Ricci solitons and geometric analysis

Matthias Wink

Green Templeton College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2018

To my parents

Acknowledgements

First and foremost I wish to thank my supervisor Andrew Dancer for his continuing support which has guided me through my DPhil studies. Many thanks for constant availability, the hours of discussions and invaluable comments on my work. Many thanks for the opportunities that I had - my mathematical adventures in different areas of geometry and the lectureship at Jesus College.

Many thanks to Stuart Hall for inviting me to Buckingham and Newcastle, for numerical experiments and discussions about Ricci flow. Many thanks to Christoph Böhm and Burkhard Wilking for inviting me back to Münster, for a stimulating discussion and for pointing out proposition 5.1.3.

I thank the Mathematical Institute of Oxford University for providing an ideal working environment, for bringing together so many researchers and research interests, and for opening up opportunities for discussions.

In particular, I thank Dominic Joyce, Yng-Ing Lee and Wei-Bo Su for introducing me to Lagrangian Mean Curvature flow and for sharing their insights.

I am grateful to Melanie Rupflin for suggesting the topic of uniform Poincaré inequalities to me, for a productive collaboration and especially for her time, advice and encouragement.

The gardens of Green Templeton College have been a place of recreation for me. I thank Michael Pirie for his dedication to keep them so beautiful and for the many interesting conversations we have had over the years during my afternoon breaks.

Special thanks go to my friends here in Oxford. To Renée, Lucas and Thomas for a productive, supportive and friendly atmosphere in the office. To Alejandro, Claudia, Claudio, Elia, Giles, Simon and Tom for countless lunches and hours in pubs. To my housemates Adriana, Cleo, Elise, Fei, Felix, Isabel, Jake, Jean-Francois and Pengjie at Banbury Road 49 and Nikhil and Ankit at RAC 47 and my neighbour Ben, as well as to everyone else who I have met along the way - all of you have made my time in Oxford so much more enjoyable.

I gratefully acknowledge the financial support which I have received from the Oxford-Aarhus QGM collaboration, the Studienstiftung des deutschen Volkes e.V. and the EPSRC. Oxford University, Green Templeton College and conference funds have brought me to Chicago, Warwick, Münster, London, Zürich and Toronto.

Above all, I thank my parents, without whose support none of this would have been possible.

Abstract

This thesis studies Ricci solitons of cohomogeneity one and uniform Poincaré inequalities for differentials on Riemann surfaces.

In the two summands case, which assumes that the isotropy representation of the principal orbit G/K consists of two inequivalent Ad_K-invariant irreducible summands, complete steady and expanding Ricci solitons have been detected numerically by Buzano-Dancer-Gallaugher-Wang [BDGW15b, BDGW15a]. This work provides a rigorous construction thereof.

A Lyapunov function is introduced to prove that the Ricci soliton metrics lie in a bounded region of an associated phase space. This also gives an alternative construction of non-compact Einstein metrics of non-positive scalar curvature due to Böhm [Böh99]. It is explained how the asymptotics of the Ricci flat trajectories induce Böhm's [Böh98] Einstein metrics on spheres and other low dimensional spaces. A numerical study suggests that all other Einstein metrics of positive scalar curvature which are induced by the generalised Hopf fibrations occur in an entirely non-linear regime of the Einstein equations.

Extending the theory of cohomogeneity one steady and expanding Ricci solitons, an estimate which allows to prescribe the growth rate of the soliton potential at any given time is shown. As an application, continuous families of Ricci solitons on complex line bundles over products of Fano Kähler Einstein manifolds are constructed. This generalises work of Appleton [App18] and Stolarski [Sto17]. The method also applies to the Lü-Page-Pope set-up [LPP04] and allows to cover an optimal parameter range in the two summands case.

The Ricci soliton equation on manifolds foliated by torus bundles over products of Fano Kähler Einstein manifolds is discussed. A rigidity theorem is obtained and a preserved curvature condition is discovered.

The cohomogeneity one initial value problem is solved for m-quasi-Einstein metrics and complete metrics are described.

L^p-Poincaré inequalities for k-differentials on closed Riemann surfaces are shown. The estimates are uniform in the sense that the Poincaré constant only depends on $p \geq 1$, $k \geq 2$ and the genus $\gamma \geq 2$ of the surface but not on its complex structure. Examples show that the analogous estimate for 1-differentials cannot be uniform. This part is based on joint work with Melanie Rupflin.

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I also declare that the work contained in this thesis has not been submitted towards any other degree at this institution or at any other institution.

Chapter 3 and section 6.2.3 are based on my arXiv preprint [Win17a], chapter 5 is based on my arXiv preprint [Win17b].

Chapter 7 is based on joint work with Melanie Rupflin.

Matthias Wink

Oxford, 24th June 2018

Contents

List of Figures

Chapter 1

Introduction

This thesis studies *cohomogeneity one Ricci solitons and Einstein metrics,* with a particular emphasis on the construction of complete non-Kähler examples.

A different topic in geometric analysis, namely *uniform L^p-Poincaré inequalities for differentials on Riemann surfaces*, is discussed in an additional chapter. This is based on joint work with Melanie Rupflin.

1.1 Motivation and Contents

A *Ricci soliton* is a Riemannian manifold (M, g) together with a smooth vector field X on M satisfying the equation

$$\operatorname{Ric} + \frac{1}{2} L_X g + \frac{\varepsilon}{2} g = 0$$

for some $\varepsilon \in \mathbb{R}$, where $L_X g$ denotes the Lie derivative of the metric. It is called *trivial* if X is a Killing vector field and *gradient* if $X = \operatorname{grad} u$ for some $u \in C^\infty(M)$, which implies $\frac{1}{2} L_X g = \operatorname{Hess} u$. If M has a complex structure such that g is Kähler and X is holomorphic, then M is called a *Kähler Ricci soliton.* As Einstein manifolds evolve by the scaling $g(t) = (1 + \varepsilon t) g$ under Hamilton's Ricci flow, Ricci solitons are called *shrinking*, *steady* or *expanding* depending on whether $\varepsilon < 0$, $\varepsilon = 0$, or $\varepsilon > 0$.

More precisely, Ricci solitons characterise *self-similar* solutions to the Ricci flow, i.e. Ricci flows which evolve the metric only by scaling and by a family of diffeomorphisms, and they are of particular importance because all known *singularity models* in Ricci flow are Ricci solitons.

Ricci solitons are also natural generalisations of Einstein metrics and are hence candidates for a canonical metric on a manifold. For example, there are Kähler manifolds without any Kähler Einstein metric but which have a Kähler Ricci soliton structure, cf. [WZ04].

Though Ricci solitons have been studied extensively since they were introduced by Hamilton, the number of examples is still limited, in particular outside of the class of Kähler manifolds. In fact, all *known* examples of non-trivial shrinking Ricci solitons are either Kähler, the shrinking Gaussian, or decompose into products of Riemannian manifolds of smaller dimensions, and the task of constructing a compact non-trivial *non*-Kähler shrinking Ricci soliton has often been raised, cf. [Cao10], [Der12].

This thesis studies the construction problem for Ricci solitons and Einstein metrics under a cohomogeneity one symmetry assumption, which simplifies the Ricci soliton equation by turning it into a singular ordinary differential equation. With regard to an ansatz with an even higher degree of symmetry one has to note that non-trivial homogeneous Ricci solitons are either expanding, non-compact and non-gradient or shrinking and finitely covered by a product of an Einstein manifold and the shrinking Gaussian, cf. [Nab10], [PW09a], [Jab15]. Furthermore, only gradient Ricci solitons will be considered because non-trivial compact Ricci solitons must be shrinking [Ham88], [Ive93] and of gradient type [Per02].

1.2 Main results and discussion

The first main theorem of this thesis establishes the existence of complete steady and expanding Ricci soliton metrics on spaces which inherit a cohomogeneity one structure from the generalised Hopf fibrations.

Theorem 1.2.1 *On $CaP^2 \setminus \{ \text{ point } \}$, $\mathbb{H}P^{m+1} \setminus \{ \text{ point } \}$ for $m \geq 1$ and on the vector bundle associated to $(G, H, K) = (Sp(m+1), Sp(1) \times Sp(m), U(1) \times Sp(m))$ for $m \geq 3$, there exist a 1-parameter family of non-homothetic complete steady and a 2-parameter family of non-homothetic complete expanding Ricci solitons.*

The steady Ricci solitons are asymptotically conical whereas the expanding Ricci solitons are asymptotically paraboloid.

Notice that indeed these spaces are diffeomorphic to open disc bundles $G \times_H D^{d_S+1}$ associated to group diagrams $K \subset H \subset G$ where the naturally induced Riemannian submersions $\pi\colon G/K \to G/H,\ gK \mapsto gH$ are the generalised Hopf fibrations, see [Bes87]. Furthermore, they are examples of the *two summands case*, that is, the isotropy representation of the principal orbit G/K splits into two inequivalent Ad_K-invariant irreducible real summands. Hence, away from the singular orbit, the metric can be described by

$$g_{M\setminus Q} = dt^2 + f_1(t)^2 g_S + f_2(t)^2 g_Q,$$

where $S = H/K$ is the collapsing sphere and $Q = G/H$ is the singular orbit. The corresponding Ricci soliton equations take the form

$$\frac{d}{dt}\frac{\dot{f}_1}{f_1} = -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}_1}{f_1} + \frac{\varepsilon}{2} + \frac{A_1}{d_1}\frac{1}{f_1^2} + \frac{A_3}{d_1}\frac{f_1^2}{f_2^4}, \tag{1.1}$$

$$\frac{d}{dt}\frac{\dot{f}_2}{f_2} = -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}_2}{f_2} + \frac{\varepsilon}{2} + \frac{A_2}{d_2}\frac{1}{f_2^2} - 2\frac{A_3}{d_2}\frac{f_1^2}{f_2^4}. \tag{1.2}$$

The constants d_1, d_2 are the dimensions of the collapsing sphere and of the singular orbit, respectively. In geometric applications one has $A_1 = d_1(d_1 - 1)$, $A_2 = d_2 \operatorname{Ric}^Q$ and A_3 is the norm of an O'Neill tensor associated to the Riemannian submersion $G/K \to G/H$. In the following discussion, d_1, $d_2 > 1$, A_1, $A_2 > 0$, and $A_3 \geq 0$ shall be assumed.

It is important to note that the examples in theorem 1.2.1 in fact satisfy $A_3 > 0$.

To construct a complete metric, long time existence of solutions needs to be established. The key step is to obtain good control on the components $\dot{f}_i/f_i$ of the shape operator. Due to (1.1), positivity of $\dot{f}_1/f_1$ is preserved if $\varepsilon \geq 0$, which corresponds to the case of steady and expanding Ricci solitons.

However, as $A_3 > 0$ holds in the situation of theorem 1.2.1, a similar statement for $\dot{f}_2/f_2$ does not follow automatically. In fact, there are configurations for the parameters d_i and A_i such that solutions to (1.1), (1.2) must blow up in finite time and thus necessarily satisfy $\dot{f}_2/f_2 < 0$ after some time. (If $\varepsilon \geq 0$, a maximal solution to the cohomogeneity one Einstein or Ricci soliton equation with positive definite shape operator induces a complete metric.)

In previous work, this issue did not arise because the doubly warped product type examples of Ivey [Ive94], Gastel-Kronz [GK04] and Dancer-Wang [DW09a, DW09b] all satisfy $A_3 = 0$.

Notice on the other hand that if the estimate

$$\frac{f_1}{f_2} < \sqrt{\frac{A_2}{2A_3}}$$

holds along the entire solution, then positivity of $\dot{f}_2/f_2$ is again preserved and a complete metric is induced.

This thesis gives two approaches to control the quotient

$$\omega = \frac{f_1}{f_2}.$$

The first approach is based on a Lyapunov function, which is specific to the two summands case, and **proposition 3.2.5** shows that if $d_1 > 1$, $A_1 = d_1(d_1 - 1)$ and $(d_1 + 1)A_2^2 > 4d_1 d_2(2d_1 + d_2)A_3 > 0$, then $\omega^2 < \hat{\omega}_1^2 < \frac{A_2}{2A_3}$ for a constant $\hat{\omega}_1 > 0$.

Moreover, this argument carries over to the Einstein case and the Einstein metrics of non-positive scalar curvature due to Böhm [Böh99] on the spaces in theorem 1.2.1 are recovered, see **proposition 3.2.9** and **corollary 3.2.12**.

Remark 1.2.2 More generally, the argument can be adjusted to yield m-quasi-Einstein metrics, which are metrics that satisfy the curvature condition

$$\mathrm{Ric} + \mathrm{Hess}\, u - \frac{1}{m} du \otimes du + \frac{\varepsilon}{2} g = 0$$

for a smooth function u and $m \in (0, \infty]$. In particular, **theorem 6.2.19** provides an analogue of theorem 1.2.1 for m-quasi-Einstein metrics.

In order to obtain a smooth metric from a solution to the two summands Ricci soliton ODE, the smoothness conditions

$$\begin{aligned} f_1(0) = 0,\ \dot{f}_1(0) = 1 \ \text{ and } \ f_2(0) = \bar{f} > 0,\ \dot{f}_2(0) = 0, \\ u(0) = 0,\ \dot{u}(0) = 0,\ (d_1 + 1)\ddot{u}(0) = C \leq 0 \end{aligned} \tag{1.3}$$

are imposed. However, notice that these are *singular* initial conditions for the Ricci soliton equations (1.1), (1.2), but short time existence of solutions to this initial value problem follows from the work of Buzano [Buz11].

To resolve this coordinate singularity, one may consider the rescaled variables

$$X_i = \frac{1}{-\dot{u} + \mathrm{tr}(L)} \frac{\dot{f}_i}{f_i} \ \text{ and } \ Y_i = \frac{1}{-\dot{u} + \mathrm{tr}(L)} \frac{1}{f_i}$$

in the time slice defined by $\frac{d}{ds} = \frac{1}{-\dot{u} + \mathrm{tr}(L)} \frac{d}{dt}$. Then the smoothness conditions (1.3) correspond to the stationary point

$$X_1 = Y_1 = \frac{1}{d_1} \ \text{ and } \ X_2 = Y_2 = 0$$

of the rescaled Ricci soliton ODE.

Notice that these conditions are independent of the parameter $\bar{f} = f_2(0) > 0$. In the Einstein case, this observation is used to analyse the behaviour of the solutions as the volume of the singular orbit tends to zero, i.e. as $\bar{f} \to 0$. In the rescaled coordinate system one obtains a well-defined limit trajectory, which moreover must solve the (rescaled) Ricci flat equation. A simple analysis of the long time behaviour then shows that as $\bar{f} \to 0$ a solution to the two summands Einstein equation converges to a *cone solution*, that is a solution of the form

$$f_i(t) = c_i \sin(t) \quad \text{for} \quad t \in (0, \pi) \quad \text{or}$$
$$f_i(t) = c_i t \quad \text{and} \quad f_i(t) = c_i \sinh(t) \quad \text{for} \quad t > 0,$$

for positive constants $c_i > 0$, depending on whether the Einstein constant is positive, zero or negative. That is, the metrics are desingularisations of the cone solutions.

In particular, **proposition 3.5.2** gives a different approach to the convergence results [Böh98, Theorem 5.7] and [Böh99, Theorem 11.1] due to Böhm, which are crucial for Böhm's construction of Einstein metrics of positive scalar curvature on low dimensional spheres and other low dimensional spaces in [Böh98] and his Einstein metrics with non-positive scalar curvature on the spaces in theorem 1.2.1 in [Böh99].

Remark 1.2.3 In his original proof, Böhm uses a different coordinate system to construct a limit trajectory as $\bar{f} \to 0$, which is in particular a solution to a planar ODE system. Then the Poincaré-Bendixson Theorem can be applied. In contrast, in the Ricci soliton case, the extra degree of freedom of the Ricci soliton potential does not allow this reduction and thus theorem 1.2.1 requires a different proof.

Furthermore, the new approach in this thesis also carries over to the case of complex line bundles over a single Fano Kähler Einstein manifold, which is not covered by Böhm's work because the associated cone solution is unstable. **Theorem 3.3.4** gives the details and theorem 1.2.4 below discusses a more general result in the case of Ricci solitons.

A computationally simplified construction of Böhm's Einstein metrics is included in this thesis though the overall strategy of the construction due to Böhm [Böh98] remains unchanged.

Furthermore, Böhm [Böh98] could also construct an Einstein metric of positive scalar curvature on $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$ by studying the linearisation of the Einstein equation along the cone solution and found numerical evidence for the existence of two symmetric Einstein metrics of positive scalar curvature on $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$ for $m \geq 3$.

In an independent numerical analysis, the long time behaviour of the solutions to the associated initial value problem (1.3) is studied with a particular emphasis on the dependence on the dimensional parameter m and the approximation of the cone solution in the limit as $\bar{f} \to 0$. However, both studies suggest that the solutions which correspond to Einstein metrics on $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$, $m \geq 3$, are very far away from the cone solutions and their existence may not be deduced from the dynamics of the Einstein equation close to the cone solutions. In this sense, these metrics occur in an entirely non-linear regime of the Einstein equations.

Similar statements can be made for symmetric Einstein metrics on $F^m \# \overline{F}^m$ for $m \geq 2$. These spaces are diffeomorphic to two disc bundles $G \times_H D^{d_S+1}$ with opposite orientations which are glued together along their common boundary G/K in the case $(G, H, K) = (Sp(m+1), Sp(1) \times Sp(m), U(1) \times Sp(m))$.

A second approach to obtain control on the quotient $\omega = f_1/f_2$ uses estimates on the soliton potential u and is thus specific to the Ricci soliton case.

In the example of the two summands case, ω satisfies the ODE

$$\frac{d^2}{dt^2}\omega = \dot{\omega}\left\{\frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2} - (-\dot{u} + \operatorname{tr}(L))\right\} + \frac{\omega}{f_1^2} f(\omega),$$

where $f(\omega) = \frac{A_1}{d_1} - \frac{A_2}{d_2}\omega^2 + A_3\left(\frac{1}{d_1} + \frac{2}{d_2}\right)\omega^4$. If $f(\omega) \leq 0$ then this ODE can be integrated explicitly and one can deduce that

$$\dot{\omega}(t) \leq \frac{1}{\bar{f}} \exp(u(t) - u(t_*))$$

for $t \geq t_*$ and as long as $\omega < \sqrt{\frac{A_2}{2A_3}}$, which in practice will clearly hold at t_*. In particular, the growth of ω is controlled by e^u. Thus if u decays fast enough, ω can never exceed the bound $\sqrt{\frac{A_2}{2A_3}}$.

However, tautologically, previous estimates on the long time behaviour of the soliton potential are not helpful as the solution may not even correspond to a complete metric. In contrast, given any $\tau > t_*$ and $c > 0$, **theorem 5.1.1** guarantees $-\dot{u}(t) \geq c$ for $t \geq \tau$ provided that $\ddot{u}(0)$ is sufficiently negative. Moreover, a similar estimate applies to any local solution to the steady or expanding Ricci soliton equation on a cohomogeneity one manifold with a singular orbit provided that the metric stays in diagonal form, the shape operator of the principal orbit remains positive definite and there is a uniform bound on the norm of the Killing form $-\operatorname{tr}_{g_{t_*}}(B_{|\mathfrak{p}})$ at t_*. (Positive definiteness of the shape operator is natural in the sense that it holds for all currently known examples.)

With this technique, the missing low dimensional examples in theorem 1.2.1 of the two summands case can be settled, see **theorem 5.2.1**. In fact, according to **proposition 5.2.4**, the methods apply in an optimal parameter range, that is, as long as the associated cone solutions are real valued. This is indeed a necessary condition as the asymptotics of the metric are related to the cone solutions.

As a further application, **theorem 5.2.9** shows that the examples of Kähler Ricci solitons due to Cao [Cao96], Feldman-Ilmanen-Knopf [FIK03] and Dancer-Wang [DW11] occur amongst continuous families of non-Kähler Ricci solitons:

Theorem 1.2.4 *Let L be the total space of a complex line bundle over a product of Fano Kähler Einstein manifolds with $m \geq 1$ factors whose Euler class is a rational linear combination of the first Chern classes of the base manifolds. Then there exist an m-parameter family of complete non-trivial steady and an $(m+1)$-parameter family of complete non-trivial expanding gradient Ricci solitons on L.*

This also generalises results of [App18] and Stolarski [Sto17] who construct steady Ricci solitons on line bundles over a single Fano Kähler Einstein manifold.

Furthermore, Lü-Page-Pope [LPP04] have constructed explicit Einstein metrics on warped products of spheres with total spaces of complex line bundles over Fano Kähler Einstein manifolds whose Euler class is a rational multiple of the first Chern class of the base manifold. **Theorem 5.2.23** shows that again there are continuous families of complete steady and expanding non-Kähler Ricci solitons on these spaces.

In contrast to the above examples, the Ricci soliton equation on manifolds which are foliated by r-torus bundles over products of Fano Kähler Einstein manifolds for $r \geq 2$ does not preserve the diagonal ansatz for the metric. In this case the rigidity results of **proposition 6.1.7** say that the specific ansatz that was used by Cao [Cao96], Feldman-Ilmanen-Knopf [FIK03] and Dancer-Wang [DW11] in the case of circle bundles does not generalise to torus bundles.

Recall that on a Kähler manifold, the Kähler condition is preserved by the Ricci soliton ODE. In a similar spirit, **proposition 6.1.9** describes a curvature condition which is preserved by the Ricci soliton ODE on torus bundles over a single Fano Kähler Einstein manifold, which implies long time existence provided that the shape operator becomes positive definite. Explicit examples due to Chen [Che11] of Ricci flat metrics show that there are indeed trajectories that satisfy this condition.

For an introduction to and results on *uniform L^p-Poincaré inequalities for k-differentials on closed hyperbolic surfaces*, the reader is directed to chapter 7, which is self-contained and is based on joint work with Melanie Rupflin.

1.3 Structure of this thesis

Section 1.4 gives a more detailed literature review. It explains how a Ricci soliton induces a self-similar Ricci flow and how Ricci solitons arise as singularity models for the Ricci flow. Perelman's no local collapsing theorem and monotonicity formulas are stated.

Constructions of Kähler Ricci solitons via complex Monge-Ampère equations and K-stability as well as the Calabi ansatz with applications to Ricci flows through singularities are discussed. In the non-Kähler case, the dynamical systems approach, which is based on a cohomogeneity one assumption, is reviewed.

Rigidity theorems based on convergence results for the Ricci flow, symmetry assumptions and curvature assumptions are recalled.

Chapter 2 summarises the existing theory of cohomogeneity one Ricci solitons. It recalls the topological and geometric structure of cohomogeneity one manifolds and sets up the notation that will be used throughout the thesis. Particular emphasis is given to singular orbits as these will always be assumed to exist. The gradient Ricci soliton equation as well as Hamilton's conservation law are formulated as ordinary differential equations. The solution of the corresponding initial value problem at a singular orbit due to Buzano is explained and it is shown that the soliton potential of steady and expanding Ricci solitons is monotonic and often concave along the flow of the Ricci soliton ODE. If the solitons are complete, estimates on the scalar curvature and the volume growth imply restrictions on the asymptotic behaviour. Finally, the Böhm functional, an important monotonic quantity of the cohomogeneity one Ricci soliton equations, is introduced.

In chapter 3 the Ricci soliton equation is studied under the assumption that the isotropy representation of the principal orbit decomposes into two inequivalent irreducible summands. This is known as the two summands case. Geometric examples are induced by the generalised Hopf fibrations and include in particular $\mathbb{H}P^m \setminus \{ \text{ point } \}$ and complex line bundles over Fano Kähler Einstein manifolds.

As in previous work, the coordinate singularity of the Ricci soliton equation is smoothed by a rescaling with the inverse of the mean curvature. In the case of warped products, Dancer-Wang have observed that Hamilton's conservation law provides an priori estimate that implies long time existence. However, this is not the case in geometrically more complicated situations, but in joint work with Buzano and

Gallaugher, they have found numerical evidence for the existence of complete Ricci solitons in these examples too. Chapter 3 gives a rigorous construction thereof.

With the help of a Lyapunov function it is shown that in many instances the steady and expanding Ricci soliton trajectories lie in a bounded region of phase space and thus induce complete metrics. In fact, the arguments carry over to the Einstein case and provide a different construction of examples due to Böhm. Böhm's approach on the other hand is based on a reduction to a planar system and the Poincaré-Bendixson theorem. Due to the extra degree of freedom of the soliton potential, this approach does not carry over to the Ricci soliton case.

An analysis of the asymptotic behaviour of the metrics is given. It is explained how the asymptotics of the Ricci flat metrics induce Einstein metrics of positive scalar curvature, which simplifies Böhm's construction of Einstein metrics on spheres and products of spheres.

Chapter 4 contains a numerical study of the two summands case with regard to the existence of Einstein metrics of positive scalar curvature. An analysis of the matching condition to symmetrically glue together two disc bundles with opposite orientations recovers all previously detected solutions and suggests that these are in fact all symmetric cohomogeneity one Einstein metrics on these spaces. A long time analysis of the solution is performed to study the equations in a parameter range where the qualitative behaviour of the solution changes. By considering a suitable rescaling of the associated Ricci flat trajectories, one can study how the trajectories corresponding to smooth metrics with a small singular orbit approach the first cone solution, a canonical solution with conical singularities at the singular orbit. However, both studies suggest that the numerically detected Einstein metrics on $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$ for $m \geq 3$ and $F^m \# \overline{F}^m$ for $m \geq 2$ occur in an entirely non-linear regime of the Einstein equations. That is, these solutions seem to be too far away from the cone solutions in order to construct them via an analysis of the dynamics of the Einstein equations close to the cone solutions.

Chapter 5 introduces a new estimate on the soliton potential for steady and expanding cohomogeneity one Ricci solitons, generalising an observation of Appleton for steady Ricci solitons on complex line bundles over a single Fano Kähler Einstein manifold. It allows to prescribe the growth rate of the soliton potential provided that the metric remains in diagonal form and the shape operator of the principal orbit is positive definite.

