{\Phi} + \nu \frac{x^2}{\Phi^2} \right\}. \quad (3)$$

This extension was motivated by the fact that in systems with long-range interactions, it has been shown useful to extend the standard Boltzmann-Gibbs thermodynamics to generalize the concept of entropy to nonextensive regimes; see work by Tsallis and others [14, 15, 16]. Here \tilde{v} is the Tsallis correction term which is proportional to T , the temperature, while ν is the usual thermal correction term proportional to T^2 [17]. (For a full derivation and dimensional forms of these terms, see [13].) In the low temperature limit, we therefore have $\nu \ll \tilde{v} \ll 1$.

Motivated by the work of Martinenko and Shivamoggi [13], there have been several extensions. Equation (3) was generalized to n -dimensional space by [18]. The same authors considered the Thomas-Fermi approach for self-gravitating fermions within the framework of q -statistics [19]. Starting from the q -deformation of the Fermi-Dirac distribution function, they obtained a generalized Thomas-Fermi equation. They then show that the Tsallis entropy preserves a scaling property of this equation. As well as being a correction to the standard Thomas-Fermi theory, models such as (3) can be used in the study of hot plasmas [20].

Equation (3) was later generalized to include both nonextensive and relativistic effects by [21], which resulted in the equation

$$\frac{d^2\Phi}{dx^2} = \frac{\Phi^{3/2}}{\sqrt{x}} \left(1 + \lambda \frac{\Phi}{x} \right)^{3/2} \left\{ 1 + \tilde{v} \frac{x}{\Phi} \left(1 + \lambda \frac{\Phi}{x} \right)^{-1} + \nu \frac{x^2}{\Phi^2} \left(1 + \lambda \frac{\Phi}{x} \right)^{-2} \right\}. \quad (4)$$

Here λ is a relativistic correction term which scales like c^{-2} , where c is the speed of light (see [21] for full derivation and for dimensional forms of the quantities). Therefore, in the non-relativistic limit $c \rightarrow \infty$, we have that $\lambda \rightarrow 0$ and hence that (4) reduces to (3) in the non-relativistic limit. It is therefore natural to consider $0 \leq \lambda < 1$. Meanwhile, (1) is recovered when we set all three parameters $\tilde{v}, \nu, \lambda = 0$. Equation (4) is still solved subject to the boundary conditions (2). Physically, this relativistic model may be useful in understanding the screening process in relativistic dense astrophysical plasmas [4, 22].

Despite the interest in these generalized Thomas-Fermi models, solutions of these generalized models have not been studied, in contrast to the many solution methods which have been applied to the case of the standard Thomas-Fermi model. In the present paper, we study asymptotic solutions of the generalization which includes both nonextensive statistical mechanics and relativistic effects. One method which was quite useful for the construction of qualitatively useful asymptotic solutions to the standard Thomas-Fermi equation (1) was the δ -expansion method [11, 23, 24]. In [11], the method was used to obtain solutions to the boundary value problem (1)-(2), and we shall therefore apply the method to the more general problem of (4) with boundary conditions (2).

The paper is organized as follows. In Section 2, we apply the δ -expansion method to obtain a general asymptotic solution of (4) valid for various values of the parameters \tilde{v} , ν , and λ . As the critical slope $\Phi'(0)$ is of physical relevance (in particular, it tells us how rapidly the charge density will decay near the origin, which in turn will strongly influence the entire potential function for $x > 0$), in Section 3 we shall give an asymptotic approximation to its value for various combinations of the parameters \tilde{v} , ν , and λ . Finally, we shall give concluding remarks in Section 4.

2. Asymptotic solutions for (4)

In this section we shall apply the δ -expansion method to obtain asymptotic solutions for the generalized Thomas-Fermi equation (4). For examples of the method and more background on theoretical developments, see [11, 25, 26, 27, 28, 29]. To make use of this approach, we assume that $\Phi(x)$ can be written as a series in powers of δ [11]

$$\Phi(x) = \Phi_0(x) + \delta\Phi_1(x) + \delta^2\Phi_2(x) + O(\delta^3). \quad (5)$$

In applying this method (4) becomes

$$\frac{d^2\Phi}{dx^2} = \Phi \left(\frac{\Phi}{x} \right)^\delta \left(1 + \lambda \left(\frac{\Phi}{x} \right)^{2\delta} \right)^{\frac{3}{2}} \left\{ 1 + \tilde{v} \left(\frac{\Phi}{x} \right)^{-2\delta} \left(1 + \lambda \left(\frac{\Phi}{x} \right)^{2\delta} \right)^{-1} + \nu \left(\frac{\Phi}{x} \right)^{-4\delta} \left(1 + \lambda \left(\frac{\Phi}{x} \right)^{2\delta} \right)^{-2} \right\}. \quad (6)$$

