Universal Solutions in Nonlinear Anelasticity

Christian Goodbrake
St Peter's College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2020

Soli Deo Gloria

Acknowledgements

In my wildest dreams I would not have expected to be where I am now. Had I been asked even five years ago where I would be now, I would not have been able to predict the blessings and fortunes that have come my way; Oxford would not be on my tongue. As such, I can only credit my current place to God's providence, because without Him directing my life at those times when I couldn't see a way forward, I wouldn't be alive today, let alone where I am now. I am grateful for all the people He has placed in my life and for ensuring my safety thus far, in all my travels, and for His faithfulness.

This work is the culmination of nearly two and a half decades of formation, and the majority of that formation can be directly credited to my parents, Barney and Carol Goodbrake. Each act of love they showed me, each childhood night of reading, each camping trip, and every family dinner pushed me closer to the person I am now. The structure and discipline they gave to me allowed me to develop the tools I needed to flourish in this academic environment. I'm grateful for the love and guidance they've provided me as my parents, and the years of discipline, care, and joy they've given me are an invaluable treasure for me to draw on in every circumstance in my life.

While in Oxford, the congregation of the Oxford Evangelical Presbyterian Church has been a home away from home for me, and I have been blessed greatly by their fellowship. This community has been a huge encouragement and has shared in my joy in our corporate worship. The many conversations and gatherings we've shared have been some of the most delightful and stimulating experiences I've had, both in and out of Oxford. Pastor Andy Young's faithful preaching and mentorship were a great benefit, especially in the advent of global pandemic.

This research would not be possible for the support provided by the Mathematical Institute. I was generously supported by a departmental scholarship that covered all of my university and college fees while still providing a stipend so that I could live in Oxford these past three years. Furthermore, the department has cultivated an environment full of friendliness and collegiality that simultaneously pushes those in it to perform at their best without crumbling under the pressure of excellency. This is not only true from a top-down perspective, but the faculty and fellow students share a zeal and excitement for inquiry that is infectious, and this environment has shaped my mathematical thinking.

My thinking on this problem was greatly assisted by my collaboration with Arash Yavari during his Oxford visit. My discussions with him helped me solidify my understanding of the role symmetry plays in this problem and it was this understanding that underpins all of my results.

I am finally indebted to my supervisor Alain Goriely, for the direction he's provided during the course of my DPhil. I would not have known about this interesting problem, nor would I have access to the tools to address it had Alain not pointed me to these things. Alain's mentorship has led to both my personal and professional development, and I would not have been able to complete this research without his guidance.

Abstract

In this work, we examine the existence and properties of universal solutions in nonlinear incompressible isotropic anelasticity. Universal solutions are those that exist for all members of a class of materials under the imposition of suitable boundary tractions. To this end, we provide a framework under which a wide array of different anelasticity theories can be recast, and use this framework to classify all anelastic universal solutions exhibiting particular symmetries, using known families of universal solutions in classical nonlinear elasticity as a starting point. We demonstrate that all known universal solutions possess one of these particular symmetries, prove that the classical universal solution families merge according to their symmetry groups once extended to the anelastic setting, and conjecture that such symmetries are necessary features of universal solutions, and hence that our classification is complete. In the process of doing this, we discover that two of these families not only possess generic solution branches depending on arbitrary functions, but also anomalous branches outside of these whose forms are fixed up to a finite number of constants. We provide graphical representations of examples from these anomalous branches, and discuss their possible applications.

Contents

List of Figures

Symbols

M	Generic Topological Space
τ	Topology
$\emptyset$	Empty Set
X, x	Generic Points
U	Generic Open Set/Neighborhood
$\mathbb{N}$	Natural Numbers
$\mathbb{R}$	Real Numbers
$\mathbb{R}^N$	Real Vector Space of Dimension N
$\mathcal{M}$	Generic Smooth Manifold
ζ_α	Chart for the Neighborhood U_α
$\zeta_{\alpha\beta}$	Transition Map From $\zeta_\alpha (U_\alpha \cap U_\beta)$ to $\zeta_\beta (U_\alpha \cap U_\beta)$
Q^A, q^a	Local Curvilinear Coordinates
$C^\infty (U)$	Smooth Functions $U \to \mathbb{R}$
I_X	The Ideal of Functions Vanishing at X
$T_X^* \mathcal{M}$	Cotangent Space of $\mathcal{M}$ at the Point X
$T_X \mathcal{M}$	Tangent Space of $\mathcal{M}$ at the Point X
dQ^A, dq^a	Coordinate 1-Forms
$\frac{\partial}{\partial Q^A}, \frac{\partial}{\partial q^a}$	Coordinate Derivatives
$v, \mathbf{v}, v^a$	Generic Vector Field
$w, \mathbf{w}, w_a$	Generic Covector Field
$\mathbf{Z}, Z^{a_1 \dots a_p}_{b_1 \dots b_q}$	Generic (p, q) Tensor Field
V	Generic Vector Space/Bundle
V^*	Dual Space/Bundle of V
$\aleph_V$	Index Alphabet for Vector Space/Bundle V/V^*
(E, π, B, F)	Generic Fiber Bundle
$T\mathcal{M}$	Tangent Bundle of $\mathcal{M}$
$T^*\mathcal{M}$	Cotangent Bundle of $\mathcal{M}$
$T\varphi$	Tangent Map of φ
T	Tangent Functor
$\Gamma (\mathcal{B})$	Space of Sections of the bundle $\mathcal{B}$
φ	Generic Diffeomorphism
φ^*	Pull-Back Under φ
φ_*	Push-Forward Under φ
$(\mathcal{M}, \mathbf{K})$	Generic Riemannian Manifold with Metric $\mathbf{K}$

$v^\flat, \mathbf{v}^\flat$	Flat of the Vector Field v
$w^\sharp, \mathbf{w}^\sharp$	Sharp of the Covector Field w
$\Gamma_{AB}{}^C, \gamma_{ab}{}^c$	Affine Connection
$\nabla_a, \nabla_a{}^{\mathbf{K}}$	Covariant Derivative (Induced by Metric $\mathbf{K}$)
$G, \mathfrak{g}$	Lie Group and it's Lie Algebra
$\rho, \rho_{\mathcal{M}} : G \times \mathcal{M} \to \mathcal{M}$	Group Action on a Manifold $\mathcal{M}$
$\rho_g, \rho\,(g, -)$	Action-Induced Diffeomorphism Under $g \in G$
$\mathbb{E}^n$	Euclidean Space of Dimension n
$\mathrm{SE}(n)$	Group of Orientation-Preserving Isometries of $\mathbb{E}^n$
$\mathrm{SO}(n)$	Group of Rotations in $\mathbb{E}^n$
$\mathrm{T}(n)$	Group of Translations of $\mathbb{E}^n$
$(\mathcal{S}, \mathbf{m})$	Ambient Space
$(\mathcal{B}, \mathbf{M})$	Reference Configuration
$(\mathcal{C}, \mathbf{m})$	Current Configuration
$(\mathcal{B}, \mathbf{H})$	Intermediate Configuration
$\mathbf{F}, F^a{}_A$	Deformation Gradient
$\mathbf{C}, C^A{}_B$	Right Cauchy-Green Stretch Tensor
J	Jacobian
$\mathbf{G}, G^\alpha{}_A$	Anelastic Factor
$\mathbf{A}, A^a{}_\alpha$	Elastic Factor
γ	Anelastic Deformation
α	Elastic Deformation
$\mathbf{E}$	Lagrange Strain
$\{\mathbf{e}_\alpha\}$	Anholonomic Frame
$\{\vartheta^\alpha\}$	Anholonomic Coframe
ι	Inclusion Map
π	Projection Map
ε	Embedding Map
ρ, ρ_0	Mass Density
W	Strain Energy Density
$\mathcal{L}$	Lagrangian Density
S	Action
$\mathbf{P}, P_a{}^A$	Piola Stress
$\boldsymbol{\sigma}, \sigma_a{}^b$	Cauchy Stress
$\mathbf{B}, B^A{}_B$	Left Cauchy-Green Stretch Tensor
$\mathbf{c}, c^a{}_b$	Cauchy Deformation Tensor
$\mathbf{b}, b^a{}_b$	Finger Deformation Tensor
I_1, I_2, I_3	Invariants of $\mathbf{b}$
$\lambda_1, \lambda_2, \lambda_3$	Eigenvalues of $\mathbf{b}$
e_1, e_2	Elementary Symmetric Polynomials in λ_1, and λ_2
$\mathcal{U}_A$	Universal Solutions for Family A

Chapter 1

Prefatory Materials

This work sits at the intersection of two subfields of nonlinear continuum mechanics: anelasticity theory and the theory of universal solutions. As such we seek to provide the reader with the relevant background material in these areas to appreciate the main results provided later.

1.1 Classical Nonlinear Elasticity

First, we briefly present a classical treatment of nonlinear elasticity, which we will later generalize in chapter 3 to accommodate anelastic strains: those that do not store strain energy.

1.1.1 Kinematics

We model bodies as continua, identifying them with subsets of three-dimensional Euclidean space, and refer to such an identification as a *configuration*. When examining the motion of a body, we employ two such configurations, one fixed, called the reference configuration, and one changing with time, called the current configuration.

Denoting the reference configuration as $\mathcal{B}$ with associated position vector X, and the current configuration at time t as $\mathcal{C}(t)$ with associated position vector x, our motion becomes the map

$$\varphi_t : \mathcal{B} \to \mathcal{C}(t)\,, \tag{1.1}$$

and the position of a point of our body at any given time t is

$$x(X,t) = \varphi_t(X)\,. \tag{1.2}$$

We shall suppose that the dynamics of our body shall depend on the local properties of its deformation, hence we define the deformation gradient

$$\mathbf{F} = \operatorname{Grad} x; \quad dx = \mathbf{F}dX. \tag{1.3}$$

This two-point tensor field maps tangent vectors of $\mathcal{B}$ to corresponding tangent vectors of $\mathcal{C}$. Since we want φ to be invertible, we demand a priori that $J \equiv \det \mathbf{F} > 0$. The tensor $\mathbf{F}$ possesses a uniquely defined polar decomposition

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{1.4}$$

where $\mathbf{R}$ is a rotation, and $\mathbf{U}$, and $\mathbf{V}$ are positive definite symmetric tensor fields, with $\mathbf{U}$ living entirely in the reference configuration, and $\mathbf{V}$ living entirely in the current configuration.

We also consider the right and left Cauchy Green stretch tensors, which are respectively

$$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} = \mathbf{U}^2, \tag{1.5}$$

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathsf{T}} = \mathbf{V}^2. \tag{1.6}$$

Unlike $\mathbf{F}$, these tensors are not two-point tensors; $\mathbf{C}$ maps the tangent spaces of $\mathcal{B}$ into themselves, and $\mathbf{b}$ maps the tangent spaces of $\mathcal{C}$ into themselves. These tensors are positive definite and symmetric, hence they have positive real eigenvalues, and orthonormal eigenvectors. These tensors share the same eigenvalues, which are called the principal stretches, since when we examine the lengths of deformed tangent vectors, we have

$$|dx|^2 = dx \cdot dx = \mathbf{F}dX \cdot \mathbf{F}dX = dX \cdot \mathbf{C}dX, \tag{1.7}$$

hence, if dX is an eigenvector of $\mathbf{C}$ with eigenvalue λ^2, we have

$$|dx|^2 = \lambda^2 |dX|^2. \tag{1.8}$$

With these relations, we see that $\mathbf{C}$ is a tensor field on the reference configuration that encapsulates information about distances in the current configuration. A similar tensor field that encapsulates information about distances in the reference configuration while living in the current configuration exists, called the Cauchy deformation tensor. This tensor is typically denoted $\mathbf{c}$, and is defined as

$$\mathbf{c} = \mathbf{b}^{-1}. \tag{1.9}$$

Because $\mathbf{b}$ is positive definite and symmetric, so is $\mathbf{c}$, and the eigenvalues of $\mathbf{c}$ are the inverses of those of $\mathbf{b}$; these tensors share the same eigenvectors.

1.1.2 Dynamics

We now require our body to obey the laws of physics, namely the conservation of mass, linear momentum, angular momentum, and energy.

The total mass of our body is

$$M = \int_{\mathcal{C}} \rho \, dv, \tag{1.10}$$

where ρ is the body's mass density. In the absence of mass generation or decay, we demand

$$\frac{dM}{dt} = \frac{d}{dt} \int_{\mathcal{C}(t)} \rho \, dv = 0. \tag{1.11}$$

To avoid having to differentiate an integral over a moving domain, we pull this equation back to the reference configuration, which lets us write

$$\frac{dM}{dt} = \int_{\mathcal{B}} \frac{\partial}{\partial t} (\rho J) \, dV = 0, \tag{1.12}$$

where the partial time derivative is taken holding X fixed. However, this must not only hold for $\mathcal{B}$, but for all subsets of $\mathcal{B}$, so provided that $\frac{\partial}{\partial t}(\rho J)$ is continuous, we can localize this to obtain

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} v = 0, \tag{1.13}$$

where v is the velocity field of the body, and we have employed the identity $\frac{\partial J}{\partial t} = J \operatorname{div} v$.

Next, we postulate two types of forces acting on our body, surface forces, and body forces. The total force on our body is then

$$F = F_b + F_s = \int_{\mathcal{C}} \rho b \, dv + \int_{\partial \mathcal{C}} t \, da, \tag{1.14}$$

where b is a force per unit mass, and t is a force per unit area, called the traction. These forces have to balance the change in momentum, hence, we have

$$\frac{d}{dt} \int_{\mathcal{C}} \rho v \, dv = \int_{\mathcal{C}} \rho b \, dv + \int_{\partial \mathcal{C}} t \, da. \tag{1.15}$$

The Cauchy-Noll theorem states that the traction on a surface at a point depends linearly on the normal vector of that surface n, hence we can write

$$t = \boldsymbol{\sigma} n \tag{1.16}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. With this, we can employ the divergence theorem to obtain

$$\frac{d}{dt} \int_{\mathcal{C}} \rho v \, dv = \int_{\mathcal{C}} \rho b + \operatorname{div} \boldsymbol{\sigma} \, dv. \tag{1.17}$$

We then pull the left hand side back to the reference configuration, use conservation of mass, and push the result forward to the current configuration to obtain

$$\int_{\mathcal{C}} \rho b + \operatorname{div} \boldsymbol{\sigma} - \rho \frac{\partial v}{\partial t} \, dv = 0. \tag{1.18}$$

We localize this to obtain Cauchy's equation of motion

$$\operatorname{div} \boldsymbol{\sigma} + \rho b = \rho a \tag{1.19}$$

where $a = \frac{\partial v}{\partial t}$ is taken fixing the position in the reference configuration i.e. we consider $v(X,t)$, rather than $v(x,t)$. Often, we seek equilibrium solutions in the absence of body forces, hence we solve

$$\operatorname{div} \boldsymbol{\sigma} = 0. \tag{1.20}$$

Next, we consider the angular momentum of the body about a point, and the moments generated by the forces we have considered.

The angular momentum H_0 about the origin is

$$H_0 = \int_{\mathcal{C}} x \wedge \rho v \, dv. \tag{1.21}$$

The rate of change of this quantity is balanced by the moments provided by the body forces and tractions, hence

$$\int_{\mathcal{C}} x \wedge (\rho a - \rho b) \, dv - \int_{\partial\mathcal{C}} x \wedge \boldsymbol{\sigma} n \, da = 0. \tag{1.22}$$

Applying balance of linear momentum and the divergence theorem yields

$$\int_{\mathcal{C}} x \wedge \operatorname{div} \boldsymbol{\sigma} - \operatorname{div}(x \wedge \boldsymbol{\sigma}) \, dv = 0. \tag{1.23}$$

The product rule reduces this to

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathsf{T}} = 0, \tag{1.24}$$

hence the conservation of angular momentum demands that the Cauchy stress be symmetric.

Finally, we consider the balance of energy. We know that the rate of change of the kinetic energy of our body is balanced by the power of the forces, hence we have

$$\frac{d}{dt}\mathrm{KE} = \frac{d}{dt}\frac{1}{2}\int_{\mathcal{C}} \rho v \cdot v \, dv = P_b + P_s + P_{\mathrm{int}} = \int_{\mathcal{C}} \rho b \cdot v \, dv + \int_{\partial\mathcal{C}} v \cdot \boldsymbol{\sigma} n \, da + P_{\mathrm{int}}, \tag{1.25}$$

where P_b, P_s, and P_{int} are the respective powers of the body forces, tractions, and internal forces. Applying the divergence theorem and the previous conservation laws, this ultimately yields

$$P_{\text{int}} = -\int_{\mathcal{C}} \operatorname{tr}\left(\boldsymbol{\sigma}^{\mathsf{T}} \mathbf{D}\right) \, dv, \tag{1.26}$$

where only $\mathbf{D} = \frac{1}{2}\left(\operatorname{grad} v + (\operatorname{grad} v)^{\mathsf{T}}\right)$, the symmetric part of the velocity gradient, appears due to the symmetry of $\boldsymbol{\sigma}$.

If we then define an internal energy density w such that

$$P_{\text{int}} = -\frac{dE_{\text{int}}}{dt} = -\frac{d}{dt}\int_{\mathcal{C}} w \, dv, \tag{1.27}$$

we can combine these two equations and localize to obtain

$$\frac{\partial}{\partial t}(Jw) = J \operatorname{tr}\left(\boldsymbol{\sigma}^{\mathsf{T}} \mathbf{D}\right). \tag{1.28}$$

Defining the referential quantities $\rho_0 = J\rho$, $W = Jw$, and $\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathsf{T}}$, we can pull these equations back to the reference configuration to obtain

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{1.29}$$

$$\operatorname{Div} \mathbf{P} + \rho_0 b = \rho_0 a, \tag{1.30}$$

$$\mathbf{P}\mathbf{F}^{\mathsf{T}} = \mathbf{F}\mathbf{P}^{\mathsf{T}}, \tag{1.31}$$

$$\frac{\partial W}{\partial t} = \operatorname{tr}\left(\mathbf{P}^{\mathsf{T}} \frac{d\mathbf{F}}{dt}\right). \tag{1.32}$$

1.1.3 Constitutive Laws

Up until now, all of our work has applied equally to all continua; nothing we have done is particular to solids. For hyperelastic solids, we postulate that the internal forces are conservative. We therefore define the strain energy density at each point to be a function of $\mathbf{F}$ at that point alone. By applying the chain rule to equation 1.32, this reveals that the Piola stress $\mathbf{P}$ is

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}. \tag{1.33}$$

Additionally, we suppose that following any particular motion by a rigid motion leaves the energy unchanged; we assume the strain energy only depends on the strain the material experiences, not the local orientation of any given bit of our material. Hence, for all $\mathbf{Q} \in \mathrm{SO}(3)$, we have

$$W(\mathbf{F}) = W(\mathbf{QF}). \tag{1.34}$$

Choosing $\mathbf{Q} = \mathbf{R}^\mathsf{T}$, where $\mathbf{R}$ is the rotation appearing in the polar decomposition of $\mathbf{F}$, this means that W must depend on $\mathbf{F}$ through $\mathbf{U}$ alone. Additionally, since $\mathbf{U}$ is uniquely determined by $\mathbf{C}$, this is equivalent to W depending solely on $\mathbf{C}$.

Similarly, for isotropic materials, we impose that for any rotation $\mathbf{Q}$,

$$W(\mathbf{F}) = W(\mathbf{FQ}). \tag{1.35}$$

This is because the material response of isotropic materials does not depend on the local orientation of the material prior to being strained. In terms of $\mathbf{C}$, this becomes

$$W(\mathbf{C}) = W\left(\mathbf{Q}^\mathsf{T}\mathbf{CQ}\right). \tag{1.36}$$

Since $\mathbf{C}$ is symmetric, we can choose $\mathbf{Q}$ to diagonalize it, hence W must only depend on the invariants of $\mathbf{C}$. Therefore we can write

$$W = W(I_1, I_2, I_3), \tag{1.37}$$

where

$$I_1 = \operatorname{tr}\mathbf{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{1.38}$$

$$\frac{1}{2}\left(\operatorname{tr}(\mathbf{C})^2 - \operatorname{tr}\left(\mathbf{C}^2\right)\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \tag{1.39}$$

$$I_3 = \det\mathbf{C} = \lambda_1^2\lambda_2^2\lambda_3^2, \tag{1.40}$$

and λ_1^2, λ_2^2, and λ_3^2 are the eigenvalues of $\mathbf{C}$.

We shall also consider constrained materials. For a material under a constraint of the form

$$\phi(\mathbf{F}) = 0, \tag{1.41}$$

we replace $W(\mathbf{F})$ with $W(\mathbf{F}) - p\phi(\mathbf{F})$, where p is a Lagrange multiplier associated with the material constraint. For incompressibility, we take

$$\phi = \det\mathbf{F} - 1. \tag{1.42}$$

Combining all of these, we obtain the Cauchy stress for an isotropic, hyperelastic, incompressible material

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\frac{\partial W}{\partial I_1}\mathbf{b} - 2\frac{\partial W}{\partial I_2}\mathbf{c}, \tag{1.43}$$

where $\mathbf{I}$ is the identity tensor, and p is interpreted as a constitutively indeterminate pressure field. Notice that this form of the Cauchy stress automatically satisfies angular momentum conservation by virtue of the symmetry of $\mathbf{I}$, $\mathbf{b}$, and $\mathbf{c}$.

1.2 Introduction to Anelasticity

Anelasticity theories seek to broaden the applicability of conventional nonlinear elasticity to situations involving strain that does not contribute energetically. This makes anelasticity more of a broad collection of theories, rather than a single theory, generalizing conventional nonlinear elasticity [Epstein and Maugin, 1996a] to situations such as thermal effects [Stojanović et al., 1964, Ozakin and Yavari, 2010], plastic flows [Kondo, 1949], dislocations and defects [Nye, 1953, Bilby et al., 1955, Yavari and Goriely, 2012], growth and remodeling [Goriely, 2017, Naumov, 1994, Rodriguez et al., 1994, Takamizawa, 1991, Yavari, 2010], and swelling [Pence and Tsai, 2006, 2005, Templet and Steigmann, 2013]. Such processes are characterized by the presence of *eigenstrains* [Mura, 1982] which are those strains that do not produce elastic stress, since they do not contribute to the elastic strain energy. Because these strains are generally incompatible, further elastic deformation is typically required to embed bodies with nontrivial eigenstrains in space, with the resulting self-equilibriating elastic stresses referred to as *residual stresses*. In the literature, we see numerous different formulations of anelasticity theories, divided into two main categories: those governed by local data, and those formulated geometrically.

1.2.1 Theories Based on Local Data

One general approach is to introduce anelastic parameters as fields that then appear in governing equations. The specific fields introduced depend on the formulation, and the nature of these fields, and exactly which fields should be utilized have been contested in the literature [Casey and Naghdi, 1980]. At their simplest, these theories may only introduce a single scalar field, representing an isotropic volume dilatation [Pence and Tsai, 2006, 2005, Templet and Steigmann, 2013], though for anisotropic strains, higher order tensor fields are typically required.

1.2.1.1 Primitive Strain Theories

Some formulations introduce a primitive anelastic strain field, which is analogous to the Lagrange strain in conventional nonlinear elasticity, but is devoted to capturing the anelastic strain. This was the approach taken for example, in Green and Naghdi [1965], who introduced a plastic strain in addition to the total Lagrange strain. These parameters are independent (subject to a positive definiteness condition on the associated plastic stretch tensor) and appear in the expression for the strain energy density. This field is primitive in the sense that it is introduced as a fundamental quantity rather than being derived from a more fundamental principle.

1.2.1.2 Multiplicative Decompositions

Other theories have introduced a factorization of the deformation gradient as the primitive anelastic quantity; the development of this formulation is more thoroughly documented in Sadik and Yavari [2017]. Theories of this type essentially treat deformations locally as the composition of a purely anelastic motion and a purely elastic motion, and accordingly factor the deformation gradient $\mathbf{F}$ into the form

$$\mathbf{F} = \mathbf{A}\mathbf{G} \tag{1.44}$$

where $\mathbf{G}$ governs the anelastic portion of the deformation and $\mathbf{A}$ governs the elastic portion. This factorization is purely local in the following sense: while $\mathbf{F}$ is the gradient of a deformation, and hence is integrable, neither $\mathbf{A}$ nor $\mathbf{G}$ are gradients, hence there is not a corresponding "anelastic position" to be obtained by integrating $\mathbf{G}$.

1.2.2 Geometric Theories

In contrast to the previous local theories, where anelastic features are encoded in fields, there are a host of geometric theories that introduce a new configuration and encode anelastic data in the intrinsic geometry of this configuration. In these theories, geometric structures like metric and torsion tensor fields are imposed on the body to create this configuration to capture various anelastic phenomena [Kondo, 1949, Noll, 1967, Epstein and Maugin, 1996b, Yavari and Goriely, 2012, Goriely, 2017]. The body equipped with these structures is typically called the *material manifold*, and is typically non-Euclidean, and sometimes non-Riemannian. The residual stress we observe is then interpreted as the result of embedding the material manifold in the ambient Euclidean space, which requires geometrical changes since Euclidean space has vanishing curvature and torsion.

The advantage to these theories is that the material manifold acts as a configuration, much like the usual reference configuration, though with nontrivial geometry. The decomposition of anelastic and elastic deformation is typically captured in a factoring of the total *deformation* as a map from the reference configuration to the material manifold, followed by a map from the material manifold to the current configuration, with each factor of the map governed by the appropriate physics. However, the requirement of the existence of a material manifold raises concerns of the range of applicability of these theories, since the possible anelastic features are constrained to be compatible with the structure of a manifold, perhaps one with highly nontrivial geometric features like torsion and nonmetricity, but a manifold nonetheless.

1.3 Introduction to Ericksen's Problem

Universal deformations in nonlinear elasticity are deformations that exist for all members of a particular class of materials in the absence of body forces. Specifically, for any particular material, an equilibrium deformation that can be sustained solely by the application of surface tractions is called a controllable motion for that material. A deformation is therefore a universal deformation for a class of materials if it is a controllable motion for every material in the class. For instance in unconstrained isotropic elastic materials, only homogeneous deformations are universal. However, adding material constraints, i.e., restricting the class under consideration, expands the set of universal solutions. In particular, under the imposition of incompressibility, there are five known families of universal deformations in addition to the universal homogeneous deformations, which is now restricted to isochoric homogeneous deformations in keeping with the material constraint.

In the determination of universal solutions for constrained materials, both the specific constraint and the functional form of the considered energy densities play key roles, as there are deformations that may be universal for some classes of constrained materials, but not others. For example, anti-plane shear of the form

$$x = X + f(Y, Z), \qquad y = Y, \qquad z = Z, \tag{1.45}$$

is not generally controllable for strain energies of the form $W(I_1, I_2)$, but is for the subclass of generalized neo-Hookean materials whose energies take the form $W(I_1)$. Additionally, motion of the form

$$x = AX + \sin(Y), \qquad y = DY, \qquad z = AZ - \cos(Y), \tag{1.46}$$

with A and D being constants is not universal for incompressible materials, but is universal for other constrained materials.

Therefore, for the incompressible version of this problem, we seek all deformations φ such that for arbitrary strain energies $W(I_1, I_2)$, the equation

$$\operatorname{div} \boldsymbol{\sigma} = 0 \tag{1.47}$$

can be solved with suitable boundary tractions and a suitable Lagrange multiplier field p, where $\boldsymbol{\sigma}$ is defined as in equation 1.43. The difficulty here is that for such a universal solution, the undetermined pressure field depends on the strain energy density. Ordinarily the motion and the pressure field are solved for simultaneously, but here we seek all motions such that after inserting the motion into the governing equations, the remaining problem of determining the pressure field is always *solvable* for arbitrary strain energies, even if the particular pressure solution depends on the strain energy.

It is important to note that when we speak of a deformation here, we are specifically concerned with the kinematics. To illustrate why this is important, consider rectilinear shear of the form (using Cartesian coordinates)

$$x = X + f(Y), \qquad y = Y, \qquad z = Z. \tag{1.48}$$

Inserting this deformation into the equations of motion yields a pressure that is linear in X and independent of Z, which in turn gives an equation determining the unknown function $f(Y)$ depending on $\frac{\partial p}{\partial X}$, and an equation determining $\frac{\partial p}{\partial Y}$ that can be solved once $f(Y)$ is determined. The solution to the equation determining $f(Y)$ depends on the specific material properties, unless the pressure is constant, in which case we obtain simple shear. Therefore, while this motion is controllable for arbitrary incompressible isotropic materials, because the specific kinematics depend on material properties, this motion is not universal, except in the case of simple shear.

The process of obtaining and classifying all universal solutions is a highly non-trivial task. This line of research originates in the seminal work of Jerald Ericksen. In 1954, he made the first systematic attempt to classify all universal deformations in isotropic incompressible elasticity [Ericksen, 1954]. His work revealed four families of universal solutions in addition to homogeneous solutions. In 1955, he completely solved the analogous problem for unconstrained elastic bodies, proving that the only compressible universal solutions are homogeneous deformations [Ericksen, 1955]. In the case of incompressible elasticity, another family of universal solutions was then discovered by Singh and Pipkin [1965], with a special case of this family discovered by Klingbeil and Shield [1966]. Additionally, Fosdick [1966] noted that a different special case of this deformation represented a universal solution with constant invariants, a special case not addressed by Ericksen's initial work. Further contributions and specializations of this problem were made by a number of authors [Fosdick and Schuler, 1969, Kafadar, 1972, Knowles, 1979, Marris, 1982, Martin and Carlson, 1976] and the current conjecture is that no other solution to the Ericksen's problem exists but a proof of it remains an outstanding open problem of rational mechanics [Antman, 1995].

1.3.1 Ericksen's Seminal Work

Ericksen's initial investigation relied heavily on the invariants of the left Cauchy Green stretch tensor, **b**. Specifically, he first showed that for universal deformations, these invariants are functionally dependent, and hence their gradients are linearly dependent vector fields. Ericksen then examines the surfaces on which these invariants are constant, and derives from the universal equilibrium equations that these surfaces must have constant Gaussian and mean curvature, and hence their principal curvatures are constant. The only such surfaces are parallel planes, right concentric cylinders, and concentric spheres. Ericksen proceeds to exhaust these cases and fully classify the universal solutions that fall under the scope of his analysis.

There are two edge cases that Ericksen's work does not touch upon. Because Ericksen uses the gradients of the strain invariants, and relies on these being nonzero, his work does not consider deformations whose invariants are spatially constant. Secondly, when two of the eigenvalues of **b** are equal, Ericksen only considered the special case when the distinguished eigenvector field **n**, corresponding to the distinct eigenvalue, is *complex-lamellar*, namely that it satisfies

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{n} = 0. \tag{1.49}$$

This later case was eventually reduced to the first, when Marris and Shiau [1970] proved that if two eigenvalues of **b** are equal, and the eigenvalues are not spatially constant, then the solution must be one of the already discovered families.

1.3.2 The Mysterious Fifth Family

After the publication of Ericksen's initial work, investigation into universal solutions in nonlinear isotropic incompressible elasticity slowed, as even though Ericksen did not rule out the existence of other solutions, there was also no indication that other solutions exists. This changed when Klingbeil and Shield [1966] and Singh and Pipkin [1965] independently discovered a fifth family of solutions possessing constant invariants that was not contained in any of Ericksen's families. This sparked renewed interest in completing this classification, as it revealed that there exist nontrivial constant-invariant universal motions, hence this special case not treated by Ericksen's initial work, in contrast to the second case not treated, contains solutions not found in other families. This special case is still unsolved, though it has been further restricted by more recent developments.

1.3.3 The Anholonomic Frame Technique

Tackling the constant invariant case has been a series of small incremental gains. The common technique has been to acknowledge that the strain tensor field is positive definite and symmetric, and hence its eigenvectors at each point form an orthonormal frame. When expressed on this frame, the strain tensor appears constant and diagonal, which dramatically simplifies the equilibrium equations at the cost of having to introduce non-commuting directional derivatives.

The first set of results of this technique came from Kafadar [1972], who further reduced the space of possible undiscovered universal solutions, proving that undiscovered universal solutions cannot possess complex-lamellar eigenvector fields. Indeed he proved that if any one eigenvector field of the strain tensor is complex-lamellar, then all of them must be, and if all of the eigenvector fields are complex-lamellar, then at least one must be constant. Kafadar also proved that any new solutions must possess distinct eigenvalues; if any two were equal, then the solution must belong to an already discovered family.

Marris went on to further restrict this unsolved case, proving that undiscovered solutions cannot have more than one of eigenvector field with constant abnormality [Marris, 1975], and later that if the ratios of the abnormalities of the eigenvector fields are necessarily constant, then there are no new solutions, and that the curvatures and abnormalities of the eigenvector fields, and the stretch tensor's eigenvalues are functionally related. [Marris, 1982]. In a similar spirit, Adeleke [1984] proved that any undiscovered universal solutions lie in at most a 19 parameter set, and using this demonstrated that if the first and second derivatives of $\mathbf{c}$ vanish at a point, then the resulting deformation must be homogeneous.

Additionally, both Marris [1982] and Adeleke obtain a systems of differential equations that can be in principle analyzed by computer. The systems differ largely because Marris made use of an anholonomic frame, and hence expresses his equations in terms of directional derivatives that do not commute. The mathematical framework for developing algorithms to address such systems was developed by Hubert [2005], hence solving Ericksen's problem algorithmically is possible in principle, though the run time and memory requirements of the algorithms that characterize all solutions to such systems can be prohibitive.

These restrictions suggest that if an undiscovered universal solution exists, its form is particularly complicated. Unfortunately, this does not rule out the possibility of new solutions, just the possibility of *easily* discovering new solutions.

Chapter 2

Prerequisite Mathematics

Having briefly presented the relevant background of this work in the larger literature, we now seek to provide the reader with the relevant mathematical background. In the hopes of making this work self-contained, all necessary mathematical definitions will be briefly presented here.

2.1 Differential Geometry

Modern elasticity, like general relativity, Lagrangian mechanics, and Hamiltonian mechanics, is a fundamentally geometric theory. The central objects in these theories are geometric in nature; bodies are represented by Riemannian manifolds, and fields behave tensorially. However, for those not fluent in Riemannian geometry, this formalism may appear daunting and opaque. Here we shall build up the definitions of Riemannian manifolds, starting from more primitive objects.

2.1.1 Manifolds

Bodies in modern theories of elasticity are modeled as Riemannian manifolds and deformations are modeled as diffeomorphisms. These are relatively sophisticated mathematical objects, allowing us to formulate this theory rather elegantly. Riemannian manifolds have at their core simpler objects, augmented with additional structures. Here we progressively build up these structures, beginning with the core objects: topological spaces.

2.1.1.1 Topological Spaces

Topological spaces are the most general conceptualization of space that allow us to speak of connectedness, continuity, and convergence. They form the backbone of the types of spaces to which we will generalize elasticity theory. In essence, equipping a set of points with a topology gives that set a notion of "closeness" that we can later use to build up sensible notions of limits, and derivatives.

Definition 2.1.1 (Topological Space)**.** A *topological space* is a set of points M, and a collection of subsets of M, τ, that satisfies the following axioms:

- $\emptyset \in \tau$ and $M \in \tau$,
- τ is closed under arbitrary (finite or infinite) unions of its members,
- τ is closed under finite intersections of its members.

The elements of τ are called "open sets", and τ is called a "topology" on M.

Topological spaces are objects in the category Top, whose morphisms are continuous maps.

This definition is incredibly general, in fact it is too general for our purposes, as we have no ability to define concepts of distance, strain, or curvature at the moment. Because of this we will impose enough additional structure to carry over some of the key formalism from elasticity theory. Firstly, we need some separability conditions on our spaces, which allows us to uniquely define limit points.

Definition 2.1.2 (Hausdorff Space)**.** A topological space is called *Hausdorff* or a *Hausdorff space* if for any two distinct points $X_1 \neq X_2$, there are open sets U_1 and U_2 such that

- $X_1 \in U_1$ and $X_2 \in U_2$,
- $U_1 \cap U_2 = \emptyset$.

Informally, Hausdorff spaces are "separable by open sets" in that we can surround any two distinct points by two non-intersecting open sets. Additionally, we wish to impose certain properties on the types of open sets that are necessary to generate our topologies through unions.

Definition 2.1.3 (Second Countability)**.** A topological space M is *second countable* if there is a countable collection of open subsets of M, $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$, such that any open subset of M can be written as a union of elements of $\mathcal{U}$.

Any collection (countable or not) of open subsets that can be used in such a manner is called a "base," and therefore second countability requires the existence of a countable base. This countable base may or may not be the base typically used; second countability only requires its existence.

Example 1

We can give $\mathbb{R}$ a topology by defining a base of open subsets to be open intervals centered at each point. This base is not countable, but we can restrict it to a countable base consisting only of those intervals centered at rational numbers with rational radii. $\mathbb{R}$ is therefore a second-countable space, since the rationals are dense in the reals. It is also Hausdorff, as for any two distinct points X_1 and X_2, we can surround each with an open interval U_1 and U_2, which are disjoint so long as $r(U_1) + r(U_2) \leq |X_2 - X_1|$, where if $U = (x_0, x_1)$, $r(U) = \frac{|x_1 - x_0|}{2}$ is the radius of U. $\mathbb{R}^N$ can be shown to be a second countable Hausdorff space by a similar construction utilizing open balls of rational radius.

2.1.1.2 Smooth Manifolds, Atlases, and Charts

Smooth manifolds are a type of topological space that is similar enough to Euclidean spaces that we can do calculus on them. In particular, a manifold is a topological space that is "locally Euclidean" in the sense that it is locally homeomorphic to subsets of $\mathbb{R}^N$.

Definition 2.1.4 ((Smooth) Manifold)**.** A manifold $\mathcal{M}$ is a second countable Hausdorff space equipped with a collection $\{U_\alpha, \zeta_\alpha\}$ called an atlas, where U_α are open sets covering $\mathcal{M}$, and $\zeta_\alpha : U_\alpha \to \mathbb{R}^N$ are homeomorphisms onto open subsets of $\mathbb{R}^N$ called coordinate charts. Additionally we define

$$\zeta_{\alpha\beta} = \zeta_\beta \circ \zeta_\alpha^{-1}|_{\zeta_\alpha(U_\alpha \cap U_\beta)} : \zeta_\alpha (U_\alpha \cap U_\beta) \to \zeta_\beta (U_\alpha \cap U_\beta), \tag{2.1}$$

the *transition maps*. We say a manifold is smooth if the transition maps are smooth (see Figure 2.1).

Smooth manifolds are objects in the categories Man^p, with p being an integer, ∞, or ω, whose objects are p times continuously differentiable manifolds, i.e. manifolds with p times continuously differentiable transition maps, and whose morphisms are p times continuously differentiable maps, or for Man^ω, these maps must be analytic. As a convention, we shall always take $p > 0$ unless explicitly specified to the contrary.

Example 2

$\mathbb{R}$ with its standard topology is an analytic manifold, and hence is an object in every category Man^p. Consider the map $\varphi : \mathbb{R} \to \mathbb{R}$ sending $x \in \mathbb{R}$ to $x^3 \in \mathbb{R}$. This continuous map is an isomorphism in Man^0, since its inverse, the cube root, is continuous. However, this inverse function is not continuously differentiable at 0, hence this map is not an isomorphism in Man^p with $p > 0$.

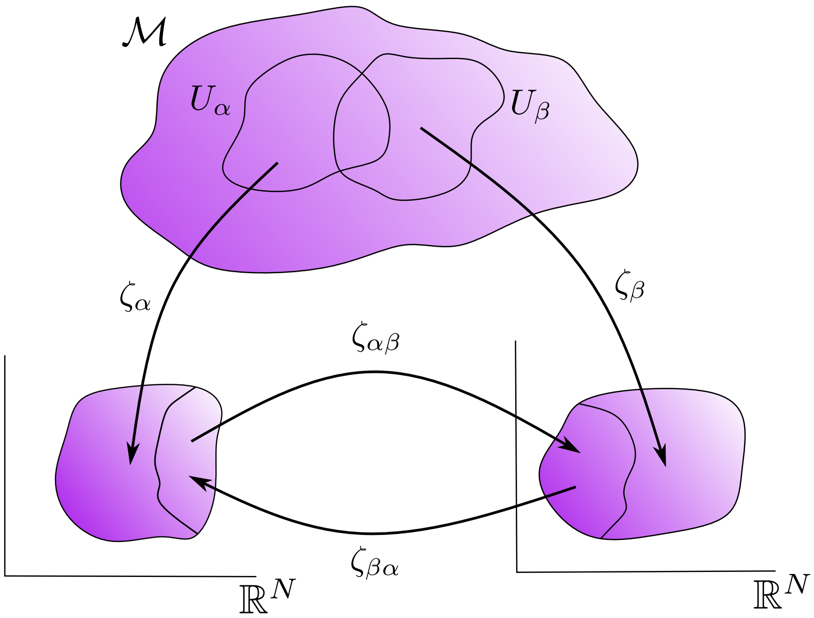


Figure 2.1: A manifold $\mathcal{M}$ equipped with coordinate charts ζ_α and transition maps $\zeta_{\alpha\beta}$.

2.1.1.3 Local Coordinates

Because $\mathbb{R}^N$ is the N-fold Cartesian product of $\mathbb{R}$ with itself, it carries with it a projection onto each factor. We can append any of these projection maps onto our charts to obtain the i-th local coordinate.

Definition 2.1.5 (Local Coordinates)**.** A set of local coordinates $Q^A(p) \in \mathbb{R}^N$ at a point $X \in U_\alpha$ is given by

$$\left(Q^1(X), ..., Q^N(X)\right) = \zeta_\alpha(X). \tag{2.2}$$

The i-th local coordinate is obtained by projecting $\left(Q^1(X), ..., Q^N(X)\right)$ onto the i-th factor. Likewise, given N functionally independent maps $\{Q^A(X)\}$ from an open set U to $\mathbb{R}$, the N-tuple $\left(Q^1(X), ..., Q^N(X)\right)$ forms a local system of coordinates on U.

Notice in particular, that the charts ζ_α do not have to be defined globally, and often cannot be. This is not a problem, as the transition maps $\zeta_{\alpha\beta}$ allow us to switch from one set of local coordinates to another on areas where more than one set is defined, and because the open sets in our atlas cover $\mathcal{M}$, we can always locally find a homeomorphism to $\mathbb{R}^N$. With these homeomorphisms, we can carry local information from our manifold to $\mathbb{R}^N$ and back, allowing us to consider limits and hence derivatives on $\mathcal{M}$ in terms limits and derivatives on $\mathbb{R}^N$. The coordinate charts are not unique, as we can append any homeomorphism from $\mathbb{R}^N$ to itself to a coordinate chart to obtain another valid coordinate chart.

Since we are considering smooth manifolds, given a local coordinate chart, we can invert these functions and identify a spatial point given the values of its coordinates. We can therefore (at least locally) write the position $X\left(Q^A\right)$ as a function of our coordinates, in the usual way.

2.1.1.4 Tangent Spaces

In order to speak about local features of a manifold, we need to first define the tangent space of a manifold at a point. Geometrically, the tangent space of a manifold at a point is the vector space that "best approximates" the manifold, where the manifold is locally thought of as a graph of a function. This interpretation, while useful for surfaces embedded in three-dimensional Euclidean space, becomes difficult to grapple with for higher dimensional manifolds. Instead, we shall first construct the cotangent space at a point in terms of smooth functions defined on our manifold, and obtain the tangent space by taking the dual of this space.

We begin by choosing a point X and a neighborhood U of X. We define $C^\infty(U)$ to be the set of functions $f : U \to \mathbb{R}$ where, for any coordinate chart $\zeta : U \to \mathbb{R}^N$, the function $f \circ \zeta^{-1} : \zeta(U) \subset \mathbb{R}^N \to \mathbb{R}$ is infinitely differentiable. This set is a real associative algebra under point-wise addition and multiplication. We define the ideal I_X to be the subset of functions that vanish at X.

The *cotangent space* of $\mathcal{M}$ at X, denoted $T^*_X\mathcal{M}$, is the quotient space I_X/I_X^2: the set of linear real-valued functions vanishing at X. A function in this space takes the form

$$f\left(Q^A\right) = \sum_{A=1}^{N} w_A Q^A; \tag{2.3}$$

where our coordinates are chosen such that $Q^A(X) = 0$; $T^*_X\mathcal{M}$ is the span of $\{Q^A\}$, a real vector space. When restricted to the quotient space I_X/I_X^2, the exterior derivative is an isomorphism, connecting this presentation of the cotangent space to the usual presentation in terms of the coordinate differentials dQ^A, which we will utilize later. The dual of this space is then the space of linear functions from $T^*_X\mathcal{M} \to \mathbb{R}$, which is equal to the span of the N linear maps taking each of $\{Q^A\}$ to 1, and the remainder of $\{Q^A\}$ to 0. These N maps are the coordinate partial derivatives $\frac{\partial}{\partial Q^A}$ at X, hence we define the tangent space $T_X\mathcal{M}$ to be the span of these partial derivatives. An element of this space takes the form

$$v = \sum_{A=1}^{N} v^A \frac{\partial}{\partial Q^A}. \tag{2.4}$$

At this point one may notice that we have placed our indices in a rather peculiar way; this is to help us introduce the Einstein summation convention and its abstract interpretation. Firstly, notice that in all cases where we have expressed elements of vector spaces as linear combinations of basis elements, we have had one raised index and one lowered index. This will always be the case, and hence from now on we will omit the summation sigma, and simply state that summation is implied whenever we have a repeated index appearing in product, where one instance is raised and the other is lowered.

With regard to the particular placement of indices, we have to first examine how our choice of coordinates affects our representation of a tangent vector v.

Suppose for $U \ni X$ we have a different system of coordinates $\{\tilde{Q}^B\}$. Since both of these systems of coordinates arise as a result of homeomorphisms with U, we can write one system of coordinates in terms of the other, namely we can write $\tilde{Q}^B\left(Q^A\right)$. We can then write v in terms of both of these coordinates, and obtain

$$v = v^A \frac{\partial}{\partial Q^A} = v^A \frac{\partial \tilde{Q}^B}{\partial Q^A} \frac{\partial}{\partial \tilde{Q}^B} = \tilde{v}^B \frac{\partial}{\partial \tilde{Q}^B}, \tag{2.5}$$

hence, if we want our representation to be independent of our choice of coordinates, we require the components v^A to transform by the following transformation law

$$\tilde{v}^B = \frac{\partial \tilde{Q}^B}{\partial Q^A} v^A. \tag{2.6}$$

Dually, we want elements of the cotangent space to be coordinate independent as well. Our new coordinates $\tilde{Q}^B$ are not necessarily linear functions of our old coordinates, but in the quotient I_X/I_X^2, all nonlinear terms map to 0, hence at the level of the cotangent space, our new coordinates are linear functions of the old coordinates. Therefore in the cotangent space, we have

$$\tilde{Q}^B = \frac{\partial \tilde{Q}^B}{\partial Q^A} Q^A, \tag{2.7}$$

or, utilizing the exterior derivative as a an isomorphism on the cotangent space, we have the more familiar

$$d\tilde{Q}^B = \frac{\partial \tilde{Q}^B}{\partial Q^A} dQ^A. \tag{2.8}$$

Writing an element of the cotangent space in terms of both of these coordinates, we have

$$w = \tilde{w}_B d\tilde{Q}^B = \tilde{w}_B \frac{\partial \tilde{Q}^B}{\partial Q^A} dQ^A = w_A dQ^A, \tag{2.9}$$

and we obtain the transformation law for covector components:

$$w_A = \frac{\partial \tilde{Q}^B}{\partial Q^A} \tilde{w}_B. \tag{2.10}$$

This reveals why we have chosen to place our indices in the way we have; the transformation laws automatically place our indices in the proper places, and so we raise indices for vector components and lower indices for covector components to be consistent with the transformation laws and the summation convention.

For these constructions, we chose coordinate systems, but then structured the transformation laws for our components so that this choice does not matter. Therefore, we take an abstract view of this index notation, where we do not view v^A as the A-th component of the vector v in some *particular* system of coordinates, but rather we view v^A as indicating which space contains v, and the appropriate transformation law. With this, we do not chose a particular basis, and we represent v as v^A. Additionally, the natural pairing of a vector v and a covector w is particularly nice notated this way: $v^A w_A$ not only represents the abstract pairing of v^A and w_A, but also upon choosing any particular coordinate system it tells us how to compute this pairing. This action extends to higher order tensors by simply introducing new indices; the tensor product of vectors v and u is simply $v^A u^B$. Because we can take tensor products of elements from different vector spaces, for each vector space V and its corresponding dual space V^*, we will introduce an index alphabet $\aleph_V$. With this, we can instantly recognize which space an arbitrary tensor belongs to simply by the placement of its indices and which alphabet they are selected from.

Example 3

Suppose we have two vector spaces V, and W, where $\aleph_V$ is capital Latin letters, and $\aleph_W$ is lower case Latin letters. We can instantly tell that the tensor $Z^{AB}{}_{C}{}^{a}{}_{b}$ is an element of the space $V \otimes V \otimes V^* \otimes W \otimes W^*$.

Notice that this convention is motivated by coordinates on the manifold $\mathcal{M}$, but so far we have only spoken of individual (co)-tangent spaces of $\mathcal{M}$. Since points in $\mathcal{M}$ are in some sense "smoothly connected" to each other, we likewise want to be able to connect these (co)-tangent spaces, which forces us to introduce the notion of (co)-tangent *bundles*.

2.1.2 The Functorial Nature of the Tangent Bundle

Here we build up the structure of the tangent bundle, with a particular emphasis on the functoriality of this construction. This is not intended to be a crash course in category theory, but rather the natural perspective to take in order to consider both the tangent bundle and the tangent map cohesively, and will prove to be useful when discussing group action on manifolds. With this perspective, the principal object to view is the tangent functor, which takes a manifold to its tangent bundle, and maps between manifolds to their induced tangent maps.

2.1.2.1 Fiber Bundles

We begin by discussing the tangent bundle of $\mathbb{R}^N$, which will provide a natural bridge to the tangent bundles of more general manifolds, since manifolds are locally homeomorphic to $\mathbb{R}^N$. First, however, we must define the type of object that the tangent bundle is, namely it is an example of a fiber bundle.

Definition 2.1.6 (Fiber Bundles)**.** A *fiber bundle* is a quadruple (E, π, B, F), where $\pi : E \to B$ is a continuous surjective map, and E, B, and F are topological spaces. E is called that total space, B, is called the base space, and F is called the fiber. Additionally, we have the following *local triviality* condition:

For each $X \in B$, there exists a contractible neighborhood U of X such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$.

A fiber bundle is *trivial* if $E = B \times F$, with π being the projection onto the first factor.

Fiber bundles make up the objects of the category Bun, whose morphisms are fiber bundle morphisms, which are given by pairs of continuous maps $f : E_1 \to E_2$, and $g : B_1 \to B_2$ satisfying

$$\pi_2 \circ f = g \circ \pi_1. \tag{2.11}$$

Notice in particular that because π_1 is surjective, the function g is determined by the function f.

Definition 2.1.7 (Vector Bundles)**.** A *vector bundle* is a fiber bundle whose fibers carry the structure of a finite dimensional real vector space.

All of the bundles we will consider are vector bundles. These bundles are the objects of the category Vec, which is the subcategory of Bun whose objects are vector bundles, and whose morphisms are *vector bundle morphisms*: fiber bundle morphisms where f is linear when restricted to each fiber of E_1.

A vector space can be considered a vector bundle whose base space is a single point. Vector fields can then be viewed as *sections* of a vector bundle.

Definition 2.1.8 (section)**.** A *section* of the vector bundle (E, π, B, F) is a continuous map $s : B \to E$ satisfying $\pi \circ s = \mathrm{id}_B$.

2.1.2.2 The Tangent Bundles

As a set, the tangent bundle of a manifold is the disjoint union of all of its tangent spaces, but equipped with a topology induced by the topology of the manifold. In the case of the tangent bundle $T\mathbb{R}^N$, the base space is $\mathbb{R}^N$ itself, and the fiber at each point is the tangent space of $\mathbb{R}^N$, which is itself isomorphic to $\mathbb{R}^N$. We then see that, because $\mathbb{R}^N$ itself is contractible, we can take it to be a neighborhood of all of its own points, and hence $T\mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$.

While the tangent bundle of $\mathbb{R}^N$ is trivial, not all tangent bundles are trivial. For a general manifold $\mathcal{M}$ of dimension N, the tangent space at each point is isomorphic to $\mathbb{R}^N$, and hence $T\mathcal{M}$, the tangent bundle of $\mathcal{M}$, should be locally homeomorphic to $\mathbb{R}^N \times \mathbb{R}^N$, since $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^N$.

Demonstrating this, for each open set in the atlas (U_α, ζ_α), the diffeomorphism ζ_α induces a diffeomorphism $\tilde{\zeta}_\alpha : TU_\alpha \approx U_\alpha \times \mathbb{R}^N \to \mathbb{R}^{2N}$ sending $\left(X, v^A \frac{\partial}{\partial Q^A}\right)$ to $\left(Q^1, ..., Q^N, v^1, ..., v^N\right)$. A set $S \subset T\mathcal{M}$ is open if $\tilde{\zeta}_\alpha \left(S \cap \pi^{-1}\left(U_\alpha\right)\right)$ is open in $\mathbb{R}^{2N}$ for all α, where π here is the projection $\pi : TU_\alpha \to U_\alpha$.

Essentially, we recognize that $T\mathcal{M}$ is a manifold, and we seek to construct an atlas for it from the atlas for $\mathcal{M}$. Because each U_α in $\mathcal{M}$'s atlas is homeomorphic to a subset of $\mathbb{R}^N$, $TU_\alpha = U_\alpha \times \mathbb{R}^N$ is homeomorphic to a subset of $\mathbb{R}^{2N}$. We define the open sets in the atlas of $T\mathcal{M}$ to be $\tilde{U}_\alpha = U_\alpha \times \mathbb{R}^N$. The intersection of these sets is clearly $\tilde{U}_\alpha \cap \tilde{U}_\beta = \left(U_\alpha \times \mathbb{R}^N\right) \cap \left(U_\beta \times \mathbb{R}^N\right) = (U_\alpha \cap U_\beta) \times \mathbb{R}^N$. Extending the charts, we define $\tilde{\zeta}_\alpha : \tilde{U}_\alpha \to \mathbb{R}^{2N}$ sending $\left(X, v^A \frac{\partial}{\partial Q^A}\right)$ to $\left(\zeta_\alpha\left(X\right), v^A\right)$. We now only have to extend the transition maps. We define the transition map $\tilde{\zeta}_{\alpha\beta} : \tilde{\zeta}_\alpha \left(\tilde{U}_\alpha \cap \tilde{U}_\beta\right) \to \tilde{\zeta}_\beta \left(\tilde{U}_\alpha \cap \tilde{U}_\beta\right)$ as the map $\left(\zeta_{\alpha\beta}, \frac{\partial \zeta^B_{\alpha\beta}}{\partial Q^A}\right)$, where $\zeta^B_{\alpha\beta}$ is $\pi_B \circ \zeta_{\alpha\beta}$, the projection of the image of $\zeta_{\alpha\beta}$ onto the B-th factor, and the linear map $\frac{\partial \zeta^B_{\alpha\beta}}{\partial Q^A}$ is left multiplication by the matrix whose (B, A)th component is $\frac{\partial \zeta^B_{\alpha\beta}}{\partial Q^A}$. Concretely the Bth component of $\frac{\partial \zeta^B_{\alpha\beta}}{\partial Q^A}\left(v^1, ..., v^N\right)$ is $\frac{\partial \zeta^B_{\alpha\beta}}{\partial Q^A} v^A$.

With this, we have constructed an atlas for $T\mathcal{M}$. Additionally, the above construction is functorial, hence we can define the covariant functor T from the category of smooth manifolds and smooth maps, Man^p, to the category of vector bundles and vector bundle morphisms Vec that sends manifolds to their tangent bundles and smooth maps to their induced tangent maps.

This functor takes the smooth map $\varphi : \mathcal{M} \to \mathcal{N}$, expressed in coordinates $\{Q^A\}$ on $\mathcal{M}$ and $\{q^a\}$ on $\mathcal{N}$ as $q^a = \varphi^a\left(Q^A\right)$, to the vector bundle morphism $T\varphi : T\mathcal{M} \to T\mathcal{N} = \left(\varphi, \frac{\partial \varphi^a}{\partial Q^A} \frac{\partial}{\partial q^a} \otimes dQ^A\right)$.

