



RBF multiscale collocation for second order elliptic boundary value problems

by

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RBF MULTISCALE COLLOCATION FOR SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

PATRICIO FARRELL* AND HOLGER WENDLAND†

Abstract. In this paper, we discuss multiscale radial basis function collocation methods for solving elliptic partial differential equations on bounded domains. The approximate solution is constructed in a multi-level fashion, each level using compactly supported radial basis functions of smaller scale on an increasingly fine mesh. On each level, standard symmetric collocation is employed. A convergence theory is given, which builds on recent theoretical advances for multiscale approximation using compactly supported radial basis functions. We are able to show that the convergence is linear in the number of levels. We also discuss the condition numbers of the arising systems and the effect of simple, diagonal preconditioners, now proving rigorously previous numerical observations.

Key words. partial differential equation, multiscale collocation, radial basis functions

AMS subject classifications. 65N35, 65N55

1. Introduction. Multiscale phenomena occur in such diverse disciplines as fluid flow, weather forecasting, geophysics, plasma physics and even operations research. However, in real-life applications the data are often unstructured and randomly scattered in space, which makes it hard to build natural multiscale spaces based on this scattered information.

For such data sets, approximation methods using positive definite functions or more specifically *radial basis functions (RBF)* have proven particularly powerful, see [3, 26]. The main reason is that these methods can handle scattered data without the expensive generation of a mesh. Apart from its relatively well studied applications to multivariate interpolation, RBF techniques have extensively been used to solve partial differential equations as well, see for example [14, 5, 9, 10, 28]. While Galerkin methods have been suggested ([25]), the predominant method for solving partial differential equations using radial basis functions is based upon collocation. The advantage here is that no numerical integration is required to set up the linear system of the discretised equations. In general, collocation is also easier to implement when it comes to nonlinear problems. Though collocation requires higher regularity of the solution than a Galerkin approach, radial basis functions usually target problems with smoother solutions anyway since they easily allow to build smooth approximation spaces and can be considered as high-order methods, see also [7, 8].

Though designed as high-order methods, radial basis functions have to deal with the problem of full, ill-conditioned discretisation matrices, see [20, 27]. To overcome this problem different strategies have been devised. Here, we want to discuss a method related to *multiscale approximation*.

Multiscale or multilevel approximation by compactly supported radial basis functions employs the shifts of differently scaled compactly supported basis functions. The obvious idea behind it is that the shifts of a coarse function capture the “low frequencies” of the target functions, while increasingly finer scaled basis functions capture more and more details. These method were numerically introduced for pure function reconstruction in [6, 19]. They showed remarkable results for pure function reconstruction, such as good approximation properties and a constant condition number

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on each level, being independent of the mesh size. Despite some theoretical work in [17, 12], proofs of these remarkable properties were only recently given – first on the sphere in [15, 16] and then on bounded domains in [29].

Shortly after the initial RBF multilevel method had been introduced, the multilevel idea was also employed in the context of solving partial differential equations by collocation, see [5]. However, here the numerical results were diverse. To be more precise, in [5], the following observations were made:

- There is no convergence in the stationary setting, i.e. if the support radius at a given level is chosen proportional to the mesh norm of that level.
- There is convergence, if the support radii go slower to zero than the mesh norms.
- In contrast to pure interpolation, even in the stationary setting, the condition numbers of the collocation matrices depend on the level.
- In the stationary case, a simple preconditioning $PAP\alpha = Py$ with a diagonal matrix P leads to a level-independent condition number.
- This preconditioning technique does not lead to a level-independent condition number in the non-stationary setting, where convergence occurs.

It is the goal of this paper to rigorously prove Fasshauer’s numerical observations for multiscale RBF collocation. We will rely on ideas of the papers previously mentioned and the paper [16], which discussed multiscale RBF collocation on the sphere. However, in this paper, we have to address the issue of bounded domains and boundary value problems which was not the case on the sphere.

Finally, the multilevel strategy presented in this paper is built on “full” levels, which is particularly interesting for data sets containing a limited range of different scales. If very different scales are present, full levels may become too expensive. Here, the combination with adaptive strategies (see for example [22, 4]) should be an alternative.

After quickly introducing some of the notation we use, we will discuss in the second section generalised interpolation. In the third section, we introduce the multilevel collocation algorithm. Section 4 is then dedicated to our main convergence result. Section 5 is devoted to the analysis of the condition number of the involved collocation matrices. Though [5] already contains several numerical examples, we end this paper by providing an additional example with special emphasis on our theoretical findings.

1.1. Notation. We start by introducing some notation. We measure the error of arbitrarily scattered data in terms of the so called mesh norm and the separation distance. For a bounded domain $\Omega \subseteq \mathbb{R}^d$ and data points $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ these measures are defined as follows

$$h_{X,\Omega} = \sup_{x \in \Omega} \min_{\mathbf{x}_j \in X} \|\mathbf{x} - \mathbf{x}_j\|_2,$$

$$q_X = \min_{j \neq k} \|\mathbf{x}_j - \mathbf{x}_k\|_2.$$

That is, the mesh norm measures the radius of the largest data-free hole contained in Ω and the separation distance is the smallest distance between two points in X . Sometimes we will simplify the notation to h and q if it is obvious from the context what is meant.

Moreover, for non-negative integer k and $1 \leq p \leq \infty$ let $W_p^k(\Omega)$ denote the Sobolev space with differentiability order k and integrability power p . Define for $u \in W_p^k(\Omega)$

and finite p the Sobolev (semi-)norms

$$|u|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p} \quad \text{and} \quad \|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p}.$$

For $p = \infty$ the Sobolev (semi-)norm is given by

$$|u|_{W_\infty^k(\Omega)} = \sup_{|\alpha|=k} \|D^\alpha u\|_{L_\infty(\Omega)} \quad \text{and} \quad \|u\|_{W_\infty^k(\Omega)} = \sup_{|\alpha| \leq k} \|D^\alpha u\|_{L_\infty(\Omega)}.$$

In the the case $p = 2$, we have a Hilbert space and can also introduce a norm via Fourier transforms which has the advantage that it can be generalised to non-integer values $0 < \sigma < \infty$ and yields an equivalent norm to the one defined above if we choose σ to be an integer. We can describe the functions in the fractional Sobolev space $W_2^\sigma(\mathbb{R}^d)$ as precisely those square-integrable functions that are finite in the norm

$$\|f\|_{W_2^\sigma(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\sigma d\boldsymbol{\omega}.$$

Here, \widehat{f} is the usual Fourier transform

$$\widehat{f}(\boldsymbol{\omega}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i\mathbf{x}^T \boldsymbol{\omega}} d\mathbf{x}.$$

Throughout this paper, the index σ will indicate fractional or non-fractional order of smoothness, while the index k will be reserved for integer order smoothness only.

Fractional Sobolev spaces on bounded domains can also be introduced in different ways. One way would be interpolation between integer order Sobolev spaces, see [1]. Another one would simply define

$$W_p^\sigma(\Omega) = \{f|_\Omega : f \in W_p^\sigma(\mathbb{R}^d)\}.$$

These definitions are equivalent under moderate assumption on the domain; to be more precise, this is in particular the case if the domain allows an extension operator, see Lemma 3.4.

It is obvious that when discussing boundary value problems, we need to treat the boundary differently from the interior. In particular, we need to define the Sobolev norm on the boundary of Ω . In order to do this, we need to impose some smoothness restrictions on the boundary. Let us assume Ω has a $C^{k,s}$ -boundary where $k \in \mathbb{N}_0$ and $s \in [0, 1)$. A proper definition can be found in [30, Definition 2.7].

If we assume that Ω has a $C^{k,s}$ -boundary, then we are able to represent the boundary $\partial\Omega$ by a finite number of $C^{k,s}$ -diffeomorphisms

$$\psi_j : B \rightarrow V_j \quad \text{for} \quad 1 \leq j \leq K,$$

where $B = B(\mathbf{0}, 1)$ is the unit ball in \mathbb{R}^{d-1} and the $V_j \subseteq \mathbb{R}^d$ are open sets such that $\partial\Omega \subset \bigcup_{j=1}^K V_j$. With the help of a partition of unity $\{\omega_j\}$ with respect to V_j we can then define the Sobolev norm on the boundary to be

$$\|u\|_{W_p^\sigma(\partial\Omega)}^p = \sum_{j=1}^K \|(u\omega_j) \circ \psi_j\|_{W_p^\sigma(B)}^p.$$

Even though in our case the atlas is fixed, it is noteworthy that in general the definition is independent of the chosen atlas $\{V_j, \psi_j\}$, in the sense that a different atlas would yield a norm equivalent to the one above. Introducing the Sobolev norm on the boundary via charts, enables us also to define the mesh norm on the boundary. For some point set $Y \subseteq \partial\Omega$, we define

$$h_{Y, \partial\Omega} = \max_{1 \leq j \leq K} h_{T_j, B}, \quad (1.1)$$

where $T_j = \psi_j^{-1}(Y \cap V_j) \subseteq B$. Lastly, we introduce notation for the floor function $\lfloor x \rfloor$ which yields the largest integer less than or equal to $x \in \mathbb{R}$.

2. Generalised interpolation. In the context of (generalised) interpolation, reproducing kernel Hilbert spaces play an important role. Suppose $H \subseteq C(\Omega)$ denotes a real Hilbert space of continuous functions $f: \Omega \rightarrow \mathbb{R}$, then $\Phi: \Omega \times \Omega \rightarrow \mathbb{R}$ is said to be a reproducing kernel for H if

- (1) $\Phi(\cdot, \mathbf{y}) \in H$ for all $\mathbf{y} \in \Omega$,
- (2) $f(\mathbf{y}) = (f, \Phi(\cdot, \mathbf{y}))_H$ for all $f \in H$ and all $\mathbf{y} \in \Omega$.