As an application, continuous families of Ricci solitons on complex line bundles over products of Fano Kähler Einstein manifolds are constructed. The two summands case is revisited and the new method applies in an optimal parameter range. In particular, all examples which are induced by the Hopf fibrations are covered. Complete Ricci solitons are also constructed in the set-up of Lü-Page-Pope, who have discovered explicit Ricci flat metrics on warped products of total spaces of complex line bundles with spheres.

Chapter 6 discusses generalisations of the ideas of the previous chapters to two different, related geometries.

In the first part, the Ricci soliton equation on manifolds foliated by r-torus bundles over products of Fano Kähler Einstein manifolds is considered. In this case the metric does not remain in diagonal form along the flow of the Ricci soliton ODE if $r \geq 2$. This class of manifolds generalises the set-up of Cao, Feldman-Ilmanen-Knopf and Dancer-Wang who provide Ricci solitons in the case of circle bundles. In their examples, a common coordinate change which goes back to Bérard-Bergery ensures that the soliton potential and the scaling functions of the base manifolds become linear.

However, it is shown that this ansatz does not yield any expanding Ricci solitons if $r \geq 2$ and steady Ricci solitons cannot exist either if the manifold is foliated by a 2-torus bundle over a single Fano Kähler Einstein manifold and the singular orbit is a non-trivial circle bundle.

In contrast, Chen [Che11] has found Einstein metrics in the case of 2-torus bundles over a single Fano Kähler Einstein manifold by assuming that the scaling function of the base manifold is linear. A more general curvature condition which is satisfied by solutions of this type and which is preserved by the flow of the Ricci soliton ODE is described.

The second part considers m-quasi-Einstein metrics, which have gained attention due to their close relation to Einstein warped products. A separate introduction to this topic, including a literature review, is given at the beginning of the chapter. In the cohomogeneity one case, the initial value problem at a singular orbit is solved, following the work of Buzano and Eschenburg-Wang, and the methods from chapter 3 are adapted to the two summands m-quasi-Einstein case. Examples of multiple warped product type are also discussed.

Chapter 7 covers a different topic in geometric analysis and is based on joint work with Melanie Rupflin. The main theorem establishes uniform L^p-Poincaré estimates for k-differentials on compact hyperbolic surfaces if $k \geq 2$. That is, the Poincaré constant only depends on $p \geq 1$, $k \geq 2$ and the genus of the Riemann surface but not on the complex structure. The case $p = 1$ and $k = 2$ is due to Rupflin-Topping [RT15] and the proof is based on ideas of that paper. On the other hand, examples show that in the case of 1-differentials the corresponding estimates cannot be uniform.

1.4 Ricci flow, examples and rigidity

This section gives a brief literature review and explains the role of Ricci solitons in Ricci flow, previous constructions of Ricci solitons and rigidity results.

1.4.1 Ricci solitons and Ricci flow

Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2 \operatorname{Ric}(g(t))$$

has been a central topic in geometric analysis since its introduction in [Ham82]. It features most prominently in Perelman's resolution of Thurston's geometrisation conjecture for 3-manifolds [Per02, Per03b], which includes the Poincaré conjecture as a special case [Per03a], or in the proof of the differentiable sphere theorem due to Brendle-Schoen [BS09].

Self-similar solutions are Ricci flows of the form

$$g(t) = \sigma(t) \varphi_t^* g(0), \tag{1.4}$$

where $\sigma(t)$ is a smooth positive function and $\varphi_t \colon M \to M$ is a 1-parameter family of diffeomorphisms. Hamilton [Ham88] has observed that self-similar Ricci flows are characterised by the Ricci soliton equation. Indeed, given a self-similar Ricci flow, define $X = \frac{\partial}{\partial t}\big|_{t=0} \varphi_t$ and $\varepsilon = \sigma'(0)$ to obtain the corresponding Ricci soliton. Conversely, set $\sigma(t) = 1 + \varepsilon t$ and φ_t to be the flow of $\frac{1}{1+\varepsilon t} X$. Then $\sigma(t)$ and φ_t induce a self-similar Ricci flow.

An important aspect of Ricci solitons is their appearance as blow-up limits in Ricci flow. For example, suppose that $(M, g(t))$ is a complete Ricci flow with maximal existence time $T < \infty$. Then the singularity is of type-I if there is a constant $C > 0$ such that

$$\sup_M |\operatorname{Rm}(g(t))|_{g(t)} \leq \frac{C}{T-t}$$

for all t, otherwise it is of type-II. In the case of type-I singularities, according to results of Sesum [Ses06], Naber [Nab10] and Enders-Müller-Topping [EMT11], there exists a suitable blow up sequence of Ricci flows which converges to a non-flat gradient shrinking Ricci soliton.

On the other hand, the known type-II singularities are modelled on the Bryant soliton [Bry05], which is the unique rotationally symmetric steady Ricci soliton on $\mathbb{R}^n$, $n \geq 3$. It indeed occurs as a type-II blow up limit of Ricci flow solutions due to the work of Gu-Zhu [GZ08], Angenent-Isenberg-Knopf [AIK11, AIK15] and Wu [Wu14].

An important structural result on the formation of singularities is Perelman's [Per02] no local collapsing theorem. A metric g on M is called κ-non-collapsed below the scale $\rho > 0$ if for all $p \in M$ and all $r \in (0, \rho)$ with $|\operatorname{Rm}| \leq \frac{1}{r^2}$ on $B(p,r)$ there holds

$$\frac{\operatorname{Vol}(B(p,r))}{r^n} \geq \kappa > 0.$$

If $(M, g(t))_{t\in[0,T)}$ is a Ricci flow on a compact manifold with $T < \infty$ and $\rho > 0$ then due to Perelman's result [Per02] there exists $\kappa = \kappa(n, g(0), T, \rho) > 0$ such that $g(t)$ is κ-non-collapsed below the scale ρ for all $t \in [0, T)$. This implies in particular that blow-up limits must be non-collapsed. For example, this rules out Hamilton's cigar as a blow up limit.

The proof of the no local collapsing theorem relies on monotonic quantities whose discovery has been a major breakthrough in Ricci flow theory. Recall that Perelman's $\mathcal{F}$-functional $\mathcal{F}(g,f) = \int_M (R + |\nabla f|^2) e^{-f} d\operatorname{Vol}$ satisfies

$$\frac{d}{dt}\mathcal{F}(g,f) = 2\int_M |\operatorname{Ric} + \operatorname{Hess} f|^2 e^{-f} d\operatorname{Vol} \geq 0$$

if g and f evolve according to the coupled PDE

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\operatorname{Ric}(g), \\ \frac{\partial}{\partial t} f &= -\Delta f + |\nabla f|^2 - R \end{aligned}$$

and thus is stationary precisely on gradient steady Ricci solitons.

Similarly, along any solution (g, f, τ) of

$$\begin{aligned}
\frac{\partial}{\partial t} g &= -2 \operatorname{Ric}(g), \\
\frac{\partial}{\partial t} f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\
\frac{\partial}{\partial t} \tau &= -1,
\end{aligned}$$

Perelman's $\mathcal{W}$-functional $\mathcal{W}(g, f, \tau) = \int_M (\tau(R + |\nabla f|^2) + f - n)(4\pi\tau)^{-\frac{n}{2}} e^{-f} d\operatorname{Vol}$ is monotonic, i.e.

$$\frac{d}{dt} \mathcal{W}(g, f, \tau) = 2\tau \int_M |\operatorname{Ric} + \operatorname{Hess} f - \frac{g}{2\tau}|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f} d\operatorname{Vol} \geq 0,$$

and is stationary precisely on gradient shrinking Ricci solitons.

1.4.2 Constructions of Ricci solitons

There are two main approaches to constructing Ricci solitons. The first uses Kähler geometry and its connection to complex Monge-Ampère equations and K-stability in algebraic geometry.

Due to a result of Tian-Zhu [TZ00] any Kähler Ricci soliton structure on a compact Kähler manifold is unique, if it exists. On the other hand, there are compact Kähler manifolds without any Kähler Ricci soliton structure, cf. [WZ04]. This follows from obstructions to Kähler Einstein metrics and from the non-existence of non-trivial holomorphic vector fields on these manifolds, cf. [Tia99].

Wang-Zhu [WZ04] have proved the existence of Kähler Ricci solitons on toric Kähler manifolds with positive first Chern class via the theory of complex Monge-Ampère equations. A different proof of their result has later been given by Donaldson [Don08]. New techniques have been introduced in the proof of the Donaldson-Tian-Yau conjecture due to Chen-Donaldson-Sun [CDS15a, CDS15b, CDS15c]. Datar and Székelyhidi [DS16] improved the result for Fano manifolds with a holomorphic action of a compact group G by characterising the existence of Kähler-Einstein metrics in terms of G-equivariant K-stability. Their proof uses a smooth continuity method which extends to the Kähler Ricci soliton situation. This has been used by Ilten-Süß [IS17] and Cable-Süß [CS18] to provide new examples of Kähler Einstein manifolds and Kähler Ricci solitons.

The second approach relies on symmetry assumptions. Interesting homogeneous examples have been found by Lauret [Lau01] and they are necessarily expanding, non-compact and non-gradient, cf. [Nab10], [PW09a], [Jab15].

In the cohomogeneity one case, the first examples rely on the Calabi ansatz and an explicit integration of the corresponding Kähler Ricci soliton equations.

The $U(1)$-invariant shrinking Kähler Ricci soliton on $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is due to Cao [Cao96] and Koiso [Koi90]. Furthermore, Cao [Cao96] has found $U(n)$-invariant steady Kähler Ricci solitons on $\mathbb{C}^n$ and on the blow-up of $\mathbb{C}^n/\mathbb{Z}^n$ at the origin. In [Cao97], Cao has also obtained the corresponding $U(n)$-invariant expanding Kähler Ricci soliton on $\mathbb{C}^n$. These examples have been complemented by Feldman-Ilmanen-Knopf [FIK03] who provided $U(n)$-invariant, non-compact shrinking and expanding Kähler Ricci solitons on non-trivial complex line bundles over $\mathbb{C}P^n$. In contrast, a result of Feldman-Ilmanen-Knopf [FIK03] says that any $U(n)$-invariant complete shrinking Kähler Ricci soliton on $\mathbb{C}^n$ is flat.

By concatenating the induced Ricci flows of the shrinking Ricci soliton on the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^n$ for $t < 0$ with the associated asymptotic cone on $\mathbb{C}^n \setminus \{0\}$ at $t = 0$ and the expanding Kähler Ricci soliton on $\mathbb{C}^n$ which is asymptotic to the same cone for $t > 0$, Feldman-Ilmanen-Knopf [FIK03] have obtained a Ricci flow which is smooth except for an isolated singularity at $t = 0$. Intuitively, the Ricci flow shrinks a $\mathbb{C}P^n$ in $\mathcal{O}(-1)$ to a point and smooths the singularity in a continuous way. This is an example of a phenomenon called Ricci flow through singularities. In this case, the effect is a blowdown in the sense of algebraic geometry and therefore the shrinking Ricci soliton on $\mathcal{O}(-1)$ over $\mathbb{C}P^n$ is often called blowdown soliton. Máximo [Máx14] and Isenberg-Knopf-Sesum [IKS18] have also shown that the blowdown soliton occurs as a singularity model of type-I singularities for Ricci flows on certain compact Kähler and non-Kähler manifolds, respectively.

This set of examples has been generalised in various directions and by several authors, cf. [Gua95], [CV96], [Li11], [DW11] or [Yan12]. For example, Li [Li11] has constructed Kähler Ricci solitons on the total space of direct sums of a fixed line bundle over a Fano Kähler Einstein manifold. These are also used in a construction of Kähler-Ricci flows through singularities due to Song [Son13], see also [ST17].

In [DW11], Dancer-Wang have initiated a systematic study of the cohomogeneity one Ricci soliton equation and as an application, they have given a unified approach to constructing expanding, steady and shrinking Kähler Ricci solitons on manifolds foliated by S^1- and $\mathbb{C}P^1$-bundles over products of Fano Kähler Einstein manifolds. This has also established a strong connection between this class of Ricci solitons and a construction of Einstein manifolds due to Wang-Wang [WW98]. Notice also that the existence of Einstein metrics on total spaces of complex line bundles over

Fano Kähler Einstein manifolds has already been discussed in the seminal work of Bérard-Bergery [BB82].

The unique rotationally symmetric steady Ricci soliton on $\mathbb{R}^2$ is known as Hamilton's cigar and it is also Kähler, but the Bryant soliton [Bry05], its higher dimensional analogue, is non-Kähler. Generalising these examples, Ivey [Ive94] has constructed steady Ricci solitons of doubly warped product type and the corresponding expanding Ricci solitons are due to Gastel-Kronz [GK04]. The case of multiple warped products has been covered in a series of papers by Dancer-Wang [DW09a, DW09b] and then also in joint work with Buzano and Gallaugher [BDGW15a, BDW15].

In contrast to the explicit Kähler Ricci solitons that arise from the Calabi ansatz, these examples are obtained via a qualitative study of the associated Ricci soliton ODE. A common feature in this approach is to carry out a coordinate change which traps the solutions in a bounded region of phase space to conclude long time existence. However, the geometrically simple structure of warped products is reflected by the fact that a known conservation law due to Hamilton [Ham95] already provides the required estimates. This technique extends to the Einstein case, where it recovers non-compact Einstein metrics of non-positive scalar curvature due to Böhm [Böh99]. Böhm's initial approach is based on the observation that certain canonical metrics with conical singularities are often attractors for the Einstein equations.

In the case where the isotropy representation of the principal orbit decomposes into two inequivalent Ad-invariant irreducible real summands, numerical evidence for the existence of continuous families of complete steady and expanding Ricci solitons on certain non-trivial vector bundles has been provided by Buzano-Dancer-Gallaugher-Wang [BDGW15b, BDGW15a]. These examples have been constructed analytically by the author in [Win17a], which also includes the first examples of non-Kähler steady and expanding Ricci solitons on complex line bundles over Fano Kähler Einstein manifolds. In the steady case, and with different methods, Appleton [App18] could extend the class of line bundles and in particular prove the existence of a non-collapsed Ricci soliton under topological constraints. The existence of complete, non-Kähler steady Ricci solitons on complex line bundles is independently discussed by Stolarski [Sto17]. A generalisation of this work to the expanding case has appeared in [Win17b].

Another important Ricci soliton is the *Gaussian* on $(\mathbb{R}^n, g_{\text{flat}})$. It shows that the same Riemannian manifold can be of expanding or shrinking type, depending on the choice of the soliton potential $u(x) = \pm\frac{|x|^2}{2}$.

1.4.3 Rigidity results

This thesis focusses on constructions of non-trivial gradient Ricci solitons. In contrast, this section discusses a partial converse to this problem. It collects rigidity results which illustrate that in many geometric situations Ricci solitons either have a special structure or must be Einstein.

With regard to compact Ricci solitons, Hamilton [Ham88] and Ivey [Ive93] have shown that compact gradient Ricci solitons must either be shrinking or Einstein, and that in dimensions $n = 2, 3$ they are always Einstein. Moreover, by the work of Perelman [Per02], any compact Ricci soliton is in fact of gradient type.

A celebrated result of Böhm-Wilking [BW08] asserts that the Ricci flow evolves any metric with positive curvature operator on a compact manifold to a limit metric of constant sectional curvature and thus compact shrinking Ricci solitons with positive curvature operator are space forms. This has been generalised by Brendle-Schoen [BS09] to manifolds whose product with $\mathbb{R}^2$ has positive isotropic curvature (PIC_2). This follows from their result that PIC_2 is a Ricci flow preserved curvature condition, which also implies the differential sphere theorem.

Recall that all *known* non-trivial compact shrinking Ricci solitons are either Kähler or decompose into products of Riemannian manifolds of smaller dimensions.

By answering a question of Perelman [Per02], Brendle [Bre13] has shown that any 3-dimensional, non-flat, κ-non-collapsed complete steady gradient Ricci soliton is rotationally symmetric and therefore, up to scaling, isometric to the Bryant soliton. In dimensions $n \geq 4$, Brendle [Bre14] could obtain the same conclusion for steady gradient Ricci solitons with positive sectional curvature which are asymptotically cylindrical.

As defined by Petersen-Wylie [PW09b], a gradient Ricci soliton is called rigid if it is isometric to a quotient of $N \times \mathbb{R}^k$ where N is Einstein with Einstein constant λ and $\mathbb{R}^k$ is equipped with the Euclidean metric and the soliton potential $\frac{\lambda}{2}|x|^2$. Petersen-Wylie [PW09a] have shown that any homogeneous gradient Ricci soliton is rigid. Non-rigid homogeneous Ricci solitons must in fact be expanding, non-compact and non-gradient, cf. [Nab10], [PW09a], [Jab15]. In the cohomogeneity one case, all complete non-compact shrinking gradient Ricci solitons with non-negative sectional curvature are rigid [PW09a].

In [PW09b], Petersen-Wylie have shown that a shrinking or expanding gradient Ricci soliton is rigid if and only if it has constant scalar curvature and is radially flat, that is $\sec(\cdot, \nabla u) = 0$. For example, these conditions are satisfied if the Ricci soliton has harmonic curvature tensor. In the steady case, it is already sufficient for rigidity that the scalar curvature achieves a minimum, in particular if it is constant. Furthermore, due to a result of Munteanu-Sesum [MS13], any complete gradient shrinking Ricci soliton with harmonic Weyl tensor is a finite quotient of $\mathbb{R}^n$, $S^{n-1} \times \mathbb{R}$ or S^n, see also [FLGaRo11] for the compact case. In particular, this applies to locally conformally flat gradient shrinking Ricci solitons. Similarly, Cao-Chen [CC13] have shown that any complete Bach flat shrinking gradient Ricci soliton of dimension $n \geq 4$ is either Einstein or a finite quotient of the Gaussian on $\mathbb{R}^n$ or of $N \times \mathbb{R}$, where N is an $(n-1)$-dimensional Einstein manifold with positive scalar curvature.

Chapter 2

Background

This chapter recalls the theory of cohomogeneity one gradient Ricci solitons with singular orbits.

It starts with a general description of cohomogeneity one manifolds in section 2.1, discusses properties of (possibly incomplete) solutions to the Ricci solution equation on non-compact cohomogeneity one manifolds with a singular orbit in sections 2.2 and 2.3, and recalls necessary curvature conditions and asymptotics of the soliton potential on complete Ricci solitons in section 2.4. Finally, section 2.5 reviews the Böhm functional, which is a useful tool to analyse the asymptotic behaviour of the metric.

2.1 Cohomogeneity one manifolds

A general concept for constructions of manifolds with prescribed curvature conditions is to impose symmetry assumptions in order to reduce the complexity of the equations.

However, with regard to *homogeneous* Ricci solitons, it is important to note that, by the work of Naber [Nab10] and Petersen-Wylie [PW09a], see also Jablonski [Jab14, Jab15], a non-trivial homogeneous Ricci soliton is either expanding, non-compact and non-gradient or shrinking and finitely covered by a product of an Einstein manifold and Euclidean space.

To be more flexible, it is convenient to investigate *cohomogeneity one manifolds.* In this case an $(n+1)$-dimensional connected manifold M carries an action by a compact connected Lie group G such that the orbit space M/G is one dimensional. Then, up to diffeomorphism, the following cases can occur:

1. $M/G = \mathbb{R}$ is the real line
2. $M/G = S^1$ is a circle
3. $M/G = [0, \infty)$ is a half line
4. $M/G = [-1, 1]$ is a closed interval.

It is well known, cf. [Ham88], [Ive93], [Per02], that compact Ricci solitons are Einstein or shrinking and of gradient type. Furthermore, according to a result of Wylie [Wyl08], any complete shrinking Ricci soliton has finite fundamental group. Therefore, the case $M/G = S^1$ cannot occur amongst non-trivial Ricci solitons.

Following [Zil13], in order to describe the topological structure of a cohomogeneity one manifold more concretely, let $G_p = \{g \in G \mid g \cdot p = p\}$ be the isotropy group of G at $p \in M$. By definition, two points p and q have the same isotropy type if the corresponding isotropy groups are conjugate. Notice that being conjugate to a subgroup of an isotropy group defines a partial ordering on the set of isotropy types. In fact, there exists a unique minimal isotropy type and the corresponding orbits G/G_p are called *principal orbits*. The other orbits are called *exceptional* or *singular* depending on whether or not they have the same dimension as the principal orbit. In the above description, boundary points of the intervals exactly correspond to non-principal orbits.

Remark 2.1.1 The proof of the principal orbit theorem in [Zil13] is based on the slice theorem and Kleiner's lemma [Kle90], which asserts that the principal isotropy group is constant:

Let G be a compact Lie group acting isometrically on a Riemannian manifold (M, g) and let $\gamma\colon [0,1] \to M$ be a shortest path from $G \cdot \gamma(0)$ to $G \cdot \gamma(1)$, then

$$G_{\gamma(t)} = \{g \in G \mid g \cdot \gamma(s) = \gamma(s) \text{ for all } s \in [0,1]\} = \bigcap_{s\in[0,1]} G_{\gamma(s)}$$

for all $t \in (0,1)$. In particular, $G_{\gamma(t)}$ is constant along the interior of $\mathrm{im}(\gamma)$ and is a subgroup of both $G_{\gamma(0)}$ and $G_{\gamma(1)}$.

In this thesis all cohomogeneity one manifolds will have at least one singular orbit. In the non-compact case, the orbit space is hence a half open interval and let $K \subset H$ denote the isotropy group of the principal and singular orbit, respectively. By the slice theorem, the manifold M is then diffeomorphic to the total space of the open disc bundle $G \times_H D^{d_S+1} \to G/H$, where D^{d_S+1} denotes the normal disc to the singular orbit G/H and $\partial D^{d_S+1} = S^{d_S} = K/H$ is the collapsing sphere.

Conversely, let G be a compact connected Lie group and let $K \subset H$ be closed subgroups such that H/K is a sphere. Then $G \times_H \mathbb{R}^{d_S+1}$ is a cohomogeneity one manifold with principal orbit G/K and the non-principal orbit G/H is singular if it has a strictly smaller dimension than G/K.

The orbit space of any non-trivial compact Ricci soliton is diffeomorphic to $[-1,1]$ and the manifold M is the union of two disc bundles $G \times_{H^\pm} D^{d_S+1}$ glued together along their common boundary G/K. Notice that the isotropy groups of the singular orbits may be different and M corresponds to a group diagram $K \subset \{H^-, H^+\} \subset G$.

Now the geometric structure of M will be explained. To this end, suppose furthermore that G acts isometrically on the Riemannian manifold (M, g). In fact, since G is compact, there always exists a G-invariant metric g on M.

Choose a horizontal lift of M/G to obtain a normal geodesic γ perpendicular to all principal orbits and denote by K the principal isotropy group, which is constant along γ. Observe that there is a G-equivariant diffeomorphism

$$\begin{aligned} \Phi\colon I \times G/K &\to M_0, \\ (t, gK) &\mapsto g \cdot \gamma(t), \end{aligned} \tag{2.1}$$

where $M_0 \subset M$ is the open dense subset consisting of the union of all principal orbits and $I \subset \mathbb{R}$ is an open interval. Then the metric $\Phi^* g$ on $I \times G/K$ is given by $\Phi^* g = dt^2 + g_t$, where g_t is a one-parameter family of G-invariant metrics on $P = G/K$.

Let $N = \Phi_*(\frac{\partial}{\partial t})$ be a unit normal vector field and let $L_t = \nabla N$ denote the shape operator of the hypersurface $\Phi(\{t\} \times P)$. Via Φ, L_t can be regarded as a one-parameter family of G-equivariant, g_t-symmetric endomorphisms of TP which satisfies the equation $\dot{g}_t = 2 g_t L_t$. Similarly, let Ric_t be the Ricci curvature corresponding to g_t. Eschenburg-Wang [EW00] have calculated the Ricci curvature of the cohomogeneity one manifold (M, g) and it is given by

$$\begin{aligned} \mathrm{Ric}(N,N) &= -\operatorname{tr}(\dot{L}_t) - \operatorname{tr}(L_t^2), \\ \mathrm{Ric}(X,N) &= -g_t(\delta^{\nabla^t} L_t, X) - d(\operatorname{tr}(L_t))(X), \\ \mathrm{Ric}(X,Y) &= \mathrm{Ric}_t(X,Y) - \operatorname{tr}(L_t) g_t(L_t(X), Y) - g_t(\dot{L}_t(X), Y), \end{aligned}$$

where $X, Y \in TP$, $\delta^{\nabla^t}\colon T^*P \otimes TP \to TP$ is the codifferential, and L_t is regarded as a TP-valued 1-form on TP.

2.2 The cohomogeneity one Ricci soliton equation

The framework of cohomogeneity one Ricci solitons was set up by Dancer-Wang [DW11]. As the Lie group G is compact, any cohomogeneity one Ricci soliton induces a Ricci soliton with a G-invariant vector field. Hence, in the case of gradient Ricci solitons, the soliton potential can be assumed to be G-invariant and the gradient Ricci soliton equation $\mathrm{Ric} + \mathrm{Hess}\, u + \frac{\varepsilon}{2} g = 0$ then takes the form

$$-(\delta^{\nabla^t} L_t)^\flat - d(\mathrm{tr}(L_t)) = 0, \tag{2.2}$$
$$-\mathrm{tr}(\dot{L}_t) - \mathrm{tr}(L_t^2) + \ddot{u} + \frac{\varepsilon}{2} = 0, \tag{2.3}$$
$$-\dot{L}_t - (-\dot{u} + \mathrm{tr}(L_t))L_t + r_t + \frac{\varepsilon}{2}\mathbb{I} = 0, \tag{2.4}$$

where $r_t = g_t \circ \mathrm{Ric}_t$ is the Ricci endomorphism. Conversely, the above system induces a gradient Ricci soliton on $I \times P$ provided that the metric g_t is defined via $\dot{g}_t = 2g_t L_t$. The special case of constant u recovers the cohomogeneity one Einstein equations.