We notice especially that we use same system of coordinates for all points in a neighborhood, hence we shall associate our index alphabets not simply to an individual tangent space, and its dual space, but to the entire tangent bundle, and its dual bundle, the cotangent bundle, hence v^A is not simply thought of as a tangent vector, but as a tangent vector field. With this, we introduce the *tensor product bundle*.

Definition 2.1.9 (tensor product bundle)**.** Given two vector bundles E_1 and E_2 over the same base space B, with respective fibers F_1 and F_2, the tensor product bundle $E_1 \otimes E_2$ is the vector bundle over B whose fibers are $F_1 \otimes F_2$, where the tensor product here is the usual tensor product of vector spaces.

With this definition, given a tensor field on a manifold, we can immediately recognize which vector bundle that tensor field is a section of by our index notation, as we did before when we were only considering vector spaces, not vector bundles.

2.1.2.3 The Cotangent Bundle

The cotangent bundle $T^*\mathcal{M}$ is the dual bundle of $T\mathcal{M}$, where the fibers of $T^*\mathcal{M}$ are the dual spaces of those of $T\mathcal{M}$. These two bundles are isomorphic, but they are not naturally isomorphic, because the specific isomorphism depends on the choice of coordinates, in the same way that finite dimensional vector spaces are isomorphic to their dual spaces, but not in any canonical way.

One might suspect that like the construction of the tangent bundle, the construction of the cotangent bundle is a functor of some kind. While *fibers* of $T^*\mathcal{M}$ can be pulled back, the base space is pushed forward, hence from a categorical point of view, taking the cotangent bundle is a more complicated operation. When morphisms in Man^p are restricted to local diffeomorphisms, a covariant functor can be created since fibers can be pulled back under the (local) inverse morphism; we will not be ordinarily working in this category, except when speaking of induced group actions on the cotangent bundles of manifolds, since the maps induced by group actions are necessarily automorphisms.

Taking sections of both the tangent and cotangent bundle are functorial operations, covariant and contravariant respectively, as shown by the following diagram:

$$\begin{array}{ccc}
\Gamma\left(T\mathcal{M}\right) & \xrightarrow{\varphi_*} & \Gamma\left(T\mathcal{N}\right) \\
\Gamma_T \Big\Uparrow & & \Big\Uparrow \Gamma_T \\
\mathcal{M} & \xrightarrow{\varphi} & \mathcal{N} \\
\Gamma_{T^*} \Big\Downarrow & & \Big\Downarrow \Gamma_{T^*} \\
\Gamma\left(T^*\mathcal{M}\right) & \xleftarrow{\varphi^*} & \Gamma\left(T^*\mathcal{N}\right)
\end{array}$$

Here we have denoted the space of sections of the bundle $\mathcal{B}$ as $\Gamma\left(\mathcal{B}\right)$. These functors, Γ_T, and Γ_{T^*}, associate a smooth manifold $\mathcal{M}$ with the spaces of sections of $T\mathcal{M}$, and $T^*\mathcal{M}$ respectively, i.e. $\Gamma_T\left(\mathcal{M}\right) = \Gamma\left(T\mathcal{M}\right)$, and $\Gamma_{T^*}\left(\mathcal{M}\right) = \Gamma\left(T^*\mathcal{M}\right)$. The first associates the map $\varphi : \mathcal{M} \to \mathcal{N}$ with the pushforward map φ_* which sends the vector field v^A on $\mathcal{M}$ to the vector field $\frac{\partial \varphi^a}{\partial Q^A} v^A$ on $\mathrm{im}\left(\varphi\right) \subset \mathcal{N}$, and the second associates the map φ with the pullback map φ^*, which sends the covector field w_a on $\mathcal{N}$ to the covector field $\frac{\partial \varphi^a}{\partial Q^A} w_a$ on $\mathcal{M}$.

In the literature, sections of $T\mathcal{M}$ are sometimes called "contravariant vector fields," and those of $T^*\mathcal{M}$ are sometimes called "covariant vector fields," which is rather unfortunate, since the functor Γ_T is covariant, and the functor Γ_{T^*} is contravariant, but this convention is too deeply entrenched to be changed at this point. Hence we shall refer to them respectively as "vector fields" and "covector fields."

When the vector space V has an inner product $\langle \cdot, \cdot \rangle$, there is an obvious choice of isomorphism connecting V and V^* that "agrees" with the inner product. This isomorphism identifies each vector v with the covector $w \in V^*$ satisfying

$$w\left(u\right) = \langle v, u \rangle \qquad \forall u \in V. \tag{2.12}$$

In order for us to do this for vector bundles, we need to establish the appropriate notion of an inner product for vector bundles; Riemannian manifolds provide this structure.

2.1.3 Riemannian Manifolds

Riemannian manifolds are smooth manifolds with additional structure that endows them with sensible notions of length and angle.

Definition 2.1.10 (Riemannian Manifold)**.** A *Riemannian manifold* is a smooth manifold $\mathcal{M}$ equipped with a smoothly varying positive definite symmetric tensor field $\mathbf{K}(X) : T_X\mathcal{M} \otimes T_X\mathcal{M} \to \mathbb{R}$ called the *metric tensor* field.

Once equipped with a metric tensor, we shall explicitly denote a Riemannian manifold as the pair $(\mathcal{M}, \mathbf{K})$. We see that the metric tensor is a section of the vector bundle $T^*\mathcal{M} \otimes T^*\mathcal{M}$, since at each point it is a bilinear map taking pairs of tangent vectors to the reals. This allows us to construct what is known as the *musical isomorphism*.

2.1.3.1 The Musical Isomorphism

Now that we have an inner product on tangent spaces, we can identify vectors with covectors in a way that "agrees" with the metric tensor. This is known as the musical isomorphism.

Definition 2.1.11 (Flat)**.** The *flat* of a vector v^A is the covector $\left(v^\flat\right)_A = K_{AB} v^B$.

The applying the inverse of this isomorphism is called taking the "sharp" of a covector.

Definition 2.1.12 (Sharp)**.** The *sharp* of a covector w_A is the vector $\left(w^\sharp\right)^A = K^{AB} w_B$, where K^{AB} is the inverse of K_{AB}.

This action is sometimes referred to as *lowering* and *raising* indices, especially when the musical isomorphism is used implicitly.

2.1.3.2 Covariant Derivatives and the Levi-Civita Connection

Many physical theories are based on gradients of physical quantities. We want these gradients to be tensorial in the same way that the quantities themselves to be tensorial.

The problem we run into is that simply taking partial derivatives of components is not a tensorial operation; it inherently depends on the coordinates chosen.

Explicitly, under a change of coordinates, we have

$$\frac{\partial \tilde{v}^A}{\partial \tilde{Q}^B} = \frac{\partial Q^C}{\partial \tilde{Q}^B} \frac{\partial}{\partial Q^C} \left(\frac{\partial \tilde{Q}^A}{\partial Q^D} v^D \right) = \frac{\partial Q^C}{\partial \tilde{Q}^B} \frac{\partial \tilde{Q}^A}{\partial Q^D} \frac{\partial v^D}{\partial Q^C} + v^D \frac{\partial^2 \tilde{Q}^A}{\partial \tilde{Q}^B \partial Q^D}. \tag{2.13}$$

Here, we see that the first term obeys the proper tranformation law, but the Leibniz rule introduces a second term that results in a non-tensorial transformation.

This problem arises because there is no canonical way to compare tangent vectors lying in different tangent spaces. Therefore, we impose this additional structure, defining an *affine connection* on $\mathcal{M}$, which lets us prescribe a parallel transport law

$$dv^C = -\Gamma_{AB}{}^C v^B dQ^A, \tag{2.14}$$

where $\Gamma_{AB}{}^C$ are the called the connection coefficients. This defines a notion of parallel transport on our manifold, in that we say a vector is parallel transported in the direction dQ^A with respect to the connection $\Gamma_{AB}{}^C$ if it obeys equation (2.14).

This lets us define the covariant derivative, which measures how the directional change of a vector field deviates from a parallel transported vector field with the same initial value, and is defined to be

$$\nabla_B v^A = \frac{\partial v^A}{\partial Q^B} + \Gamma_{BC}{}^A v^C. \tag{2.15}$$

Clearly, if a vector is parallel transported in a particular direction, its covariant derivative in that direction vanishes. We know that partial derivatives do not transform as a tensor, hence we require that $\Gamma_{AB}{}^C$ also not transform as a tensor so that $\nabla_B v^A$ does transform tensorially.

We still have much flexibility in how $\Gamma_{AB}{}^C$ are defined, but once we impose a metric K_{AB}, there is a unique connection that satisfies two criteria:

- the connection is metric compatible, i.e. $\nabla_A K_{BC} = 0$,
- the connection is torsion free, i.e. $\Gamma_{AB}{}^C - \Gamma_{BA}{}^C = 0$.

The first of these demands that parallel transport does not change lengths or angles, and the second requires that vectors do not twist as they are parallel transported. This is endows the "imposed parallelism" defined by (2.14) with our intuitive notions of how parallel transport "ought" to behave, since without these criteria, we have simply defined a particular way of transporting vectors, and decreed that this method is by definition "parallel." This particular connection is called the *Levi-Civita* connection, and its components are determined by the metric tensor as follows:

$$\Gamma_{AB}{}^C = \frac{1}{2} K^{CD} \left(\frac{\partial K_{BD}}{\partial Q^A} + \frac{\partial K_{AD}}{\partial Q^B} - \frac{\partial K_{AB}}{\partial Q^D} \right). \tag{2.16}$$

For the purposes of this work, we shall only need to use the Levi-Civita connection, though it is by no means the only useful connection in the differential geometry literature. Since the Levi-Civita connection depends explicitly on the metric chosen, and we shall be simultaneously working with several different Riemannian manifolds, we shall denote the Levi-Civita connection of the reference configuration as ${\Gamma_{AB}}^C$ and the Levi-Civita connection of the current configuration as ${\gamma_{ab}}^c$. Additionally, when necessary, we shall denote the covariant derivative with respect to the Levi-Civita connection determined by the metric $\mathbf{K}$ as $\nabla^{\mathbf{K}}$.

The existence and uniqueness of this connection is the content of the Fundamental Theorem of Riemannian Geometry. Since some of our work shall consider Riemannian manifolds with metric tensors of various degrees of smoothness, we will impose by definition that a Riemannian manifold has at least a C^1 smooth metric tensor field, so that this connection can be defined, though in practice, the manifolds we will consider will be analytic at least almost everywhere.

2.1.4 Lie Groups

There are other manifolds that we will be working with, namely Lie Groups.

Definition 2.1.13 (Lie Group)**.** A *Lie Group G* is a group G that also possesses the structure of a manifold such that multiplication maps by group elements are diffeomorphisms.

Because there is a distinguished point in a Lie Group, namely the identity e, there is a distinguished tangent space, namely the tangent space at the identity T_eG. Elements of this tangent space are often denoted "infinitesimal generators" of G, and themselves form an algebra, denoted $\mathfrak{g}$ where addition is the usual vector addition in T_eG, and the multiplication is the linearization of commutation in G. This algebra, called a *Lie algebra*, is not associative; its product $[-,-]$ is antisymmetric, and satisfies the Jacobi identity

$$[v_1,[v_2,v_3]]+[v_2,[v_3,v_1]]+[v_3,[v_1,v_2]]=0,\ \forall v_1,v_2,v_3\in\mathfrak{g}. \tag{2.17}$$

Additionally, there is a map, called the *exponential map*, that identifies one dimensional subspaces of $\mathfrak{g}$ with one dimensional subgroups of G, or equivalently elements of $\mathfrak{g}$ with *uniformly parameterized* one dimensional subgroups of G, where the magnitude of the chosen element is identified with the "speed" at which this subgroup is traversed.

2.2 Group Actions On Manifolds

Central to our construction is the notion of symmetry, hence we must examine how groups can act on manifolds. We will identify elements of a group with automorphisms of our manifolds in a way that agrees with the group's structure. This identification is called a "group action."

Definition 2.2.1 (Group Action)**.** A left[1] action of a group G on a manifold $\mathcal{M}$ is a differentiable map $\rho : G \times \mathcal{M} \to \mathcal{M}$ satisfying

- $\rho(e, X) = X$ for all $X \in \mathcal{M}$, where e is the identity of G.
- $\rho(g_2, \rho(g_1, X)) = \rho(g_2 g_1, X)$ for all g_1, $g_2 \in G$ and for all $X \in \mathcal{M}$.

A manifold equipped with a left action of the group G is called a G-manifold.

We can then fix any particular group element g, and consider the map $\rho_g = \rho(g, -) : \mathcal{M} \to \mathcal{M}$. This is a map between manifolds, namely from $\mathcal{M}$ to itself, and hence induces a pushforward map on sections of the tangent bundle, and a pullback map on sections of the cotangent bundle. Since this map is by definition invertible, its inverse also induces pullback and pushforward maps, but now in the opposite directions, allowing us to move vector fields and covector fields freely back and forth.

In this way, given a group action on a manifold, there is a natural induced "action" on the tangent bundle, and likewise on the cotangent bundle. The question of whether or not this "action" is a proper group action depends on the action on the underlying manifold. Specifically, by definition, group actions need to be invertible, hence if the group action on our manifold has critical points, its "action" on the tangent bundle will not be invertible at these critical points, and hence will not be a group action. However, provided our group action is regular (i.e. it lacks critical points), then the induced maps on the tangent bundle and the cotangent bundle will be bona fide group actions. Therefore, if G's action on $\mathcal{M}$ agrees with the G's group structure, so will its induced action on $T\mathcal{M}$ and $T^*\mathcal{M}$ so long as its action on $\mathcal{M}$ is regular, since the identity map on $\mathcal{M}$ prolongs to the identity map on both $T\mathcal{M}$ and $T^*\mathcal{M}$.

This distinction highlights the importance of the categorical view of these objects. The induced map $\rho(g, -)$ must not only be invertible, it must also be a diffeomorphism in Man^p, which rules out the possibility of generating singular points as in

[1] There is an equivalent theory of right group actions, where the second axiom is modified to agree with right group multiplication rather than left multiplication; we shall only be using left actions, as any right action can be recast as a left action and vice versa.

example 2, since all derivatives of order p or lower must have continuous inverses. In general categorical language, in order to prolong a group action on a manifold to its tangent or cotangent bundles, the induced maps $\rho(g,-) : \mathcal{M} \to \mathcal{M}$ must be automorphisms in the appropriate category, namely Man^p for the tangent bundle, or for the cotangent bundle, Man^p with its morphisms restricted to local diffeomorphisms. It is from these categories that the covariant tangent and cotangent functors are defined, hence it would be inconsistent to attempt to prolong a group action to the tangent or cotangent bundle, unless the maps induced by the group action are automorphisms in these categories.

Example 4 (The special Euclidean group: SE(n))

The set of orientation-preserving isometries of Euclidean space $\mathbb{E}^n$ forms a group, called the special Euclidean group, denoted SE(n).

Such a transformation is an affine transformation, and thus can be written

$$X \to \mathbf{Q}X + c, \tag{2.18}$$

where $\mathbf{Q} \in \mathrm{SO}(n)$ is a rotation, and c is a translation. Therefore, we can specify an element of SE(n) by specifying the tuple $(\mathbf{Q}|c)$. Clearly, both SO(n) and T(n), the groups of n-dimensional rotations and n-dimensional translations respectively, are subgroups of SE(n), since $(\mathbf{Q},0)$ is a pure rotation and $(\mathbf{I},c)$ is a pure translation. One might expect that SE(n) =SO(n) ×T(n), but this is not true.

We must examine the group operation of SE(n); taking the composition of two special Euclidean motions, we have

$$X \to \mathbf{Q}_1 X + c_1 \to \mathbf{Q}_2 (\mathbf{Q}_1 X + c_1) + c_2 = \mathbf{Q}_2\mathbf{Q}_1 X + \mathbf{Q}_2 c_1 + c_2. \tag{2.19}$$

Therefore, it is clear that this group operation $\star$ takes the form

$$(\mathbf{Q}_2, c_2) \star (\mathbf{Q}_1, c_1) = (\mathbf{Q}_2\mathbf{Q}_1, \mathbf{Q}_2 c_1 + c_2), \tag{2.20}$$

and that the inverse of $(\mathbf{Q}, c)$ is $(\mathbf{Q}^{-1}, -\mathbf{Q}^{-1}c)$.

This group operation is not the group operation induced by the direct product of SO(n) and T(n), despite the fact that both SO(n) and T(n) are subgroups of SE(n). However, SE(n) is *a* product of SO(n) and T(n), since any element $(\mathbf{Q}, c)$ of SE(n) can be written either as the composition $(\mathbf{Q}|0) \star (\mathbf{I}|\mathbf{Q}^{-1}c)$ or $(\mathbf{I}|c) \star (\mathbf{Q}, 0)$, and the intersection of the subgroups SO(n) and T(n) is the trivial intersection $\{(\mathbf{I}, 0)\}$.

Examining the commutator of two elements of SE(n), we have

$$\left(\mathbf{Q}_2^{-1}, -\mathbf{Q}_2^{-1}c_2\right) \star \left(\mathbf{Q}_1^{-1}, -\mathbf{Q}_1^{-1}c_1\right) \star (\mathbf{Q}_2, c_2) \star (\mathbf{Q}_1, c_1)$$
$$= \left(\mathbf{Q}_2^{-1}\mathbf{Q}_1^{-1}\mathbf{Q}_2\mathbf{Q}_1, \mathbf{Q}_2^{-1}\mathbf{Q}_1^{-1}\mathbf{Q}_2 c_1 + \mathbf{Q}_2^{-1}\mathbf{Q}_1^{-1}c_2 - \mathbf{Q}_2^{-1}\mathbf{Q}_1^{-1}c_1 - \mathbf{Q}_2^{-1}c_2\right). \quad (2.21)$$

If we take $c_1 = 0$, this becomes $\left(\mathbf{Q}_2^{-1}\mathbf{Q}_1^{-1}\mathbf{Q}_2\mathbf{Q}_1, \mathbf{Q}_2^{-1}\mathbf{Q}_1^{-1}c_2 - \mathbf{Q}_2^{-1}c_2\right)$, which is not a pure rotation. Hence SO(n) is not a normal subgroup of SE(n). However, if we take $\mathbf{Q}_1 = \mathbf{I}$, we obtain $\left(\mathbf{I}, c_1 - \mathbf{Q}_2^{-1}c_1\right)$, which is a pure translation, hence T(n) is a normal subgroup of SE(n). The product structure compatible with this is called a "semi-direct" product, and is denoted SE(n)=SO(n) $\ltimes$T(n).

2.2.1 Group Equivariant Maps

We have seen how a group can act on a manifold in a way that agrees with the group's structure. We now wish to consider maps between manifolds, each equipped with a group action, and seek to define maps that agree with this added structure.

Definition 2.2.2 (G-equivariant map)**.** Fix a group G, and let f be a map from the G-manifold $\mathcal{M}$ with G-action $\rho_{\mathcal{M}}$ to the G-manifold $\mathcal{N}$ with G-action $\rho_{\mathcal{N}}$. The map f is *G-equivariant* if $\forall g \in G$

$$f \circ \rho_{\mathcal{M}}(g, -) = \rho_{\mathcal{N}}(g, -) \circ f, \quad (2.22)$$

i.e. f "commutes" with the group actions on $\mathcal{M}$ and $\mathcal{N}$.

Equivariant maps will play a key role in our discussion of symmetric deformations. The deformations themselves will not be the maps in question, but rather we shall examine the induced actions of SE(3) on various vector bundles over $\mathbb{E}^3$, and look at G-equivariant sections of these bundles, where G is a Lie subgroup of SE(3).

Example 5 (SE(3) equivariant sections of $T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$)

We shall be looking at right Cauchy Green stretch tensor fields, which are sections of $T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$. Specifically, we shall be considering sections of this bundle that are equivariant maps with respect to subgroups of SE(3), so we should examine the structure of such symmetric fields. A section being a map from the base space to the fiber above it, such a field is a map $\mathbf{C} : \mathbb{E}^3 \to T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$ satisfying $\mathrm{id}_{\mathbb{E}^3} = \pi \circ \mathbf{C}$.

For $\mathbb{E}^3$, we have the action

$$\rho_{\mathbb{E}^3}((\mathbf{Q}|c), X) = \mathbf{Q}X + c, \quad (2.23)$$

or in index notation,

$$\rho_{\mathbb{E}^3}\left(\left(Q^A{}_B|c^D\right), X^E\right) = Q^E{}_F X^F + c^E. \quad (2.24)$$

This induces the following actions on $T\mathbb{E}^3$ and $T^*\mathbb{E}^3$ respectively,

$$\rho_{T\mathbb{E}^3}\left(\left({Q^A}_B|c^D\right), \left(X^E, v^F\right)\right) = \left({Q^E}_A X^A + c^E, {Q^F}_B v^B\right), \tag{2.25}$$

$$\rho_{T^*\mathbb{E}^3}\left(\left({Q^A}_B|c^D\right), \left(X^E, w_F\right)\right) = \left({Q^E}_A X^A + c^E, M_{FD}{Q^D}_G M^{GB} w_B\right), \tag{2.26}$$

where M_{AB} is the Euclidean metric, which means the corresponding action on $T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$ is

$$\rho_{T\mathbb{E}^3\otimes T^*\mathbb{E}^3}\left(\left({Q^A}_B|c^D\right), \left(X^E, {C^G}_H\right)\right) = \left({Q^E}_A X^A + c^E, {Q^G}_A {C^A}_B M^{BD} {Q^F}_D M_{FH}\right). \tag{2.27}$$

The equivariance requirement demands that

$$ {C^A}_B\left({Q^D}_E X^E + c^D\right) = {Q^A}_E {C^E}_F\left(X^D\right) M^{FG} {Q^H}_G M_{HB}, \tag{2.28}$$

which in index free notation this is simply $\mathbf{C}\left(\mathbf{Q}X + c\right) = \mathbf{QC}\left(X\right)\mathbf{Q}^{\mathsf{T}}$, for $(\mathbf{Q}|c)$ in the symmetry subgroup $G \subset \mathrm{SE}\left(3\right)$.

Notice the rich interplay between the different structures here: $\mathbb{E}^3$ has an isometry group action determined by its metric tensor field, which is a symmetric section of $T^*\mathbb{E}^3 \otimes T^*\mathbb{E}^3$. We prolong this action to the bundle $T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$, making this bundle a G-manifold, with a specific induced G action. We then recognize sections as maps from $\mathbb{E}^3$ to $T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$ with additional structure, and seek such maps that are G-equivariant maps with respect to this specific induced G-action.

Chapter 3

Riemannian Elasticity and Its Role in Anelasticity

In the geometric formulation of nonlinear elasticity, we define the *ambient space* to be a Riemannian manifold $(\mathcal{S}, \mathbf{m})$, where $\mathbf{m}$ is a fixed background metric. Since we are interested in universal deformations that take place in Euclidean space we assume that the ambient space is Euclidean, i.e. that $\nabla^{\mathbf{m}}$ generates no curvature. We then define a *body* as a Riemannian manifold $(\mathcal{B}, \mathbf{M})$. We define a *motion* as an isotopy φ

$$\varphi : \mathcal{B} \times \mathbb{R} \to \mathcal{C} \subset \mathcal{S}. \tag{3.1}$$

parameterized by time t that gives a homeomorphism at each time t between the *reference configuration* $\mathcal{B}$ and the *physical configuration* at time t defined by $\varphi(\mathcal{B}, t)$ (see Figure 3.1): We use coordinate charts $\{Q^A\}$ and $\{q^a\}$ for $\mathcal{B}$ and $\mathcal{C}$, respectively. We utilize uppercase Latin letters to denote most quantities and indices in the reference configuration, and lowercase Latin letters to denote most quantities and indices in the physical configuration, in accordance with our previous discussion on vector bundles.

The homeomorphism at time t, $\varphi(-, t)$, can be interpreted physically as a map determining the position of a small piece of material at time t, given its position in $\mathcal{B}$, which is often interpreted as an initial position, though the important feature of the reference configuration is that it specifies the relaxed geometry of the body; it is not necessarily the initial configuration of the body. While in elasticity, $(\mathcal{B}, \mathbf{M})$ is Euclidean, for an anelastic system, $(\mathcal{B}, \mathbf{M})$ may not be Euclidean and, in such case, $\varphi_0 \equiv \varphi(-, 0)$ is not the identity map. Since $\varphi_0(\mathcal{B})$ and $\varphi(\mathcal{B}, t)$ correspond to physical configurations, we model them as subsets of Euclidean space, and hence, we identify the positions in the respective configurations with the position vectors X, and x.

Since we are interested in equilibrium states, we restrict our attention to a finite time $t^* > 0$ and suppress the explicit time-dependence so that we define, with a slight abuse of notation, $\varphi(\mathcal{B}) = \varphi(\mathcal{B}, t^*)$.

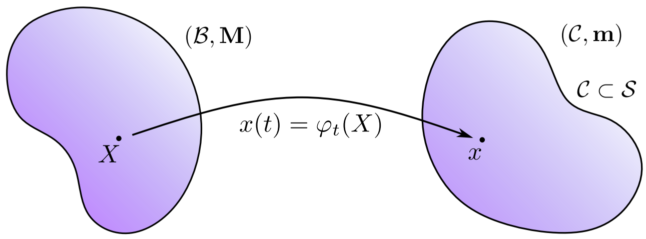


Figure 3.1: General motion from the reference configuration $\mathcal{B}$ to the current configuration $\mathcal{C}$.

The local properties of deformations are encapsulated in the derivative of the map φ that we explore next.

3.1 Kinematics

With our abuse of notation, we have deformations represented by diffeomorphisms between n-dimensional smooth manifolds

$$\varphi : \mathcal{B} \to \mathcal{C}, \tag{3.2}$$

where $\mathcal{B}$ and $\mathcal{C}$ represent the reference and the current configurations, respectively. To examine the dynamics of such deformations, one examines the induced tangent map

$$T\varphi : T\mathcal{B} \to T\mathcal{C}, \tag{3.3}$$

which sends the point $(X, v) \in T\mathcal{B}$ to $(\varphi(X), \mathbf{F}v) \in T\mathcal{C}$, where in coordinates $\{Q^A\} : \mathcal{B} \to \mathbb{R}^n$ and $\{q^a\} : \mathcal{C} \to \mathbb{R}^n$, one has

$$\mathbf{F}(X) = \left(\left. \frac{\partial \varphi^a}{\partial Q^A} \right|_X \right) \frac{\partial}{\partial q^a} \otimes dQ^A. \tag{3.4}$$

With this, one writes the vector bundle morphism $T\varphi$ as

$$T\varphi = (\varphi, \mathbf{F}), \tag{3.5}$$

i.e., the deformation φ maps points in $\mathcal{B}$ to points in $\mathcal{C}$, and the *deformation gradient* $\mathbf{F}$ maps tangent vectors of $\mathcal{B}$ at the point X to tangent vectors of $\mathcal{C}$ at the point $\varphi(X)$. Additionally it is often required, as we will here, that $\det \mathbf{F} > 0$ everywhere, which ensures that φ is a isomorphism in Man^1, and is orientation preserving.

To examine the adjoint of $\mathbf{F}$, we fix metrics $\mathbf{M}$ and $\mathbf{m}$ respectively on $\mathcal{B}$ and $\mathcal{C}$. The adjoint of $\mathbf{F}$ is then

$$\mathbf{F}^{\mathsf{T}}(X,t) : T_{\varphi(X)}\mathcal{C} \to T_X\mathcal{B}\,, \qquad \mathbf{m}\left(\mathbf{F}\mathbf{V},\mathbf{v}\right) = \mathbf{M}\left(\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\right)\,, \quad \forall\,\mathbf{V} \in T_X\mathcal{B},\, \mathbf{v} \in T_{\varphi(X)}\mathcal{C}\,. \tag{3.6}$$

In components, $(F^{\mathsf{T}})^A{}_a = M^{AB} m_{ab} F^b{}_B$. Note that the adjoint explicitly depends on the metrics $\mathbf{M}$ and $\mathbf{m}$. The right Cauchy Green stretch tensor is defined as

$$\mathbf{C}(X) = \mathbf{F}^{\mathsf{T}}(X)\mathbf{F}(X) : T_X\mathcal{B} \to T_X\mathcal{B}. \tag{3.7}$$

It is straightforward to see that $\mathbf{C}^\flat = \varphi^*(\mathbf{m})$, which has components $C_{AB} = F^a{}_A F^b{}_B m_{ab}$. The Jacobian relates the material and spatial Riemannian volume measures via the relation $dv(x,\mathbf{m}) = JdV(X,\mathbf{M})$, where J is given by

$$J = \sqrt{\frac{\det \mathbf{m}}{\det \mathbf{M}}}\,\det \mathbf{F}\,. \tag{3.8}$$

Over the years as additional physics were considered, theories accounting for strain other than that due to elastic deformations, called anelastic strain,[1] were formulated, making up the broad field of nonlinear anelasticity. Various theories of anelasticity, as documented in Sadik and Yavari [2017], have been developed in which the deformation gradient $\mathbf{F}$ is decomposed multiplicatively as

$$\mathbf{F} = \mathbf{A}\mathbf{G}, \tag{3.9}$$

where $\mathbf{G}$ generates purely anelastic strain and $\mathbf{A}$ generates purely elastic strain. These theories are in contrast to theories introducing a so-called "intermediate configuration," where the map φ is decomposed as the compositions of maps as $\varphi = \alpha \circ \gamma$. The practice of decomposing the deformation gradient has been met with some criticism [Casey and Naghdi, 1980], for while $\mathbf{F}$ is the derivative of φ, in general neither $\mathbf{A}$ nor $\mathbf{G}$ are derivatives of maps. Additionally a decomposition generating the required strain is not unique, as for any local isometry $\mathbf{Q}$, the alternative decomposition $\mathbf{F} = (\mathbf{A}\mathbf{Q})(\mathbf{Q}^{-1}\mathbf{G})$ generates the same strain.

Additionally, the multiplicative decomposition is often difficult to interpret. Because $\mathbf{G}$ is not generally integrable, the collection of deformed tangent spaces is typically treated as a disjoint union of vector spaces absent a topology connecting

[1] In the literature several other terms are used for anelastic strain. Examples are: *eigenstrain* [Mura, 1982], *initial strain* [Kondo, 1949], *inherent strain* [Ueda et al., 1975], and *transformation strain* [Eshelby, 1957].

them, which given the extensive development of elasticity in terms of manifolds and vector bundles on them, seems unnatural.

We seek to simultaneously resolve all of these problems, using the nonuniqueness of the multiplicative decomposition to attempt to construct a different decomposition $\mathbf{F} = \tilde{\mathbf{A}}\tilde{\mathbf{G}}$ satisfying the following two properties:

i) The new decomposition generates the same anelastic strain as the original, i.e.,

$$\mathbf{G}^{\mathsf{T}}\mathbf{G} = \tilde{\mathbf{G}}^{\mathsf{T}}\tilde{\mathbf{G}}. \tag{3.10}$$

ii) The decomposition is induced by the composition of maps between Riemannian manifolds, i.e.,

$$\varphi = \alpha \circ \gamma, \quad \text{with} \quad T\alpha = (\alpha, \tilde{\mathbf{A}}), \quad \text{and} \quad T\gamma = (\gamma, \tilde{\mathbf{G}}). \tag{3.11}$$

We seek to identify when this is possible, and when it is, to what extent the maps α and γ are uniquely defined.

Here, we demonstrate that we can start with a multiplicative decomposition of the deformation gradient and under suitable circumstances, construct an intermediate configuration. Given a factorization of φ through an intermediate configuration, we can trivially construct a multiplicative decomposition of the deformation gradient by applying the tangent functor to this factorization. Therefore, we derive the conditions under which these two formulations are equivalent, hence any when results derived from one formulation equally apply to the other.

3.1.1 Implicit Assumptions

We seek to make explicit the assumptions present in the initial posing of this question, and deduce as much as we can from these assumptions. For now, we restrict the discussion to each point, and suppress the dependence of $\mathbf{F}$ on X. We recognize first, that simply writing a factorization like this can be done while assuming very little. $\mathbf{G}$ and $\mathbf{A}$ do not even have to be square, and hence we, as of now, cannot even speak about their invertibility. In fact, the dimensions of $\mathbf{A}$ and $\mathbf{G}$ do not even have to be the same from point to point, provided that their product remains well defined. We do know that $\mathbf{G}$ has a left inverse and $\mathbf{A}$ has a right inverse, because $\mathbf{F}$ is invertible; $\mathbf{G}$ is injective and $\mathbf{A}$ is surjective. Explicitly, the right inverse of $\mathbf{A}$ and the left inverse of $\mathbf{G}$ are

$$(\mathbf{A})_R^{-1} = \mathbf{G}\mathbf{F}^{-1}, \quad (\mathbf{G})_L^{-1} = \mathbf{F}^{-1}\mathbf{A}. \tag{3.12}$$

We do, however, speak of strain, which requires that the manifolds $\mathcal{B}$ and $\mathcal{C}$ be equipped with metrics $\mathbf{M}$ and $\mathbf{m}$, respectively. Therefore, we shall replace the smooth manifold $\mathcal{B}$ with the Riemannian manifold $(\mathcal{B}, \mathbf{M})$, and the smooth manifold $\mathcal{C}$ with the Riemannian manifold $(\mathcal{C}, \mathbf{m})$. With this, we consider φ as the diffeomorphism

$$\varphi : (\mathcal{B}, \mathbf{M}) \to (\mathcal{C}, \mathbf{m}), \tag{3.13}$$

which lets us compute the Lagrangian strain induced by φ as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{I}\right), \tag{3.14}$$

where $\mathbf{I}$ is the identity tensor.

Since we want $\mathbf{G}$ to also generate strain, we require that its codomain have an inner product.[2] Labeling the codomain of $\mathbf{G}$ as U_X, we have $\mathbf{F} = \mathbf{AG}$, with

$$\mathbf{G} : T_X(\mathcal{B}, \mathbf{M}) \to U_X, \tag{3.15}$$

from the tangent space at X to some inner product space U_X and

$$\mathbf{A} : U_X \to T_{\varphi(X)}(\mathcal{C}, \mathbf{m}). \tag{3.16}$$

Because $\mathbf{G}$ is injective, its image is a subspace of U_X with the same dimension as $T_X(\mathcal{B}, \mathbf{M})$. If we consider

$$\hat{\mathbf{G}} : T_X(\mathcal{B}, \mathbf{M}) \to \mathrm{im}(\mathbf{G}), \tag{3.17}$$

such that $\mathbf{G} = \iota \circ \hat{\mathbf{G}}$ (ι being the inclusion map $\mathrm{im}(\mathbf{G}) \hookrightarrow U_X$), we have an invertible linear transformation. If we then restrict the domain of $\mathbf{A}$ to $\mathrm{im}(\mathbf{G})$, we obtain another invertible linear transformation

$$\hat{\mathbf{A}} : \mathrm{im}(\mathbf{G}) \to T_{\varphi(X)}(\mathcal{C}, \mathbf{m}). \tag{3.18}$$

The composition $\hat{\mathbf{A}}\hat{\mathbf{G}}$ is another multiplicative decomposition of $\mathbf{F}$, but consisting of invertible factors. Each factor is still not generally integrable, so this is not the decomposition we ultimately seek. Next, we turn our attention to the strain induced by $\mathbf{G}$. The inner product structure on U_X, being a positive-definite symmetric, bilinear map, admits a representation as a positive-definite symmetric tensor

$$\mathbf{h} : U_X \otimes U_X \to \mathbb{R}. \tag{3.19}$$

[2] This product is implicit in the definition of $\mathbf{G}^{\mathsf{T}}$. Therefore, it must be initially prescribed along with the decomposition $\mathbf{F} = \mathbf{AG}$, though in practice it is often implicitly taken to be the standard inner product on $\mathbb{R}^m$.

With this, we can now choose a basis for U_X, $\{\mathbf{e}_\alpha\}$, and its corresponding dual basis $\{\boldsymbol{\vartheta}^\alpha\}$, and express the requirement (3.10) in components as

$$M^{AD} G^\alpha{}_D h_{\alpha\beta} G^\beta{}_B = M^{AD} \tilde{G}^\alpha{}_D h_{\alpha\beta} \tilde{G}^\beta{}_B, \tag{3.20}$$

explicitly showing how the inner product structure on U_X is implicitly required to examine the strain induced by $\mathbf{G}$.[3] This structure generally varies from point to point. For now, we make no assumptions about the smoothnesses of $\mathbf{h}$, $\mathbf{G}$, or $\mathbf{A}$, other than noticing that the product $\mathbf{AG}$ is $\mathbf{F}$, which is smooth, being the induced tangent map of the diffeomorphism φ. Using $\mathbf{h}$, we can write U_X as the direct sum of orthogonal subspaces. In particular,

$$U_X = \mathrm{im}(\mathbf{G}) \oplus \mathrm{im}(\mathbf{G})^\perp, \tag{3.21}$$

and we can construct the orthogonal projection

$$\boldsymbol{\pi} : U_X \to \mathrm{im}(\mathbf{G}). \tag{3.22}$$

If we denote the inclusion $\iota^\perp : \mathrm{im}(\mathbf{G})^\perp \to U_X$ and the inclusion $\iota : \mathrm{im}(\mathbf{G}) \to U_X$ as before, this projection satisfies $\boldsymbol{\pi} \circ \iota = \mathrm{id}_{\mathrm{im}(\mathbf{G})}$, $\iota \circ \boldsymbol{\pi} \circ \mathbf{G} = \mathbf{G}$, and $\boldsymbol{\pi} \circ \iota^\perp = \mathbf{0}$. With these definitions, we have

$$\hat{\mathbf{G}} = \boldsymbol{\pi} \circ \mathbf{G}, \qquad \hat{\mathbf{A}} = \mathbf{A} \circ \iota. \tag{3.23}$$

The strain induced by $\mathbf{G}$ at each point X is then

$$\mathbf{E}_{\mathbf{G}} = \frac{1}{2}\left(\mathbf{G}^{\mathsf{T}}\mathbf{G} - \mathbf{I}\right). \tag{3.24}$$

Since $\mathbf{G}$ is injective, $\mathbf{E}_{\mathbf{G}}$ is also equal to the strain induced by $\hat{\mathbf{G}}$, which is given by the expression (3.24) with $\mathbf{G}$ replaced by $\hat{\mathbf{G}}$.

3.1.2 Construction of an Intermediate Configuration

Now that we have essentially taken full advantage of the assumptions implicit in the posing of the question, we identify the salient features used in the construction of the intermediate configuration. Given $\mathbf{G}$ and $\mathbf{h}$, we can construct a positive-definite, symmetric tensor field on $\mathcal{B}$ in a natural way. Denoting this tensor field by $\mathbf{H}$, we require that

$$\mathbf{H}(u, v) = \mathbf{h}(\mathbf{G}u, \mathbf{G}v), \quad \forall u, v \in T_X(\mathcal{B}, \mathbf{M}), \tag{3.25}$$

[3] The multiplicative decomposition of the deformation gradient in anelasticity and the geometric structures induced by moving frames were recently discussed in Sozio and Yavari [2019].

i.e., $\mathbf{H}$ is the pull-back of $\mathbf{h}$ under $\mathbf{G}$. Because $\mathbf{G}$ is injective, and $\mathbf{h}$ is positive-definite and symmetric, $\mathbf{H}$ is positive-definite and symmetric. If $\mathbf{H}$ is smooth with respect to X, it satisfies the definition of a metric tensor, and we can consider the Riemannian manifold $(\mathcal{B}, \mathbf{H})$; this will ultimately be the intermediate configuration.

Remark. As we consider consequences of Riemannian metrics with various degrees of smoothness, we require by definition here as before that a Riemannian manifold have a C^1 metric, which seems appropriate, as this is the minimal degree of smoothness required for the fundamental theorem of Riemannian geometry to hold.

The degree of smoothness of $\mathbf{H}$ will determine which geometric structures are well defined in the intermediate configuration. Specifically, if $\mathbf{H}(X)$ is C^1, we can construct the Levi-Civita connection on $(\mathcal{B}, \mathbf{H})$, if it is C^2, we can define the Riemann curvature, and if it is C^k with $k \geq 3$, we can construct an isometric embedding in a sufficiently high dimensional Euclidean space that is also C^k [Nash, 1956]. We will consider $\mathbf{H}$ to be at least C^1 for the remainder of this work. Notice also that $\mathbf{H} = \left(\mathbf{G}^\intercal\mathbf{G}\right)^\flat$, which, since $\mathbf{M}$ is at least C^1, means that $\mathbf{H}$ is C^1, if and only if $\mathbf{G}^\intercal\mathbf{G}$ is C^1 as well.[4]

In general, $(\mathcal{B}, \mathbf{M})$ and $(\mathcal{B}, \mathbf{H})$ are different Riemannian manifolds, since their specific geometries are defined by the distinct metrics $\mathbf{M}$ and $\mathbf{H}$. However, their topologies are the same, since the metric topology of any Riemannian manifold agrees with its underlying manifold topology [Lee, 2001], which is the same for both $(\mathcal{B}, \mathbf{M})$ and $(\mathcal{B}, \mathbf{H})$. Note that this result relies on the positive definiteness of the metric tensor field everywhere. In the case where the metric tensor $\mathbf{H}$ either becomes singular or loses positive definiteness on some subset, the topologies of $(\mathcal{B}, \mathbf{M})$ and $(\mathcal{B}, \mathbf{H})$ no longer necessarily agree.

Next we consider the map $\mathrm{id}_\mathcal{B} : (\mathcal{B}, \mathbf{M}) \to (\mathcal{B}, \mathbf{H})$, the unique identity map on the smooth manifold $\mathcal{B}$, where the domain is equipped with the metric $\mathbf{M}$, and the codomain is equipped with the metric $\mathbf{H}$. Essentially we map all subsets of points to themselves, but now consider these sets with a different geometry. Equivalently, we merely replace the metric tensor $\mathbf{M}$ with the metric tensor $\mathbf{H}$. This map is particularly nice, because it is smooth and defined at the level of points.

The total deformation then factors as

$$(\mathcal{B}, \mathbf{M}) \xrightarrow{\mathrm{id}_\mathcal{B}} (\mathcal{B}, \mathbf{H}) \xrightarrow{\tilde{\varphi}} (\mathcal{C}, \mathbf{m}), \tag{3.26}$$

where $\tilde{\varphi}$ is the same map as φ at the level of points, but with the manifold $(\mathcal{B}, \mathbf{H})$ as its domain. Additionally, by construction, the map $\mathrm{id}_\mathcal{B}$ defined above induces the

[4] In the case where the anelastic strain $\mathbf{E}_\mathbf{G}$ is directly prescribed instead of a multiplicative decomposition, we instead examine the smoothness of $\mathbf{H} = (2\mathbf{E}_\mathbf{G} + \mathbf{I})^\flat$, and proceed mutatis mutandis.

same strain as $\mathbf{G}$, since $\mathbf{H}$ is the pullback of $\mathbf{h}$ under $\mathbf{G}$, as can be explicitly checked in components

$$\delta^A{}_B H_{AC} \delta^C{}_D = G^\alpha{}_B h_{\alpha\beta} G^\beta{}_D. \tag{3.27}$$

Remark. Note that this is written in terms of the disjoint union of the frames $\{\mathbf{e}_\alpha\}$; the interpretation of these as a moving frame depends on the smoothness of $\mathbf{G}$. Specifically, $\mathbf{G}$ can be (highly) discontinuous in which case we just have a collection of frames, one frame for each tangent space, with no general relationship between the frames at different points. As a pathological example, $\mathbf{G}$ could be square and invertible on points with rational coordinates, and non-square (but still full rank) on the other points, in which case $\{\mathbf{e}_\alpha\}$ at each point would not even have a consistent number of elements, let alone be interpretable as a moving frame. In this case, $\mathbf{h}$ would not be consistently the same size, but $\mathbf{H}$, its pull back, would be, which is why the smoothness of $\mathbf{H}$ is the important criteria, not the smoothness of $\mathbf{h}$ or $\mathbf{G}$ separately.

Note that the induced factorization $T\varphi = T\tilde{\varphi} \circ T\,\mathrm{id}_{\mathcal{B}}$ generates the same strain as the original factorization $\mathbf{F} = \mathbf{AG}$. As desired, it is induced by the composition of maps between Riemannian manifolds. Therefore, taking

$$\gamma = \mathrm{id}_{\mathcal{B}}, \qquad \alpha = \tilde{\varphi}, \tag{3.28}$$

satisfies both of our desired conditions (3.10) and (3.11), since

$$(\varphi, \mathbf{F}) = T\varphi = T\,(\alpha \circ \gamma) = T\alpha \circ T\gamma = (\alpha, \tilde{\mathbf{A}}) \circ (\gamma, \tilde{\mathbf{G}}) = (\alpha \circ \gamma, \tilde{\mathbf{A}}\tilde{\mathbf{G}}). \tag{3.29}$$

Here, $\tilde{\mathbf{G}}$ maps a vector in $T_X(\mathcal{B}, \mathbf{M})$ to the corresponding vector in $T_X(\mathcal{B}, \mathbf{H})$, which despite appearing like the identity, generates strain because the inner product $\mathbf{H}$ is typically different than the inner product $\mathbf{M}$. If these inner product structures are dropped, then $\tilde{\mathbf{G}}$ does become the identity on $T_X\mathcal{B}$. The other factor is therefore $\tilde{\mathbf{A}} = \mathbf{F}\tilde{\mathbf{G}}^{-1}$, which clearly exists since $\tilde{\mathbf{G}}$ is trivially invertible.

We now wish to determine the precise relationship between $\mathbf{G}$ and $\tilde{\mathbf{G}}$. We have already seen that they generate the same strain, but $\tilde{\mathbf{G}}$ is induced by a map between Riemannian manifolds, while $\mathbf{G}$ is simply a linear map applied to each tangent space of $(\mathcal{B}, \mathbf{M})$ separately. Given a basis $\{\mathbf{E}_i\},\ i = 1, ..., n$ for $T_X(\mathcal{B}, \mathbf{M})$, there is a natural induced basis for $\mathrm{im}(\mathbf{G})$ given as

$$\bar{\mathbf{e}}_i = \mathbf{G}\mathbf{E}_i, \tag{3.30}$$

i.e., we use the images of the basis vectors of $T_X(\mathcal{B}, \mathbf{M})$ as the basis for $\mathrm{im}(\mathbf{G})$. On these bases, $\mathbf{G}$ has the particularly nice form

$$\mathbf{G} = \delta^{\alpha}{}_{A} \bar{\mathbf{e}}_{\alpha} \otimes \mathbf{E}^{A}, \tag{3.31}$$

where $\{\mathbf{E}^A\}$ is the dual basis of $\{\mathbf{E}_A\}$. The linear map

$$\begin{aligned} \boldsymbol{\varepsilon} : \mathrm{im}(\mathbf{G}) &\to T_X(\mathcal{B}, \mathbf{H}), \\ \bar{\mathbf{e}}_i &\mapsto \mathbf{E}_i, \end{aligned} \tag{3.32}$$

is then an isometry. Note that $\boldsymbol{\varepsilon}$ depends on the specific decomposition, but the composition $\boldsymbol{\varepsilon} \circ \boldsymbol{\pi} \circ \mathbf{G} = \tilde{\mathbf{G}}$, only depends on the strain generated. Therefore, provided that $\mathbf{H}$ is smooth enough for $(\mathcal{B}, \mathbf{H})$ to be a Riemannian manifold, we can take the disjoint images of the tangent spaces of the manifold $(\mathcal{B}, \mathbf{M})$ under the map $\mathbf{G}$, and embed them *isometrically* into the tangent bundle $T(\mathcal{B}, \mathbf{H})$ via the maps $\boldsymbol{\varepsilon}$. These images then inherit the unique Levi-Civita connection based on the metric tensor $\mathbf{H}$. If $\boldsymbol{\varepsilon}$ is interpreted passively as a change of basis, the decomposition $\mathbf{F} = \hat{\mathbf{A}}\hat{\mathbf{G}}$ can be interpreted as the same decomposition as $\mathbf{F} = \tilde{\mathbf{A}}\tilde{\mathbf{G}} = (\hat{\mathbf{A}} \circ \boldsymbol{\varepsilon}^{-1})(\boldsymbol{\varepsilon} \circ \hat{\mathbf{G}})$, expressed in terms of an anholonomic basis on $(\mathcal{B}, \mathbf{H})$. Notice that postcomposition by $\boldsymbol{\varepsilon} \circ \boldsymbol{\pi}$ effectively removes any nonsmoothness or discontinuities present in $\mathbf{G}$. If $\hat{\mathbf{G}}$ is C^1, we can determine the object of anholonomicity for the anholonomic basis on $(\mathcal{B}, \mathbf{H})$ described above, though this is superfluous to the main result. The construction of the intermediate configuration and its associated maps is schematically shown in the diagram in Figure 3.2. Starting from $\mathbf{F} = \mathbf{A}\mathbf{G}$, and knowledge of the inner products on U_X, provided the pullback of $\mathbf{h}$ is C^1, this diagram can be constructed. The maps $\boldsymbol{\varepsilon}$ and ι are isometries, and all paths commute, apart from those starting at U_X.

Next, we wish to see to what extent the factorization $\varphi = \alpha \circ \gamma$ is unique. Suppose we have a different intermediate configuration, $(\mathcal{M}, \mathbf{K})$ that satisfies the two requirements (3.10) and (3.11). We then write $\varphi = \alpha' \circ \gamma'$ with

$$\begin{aligned} \gamma' &: (\mathcal{B}, \mathbf{M}) \to (\mathcal{M}, \mathbf{K}), \\ \alpha' &: (\mathcal{M}, \mathbf{K}) \to (\mathcal{C}, \mathbf{m}). \end{aligned} \tag{3.33}$$

Denoting the tangent maps as

$$T\gamma' = (\gamma', \mathbf{G}'), \quad T\alpha' = (\alpha', \mathbf{A}'), \tag{3.34}$$

the anelastic strain requirement (3.10) demands

$$\tilde{\mathbf{G}}^{\mathsf{T}}\tilde{\mathbf{G}} = \mathbf{G}'^{\mathsf{T}}\mathbf{G}', \tag{3.35}$$

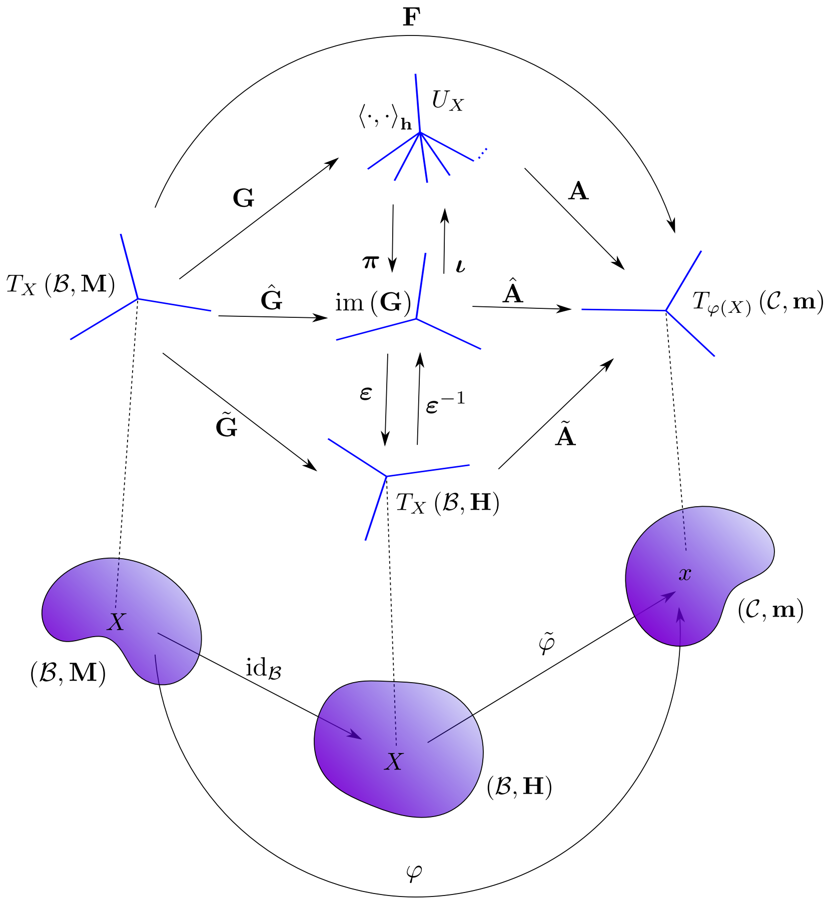


Figure 3.2: Construction of the intermediate configuration.

which in components reads

$$\delta^A{}_B H_{AC} \delta^C{}_D = G'^A{}_B K_{AC} G'^C{}_D. \tag{3.36}$$

This, however, is exactly the condition for $(\mathcal{B}, \mathbf{H})$ and $(\mathcal{M}, \mathbf{K})$ to be isometric under the map $\gamma' \circ \gamma^{-1}$. Hence, the intermediate configuration we constructed is unique up to isometry. This lets us finally state our main result:

Theorem 3.1.1 (Isometric Integrability)**.** *Given the deformation* $\varphi : (\mathcal{B}, \mathbf{M}) \to (\mathcal{C}, \mathbf{m})$ *with the tangent map* $T\varphi = (\varphi, \mathbf{F})$ *satisfying* $\det \mathbf{F} > 0$*, and a multiplicative decomposition* $\mathbf{F} = \mathbf{AG}$ *with* $\mathbf{G}$ *mapping into an inner product space, if* $\mathbf{G}^\intercal \mathbf{G}$ *is at least* C^1*, there exists a Riemannian manifold* $(\mathcal{B}, \mathbf{H})$*, unique up to isometry, such that the composition of maps* $\varphi = \alpha \circ \gamma : (\mathcal{B}, \mathbf{M}) \xrightarrow{\gamma} (\mathcal{B}, \mathbf{H}) \xrightarrow{\alpha} (\mathcal{C}, \mathbf{m})$ *induces a factorization* $\mathbf{F} = \tilde{\mathbf{A}}\tilde{\mathbf{G}}$ *satisfying* $\mathbf{G}^\intercal \mathbf{G} = \tilde{\mathbf{G}}^\intercal \tilde{\mathbf{G}}$.

3.2 Equilibrium Equations

Having established that we can treat the elastic portion of anelastic deformations via a Riemannian intermediate configuration, we now need to establish the relevant equilibrium equations. From this point forth, we shall ignore the initial reference configuration since its geometry does not play a role in the dynamics. We shall simply consider the portion of the deformation governed by elasticity; essentially we shall consider our constructed Riemannian intermediate configuration as the reference configuration. As we treat our bodies and ambient spaces intrinsically as abstract Riemannian manifolds, we will derive the governing laws for elastic Riemannian bodies, and specialize to particular spaces of interest, e.g. the ambient Euclidean space. This specialization can be eschewed if one is interested in solid mechanics in curved space, for example Banner, Brunnhilde, and Odinson's discussion of "geodetic strain" near spacetime singularities [Waititi, 2017].

3.2.1 Action Functional

Until now, we have made little distinction between a body, and the space it occupies. We must now make this difference explicit, since physical quantities like energy and momentum will be stored in our body by its own deformation, i.e., changes in its intrinsic geometry, but our body moves through space, and hence its geometry is dictated by the geometry of the space it occupies.

We define density fields over our body, namely the mass density $\rho_0\left(Q^A;t\right)$, and the strain-energy density function $W\left(\frac{\partial q^a}{\partial Q^A}, m_{ab}, M_{AB}, Q^A\right)$. We consider the reference configuration to be fixed, and hence, for mass to be conserved, we require $\int_{m\subset\mathcal{M}} \rho_0 dQ^1\wedge ... \wedge dQ^N$ to be constant in time for all submanifolds $m\subset\mathcal{M}$.

When taking partial derivatives, we consider the quantities Q^A and t to be independent, and so we have

$$\frac{d}{dt}\int_{m\subset\mathcal{M}} \rho_0 dQ^1\wedge ... \wedge dQ^N = \int_{m\subset\mathcal{M}} \frac{\partial\rho_0}{\partial t} dQ^1\wedge ... \wedge dQ^N = 0, \tag{3.37}$$

which localizes to become $\frac{\partial\rho_0}{\partial t}=0$, which is trivially satisfied for $\rho_0\left(Q^A;t\right)=\rho_0\left(Q^A\right)$.