Suppose we are given N linearly independent functionals $\lambda_1, \dots, \lambda_N$ in the dual space of H as well as $f \in H$, of which we only know discrete values $f_j = \lambda_j(f) \in \mathbb{R}$ for $1 \leq j \leq N$. For an interpolant of the form

$$s_{f, \Lambda}(\mathbf{x}) = \sum_{j=1}^N \alpha_j \lambda_j^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y}), \quad (2.1)$$

we can then show well-posedness of the generalised interpolation problem as well as a minimisation property.

THEOREM 2.1. *There is a unique interpolant of the form (2.1) which solves the generalised interpolation problem,*

$$\lambda_j(s_{f, \Lambda}) = f_j, \quad 1 \leq j \leq N, \quad (2.2)$$

and is norm-minimal in the Hilbert space norm, i. e.

$$\|s_{f, \Lambda}\|_H \leq \|s\|_H$$

for all $s \in H$ with $\lambda_j(s) = f_j$ for $1 \leq j \leq N$.

For a proof of this theorem see for example [26, Theorem 16.1]. If the functionals are chosen to be the point evaluation functionals at the data sites, $\lambda_j = \delta_{\mathbf{x}_j}$, then the generalized interpolation condition (2.2) reduces to (pure) interpolation.

The reproducing property is so important to (generalised) interpolation because every positive definite kernel generates a reproducing kernel Hilbert space, the so called native space $\mathcal{N}_{\Phi}(\Omega)$ of the kernel Φ . For a translation invariant kernel, i. e. a kernel with the property $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$, the native space $\mathcal{N}_{\Phi}(\mathbb{R}^d)$ consists of all functions $g \in L_2(\mathbb{R}^d)$ with

$$\|g\|_{\Phi}^2 = \int_{\mathbb{R}^d} \frac{|\widehat{g}(\boldsymbol{\omega})|^2}{\widehat{\Phi}(\boldsymbol{\omega})} d\boldsymbol{\omega} < \infty,$$

see [26, Theorem 10.12]. Assuming additionally $\sigma > d/2$ as well as the algebraic decay condition of the Fourier transform

$$c_1(1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma} \leq \widehat{\Phi}(\boldsymbol{\omega}) \leq c_2(1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma}, \quad \boldsymbol{\omega} \in \mathbb{R}^d, \quad (2.3)$$

for two fixed constants $0 < c_1 \leq c_2$, the native space is norm-equivalent to the Sobolev space $W_2^\sigma(\mathbb{R}^d)$. This is due to two facts: firstly, by the Sobolev embedding theorem the condition on the differentiability order implies that the Sobolev space is a subset of the continuous functions and, secondly, the algebraic decay condition leads to norm equivalence between the native space and the fractional Sobolev norm.

For the rest of our discussion, we will consider translation-invariant kernels with Fourier transforms which satisfy (2.3). Examples of such functions are the compactly supported basis functions, that are commonly referred to as Wendland's functions, see [24]. By construction they are radial and translation-invariant. The algebraic decay of their Fourier transform is proven in [26, Theorem 10.35].

We obtain the coefficients of the interpolant (2.1) by applying (2.2) to it and then solving the linear system

$$\begin{pmatrix} \lambda_1^x \lambda_1^y \Phi(\mathbf{x} - \mathbf{y}) & \dots & \lambda_1^x \lambda_N^y \Phi(\mathbf{x} - \mathbf{y}) \\ \vdots & & \vdots \\ \lambda_N^x \lambda_1^y \Phi(\mathbf{x} - \mathbf{y}) & \dots & \lambda_N^x \lambda_N^y \Phi(\mathbf{x} - \mathbf{y}) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}. \quad (2.4)$$

We abbreviate the system with

$$A_\Lambda \boldsymbol{\alpha} = \mathbf{f}.$$

We will look at functionals that consist of a combination of point evaluation functionals as well as a linear PDE operator composed with point evaluation functionals. With the help of these functionals we will approximate the solution of an elliptic boundary value problem of second order through collocation.

We will need to be able to bound weaker Sobolev norms by stronger ones in terms of the mesh norm $h_{X,\Omega}$. This is achieved by so called *sampling inequalities*. The first of these was proven in [18]. We will need a version of the sampling inequality which allows fractional values for the integrability power. This can be achieved by interpolation of Sobolev spaces. The following modified sampling inequality can be found in [28, Theorem 4.6].

THEOREM 2.2 (Sampling inequality for fractional norms). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. Let $1 < p, q < \infty$ and let $\tau > d/p$. Let $X \subseteq \Omega$ be a discrete set with a sufficiently small mesh norm $h_{X,\Omega}$. For each $u \in W_p^\tau(\Omega)$ with $u|_X = 0$, we have for $0 \leq \sigma \leq \tau - d(1/p - 1/q)_+$ the estimate*

$$|u|_{W_q^\sigma(\Omega)} \leq C h_{X,\Omega}^{\tau - \sigma - d(1/p - 1/q)_+} |u|_{W_p^\tau(\Omega)},$$

where $C > 0$ is a constant independent of u and $h_{X,\Omega}$.

3. Multiscale analysis. In this section, we define the elliptic operator that we use in this paper. After describing the collocation method, we state the multilevel collocation algorithm.

3.1. Second order elliptic boundary value problems. For a solution $u \in W_2^\sigma(\Omega)$, we consider a boundary value problem of the form

$$Lu = f \quad \text{in } \Omega, \quad (3.1)$$

$$u = F \quad \text{on } \partial\Omega, \quad (3.2)$$

where L is an elliptic linear differential operator of second order

$$Lu(\mathbf{x}) = \sum_{i,j=1}^d a_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u(\mathbf{x}) + \sum_{i=1}^d b_i(\mathbf{x}) \frac{\partial}{\partial x_i} u(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}), \quad (3.3)$$

which is strictly elliptic on Ω . That is, there exists a constant $c_E > 0$ such that

$$c_E \|\boldsymbol{\xi}\|_2^2 \leq \sum_{i,j=1}^d a_{ij}(\mathbf{x}) \xi_i \xi_j$$

for all $\mathbf{x} \in \Omega$ and $\boldsymbol{\xi} \in \mathbb{R}^d$.

If we assume that $u \in W_2^\sigma(\Omega)$ with $\sigma > d/2 + 2$, then Lu is in fact well-defined since we know by the Sobolev embedding theorem (see e.g. [2]) that $W_2^\sigma(\Omega) \subseteq C^2(\Omega)$.

So far we have not imposed any restrictions on the coefficients. If we assume for $k := \lfloor \sigma \rfloor > 2 + d/2$ that $a_{ij}, b_i, c \in W_\infty^{k-1}(\Omega)$, then one can verify that L is a bounded operator from $W_2^\sigma(\Omega)$ to $W_2^{\sigma-2}(\Omega)$, a fact we state more carefully in Lemma 3.1. However, since $(k-1) - 1 > d/2$, we can again use the Sobolev embedding theorem to conclude that $W_\infty^{k-1}(\Omega) \subseteq W_2^{k-1}(\Omega) \subseteq C^1(\Omega)$. The proof of the following lemma can be found in a more general context in [10, Lemma 3.4].

LEMMA 3.1. *Let $\sigma \in \mathbb{R}$ with $k := \lfloor \sigma \rfloor > 2 + d/2$. Suppose that the coefficients a_{ij}, b_i and c of the operator defined in (3.3) are in $W_\infty^{k-1}(\Omega)$. Then L is a bounded operator from $W_2^\sigma(\Omega)$ to $W_2^{\sigma-2}(\Omega)$, i. e.*

$$\|Lu\|_{W_2^{\sigma-2}(\Omega)} \leq C \|u\|_{W_2^\sigma(\Omega)}$$

for $u \in W_2^\sigma(\Omega)$.

Note that since the principal coefficients a_{ij} of the operator L are differentiable, we can write the operator in *divergence* form

$$Lu(\mathbf{x}) = \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(\tilde{a}_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} u(\mathbf{x}) \right) + \sum_{i=1}^d \tilde{b}_i(\mathbf{x}) \frac{\partial}{\partial x_i} u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}), \quad (3.4)$$

where the coefficients \tilde{a}_{ij} and \tilde{b}_i are not the same as the a_{ij} and b_i in (3.3). The divergence form of the elliptic operator has the advantage that it can be applied in a *weak* sense to more functions than twice continuously differentiable ones, see for instance [11, Chapter 8]. For our purposes this is not so important as we will work with functions in $W_2^\sigma(\Omega)$, which can be embedded in $C^2(\Omega)$.

3.2. Collocation. Finally, we choose a point set $X = X_1 \cup Y_1$ consisting of interior points $X_1 = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq \Omega$ and boundary points $Y_1 = \{\mathbf{x}_{n+1}, \dots, \mathbf{x}_N\} \subseteq \partial\Omega$. Then, we can define linear functionals

$$\lambda_j = \begin{cases} \delta_{\mathbf{x}_j} \circ L, & 1 \leq j \leq n, \\ \delta_{\mathbf{x}_j} \circ I, & n+1 \leq j \leq N, \end{cases} \quad (3.5)$$

where L is the operator defined in (3.3) and I denotes the identity. Assuming these functionals $\Lambda = \{\lambda_1, \dots, \lambda_N\}$ are linearly independent, we can, according to Theorem 2.1, construct a unique $\|\cdot\|_\Phi$ -norm-minimal interpolant of the form

$$s_{u,\Lambda} = \sum_{j=1}^N \alpha_j \lambda_j^Y \Phi(\cdot, \mathbf{y}) = \sum_{j=1}^n \alpha_j L_2 \Phi(\cdot, \mathbf{x}_j) + \sum_{j=n+1}^N \alpha_j \Phi(\cdot, \mathbf{x}_j),$$

that satisfies the generalised interpolation conditions

$$\begin{aligned} L s_{u,\Lambda}(\mathbf{x}_j) &= f(\mathbf{x}_j) & 1 \leq j \leq n, \\ s_{u,\Lambda}(\mathbf{x}_j) &= F(\mathbf{x}_j) & n+1 \leq j \leq N. \end{aligned}$$

The subscript 2 in $L_2\Phi$ denotes the action of L with respect to the second argument of Φ . Lastly, we need to make sure that the functionals defined in (3.5) are linearly independent. For this we introduce the notion of a singular point of a differential operator.