Boundary conditions become

$$\Phi_0(0) = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \Phi_0(x) = 0 \quad (7)$$

and

$$\Phi_k(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \Phi_k(x) = 0 \quad \text{for} \quad k \geq 1. \quad (8)$$

Note that equation (4) is recovered if $\delta = \frac{1}{2}$ is substituted into equation (6). Therefore, we shall expand our approximation in a Taylor series in parameter δ centered about $\delta = 0$, and attempt to extend the result to $\delta = \frac{1}{2}$. If the approximation exists for $\delta = \frac{1}{2}$, then we will have obtained an approximation, up to the order of δ at which we truncate the series (5). In what follows we shall determine the first few functions $\Phi_k(x)$ in the expansion (5).

2.1. δ^0 solution

At $\delta = 0$ we obtain from (6) the zeroth order delta equation

$$\Phi_0''(x) = \Phi_0(1 + \lambda)^{\frac{3}{2}} \left\{ 1 + \tilde{v}(1 + \lambda)^{-1} + v(1 + \lambda)^{-2} \right\}. \quad (9)$$

Applying the boundary conditions $\Phi_0(0) = 1$ and $\Phi_0(\infty) = 0$ the solution is of the form

$$\Phi_0(x) = e^{-(1+\lambda)^{\frac{3}{2}} \sqrt{1 + \tilde{v}(1+\lambda)^{-1} + v(1+\lambda)^{-2}} x}. \quad (10)$$

Defining the constant

$$\Lambda(\tilde{v}, v, \lambda) = (1 + \lambda)^{\frac{3}{2}} \sqrt{1 + \tilde{v}(1 + \lambda)^{-1} + v(1 + \lambda)^{-2}}, \quad (11)$$

allows (10) to take the simpler form

$$\Phi_0(x) = e^{-\Lambda x}. \quad (12)$$

2.2. δ^1 solution

Substituting (5) into (6), expanding series in δ , and then matching powers of δ^1 , we obtain the second order ODE

$$\Phi_1''(x) - \Lambda^2 \Phi_1(x) = \Psi \Phi_0(x) \ln \frac{\Phi_0(x)}{x}, \quad (13)$$

with conditions $\Phi_1(0) = 0$ and $\Phi_1(\infty) = 0$. The constant Ψ is equal to

$$\Psi(\tilde{v}, v, \lambda) = \frac{4\lambda^3 + 9\lambda^2 + (6 - 4v - \tilde{v})\lambda - \tilde{v} - 3v + 1}{(1 + \lambda)^{\frac{3}{2}}}. \quad (14)$$

Substituting (12) into (13), the ODE becomes

$$\Phi_1''(x) - \Lambda^2 \Phi_1(x) = \Psi e^{-\Lambda x} \ln \frac{e^{-\Lambda x}}{x}. \quad (15)$$

This can be solved by assuming a solution of the form

$$\Phi_1(x) = e^{-\Lambda x} u_1(x). \quad (16)$$

Determining first and second derivatives of (16) and applying the conditions $\Phi_1(0) = u_1(0) = 0$ and $\Phi_1'(0) = u_1'(0)$ results in the equation

$$u_1''(x) - 2\Lambda u_1'(x) = \Psi \ln \frac{e^{-\Lambda x}}{x}, \quad (17)$$

which can be integrated once using the integrating factor $e^{-2\Lambda x}$ to yield

$$u_1'(x) = e^{2\Lambda x} \left(u_1'(0) + \Psi \int_0^x e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds \right). \quad (18)$$

Integrating (18) once more and substituting into (16) yields

$$\Phi_1(x) = e^{-\Lambda x} \int_0^x e^{2\Lambda t} \left(u_1'(0) + \Psi \int_0^t e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds \right) dt. \quad (19)$$

In order to satisfy the condition $\Phi_1(\infty) = 0$ the integrand of (19) must tend to zero, implying that

$$u_1'(0) = -\Psi \int_0^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds. \quad (20)$$

Substituting $u_1'(0)$ into (19) results in the first order correction

$$\Phi_1(x) = -\Psi e^{-\Lambda x} \int_0^x e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt. \quad (21)$$

2.3. δ^2 solution

The second order correction is found by substituting (5) into (6), expanding in δ , and then matching powers of δ^2 . In grouping like terms, the second order ODE becomes

$$\Phi_2''(x) - \Lambda^2 \Phi_2(x) = \Omega \left(\ln \frac{\Phi_0(x)}{x} \right)^2 \Phi_0(x) + \Psi \left(\ln \frac{\Phi_0(x)}{x} + 1 \right) \Phi_1(x), \quad (22)$$

where the constant Ω is

$$\Omega(\tilde{v}, \nu, \lambda) = \frac{16\lambda^4 + 46\lambda^3 + (2\tilde{v} + 16\nu + 45)\lambda^2 + (3\tilde{v} + 22\nu + 16)\lambda + \tilde{v} + 9\nu + 1}{2(1 + \lambda)^{\frac{5}{2}}}. \quad (23)$$