We seek to obtain the relevant balance laws for these systems, and to do so, we consider Lagrangian densities of the form

$$\begin{aligned}\mathcal{L}&\left(q^a, \frac{\partial q^a}{\partial t}, \frac{\partial q^a}{\partial Q^A}, m_{ab}, M_{AB}\right)\\ &= \frac{1}{2}\rho_0\left(Q^A\right) m_{ab}\left(q^a\right)\frac{\partial q^a}{\partial t}\frac{\partial q^b}{\partial t} - U\left(q^a\right) - W\left(\frac{\partial q^a}{\partial Q^A}, m_{ab}, M_{AB}, Q^A\right),\end{aligned} \tag{3.38}$$

and demand that physical solutions $q^a\left(Q^A;t\right)$ render the action

$$S=\int_{t_0}^{t_f}\int_{\mathcal{M}}\mathcal{L}\left(q^a,\frac{\partial q^a}{\partial t},\frac{\partial q^a}{\partial Q^A},m_{ab},M_{AB}\right)dQ^1\wedge...\wedge dQ^N dt, \qquad (3.39)$$

stationary.

We consider such a state $q^a=\varphi^a\left(Q^A;t\right)$, and consider nearby states of the form

$$\tilde{q}^a=q^a+\epsilon\xi^a, \quad \tilde{dq}^a=dq^a+\epsilon d\xi^a+\epsilon\Omega^a{}_b dq^b, \qquad (3.40)$$

where $\Omega^a{}_b$ is skew symmetric, i.e. $m_{ac}\Omega^c{}_b+m_{bc}\Omega^c{}_a=0$, and we take the nearby state to agree with our solution on the boundary of our domain in both space and time. Here ξ^a is an infinitesimal translation and $\Omega^a{}_b$ is an infinitesimal rotation. Because q^a leaves S stationary, for the nearby state $\tilde{q}^a$ we have

$$\tilde{S}-S=\int_{t_0}^{t_f}\int_{\mathcal{M}}\tilde{\mathcal{L}}-\mathcal{L}dQ^1\wedge...\wedge dQ^N dt, \qquad (3.41)$$

where $\tilde{\mathcal{L}}=\mathcal{L}\left(\tilde{q}^a,\frac{\partial\tilde{q}^a}{\partial t},\frac{\partial\tilde{q}^a}{\partial Q^A},\tilde{m}_{ab},M_{AB}\right)$. We consider the limit as our nearby states approach the stationary solution and demand for all ξ^a and $\Omega^a{}_b$,

$$\delta S=\lim_{\epsilon\to 0}\frac{\tilde{S}-S}{\epsilon}=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0}^{t_f}\int_{\mathcal{M}}\tilde{\mathcal{L}}-\mathcal{L}dQ^1\wedge...\wedge dQ^N dt=0. \qquad (3.42)$$

We Taylor expand $\tilde{\mathcal{L}}$ about the state q^a to obtain to order ϵ

$$\tilde{\mathcal{L}}-\mathcal{L}\approx\epsilon\Bigg(\frac{\partial\mathcal{L}}{\partial q^a}|_{q^i}\xi^a+2\frac{\partial\mathcal{L}}{\partial m_{ab}}|_{q^i}m_{ad}\gamma_{cb}{}^d\xi^c+\frac{\partial\mathcal{L}}{\partial\frac{\partial q^a}{\partial t}}|_{q^i}\left(\frac{\partial\xi^a}{\partial t}+\Omega^a{}_b\frac{\partial q^b}{\partial t}\right)$$
$$+\frac{\partial\mathcal{L}}{\partial\frac{\partial q^a}{\partial Q^A}}|_{q^i}\left(\frac{\partial\xi^a}{\partial Q^A}+\Omega^a{}_b\frac{\partial q^b}{\partial Q^A}\right)\Bigg). \qquad (3.43)$$

We used the fact that $\frac{\partial\mathcal{L}}{\partial m_{ab}}$ is symmetric and we evaluate the partial derivative of the metric m_{ab} by means of the metric compatibility condition of the Levi-Civita connection $\gamma_{ab}{}^c$.

$$\nabla_a m_{bc}=\frac{\partial m_{bc}}{\partial q^a}-m_{dc}\gamma_{ab}{}^d-m_{bd}\gamma_{ac}{}^d=0. \qquad (3.44)$$

We insert this into $\delta S=0$, abbreviate $dQ^1\wedge...\wedge dQ^N\equiv dV$, and evaluate the limit to obtain

$$\int_{t_0}^{t_f}\int_{\mathcal{M}}\Bigg(\frac{\partial\mathcal{L}}{\partial q^a}\xi^a+2\frac{\partial\mathcal{L}}{\partial m_{cb}}m_{cd}\gamma_{ab}{}^d\xi^a+\frac{\partial\mathcal{L}}{\partial\frac{\partial q^a}{\partial t}}\left(\frac{\partial\xi^a}{\partial t}+\Omega^a{}_b\frac{\partial q^b}{\partial t}\right)$$
$$+\frac{\partial\mathcal{L}}{\partial\frac{\partial q^a}{\partial Q^A}}\left(\frac{\partial\xi^a}{\partial Q^A}+\Omega^a{}_b\frac{\partial q^b}{\partial Q^A}\right)\Bigg)dV dt=0. \qquad (3.45)$$

We seek to shift the derivatives off of the term ξ^a using the product rule, so first we examine the quantity $\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}} \frac{\partial \xi^a}{\partial t}$. By the product rule we write this as

$$\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}} \frac{\partial \xi^a}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}} \xi^a \right) - \xi^a \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}}. \tag{3.46}$$

The first term can be integrated out; because ξ^a vanishes at t_f and t_0, this integral is 0.

Next we look at the term $\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \frac{\partial \xi^a}{\partial Q^A}$. We have by the product rule

$$\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \frac{\partial \xi^a}{\partial Q^A} = \frac{\partial}{\partial Q^A} \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \xi^a \right) - \xi^a \frac{\partial}{\partial Q^A} \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}}. \tag{3.47}$$

Next, we have the divergence of $\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \xi^a$ as

$$\nabla_A \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \xi^a \right) = \frac{\partial}{\partial Q^A} \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \xi^a \right) + \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^B}} \xi^a {\Gamma_{AB}}^A, \tag{3.48}$$

and we have the divergence of $\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}}$ as

$$\nabla_A \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \right) = \frac{\partial}{\partial Q^A} \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \right) - \frac{\partial \mathcal{L}}{\partial \frac{\partial q^c}{\partial Q^A}} \frac{\partial q^b}{\partial Q^A} {\gamma_{ab}}^c + \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^B}} {\Gamma_{AB}}^A. \tag{3.49}$$

With these, we can write

$$\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \frac{\partial \xi^a}{\partial Q^A} = \nabla_A \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \xi^a \right) - \xi^a \nabla_A \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \right) - \xi^a \frac{\partial \mathcal{L}}{\partial \frac{\partial q^c}{\partial Q^A}} \frac{\partial q^b}{\partial Q^A} {\gamma_{ab}}^c. \tag{3.50}$$

We can use the divergence theorem to remove this first term, which then vanishes. Therefore we can write the stationary action condition as

$$\int_{t_0}^{t_f} \int_{\mathcal{M}} \left(\frac{\partial \mathcal{L}}{\partial q^a} + 2 \frac{\partial \mathcal{L}}{\partial m_{cb}} m_{cd} {\gamma_{ab}}^d - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}} - \nabla_A \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \right) - \frac{\partial \mathcal{L}}{\partial \frac{\partial q^c}{\partial Q^A}} \frac{\partial q^b}{\partial Q^A} {\gamma_{ab}}^c \right) \xi^a + \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}} \frac{\partial q^b}{\partial t} + \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \frac{\partial q^b}{\partial Q^A} \right) {\Omega^a}_b dQ^1 \wedge ... \wedge dQ^N dt = 0. \tag{3.51}$$

We require this to be true for all ξ^a and all skew symmetric ${\Omega^a}_b$, which lets us obtain the balance laws

$$\frac{\partial \mathcal{L}}{\partial q^a} + 2 \frac{\partial \mathcal{L}}{\partial m_{cb}} m_{cd} {\gamma_{ab}}^d - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}} - \nabla_A \left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}} \right) - \frac{\partial \mathcal{L}}{\partial \frac{\partial q^c}{\partial Q^A}} \frac{\partial q^b}{\partial Q^A} {\gamma_{ab}}^c = 0, \tag{3.52}$$

and

$$\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}}\frac{\partial q^b}{\partial t}+\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}}\frac{\partial q^b}{\partial Q^A}\right)m_{bc}=\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial q^c}{\partial t}}\frac{\partial q^b}{\partial t}+\frac{\partial \mathcal{L}}{\partial \frac{\partial q^c}{\partial Q^A}}\frac{\partial q^b}{\partial Q^A}\right)m_{ba}. \tag{3.53}$$

We then calculate the necessary partial derivatives of the Lagrangian density considered above.

$$\frac{\partial \mathcal{L}}{\partial q^a}=\rho_0\frac{\partial q^b}{\partial t}\frac{\partial q^c}{\partial t}m_{bd}\gamma_{ac}{}^{d}-\frac{\partial U}{\partial q^a}, \tag{3.54}$$

$$\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}}=\rho_0 m_{ab}\frac{\partial q^b}{\partial t}, \tag{3.55}$$

$$\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial Q^A}}=-\frac{\partial W}{\partial \frac{\partial q^a}{\partial Q^A}}\equiv -P_a{}^{A}, \tag{3.56}$$

$$\frac{\partial \mathcal{L}}{\partial m_{ab}}=\frac{1}{2}\rho_0\frac{\partial q^a}{\partial t}\frac{\partial q^b}{\partial t}-\frac{\partial W}{\partial m_{ab}}, \tag{3.57}$$

where we have defined the Piola stress $P_a{}^{A}$.

With this, we compute the quantities

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial q^a}{\partial t}}=\rho_0\frac{\partial m_{ab}}{\partial q^c}\frac{\partial q^b}{\partial t}\frac{\partial q^c}{\partial t}+\rho_0 m_{ab}\frac{\partial^2 q^b}{\partial t^2}=\rho_0\left(m_{db}\gamma_{ca}{}^{d}+m_{da}\gamma_{cb}{}^{d}\right)\frac{\partial q^b}{\partial t}\frac{\partial q^c}{\partial t}+\rho_0 m_{ab}\frac{\partial^2 q^b}{\partial t^2}, \tag{3.58}$$

and hence we have the balance of linear momentum

$$\nabla_A\left(P_a{}^{A}\right)+\frac{\partial \mathcal{L}}{\partial q^a}+\left(P_c{}^{A}F^b{}_A-2\frac{\partial W}{\partial m_{bd}}m_{dc}\right)\gamma_{ab}{}^{c}=\rho_0 m_{ab}\left(\frac{\partial^2 q^b}{\partial t^2}+\frac{\partial q^c}{\partial t}\frac{\partial q^d}{\partial t}\gamma_{cd}{}^{b}\right), \tag{3.59}$$

where we have labeled $\frac{\partial q^a}{\partial Q^A}=F^a{}_A$. With this, the angular momentum equation then becomes

$$P_a{}^{A}F^b{}_A m_{bc}=P_c{}^{A}F^b{}_A m_{ba}. \tag{3.60}$$

If we suppose that the strain energy density W only depends on the deformation through deformed lengths in the current configuration, we demand that W only depends on $F^a{}_A$ and m_{ab} through the quantity $C_{AB}=F^a{}_A m_{ab}F^b{}_B=\left(\varphi^*\left(\mathbf{m}\right)\right)_{AB}$, the pull back of the metric tensor. We can compute $\frac{\partial W}{\partial m_{ab}}$, and $P_a{}^{A}=\frac{\partial W}{\partial F^a{}_A}$ in terms of $\frac{\partial W}{\partial C_{AB}}$ through simple application of the chain rule giving

$$P_a{}^{A}=2\frac{\partial W}{\partial C_{AB}}F^c{}_B m_{ca}, \tag{3.61}$$

$$\frac{\partial W}{\partial m_{ab}}=\frac{\partial W}{\partial C_{AB}}F^a{}_A F^b{}_B. \tag{3.62}$$

This causes the linear momentum equation to simplify to become

$$\rho_0 m_{ab}\left(\frac{\partial^2 q^b}{\partial t^2}+\frac{\partial q^c}{\partial t}\frac{\partial q^d}{\partial t}\gamma_{cd}{}^b\right)=\nabla_A\left(P_a{}^A\right)+\rho_0 b_a, \tag{3.63}$$

where we have defined the body force $\frac{\partial \mathcal{L}}{\partial q^a}=\rho_0 b_a$. Notice that these are covariant forms of the usual balance laws

$$\rho_0 a=\operatorname{Div}\mathbf{P}+\rho_0 b, \tag{3.64}$$

$$\mathbf{P}\mathbf{F}^{\mathsf{T}}=\mathbf{F}\mathbf{P}^{\mathsf{T}}. \tag{3.65}$$

Additionally, we reformulate this in terms of the Cauchy stress $\sigma_a{}^b=J^{-1}P_a{}^A F^b{}_A$.

$$\nabla_A P_a{}^A=\left(\nabla_b P_a{}^A\right)F^b{}_A=J\left(\left(\nabla_b P_a{}^A\right)F^b{}_A J^{-1}\right)=J\nabla_b\left(P_a{}^A F^b{}_A J^{-1}\right)=J\nabla_b\sigma_a{}^b, \tag{3.66}$$

where we have made use of the Piola identity in the form $\nabla_b\left(J^{-1}F^b{}_A\right)=0$. With this, we can divide the linear momentum equation by J, and using the current configuration mass density $\rho=\frac{\rho_0}{J}$ we have the conservation of linear momentum equation

$$\rho a=\operatorname{div}\boldsymbol{\sigma}+\rho b, \tag{3.67}$$

or in coordinates

$$\rho m_{ab}\left(\frac{\partial^2 q^b}{\partial t^2}+\frac{\partial q^c}{\partial t}\frac{\partial q^d}{\partial t}\gamma_{cd}{}^b\right)=\nabla_b\left(\sigma_a{}^b\right)+\rho b_a, \tag{3.68}$$

and the conservation of angular momentum equation

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^{\mathsf{T}} \tag{3.69}$$

or in components

$$\sigma_a{}^d m_{db}=\sigma_b{}^d m_{da}. \tag{3.70}$$

We seek equilibrium states without body forces, so we accordingly set a and b to 0, and have

$$\operatorname{div}\boldsymbol{\sigma}=0. \tag{3.71}$$

3.2.2 Constitutive equations

In this paper we restrict ourselves to bodies that are isotropic in the absence of eigenstrains. We also assume that the elastic deformations are incompressible. The referential left Cauchy Green stretch tensor is defined as $\mathbf{B}^\sharp=\left(\varphi^{-1}\right)_*\left(\mathbf{m}^{-1}\right)$ and has components $B^{AB} = (F^{-1})^A{}_a (F^{-1})^B{}_b \, m^{ab}$, where m^{ab} are components of $\mathbf{m}^{-1}$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are defined as

$$\mathbf{c}^\flat = \left(\varphi^{-1}\right)^* \mathbf{M}, \; c_{ab} = \left(F^{-1}\right)^A{}_a \left(F^{-1}\right)^B{}_b \, M_{AB}, \tag{3.72}$$

and

$$\mathbf{b}^\sharp = \varphi_*(\mathbf{M}^{-1}), \; b^{ab} = F^a{}_A F^b{}_B M^{AB}, \tag{3.73}$$

where $\mathbf{b}^\sharp$ is often called the Finger deformation tensor. The two tensors $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants, which are denoted by I_1, I_2, and I_3 [Ogden, 1984]. In the case of an isotropic solid the energy function W depends only on I_1, I_2, and I_3. If the material is incompressible $I_3 = 1$, $W = W(I_1, I_2)$, and the Cauchy stress has the following representation [Simo and Marsden, 1984]

$$\boldsymbol{\sigma}^\sharp = \left(-p + 2I_2 \frac{\partial W}{\partial I_2}\right) \mathbf{m}^{-1} + 2\frac{\partial W}{\partial I_1} \mathbf{b}^\sharp - 2\frac{\partial W}{\partial I_2} \mathbf{c}^\sharp, \tag{3.74}$$

where p is a Lagrange multiplier that is associated with the incompressibility constraint $J = 1$. Additionally, since p is a Lagrange multiplier to be determined, we can redefine it to express the Cauchy stress as

$$\boldsymbol{\sigma}^\sharp = -p\mathbf{m}^{-1} + 2\frac{\partial W}{\partial I_1} \mathbf{b}^\sharp - 2\frac{\partial W}{\partial I_2} \mathbf{c}^\sharp. \tag{3.75}$$

3.3 Ericksen's problem

With this reformulation, we can now consider the generalized version of Ericksen's problem applicable to anelastic systems. The *anelastic Ericksen problem* relaxes the requirement that $\mathcal{B} \subset \mathbb{E}^3$ and is stated as follows: *Determine all Riemannian manifolds* $(\mathcal{B}, \mathbf{M})$*, and all maps* $\varphi : \mathcal{B} \to \mathcal{C}$ *with* $\mathcal{C} \subset \mathbb{E}^3$ *that can be sustained by an arbitrary incompressible isotropic hyperelastic solid with suitable boundary tractions.*

From this, we recover the classical Ericksen problem by appending the condition that the reference configuration $(\mathcal{B}, \mathbf{M})$ is Euclidean i.e. the Riemann-Christoffel curvature tensor based on $\mathbf{M}$ vanishes. Equivalently, this requires the Riemann-Christoffel curvature tensor based on $\mathbf{c}^\flat$ to vanish since $\mathbf{M}$ is the pullback of $\mathbf{c}^\flat$ under the diffeomorphism φ.

These problem are treated locally in the sense that the equilibrium equations are locally satisfied by these deformations for arbitrary incompressible isotropic hyperelastic materials. We do not address non-local problems such as self-intersection or self-contact beyond requiring that our solutions locally be homeomorphisms, which is guaranteed by the condition $\det \mathbf{F} > 0$. This condition ensures that in a neighborhood our solution is an embedding, rather than an immersion.

Chapter 4

The Anelastic Ericksen Problem

Here, we are interested in generalizing Ericksen's problem to nonlinear anelasticity. The first step in this research program was to generalize Ericksen's theorem for compressible isotropic materials to anelastic deformations. By using a geometric formulation, it was proved that only covariantly homogeneous deformations are universal [Yavari and Goriely, 2016]. The second step, considered here, is to extend the current classification of isotropic incompressible nonlinear elasticity to isotropic incompressible nonlinear anelasticity. This general problem is more involved than the classical Ericksen problem as we have to determine simultaneously both the anelastic and elastic components that render the solutions universal.

While anelastic deformations can be modeled through a multiplicative decomposition, we have demonstrated that it is equivalent but more appropriate in our context to model them as a stress-free evolution into a general Riemannian manifold, the *material manifold*, via some anelastic process. This non-Euclidean material configuration contains all the information of the anelastic processes. It can then be mapped by an elastic deformation into the current Euclidean configuration. Only the strain induced by the elastic component appears in the strain-energy formulation, which models the notion that the anelastic deformation changes the relaxed geometry of the material, and that only the further elastic deformation stores energy by straining the material. In looking for universal solutions, both the Riemannian metric of the material manifold and the elastic deformation are unknowns to be determined.

Here, we extend the currently known families of incompressible universal solutions to the anelastic setting in an appropriately symmetric way, which will be precisely defined shortly. In the process of doing this, we discover that under these symmetry conditions, all families exhibit a branch of *generic universal solutions* that contain arbitrary functions as parameters, but some families also contain *anomalous universal* *solution* branches outside of these, whose form is fixed up to a finite number of constants.

First, we summarize the known universal solution families. Then, we compute the appropriate symmetry group for each family and impose this symmetry on the metric tensor field a priori. We then formulate the problem of extending the universal families to the anelastic setting and outline techniques used in our analysis. We then derive the form of the generic solutions for each family and obtain the constant invariant conditions that are necessary for anomalous solutions to exist. We present the general form of the anomalous solutions, relegating the explicit calculations to the appendix. Next, we examine the overlap of these families of solutions, showing that the six classical families become redundant upon moving to the anelastic setting. Finally, we present some visualizations of the Riemannian geometry of strains and stresses induced by these anomalous solutions, and summarize our results.

4.1 The Known Universal Deformations

We begin with the known families of universal solutions in the absence of eigenstrains. We merely present them and direct the reader to the original papers for their derivation [Ericksen, 1955, 1954, Klingbeil and Shield, 1966, Singh and Pipkin, 1965]. The corresponding deformation gradients are derived explicitly in Goriely [2017]. The emphasis and novelty here is in identifying the particular type of symmetry associated to each family as these will play a key role in the generalization of the problem to anelastic systems. Expressed in terms of the standard Cartesian coordinates $\{x, y, z\}$, cylindrical polar coordinates $\{r, \theta, z\}$, or spherical polar coordinates $\{r, \theta, \varphi\}$ (letting capital letters denote reference configuration coordinates, and lower case letters denote current configuration coordinates), we have the following six universal families.

Family 0: Homogeneous Deformations. Using Cartesian coordinates $\{q^a\} = \{x, y, z\}$ and $\{Q^A\} = \{X, Y, Z\}$, this family is most compactly expressed as

$$q^a = F^a{}_A \, Q^A + c^a, \tag{4.1}$$

where $F^a{}_A$ is a constant tensor with $\det F^a{}_A = 1$, and c^a is a constant vector. All homogeneous deformations amount to a combination of stretching, shearing, and rotation. The shearing vanishes on a specific set of principal basis vectors by virtue of the polar decomposition of $F^a{}_A$; a deformation of this type is depicted in Figure 4.1. The form of equation (4.1) immediately reveals that the deformation gradient is

${F^a}_A$, as evidenced by the induced tangent map $dq^a = {F^a}_A\, dQ^A$. Since the deformation gradient $\mathbf{F}$ is spatially constant, ${F^a}_A(Q^A) = {F^a}_A$, the transformation

$$Q^A \to \bar{Q}^A = Q^A + C^A, \quad \forall C^A \in \mathbb{R}, \tag{4.2}$$

leaves the deformation gradient unchanged, and hence, C_{AB} remains unchanged. In terms of symmetry group, we notice that the action $Q^A \to Q^A + C^A$ is precisely the action of $\mathrm{T}(3) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$, and the constancy of C_{AB} is equivalent to C_{AB} being a $\mathrm{T}(3)$-equivariant section of $T^*\mathbb{E}^3 \otimes T^*\mathbb{E}^3$.

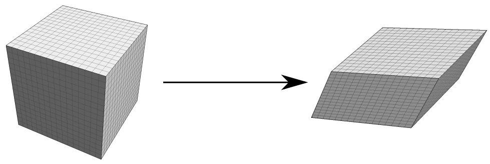


Figure 4.1: Family 0: homogeneous deformations.

Family 1: Bending, Stretching, and Shearing of a Rectangular Block. When expressed using cylindrical polar coordinates $\{q^a\} = \{r, \theta, z\}$ and Cartesian coordinates $\{Q^A\} = \{X, Y, Z\}$ in the current and reference configurations, respectively, universal deformations in this family take the form

$$r = \sqrt{A\,(2X + D)}, \quad \theta = B\,(Y + E)\,, \quad z = \frac{Z}{AB} - BCY + F, \tag{4.3}$$

though the parameters E and F only correspond to rigid motions, and hence, can be safely disregarded. These parameters generate rotation around and translation along the $r = 0$ axis, as seen in Figure 4.2. We can compute the deformation gradient, which when expressed on mixed orthonormal frames, takes the form

$$\mathbf{F} = \frac{A}{r}\mathbf{e}_r \otimes \mathbf{E}_X + Br\mathbf{e}_\theta \otimes \mathbf{E}_Y - BC\mathbf{e}_z \otimes \mathbf{E}_Y + \frac{1}{AB}\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{4.4}$$

or in terms of a mixed coordinate bases, we have the components

$$\left[{F^a}_A = \frac{\partial q^a}{\partial Q^A}\right] = \begin{bmatrix} \frac{A}{r} = \sqrt{\frac{A}{2X+D}} & 0 & 0 \\ 0 & B & 0 \\ 0 & -BC & \frac{1}{AB} \end{bmatrix}. \tag{4.5}$$

Working in terms of the mixed bases will often be advantageous when computing symmetries and the components of arbitrary metric tensors, as these bases are automatically induced by the coordinate map. Additionally, we demand $AB \neq 0$ to ensure the deformation is invertible; the incompressibility condition is automatically satisfied. We compute C_{AB} as

$$[C_{AB}] = \begin{bmatrix} \frac{A}{2X+D} & 0 & 0 \\ 0 & B^2\left(A\left(2X+D\right)+C^2\right) & -\frac{C}{A} \\ 0 & -\frac{C}{A} & \frac{1}{A^2B^2} \end{bmatrix}, \tag{4.6}$$

and note that this remains unchanged under the transformation

$$Y \to \bar{Y} = Y + C_1, \quad Z \to \bar{Z} = Z + C_2, \quad \forall C_1, C_2 \in \mathbb{R}. \tag{4.7}$$

This is precisely the action of $\mathrm{T}(2) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$, and is equivalent to C_{AB} being a $\mathrm{T}\,(2)$-equivariant section of $T^*\mathbb{E}^3 \otimes T^*\mathbb{E}^3$.

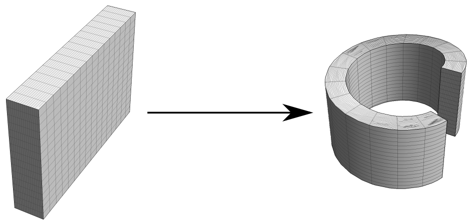


Figure 4.2: Family 1: stretching, shearing, and bending of a rectangular block around a cylinder.

Family 2: Straightening, Stretching, and Shearing of an Annular Wedge. Deformations in this family are most naturally expressed using Cartesian and cylindrical polar coordinates $\{q^a\} = \{x, y, z\}$ and $\{Q^A\} = \{R, \Theta, Z\}$ in the current and reference configurations respectively, and take the form

$$x = \frac{1}{2}AB^2R^2 + D, \quad y = \frac{\Theta}{AB} + E, \quad z = \frac{Z}{B} + \frac{C\Theta}{AB} + F. \tag{4.8}$$

An example of one of these deformations is depicted in Figure 4.3. The corresponding deformation gradient is

$$\mathbf{F} = AB^2R\mathbf{e}_x \otimes \mathbf{E}_R + \frac{1}{ABR}\mathbf{e}_y \otimes \mathbf{E}_\Theta + \frac{C}{ABR}\mathbf{e}_z \otimes \mathbf{E}_\Theta + \frac{1}{B}\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{4.9}$$

or, in terms of the induced coordinate bases

$$[F^a{}_A] = \begin{bmatrix} AB^2R = \sqrt{2AB^2\left(x-D\right)} & 0 & 0 \\ 0 & \frac{1}{AB} & 0 \\ 0 & \frac{C}{AB} & \frac{1}{B} \end{bmatrix}. \tag{4.10}$$

We demand $AB \neq 0$ to ensure the deformation is invertible, and as in the previous case, the incompressibility condition is automatically satisfied. Thus

$$[C_{AB}] = \begin{bmatrix} A^2B^4R^2 & 0 & 0 \\ 0 & \frac{1+C^2}{A^2B^2} & \frac{C}{AB^2} \\ 0 & \frac{C}{AB^2} & \frac{1}{B^2} \end{bmatrix}. \tag{4.11}$$

The transformation

$$\Theta \to \bar{\Theta} = \Theta + \Phi, \quad Z \to \bar{Z} = Z + K, \quad \forall \Phi, K \in \mathbb{R}, \tag{4.12}$$

leaves these components unchanged. This is the action of $\mathrm{SO}(2) \times \mathrm{T}(1) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$, and C_{AB} is a $\mathrm{SO}(2) \times \mathrm{T}(1)$-equivariant section of $T^*\mathbb{E}^3 \otimes T^*\mathbb{E}^3$.

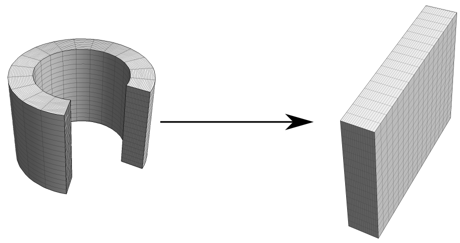


Figure 4.3: Family 2: straightening and subsequent stretching and shearing of an annular wedge.

Family 3: Torsion, Extension, and Shearing of an Annular Wedge. When expressed in cylindrical polar coordinates, deformations in this family take the form

$$r^2 = AR^2 + B, \quad \theta = C\Theta + DZ + G, \quad z = E\Theta + FZ + H, \tag{4.13}$$

and an example of a deformation from this family is depicted in Figure 4.4. The deformation gradient can be naturally expressed on orthonormal cylindrical polar bases as

$$\mathbf{F} = \frac{AR}{r}\mathbf{e}_r \otimes \mathbf{E}_R + \frac{Cr}{R}\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + Dr\mathbf{e}_\theta \otimes \mathbf{E}_Z + \frac{E}{R}\mathbf{e}_z \otimes \mathbf{E}_\Theta + F\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{4.14}$$

or equivalently on the coordinate bases, it has components

$$[F^a{}_A] = \begin{bmatrix} \sqrt{\frac{A(r^2-B)}{r^2}} = \sqrt{\frac{A^2R^2}{AR^2+B}} & 0 & 0 \\ 0 & C & D \\ 0 & E & F \end{bmatrix}. \tag{4.15}$$

We have the incompressibility condition $A\left(CF-DE\right)=1$, which also ensures invertibility. Thus, C_{AB} is written as

$$[C_{AB}]=\begin{bmatrix}\frac{A^2R^2}{AR^2+B} & 0 & 0\\ 0 & C^2\left(AR^2+B\right)+E^2 & CD\left(AR^2+B\right)+EF\\ 0 & CD\left(AR^2+B\right)+EF & D^2\left(AR^2+B\right)+F^2\end{bmatrix}, \tag{4.16}$$

and notice that C_{AB} does not depend on Θ or Z, which again means that C_{AB} is a $\mathrm{SO}(2)\times\mathrm{T}(1)$-equivariant section of $T^*\mathbb{E}^3\otimes T^*\mathbb{E}^3$.

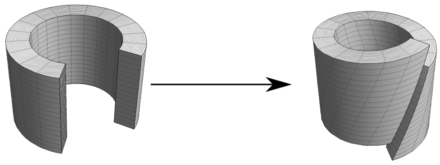


Figure 4.4: Family 3: twisting, extension, and axial shearing of a cylindrical sector.

Family 4: Inflation/Eversion of a Sphere. In spherical coordinates $\{q^a\}=\{r,\theta,\phi\}$ and $\{Q^A\}=\{R,\Theta,\Phi\}$, deformations in this family take the form

$$r^3=\pm R^3+A^3,\quad \theta=\pm\Theta,\quad \phi=\Phi. \tag{4.17}$$

An example of one of these deformations is depicted in Figure 4.5. The deformation gradient on orthonormal bases reads

$$\mathbf{F}=\pm\frac{R^2}{r^2}\mathbf{e}_r\otimes\mathbf{E}_R+\frac{r}{R}\left(\mathbf{e}_\phi\otimes\mathbf{E}_\Phi\pm\mathbf{e}_\theta\otimes\mathbf{E}_\Theta\right). \tag{4.18}$$

or, in terms of the coordinate bases, we have the components

$$[F^a{}_A]=\begin{bmatrix}\frac{R^2}{(\pm R^3+A^3)^{\frac{2}{3}}}=\frac{\left(\pm r^3\mp A^3\right)^{\frac{2}{3}}}{r^2} & 0 & 0\\ 0 & \pm1 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{4.19}$$

Incompressibility and invertibility are trivially satisfied. Thus

$$[C_{AB}]=\begin{bmatrix}\frac{R^4}{(\pm R^3+A^3)^{4/3}} & 0 & 0\\ 0 & \left(\pm R^3+A^3\right)^{2/3}\sin^2\Phi & 0\\ 0 & 0 & \left(\pm R^3+A^3\right)^{2/3}\end{bmatrix}. \tag{4.20}$$

We can then represent this tensor on an orthonormal spherical basis (using the standard Euclidean metric) as

$$\mathbf{C}(X)=\frac{R^4}{\left(\pm R^3+A^3\right)^{4/3}}\mathbf{E}_R\otimes\mathbf{E}_R+\frac{\left(\pm R^3+A^3\right)^{2/3}}{R^2}\left(\mathbf{I}-\mathbf{E}_R\otimes\mathbf{E}_R\right), \tag{4.21}$$

where $\mathbf{I}$ is the identity tensor, and $\mathbf{E}_R = \frac{X}{|X|}$. This obeys the symmetry transformation

$$\mathbf{C}(\mathbf{Q}X) = \mathbf{Q}\mathbf{C}(X)\mathbf{Q}^{\mathsf{T}}, \quad \forall \mathbf{Q} \in \mathrm{SO}(3), \tag{4.22}$$

which means $\mathbf{C}$ is an $\mathrm{SO}(3) \subset \mathrm{SE}(3)$-equivariant section of $T\mathbb{E}^3 \otimes T^*\mathbb{E}^3$.

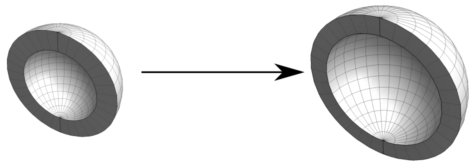


Figure 4.5: Family 4: inflation of a spherical cap.

Family 5: Inflation, Bending, Extension, and Azimuthal Shearing of an Annular Wedge. When expressed in cylindrical polar coordinates $\{q^a\} = \{r, \theta, z\}$ and $\{Q^A\} = \{R, \Theta, Z\}$, deformations in this family take the form

$$r = AR, \quad \theta = B \log R + C\Theta + D, \quad z = EZ + F. \tag{4.23}$$

An example of one of these deformations is presented in Figure 4.6. The deformation gradient expressed on orthonormal bases is written as

$$\mathbf{F} = A\mathbf{e}_r \otimes \mathbf{E}_R + AB\mathbf{e}_\theta \otimes \mathbf{E}_R + AC\mathbf{e}_\theta \otimes \mathbf{E}_\Theta + E\mathbf{e}_z \otimes \mathbf{E}_Z, \tag{4.24}$$

or, on the coordinate bases

$$[F^a{}_A] = \begin{bmatrix} A & 0 & 0 \\ \frac{B}{R} & C & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ \frac{AB}{r} & C & 0 \\ 0 & 0 & E \end{bmatrix}. \tag{4.25}$$

In order to satisfy incompressibility, we have $A^2CE = 1$. This family is peculiar, as the stretch generated by the other inhomogeneous families has an eigenvector along the direction of inhomogeneity, but this one does not. Additionally, the invariants of $\mathbf{b}$ for this family are spatially constant.

When we generalize these deformations to include an anelastic component, we have to change the incompressibility condition from $\det \mathbf{F} = 1$ to

$$J = \sqrt{\frac{\det \mathbf{m}}{\det \mathbf{M}}} \det \mathbf{F} = 1, \tag{4.26}$$

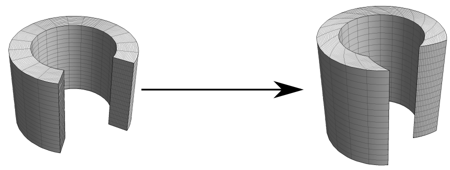


Figure 4.6: Family 5: inflation, bending, extension, and azimuthal shearing of an annular wedge.

to reflect the fact that we are only constraining the elastic component of the deformation to be isochoric. It is easier however, to consider square of this equation in the form

$$\det \mathbf{b} = 1, \tag{4.27}$$

which in components reads

$$\det \left(F^a{}_A M^{AB} F^b{}_B \right) = \det m^{ab}. \tag{4.28}$$

This is because when we move to the anelastic setting, the stretch $\mathbf{b}$ is a more natural object to work with, since it captures geometric data about the material manifold, but itself lives in Euclidean space. The right Cauchy Green strain reads

$$[C_{AB}] = \begin{bmatrix} A^2 + A^2B^2 & A^2BCR & 0 \\ A^2BCR & A^2C^2R^2 & 0 \\ 0 & 0 & E^2 \end{bmatrix}, \tag{4.29}$$

which is invariant under changes in Z or Θ, so C_{AB} is a $\mathrm{SO}(2) \times \mathrm{T}(1)$-equivariant section of $T^*\mathbb{E}^3 \otimes T^*\mathbb{E}^3$.

4.1.1 Summary of the symmetry groups.

The symmetries we have calculated are all generated by the usual action of Lie subgroups of the special Euclidean group on the reference configuration. For each family, there is some continuous family of rotations and/or translations, which once prolonged, leaves the right Cauchy Green stretch tensor field unchanged. In a similar manner, we can compute symmetries of the left Cauchy Green stretch tensor field, which also happen to be Lie subgroups of the special Euclidean group but, acting now on the current configuration. These groups are summarized in Table 4.1, expressed in terms of the group of n-dimensional rotations $\mathrm{SO}(n)$ and the group of n-dimensional translations $\mathrm{T}(n)$.

Interestingly enough, for each family, the symmetry group of $\mathbf{C}$ does not necessarily match the symmetry group of $\mathbf{b}$, but the dimensions of these two groups

Family	$\mathbf{C}^\flat = \left(\mathbf{F}^\mathsf{T}\mathbf{F}\right)^\flat$	Dimension	$\mathbf{b} = \mathbf{F}\mathbf{F}^\mathsf{T}$	Dimension
0	T(3)	3	T(3)	3
1	T(2)	2	SO(2) × T(1)	2
2	SO(2) × T(1)	2	T(2)	2
3	SO(2) × T(1)	2	SO(2) × T(1)	2
4	SO(3)	3	SO(3)	3
5	SO(2) × T(1)	2	SO(2) × T(1)	2

Table 4.1: Symmetry of the universal stretch tensor fields; these are subgroups of $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathrm{T}(3)$.

do match, as their actions are related through the maps relating the coordinates in the two configurations. Additionally, when we impose the symmetries of $\mathbf{C}$ on the material manifold, families with 3-dimensional Lie symmetries automatically satisfy the equilibrium conditions, but those only containing a 2-dimensional Lie symmetry require additional constraints to satisfy equilibrium. This seems to suggest that the dimension of the symmetry group plays a role in constraining the material response, and sufficiently high dimensional symmetries are sufficient for guaranteeing equilibrium.

These symmetries can be summarized as a single key property, namely that *for a given universal deformation and its associated deformation gradient* $\mathbf{F}$ *and right Cauchy Green tensor,* $\mathbf{C} = \mathbf{F}^\mathsf{T}\mathbf{F} = \mathbf{M}^{-1}\varphi^*(\mathbf{m})$*, we have that* $\mathbf{C}^\flat = \varphi^*(\mathbf{m})$ *is an equivariant section under the prolonged action of a Lie subgroup of the special Euclidean group acting on the reference configuration.* We will use this key property when studying the anelastic Ericksen problem, and accordingly refer to the symmetries of $\mathbf{C}^\flat$ as 'universal symmetries'.

4.2 General construction

The previous section demonstrated a remarkable symmetry property of the known universal solutions in the absence of eigenstrain. We can use these symmetries to generalize Ericksen's problem to anelastic systems. The problem is then to find a suitable metric on the material manifold that preserves both the symmetry and the general functional form of the universal deformations. The eigenstrains and metric associated with this new metric are referred to as 'universal'. This can be achieved by the following construction:

- First, in the absence of eigenstrains, the body is isometrically embedded in the ambient space with an induced metric $\bar{\mathbf{M}}$. The material manifold is the flat Riemaniann manifold $(\mathcal{B}, \bar{\mathbf{M}})$ and the deformation is a map from this manifold to the ambient space.

- Second, in the presence of eigenstrains the natural configuration of the body is a Riemannian manifold $(\mathcal{B}, \mathbf{M})$, where $\mathbf{M}$ has non-vanishing curvature [Yavari and Goriely, 2013, 2015, Golgoon and Yavari, 2018]. In this case, the deformation is a map from $(\mathcal{B}, \mathbf{M})$ to the ambient space $(\mathcal{S}, \mathbf{m})$.

- Third, we choose curvilinear coordinates $\{Q^A\}$ on $\mathcal{B}$ and curvilinear coordinates $\{q^a\}$ on $\mathcal{C} \subset \mathcal{S}$. These coordinates are not necessarily associated with the metrics $\bar{\mathbf{M}}$ and $\mathbf{M}$. We know some classes of universal deformations $q^a = \varphi(Q^A)$ for $(\mathcal{B}, \bar{\mathbf{M}})$. We fix this functional dependence φ and seek to determine the the metrics $\mathbf{M}$ compatible with these solutions.

- Fourth, we pull back $\mathbf{m}$ under the deformation φ, and consider the three manifolds: $(\mathcal{B}, \bar{\mathbf{M}})$, $(\mathcal{B}, \mathbf{M})$, and $(\mathcal{B}, \varphi^*(\mathbf{m}))$. We have two candidates for determining the symmetry to apply to $\mathbf{M}$: $\bar{\mathbf{M}}$ and $\varphi^*(\mathbf{m})$. We use $\varphi^*(\mathbf{m})$ since it captures information about the deformation, compute its symmetries, and demand $\mathbf{M}$ to have these same symmetries. Explicitly, since both $\bar{\mathbf{M}}$ and $\mathbf{m}$ are Euclidean metrics, both are equivariant under the full induced action of SE(3) acting on their respective base spaces. By considering Euclidean symmetries of $\varphi^*(\mathbf{m})$, we are identifying Euclidean symmetries in the current configuration that are mapped to other Euclidean symmetries in the reference configuration when pulled back under the classical universal deformation in question.

- Fifth, we compute the deformation mapping $(\mathcal{B}, \mathbf{M})$ to $(\mathcal{C}, \mathbf{m})$, where now $\mathbf{M}$ is required to obey the derived symmetries, and compute the specific symmetric metrics $\mathbf{M}$ that generate universal eigenstrains.

4.2.1 Universal equilibrium equations

We fix the coordinate labels for the anelastic component of the local deformation; if Q^A are coordinates for the material manifold, we write the anelastic deformation in terms of coordinates as

$$Q^A = \delta^A_{\bar{A}} \bar{Q}^{\bar{A}}, \tag{4.30}$$

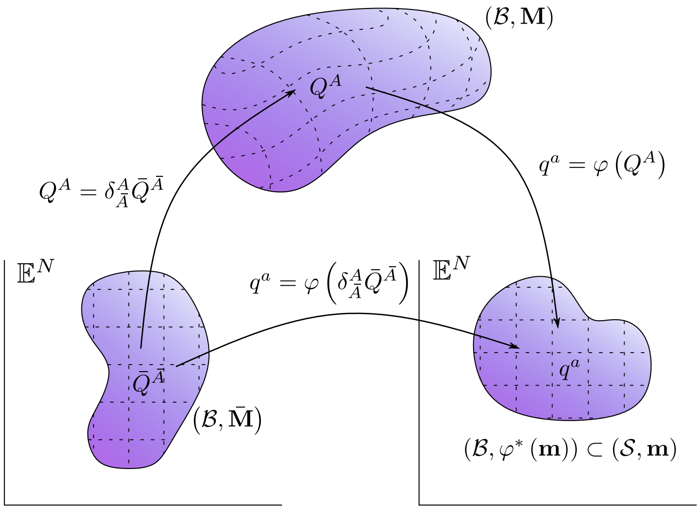


Figure 4.7: A universal deformation where the anelastic strain is captured by the changing of the metric $\bar{\mathbf{M}}$ to $\mathbf{M}$, and the elastic deformation is captured by the coordinate map from the classical family.

where $\delta^A_{\bar{A}}$ is the Kronecker delta, and use the metric tensor field to reflect the change in geometry (see Figure 4.7). Here we have prescribed the coordinate transformations for each family at every stage of the motion, and the unknown quantity to be determined is the metric tensor field $\mathbf{M}$ of the material manifold, which dictates the stress-free geometry. This allows us to trivially pull the material metric tensor back to the reference configuration using

$$M_{\bar{A}\bar{B}} = \delta^A_{\bar{A}}\, M_{AB}\, \delta^B_{\bar{B}}, \tag{4.31}$$

and treat $M_{\bar{A}\bar{B}}$ as our unknown quantity. Since $M_{\bar{A}\bar{B}}$ and $C_{\bar{A}\bar{B}}$ are sections of the same vector bundle, we can immediately impose the symmetry of $C_{\bar{A}\bar{B}}$ on $M_{\bar{A}\bar{B}}$, which naturally imposes a symmetry on M_{AB} via (4.31).

After determining the most general form of a metric tensor field obeying these symmetries, we can compute the general form of the elastic left Cauchy Green tensor

$$b^{ab} = F^a{}_A\, M^{AB} F^b{}_B\,, \tag{4.32}$$

and its inverse in terms of the undetermined components of the metric tensor field. Both of these appear in the Cauchy stress of an incompressible isotropic elastic solid, which has the following representation

$$\sigma^{ab} = -pm^{ab} + 2W_1 b^{ab} - 2W_2 c^{ab}, \tag{4.33}$$

in terms of $W_i = \partial W/\partial I_i$, where I_1 and I_2 are the two non-trivial invariants of $\mathbf{b}$, and p is the Lagrange multiplier corresponding to the incompressibility constraint. We seek equilibrium solutions, hence we must satisfy

$$\nabla_b \sigma^{ab} = -m^{ab}\nabla_b p + 2\nabla_b \left(W_1 b^{ab}\right) - 2\nabla_b \left(W_2 c^{ab}\right) = 0. \tag{4.34}$$

We wish to eliminate the pressure field from the analysis, so we take a second covariant derivative, lower the upper free index, and compute the antisymmetric part. This operation eliminates the pressure identically and yields the condition

$$m_{a[d}\nabla_{c]}\nabla_b \sigma^{ab} = \frac{1}{2} m_{ad}\nabla_c\nabla_b\sigma^{ab} - \frac{1}{2} m_{ac}\nabla_d\nabla_b\sigma^{ab} = 0, \tag{4.35}$$

which must be satisfied for arbitrary choices of the strain energy function W. Because W is arbitrary, we can freely vary its partial derivatives independently, so in order to satisfy (4.35) for any W, we require each of the terms multiplying a partial derivative of W to vanish independently. This yields nine independent conditions that must be satisfied, corresponding to the nine independent mixed partial derivatives of W that appear: W_1, W_2, W_{11}, W_{12}, W_{22}, W_{111}, W_{112}, W_{122}, and W_{222}. These *universal equilibrium equations* are

$$m_{ad}\nabla_c\nabla_b b^{ab} - m_{ac}\nabla_d\nabla_b b^{ab} = 0, \tag{4.36}$$

$$m_{ad}\nabla_c\nabla_b c^{ab} - m_{ac}\nabla_d\nabla_b c^{ab} = 0, \tag{4.37}$$

$$\begin{aligned} &m_{ad}\nabla_b b^{ab}\nabla_c I_1 + m_{ad}\nabla_c b^{ab}\nabla_b I_1 + m_{ad} b^{ab}\nabla_c\nabla_b I_1 \\ &- m_{ac}\nabla_b b^{ab}\nabla_d I_1 - m_{ac}\nabla_d b^{ab}\nabla_b I_1 - m_{ac} b^{ab}\nabla_d\nabla_b I_1 = 0, \end{aligned} \tag{4.38}$$

$$\begin{aligned} &m_{ad}\nabla_b b^{ab}\nabla_c I_2 - m_{ad}\nabla_b c^{ab}\nabla_c I_1 + m_{ad}\nabla_c b^{ab}\nabla_b I_2 \\ &- m_{ad}\nabla_c c^{ab}\nabla_b I_1 + m_{ad} b^{ab}\nabla_c\nabla_b I_2 - m_{ad} c^{ab}\nabla_c\nabla_b I_1 \\ &- m_{ac}\nabla_b b^{ab}\nabla_d I_2 + m_{ac}\nabla_b c^{ab}\nabla_d I_1 - m_{ac}\nabla_d b^{ab}\nabla_b I_2 \\ &+ m_{ac}\nabla_d c^{ab}\nabla_b I_1 - m_{ac} b^{ab}\nabla_d\nabla_b I_2 + m_{ac} c^{ab}\nabla_d\nabla_b I_1 = 0, \end{aligned} \tag{4.39}$$

$$\begin{aligned} &m_{ad}\nabla_b c^{ab}\nabla_c I_2 + m_{ad}\nabla_c c^{ab}\nabla_b I_2 + m_{ad} c^{ab}\nabla_c\nabla_b I_2 \\ &- m_{ac}\nabla_b c^{ab}\nabla_d I_2 - m_{ac}\nabla_d c^{ab}\nabla_b I_2 - m_{ac} c^{ab}\nabla_d\nabla_b I_2 = 0, \end{aligned} \tag{4.40}$$

$$m_{ad} b^{ab}\nabla_c I_1\nabla_b I_1 - m_{ac} b^{ab}\nabla_d I_1\nabla_b I_1 = 0, \tag{4.41}$$

$$\begin{aligned} &m_{ad} b^{ab}\nabla_c I_2\nabla_b I_1 + m_{ad} b^{ab}\nabla_c I_1\nabla_b I_2 - m_{ad} c^{ab}\nabla_c I_1\nabla_b I_1 \\ &- m_{ac} b^{ab}\nabla_d I_2\nabla_b I_1 - m_{ac} b^{ab}\nabla_d I_1\nabla_b I_2 + m_{ac} c^{ab}\nabla_d I_1\nabla_b I_1 = 0, \end{aligned} \tag{4.42}$$

$$\begin{aligned} &m_{ad} b^{ab} \nabla_c I_2 \nabla_b I_2 - m_{ad} c^{ab} \nabla_c I_2 \nabla_b I_1 - m_{ad} c^{ab} \nabla_c I_1 \nabla_b I_2 \\ &\quad - m_{ac} b^{ab} \nabla_d I_2 \nabla_b I_2 + m_{ac} c^{ab} \nabla_d I_2 \nabla_b I_1 + m_{ac} c^{ab} \nabla_d I_1 \nabla_b I_2 = 0, \end{aligned} \tag{4.43}$$

$$m_{ad} c^{ab} \nabla_c I_2 \nabla_b I_2 - m_{ac} c^{ab} \nabla_d I_2 \nabla_b I_2 = 0. \tag{4.44}$$

Each of these nine equations is antisymmetric in the two free indices. Therefore, each equation has three independent components providing an overdetermined system of 27 scalar conditions that must be satisfied.

Augmenting these equations with the additional condition that the Riemann-Christoffel curvature tensor based on $\mathbf{c}^\flat$ vanishes recovers the classical Ericksen problem, hence any classical universal solution must of necessity also be an anelastic universal solution. We do not seek here to study these equations in their full generality. Rather, we will impose symmetries motivated by the known classical families on these equations, and then solve the reduced symmetric systems. The reason for this is twofold.

Firstly, Ericksen's initial work showing that surfaces of constant invariants must be right concentric cylinders, parallel planes, or concentric spheres does not rely on the curvature generated by $\mathbf{c}$ vanishing, hence it is equally applicable here. Ericksen only uses this requirement when explicitly solving for specific solutions later in his initial paper. This is a weaker form of the group equivariant symmetries we impose, since if a tensor field obeys the group equivariant symmetries, its invariants will be constant on the invariant sets of these group actions, which are precisely the surfaces Ericksen utilized. All known universal solutions also obey this stronger symmetry, not just the weaker symmetry based on the strain invariants, and it is this symmetry we seek to explore. Secondly, while Ericksen's analysis assumed that the strain invariants were not both constant, and hence was unable to capture the fifth family, we will see that the group equivariant approach manages to capture not only how the strain invariants vary spatially, but also how the orientation of the eigenvectors of $\mathbf{c}$ vary. This allows this approach to treat the classical fifth family even though it has constant invariants.

Imposing the universal symmetries lead to two different situations depending on the symmetry:

Case I: For two universal families (the homogeneous and spherical deformations) these equations are trivially satisfied by virtue of the symmetry alone and do not lead to any new conditions.

Case II: For the other families, these equations will either require particular off-diagonal components of the metric tensor to vanish[1] (Case IIa) or the invariants to be constant (Case IIb). For Case IIa the equilibrium equations are trivially satisfied. This is the so-called *generic* case for all the remaining families analysed in section 4.4. Case IIb corresponds to the anomalous solutions. In this case the symmetry condition generates a set of ordinary differential equations constraining these components in terms of a single independent variable. In addition to satisfying these, we also must satisfy three algebraic constraints, namely that the invariants of $\mathbf{b}$ are spatially constant. This leaves us with an overdetermined system of four linear differential equations, one linear algebraic equation and two nonlinear algebraic equations for the six unknown components of the metric tensor field. We can integrate the differential equations, and use the linear equation to express the two algebraic equations in terms of the following $15 \leq n \leq 18$ variables: a single unknown component of the metric tensor, the remaining independent variable in space (e.g. radius), the integration constants introduced by our integration of the ordinary differential equations, the deformation parameters, and the two constant invariants. Therefore, the remaining algebraic conditions are polynomial equations in these n variables that are quadratic in the unknown component of the material metric tensor field. We compute the resultant of these polynomials in this component, and demand it vanish for the two equations to have a common solution, since our metric tensor must simultaneously satisfy both. This resultant is itself a polynomial in the dependent variable that must vanish. Because we seek solutions that are universal over an open set, we can send each of the coefficients of the resultant to zero identically. This leaves us with a set of algebraic equations relating the integration constants, the deformation parameters, and the invariants that must be necessarily satisfied in order for these anomalous solutions to exist. These algebraic equations are solved in appendix A, and the solutions themselves are presented in section 4.5.

4.3 Symmetries of the Material Metric

Family 0: Homogeneous Deformations. Recall for Cartesian coordinates $\{Q^A\}$, that the action $Q^A \to Q^A + C^A$ is the action of $\mathrm{T}(3) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$. We seek to

[1]More precisely, in Family 5 one off diagonal component vanishes, and another becomes fully determined by the other metric components; it does not vanish, but its indeterminacy is eliminated nonetheless.

impose this symmetry on the material metric tensor field, and hence demand

$$M_{AB}\left(\bar{Q}^D\right) = M_{AB}\left(Q^D + C^D\right) = M_{AB}\left(Q^D\right). \tag{4.45}$$

Taking the derivative of this with respect to C^D gives

$$\frac{\partial M_{AB}}{\partial C^D} = \frac{\partial M_{AB}}{\partial \bar{Q}^E}\frac{\partial \bar{Q}^E}{\partial C^D} = \frac{\partial M_{AB}}{\partial \bar{Q}^D} = 0, \tag{4.46}$$

and hence, we consider constant metric tensor fields.

It is however, more useful to express this condition in terms of the current configuration variables, which we can do via the chain rule

$$\frac{\partial M_{AB}}{\partial Q^D} = \frac{\partial M_{AB}}{\partial q^a}\frac{\partial q^a}{\partial Q^D} = \frac{\partial M_{AB}}{\partial q^a} F^a{}_D = 0. \tag{4.47}$$

Since $\mathbf{F}$ is invertible, this implies that M_{AB} is constant when expressed in terms of the current configuration coordinates as well. Additionally, it is more useful to consider the inverse metric tensor field M^{AB}, which also must be constant in order for the identity

$$M_{AB}M^{BC} = \delta^A_C, \tag{4.48}$$

to hold.

Family 1: Bending, Stretching, and Shearing of a Rectangular Block. Recall, that the symmetry associated with this deformation is the action of $\mathrm{T}(2) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$. We therefore require the same equivariance for M_{AB}:

$$M_{AB}\left(X, Y + C_1, Z + C_2\right) = M_{AB}\left(X, Y, Z\right). \tag{4.49}$$

Taking the derivative with respect to C_1 and C_2 independently gives the conditions

$$\frac{\partial M_{AB}\left(X,\bar{Y},\bar{Z}\right)}{\partial C_1} = \frac{\partial M_{AB}\left(X,\bar{Y},\bar{Z}\right)}{\partial \bar{Y}}\frac{\partial \bar{Y}}{\partial C_1} = \frac{\partial M_{AB}\left(X,\bar{Y},\bar{Z}\right)}{\partial \bar{Y}} = 0,$$
$$\frac{\partial M_{AB}\left(X,\bar{Y},\bar{Z}\right)}{\partial C_2} = \frac{\partial M_{AB}\left(X,\bar{Y},\bar{Z}\right)}{\partial \bar{Z}}\frac{\partial \bar{Z}}{\partial C_2} = \frac{\partial M_{AB}\left(X,\bar{Y},\bar{Z}\right)}{\partial \bar{Z}} = 0.$$

Therefore, we assume that the metric tensor is of the form $M_{AB}(X)$, which because of the form of the deformation, can be recast into the form $M_{AB}(r)$, and equivalently $M^{AB}(r)$.