From now on, for simplicity, the t-dependence may not be stated explicitly.

It is an immediate consequence of (2.3) that the mean curvature with respect to the volume element $e^{-u} d\,\mathrm{Vol}_g$ is a Lyapunov function if $\varepsilon \leq 0$.

Proposition 2.2.1 *Fix $\varepsilon \leq 0$. Then the generalised mean curvature $-\dot{u} + \mathrm{tr}(L)$ is monotonically decreasing along the flow of the cohomogeneity one Ricci soliton equation.*

If the Ricci soliton metric is at least C^3-regular, then the second Bianchi identity implies that the *conservation law*

$$\ddot{u} + (-\dot{u} + \mathrm{tr}(L))\dot{u} = C + \varepsilon u \tag{2.5}$$

has to be satisfied for some constant $C \in \mathbb{R}$. Using the equations (2.3) and (2.4) it can be reformulated as

$$\mathrm{tr}(r) + \mathrm{tr}(L^2) - (-\dot{u} + \mathrm{tr}\,(L))^2 + (n-1)\frac{\varepsilon}{2} = C + \varepsilon u. \tag{2.6}$$

Recall that the scalar curvature R of a cohomogeneity one Riemannian manifold (M^{n+1}, g) is given by $R = \mathrm{tr}(r) - \mathrm{tr}(L^2) - \mathrm{tr}(L)^2 - 2\,\mathrm{tr}(\dot{L})$. It hence follows with (2.4) that the conservation law (2.6) is just the cohomogeneity one version of Hamilton's [Ham95] general identity $R + |\nabla u|^2 + \varepsilon u = \overline{C}$ for gradient Ricci solitons (where $\overline{C} = -C - \frac{n+1}{2}\varepsilon$). This also provides a formula for the scalar curvature in terms of the soliton potential:

$$R = -C - \varepsilon u - \dot{u}^2 - (n+1)\frac{\varepsilon}{2}. \tag{2.7}$$

2.3 Ricci solitons with a singular orbit

Suppose that $\Phi\colon (0,T) \times P \to M_0$ is a G-equivariant parametrisation of the union of all principal orbits as in (2.1). In the following, this will correspond to a local solution of the Ricci soliton equations and from now on, assume that there is a singular orbit $Q = G/H$ at $t = 0$.

Building up on an idea of Back [Bac86], see also [EW00], Dancer-Wang [DW11] have shown that in the presence of a singular orbit, equation (2.4) implies (2.2) automatically, provided that the metric is at least C^2-regular and the soliton potential is of class C^3. Moreover, if in this case the conservation law (2.5) is satisfied, then equation (2.3) holds as well.

Conversely, any trajectory of the Ricci soliton equations (2.3), (2.4) that describes a C^3-regular metric with a singular orbit has to satisfy the conservation law (2.6).

The initial value problem for gradient cohomogeneity one Ricci solitons has been considered by Buzano [Buz11]. Extending Eschenburg-Wang's work [EW00] in the Einstein case, under the technical assumption that the representations of the principal isotropy group K on the tangent space and the normal space of the singular orbit Q have no common subrepresentations, the initial value problem can be solved close to a singular orbit regardless of the soliton being shrinking, steady or expanding. However, the solution may not be unique. The strategy of the proof is to construct a formal power series solution starting from (2.4), (2.5) and then a general theorem of Malgrange [Mal74] guarantees the existence of a genuine solution. Alternatively, a Picard iteration argument can be applied as in [EW00]. The argument also shows that, if sufficiently many initial conditions are fixed (i.e. possibly conditions for higher derivatives), the then unique trajectory with these initial conditions corresponds to a smooth, possibly incomplete cohomogeneity one Ricci soliton.

The existence of a singular orbit at $t = 0$ imposes the smoothness condition $\dot{u}(0) = 0$ on the soliton potential u. Notice also that without loss of generality $u(0) = 0$ can be assumed, as the Ricci soliton equation is invariant under changing the potential by an additive constant. If d_S denotes the dimension of the collapsing sphere at the singular orbit, then the trace of the shape operator grows like $\mathrm{tr}(L) = \frac{d_S}{t} + O(t)$ as $t \to 0$. Therefore the conservation law (2.5) implies $\ddot{u}(0) = \frac{C}{d_S+1}$. To summarize:

$$u(0) = 0, \quad \dot{u}(0) = 0, \quad \ddot{u}(0) = \frac{C}{d_S + 1}. \tag{2.8}$$

The existence of a singular orbit has consequences for the behaviour of the soliton potential. Proposition 2.3.1 below actually follows from [BDW15, Propositions 2.3

and 2.4] and [BDGW15a, Proposition 1.11]. To emphasise that the properties hold along the flow of the Ricci soliton equation and completeness of the metric is not required, the proofs are summarised here in a single argument.

Proposition 2.3.1 *Along any Ricci soliton trajectory with $\varepsilon \geq 0$ and $C < 0$ in (2.8) that corresponds to a cohomogeneity one manifold of dimension $n+1$ with a singular orbit at $t = 0$, for $t > 0$ and as long as the solution exists, the soliton potential satisfies $u(t)$, $\dot{u}(t) < 0$ and also $\ddot{u}(t) < 0$ if $\varepsilon > 0$ or $\varepsilon = 0$ and $L_t \neq 0$.*

Furthermore, if $\varepsilon = 0$ and $C \leq 0$, there holds $\operatorname{tr}(L_t) \leq \frac{n}{t}$ for $t > 0$ and as long as the solution exists.

Proof: It follows from the initial conditions (2.8), that the claim is true for $t > 0$ small enough. Suppose that $\dot{u}(t_0) = 0$ and $\dot{u}(t) < 0$ for $t \in (0, t_0)$. Then it follows that $u(t_0) < 0$ and $\ddot{u}(t_0) \geq 0$. However, the conservation law (2.5) implies the contradiction

$$0 \leq \ddot{u}(t_0) = C + \varepsilon u(t_0) < 0.$$

Therefore there holds $\dot{u}(t) < 0$ for $t > 0$ and hence also $u(t) < 0$ for $t > 0$. Finally, to prove concavity, differentiate the conservation law (2.5) and use (2.3) to get

$$\begin{aligned} 0 &= \frac{d^3}{dt^3}u + (-\dot{u} + \operatorname{tr}(L))\ddot{u} - \varepsilon\dot{u} + (-\ddot{u} + \operatorname{tr}(\dot{L}))\dot{u} \\ &= \frac{d^3}{dt^3}u + (-\dot{u} + \operatorname{tr}(L))\ddot{u} - \varepsilon\dot{u} + (-\operatorname{tr}(L^2) + \frac{\varepsilon}{2})\dot{u} \\ &= \frac{d^3}{dt^3}u + (-\dot{u} + \operatorname{tr}(L))\ddot{u} - (\frac{\varepsilon}{2} + \operatorname{tr}(L^2))\dot{u}. \end{aligned}$$

Therefore, if there is $t_1 > 0$ such that $\ddot{u}(t_1) = 0$ then $\frac{d^3}{dt^3}u(t_1) < 0$. By continuity this in turn implies that $\ddot{u}(t) < 0$ for all $t > 0$.

Finally, restrict to the case $\varepsilon = 0$ and recall that $\frac{1}{n}\operatorname{tr}(L)^2 \leq \operatorname{tr}(L^2)$ follows from $\operatorname{tr}((L^{(0)})^2) \geq 0$, where $L^{(0)} = L - \frac{1}{n}\operatorname{tr}(L)\mathbb{I}$ is the trace-less shape operator. Therefore the Ricci soliton equation (2.3) implies

$$\frac{d}{dt}\operatorname{tr}(L) = -\operatorname{tr}(L^2) + \ddot{u} \leq -\frac{1}{n}\operatorname{tr}(L)^2.$$

Note that $f(t) = \left(\frac{1}{f(t_0)} + \frac{1}{n}(t - t_0)\right)^{-1}$ solves $f' = -\frac{1}{n}f^2$. Since $\operatorname{tr}(L) = \frac{d_S}{t} + O(t)$, where d_S is the dimension of the collapsing sphere, the claim follows. $\square$

Remark 2.3.2 Let $\frac{d}{ds} = \frac{1}{-\dot{u}+\mathrm{tr}(L)}\frac{d}{dt}$ and $\mathcal{L} = \frac{1}{-\dot{u}+\mathrm{tr}(L)}$. Then a direct calculation based only on the Ricci soliton equations shows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\frac{C+\varepsilon u}{(-\dot{u}+\mathrm{tr}(L))^2} &= \left(\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}-1\right)\frac{\varepsilon}{2}\mathcal{L}^2 \\
&\quad + \frac{C+\varepsilon u}{(-\dot{u}+\mathrm{tr}(L))^2}\left(\frac{\mathrm{tr}(L^2)}{(-\dot{u}+\mathrm{tr}(L))^2}-\frac{\varepsilon}{2}\mathcal{L}^2\right), \\
\frac{d}{ds}\left(\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}-1\right) &= \left(\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}-1\right)\left(\frac{\mathrm{tr}(L^2)}{(-\dot{u}+\mathrm{tr}(L))^2}-\frac{\varepsilon}{2}\mathcal{L}^2-1\right) \\
&\quad + \frac{\mathrm{tr}(L^2)+\mathrm{tr}(r_t)}{(-\dot{u}+\mathrm{tr}(L))^2}+(n-1)\frac{\varepsilon}{2}\mathcal{L}^2-1.
\end{aligned}$$

The conservation law (2.6) connects the two equations and identifies two loci inside phase space: The two loci

$$\begin{aligned}
&\left\{\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}<1 \text{ and } \frac{\mathrm{tr}(L^2)+\mathrm{tr}(r_t)}{(-\dot{u}+\mathrm{tr}(L))^2}+(n-1)\frac{\varepsilon}{2}\mathcal{L}^2<1\right\}, \\
&\left\{\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}=1 \text{ and } \frac{\mathrm{tr}(L^2)+\mathrm{tr}(r_t)}{(-\dot{u}+\mathrm{tr}(L))^2}+(n-1)\frac{\varepsilon}{2}\mathcal{L}^2=1\right\}
\end{aligned}$$

are preserved by the Ricci soliton ODE if $\varepsilon \geq 0$. It will be explained in section 2.4 that the first locus contains all trajectories corresponding to complete steady or expanding Ricci solitons and the second locus corresponds to Einstein trajectories with non-positive scalar curvature.

Remark 2.3.3 The equation for $\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}$ can also be stated differently:

$$\frac{d}{dt}\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)} = \frac{1}{-\dot{u}+\mathrm{tr}(L)}\left\{\left(\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}-1\right)\left(\mathrm{tr}(L^2)-\frac{\varepsilon}{2}\right)+\ddot{u}\right\}.$$

In particular, in the steady case, proposition 2.3.1 shows that $\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)}$ is monotonically decreasing as long as $-\dot{u}+\mathrm{tr}(L)>0$. According to proposition 2.4.1 below, this is always true if the metric corresponds to a complete steady Ricci soliton. In this case, moreover, it follows that $\frac{\mathrm{tr}(L)}{-\dot{u}+\mathrm{tr}(L)} \to 0$ as $t \to \infty$.

2.4 Consequences of completeness

If the trajectories correspond to a non-trivial *complete* Ricci soliton metric, further restrictions on the asymptotics of the soliton potential and the metric are known.

In the steady case, according to a result of Chen [Che09a], the ambient scalar curvature of steady Ricci solitons satisfies $R \geq 0$ with equality if and only if the metric is Ricci flat. Then (2.7) implies that $C \le 0$ is a necessary for completeness and $C = 0$ precisely corresponds to the Ricci flat case. Munteanu-Sesum [MS13] have shown that non-trivial complete steady Ricci solitons have at least linear volume growth and Buzano-Dancer-Wang used this to show [BDW15, Proposition 2.4 and Corollary 2.6]:

Proposition 2.4.1 *Along any trajectory which corresponds to a non-trivial* complete *steady cohomogeneity one Ricci soliton of dimension $n+1$ with a singular orbit at $t = 0$ and integrability constant $C < 0$, the estimates*

$$0 < \operatorname{tr}(L) < \frac{n}{t} \quad \textit{and} \quad 0 < -\dot{u}\operatorname{tr}(L) < R < 2\sqrt{-C}\frac{n}{t} + \frac{n^2}{t^2}$$

hold for $t > 0$ and the soliton potential satisfies

$$-\dot{u}(t) \to \sqrt{-C} \quad \textit{and} \quad \ddot{u}(t) \to 0$$

as $t \to \infty$.

In the case of expanding Ricci solitons, a similar result of Chen [Che09a] implies that the scalar curvature R of a non-trivial, complete expanding Ricci soliton satisfies $R > -\frac{\varepsilon}{2}(n+1)$. It follows from (2.7) that $0 \ge -\dot{u}^2 > C + \varepsilon u$ holds on any complete expanding Ricci soliton. The smoothness condition (2.8) at the singular orbit therefore requires $C < 0$ as a *necessary* condition to construct non-trivial, complete expanding Ricci solitons. Conversely, Einstein metrics with negative scalar curvature correspond to trajectories with $C = 0$.

According to results of Buzano-Dancer-Gallaugher-Wang [BDGW15a], any non-trivial, complete gradient expanding Ricci soliton has at least logarithmic volume growth, which has consequences for the asymptotic behaviour of the soliton, see [BDGW15a, Equation (1.10) and Proposition 1.18]:

Proposition 2.4.2 *Consider a non-trivial* complete *cohomogeneity one gradient expanding Ricci soliton of dimension $n+1$ with a singular orbit at $t = 0$ and integrability constant $C < 0$ in (2.8). Then given any $t_0 > 2\sqrt{5/\varepsilon}$ one has*

$$\frac{9}{10}\frac{\frac{\varepsilon}{2}t + \sqrt{n\frac{\varepsilon}{2}} + \sqrt{-C}}{\frac{\varepsilon}{2}t_0 + \sqrt{n\frac{\varepsilon}{2}} + \sqrt{-C}}(-\dot{u}(t_0)) < -\dot{u}(t) < \frac{\varepsilon}{2}t + \sqrt{-C}$$

for all $t \ge t_0$. The second inequality holds in fact for all $t \ge 0$.

Moreover, the mean curvature eventually satisfies the bound $|\operatorname{tr}(L)(t)| < \sqrt{\frac{n}{2}\varepsilon}$.

2.5 The Böhm functional

Böhm [Böh99] has introduced the functional $\mathcal{F}_0$ to the study of Einstein manifolds of cohomogeneity one. Provided the mean curvature of the principal orbit remains positive, $\mathcal{F}_0$ is a monotonic quantity along trajectories of cohomogeneity one Einstein equations and critical points correspond to canonical cone solutions, cf. section 3.4.1. Subsequently it was considered by Dancer-Wang and their collaborators Buzano, Gallaugher and Hall in the context of cohomogeneity one Ricci solitons [BDW15, BDGW15a, DHW13].

To define the Böhm functional $\mathcal{F}_0$, let $v(t) = \sqrt{\det g_t}$ denote the relative volume of the principal orbits and let $L^{(0)} = L - \frac{1}{n}\operatorname{tr}(L)\mathbb{I}$ denote the trace less part of the shape operator. Then the Böhm functional is given by

$$\mathcal{F}_0 = v^{\frac{2}{n}} \left(\operatorname{tr}(r_t) + \operatorname{tr}((L^{(0)})^2)\right). \tag{2.9}$$

Due to the behaviour of the soliton potential discussed in proposition 2.3.1 and section 2.4, the Böhm functional also enjoys strong monotonicity properties along trajectories of the gradient Ricci soliton equation:

Proposition 2.5.1 ([DHW13, Proposition 2.17]) *Along the flow of a C^3-regular cohomogeneity one gradient Ricci soliton the Böhm functional $\mathcal{F}_0$ satisfies*

$$\frac{d}{dt}\mathcal{F}_0 = -2v^{\frac{n}{2}}\operatorname{tr}((L^{(0)})^2)\left(-\dot{u} + \frac{n-1}{n}\operatorname{tr}(L)\right). \tag{2.10}$$

Remark 2.5.2 The C^3-regularity condition guarantees that the conservation law (2.5) is satisfied. On the other hand the existence of a singular orbit along the trajectory is not required to prove (2.10).

Chapter 3

The two summands case

This chapter discusses the Ricci soliton equation in the *two summands case*, which considers the case that the space of invariant metrics on the principal orbit is two dimensional.

After reviewing the general set-up in section 3.1, a qualitative analysis of the Ricci soliton ODE is carried out in section 3.2 and the existence of continuous families of complete steady and expanding Ricci solitons is shown. The methods also recover constructions of non-compact Einstein manifolds due to Böhm and can be adapted to provide examples of Ricci solitons on complex line bundles over Fano Kähler Einstein manifolds, see section 3.3. Section 3.4 analyses the asymptotic behaviour of the metrics and in section 3.5 this is used to reconstruct Böhm's Einstein manifolds of positive scalar curvature on spheres and other low dimensional spaces from the asymptotics of the Ricci flat trajectories.

3.1 The geometric set-up

Recall the construction of a cohomogeneity one manifold from a group diagram from section 2.1: Let G be a compact connected Lie group and let $K \subset H$ be closed subgroups such that H/K is a sphere. Then $G \times_H \mathbb{R}^{d_1+1}$ is a cohomogeneity one manifold with principal orbit G/K. Suppose that the non-principal orbit G/H is singular, i.e. of strictly less dimension then G/K, then the orbit space is a half open interval.

Choose a bi-invariant metric b on G which induces the metric of constant curvature 1 on H/K. The *two summands case* assumes that the space of G-invariant metrics on the principal orbit is two dimensional: Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an $\mathrm{Ad}(K)$-invariant decomposition of the Lie algebra of G and suppose furthermore that $\mathfrak{p}$ decomposes into two inequivalent, b-orthogonal, irreducible K-modules, $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. In fact, $\mathfrak{p}_1$

can be identified with the tangent space to the collapsing sphere $S = H/K$ and $\mathfrak{p}_2$ with the tangent space of the singular orbit $Q = G/H$. Let $g_S = b_{|\mathfrak{p}_1}$ and $g_Q = b_{|\mathfrak{p}_2}$ denote the induced metrics. Then, away from the singular orbit Q, the metric on M is given by

$$g_{M\setminus Q} = dt^2 + f_1(t)^2 g_S + f_2(t)^2 g_Q. \tag{3.1}$$

If the functions f_1, f_2 satisfy the smoothness conditions

$$f_1(0) = 0,\ \dot{f}_1(0) = 1 \quad \text{and} \quad f_2(0) = \bar{f} > 0,\ \dot{f}_2(0) = 0, \tag{3.2}$$

then due to the work of Eschenburg-Wang [EW00] the metric extends smoothly over the singular orbit. Recall that the smoothness conditions of the associated G-invariant soliton potential are considered in (2.8).

In order to state the induced Ricci soliton equations, explicit formulae for the shape operator of the principal orbit and the Ricci endomorphism are required. As the metric remains in diagonal form, the relation $\dot{g}_t = 2 g_t L_t$ yields that the shape operator is given by $L_t = \left(\frac{\dot{f}_1}{f_1} \mathbb{I}_{d_1}, \frac{\dot{f}_2}{f_2} \mathbb{I}_{d_2} \right)$. Furthermore, it follows from the theory of Riemannian submersions and the O'Neill calculus, cf. [Böh98], that the Ricci endomorphism takes the form

$$r_t = \left(\left\{ \frac{A_1}{d_1} \frac{1}{f_1^2} + \frac{A_3}{d_1} \frac{f_1^2}{f_2^4} \right\} \mathbb{I}_{d_1}, \left\{ \frac{A_2}{d_2} \frac{1}{f_2^2} - \frac{2A_3}{d_2} \frac{f_1^2}{f_2^4} \right\} \mathbb{I}_{d_2} \right) \tag{3.3}$$

and therefore the Ricci soliton equations (2.3) and (2.4) take the form

$$\begin{aligned}
\ddot{u} &= \sum_{i=1}^{2} d_i \frac{\ddot{f}_i}{f_i} - \frac{\varepsilon}{2}, \\
\frac{d}{dt} \frac{\dot{f}_1}{f_1} &= -(-\dot{u} + \operatorname{tr}(L)) \frac{\dot{f}_1}{f_1} + \frac{\varepsilon}{2} + \frac{A_1}{d_1} \frac{1}{f_1^2} + \frac{A_3}{d_1} \frac{f_1^2}{f_2^4}, \\
\frac{d}{dt} \frac{\dot{f}_2}{f_2} &= -(-\dot{u} + \operatorname{tr}(L)) \frac{\dot{f}_2}{f_2} + \frac{\varepsilon}{2} + \frac{A_2}{d_2} \frac{1}{f_2^2} - 2 \frac{A_3}{d_2} \frac{f_1^2}{f_2^4}.
\end{aligned}$$

In this context the appearing constants are defined as follows: d_1 is the dimension of the collapsing sphere and d_2 is the dimension of the singular orbit. Furthermore, $A_1 = d_1(d_1 - 1)$ and $A_2 = d_2 \operatorname{Ric}^Q$, where Ric^Q is the Einstein constant of the isotropy irreducible space (Q, g_Q). Finally, $A_3 = d_2 ||A||^2$ where $||A|| \geq 0$ appears naturally in the theory of Riemannian submersions:

Following [Böh98, Section 1], fix the background metric $g_P = g_S + g_Q$ on the principal orbit P and let ∇^{g_P} be the corresponding Levi-Civita connection. Furthermore, suppose that $H_1, \ldots, H_{d_2}$ is an orthonormal basis of horizontal vector fields with respect to the Riemannian submersion $(G/K, g_P) \to (G/H, g_Q)$. Then one has $||A||^2 = \sum_{i=1}^{d_2} g_S((\nabla^{g_P}_{H_1} H_i)_{|v}, (\nabla^{g_P}_{H_1} H_i)_{|v})$, where $(\cdot)_{|v}$ denotes the projection onto the tangent space of the fibres S^{d_1}, is the norm of an O'Neill tensor associated to the above Riemannian submersion.

Warped product metrics with two homogeneous summands provide examples with $||A|| = 0$. Examples with $||A|| > 0$ are given by the total space of non-trivial disc bundles which are associated to the following group diagrams. These are induced by the Hopf fibrations, cf. [Bes87]:

The following table, which lists the group diagrams corresponding to the Hopf fibrations and the associated constants, is taken from [Böh98, Table 1].

	$\mathbb{C}P^{m+1}$	$\mathbb{H}P^{m+1}$	F^{m+1}	CaP^2
G	$U(m+1)$	$Sp(1) \times Sp(m+1)$	$Sp(m+1)$	$Spin(9)$
H	$U(1) \times U(m)$	$Sp(1) \times Sp(1) \times Sp(m)$	$Sp(1) \times Sp(m)$	$Spin(8)$
K	$U(m)$	$Sp(1) \times Sp(m)$	$U(1) \times Sp(m)$	$Spin(7)$
d_1	1	3	2	7
d_2	$2m$	$4m$	$4m$	8
$\|\|A\|\|^2$	1	3	8	7
Ric^Q	$2m+2$	$4m+8$	$4m+8$	28

Table 3.1.1: Group diagrams associated to Hopf fibrations

The generalised Hopf fibrations can be recovered from this table via the following more general procedure due to Bérard-Bergery [BB75]:

If G is a Lie group and $K \subset H \subset G$ are two compact subgroups, then there are natural metrics on G/K, G/H and H/K such that the map

$$\pi\colon G/K \to G/H,\ gK \mapsto gH$$

becomes a Riemannian submersion whose fibres are totally geodesic and are isometric to H/K.

The examples in table 3.1.1 above then give rise to the generalised Hopf fibrations $\pi\colon S^{2m+1} \to \mathbb{C}P^m$ with fibre S^1, $\pi\colon S^{4m+3} \to \mathbb{H}P^m$ with fibre S^3, $\pi\colon \mathbb{C}P^{2m+1} \to \mathbb{H}P^m$ with fibre S^2 and $\pi\colon S^{15} \to S^8$ with fibre S^7.

The corresponding disc bundles $G \times_H \mathbb{R}^{d_1+1}$ are diffeomorphic to $\mathbb{C}P^{m+1} \setminus \{$ point $\}$, $\mathbb{H}P^{m+1} \setminus \{$ point $\}$, $F^{m+1} \setminus \{$ point $\}$ and $CaP^2 \setminus \{$ point $\}$, respectively. By gluing together two copies of the same disc bundle along the principal orbits, one of which with the opposite orientation, one obtains a cohomogeneity one structure on the connected sums $\mathbb{C}P^{m+1} \# \overline{\mathbb{C}P}^{m+1}$, $\mathbb{H}P^{m+1} \# \overline{\mathbb{H}P}^{m+1}$, $F^{m+1} \# \overline{F}^{m+1}$ and $CaP^2 \# \overline{CaP}^2$.

In the following sections, the existence of Ricci solitons and Einstein metrics on these spaces will be investigated. In particular, the following theorem will be shown:

Theorem 3.1.1 *On $CaP^2 \setminus \{ \text{ point } \}$, $\mathbb{H}P^{m+1} \setminus \{ \text{ point } \}$ for $m \geq 1$ and on the vector bundle associated to $(G, H, K) = (Sp(m+1), Sp(1) \times Sp(m), U(1) \times Sp(m))$ for $m \geq 3$, there exist a 1-parameter family of non-homothetic complete steady and a 2-parameter family of non-homothetic complete expanding Ricci solitons.*

The steady Ricci solitons are asymptotically conical whereas the expanding Ricci solitons are asymptotically paraboloid.

The corresponding Einstein metrics on these spaces are due to Böhm [Böh99].