Substituting (12) and (21) into (22), and using $\Phi_2(0) = 0$ and $\Phi_2(\infty) = 0$, we obtain

$$\Phi_2''(x) - \Lambda^2 \Phi_2(x) = \Omega e^{-\Lambda x} \left(\ln \frac{e^{-\Lambda x}}{x} \right)^2 - \Psi^2 \left(\ln \frac{e^{-\Lambda x}}{x} + 1 \right) \left\{ e^{-\Lambda x} \int_0^x e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt \right\}. \quad (24)$$

Equation (24) can be solved by assuming the following solution with boundary conditions $\Phi_2(0) = u_2(0) = 0$ and $\Phi_2'(0) = u_2'(0)$,

$$\Phi_2(x) = e^{-\Lambda x} u_2(x). \quad (25)$$

Computing first and second derivatives of (25), substituting into (24), and canceling the common factor $e^{-\Lambda x}$ results in

$$u_2''(x) - 2\Lambda u_2'(x) = \Omega \left(\ln \frac{e^{-\Lambda x}}{x} \right)^2 - \Psi^2 \left(\ln \frac{e^{-\Lambda x}}{x} + 1 \right) \left\{ \int_0^x e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt \right\}. \quad (26)$$

Next, (26) is solved using the integrating factor $e^{-2\Lambda x}$ to get an equation for $u_2'(x)$:

$$u_2'(x) = e^{2\Lambda x} \int_0^\alpha \Omega e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 - \Psi^2 e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} + 1 \right) \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk. \quad (27)$$

This equation can be integrated again to find

$$u_2(x) = \int_0^x e^{2\Lambda \alpha} \left\{ \int_0^\alpha e^{-2\Lambda k} F(k) dk + u_2'(0) \right\} d\alpha, \quad (28)$$

where

$$F(k) = \Omega \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 - \Psi^2 \left(\ln \frac{e^{-\Lambda k}}{k} + 1 \right) \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt. \quad (29)$$

By substituting $u_2(x)$ from (28) into (25) results in the following equation

$$\Phi_2(x) = e^{-\Lambda x} \int_0^x e^{2\Lambda \alpha} \left\{ \int_0^\alpha \Omega e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 - \Psi^2 \left(\ln \frac{e^{-\Lambda k}}{k} + 1 \right) \left\{ \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk + u_2'(0) \right\} d\alpha. \quad (30)$$

To satisfy the condition $\Phi_2(\infty) = 0$, the integrand of (30) must tend to zero, implying

$$u'_2(0) = - \int_0^\infty \Omega e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 - \Psi^2 e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} + 1 \right) \left\{ \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk \right\}. \quad (31)$$

Finally, replacing the $u'_2(0)$ in (30) yields the second order correction

$$\Phi_2(x) = -e^{-\Lambda x} \int_0^x e^{2\Lambda \alpha} \left\{ \int_\alpha^\infty \Omega e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 - \Psi^2 e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} + 1 \right) \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk \right\} d\alpha. \quad (32)$$

2.4. Determining the solution $\Phi(x)$

The solution of $\Phi(x)$ to second order is found by substituting the zeroth, first, and second order corrections from equations (12), (21), and (32) into (5). Doing this, we obtain

$$\begin{aligned} \Phi(x) &= \Phi_0(x) + \delta \Phi_1(x) + \delta^2 \Phi_2(x) \\ &= e^{-\Lambda x} \left(1 - \delta \Psi \int_0^x e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt - \delta^2 \int_0^x e^{2\Lambda \alpha} \left\{ \int_\alpha^\infty \Omega e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 \right. \right. \\ &\quad \left. \left. - \Psi^2 e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} + 1 \right) \left\{ \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk \right\} d\alpha \right\} \right. \\ &\quad \left. + O(\delta^3) \right). \end{aligned} \quad (33)$$

Taking $\delta = 1/2$ in (33) therefore will give the second order approximation to the solution of (4). With this, we have determined the asymptotic solution, up to second order in δ . In what follows, we shall use this asymptotic solution to study the value of $\Phi'(0)$ as a function of the parameters \tilde{v} , v , and λ .