Family 2: Straightening, Stretching, and Shearing of a Sector of a Cylinder. Here, we require that the metric is equivariant under the action of $\mathrm{SO}(2) \times \mathrm{T}(1) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$:

$$M_{AB}\left(R, \Theta + \Phi, Z + K\right) = M_{AB}\left(R, \Theta, Z\right), \tag{4.50}$$

and hence, by the same reasoning as before one finds

$$\frac{\partial M_{AB}\left(R, \bar{\Theta}, \bar{Z}\right)}{\partial \bar{\Theta}} = \frac{\partial M_{AB}\left(R, \bar{\Theta}, \bar{Z}\right)}{\partial \bar{Z}} = 0. \tag{4.51}$$

Note that this does not imply $\frac{\partial \mathbf{M}}{\partial \Theta} = \mathbf{0}$, since the basis vectors of $\mathbf{C}$ do change with Θ. Rather, the components M_{AB} do not change with Θ when $\mathbf{M}$ is represented with respect to a cylindrical polar basis. We can use the equation $M_{AB} M^{BC} = \delta_A^C$ to show that $\frac{\partial M^{AB}}{\partial \Theta} = \frac{\partial M^{AB}}{\partial Z} = 0$. Hence, we write $M^{AB}(R)$ or equivalently $M^{AB}(x)$. This symmetry of M^{AB} is precisely equivariance the action $\mathrm{SO}(2) \times \mathrm{T}(1) \subset \mathrm{SE}(3)$ prolonged to $T\mathbb{E}^3 \otimes T\mathbb{E}^3$.

Family 3: Inflation, Bending, Torsion, Extension, and Shearing of an Annular Wedge. As with Family 2, we demand that M_{AB} be equivariant under the actions induced by the transformation

$$\Theta \to \bar{\Theta} = \Theta + \Phi, \quad Z \to \bar{Z} = Z + K, \quad \forall \Phi, K \in \mathbb{R}, \tag{4.52}$$

which renders the conditions

$$\frac{\partial M_{AB}\left(R, \bar{\Theta}, \bar{Z}\right)}{\partial \bar{\Theta}} = \frac{\partial M_{AB}\left(R, \bar{\Theta}, \bar{Z}\right)}{\partial \bar{Z}} = 0. \tag{4.53}$$

So we consider metric tensor fields of the form $M_{AB}(R)$, or equivalently, $M^{AB}(r)$.

Family 4: Inflation/Eversion of a Sphere. Here, we demand the equivariance of $\mathbf{M}(X)$ under the actions of $\mathrm{SO}(3) \subset \mathrm{SE}(3)$ on $\mathbb{E}^3$ and $T^*\mathbb{E}^3 \otimes T^*\mathbb{E}^3$. We therefore seek the most general positive-definite symmetric tensor field that satisfies

$$\mathbf{M}\left(\mathbf{Q}X\right) = \mathbf{Q}\mathbf{M}(X)\mathbf{Q}^{\mathsf{T}}, \quad \forall \mathbf{Q} \in \mathrm{SO}(3). \tag{4.54}$$

Because $\mathbf{M}(X)$ is positive-definite and symmetric, we can represent it in spectral form on an orthonormal basis $\{\mathbf{u}_a\}$ as

$$\mathbf{M}(X) = \sum_{i=1}^{3} m_i^2(X)\mathbf{u}_i(X) \otimes \mathbf{u}_i(X). \tag{4.55}$$

We then consider the subgroup of rotations leaving X fixed. Under this one-parameter family, we have the symmetry condition

$$\mathbf{M}(X) = \mathbf{Q}\mathbf{M}(X)\mathbf{Q}^{\mathsf{T}}, \;\; \forall\, \mathbf{Q} \;\text{ such that }\; \mathbf{Q}X = X. \tag{4.56}$$

This implies that (suppressing the X dependence)

$$m_i^2 \mathbf{u}_i = \mathbf{M}\mathbf{u}_i = \mathbf{Q}\mathbf{M}\mathbf{Q}^{\mathsf{T}}\mathbf{u}_i \quad \Rightarrow \quad m_i^2 \mathbf{Q}^{\mathsf{T}}\mathbf{u}_i = \mathbf{M}\mathbf{Q}^{\mathsf{T}}\mathbf{u}_i, \tag{4.57}$$

i.e., the rotated vector $\mathbf{Q}^{\mathsf{T}}\mathbf{u}_i$ lies in the same eigenspace as the eigenvector $\mathbf{u}_i$. For this to hold for all $\mathbf{Q}$ in this one-parameter family, the eigenspaces of $\mathbf{M}$ at the point X must be unchanged by these rotations, i.e., the swept vector $\mathbf{Q}^{\mathsf{T}}\mathbf{u}_i$ remains in the eigenspace. Generally, a rotating vector sweeps out a cone, which not being an affine space, cannot be the eigenspace of a linear operator, as depicted in Figure 4.8.

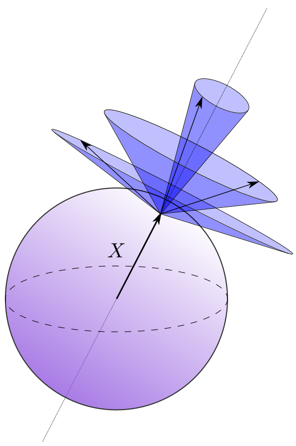


Figure 4.8: The swept eigenspaces of a general symmetric tensor.

However, there are two degenerate cases where cones become affine spaces, where the rotation axis and swept vector are coincident, and where they are perpendicular, which means that the eigenspaces of $\mathbf{M}$ at X must contain the axis of the rotation, and the plane orthogonal to it, as depicted in Figure 4.9, forcing $\mathbf{M}(X)$ to be of the form

$$\mathbf{M}(X) = m_1^2(X)\mathbf{E}_R(X) \otimes \mathbf{E}_R(X) + m_2^2(X)\left(\mathbf{I} - \mathbf{E}_R(X) \otimes \mathbf{E}_R(X)\right), \tag{4.58}$$

because for each X, the axis of rotation is $\mathbf{E}_R(X)$. Imposing the more general symmetry condition (4.54), on this spectral form, we get the condition

$$\begin{aligned} &m_1^2\left(\mathbf{Q}X\right)\mathbf{E}_R(X) \otimes \mathbf{E}_R(X) + m_2^2\left(\mathbf{Q}X\right)\left(\mathbf{I} - \mathbf{E}_R(X) \otimes \mathbf{E}_R(X)\right) \\ &= m_1^2(X)\mathbf{E}_R(X) \otimes \mathbf{E}_R(X) + m_2^2(X)\left(\mathbf{I} - \mathbf{E}_R(X) \otimes \mathbf{E}_R(X)\right), \end{aligned} \tag{4.59}$$

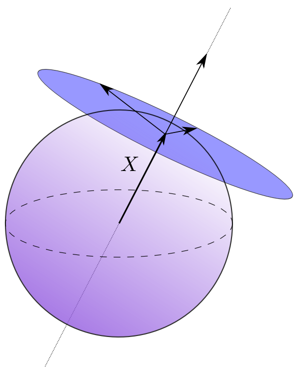


Figure 4.9: The specific orientation required for eigenspaces to be preserved under point-preserving rotations.

which implies that

$$m_1^2(\mathbf{Q}X) = m_1^2(X), \quad m_2^2(\mathbf{Q}X) = m_2^2(X), \quad \forall\, \mathbf{Q} \in \mathrm{SO}(3). \tag{4.60}$$

This ultimately requires that m_1 and m_2 depend on X solely through its norm, $R = |X|$, since for any two points, $X_1 = R\mathbf{n}_1$, and $X_2 = R\mathbf{n}_2$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors, we can construct orthogonal transformations such that $\mathbf{n}_2 = \mathbf{Q}\mathbf{n}_1$. Hence, because the functions m_1 and m_2 take on the same values whenever their arguments have the same norm, these functions must only depend on their argument through its norm, as shown in Figure 4.10. We then have

$$\mathbf{M}(X) = m_1^2(R)\mathbf{E}_R(X) \otimes \mathbf{E}_R(X) + m_2^2(R)\left(\mathbf{I} - \mathbf{E}_R(X) \otimes \mathbf{E}_R(X)\right), \tag{4.61}$$

which is the general form of the pullback of our material metric tensor.[2] Computing the components of M_{AB} then gives

$$[M_{AB}(R)] = \begin{bmatrix} m_1^2(R) & 0 & 0 \\ 0 & m_2^2(R)R^2\sin^2\Phi & 0 \\ 0 & 0 & m_2^2(R)R^2 \end{bmatrix}. \tag{4.62}$$

Since m_1 and m_2 are arbitrary, and R is only a function of r, this can be rewritten as

$$[M_{AB}(R)] = [M_{AB}(R(r))] = \begin{bmatrix} m_1^2(r) & 0 & 0 \\ 0 & m_2^2(r)\sin^2\phi & 0 \\ 0 & 0 & m_2^2(r) \end{bmatrix}, \tag{4.63}$$

[2]The methodological difference in the symmetry calculation for this family is due to the topology of SO(3). This group is not the product of Lie groups, and hence, we cannot express its action by independently varying coordinates as we do in the other families; while $\Theta \to \Theta + \Psi_1$ is a rotation, $\Phi \to \Phi + \Psi_2$ is not.

and equivalently

$$\left[M^{AB}(r)\right] = \begin{bmatrix} m_1^2(r) & 0 & 0 \\ 0 & \frac{m_2^2(r)}{\sin^2\phi} & 0 \\ 0 & 0 & m_2^2(r) \end{bmatrix}. \tag{4.64}$$

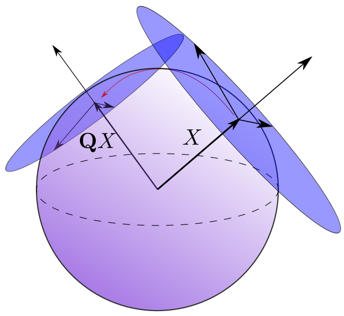


Figure 4.10: Eigenspaces transformed under arbitrary rotations relating the eigenvalues at different points on spherical surfaces.

These metrics are precisely the form considered by Ben Amar and Goriely [2005], and Yavari and Goriely [2013], though the first represents this tensor on an orthonormal spherical basis, and the second works with the components of the metric tensor rather than its inverse, as we have done.

Family 5: Inflation, Bending, Extension, and Azimuthal Shearing of an Annular Wedge. As in other cases, we have the following symmetry relations for **M**:

$$\frac{\partial M_{AB}\left(R, \bar{\Theta}, \bar{Z}\right)}{\partial \bar{\Theta}} = \frac{\partial M_{AB}\left(R, \bar{\Theta}, \bar{Z}\right)}{\partial \bar{Z}} = 0, \tag{4.65}$$

so we ultimately consider inverse metric tensor fields of the form $M^{AB}(r)$.

In conclusion, we note that the application of the symmetry condition leads to a reduction of the independent variables to a single one (either the radial variable r in cylindrical or spherical coordinates, or x for the deformation of rectangular blocks).

4.4 Generic Universal Solutions

Having established the symmetry conditions on the material metric, we can express the universal equilibrium equations under these restrictions. For all families (Case I and IIa), these equations will have generic solutions, and in some cases, these solutions are the only ones satisfying the symmetry conditions. Other families also have anomalous solutions (Case IIb) outside of these generic branches that occur only when the invariants of the tensor **b**, or equivalently **C**, are constant; these will be addressed in the next section. The nature of the anomalous solutions differs markedly from the generic solutions found here: generic solutions contain arbitrary functions as free parameters, while the form of the anomalous solutions is determined up to a finite number of undetermined constants. Additionally, for generic solutions, the eigenvectors of **b** are contained within or perpendicular to the span of the infinitesimal generators of the symmetry group, while for anomalous solutions, this alignment does not occur. While the invariants, and hence, their gradients could be calculated explicitly in terms of the unknown inverse metric components, it is easier to keep these functions unevaluated at the moment, because we will ultimately show that they must be constant for the anomalous solution to exist.

Family 0: Homogeneous Deformations. The deformation mapping written in Cartesian coordinates is given by (4.1), and the deformation gradient ${F^a}_A$ is constant. We compute the left Cauchy Green tensor as

$$b^{ab} = {F^a}_A \, M^{AB} {F^b}_B \,, \tag{4.66}$$

which is also constant, and hence its invariants are constant. The Cauchy stress takes the form

$$\sigma^{ab}\left(x,y,z\right) = -p\left(x,y,z\right) m^{ab} + 2W_1 b^{ab} - 2W_2 c^{ab}, \tag{4.67}$$

and the equilibrium equations read

$$\nabla_b \sigma^{ab} = -m^{ab} \nabla_b p\left(x,y,z\right) = 0. \tag{4.68}$$

Because m^{ab} is invertible, this implies that p is constant, and the equilibrium equations are satisfied simply because of the assumed form of M^{AB}.

The only remaining condition is the incompressibility condition which, in the chosen coordinate systems, reads

$$\det b^{ab} = \det m^{ab} = 1. \tag{4.69}$$

We can express this constraint as a condition on M^{AB}, or as a condition on ${F^a}_A$. In reality, these conditions are equivalent, since we have

$$\left(\det {F^a}_A\right)^2 \det M^{MN} = 1, \tag{4.70}$$

and one can be freely transformed into the other by changing coordinates. However, for the other families, it will be easier to express this condition as a restriction on the inverse metric tensor, so for consistency we choose the invertible tensor $F^a{}_A$ and then enforce

$$\det M^{AB} = \frac{1}{(\det F^a{}_A)^2}, \tag{4.71}$$

which ensures that the volume form in the material manifold agrees with that in the current configuration.

Because the material metric is constant, its Levi-Civita connection produces no curvature, and thus the material manifold is Euclidean. This is useful when using a multiplicative decomposition of the deformation gradient into elastic and anelastic factors $\mathbf{F} = \mathbf{A}\mathbf{G}$, as we can choose a Cartesian frame $\{\mathbf{e}_\alpha\}$ in the material manifold and its corresponding coframe $\{\vartheta^\alpha\}$, in which case the anelastic factor must satisfy $G^\alpha{}_A G^\beta{}_B \delta_{\alpha\beta} = M_{AB}$. Since the matrix of components M_{AB} is positive definite and symmetric, we can take the matrix of components $G^\alpha{}_A$ to be its unique positive-definite symmetric square root, in which case we satisfy $G^\alpha{}_A G^\beta{}_B \delta_{\alpha\beta} = M_{AB}$. Alternatively, we may prescribe the anelastic factor in such a way that the induced material metric is valid. In this case, since M_{AB} is constant, any constant invertible anelastic factor will furnish a valid material metric. The incompressibility constraint becomes $\det(A^a{}_\alpha A^b{}_\beta \delta^{\alpha\beta} m_{bc}) = 1$, which furnishes a differential equation constraining the volume in the current configuration to agree with that in the material manifold.

Family 1: Bending, Stretching, and Shearing of a Rectangular Block. The deformation for this family is given in (4.3) with the deformation gradient (4.4). We compute the quantity $m_{a[l}\nabla_{k]}\nabla_b\sigma^{ab} = 0$, which for this family, only has two independent nonzero components, and we take the coefficients of the partial derivatives of W to vanish independently. The W_{111} coefficient of this equation gives the conditions

$$\begin{bmatrix} ABrM^{12}(r) \\ \frac{B\left(M^{13}(r) - ACM^{12}(r)\right)}{r} \end{bmatrix} I_1'(r)^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.72}$$

The first equation implies that either $M^{12}(r) = 0$, or I_1 is constant, because $AB \neq 0$ to ensure the invertibility of the deformation. If I_1 is not constant, we have $M^{12}(r) = 0$. The second component then becomes $M^{13}(r) = 0$. Therefore, if I_1 is not constant, we have that $M^{12}(r) = M^{13}(r) = 0$.

If I_1 is constant, we can examine the W_{122} component, which implies the conditions

$$\begin{bmatrix} ABrM^{12}(r) \\ \frac{B\left(M^{13}(r) - ACM^{12}(r)\right)}{r} \end{bmatrix} I_2'(r)^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.73}$$

Therefore, I_2 is constant, or $M^{12}(r) = M^{13}(r) = 0$ and we have established that either $M^{12}(r) = M^{13}(r) = 0$, or all the invariants of $\mathbf{b}$ are constant, which is the condition for the anomalous solution. Hence, in this section, we take $M^{12}(r) = M^{13}(r) = 0$ and we consider the constant invariant case in the next section. With $M^{12}(r) = M^{13}(r) = 0$, we have the equilibrium equation satisfied, i.e., all of its terms identically vanish. We only have to satisfy incompressibility. Demanding $\det b^{ab} = \det m^{ab}$, we have the condition

$$M^{11}(r)\left[M^{22}(r)M^{33}(r) - \left(M^{23}(r)\right)^2\right] = 1. \tag{4.74}$$

Since $M^{11}(r) \neq 0$, and $M^{22}(r) \neq 0$, the relation

$$M^{33}(r) = \frac{1}{M^{11}(r)M^{22}(r)} + \frac{\left(M^{23}(r)\right)^2}{M^{22}(r)}, \tag{4.75}$$

ensures that the incompressibility is satisfied. Hence, we have the generic solution

$$\left[M^{AB}(r)\right] = \begin{bmatrix} M^{11}(r) & 0 & 0 \\ 0 & M^{22}(r) & M^{23}(r) \\ 0 & M^{23}(r) & \frac{1}{M^{11}(r)M^{22}(r)} + \frac{\left(M^{23}(r)\right)^2}{M^{22}(r)} \end{bmatrix}, \tag{4.76}$$

which, by writing $M^{AB}(X) = M^{AB}\left(r(X)\right)$, can finally be written in terms of the referential variables as

$$\left[M^{AB}(X)\right] = \begin{bmatrix} M^{11}(X) & 0 & 0 \\ 0 & M^{22}(X) & M^{23}(X) \\ 0 & M^{23}(X) & \frac{1}{M^{11}(X)M^{22}(X)} + \frac{\left(M^{23}(X)\right)^2}{M^{22}(X)} \end{bmatrix}. \tag{4.77}$$

The current form of M_{AB} automatically captures incompressibility because we imposed a particular form of $r(X)$. If we leave this unspecified, we can simply say that M_{AB} is of the form

$$\left[M_{AB}(X)\right] = \begin{bmatrix} M_{11}(X) & 0 & 0 \\ 0 & M_{22}(X) & M_{23}(X) \\ 0 & M_{23}(X) & M_{33}(X) \end{bmatrix}, \tag{4.78}$$

and use the incompressibility constraint to determine $r(X)$. This is equivalent to introducing a change in coordinates rescaling X.

When using a multiplicative decomposition, $\mathbf{F} = \mathbf{A}\mathbf{G}$, we can choose an orthonormal frame $\{\mathbf{e}_\alpha\}$ and its coframe $\{\vartheta^\alpha\}$ in the material manifold, and require $G^\alpha{}_A G^\beta{}_B \delta_{\alpha\beta} = M_{AB}$. Since M_{AB} is block diagonal, positive definite, and symmetric, we can take ${G^\alpha}_A$ to be its unique positive definite symmetric square root. Because the components M_{AB} are arbitrary functions of X, we can take ${G^\alpha}_A$ to be of the form

$$[{G^\alpha}_A(X)] = \begin{bmatrix} {G^1}_1(X) & 0 & 0 \\ 0 & {G^2}_2(X) & {G^2}_3(X) \\ 0 & {G^2}_3(X) & {G^3}_3(X) \end{bmatrix}, \tag{4.79}$$

which will yield a suitable M_{AB}. Additionally, we can multiply $\mathbf{G}$ by an arbitrary local rotation $\mathbf{Q}$ yielding $\mathbf{QG}$, which may be more useful depending on the particular problem.[3] This is equivalent to choosing a different orthonormal frame in the material manifold, which being non-Euclidean in general, does not possess a preferred orthonormal frame to begin with.

Family 2: Straightening, Stretching, and Shearing of a Sector of a Cylinder. Recall that any deformation in this family is given by (4.8) with deformation gradient (4.9). We compute the equilibrium condition $m_{a[l}\nabla_{k]}\nabla_b\sigma^{ab} = 0$, and to aid computations, we use the incompressibility constraint $\det b^{ab} = \det m^{ab}$ by evaluating c^{ab} as $c^{ab}\det\left({b^n}_m\right) = c^{ab}\det\left(b^{np}m_{pm}\right)$.

The W_{111} coefficient of the equilibrium equation has two independent components giving the conditions

$$\sqrt{\frac{2\,(x-D)}{A}}\begin{bmatrix} M^{12}(x) \\ CM^{12}(x) + AM^{13}(x) \end{bmatrix} I_1'(x)^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.80}$$

If I_1 is constant, we satisfy these equations, but if I_1 is not constant, we have $M^{12}(x) = M^{13}(x) = 0$, since $A\,(x-D) \neq 0$.

If I_1 is constant, we then consider the W_{122} component of the equilibrium condition to obtain

$$\sqrt{\frac{2\,(x-D)}{A}}\begin{bmatrix} M^{12}(x) \\ CM^{12}(x) + AM^{13}(x) \end{bmatrix} I_2'(x)^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.81}$$

which again implies that either $M^{12}(x) = M^{13}(x) = 0$, or I_2 is constant. Therefore, unless the invariants of $\mathbf{b}$ are constant, we have $M^{12}(x) = M^{13}(x) = 0$. Setting these components to 0 satisfies equilibrium, and so we compute the incompressibility condition $\det b^{ab} = \det m^{ab}$. This becomes

$$\frac{2\,(x-D)}{AB^2} M^{11}(x)\left[M^{22}(x)M^{33}(x) - \left(M^{23}(x)\right)^2\right] = 1, \tag{4.82}$$

[3] For example, if the eigenstrain corresponds to anelastic simple shear, it may be more natural to express $\mathbf{G}$ in an upper triangular form, rather than a symmetric form.

which implies that

$$M^{33}(x) = \frac{AB^2}{2\,(x-D)\,M^{11}(x)M^{22}(x)} + \frac{\left(M^{23}(x)\right)^2}{M^{22}(x)}. \tag{4.83}$$

Bringing all of these together we have

$$\left[M^{AB}(x)\right] = \begin{bmatrix} M^{11}(x) & 0 & 0 \\ 0 & M^{22}(x) & M^{23}(x) \\ 0 & M^{23}(x) & \frac{AB^2}{2(x-D)M^{11}(x)M^{22}(x)} + \frac{\left(M^{23}(x)\right)^2}{M^{22}(x)} \end{bmatrix}, \tag{4.84}$$

or in terms of referential coordinates, writing $M^{AB}(R) = M^{AB}\left(x(R)\right)$,

$$\left[M^{AB}(R)\right] = \begin{bmatrix} M^{11}(R) & 0 & 0 \\ 0 & M^{22}(R) & M^{23}(R) \\ 0 & M^{23}(R) & \frac{1}{R^2M^{11}(R)M^{22}(R)} + \frac{\left(M^{23}(R)\right)^2}{M^{22}(R)} \end{bmatrix}. \tag{4.85}$$

This is the generic solution, and we have set up the conditions for the anomalous solution, namely that the invariants of $\mathbf{b}$ must be constant.

As before, we can introduce a coordinate rescaling, treating x as an unknown function of R, which allows the tensor M^{AB} to take the form

$$\left[M^{AB}(R)\right] = \begin{bmatrix} M^{11}(R) & 0 & 0 \\ 0 & M^{22}(R) & M^{23}(R) \\ 0 & M^{23}(R) & M^{33}(R) \end{bmatrix}, \tag{4.86}$$

and turns the incompressibility constraint into a differential equation that can be integrated to determine $x(R)$. If a multiplicative decomposition of $\mathbf{F} = \mathbf{AG}$ is used, we can express $\mathbf{G}$ on an orthonormal frame in the form

$$\left[G^{\alpha}{}_{A}(R)\right] = \begin{bmatrix} G^1{}_1(R) & 0 & 0 \\ 0 & G^2{}_2(R) & G^2{}_3(R) \\ 0 & G^2{}_3(R) & G^3{}_3(R) \end{bmatrix}, \tag{4.87}$$

which guarantees that $M_{AB} = G^{\alpha}{}_{A}G^{\beta}{}_{B}\delta_{\alpha\beta}$ is of the proper form. As before, an arbitrary local rotation $\mathbf{Q}$ can be imposed yielding the factor $\mathbf{QG}$, where $\mathbf{G}$ is as above, and this new factorization will yield a material metric of the proper form.

Family 3: Inflation, Bending, Torsion, Extension, and Shearing of an Annular Wedge. This family of deformations can be written using cylindrical polar coordinates in both configurations as given in (4.13) with deformation gradient (4.14).

As before, we compute the W_{111} coefficient of the equilibrium equation and obtain

$$\sqrt{A\left(r^2-B\right)}I_1'(r)^2 \begin{bmatrix} Cr & Dr \\ \frac{E}{r} & \frac{F}{r} \end{bmatrix} \begin{bmatrix} M^{12}(r) \\ M^{13}(r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.88}$$

The matrix on the left-hand side is invertible, since its determinant, $CF - DE$, being a factor of the determinant of $\mathbf{F}$, is nonzero to ensure invertibility. Therefore, we have either I_1 being constant, or both $M^{12}(r)$ and $M^{13}(r)$ must be 0.

If I_1 is constant, we examine the W_{122} coefficient of the equilibrium equation and obtain

$$\sqrt{A\left(r^2 - B\right)} I_2'(r)^2 \begin{bmatrix} Cr & Dr \\ \frac{E}{r} & \frac{F}{r} \end{bmatrix} \begin{bmatrix} M^{12}(r) \\ M^{13}(r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.89}$$

which as before implies that $M^{12}(r) = M^{13}(r) = 0$, or I_2 is constant. Therefore, to satisfy equilibrium, we must have all of the invariants of $\mathbf{b}$ being constant, or $M^{12}(r) = M^{13}(r) = 0$. The latter of these conditions is also sufficient to guarantee equilibrium. We only have to satisfy incompressibility, which amounts to the equation

$$A\left(CF - DE\right)^2 \left(r^2 - B\right) M^{11}(r) \left[M^{22}(r)M^{33}(r) - \left(M^{23}(r)\right)^2\right] = 1, \tag{4.90}$$

which we can do by setting

$$M^{33}(r) = \frac{1}{A\left(CF - DE\right)^2 \left(r^2 - B\right) M^{11}(r)M^{22}(r)} + \frac{\left(M^{23}(r)\right)^2}{M^{22}(r)}. \tag{4.91}$$

This gives the generic solution

$$\left[M^{AB}(r)\right] = \begin{bmatrix} M^{11}(r) & 0 & 0 \\ 0 & M^{22}(r) & M^{23}(r) \\ 0 & M^{23}(r) & \frac{1}{A(CF-DE)^2(r^2-B)M^{11}(r)M^{22}(r)} + \frac{\left(M^{23}(r)\right)^2}{M^{22}(r)} \end{bmatrix}, \tag{4.92}$$

or in referential variables, writing $M^{AB}(R) = M^{AB}(r(R))$,

$$\left[M^{AB}(R)\right] = \begin{bmatrix} M^{11}(R) & 0 & 0 \\ 0 & M^{22}(R) & M^{23}(R) \\ 0 & M^{23}(R) & \frac{1}{A^2R^2(CF-DE)^2M^{11}(R)M^{22}(R)} + \frac{\left(M^{23}(R)\right)^2}{M^{22}(R)} \end{bmatrix}. \tag{4.93}$$

As in the other families, we can introduce a coordinate rescaling to express the material metric in the form

$$\left[M_{AB}(R)\right] = \begin{bmatrix} M_{11}(R) & 0 & 0 \\ 0 & M_{22}(R) & M_{23}(R) \\ 0 & M_{23}(R) & M_{33}(R) \end{bmatrix}, \tag{4.94}$$

which means that on some orthonormal frame, the anelastic factor of a multiplicative decomposition $\mathbf{F} = \mathbf{AG}$ takes the form

$$\left[G^{\alpha}{}_{B}(R)\right] = \begin{bmatrix} G^1{}_1(R) & 0 & 0 \\ 0 & G^2{}_2(R) & G^2{}_3(R) \\ 0 & G^2{}_3(R) & G^3{}_3(R) \end{bmatrix}. \tag{4.95}$$

Doing this turns the incompressibility condition into a differential equation for the unknown function $r(R)$, and as before, any other compatible anelastic factor can be expressed as $\mathbf{QG}$, where $\mathbf{Q}$ is an arbitrary local rotation and $\mathbf{G}$ is as above.

Family 4: Inflation/Eversion of a Sphere. For this family, the symmetry enforced on the metric tensor automatically satisfies the universal equilibrium equations without additional restrictions. Demonstrating this, under this symmetry, the left Cauchy Green tensor reads

$$\left[b^{ab}\right] = \begin{bmatrix} \frac{\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}}}{r^4} m_1^2(r) & 0 & 0 \\ 0 & \frac{m_2^2(r)}{\sin^2(\phi)} & 0 \\ 0 & 0 & m_2^2(r) \end{bmatrix}, \tag{4.96}$$

and its inverse is

$$\left[c^{ab}\right] = \begin{bmatrix} \frac{r^4}{\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^2(r)} & 0 & 0 \\ 0 & \frac{1}{m_2^2(r) r^4 \sin^2(\phi)} & 0 \\ 0 & 0 & \frac{1}{m_2^2(r) r^4} \end{bmatrix}. \tag{4.97}$$

We can compute the invariants of $\mathbf{b}$ as

$$\begin{aligned} I_1 &= \frac{(\pm r^3 \mp A^3)^{\frac{4}{3}}}{r^4} m_1^2(r) + 2r^2 m_2^2(r), \\ I_2 &= \frac{m_2^2(r)\left[2\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^2(r) + r^6 m_2^2(r)\right]}{r^2}, \\ I_3 &= \left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^2(r) m_2^4(r) = 1. \end{aligned} \tag{4.98}$$

Notice in particular, that these invariants only depend on r. The Cauchy stress is diagonal with components

$$\begin{aligned} \sigma^{11} &= -p + \frac{2\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^2(r) W_1}{r^4} - \frac{2r^4 W_2}{\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^2(r)}, \\ \sigma^{22} &= \frac{2r^4 m_2^4(r) W_1 - 2W_2 - r^2 m_2^2(r) p}{m_2^2(r) r^4 \sin^2(\phi)}, \\ \sigma^{33} &= -\frac{p}{r^2} + 2m_2^2(r) W_1 - \frac{2W_2}{r^4 m_2^2(r)}. \end{aligned}$$

Taking the divergence of this tensor and setting it equal to 0 gives

$$\nabla_b \sigma^{1b} = \frac{4W_2}{r^3 m_2^2(r)} + \frac{4r^4 m_1'(r) W_2}{\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^3(r)} - 4r m_2^2(r) W_1 - \frac{\partial p}{\partial r}$$

$$
\begin{aligned}
&- \frac{2r^3\left(\left(\pm 2r^2 \mp 6A^3\right)W_2 + r\left(\pm r^3 \mp A^3\right)\left(I_2'(r)W_{22} + I_1'(r)W_12\right)\right)}{\left(\pm r^3 \mp A^3\right)^{\frac{7}{3}} m_1^2(r)} \\
&\pm \frac{2\left(r^3 - A^3\right)^{\frac{1}{3}} m_1^2(r)\left(2\left(r^3 + A^3\right)W_1 \pm r\left(r^3 - A^3\right)\left(I_2'(r)W_{12} + I_1'(r)W_11\right)\right)}{r^5} \\
&+ \frac{4\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1(r)m_1'(r)W_1}{r^4} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\nabla_b \sigma^{2b} &= -\frac{1}{r^2\sin^2\phi}\frac{\partial p}{\partial \theta} = 0, \\
\nabla_b \sigma^{3b} &= -\frac{1}{r^2}\frac{\partial p}{\partial \phi} = 0.
\end{aligned}
$$

Therefore, the undetermined pressure must only depend on r, hence the constitutively determined components of $\nabla_b \sigma^{ab}$ only depend on r. Using $\frac{\partial p}{\partial \theta} = 0$ and $\frac{\partial p}{\partial \phi} = 0$, and defining $v^a = \nabla_b \sigma^{ab}$, we can compute $V^a_{\ c} = \nabla_c v^a$. For simplicity, we will only compute the off-diagonal components of this tensor. Note that

$$
v^a = \begin{bmatrix} v^1(r) \\ 0 \\ 0 \end{bmatrix}, \tag{4.99}
$$

and $V^a_{\ c} = \partial_c v^a + v^d \gamma_{cd}{}^a$. Computing the off-diagonal components, we get

$$
\begin{aligned}
V^1_{\ 2} &= \frac{\partial v^1}{\partial \theta} + v^1\gamma_{21}{}^1 = 0, \quad V^2_{\ 3} = \frac{\partial v^2}{\partial \varphi} + v^1\gamma_{31}{}^2 = 0, \quad V^3_{\ 1} = \frac{\partial v^3}{\partial r} + v^1\gamma_{11}{}^3 = 0, \\
V^2_{\ 1} &= \frac{\partial v^2}{\partial r} + v^1\gamma_{11}{}^2 = 0, \quad V^3_{\ 2} = \frac{\partial v^3}{\partial \theta} + v^1\gamma_{21}{}^3 = 0, \quad V^1_{\ 3} = \frac{\partial v^1}{\partial \varphi} + v^1\gamma_{31}{}^1 = 0.
\end{aligned} \tag{4.100}
$$

Therefore, $V^a_{\ c}$ is diagonal. Because m_{ab} is also diagonal, we conclude that V_{bc} is diagonal, and hence, is identically symmetric. Recognizing V_{bc} as $m_{ba}V^a_{\ c} = m_{ba}\left(\nabla_c \nabla_d \sigma^{ad}\right)$, this means that the equilibrium equations are automatically satisfied for an appropriate pressure field, because the antisymmetric part of $m_{ba}\left(\nabla_c \nabla_d \sigma^{ad}\right)$ vanishes simply due to the symmetry of the tensor field M^{AB}.

We now only need to satisfy the incompressibility condition $\det \mathbf{b} = 1$. Computing this yields

$$
\left(\pm r^3 \mp A^3\right)^{\frac{4}{3}} m_1^2(r) m_2^4(r) = 1, \tag{4.101}
$$

or in referential variables, writing $m_1(R) = m_1(r(R))$, $R^4 m_1^2(R) m_2^4(R) = 1$. Therefore, the final form of the inverse material metric tensor for this family is

$$
\left[M^{AB}\right] = \begin{bmatrix} m_1^2(R) & 0 & 0 \\ 0 & \frac{1}{m_1(R)R^2\sin^2\Phi} & 0 \\ 0 & 0 & \frac{1}{m_1(R)R^2} \end{bmatrix}. \tag{4.102}
$$

Alternatively, introducing a coordinate rescaling as before, the material metric tensor takes the form

$$[M_{AB}] = \begin{bmatrix} m_1(R) & 0 & 0 \\ 0 & m_2(R)\sin^2\Phi & 0 \\ 0 & 0 & m_2(R) \end{bmatrix}, \tag{4.103}$$

and we obtain a differential equation that can be integrated to determine $r(R)$.

Then, taking a multiplicative decomposition of $\mathbf{F} = \mathbf{AG}$, we can express $\mathbf{G}$ using an orthonormal frame in the material manifold yielding

$$[G^\alpha{}_A] = \begin{bmatrix} G_1(R) & 0 & 0 \\ 0 & G_2(R)\sin\Phi & 0 \\ 0 & 0 & G_2(R) \end{bmatrix}, \tag{4.104}$$

which again can be left multiplied by an arbitrary local rotation $\mathbf{Q}$ if desired.

Family 5: Inflation, Bending, Extension, and Azimuthal Shearing of an Annular Wedge. For this family, we have the deformation as given in (4.23) with deformation gradient (4.24). Again, we compute the equilibrium condition $m_{a[l}\nabla_{k]}\nabla_b\sigma^{ab} = 0$, and look at the W_{111} coefficient. This contains two independent equations:

$$\begin{bmatrix} AEM^{13}(r) \\ Ar\left(ABM^{11}(r) + CrM^{12}(r)\right) \end{bmatrix} I_1'(r)^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.105}$$

If I_1 is constant, this equation is satisfied, and if I_1 is not constant, we require $M^{13}(r) = 0$, and $M^{12}(r) = -\frac{ABM^{11}(r)}{Cr}$. If I_1 is constant, we examine the W_{122} coefficient in the equilibrium equation and obtain

$$\begin{bmatrix} AEM^{13}(r) \\ Ar\left(ABM^{11}(r) + CrM^{12}(r)\right) \end{bmatrix} I_2'(r)^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.106}$$

which implies either I_2 is constant, or $M^{13}(r) = 0$ and $M^{12}(r) = -\frac{ABM^{11}(r)}{Cr}$. Hence, we either have all of the invariants of $\mathbf{b}$ constant, or we have $M^{13}(r) = 0$ and $M^{12}(r) = -\frac{ABM^{11}(r)}{Cr}$, which characterizes the generic solution.

The conditions on the components of the metric are sufficient to satisfy equilibrium, so we only have to satisfy incompressibility. The incompressibility condition in this case reads

$$A^2E^2M^{11}(r)\left[\left(C^2r^2M^{22}(r) - A^2B^2M^{11}(r)\right)M^{33}(r) - C^2r^2M^{23}(r)^2\right] = 1. \tag{4.107}$$

We solve this for $M^{22}(r)$ to obtain

$$M^{22}(r) = \frac{1}{A^2C^2E^2r^2M^{11}(r)M^{33}(r)} + \frac{\left(M^{23}(r)\right)^2}{M^{33}(r)} + \frac{A^2B^2M^{11}(r)}{C^2r^2}, \tag{4.108}$$

which gives the generic solution

$$\left[M^{AB}(r)\right]=\begin{bmatrix} M^{11}(r) & -\frac{ABM^{11}(r)}{Cr} & 0 \\ -\frac{ABM^{11}(r)}{Cr} & \frac{1+A^2C^2E^2r^2\left(M^{23}(r)\right)^2M^{11}(r)}{A^2C^2E^2r^2M^{11}(r)M^{33}(r)}+\frac{A^2B^2M^{11}(r)}{C^2r^2} & M^{23}(r) \\ 0 & M^{23}(r) & M^{33}(r) \end{bmatrix}, \tag{4.109}$$

or in referential variables, writing $M^{AB}(R)=M^{AB}(r(R))$,

$$\left[M^{AB}(R)\right]=\begin{bmatrix} M^{11}(R) & -\frac{BM^{11}(R)}{CR} & 0 \\ -\frac{BM^{11}(R)}{CR} & \frac{1+A^4C^2E^2R^2\left(M^{23}(R)\right)^2M^{11}(R)}{A^4C^2E^2R^2M^{11}(R)M^{33}(R)}+\frac{B^2M^{11}(R)}{C^2R^2} & M^{23}(R) \\ 0 & M^{23}(R) & M^{33}(R) \end{bmatrix}. \tag{4.110}$$

Unlike the other families, the standard Euclidean inverse metric $M^{AB}(R)=\operatorname{diag}\{1,R^{-2},1\}$ is not a member of the generic solution branch for this family. This is because this Euclidean metric yields a special case of the anomalous solution, having constant invariants.

In principle, we can rescale our coordinates and compute the form of the anelastic factor in a multiplicative decomposition for a member of this family as we have done for the previous families. However we will not concern ourselves with the multiplicative decomposition for this family, because its generic solution branch does not contain the solution without eigenstrain. As such, any continuous process based on this family beginning with zero eigenstrain will not lie in this solution branch, but rather on the anomalous branch, and so the multiplicative decomposition associated with this generic branch is of limited use.

4.5 The Anomalous Universal Solutions

The analysis for each family follows the same general pattern, so we will merely outline these steps here on an example appearing in Family 5, then present the results. Details are given in Appendix A.

Step 1: For the anomalous solution, we start with the equations derived from the equilibrium conditions: four second-order linear differential equations for each family, involving the six undetermined components of the inverse metric tensor. By integrating the equilibrium conditions, up to two of these components can be expressed in terms of the other variables.

We take, as an example the deformation

$$r = R, \quad \theta = \log R + \Theta, \quad z = Z, \tag{4.111}$$

for which we compute the components of b^{ab} as

$$\left[b^{ab}\right] = \begin{bmatrix} M^{11} & M^{12} + \frac{M^{11}}{r} & M^{13} \\ M^{12} + \frac{M^{11}}{r} & M^{22} + 2\frac{M^{12}}{r} + \frac{M^{11}}{r^2} & M^{23} + \frac{M^{13}}{r} \\ M^{13} & M^{23} + \frac{M^{13}}{r} & M^{33} \end{bmatrix}. \tag{4.112}$$

The first universal equilibrium equation is $m_{a[l}\nabla_{k]}\nabla_b b^{ab} = 0$, which in these coordinates amounts to the two equations

$$M^{13}(r)'' + \frac{M^{13}(r)'}{r} - \frac{M^{13}(r)}{r^2} = 0,$$
$$r\left(5M^{12}(r)' + M^{11}(r)'' + rM^{12}(r)''\right) + 3M^{12}(r) + 3M^{11}(r)' = 0.$$

The general solution of these equations is

$$M^{13}(r) = \alpha_1 r + \frac{\alpha_2}{r}, \quad M^{12}(r) = \frac{\gamma_1}{r} + \frac{\gamma_2}{r^3} - \frac{M^{11}(r)}{r}. \tag{4.113}$$

For the purposes of our example, we will take $\gamma_2 = 0$, $\gamma_1 = 0$, and $M^{13}(r) = r$. With this, b^{ab} becomes

$$\left[b^{ab}\right] = \begin{bmatrix} M^{11} & 0 & r \\ 0 & M^{22} - \frac{M^{11}}{r^2} & M^{23} + 1 \\ r & M^{23} + 1 & M^{33} \end{bmatrix}. \tag{4.114}$$

Next, we compute the equilibrium condition $m_{a[l}\nabla_{k]}\nabla_b c^{ab} = 0$, which is simplified by multiplying c^{ab} by the condition $\det \mathbf{b} = 1$. This condition yields the equations

$$r\left(r^2 M^{23}(r)'' + 7rM^{23}(r)' + 8M^{23}(r) + 8\right) = 0,$$
$$3M^{12}(r) + 3M^{11}(r)' + r\left(5M^{12}(r)' + M^{11}(r)'' + rM^{12}(r)''\right) = 0.$$

Integrating these equations gives the solutions

$$r^4 M^{23}(r) = r^2\mu_1 + \mu_2 - r^4, \quad r^6 M^{22}(r) = r^4 M^{11}(r) + r^4\beta_1 + r^2\beta_2, \tag{4.115}$$

which gives b^{ab} as

$$\left[b^{ab}\right] = \begin{bmatrix} M^{11} & 0 & r \\ 0 & \frac{\beta_1}{r^2} + \frac{\beta_2}{r^4} & \frac{\mu_1}{r^2} + \frac{\mu_2}{r^4} \\ r & \frac{\mu_1}{r^2} + \frac{\mu_2}{r^4} & M^{33} \end{bmatrix}. \tag{4.116}$$

Step 2: After integrating these equations, we have the three constant invariant conditions for each family to solve. The constant trace condition is linear in the unknown components of the inverse metric, so we can use it to solve for one undetermined inverse metric component in exchange for introducing the trace of $\mathbf{b}$ as a parameter.

We have the constant trace condition

$$I_1 = b^{ab} m_{ab} = M^{11} + M^{33} + \beta_1 + \frac{\beta_2}{r^2}, \tag{4.117}$$

hence,

$$M^{33} = I_1 - M^{11} - \beta_1 - \frac{\beta_2}{r^2}. \tag{4.118}$$

The incompressibility condition $\det \mathbf{b} = 1$ then can be written as

$$\begin{aligned}&\left(\beta_1 r^6 + \beta_2 r^4\right) M^{11}(r)^2 + \left[\left(r^2\mu_1 + \mu_2\right)^2 + \left(r^3\beta_1 + r\beta_2\right)^2 - I_1\left(\beta_1 r^6 + \beta_2 r^4\right)\right] M^{11}(r)\\&\quad + r^6\left(1 + \beta_1 r^2 + \beta_2\right) = 0,\end{aligned} \tag{4.119}$$

and the constant second invariant condition can be written as

$$\begin{aligned}&r^6 M^{11}(r)^2 - r^4\left(I_1 r^2 - \beta_1 r^2 - \beta_2\right) M^{11}(r) + \left(r^2\mu_1 + \mu_2\right)^2 + \left(\beta_1 r^3 + \beta_2 r\right)^2\\&\quad + r^8 + I_2 r^6 - I_1 r^4\left(\beta_1 r^2 + \beta_2\right).\end{aligned} \tag{4.120}$$

Step 3: We are left with two nonlinear algebraic equations. The first is the incompressibility condition $\det \mathbf{b} = 1$, and the second is the constancy of the second invariant of $\mathbf{b}$. Both are quadratic equations in the remaining component of the inverse metric tensor, which creates an overdetermined system. We compute the resultant of these two equations in this component, and demand this resultant vanish to ensure that these two equations have a common root. The resultant of these equations is itself a polynomial in the other undetermined integration constants: the invariants of $\mathbf{b}$, and the remaining independent spatial coordinate.

Step 4: Therefore, the resultant is a polynomial equation of the form $p(q) = 0$, which must hold for all values of the independent variable q (which is either r or x depending on the family). Accordingly, we set each coefficient to zero independently, and obtain an overdetermined system of nonlinear polynomial equations for the undetermined constants. We wish to find all the solutions to these equations, and so we compute a primary decomposition of the radical ideal generated by these equations. These equations are simple enough that this can be done with the assistance of a symbolic algebra package, though even then the computations are rather cumbersome (see Appendix A). After we have done this, we are left with a set of conditions on the undetermined constants that are necessary and sufficient for the existence of a common root of the original quadratic equations in the undetermined inverse metric component over an open set. We substitute these constants into these equations and use them to solve for the final component of the inverse metric tensor, which gives us the general form of the anomalous solution. In all of these cases, despite encountering branching conditions in the course of analyzing the conditions on the constants, the separate branches ultimately are redundant, and we are left with a single anomalous solution branch for each family.

Even in our simplified example, the resultant of these equations in $M^{11}(r)$ yields the condition a relatively lengthy polynomial equation $p(r) = 0$. It can be immediately simplified by noticing that one of the coefficients is simply μ_2^6, so $\mu_2 = 0$ is a necessary condition for there to be a common solution to these two equations. It can be further simplified by noting that after using $\mu_2 = 0$, one of the coefficients becomes β_2^6, so we demand $\beta_2 = 0$. With this, a different coefficient becomes μ_1^6, and hence $\mu_1 = 0$ as well. Using these conditions $\mu_2 = 0$, $\beta_2 = 0$, and $\mu_1 = 0$, the polynomial simply becomes $\left(\beta_1^3 - I_1\beta_1^2 + I_2\beta_1 - 1\right)^2 r^{18} = 0$, which demands $\beta_1^3 - I_1\beta_1^2 + I_2\beta_1 - 1 = 0$, because $r > 0$. We recognize that this is the eigenvalue equation for the tensor $\mathbf{b}$, so we require β_1 to be an eigenvalue of $\mathbf{b}$. We can satisfy this by writing $I_1 = \beta_1 + e_1$ and $I_2 = \beta_1 e_1 + \frac{1}{\beta_1}$, where $e_1 = \lambda_1^2 + \lambda_2^2$ is the sum of the other two eigenvalues of $\mathbf{b}$ and we have used incompressibility in the form $\lambda_1^2\lambda_2^2\beta_1 = 1$. When we substitute these conditions back into the original equations for M^{11}, they both become (up to some nonzero constant)

$$\beta_1 M^{11}(r)^2 - e_1\beta_1 M^{11}(r) + r^2\beta_1 + 1 = 0. \tag{4.121}$$

This equation can be solved for M^{11} and we obtain using $r = R$

$$M^{11}(r = R) = \frac{1}{2}\left(e_1 \pm \sqrt{e_1^2 - 4\left(R^2 + \frac{1}{\beta_1}\right)}\right), \tag{4.122}$$

which gives one example of M^{AB} as

$$\left[M^{AB}\right] = \begin{bmatrix} M^{11}(R) & -\frac{1}{R}\left(M^{11}(R)\right) & R \\ -\frac{1}{R}\left(M^{11}(R)\right) & \frac{1}{R}\left(M^{11}(R)\right) + \frac{\beta_1}{R^2} & -1 \\ R & -1 & \frac{e_1}{2} \mp \sqrt{\frac{e_1^2}{4} - R^2 - \frac{1}{\beta_1}} \end{bmatrix}, \tag{4.123}$$

This lets us compute the corresponding elastic left Cauchy Green stretch tensor as

$$
\left[b^{ab}\right] = \begin{bmatrix} \frac{1}{2}\left(e_1 \pm \sqrt{e_1^2 - 4\left(r^2 + \frac{1}{\beta_1}\right)}\right) & 0 & r \\ 0 & \frac{\beta_1}{r^2} & 0 \\ r & 0 & \frac{1}{2}\left(e_1 \mp \sqrt{e_1^2 - 4\left(r^2 + \frac{1}{\beta_1}\right)}\right) \end{bmatrix}. \tag{4.124}
$$

We can verify that b^{ab} satisfies the equilibrium conditions and the constant invariant conditions. Completely determining the universal anelastic extensions of these families amounts to doing a similar analysis for each of the remaining families, but in full generality, i.e., not assuming particular values for the parameters appearing in the deformation, nor selecting values for the integration constants a priori. These computations are included in the appendix, only the results are presented next.

Family 1: Bending, Stretching, and Shearing of a Rectangular Block. Integrating the equilibrium equations and solving the algebraic equations gives the following anomalous solution branch for this family:

$$
\begin{aligned}
M^{12}(r) &= \frac{\alpha_1}{r^2} + \alpha_2, \\
M^{13}(r) &= \frac{AB^2C\alpha_1}{r^2} + \gamma_1 r^2 + \gamma_2, \\
M^{11}(r) &= \frac{e_1 r^2}{2A^2} \\
&\pm \frac{r\sqrt{B^2 e_1^2 r^2 - 4\left(B^2 r^2\left(e_2 + A^2B^2M^{12}(r)^2\right) + \left(AB^2CM^{12}(r) - M^{13}(r)\right)^2\right)}}{2A^2B}, \\
M^{22}(r) &= \frac{A^2B^4e_2\left(e_1r^2 - A^2M^{11}(r)\right)M^{12}(r)^2 + \left[AB^2CM^{12}(r) - M^{13}(r)\right]^2}{B^2e_2r^2\left[\left(AB^2CM^{12}(r) - M^{13}(r)\right)^2 + r^2M^{12}(r)^2\right]}, \\
M^{23}(r) &= \frac{AC\left(AB^2CM^{12}(r) - M^{13}(r)\right)^2}{e_2r^2\left[\left(AB^2CM^{12}(r) - M^{13}(r)\right)^2 + r^2M^{12}(r)^2\right]} \\
&\quad + \frac{A^2B^2\left(e_1e_2r^2 - r^2 - A^2e_2M^{11}(r)\right)M^{12}(r)M^{13}(r)}{e_2r^2\left[\left(AB^2CM^{12}(r) - M^{13}(r)\right)^2 + r^2M^{12}(r)^2\right]}, \\
M^{33}(r) &= \frac{A^2B^2}{e_2r^2}\times \\
&\frac{\left[AB^2\left(C^2 + r^2\right)M^{12}(r) - CM^{13}(r)\right]^2 + e_2\left(e_1r^2 - A^2M^{11}(r)\right)M^{13}(r)^2}{\left(AB^2CM^{12}(r) - M^{13}(r)\right)^2 + r^2M^{12}(r)^2}.
\end{aligned}
$$

Here the constants e_1 and e_2 are the elementary symmetric polynomials in two of the three free eigenvalues of $\mathbf{b}$

$$
e_1 = \lambda_1^2 + \lambda_2^2, \quad e_2 = \lambda_1^2\lambda_2^2, \tag{4.125}
$$

with the incompressibility condition determining the third eigenvalue as $\lambda_3^2 = \frac{1}{\lambda_1^2\lambda_2^2}$. The parameters e_1 and e_2 must be positive with $e_1^2 > 4e_2$, since $\mathbf{b}$ is positive definite. The remaining constants, α_1, α_2, γ_1, and γ_2 are arbitrary, subject to the condition that the choice of e_1, e_2, α_1, α_2, γ_1, and γ_2 must yield a positive-definite metric tensor. One can explicitly verify that the invariants of $\mathbf{b}$ generated by this metric are

$$\begin{aligned} I_1 &= e_1 + \frac{1}{e_2} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{e_1}{e_2} + e_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 = 1. \end{aligned} \tag{4.126}$$

Additionally, we can express this in terms of the referential variables by expressing r in terms of X by the relation $r = \sqrt{A\,(2X + D)}$.

Family 2: Straightening, Stretching, and Shearing of a Sector of a Cylinder. With this family, it is prudent to make the substitution $\xi = x - D$, which allows us to express the anomalous solution branch as

$$\begin{aligned} M^{12}(\xi) &= \frac{\alpha_1\xi + \alpha_2}{\sqrt{\xi}}, \\ M^{13}(\xi) &= \frac{\gamma_1\xi + \gamma_2}{\sqrt{\xi}}, \\ M^{11}(\xi) &= \frac{\sqrt{A}e_1 \pm \sqrt{Ae_1^2 - 4\left[Ae_2 + 2\xi\left(\left(AM^{13}(\xi) + CM^{12}(\xi)\right)^2 + M^{12}(\xi)^2\right)\right]}}{4A^{\frac{3}{2}}B^2\xi}, \\ M^{22}(\xi) &= \frac{A^2B^2\left[\left(CM^{12}(\xi) + AM^{13}(\xi)\right)^2 + e_2\left(e_1 - 2AB^2\xi M^{11}(\xi)\right)M^{12}(\xi)^2\right]}{e_2\left[M^{12}(\xi)^2 + \left(AM^{13}(\xi) + CM^{12}(\xi)\right)^2\right]}, \\ M^{23}(\xi) &= -\frac{AB^2\left[C\left(CM^{12}(\xi) + AM^{13}(\xi)\right)^2 + CM^{12}(\xi)^2\right]}{e_2\left(M^{12}(\xi)^2 + \left(AM^{13}(\xi) + CM^{12}(\xi)\right)^2\right)}, \\ &\quad - \frac{AB^2\left[\left(A - e_1e_2 + 2AB^2e_2\xi M^{11}(\xi)\right)M^{12}(\xi)M^{13}(\xi)\right]}{e_2\left[M^{12}(\xi)^2 + \left(AM^{13}(\xi) + CM^{12}(\xi)\right)^2\right]}, \\ M^{33}(\xi) &= \frac{B^2}{e_2}\frac{\left(\left(1 + C^2\right)M^{12}(\xi) + ACM^{13}(\xi)\right)^2 + A^2e_2\left(e_1 - 2AB^2\xi M^{11}(\xi)\right)M^{13}(\xi)^2}{M^{12}(\xi)^2 + \left(AM^{13}(\xi) + CM^{12}(\xi)\right)^2}. \end{aligned}$$

Alternatively, we can express this in terms of referential variables using the equation $\xi = \frac{1}{2}AB^2R^2$. As in the previous case, e_1 and e_2 are the elementary symmetric polynomials in the free eigenvalues of $\mathbf{b}$, λ_1^2 and λ_2^2.

$$e_1 = \lambda_1^2 + \lambda_2^2, \quad e_2 = \lambda_1^2\lambda_2^2. \tag{4.127}$$

With this, the third eigenvalue is determined via the incompressibility condition $\lambda_3^2 = \frac{1}{\lambda_1^2\lambda_2^2}$. This ensures that the invariants of $\mathbf{b}$ are

$$\begin{aligned} I_1 &= e_1 + \frac{1}{e_2} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{e_1}{e_2} + e_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 = 1. \end{aligned} \tag{4.128}$$

The constants e_1, e_2, α_1, α_2, γ_1, γ_2 are largely arbitrary, apart from the condition that $e_1 > 0$, and $e_1^2 > 4e_2 > 0$, and that the constants are chosen such that the metric tensor is positive definite.