3.2 Qualitative ODE analysis

This section discusses the long time existence of solutions to the two summands Ricci soliton equation and hence completeness of the corresponding metrics.

Recall that, in the coordinate system of section 3.1, the Ricci soliton equations (2.3) and (2.4) become singular at the singular orbit. To have good control on the trajectories even close to the singular orbit, a rescaling as in remark 2.3.2 will be introduced which smooths the Ricci soliton equations. In particular, the corresponding Einstein and Ricci soliton loci will correspond to *bounded* regions in phase space. Long time existence of the rescaled system will then be immediate from the structure of the Ricci soliton ODE and propositions 3.2.9 and 3.2.12 provide a general principle to conclude completeness of the metrics.

In chapter 5, a different argument for completeness, which is based on the positive definiteness of the shape operator, will be presented, see proposition 5.1.3 for details.

The following coordinates have effectively been used by Dancer-Wang [DW09b] and are motivated by Ivey's work [Ive93]. Notice that under the coordinate change

$$\begin{aligned} X_i &= \frac{1}{-\dot{u} + \operatorname{tr}(L)} \frac{\dot{f}_i}{f_i}, \quad & Y_i &= \frac{1}{-\dot{u} + \operatorname{tr}(L)} \frac{1}{f_i}, \text{ for } \ i = 1, 2, \\ \mathcal{L} &= \frac{1}{-\dot{u} + \operatorname{tr}(L)}, \quad & \frac{d}{ds} &= \frac{1}{-\dot{u} + \operatorname{tr}(L)} \frac{d}{dt} \end{aligned} \tag{3.4}$$

the cohomogeneity one two summands Ricci soliton equations reduce to the ODE system

$$\begin{aligned}
X_1' &= X_1\left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - 1\right) + \frac{A_1}{d_1}Y_1^2 + \frac{\varepsilon}{2}\mathcal{L}^2 + \frac{A_3}{d_1}\frac{Y_2^4}{Y_1^2}, \\
X_2' &= X_2\left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - 1\right) + \frac{A_2}{d_2}Y_2^2 + \frac{\varepsilon}{2}\mathcal{L}^2 - \frac{2A_3}{d_2}\frac{Y_2^4}{Y_1^2}, \\
Y_j' &= Y_j\left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - X_j\right), \\
\mathcal{L}' &= \mathcal{L}\left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2\right).
\end{aligned} \tag{3.5}$$

Here and in the following, the $\frac{d}{ds}$ derivative is denoted by a prime $'$. On the other hand, the $\frac{d}{dt}$ derivative will always correspond to a dot $\dot{}$.

Notice that time, metric and soliton potential can be recovered from the ODE via

$$t(s) = t(s_0) + \int_{s_0}^{s} \mathcal{L}(\tau)d\tau \quad \text{and} \quad f_i = \frac{\mathcal{L}}{Y_i}, \text{ for } \ i = 1,2, \quad \text{and} \quad \dot{u} = \frac{\sum_{i=1}^{2} d_i X_i - 1}{\mathcal{L}}.$$

The term *two summands case* will refer to a set of parameters with $d_1 > 1, d_2 > 0$, $A_1, A_2 > 0$ and $A_3 \geq 0$. In geometric examples one has in fact $d_2 > 1$ too, as this is necessary for the singular orbit to have positive Ricci curvature, which implies $A_2 = d_2 \operatorname{Ric}^Q > 0$. The condition $d_1 > 1$ also has significance from a purely dynamical systems point of view as it is necessary to carry out the argument in proposition 3.2.5.

The term S^1*-bundle case* will be used to indicate that parameters $d_1 = 1$, $d_2 > 0$, $A_1 = 0$ and $A_2, A_3 > 0$ are considered and this case will be discussed in more detail in section 3.3. This case is special as *explicit* solutions are known, see [Cao96], [FIK03] or [DW11]. However, in the case of Ricci solitons, all of these are Kähler, whereas theorem 3.3.4 gives rise to *non*-Kähler metrics. In the Einstein case, explicit non-Kähler metrics are due to Wang-Wang [WW98] and, with different methods, Appleton [App18] and Stolarski [Sto17] independently construct complete, non-trivial, non-Kähler steady Ricci solitons too.

Remark 3.2.1 (a) The two summands case is also the set-up for Böhm's work [Böh98, Böh99] on Einstein manifolds. In fact, some key ideas carry over to the Ricci soliton case. However, Böhm uses different coordinate systems to analyse the trajectories. In particular, in order to prove completeness, Böhm considers a planar ODE and invokes the Poincaré-Bendixson theorem. Section 3.5.1 explains that

Böhm's ODE system in fact describes the Ricci flat trajectories. A reduction of the Ricci flat system to a planar ODE is discussed in section 3.4.2.1. In the pure Ricci soliton case, the extra degree of freedom of the soliton potential does not allow this reduction and a new proof is required.

(b) The case $A_3 = 0$ is already well understood from works on multiple warped products, cf. [DW09a, DW09b], [BDGW15a] or [BDW15]. In this case, the conservation law (3.7) shows that the variables are bounded if $d_1, d_2 > 0$, $A_1, A_2 > 0$ and completeness of the metric directly follows as in proposition 3.2.9 and corollary 3.2.12. However, if $A_3 > 0$, a refined analysis is necessary to ensure that the variables remain bounded.

In the new coordinate system, the smoothness conditions for the metric and the soliton potential correspond to the stationary point

$$X_1 = Y_1 = \frac{1}{d_1} \quad \text{and} \quad X_2 = Y_2 = 0 \quad \text{and} \quad \mathcal{L} = 0. \tag{3.6}$$

Trajectories emanating from (3.6) will be parametrised so that (3.6) corresponds to $s = -\infty$.

It follows from the ODE for Y_1 and $\lim_{s \to -\infty} Y_1(s) = 1$ that $Y_1 > 0$ holds along any such trajectory. Since the ODE system is invariant under changing the sign of the variables, $Y_2 > 0$ and similarly $\mathcal{L} > 0$ can also be assumed.

The conservation law (2.6) takes the form

$$\sum_{i=1}^{2} d_i X_i^2 + \sum_{i=1}^{2} A_i Y_i^2 - A_3 \frac{Y_2^4}{Y_1^2} + (n-1)\frac{\varepsilon}{2}\mathcal{L}^2 = 1 + (C + \varepsilon u)\,\mathcal{L}^2. \tag{3.7}$$

By identifying the Einstein and Ricci soliton loci in phase space, remark 2.3.2 explains how the conservation law enters the rescaled system, even though the soliton potential does not explicitly occur in the Ricci soliton ODE:

Consider the functions

$$\begin{aligned}
\mathcal{S}_1 &= \sum_{i=1}^{2} d_i X_i^2 + \sum_{i=1}^{2} A_i Y_i^2 - A_3 \frac{Y_2^4}{Y_1^2} + (n-1)\frac{\varepsilon}{2}\mathcal{L}^2 - 1, \\
\mathcal{S}_2 &= \sum_{i=1}^{2} d_i X_i - 1.
\end{aligned}$$

Notice that $\mathcal{S}_1$ occurs in the conservation law and $\mathcal{S}_2 = \frac{\dot{u}}{-\dot{u} + \mathrm{tr}(L)}$ encodes the derivative of the soliton potential in the rescaled coordinates.

Fix $\varepsilon \geq 0$ and recall from section 2.4 that $C \leq 0$ is a necessary condition to obtain trajectories that correspond to complete steady or expanding Ricci solitons and that $C = 0$ is the Einstein case. Due to the initial conditions (2.8) and proposition 2.3.1, the soliton potential satisfies $u, \dot{u} \leq 0$ if $C \leq 0$ and away from the singular orbit equality can only occur in the Einstein case.

Therefore, any trajectory with $\varepsilon \geq 0$ and $C \leq 0$ satisfies $\mathcal{S}_1, \mathcal{S}_2 \leq 0$. Equality occurs at the initial critical point and then *Einstein* trajectories lie in the locus

$$\{\mathcal{S}_1 = 0\} \cap \{\mathcal{S}_2 = 0\} \tag{3.8}$$

whereas trajectories of *complete non-trivial* Ricci solitons are contained in the locus

$$\{\mathcal{S}_1 < 0\} \cap \{\mathcal{S}_2 < 0\}. \tag{3.9}$$

Conversely, trajectories in these loci correspond to Einstein metrics and non-trivial Ricci solitons. In particular, these are exactly the loci described in remark 2.3.2.

In fact, the invariance of the above loci for $\varepsilon \geq 0$ also follows from the Ricci soliton ODE as a direct calculation verifies

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\mathcal{S}_1 &= \left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2\right)\mathcal{S}_1 + \frac{\varepsilon}{2}\mathcal{L}^2 \cdot \mathcal{S}_2, \\
\frac{d}{ds}\mathcal{S}_2 &= \mathcal{S}_1 + \left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - 1\right)\mathcal{S}_2.
\end{aligned}$$

Now the existence of trajectories which lie in one of the above loci and in the unstable manifold of the critical point (3.6) will be discussed. Different trajectories will correspond to non-homothetic Einstein or Ricci soliton metrics.

The linearisation of the Ricci soliton ODE at the initial stationary point (3.6) is given by

$$\begin{pmatrix}
\frac{3}{d_1} - 1 & 0 & \frac{2(d_1-1)}{d_1} & 0 & 0 \\
0 & \frac{1}{d_1} - 1 & 0 & 0 & 0 \\
\frac{1}{d_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{d_1} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{d_1}
\end{pmatrix}.$$

The corresponding eigenvalues are hence $\frac{2}{d_1}$, and both $\frac{1}{d_1} - 1$ and $\frac{1}{d_1}$ appear twice. In particular, the critical point is *hyperbolic* if $d_1 > 1$. The corresponding eigenspaces are given by $E_{\frac{2}{d_1}} = \mathrm{sp}\left\{(2,0,1,0,0)\right\}$, $E_{\frac{1}{d_1}-1} = \mathrm{sp}\left\{(0,1,0,0,0),(d_1-1,0,-1,0,0)\right\}$ and $E_{\frac{1}{d_1}} = \mathrm{sp}\left\{(0,0,0,1,0),(0,0,0,0,1)\right\}.$

Notice that the stationary point (3.6) lies in the set $\{\mathcal{S}_1 = 0\} \cap \{\mathcal{S}_2 = 0\}$. Moreover, $\{\mathcal{S}_1 = 0\}$ is a submanifold of $\mathbb{R}^5$ if $Y_1 \neq 0$ and its tangent space at (3.6) is $\mathrm{sp}\{(1, 0, d_1 - 1, 0, 0)\}^{\perp}$. Similarly, $\{\mathcal{S}_2 = 0\}$ is a submanifold with tangent space $\mathrm{sp}\{(d_1, d_2, 0, 0, 0)\}^{\perp}$ at (3.6). Notice that both tangent spaces contain $E_{\frac{1}{d_1}}$ but not $E_{\frac{2}{d_1}}$ and that $E_{\frac{1}{d_1}} \oplus E_{\frac{2}{d_1}}$ is the tangent space to the unstable manifold.

According to the above discussion, trajectories in the unstable manifold of (3.6) that either remain in the set $\{\mathcal{S}_1 = 0\} \cap \{\mathcal{S}_2 = 0\}$ or flow into $\{\mathcal{S}_1 < 0\} \cap \{\mathcal{S}_2 < 0\}$ need to be considered. Notice, however, that if $\varepsilon = 0$ the ODE for $\mathcal{L}$ decouples. Hence, the soliton system effectively reduces to a system in X_i, Y_i for $i = 1, 2$. Counting trajectories with respect to the possibly reduced system then gives the following result.

Proposition 3.2.2 *Suppose that $d_1 > 1$. If $\varepsilon \neq 0$, then there exists a 1-parameter family of trajectories lying both in the unstable manifold of* (3.6) *and the Einstein locus* (3.8) *and a 2-parameter family of trajectories lying both in the unstable manifold of* (3.6) *and the Ricci soliton locus* (3.9)*.*

If $\varepsilon = 0$, then the unstable manifold of (3.6) *with respect to the reduced two summands ODE in X_1, X_2 and Y_1, Y_2 contains a unique trajectory lying in the Einstein locus* (3.8) *and a 1-parameter family of trajectories lying in the Ricci soliton locus* (3.9)*. These give rise to an (up to scaling) unique Ricci flat metric and a 1-parameter family of Ricci solitons with soliton potential $u = 0$ at the singular orbit.*

Proposition 3.2.2 is in agreement with the theory of solutions to the initial value problem for cohomogeneity one Ricci solitons and Einstein metrics developed by Buzano [Buz11] and Eschenburg-Wang [EW00], respectively. Notice that their methods also carry over to the case $d_1 = 1$.

The following lemma shows a basic dynamical property of the Ricci soliton ODE and sets up the discussion of the long time behaviour.

Lemma 3.2.3 *Let $\varepsilon \geq 0$ and consider a trajectory of the two summands Ricci soliton ODE that emanates from* (3.6) *at $s = -\infty$ and enters either* (3.8) *or* (3.9)*.*

Then there holds $X_1 > 0$ for all finite s and X_2 is positive for sufficiently negative s. Moreover, suppose there is an $s_0 \in \mathbb{R}$ such that $X_2(s_0) < 0$. Then $X_2(s) < 0$ for all $s \geq s_0$.

Proof: Recall that $\lim_{s\to-\infty} X_1 = 1/d_1 > 0$ and in particular X_1 is positive initially. If there is an $s \in \mathbb{R}$ such that $X_1(s) = 0$, then $X_1'(s) > 0$. By continuity this implies $X_1 > 0$ everywhere.

The conservation law (3.7) implies that $\sum_{i=1}^2 d_i X_i^2 - 1 \leq A_3 \frac{Y_2^4}{Y_1^2} - \sum_{i=1}^2 A_i Y_i^2 < 0$ close to the initial point as $Y_1 \to \frac{1}{d_1}$ and $Y_2 \to 0$. Similarly, $\frac{A_2}{d_2} Y_2^2 - \frac{2A_3}{d_2} \frac{Y_2^4}{Y_1^2} > 0$ for sufficiently negative times. If $X_2(s_0) < 0$ in this region, then the ODE

$$X_2' = X_2 \left(\sum_{i=1}^2 d_i X_i^2 - \frac{\varepsilon}{2} \mathcal{L}^2 - 1 \right) + \frac{A_2}{d_2} Y_2^2 + \frac{\varepsilon}{2} \mathcal{L}^2 - \frac{2A_3}{d_2} \frac{Y_2^4}{Y_1^2}$$

implies that $X_2'(s_0) > 0$ as $\varepsilon \geq 0$. In particular $X_2(s) \leq X_2(s_0) < 0$ for all $s \leq s_0$. This contradicts $X_2 \to 0$ as $s \to -\infty$.

If the last statement is not true, then there exist $s_* < s^*$ such that $X_2 < 0$ on (s_*, s^*) and

$$\begin{aligned} X_2(s_*) = 0 \ \text{ and } \ X_2'(s_*) \leq 0, \\ X_2(s^*) = 0 \ \text{ and } \ X_2'(s^*) \geq 0. \end{aligned}$$

Then it follows that $\frac{A_2}{d_2} Y_2^2(s_*) + \frac{\varepsilon}{2} \mathcal{L}^2(s_*) - 2\frac{A_3}{d_2} \frac{Y_2^4}{Y_1^2}(s_*) \leq 0$ which is equivalent to

$$\frac{A_2}{d_2} \leq \left[2\frac{A_3}{d_2} \left(\frac{Y_2}{Y_1} \right)^2 - \frac{\varepsilon}{2} \left(\frac{\mathcal{L}}{Y_2} \right)^2 \right] (s_*).$$

Similarly, the second condition gives the reverse inequality at s^*. Therefore,

$$\begin{aligned} 0 &\leq \left[2\frac{A_3}{d_2} \left(\frac{Y_2}{Y_1} \right)^2 - \frac{\varepsilon}{2} \left(\frac{\mathcal{L}}{Y_2} \right)^2 \right] (s_*) - \left[2\frac{A_3}{d_2} \left(\frac{Y_2}{Y_1} \right)^2 - \frac{\varepsilon}{2} \left(\frac{\mathcal{L}}{Y_2} \right)^2 \right] (s^*) \\ &= \frac{d}{ds} \left[2\frac{A_3}{d_2} \left(\frac{Y_2}{Y_1} \right)^2 - \frac{\varepsilon}{2} \left(\frac{\mathcal{L}}{Y_2} \right)^2 \right] (\xi) \cdot (s_* - s^*) \end{aligned}$$

holds for some $\xi \in (s_*, s^*)$. On the other hand, observe that

$$\frac{d}{ds} \frac{Y_2}{Y_1} = \frac{Y_2}{Y_1}(X_1 - X_2) \ \text{ and } \ \frac{d}{ds} \frac{\mathcal{L}}{Y_2} = \frac{\mathcal{L}}{Y_2} X_2.$$

Therefore, $X_2(\xi) < 0$, $\varepsilon \geq 0$ and $s_* < s^*$ imply

$$\begin{aligned} 0 &\leq \frac{d}{ds} \left[2\frac{A_3}{d_2} \left(\frac{Y_2}{Y_1} \right)^2 - \frac{\varepsilon}{2} \left(\frac{\mathcal{L}}{Y_2} \right)^2 \right] (\xi) \cdot (s_* - s^*) \\ &= 2 \left[2\frac{A_3}{d_2} \left(\frac{Y_2}{Y_1} \right)^2 (X_1 - X_2) - \frac{\varepsilon}{2} \left(\frac{\mathcal{L}}{Y_2} \right)^2 X_2 \right] (\xi) \cdot (s_* - s^*) < 0, \end{aligned}$$

which is a contradiction. □

Remark 3.2.4 In fact, the possibility that $X_2 < 0$ is the only obstruction to long time existence. Geometrically this says that along the trajectory of an incomplete metric the shape operator cannot remain positive definite.

Proposition 5.1.3 shows conversely that, if $\varepsilon \geq 0$, any maximal Einstein or Ricci soliton trajectory on a cohomogeneity one manifold with a singular orbit and positive definite shape operator induces a complete metric.

However, if $A_3 = 0$, then $X_2 > 0$ is immediate and the Einstein and Ricci soliton loci (3.8) and (3.9), respectively, are bounded regions in phase space. Completeness of the metric then follows as in propositions 3.2.9 and 3.2.12 below. Geometrically, the case $A_3 = 0$ corresponds to the doubly warped product situation and it has been considered by Dancer-Wang [DW09a, DW09b].

If $A_3 > 0$, notice that $X_2 > 0$ clearly holds as long as $\frac{Y_2}{Y_1} < \sqrt{\frac{A_2}{2A_3}}$. Therefore, the quotient

$$\omega = \frac{Y_2}{Y_1}$$

plays a central role in the discussion. Observe that ω satisfies

$$\omega' = \omega(X_1 - X_2) \tag{3.10}$$

and in particular, the Ricci soliton equation is equivalent to an ODE system with polynomial right hand side.

The following strategy to obtain a bound on ω goes back to Böhm [Böh98], who considered the Einstein case. It requires the algebraic condition (3.12), which is not satisfied by all low-dimensional examples. As the discussion in this section also applies to the Einstein case, it provides a different approach to Böhm's results in [Böh99].

An alternative argument which is based on an estimate on the growth of the soliton potential will be given in chapter 5. This approach works in all geometric cases and the algebraic condition posed there is actually necessary for long time existence.

Fix $d_1 > 1$ and consider the function

$$\widehat{\mathcal{G}}(\omega) = \frac{A_1}{d_1}\frac{\omega^{2(d_1-1)}}{2(d_1-1)} - \frac{A_2}{d_2}\frac{\omega^{2d_1}}{2d_1} + A_3\left(\frac{1}{d_1} + \frac{2}{d_2}\right)\frac{\omega^{2(d_1+1)}}{2(d_1+1)}. \tag{3.11}$$

Then along the trajectories of the two summands Ricci soliton ODE there holds

$$\frac{d}{ds}\widehat{\mathcal{G}}(\omega) = \omega^{2(d_1-1)}\left\{\frac{A_1}{d_1} - \frac{A_2}{d_2}\omega^2 + A_3\left(\frac{1}{d_1} + \frac{2}{d_2}\right)\omega^4\right\}(X_1 - X_2).$$

Non-zero roots of $\widehat{\mathcal{G}}$ are of the form

$$\omega^2 = \frac{1}{2}\frac{A_2}{A_3}\frac{d_1+1}{2d_1+d_2}\left\{1 \pm \sqrt{1 - 4\frac{A_1 A_3}{A_2^2}\frac{d_2(2d_1+d_2)}{(d_1-1)(d_1+1)}}\right\}.$$

In particular, there exist two positive roots $0 < \hat{\omega}_1 < \hat{\omega}_2$ if and only if

$$\widehat{D} = \frac{A_2^2}{d_2^2} - 4\frac{A_1}{d_1(d_1-1)}\frac{A_3}{d_2}\frac{d_1}{d_1+1}(2d_1+d_2) > 0. \tag{3.12}$$

Moreover, in this case, $\hat{\omega}_1^2 < \frac{A_2}{2A_3}$.

Proposition 3.2.5 *Suppose that* $d_1 > 1$, $\widehat{D} > 0$ *and* $\varepsilon \geq 0$. *Then the set*

$$\left\{ X_2 > 0 \quad \textit{and} \quad 0 < \frac{Y_2}{Y_1} < \hat{\omega}_1 \right\}$$

contains any trajectory of the two summands Ricci soliton ODE that emanates from (3.6) *and flows into either* (3.8) *or* (3.9).

Proof: The ODE for X_2 shows that X_2 remains positive if $\frac{Y_2^2}{Y_1^2} = \omega^2 < \frac{A_2}{2A_3}$. Since $\hat{\omega}_1^2 < \frac{A_2}{2A_3}$, it suffices to show that $\omega < \hat{\omega}_1$ as long as $X_2 > 0$. Consider the function

$$\mathcal{K} = \frac{1}{2}\omega^{2(d_1-1)}\left(\frac{X_1 - X_2}{Y_1}\right)^2 - \widehat{\mathcal{G}}(\omega), \tag{3.13}$$

which was introduced by Böhm in the Einstein case [Böh98]. On the set $X_2 > 0$ it is a Lyapunov function since

$$\frac{d}{ds}\mathcal{K} = \omega^{2(d_1-1)}\left(\frac{X_1 - X_2}{Y_1}\right)^2\left\{\sum_{i=1}^{2} d_i X_i - 1 - (n-1)X_2\right\}$$

and $\sum_{i=1}^{2} d_i X_i - 1 \leq 0$ holds in both loci. Notice that $\lim_{s\to-\infty}\mathcal{K} = 0$ and $\mathcal{K} \geq 0$ if $\frac{Y_2}{Y_1} = \omega = \hat{\omega}_1$. However, $\mathcal{K}$ is non-increasing and strictly decreasing close to (3.6). This completes the proof. $\square$

Remark 3.2.6 In the case $d_1 = 1$ the argument of proposition 3.2.5 needs to be modified. However, in geometric applications one has $A_1 = d_1(d_1-1) = 0$ and with the convention that $\frac{A_1}{d_1(d_1-1)} = 1$ the arguments above carry over. In particular, this induces Ricci soliton metrics on complex line bundles over a Fano Kähler Einstein manifold. More details will be discussed in section 3.3.

However, recall that this approach is not specific to the Ricci soliton case. In the Einstein case, the equations can in fact be integrated explicitly, cf. [BB82]. In the Ricci flat case, Appleton [App18] observes that if $A_2^2 > 4d_2A_3$ this Bérard-Bergery type ansatz does not yield any complete metric.

Remark 3.2.7 The function $\mathcal{K}$ is modelled on the first derivative (3.10) of ω. Its second derivative satisfies

$$\omega'' = \omega(X_1 - X_2)^2 + \omega \left\{ (X_1 - X_2) \left(\sum_{i=1}^{2} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - 1 \right) + Y_1^2 f(\omega) \right\},$$

where

$$f(\omega) = \frac{A_1}{d_1} - \frac{A_2}{d_2}\omega^2 + A_3 \left(\frac{1}{d_1} + \frac{2}{d_2} \right) \omega^4. \tag{3.14}$$

In particular the function $\widehat{\mathcal{G}}$ is an *explicit* antiderivative of a weighted version of $f(\omega)(X_1 - X_2)$.

Corollary 3.2.8 *Suppose that* $d_1 > 1$, $\widehat{D} > 0$ *and* $\varepsilon \geq 0$. *Then along trajectories emanating from* (3.6) *and flowing into* (3.8) *or* (3.9) *there holds* $X_1,\ X_2 > 0$ *for all finite times. Moreover, the variables* X_1, X_2 *and* Y_1, Y_2 *and* ω *are bounded and if* $\varepsilon > 0$, *then* $\mathcal{L}$ *is bounded too. In particular, the rescaled flow exists for all times.*

Proof: According to lemma 3.2.3 one has $X_1,\ X_2 > 0$ initially and $X_1 > 0$ is preserved along the flow. Positivity of X_2 follows from proposition 3.2.5 and $X_1,\ X_2$ remain bounded as $0 \leq d_1 X_1 + d_2 X_2 \leq 1$ due to (3.8) and (3.9).

Then the ODE for $\mathcal{L}$ implies that $\mathcal{L}$ cannot blow up in finite time as $\varepsilon \geq 0$. By the same argument, this also holds for $Y_1,\ Y_2$.