3. Approximation of the critical slope $\Phi'(0)$

The solution of $\Phi'(0)$ using the delta expansion method up to second order delta is given by

$$\Phi'(0) = \Phi'_0(0) + \delta \Phi'_1(0) + \delta^2 \Phi'_2(0) + O(\delta^3). \quad (34)$$

We shall use the δ -expansion of Section 2 in order to approximate $\Phi'(0)$. In order to obtain more accuracy in our approximations, we shall employ a Padé approximation in δ .

3.1. The value of $\Phi'_0(0)$

Computing the first derivative of $\Phi_0(x)$ and substituting $x = 0$ results in

$$\Phi'_0(0) = -\Lambda. \quad (35)$$

3.2. The value of $\Phi'_1(0)$

The first derivative of $\Phi_1(x)$ is

$$\Phi'_1(x) = -\Psi \left(-\Lambda e^{-\Lambda x} \int_0^x e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt \right) - \Psi \left(e^{-\Lambda x} e^{2\Lambda x} \int_x^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds \right), \quad (36)$$

and taking the limit $x \rightarrow 0$ in (36) yields

$$\begin{aligned} \Phi'_1(0) &= -\Psi \int_0^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds \\ &= -\frac{\Psi}{\Lambda} \int_0^\infty e^{-2\beta} \ln \frac{\Lambda e^{-\beta}}{\beta} d\beta \\ &= -\frac{\Psi}{\Lambda} \left(\ln \Lambda \int_0^\infty e^{-2\beta} d\beta + \int_0^\infty e^{-2\beta} \ln \frac{e^{-\beta}}{\beta} d\beta \right) \\ &= -\frac{\Psi}{4\Lambda} (2\gamma + 2 \ln 2\Lambda - 1). \end{aligned} \quad (37)$$

Here γ denotes the Euler-Mascharoni constant ($\gamma \approx 0.57721$) [30, 31].

3.3. The value of $\Phi'_2(0)$

The second order term of (34) is computed by taking the first derivative of (32) and evaluating as $x \rightarrow 0$. We may express this as

$$\Phi'_2(0) = -\Omega I_1 + \Psi^2 I_2, \quad (38)$$

where we define the integrals

$$\begin{aligned} I_1 &= \int_0^\infty e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 dk, \\ I_2 &= \int_0^\infty e^{-2\Lambda k} \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk. \end{aligned} \quad (39)$$

The integral I_1 is calculated as

$$\begin{aligned} I_1 &= \int_0^\infty e^{-2\Lambda k} \left(\ln \frac{e^{-\Lambda k}}{k} \right)^2 dk = \frac{1}{\Lambda} \int_0^\infty e^{-2\sigma} \left(\ln \frac{\Lambda e^{-\sigma}}{\sigma} \right)^2 d\sigma \\ &= \frac{1}{\Lambda} \left\{ \int_0^\infty e^{-2\sigma} \left(\ln \frac{e^{-\sigma}}{\sigma} \right)^2 d\sigma + 2 \ln \Lambda \int_0^\infty e^{-2\sigma} \left(\ln \frac{e^{-\sigma}}{\sigma} \right) d\sigma + (\ln \Lambda)^2 \int_0^\infty e^{-2\sigma} d\sigma \right\} \\ &= \frac{1}{\Lambda} \left\{ \frac{3}{4} - \frac{1}{2} \gamma - \frac{1}{2} \ln 2 + \frac{1}{2} \gamma^2 + \gamma \ln 2 + \frac{1}{2} (\ln 2)^2 + \frac{\pi^2}{12} + \ln \Lambda (\gamma + \ln 2 - \frac{1}{2}) + \frac{1}{2} (\ln \Lambda)^2 \right\}. \end{aligned} \quad (40)$$

Evaluating the second integral in (39) requires more work, and we find

$$I_2 = \int_0^\infty e^{-2\Lambda k} \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) \int_0^k e^{2\Lambda t} \int_t^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dt dk = I_{21} + I_{22} + I_{23}, \quad (41)$$

where

$$\begin{aligned} I_{21} &= \frac{1}{2\Lambda} \int_0^\infty \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) \int_k^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds dk, \\ I_{22} &= -\frac{1}{2\Lambda} \left(\int_0^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds \right) \left(\int_0^\infty e^{-2\Lambda k} \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) dk \right), \\ I_{23} &= \frac{1}{2\Lambda} \int_0^\infty e^{-2\Lambda k} \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) \int_0^k \ln \frac{e^{-\Lambda s}}{s} ds dk. \end{aligned} \quad (42)$$