DEFINITION 3.2. *Let L be defined as in (3.3). The point $\mathbf{x} \in \mathbb{R}^d$ is called a **singular point** of L if $\delta_{\mathbf{x}} \circ L = 0$, i. e. $a_{ij}(\mathbf{x}) = b_i(\mathbf{x}) = c(\mathbf{x}) = 0$ for all $1 \leq i, j \leq d$. With this definition one can prove the following theorem, which is taken from [10, Proposition 3.8].*

THEOREM 3.3. *Let L be defined as in (3.3) and I the identity operator. Let $X_1 = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq \Omega$ and $Y_1 = \{\mathbf{x}_{n+1}, \dots, \mathbf{x}_N\} \subseteq \partial\Omega$ be two sets of pairwise distinct points such that X_1 contains no singular point of L . Then the functionals defined in (3.5) are linearly independent over $W_2^\sigma(\mathbb{R}^d)$ with $\sigma > 2 + d/2$.*

3.3. Multilevel collocation algorithm. The algorithm we consider here uses for each level a finer data set and a finer scaled basis function. That is, we will choose denser data sets and smaller support radii. Therefore, we assume we are given a sequence of point sets X_1, X_2, X_3, \dots in the bounded domain Ω and sequence of point sets Y_1, Y_2, Y_3, \dots on the domain's boundary $\partial\Omega$. Furthermore, for given scaling parameters $\delta_j > 0$ and a basis function Φ we define

$$\Phi_j(\mathbf{x} - \mathbf{y}) = \Phi_{\delta_j}(\mathbf{x} - \mathbf{y}) = \delta_j^{-d} \Phi\left(\frac{\mathbf{x} - \mathbf{y}}{\delta_j}\right). \quad (3.6)$$

It follows that if Φ has support radius one, the Φ_j have support radius δ_j . The scaling factor in front of the basis function is chosen such that $\widehat{\Phi_\delta}(\boldsymbol{\omega}) = \widehat{\Phi}(\delta\boldsymbol{\omega})$. In the case of pure interpolation, convergence of the corresponding subsequently stated multilevel algorithm was achieved by choosing the support radius proportional to the mesh norm, see [29, Theorem 1]. This had the useful implication that the condition numbers of the arising interpolation matrices could be bounded independently of the level, at least for *quasi-uniform* data sets, i.e. data sets, where the separation distance is comparable to the mesh norm.

Unfortunately, the generalisation to a collocation variant of this algorithm is not straight forward. One reason is that now we have to deal with two different mesh norms (corresponding to the interior and the boundary). The other reason is, that even if we choose the support radius to be proportional to the maximum of the boundary and interior mesh norm, the algorithm does not converge. We will have to introduce a non-proportional relationship between support radius and mesh norm.

Using scaled and translation-invariant Φ_j as basis functions, the local generalised interpolant comes from the approximation space

$$W_j = \text{span}\{L\Phi_j(\cdot - \mathbf{x}) \mid \mathbf{x} \in X_j\} + \text{span}\{\Phi_j(\cdot - \mathbf{y}) \mid \mathbf{y} \in Y_j\}.$$

In this setting, we obtain a multilevel algorithm of the following form.

ALGORITHM 1 (Multilevel collocation algorithm). *Given right-hand sides f and F do:*

1. Set $u_0 = 0, f_0 = f, F_0 = F$
2. For $j = 1, 2, 3, \dots$ do
 - (a) Determine the local correction s_j to f_{j-1} and F_{j-1} with

$$\begin{aligned} Ls_j(\mathbf{x}) &= f_{j-1}(\mathbf{x}), & \mathbf{x} \in X_j, \\ s_j(\mathbf{y}) &= F_{j-1}(\mathbf{y}), & \mathbf{y} \in Y_j, \end{aligned}$$

(b) Update the global approximation and the residuals

$$\begin{aligned} u_j &= u_{j-1} + s_j \\ f_j &= f_{j-1} - L(s_j|\Omega) \\ F_j &= F_{j-1} - s_j|\partial\Omega \end{aligned}$$

We define the error at level j to be the difference between the exact solution and the global approximant at level j , i.e. $e_j = u - u_j$. From the definitions one can easily establish the two identities

$$u_j = \sum_{k=1}^j s_k \quad \text{and} \quad e_j = u - \sum_{k=1}^j s_k = e_{j-1} - s_j.$$

These equations and the linearity of the differential operator allow us to rewrite the residual in the interior as well as the residual on the boundary in the following form

$$f_j = f - L(u_j|\Omega) \quad \text{and} \quad F_j = F - u_j|\partial\Omega.$$

One trick we will employ is to extend Ω to the whole \mathbb{R}^d by means of extension operators. The following result can be found in [29, Proposition 1] and comes originally from [23, Theorem 5, Section 5.3]. We need to assume the boundary to be Lipschitz, see [2, Definition 1.4.4] for a definition.

LEMMA 3.4. *Suppose $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ has a Fourier transform satisfying (2.3). Assume $\Omega \subseteq \mathbb{R}^d$ is open and has a Lipschitz boundary. Let $\sigma \geq 0$. Then, there exists a linear operator $E : W_2^\sigma(\Omega) \rightarrow W_2^\sigma(\mathbb{R}^d)$, such that, for all $f \in W_2^\sigma(\Omega)$ the two conditions hold*

1. $Ef|_\Omega = f|_\Omega$,
2. $\|Ef\|_{W_2^\sigma(\mathbb{R}^d)} \leq C_\sigma \|f\|_{W_2^\sigma(\Omega)}$

where $C_\sigma > 0$ is a constant independent of f .

Once we have extended the solution from Ω to the whole \mathbb{R}^d , we can make use of results that are known for the whole Euclidean space. In particular, the result that the native space $\mathcal{N}_\Phi(\mathbb{R}^d)$ associated to a translation-invariant kernel Φ whose Fourier transform decays algebraically is norm-equivalent to the Sobolev space $W_2^\sigma(\mathbb{R}^d)$ becomes available. This result is even true if one uses a scaled kernel Φ_δ . The following lemma, proven in [29, Lemma 1], summarises this thought.

LEMMA 3.5. *For every $\delta \in (0, 1]$ we have $\mathcal{N}_{\Phi_\delta}(\mathbb{R}^d) = W_2^\sigma(\mathbb{R}^d)$. Additionally, we have for every $g \in W_2^\sigma(\mathbb{R}^d)$ the following norm equivalence,*

$$c_1 \|g\|_{\Phi_\delta} \leq \|g\|_{W_2^\sigma(\mathbb{R}^d)} \leq c_2 \delta^{-\sigma} \|g\|_{\Phi_\delta},$$

with $c_1, c_2 > 0$.

4. Convergence result. Before we start looking at the multiscale convergence result, we need an auxiliary result for the L_2 -error between the solution of the partial differential equation (3.1) subject to the boundary condition (3.2) and its collocation approximation. Let $h_{X_1, \Omega}$ denote the mesh norm corresponding to the interior of the domain Ω and $h_{Y_1, \partial\Omega}$ denote the mesh norm corresponding to its boundary $\partial\Omega$.

In order to prove these error bounds, we need two results, well known in the theory of weak solutions to elliptic PDEs that establish a relationship between Sobolev norms for the domain Ω as well as Sobolev norms for the boundary $\partial\Omega$. The first result, taken from [30, Theorem 8.7], shows how restricting a function in $W_2^\sigma(\Omega)$, only defined almost everywhere, to the boundary $\partial\Omega$ is feasible.

LEMMA 4.1 (Trace theorem). *Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded region with a $C^{k,s}$ -boundary $\partial\Omega$ and $1/2 < \sigma \leq k + s$. Then, there exists a continuous linear operator*

$$T: W_2^\sigma(\Omega) \rightarrow W_2^{\sigma-1/2}(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ for all $u \in W_2^\sigma(\Omega)$. This means in particular that the restriction of $u \in W_2^\sigma(\Omega)$ to $\partial\Omega$ is well-defined, belongs to $W_2^{\sigma-1/2}(\partial\Omega)$ and satisfies

$$\|u\|_{W_2^{\sigma-1/2}(\partial\Omega)} \leq C\|u\|_{W_2^\sigma(\Omega)}$$

for some positive constant C , that is independent of u .

Roughly speaking, one could think of the so-called *trace operator* T as a generalisation of the restriction mapping $f \mapsto f|_{\partial\Omega}$, which in the case of Sobolev functions f is not well-defined since $\partial\Omega$ is a null set in \mathbb{R}^d . It is also possible to generalise the notion of the extension mapping $f|_{\partial\Omega} \mapsto f$ in a similar fashion with the help of an *inverse operator* Z , see [30, Theorem 8.8].

LEMMA 4.2 (Inverse trace theorem). *Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded region with a $C^{k,s}$ -boundary $\partial\Omega$ and $1/2 < \sigma \leq k + s$. Then, there exists a continuous linear operator*

$$Z: W_2^{\sigma-1/2}(\partial\Omega) \rightarrow W_2^\sigma(\Omega)$$

such that $T \circ Zu = u$ for all $u \in W_2^{\sigma-1/2}(\partial\Omega)$, where T is the trace operator from Lemma 4.1.