Alternatively, we have the case where the anelastic strain is compatible, and we have

$$[M_{AB}(R)] = \begin{bmatrix} M_{11}R^2 & M_{12}R & M_{13}R \\ M_{12}R & M_{22} & M_{23} \\ M_{13}R & M_{23} & M_{33} \end{bmatrix}, \tag{4.129}$$

where $\{M_{11}, M_{12}, M_{13}, M_{22}, M_{23}, M_{33}\}$ are constants and $\det M_{AB}(R) = R^2$. At first glance this case appears slightly more general than the previous one under the special case $\alpha_1 = \gamma_1 = 0$, because for a fixed overall deformation, there are five independent parameters determining this solution, while setting $\alpha_1 = \gamma_1 = 0$ leaves the other family with only α_2, γ_2, e_1, and e_2. However, this case causes the stretch tensor b^{ab} to be constant, which requires that the material manifold be Euclidean, and a constant isochoric stretch only depends on two independent stretches, with the remaining degrees of freedom representing a global rotation, which we can freely add or remove. This case appears to not be a special case of the previous branch, because once we eliminate the dependence on ξ, we not longer have a preferred direction, and hence we spontaneously gain additional rotational degrees of freedom that can be removed by the choice of the orientation of our current configuration Cartesian coordinates.

Physically, this amounts to the reference configuration deforming anelastically into a parallelepiped, which can be elastically deformed into the desired block, as that elastic deformation is homogeneous. Indeed, the stress required to accomplish this is always constant, and hence equilibrium conditions are trivially satisfied. One can easily verify that the only nonzero Christoffel symbol generated by this metric in these coordinates is ${\Gamma^1}_{11} = \frac{1}{R}$, which generates a vanishing curvature tensor $\mathcal{R} = \mathbf{0}$. In fact, the anelastic strain can be integrated up to an arbitrary rigid rotation and translation to obtain the position vector

$$\mathbf{x}_A = \frac{R^2}{2}\boldsymbol{\varepsilon}_1 + \Theta\boldsymbol{\varepsilon}_2 + Z\boldsymbol{\varepsilon}_3, \tag{4.130}$$

where $\boldsymbol{\varepsilon}_a$ is an arbitrary right-handed set of linearly independent vectors spanning a parallelepiped with unit volume. With this, the constants $M_{ab} = \boldsymbol{\varepsilon}_a \cdot \boldsymbol{\varepsilon}_b$, i.e., the arbitrary constants appearing in the metric tensor are given by the Euclidean inner products of the constant basis vectors.

Family 3: Inflation, Bending, Torsion, Extension, and Shearing of an Annular Wedge. For this solution, it is prudent to define the functions

$$p(r) = \gamma_1 + \frac{\gamma_2}{r^2}, \quad q(r) = \alpha_1 r^2 + \alpha_2. \tag{4.131}$$

With these definitions, we have the following anomalous solution branch

$$\begin{aligned}
M^{12}(r) &= \frac{Dq(r) - Fp(r)}{\sqrt{r^2 - B}}, \\
M^{13}(r) &= \frac{Ep(r) - Cq(r)}{\sqrt{r^2 - B}}, \\
M^{11}(r) &= r^2 \frac{e_1 \pm \sqrt{e_1^2 - 4\left[e_2 + A\left(CF - DE\right)^2\left(p(r)^2 + \frac{q(r)^2}{r^2}\right)\right]}}{2A\left(r^2 - B\right)}, \\
M^{22}(r) &= \frac{\left(Dr^2 p(r) + Fq(r)\right)^2 + e_1 e_2 r^2\left(Dq(r) - Fp(r)\right)^2}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2 r^2\right)} \\
&\quad - \frac{Ae_2\left(Fp(r) - Dq(r)\right)^2\left(r^2 - B\right)M^{11}(r)}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2 r^2\right)}, \\
M^{23}(r) &= \frac{Ae_2\left(Ep(r) - Cq(r)\right)\left(Fp(r) - Dq(r)\right)\left(r^2 - B\right)M^{11}(r)}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2 r^2\right)} \\
&\quad - \frac{e_1\left(Cq(r) - Ep(r)\right)\left(Dq(r) - Fp(r)\right)}{\left(CF - DE\right)^2\left(q(r)^2 + p(r)^2 r^2\right)} \\
&- \frac{\left(Cp(r)r^2 + Eq(r)\right)\left(Dp(r)r^2 + Fq(r)\right)}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2 r^2\right)}, \\
M^{33}(r) &= \frac{\left(Cr^2 p(r) + Eq(r)\right)^2 + e_1 e_2 r^2\left(Cq(r) - Ep(r)\right)^2}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2 r^2\right)} \\
&\quad - \frac{Ae_2\left(Ep(r) - Cq(r)\right)^2\left(r^2 - B\right)M^{11}(r)}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2 r^2\right)}.
\end{aligned}$$

As with the other families, we can use the deformation equation $r = \sqrt{AR^2 + B}$, to recast this into the referential variables. Additionally, the parameters e_1 and e_2 are the same as the previous families, and other than demanding that the eigenvalues they determine be positive, we also demand that the choice of variables α_1, α_2, γ_1, γ_2, e_1, and e_2 leaves the metric tensor positive definite.

Family 5: Inflation, Bending, Extension, and Azimuthal Shearing of an Annular Wedge. To facilitate the analysis of this family, it is useful to define the function $f(r) = \gamma_1 + \frac{\gamma_2}{r^2}$. With this, we have

$$
\begin{aligned}
M^{13}(r) &= \alpha_1 r + \frac{\alpha_2}{r}, \\
M^{11}(r) &= \frac{e_1 \pm \sqrt{e_1^2 - 4\left(e_2 + A^2 f(r)^2 + A^2 E^2 M^{13}(r)^2\right)}}{2A^2}, \\
M^{12}(r) &= \frac{f(r) - ABM^{11}(r)}{Cr}, \\
M^{22}(r) &= \frac{e_2 f(r)^2 \left(e_1 + A^2\left(B^2 - 1\right) M^{11}(r)\right)}{C^2 r^2 e_2 \left(f(r)^2 + E^2 M^{13}(r)^2\right)} \\
&\quad + \frac{E^2\left(1 + A^2 B^2 e_2 M^{11}(r)\right) M^{13}(r) - 2ABe_2 f(r)\left(f(r)^2 + E^2 M^{13}(r)^2\right)}{C^2 r^2 e_2 \left(f(r)^2 + E^2 M^{13}(r)^2\right)}, \\
M^{23}(r) &= -\frac{ABM^{13}(r)}{Cr} - \frac{M^{13}(r) f(r)\left(1 - e_1 e_2 + A^2 e_2 M^{11}(r)\right)}{Ce_2 r\left(f(r)^2 + E^2 M^{13}(r)^2\right)}, \\
M^{33}(r) &= \frac{f(r)^2 + E^2 e_2\left(e_1 - A^2 M^{11}(r)\right) M^{13}(r)^2}{E^2 e_2\left(f(r)^2 + E^2 M^{13}(r)^2\right)}.
\end{aligned}
$$

Again, the constraints on the constants appearing are as before, and are only necessary to ensure the positive definiteness of $\mathbf{b}$ and the metric tensor. We can recast this into referential variables using $r = AR$, if desired.

4.6 Graphic Representation

Because the material manifolds are generally non-Euclidean, visualizing them is difficult. A way to overcome this difficulty is to approximate their geometry as "piecewise Euclidean" and examine the deformation of each piece. This approach is similar to the three-dimensional version of approximating a curved surface with that of a polyhedron, and then representing that polyhedron in the plane by its net. The original surface can then be build up by connecting appropriate edges, but because of the curved nature of the surface, these edges cannot all be connected without distorting the pieces, or lifting them out of the plane. To demonstrate this technique, we will first start with a two-dimensional example, and then move on to a Euclidean three-dimensional example, and then finally apply the techniques to examples of anomalous material manifolds obtained from our analysis.

4.6.1 A Two-Dimensional example

We know that representing spherical geometry in the plane isometrically is an egregiously impossible task [Gauss, 1828]. To get around this, we only do this approximately, and allow for incompatibility by partitioning and separating our domain in multiple pieces. We can then stretch each piece in such a way that the deformed pieces can be approximately recombined in three-dimensional space to form an upper hemisphere. The deformed pieces are individually flat, so they can all be placed in the plane, but not in a way such that they can be pieced together without gaps (see Figure 4.11). We start with a disk, partition it, and separate the resulting pieces to allow for room for each piece to strain without overlapping its neighbors. We then strain each piece, and recombine the deformed cells to create an approximation of an upper hemisphere. Here each cell $[R_{i-1}, R_i] \times [\Theta_{j-1}, \Theta_j]$ is positioned so both the position of the point $(r, \theta) = ((R_{i-1} + R_i)/2, (\Theta_{j-1} + \Theta_j)/2)$, and the orientation of its tangent plane match that of the exact map from the disk to the hemisphere.

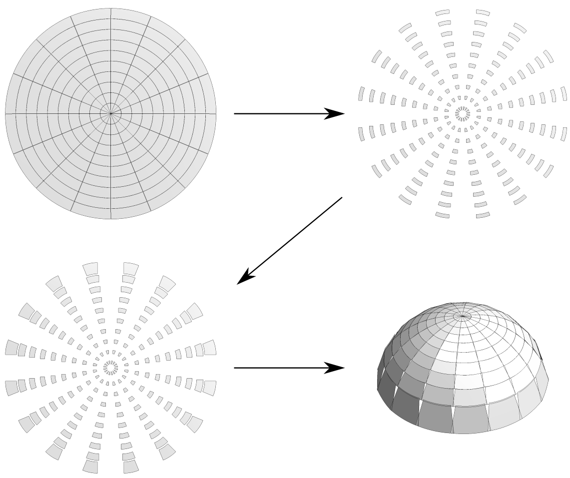


Figure 4.11: The partitioned approximation of a hemisphere.

Explicitly, we want to take the region $r \in [0, 1]$, $\theta \in [0, 2\pi)$, where r and θ are polar coordinates in the plane, and map it to the surface $z = \sqrt{1 - r^2}$ in three-dimensional space. The stretch induced by this map is described by the metric tensor with cylindrical components

$$[M_{\alpha\beta}] = \begin{bmatrix} \frac{1}{1-r^2} & 0 \\ 0 & r^2 \end{bmatrix}, \tag{4.132}$$

which we can approximate as constant on each piece, while keeping each piece in the plane, by evaluating the metric at $r^* = (r_{max} - r_{min})/2$. The deformed pieces can then be rigidly translated and rotated in three-dimensional space to approximate the desired spherical surface, with the approximation becoming better as the partition becomes finer (see Figure 4.12).

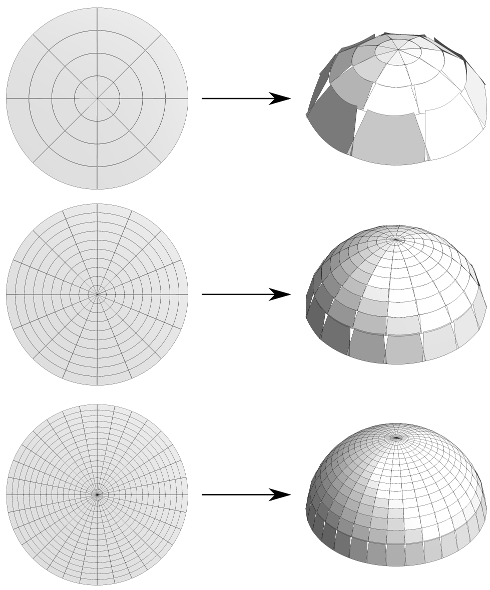


Figure 4.12: Convergence of the partitioned approximation as piece size decreases.

This two-dimensional example allows one to see the correspondence between the deformed partitioned approximation and the recombined non-Euclidean configuration, which is important because once we move up to three-dimensional examples, we are no longer able to recombine the strained pieces; we must deduce properties of its geometry from the deformed partitioned approximation alone. Additionally, while we assembled the resulting deformed pieces into a hemisphere by lifting them into a higher dimensional Euclidean space, we could have assembled them into any number of other surfaces that are isometric to the hemisphere. Because we only determine the intrinsic geometry of the material manifold, there is no preferred isometric embedding in some higher dimensional Euclidean space, unless as above, we explicitly specify the embedding.

4.6.2 A Three-Dimensional Euclidean example

Just as in the two-dimensional example, we can partition a flat three-dimensional body, explode it, and approximate the strain on each piece to represent the non-Euclidean geometry of our deformations. The only difference is that, in general, we cannot recombine the distorted pieces into a cohesive whole, because the resulting shape is not globally flat. However, if the strain that we impose is actually induced by a map between Euclidean spaces, we can apply this procedure to the partitioned pieces, and observe the local strain, while separately observing the global deformation.

We can then compare the two results to see which features are preserved by this local partitioning approach to better interpret the results of applying this procedure to the material metrics we have derived.

Consider the following map given in cylindrical polar coordinates

$$r = R, \quad \theta = \nu\Theta, \quad z = Z + \mu\nu\Theta. \tag{4.133}$$

This map produces azimuthal shear and angular stretching. Choosing $\mu = 2$ and $\nu = \frac{1}{2}$, and mapping the domain $R \in [2,3]$, $\Theta \in \left[0, \frac{2\pi}{3}\right]$, $Z \in [0,1]$, we obtain a transformation shown in Figure 4.13, where we have artificially separated our pieces to better show the deformation of internal elements. If we instead compute the stretch

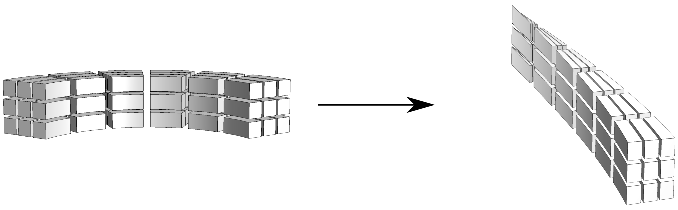


Figure 4.13: Angular stretching and azimuthal shear of an annular wedge.

tensor, and use it as a material metric for the current configuration, we obtain

$$[M_{AB}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \nu^2\left(r^2 + \mu^2\right) & \mu\nu \\ 0 & \mu\nu & 1 \end{bmatrix}. \tag{4.134}$$

Applying this stretch to our partitioned domain, we obtain the depiction shown in Figure 4.14. This side-by-side comparison shows what is happening when we do this piecewise transformation, namely we capture the strain of each piece, but we do not capture any local rotation that is present in the global deformation. This is because local rotations produce no strain, so they do not contribute to the stretch tensor, and hence, we cannot expect to be able to capture them through this reconstruction.

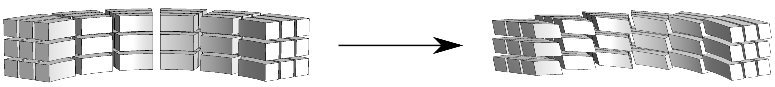


Figure 4.14: Strain-based reconstruction of the deformed state.

4.6.3 Anomalous anelastic strains

For the anomalous families, we will use this partitioning technique to attempt to visualize the deformations. We note that not all choices of parameters are valid over arbitrary domains. In particular, the parameters must be chosen such that the metric is positive definite over the chosen domain, in addition to making sure that the strain tensor $\mathbf{b}$ is positive definite, a much simpler task as this requires e_1 and e_2 to be positive with $e_1^2 > 4e_2$. We also depict the total overall deformation, coloring the current configuration by the trace of the Cauchy stress required to maintain it for a Mooney-Rivlin solid with strain energy of the form

$$W = \frac{\mu_1}{2}\left(I_1 - 3\right) + \frac{\mu_2}{2}\left(I_2 - 3\right), \tag{4.135}$$

both in the presence and absence of anelastic strain. Because the invariants of the left Cauchy Green tensor are constant for the anomalous universal eigenstrains, any choice of strain energy is indistinguishable from a Mooney-Rivlin energy, and the only invariant of the Cauchy stress that can potentially vary spatially is the pressure generated due to the constraint stress. Here we choose the two material parameters in the Mooney-Rivlin energy to each be equal to 1, though different choices of parameters would yield qualitatively similar results.

Family 1. For this family we choose the reference domain $X \in [0,1]$, $Y \in [0,6]$, and $Z \in [0,4]$, and take the deformation parameters $A = \frac{3}{2}$, $B = 1$, $C = \frac{1}{4}$, $D = 2$, $E = 0$, $F = 0$. To examine the effects of anomalous universal eigenstrain on the equilibrium stress distribution, we consider the same overall deformation, and contrast the stress generated in the presence of eigenstrain with that generated in the absence of eigenstrain. For the anelastic strain parameters, we use

$$\alpha_1 = -1, \quad \alpha_2 = \frac{1}{8}, \quad \gamma_1 = -\frac{1}{8}, \quad \gamma_2 = \frac{6}{11}, \quad \begin{array}{l} e_1 = \frac{9}{4}, \\ e_2 = \frac{9}{8}, \end{array} \Leftrightarrow \begin{array}{l} \lambda_1^2 = \frac{3}{2}, \\ \lambda_2^2 = \frac{3}{4}. \end{array} \tag{4.136}$$

One can verify that this ensures that $\mathbf{M}$ is positive definite over the chosen domain. To visualize the anelastic strain, we subdivide the domain, separate the pieces, and approximate the anelastic strain on each (see Figure 4.15). This anelastic strain is generally not compatible, i.e., the deformed pieces cannot be reassembled in Euclidean space without further deformation. We map the body into the current configuration, and color it to denote the spherical part of the Cauchy stress generated by a Mooney-Rivlin solid. This requires us to integrate the indeterminate constraint pressure field, both with and without eigenstrain. Without eigenstrain, we have the following differential equations for the constraint pressure

$$\frac{\partial p}{\partial r} = \frac{\frac{1}{B^2} + A^2\left(B^2C^2 - 1\right) - \left(B^2 + \frac{3}{A^2}\right)r^4}{r^3}, \quad \frac{\partial p}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} = 0, \tag{4.137}$$

which can be easily integrated to obtain

$$p(r,\theta,z) = p(r) = -\frac{\frac{1}{B^2} + A^2\left(B^2C^2 - 1\right)}{2r^2} - \frac{B^2 + \frac{3}{A^2}}{2}r^2. \tag{4.138}$$

Notice in particular that p does not vary with z or θ. Additionally, only the gradient of p affects the motion, which allowed us to ignore the integration constant when integrating the above equations.

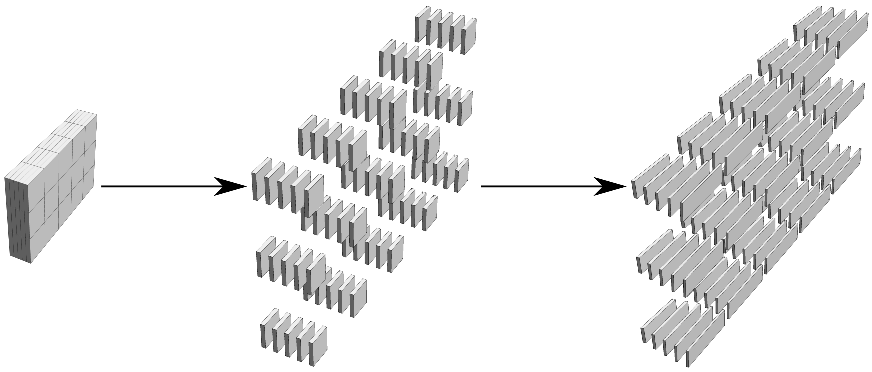


Figure 4.15: A depiction of the anomalous anelastic strain for one element of Family 1.

When there is eigenstrain, we obtain a different set of differential equations determining $p(x)$:

$$\frac{\partial p_{\text{eig}}}{\partial r} = k(r), \quad \frac{\partial p_{\text{eig}}}{\partial \theta} = \frac{2AB\left(1+e_2\right)}{e_2}\alpha_2, \quad \frac{\partial p_{\text{eig}}}{\partial z} = \frac{2\left(1+e_2\right)}{e_2}\gamma_1, \tag{4.139}$$

where $k(r)$ is an algebraic function of r alone. We can in principle integrate these to obtain

$$p_{\text{eig}}\left(r,\theta,z\right) = \int_{r_0}^{r} k\left(\hat{r}\right)d\hat{r} + \frac{2AB\left(1+e_2\right)}{e_2}\alpha_2\theta + \frac{2\left(1+e_2\right)}{e_2}\gamma_1 z. \tag{4.140}$$

In contrast with the ordinary case, when we have eigenstrain, we can generate pressure gradients that vary with θ and z. Interestingly enough, even the generic universal eigenstrain cannot generate pressure gradients in these directions; the anomalous universal eigenstrain is the only universal eigenstrain that can create pressure gradients in these directions. This suggests that the measurement of these pressure gradients can be used to partially measure the eigenstrain, and conversely these anomalous solutions can be used to generate pressure gradients in these directions to, for example, counteract the pressure generated by body forces. We then compute the first stress invariant, the trace of the Cauchy stress, or equivalently its spherical part, for the material both in the absence and presence of eigenstrain, and plot the resulting stress invariant in Figure 4.16.

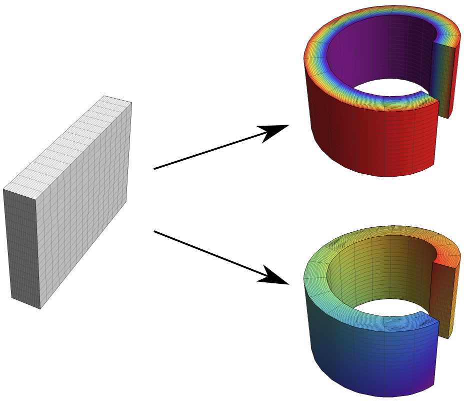


Figure 4.16: Pressure distribution for Family 1: without eigenstrain (top), and with an anomalous eigenstrain field (bottom).

Family 2. For this family, we choose the reference domain $R \in [2,3]$, $\Theta \in [0,5]$, and $Z \in [0,4]$, and take the deformation parameters $A = 1$, $B = \frac{3}{4}$, $C = \frac{1}{4}$, $D = 0$, $E = 0$, $F = 0$. These parameters define the total deformation, allowing us to examine the effects of eigenstrain on the Cauchy stress. In particular, we take the parameters appearing in the anelastic strain to be

$$\alpha_1 = \frac{1}{8}, \quad \alpha_2 = -\frac{2}{5}, \quad \gamma_1 = \frac{1}{8}, \quad \gamma_2 = -\frac{3}{13}, \quad \begin{array}{l} e_1 = \frac{21}{10}, \\ e_2 = \frac{9}{10}, \end{array} \Leftrightarrow \begin{array}{l} \lambda_1^2 = \frac{3}{2}, \\ \lambda_2^2 = \frac{3}{5}. \end{array} \tag{4.141}$$

This ensures that the metric is positive definite over the chosen domain. We subdivide and explode the domain, and apply our anelastic strain to each piece, as shown in Figure 4.17. We are then left with a set of differential equations determining the constraint stress. In the absence of eigenstrain, we have

$$\frac{\partial p}{\partial x} = 2AB^2 + \frac{1}{2AB^2\,(x-D)^2}, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = 0. \tag{4.142}$$

Upon integration, we obtain

$$p(x,y,z) = p(x) = 2AB^2\,(x-D) - \frac{1}{2AB^2\,(x-D)}. \tag{4.143}$$

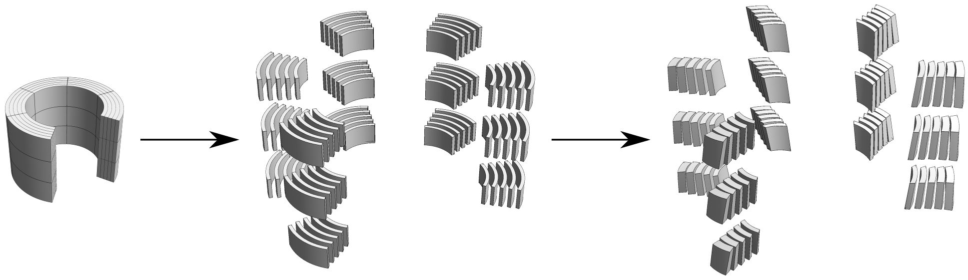


Figure 4.17: A depiction of the anomalous anelastic strain for Family 2.

In contrast, when we consider eigenstrain, we have the equations

$$\frac{\partial p_{\mathrm{eig}}}{\partial x} = k(x), \quad \frac{\partial p_{\mathrm{eig}}}{\partial y} = \frac{\sqrt{2}\,(1+e_2)}{\sqrt{A}e_2}\alpha_1, \quad \frac{\partial p_{\mathrm{eig}}}{\partial z} = \frac{\sqrt{2}\,(1+e_2)}{\sqrt{A}e_2}\left(C\alpha_1 + A\gamma_1\right), \tag{4.144}$$

where as before $k(x)$ is an algebraic function of x. We can integrate these equations to obtain

$$p_{\mathrm{eig}}\left(x,y,z\right) = \int_{x_0}^{x} k\left(\hat{x}\right) d\hat{x} + \frac{\sqrt{2}\,(1+e_2)}{\sqrt{A}e_2}\alpha_1 y + \frac{\sqrt{2}\,(1+e_2)}{\sqrt{A}e_2}\left(C\alpha_1 + A\gamma_1\right) z. \tag{4.145}$$

As before, the presence of this anomalous universal eigenstrain generates pressure gradients in directions that are not possible in their absence, in this case, the y and z directions. Again, even the generic branch of universal eigenstrains cannot generate pressure gradients in these directions, further highlighting the unique nature of the anomalous solutions. We then compute the trace of the Cauchy stress, and plot the resultant distributions both in the absence and presence of eigenstrain (see Figure 4.18).

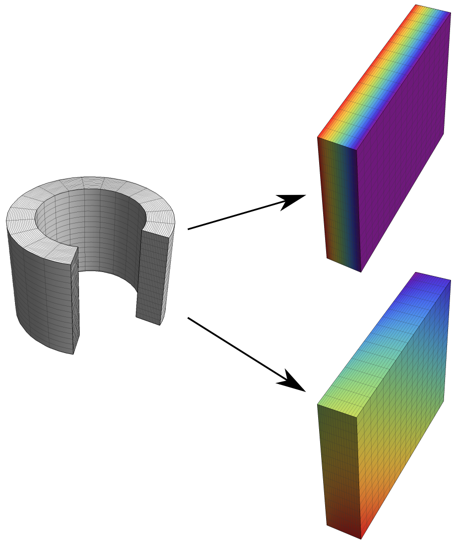


Figure 4.18: Pressure distribution for Family 2: without eigenstrain (top), and with an anomalous eigenstrain field (bottom).

Family 3. For this family, we choose the domain $R \in [2,3]$, $\Theta \in [0,5]$, and $Z \in [0,4]$, and the deformation parameters $A = 1$, $B = 0$, $C = \frac{5}{4}$, $D = \frac{1}{6}$, $E = \frac{1}{4}$, $F = \frac{5}{4}$, $G = \frac{-\pi}{4}$, $H = 0$. This completely defines the total deformation, allowing us to examine the stress generated with eigenstrain in contrast with that generated without eigenstrain. We take

$$\alpha_1 = \frac{1}{8}, \quad \alpha_2 = -\frac{2}{5}, \quad \gamma_1 = \frac{1}{8}, \quad \gamma_2 = -\frac{3}{13}, \quad \begin{matrix} e_1 = \frac{21}{10}, \\ e_2 = \frac{9}{10}, \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \lambda_1^2 = \frac{3}{2}, \\ \lambda_2^2 = \frac{3}{5}, \end{matrix} \tag{4.146}$$

as our anelastic parameters. Over the defined domain, these choices ensure that the anelastic metric tensor is positive definite. We then partition and explode the domain, approximating the eigenstrain on each piece, depicting the result in Figure 4.19. As before, we obtain differential equations for p, yielding in the elastic case

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{E^2}{(DE - CF)^2 r^3} - D^2 r + \frac{B - \frac{BF^2}{(DE-CF)^2} + r^2 + \frac{F^2 r^2}{(DE-CF)^2}}{Ar^3} \\ &+ \frac{A\left[B\left(3 + C^2\right) r - \left(1 + C^2\right) r^3\right]}{\left(r^2 - B\right)^2}, \quad \frac{\partial p}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} = 0, \end{aligned} \tag{4.147}$$

which can be integrated to obtain

$$\begin{aligned} p(r,\theta,z) = p(r) = \frac{1}{2}\Bigg[&\frac{2\left[(DE - CF)^2 + F^2\right]\log(r)}{A\,(DE - CF)^2} - \frac{2AB}{r^2 - B} - D^2 r^2 \\ &- \frac{E^2}{(DE - CF)^2 r^2} - \frac{B\left[(DE - CF)^2 - F^2\right]}{A\,(DE - CF)^2 r^2} - A\left(1 + C^2\right)\log\left(B - r^2\right)\Bigg]. \end{aligned} \tag{4.148}$$

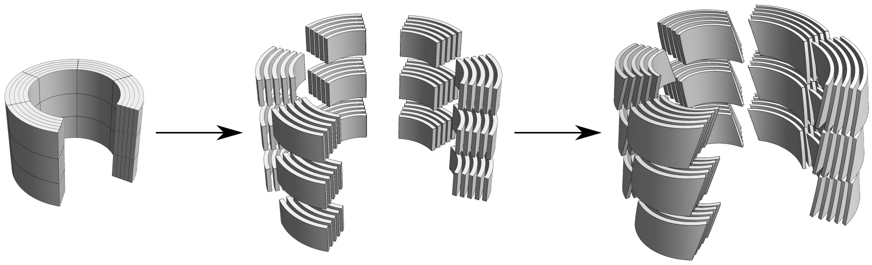


Figure 4.19: A depiction of the anomalous anelastic strain for Family 3.

In contrast, in the presence of eigenstrain, we have the differential equations

$$\begin{aligned}\frac{\partial p_{\text{eig}}}{\partial r} &= k(r),\\ \frac{\partial p_{\text{eig}}}{\partial \theta} &= \frac{2\left(A^2+e_2\right)\left(DE-CF\right)}{\sqrt{A}e_2}\gamma_1,\\ \frac{\partial p_{\text{eig}}}{\partial z} &= \frac{2\left(A^2+e_2\right)\left(DE-CF\right)}{\sqrt{A}e_2}\alpha_1,\end{aligned}$$

with $k(r)$ an algebraic expression in r as in other families. This pressure can be integrated to obtain

$$p_{\text{eig}}(r,\theta,z) = p_{\text{eig}}\left(r,\theta,z\right) = \int_{r_0}^{r} k\left(\hat{r}\right)d\hat{r} + \frac{2\left(A^2+e_2\right)\left(DE-CF\right)}{\sqrt{A}e_2}\gamma_1\theta + \frac{2\left(A^2+e_2\right)\left(DE-CF\right)}{\sqrt{A}e_2}\alpha_1 z. \quad (4.149)$$

We compute the first invariant of the Cauchy stress and color the deformation according to it in Figure 4.20. As in other families, the presence of this anomalous branch of universal eigenstrain can generate pressure gradients in directions that do not occur otherwise, specifically pressure that varies with θ and z. This property would allow one to indirectly measure the eigenstrain by measuring the pressure variation required to sustain this deformation. Likewise, if we can specify the eigenstrain, we can create specific pressure gradients that would otherwise be impossible for the generic branch.

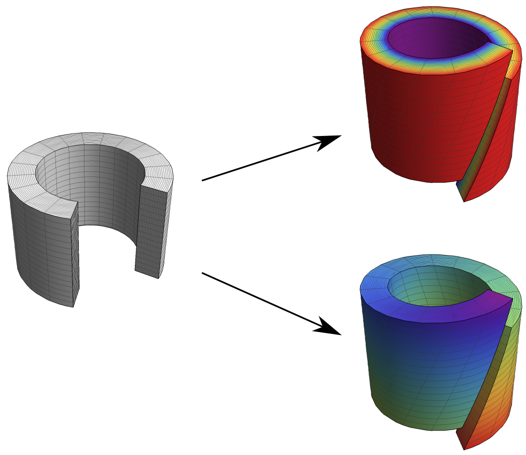


Figure 4.20: Pressure distribution for Family 3: without eigenstrain (top), and with an anomalous eigenstrain field (bottom).

Family 5. For this final family, we choose the same domain as in Family 3, i.e., $R \in [2,3]$, $\Theta \in [0,5]$, and $Z \in [0,4]$, and we take the deformation parameters to be $A = \sqrt{\frac{4}{5}}$, $B = 1$, $C = 1$, $D = -\frac{\pi}{4}$, $E = \frac{5}{4}$, $F = 0$. Choosing the anomalous eigenstrain parameters as

$$\alpha_1 = -\frac{1}{8}, \quad \alpha_2 = \frac{1}{2}, \quad \gamma_1 = -\frac{1}{10}, \quad \gamma_2 = \frac{1}{4}, \quad \begin{array}{l} e_1 = \frac{9}{4}, \\ e_2 = \frac{9}{8}, \end{array} \Leftrightarrow \begin{array}{l} \lambda_1^2 = \frac{3}{2}, \\ \lambda_2^2 = \frac{3}{4}, \end{array} \tag{4.150}$$

we subdivide and explode our domain, then approximate the eigenstrain on each piece. The result of this is depicted in Figure 4.21. As with the other families, we

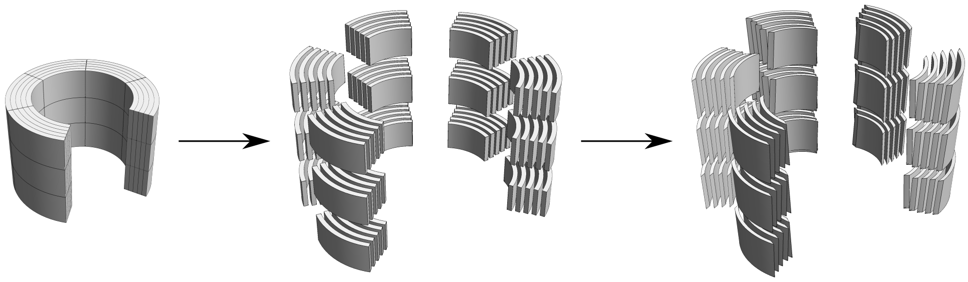


Figure 4.21: A depiction of the anomalous anelastic strain for Family 5.

are left with a set of differential equations determining the constraint stress. When eigenstrain is absent, we have the equations

$$\frac{\partial p}{\partial r} = -\frac{(B^2 + C^2 - 1)(1 + A^4 C^2)}{A^2 C^2 r}, \quad \frac{\partial p}{\partial \theta} = \frac{2B\left(A^4 + \frac{1}{C^2}\right)}{A^2}, \quad \frac{\partial p}{\partial z} = 0, \tag{4.151}$$

which can be integrated to obtain

$$p(r, \theta, z) = -\frac{(B^2 + C^2 - 1)(1 + A^4 C^2)}{A^2 C^2} \log(r) + \frac{2B\left(A^4 + \frac{1}{C^2}\right)}{A^2}\theta. \tag{4.152}$$

When we consider the anomalous solution, we have the equations

$$\frac{\partial p_{\text{eig}}}{\partial r} = k(r), \quad \frac{\partial p_{\text{eig}}}{\partial \theta} = \frac{2A\left(1+e_2\right)}{e_2}\gamma_1, \quad \frac{\partial p_{\text{eig}}}{\partial z} = \frac{2AE\left(1+e_2\right)}{e_2}\alpha_1, \tag{4.153}$$

in terms of an algebraic function $k(R)$ which can be integrated to obtain

$$p_{\text{eig}}\left(r,\theta,z\right) = \int_{r_0}^{r} k\left(\hat{r}\right)d\hat{r} + \frac{2A\left(1+e_2\right)}{e_2}\gamma_1\theta + \frac{2AE\left(1+e_2\right)}{e_2}\alpha_1 z. \tag{4.154}$$

When we compute the pressure gradient in the case of the generic universal solution, we obtain a pressure that only varies with r. We see that the anomalous branch generates pressure gradients that vary with θ and z, unlike the generic solutions. We can compute the trace of the Cauchy stress, both with eigenstrain and without eigenstrain, and use this to color the deformed configurations in Figure 4.22. Notice that unlike other families, the constraint pressure in the absence of eigenstrains can vary in a direction different than in the generic anelastic situation, specifically, for an eigenstrain in the generic solution branch, we have $\partial p/\partial\theta = 0$, but in the absence of eigenstrain, we have $\partial p/\partial\theta = 2B\left(A^4 + \frac{1}{C^2}\right)/A^2$. This is due to the fact that the standard Euclidean metric in terms of cylindrical polar coordinates lies on the anomalous solution branch, not within the generic branch. In the elastic case, when the azimuthal shearing term does not vanish, corresponding to $B \neq 0$, we have a pressure variation in θ that is necessarily nonzero. In the anomalous case, we can determine the pressure variation in both θ and z by adjusting our choices for γ_1 and α_1, allowing us to determine the pressure variation in these directions independently of the overall total deformation by choosing the anelastic strain appropriately. Specifically, we can have arbitrarily large azimuthal shearing, while also causing the azimuthal pressure variation to vanish. While the other families also allow us to select the ordinarily absent components of the pressure gradient in a similarly arbitrary way, this family is unique in having one of these pressure gradients present without eigenstrain, so the anomalous universal eigenstrain allows us to both *create* pressure variations in these directions, but also *remove* pressure variations that are ordinarily necessary to maintain the overall deformation.

Additionally, with the anomalous anelastic solution branch, the azimuthal pressure variation $\partial p/\partial\theta = 2A\left(1+e_2\right)\gamma_1/e_2$ does not depend on the degree of azimuthal shearing, i.e., it is independent of B. While the anomalous eigenstrain itself does depend explicitly on B, if the eigenstrain and the total deformation are simultaneously varied by changing B, the azimuthal pressure gradient should not change. Doing this in practice would be difficult, because fundamentally the parameter B

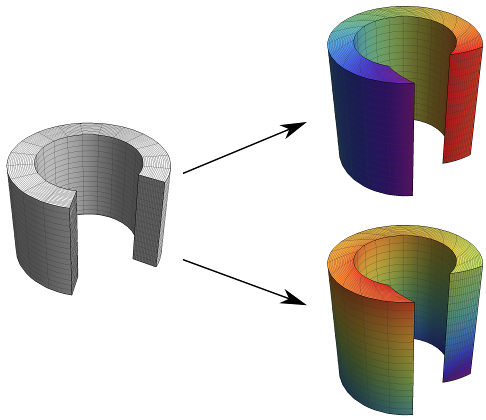


Figure 4.22: Pressure distribution for Family 5: without eigenstrain (top), and with an anomalous eigenstrain field (bottom).

partially determines the overall deformation, hence both the overall deformation and the eigenstrain would have to be simultaneously controlled in precise ways to realize this thought-experiment. Thankfully, this does not have to be done dynamically; a new value of B could be selected, the overall deformation could be controlled, and once it is established, fixed. Then the eigenstrain could be controlled until the universal eigenstrain corresponding to the chosen value of B is obtained. After this is done, the pressure variation could be measured, and this process can be repeated to establish the independence of the azimuthal pressure gradient.

Chapter 5

Merging of Universal Solution Families

After obtaining the previous results, it is natural to ask if solutions in one family correspond to solutions in another, and if so, to what extent? It is possible that the material manifolds, and the corresponding elastic deformations from two different families differ only by a change of coordinates, or equivalently, by a compatible anelastic deformation connecting the original reference configurations of the two total deformations.

5.1 Equivalent Universal Solutions

Two universal deformations, $\varphi_1 : (\mathcal{M}_R^1, \mathbf{M}^1) \to (\mathcal{M}_c^1, \mathbf{m}^1)$, and $\varphi_2 : (\mathcal{M}_R^2, \mathbf{M}^2) \to (\mathcal{M}_c^2, \mathbf{m}^2)$ are said to be *equivalent* if there exist two isometries $\Psi : (\mathcal{M}_R^1, \mathbf{M}^1) \to (\mathcal{M}_R^2, \mathbf{M}^2)$, and $\psi : (\mathcal{M}_c^1, \mathbf{m}^1) \to (\mathcal{M}_c^2, \mathbf{m}^2)$ such that

$$\psi \circ \varphi_1 = \varphi_2 \circ \Psi, \tag{5.1}$$

or, equivalently, the following diagram commutes:

$$\begin{array}{ccc} (\mathcal{M}_R^1, \mathbf{M}^1) & \xrightarrow{\varphi_1} & (\mathcal{M}_c^1, \mathbf{m}^1) \\ \downarrow \Psi & & \downarrow \psi \\ (\mathcal{M}_R^2, \mathbf{M}^2) & \xrightarrow{\varphi_2} & (\mathcal{M}_c^2, \mathbf{m}^2) \end{array} . \tag{5.2}$$

For general manifolds, this is a difficult task, as we not only have to determine whether or not two isometries exist, i.e., solve the Riemannian manifold equivalence problem twice, but also whether or not they satisfy equation (5.1). However, in our case, the current configurations of the universal deformations are both Euclidean, and hence

ψ must represent the action of an element of SE(3). Additionally, φ_1 and φ_2 are invertible and in principle known, so if we can find ψ, we can solve for $\Psi = \varphi_2^{-1} \circ \psi \circ \varphi_1$. It is then a simple matter of checking whether or not Ψ is an isometry.

We choose coordinates for all of these manifolds, writing current configuration coordinates as q^a, and material manifold coordinates as Q^A, with each set labeled by the universal deformation pertaining to it. In terms of these coordinates, these maps are

$$
\begin{aligned}
{q_1}^a &= \varphi_1\left({Q_1}^A\right), \\
{q_2}^a &= \varphi_2\left({Q_2}^A\right), \\
{q_2}^a &= \psi\left({q_1}^b\right), \\
{Q_2}^A &= \Psi\left({Q_1}^B\right),
\end{aligned}
\tag{5.3}
$$

which induce the tangent maps $(F_1)^a{}_A$, $(F_2)^a{}_A$, $\mathsf{h}^a{}_b$, $\mathsf{H}^A{}_B$, satisfying

$$
\begin{aligned}
d{q_1}^a &= (F_1)^a{}_A\, d{Q_1}^A, \\
d{q_2}^a &= (F_2)^a{}_A\, d{Q_2}^A, \\
d{q_2}^a &= \mathsf{h}^a{}_b d{q_1}^b, \\
d{Q_2}^A &= \mathsf{H}^A{}_B d{Q_1}^B.
\end{aligned}
\tag{5.4}
$$

In terms of these tangent maps, we then have the isometry conditions

$$
\left(M^1\right)_{AB} = \mathsf{H}^D{}_A \mathsf{H}^E{}_B \left(M^2\right)_{DE}, \tag{5.5}
$$

$$
\left(m^1\right)_{ab} = \mathsf{h}^c{}_a \mathsf{h}^d{}_b \left(m^2\right)_{cd}, \tag{5.6}
$$

and the prolongation of equation (5.1) as

$$
\mathsf{h}^a{}_b \left(F_1\right)^b{}_A = \left(F_2\right)^a{}_B\, \mathsf{H}^B{}_A. \tag{5.7}
$$

Because both current configurations are Euclidean, we can trivially satisfy equation (5.6) by choosing ψ to correspond to an element of SE(3), and we can then use equation (5.7) to express equation (5.5) in terms of $\mathsf{h}^a{}_b$ as

$$
\left(M^1\right)_{AB} = \left(F_1\right)^a{}_A \mathsf{h}^b{}_a \left(F_2^{-1}\right)^D{}_b \left(M^2\right)_{DE} \left(F_1\right)^c{}_B \mathsf{h}^c{}_d \left(F_2^{-1}\right)^E{}_d. \tag{5.8}
$$

We can write this expression in terms of the inverse of $b^{ab} = F^a{}_A F^b{}_B M^{AB}$ for each deformation and obtain

$$
\left(c_1\right)_{ab} = \mathsf{h}^c{}_a \left(c_2\right)_{cd} \mathsf{h}^d{}_b, \tag{5.9}
$$

and hence

$$
\left(b_2\right)^{ab} = \mathsf{h}^a{}_c \mathsf{h}^b{}_d \left(b_1\right)^{cd}. \tag{5.10}
$$

5.1.1 Redundancy of the Six Classical Universal Families

We would like to identify which families are likely to contain overlap, and take note of Table 4.1. Specifically, the left Cauchy Green tensor of each family is symmetric with respect to the prolonged action of a subgroup of SE(3). Therefore, if two universal deformations are equivalent, their corresponding strain tensors should have isomorphic symmetry groups. Denoting the symmetry group of $\mathbf{b}_1$ as $K_1 \subset \mathrm{SE}(3)$, and the symmetry group of $\mathbf{b}_2$ as $K_2 \subset \mathrm{SE}(3)$, we seek $\psi \in \mathrm{SE}(3)$ such that

$$\psi K_1 = K_2 \psi. \tag{5.11}$$

This immediately identifies a possible correspondence between Families 1, 3, and 5, because their symmetry groups are isomorphic. Additionally, we expect that there might be some universal solutions in Family 2 that are also in Family 0, since the symmetry group of Family 2 is a subgroup of that of Family 0, though we can immediately recognize that there are solutions in Family 2 that are not equivalent to any in Family 0, because not all solutions in Family 2 are invariant under the action of the full symmetry group of Family 0.

This observation immediately reveals that, up to an element of these symmetry groups, ψ must be the obvious one implied by our choice of coordinates in each family, because it must send invariant sets of K_1 to invariant sets of K_2. We recall that if a (sub)group K acts on a manifold $\mathcal{M}$, *an invariant set* of K is a set $\mathcal{S}_K \subset \mathcal{M}$ such that $\forall X \in \mathcal{S}_K$, and $\forall k \in K$, $\rho_k(X) \in \mathcal{S}_K$. Here we consider the smallest nonempty invariant sets: the orbits of single points under the action of the subgroup K_i. The invariant sets of the symmetry groups of Families 1, 3, and 5 are concentric cylinders, hence any potential ψ connecting these two families must map a family of concentric cylinders to another. The coordinates for each family were chosen such that this family of cylinders is centered on the z axis, hence we require ψ to be a Euclidean isometry mapping the z axis to itself. Apart from rotations and translation that leave the left Cauchy Green tensor fields unchanged, this restricts ψ to either be the identity, or a rotation reversing the orientation of the z axis. We will see that we can freely take ψ to be the identity.

We first show that Family 0 is contained within Family 2. To do this, we must find an equivalent deformation in Family 2 for any choice of deformation in Family 0. Identifying our coordinate systems (i.e., taking ψ to be the identity), we can express the left Cauchy Green tensor field for any deformation in Family 0 as

$$\left[b^{ab}(x,y,z)\right] = \begin{bmatrix} b^{11} & b^{12} & b^{13} \\ b^{12} & b^{22} & b^{23} \\ b^{13} & b^{23} & b^{33} \end{bmatrix}. \tag{5.12}$$

We choose a universal solution in Family 2 with material inverse metric of the form

$$\left[M^{AB}(R)\right] = \begin{bmatrix} \frac{\tilde{M}^{11}}{R^2} & \frac{\tilde{M}^{12}}{R} & \frac{\tilde{M}^{13}}{R} \\ \frac{\tilde{M}^{12}}{R} & \tilde{M}^{22} & \tilde{M}^{23} \\ \frac{\tilde{M}^{13}}{R} & \tilde{M}^{23} & \tilde{M}^{33} \end{bmatrix}, \tag{5.13}$$

with $\tilde{M}^{AB}$ being appropriate constants, which is one of the cases where the material manifold is Euclidean. Pushing this forward to the current configuration, we obtain the equations

$$\begin{bmatrix} b^{11} & b^{12} & b^{13} \\ b^{12} & b^{22} & b^{23} \\ b^{13} & b^{23} & b^{33} \end{bmatrix} = \begin{bmatrix} A^2B^4\tilde{M}^{11} & B\tilde{M}^{12} & CB\tilde{M}^{12} + AB\tilde{M}^{13} \\ B\tilde{M}^{12} & \frac{\tilde{M}^{22}}{A^2B^2} & \frac{C\tilde{M}^{22}}{A^2B^2} + \frac{\tilde{M}^{23}}{AB^2} \\ CB\tilde{M}^{12} + AB\tilde{M}^{13} & \frac{C\tilde{M}^{22}}{A^2B^2} + \frac{\tilde{M}^{23}}{AB^2} & \frac{C^2\tilde{M}^{22}}{A^2B^2} + \frac{2C\tilde{M}^{23}}{AB^2} + \frac{\tilde{M}^{33}}{B^2} \end{bmatrix}. \tag{5.14}$$

Therefore, for any given element of Family 0, the choices

$$\begin{aligned} \tilde{M}^{11} &= \frac{b^{11}}{A^2B^4}, \\ \tilde{M}^{12} &= \frac{b^{12}}{B}, \\ \tilde{M}^{13} &= \frac{b^{13} - Cb^{12}}{AB}, \\ \tilde{M}^{22} &= A^2B^2b^{22}, \\ \tilde{M}^{23} &= AB^2\left(b^{23} - Cb^{22}\right), \\ \tilde{M}^{33} &= B^2\left(b^{33} - 2Cb^{23} + C^2b^{22}\right), \end{aligned}$$

yield an equivalent member of Family 2. Also we note that these compatible material manifolds are contained as special cases of the non-homogeneous branch of Family 2 by the same argument presented earlier. Denoting $\mathcal{U}_A$ to be the set of universal deformations corresponding to Family A, we conclude that

$$\mathcal{U}_0 \subset \mathcal{U}_2. \tag{5.15}$$

We then seek to establish similar correspondences between the sets $\mathcal{U}_1$, $\mathcal{U}_3$, and $\mathcal{U}_5$. First, we consider an element of $\mathcal{U}_5$ lying in its generic branch. The left Cauchy Green tensor field of this element is fully determined by specifying three functions of

R, hence implicitly of r through $R = \frac{A}{r}$, namely $M^{11}(r)$, $M^{23}(r)$, and $M^{33}(r)$, along with values for the constants A, C, and E. Labeling these choices $\tilde{M}^{11}(r)$, $\tilde{M}^{23}(r)$, $\tilde{M}^{33}(r)$, $\tilde{A}$, $\tilde{C}$, and $\tilde{E}$, we seek elements in Families 1 and 3 that generate the same stretch tensor field.

The left Cauchy Green tensor field for the generic branch of Family 1 depends on three arbitrary functions of $X(r) = \frac{r^2}{2A} - \frac{D}{2}$, $M^{11}(X(r))$, $M^{22}(X(r))$, and $M^{23}(X(r))$ as well as the constants A, B, C. If we select the functions and constants such that

$$
\begin{aligned}
M^{11}(X(r)) &= \frac{\tilde{A}^2 r^2 \tilde{M}^{11}(r)}{A^2}, \\
M^{22}(X(r)) &= \frac{\left(\tilde{A}^2 \tilde{E}^2 r^2 \tilde{M}^{11}(r)\right)^{-1} + \tilde{C}^2 \tilde{M}^{23}(r)^2}{B^2 \tilde{M}^{33}(r)}, \\
M^{23}(X(r)) &= A\left[C \frac{\left[\tilde{A}^2 \tilde{E}^2 r^2 \tilde{M}^{11}(r)\right]^{-1} + \tilde{C}^2 \tilde{M}^{23}(r)^2}{\tilde{M}^{33}(r)} + \tilde{C}\tilde{E}\tilde{M}^{23}(r)\right],
\end{aligned}
$$

it is straightforward to verify that the stretch tensor fields generated coincide. Therefore, the generic solution branch of Family 5 is contained in the generic solution branch of Family 1, since we can find universal solutions in Family 1 that are equivalent to any universal solution in Family 5.

Similarly, the generic branch of Family 3 depends on three functions of r through $R(r) = \frac{r^2 - B}{A}$: $M^{11}(R(r))$, $M^{22}(R(r))$, and $M^{23}(R(r))$ as well as the constants A, B, C, D, E, and F. The choice

$$
\begin{aligned}
&M^{11}(R(r)) = \frac{\tilde{A}^2 r^2 \tilde{M}^{11}(r)}{A\left(r^2 - B\right)}, \\
&M^{22}(R(r)) = \\
&-\frac{EF + \tilde{A}^2 \tilde{E}^2 r^2 \tilde{M}^{11}(r)\left[C\tilde{E}\tilde{M}^{33}(r) - \tilde{C}E\tilde{M}^{23}(r)\right]\left[D\tilde{E}\tilde{M}^{33}(r) - \tilde{C}F\tilde{M}^{23}(r)\right]}{\tilde{A}^2 \tilde{E}^2 \left(CF - DE\right)^2 r^2 \tilde{M}^{11}(r)\tilde{M}^{33}(r)}, \\
&M^{23}(R(r)) = \frac{F^2 + \tilde{A}^2 \tilde{E}^2 r^2 \tilde{M}^{11}(r)\left[\tilde{C}F\tilde{M}^{23}(r) - D\tilde{E}\tilde{M}^{33}(r)\right]^2}{\tilde{A}^2 \tilde{E}^2 \left(CF - DE\right)^2 r^2 \tilde{M}^{11}(r)\tilde{M}^{33}(r)},
\end{aligned}
$$

also generates an identical stretch field, hence the generic branch of Family 5 is also contained in the generic branch of Family 3.

We have shown that the generic branch of Family 5 is contained in those of both Family 1 and Family 3. To examine the opposite direction, suppose we take an arbitrary member of the generic branch of Family 1, defined by parameters $\tilde{M}^{11}(X(r))$,

$\tilde{M}^{22}(X(r))$, $\tilde{M}^{23}(X(r))$, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, and seek to find a solution in Family 5 that generates the same stretch tensor field. Elements in Family 5 depend on the parameters $M^{11}(R(r))$, $M^{23}(R(r))$, $M^{33}(R(r))$, A, C, and E, and the choice

$$\begin{aligned}
M^{11}(R(r)) &= \frac{\tilde{A}^2\tilde{M}^{11}(X(r))}{A^2r^2},\\
M^{23}(R(r)) &= \frac{\tilde{M}^{23}(X(r))}{\tilde{A}CE} - \frac{\tilde{B}^2\tilde{C}^2\tilde{M}^{22}(X(r))}{CE},\\
M^{33}(R(r)) &= \frac{1+\tilde{M}^{11}(X(r))\left[\tilde{M}^{23}(X(r)) - \tilde{A}\tilde{B}^2\tilde{C}\tilde{M}^{22}(X(r))\right]^2}{E^2\tilde{A}^2\tilde{B}^2\tilde{M}^{11}(X(r))\tilde{M}^{22}(X(r))},
\end{aligned}$$

generates the same stretch tensor fields as the member of Family 1. Hence, the generic solution branch of Family 1 is contained in the generic branch of Family 5. Coupled with the previous result, we conclude that the generic solution branches for Families 1 and 5 are equivalent, in that every universal solution in one of these branches has at least one equivalent universal solution in the other.

Next, choosing an arbitrary universal solution in the generic branch of Family 3, we seek a universal solution in Family 5 that is equivalent. Choosing parameters $\tilde{M}^{11}(r)$, $\tilde{M}^{22}(r)$, $\tilde{M}^{23}(r)$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, and $\tilde{F}$ determining an arbitrary solution in Family 3, we can choose an element of Family 5 by specifying the parameters A, C, E, $M^{11}(R(r))$, $M^{23}(R(r))$, and $M^{33}(R(r))$, where $R(r) = \frac{r}{A}$. If we choose these such that

$$\begin{aligned}
M^{11}(R(r)) &= \frac{\tilde{A}(r^2-\tilde{B})\tilde{M}^{11}(r)}{A^2r^2},\\
M^{23}(R(r)) &= \frac{1}{CE\tilde{M}^{22}(r)}\Bigg[\frac{\tilde{D}\tilde{F}}{\tilde{A}\left(\tilde{C}\tilde{F}-\tilde{D}\tilde{E}\right)^2\left(r^2-\tilde{B}\right)\tilde{M}^{11}(r)}\\
&\quad + \left(\tilde{C}\tilde{M}^{22}(r)+\tilde{D}\tilde{M}^{23}(r)\right)\left(\tilde{E}\tilde{M}^{22}(r)+\tilde{F}\tilde{M}^{23}(r)\right)\Bigg],\\
M^{33}(R(r)) &= \frac{1}{E^2\tilde{M}^{22}(r)}\times\\
&\left[\frac{\tilde{F}^2}{\tilde{A}\left(\tilde{C}\tilde{F}-\tilde{D}\tilde{E}\right)^2(r^2-\tilde{B})\tilde{M}^{11}(r)} + \left(\tilde{E}\tilde{M}^{22}(r)+\tilde{F}\tilde{M}^{23}(r)\right)^2\right],
\end{aligned}$$

we obtain a universal solution that is equivalent to the specified solution in Family 3. Hence, the generic solution branch of Family 3 is contained within that of Family 5. Coupled with our previous results, this result means that the generic solution branches of Families 1, 3, and 5 are all equivalent to each other.