For I_{21} , we have (see Appendix A, (A.6))

$$\begin{aligned} I_{21} &= \frac{1}{2\Lambda^3} \int_0^\infty \left(1 + \ln \frac{\Lambda e^{-m}}{m} \right) \int_m^\infty e^{-2\sigma} \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dm \\ &= \frac{1}{2\Lambda^3} \int_0^\infty \int_m^\infty e^{-2\sigma} \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dm + \frac{1}{2\Lambda^3} \int_0^\infty \ln \frac{\Lambda e^{-m}}{m} \int_m^\infty e^{-2\sigma} \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dm \\ &= \frac{1}{8\Lambda^3} (\ln \Lambda + \gamma - 2 + \ln 2) + \frac{1}{2\Lambda^3} \int_0^\infty \ln \frac{\Lambda e^{-m}}{m} \int_m^\infty e^{-2\sigma} \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dm. \end{aligned} \quad (43)$$

Note (see Appendix A, (A.2) and (A.3))

$$\begin{aligned} &\int_0^\infty \ln \frac{\Lambda e^{-m}}{m} \int_m^\infty e^{-2\sigma} \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dm \\ &= \int_0^\infty m e^{-2m} \left(\ln \frac{\Lambda e^{-m}}{m} \right)^2 dm + \frac{1}{2} \int_0^\infty m^2 e^{-2m} \ln \frac{\Lambda e^{-m}}{m} dm + \int_0^\infty m e^{-2m} \left(\ln \frac{\Lambda e^{-m}}{m} \right) dm \\ &= \int_0^\infty m e^{-2m} \left(\ln \frac{e^{-m}}{m} + \ln \Lambda \right)^2 dm + \frac{1}{8} (\ln \Lambda + \ln 2 - 3 + \gamma) + \frac{1}{4} (\ln \Lambda + \ln 2 - 2 + \gamma). \end{aligned} \quad (44)$$

The remaining integral in equation (44) evaluates to (see Appendix A, (A.2) and (A.5))

$$\begin{aligned} \int_0^\infty m e^{-2m} \left(\ln \frac{e^{-m}}{m} + \ln \Lambda \right)^2 dm &= \int_0^\infty m e^{-2m} \left(\ln \frac{e^{-m}}{m} \right)^2 dm + 2 \ln \Lambda \int_0^\infty m e^{-2m} \ln \frac{e^{-m}}{m} dm + (\ln \Lambda)^2 \int_0^\infty m e^{-2m} dm \\ &= \frac{9 - 8\gamma - 4(2 - \gamma) \ln 2 + 2(\ln 2)^2 + 2\gamma^2}{8} + \frac{\pi^2}{24} + 2 \ln \Lambda \left(\frac{\ln 2 + \gamma - 2}{4} \right) + \frac{(\ln \Lambda)^2}{4}. \end{aligned} \quad (45)$$

Making use of these integrals, we find the closed-form expression for I_{21} to be

$$\begin{aligned} I_{21} &= \frac{1}{8\Lambda^3} (\ln \Lambda + \gamma - 2 + \ln 2) + \frac{1}{2\Lambda^3} \left\{ \frac{9 - 8\gamma - 4(2 - \gamma) \ln 2 + 2(\ln 2)^2 + 2\gamma^2}{8} + \frac{\pi^2}{24} \right. \\ &\quad \left. + \frac{\ln \Lambda}{2} (\ln 2 + \gamma - 2) + \frac{(\ln \Lambda)^2}{4} + \frac{1}{8} (\ln \Lambda + \ln 2 - 3 + \gamma) + \frac{1}{4} (\ln \Lambda + \ln 2 - 2 + \gamma) \right\}. \end{aligned} \quad (46)$$

For I_{22} , we find

$$\begin{aligned} I_{22} &= -\frac{1}{2\Lambda} \left(\int_0^\infty e^{-2\Lambda s} \ln \frac{e^{-\Lambda s}}{s} ds \right) \left(\int_0^\infty e^{-2\Lambda k} \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) dk \right) \\ &= -\frac{1}{2\Lambda^3} \left(\int_0^\infty e^{-2m} \ln \frac{\Lambda e^{-m}}{m} dm \right) \left(\int_0^\infty e^{-2m} \left(1 + \ln \frac{\Lambda e^{-m}}{m} \right) dm \right). \end{aligned} \quad (47)$$

Evaluating the integrals and simplifying (see Appendix A, (A.1)), we have

$$I_{22} = \frac{1 - 4(\gamma + \ln 2 + \ln \Lambda)^2}{32\Lambda^3}. \quad (48)$$