As already mentioned, we need an auxiliary result for the L_2 -error of the solution to the partial differential equation (3.1), subject to the boundary condition (3.2), and its collocation approximation (see Lemma 4.4).

One important step in the proof of this lemma is to use the solution's continuous dependence on the data. We cite the L_2 -analogue of the maximum principle taken from [11, Corollary 8.7] here.

LEMMA 4.3 (Continuous dependence on data in the L_2 -norm). *Let $u \in W_2^1(\Omega)$ satisfy (3.1) in the weak sense. Define with the help of the inverse trace operator Z the function $\phi = ZF \in W_2^1(\Omega)$ for $F \in W_2^{1/2}(\partial\Omega)$. Then we have for some $\tilde{C} > 0$ the estimate*

$$\|u\|_{W_2^1(\Omega)} \leq \tilde{C} \left\{ \|f\|_{L_2(\Omega)} + \|\phi\|_{W_2^1(\Omega)} \right\}. \quad (4.1)$$

Now bearing in mind that the inverse trace operator is a continuous and thus bounded operator, we have

$$\|\phi\|_{W_2^1(\Omega)} = \|ZF\|_{W_2^1(\Omega)} \leq c\|F\|_{W_2^{1/2}(\partial\Omega)}$$

for some positive constant c so that we can rewrite (4.1) to find

$$\|u\|_{L_2(\Omega)} \leq C \left\{ \|f\|_{L_2(\Omega)} + \|F\|_{W_2^{1/2}(\partial\Omega)} \right\}, \quad (4.2)$$

again for some $C > 0$. Here we also employed the trivial estimate $\|u\|_{L_2(\Omega)} \leq \|u\|_{W_2^1(\Omega)}$.

In order to prove our main result in this section, it is also possible to directly use the maximum principle for elliptic partial differential equations. However, as we will

see in the proof of Theorem 4.5, it is more favourable to have such a bound for the solution to (3.1) and (3.2) in the L_2 -norm than in the L_∞ -norm since the sampling inequality would in the latter case yield an extra factor of $h^{d/2}$, see Theorem 2.2.

With estimate (4.2) we can now bound the L_2 -error between the solution u and its approximation s_u .

LEMMA 4.4 (L_2 -error). *Assume that $\delta \in (0, 1]$. Let $u \in W_2^\sigma(\Omega)$ be the solution of (3.1) and (3.2). Let the domain Ω have a $C^{k,s}$ -boundary for $s \in [0, 1)$ such that $\sigma = k + s$ and $k := \lfloor \sigma \rfloor > 2 + d/2$. Then the error between the solution u and its collocation approximation s_u can be bounded in the L_2 -norm by*

$$\begin{aligned} \|u - s_u\|_{L_2(\Omega)} &\leq C\delta^{-\sigma} \{h_{X_1, \Omega}^{\sigma-2} + h_{Y_1, \partial\Omega}^{\sigma-1}\} \|Eu\|_{\Phi_\delta} \\ &\leq C\delta^{-\sigma} \left\{ h_{X_1, \Omega}^{\sigma-2} + h_{Y_1, \partial\Omega}^{\sigma-1} \right\} \|u\|_{W_2^\sigma(\Omega)} \\ &\leq C\delta^{-\sigma} h^{\sigma-2} \|u\|_{W_2^\sigma(\Omega)}, \end{aligned}$$

where $h = \max\{h_{X_1, \Omega}, h_{Y_1, \partial\Omega}\}$.

Proof. First, we note that by (4.2) we have

$$\|u - s_u\|_{L_2(\Omega)} \leq C \left\{ \|Lu - Ls_u\|_{L_2(\Omega)} + \|u - s_u\|_{W_2^{1/2}(\partial\Omega)} \right\}.$$

Now we estimate the two terms on the right-hand side separately. With the help of the sampling inequality Theorem 2.2, the boundedness of L as described in Lemma 3.1 as well as the norm equivalence for scaled kernels, Lemma 3.5, we obtain for the term corresponding to the interior of the domain the bound

$$\begin{aligned} \|Lu - Ls_u\|_{L_2(\Omega)} &\leq Ch_{X_1, \Omega}^{\sigma-2} \|Lu - Ls_u\|_{W_2^{\sigma-2}(\Omega)} \\ &\leq Ch_{X_1, \Omega}^{\sigma-2} \|u - s_u\|_{W_2^\sigma(\Omega)} \\ &\leq Ch_{X_1, \Omega}^{\sigma-2} \|Eu - s_{Eu}\|_{W_2^\sigma(\mathbb{R}^d)} \\ &\leq Ch_{X_1, \Omega}^{\sigma-2} \delta^{-\sigma} \|Eu - s_{Eu}\|_{\Phi_\delta} \\ &\leq Ch_{X_1, \Omega}^{\sigma-2} \delta^{-\sigma} \|Eu\|_{\Phi_\delta}. \end{aligned}$$

For the boundary term we use the definition of the Sobolev norm on the boundary by charts. The procedure is very similar to the proof of Theorem 3.10 in [10]. For $B = B(\mathbf{0}, 1) \subseteq \mathbb{R}^{d-1}$ we set $u_j = ((u - s_u)w_j) \circ \psi_j \in W_2^{\sigma-1/2}(B)$. Since we want to employ the sampling inequality, it is important to note that the u_j vanish on the T_j . Thus, we find, using Theorem 2.2 and Lemma 4.1,

$$\begin{aligned} \|u - s_u\|_{W_2^{1/2}(\partial\Omega)}^2 &= \sum_{j=1}^K \|u_j\|_{W_2^{1/2}(B)}^2 \\ &\leq C \sum_{j=1}^K h_{T_j, B}^{2(\sigma-1)} \|u_j\|_{W_2^{\sigma-1/2}(B)}^2 \\ &\leq Ch_{Y_1, \partial\Omega}^{2(\sigma-1)} \|u - s_u\|_{W_2^{\sigma-1/2}(\partial\Omega)}^2 \\ &\leq Ch_{Y_1, \partial\Omega}^{2(\sigma-1)} \|u - s_u\|_{W_2^\sigma(\Omega)}^2. \end{aligned}$$

From this we can deduce

$$\begin{aligned}\|u - s_u\|_{W_2^{1/2}(\partial\Omega)} &\leq Ch_{Y_1, \partial\Omega}^{\sigma-1} \|u - s_u\|_{W_2^\sigma(\Omega)} \\ &\leq Ch_{Y_1, \partial\Omega}^{\sigma-1} \delta^{-\sigma} \|Eu\|_{\Phi_\delta},\end{aligned}$$

where we used for the second inequality exactly the same last three steps as for the term $\|Lu - Ls_u\|_{L_2(\Omega)}$. Thus, by combining both results we can conclude

$$\begin{aligned}\|u - s_u\|_{L_2(\Omega)} &\leq C\delta^{-\sigma} \{h_{X_1, \Omega}^{\sigma-2} + h_{Y_1, \partial\Omega}^{\sigma-1}\} \|Eu\|_{\Phi_\delta} \\ &\leq C\delta^{-\sigma} \{h_{X_1, \Omega}^{\sigma-2} + h_{Y_1, \partial\Omega}^{\sigma-1}\} \|Eu\|_{W_2^\sigma(\mathbb{R}^d)} \\ &\leq C\delta^{-\sigma} \{h_{X_1, \Omega}^{\sigma-2} + h_{Y_1, \partial\Omega}^{\sigma-1}\} \|u\|_{W_2^\sigma(\Omega)} \\ &\leq C\delta^{-\sigma} h^{\sigma-2} \|u\|_{W_2^\sigma(\Omega)}\end{aligned}$$

using Lemma 3.5 and Lemma 3.4. \square

With the help of this lemma we are now able to prove convergence of the multilevel approximation scheme for solving elliptic partial differential equation of second order.

THEOREM 4.5. *Assume $u \in W_2^\sigma(\Omega)$ solves (3.1) and (3.2). Let $k := \lfloor \sigma \rfloor > 2 + d/2$. We define two point set sequences. Firstly, let X_1, X_2, \dots be a sequence of point sets in Ω with mesh norms $h_{X_1, \Omega}, h_{X_2, \Omega}, \dots$ such that the X_j contain no singular point of the operator L and let Y_1, Y_2, \dots be a sequence of point sets in $\partial\Omega$ with mesh norms $h_{Y_1, \partial\Omega}, h_{Y_2, \partial\Omega}, \dots$. Set $h_j = \max\{h_{X_j, \Omega}, h_{Y_j, \partial\Omega}\}$ and assume*

$$\gamma\mu h_j \leq h_{j+1} \leq \mu h_j \quad (4.3)$$

for $j = 1, 2, \dots$ and some fixed $\mu \in (0, 1)$ and $\gamma \in (0, 1)$. Let the domain Ω have a $C^{k,s}$ -boundary for $s \in [0, 1)$ such that $\sigma = k + s$ and let Φ be a kernel satisfying

$$c_1(1 + \|\omega\|_2^2)^{-\sigma} \leq \hat{\Phi}(\omega) \leq c_2(1 + \|\omega\|_2^2)^{-\sigma}, \quad \omega \in \mathbb{R}^d,$$

with two fixed constants $0 < c_1 \leq c_2$. Define $\delta_j = \left(\frac{h_j}{\mu}\right)^{1-\frac{2}{\sigma}}$ and

$$\Phi_j(\mathbf{x}, \mathbf{y}) = \Phi_{\delta_j}(\mathbf{x}, \mathbf{y}) = \delta_j^{-d} \Phi((\mathbf{x} - \mathbf{y})/\delta_j).$$

Lastly, let $h_1 \leq \mu$ be sufficiently small. Then, there exists a constant C independent of μ, j and u such that

$$\|Ee_j\|_{\Phi_{j+1}} \leq \alpha \|Ee_{j-1}\|_{\Phi_j},$$

for $j = 1, 2, \dots$ with $\alpha = C\mu^{\sigma-2}$. Thus, we have the estimate

$$\|u - u_k\|_{L_2(\Omega)} \leq C\alpha^k \|u\|_{W_2^\sigma(\Omega)}, \quad k = 1, 2, 3, \dots \quad (4.4)$$

Hence, if the constant $\mu \in (0, 1)$ has been chosen sufficiently small, so that $\alpha = C\mu^{\sigma-2} < 1$, the multiscale approximation u_k converges to u in the L_2 -norm. This convergence is linear in the number of levels.