Next we consider the anomalous solution branches for these families. First, we select an arbitrary member of Family 3 anomalous solution branch by specifying the parameters $\tilde{A}$, $\tilde{B}$, $\tilde{\eta} = \tilde{D}\tilde{E} - \tilde{C}\tilde{F}$[1], $\tilde{e_1}$, $\tilde{e_2}$, $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\gamma_1}$, and $\tilde{\gamma_2}$. We seek to find solutions in Family 1 anomalous solution branch, and Family 5 solution branches that generate equivalent solutions.

First examining Family 1, we can select values for constants α_1, α_2, γ_1, γ_2, e_1, e_2, A, B, and C. It is straightforward to verify that the choice

$$\alpha_1 = \tilde{\gamma_2}, \quad \alpha_2 = \tilde{\gamma_1}, \quad \gamma_1 = \tilde{\alpha_1}, \quad \gamma_2 = \tilde{\alpha_2} + C\tilde{\gamma_1},$$
$$B = \frac{1}{\tilde{\eta}\sqrt{\tilde{A}}}, \quad A = \tilde{A}\tilde{\eta}^2, \quad e_1 = \tilde{e_1}, \quad e_2 = \tilde{e_2},$$

generates an equivalent solution. Likewise for Family 5, we can choose values for the parameters α_1, α_2, γ_1, γ_2, e_1, e_2, A, and E, where the specific choices

$$A = \tilde{\eta}\sqrt{\tilde{A}}, \quad E = 1, \quad \alpha_1 = \tilde{\alpha_1}, \quad \alpha_2 = \tilde{\alpha_2}, \quad \gamma_1 = \tilde{\gamma_1}, \quad \gamma_2 = \tilde{\gamma_2}, \quad e_1 = \tilde{e_1}, \quad e_2 = \tilde{e_2}, \tag{5.16}$$

generate a solution that equivalent to the arbitrary solution from Family 3. Hence the anomalous branch from Family 3 is contained in both that of Family 1 and Family 5.

Conversely, we select an arbitrary member of the anomalous branch of Family 5 by specifying the parameters $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\gamma_1}$, $\tilde{\gamma_2}$, $\tilde{e_1}$, $\tilde{e_2}$, $\tilde{A}$, and $\tilde{E}$. We can verify that the choice of parameters

$$A = \tilde{A}, \quad \eta = \sqrt{\tilde{A}}, \quad e_1 = \tilde{e_1}, \quad e_2 = \tilde{e_2},$$
$$\alpha_1 = \tilde{E}\tilde{\alpha_1}, \quad \alpha_2 = \tilde{E}\tilde{\alpha_2}, \quad \gamma_1 = \tilde{\gamma_1}, \quad \gamma_2 = \tilde{\gamma_2},$$

yields a solution from Family 3 that is equivalent to the arbitrary one from Family 5.

Finally, we select an arbitrary member of the anomalous branch of Family 1 by specifying the parameters $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\gamma_1}$, $\tilde{\gamma_2}$, $\tilde{e_1}$, $\tilde{e_2}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, and seek an equivalent solution in Family 3. The parameter choices

$$\begin{aligned} &A = \tilde{A}^2, \quad B = \tilde{B}, \quad \eta = 1, \quad e_1 = \tilde{e_1}, \quad e_2 = \tilde{e_2}, \\ &\alpha_1 = \frac{\tilde{\gamma_1}}{\tilde{A}\tilde{B}}, \quad \alpha_2 = \frac{\tilde{\gamma_2}}{\tilde{A}\tilde{B}} - \tilde{B}\tilde{C}\tilde{\alpha_2}, \quad \gamma_1 = \tilde{B}\tilde{\alpha_2}, \quad \gamma_2 = \tilde{B}\tilde{\alpha_1}, \end{aligned} \tag{5.17}$$

generate such a solution. Hence, we deduce that the anomalous branches of Families 1 and 5 are contained in that of Family 3, which combined with our previous results implies that the anomalous branches of all the three families are the same.

[1]While the anomalous branch for Family 3 depends on the parameters C, D, E, and F, they only appear in the combination $DE - CF$, hence it is sufficient to only specify this value.

Therefore, having examined both the generic and anomalous branches of these families, we conclude that $\mathcal{U}_1 = \mathcal{U}_3 = \mathcal{U}_5$. Hence, in the anelastic setting, our initial six families of universal solutions have collapsed into three families $\mathcal{U}_2$, $\mathcal{U}_3$, and $\mathcal{U}_4$, one corresponding to each of the three surfaces with constant principal curvatures in 3D Euclidean space: planes, cylinders, and spheres, respectively. These surfaces are the invariant sets of the symmetry groups of the left Cauchy Green tensor fields, and they played a central role in Ericksen [1954], being the level sets of the strain invariants. Here, we see that not only are the invariants of $\mathbf{b}$ constant on these surfaces, but $\mathbf{b}$ itself is symmetric with respect to these surfaces in the manner induced by the action of the special Euclidean group. This symmetry is present even in the degenerate case when the invariants of $\mathbf{b}$ are constant, which is why we can identify the symmetry groups even in the anomalous solution branches. In the classical problem, similar surfaces can be identified in the material manifold, since in the absence of eigenstrains, the material manifold and the reference configuration coincide. These surfaces are invariant sets of the symmetry groups of the right Cauchy Green tensor fields, and prevent the identification of the classical families with each other, since the only two classical families with matching invariant sets in both configurations are Families 3 and 5. These however cannot be identified with each other because solutions in Family 5 have constant invariants, while those in Family 3 do not. Hence, it is only after the addition of eigenstrains that many of the classical families become redundant.

5.2 Standard Forms of the Three Distinct Universal Families

We note that there is some redundancy in the parameterizations we currently have, which is exhibited by observing that the parameter selections we have used to identify the families with each other are not mutual inverses. We can reparametrize to eliminate this redundancy and have a single representation for strain field of each family. Concretely, we can express the left Cauchy Green stretch field for the anomalous branches of $\mathcal{U}_3$ in the following standard form:

$$\begin{aligned}
b^{11} &= m \pm \sqrt{m^2 - \left[p + \left(b^{13}\right)^2 + \left(b^{12}r\right)^2\right]},\\
b^{22} &= \frac{\left(b^{13}\right)^2 - p\left(b^{12}r\right)^2\left(b^{11} - 2m\right)}{pr^2\left[\left(b^{13}\right)^2 + \left(b^{12}r\right)^2\right]},\\
b^{33} &= \frac{\left(b^{12}r\right)^2 - p\left(b^{13}\right)^2\left(b^{11} - 2m\right)}{p\left[\left(b^{13}\right)^2 + \left(b^{12}r\right)^2\right]},\\
b^{12} &= \frac{\gamma_2}{r^3} + \frac{\gamma_1}{r}, \quad b^{23} = -\frac{b^{12}b^{13}\left(1 + b^{11}p - 2pm\right)}{p\left[\left(b^{13}\right)^2 + \left(b^{12}r\right)^2\right]}, \quad b^{13} = r\alpha_1 + \frac{\alpha_2}{r},
\end{aligned} \tag{5.18}$$

where p is the product of the two free eigenvalues of $\mathbf{b}$, and m is the arithmetic mean of the two free eigenvalues. The inverse of this, c_{ab}, is the push forward of the material metric, and has components

$$\begin{aligned}
&c_{11} = \frac{2m - b^{11}}{p}, \quad c_{22} = \frac{\left(prb^{13}\right)^2 + b^{11}\left(r^2b^{12}\right)^2}{p\left[\left(rb^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \quad c_{33} = \frac{b^{11}\left(b^{13}\right)^2 + \left(prb^{12}\right)^2}{p\left[\left(rb^{12}\right)^2 + \left(b^{13}\right)^2\right]},\\
&c_{12} = -\frac{r^2b^{12}}{p}, \quad c_{13} = -\frac{b^{13}}{p}, \quad c_{23} = \frac{b^{12}b^{13}\left(b^{11} - p^2\right)r^2}{p\left[\left(rb^{12}\right)^2 + \left(b^{13}\right)^2\right]}.
\end{aligned} \tag{5.19}$$

The generic branch of this family likewise has a standard expression:

$$\left[b^{ab}(r)\right] = \begin{bmatrix} b^{11}(r) & 0 & 0 \\ 0 & b^{22}(r) & b^{23}(r) \\ 0 & b^{23}(r) & b^{33}(r) \end{bmatrix}, \tag{5.20}$$

with the incompressibility condition $\det \mathbf{b} = 1$ taking the form

$$r^2 b^{11}(r)\left[b^{22}(r)b^{33}(r) - \left(b^{23}(r)\right)^2\right] = 1. \tag{5.21}$$

The inverse of this form then takes the form

$$\left[c_{ab}(r)\right] = \begin{bmatrix} c_{11}(r) & 0 & 0 \\ 0 & c_{22}(r) & c_{23}(r) \\ 0 & c_{23}(r) & c_{33}(r) \end{bmatrix}, \tag{5.22}$$

with the incompressibility condition being $c_{11}(r)\left(c_{22}(r)c_{33}(r) - \left(c_{23}(r)\right)^2\right) = r^2$. The positive definiteness condition is equivalent to $c_{11}(r) > 0$, $c_{22}(r) > 0$, and $c_{33}(r) > 0$ in addition to the incompressibility condition (5.22), or in the anomalous solution, requiring $m > 0$, and $0 < p < m^2$. An example of one of these generic solutions was investigated by Yavari and Goriely [2015], with the parameter choices $c_{11}(r) = \lambda^2$, $c_{22}(r) = \lambda^2 r^2$, $c_{23}(r) = r^2\left(\psi\left(\lambda r\right) - \tau\right)$, and $c_{33}(r) = \frac{1+\lambda^2r^2(\psi(\lambda r)-\tau)^2}{\lambda^4}$.

Similarly, the left Cauchy Green tensor field for the anomalous branch of the family $\mathcal{U}_2$ takes the standard form

$$\begin{aligned}
b^{11} &= m \pm \sqrt{m^2 - \left[p + \left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \\
b^{22} &= \frac{\left(b^{13}\right)^2 - p\left(b^{12}\right)^2\left(b^{11} - 2m\right)}{p\left[\left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \\
b^{33} &= \frac{\left(b^{12}\right)^2 - p\left(b^{13}\right)^2\left(b^{11} - 2m\right)}{p\left[\left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \\
b^{12} &= \alpha_1 x + \alpha_2, \quad b^{23} = -\frac{b^{12}b^{13}\left(1 + pb^{11} - 2pm\right)}{p\left[\left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \quad b^{13} = \gamma_1 x + \gamma_2,
\end{aligned} \tag{5.23}$$

with p and m defined as previously. Inverting this to obtain c_{ab}, one obtains

$$\begin{aligned}
c_{11} &= \frac{2m - b^{11}}{p}, \quad c_{22} = \frac{b^{11}\left(b^{12}\right)^2 + \left(pb^{13}\right)^2}{p\left[\left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \quad c_{33} = \frac{b^{11}\left(b^{13}\right)^2 + \left(pb^{12}\right)^2}{p\left[\left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}, \\
c_{12} &= -\frac{b^{12}}{p}, \quad c_{13} = -\frac{b^{13}}{p}, \quad c_{23} = \frac{b^{12}b^{13}\left(b^{11} - p^2\right)}{p\left[\left(b^{12}\right)^2 + \left(b^{13}\right)^2\right]}.
\end{aligned} \tag{5.24}$$

The left Cauchy Green tensor for the generic branch of this family also has a standard form

$$\left[b^{ab}(x)\right] = \begin{bmatrix} b^{11}(x) & 0 & 0 \\ 0 & b^{22}(x) & b^{23}(x) \\ 0 & b^{23}(x) & b^{33}(x) \end{bmatrix}, \tag{5.25}$$

with the incompressibility condition becoming $b^{11}(x)\left[b^{22}(x)b^{33}(x) - \left(b^{23}(x)\right)^2\right] = 1$. The inverse of this then takes the standard form

$$\left[c_{ab}(x)\right] = \begin{bmatrix} c_{11}(x) & 0 & 0 \\ 0 & c_{22}(x) & c_{23}(x) \\ 0 & c_{23}(x) & c_{33}(x) \end{bmatrix}, \tag{5.26}$$

with the incompressibility condition

$$c_{11}(x)\left[c_{22}(x)c_{33}(x) - \left(c_{23}(x)\right)^2\right] = 1. \tag{5.27}$$

The positive-definiteness condition is equivalent to requiring $c_{11}(x) > 0$, $c_{22}(x) > 0$, and $c_{33}(x) > 0$, in addition to the incompressibility condition (5.27), or in the anomalous case, requiring $m > 0$, and $0 < p < m^2$.

Finally, the spherically-symmetric family $\mathcal{U}_4$ can be expressed in the standard form through its left Cauchy Green tensor

$$\left[b^{ab}\right] = \begin{bmatrix} g(r)^2 & 0 & 0 \\ 0 & \frac{1}{g(r)r^2\sin^2\phi} & 0 \\ 0 & 0 & \frac{1}{g(r)r^2} \end{bmatrix}, \tag{5.28}$$

which has the inverse

$$[c_{ab}] = \begin{bmatrix} g(r)^{-2} & 0 & 0 \\ 0 & g(r)r^2\sin^2\phi & 0 \\ 0 & 0 & g(r)r^2 \end{bmatrix}. \tag{5.29}$$

The incompressibility condition and positive definiteness is automatically satisfied for arbitrary functions $g(r)$ satisfying $g(r) > 0$. In terms of parameters defined by Goriely [2017] in Chapter 15.1.1, this function is $g(r) = \alpha_r = \alpha^{-2}$: it is the radial stretch.

These standard forms make it clear that universal solutions in anelasticity can be categorized by computing the tensor $\mathbf{c}^\flat$, and comparing the result with the standard forms here. As a consequence of this, the symmetry of the elastic strain in the current configuration determines which family any particular universal solution belongs to, as it is this symmetry that is reflected in $\mathbf{c}^\flat$.

Chapter 6

The Role of Symmetry

We note that the left Cauchy Green stretch tensor fields present in all known universal solutions, both classical and anelastic, are equivariant under the defining action of a Lie subgroup of the special Euclidean group. (As a brief aside, this group is the isometry group of the underlying ambient space, not the fundamental symmetry group of the governing equations discussed in Hill [1982], despite the fact that the fifth classical family obeys both symmetries.) These subgroups of the isometry group all have at least two independent generators, with the three anelasic families separated by the nature of these generators; taking two purely translational generators yields $\mathcal{U}_2$, taking one translational and one rotational yields $\mathcal{U}_3$, and taking two rotational yields $\mathcal{U}_4$. Of course, these families are not simply symmetric with respect to an arbitrary choice of generators of these natures; the translational generator in $\mathcal{U}_3$ is orthogonal to the plane of rotation determined by $\mathcal{U}_3$'s rotational generator, and both of the rotational generators for $\mathcal{U}_4$ fix a common point. It is then natural to ask if there are other universal solutions that are likewise equivariant with respect to a similar subgroup, but without these specific generator choices.

6.1 The Lie Algebra $\mathfrak{se}(n)$

To examine the subgroup structure in terms of generators, we turn our attention to $\mathfrak{se}(3)$, the Lie algebra associated to the Lie group SE(3). We can represent the group SE(n) as a subgroup of GL($n+1$) in the following way: The element $(\mathbf{Q}|c)$ $\in$SE(n) is identified with the $(n+1)\times(n+1)$ matrix

$$\begin{bmatrix} \mathbf{Q} & c \\ \mathbf{0} & 1 \end{bmatrix}, \tag{6.1}$$

where $\mathbf{0}$ is a $1 \times n$ block of 0s. It is clear that the standard matrix multiplication in $\mathrm{GL}(n+1)$ agrees with the group action determined by the defining action of $\mathrm{SE}(n)$ on $\mathbb{E}^n$. As a reminder of example 4, the defining action of the special Euclidean group induces the group action

$$(\mathbf{Q}_2|c_2) \star (\mathbf{Q}_1|c_1) = (\mathbf{Q}_2\mathbf{Q}_1|\mathbf{Q}_2c_1 + c_2)\,. \tag{6.2}$$

Translating this into the representation (6.1), we have

$$\begin{bmatrix} \mathbf{Q}_2 & c_2 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1 & c_1 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_2\mathbf{Q}_1 & \mathbf{Q}_2c_1 + c_2 \\ \mathbf{0} & 1 \end{bmatrix}, \tag{6.3}$$

which clearly captures the induced group structure in terms of standard matrix multiplication.

Taking the derivative of this representation around the identity yields a representation of the Lie algebra $\mathfrak{se}(n)$:

$$\begin{bmatrix} \mathbf{\Omega} & u \\ \mathbf{0} & 0 \end{bmatrix}, \tag{6.4}$$

where $\mathbf{\Omega}$ is a skew symmetric matrix, u is an $n \times 1$ column vector, and $\mathbf{0}$ is a $1 \times n$ block of 0s.

We seek to examine the subalgebra structure of $\mathfrak{se}(n)$, and in particular, $\mathfrak{se}(3)$, since subalgebras with two generators will directly correspond to Lie subgroups with two generators by way of the exponential map.

The defining feature of $\mathrm{SE}(n)$ being its action on $\mathbb{E}^n$, we expect there to be an analogous representation for $\mathbb{E}^n$ such that this action is reflected by the standard action of $\mathrm{GL}(n+1)$ on $\mathbb{R}^{n+1}$. Indeed the analogous representation takes the point with position vector $X \in \mathbb{R}^n$ to the vector $\begin{bmatrix} X \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}$. Under this representation, the special Euclidean group acts via matrix multiplication as follows:

$$\begin{bmatrix} \mathbf{Q} & c \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q}X + c \\ 1 \end{bmatrix}, \tag{6.5}$$

which clearly agrees with the action as defined previously.

6.1.1 Subalgebras of $\mathfrak{se}(3)$

Under the above representation, an arbitrary element of $\mathfrak{se}(3)$ takes the form

$$\begin{bmatrix} 0 & -\zeta & \epsilon & \alpha \\ \zeta & 0 & -\delta & \beta \\ -\epsilon & \delta & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.6}$$

and the Lie bracket becomes the matrix commutator.

Clearly elements of this form span a 6 dimensional subspace of $\mathbb{R}^{4\times 4}$. We are interested in Lie subalgebras generated by two generators, hence we consider two arbitrary elements of $\mathfrak{se}(3)$, examine their product, and check for linear dependence. Doing this successively will identify proper subalgebras generated by two elements.

We first want to choose our coordinates in such a way that to simplify our calculations. We seek to align our coordinate frame with the axial vector of the skew symmetric submatrix $\mathbf{\Omega}$. The axial vector of $\mathbf{\Omega}$ lies in the null space of $\mathbf{\Omega}$, which is spanned by the vector $\begin{bmatrix}\delta, \epsilon, \zeta\end{bmatrix}^\mathsf{T}$, unless $\mathbf{\Omega} = \mathbf{0}$, in which case we do not have to do anything at this step. These two options are exhaustive, since the eigenvalues of $\mathbf{\Omega}$ are $\{0, \pm\sqrt{-\delta^2 - \epsilon^2 - \zeta^2}\}$.

Provided $\mathbf{\Omega} \neq \mathbf{0}$, we can choose a Cartesian coordinate system such that $\mathbf{e}_3$ is the normalized axial vector:

$$\mathbf{e}_3 = \frac{\begin{bmatrix}\delta & \epsilon & \zeta\end{bmatrix}^\mathsf{T}}{\sqrt{\delta^2 + \epsilon^2 + \zeta^2}}. \tag{6.7}$$

We do this by considering any rotation mapping the normalized axial vector to $\mathbf{e}_3$. Denoting such a rotation $\mathbf{R}$, we change coordinates by computing

$$\begin{bmatrix}\mathbf{R} & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}\begin{bmatrix}0 & -\zeta & \epsilon & \alpha\\ \zeta & 0 & -\delta & \beta\\ -\epsilon & \delta & 0 & \gamma\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{R}^\mathsf{T} & \mathbf{0}\\ \mathbf{0} & 1\end{bmatrix}. \tag{6.8}$$

When we apply this coordinate transformation, our chosen element of the Lie algebra takes the form

$$\begin{bmatrix}0 & -\omega & 0 & \alpha\\ \omega & 0 & 0 & \beta\\ 0 & 0 & 0 & \gamma\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{6.9}$$

where $\omega = \sqrt{\delta^2 + \epsilon^2 + \zeta^2}$, and the α, β, and γ here have been relabeled, being independent linear combinations depending on $\mathbf{R}$ of the old α, β, and γ, which were arbitrary to begin with.

Next, we seek to apply a coordinate translation to simplify the translation portion of our chosen element. To do this, we seek to identify the fixed points of this action. The velocity of points under the action of the one parameter subalgebra generated by this element is given by

$$\begin{bmatrix}0 & -\omega & 0 & \alpha\\ \omega & 0 & 0 & \beta\\ 0 & 0 & 0 & \gamma\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix} = \begin{bmatrix}\alpha - \omega y\\ \omega x + \beta\\ \gamma\\ 0\end{bmatrix}. \tag{6.10}$$

Hence, if $\omega \neq 0$, we can choose a coordinate translation that sets the point $\begin{bmatrix} -\beta/\omega & \alpha/\omega & 0 \end{bmatrix}^{\mathsf{T}}$ to be the origin.

Under this transformation, our chosen element takes the form

$$\begin{bmatrix} 1 & 0 & 0 & \beta/\omega \\ 0 & 1 & 0 & -\alpha/\omega \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\omega & 0 & \alpha \\ \omega & 0 & 0 & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\beta/\omega \\ 0 & 1 & 0 & \alpha/\omega \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.11}$$

Here, $u = \gamma$, but we shall explicitly use u and ω to emphasize that we have expressed this element of $\mathfrak{se}(3)$ in a coaxial coordinate system.

In the case where $\mathbf{\Omega} = \mathbf{0}$, we simply choose our coordinate rotation so that our translation vector is aligned with $\mathbf{e}_3$, which sets our chosen Lie algebra element to the form above with $\omega = 0$.

6.1.1.1 2 Dimensional Subalgebras

Obviously, provided that the generators we select are linearly independent, they span a two-dimensional vector space, hence all subalgebras containing them are at least 2 dimensional. In order for us to identify 2 dimensional subalgebras, we simply need to establish necessary and sufficient conditions for the two generators and their bracket to be linearly dependent.

We select a coordinate system that is coaxial with one of our generators, and hence have

$$v_1 = \begin{bmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.12}$$

and select another arbitrary generator,

$$v_2 = \begin{bmatrix} 0 & -\zeta & \epsilon & \alpha \\ \zeta & 0 & -\delta & \beta \\ -\epsilon & \delta & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.13}$$

Taking the Lie bracket of these two elements, we obtain

$$[v_1, v_2] = \begin{bmatrix} 0 & 0 & \delta\omega & -u\epsilon - \beta\omega \\ 0 & 0 & \epsilon\omega & u\delta + \alpha\omega \\ -\delta\omega & -\epsilon\omega & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.14}$$

We then require the Lie bracket of our generators to be within their span, i.e. we seek all solutions to the equations

$$a_1 v_1 + a_2 v_2 + [v_1, v_2] = \mathbf{0}, \tag{6.15}$$

which explicitly become

$$\begin{aligned} a_2\alpha - u\epsilon - \beta\omega &= 0, \\ a_2\beta + u\delta + \alpha\omega &= 0, \\ a_1 u + a_2\gamma &= 0, \\ a_2\delta - \epsilon\omega &= 0, \\ a_2\epsilon + \delta\omega &= 0, \\ a_2\zeta + a_1\omega &= 0. \end{aligned}$$

Taking the combination $\delta\,(a_2\epsilon + \delta\omega = 0) - \epsilon\,(a_2\delta - \epsilon\omega = 0)$ yields the equation

$$\left(\epsilon^2 + \delta^2\right)\omega = 0, \tag{6.16}$$

which implies that either $\omega = 0$, or both $\delta = 0$ and $\epsilon = 0$.

If $\omega = 0$, we consider the combinations

$$\epsilon\,(a_2\alpha - u\epsilon = 0) - \alpha\,(a_2\epsilon = 0) = -u\epsilon^2 = 0, \tag{6.17}$$

$$\delta\,(a_2\beta + u\delta = 0) - \beta\,(a_2\delta = 0) = u\delta^2 = 0. \tag{6.18}$$

Since $v_1 \neq \mathbf{0}$, $u \neq 0$, hence we require $\delta = \epsilon = 0$.

With these substitutions, $[v_1, v_2]$ vanishes, and we have

$$v_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.19}$$

and

$$v_2 = \begin{bmatrix} 0 & -\zeta & 0 & \alpha \\ \zeta & 0 & 0 & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.20}$$

If $\zeta \neq 0$ can always reselect our origin to eliminate α and β, while leaving v_1 unchanged. Hence we obtain the symmetry of family $\mathcal{U}_3$. If $\zeta = 0$, we have two independent translational symmetries, which yields the symmetry found in family $\mathcal{U}_2$.

Now we turn our attention to the case where $\omega \neq 0$, and consider the combination

$$\alpha \left(a_2\beta + \alpha\omega = 0\right) - \beta \left(a_2\alpha - \beta\omega = 0\right) = \left(\alpha^2 + \beta^2\right)\omega = 0. \tag{6.21}$$

Since $\omega \neq 0$, we require $\alpha = \beta = 0$.

This leaves us with v_1 as initially specified and

$$v_2 = \begin{bmatrix} 0 & -\zeta & 0 & 0 \\ \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.22}$$

which means that v_1 and v_2 generate independent screw motions *about the same axis*, corresponding to the symmetry of family $\mathcal{U}_3$.

6.1.1.2 3 Dimensional Subalgebras

Defining $v_3 = [v_1, v_2]$, and provided $v_3 \neq 0$, v_1, v_2, and v_3 span a three dimensional vector space. The span of these three vectors must be closed under the Lie bracket, hence we require

$$a_1 v_1 + a_2 v_2 + a_3 v_3 + [v_1, v_3] = 0, \tag{6.23}$$

and

$$b_1 v_1 + b_2 v_2 + b_3 v_3 + [v_2, v_3] = 0. \tag{6.24}$$

We know that if $v_3 = 0$ then v_1 and v_2 generate a 2 dimensional subalgebra, hence we can freely assume $v_3 \neq 0$.

First, we recognize that if $\omega = 0$, both v_1 and v_3 are pure translations. They are linearly independent provided $\delta \neq 0$ and $\epsilon \neq 0$, in which case $v_3 = 0$. Hence, the bracket of a pure translation with a non-coaxial rotation yields another translation that is linearly independent of the original translation. Hence, $\mathfrak{t}(2)$ is contained in such a Lie subalgebra, and hence all universal solutions that are symmetric with respect to the subgroups corresponding to these subalgebras are contained in $\mathcal{U}_2$. If the rotation is coaxial with the translation, then the bracket vanishes and we are reduced to the already solved 2 dimensional case; hence from now on, we can safely assume $\omega \neq 0$.

The equations we must tackle are explicitly

$$\begin{aligned} a_1 u + a_2 \gamma &= 0 \\ a_2 \zeta + a_1 \omega &= 0 \\ b_1 u + b_2 \gamma + u\left(\delta^2 + \epsilon^2\right) + 2\omega\left(\alpha\delta + \beta\epsilon\right) &= 0 \end{aligned}$$

$$b_2\zeta + b_1\omega + \omega\left(\delta^2 + \epsilon^2\right) = 0$$
$$b_2\delta - b_3\epsilon\omega - \delta\zeta\omega = 0$$
$$b_2\epsilon + b_3\delta\omega - \epsilon\zeta\omega = 0$$
$$a_2\delta - a_3\epsilon\omega - \delta\omega^2 = 0$$
$$a_2\epsilon + a_3\delta\omega - \epsilon\omega^2 = 0$$
$$b_2\beta + b_3u\delta - u\epsilon\zeta + b_3\alpha\omega - \gamma\epsilon\omega - \beta\zeta\omega = 0$$
$$a_2\alpha - a_3u\epsilon - a_3\beta\omega - 2u\delta\omega - \alpha\omega^2 = 0$$
$$b_2\alpha - b_3u\epsilon - u\delta\zeta - b_3\beta\omega - \gamma\delta\omega - \alpha\zeta\omega = 0$$
$$a_2\beta + a_3u\delta + a_3\alpha\omega - 2u\epsilon\omega - \beta\omega^2 = 0$$

Taking the linear combination

$$\delta\left(a_2\epsilon + a_3\delta\omega - \epsilon\omega^2 = 0\right) - \epsilon\left(a_2\delta - a_3\epsilon\omega - \delta\omega^2 = 0\right) = a_3\left(\delta^2 + \epsilon^2\right)\omega = 0, \quad (6.25)$$

coupled with the condition $\omega \neq 0$ yields either $a_3 = 0$ or both $\delta = 0$ and $\epsilon = 0$. If $\delta = \epsilon = 0$, v_3 is a pure translation that is orthogonal to the axis of v_1, hence, taking $[v_1, v_3]$ generates another pure translation orthogonal to the axis of v_1 and that of v_3, hence we capture the symmetry $\mathfrak{t}(2)$ as a subgroup of our symmetry group, and hence this case is captured in family $\mathcal{U}_2$.

If either $\epsilon \neq 0$ or $\delta \neq 0$, we have $a_3 = 0$, which upon substitution yields $\delta\left(a_2 - \omega^2\right) = 0$ and $\epsilon\left(a_2 - \omega^2\right) = 0$. These equations together imply $a_2 = \omega^2$.

Substituting this new relation into our equations, two of our equations reduce to

$$-2u\delta\omega = 0$$
$$-2u\epsilon\omega = 0,$$

which together imply that $u = 0$, since δ and ϵ cannot simultaneously vanish and $\omega \neq 0$; hence v_1 must be a pure rotation, not simply a screw motion.

With this, our first equation becomes $\gamma\omega^2 = 0$, hence $\gamma = 0$ as well. When we insert this relation into our equations, we obtain

$$2\left(\alpha\delta + \beta\epsilon\right)\omega = 0, \quad (6.26)$$

which implies that the inner product of v_2's axial vector with its translation vector is 0. This implies that v_2 is also a pure rotation, since this inner product is unchanged under coordinate transformations. This can be seen by noting that the velocity field $\mathbf{u}$ induced by the action of an element of $\mathfrak{se}(3)$ is given by

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -\zeta & \epsilon & \alpha \\ \zeta & 0 & -\delta & \beta \\ -\epsilon & \delta & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \epsilon z - \zeta y + \alpha \\ \zeta x - \delta z + \beta \\ \delta y - \epsilon x + \gamma \\ 0 \end{bmatrix}. \tag{6.27}$$

Taking the inner product of this with the embedding of the axial vector $\begin{bmatrix} \delta & \epsilon & \zeta & 1 \end{bmatrix}^{\mathsf{T}}$ yields

$$\alpha\delta + \beta\epsilon + \gamma\zeta, \tag{6.28}$$

which, not depending on position, is an invariant of the velocity field. Since the velocity field is coordinate independent, we know that this invariant will be preserved under coordinate changes.

When we express our generator in a coordinate system aligned with its axis, this invariant becomes ωu, which vanishes if either our generator is a pure translation or a pure rotation.

In our analysis, we have for v_2, $\alpha\delta + \beta\epsilon = 0$ together with $\gamma = 0$, hence we know v_2 is either a pure translation or a pure rotation. We know that v_2 is not a pure translation since either δ or ϵ is nonzero. Additionally, we know that the axial vectors of v_1 and v_2 are linearly independent, since either δ or ϵ is nonzero.

Summing up our progress thus far, we have shown that both v_1 and v_2 must be pure rotations. In fact, their axes of rotation intersect, hence they generate $\mathfrak{so}(3)$, the symmetry present in $\mathcal{U}_4$, indicating that our classification captures all 3 dimensional cases. To see this, note that we have aligned our coordinates so that the axis of rotation for v_1 is the z axis. We seek to show that the axis of rotation for v_2 intersects the z axis.

First notice that for rotations about the origin, the velocity field generated is of the form

$$\mathbf{v} = \boldsymbol{\omega} \wedge X, \tag{6.29}$$

where $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$, and $\wedge$ is the standard cross product. This implies that the velocity vector at a point is orthogonal to the plane spanned by the axial vector of the rotation, and the position vector X. Since (6.29) assumes we have chosen our origin such that the axis of rotation passes through the origin, this plane is equivalently the plane containing the axis of rotation and the point X.

Therefore, for the generator v_2, we can examine the velocity generated at the origin, and recognize that it lies entirely in the x, y plane. If this velocity is nonzero, we know that the plane passing through the origin that is orthogonal to this translation contains v_2's axis of rotation. This plane also contains the z axis, since all planes passing through the origin that are orthogonal to a nonzero vector in the x, y plane contain the z axis. Therefore the axes of rotation for v_1 and v_2 are coplanar. We have already established that they are not parallel, since the axial vectors for v_1 and v_2 are linearly independent, hence they must intersect at some point.

If the velocity generated by v_2 at the origin is zero, then v_2's axis of rotation passes through the origin, and hence not only intersects the z axis, but intersects it at the origin.

We have therefore shown that all three dimensional Lie subalgebras of $\mathfrak{se}(3)$ that are generated by two linearly independent generators either contain $\mathfrak{t}(2)$ as a subalgebra, or are $\mathfrak{so}(3)$, the algebra associated with the set of rotations about a fixed point, and hence universal solutions that are equivariant with respect to the associated Lie groups are contained in one of our discovered families.

6.1.1.3 4+ Dimensional Subalgebras

Without loss of generality, we assume v_1, v_2, v_3, and $v_4 = [v_1, v_3]$ are linearly independent, since the other choice would be $v_4 = [v_2, v_3]$, which would be equivalent.

Specifically, we denote $V_2 = \text{Span}(v_1, v_2)$, and $V_3 = \text{Span}(v_1, v_2, [v_1, v_2])$. Provided that v_1, v_2, and $[v_1, v_2]$ are linearly independent, we can write $V_3 = V_2 \oplus \text{Span}([v_1, v_2])$. It suffices to take the fourth linearly independent element to be of the form

$$v_4 = [u, w], \quad u \in V_2,\ w \in \text{Span}([v_1, v_2]), \tag{6.30}$$

since for all $u, w \in V_2$, $[u, w] \in V_3$, and for all $u, w \in \text{Span}([v_1, v_2])$, $[u, w] = 0$. Since v_1 and v_2 are arbitrary, we can choose this fourth linearly independent element to be $[v_1, v_3]$.

Doing this, we have

$$v_4 = \begin{bmatrix} 0 & 0 & -\epsilon\omega^2 & -\omega(2u\delta + \alpha\omega) \\ 0 & 0 & \delta\omega^2 & -\omega(2u\epsilon + \beta\omega) \\ \epsilon\omega^2 & -\delta\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.31}$$

Notice that the axial vectors of v_1, v_3, and v_4 are $[0, 0, \omega]^\mathsf{T}$, $[-\epsilon\omega, \delta\omega, 0]^\mathsf{T}$, and $[-\delta\omega^2, -\epsilon\omega^2, 0]^\mathsf{T}$ respectively. These vectors are mutually orthogonal, hence provided $\omega \neq 0$ and that $\epsilon \neq 0$ or $\delta \neq 0$, these span $\mathbb{R}^3$, and hence the rotational components of these three generators can be used to reduce any fourth linearly independent generator to a pure translation. As shown earlier, taking the bracket of a pure translation with any other linearly independent element of $\mathfrak{se}(3)$ generates a 2 dimensional subalgebra: either $\mathfrak{t}(2)$ or $\mathfrak{so}(2) \times \mathfrak{t}(1)$. Therefore, all subalgebras of dimension 4 or higher contain one of these two dimensional subalgebras, hence universal solutions that are symmetric with respect to such a 4 dimensional subalgebra will be contained in either $\mathcal{U}_2$ or $\mathcal{U}_3$.

This lets us state our symmetry theorem:

Theorem 6.1.1 (Classification of Symmetric Universal Solutions)**.** *Any universal solution that is equivariant under the action of 2 independent* 1 *dimensional Lie subgroups of* $SE(3)$ *is contained in one of the three universal families* $\mathcal{U}_2$, $\mathcal{U}_3$, *or* $\mathcal{U}_4$.

At this point one may wonder if there exist anelastic universal solutions outside of these three families, i.e. those that break these symmetries. We recall that by appending the condition that the Riemann-Christoffel curvature tensor based on $\mathbf{c}^\flat$ vanish, we recover the classical Ericksen problem. Therefore, if there are families of anelastic universal solutions outside of those discussed here, if they contain compatible universal solutions, those with a Euclidean reference configuration, they must lie in the small edge case that has been subsequently chipped away over the decades as Ericksen's problem has been studied. Hence, any anelastic universal families that break these symmetries either do not contain any classical solutions, or the classical solutions they contain lie in the highly restricted edge case. Additionally, not all results in the classical literature utilize the Euclidean structure of the reference, hence those results equally apply to the anelastic problem, further restricting the possibility of universal families outside of these. As an example, Ericksen's initial work does make use of this flatness condition, but his use of Hamel's theorem to show that the surfaces on which the invariants of $\mathbf{c}$ are constant must be parallel planes, concentric circular cylinders, or concentric spheres does not rely on this condition, hence it applies equally well to the anelastic problem. As we have uncovered that the classical fifth universal family lies in the anomalous branch of its associated anelastic family, we have an explanation for the peculiar role it has played in the history of this problem. Therefore, we suspect that these symmetries may be necessary features of universal solutions, which if true, renders our classification here complete.

Chapter 7

Where Do We Go From Here?

We have not managed to completely rule out the possibility of other universal solutions, either in the classical Ericksen's problem, or the anelastic Ericksen's problem. However, we have shed some light as to why the fifth classical family is so qualitatively distinct from the other four (five if you include the homogeneous case) namely that it lies in the anomalous branch of the associated anelastic family. We have also managed to unite the classical families in the anelastic regime by means of their common symmetry groups, all being Lie subgroups of SE(3) with two linearly independent generators. We have also demonstrated that all such subgroups with two linearly independent generators contains one of these groups by necessity, hence our classification of *symmetric* universal solutions is complete.

A sensible trajectory for future research on this problem would be to weaken the symmetry conditions here, either by proving that these symmetries are in fact necessary, or by reducing the number of independent symmetry group generators. Examining solutions that are symmetric with respect to a one-dimensional subgroup of SE(3) would reduce the problem to a two-dimensional problem, enabling the application of methods from complex analysis to further constrain this problem, either by finding new solutions, or showing that the less symmetric case ultimately reduces to the cases examined here.

7.1 Connections to Previous Research

As mentioned in section 1.3.3, other authors have made use of the orthonormal eigenvectors of the Cauchy deformation tensor as an anholonomic frame when examining constant invariant universal solutions. The advantage of this frame is that the strain is constant and diagonal with respect to this frame, though at the cost of having to introduce non-commuting differential operators. If our symmetry conjecture is correct, a different orthonormal frame exists with the property that the Cauchy deformation tensor's components are all functions of a single spatial coordinate, with the gradient of this coordinate corresponding to one of the basis vector fields in the orthonormal frame. Furthermore, this coordinate can be identified as either a Cartesian coordinate, the cylindrical polar radius, or the spherical polar radius. Note that this does not imply that the vectors in this frame are eigenvectors of the Cauchy deformation tensor, just that in this frame, our universal equilibrium equations necessarily reduce to ODEs. Additionally, while the Cauchy deformation tensor is not constant (or diagonal) in this frame, it is in a sense covariantly constant on the coordinate surfaces, i.e. those on which the distinguished coordinate is constant. When we say "in a sense covariantly constant," we mean precisely that we consider these surfaces as embedded surfaces in $\mathbb{E}^3$, and examine the deformation tensor field restricted to these surfaces. We can then define the parallel transport of tangent vectors to these embedded surfaces using the induced Levi-Civita connection induced by the ambient space's Euclidean metric. We can then supplement this connection by defining the parallel transport of normal vectors to these surfaces by requiring transported normal vectors to remain normal to the surface. Taken together, these parallel transport laws allow us to transport arbitrary vector fields along these surfaces, and it is this transport that preserves our universal solution's Cauchy deformation tensors. This frame shares some of the advantages of the eigenframe, in that it simplifies the form of the deformation tensor, but provides for greater flexibility by not being bound to the eigenframe itself, but rather tied to how the eigenframe varies from point to point, once restricted to these coordinate surfaces.

The existence of this frame explains why the spherically symmetric family does not possess an anomalous branch. Because the distinguished surfaces of this family have nonzero Gaussian curvature, the induced parallel transport depends fundamentally on the path taken. Because of this, the Cauchy deformation tensor's eigenvectors must be either tangent to or normal to a family of concentric spheres, and the tangent planes of these spheres must be eigenspaces of the deformation tensor, as explained in section 4.3.

Furthermore, there is a clear connection between this frame, and the surfaces used in Ericksen's initial paper. Specifically, in Ericksen's seminal work, he proved that the strain invariants are functionally dependent, and that surfaces on which these invariants are constant have constant principal curvatures, and hence must either be parallel planes, concentric right cylinders, or concentric spheres. These surfaces are the orbits of points under the symmetry groups we have discovered here, and it is clear that if the Cauchy deformation tensor is “covariantly constant” in these surfaces, then the strain invariants must be constant on these surfaces. Ericksen’s method breaks down when the strain invariants are spatially constant, since these surfaces can no longer be identified, but this problem does not plague this generalized notion of “covariant constancy” on surfaces, since the only tensor fields that are “covariantly constant” on arbitrary surfaces are multiples of the identity. Hence if the Cauchy deformation tensor has to be “covariantly constant” on particular surfaces, an argument similar to Ericksen’s initial argument could be used to provide a complete classification, since the only edge case not covered would be deformations whose strain is a constant multiple of the identity, which by the incompressibility constraint, must of necessity be rigid motions. A piece doing this would be a fitting way to close out Ericksen’s problem, bringing nearly 70 years of work full circle.

More recently, the linear version of Ericksen’s problem has been tackled by Yavari et al. [2020], who classified universal displacements for anisotropic linear elastic solids, providing a complete description of universal solutions for each of the crystallographic anisotropy classes. One could consider the related nonlinear problem, either restricted to the purely elastic setting, or the anelastic setting. Doing so may shed light on the connection between the symmetry groups of the universal deformations and the material symmetry groups. In practice this must be done thoughtfully, since material symmetry groups, describing the material around a single point, are inherently local, while the universal symmetry groups are global, since they compare strains around different points.

7.2 Further Research Questions

We have seen that when the above-mentioned symmetries are present, we can fully classify the associated symmetric solutions. Additionally, every universal solution known possesses these symmetries, which raises the question: are these symmetries necessary for some reason, or do they just simplify otherwise difficult calculations enough to make them tractable? Additionally, the symmetry groups are subgroups of the ambient space’s isometry group, acting in the usual way. Is this merely a coincidence, or is the existence of universal solutions fundamentally connected to isometry subgroups of the ambient space? Furthermore, what roles do the incompressibility constraint and material symmetry play in this discussion? Answers to these questions would perhaps not only allow us to complete Ericksen’s initial classification; either by proving that all universal solutions are accounted for, or by discovering new and exotic universal solutions; but may also allow us to extend Ericksen's problem, and our group equivariant analysis to curved ambient spaces, or more usefully, different material constraints, since we would be able to systematically determine which group actions play key roles in the existence of universal solutions.

Appendix A

Explicit Calculations of the Anomalous Solutions

For families containing anomalous solution branches, we have the necessary (but not sufficient) condition that the invariants of $\mathbf{b}$ are constant, with $\det \mathbf{b} = 1$ for incompressibility. For each of these families, we have four linear differential equations, one linear algebraic equation, and two nonlinear algebraic equations for the six unknown functions comprising the components of M^{AB}. We will use the linear equations to solve for five of these unknown functions in terms of the sixth, and then characterize the common solutions to the remaining two equations to determine the final component.

A.1 Family 1

For this family, we consider the case where the invariants of $\mathbf{b}$ are constant. This gives the equation

$$m_{a[l}\nabla_{k]}\nabla_b\sigma^{ab} = 0, \tag{A.1}$$

which, when applying the constant invariant condition and forcing this to hold for all energy functions requires

$$m_{a[k}\nabla_{l]}\nabla_b b^{ab} = 0, \quad m_{a[k}\nabla_{l]}\nabla_b c^{ab} = 0. \tag{A.2}$$

The first of these has two nonzero components after substituting the form of M^{AB}. One of these components yields the differential equation $3M^{12}(r)' + rM^{12}(r)'' = 0$. This equation is readily integrated to obtain

$$M^{12}(r) = \frac{\alpha_1}{r^2} + \alpha_2. \tag{A.3}$$

Applying this to the second nonzero component of the condition on the divergence of $\mathbf{b}$ yields the differential equation $r^4 M^{13}(r)'' - r^3 M^{13}(r)' - 8AB^2C\alpha_1 = 0$, which again can be integrated to obtain

$$M^{13}(r) = \frac{AB^2C\alpha_1}{r^2} + \gamma_1 r^2 + \gamma_2. \tag{A.4}$$

After this, we must compute the condition on $\mathbf{c}$, which we can simplify first by noting that $\det \mathbf{b} = 1$, so we can utilize $\mathbf{c} = (\det \mathbf{b})\, \mathbf{c}$, which gives

$$[c_{ab}] = \begin{bmatrix} b^{22}b^{33} - \left(b^{23}\right)^2 & b^{13}b^{23} - b^{12}b^{33} & b^{12}b^{23} - b^{13}b^{22} \\ b^{13}b^{23} - b^{12}b^{33} & b^{11}b^{33} - \left(b^{13}\right)^2 & b^{12}b^{13} - b^{23}b^{11} \\ b^{12}b^{23} - b^{13}b^{22} & b^{12}b^{13} - b^{23}b^{11} & b^{11}b^{22} - \left(b^{12}\right)^2 \end{bmatrix}, \tag{A.5}$$

i.e., the cofactor tensor of $\mathbf{b}$ equals the inverse of $\mathbf{b}$. When we use this, we obtain the two differential equations, which can be expressed as

$$\begin{aligned} \left(r\alpha_1 + r^3\alpha_2\right) M^{23}(r)'' - \left(\alpha_1 - 3r^2\alpha_2\right) M^{23}(r)' - \left(AB^2Cr\alpha_1 + r^5\gamma_1 + r^3\gamma_2\right) M^{22}(r)'' \\ + \left(AB^2C\alpha_1 - 7r^4\gamma_1 - 3r^2\gamma_2\right) M^{22}(r)' - 8r^3\gamma_1 M^{22}(r) = 0, \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} \left(A^2B^4C^2r\alpha_1 + AB^2Cr^5\gamma_1 + AB^2Cr^3\gamma_2\right) M^{22}(r)'' + 8AB^2Cr^3\gamma_1 M^{22}(r) \\ + \left(-2AB^2Cr\alpha_1 - AB^2Cr^3\alpha_2 - r^5\gamma_1 - r^3\gamma_2\right) M^{23}(r)'' - 8r^3\gamma_1 M^{23}(r) \\ + \left(-A^2B^4C^2\alpha_1 + 7AB^2Cr^4\gamma_1 + 3AB^2Cr^2\gamma_2\right) M^{22}(r)' + \left(r\alpha_1 + r^3\alpha_2\right) M^{33}(r)'' \\ + \left(2AB^2C\alpha_1 - 3AB^2Cr^2\alpha_2 - 7r^4\gamma_1 - 3r^2\gamma_2\right) M^{23}(r)' + \left(-\alpha_1 + 3r^2\alpha_2\right) M^{33}(r)' = 0. \end{aligned} \tag{A.7}$$

These equations can be integrated to obtain the conditions

$$\left(\alpha_1 + r^2\alpha_2\right) M^{33}(r) - \left(AB^2C\alpha_1 + r^4\gamma_1 + r^2\gamma_2\right) M^{23}(r) = \beta_1 + r^2\beta_2, \tag{A.8}$$

$$\left(AB^2C\alpha_1 + r^4\gamma_1 + r^2\gamma_2\right) M^{22}(r) - \left(\alpha_1 + r^2\alpha_2\right) M^{23}(r) = \mu_1 + r^2\mu_2. \tag{A.9}$$

We also have the constant trace condition on $\mathbf{b}$, which becomes

$$A^4B^2M^{11}(r) + A^2B^4r^2\left(C^2 + r^2\right) M^{22}(r) - 2AB^2Cr^2M^{23}(r) + r^2M^{33}(r) = A^2B^2r^2I_1. \tag{A.10}$$

We can express this system of equations, linear in $M^{22}(r)$, $M^{23}(r)$, and $M^{33}(r)$, as the matrix equation

$$\begin{bmatrix} 0 & -r^2M^{13}(r) & r^2M^{12}(r) \\ r^2M^{13}(r) & -r^2M^{12} & 0 \\ A^2B^4r^2\left(r^2+C^2\right) & -2AB^2Cr^2 & r^2 \end{bmatrix} \begin{bmatrix} M^{22}(r) \\ M^{23}(r) \\ M^{33}(r) \end{bmatrix}$$
$$= \begin{bmatrix} \beta_1 + r^2\beta_2 \\ \mu_1 + r^2\mu_2 \\ A^2B^2r^2I_1 - A^4B^2M^{11}(r) \end{bmatrix}, \quad \text{(A.11)}$$

which is invertible, because the determinant of the matrix on the left hand side is

$$r^6\left[\left(M^{13}(r) - AB^2CM^{12}(r)\right)^2 + \left(AB^2rM^{12}(r)\right)^2\right] > 0, \quad \text{(A.12)}$$

since $M^{12}(r)$ and $M^{13}(r)$ cannot simultaneously vanish. We invert these equations to obtain expressions for these components of the inverse metric in terms of $M^{11}(r)$, r, and various constants.

$$M^{22} = \frac{A^2B^2\left(I_1r^2 - A^2M^{11}\right)\left(M^{12}\right)^2 + \left(\mu_1 + r^2\mu_2\right)M^{13}}{r^2\left[\left(M^{13} - AB^2CM^{12}\right)^2 + \left(AB^2rM^{12}\right)^2\right]} \quad \text{(A.13)}$$
$$- \frac{\left(\beta_1 + 2AB^2C\mu_1 + r^2\left(\beta_2 + 2AB^2C\mu_2\right)\right)M^{12}}{r^2\left[\left(M^{13} - AB^2CM^{12}\right)^2 + \left(AB^2rM^{12}\right)^2\right]}, \quad \text{(A.14)}$$
$$M^{23} = \quad \text{(A.15)}$$
$$- \frac{\left(\beta_1 + r^2\beta_2\right)M^{13} + A^2B^2M^{12}\left[B^2\left(C^2 + r^2\right)\left(\mu_1 + r^2\mu_2\right) + \left(A^2M^{11} - I_1r^2\right)M^{13}\right]}{r^2\left[\left(M^{13} - AB^2CM^{12}\right)^2 + \left(AB^2rM^{12}\right)^2\right]}, \quad \text{(A.16)}$$
$$M^{33} = \frac{A^2B^2}{r^2}\frac{\left(\beta_1 + r^2\beta_2\right)\left[B^2\left(C^2 + r^2\right)M^{12} - 2\frac{C}{A}M^{13}\right]}{\left(M^{13} - AB^2CM^{12}\right)^2 + \left(AB^2rM^{12}\right)^2} \quad \text{(A.17)}$$
$$- \frac{A^2B^2}{r^2}\frac{M^{13}\left[B^2\left(C^2 + r^2\right)\left(\mu_1 + r^2\mu_2\right) + \left(A^2M^{11} - I_1r^2\right)M^{13}\right)}{\left(M^{13} - AB^2CM^{12}\right)^2 + \left(AB^2rM^{12}\right)^2}. \quad \text{(A.18)}$$

We then have the other two restrictions, the constancy of the second invariant of **b** and the incompressibility constraint. After substituting the above expressions into these conditions, they become

$$\begin{aligned} p_1 = {} & A^6B^2\left[A^2B^4\left(\alpha_1 + r^2\alpha_2\right)^2 + r^2\left(r^2\gamma_1 + \gamma_2 - AB^2C\alpha_2\right)^2\right]M^{11}(r)^2 \\ & + A^4B^2r^2\Big[B^2\left(\left(\alpha_1 + r^2\alpha_2\right)\left(\beta_1 + r^2\beta_2\right) + r^2\left(r^2\gamma_1 + \gamma_2 - AB^2C\alpha_2\right)\left(\mu_1 + r^2\mu_2\right)\right) \\ & \qquad + AB^4C\left(\alpha_1 + r^2\alpha_2\right)\left(\mu_1 + r^2\mu_2\right) \\ & \qquad - A^2B^4I_1\left(\alpha_1 + r^2\alpha_2\right)^2 - I_1r^2\left(r^2\gamma_1 + \gamma_2 - AB^2C\alpha_2\right)^2\Big]M^{11}(r) \\ & \qquad + A^6B^8\left(\alpha_1 + r^2\alpha_2\right)^4 + B^2r^4\left(\beta_1 + r^2\beta_2\right)^2 \\ & \qquad + A^4B^4r^2\left(\alpha_1 + r^2\alpha_2\right)^2\left[B^2I_2r^2 + 2\left(r^2\gamma_1 + \gamma_2 - AB^2C\alpha_2\right)^2\right] \end{aligned}$$

$$
\begin{aligned}
&- A^3B^6CI_1r^4\left(\alpha_1+r^2\alpha_2\right)\left(\mu_1+r^2\mu_2\right)+2AB^4Cr^4\left(\beta_1+r^2\beta_2\right)\left(\mu_1+r^2\mu_2\right)\\
&+A^2r^4\Big[B^2I_2r^2\left(r^2\gamma_1+\gamma_2-AB^2C\alpha_2\right)^2+\left(r^2\gamma_1+\gamma_2-AB^2C\alpha_2\right)^4\\
&+B^6\left(C^2+r^2\right)\left(\mu_1+r^2\mu_2\right)^2\\
&-B^4I_1\left(\left(\alpha_1+r^2\alpha_2\right)\left(\beta_1+r^2\beta_2\right)+r^2\left(r^2\gamma_1+\gamma_2-AB^2C\alpha_2\right)\left(\mu_1+r^2\mu_2\right)\right)\Big]=0,
\end{aligned}
\tag{A.19}
$$

and

$$
\begin{aligned}
&p_2=\\
&A^4B^2\left(\left(\alpha_1+r^2\alpha_2\right)\left(\beta_1+r^2\beta_2\right)+\left(r^4\gamma_1+r^2\gamma_2+AB^2C\alpha_1\right)\left(\mu_1+r^2\mu_2\right)\right)M^{11}(r)^2\\
&+r^2\Big[\left(\beta_1+r^2\beta_2\right)^2+A^2B^4\left(C^2+r^2\right)\left(\mu_1+r^2\mu_2\right)^2\\
&-A^2B^2I_1\left(AB^2C\alpha_1+r^4\gamma_1+r^2\gamma_2\right)\left(\mu_1+r^2\mu_2\right)\\
&+AB^2\left(\beta_1+r^2\beta_2\right)\left(2C\left(\mu_1+r^2\mu_2\right)-AI_1\left(\alpha_1+r^2\alpha_2\right)\right)\Big]M^{11}(r)\\
&+\left[A^2B^4\left(\alpha_1+r^2\alpha_2\right)^2+r^2\left(r^2\gamma_1+\gamma_2-AB^2C\alpha_2\right)^2\right]\\
&\times\left[r^4+\left(\beta_1+r^2\beta_2\right)\left(\alpha_1+r^2\alpha_2\right)+\left(\mu_1+r^2\mu_2\right)\left(r^4\gamma_1+r^2\gamma_2+AB^2C\alpha_1\right)\right]=0.
\end{aligned}
\tag{A.20}
$$

We then compute[1] the resultant of these two equations in $M^{11}(r)$, yielding a polynomial in r that must identically be equal to 0. In order for this to be satisfied, each of its coefficients must vanish independently. This can be shown by repeatedly taking derivatives of the equation in r, which will ultimately require each of the coefficients to vanish independently. Computing this resultant we obtain

$$
\mathsf{Res}_{M^{11}(r)}\left(p_1,p_2\right)=A^6B^2r^8\left[A^2B^4\left(\alpha_1+r^2\alpha_2\right)^2+r^2\left(r^2\gamma_1+\gamma_2-AB^2C\alpha_2\right)^2\right](...)\,.
\tag{A.21}
$$

The factors explicitly shown are identically nonzero, since A and B are nonzero for the deformation to be invertible, $r>0$, and the other factor only vanishes if both $M^{12}(r)$ and $M^{13}(r)$ vanish, in which case we are no longer on the anomalous solution branch. Therefore, we take $(...)=0$. This factor is massive, being approximately 8000 terms, so it it far too large to print here, but enough information has been provided to compute it explicitly if the reader desires. We next take its coefficients to vanish independently, and factor each coefficient. The shortest of these factors is

$$
A^6B^2\left(B^6\mu_2^3-B^4I_1\gamma_1\mu_2^2+B^2I_2\gamma_1^2\mu_2-\gamma_1^3\right)^2=0,
\tag{A.22}
$$

[1] Symbolic computations were done with Mathematica Version 12.0.0.0, Wolfram Research, Champaign, IL.

which can be satisfied in one of two ways. Either both μ_2 and γ_1 are 0, or $\nu = \frac{B^2\mu_2}{\gamma_1}$ is an eigenvalue of $\mathbf{b}$. In the first case, after simplification of the other coefficients, we obtain another equation

$$B^2\left(\beta_2^3 - I_1\beta_2^2A^2B^2\alpha_2 + I_2\beta_2A^4B^4\alpha_2^2 - A^6B^6\alpha_2^3\right)^2 = 0, \tag{A.23}$$

which implies either $\alpha_2 = \beta_2 = 0$, or $\nu = \frac{\beta_2}{A^2B^2\alpha_2}$ is an eigenvalue of $\mathbf{b}$.