Finally, for I_{23} of equation (42), we have

$$\begin{aligned} I_{23} &= \frac{1}{2\Lambda^2} \int_0^\infty e^{-2\Lambda k} \left(1 + \ln \frac{e^{-\Lambda k}}{k} \right) \int_0^{\Lambda k} \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dk \\ &= \frac{1}{2\Lambda^3} \int_0^\infty e^{-2m} \left(1 + \ln \frac{\Lambda e^{-m}}{m} \right) \int_0^m \ln \frac{\Lambda e^{-\sigma}}{\sigma} d\sigma dm, \end{aligned} \quad (49)$$

when evaluates to (see Appendix A, (A.1))

$$I_{23} = \frac{1}{16\Lambda^3} \left(2(\ln 2)^2 + \ln \Lambda (4\gamma + 4 \ln 2 - 3) + (4 \ln 2 - 3)\gamma + 2\gamma^2 - 3 \ln 2 + 2(\ln 2)^2 + \frac{\pi^2}{3} + 1 \right). \quad (50)$$

Making use of all of these integrations, we recover

$$\begin{aligned} \Phi'_2(0) &= -\frac{\Omega}{\Lambda} \left\{ \frac{1}{2} (\gamma^2 - \gamma + \ln \Lambda^2 - \ln \Lambda + \ln 2^2 - \ln 2) + \gamma \ln \Lambda + \gamma \ln 2 + \ln \Lambda \ln 2 + \frac{\pi^2}{12} + \frac{3}{4} \right\} \\ &\quad + \frac{\Psi^2}{\Lambda^3} \left\{ \frac{1}{8} (\gamma^2 + (\ln 2)^2 + (\ln \Lambda)^2 - 3\gamma - 3 \ln 2 - 3 \ln \Lambda) + \frac{1}{4} (\gamma \ln 2 + \gamma \ln \Lambda + \ln \Lambda \ln 2) + \frac{\pi^2}{24} - \frac{1}{32} \right\}. \end{aligned} \quad (51)$$

3.4. The approximation of $\Phi'(0)$ to order δ^2

Making use of the terms calculated earlier in this section, we obtain

$$\begin{aligned} \Phi'(0) &= -\Lambda - \delta \frac{\Psi}{4\Lambda} (2\gamma + 2 \ln 2\Lambda - 1) \\ &\quad - \delta^2 \frac{\Omega}{\Lambda} \left\{ \frac{1}{2} (\gamma^2 - \gamma + \ln \Lambda^2 - \ln \Lambda + \ln 2^2 - \ln 2) + \gamma \ln \Lambda + \gamma \ln 2 + \ln \Lambda \ln 2 + \frac{\pi^2}{12} + \frac{3}{4} \right\} \\ &\quad - \delta^2 \frac{\Psi^2}{\Lambda^3} \left\{ \frac{1}{8} (\gamma^2 + (\ln 2)^2 + (\ln \Lambda)^2 - 3\gamma - 3 \ln 2 - 3 \ln \Lambda) + \frac{1}{4} (\gamma \ln 2 + \gamma \ln \Lambda + \ln \Lambda \ln 2) + \frac{\pi^2}{24} - \frac{1}{32} \right\} \\ &\quad + O(\delta^3). \end{aligned} \quad (52)$$

(m, n)	$\Phi'(0)$	Relative Error
(1, 2)	-1.429795	9.97%
(1, 3)	-1.347813	15.1%
(1, 4)	-1.373448	13.5%
(1, 5)	-1.388622	12.6%
(2, 1)	-1.384283	12.8%
(2, 2)	-1.384283	12.8%
(3, 1)	-1.384283	12.8%
(3, 2)	-1.384283	12.8%
(4, 1)	-1.384283	12.8%
(4, 2)	-1.384283	12.8%
(5, 1)	-1.384283	12.8%

Table 1: (m,n) - Padé approximation values of $\Phi'(0)$ for $\nu = \tilde{\nu} = \lambda = 0$. We compare these to the “exact” value of $\Phi'(0) = -1.5880710$ [11], and list the relative error. While these errors are not particularly low, note that they are low enough for us to extract qualitative information about the solutions. Similar error was obtained in [11] in the study of the standard Thomas-Fermi equation (1).

If we evaluate at $\delta = 1/2$, we will recover the desired solution. However, although we have obtained the first three terms in the approximation to $\Phi'(0)$, note that the accuracy of the approximation can still be improved if we take an approximation using a specific Padé approximant (for background and examples of Padé approximants, see [32]) in δ . This allows us to obtain a rational function approximation, as opposed to a polynomial approximation, which can sometimes improve the accuracy of the approximate solutions. This approach was discussed for the standard Thomas-Fermi equation (1) in [11, 8].

To begin with, Padé approximants of (52) were computed with $\tilde{\nu}, \nu, \lambda = 0$, so that we can compare the approximation with known results. In particular, the results can be compared with the correct value of the standard Thomas-Fermi equation $\Phi'(0) = -1.5880710$ [11]. The results of the various Padé approximations are displayed in Table 1. From Table 1, we observe that the (1,2) - Padé approximation gives the most accurate result, relative to the known value. Although approximate or numerical solutions do not exist in the literature for non-zero values of $\nu, \tilde{\nu}$, or λ , we hypothesize that the same Padé approximation will be useful for those cases, as well, and hence we keep the (1,2) - Padé approximant as the rational approximation of choice for all parameter values. This makes sense, as the parameters in the problem should all remain sufficiently small.