Alternatively, it follows from the bound $\frac{Y_2^2}{Y_1^2} < \hat{\omega}_1^2 < \frac{A_2}{2A_3}$ that $A_2 - A_3 \frac{Y_2^2}{Y_1^2} > \frac{A_2}{2}$. The Einstein and Ricci soliton loci (3.8) and (3.9) are therefore contained in the bounded region $\left\{ \sum_{i=1}^{2} d_i X_i^2 + A_1 Y_1^2 + \frac{A_2}{2} Y_2^2 + (n-1)\frac{\varepsilon}{2}\mathcal{L}^2 \leq 1 \right\}$. By considering $\omega = \frac{Y_2}{Y_1}$ as an independent variable, one obtains an ODE system with polynomial right hand side. Since $\omega < \hat{\omega}_1$ is bounded, standard ODE theory implies that the flow exists for all times. □

In order to prove that the corresponding metrics are complete, it suffices to show that $t_{\max} = \infty$. Recall from the coordinate change that

$$t(s) = t(s_0) + \int_{s_0}^{s} \mathcal{L}(\tau) d\tau. \tag{3.15}$$

Therefore it is necessary to estimate the asymptotic behaviour of $\mathcal{L}$. This needs to be considered separately for the cases $\varepsilon = 0$ and $\varepsilon > 0$.

Proposition 3.2.9 *Suppose that* $d_1 > 1$ *and* $\widehat{D} > 0$. *Then the corresponding steady Ricci soliton and Ricci flat metrics are complete.*

Proof: A special feature of the case $\varepsilon = 0$ is that $\mathcal{L} = \frac{1}{-\dot{u}+\mathrm{tr}(L)}$ is in fact a *Lyapunov* function. As $\mathcal{L}$ becomes positive initially and is therefore monotonically increasing, it is bounded away from zero for $s \geq s_0$ and any $s_0 \in \mathbb{R}$. Then the time rescaling (3.15) shows that $t \to \infty$ as $s \to \infty$, i.e. the metrics are complete. □

Remark 3.2.10 The Ricci flat metrics in proposition 3.2.9 have first been constructed by Böhm [Böh99] by different means, see remark 3.4.8 and section 3.5.1.

The cases of expanding Ricci solitons and Einstein metrics with negative scalar curvature correspond to $\varepsilon > 0$. It will be sufficient to have an upper bound on $\mathcal{L}$ to prove completeness of the metric.

Lemma 3.2.11 *Let $d_1 > 1$, $\widehat{D} > 0$ and $\varepsilon > 0$. Then along trajectories emanating from* (3.6) *and flowing into* (3.8) *or* (3.9) *there holds*

$$0 < \frac{\varepsilon}{2}\mathcal{L}^2 \leq \max\left\{\frac{1}{d_1}, \frac{1}{d_2}\right\}.$$

Moreover, in the Einstein case, given $s_0 \in \mathbb{R}$ there holds

$$\frac{\varepsilon}{2}\mathcal{L}^2(s) \geq \min\left\{\frac{\varepsilon}{2}\mathcal{L}^2(s_0), \frac{1}{n}\right\}$$

for all $s \geq s_0$. In particular, $\mathcal{L}(s)$ is bounded away from zero for all $s \geq s_0$.

Proof: Notice that $0 \leq \sum_{i=1}^2 d_i X_i^2 \leq \max\left\{\frac{1}{d_1}, \frac{1}{d_2}\right\}$ on $X_1, X_2 \geq 0$ and $\sum_{i=1}^2 d_i X_i \leq 1$. Therefore, if there is an s_0 such that $\frac{\varepsilon}{2}\mathcal{L}^2(s_0) > \max\left\{\frac{1}{d_1}, \frac{1}{d_2}\right\}$ then $\mathcal{L}'(s_0) < 0$. This yields $\mathcal{L}(s) \geq \mathcal{L}(s_0)$ for all $s \leq s_0$, which contradicts $\lim_{s\to-\infty}\mathcal{L} = 0$.

To prove the second statement, suppose that $\frac{\varepsilon}{2}\mathcal{L}^2(s_0) < \frac{1}{n}$. Since $\sum_{i=1}^2 d_i X_i = 1$ in the Einstein locus, one has $\sum_{i=1}^2 d_i X_i^2 \geq \frac{1}{n}$ and therefore $\mathcal{L}'(s_0) > 0$. Hence, $\frac{\varepsilon}{2}\mathcal{L}^2$ is monotonically increasing whenever it is less then $\frac{1}{n}$. □

Corollary 3.2.12 *Suppose that $d_1 > 1$ and $\widehat{D} > 0$. Then the corresponding expanding Ricci solitons and Einstein metrics of negative scalar curvature are complete.*

Proof: Suppose for contradiction that $t_{\max} < \infty$. By (3.15) this is equivalent to saying that $\|\mathcal{L}\|_{L^1(0,\infty)} < \infty$. However, since $\mathcal{L}$ is bounded due to lemma 3.2.11, this implies $\mathcal{L} \in L^2(0,\infty)$. Hence the ODE for $\mathcal{L}$ yields $\mathcal{L}(s) \geq \mathcal{L}(0)\exp\left(-\frac{\varepsilon}{2}\|\mathcal{L}\|_{L^2(0,\infty)}\right) > 0$ and $\mathcal{L}$ is bounded away from zero for $s \geq 0$. However, this contradicts $\mathcal{L} \in L^1(0,\infty)$. □

Remark 3.2.13 The Einstein metrics of negative scalar curvature in corollary 3.2.12 have first been constructed by Böhm [Böh99] by different means, see remark 3.4.13.

3.3 Ricci solitons from circle bundles

The two summands case allows the possibility $d_1 = 1$ where examples of *Kähler* Ricci solitons have been found by Cao-Koiso [Cao96], [Koi90] and Feldman-Ilmanen-Knopf [FIK03]. These examples are constructed on manifolds foliated by principal circle bundles over a Fano Kähler-Einstein manifold (V, J, g).

The precise geometric set-up is as follows: Recall that due to a theorem of Kobayashi [Kob61] any Fano manifold V is simply connected and hence $H^2(V_i, \mathbb{Z})$ is torsion free. Therefore the first Chern class is $c_1(V, J) = p\rho$ for a positive integer p and an indivisible class $\rho \in H^2(V, \mathbb{Z})$. Suppose that the Ricci curvature of (V, g) is normalised to be $\mathrm{Ric} = pg$. If $\pi\colon P \to V$ is the principal circle bundle with Euler class $q\pi^*\rho$ for a non-zero integer $q \in \mathbb{Z} \setminus \{0\}$ and θ the principal S^1-connection with curvature form $\Omega = q\pi^*\eta$, where η is the Kähler form associated to g, then the Ricci soliton equation on $I \times P$ corresponding to the metric

$$dt^2 + f_1^2(t)\theta \otimes \theta + f_2^2(t)\pi^* g$$

is described by the two summands system with $d_1 = 1$, $d_2 = d = \dim_{\mathbb{R}} V$ and $A_1 = 0$, $A_2 = d_2 p$, $A_3 = \frac{d_2 q^2}{4}$. Notice also that the structure of the ODE has changed since $A_1 = 0$. If the smoothness conditions (3.2) are satisfied, this construction induces a smooth metric on the associated complex line bundle over V.

Metrics whose curvature tensor is invariant under the complex structure are considered by Dancer-Wang [DW11] in the Ricci soliton case and by Wang-Wang [WW98] in the Einstein case. This condition is equivalent to saying that

$$\frac{\dot{f}_2^2}{f_2^2} - \frac{q^2}{4}\frac{f_1^2}{f_2^4} = \left(-\dot{u} + \mathrm{tr}(L) + \frac{\dot{f}_1}{f_1}\right)\frac{\dot{f}_2}{f_2} - \frac{p}{f_2^2} + \frac{\varepsilon}{2}.$$

As a special case, the *Kähler* condition reads

$$\frac{\dot{f}_2}{f_2} = -\frac{q}{2}\frac{f_1}{f_2^2}$$

and it is preserved by the flow. In both cases, the equations can actually be integrated *explicitly.* In order to investigate *non-Kähler* trajectories, the Ricci soliton ODE will be studied qualitatively as before. To adjust the argument in proposition 3.2.5 to the conditions $d_1 = 1$ and $A_1 = 0$, adopt the convention $\frac{A_1}{d_1(d_1-1)} = 1$. That is, consider

$$\widehat{\mathcal{G}}(\omega) = \frac{1}{2} - \frac{p}{2}\omega^2 + \frac{d+2}{16}q^2\omega^4,$$
$$\mathcal{K} = \frac{1}{2}\left(\frac{X_1 - X_2}{Y_1}\right)^2 - \widehat{\mathcal{G}}(\omega)$$

and note that $\widehat{\mathcal{G}}$ has two positive roots $0 < \hat{\omega}_1 < \hat{\omega}_2$ if $2p^2 > (d+2)q^2$. Then the proof of proposition 3.2.5 shows

Proposition 3.3.1 *Suppose that $d_1 = 1$, $A_1 = 0$ and $2p^2 > (d+2)q^2 > 0$. If $\varepsilon \geq 0$, the set*

$$\left\{ X_2 > 0 \quad \textit{and} \quad 0 < \frac{Y_2}{Y_1} < \hat{\omega}_1 \right\}$$

contains any trajectory of the S^1-bundle Ricci soliton ODE that emanates from (3.6) *and flows into either* (3.8) *or* (3.9).

Completeness of the metric can then be established as in proposition 3.2.9 and corollary 3.2.12. Notice in particular that long time existence still follows from corollary 3.2.8. As the proof shows, even though Y_1 is not controlled by the conservation law (3.7) anymore since $A_1 = 0$, it cannot blow up in finite time.

Corollary 3.3.2 *Let $d_1 = 1$, $A_1 = 0$, $2p^2 > (d+2)q^2 > 0$ and $\varepsilon \geq 0$. Then any trajectory of the Ricci soliton ODE emanating from the critical point* (3.6) *and lying in the Einstein locus* (3.8) *or Ricci soliton locus* (3.9) *corresponds to a complete Einstein or Ricci soliton metric, respectively.*

Remark 3.3.3 (a) Notice on the contrary that the construction of Kähler Ricci solitons due to Feldman-Ilmanen-Knopf [FIK03] requires the condition $-q = p$ in the steady case and $-q > p$ in the expanding case, see also [DW11, Theorem 4.20 and Remark 4.21]. For example, in the case of $\mathbb{C}P^n$ one has $p = \frac{d+2}{2}$ and one thus requires $p > q^2 > 0$ for the argument of proposition 3.3.1 to work. In particular, the Kähler examples due to Feldman-Ilmanen-Knopf are not covered by the corollary. In the case of $\mathbb{C}P^n$, these Kähler Ricci soliton metrics have also been investigated by Chave-Valent [CV96].

(b) Explicit Kähler and non-Kähler Einstein metrics have already been described by Bérard-Bergery [BB82].

(c) If $q > p$, Appleton [App18] proves that there cannot exist a complete Ricci flat metric on the associated complex line bundle. In particular, the corresponding trajectory cannot satisfy the bound $\omega < \frac{4p}{(d+2)q^2}$ for all times.

Observe that the initial stationary point (3.6) is not hyperbolic in the case $d_1 = 1$. Therefore a center manifold exists and the analysis before proposition 3.2.2 does not carry over. However, the work of Buzano [Buz11] and Eschenburg-Wang [EW00] still applies and the existence of Ricci soliton trajectories can be deduced, see also [Sto17] or [App18] for different arguments.

Theorem 3.3.4 *Suppose that $d_1 = 1$, $A_1 = 0$ and $2p^2 > (d+2)q^2 > 0$.*

If $\varepsilon = 0$ there exists a 1-parameter family and if $\varepsilon \neq 0$ a 2-parameter family of trajectories lying in both the unstable manifold of (3.6) *and the Ricci soliton locus* (3.9)*. In particular, these give rise to complete Ricci soliton metrics on the total spaces of the corresponding complex line bundles over Fano Kähler-Einstein manifolds.*

Similarly, there exist a (up to homotheties) unique complete Ricci flat metric and a 1-parameter family of complete Einstein metrics with negative scalar curvature on these spaces.

3.4 Asymptotics

This section discusses the asymptotic behaviour of the metrics. It will be shown that the steady Ricci solitons are asymptotically paraboloid and the expanding Ricci solitons as well as the Ricci flat metrics are asymptotically conical.

3.4.1 Cone solutions

The following construction of cohomogeneity one Einstein metrics from homogeneous ones is well known, cf. [Böh99], and partially carries over to the Ricci soliton case.

Proposition 3.4.1 *Let (P, g_E) be a homogeneous space with $Ric = (n-1)g_E$. Then the metrics*

$$dt^2 + \sin^2(t) g_E \quad \text{for} \quad t \in (0, \pi),$$
$$dt^2 + t^2 g_E \quad \text{and} \quad dt^2 + \sinh^2(t) g_E \quad \text{for} \quad t > 0$$

define cohomogeneity one Einstein metrics on $(0, \frac{\pi}{2}) \times P$ and $(0, \infty) \times P$ with Einstein constant $-\frac{\varepsilon}{2} = n, 0, -n$, respectively. Any of these solutions will be called cone solution.

Furthermore, the Ricci flat metrics together with the soliton potential $-\dot{u}(t) = \frac{\varepsilon}{2}t$ induce a shrinking or expanding Ricci soliton on $(0, \infty) \times P$ depending on whether $\varepsilon < 0$ or $\varepsilon > 0$. If $\varepsilon < 0$ these solutions are called conical Gaussians.

If P is the principal orbit of a cohomogeneity one manifold with one or two singular orbits, then proposition 3.4.1 defines canonical Einstein metrics on the union of all principal orbits. These have conical singularities at the singular orbits unless each singular orbit consists of a point. In this case the metrics correspond to the standard metrics on S^{n+1}, $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$, respectively.

Remark 3.4.2 In the two summands case, according to proposition 3.4.19, the steady Ricci solitons satisfy asymptotically

$$-\dot{u}(t) \sim \sqrt{-C} \quad \text{and} \quad g_t \sim \frac{2(n-1)}{\sqrt{-C}} \cdot t g_E$$

for large t. However, assuming the above exact paraboloid geometry of the metric, the Ricci soliton equations cannot be coherently solved for the potential u.

The Ricci endomorphism of the principal orbit in the two summands case is given by (3.3) and its normalisation to $(n-1)\mathbb{I}_n$ leads to the following definition:

Definition 3.4.3 *Positive solutions* (c_1, c_2) *to the equations*

$$(n-1)d_1 = \frac{A_1}{c_1^2} + A_3\frac{c_1^2}{c_2^4} \quad \textit{and} \quad (n-1)d_2 = \frac{A_2}{c_2^2} - 2A_3\frac{c_1^2}{c_2^4} \tag{3.16}$$

are called cone solutions.

Remark 3.4.4 If $A_3 > 0$, the cone solutions take the explicit form

$$\begin{aligned} c_1^2 &= \frac{1}{2d_1+d_2}\left(\frac{A_2^2 d_1 + 4A_1A_3(2d_1+d_2)}{2A_3(n-1)(2d_1+d_2)} \mp \sqrt{D}\right), \\ c_2^2 &= \frac{1}{2d_1+d_2}\left(A_2 n \pm 2A_3(2d_1+d_2)\sqrt{D}\right), \end{aligned}$$

where the discriminant D is given by

$$D = \left(\frac{A_2}{2A_3}\frac{d_1}{2d_1+d_2}\right)^2 - \frac{A_1}{A_3}\frac{d_2}{2d_1+d_2}. \tag{3.17}$$

Inserting the geometric definitions of the constants A_1, A_2, A_3 into (3.17), one obtains

$$D \geq 0 \ \text{ if and only if } \ \frac{(\mathrm{Ric}^{G/H})^2}{4||A||^2} \geq (2d_1+d_2)\frac{d_1-1}{d_1}.$$

Suppose that there are two real cone solutions. For a cone solution (c_1, c_2), set $\omega = \frac{c_1}{c_2}$. Then the ordering $\omega_1 < \omega_2$ defines the *first* and *second* cone solution.

In particular, if $\widehat{D} > 0$, cf. (3.12), there exist two cone solutions and it is easy to check that $\omega_1 < \hat{\omega}_1 < \omega_2 < \hat{\omega}_2$ holds in this case. This has also been observed by Böhm [Böh98].

However, notice that if $D < 0$ the cone solutions are complex. In this case the solution of the corresponding ODE system does not induce a complete metric. For example, if X_2 remains positive along the flow, proposition 3.4.7 below shows that in the Ricci flat case $(nY_i)^{-1}$ tends to the first cone solution. As the real ODE system cannot develop complex solutions, it follows that if the cone solutions defined by the parameters $d_1, d_2,\ A_1, A_2, A_3$ are complex, X_2 becomes negative and in fact the solution to the ODE system blows up in finite time. Similar statements also hold in the other cases.

Let $D \geq 0$ and let (c_1, c_2) be a cone solution as in definition 3.4.3. With the normalisation of the Einstein constant $-\frac{\varepsilon}{2} \in \{-n, 0, n\}$, the two summands Einstein cone solutions of proposition 3.4.1 take the form

$$f_i(t) = c_i \sin(t) \quad \text{for} \quad t \in (0, \pi)$$

in the case of positive scalar curvature and

$$f_i(t) = c_i t \quad \text{and} \quad f_i(t) = c_i \sinh(t) \quad \text{for} \quad t > 0$$

in the Ricci flat and negative scalar curvature case, respectively. Any cone solution is called *first* cone solution if that is the case for the pair (c_1, c_2) as remark 3.4.4.

Example 3.4.5 Recall the examples of group diagrams in table 3.1.1, which induce the Hopf fibrations. In the $\mathbb{H}P^{m+1}$-example the cone solutions are

$$\begin{aligned} c_1^2 &= \frac{9 + 14m + 4m^2}{(1 + 2m)(3 + 2m)^2} \quad \text{and} \quad c_2^2 = \frac{9 + 14m + 4m^2}{(1 + 2m)(3 + 2m)}, \\ c_1^2 &= c_2^2 = 1, \end{aligned}$$

in the F^{m+1}-example they are given by

$$\begin{aligned} c_1^2 &= \frac{(1 + m)^2 + m}{(1 + m)^2(1 + 4m)} \quad \text{and} \quad c_2^2 = 4\frac{(1 + m)^2 + m}{(2m + 1)^2 + m}, \\ c_1^2 &= \frac{1 + m}{1 + 4m} \quad \text{and} \quad c_2^2 = 4c_1^2. \end{aligned}$$

and in the CaP^2-example they are $c_1^2 = \frac{57}{121}$, $c_2^2 = \frac{19}{11}$ and $c_1^2 = c_2^2 = 1$. In all cases, the first pair also describes the first cone solution.

Recall from remark 3.4.4 that if $D > 0$ there are two cone solutions and the ordering $\omega_1 < \omega_2$ of the associated ratios $\omega = \frac{c_1}{c_2}$ defines the first and second cone solution. The following characterisation of ω_1 and ω_2 is elementary but important.

Proposition 3.4.6 *Let $D > 0$. Then the two positive roots of the function*

$$f(\omega) = \frac{A_1}{d_1} - \frac{A_2}{d_2}\omega^2 + A_3\left(\frac{1}{d_1} + \frac{2}{d_2}\right)\omega^4 \tag{3.14}$$

are the ratios ω_1, ω_2 of the first and second cone solution, respectively, i.e.

$$\omega_1^2 = \frac{A_2}{2A_3}\frac{d_1}{2d_1+d_2} - \sqrt{D},$$
$$\omega_2^2 = \frac{A_2}{2A_3}\frac{d_1}{2d_1+d_2} + \sqrt{D},$$

In particular, it follows that $\omega_1^2 < \frac{A_2}{4A_3}$ and $\omega_2^2 < \frac{A_2}{2A_3}$.

Proof: This is immediate from definition 3.4.3. □

In the next section it will be shown that the two summands cone solutions are attained asymptotically as $t \to \infty$. Moreover, in the Einstein case, proposition 3.5.2 also shows that the cone solutions yield the limiting behaviour of the metrics as the volume of the singular orbit tends to zero. This is a result due to Böhm [Böh98]. However, the new choice of the coordinate system simplifies the proof considerably as it does not rely on the Poincaré-Bendixson Theorem.

3.4.2 Ricci flat metrics

The Ricci flat trajectories are particularly easy to analyse in the chosen coordinate system. This will also be crucial for the reconstruction of Böhm's Einstein metrics of positive scalar curvature in section 3.5. It will be shown there that the asymptotics of the *Ricci flat* metrics induce Einstein metrics of positive scalar curvature on spheres and other low dimensional spaces.

It will follow directly from proposition 3.4.7 below that the Ricci flat metrics are asymptotically conical.

Proposition 3.4.7 *Let $d_1 > 1$ and $\widehat{D} > 0$. Then along the trajectories of the Ricci flat system the variables X_1, X_2, Y_1, Y_2 converge to finite positive values as $s \to \infty$. More precisely, $X_i \to \frac{1}{n}$ and $Y_i \to \frac{1}{nc_i}$ as $s \to \infty$, where (c_1, c_2) denotes the first cone solution.*

Proof: Recall from corollary 3.2.8 that the variables X_i, Y_i for $i = 1, 2$, are all positive and bounded along the flow since $d_1 > 1$ and $\widehat{D} > 0$. To deduce the asymptotics, consider the function

$$\mathcal{G} = Y_1^{d_1} Y_2^{d_2},$$

which is in fact the inverse of Böhm's Lyapunov (2.9) in the X-Y-coordinates. Its derivative is given by

$$\mathcal{G}' = n\mathcal{G}\left\{\sum_{i=1}^{2} d_i X_i^2 - \frac{1}{n}\right\}$$

and hence it is non-decreasing and bounded. Thus, it converges to a finite positive limit as $s \to \infty$. This also shows that Y_1, Y_2 are bounded away from zero as $s \to \infty$. Standard ODE theory now implies that the ω-limit set Ω of the flow of X_1, X_2, Y_1, Y_2 is non-empty, compact, connected and flow invariant. As $\mathcal{G}$ is monotonic and bounded, it must be constant on Ω. But since $d_1 X_1 + d_2 X_2 = 1$ there holds $\mathcal{G}' = 0$ if and only if $X_1 = X_2 = \frac{1}{n}$. Moreover, this yields

$$
\begin{aligned}
0 = X_1' &= \frac{1}{n}\left(\frac{1}{n} - 1\right) + \frac{A_1}{d_1} Y_1^2 + \frac{A_3}{d_1}\frac{Y_2^4}{Y_1^2}, \\
0 = X_2' &= \frac{1}{n}\left(\frac{1}{n} - 1\right) + \frac{A_2}{d_2} Y_2^2 - 2\frac{A_3}{d_2}\frac{Y_2^4}{Y_1^2}
\end{aligned}
$$

on the ω-limit set. In particular, the pair $((nY_1)^{-1}, (nY_2)^{-1})$ satisfies the equations (3.16) of the cone solutions. Since the bound $Y_2/Y_1 < \hat{\omega}_1$ holds along the flow and $\omega_1 < \hat{\omega}_1 < \omega_2 < \hat{\omega}_2$, it follows that $Y_i \to \frac{1}{nc_i}$ as $s \to \infty$, where (c_1, c_2) describes the first cone solution. This completes the proof. $\square$

The asymptotic behaviour of the metric can be deduced from $\dot{f}_i = \frac{X_i}{Y_i} \to c_i$ as $t \to \infty$. The metric is therefore approximately conical at infinity.

Remark 3.4.8 Böhm's construction of the above Ricci flat metrics is based on the fact that the Ricci flat trajectories approach the cone solution as $f_2(0) \to 0$, see [Böh99, Theorem 11.1] and [Böh98, Theorem 5.7].

However, the Ricci flat metric is unique up to scaling, hence it is asymptotically conical for any initial value $f_2(0) > 0$.

Conversely, proposition 3.5.2 gives an alternative proof of Böhm's convergence result which is based on proposition 3.4.7.

3.4.2.1 Explicit trajectories and rotational behaviour

It is a special feature of the Ricci flat equation that it reduces to a planar system for the variables X_1, Y_1 as the variable $\mathcal{L}$ decouples completely. More generally, in the Einstein case there holds $X_2 = \frac{1}{d_2}(1 - d_1 X_1)$ and the conservation law then determines Y_2 in terms of X_1, Y_1 and $\frac{\varepsilon}{2}\mathcal{L}^2$. Explicitly, it is given by

$$
Y_2^2 = \frac{A_2}{2A_3} Y_1^2 \pm \frac{1}{2A_3}\sqrt{A_2^2 Y_1^4 + 4A_3\left(\sum_{i=1}^{2} d_i X_i^2 + (n-1)\frac{\varepsilon}{2}\mathcal{L}^2 - 1 + A_1 Y_1^2\right) Y_1^2}.
$$

Initially, Y_2 is given by the solution corresponding to '$-$' as $\lim_{s \to -\infty} Y_2 = 0$. Notice that the discriminant vanishes if and only if $Y_2^2/Y_1^2 = \frac{A_2}{2A_3}$ and recall that if $d_1 > 1$,

$\widehat{D} > 0$ and $\varepsilon \geq 0$ the estimate $Y_2^2/Y_1^2 < \hat{\omega}_1^2 < \frac{A_2}{2A_3}$ has been established. Hence, in this case, only the '$-$' solution is realised by the flow.

Therefore, consider the ODE system

$$\begin{aligned}
X_1' = &\left(X_1 + \frac{1}{d_1}\right)\left(n\frac{d_1}{d_2}X_1^2 - 2\frac{d_1}{d_2}X_1 + \frac{1}{d_2} - \frac{\varepsilon}{2}\mathcal{L}^2 - 1\right) \\
&+ \left(2A_1 + \frac{A_2^2}{2A_3}\right)\frac{Y_1^2}{d_1} + \frac{\varepsilon}{2}\left(1 + \frac{n}{d_1}\right)\mathcal{L}^2 \\
&- \frac{A_2}{2d_1A_3}\sqrt{A_2^2Y_1^4 + 4A_3\left(\sum_{i=1}^{2} d_iX_i^2 + (n-1)\frac{\varepsilon}{2}\mathcal{L}^2 - 1 + A_1Y_1^2\right)Y_1^2}, \\
Y_1' = &\, Y_1\left(n\frac{d_1}{d_2}X_1^2 - 2\frac{d_1}{d_2}X_1 + \frac{1}{d_2} - \frac{\varepsilon}{2}\mathcal{L}^2 - X_1\right), \\
\mathcal{L}' = &\, \mathcal{L}\left(n\frac{d_1}{d_2}X_1^2 - 2\frac{d_1}{d_2}X_1 + \frac{1}{d_2} - \frac{\varepsilon}{2}\mathcal{L}^2\right).
\end{aligned}$$

In the Ricci flat case this yields indeed a 2-dimensional system for X_1 and Y_1. Moreover, one has $\mathcal{L}(s) = \mathcal{L}(s_0)\exp\left[\int_{s_0}^{s}\left(n\frac{d_1}{d_2}X_1^2 - 2\frac{d_1}{d_2}X_1 + \frac{1}{d_2}\right)d\tau\right]$.