Taking $\alpha_2 = \beta_2 = 0$, we obtain another similar eigenvalue equation that implies $\gamma_2 = \mu_1 = 0$ or $\nu = \frac{B^2\mu_1}{\gamma_2}$ is an eigenvalue of $\mathbf{b}$.

Taking $\gamma_2 = \mu_1 = 0$, we obtain another eigenvalue equation, but this equation demands $\nu = \frac{\beta_1}{A^2B^2\alpha_1}$, because we already have $\alpha_2 = \gamma_1 = \gamma_2 = 0$; $\alpha_1 = 0$ would result in both $M^{12}(r) = 0$ and $M^{13}(r) = 0$, a contradiction. This condition is sufficient for solving all of the necessary conditions.

Backing up a branch, we can take $\nu = \frac{B^2\mu_1}{\gamma_2}$ as an eigenvalue of $\mathbf{b}$. We then perform the substitutions $I_1 = e_1 + \frac{1}{e_2}$ and $I_2 = \frac{e_1}{e_2} + e_2$ with $\nu = \frac{1}{e_2}$, which expresses the invariants of $\mathbf{b}$ in terms of the elementary symmetric polynomials in the other two eigenvalues. This reveals an equation with $\beta_1 - \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2}$ as a factor. If this factor is 0, we satisfy all of the necessary equations. If this factor is not 0, we have either $e_2^3 - e_2e_1 + 1 = 0$, or $\alpha_1\left(e_2\beta_1 + AC\gamma_2\right) = 0$. In the later case, plugging in $\alpha_1 = 0$ we obtain $\beta_1 = \frac{-AC\gamma_2}{e_2}$, which corresponds to the vanishing of the other factor. Likewise, if we take $e_2\beta_1 + AC\gamma_2 = 0$, we obtain $\alpha_1 = 0$ as a condition. Both of these together however imply that $\beta_1 - \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2} = 0$, which is a contradiction.

If $e_2^3 - e_1e_2 + 1 = 0$, this implies that $\lambda_1^2 = \frac{1}{\lambda_2^4}$ or $\lambda_2^2 = \frac{1}{\lambda_1^4}$. In either case we can express the remaining equation in terms of only one remaining eigenvalue. This equation has one factor that we know is nonzero because it corresponds to $\beta_1 = \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2}$, which would yield a contradiction. So we take the remaining factor to vanish. This factor is quadratic in α_1. Taking the discriminant of this equation in α_1, we obtain

$$\Delta_{\alpha_1} = -4A^6B^6\gamma_2^4\lambda_a^8, \tag{A.24}$$

where λ_a^2 is the repeated eigenvalue. This discriminant must be non-negative in order for the factor to vanish with real values of α_1. However, this discriminant is identically non-positive, which means it must be 0. However, the only way for this to happen would be for $\gamma_2 = 0$, which is a contradiction.

Having exhausted the options corresponding to $\alpha_2 = \beta_2 = 0$, we consider $\nu = \frac{\beta_2}{A^2B^2\alpha_2}$ as an eigenvalue of $\mathbf{b}$, and perform the substitutions on the invariants to express the invariants in terms of elementary symmetric polynomials in λ_1^2 and λ_2^2. Doing this gives five remaining polynomial equations, which are still rather long and complicated. Ordering these equations by their length, and taking the second shortest one, we note that this equation is quadratic in e_1. Taking the discriminant of this equation in e_1, and demanding it be non-negative, we obtain

$$\begin{aligned}\Delta_{e_1} = &-4A^6B^6e_2^2\alpha_1^2\left(\beta_1 + AB^2C\mu_1\right)^2\left(A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1\right)^2 \\ &\times\left(AB^2C\alpha_2\beta_1 - \beta_1\gamma_2 + A^2B^4\alpha_1\mu_1 + A^2B^4C^2\alpha_2\mu_1 - AB^2C\gamma_2\mu_1\right)^4 \geq 0. \quad \text{(A.25)}\end{aligned}$$

This quantity is identically non-positive, and so the only way we can have solutions with real values for e_1 is if this quantity is 0. There are four factors that can possibly be 0: α_1, $\beta_1 + AB^2C\mu_1$, $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1$, and $AB^2C\alpha_2\beta_1 - \beta_1\gamma_2 + A^2B^4\alpha_1\mu_1 + A^2B^4C^2\alpha_2\mu_1 - AB^2C\gamma_2\mu_1$.

First consider $\alpha_1 = 0$. Inserting this, many of the remaining equations have a factor $\beta_1 + AB^2C\mu_1$. If this factor vanishes, the remaining equations both contain the factor $AB^2C\alpha_2 - \gamma_2 + B^2e_2\mu_1$. The vanishing of this factor satisfies all of the equations. If this factor does not vanish, we can take the resultant of the remaining factors in e_1, and obtain

$$A^2B^6\alpha_2^2\left(AB^2C\alpha_2 - \gamma_2\right)^2\mu_1^2\left(AB^2C\alpha_2 - \gamma_2 + B^2e_2\mu_1\right)^2 = 0. \quad \text{(A.26)}$$

Only two factors here can vanish, namely μ_1 and $AB^2C\alpha_2-\gamma_2$. If we take $\mu_1 = 0$ and insert it into the remaining two equations, one simplifies to imply $AB^2C\alpha_2 - \gamma_2 = 0$. Likewise, if we instead take $AB^2C\alpha_2 - \gamma_2$ to vanish, we obtain $\mu_1 = 0$, hence both must vanish. However, if both of these vanish, the original term $AB^2C\alpha_2-\gamma_2+B^2e_2\mu_1$ vanishes, a contradiction.

We can then consider the case when $\beta_1 \neq -AB^2C\mu_1$. Taking the remaining factor of the shortest equation, we have

$$A^2\left(AB^2C\alpha_2 - \gamma_2\right)^4 + B^2\left(\beta_1 + AB^2C\mu_1\right)^2 = 0. \quad \text{(A.27)}$$

This requires $AB^2C\alpha_2 - \gamma_2 = 0$ and $\beta_1 + AB^2C\mu_1 = 0$, but the second of these is a contradiction. This exhausts the case where $\alpha_1 = 0$, so we take $\alpha_1 \neq 0$, and consider the next factor $\beta_1 + AB^2C\mu_1 = 0$. Inserting this into the equations, we obtain

$$A^{10}B^{14}\alpha_1^4\left(A^2\alpha_1^2 + B^2\mu_1^4\right) = 0. \quad \text{(A.28)}$$

This cannot vanish for $\alpha_1 \neq 0$, so we come to a contradiction.

Next, we consider $\alpha_1 \neq 0$ and $\beta_1 + AB^2C\mu_1 \neq 0$, and take $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 = 0$. Solving this for β_1, and inserting this into the equations, we obtain the condition

$$AB^2C\alpha_2 - \gamma_2 + B^2e_2\mu_1 = 0, \quad \text{(A.29)}$$

which can be solved for μ_1 and is sufficient to satisfy the remaining equations. Finally, we consider the remaining option with $\alpha_1 \neq 0$, $\beta_1 + AB^2C\mu_1 \neq 0$, and $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 \neq 0$ with $AB^2C\alpha_2\beta_1 - \beta_1\gamma_2 + A^2B^4\alpha_1\mu_1 + A^2B^4C^2\alpha_2\mu_1 - AB^2C\gamma_2\mu_1 = 0$. This equation can be solved for either β_1 or μ_1. If $\gamma_2 \neq AB^2C\alpha_2$, we can solve this for β_1 and obtain

$$\beta_1 = \frac{AB^2C\gamma_2 - A^2B^4\left(\alpha_1 + C^2\alpha_2\right)}{AB^2C\alpha_2 - \gamma_2}\mu_1. \tag{A.30}$$

If we insert this, we obtain

$$\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}, \tag{A.31}$$

as a necessary and sufficient condition for the remaining equations to be satisfied. This yields

$$\beta_1 = \frac{-AB^2C\gamma_2 + A^2B^4\left(\alpha_1 + C^2\alpha_2\right)}{B^2e_2}. \tag{A.32}$$

However, with these, the equation $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 = 0$ is satisfied, a contradiction. Hence, we consider $\gamma_2 = AB^2C\alpha_2$, which requires $\mu_1 = 0$. With this, we have the conditions $\beta_1 \neq 0$, $\alpha_1 \neq 0$, and $A^2B^2\alpha_1 - e_2\beta_1 \neq 0$. With these, the remaining equations demand

$$e_2^3 - e_1e_2 + 1 = 0, \tag{A.33}$$

which in turn demands either $\lambda_1^2 = \frac{1}{\lambda_2^4}$ or $\lambda_2^2 = \frac{1}{\lambda_1^4}$. Denoting the repeated eigenvalue as λ_a^2, we obtain the necessary condition $\lambda_a^2 = 1$. However, with this we have $\beta_1 = A^2B^2\alpha_1$, which contradicts the above non-equality conditions, i.e., this case is already accounted for in the previous cases.

This exhausts the options with $\mu_2 = \gamma_1 = 0$, so we consider $\nu = \frac{B^2\mu_2}{\gamma_1}$ as an eigenvalue of $\mathbf{b}$. Inserting this into our equations yields

$$\beta_2 = \frac{A^2B^2\alpha_2 - AC\gamma_1}{e_2}. \tag{A.34}$$

This requires $e_2^3 - e_1e_2 + 1 = 0$, or if not, $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}$.

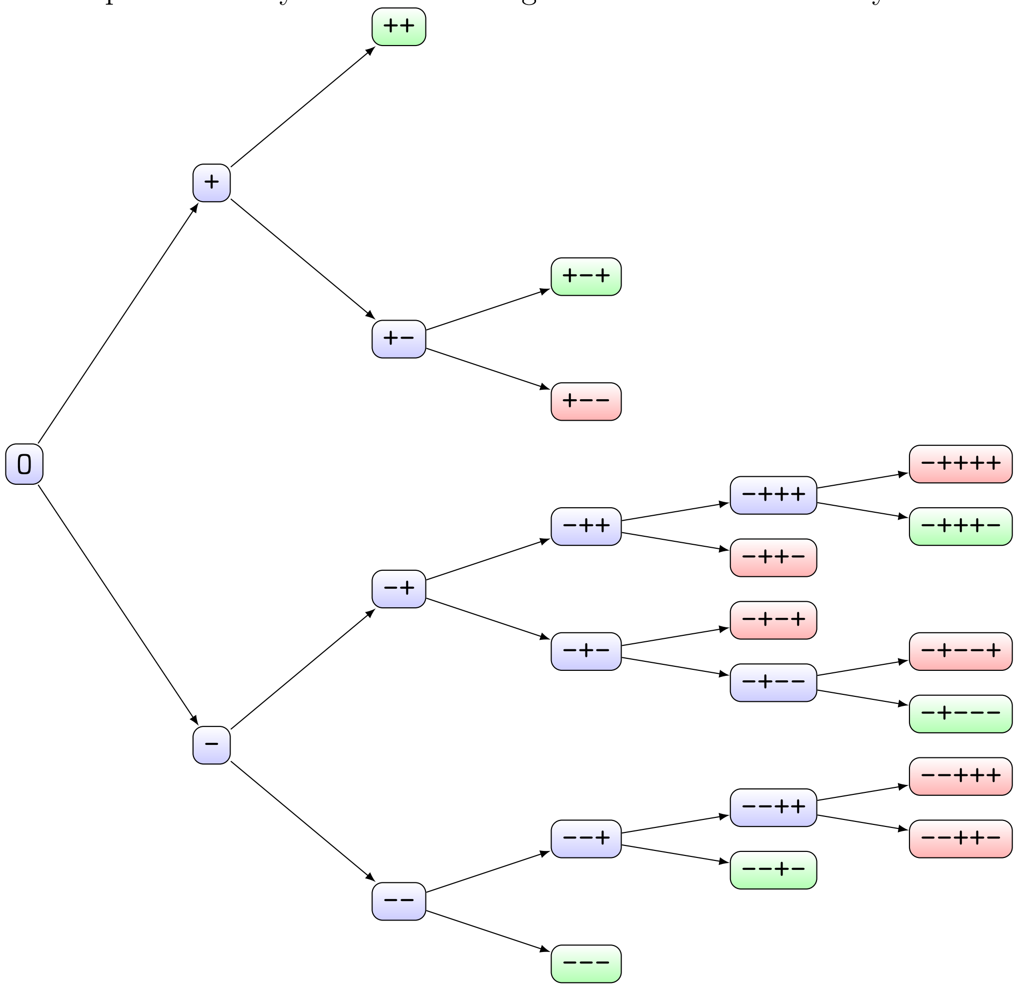
In the later case, we have $\alpha_2 = 0$ or $\beta_1 = \frac{A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2}{e_2}$. This second option satisfies the remaining equation, so we then consider $\alpha_2 = 0$. With this, we obtain $\beta_1 = \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2}$, which is a special case of the previous option.

We then consider $e_2^3 - e_1e_2 + 1 = 0$, which demands a repeated eigenvalue. Upon further substitution, the remaining equations demand that this repeated eigenvalue $\lambda_a^2 = 1$, which means that all eigenvalues are the same. This allows us to solve for β_1 and μ_1 as

$$\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2}, \quad \beta_1 = A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2. \tag{A.35}$$

This satisfies the remaining conditions, and so our analysis is complete, having exhausted all possible branches of solutions.

In order to depict this branching set of conditions, we can express the steps of the analysis in a tree. Each node on these trees represents a system of equations, and each edge represents one partial solution to these equations. Terminal nodes are color coded corresponding to whether they are consistent (green) or not (red). Nodes and edges are then labelled below the tree with the relevant equation/solutions utilized at each step of the analysis. The following is the tree for this family:



Nodes

- 0: $B^6\mu_2^3 - I_1B^4\mu_2^2\gamma_1 + I_2B^2\mu_2\gamma_1^2 - \gamma_1^3 = 0$
- −: $\beta_2^3 - I_1A^2B^2\beta_2^2\alpha_2 + I_2A^4B^4\beta_2\alpha_2^2 - A^6B^6\alpha_2^3 = 0$
- +: $\beta_2 = \frac{A^2B^2\alpha_2 - AC\gamma_1}{e_2}$, and $(B^2\mu_1e_2 - \gamma_2 + AB^2C\alpha_2)(e_2^3 - e_2e_1 + 1) = 0$
- −−: $B^6\mu_1^3 - I_1B^4\mu_1^2\gamma_2 + I_2B^2\mu_1\gamma_2^2 - \gamma_2^3 = 0$

- $-+$: $\alpha_1 \left(\beta_1 + AB^2C\mu_1\right) \left(A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1\right)$
  $\left(AB^2C\alpha_2\beta_1 - \beta_1\gamma_2 + A^2B^4\alpha_1\mu_1 + A^2B^4C^2\gamma_2\mu_1\right) = 0$
- $+-$: $\alpha_2 \left(A^2B^2 \left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2 - \beta_1 e_2\right) = 0$
- $++$: $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2}$, and $\beta_1 = A^2B^2 \left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2$
- $---$: $\beta_1^3 - I_1A^2B^2\beta_1^2\alpha_1 + I_2A^4B^4\beta_1\alpha_1^2 - A^6B^6\alpha_1^3 = 0$
- $--+$: $\frac{B^2\mu_1}{\gamma_2}$ is an eigenvalue of $\mathbf{b}$
- $-+-$: $\left(\beta_1 + AB^2C\mu_1\right)(...) = 0$
- $-++$: $\left(\beta_1 + AB^2C\mu_1\right) \left(A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1\right)$
  $\left(AB^2C\alpha_2\beta_1 - \beta_1\gamma_2 + A^2B^4\alpha_1\mu_1 + A^2B^4C^2\gamma_2\mu_1\right) = 0$
- $+--$: $\beta_1 = \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2}$, and $\beta_1 \neq \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2}$
- $+-+$: $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}$, and $\beta_1 = \frac{A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2}{e_2}$
- $--+-$: $\beta_1 = \frac{A^2B^2\alpha_1 - AC\gamma_2}{e_2}$
- $--++$: $\alpha_1 \left(e_2^3 - e_2e_1 + 1\right) \left(e_2\beta_1 + AC\gamma_2\right) = 0$
- $-+--$: $\left(\gamma_2 - AB^2C\alpha_2 - B^2e_2\mu_1\right)(...) = 0$
- $-+-+$: $\gamma_2 = AB^2C\alpha_2 \Rightarrow \beta_1 = -AB^2C\mu_1$, and $\beta_1 \neq -AB^2C\mu_1$
- $-++-$: $\alpha_1 = 0$, and $\alpha_1 \neq 0$
- $-+++$: $\left(A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1\right)$
  $\left(AB^2C\alpha_2\beta_1 - \beta_1\gamma_2 + A^2B^4\alpha_1\mu_1 + A^2B^4C^2\gamma_2\mu_1\right) = 0$
- $--++-$: $\gamma_2 = 0$, and $\gamma_2 \neq 0$
- $--+++$: $\beta_1e_2 - A^2B^2\alpha_1 - AC\gamma_2 = 0$, and $\beta_1e_2 - A^2B^2\alpha_1 - AC\gamma_2 \neq 0$
- $-+---$: $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}$
- $-+--+$: $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}$, and $\mu_1 \neq \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}$
- $-+++-$: $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}$

- $-++++$: $\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2 e_2}$, and $\beta_1 = \frac{A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2}{e_2} \Rightarrow$ $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 = 0$, and $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 \neq 0$

Edges (labelled by child node):

- $-$: $\gamma_1 = 0$, and $\mu_2 = 0$
- $+$: $\frac{B^2\mu_2}{\gamma_1}$ is an eigenvalue of $\mathbf{b}$
- $--$: $\beta_2 = 0$, and $\alpha_2 = 0$
- $-+$: $\frac{\beta_2}{A^2B^2\alpha_2}$ is an eigenvalue of $\mathbf{b}$
- $+-$: $e_2^3 - e_1e_2 + 1 \neq 0$
- $++$: $e_2^3 - e_1e_2 + 1 = 0 \Rightarrow \lambda_a^2 = 1$
- $---$: $\gamma_2 = 0$, and $\mu_1 = 0$
- $--+$: $\frac{B^2\mu_1}{\gamma_2}$ is an eigenvalue of $\mathbf{b}$
- $-+-$: $\alpha_1 = 0$
- $-++$: $\alpha_1 \neq 0$
- $+--$: $\beta_1 \neq \frac{A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2}{e_2}$, and $\alpha_2 = 0$
- $+-+$: $\beta_1 = \frac{A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2}{e_2}$
- $--+-$: $\beta_1e_2 - A^2B^2\alpha_1 - AC\gamma_2 = 0$
- $--++$: $\beta_1e_2 - A^2B^2\alpha_1 - AC\gamma_2 \neq 0$
- $-+--$: $\beta_1 = -AB^2C\mu_1$
- $-+-+$: $\beta_1 \neq -AB^2C\mu_1$
- $-++-$: $\beta_1 + AB^2C\mu_1 = 0$
- $-+++$: $\beta_1 + AB^2C\mu_1 \neq 0$
- $--++-$: $e_2^3 - e_2e_1 + 1 = 0$
- $--+++$: $\alpha_1 = 0 \Leftrightarrow e_2\beta_1 + AC\gamma_2 = 0$

- $-+---$: $\gamma_2 = AB^2C\alpha_2 + B^2e_2\mu_1$
- $-+--+$: $\gamma_2 \neq AB^2C\alpha_2 + B^2e_2\mu_1 \Rightarrow \mu_1 = 0$, and $\gamma_1 = AB^2C\alpha_2$
- $-+++-$: $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 = 0$
- $-++++$: $A^2B^2\alpha_1 - e_2\beta_1 - AB^2Ce_2\mu_1 \neq 0$

The reader should note that this tree is not unique; there are potentially numerous ways we could have performed these algebraic eliminations, but ultimately, we have derived these conditions as necessary, and we have shown they are sufficient as well, so any other sequence of algebraic reductions would yield equivalent results. The branches of this tree are all special cases of the conditions

$$\mu_1 = \frac{\gamma_2 - AB^2C\alpha_2}{B^2e_2}, \tag{A.36}$$

$$\beta_2 = \frac{A^2B^2\alpha_2 - AC\gamma_1}{e_2}, \tag{A.37}$$

$$\mu_2 = \frac{\gamma_1}{B^2e_2}, \tag{A.38}$$

$$\beta_1 = \frac{A^2B^2\left(\alpha_1 + C^2\alpha_2\right) - AC\gamma_2}{e_2}. \tag{A.39}$$

Note that these values are always well defined, since $B \neq 0$, and $e_2 = \lambda_1^2\lambda_2^2 > 0$, even though in their derivation we considered branching cases that are mutually exclusive. After these substitutions, both of the constant invariant conditions become

$$A^4B^2M^{11}(r)^2 - A^2B^2e_1r^2M^{11}(r) + A^2B^4\left(\alpha_1 + r^2\alpha_2\right)^2 \\ + r^2\left(r^2\gamma_1 + \gamma_2 - AB^2C\alpha_2\right)^2 + B^2e_2r^4 = 0, \tag{A.40}$$

which let us solve for $M^{11}(r)$ as

$$M^{11}(r) = \frac{e_1r^2}{2A^2} \\ \pm \frac{r\sqrt{B^2e_1^2r^2 - 4\left[B^2r^2\left(e_2 + A^2B^2M^{12}(r)^2\right) + \left(AB^2CM^{12}(r) - M^{13}(r)\right)^2\right]}}{2A^2B}, \tag{A.41}$$

and hence, completely determine the anomalous inverse metric tensor field for this family.

$$M^{12}(r) = \frac{\alpha_1}{r^2} + \alpha_2, \tag{A.42}$$

$$M^{13}(r) = \frac{AB^2C\alpha_1}{r^2} + \gamma_1 r^2 + \gamma_2, \tag{A.43}$$

$$M^{11}(r) = \frac{e_1 r^2}{2A^2} \tag{A.44}$$

$$\pm \frac{r\sqrt{B^2 e_1^2 r^2 - 4\left[B^2 r^2\left(e_2 + A^2 B^2 M^{12}(r)^2\right) + \left(AB^2 C M^{12}(r) - M^{13}(r)\right)^2\right]}}{2A^2 B}, \tag{A.45}$$

$$M^{22}(r) = \frac{A^2 B^4 e_2\left(e_1 r^2 - A^2 M^{11}(r)\right) M^{12}(r)^2 + \left[AB^2 C M^{12}(r) - M^{13}(r)\right]^2}{B^2 e_2 r^2\left[\left(AB^2 C M^{12}(r) - M^{13}(r)\right)^2 + r^2 M^{12}(r)^2\right]}, \tag{A.46}$$

$$M^{23}(r) = \frac{AC\left[AB^2 C M^{12}(r) - M^{13}(r)\right]^2}{e_2 r^2\left[\left(AB^2 C M^{12}(r) - M^{13}(r)\right)^2 + r^2 M^{12}(r)^2\right]} + \frac{A^2 B^2\left(e_1 e_2 r^2 - r^2 - A^2 e_2 M^{11}(r)\right) M^{12}(r) M^{13}(r)}{e_2 r^2\left[\left(AB^2 C M^{12}(r) - M^{13}(r)\right)^2 + r^2 M^{12}(r)^2\right]}, \tag{A.47}$$

$$M^{33}(r) = \frac{A^2 B^2}{e_2 r^2} \times \tag{A.48}$$

$$\frac{\left[AB^2\left(C^2 + r^2\right) M^{12}(r) - C M^{13}(r)\right]^2 + e_2\left(e_1 r^2 - A^2 M^{11}(r)\right) M^{13}(r)^2}{\left[AB^2 C M^{12}(r) - M^{13}(r)\right]^2 + r^2 M^{12}(r)^2}. \tag{A.49}$$

One can check that in all cases, the equilibrium conditions are satisfied, and that the invariants of $\mathbf{b}$ are

$$I_1 = e_1 + \frac{1}{e_2}, \quad I_2 = \frac{e_1}{e_2} + e_2, \quad I_3 = 1, \tag{A.50}$$

as expected.

A.2 Family 2

As above, we compute the equations

$$m_{a[k}\nabla_{l]}\nabla_b b^{ab} = 0, \quad m_{a[k}\nabla_{l]}\nabla_b c^{ab} = 0. \tag{A.51}$$

The first of these contains a component

$$\frac{4\xi^2 M^{12}(\xi)'' + 4\xi M^{12}(\xi)' - M^{12}(\xi)}{2\sqrt{2}\xi^{\frac{3}{2}} A^{\frac{1}{2}}} = 0, \tag{A.52}$$

where here we have made the substitution $x = \xi + D$. The denominator is nonzero, so we can simply take the numerator to be zero, and integrate it. This yields

$$M^{12}(\xi) = \frac{\alpha_1 \xi + \alpha_2}{\sqrt{\xi}}. \tag{A.53}$$

Substituting this into the other component of this equation, we obtain a differential equation for $M^{13}(\xi)$:

$$\frac{\sqrt{A}\left[4\xi^2 M^{13}(\xi)'' + 4\xi M^{13}(\xi)' - M^{13}(\xi)\right]}{2\sqrt{2}\xi^{\frac{3}{2}}} = 0, \tag{A.54}$$

which has the same general solution as the equation for $M^{12}(\xi)$, and hence

$$M^{13}(\xi) = \frac{\gamma_1\xi + \gamma_2}{\sqrt{\xi}}. \tag{A.55}$$

The first of the remaining differential equations is

$$\frac{\sqrt{2}\left(\left(\alpha_1\xi + \alpha_2\right) M^{23}(\xi)'' - \left(\gamma_1\xi + \gamma_2\right) M^{22}(\xi)'' + 2\alpha_1 M^{23}(\xi)' - 2\gamma_1 M^{22}(\xi)'\right)}{A^{\frac{3}{2}} B^2} = 0, \tag{A.56}$$

which equivalently reads $\left[\left(\alpha_1\xi + \alpha_2\right) M^{23}(\xi) - \left(\gamma_1\xi + \gamma_2\right) M^{22}(\xi)\right]'' = 0$, and integrates to

$$\left(\alpha_1\xi + \alpha_2\right) M^{23}(\xi) - \left(\gamma_1\xi + \gamma_2\right) M^{22}(\xi) = \mu_1\xi + \mu_2. \tag{A.57}$$

The second equation is

$$\begin{aligned}&\frac{\sqrt{2}\left(2C\gamma_1 M^{22}(\xi)' + 2\left(A\gamma_1 - C\alpha_1\right) M^{23}(\xi)' - 2A\alpha_1 M^{33}(\xi)'\right)}{A^{\frac{3}{2}} B^2} \\ &\quad + \frac{\sqrt{2}\left(C\left(\gamma_1\xi + \gamma_2\right) M^{22}(\xi)'' - \left(A\left(\gamma_1\xi + \gamma_2\right) - C\left(\alpha_1\xi + \alpha_2\right)\right) M^{23}(\xi)''\right)}{A^{\frac{3}{2}} B^2} \\ &\qquad + \frac{\sqrt{2}A\left(\alpha_1\xi + \alpha_2\right) M^{33}(\xi)''}{A^{\frac{3}{2}} B^2} = 0,\end{aligned} \tag{A.58}$$

or equivalently

$$\left[\left(\gamma_1\xi + \gamma_2\right)\left(AM^{23}(\xi) + CM^{22}(\xi)\right) - \left(\alpha_1\xi + \alpha_2\right)\left(AM^{33}(\xi) + CM^{23}(\xi)\right)\right]'' = 0. \tag{A.59}$$

We can add C times the previous differential equation, and divide by A to obtain

$$\left[\left(\gamma_1\xi + \gamma_2\right) M^{23}(\xi) - \left(\alpha_1\xi + \alpha_2\right) M^{33}(\xi)\right]'' = 0. \tag{A.60}$$

which integrates to

$$\left(\gamma_1\xi + \gamma_2\right) M^{23}(\xi) - \left(\alpha_1\xi + \alpha_2\right) M^{33}(\xi) = \beta_1\xi + \beta_2. \tag{A.61}$$

Finally, we have the constant trace condition on $\mathbf{b}$, which reads

$$\left(1 + C^2\right) M^{22}(\xi) + 2ACM^{23}(\xi) + A^2 M^{33}(\xi) = -2A^3B^4\xi M^{11}(\xi) + A^2B^2 I_1. \tag{A.62}$$

These equations can be solved for $M^{22}(\xi)$, $M^{23}(\xi)$, and $M^{33}(\xi)$, as the determinant of these equations is

$$-\left(\alpha_1\xi+\alpha_2\right)^2-\left(C\left(\alpha_1\xi+\alpha_2\right)+A\left(\gamma_1\xi+\gamma_2\right)\right)^2<0, \tag{A.63}$$

which only vanishes if both $M^{12}(\xi)$ and $M^{13}(\xi)$ vanish.

We then have the solution in terms of the yet-to-be determined component $M^{11}(\xi)$, and numerous undetermined constants. Denoting $\alpha=\alpha_1\xi+\alpha_2$, $\gamma=\gamma_1\xi+\gamma_2$, $\mu=\mu_1\xi+\mu_2$, $\beta=\beta_1\xi+\beta_2$, and $h=A^2B^2I_1-2A^3B^4\xi M^{11}(\xi)$, we have

$$\begin{bmatrix} M^{22}(\xi) \\ M^{23}(\xi) \\ M^{33}(\xi) \end{bmatrix} = \frac{1}{\alpha^2+\left(C\alpha+A\gamma\right)^2}\begin{bmatrix} h\alpha^2+A\left(A\alpha\beta-2C\alpha\mu-A\gamma\mu\right) \\ h\alpha\gamma+A^2\beta\gamma+\left(1+C^2\right)\alpha\mu \\ -\left(1+C^2\right)\alpha\beta+\gamma\left(-2AC\beta+h\gamma+\left(1+C^2\right)\mu\right) \end{bmatrix}. \tag{A.64}$$

Under these substitutions, the constant second invariant condition is written as

$$\begin{aligned} 4A^2B^4\xi^2M^{11}(\xi)^2-\frac{2A\xi\left[B^2I_1\left[\alpha^2+\left(C\alpha+A\gamma\right)^2\right]+\alpha\beta+\gamma\mu\right]}{\alpha^2+\left(C\alpha+A\gamma\right)^2}M^{11}(\xi)= \\ -\frac{2A^5B^4\gamma^4+A^4B^4\gamma^2\left(I_2+8C\alpha\gamma\right)+2A^3B^4\alpha\gamma\left(CI_2+2\alpha\gamma+6C^2\gamma\alpha\right)+\left(1+C^2\right)\mu^2}{A^2B^4\left[\alpha^2+\left(C\alpha+A\gamma\right)^2\right]} \\ -\frac{2\left[B^4\left(1+C^2\right)^2\alpha^4-C\beta\mu\right]}{AB^4\left[\alpha^2+\left(C\alpha+A\gamma\right)^2\right]} \\ -\frac{A\left[\beta^2+B^4\left(1+C^2\right)\alpha^2\left(I_2+8C\alpha\gamma\right)+B^2I_1\left(\alpha\beta+\gamma\mu\right)\right]}{AB^4\left[\alpha^2+\left(C\alpha+A\gamma\right)^2\right]}, \end{aligned} \tag{A.65}$$

and the incompressibility condition reads

$$\begin{aligned} \frac{4A^2B^2\xi^2\left(\alpha\beta+\gamma\mu\right)}{\alpha^2+\left(C\alpha+A\gamma\right)^2}M^{11}(\xi)^2-\frac{2\xi\left[\mu^2+B^2I_1\left(\alpha\beta+\gamma\mu\right)+\left(C\mu-A\beta\right)^2\right]}{\alpha^2+\left(C\alpha+A\gamma\right)^2}M^{11}(\xi) \\ =\frac{AB^2-2\alpha\beta-2\gamma\mu}{AB^2}. \end{aligned} \tag{A.66}$$

Clearing denominators and computing the resultant of these equations in $M^{11}(\xi)$, we obtain

$$\mathsf{Res}_{M^{11}(\xi)}\left(p_1,p_2\right)=16A^4B^8\xi^4\left(\alpha^2+\left(C\alpha+A\gamma\right)^2\right)\left(...\right). \tag{A.67}$$

This must vanish for solutions to exist, so we take $(...)=0$, since the remaining factors are nonzero.

We then take the coefficients of this polynomial in ξ to vanish independently, factor these coefficients, and order them by length. The first of these demands

$$A\left(C\alpha_1+A\gamma_1\right)\beta_1=\left(AC\gamma_1+\left(1+C^2\right)\alpha_1\right)\mu_1. \tag{A.68}$$

If this equation can be solved for β_1, we can perform this substitution and obtain the condition

$$\mu_1^3 - I_1\mu_1^2\left(AB^2\left(C\alpha_1 + A\gamma_1\right)\right) + I_2\mu_1\left(AB^2\left(C\alpha_1 + A\gamma_1\right)\right)^2 - \left(AB^2\left(C\alpha_1 + A\gamma_1\right)\right)^3 = 0, \tag{A.69}$$

which requires that $\nu = -\frac{\mu_1}{AB^2(C\alpha_1 + A\gamma_1)}$ be an eigenvalue of $\mathbf{b}$. We perform the usual substitutions with the invariants to express equations in terms of e_1 and e_2, then take discriminants of the resulting equations in e_1, and demand non-negativity. This yields the condition

$$\beta_2 = \frac{A^2B^2\left(\alpha_1\gamma_2 - \alpha_2\gamma_1\right) + \left(1 + C^2\right)e_2\alpha_1\mu_2 + ACe_2\gamma_1\mu_2}{Ae_2\left(C\alpha_1 + A\gamma_1\right)}, \tag{A.70}$$

or

$$e_2\alpha_1\beta_2 = -\left(B^2\left(A\gamma_1\left(C\alpha_2 + A\gamma_2\right) + \alpha_1\left(\alpha_2 + C^2\alpha_2 + AC\gamma_2\right)\right) + e_2\gamma_1\mu_2\right). \tag{A.71}$$

In the first case, substitution yields either

$$\mu_2 = -\frac{AB^2\left(C\alpha_2 + A\gamma_2\right)}{e_2}, \tag{A.72}$$

or not, in which case $e_2^3 - e_1e_2 + 1 = 0$. In the first case, we satisfy the equations and obtain

$$\beta_2 = -\frac{B^2\left(\alpha_2 + C^2\alpha_2 + AC\gamma_2\right)}{e_2}. \tag{A.73}$$

Otherwise, we take $e_2^3 - e_2e_1 + 1 = 0$.

Substituting this yields $\lambda_1^2 = \lambda_2^2 = 1$, which reduces the equations to only one. After removing nonzero factors of this equation, we obtain something quadratic in μ_2, which after taking the discriminant and demanding non-negativity yields $\alpha_2\gamma_1 = \alpha_1\gamma_2$. With this, if $\alpha_1 \neq 0$, we obtain the final necessary result $\mu_2 = -AB^2\left(C\alpha_2 + A\gamma_2\right)$, which is a contradiction, since it is the case considered earlier with $\lambda_1^2 = \lambda_2^2 = 1$. If $\alpha_1 = 0$, we require $\alpha_2 = 0$, since $C\alpha_1 + A\gamma_1 \neq 0$. With this, we require $\mu_2 = -A^2B^2\gamma_2$, again, a contradiction.

Next, we consider the case where

$$e_2\alpha_1\beta_2 = -\left(B^2\left(A\gamma_1\left(C\alpha_2 + A\gamma_2\right) + \alpha_1\left(\alpha_2 + C^2\alpha_2 + AC\gamma_2\right)\right) + e_2\gamma_1\mu_2\right). \tag{A.74}$$

If $\alpha_1 \neq 0$, we solve this expression for β_2 and obtain the necessary and sufficient condition

$$\mu_2 = -\frac{AB^2\left(C\alpha_2 + A\gamma_2\right)}{e_2}, \tag{A.75}$$

with

$$\beta_2 = -\frac{B^2\left(\left(1+C^2\right)\alpha_2 + AC\gamma_2\right)}{e_2}. \tag{A.76}$$

If $\alpha_1 = 0$, we can solve for μ_2 to obtain

$$\mu_2 = \frac{-AB^2\left(C\alpha_2 + A\gamma_2\right)}{e_2}. \tag{A.77}$$

This requires

$$\beta_2 = \frac{-B^2\left(\left(1+C^2\right)\alpha_2 + AC\gamma_2\right)}{e_2}. \tag{A.78}$$

Alternatively, if $C\alpha_1 + A\gamma_1 = 0$, we obtain $\gamma_1 = \frac{-C\alpha_1}{A}$, which implies $\alpha_1\mu_1 = 0$. If we assume $\alpha_1 = 0$, and insert this relation into the equations, we obtain $\mu_1^2 + \left(A\beta_1 - C\mu_1\right)^2 = 0$, so we can freely take $\mu_1 = 0$. With this, we obtain an eigenvalue equation that demands that either $\nu = -\frac{\beta_1}{B^2\alpha_1}$ is an eigenvalue of $\mathbf{b}$, or that $\beta_1 = \alpha_1 = 0$. In the first case, we perform the usual substitutions and take discriminants in e_1 and demand non-negativity, which yields

$$\mu_2 = -\frac{AB^2\left(C\alpha_2 + A\gamma_2\right)}{e_2}, \tag{A.79}$$

or not, in which case

$$\beta_2 = \frac{Ce_2\mu_2 - AB^2\alpha_2}{Ae_2}. \tag{A.80}$$

In the first case, we have the sufficient condition

$$\beta_2 = -\frac{B^2\left[\left(1+C^2\right)\alpha_2 + AC\gamma_2\right]}{e_2}, \tag{A.81}$$

or not, in which case $e_2^3 - e_1e_2 + 1 = 0$. With this, we then obtain the condition $\lambda_1^2 = \lambda_2^2 = 1$, and then requiring discriminants in β_2 to be non-negative, $C\alpha_2 + A\gamma_2 = 0$. With this, we take $\gamma_2 = -\frac{C}{A}\alpha_2$, which requires $\beta_2 = -B^2\alpha_2$, which is a special case of the previous solution.

Next, we consider $\beta_2 = \frac{Ce_2\mu_2 - AB^2\alpha_2}{Ae_2}$. This requires the necessary and sufficient condition

$$\mu_2 = -\frac{AB^2\left(A\gamma_2 + C\alpha_2\right)}{e_2}, \tag{A.82}$$

which is the same as before.

The previously examined cases are all particular instances of the following anomalous solution:

$$M^{12}(\xi) = \frac{\alpha_1\xi + \alpha_2}{\sqrt{\xi}}, \tag{A.83}$$

$$M^{13}(\xi) = \frac{\gamma_1 \xi + \gamma_2}{\sqrt{\xi}}, \tag{A.84}$$

$$M^{11}(\xi) = \frac{\sqrt{A} e_1 \pm \sqrt{A e_1^2 - 4\left[A e_2 + 2\xi \left((A M^{13}(\xi) + C M^{12}(\xi))^2 + M^{12}(\xi)^2\right)\right]}}{4 A^{\frac{3}{2}} B^2 \xi}, \tag{A.85}$$

$$M^{22}(\xi) = \frac{A^2 B^2 \left[(C M^{12}(\xi) + A M^{13}(\xi))^2 + e_2 \left(e_1 - 2 A B^2 \xi M^{11}(\xi)\right) M^{12}(\xi)^2\right]}{e_2 \left[M^{12}(\xi)^2 + (A M^{13}(\xi) + C M^{12}(\xi))^2\right]}, \tag{A.86}$$

$$\begin{aligned} M^{23}(\xi) = &-\frac{A B^2 \left[C \left(C M^{12}(\xi) + A M^{13}(\xi)\right)^2 + C M^{12}(\xi)^2\right]}{e_2 \left[M^{12}(\xi)^2 + (A M^{13}(\xi) + C M^{12}(\xi))^2\right]}, \\ &- \frac{A B^2 \left[\left(A - e_1 e_2 + 2 A B^2 e_2 \xi M^{11}(\xi)\right) M^{12}(\xi) M^{13}(\xi)\right]}{e_2 \left[M^{12}(\xi)^2 + (A M^{13}(\xi) + C M^{12}(\xi))^2\right]}, \end{aligned} \tag{A.87}$$

$$M^{33}(\xi) = \tag{A.88}$$

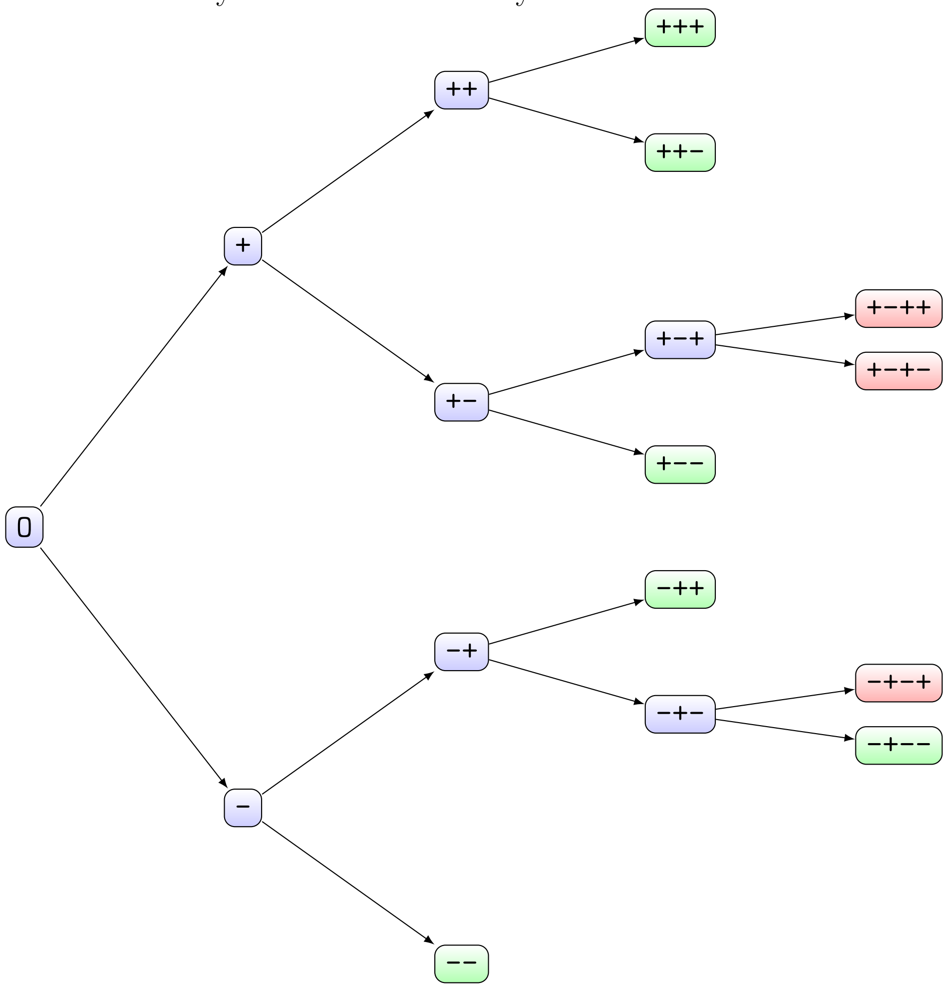
$$\frac{B^2}{e_2} \frac{\left(\left(1 + C^2\right) M^{12}(\xi) + A C M^{13}(\xi)\right)^2 + A^2 e_2 \left(e_1 - 2 A B^2 \xi M^{11}(\xi)\right) M^{13}(\xi)^2}{M^{12}(\xi)^2 + (A M^{13}(\xi) + C M^{12}(\xi))^2}. \tag{A.89}$$

Finally, when $\beta_1 = \alpha_1 = 0$, we are left with only one equation. In principle it can be solved for I_2, but because I_2 does not appear in any of the other equations, we can simply use the remaining constants as a transcendence basis in lieu of actually solving it. When we do this, we recognize that with $\gamma_1 = \alpha_1 = \beta_1 = \mu_1 = 0$, we obtain b^{ab} being constant. Hence, we can take any constant positive-definite tensor with $\det \mathbf{b} = 1$, and obtain M^{AB} via $M^{AB} = (F^{-1})^A{}_a b^{ab} (F^{-1})^B{}_b$, which one can easily see generates a positive-definite symmetric metric tensor, since $\mathbf{b}$ is positive-definite and symmetric. This, however, implies that the material manifold is Euclidean, since $\mathbf{c}$ is the push forward of the material metric tensor, and it is constant in Cartesian coordinates. Therefore, the curvature tensor based on $\mathbf{c}$ vanishes, and the anelastic deformation is stress free, i.e., Euclidean.

In general, a metric $M_{AB}(R)$ arising from this case has the form

$$[M_{AB}(R)] = \begin{bmatrix} M_{11} R^2 & M_{12} R & M_{13} R \\ M_{12} R & M_{22} & M_{23} \\ M_{13} R & M_{23} & M_{33} \end{bmatrix}, \tag{A.90}$$

where the constants $\{M_{11}, M_{12}, M_{13}, M_{22}, M_{23}, M_{33}\}$ satisfy the incompressibility constraint $M_{11} M_{22} M_{33} + 2 M_{12} M_{13} M_{23} - {M_{13}}^2 M_{22} - {M_{23}}^2 M_{11} - {M_{12}}^2 M_{33} = 1$. Any choice of these constants that yields a positive-definite metric generates an admissible constant tensor $\mathbf{b}$. Though not immediately obvious, all of the above solutions are part of the same branch, apart from a global rigid rotation, which can be freely removed. The analysis tree for this family is:



Nodes:

- 0: $A\left(C\alpha_1 + A\gamma_1\right)\beta_1 = \left(AC\gamma_1 + \left(1 + C^2\right)\alpha_1\right)\mu_1$
- −: $\beta_1^3 + I_1 B^2 \alpha_1 \beta_1^2 + I_2 B^4 \alpha_1^2 \beta_1 + B^6 \alpha_1^3$
- +: $\left(A\beta_2 e_2\left(C\alpha_1 + A\gamma_1\right) - A^2 B^2\left(\gamma_2\alpha_1 - \alpha_2\gamma_1\right) - \left(1 + C^2\right)e_2\alpha_1\mu_2 - ACe_2\gamma_1\mu_2\right)$ $\left(e_2\alpha_1\beta_2 + B^2\left(A\gamma_1\left(C\alpha_2 + A\gamma_2\right) + \alpha_1\left(\alpha_2 + C^2\alpha_2 + AC\gamma_2\right)\right) + e_2\gamma_1\mu_2\right) = 0$
- −−: b^{ab} is constant
- −+: $\left(\mu_2 e_2 + AB^2\left(C\alpha_2 + A\gamma_2\right)\right)\left(A\beta_2 e_2 + AB^2\alpha_2 - Ce_2\mu_2\right) = 0$
- +−: $\left(\mu_2 e_2 + AB^2\left(C\alpha_2 + A\gamma_2\right)\right)\left(e_2^3 - e_1 e_2 + 1\right) = 0$
- ++:$e_2\alpha_1\beta_2 + B^2\left(A\gamma_1\left(C\alpha_2 + A\gamma_2\right) + \alpha_1\left(\alpha_2 + C^2\alpha_2 + AC\gamma_2\right)\right) + e_2\gamma_1\mu_2 = 0$

- $-+-$: $(\beta_2 e_2 + B^2((1+C^2)\alpha_2 + AC\gamma_2))(e_2^3 - e_1 e_2 + 1) = 0$
- $-++$: $\mu_2 = \frac{-AB^2(C\alpha_2 + A\gamma_2)}{e_2}$
- $+--$: $\mu_2 = \frac{-AB^2(C\alpha_2 + A\gamma_2)}{e_2}$
- $+-+$: $\alpha_1\gamma_2 = \alpha_2\gamma_1$
- $++-$: $\mu_2 = \frac{-AB^2(C\alpha_2 + A\gamma_2)}{e_2}$
- $+++$: $\beta_2 = \frac{-B^2\left(\left(1+C^2\right)\alpha_2 + AC\gamma_2\right)}{e_2}$
- $-+--$: $\mu_2 = \frac{-AB^2(C\alpha_2 + A\gamma_2)}{e_2}$ and $\beta_2 = -\frac{B^2\left(\left(1+C^2\right)\alpha_2 + AC\gamma_2\right)}{e_2}$
- $-+-+$: $\lambda_a^2 = 1 \Rightarrow C\alpha_2 + A\gamma_2 = 0 \Rightarrow \beta_2 = -B^2\alpha_2 \& \beta_2 \neq -B^2\alpha_2$
- $+-+-$: $\mu_2 = -AB^2(C\alpha_2 + A\gamma_2) \neq -AB^2(C\alpha_2 + A\gamma_2)$
- $+-++$: $\mu_2 = -A^2B^2\gamma_2 \neq -A^2B^2\gamma_2$

Edges (labelled by child node):

- $-$: $\gamma_1 = -\frac{C}{A}\alpha_1 \Rightarrow \mu_1 = 0$
- $+$: $\gamma_1 \neq \frac{C}{A}\alpha_1 \Rightarrow \nu = -\frac{\mu_1}{AB^2(C\alpha_1 + A\gamma_1)}$
- $--$: $\beta_1 = \alpha_1 = 0$
- $-+$: $\nu = -\frac{\beta_1}{B^2\alpha_1}$
- $+-$: $\beta_2 = \frac{A^2B^2(\alpha_1\gamma_2 - \alpha_2\gamma_1) + \left(1+C^2\right)e_2\alpha_1\mu_2 + ACe_2\gamma_1\mu_2}{Ae_2(C\alpha_1 + A\gamma_1)}$
- $++$: $e_2\alpha_1\beta_2 = -(B^2(A\gamma_1(C\alpha_2 + A\gamma_2) + \alpha_1(\alpha_2 + C^2\alpha_2 + AC\gamma_2)) + e_2\gamma_1\mu_2)$
- $-+-$: $\mu_2 = \frac{-AB^2(C\alpha_2 + A\gamma_2)}{e_2}$
- $-++$: $\beta_2 = \frac{Ce_2\mu_2 - AB^2\alpha_2}{Ae_2}$
- $+--$: $\mu_2 e_2 + AB^2(C\alpha_2 + A\gamma_2) = 0$
- $+-+$: $\mu_2 e_2 + AB^2(C\alpha_2 + A\gamma_2) \neq 0$ and $e_2^3 - e_1 e_2 + 1 = 0$
- $++-$: $\beta_2 = \frac{-\left(B^2\left(A\gamma_1(C\alpha_2 + A\gamma_2) + \alpha_1\left(\alpha_2 + C^2\alpha_2 + AC\gamma_2\right)\right) + e_2\gamma_1\mu_2\right)}{e_2\alpha_1}$
- $+++$: $\alpha_1 = 0$ and $\mu_2 = \frac{-AB^2\left(\left(1+C^2\right)\alpha_2 + AC\gamma_2\right)}{e_2}$

- $-+--$: $\beta_2 = -\frac{B^2\left(\left(1+C^2\right)\alpha_2+AC\gamma_2\right)}{e_2}$
- $-+-+$: $\beta_2 \neq -\frac{B^2\left(\left(1+C^2\right)\alpha_2+AC\gamma_2\right)}{e_2} \& e_2^3 - e_1 e_2 + 1 = 0$
- $+-+-$: $\gamma_2 = \frac{\alpha_2\gamma_1}{\alpha_1}$
- $+-++$: $\alpha_1 = \alpha_2 = 0$

A.3 Family 3

For this family, we can write the remaining equations corresponding to the coefficients of terms involving W_1 and W_2, and solve them before recasting the results in terms of the components of $\mathbf{M}$. First, we take the equation $m_{a[l}\nabla_{k]}\nabla_b b^{ab} = 0$, which has two nonzero components yielding

$$r^2 b^{12\prime\prime}(r) + 5rb^{12\prime}(r) + 3b^{12}(r) = 0, \tag{A.91}$$
$$r^2 b^{13\prime\prime}(r) + rb^{13\prime}(r) - b^{13}(r) = 0. \tag{A.92}$$

Solving these yields

$$b^{12}(r) = a_1 r^{-1} + a_2 r^{-3} = \frac{\sqrt{A\left(r^2-B\right)}}{r}\left(CM^{12} + DM^{13}\right), \tag{A.93}$$
$$b^{13}(r) = b_1 r + b_2 r^{-1} = \frac{\sqrt{A\left(r^2-B\right)}}{r}\left(EM^{12} + FM^{13}\right). \tag{A.94}$$

Reparameterizing these equations to simplify constants yields the equation

$$\begin{bmatrix} C & D \\ E & F \end{bmatrix}\begin{bmatrix} M^{12} \\ M^{13} \end{bmatrix} = \begin{bmatrix} \gamma_1 + \gamma_2 r^{-2} \\ \alpha_1 r^2 + \alpha_2 \end{bmatrix}\frac{DE-CF}{\sqrt{r^2-B}}, \tag{A.95}$$

which gives

$$\begin{bmatrix} M^{12} \\ M^{13} \end{bmatrix} = \begin{bmatrix} -F & D \\ E & -C \end{bmatrix}\begin{bmatrix} \gamma_1 + \gamma_2 r^{-2} \\ \alpha_1 r^2 + \alpha_2 \end{bmatrix}\left(r^2 - B\right)^{-\frac{1}{2}}. \tag{A.96}$$

With this, we have the W_2 equations involving $\mathbf{c}$, which are

$$\begin{aligned} 8r^3\alpha_1 b^{23}(r) + 7r^4\alpha_1 b^{23\prime}(r) + 3r^2\alpha_2 b^{23\prime}(r) - 3r^2\gamma_1 b^{33\prime}(r) + \gamma_2 b^{33\prime}(r) \\ + r^5\alpha_1 b^{23\prime\prime}(r) + r^3\alpha_2 b^{23\prime\prime}(r) - r^3\gamma_1 b^{33\prime\prime}(r) - r\gamma_2 b^{33\prime\prime}(r) = 0, \end{aligned} \tag{A.97}$$

and

$$\begin{aligned} 8r^3\alpha_1 b^{22}(r) + 7r^4\alpha_1 b^{22\prime}(r) + 3r^2\alpha_2 b^{22\prime}(r) - 3r^2\gamma_1 b^{23\prime}(r) + \gamma_2 b^{23\prime}(r) \\ + r^5\alpha_1 b^{22\prime\prime}(r) + r^3\alpha_2 b^{22\prime\prime}(r) - r^3\gamma_1 b^{23\prime\prime}(r) - r\gamma_2 b^{23\prime\prime}(r) = 0, \end{aligned} \tag{A.98}$$

which have the common solution

$$\left(r^2\gamma_1+\gamma_2\right)b^{33}(r)-r^2\left(r^2\alpha_1+\alpha_2\right)b^{23}(r)=\mu_1+\mu_2r^2, \tag{A.99}$$

$$\left(r^2\gamma_1+\gamma_2\right)b^{23}(r)-r^2\left(r^2\alpha_1+\alpha_2\right)b^{22}(r)=\beta_1+\beta_2r^2. \tag{A.100}$$

We also have the constant trace condition that amounts to $b^{11}(r)+r^2b^{22}(r)+b^{33}(r)=I_1$, which lets us solve for $b^{22}(r)$, $b^{23}(r)$, and $b^{33}(r)$, as these equations are linear in these components, and the determinant of the linear system is identically nonzero, being $-\left[\left(r\left(r^2\gamma_1+\gamma_2\right)\right)^2+\left(r^2\left(r^2\alpha_1+\alpha_2\right)\right)^2\right]$. Additionally, we have the following relations between the components of $\mathbf{b}$ and the components of $\mathbf{M}$

$$\begin{bmatrix}C^2 & 2CD & D^2\\ CE & DE+CF & DF\\ E^2 & 2EF & F^2\end{bmatrix}\begin{bmatrix}M^{22}(r)\\ M^{23}(r)\\ M^{33}(r)\end{bmatrix}=\begin{bmatrix}b^{22}(r)\\ b^{23}(r)\\ b^{33}(r)\end{bmatrix}, \tag{A.101}$$

which can be inverted, as the determinant of this system is $(CF-DE)^3\neq 0$. Doing this, we define

$$p(r)=\gamma_1+\frac{\gamma_2}{r^2},\quad q(r)=\alpha_1r^2+\alpha_2, \tag{A.102}$$

and obtain

$$M^{12}(r)=\frac{Dq(r)-Fp(r)}{\sqrt{r^2-B}}, \tag{A.103}$$

$$M^{13}(r)=\frac{Ep(r)-Cq(r)}{\sqrt{r^2-B}}, \tag{A.104}$$

$$\begin{aligned}M^{22}(r)=&\frac{F^2\left(I_1p(r)^2r^2-q(r)\left(\beta_1+r^2\beta_2\right)-p(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\\&+\frac{D^2r^2\left(I_1q(r)^2+q(r)\left(\beta_1+r^2\beta_2\right)+p(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\\&-\frac{2DF\left(I_1p(r)r^2\left(\beta_1+r^2\beta_2\right)-q(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\end{aligned} \tag{A.105}$$

$$-\frac{A\left(r^2-B\right)\left(Fp(r)-Dq(r)\right)^2M^{11}(r)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}, \tag{A.106}$$

$$\begin{aligned}M^{23}(r)=&\frac{-EF\left(I_1p(r)^2r^2-q(r)\left(\beta_1+r^2\beta_2\right)-p(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\\&+\frac{CDr^2\left(I_1q(r)^2+q(r)\left(\beta_1+r^2\beta_2\right)+p(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\\&+\frac{DE\left(I_1p(r)r^2q(r)+p(r)r^2\left(\beta_1+r^2\beta_2\right)-q(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\\&+\frac{CF\left(I_1p(r)r^2\left(\beta_1+r^2\beta_2\right)-q(r)\left(\mu_1+r^2\mu_2\right)\right)}{(CF-DE)^2r^2\left(p(r)^2r^2+q(r)^2\right)}\end{aligned}$$

$$+\frac{A\left(r^2-B\right)\left(Fp(r)-Dq(r)\right)\left(Ep(r)-Cq(r)\right)M^{11}(r)}{\left(CF-DE\right)^2 r^2\left(p(r)^2r^2+q(r)^2\right)}, \tag{A.107}$$

$$M^{33}(r)=\frac{E^2\left(I_1p(r)^2r^2-q(r)\left(\beta_1+r^2\beta_2\right)-p(r)\left(\mu_1+r^2\mu_2\right)\right)}{\left(CF-DE\right)^2 r^2\left(p(r)^2r^2+q(r)^2\right)}$$
$$+\frac{C^2r^2\left(I_1q(r)^2+q(r)\left(\beta_1+r^2\beta_2\right)+p(r)\left(\mu_1+r^2\mu_2\right)\right)}{\left(CF-DE\right)^2 r^2\left(p(r)^2r^2+q(r)^2\right)}$$
$$-\frac{2EC\left(I_1p(r)r^2\left(\beta_1+r^2\beta_2\right)-q(r)\left(\mu_1+r^2\mu_2\right)\right)}{\left(CF-DE\right)^2 r^2\left(p(r)^2r^2+q(r)^2\right)} \tag{A.108}$$

$$-\frac{A\left(r^2-B\right)\left(Ep(r)-Cq(r)\right)^2M^{11}(r)}{\left(CF-DE\right)^2 r^2\left(p(r)^2r^2+q(r)^2\right)}. \tag{A.109}$$

We are then left with the constant invariant condition for I_2 and the incompressibility condition. We simplify notation by writing $\mu=\mu_1+r^2\mu_2$ and $\beta=\beta_1+r^2\beta_2$. These equations are

$$\frac{A^2\left(r^2-B\right)^2\left(q(r)\beta-p(r)\mu\right)M^{11}(r)^2}{r^4\left(q(r)^2+p(r)^2r^2\right)}$$
$$-\frac{A\left(r^2-B\right)\left(I_1q(r)r^2\beta+r^2\beta^2-I_1p(r)r^2\mu+\mu^2\right)M^{11}(r)}{r^4\left(q(r)^2+p(r)^2r^2\right)}$$
$$+\frac{A\left(DE-CF\right)^2\left(q(r)\beta-p(r)\mu\right)}{r^2}=1, \tag{A.110}$$

$$\frac{A^2\left(r^2-B\right)^2M^{11}(r)^2}{r^4}-\frac{A\left(r^2-B\right)\left(I_1\left(q(r)^2+p(r)^2r^2\right)+q(r)\beta-p(r)\mu\right)M^{11}(r)}{r^2\left(q(r)^2+p(r)^2r^2\right)}$$
$$+I_2+\frac{A\left(CF-DE\right)^2\left(q(r)^2+p(r)^2r^2\right)}{r^2}+\frac{I_1r^2\left(q(r)\beta-p(r)\mu\right)+r^2\beta^2+\mu^2}{r^2\left(q(r)^2+p(r)^2r^2\right)}=0 \tag{A.111}$$

We take the resultant of these equations in $M^{11}(r)$, and demand this vanish. We remove nonzero factors, and then demand each coefficient in r vanish independently. The first condition we obtain is $\alpha_1^3+I_2\alpha_1^2\beta_2+I_1\alpha_1\beta_2^2+\beta_2^3=0$, which implies that either $\nu=-\frac{\beta_2}{\alpha_1}$ is an eigenvalue of $\mathbf{b}$, or $\alpha_1=\beta_2=0$. In the first case, we obtain $\mu_2=\frac{\gamma_1}{e_2}$. This yields two subcases:

$$\beta_1=-\frac{\alpha_2}{e_2}, \tag{A.112}$$

or alternatively

$$e_2^3-e_1e_2+1=0. \tag{A.113}$$

In this first subcase, we obtain the sufficient condition $\mu_1=\frac{\gamma_2}{e_2}$, or $e_2^3-e_1e_2+1=0$. However, when we take $e_2^3-e_1e_2+1=0$, we obtain the same value for μ_1, and hence, this second condition is redundant. If we take the subcase, $e_2^3 - e_1 e_2 + 1 = 0$ and $\beta_1 \neq -\frac{\alpha_2}{e_2}$, we obtain $\lambda_1^2 = \lambda_2^2 = 1$.