We now use the (1,2) - Padé approximation to calculate $\Phi'(0)$ for various values of $\nu < \tilde{\nu} < 1$ and $0 < \lambda < 1$. Recall that $\lambda \propto \frac{1}{c^2}$ is the relativistic parameter, $\nu \propto T^2$ is the thermal-correction term, and $\tilde{\nu} \propto T$ is the Tsallis correction term [21]. These results are listed in Table 2.

4. Discussion

From the results listed in Table 2, we find that $-\Phi'(0) > 0$ increases with an increase in λ or in ν . The behavior in $\tilde{\nu}$ is less clear. When λ is sufficiently small, $-\Phi'(0)$ increases with an increase $\tilde{\nu}$, while for large values of λ , $-\Phi'(0)$ decreases with an increase in $\tilde{\nu}$. For intermediate values of λ , the relation between $-\Phi'(0)$ and $\tilde{\nu}$ is not monotone.

Since $\Psi'(0)$ determines the shape of the charge density function as it tends from unity at $x = 0$ to zero as $x \rightarrow \infty$, there is a sharp decline moving away from the origin when either the standard thermal correction parameter, ν , or the relativistic correction parameter, λ , is increased. On the other hand, the behavior of the solution with the Tsallis correction term is more ambiguous. If relativistic effects are small, an increase in the Tsallis correction term results in a more sharp decline of the solution profiles moving away from the origin. This is the regime where (3) is valid, and hence the non-relativistic generalization of the Thomas-Fermi model (3) has solutions which more sharply decline in density moving away from the origin, if either of the two correction terms are increased in size. However, when the relativistic correction becomes more prominent, firmly away from the regime where (3) is valid, we notice a reversal of this behavior with an increase in the Tsallis correction term.

If we consider the neutron star or relativistic dense astrophysical plasmas applications, an increase in the relativistic correction λ results in more density near the origin (due to the sharper drop-off, $\Phi'(0)$), which means a tighter region

$\tilde{\nu}$	ν	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$
0.00	0.00	1.430	2.134	3.284	5.430	10.678
0.05	0.00	1.463	2.166	3.303	5.388	10.304
0.10	0.00	1.494	2.196	3.321	5.350	9.978
0.15	0.00	1.523	2.225	3.337	5.315	9.693
0.20	0.00	1.551	2.252	3.354	5.188	9.442
0.25	0.00	1.578	2.278	3.370	5.257	9.221
0.30	0.00	1.604	2.304	3.386	5.233	9.025
0.50	0.00	1.699	2.399	3.448	5.165	8.431
0.70	0.00	1.787	2.487	3.513	5.134	8.046
0.90	0.00	1.869	2.572	3.580	5.130	7.791
0.10	0.05	1.651	2.451	3.774	6.308	12.910
0.15	0.05	1.678	2.475	3.779	6.230	12.359
0.20	0.05	1.704	2.498	3.784	6.161	11.887
0.25	0.05	1.729	2.521	3.790	6.010	11.480
0.30	0.05	1.754	2.543	3.797	6.045	11.126
0.50	0.05	1.773	2.629	3.832	5.883	10.089
0.70	0.05	1.933	2.712	3.878	5.788	9.437
0.90	0.05	2.016	2.795	3.934	5.740	9.008
0.15	0.10	1.862	2.783	4.351	7.528	17.089
0.20	0.10	1.887	2.801	4.340	7.391	16.083
0.25	0.10	1.910	2.819	4.331	7.271	15.248
0.30	0.10	1.934	2.837	4.324	7.164	14.546
0.50	0.10	2.023	2.910	4.318	6.845	12.599
0.70	0.10	2.108	2.986	4.337	6.649	11.447
0.90	0.10	2.192	3.065	4.374	6.532	10.711
0.20	0.15	2.111	3.185	5.088	9.250	24.981
0.25	0.15	2.132	3.196	5.055	9.015	22.812
0.30	0.15	2.154	3.208	5.027	8.810	21.105
0.50	0.15	2.239	3.264	4.955	8.207	16.852
0.70	0.15	2.324	3.329	4.931	7.835	14.612
0.90	0.15	2.410	3.403	4.940	7.601	13.265
0.25	0.20	2.413	3.695	6.081	11.900	45.814
0.30	0.20	2.432	3.697	6.015	11.479	38.851
0.50	0.20	2.512	3.724	5.829	10.287	25.646
0.70	0.20	2.596	3.773	5.733	9.575	20.335
0.90	0.20	2.686	3.839	5.694	9.127	17.527
0.30	0.25	2.801	4.373	7.512	16.572	264.721
0.50	0.25	2.872	4.352	7.109	13.866	54.672
0.70	0.25	2.955	4.373	6.878	12.384	33.850
0.90	0.25	3.049	4.423	6.753	11.487	26.083
0.50	0.30	3.370	5.263	9.167	21.499	-
0.70	0.30	3.451	5.230	8.653	17.699	105.248
0.90	0.30	3.551	5.252	8.353	15.634	52.060
0.70	0.50	14.699	39.330	-	-	-
0.90	0.50	14.897	31.823	-	-	-