Recall that one expects $(X_1, Y_1) \to (\frac{1}{n}, \frac{1}{nc_1})$ as $s \to \infty$ if the cone solutions are real. To study the dynamics of the planar (X_1, Y_1)-system close to the stationary point $(\frac{1}{n}, \frac{1}{nc_1})$, consider its linearisation at that point. It is described by the matrix

$$\begin{pmatrix} -\frac{n-1}{n} & 2\frac{c_1}{n}\left[n - 1 - 2A_3\frac{c_1^2}{c_2^4}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\right] \\ -\frac{1}{c_1 n} & 0 \end{pmatrix}.$$

The eigenvalues are the solutions to the quadratic equation

$$\lambda^2 + \frac{n-1}{n}\lambda + \frac{2}{n^2}\left[n - 1 - 2A_3\frac{c_1^2}{c_2^4}\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\right] = 0$$

and it is therefore easy to deduce:

Corollary 3.4.9 *The limiting point of the Ricci flat trajectories is a stable spiral if and only if*

$$\frac{(n-1)(n-9)}{8} + 2A_3\frac{c_1^2}{c_2^4}\left(\frac{1}{d_1} + \frac{1}{d_2}\right) < 0. \tag{3.18}$$

In particular, if $A_3 = 0$ this is equivalent to $2 \leq n \leq 8$. Otherwise, it is a stable node.

The case $A_3 = 0$ will be particularly important for applications to Böhm's Einstein metrics of positive curvature. In fact, in his construction, Böhm necessarily considered a system with the same qualitative behaviour [Böh98].

The reduction to the planar (X_1, Y_1)-system can also be used to describe explicit trajectories. The key idea is to introduce polar coordinates centred at $(\frac{1}{n}, \frac{1}{nc_1})$. To this end, let $\overline{X} = X_1 - \frac{1}{n}$ and $\overline{Y} = Y_1 - \frac{1}{nc_1}$. Then trajectories which correspond to smooth complete Ricci flat metrics emanate from $(\frac{1}{d_1} - \frac{1}{n}, \frac{1}{d_1} - \frac{1}{nc_1})$ and are expected to converge to the origin. In low dimensional examples, these trajectories are actually realised by straight lines!

This provides a new coordinate representation of metrics of special holonomy considered in [BS89] and [GPP90].

Theorem 3.4.10 *On the open disc bundles which are associated to the group diagrams $G = Sp(1) \times Sp(2)$, $H = Sp(1) \times Sp(1) \times Sp(1)$, $K = Sp(1) \times Sp(1)$ and $G = Sp(2)$, $H = Sp(1) \times Sp(1)$, $K = U(1) \times Sp(1)$ the trajectories of the complete Ricci flat two summands metrics are line segments when represented in the above coordinate system.*

Indeed, if polar coordinates are introduced in the standard way, then a straightforward calculation reveals that in the above instances the angle in fact remains constant.

To at least state the equations in the new coordinates, let

$$\begin{aligned}
a_0 &= \frac{a_4}{n^4 c_1^4} - \frac{4}{n^2 c_1^2} A_3 \frac{n-1}{n} \quad \text{and} \quad a_1 = \frac{3a_4}{n^3 c_1^3} - \frac{8}{nc_1} A_3 \frac{n-1}{n}, \\
a_2 &= \frac{6a_4}{n^2 c_1^2} - 4A_3 \frac{n-1}{n} \qquad \text{and} \quad a_3 = \frac{4a_4}{nc_1}, \\
a_4 &= A_2^2 + 4A_1 A_3,
\end{aligned}$$

and observe that the identities

$$\sum_{i=1}^{2} d_i X_i^2 = n\frac{d_1}{d_2} X_1^2 - 2\frac{d_1}{d_2} X_1 + \frac{1}{d_2} = n\frac{d_1}{d_2}\overline{X}^2 + \frac{1}{n}$$

and

$$a_4 Y_1^4 + 4A_3 Y_1^2 \left(\sum_{i=1}^{2} d_i X_i^2 - 1 \right) = \sum_{i=0}^{4} a_i \overline{Y}^i + 4A_3 n \frac{d_1}{d_2} \overline{X}^2 \left(\overline{Y}^2 + \frac{2}{nc_1}\overline{Y} + \frac{1}{n^2 c_1^2} \right)$$

imply that, in the case $A_3 > 0$, the ODE system can be rewritten as

$$\begin{aligned}
\overline{X}' &= \overline{X}\left(n\frac{d_1}{d_2}\overline{X}^2 + \frac{2d_1+d_2}{d_2}\overline{X} - \frac{n-1}{n}\right) + \left(2A_1 + \frac{A_2^2}{2A_3}\right)\frac{\overline{Y}}{d_1}\left(\overline{Y} + \frac{2}{nc_1}\right) \qquad (3.19)\\
&\quad + \left(-(d_1+n)(n-1) + \left(2A_1 + \frac{A_2^2}{2A_3}\right)\frac{1}{c_1^2}\right)\frac{1}{n^2 d_1}\\
&\quad - \frac{A_2}{2d_1A_3}\sqrt{\sum_{i=0}^{4} a_i\overline{Y}^i + 4A_3 n\frac{d_1}{d_2}\overline{X}^2\left(\overline{Y}^2 + \frac{2}{nc_1}\overline{Y} + \frac{n^2}{c_1^2}\right)},\\
\overline{Y}' &= \overline{X}\left(n\frac{d_1}{d_2}\overline{X} - 1\right)\left(\overline{Y} + \frac{1}{nc_1}\right),
\end{aligned}$$

whereas if $A_3 = 0$ the ODE in $\overline{X}$, $\overline{Y}$ takes the form

$$\begin{aligned}
\overline{X}' &= \overline{X}\left(n\frac{d_1}{d_2}\overline{X}^2 + \frac{d_1}{d_2}\overline{X} - \frac{n-1}{n}\right) + \frac{A_1}{d_1}\overline{Y}\left(\overline{Y} + \frac{2}{c_1 n}\right),\\
\overline{Y}' &= \overline{X}\left(n\frac{d_1}{d_2}\overline{X} - 1\right)\left(\overline{Y} + \frac{1}{nc_1}\right).
\end{aligned}$$

Remark 3.4.11 In geometrically relevant cases, the new variables $\overline{X}, \overline{Y}$ are expected to tend to zero as $s \to \infty$. Then the occurring terms of order zero have to cancel in the above system. Indeed, notice that the equations in (3.16) yield the formula

$$\frac{1}{n^2c_2^2} = \frac{A_2}{2A_3}\frac{1}{n^2c_1^2} - \frac{1}{2A_3}\sqrt{\frac{A_2^2}{n^4c_1^4} + 4A_3\left(\frac{1}{n} - 1 + \frac{A_1}{n^4c_1^4}\right)}.$$

Then the identity

$$\frac{A_2}{2d_1A_3}\sqrt{a_0} = \frac{A_2}{d_1}\left\{\frac{A_2}{2A_3}\frac{1}{n^2c_1^2} - \frac{1}{n^2c_2^2}\right\} \stackrel{!}{=} -\left(\frac{1}{n} + \frac{1}{d_1}\right)\frac{n-1}{n} + \left(2A_1 + \frac{A_2^2}{2A_3}\right)\frac{1}{d_1n^2c_1^2}$$

can easily be reduced to the definition (3.16) of the cone solutions.

3.4.3 Einstein metrics with negative scalar curvature

The asymptotic behaviour of the metric will again be derived from the long time behaviour of the rescaled system. It will follow that both f_1, f_2 eventually grow exponentially.

With the help of the Böhm functional one can then deduce that the metric asymptotically approaches the first cone solution of proposition 3.4.1.

Proposition 3.4.12 *Let $d_1 > 1$ and $\widehat{D} > 0$. Then the asymptotic behaviour of the trajectories of complete Einstein metrics with negative scalar curvature is given by*

$$X_1, X_2 \to \frac{1}{n} \quad \textit{and} \quad Y_1, Y_2 \to 0,\ \frac{Y_2}{Y_1} \to \omega_1 \quad \textit{and} \quad \mathcal{L} \to \sqrt{\frac{2}{n\varepsilon}}$$

as $s \to \infty$, where $\omega_1 = \frac{c_1}{c_2}$ is the ratio of the first cone solution.

Proof: As in the proof of corollary 3.2.8, introduce the variable $\omega = \frac{Y_2}{Y_1}$ in order to view the Ricci soliton equation as an ODE with polynomial right hand side. Furthermore, all variables remain bounded along the flow and hence the ω-limit set Ω is non-empty, connected, compact and flow-invariant.

Recall from lemma 3.2.11 that $\mathcal{L}(s)$ is bounded away from zero for $s \geq 0$. As the quotients $\frac{Y_i}{\mathcal{L}}$ satisfy $\frac{d}{ds}\frac{Y_i}{\mathcal{L}} = -\frac{Y_i}{\mathcal{L}} X_i$, they are monotonically decreasing and hence converge as $s \to \infty$. Moreover, the quotients are well-defined on Ω. Therefore their derivatives vanish, which implies $Y_i \cdot X_i = 0$ on Ω. But due to the bounds on $\mathcal{L}$ in lemma 3.2.11, X_1 is bounded away from zero and in particular is non-zero on Ω. This implies $0 < Y_2 < \hat{\omega}_1 Y_1 \to 0$ as $s \to \infty$.

Now consider the evolution of the Böhm functional $\mathcal{F}_0 = v^{\frac{2}{n}} \left(\operatorname{tr}(r_t) + \operatorname{tr}((L^{(0)})^2)\right)$, which was introduced in (2.9). In the current coordinate system it is given by

$$\begin{aligned}\mathcal{F}_0 &= \prod_{i=1}^{2} Y_i^{-2d_i/n} \left\{ \sum_{i=1}^{2} A_i Y_i^2 - A_3 \frac{Y_2^4}{Y_1^2} + \sum_{i=1}^{2} d_i X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{2} d_i X_i\right)^2 \right\} \\ &= \prod_{i=1}^{2} Y_i^{-2d_i/n} \left\{ A_1 Y_1^2 + Y_2^2 \left(A_2 - A_3 \frac{Y_2^2}{Y_1^2}\right) + \sum_{i=1}^{2} d_i X_i^2 - \frac{1}{n} \right\}.\end{aligned}$$

Observe that it is bounded from below by zero as $\frac{Y_2}{Y_1} = \omega < \hat{\omega}_1 < \frac{A_2}{2A_3}$. Furthermore, according to (2.10), $\mathcal{F}_0$ is non-increasing and therefore converges as $s \to \infty$. However, for $\mathcal{F}_0$ to be finite on the ω-limit set Ω, one has to have $\sum_{i=1}^{2} d_i X_i^2 = \frac{1}{n}$, which forces $X_1 = X_2 = \frac{1}{n}$ in the Einstein locus $\sum_{i=1}^{2} d_i X_i = 1$ as $X_1, X_2 \geq 0$. Therefore X_1 is constant on Ω and then also $\mathcal{L}$ due to the ODE for X_1. Finally, the ODE for $\mathcal{L}$ itself shows that $\frac{\varepsilon}{2}\mathcal{L}^2 = \frac{1}{n}$ on Ω.

The analysis of the Böhm functional can be refined to deduce the asymptotic behaviour of ω. The monotonicity of $\mathcal{F}_0$ and

$$\begin{aligned}\mathcal{F}_0 &= \frac{1}{\omega^{2d_2/n}} \left(A_1 + A_2\omega^2 - A_3\omega^4\right) + v^{2/n} \operatorname{tr}((L^{(0)})^2) \\ &= \frac{A_1}{\omega^{2d_2/n}} + A_2 \omega^{2d_1/n} - A_3 w^{2(2d_1+d_2)/n} + v^{2/n} \operatorname{tr}((L^{(0)})^2)\end{aligned}$$

imply that ω is bounded away from zero for $t \geq t_0 > 0$. Notice furthermore that

$$\begin{aligned}\frac{d}{dt} v^{2/n} \operatorname{tr}((L^{(0)})^2) &= \frac{d}{dt}\mathcal{F}_0 - \frac{d}{dt}\left\{ \frac{A_1}{\omega^{2d_2/n}} + A_2\omega^{2d_1/n} - A_3 w^{2(2d_1+d_2)/n} \right\} \\ &= -2\frac{n-1}{n} v^{2/n} \operatorname{tr}((L^{(0)})^2) - \frac{2d_1 d_2}{n} \omega^{-2d_2/n-1} f(\omega),\end{aligned}$$

where the polynomial $f(\omega)$ is defined in (3.14). Therefore, $v^{2/n} \operatorname{tr}((L^{(0)})^2)$ can be treated as an independent variable, which is nonnegative, bounded by $\mathcal{F}_0$ and satisfies a well-defined ODE on the ω-limit set Ω.

Since $\mathcal{F}_0$ takes a finite value on Ω and $\frac{d}{dt}\mathcal{F}_0 = -2\frac{n-1}{n} v^{2/n} \operatorname{tr}((L^{(0)})^2)$, it follows that $v^{2/n} \operatorname{tr}((L^{(0)})^2) \to 0$ as $t \to \infty$. This in turn implies $f(\omega) \to 0$ and thus $\omega \to \omega_1$ as $t \to \infty$ due to proposition 3.4.6. □

The proof shows that the Böhm functional $\mathcal{F}_0$ evaluated on the trajectories corresponding to Einstein metrics of negative scalar curvature asymptotically approaches the value $\mathcal{F}_0 = n(n-1)c_1^{2d_1/n} c_2^{2d_2/n}$ of the first cone solution.

However, if the scalar curvature of the principal orbit is positive, any Einstein trajectory that takes a constant value on $\mathcal{F}_0$ is a cone solution. This was first observed by Böhm in the general context of cohomogeneity one Einstein manifolds [Böh99, Corollary 2.4]. In the two summands case, one can simply note that if $A_3 = 0$ the cone solution is unique and hence the minimum. Otherwise a direct calculation confirms that the first cone solution realises the minimum in the geometric examples in 3.4.5.

Therefore, the metric asymptotically approaches the cone solution in proposition 3.4.1.

Remark 3.4.13 Böhm's [Böh99] construction of the Einstein metrics relies on the fact that as $f_2(0) \to 0$ the solution converges to the first cone solution, see proposition 3.5.2. Then an attractor property of the cone solution implies that for $f_2(0) > 0$ small enough the Einstein trajectories remain close to the cone solution and are hence complete.

Proposition 3.4.12 implies that even for large $f_2(0) > 0$ the corresponding solution to the Einstein equation approaches the cone solution asymptotically.

Remark 3.4.14 To see that the metric grows exponentially for large t it is enough to note that $\frac{\dot{f}_i}{f_i} = \frac{X_i}{\mathcal{L}} \to \sqrt{\frac{\varepsilon}{2n}}$ as $t \to \infty$.

3.4.4 Steady Ricci solitons

The rotationally symmetric Bryant soliton on $\mathbb{R}^n$, $n \geq 3$, is asymptotically paraboloid and therefore non-collapsed. This section generalises this observation to all the non-trivial steady Ricci solitons that arise in the two summands case.

Recall from proposition 2.4.1 that on a complete, non-trivial cohomogeneity one steady Ricci soliton there holds $-\dot{u}(t) \to \sqrt{-C}$ as $t \to \infty$ and $0 < \operatorname{tr}(L) < \frac{n}{t}$ for $t > 0$. Therefore, if the shape operator remains positive definite, it follows that

$\frac{\dot f_i}{f_i} \to 0$ as $t \to \infty$. According to corollary 3.2.8, this automatically holds in the two summands case if $d_1 > 1$ and $\widehat{D} > 0$.

Proposition 3.4.15 *Let $d_1 > 1$, $\widehat{D} > 0$ and $\varepsilon = 0$. Along trajectories of non-trivial complete steady Ricci solitons there holds*

$$X_1, X_2 \to 0 \quad \textit{and} \quad Y_1, Y_2 \to 0 \quad \textit{and} \quad \mathcal{L} \to \frac{1}{\sqrt{-C}}$$

as $s \to \infty$.

Proof: The first and third statement follow directly from general theory as discussed above.

Recall from corollary 3.2.8 that also the variables Y_1, Y_2 and $\omega = \frac{Y_2}{Y_1}$ remain bounded along the Ricci soliton ODE. With (3.10) it follows that the ω-limit set Ω of the associated flow is non-empty, compact, connected and flow invariant. Then $X_1' \equiv 0$ on Ω implies that $Y_1 = 0$ holds on Ω. Since $\omega^2 \leq \hat{\omega}_1^2 < \frac{A_2}{2A_3}$, one similarly concludes $Y_2 \to 0$ as $s \to \infty$. □

In order to deduce the asymptotic behaviour of the metric, one would like to consider the quotient $\frac{X_i}{Y_i^2} \sim \sqrt{-C} f_i \dot{f}_i$, which is expected to converge to $\frac{-C}{4}(n-1)c_i^2$. If $i = 1$, it satisfies the ODE

$$\frac{d}{ds}\frac{X_1}{Y_1^2} = \frac{X_1}{Y_1^2}\left(-\sum_{i=1}^{2} d_i X_i^2 - 1 - 2X_1\right) + \frac{A_1}{d_1} + \frac{A_3}{d_1}\frac{Y_2^4}{Y_1^4}$$

and, if furthermore $A_3 = 0$, one can proceed as in [DW09b, Lemma 3.8] to confirm the expected asymptotics. However, if $A_3 > 0$, one requires more detailed information about the quotient $\omega = \frac{Y_2}{Y_1}$.

Proposition 3.4.16 *Let $d_1 > 1$, $\widehat{D} > 0$ and $\varepsilon = 0$. Then along trajectories of non-trivial steady Ricci solitons the limit $\omega_\infty = \lim_{t\to\infty} \omega(t)$ exists and $\lim_{t\to\infty} \dot{\omega}(t) = 0$.*

Proof: Let $v(t) = \sqrt{\det g_t} = f_1^{d_1}(t) f_2^{d_2}(t)$ denote the relative volume of the principal orbit and consider the variables $\frac{v^{1/n}}{f_i}$ and $\frac{v^{2/n}\dot{f}_i}{f_i}$ for $i = 1, 2$. Observe that $\frac{v^{1/n}}{f_1} = \frac{1}{\omega^{d_2/n}}$ and $\frac{v^{1/n}}{f_2} = \omega^{d_1/n}$.

Therefore, the Böhm functional has the lower bound

$$\mathcal{F}_0 = v^{\frac{2}{n}}\left(\mathrm{tr}(r_t) + \mathrm{tr}((L^{(0)})^2)\right) \geq v^{\frac{2}{n}}\,\mathrm{tr}(r_t) = \frac{A_1}{\omega^{2d_2/n}} + A_2\omega^{2d_1/n} - A_3 w^{2(2d_1+d_2)/n}.$$

Since $\mathcal{F}_0$ is non-increasing, $v^{\frac{2}{n}}\operatorname{tr}(r_t)$ is bounded from above for $t \geq t_0 > 0$ and hence ω is bounded away from zero for these t. As $\omega < \hat{\omega}_1$, the variables $\frac{v^{1/n}}{f_i}$ are hence bounded for $t \geq t_0$.

Furthermore, the variables $\frac{v^{2/n}\dot{f}_i}{f_i}$ satisfy the ODE system

$$\begin{aligned}
\frac{d}{dt}\frac{v^{2/n}\dot{f}_1}{f_1} &= -(-\dot{u} + \frac{n-2}{n}\operatorname{tr}(L))\frac{v^{2/n}\dot{f}_1}{f_1} + \frac{A_1}{d_1}\left(\frac{v^{1/n}}{f_1}\right)^2 + \frac{A_3}{d_1}\omega^2\left(\frac{v^{1/n}}{f_2}\right)^2, \\
\frac{d}{dt}\frac{v^{2/n}\dot{f}_2}{f_2} &= -(-\dot{u} + \frac{n-2}{n}\operatorname{tr}(L))\frac{v^{2/n}\dot{f}_2}{f_2} + \frac{A_2}{d_2}\left(\frac{v^{1/n}}{f_2}\right)^2 - \frac{2A_3}{d_2}\omega^2\left(\frac{v^{1/n}}{f_2}\right)^2.
\end{aligned}$$

Due to the known asymptotics, the coefficient of $\frac{v^{2/n}\dot{f}_i}{f_i}$ tends to $-\sqrt{-C}$ and the remaining polynomial terms are bounded. Hence, by comparison, the variables $\frac{v^{2/n}\dot{f}_i}{f_i}$ remain bounded. Therefore, one can pass to the ω-limit set Ω. Due to its monotonicity, $\mathcal{F}_0$ converges. Its derivative (2.10) has to vanish on Ω and therefore the limiting value $\mathcal{F}_0 = (v^{\frac{2}{n}}\operatorname{tr}(r))_\infty$ can be expressed in terms of ω as above. In particular, ω converges.

The asymptotics of $\dot{\omega}$ simply follow from the ODE $\dot{\omega} = \omega\left\{\frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2}\right\}$ and the fact that $\frac{\dot{f}_i}{f_i} \to 0$ as $t \to \infty$. □

Remark 3.4.17 It is also possible to derive an integral formula for $\dot{\omega}$. Indeed, it is easy to check that

$$\frac{d}{dt}\left\{\dot{\omega}e^{-u}f_1^{d_1-1}f_2^{d_2+1}\right\} = f(\omega)e^{-u}f_1^{d_1-2}f_2^{d_2},$$

where $f(\omega)$ is defined in (3.14). If $d_1 > 1$, it follows that

$$\dot{\omega}(t) = \frac{e^u(t)}{f_1^{d_1-1}(t)f_2^{d_2+1}(t)} \cdot \int_0^t f(\omega(s))e^{-u(s)}f_1^{d_1-2}(s)f_2^{d_2}(s)ds.$$

Since f_1, f_2 are monotonic and $e^{u(t)}\int_0^t e^{-u(s)}ds \to \frac{1}{\sqrt{-C}}$ as $t \to \infty$ by L'Hôpital's rule, one has the bound $\dot{\omega}(t) \leq \overline{C} \cdot \frac{1}{f_1(t)f_2(t)}$ for some constant $\overline{C} > 0$.

Now the asymptotics of the metric can be deduced from [App18, Lemma 6.3], where the following lemma is proved:

Lemma 3.4.18 ([App18]) *Let $\varepsilon > 0$ and $c_i : [0,\infty) \to \mathbb{R}$, $i = 1, 2$, be smooth functions such that $|c_i(t) - c_i^*| < \varepsilon$ for constants $c_i^* > 0$, $i = 1, 2$. Then for a solution $f : [0,\infty) \to \mathbb{R}$ to the ODE*

$$\ddot{f} = -c_1\dot{f} - (n-1)\frac{\dot{f}^2}{f} + \frac{c_2}{2f}$$

with initial conditions $f(0), \dot{f}(0) > 0$ *there exists* $t_0 > 0$ *such that for all* $t > t_0$ *there holds*

$$f(t_0)^2 + \gamma_- (1+\varepsilon)^{-1} (t - t_0) \leq f^2(t) \leq f(t_0)^2 + \gamma_+ (t - t_0)\,,$$

where $\gamma_\pm = \frac{c_2^* \pm \varepsilon}{c_1^* \mp \varepsilon}$.

Observe that f_1, f_2 satisfy the ODE

$$\ddot{f}_1 = -(-\dot{u} - d_2 \frac{\dot{\omega}}{\omega})\dot{f}_1 - (n-1)\frac{\dot{f}_1^2}{f_1} + \frac{A_1 + A_3\omega^4}{d_1 f_1},$$
$$\ddot{f}_2 = -(-\dot{u} + d_1 \frac{\dot{\omega}}{\omega})\dot{f}_2 - (n-1)\frac{\dot{f}_2^2}{f_2} + \frac{A_2 - 2A_3\omega^2}{d_2 f_2}.$$

As $\omega < \hat{\omega}_1 < \sqrt{\frac{A_2}{2A_3}}$, $A_1 > 0$ and ω converges, the lemma is applicable and yields the claimed paraboloid asymptotics.

Proposition 3.4.19 *Let* $d_1 > 1$, $A_1, \widehat{D} > 0$ *and suppose that* $(d_1+1)\ddot{u}(0) = C < 0$. *Then the corresponding two summands steady Ricci soliton metrics satisfy*

$$-\dot{u}(t) \to \sqrt{-C} \quad \text{and} \quad \frac{f_i^2(t)}{t} \to \frac{2}{\sqrt{-C}}(n-1)c_i^2$$

as $t \to \infty$, *where* (c_1, c_2) *denotes the first cone solution.*

In particular, $\omega \to \omega_1$ *as* $t \to \infty$.

Proof: It remains to calculate $\omega_\infty = \lim_{t\to\infty} \omega(t)$ as the formulae for f_i then follow from definition 3.4.3.

Notice that lemma 3.4.18 provides an asymptotic formula for ω which implies $\frac{\gamma_{1,-}}{\gamma_{2,+}} \leq \omega_\infty^2 \leq \frac{\gamma_{1,+}}{\gamma_{2,-}}$ for any $\varepsilon > 0$. In the limit as $\varepsilon \to 0$ one obtains equality and thus

$$\omega_\infty^2 = \frac{d_2}{d_1}\frac{A_1 + A_3\omega_\infty^4}{A_2 - 2A_3\omega_\infty^2}.$$

In particular, $0 < \omega_\infty \leq \hat{\omega}_1$ is a root of $f(\omega)$ and due the characterisation 3.4.6 of the cone solutions, it follows that $\omega_\infty = \omega_1$ is the ratio of the first cone solution. □

3.4.5 Expanding Ricci solitons

The expanding Ricci solitons are asymptotically conical at infinity and the soliton potential grows quadratically at infinity.