We then obtain the sufficient condition

$$\mu_1 = \frac{\alpha_1 \gamma_2 - \gamma_1 (\alpha_2 + \beta_1)}{\alpha_1}, \tag{A.114}$$

which further yields $\beta_1 = -\alpha_2$, which is again a special case of the already discovered solutions. Next, we consider $\alpha_1 = \beta_2 = 0$. Inserting this, we obtain the equation $\mu_2^3 - I_1 \mu_2^2 \gamma_1 + I_2 \mu_2 \gamma_1^2 - \gamma_1^3 = 0$, which implies $\nu = \frac{\mu_2}{\gamma_1}$ is an eigenvalue of $\mathbf{b}$, or $\mu_2 = \gamma_1 = 0$. In the first case, we take discriminants in e_1 and demand non-negativity, which yields the condition

$$\gamma_2 \mu_1 (e_2 \mu_1 - \gamma_2) (\beta_1 \gamma_2 + \alpha_2 \mu_1) = 0. \tag{A.115}$$

Beginning with $\gamma_2 = 0$, we obtain

$$\mu_1^4 \left[A (CF - DE)^2 \alpha_2^4 + \mu_1^2 \right] = 0, \tag{A.116}$$

which requires that $\mu_1 = 0$. With this, we then obtain equations with the common factor $\alpha_2 + e_2 \beta_1$. Hence, it is sufficient if this factor vanishes. If it does not, we can take the resultant of the remaining two equations in β_1 to obtain $(e_2^3 - e_1 e_2 + 1) \alpha_2 = 0$. If $\alpha_2 = 0$, we obtain that $\beta_1 = 0$ as well, which is a case of the previously solved condition. If we instead take $e_2^3 - e_1 e_2 + 1 = 0$, we obtain the condition $\beta_1 = -\alpha_2 \lambda_2^2$, which is a special case of the previously solved condition, so we can always take $\beta_1 = -\frac{\alpha_2}{e_2}$.

Next, we assume $\gamma_2 \neq 0$ and take $\mu_1 = 0$. Doing this yields

$$\gamma_2^4 \left[\gamma_2^2 + A (CF - DE)^2 \beta_1^4 \right] = 0, \tag{A.117}$$

which demands $\gamma_2 = 0$, a contradiction. We then assume $\gamma_2 \neq 0$, $\mu_1 \neq 0$, and take $\mu_1 = \frac{\gamma_2}{e_2}$. This requires $\beta_1 = -\frac{\alpha_2}{e_2}$, which is also sufficient. Finally, we assume $\mu_1 e_2 - \gamma_2 \neq 0$ in addition to $\gamma_2 \neq 0$ and $\mu_1 \neq 0$, and take $\beta_1 = -\frac{\alpha_2 \mu_1}{\gamma_2}$. This demands

$$\left(\mu_1 - \lambda_1^2 \gamma_2 \right) \left(\mu_1 - \lambda_2^2 \gamma_2 \right) = 0. \tag{A.118}$$

Making the first factor vanish reveals $\lambda_1^2 = \lambda_2^2$ or $\lambda_2^2 = \frac{1}{\lambda_1^4}$. The first case ultimately requires $\lambda_2^2 = 1$, but this also means $e_2 = 1$, and hence $\mu_1 e_2 - \gamma_2 = 0$, a contradiction. In the case where $\lambda_2^2 = \frac{1}{\lambda_1^4}$, this contradiction is immediate. The steps are identical for the case where $\mu_1 = \lambda_2^2 \gamma_2$, and so we have exhausted this branch of solutions.

Next, we take $\mu_2 = \gamma_1 = 0$. This returns

$$\alpha_2^3 + I_2\alpha_2^2\beta_1 + I_1\alpha_2\beta_1^2 + \beta_1^3 = 0, \tag{A.119}$$

which requires $\nu = -\frac{\beta_1}{\alpha_2}$ be an eigenvalue of $\mathbf{b}$, or $\alpha_2 = \beta_1 = 0$. In the first case, we obtain a sufficient condition $\mu_1 = \frac{\gamma_2}{e_2}$. If this condition is not met, we can take the resultant of the remaining two equations in γ_2 and obtain the necessary condition

$$\left(e_2^3 - e_1e_2 + 1\right)\mu_1 = 0. \tag{A.120}$$

Taking $\mu_1 = 0$ yields $\gamma_2 = 0$. Taking $e_2^3 - e_1e_2 + 1 = 0$, i.e., $\lambda_1^2 = \frac{1}{\lambda_2^4}$ yields $\mu_1 = \gamma_2\lambda_2^2 = \frac{\gamma_2}{e_2}$, again obtaining the previous sufficient condition. Finally, if $\alpha_2 = \beta_1 = 0$, we obtain

$$\mu_1^3 - I_1\gamma_2\mu_1^2 + I_2\gamma_2^2\mu_1 - \gamma_2^3 = 0, \tag{A.121}$$

which demands that $\mu_1 = \frac{\gamma_2}{e_2}$, since γ_2 cannot vanish without leaving the anomalous solution branch. This exhausts all solution branches of this family and reveals all solutions to be cases of the anomalous solution

$$\begin{aligned} \beta_1 &= -\frac{\alpha_2}{e_2}, \\ \beta_2 &= -\frac{\alpha_1}{e_2}, \\ \mu_1 &= \frac{\gamma_2}{e_2}, \\ \mu_2 &= \frac{\gamma_1}{e_2}. \end{aligned} \tag{A.122}$$

This lets us solve for $M^{11}(r)$, and finally obtain the anomalous solution

$$p(r) = \gamma_1 + \frac{\gamma_2}{r^2}, \tag{A.123}$$

$$q(r) = \alpha_1 r^2 + \alpha_2, \tag{A.124}$$

$$M^{12}(r) = \frac{Dq(r) - Fp(r)}{\sqrt{r^2 - B}}, \tag{A.125}$$

$$M^{13}(r) = \frac{Ep(r) - Cq(r)}{\sqrt{r^2 - B}}, \tag{A.126}$$

$$M^{11}(r) = r^2 \frac{e_1 \pm \sqrt{e_1^2 - 4\left[e_2 + A\left(CF - DE\right)^2\left(p(r)^2 + \frac{q(r)^2}{r^2}\right)\right]}}{2A\left(r^2 - B\right)}, \tag{A.127}$$

$$\begin{aligned} M^{22}(r) = &\frac{\left(Dr^2p(r) + Fq(r)\right)^2 + e_1e_2r^2\left(Dq(r) - Fp(r)\right)^2}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2r^2\right)} \\ &- \frac{Ae_2\left(Fp(r) - Dq(r)\right)^2\left(r^2 - B\right)M^{11}(r)}{e_2\left(CF - DE\right)^2 r^2\left(q(r)^2 + p(r)^2r^2\right)}, \end{aligned} \tag{A.128}$$

$$M^{23}(r) = \frac{Ae_2 \left(Ep(r) - Cq(r)\right) \left(Fp(r) - Dq(r)\right) \left(r^2 - B\right) M^{11}(r)}{e_2 \left(CF - DE\right)^2 r^2 \left(q(r)^2 + p(r)^2 r^2\right)}$$

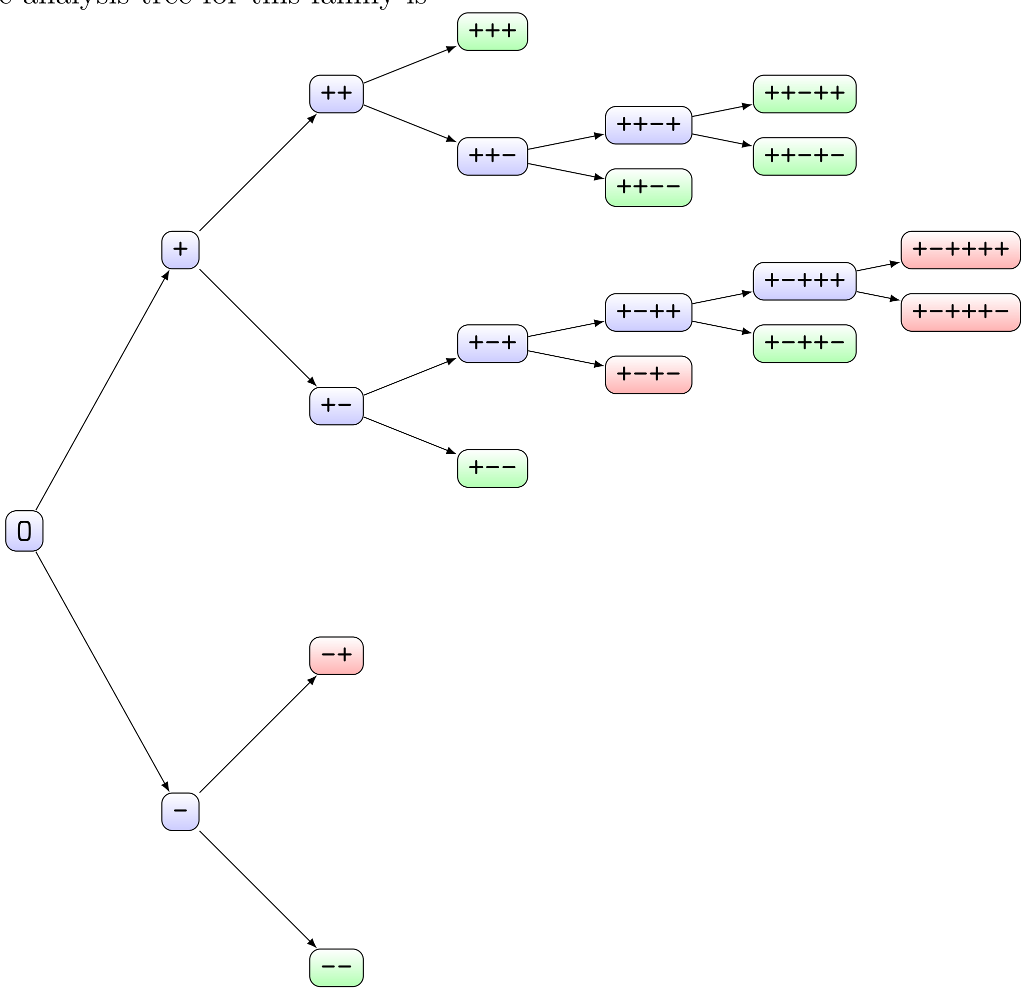
$$- \frac{e_1 \left(Cq(r) - Ep(r)\right) \left(Dq(r) - Fp(r)\right)}{\left(CF - DE\right)^2 \left(q(r)^2 + p(r)^2 r^2\right)} \tag{A.129}$$

$$- \frac{\left(Cp(r) r^2 + Eq(r)\right) \left(Dp(r) r^2 + Fq(r)\right)}{e_2 \left(CF - DE\right)^2 r^2 \left(q(r)^2 + p(r)^2 r^2\right)}, \tag{A.130}$$

$$M^{33}(r) = \frac{\left(Cr^2 p(r) + Eq(r)\right)^2 + e_1 e_2 r^2 \left(Cq(r) - Ep(r)\right)^2}{e_2 \left(CF - DE\right)^2 r^2 \left(q(r)^2 + p(r)^2 r^2\right)}$$

$$- \frac{Ae_2 \left(Ep(r) - Cq(r)\right)^2 \left(r^2 - B\right) M^{11}(r)}{e_2 \left(CF - DE\right)^2 r^2 \left(q(r)^2 + p(r)^2 r^2\right)}. \tag{A.131}$$

The analysis tree for this family is



Nodes:

- 0: $\alpha_1^3 + I_2 \alpha_1^2 \beta_2 + I_1 \alpha_1 \beta_2^2 + \beta_2^3 = 0$
- −: $\left(\beta_1 e_2 + \alpha_2\right) \left(e_2^3 - e_1 e_2 + 1\right) = 0$

- $+$: $\gamma_1^3 - I_2\gamma_1^2\mu_2 + I_1\mu_2^2\gamma_1 - \mu_2^3 = 0$
- $--$: $\mu_1 = \frac{\gamma_2}{e_2}$
- $-+$: $\beta_1 = -\alpha_2 \neq -\alpha_2$
- $+-$: $\gamma_2\mu_1 \left(e_2\mu_1 - \gamma_2\right)\left(\beta_1\gamma_2 + \alpha_2\mu_1\right) = 0$
- $++$: $\beta_1^3 + I_1\beta_1^2\alpha_2 + I_2\beta_1\alpha_2^2 + \alpha_2^3 = 0$
- $+--$: $\mu_1 = 0$ and $\beta_1 = -\frac{\alpha_2}{e_2}$
- $+-+$: $\mu_1 \left(e_2\mu_1 - \gamma_2\right)\left(\beta_1\gamma_2 + \alpha_2\mu_1\right) = 0$
- $++-$: $\left(\mu_1 e_2 - \gamma_2\right)\left(...\right)_1 = 0$ and $\left(\mu_1 e_2 - \gamma_2\right)\left(...\right)_2 = 0$
- $+++$: $\mu_1 = \frac{\gamma_2}{e_2}$
- $+-+-$: $\gamma_2 = 0 \neq 0$
- $+-++$: $\left(e_2\mu_1 - \gamma_2\right)\left(\beta_1\gamma_2 + \alpha_2\mu_1\right) = 0$
- $++--$: $\mu_1 = \frac{\gamma_2}{e_2}$
- $++-+$: $\left(e_2^3 - e_1e_2 + 1\right)\mu_1 = 0$
- $+-++-$: $\beta_1 = -\frac{\alpha_2}{e_2}$
- $+-+++$: $\left(\mu_1 - \lambda_1^2\gamma_2\right)\left(\mu_1 - \lambda_2^2\gamma_2\right) = 0$
- $++-+-$: $\mu_1 = \gamma_2 = 0$
- $++-++$: $\mu_1 = \frac{\gamma_2}{e_2}$
- $+-+++-$: $\mu_1 = \frac{\gamma_2}{e_2} \neq \frac{\gamma_2}{e_2}$
- $+-++++$: $\mu_1 = \frac{\gamma_2}{e_2} \neq \frac{\gamma_2}{e_2}$

Edges (labelled by child node):

- $-$: $\nu = -\frac{\beta_2}{\alpha_1} \Rightarrow \mu_2 = \frac{\gamma_1}{e_2}$
- $+$: $\alpha_1 = \beta_2 = 0$
- $--$: $\beta_1 = -\frac{\alpha_2}{e_2}$
- $-+$: $\beta_1 \neq -\frac{\alpha_2}{e_2}$ and $e_2^3 - e_1e_2 + 1 = 0 \Rightarrow \lambda_a^2 = 1$

- $+-$: $\nu = \frac{\mu_2}{\gamma_1}$
- $++$: $\mu_2 = \gamma_1 = 0$
- $+--$: $\gamma_2 = 0$
- $+-+$: $\gamma_2 \neq 0$
- $++-$: $\nu = \frac{-\beta_1}{\alpha_2}$
- $+++$: $\beta_1 = \alpha_2 = 0 \Rightarrow \mu_1^3 - I_1\gamma_2\mu_1^2 + I_2\gamma_2^2\mu_1 - \gamma_2^3 = 0$
- $+-+-$: $\mu_1 = 0$
- $+-++$: $\mu_1 \neq 0$
- $++--$: $\mu_1 e_2 - \gamma_2 = 0$
- $++-+$: $\text{Res}_{\gamma_2}\left((...)_1, (...)_2\right) = 0$
- $+-++-$: $\mu_1 = \frac{\gamma_2}{e_2}$
- $+-+++$: $\mu_1 \neq \frac{\gamma_2}{e_2}$, $\mu_1 \neq 0$, and $\gamma_2 \neq 0$
- $++-+-$: $\mu_1 = 0$
- $++-++$: $\mu_1 \neq 0$
- $+-+++-$: $\mu_1 = \lambda_1^2\gamma_2$
- $+-++++$: $\mu_1 = \lambda_2^2\gamma_2$

A.4 Family 5

First addressing the equilibrium equations involving **b**, we obtain the equations

$$r^2 M^{13\prime\prime}(r) + rM^{13\prime}(r) - M^{13}(r) = 0, \tag{A.132}$$

$$3CM^{12}(r) + 3ABM^{11\prime}(r) + 5CrM^{12\prime}(r) + ABrM^{11\prime\prime}(r) + Cr^2M^{12\prime\prime}(r) = 0, \tag{A.133}$$

which simplify upon defining the auxiliary function $f = CrM^{12}(r) + ABM^{11}(r)$, which makes the second equilibrium equation $rf''(r) + 3f'(r) = 0$. These equations can be integrated to obtain

$$f(r) = \gamma_1 + \frac{\gamma_2}{r^2} \Rightarrow M^{12}(r) = \frac{\gamma_1 + \frac{\gamma_2}{r^2} - ABM^{11}(r)}{Cr}, \tag{A.134}$$

$$M^{13}(r) = \alpha_1 r + \frac{\alpha_2}{r}. \tag{A.135}$$

Next, we have the equilibrium equations derived from $\mathbf{c}$, which are

$$\begin{aligned} 8ABr^3\alpha_1^2 + 8Cr^3\alpha_1 M^{23}(r) + Cr^2\left(7r^2\alpha_1 + 3\alpha_2\right) M^{23\prime}(r) + \left(-3r^2\gamma_1 + \gamma_2\right) M^{33\prime}(r) \\ + Cr^3\left(r^2\alpha_1 + \alpha_2\right) M^{23\prime\prime}(r) - r\left(r^2\gamma_1 + \gamma_2\right) M^{33\prime\prime}(r) = 0, \end{aligned} \tag{A.136}$$

$$\begin{aligned} &- 8AB\alpha_2\gamma_2 - 8C^2r^6\alpha_1 M^{22}(r) + A^2B^2r^3\left(3r^2\alpha_1 - \alpha_2\right) M^{11\prime}(r) \\ &\quad - C^2r^5\left(7r^2\alpha_1 + 3\alpha_2\right) M^{22\prime}(r) + Cr^3\left(3r^2\gamma_1 - \gamma_2\right) M^{23\prime}(r) \\ &\quad + A^2B^2r^4\left(r^2\alpha_1 + \alpha_2\right) M^{11\prime\prime}(r) - C^2r^6\left(r^2\alpha_1 + \alpha_2\right) M^{22\prime\prime}(r) \\ &\quad + Cr^4\left(r^2\gamma_1 + \gamma_2\right) M^{23\prime\prime}(r) = 0. \end{aligned} \tag{A.137}$$

These equations can be integrated to obtain

$$\left(r^2\gamma_1 + \gamma_2\right) M^{33}(r) - Cr^2\left(r^2\alpha_1 + \alpha_2\right) M^{23}(r) = r^2\mu_1 + \mu_2 + ABr^4\alpha_1^2, \tag{A.138}$$

$$\begin{aligned} C^2r^4\left(r^2\alpha_1 + \alpha_2\right) M^{22}(r) - Cr^2\left(r^2\gamma_1 + \gamma_2\right) M^{23}(r) = A^2B^2r^2\left(r^2\alpha_1 + \alpha_2\right) M^{11}(r) \\ + r^4\beta_1 + r^2\beta_2 - AB\alpha_2\gamma_2, \end{aligned} \tag{A.139}$$

which coupled with the constant trace condition

$$E^2r^2M^{33}(r) + c^2r^4M^{22}(r) = -A^2\left(1 - B^2\right) M^{11}(r) - 2AB\left(r^2\gamma_1 + \gamma_2\right) + I_1 r^2, \tag{A.140}$$

lets us solve for $M^{22}(r)$, $M^{23}(r)$, and $M^{33}(r)$ in terms of undetermined constants, and $M^{11}(r)$. The determinant of this system is $-C^3r^6\left[E^2r^2\left(r^2\alpha_1 + \alpha_2\right)^2 + \left(r^2\gamma_1 + \gamma_2\right)^2\right]$, which is identically nonzero, so we can always invert these equations. Denoting $\gamma = \gamma_1 r^2 + \gamma_2$, $\alpha = \alpha_1 r^2 + \alpha_2$, $\mu = \mu_1 r^2 + \mu_2$, and $\beta = \beta_1 r^2 + \beta_2$, we obtain

$$\begin{aligned} M^{22}(r) = &\frac{\gamma^2\left(I_1 r^2 - 2AB\gamma\right) + E^2r^2\left(r^2\alpha\beta - AB\left(\alpha^2\gamma + \alpha_2^2\gamma + \alpha\alpha_2\left(\gamma_2 - 2\gamma\right)\right) - \gamma\mu\right)}{C^2r^4\left(E^2r^2\alpha^2 + \gamma^2\right)} \\ &+ \frac{A^2\left(B^2 - \frac{\gamma^2}{E^2r^2\alpha^2 + \gamma^2}\right) M^{11}(r)}{C^2r^2}, \end{aligned} \tag{A.141}$$

$$\begin{aligned} M^{23}(r) = &\frac{-AB\left(E^2r^2\alpha\left(\alpha - \alpha_2\right)^2 + \gamma\left(2\alpha\gamma - \alpha_2\gamma_2\right)\right) + r^2\left(I_1\alpha\gamma - \beta\gamma - E^2\alpha\mu\right)}{Cr^2\left(E^2r^2\alpha^2 + \gamma^2\right)} \\ &- \frac{A^2\alpha\gamma M^{11}(r)}{CE^2r^2\alpha^2 + C\gamma^2}, \end{aligned} \tag{A.142}$$

$$\begin{aligned} M^{33}(r) = &\frac{I_1 r^2\alpha^2 - r^2\alpha\beta - AB\alpha^2\gamma - 2AB\alpha\alpha_2\gamma + AB\alpha_2^2\gamma + AB\alpha\alpha_2\gamma_2 + \gamma\mu}{E^2r^2\alpha^2 + \gamma^2} \\ &- \frac{A^2r^2\alpha^2 M^{11}(r)}{E^2r^2\alpha^2 + \gamma^2}. \end{aligned} \tag{A.143}$$

This leaves us with the incompressibility condition, and the constant second invariant of $\mathbf{b}$ to satisfy. As in the other cases, these equations are quadratic in $M^{11}(r)$, and we compute their resultant in $M^{11}(r)$, factor each coefficient in r and demand they all vanish independently.

The first of these is the equation

$$E^6\left[(\beta_1+AB\alpha_1\gamma_1)^3-I_1(\beta_1+AB\alpha_1\gamma_1)^2\alpha_1+I_2(\beta_1+AB\alpha_1\gamma_1)\alpha_1^2-\alpha_1^3\right]^2=0, \tag{A.144}$$

which implies that either $\nu=\frac{\beta_1+AB\alpha_1\gamma_1}{\alpha_1}$ is an eigenvalue of $\mathbf{b}$, or $\alpha_1=0$ and $\beta_1=0$. In the first case, we immediately obtain $\mu_1=2AB\alpha_1\alpha_2+\frac{\gamma_1}{E^2e_2}$, which upon substitution yields

$$\left(e_2^3-e_1e_2+1\right)\left(e_2\beta_2-\alpha_2+ABe_2\left(\alpha_2\gamma_1+\alpha_1\gamma_2\right)\right)=0, \tag{A.145}$$

which requires $\beta_2=\frac{\alpha_2}{e_2}-AB\left(\alpha_2\gamma_1+\alpha_1\gamma_2\right)$, or if not, $\lambda_1^2=\frac{1}{\lambda_2^4}$. Tackling this latter case first, after substitution we demand $\lambda_2^2=1$ to avoid a contradiction and obtain

$$\mu_2=\frac{-\alpha_2\gamma_1+\beta_2\gamma_1+\alpha_1\gamma_2+AB\left[E^2\alpha_1\alpha_2^2+\gamma_1\left(\alpha_2\gamma_1+\alpha_1\gamma_2\right)\right]}{E^2\alpha_1}. \tag{A.146}$$

This reduces the remaining equations to requiring $\beta_2=\frac{\alpha_2}{e_2}-AB\left(\alpha_2\gamma_1+\alpha_1\gamma_2\right)$, so we can immediately consider this case. One of the remaining equations then implies

$$\gamma_1\left(ABE^2e_2\alpha_2^2+\gamma_2-E^2e_2\mu_2\right)\left(e_2^3-e_1e_2+1\right)=0. \tag{A.147}$$

Taking $\mu_2=\frac{ABE^2e_2\alpha_2^2+\gamma_2}{E^2e_2}$ is sufficient, so then we consider the case where $\mu_2\neq\frac{ABE^2e_2\alpha_2^2+\gamma_2}{E^2e_2}$. With this, we first take $\gamma_1=0$, but this leads immediately to a contradiction, so we then consider $e_2^3-e_1e_2+1=0$. However, this case also immediately leads to a contradiction, so we next consider $\beta_1=\alpha_1=0$. This yields the equation

$$E^6\mu_1^3-I_1\gamma_1E^4\mu_1^2+I_2\gamma_1^2E^2\mu_1-\gamma_1^3=0, \tag{A.148}$$

which implies that $\nu=\frac{E^2\mu_1}{\gamma_1}$ is an eigenvalue of $\mathbf{b}$, or $\gamma_1=\mu_1=0$. In the first case, we the take discriminants in e_1 and demand non-negativity to obtain

$$\begin{aligned}\gamma_2\left(AB\alpha_2^2-\mu_2\right)&\left(ABE^2e_2\alpha_2^2+\gamma_2-E^2e_2\mu_2\right)\\&\times\left(ABE^2e_2\alpha_2^3+\beta_2\gamma_2+AB\alpha_2\gamma_1\gamma_2-E^2\alpha_2\mu_2\right)=0.\end{aligned} \tag{A.149}$$

Examining these factors one at a time, we first consider $\gamma_2=0$. With this, we get the equation

$$\left(AB\alpha_2^2-\mu_2\right)\left[A^2\alpha_2^4+\left(AB\alpha_2^2-\mu_2\right)^2\right]=0, \tag{A.150}$$

hence, we take $\mu_2 = AB\alpha_2^2$. Inserting this yields either the sufficient condition

$$\beta_2 = \frac{\alpha_2}{e_2} - AB\alpha_2\gamma_1, \tag{A.151}$$

or $\beta_2 \neq \frac{\alpha_2}{e_2} - AB\alpha_2\gamma_1$, in which case we can take the resultant of the remaining nonzero factors in β_2 to require

$$\alpha_2 \left(e_2^3 - e_1 e_2 + 1\right) = 0. \tag{A.152}$$

If we take $\alpha_2 = 0$ we obtain $\beta_2 = 0$, which is a special case of the sufficient condition above. If we take $e_2^3 - e_1 e_2 + 1 = 0$, we ultimately see that the sufficient condition is also necessary. Considering $\gamma_2 \neq 0$, we then take $\mu_2 = AB\alpha_2^2$. However, substituting this yields an expression that is positive-definite in γ_2 being equal to 0, so we have a contradiction.

Next, considering $\gamma_2 \neq 0$ and $\mu_2 \neq AB\alpha_2^2$, we take $ABE^2 e_2 \alpha_2^2 + \gamma_2 - E^2 e_2 \mu_2 = 0$. Doing this reveals the necessary and sufficient condition $\beta_2 = \frac{\alpha_2}{e_2} - AB\alpha_2\gamma_1$. In the last case, we have

$$\beta_2 = \frac{-ABE^2\alpha_2^3 - AB\alpha_2\gamma_1\gamma_2 + E^2\alpha_2\mu_2}{\gamma_2}, \tag{A.153}$$

which implies either the sufficient condition

$$\mu_2 = \frac{ABE^2 e_2 \alpha_2^2 + \gamma_2}{E^2 e_2}, \tag{A.154}$$

or if this condition is not satisfied, we can take resultants in μ_2 with the remaining nonzero factors to obtain the necessary condition

$$\left(e_1^2 - 4e_2\right)\left(e_2^3 - e_1 e_2 + 1\right) = 0, \tag{A.155}$$

which requires either $\lambda_1^2 = \lambda_2^2$ or $\lambda_1^2 = \frac{1}{\lambda_2^4}$, i.e., $\mathbf{b}$ has a repeated eigenvalue. If we take $\lambda_1^2 = \lambda_2^2$, and look for different values of μ_2, we obtain

$$\mu_2 = \frac{ABE^2\alpha_2^2 + \gamma_2\lambda_2^2}{E^2}. \tag{A.156}$$

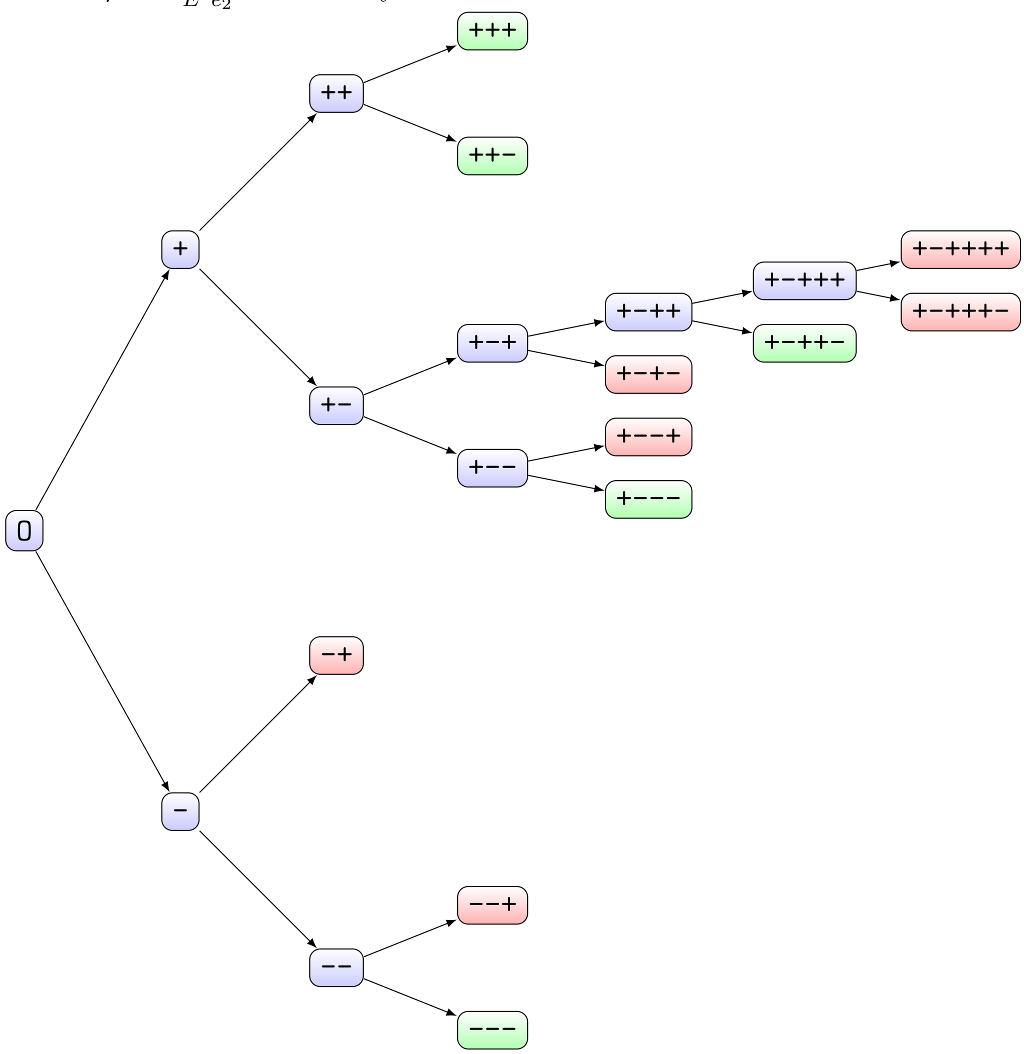
Substituting this into the remaining equations, we see that either $\lambda_2^2 = 1$ or $\alpha_2 = 0$, both of which reduce this value of μ_2 to the previous sufficient condition. Taking $\lambda_1^2 = \frac{1}{\lambda_2^4}$ and looking for different solutions immediately requires $\alpha_2 = 0$, which then further implies $\mu_2 = \frac{\gamma_2}{E^2\lambda_2^4}$. This, however, demands $\lambda_2^2 = 1$, and we realize that the previous sufficient condition was also necessary.

Next, considering $\mu_1 = \gamma_1 = 0$, we obtain the eigenvalue equation $\beta_2^3 - I_1\alpha_2\beta_2^2 + I_2\alpha_2^2\beta_2 - \alpha_2^3 = 0$, which demands $\mathbf{b}$ have the eigenvalue $\nu = \frac{\beta_2}{\alpha_2}$, or $\beta_2 = \alpha_2 = 0$. In the first case, we obtain the sufficient condition

$$\mu_2 = \frac{ABE^2 e_2\alpha_2^2 + \gamma_2}{E^2 e_2}. \tag{A.157}$$

Taking the resultant of the remaining factors yields $\gamma_2\left(e_2^3 - e_1e_2 + 1\right) = 0$, which has solutions $\gamma_2 = 0$ or $\lambda_1^2 = \frac{1}{\lambda_2^4}$. Taking $\gamma_2 = 0$ shows that in this case the sufficient condition is also necessary, and taking $\lambda_1^2 = \frac{1}{\lambda_2^4}$ reveals that the sufficient condition is necessary in all cases.

Finally, we take $\alpha_2 = \beta_2 = 0$. This reveals only one remaining equation $E^6\mu_2^3 - I_1\gamma_2E^4\mu_2^2 + I_2\gamma_2^2E^2\mu_2 - \gamma_2^3 = 0$, which, since $\gamma_2 \neq 0$, requires the eigenvalue $\nu = \frac{E^2\mu_2}{\gamma_2}$, and hence $\mu_2 = \frac{\gamma_2}{E^2 e_2}$. The analysis tree is then



Nodes:

- 0: $\left(\beta_1 + AB\alpha_1\gamma_1\right)^3 - I_1\left(\beta_1 + AB\alpha_1\gamma_1\right)^2\alpha_1 + I_2\left(\beta_1 + AB\alpha_1\gamma_1\right)\alpha_1^2 - \alpha_1^3 = 0$

- $-$: $(e_2^3 - e_1 e_2 + 1)(e_2 \beta_2 - \alpha_2 + ABe_2(\alpha_2\gamma_1 + \alpha_1\gamma_2)) = 0$
- $+$: $(E^2\mu_1)^3 - I_1 (E^2\mu_1)^2 \gamma_1 + I_2 (E^2\mu_1) \gamma_1^2 - \gamma_1^3 = 0$
- $--$: $(E^2 e_2 \mu_2 - ABE^2 e_2 \alpha_2^2 - \gamma_2)(e_2^3 - e_1 e_2 + 1) = 0$
- $-+$: $\beta_2 = \frac{\alpha_2}{e_2} - AB(\alpha_2\gamma_1 + \alpha_1\gamma_2) \neq \frac{\alpha_2}{e_2} - AB(\alpha_2\gamma_1 + \alpha_1\gamma_2)$
- $+-$: $\gamma_2 (AB\alpha_2^2 - \mu_2)(ABE^2 e_2 \alpha_2^2 + \gamma_2 - E^2 e_2 \mu_2)$
  $(ABE^2 e_2 \alpha_2^3 + \beta_2\gamma_2 + AB\alpha_2\gamma_1\gamma_2 - E^2\alpha_2\mu_2) = 0$
- $++$: $\beta_2^3 - I_1 \alpha_2 \beta_2^2 + I_2 \alpha_2 \beta_2 - \alpha_2^3 = 0$
- $---$: $\mu_2 = \frac{ABE^2 e_2 \alpha_2^2 + \gamma_2}{E^2 e_2}$
- $--+$: $\mu_2 = \frac{ABE^2 e_2 \alpha_2^2 + \gamma_2}{E^2 e_2} \neq \frac{ABE^2 e_2 \alpha_2^2 + \gamma_2}{E^2 e_2}$
- $+--$: $AB\alpha_2^2 - \mu_2 = 0$ and $(\beta_2 e_2 - \alpha_2 + ABe_2\alpha_2\gamma_1)(...)_a = 0$
- $+-+$: $\gamma_2 = 0 \neq 0$
- $++-$: $\mu_2 = \frac{ABE^2 e_2 \alpha_2^2 + \gamma_2}{E^2 e_2}$
- $+++$: $\mu_2 = \frac{\gamma_2}{E^2 e_2}$
- $+---$: $\beta_2 = \frac{\alpha_2}{e_2} - AB\alpha_2\gamma_1$
- $+--+$: $\alpha_2 (e_2^3 - e_1 e_2 + 1) = 0$
- $+-+-$: $\gamma_2 = 0 \neq 0$
- $+-++$: $(ABE^2 e_2 \alpha_2^2 + \gamma_2 - E^2 e_2 \mu_2)$
  $\times (ABE^2 e_2 \alpha_2^3 + \beta_2\gamma_2 + AB\alpha_2\gamma_1\gamma_2 - E^2\alpha_2\mu_2) = 0$
- $+-++-$: $\beta_2 = \frac{\alpha_2}{e_2} - AB\alpha_2\gamma_1$
- $+-+++$: $(e_1^2 - 4e_2)(e_2^3 - e_1 e_2 + 1) = 0$
- $+-+++-$: $ABE^2 e_2 \alpha_2^2 + \gamma_2 - E^2 e_2 \mu_2 = 0 \neq 0$
- $+-++++$: $ABE^2 e_2 \alpha_2^2 + \gamma_2 - E^2 e_2 \mu_2 = 0 \neq 0$

Edges (labelled by child node):

- $-$: $\nu = \frac{\beta_1 + AB\alpha_1\gamma_1}{\alpha_1}$

- $+$: $\alpha_1 = \beta_1 = 0$
- $--$: $\beta_2 = \frac{\alpha_2}{e_2} - AB\left(\alpha_2\gamma_1 + \alpha_1\gamma_2\right)$
- $-+$: $\lambda_1^2 = \frac{1}{\lambda_2^4}$
- $+-$: $\nu = \frac{E^2\mu_1}{\gamma_1}$
- $++$: $\gamma_1 = \mu_1 = 0$
- $---$: $\mu_2 = \frac{ABE^2e_2\alpha_2^2+\gamma_2}{E^2e_2}$
- $--+$: $\mu_2 \neq \frac{ABE^2e_2\alpha_2^2+\gamma_2}{E^2e_2}$ and $e_2^3 - e_1e_2 + 1 = 0$
- $+--$: $\gamma_2 = 0$
- $+-+$: $\gamma_2 \neq 0$
- $++-$: $\nu = \frac{\beta_2}{\alpha_2}$
- $+++$: $\beta_2 = \alpha_2 = 0$
- $+---$: $\mu_2 = AB\alpha_2^2$ and $\beta_2 = \frac{\alpha_2}{e_2} - AB\alpha_2\gamma_1$
- $+--+$: $\mathrm{Res}_{\beta_2}\left(\left(...\right)_a, \left(...\right)_b\right) = 0$
- $+-+-$: $\mu_2 = AB\alpha_2^2$
- $+-++$: $\mu_2 \neq AB\alpha_2^2$
- $+-++-$: $ABE^2e_2\alpha_2^2 + \gamma_2 - E^2e_2\mu_2 = 0$
- $+-+++$: $ABE^2e_2\alpha_2^2 + \gamma_2 - E^2e_2\mu_2 \neq 0$
- $+-+++-$: $e_1^2 = 4e_2$
- $+-++++$: $e_2^3 - e_1e_2 + 1 = 0$

These are all special cases of the solution

$$\mu_1 = 2AB\alpha_1\alpha_2 + \frac{\gamma_1}{E^2e_2}, \tag{A.158}$$

$$\mu_2 = \frac{ABE^2e_2\alpha_2^2 + \gamma_2}{E^2e_2}, \tag{A.159}$$

$$\beta_1 = \frac{\alpha_1}{e_2} - AB\alpha_1\gamma_1, \tag{A.160}$$

$$\beta_2 = \frac{\alpha_2}{e_2} - AB\left(\alpha_2\gamma_1 + \alpha_1\gamma_2\right), \tag{A.161}$$

which lets us solve for $M^{11}(r)$ and obtain the anomalous solution

$$f(r) = \gamma_1 + \frac{\gamma_2}{r^2}, \tag{A.162}$$

$$M^{13}(r) = \alpha_1 r + \frac{\alpha_2}{r}, \tag{A.163}$$

$$M^{11}(r) = \frac{e_1 \pm \sqrt{e_1^2 - 4\left(e_2 + A^2 f(r)^2 + A^2E^2M^{13}(r)^2\right)}}{2A^2}, \tag{A.164}$$

$$M^{12}(r) = \frac{f(r) - ABM^{11}(r)}{Cr}, \tag{A.165}$$

$$M^{22}(r) = \frac{e_2 f(r)^2\left(e_1 + A^2\left(B^2-1\right)M^{11}(r)\right)}{C^2r^2e_2\left(f(r)^2 + E^2M^{13}(r)^2\right)} + \frac{E^2\left(1 + A^2B^2e_2M^{11}(r)\right)M^{13}(r) - 2ABe_2 f(r)\left(f(r)^2 + E^2M^{13}(r)^2\right)}{C^2r^2e_2\left(f(r)^2 + E^2M^{13}(r)^2\right)}, \tag{A.166}$$

$$M^{23}(r) = -\frac{ABM^{13}(r)}{Cr} - \frac{M^{13}(r)f(r)\left(1 - e_1e_2 + A^2e_2M^{11}(r)\right)}{Ce_2r\left(f(r)^2 + E^2M^{13}(r)^2\right)}, \tag{A.167}$$

$$M^{33}(r) = \frac{f(r)^2 + E^2e_2\left(e_1 - A^2M^{11}(r)\right)M^{13}(r)^2}{E^2e_2\left(f(r)^2 + E^2M^{13}(r)^2\right)}. \tag{A.168}$$

References

S. Adeleke. On ericksen's problem of elasticity. *Mechanics Research Communications*, 11(1):21 – 27, 1984. ISSN 0093-6413. doi: https://doi.org/10.1016/0093-6413(84)90090-9. URL http://www.sciencedirect.com/science/article/pii/0093641384900909.

S. S. Antman. *Nonlinear problems of elasticity*. Springer Verlag, New York, 1995.

M. Ben Amar and A. Goriely. Growth and instability in elastic tissues. *Journal of the Mechanics and Physics of Solids*, 53:2284–2319, 2005.

B. A. Bilby, R. Bullough, and E. Smith. Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc. Roy. Soc. Lond. A*, 231(1185):263–273, 1955.

J. Casey and P. M. Naghdi. A Remark on the Use of the Decomposition $F = F_e F_p$ in Plasticity. *Journal of Applied Mechanics*, 47(3):672–675, 09 1980.

M. Epstein and G. A. Maugin. Geometric material structure of finite-strain elasticity and anelasticity. *ZAMM*, 76:125–128, 1996a.

M. Epstein and G. A. Maugin. On the geometrical material structure of anelasticity. *Acta Mech.*, 115(1):119–131, 1996b.

J. L. Ericksen. Deformations possible in every isotropic, incompressible, perfectly elastic body. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 5(6):466–489, 1954.

J. L. Ericksen. Deformations possible in every compressible, isotropic, perfectly elastic material. *Studies in Applied Mathematics*, 34(1-4):126–128, 1955.

J. D. Eshelby. The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the Royal Society of London A*, 241(1226):376–396, 1957.

R. L. Fosdick. Remarks on Compatibility. *Modern Developments in the Mechanics of Continua*, pages 109–127, 1966.

R. L. Fosdick and K. W. Schuler. On Ericksen's problem for plane deformations with uniform transverse stretch. *International Journal of Engineering Science*, 7 (2):217–233, 1969.

C. F. Gauss. *Disquisitiones generales circa superficies curvas.* Typis Ditericianis, 1828.

A. Golgoon and A. Yavari. Nonlinear elastic inclusions in anisotropic solids. *Journal of Elasticity*, 130(2):239–269, 2018.

A. Goriely. *The Mathematics and Mechanics of Biological Growth*, volume 45. Springer, 2017.

A. Green and P. Naghdi. A general theory of an elastic-plastic continuum. *Archive for Rational Mechanics and Analysis*, 18, 01 1965.

J. Hill. On static similarity deformations for isotropic materials. *Quarterly of Applied Mathematics*, 40:287–291, 1982.

E. Hubert. Differential algebra for derivations with nontrivial commutation rules. *Journal of Pure and Applied Algebra*, 200:163–190, 2005.

C. B. Kafadar. On Ericksen's problem. *Archive for Rational Mechanics and Analysis*, 47(1):15–27, 1972.

W. W. Klingbeil and R. T. Shield. On a class of solutions in plane finite elasticity. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 17(4):489–511, 1966.

J. K. Knowles. Universal states of finite anti-plane shear: Ericksen's problem in miniature. *The American Mathematical Monthly*, 86(2):109–113, 1979.

K. Kondo. A proposal of a new theory concerning the yielding of materials based on Riemannian geometry. *The Journal of the Japan Society of Aeronautical Engineering*, 2(8):29–31, 1949.

J. M. Lee. *Introduction to Smooth Manifolds.* Springer, 2001.

A. Marris. Universal deformations in incompressible isotropic elastic materials. *Journal of Elasticity*, 5(2):111–128, 1975.

A. Marris. Two new theorems on Ericksen's problem. *Archive for Rational Mechanics and Analysis*, 79(2):131–173, 1982.

A. Marris and J. Shiau. Universal deformation in isotropic incompressible hyperelastic materials when the deformation tensor has equal proper values. *Archive for Rational Mechanics and Analysis*, 36:135–160, 1970.

S. E. Martin and D. E. Carlson. A note on Ericksen's problem. *Journal of Elasticity*, 6(1):105–108, 1976.

T. Mura. *Micromechanics of Defects in Solids.* Martinus Nijhoff, 1982.

J. Nash. The imbedding problem for Riemannian manifolds. *Annals of Mathematics*, 63(1):20–63, 1956.

V. E. Naumov. Mechanics of growing deformable solids: a review. *Journal of Engineering Mechanics*, 120(2):207–220, 1994.

W. Noll. Materially uniform simple bodies with inhomogeneities. *Arch. Rat. Mech. Anal*, 27(1):1–32, 1967.

J. F. Nye. Some geometrical relations in dislocated crystals. *Acta metallurgica*, 1(2): 153–162, 1953.

R. W. Ogden. *Non-Linear Elastic Deformations.* Dover, 1984.

A. Ozakin and A. Yavari. A geometric theory of thermal stresses. *Journal of Mathematical Physics*, 51(3):032902, 2010.

T. J. Pence and H. Tsai. Swelling-induced microchannel formation in nonlinear elasticity. *IMA journal of applied mathematics*, 70(1):173–189, 2005.

T. J. Pence and H. Tsai. Swelling-induced cavitation of elastic spheres. *Mathematics and mechanics of solids*, 11(5):527–551, 2006.

E. K. Rodriguez, A. Hoger, and A. D. McCulloch. Stress-dependent finite growth in soft elastic tissues. *Journal of Biomechanics*, 27(455-467), 1994.

S. Sadik and A. Yavari. On the origins of the idea of the multiplicative decomposition of the deformation gradient. *Mathematics and Mechanics of Solids*, 22:771–772, 2017.

J. C. Simo and J. E. Marsden. Stress tensors, Riemannian metrics and the alternative descriptions in elasticity. In *Trends and Applications of Pure Mathematics to Mechanics*, pages 369–383. Springer, 1984.

M. Singh and A. C. Pipkin. Note on ericksen's problem. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 16(5):706–709, 1965.

F. Sozio and A. Yavari. Riemannian and Euclidean material structures in anelasticty. *Mathematics and Mechanics of Solids*, 2019. doi: 10.1177/1081286519884719.

R. Stojanović, S. Djurić, and L. V. sević. On finite thermal deformations. *Arch. Mech. Stosow*, 16:103–108, 1964.

K. Takamizawa. Stress-free configuration of a thick-walled cylindrical model of the artery - an application of riemann geometry to the biomechanics of soft tissues. *Journal of Applied Mechanics*, 58:840–842, 1991.

G. Templet and D. Steigmann. On the theory of diffusion and swelling in finitely deforming elastomers. *Mathematics and Mechanics of Complex Systems*, 1, 02 2013. doi: 10.2140/memocs.2013.1.105.

Y. Ueda, K. Fukuda, K. Nakacho, and S. Endo. A new measuring method of residual stresses with the aid of finite element method and reliability of estimated values. *Transactions of JWRI*, 4(2):123–131, 1975.

T. Waititi. *Thor: Ragnarok*. United States: Marvel Studios., 2017.

A. Yavari. A geometric theory of growth mechanics. *Journal of Nonlinear Science*, 20(6):781–830, 2010.

A. Yavari and A. Goriely. Riemann–Cartan geometry of nonlinear dislocation mechanics. *Archive for Rational Mechanics and Analysis*, 205(1):59–118, 2012.

A. Yavari and A. Goriely. Nonlinear elastic inclusions in isotropic solids. *Proc. R. Soc. A*, 469(2160):20130415, 2013.

A. Yavari and A. Goriely. The twist-fit problem: finite torsional and shear eigenstrains in nonlinear elastic solids. *Proc. R. Soc. A*, 2015.

A. Yavari and A. Goriely. The anelastic Ericksen problem: Universal eigenstrains and deformations in compressible isotropic elastic solids. *Proc. R. Soc. A*, 472(2196): 20160690, 2016.

A. Yavari, C. Goodbrake, and A. Goriely. Universal displacements in linear elasticity. *Journal of the Mechanics and Physics of Solids*, 135:103782, 2020. ISSN 0022-5096. doi: https://doi.org/10.1016/j.jmps.2019.103782.