Table 2: Approximate values of $-\Phi'(0)$ for various combinations of the parameters $\tilde{\nu}$, ν , and λ . These results were obtained by way of the (1,2) - Padé approximation evaluated at $\delta = 1/2$. We use '-' to denote values for which the solutions likely loose validity and are no longer accurate.

of mass. Similarly, in the regime where $-\Phi'(0)$ increases with an increase in either the Tsallis or standard thermal correction, this results in a tighter density. For such applications, a higher temperature would therefore correspond to a tighter density distribution.

Of course, one could attempt other solution methods, as there are certainly more accurate solution methods available for the standard Thomas-Fermi model (1). However, these solution methods are often very involved or computationally depending. For the sake of our interests, and since this is the first study on solution behavior, a qualitative study of the generalized Thomas-Fermi model (4) solutions via the δ -expansion method is sufficient. There are also a number of mathematically interesting questions, such as whether solutions to (4) will exist for any parameter values or only over a fixed range of parameters values, or if they are parameter regimes with multiple or non-unique solutions possible. Since (4) is nonlinear, involves a singularity at $x = 0$, and is solved as a boundary value problem over $0 < x < \infty$, the rigorous analysis of solutions to (4) will be challenging, and could motivate future interesting work.

Appendix A. Calculation of integrals

We list the special integral formulas used in the calculation of $\Phi'(0)$:

$$\int_0^\infty e^{-2\sigma} \ln \frac{e^{-\sigma}}{\sigma} d\sigma = \frac{2\gamma - 1 + 2 \ln 2}{4} \approx 0.38519, \quad (\text{A.1})$$

$$\int_0^\infty \sigma e^{-2\sigma} \ln \frac{e^{-\sigma}}{\sigma} d\sigma = \frac{\gamma - 2 + \ln 2}{4} \approx -0.18241, \quad (\text{A.2})$$

$$\int_0^\infty \sigma^2 e^{-2\sigma} \ln \frac{e^{-\sigma}}{\sigma} d\sigma = \frac{-3 + \gamma + \ln 2}{4} \approx -0.43241, \quad (\text{A.3})$$

$$\int_0^\infty e^{-2\sigma} \left(\ln \frac{e^{-\sigma}}{\sigma} \right)^2 d\sigma = \frac{3}{4} - \frac{\ln 2}{2} + \frac{\ln^2 2}{2} + \frac{\pi^2}{12} + \left(\ln 2 - \frac{1}{2} \right) \gamma + \frac{\gamma^2}{2} \approx 1.7442, \quad (\text{A.4})$$

$$\int_0^\infty \sigma e^{-2\sigma} \left(\ln \frac{e^{-\sigma}}{\sigma} \right)^2 d\sigma = \frac{9}{8} - \ln 2 + \frac{\ln^2 2}{4} + \frac{\pi^2}{24} + \left(\frac{\ln 2}{2} - 1 \right) \gamma + \frac{\gamma^2}{4} \approx 0.66934, \quad (\text{A.5})$$

$$\int_0^\infty \int_\tau^\infty e^{-2\sigma} \ln \frac{e^{-\sigma}}{\sigma} d\sigma d\tau = -\frac{2 - \gamma - \ln 2}{4} \approx -0.18241, \quad (\text{A.6})$$

$$\int_0^\infty e^{-2\tau} \int_0^\tau \ln \frac{e^{-\sigma}}{\sigma} d\sigma d\tau = \frac{2\gamma - 1 + 2 \ln 2}{8} \approx 0.19259, \quad (\text{A.7})$$

$$\int_0^\infty e^{-2\tau} \ln \frac{e^{-\tau}}{\tau} \int_0^\tau \ln \frac{-\sigma}{\sigma} d\sigma d\tau = \frac{1}{4} - \frac{5 \ln 2}{8} + \frac{\ln^2 2}{4} + \frac{\pi^2}{24} + \left(\frac{\ln 2}{2} - \frac{5}{8} \right) \gamma + \frac{\gamma^2}{4} \approx 0.27072. \quad (\text{A.8})$$

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