Recall from proposition 2.4.2 that on a complete, non-trivial cohomogeneity one expanding Ricci soliton $-\dot{u}$ is asymptotically linear and the mean curvature of the principal orbit is bounded. Therefore, positive definiteness of the shape operator in the two summands case implies:

Proposition 3.4.20 *Suppose that $d_1 > 1$ and $\widehat{D} > 0$ and consider the flow of the Ricci soliton ODE in the phase space of expanding Ricci solitons. Then*

$$X_1, X_2 \to 0 \quad \text{and} \quad Y_1, Y_2 \to 0 \quad \text{and} \quad \mathcal{L} \to 0$$

as $s \to \infty$.

Proof: This is immediate from the definition of the rescaled variables and the above discussion. □

A modification of the discussion in [DW09a] can now be used to deduce the claimed asymptotically conical geometry at infinity:

Proposition 3.4.21 *Suppose that $d_1 > 1$ and $\widehat{D} > 0$. Then the trajectories corresponding to non-trivial expanding Ricci soliton metrics satisfy*

$$\frac{-\dot{u}(t)}{t} \to \frac{\varepsilon}{2} \quad \text{and} \quad \frac{L_t}{t} \to \mathbb{I}_n$$

as $t \to \infty$.

Proof: Consider the ODE system

$$\frac{d}{ds}\frac{X_1}{\mathcal{L}^2} = \left(-\sum_{i=1}^{2} d_i X_i^2 - 1\right)\frac{X_1}{\mathcal{L}^2} + \frac{\varepsilon}{2}(1+X_1) + \frac{A_1}{d_1}\left(\frac{Y_1}{\mathcal{L}}\right)^2 + \frac{A_3}{d_1}\left(\frac{Y_2}{\mathcal{L}}\right)^2\left(\frac{Y_2}{Y_1}\right)^2,$$
$$\frac{d}{ds}\frac{X_2}{\mathcal{L}^2} = \left(-\sum_{i=1}^{2} d_i X_i^2 - 1\right)\frac{X_2}{\mathcal{L}^2} + \frac{\varepsilon}{2}(1+X_2) + \frac{A_2}{d_2}\left(\frac{Y_2}{\mathcal{L}}\right)^2 - \frac{2A_3}{d_2}\left(\frac{Y_2}{\mathcal{L}}\right)^2\left(\frac{Y_2}{Y_1}\right)^2$$

and notice that $\frac{d}{ds}\frac{Y_i}{\mathcal{L}} = -\frac{Y_i}{\mathcal{L}}X_i$ implies that both limits $\hat{y}_i = \lim \frac{Y_i}{\mathcal{L}} \in [0, \infty)$ exist.

If $\hat{y}_1 = 0$ then as $\omega = \frac{Y_2}{Y_1}$ remains bounded one necessarily also has $\hat{y}_2 = 0$. In this case one can proceed as in [DW09a] to show that $\frac{X_i}{\mathcal{L}} \to \frac{\varepsilon}{2}$ as $s \to \infty$ because the extra terms involving A_3 tend to zero. This implies $\mathcal{L} \cdot t \to \frac{2}{\varepsilon}$ as $t \to \infty$ and then the claim.

It remains to rule out that possibly $\hat{y}_1 > 0$. However, in this case the existence of $\omega_\infty = \lim_{s\to\infty} \frac{Y_2}{Y_1}$ is immediate. (This independent fact can also be shown by an analysis of the Böhm functional as in the previous sections.) Then the ODE implies that

$$\frac{X_1}{\mathcal{L}^2} \to \frac{\varepsilon}{2} + \frac{1}{d_1}\left(A_1\hat{y}_1^2 + A_3\hat{y}_2^2\omega_\infty^2\right) \quad \text{and} \quad \frac{X_2}{\mathcal{L}^2} \to \frac{\varepsilon}{2} + \frac{1}{d_2}\left(A_2\hat{y}_2^2 - 2A_3\hat{y}_2^2\omega_\infty^2\right)$$

as $s \to \infty$ and both limits are positive as $\varepsilon > 0$ and $\omega_\infty^2 < \frac{A_2}{2A_3}$. For brevity, set $\Lambda_i = \lim_{s\to\infty} \frac{X_i}{\mathcal{L}^2} > 0$. Then it follows that

$$\frac{Y_i'}{\mathcal{L}'} = \frac{Y_i\left(\sum_{i=1}^2 d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - X_i\right)}{\mathcal{L}\left(\sum_{i=1}^2 d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2\right)} \to \hat{y}_i \cdot \frac{\varepsilon + 2\Lambda_i}{\varepsilon}$$

as $s \to \infty$, but if $\hat{y}_1 > 0$ L'Hôpital's rule implies $\Lambda_1 = 0$ and thus a contradiction. □

3.5 Böhm's Einstein metrics of positive scalar curvature

In this section, the construction of Einstein metrics with positive scalar curvature on spheres and other low dimensional spaces due to Böhm [Böh98] will be revisited. The key simplification is the new proof of proposition 3.5.2 below. It is based on the observation that the coordinate system used in the previous sections allows good control of the trajectories as the volume of the singular orbit tends to zero. In fact, the limit trajectory turns out to satisfy the *Ricci flat* system, which was analysed in section 3.4.2. Nonetheless, the general strategy of the construction due to Böhm remains the same.

After recalling the general set-up, convergence to the cone solutions as the volume of the singular orbit tends to zero will be proven in section 3.5.1. The application of a general counting argument, which already appears in Böhm's work [Böh98], will then yield Böhm's Einstein metrics on products of spheres and other low dimensional spaces in section 3.5.2. Böhm's Einstein metrics on low dimensional spheres will be described in section 3.5.3.

Recall from (3.1) that the metric is given by

$$g_{M\setminus Q} = dt^2 + f_1(t)^2 g_S + f_2(t)^2 g_Q$$

away from the singular orbits. It follows from the results of Eschenburg-Wang [EW00] that there exists a unique one parameter family $c_{\bar{f}}(t) = (f_1, \dot{f}_1, f_2, \dot{f}_2)(t)$ of solutions to the Einstein equations (2.3), (2.4) with initial condition $c_{\bar{f}}(0) = (0, 1, \bar{f}, 0)$ for any $\bar{f} > 0$. Moreover, Böhm [Böh98] has observed that it depends *continuously* on the initial condition $\bar{f} > 0$. Notice also that (2.4) implies $(d_1 + 1)\ddot{f}_2(0) = \frac{\varepsilon}{2}\bar{f} + \frac{A_2}{d_2}\frac{1}{\bar{f}} > 0$

if either $\varepsilon \geq 0$ or $\bar{f}^2 < -\frac{2}{\varepsilon}\frac{A_2}{d_2}$ and $\varepsilon < 0$. However, the equations are a priori not well defined if $\bar{f} = 0$. This singular condition corresponds geometrically to the collapse of the full principal orbit.

Observe that the volume V of the principal orbit satisfies $V' = V\,\mathrm{tr}(L)$, where $\mathrm{tr}(L)$ is the mean curvature. Along trajectories corresponding to Einstein metrics with positive scalar curvature, every critical point of V is a maximum or a singular orbit is reached. Therefore, if the maximal volume orbit exists, it is unique and characterised by $\mathrm{tr}(L) = 0$.

If $A_3 = 0$, the maximal volume orbit always exists [Böh98, section 4, (e)]. An alternative argument is discussed below, mainly to introduce a natural coordinate system which extends past the maximal volume orbit.

With the same notation as in the previous sections, the Einstein constant is $-\frac{\varepsilon}{2}$.

Lemma 3.5.1 *Suppose that $A_3 = 0$ and $\varepsilon < 0$, then any Einstein trajectory has a maximal volume orbit.*

Proof: In analogy to the previous sections, introduce the variables

$$\widehat{X}_i = \frac{\dot{f}_i}{f_i},\ \widehat{Y}_i = \frac{1}{f_i},\quad \text{for} \quad i = 1, 2, \quad \text{and} \quad \widehat{\mathcal{L}} = -\dot{u} + \mathrm{tr}(L). \tag{3.20}$$

In fact, these differ from the previous variables only by the factor $\frac{1}{-\dot{u}+\mathrm{tr}(L)}$ which was introduced to obtain a smoothing at the singular orbit. Clearly, in this section only the Einstein case $-\dot{u} \equiv 0$ is considered. However, in the above variables, the Ricci soliton and Einstein equations have the same form

$$\begin{aligned}
\frac{d}{dt}\widehat{X}_1 &= -\widehat{X}_1\widehat{\mathcal{L}} + \frac{A_1}{d_1}\widehat{Y}_1^2 + \frac{\varepsilon}{2} + \frac{A_3}{d_1}\frac{\widehat{Y}_2^4}{\widehat{Y}_1^2},\\
\frac{d}{dt}\widehat{X}_2 &= -\widehat{X}_2\widehat{\mathcal{L}} + \frac{A_2}{d_2}\widehat{Y}_2^2 + \frac{\varepsilon}{2} - \frac{2A_3}{d_2}\frac{\widehat{Y}_2^4}{\widehat{Y}_1^2},\\
\frac{d}{dt}\widehat{Y}_i &= -\widehat{X}_i\widehat{Y}_i,\\
\frac{d}{dt}\widehat{\mathcal{L}} &= \frac{\varepsilon}{2} - \sum_{i=1}^{2} d_i\widehat{X}_i^2.
\end{aligned}$$

Notice that the time slice has not been rescaled. The conservation law takes the form

$$\sum_{i=1}^{2} d_i\widehat{X}_i^2 + \sum_{i=1}^{2} A_i\widehat{Y}_i^2 - A_3\frac{\widehat{Y}_2^4}{\widehat{Y}_1^2} + (n-1)\frac{\varepsilon}{2} = \widehat{\mathcal{L}}^2 \tag{3.21}$$

and notice also that $\sum_{i=1}^{2} d_i \widehat{X}_i = \widehat{\mathcal{L}}$. These two equations describe the rescaled Einstein locus (3.8). If $A_3 = 0$, then the above system is an ODE system with polynomial right hand side. In particular, a solution can only develop a finite time singularity if the norm of $(\widehat{X}_i, \widehat{Y}_i, \widehat{\mathcal{L}})$ blows up. However, the conservation law (3.21) shows that this can only be the case if $\widehat{\mathcal{L}}$ blows up. At the first singular orbit, i.e. at time $t = 0$, one has $\widehat{\mathcal{L}} = +\infty$ and $\widehat{\mathcal{L}}$ is strictly decreasing for all $t > 0$ as $\varepsilon < 0$. Hence, the finite time singularity corresponds to $\widehat{\mathcal{L}} = -\infty$ and in particular there exists a time with $\mathrm{tr}(L) = \widehat{\mathcal{L}} = 0$, the maximal volume orbit. □

3.5.1 Convergence to cone solutions

To describe the behaviour of the Einstein equations as the volume of the singular orbit tends to zero, the following observation is key: In the $(X_i, Y_i, \mathcal{L})$-coordinate system of the previous sections, the initial condition $(0, 1, \bar{f}, 0)$ of the trajectory $c_{\bar{f}}$ corresponds to the stationary point (3.6), which is independent of $\bar{f}$. Furthermore, the initial condition $\bar{f}$ of the scaling function f_2 of the singular orbit can be recovered via $\bar{f} = \lim_{s \to -\infty} \frac{\mathcal{L}}{Y_2}$. In particular, $\bar{f} = 0$ is the limit of trajectories with $\mathcal{L} \equiv 0$.

However, the two coordinate systems are only equivalent along trajectories with $\mathcal{L} > 0$. Nonetheless, due to the continuous dependence on the initial condition, any trajectory with $\mathcal{L} \equiv 0$ can hence be viewed as a continuous limit of Einstein, or more generally Ricci soliton, trajectories. Hence, the collapse $\bar{f} \to 0$ is described in the rescaled system by the solutions to the *steady* Ricci soliton or *Ricci flat* equations!

In the Einstein case, the limiting process as $\bar{f} \to 0$ will be described even more explicitly. This recovers a result of Böhm [Böh98, Theorem 5.7], which is central to his construction of Einstein metrics of positive scalar curvature.

From now on fix the normalisation

$$-\frac{\varepsilon}{2} \in \{-n, 0, n\}$$

of the Einstein constant $-\frac{\varepsilon}{2}$ and recall from proposition 3.4.1 and definition 3.4.3 that

$$f_i(t) = c_i \sin(t), \text{ for } t \in (0, \pi),$$
$$f_i(t) = c_i t, \quad \text{and} \quad f(t) = c_i \sinh(t) \quad \text{for} \quad t > 0$$

describe the corresponding cone solutions, which are degenerate solutions with initial condition $\bar{f} = 0$. Since the trajectory $c_{\bar{f}}$ of the Einstein equations with initial condition $c_{\bar{f}}(0) = (0, 1, \bar{f}, 0)$ is unique, one might expect that the limit as $\bar{f} \to 0$ is a cone solution. This intuition is confirmed by the following result.

Proposition 3.5.2 *Suppose that $d_1 > 1$ and $A_3 = 0$. As $\bar{f} \to 0$, the solution $c_{\bar{f}}$ to the two summands Einstein equations converges to the first cone solution on every relatively compact subset of $(0, \pi)$ if $\frac{\varepsilon}{2} = -n$ and $(0, \infty)$ if $\frac{\varepsilon}{2} \in \{0, n\}$, respectively.*

Proof: Recall that the limit trajectory with $\bar{f} = 0$ corresponds to the trajectory with $\mathcal{L} \equiv 0$, which is described by the Ricci flat system in X_i, Y_i. According to proposition 3.4.7, the Ricci flat trajectory asymptotically approaches the first cone solution, which takes the constant value $X_i = \frac{1}{n}$ and $Y_i = \frac{1}{nc_i}$ for $i = 1, 2$. Notice that this is in fact the value at $t = 0$ of all cone solutions. Therefore it will be called *base point* of the cone solution.

If $\varepsilon \geq 0$, notice, as in the proof of proposition 3.4.12, that the variables X_i, Y_i are bounded, that the Böhm functional $\mathcal{F}_0$ is bounded from below and non-increasing, and that it has a critical point on the cone solution. In fact, any Einstein trajectory that takes a constant value on $\mathcal{F}_0$ is a cone solution and $\mathcal{F}_0 = n(n-1)c_1^{2d_1/n}c_2^{2d_2/n}$. However, since $A_3 = 0$, the cone solution is unique and hence the minimum.

If $\varepsilon = -n$, then (2.10) implies that $\mathcal{F}_0$ achieves its minimum along a trajectory $c_{\bar{f}}$ on the maximal volume orbit. On any maximal volume orbit the coordinates (3.20) satisfy $\sum_{i=1}^{2} d_i \widehat{X}_i = \widehat{\mathcal{L}} = 0$ and the conservation law (3.21) then implies that the variables $\widehat{X}_i, \widehat{Y}_i$ are bounded, since $A_3 = 0$. Hence, $\mathcal{F}_0 = n(n-1)\prod_{i=1}^{2} \widehat{Y}_i^{-2d_i/n}$ has a minimum on the maximal volume orbit, which is achieved by the value of cone solution.

However, $\mathcal{F}_0$ is constant on the cone solution and since the solution $c_{\bar{f}}$ approaches the base point of the cone solution as $\bar{f} \to 0$, the claim follows. $\square$

Remark 3.5.3 (a) The simplifying assumption $A_3 = 0$ can be relaxed. For the geometric examples in 3.4.5, one can calculate directly that the first cone solution realises the minimum. So the exact same proof works if $\widehat{D} > 0$ and $\varepsilon \geq 0$ due to proposition 3.2.5.

(b) The behaviour of the Böhm functional close to cone solutions was studied in a more general context in [Böh99]. In particular, it can be used to show that any *stable* cone solution is a local attractor of the cohomogeneity one Einstein equations.

(c) In the original proof, Böhm [Böh98] uses a different coordinate system, which still enables him to find a limit trajectory and observes that it can be considered in a compact planar domain. To prove convergence to the base point of the cone solution, non-trivial closed trajectories are ruled out and then the Poincaré-Bendixson theorem is applied.

The stability of the first cone solution in the two summands case for $d_1 > 1$ and $\widehat{D} > 0$ is then established via the attractor function (3.13), but considered in a coordinate system that is defined along the entire cone solution, except at the conical singularities.

3.5.2 Symmetric solutions

Recall that trajectories $c_{\bar{f}} = (f_1, \dot{f}_1, f_2, \dot{f}_2)$ of the Einstein equations with initial condition $c_{\bar{f}}(0) = (0, 1, \bar{f}, 0)$ are considered. By requiring the existence of $\bar{f}', \tau > 0$ such that $c_{\bar{f}}(\tau) = (0, -1, \bar{f}', 0)$ one closes the cohomogeneity one manifold with the same singular orbit. Any such solution with $\bar{f} = \bar{f}'$ is called *symmetric*. In fact, $c_{\bar{f}}$ is symmetric if and only if there exists $t_0 > 0$ such that $c_{\bar{f}}(t_0) = (f_1(t_0), 0, f_2(t_0), 0)$ with $f_1(t_0), f_2(t_0) > 0$.

Furthermore, recall that if the maximal volume orbit exists, it is unique and determined by the condition $\mathrm{tr}(L) = 0$. In particular, symmetric solutions are characterised by the condition $\dot{f}_1 = \dot{f}_2 = 0$ on the maximal volume orbit. Notice that in this case reflection along normal geodesics of the maximal volume orbit becomes an isometry, i.e. if $c_{\bar{f}}$ satisfies $f_1(t_0), f_2(t_0) > 0$ and $\dot{f}_1(t_0) = \dot{f}_2(t_0) = 0$ at some $t_0 > 0$, then

$$\tilde{c}_{\bar{f}}(t) = \begin{cases} (f_1, \dot{f}_1, f_2, \dot{f}_2)(t) & \text{for } \ 0 \leq t \leq t_0 \\ (f_1, -\dot{f}_1, f_2, -\dot{f}_2)(2t_0 - t) & \text{for } \ t_0 \leq t \leq 2t_0 \end{cases}$$

is a smooth, symmetric solution to the Einstein equations.

Conversely, given a symmetric Einstein solution $c_{\bar{f}} = (f_1, \dot{f}_1, f_2, \dot{f}_2)$, consider the reversed trajectory $\tilde{c}_{\bar{f}}(t) = (f_1, -\dot{f}_1, f_2, -\dot{f}_2)(\tau - t)$. Then $\tilde{c}_{\bar{f}}$ is again a solution to the Einstein equations and satisfies $\tilde{c}_{\bar{f}}(0) = c_{\bar{f}}(0)$. By uniqueness of solutions to initial value problems there holds $f_i(t) = f_i(\tau - t)$ for $i = 1, 2$ and thus $\dot{f}_i(\tau/2) = 0$ as claimed.

Examples of cohomogeneity one manifolds that arise from this construction include products $S^{d_1+1} \times Q$, where $Q = G/H$ is a compact connected isotropy irreducible homogeneous space, or connected sums such as $\mathbb{H}P^{m+1} \# \overline{\mathbb{H}P}^{m+1}$, $F^{m+1} \# \overline{F}^{m+1}$ and $CaP^2 \# \overline{CaP}^2$. Notice that the products satisfy $A_3 = 0$ whereas the connected sums have $A_3 > 0$. In order to have $A_2 = d_2 \,\mathrm{Ric}^Q > 0$, one needs to assume that Q is not a torus.

The following observation is key for the construction. Notice that the function $\omega = \frac{f_1}{f_2}$ satisfies $\dot{\omega} = \omega(\frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2})$. Hence, any symmetric solution is characterised by a critical point of ω on the maximal volume orbit.

Lemma 3.5.4 *Critical points of ω are non-degenerate along Einstein trajectories.*

Proof: Following [Böh98, Lemma 4.2.1], notice that any solution with $\dot{\omega} = \ddot{\omega} = 0$ at some fixed time in fact satisfies the equations (3.16) of a cone solution at that time. However, cone solutions are stationary points of the flow. In particular, the solution could not emanate from the stationary point (3.6), which corresponds to the initial condition $(0, 1, \bar{f}, 0)$. A contradiction. $\square$

The non-degeneracy of the critical points of ω allows the application of the following general counting principle.

Lemma 3.5.5 *Let $T_{\bar{f}}, \varepsilon_{\bar{f}}$ be continuous, positive functions of the real parameter $\bar{f}$. Suppose that $c_{\bar{f}}\colon [0, T_{\bar{f}}+\varepsilon_{\bar{f}}) \to \mathbb{R}^n$ is a family of C^1-maps which depends continuously on $\bar{f}$ and $\omega \in C^1$ a real valued map such that any critical point of $\omega = \omega \circ c_{\bar{f}}$ is non-degenerate and $\dot{\omega}(0) > 0$ for all $\bar{f}$.*

Let $\mathcal{C}(\bar{f}) = \mathcal{C}(\bar{f}, T_{\bar{f}})$ denote the number of critical points of ω along c_f before $T_{\bar{f}}$. Fix $\bar{f}_1 < \bar{f}_2$. Then the following statements hold:

1. *If $\dot{\omega}(T_{\bar{f}}) \neq 0$ for all $\bar{f} \in [\bar{f}_1, \bar{f}_2]$ then $\mathcal{C}(\bar{f})$ is constant on $[\bar{f}_1, \bar{f}_2]$.*
2. *If $\bar{f}^*$ is the unique value of $\bar{f} \in [\bar{f}_1, \bar{f}_2]$ with $\dot{\omega}(T_{\bar{f}}) = 0$, then*
$$|\mathcal{C}(\bar{f}') - \mathcal{C}(\bar{f}'')| \leq 1$$
for all $\bar{f}', \bar{f}'' \in [\bar{f}_1, \bar{f}_2]$ with $\bar{f}' < \bar{f}^* < \bar{f}''$.

In particular, for any $\bar{f}', \bar{f}'' \in [\bar{f}_1, \bar{f}_2]$ there exist at least $|\mathcal{C}(\bar{f}') - \mathcal{C}(\bar{f}'')|$ solutions with $\dot{\omega}(T_{\bar{f}}) = 0$ for $\bar{f} \in [\bar{f}', \bar{f}'']$.

Remark 3.5.6 (a) In fact, if $c_{\bar{f}}$ is just continuous, the lemma can still be used to count roots of continuous functions along $c_{\bar{f}}$. In this case $c_{\bar{f}}$ has to intersect the zero set of the function transversally.

(b) Böhm proved the counting principle explicitly in the case where $T_{\bar{f}}$ is the time when the maximal volume orbit is reached, [Böh98, Lemmas 4.4 and 4.5]. More recently it was used by Foscolo-Haskins in the construction of nearly Kähler metrics, [FH17, Lemma 7.2].

Theorem 3.5.7 ([Böh98, Theorem 3.4]) *Let $d_1 > 1$, $A_3 = 0$ and $-\frac{\varepsilon}{2} = n$. If the dimension of the principal orbit satisfies $2 \leq n \leq 8$, there exist infinitely many symmetric solutions to the two summands Einstein equations.*

Proof: Normalise the Ricci curvature of the singular orbit to be $\mathrm{Ric}^Q = d_2 - 1$, then the metric of the round sphere $(f_1, f_2)(t) = (\sin(t), \cos(t))$ induces a solution to the two summands Einstein equations without any critical point of ω before the maximal volume orbit.

According to lemma 3.5.5, it suffices to show that there are trajectories with an arbitrarily high number of critical points of ω before the maximal volume orbit. Recall that the maximal volume orbit of a trajectory is achieved exactly when $\mathrm{tr}(L) = 0$. In the variables of section 3.2 this corresponds to the blow up time of $\mathcal{L}$. In particular, critical points of ω which are detected by the rescaled system happen to be before the maximal volume orbit. Recall that $\omega' = \omega\,(X_1 - X_2)$ and that every critical point in these rescaled variables also corresponds to a critical point of ω in the original time frame t. Since the Einstein trajectories lie in the subvariety $d_1X_1 + d_2X_2 = 1$, critical points occur if and only if $X_1 = \frac{1}{n}$.

Recall that by proposition 3.4.7 the trajectory of the *Ricci flat* system satisfies $X_i \to \frac{1}{n}$ and $Y_i \to \frac{1}{c_i n}$ where (c_1, c_2) denotes the first cone solution. Moreover, observe that the Ricci flat system is realised by solutions to the two summands system for any value of $\varepsilon \in \mathbb{R}$ by the trajectory with $\mathcal{L} \equiv 0$, as ε and $\mathcal{L}$ only occur in the combination $\frac{\varepsilon}{2}\mathcal{L}^2$. However, as explained in section 3.5.1, the limit $\mathcal{L} \equiv 0$ exactly corresponds to a smoothing of the trajectory $c_{\bar{f}}$ in the limit $\bar{f} = 0$. Due to the continuous dependence of the solution on the initial condition, for any $\varepsilon \in \mathbb{R}$ and $\bar{f} > 0$ small enough, the solution to the two summands system approaches the base point of the first cone solution along a trajectory which is C^0-close to the Ricci flat trajectory γ_{RF} with $\mathcal{L} \equiv 0$, and then remains close to the actual cone solution in the sense of proposition 3.5.2.

The dimension assumption and corollary 3.4.9 imply that the projection of the Ricci flat trajectory γ_{RF} onto the (X_1, Y_1)-plane rotates infinitely often around the stationary point $(\frac{1}{n}, \frac{1}{c_1 n})$, which is the base point of the first cone solution. Hence, the variable X_1 takes the value $X_1 = \frac{1}{n}$ arbitrarily often, which implies that $\mathcal{C}(\bar{f}, T_{\bar{f}}) \to \infty$ as $\bar{f} \to 0$, where $T_{\bar{f}}$ denotes the time of the maximal volume orbit.