Proof. First we see that

$$\delta_j = \left(\frac{h_j}{\mu}\right)^{1-\frac{2}{\sigma}} \leq \left(\frac{h_j h_{j-1}}{h_j}\right)^{1-\frac{2}{\sigma}} < \left(\frac{h_{j-1}}{\mu}\right)^{1-\frac{2}{\sigma}} = \delta_{j-1}.$$

That is $\delta_j < \delta_1 \leq 1$ since $h_1 \leq \mu$ and $1 - \frac{2}{\sigma} > 0$ due to our assumptions. Hence we can apply Lemma 3.5. We use the definition of the native space norm to obtain

$$\begin{aligned}
\|Ee_j\|_{\Phi_{j+1}}^2 &= \int_{\mathbb{R}^d} \frac{|\widehat{Ee_j}(\omega)|^2}{\widehat{\Phi}_{j+1}(\omega)} d\omega = \int_{\mathbb{R}^d} \frac{|\widehat{Ee_j}(\omega)|^2}{\widehat{\Phi}(\delta_{j+1}\omega)} d\omega \\
&\leq \frac{1}{c_1} \int_{\|\omega\|_2 \leq 1/\delta_{j+1}} |\widehat{Ee_j}(\omega)|^2 (1 + \delta_{j+1}^2 \|\omega\|_2^2)^\sigma d\omega \\
&\quad + \frac{1}{c_1} \int_{\|\omega\|_2 \geq 1/\delta_{j+1}} |\widehat{Ee_j}(\omega)|^2 (1 + \delta_{j+1}^2 \|\omega\|_2^2)^\sigma d\omega \\
&=: \frac{1}{c_1} (I_1 + I_2).
\end{aligned}$$

For the first term I_1 , we use Parseval's identity, Lemma 3.4 and Lemma 4.4 to obtain

$$\begin{aligned}
I_1 &\leq 2^\sigma \int_{\mathbb{R}^d} |\widehat{Ee_j}(\omega)|^2 d\omega = 2^\sigma \|Ee_j\|_{L_2(\mathbb{R}^d)}^2 \\
&\leq C \|e_j\|_{L_2(\Omega)}^2 \\
&\leq Ch_j^{2(\sigma-2)} \delta_j^{-2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2 \\
&= C\mu^{2(\sigma-2)} \|Ee_{j-1}\|_{\Phi_j}^2,
\end{aligned}$$

where in the last step we have used

$$\mu^{2(\sigma-2)} = \frac{h_j^{2(\sigma-2)}}{\delta_j^{2\sigma}}.$$

For the second term I_2 , we obtain

$$\begin{aligned}
I_2 &\leq 2^\sigma \delta_{j+1}^{2\sigma} \int_{\|\omega\|_2 \geq 1/\delta_{j+1}} |\widehat{Ee_j}(\omega)|^2 \|\omega\|_2^{2\sigma} d\omega \\
&\leq 2^\sigma \delta_{j+1}^{2\sigma} \int_{\|\omega\|_2 \geq 1/\delta_{j+1}} |\widehat{Ee_j}(\omega)|^2 (1 + \|\omega\|_2^2)^\sigma d\omega \\
&\leq 2^\sigma \delta_{j+1}^{2\sigma} \|Ee_j\|_{W_2^\sigma(\mathbb{R}^d)}^2 \\
&\leq C \delta_{j+1}^{2\sigma} \|e_j\|_{W_2^\sigma(\Omega)}^2 \\
&= C \delta_{j+1}^{2\sigma} \|Ee_{j-1} - s_{Ee_{j-1}}\|_{W_2^\sigma(\Omega)}^2 \\
&\leq C \delta_{j+1}^{2\sigma} \|Ee_{j-1} - s_{Ee_{j-1}}\|_{W_2^\sigma(\mathbb{R}^d)}^2 \\
&\leq C \delta_{j+1}^{2\sigma} \delta_j^{-2\sigma} \|Ee_{j-1} - s_{Ee_{j-1}}\|_{\Phi_j}^2 \\
&\leq C \delta_{j+1}^{2\sigma} \delta_j^{-2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2 \\
&\leq C\mu^{2(\sigma-2)} \|Ee_{j-1}\|_{\Phi_j}^2,
\end{aligned}$$

where in the last step we have used

$$\left(\frac{\delta_{j+1}}{\delta_j}\right)^{2\sigma} = \left(\frac{h_{j+1}}{h_j}\right)^{2(\sigma-2)} \leq \mu^{2(\sigma-2)}.$$

Thus, in total we have

$$\|Ee_j\|_{\Phi_{j+1}} \leq C\mu^{\sigma-2} \|Ee_{j-1}\|_{\Phi_j} = \alpha \|Ee_{j-1}\|_{\Phi_j}, \quad (4.5)$$

for $\alpha := C\mu^{\sigma-2}$. From this we can conclude

$$\begin{aligned}\|u - u_k\|_{L_2(\Omega)} &= \|e_k\|_{L_2(\Omega)} \\ &\leq Ch_k^{\sigma-2} \|Ee_k\|_{W_2^\sigma(\mathbb{R}^d)} \\ &\leq Ch_k^{\sigma-2} \delta_{k+1}^{-\sigma} \|Ee_k\|_{\Phi_{k+1}} \\ &\leq C \|Ee_k\|_{\Phi_{k+1}},\end{aligned}$$

proceeding as in the proof of Lemma 4.4 and using Lemma 3.5 and

$$\frac{h_k^{\sigma-2}}{\delta_{k+1}^\sigma} = \frac{h_k^{\sigma-2}}{h_{k+1}^{\sigma-2}} \mu^{\sigma-2} \leq \frac{1}{(\mu\gamma)^{\sigma-2}} \mu^{\sigma-2} = \gamma^{2-\sigma}.$$

Now applying (4.5) k times, we find

$$\|u - u_k\|_{L_2(\Omega)} \leq C \|Ee_k\|_{\Phi_{k+1}} \leq C\alpha^k \|Eu\|_{W_2^\sigma(\mathbb{R}^d)} \leq C\alpha^k \|u\|_{W_2^\sigma(\Omega)}$$

for $\alpha = C\mu^{\sigma-2}$. \square

5. Stability results. In this section, we investigate how the condition numbers of the level collocation matrices depend on the support radius as well as the separation distance. Throughout this section, we will use the separation distance rather than the mesh norm since it is more natural in the context of condition numbers. Hence, if we want to have statements on both convergence and conditioning we will implicitly assume that the data sets are quasi-uniform, i.e. that the separation distance and the mesh norm are of comparable size.

To simplify the analysis, we will drop the index for the current level. We will continue to use the notation introduced at the beginning of Section 3.2. We will denote the interior collocation points by $\mathbf{x}_1, \dots, \mathbf{x}_n \in \Omega$ and the boundary collocation points by $\mathbf{x}_{n+1}, \dots, \mathbf{x}_N \in \partial\Omega$. The set of *all* collocation points will now be denoted by $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ and its separation distance by q_X . The functionals λ_j are then given by (3.5) and the collocation matrix is given by

$$A_\Lambda = (\lambda_j^\mathbf{x} \lambda_k^\mathbf{y} \Phi_\delta(\mathbf{x} - \mathbf{y})) \in \mathbb{R}^{N \times N}.$$

However, since any constant factor in the definition of the scaled basis function will cancel out when it comes to the condition number, from now on, we simply define the scaled basis function by

$$\Phi_\delta := \Phi(\cdot/\delta).$$

for $\delta > 0$.

It will turn out that a linear relationship between support radius and separation distance leads to condition numbers that are not independent of the level – unlike for pure interpolation, see [29, Theorem 4]. For this reason, we investigate also a simple diagonal preconditioner as suggested in [5].

To follow up the ideas of that paper, we will derive estimates for general scalings and then specify the results for the two specific choices $\delta = cq_X$ (stationary setting) and $\delta = cq_X^{1-2/\sigma}$ (non-stationary setting). While the first choice does not lead to a convergent scheme, it is still interesting in itself since Fasshauer observed that the simple preconditioning strategy leads to level-independent condition numbers, which

we will prove here. The latter choice corresponds to the non-stationary situation for which we proved convergence of the scheme in Theorem 4.5.

In this section, we will follow ideas developed in [27]. In that paper, a Sobolev space $W_2^\sigma(\mathbb{R}^d)$ was called *feasible* if it contained a compactly supported function Φ satisfying (2.3). This was necessary at that time. Since then, a series of new compactly supported radial basis functions have emerged, so that, for example, it is now known that every $H^\sigma(\mathbb{R}^d)$ with $\sigma \in \mathbb{N}$, $\sigma > d/2$ has a compactly supported reproducing kernel, see [13, 21].

In our context, however, we are only working in Sobolev spaces with a compactly supported reproducing kernel, since we are using this kernel to build our approximation spaces. Nonetheless, we will state and prove our stability estimates more generally also for non-compactly supported radial basis functions, as long as they are reproducing kernels to a Sobolev space, which also has a compactly supported reproducing kernel.

5.1. Condition number of the collocation matrix. We have the following first result.

THEOREM 5.1. *Suppose $W_2^\sigma(\mathbb{R}^d)$ with $\sigma > d/2 + 2$ has a compactly supported reproducing kernel. Suppose Φ has a Fourier transform which satisfies (2.3). Let $\Phi_\delta = \Phi(\cdot/\delta)$ with $0 < \delta \leq 1$. Suppose L is a linear, strictly elliptic, bounded, second order differential operator. Then, for sufficiently small δ the condition number of the collocation matrix A_Λ can be bounded by*

$$\text{cond}(A_\Lambda) \leq C\delta^{-4} \left(1 + \frac{2\delta}{q_X}\right)^d \left(\frac{\delta}{q_X}\right)^{2\sigma-d},$$

with a constant $C > 0$ independent of X and δ .

Proof. We estimate the smallest and the largest eigenvalue of A_Λ . We first prove for the smallest eigenvalue that

$$\lambda_{\min}(A_\Lambda) \geq C \left(\frac{q_X}{\delta}\right)^{2\sigma-d}$$

holds for a constant $C > 0$ independent of X and δ . Using the functionals λ_j introduced in (3.5), we are done once we can prove that the following bound for the quadratic form exists

$$\boldsymbol{\beta}^T A_\Lambda \boldsymbol{\beta} = \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_\delta(\mathbf{x} - \mathbf{y}) \geq C \left(\frac{q_X}{\delta}\right)^{2\sigma-d} \|\boldsymbol{\beta}\|_2^2$$

for all $\boldsymbol{\beta} \in \mathbb{R}^N$.

We can choose a function Ψ with compact support in the unit ball $B(\mathbf{0}, 1)$, whose Fourier transform satisfies for some $c_1, c_2 > 0$ the decay condition

$$c_1(1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma} \leq \widehat{\Psi}(\boldsymbol{\omega}) \leq c_2(1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma}.$$

for all $\boldsymbol{\omega} \in \mathbb{R}^d$. If we set $\Psi_\varepsilon = \Psi(\cdot/\varepsilon)$, it follows for $\varepsilon \leq 1$ that

$$\widehat{\Psi_\varepsilon}(\boldsymbol{\omega}) = \varepsilon^d \widehat{\Psi}(\varepsilon \boldsymbol{\omega}) \leq c_2 \varepsilon^d (1 + \|\varepsilon \boldsymbol{\omega}\|_2^2)^{-\sigma} \leq c_2 \varepsilon^{d-2\sigma} (1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma}.$$

Combining this result with (2.3), we obtain

$$\widehat{\Phi}(\boldsymbol{\omega}) \geq \frac{c_1(\Phi)}{c_2(\Psi)} \varepsilon^{2\sigma-d} \widehat{\Psi_\varepsilon}(\boldsymbol{\omega}). \quad (5.1)$$

Now we compute

$$\begin{aligned}
\boldsymbol{\beta}^T A_\Lambda \boldsymbol{\beta} &= \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_\delta(\mathbf{x} - \mathbf{y}) \\
&= \frac{1}{(2\pi)^{d/2}} \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Phi}_\delta(\boldsymbol{\omega}) e^{-i(\mathbf{x}-\mathbf{y})^T \boldsymbol{\omega}} d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^{d/2}} \sum_{j,k=1}^N \delta^d \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Phi}(\delta \boldsymbol{\omega}) e^{-i(\mathbf{x}-\mathbf{y})^T \boldsymbol{\omega}} d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^{d/2}} \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Phi}(\boldsymbol{\eta}) e^{-i(\mathbf{x}-\mathbf{y})^T \boldsymbol{\eta}/\delta} d\boldsymbol{\eta} \\
&= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{\Phi}(\boldsymbol{\eta}) \left| \sum_{j=1}^N \beta_j \lambda_j^{\mathbf{x}} e^{-i \frac{\mathbf{x}^T \boldsymbol{\eta}}{\delta}} \right|^2 d\boldsymbol{\eta}.
\end{aligned}$$

Using estimate (5.1), we can now derive the following bound:

$$\begin{aligned}
\boldsymbol{\beta}^T A_\Lambda \boldsymbol{\beta} &\geq \frac{1}{(2\pi)^{d/2}} \frac{c_1(\Phi)}{c_2(\Psi)} \varepsilon^{2\sigma-d} \int_{\mathbb{R}^d} \widehat{\Psi}_\varepsilon(\boldsymbol{\eta}) \left| \sum_{j=1}^N \beta_j \lambda_j^{\mathbf{x}} e^{-i \frac{\mathbf{x}^T \boldsymbol{\eta}}{\delta}} \right|^2 d\boldsymbol{\eta} \\
&= \frac{1}{(2\pi)^{d/2}} \frac{c_1(\Phi)}{c_2(\Psi)} \varepsilon^{2\sigma-d} \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Psi}_\varepsilon(\boldsymbol{\eta}) e^{-i \frac{(\mathbf{x}-\mathbf{y})^T \boldsymbol{\eta}}{\delta}} d\boldsymbol{\eta} \\
&= \frac{1}{(2\pi)^{d/2}} \frac{c_1(\Phi)}{c_2(\Psi)} \varepsilon^{2\sigma-d} \delta^d \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Psi}_\varepsilon(\delta \boldsymbol{\omega}) e^{-i(\mathbf{x}-\mathbf{y})^T \boldsymbol{\omega}} d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^{d/2}} \frac{c_1(\Phi)}{c_2(\Psi)} (\varepsilon \delta)^d \varepsilon^{2\sigma-d} \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Psi}(\varepsilon \delta \boldsymbol{\omega}) e^{-i(\mathbf{x}-\mathbf{y})^T \boldsymbol{\omega}} d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^{d/2}} \frac{c_1(\Phi)}{c_2(\Psi)} \varepsilon^{2\sigma-d} \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \int_{\mathbb{R}^d} \widehat{\Psi}_{\varepsilon \delta}(\boldsymbol{\omega}) e^{-i(\mathbf{x}-\mathbf{y})^T \boldsymbol{\omega}} d\boldsymbol{\omega} \\
&= \frac{c_1(\Phi)}{c_2(\Psi)} \varepsilon^{2\sigma-d} \sum_{j,k=1}^N \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Psi_{\varepsilon \delta}(\mathbf{x} - \mathbf{y}).
\end{aligned}$$

If we now choose $\varepsilon \delta = q_X$, we only sum over diagonal elements. Thus we find

$$\boldsymbol{\beta}^T A_\Lambda \boldsymbol{\beta} \geq \frac{c_1(\Phi)}{c_2(\Psi)} \left(\frac{q_X}{\delta} \right)^{2\sigma-d} \left\{ \sum_{j=1}^n \beta_j^2 \lambda_j^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y}) + \sum_{j=n+1}^N \beta_j^2 \Psi_{q_X}(\mathbf{0}) \right\}. \quad (5.2)$$

Now we have a closer look at the first sum. For $1 \leq m \leq n$ we compute

$$\begin{aligned}\lambda_m^{\mathbf{x}} \lambda_m^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y}) &= q_X^{-4} \sum_{i,j=1}^d \sum_{k,\ell=1}^d a_{ij}(\mathbf{x}_m) a_{k\ell}(\mathbf{x}_m) \frac{\partial^4 \Psi(\mathbf{0})}{\partial x_i \partial x_j \partial x_k \partial x_\ell} \\ &\quad + q_X^{-2} \sum_{i,j=1}^d [2a_{ij}(\mathbf{x}_m) c(\mathbf{x}_m) - b_i(\mathbf{x}_m) b_j(\mathbf{x}_m)] \frac{\partial^2 \Psi(\mathbf{0})}{\partial x_i \partial x_j} \\ &\quad + c(\mathbf{x}_m)^2 \Psi(\mathbf{0}).\end{aligned}$$

For the first term on the right-hand side note, that expressing the derivatives using the inverse Fourier transform and then the strict ellipticity of the operator yields

$$\begin{aligned}&\sum_{i,j=1}^d \sum_{k,\ell=1}^d a_{ij}(\mathbf{x}_m) a_{k\ell}(\mathbf{x}_m) \frac{\partial^4 \Psi(\mathbf{0})}{\partial x_i \partial x_j \partial x_k \partial x_\ell} \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{\Psi}(\omega) \left(\sum_{i,j=1}^d a_{ij}(\mathbf{x}_m) \omega_i \omega_j \right) \left(\sum_{k,\ell=1}^d a_{k\ell}(\mathbf{x}_m) \omega_k \omega_\ell \right) d\omega \\ &\geq \frac{c_E^2}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|\omega\|_2^4 \widehat{\Psi}(\omega) d\omega \\ &\geq \frac{c_E^2 c_1(\Psi)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \|\omega\|_2^4 (1 + \|\omega\|_2^2)^{-\sigma} d\omega. \\ &=: C_1 > 0.\end{aligned}$$

Note that the last integral is indeed finite since $\sigma > d/2 + 2$. We introduce the abbreviations

$$A := \max_{1 \leq i,j \leq d} \|a_{ij}\|_{L^\infty(\Omega)}, \quad B := \max_{1 \leq k \leq d} \|b_k\|_{L^\infty(\Omega)} \quad \text{and} \quad C := \|c\|_{L^\infty(\Omega)}.$$

Using the inverse triangle inequality, the previous estimates and the boundedness of the coefficients, we obtain

$$|\lambda_m^{\mathbf{x}} \lambda_m^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y})| \geq q_X^{-4} C_1 - q_X^{-2} d^2 \max_{1 \leq i,j \leq d} \left| \frac{\partial^2}{\partial x_i \partial x_j} \Psi(\mathbf{0}) \right| [2AC + B^2] - C \Psi(\mathbf{0}).$$

For sufficiently small q_X the leading term dominates the rest and we can derive

$$|\lambda_m^{\mathbf{x}} \lambda_m^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y})| \geq \widetilde{C}_1 q_X^{-4} \quad (5.3)$$

with a $\widetilde{C}_1 > 0$, which remains true for all $q_X \in (0, 1)$.

Thus, using $\Psi_{q_X}(\mathbf{0}) = \Psi(\mathbf{0})$ and putting everything together we have for q_X small enough

$$\begin{aligned}\beta^T A_\Lambda \beta &\geq \frac{c_1(\Phi)}{c_2(\Psi)} \min\{\Psi(\mathbf{0}), \widetilde{C}_1 q_X^{-4}\} \left(\frac{q_X}{\delta} \right)^{2\sigma-d} \|\beta\|_2^2 \\ &= C \Psi(\mathbf{0}) \left(\frac{q_X}{\delta} \right)^{2\sigma-d} \|\beta\|_2^2.\end{aligned}$$

Next, we will investigate the behaviour of the largest eigenvalue $\lambda_{\max} = \lambda_{\max}(A_\Lambda)$. The Gershgorin theorem asserts that we can find at least one index j such that

$$|\lambda_{\max} - \lambda_j^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Phi_\delta(\mathbf{x} - \mathbf{y})| \leq \sum_{\substack{k=1 \\ k \neq j}}^N |\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_\delta(\mathbf{x} - \mathbf{y})|$$

holds, which yields by the inverse triangle inequality for the modulus

$$|\lambda_{\max}| \leq \sum_{k=1}^N |\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})|.$$

Now bearing in mind that Φ_{δ} has support in the ball $B(\mathbf{0}, \delta)$, we notice that some of the terms vanish. More precisely we only need to sum over the amount of nonzero entries in the j^{th} row

$$n_j := |\{k \in \mathbb{N} : \|\mathbf{x}_j - \mathbf{x}_k\|_2 < \delta\}|.$$

In order to find an upper bound on n_j , we define balls with radius $q_X/2$ and centre \mathbf{x}_k

$$B(\mathbf{x}_k, q_X/2) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{x}_k\| < q_X/2\}.$$

If we only consider k taken from the index set

$$I_j = \{k \in \mathbb{N} : \|\mathbf{x}_j - \mathbf{x}_k\| < \delta\},$$

the balls $B(\mathbf{x}_k, q_X/2)$ are disjoint from one another and

$$\bigcup_{k \in I_j} B(\mathbf{x}_k, q_X/2) \subseteq B(\mathbf{x}_j, \delta + q_X/2).$$

Thus, we can conclude

$$\begin{aligned} n_j (q_X/2)^d \text{vol}(B(\mathbf{0}, 1)) &= \sum_{k \in I_j} \text{vol}(B(\mathbf{x}_k, q_X/2)) = \text{vol}\left(\bigcup_{k \in I_j} B(\mathbf{x}_k, q_X/2)\right) \\ &\leq \text{vol}(B(\mathbf{x}_j, \delta + q_X/2)) = (\delta + q_X/2)^d \text{vol}(B(\mathbf{0}, 1)). \end{aligned}$$

This leads to the estimate

$$n_j \leq \left(\frac{\delta + q_X/2}{q_X/2}\right)^d = \left(1 + \frac{2\delta}{q_X}\right)^d.$$

Now we can bound the largest eigenvalue by

$$\begin{aligned} |\lambda_{\max}| &\leq \sum_{k=1}^N |\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})| \\ &\leq n_j \max_{1 \leq k \leq N} \{|\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})|\} \\ &\leq \left(1 + \frac{2\delta}{q_X}\right)^d \max\{\Phi(\mathbf{0}), \delta^{-2} L \Phi(\mathbf{0}), \delta^{-4} L^2 \Phi(\mathbf{0})\}. \end{aligned} \tag{5.4}$$

That is, for sufficiently small support radius δ , the largest eigenvalue can be bounded by

$$|\lambda_{\max}| \leq C \delta^{-4} \left(1 + \frac{2\delta}{q_X}\right)^d$$

for some $C > 0$ which is independent of X and δ . Combining the result for the smallest and the largest eigenvalue we find for the condition number

$$\text{cond}(A_\Lambda) = \frac{\lambda_{\max}(A_\Lambda)}{\lambda_{\min}(A_\Lambda)} \leq C\delta^{-4} \left(1 + \frac{2\delta}{q_X}\right)^d \left(\frac{\delta}{q_X}\right)^{2\sigma-d}$$

for some $C > 0$ which is independent of X and δ . \square

As a consequence of this general result, we can now discuss the two cases $\delta = cq_X$ and $\delta = cq_X^{1-2/\sigma}$.

COROLLARY 5.2.

1. *In the stationary setting $\delta = cq_X$, where the support radius is proportional to the separation distance, the condition number of the collocation matrix can be bounded by*

$$\text{cond}(A_\Lambda) \leq C\delta^{-4} = Cq_X^{-4}.$$

2. *In the non-stationary setting $\delta = cq_X^{1-2/\sigma}$, where the support radius goes slower to zero than the separation distance, the condition number of the collocation matrix can be bounded by*

$$\text{cond}(A_\Lambda) \leq C\delta^{-4}q_X^{-4} = Cq_X^{-8+\frac{8}{\sigma}}.$$

While a condition number growing like q^{-4} for a symmetric collocation matrix employing the application of a second order differential operator is not too surprising, the growth of the condition number in the non-stationary setting is unacceptable. Hence, in the next section, we will discuss a simple, but nonetheless to a certain extent successful preconditioning technique.

5.2. Diagonal preconditioner. In [5], it was suggested that one reason for the ill conditioning was the different scaling of the different parts of the collocation matrix with respect to the support radius. While the diagonal part corresponding to the inner points scales like $\mathcal{O}(\delta^{-4})$, the diagonal part corresponding to the boundary points scales like $\mathcal{O}(1)$ and the off-diagonal entries scale like $\mathcal{O}(\delta^{-2})$.

An easy way of resolving this problem is to employ a diagonal preconditioner. Hence, we define the diagonal matrix $P = (p_{jk})_{1 \leq j, k \leq N}$ by

$$\begin{cases} p_{jk} = 0 & j \neq k \\ p_{jj} = p_j = \delta^2 & 1 \leq j \leq n \\ p_{jj} = p_j = 1 & n+1 \leq j \leq N \end{cases}$$

THEOREM 5.3. *Suppose $W_2^\sigma(\mathbb{R}^d)$ with $\sigma > d/2 + 2$ has a compactly supported reproducing kernel. Suppose Φ has a Fourier transform which satisfies (2.3). Let $\Phi_\delta = \Phi(\cdot/\delta)$ with $0 < \delta \leq 1$. Suppose L is a linear, strictly elliptic, bounded, second order differential operator. Assume that $q_X \leq \delta$. Then, for sufficiently small δ and q_X , the condition number of the preconditioned collocation matrix can be bounded by*

$$\text{cond}(PA_\Lambda P) \leq C \left(1 + \frac{2\delta}{q_X}\right)^d \left(\frac{\delta}{q_X}\right)^{2\sigma-d}$$

with a constant $C > 0$ independent of X , q_X and δ .

Proof. Once again, we estimate the smallest and the largest eigenvalue of the matrix, which this time is given by $PA_\Lambda P$. Noting that

$$\boldsymbol{\beta}^T PA_\Lambda P \boldsymbol{\beta} = \sum_{j,k=1}^N (p_j \beta_j)(p_k \beta_k) \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_\delta(\mathbf{x} - \mathbf{y}),$$

we can directly use most of the computations from the proof of Theorem 5.1.

First of all, if we follow along the lines to estimate the smallest eigenvalue, (5.2) becomes

$$\begin{aligned} & \boldsymbol{\beta}^T PA_\Lambda P \boldsymbol{\beta} \\ & \geq \frac{c_1(\Phi)}{c_2(\Psi)} \left(\frac{q_X}{\delta} \right)^{2\sigma-d} \left\{ \sum_{j=1}^n p_j^2 \beta_j^2 \lambda_j^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y}) + \sum_{j=n+1}^N p_j^2 \beta_j^2 \Psi_{q_X}(\mathbf{0}) \right\}. \end{aligned}$$

As done previously, we have a closer look at the first sum. Taking into account that we have here $p_j = \delta^2$, we can conclude for $1 \leq m \leq n$ that

$$\begin{aligned} p_m^2 \lambda_m^{\mathbf{x}} \lambda_m^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y}) &= \left(\frac{\delta}{q_X} \right)^4 \sum_{i,j=1}^d \sum_{k,\ell=1}^d a_{ij}(\mathbf{x}_m) a_{k\ell}(\mathbf{x}_m) \frac{\partial^4 \Psi(\mathbf{0})}{\partial x_i \partial x_j \partial x_k \partial x_\ell} \\ &+ \delta^4 q_X^{-2} \sum_{i,j=1}^d [2a_{ij}(\mathbf{x}_m) c(\mathbf{x}_m) - b_i(\mathbf{x}_m) b_j(\mathbf{x}_m)] \frac{\partial^2 \Psi(\mathbf{0})}{\partial x_i \partial x_j} \\ &+ \delta^4 c(\mathbf{x}_m)^2 \Psi(0), \end{aligned}$$

which then leads to the bound

$$p_m^2 |\lambda_m^{\mathbf{x}} \lambda_m^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y})| \geq C_1 \left(\frac{\delta}{q_X} \right)^4 - C_2 \delta^2 \left[\left(\frac{\delta}{q_X} \right)^2 + \delta^2 \right].$$

If both δ and q_X go to zero, this leads eventually to a bound of the form

$$p_m^2 |\lambda_m^{\mathbf{x}} \lambda_m^{\mathbf{y}} \Psi_{q_X}(\mathbf{x} - \mathbf{y})| \geq \tilde{C}_1 \left(\frac{\delta}{q_X} \right)^4 \geq \tilde{C}_1$$

as long as q_X/δ remains bounded, which means that δ must not go faster to zero than q_X . In this situation, we find for the quadratic form

$$\begin{aligned} \sum_{j,k=1}^N p_j p_k \beta_j \beta_k \lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_\delta(\mathbf{x} - \mathbf{y}) &\geq \frac{c_1(\Phi)}{c_2(\Psi)} \min\{\Psi(\mathbf{0}), \tilde{C}_1\} \left(\frac{q_X}{\delta} \right)^{2\sigma-d} \|\boldsymbol{\beta}\|_2^2 \\ &= C \left(\frac{q_X}{\delta} \right)^{2\sigma-d} \|\boldsymbol{\beta}\|_2^2, \end{aligned}$$

and hence for the smallest eigenvalue:

$$\lambda_{\min}(PA_\Lambda P) \geq C \left(\frac{q_X}{\delta} \right)^{2\sigma-d}.$$

The investigation of the largest eigenvalue $\lambda_{\max} = \lambda_{\max}(PA_\Lambda P)$ is again done as in the proof of Theorem 5.1. Hence, equation (5.4) now becomes

$$\begin{aligned}
|\lambda_{\max}| &\leq \sum_{k=1}^N p_j p_k |\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})| \\
&\leq n_j \max_{1 \leq k \leq N} \{p_j p_k |\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})|\} \\
&\leq \left(1 + \frac{2\delta}{q_X}\right)^d \max_{1 \leq k \leq N} \{p_j p_k |\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})|\}.
\end{aligned}$$

We need to distinguish three cases to bound the maximum: Both indices j and k lie between 1 and n , only one of them and none of them does. In the first case, since $\Phi \in C^4(\mathbb{R}^d)$ we find a constant $c_{LL} > 0$ such that

$$\begin{aligned}
|\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})| &\leq \delta^{-4} \left| \sum_{\ell, m=1}^d \sum_{r, s=1}^d a_{\ell m}(\mathbf{x}_j) a_{rs}(\mathbf{x}_k) \frac{\partial^4 \Phi((\mathbf{x}_j - \mathbf{x}_k)/\delta)}{\partial x_{\ell} \partial x_m \partial x_r \partial x_s} \right| + \mathcal{O}(\delta^{-3}) \\
&\leq \delta^{-4} c_{LL} + \mathcal{O}(\delta^{-3}).
\end{aligned}$$

Similarly, in the second case, we find a constant $c_L > 0$ such that

$$|\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})| \leq \delta^{-2} c_L + \mathcal{O}(\delta^{-1}).$$

In the third case, we just need to remember that $|\Phi(\mathbf{x})| \leq \Phi(\mathbf{0})$ for all $\mathbf{x} \in \mathbb{R}^d$, i. e.

$$|\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})| = |\Phi(\mathbf{x}_j - \mathbf{x}_k)| \leq \Phi(\mathbf{0}).$$

All three bounds on $|\lambda_j^{\mathbf{x}} \lambda_k^{\mathbf{y}} \Phi_{\delta}(\mathbf{x} - \mathbf{y})|$ are independent of the indices j and k . Thus, we find

$$\begin{aligned}
|\lambda_{\max}| &\leq \left(1 + \frac{2\delta}{q_X}\right)^d \max \{ \Phi(\mathbf{0}), \delta^2 [\delta^{-2} c_L + \mathcal{O}(\delta^{-1})], \delta^4 [\delta^{-4} c_{LL} + \mathcal{O}(\delta^{-3})] \} \\
&= \left(1 + \frac{2\delta}{q_X}\right)^d \max \{ \Phi(\mathbf{0}), c_L + \mathcal{O}(\delta), c_{LL} + \mathcal{O}(\delta) \}.
\end{aligned}$$

That is, for a sufficiently small support radius δ , the largest eigenvalue can be bounded by

$$|\lambda_{\max}| \leq C \left(1 + \frac{2\delta}{q_X}\right)^d$$

for some $C > 0$ which is independent of X and $\delta \in (0, 1]$. Combining the result for the smallest and the largest eigenvalue we find for the condition number

$$\text{cond}(PA_{\Lambda}P) = \frac{\lambda_{\max}(PA_{\Lambda}P)}{\lambda_{\min}(PA_{\Lambda}P)} \leq C \left(1 + \frac{2\delta}{q_X}\right)^d \left(\frac{\delta}{q_X}\right)^{2\sigma-d}$$

for some $C > 0$ which is independent of X and δ . \square

As before, we take a closer look at the two cases $\delta = cq_X$ and $\delta = cq_X^{1-2/\sigma}$. In both cases we can assume that $q_X \leq \delta$ such that we can apply the theorem.

COROLLARY 5.4.

1. In the stationary setting $\delta = cq_X$, where the support radius is proportional to the separation distance, the condition number of the collocation matrix can be bounded by

$$\text{cond}(PA_\Lambda P) \leq C.$$

2. In the non-stationary setting $\delta = cq_X^{1-2/\sigma}$, where the support radius goes slower to zero than the separation distance, the condition number of the collocation matrix can be bounded by

$$\text{cond}(PA_\Lambda P) \leq Cq_X^{-4}.$$

6. Numerical example. Since there have already been extensive numerical examples in [5], we will restrict ourselves here to only one example, in which we will concentrate on the predicted order of the condition number of the collocation matrices in the non-stationary setting.

We are looking at the following model problem, consisting of a Poisson problem on the unit square

$$\begin{aligned} \Delta u &= -\frac{5}{4} \sin(\pi x) \cos(\pi y/2) \quad \text{on } \Omega = (0, 1)^2, \\ u &= \begin{cases} \sin(\pi x) & \text{on } 0 \leq x \leq 1, y = 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As basis function, we use Wendland's compactly supported radial basis function $\phi_{2,3}(r) = (1-r)_+^8 (32r^3 + 25r^2 + 8r + 1) \in C^6(\mathbb{R}^2)$, which generates $W_2^\sigma(\mathbb{R}^2)$ with $\sigma = 4.5$, and support radii of the form

$$\delta_j = \nu (h_j/\mu)^{1-2/4.5}$$

with $\mu = 0.5$ and $\nu = 2.4$. The theoretical results from Corollaries 5.2 and 5.4 suggest that the condition number of the collocation matrix should behave like $q_X^{-8+8/\sigma} = q_X^{-6.2}$, while the condition number of the preconditioned collocation matrix should behave like q_X^{-4} .

Table 6.1 shows the results of our computation, where we have the point sets chosen to be on nested uniform grids. We have computed the condition number of the unpreconditioned matrix, $\kappa(A)$ and the corresponding order as well as the condition number of the preconditioned matrix $\kappa(PAP)$ and its order. In the latter case, the actual condition number can further be reduced by using a slightly changed preconditioner \tilde{P} , which is also a diagonal matrix. This time, however, the diagonal entries are given by $p_j = 1/\sqrt{a_{jj}}$. Note that the analysis given in Theorems 5.1 and 5.3 remains valid since this preconditioner differs from the other one just by constant factors. Thus, the order achieved by this preconditioner remains the same as the order achieved by the other one.

Lastly, Table 6.2 gives an overview of the L_2 -errors for the closed domain as well as separate errors for the interior and the boundary. The order refers to the error on the closed domain $\bar{\Omega} = [0, 1]^2$. As Lemma 4.4 suggests, the error on the boundary is smaller than the error in the interior.

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Level	N	$\kappa(A)$	order	$\kappa(PAP)$	order	$\kappa(\tilde{P}A\tilde{P})$	order
1	9	1.5e+03	—	2.20e+04	—	1.51e+01	—
2	25	1.0e+05	6.12	3.36e+05	3.93	1.72e+02	3.51
3	81	2.3e+07	7.78	1.60e+07	5.57	4.60e+03	4.74
4	289	2.3e+09	6.65	3.44e+08	4.43	8.34e+04	4.18
5	1089	1.7e+11	6.25	5.62e+09	4.03	1.35e+06	4.01
6	4225	1.3e+13	6.21	8.92e+10	3.99	2.12e+07	3.98
7	16641	9.5e+14	6.21	1.42e+12	3.99	3.38e+08	3.99
8	66049	7.1e+16	6.22	2.26e+13	3.99	5.37e+09	3.99

TABLE 6.1

Condition numbers and their orders for the multiscale algorithm applied to a Poisson problem on uniform grids.

Level	N	$L_2(\bar{\Omega})$	order	$L_2(\Omega)$	$L_2(\partial\Omega)$
1	9	8.00e-02	—	8.03e-02	2.15e-02
2	25	2.09e-02	1.94	2.10e-02	3.99e-03
3	81	4.59e-03	2.19	4.61e-03	5.80e-04
4	289	8.81e-04	2.38	8.85e-04	6.53e-05
5	1089	1.44e-04	2.62	1.44e-04	6.16e-06
6	4225	2.00e-05	2.85	2.01e-05	5.02e-07
7	16641	2.41e-06	3.05	2.42e-06	3.58e-08
8	66049	2.61e-07	3.21	2.62e-07	2.14e-09

TABLE 6.2

L_2 -errors for the multiscale algorithm applied to a Poisson problem on uniform grids.

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