A direct computation of curvatures shows that the metrics are inhomogeneous and non-isometric, cf. [Böh98, section 6]. $\square$

As an explicit application, theorem 3.5.7 recovers Böhm's Einstein metrics on certain low dimensional spaces.

Corollary 3.5.8 ([Böh98, Theorem 3.4]) *Let $d_S > 1$ and suppose that Q is a compact, connected, isotropy irreducible homogeneous space of positive Ricci curvature* *and of dimension d_Q. If $2 \leq d_S, d_Q$ and $d_S + d_Q \leq 8$, then there exist infinitely many non-isometric cohomogeneity one Einstein metrics of positive scalar curvature on $S^{d_S+1} \times Q$.*

Remark 3.5.9 By considering the linearisation of the Einstein equations along the cone solutions, Böhm was also able to construct a symmetric cohomogeneity one Einstein metric on $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$.

3.5.3 Böhm's Einstein metrics on low dimensional spheres

It is also possible to close the cohomogeneity one manifold with a different singular orbit. These solutions correspond to trajectories $c_{\bar{f}}$ with $c_{\bar{f}}(0) = (0, 1, \bar{f}, 0)$ and $c_{\bar{f}}(\tau) = (\bar{f}', 0, 0, -1)$ for some $\bar{f}, \bar{f}', \tau > 0$. In the case of $SO(d_1+1) \times SO(d_2+1)$-invariant doubly warped product metrics on spheres, the convergence theory from the previous sections can be applied to give Böhm's inhomogeneous Einstein metrics on $S^5, \ldots, S^9$.

For the rest of the section, fix $A_3 = 0$ and normalise the Ricci curvature of the singular orbit to be $\mathrm{Ric}^Q = d_2 - 1$, so that $A_i = d_i(d_i - 1)$ holds for $i = 1, 2$. As before, the trajectory $c_{\bar{f}}$ will always correspond to an Einstein metric on a tubular neighbourhood of a singular orbit of dimension d_1 and with a principal orbit of dimension $n = d_1 + d_2$.

As in section 3.5.1, the limit $\bar{f} \to 0$ can be described in the $(X_i, Y_i, \mathcal{L})$ coordinates by the trajectory corresponding to $\mathcal{L} \equiv 0$, which satisfies the Ricci flat equations. Under the dimension assumptions $d_1 > 1$ and $2 \leq n \leq 8$, the base point $(\frac{1}{n}, \frac{1}{nc_1})$ of the cone solution is a stable spiral due to corollary 3.4.9. As in the proof of theorem 3.5.7, it follows from the continuous dependence on the initial value, that also $c_{\bar{f}}$, for $\bar{f} > 0$ small enough, exhibits a rotational behaviour as it approaches the base point at $t = 0$ of the first cone solution γ. Proposition 3.5.2 then says that given any compact set $K \subset\subset (0, \pi)$, the trajectory $c_{\bar{f}}$ remains C^0-close to γ on K if $\bar{f} > 0$ is small enough. In fact, $c_{\bar{f}}$ obeys a rotational behaviour in every slice around the cone solution as $\bar{f} \to 0$. To make this precise, notice that the variables $(\widehat{X}_1, \widehat{Y}_1, \widehat{\mathcal{L}})$ of (3.20) form a local coordinate system along the Einstein trajectories away from the singular orbits. For example, the first cone solution has coordinates $(\cot(t), \frac{1}{c_1 \sin(t)}, n\cot(t))$. One should think of $\widehat{\mathcal{L}}$ as the time variable. For any fixed value $\widehat{\mathcal{L}}$ and for $\bar{f} > 0$ sufficiently small, the trajectory $c_{\bar{f}}$ intersects the $(\widehat{X}_1, \widehat{Y}_1, \widehat{\mathcal{L}})$-plane $P_{\widehat{\mathcal{L}}}$ in a unique point. As $\bar{f} > 0$ varies, the intersection points describe a continuous curve in this plane.

Proposition 3.5.10 *Let $A_3 = 0$ and $2 \leq n \leq 8$. Then in any coordinate slice $P_{\widehat{\mathcal{L}}}$, the intersection points of $c_{\bar{f}}$ with a disc around the first cone solution γ in $P_{\widehat{\mathcal{L}}}$ exhibit the same rotational behaviour as $\bar{f} \to 0$.*

This follows from the general counting principle 3.5.5 applied to the time $T_{\bar{f}}$ when $c_{\bar{f}}$ intersects the disc, the observation that $\mathcal{C}(c_{\bar{f}}, T_{\bar{f}}) \to \infty$ as $\bar{f} \to 0$, and lemma 3.5.11.

Lemma 3.5.11 *Along any trajectory $c_{\bar{f}}$, critical points of ω occur if and only if $\widehat{X}_1 = \frac{\widehat{\mathcal{L}}}{n}$ and ω is increasing if $\widehat{X}_1 > \frac{\widehat{\mathcal{L}}}{n}$ and decreasing if $\widehat{X}_1 < \frac{\widehat{\mathcal{L}}}{n}$.*

Moreover, if $A_3 = 0$, then the $\widehat{Y}_1$-coordinate of ω in $P_{\widehat{\mathcal{L}}}$ satisfies $\widehat{Y}_1(\omega) > \widehat{Y}_1(\gamma)$ at any maximum and $\widehat{Y}_1(\omega) < \widehat{Y}_1(\gamma)$ at any minimum, where γ is the first cone solution.

Proof: The first statement follows from $\dot{\omega} = \omega(\widehat{X}_1 - \widehat{X}_2)$ and $\sum_{i=1}^{2} d_i \widehat{X}_i = \widehat{\mathcal{L}}$. If $A_3 = 0$, the identity $\ddot{\omega} = \frac{A_1}{d_1}\widehat{Y}_1^2 - \frac{A_2}{d_2}\widehat{Y}_2^2$ holds at any critical point of ω. However, as $\dot{\omega} = \ddot{\omega} = 0$ only occurs on the cone solution, the claim follows. □

Remark 3.5.12 More generally, due to (5.7) and proposition 3.4.6, the value of ω relative to ω_1, ω_2 determines whether a critical point is a maximum or a minimum, cf. [Böh98, Lemma 4.2].

Suppose that $d_{\overline{F}} = (F_1, \dot{F}_1, F_2, \dot{F}_2)$ is also an Einstein trajectory which satisfies $d_{\overline{F}}(0) = (0, 1, \overline{F}, 0)$ but instead induces a metric on a tubular neighbourhood of a singular orbit of dimension d_2 and with a principal orbit of dimension $n = d_1 + d_2$. Then the above considerations also apply to $d_{\overline{F}}$. Clearly, the trajectories depend continuously on the parameters $d_1, d_2 > 1$. Hence, in the dimension range $2 \leq n \leq 8$, the trajectories $c_{\bar{f}}$ and $d_{\overline{F}}$ have the *same* rotational behaviour as they approach their respective base point of the cone solution at $t = 0$.

Now consider the twisted trajectory $d_{\overline{F}}^{\text{twisted}}(t) = (F_2, -\dot{F}_2, F_1, -\dot{F}_1)(t)$. If $\tau > 0$ is small enough, then $d_{\overline{F}}^{\text{twisted}}(\tau - t)$ is an actual solution to the Einstein equations due to the symmetries of the equations in d_1, d_2 as $A_3 = 0$. That is, $d_{\overline{F}}^{\text{twisted}}(t)$ runs through the Einstein equations in 'opposite direction', starting at $d_{\overline{F}}^{\text{twisted}}(0) = (\overline{F}, 0, 0, -1)$ and then approaching the same cone solution as $c_{\bar{f}}$ but at the base point corresponding to $t = \pi$. In particular, it has the *opposite* rotational behaviour to $c_{\bar{f}}$. Due to proposition 3.5.2, for $\bar{f}, \overline{F} > 0$ small enough, both $c_{\bar{f}}$ and $d_{\overline{F}}^{\text{twisted}}$ intersect the plane $\{(\widehat{X}_1, \widehat{Y}_1, n)\}$, which is the slice of the maximal volume orbit of the cone solution, in a unique point. Since both trajectories in fact wind around the cone solution arbitrarily often as $\bar{f}, \overline{F} \to 0$, respectively, and $c_{\bar{f}}$, $d_{\overline{F}}^{\text{twisted}}$ have the opposite rotational behaviour, there are infinitely many intersection points in this (or any other) slice. For any such, there exist $t_0, t_1 > 0$ such that the matching condition $c_{\bar{f}}(t_0) = d_{\overline{F}}^{\text{twisted}}(t_1)$ holds. Then

$$\tilde{c}_{\bar{f},\overline{F}}(t) = \begin{cases} c_{\bar{f}}(t) & \text{for } \ 0 \leq t \leq t_0 \\ d_{\overline{F}}^{\text{twisted}}(t_0 + t_1 - t) & \text{for } \ t_0 \leq t \leq t_0 + t_1 \end{cases}$$

satisfies $\tilde{c}_{\bar{f},\overline{F}}(0) = (0, 1, \bar{f}, 0)$ and $\tilde{c}_{\bar{f},\overline{F}}(t_0+t_1) = (\overline{F}, 0, 0, -1)$ and it is a *smooth* solution to the Einstein equation as required. Smoothness indeed follows from the uniqueness of solutions to ODEs with fixed initial conditions, since $\tilde{c}_{\bar{f},\overline{F}}$ clearly solves the Einstein equations on both intervals. In particular, any such pair $(\bar{f}, \overline{F})$ induces an Einstein metric $g(\bar{f}, \overline{F})$ on S^{n+1}.

Remark 3.5.13 A direct curvature computation shows that the metrics are indeed inhomogeneous. Moreover, if the metrics $g(\bar{f}, \overline{F})$ and $g(\bar{f}', \overline{F}')$ on S^{n+1} are isometric, it follows that $\bar{f} = \bar{f}'$ if $d_1 \neq d_2$ and $\bar{f} = \bar{f}'$ or $\bar{f} = \overline{F}'$ if $d_1 = d_2$, since isometries must map orbits onto orbits, see [Böh98, Section 7].

This proves:

Corollary 3.5.14 ([Böh98, Theorem 3.6]) *On S^5 and S^6 there exists one, on S^7 and S^8 there exist two, and on S^9 there exist three infinite families of non-isometric, strictly cohomogeneity one Einstein metrics of positive scalar curvature.*

Chapter 4

A numerical study in the two summands Einstein case

In section 3.5, Böhm's Einstein metrics on spheres and products of spheres have been reconstructed, which are solutions to the two summands Einstein equations of warped product type, that is with $A_3 = 0$. Gibbons-Hartnoll-Pope [GHP03] provide plots of these solutions which illustrate the convergence to the cone solution very well.

On the other hand, the only Einstein metric which has rigorously been shown to exist in the case $A_3 > 0$ is Böhm's [Böh98] Einstein metric on $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$. However, numerical studies in [PP86], [GPP90], [Böh98] and [DHW13] suggest that symmetric cohomogeneity one Einstein metrics on $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$ and in the F^m-examples do exist for $m \geq 2$ and the corresponding initial conditions $\bar{f} > 0$ are listed there. This chapter refines these results by providing an analysis of the matching condition similar to [FH17]. The results suggest that the previously found initial conditions in fact cover all symmetric cohomogeneity one Einstein metrics on these spaces and that the solutions are entirely non-linear phenomena. In particular, they cannot be observed via the linearisation along the cone solution or by studying how the cone solution is approached.

The study in [DHW13] suggests that non-trivial compact cohomogeneity one Ricci solitons do not exist on any of the examples induced by the Hopf fibrations.

4.1 A numerical study of the matching condition

This section builds up on the ideas of Foscolo-Haskins' [FH17] numerical analysis of cohomogeneity one nearly Kähler metrics and provides plots of the matching condition in the case $A_3 > 0$ and of the long time behaviour of $\dot{\omega}$. The main focus will be on the

$\mathbb{H}P^m$-examples. The F^m-examples have a similar behaviour and in agreement with earlier searches, no Einstein metric could be found on $CaP^2\#\overline{CaP}^2$.

Recall that in the two summands case the metric on the principal orbit is given by $g_t = f_1^2(t)g_S + f_2^2(t)g_Q$, where g_S and g_Q are fixed metrics on the collapsing sphere and the singular orbit, respectively. Due to [EW00], [Böh98], the smoothness conditions $f_1(0) = 0$, $\dot{f}_1(0) = 1$ and $f_2(0) = \bar{f} > 0$, $\dot{f}_2(0) = 0$ give rise to a unique Einstein trajectory which depends continuously on $\bar{f} > 0$. Any such is called *symmetric* if it is symmetric under reflection along the unique maximal volume orbit, which is exactly the principal orbit with vanishing mean curvature. The unique time with $\mathrm{tr}(L) = 0$ will be denoted by $\tau_{\bar{f}} > 0$.

Set $\omega = \frac{f_1}{f_2}$. Then symmetric solutions are characterised by the *matching condition*

$$\dot{\omega}_{\bar{f}} = 0 \quad \text{at} \quad \tau_{\bar{f}} > 0.$$

This section numerically studies the functions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$ and $t \mapsto \dot{\omega}_{\bar{f}}(t)$ to obtain information about the number $\mathcal{C}(\mathcal{M}, \tau_{\bar{f}})$ and the structure of the critical points of $\omega_{\bar{f}}$ before $\tau_{\bar{f}}$, in any of the $\mathcal{M} \in \{\mathbb{H}P^m, F^m, CaP^2\}$-examples.

As the cohomogeneity one Ricci soliton equations (2.2) - (2.4) become singular at $t = 0$, a symbolic calculation scheme has been set up with the Symbolic Math Toolbox in MATLAB to calculate the coefficients of the Taylor series of f_1, f_2, ω and other relevant quantities to arbitrarily high order. This has been done by an algorithm based on the existence proof in [EW00]. In practice, all non-zero coefficients up to order 20 have been calculated. However, general explicit formulae become complicated very quickly:

Proposition 4.1.1 *Fix the Einstein constant to be $-\frac{\varepsilon}{2} = d_1 + d_2$.*

Then the Taylor series of f_1 and f_2 begin with the following expansions:

$$\begin{aligned} f_1(t) = t &- \frac{d_2 \operatorname{Ric}^Q + (d_1+d_2)(d_1-d_2+1)\bar{f}^2}{6d_1(d_1+1)\bar{f}} \cdot t^3 \\ &+ \{\ 36d_1(d_1+1)^2 A_3 + (12d_1+13d_2)d_1 d_2 (\operatorname{Ric}^Q)^2 + 3d_2^2(\operatorname{Ric}^Q)^2 \\ &\quad + \{\ 2d_1(3+7d_2-13d_2^2) - 4d_1^2(-5+9d_2) - 10d_1^3 \\ &\qquad - 6d_2(d_2-1)\ \}\ d_2 \operatorname{Ric}^Q \bar{f}^2 \\ &\quad + \{\ d_1(6d_2 - 5d_2^2 - 14d_2^3 + 13d_2^4) + d_1^2(3+8d_2-32d_2^2+24d_2^3) \\ &\qquad + d_1^3((7-10d_2+10d_2^2) + 5d_1^4 + d_1^5 + 3d_2^2(d_2-1)^2\ \}\ \bar{f}^4\ \} \cdot \\ &\cdot \frac{t^5}{(120d_1^2(d_1+1)^2(d_1+3)\bar{f}^4)} + O(t^7), \end{aligned}$$

$$
\begin{aligned}
f_2(t) = \bar{f} + \left(\mathrm{Ric}^Q - (d_1+d_2)\bar{f}^2\right)\frac{t^2}{2(d_1+1)\bar{f}} +& \\
+ \{ -12\frac{(d_1+1)^2}{d_2}A_3 + (3-3d_1-4d_2)(\mathrm{Ric}^Q)^2 +& \\
+ \{ 7d_1d_2(2-d_2) + d_1^2(7-2d_2) + d_1^3 + 7d_2^2 - 4d_2^3 \} \bar{f}^4 +& \\
+ \{ 10d_1(d_2-1) + 2d_1^2 - 10d_2 + 8d_2^2 \} \mathrm{Ric}^Q \bar{f}^2 \} \cdot& \\
\cdot \frac{t^4}{24(d_1+3)(d_1+1)^2\bar{f}^3} + O(t^6).&
\end{aligned}
$$

Remark 4.1.2 By setting $\mathrm{Ric}^Q = d_2 - 1$, $A_3 = 0$ and $\bar{f} = 1$, the Taylor expansions of $\sin(t)$ and $\cos(t)$ are recovered, which correspond to the standard metric on S^{n+1}.

The Taylor expansions of f_1, f_2, and $\dot{f}_1/f_1$, $\dot{f}_2/f_2$ are evaluated at some small $t_* > 0$, typically $t_* = 10^{-3}$, to obtain an approximate solution of the Einstein equations at that time. Then this is used as a set of initial data to solve the Einstein equations numerically with the MATLAB ODE solver ode45. To avoid substantial numerical errors, it turns out that the parameter $\bar{f} > 0$ should satisfy $\bar{f} \geq 10 \cdot t_*$. In particular this gives a useful lower bound on the smallest value $\bar{f}_{\min}$ which should be used in the numerical simulations.

4.1.1 The $\mathbb{H}P^m$-example

The diagrams in figures 4.1.1 and 4.1.2 show the functions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$ for parameters $2 \leq m \leq 10$ and data according to the $\mathbb{H}P^m$-example in table 3.1.1. The green curves correspond to the values $m = 2, 3, 4$, the blue curves to $m = 5, 6, 7$ and the black curves to $m = 8, 9, 10$. Each diagram shows a different range of initial values $\bar{f}$ or considers a different scaling of the y-axis. The step size is always $\bar{f}_{\text{step}} = 0.01$. The Taylor approximation is of order $O(t^{15})$ and it has been evaluated at $t_* = 10^{-3}$.

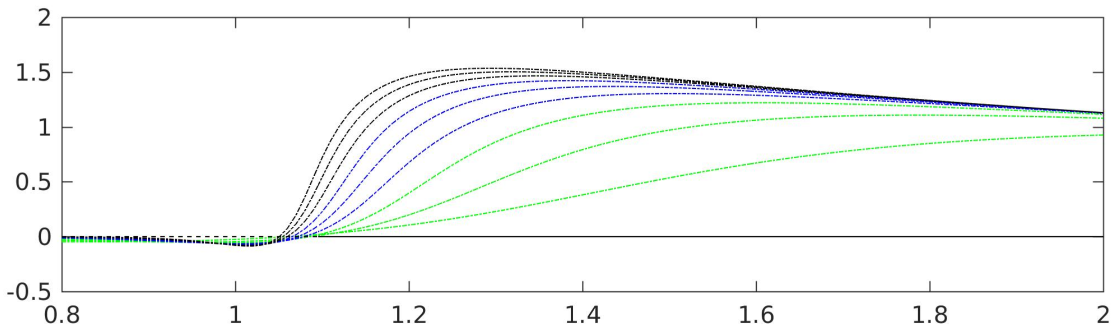

Figure 4.1.1: $\mathbb{H}P^m$-example: Matching conditions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$

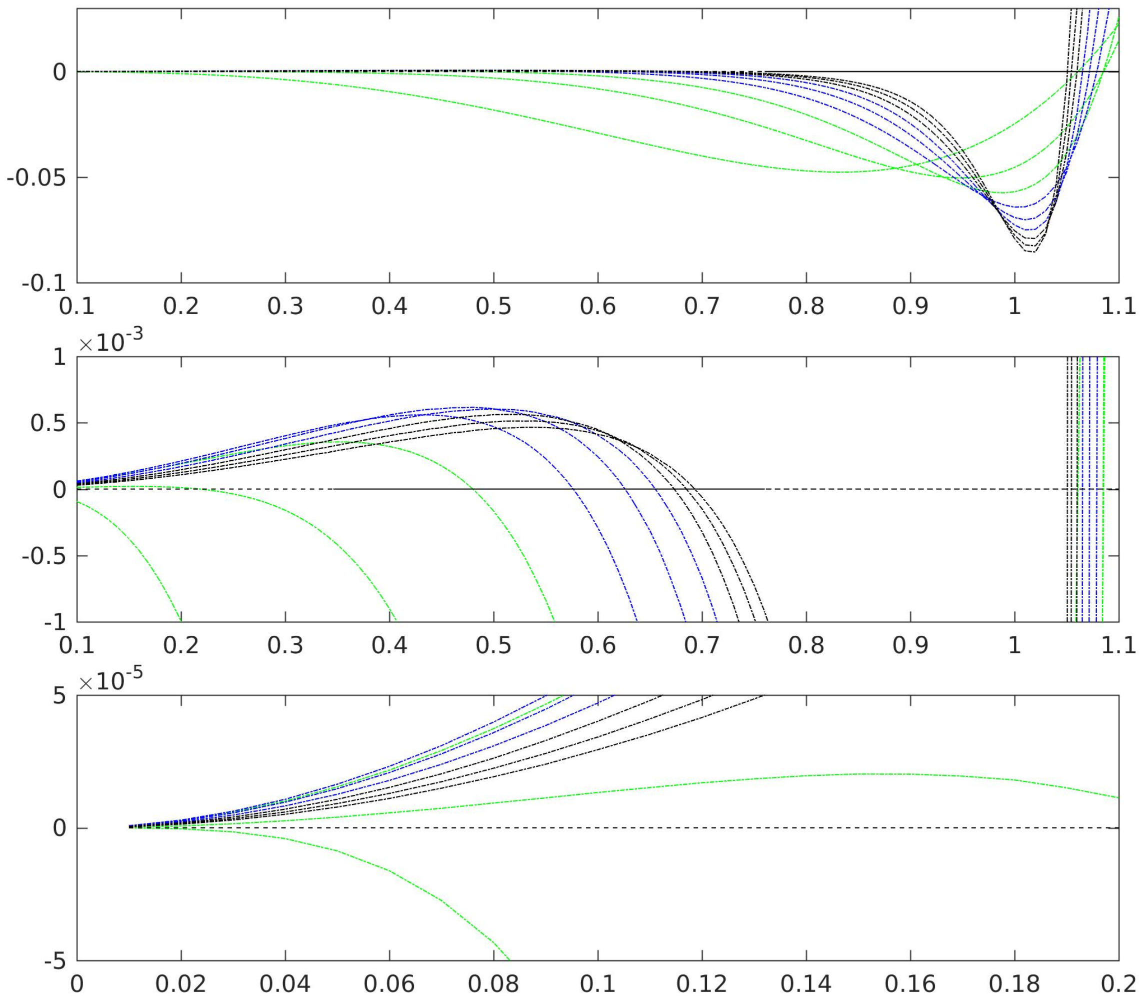

Figure 4.1.2: $\mathbb{H}P^m$-example: Matching conditions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$

In the above figures, a qualitatively different behaviour is exhibited in the $\mathbb{H}P^2$-example as opposed to the $\mathbb{H}P^m$-examples for $m \geq 3$. In particular, only in the case $m = 2$ a single root is observed. All other curves have two roots. The values $\bar{f}_*$ such that $\dot{\omega}_{\bar{f}_*}(\tau_{\bar{f}_*}) = 0$ agree with those calculated by Böhm [Böh98].

Recall that initially there holds $\dot{\omega}_{\bar{f}}(0) = \frac{1}{\bar{f}} > 0$. Therefore the analysis suggests that $\lim_{\bar{f}\to 0} \mathcal{C}(\mathbb{H}P^2, \tau_{\bar{f}}) = 1$. By using the linearisation of the Einstein equation along the cone solution, Böhm [Böh98] could prove that $\dot{\omega}_{\bar{f}}$ has at least one critical point before $\tau_{\bar{f}} > 0$ in the limit as $\bar{f} \to 0$ and that it gives rise to an Einstein metric of positive scalar curvature on $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$.

The figures 4.1.1 and 4.1.2 suggest the following conjecture.

Conjecture 4.1.3 *Böhm's [Böh98] symmetric cohomogeneity one Einstein metric on* $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$ *is unique and there exist exactly two symmetric Einstein metrics on*

$\mathbb{H}P^m\#\overline{\mathbb{H}P}^m$ *for* $m \geq 3$.

The number of critical points of $\omega_{\bar{f}}$ *before the maximal volume orbit satisfies*

$$\lim_{\bar{f}\to 0} \mathcal{C}(\mathbb{H}P^2, \tau_{\bar{f}}) = 1 \quad \textit{and} \quad \lim_{\bar{f}\to 0} \mathcal{C}(\mathbb{H}P^m, \tau_{\bar{f}}) = 0$$

for $m \geq 3$ *and* $\lim_{\bar{f}\to\infty} \mathcal{C}(\mathbb{H}P^m, \tau_{\bar{f}}) = 0$ *for* $m \geq 2$.

This is consistent with the non-degeneracy of the critical points of $\omega_{\bar{f}}$, which is essential for the construction in section 3.5. Notice in particular, in order to prove the existence of *both* Einstein metrics on $\mathbb{H}P^m\#\overline{\mathbb{H}P}^m$, for $m \geq 3$, it is sufficient to find a *single* trajectory $c_{\bar{f}}$ such that $\dot{\omega}_{\bar{f}}$ has a root before $\tau_{\bar{f}}$ and $\mathcal{C}(\mathbb{H}P^m, \tau_{\bar{f}}) \to 0$ as $\bar{f} \to 0$ and $\bar{f} \to \infty$.

The following series of figures studies the long time behaviour $t \mapsto \dot{\omega}_{\bar{f}}(t)$ in the $\mathbb{H}P^m$-example for $m \approx 2$. Of course, for non-integer values of m the equations do not have a geometric interpretation. In this case the term $\mathbb{H}P^m$-example shall simply refer to the set of data obtained from table 3.1.1 in the $\mathbb{H}P^m$-column.

The blue curves correspond to initial conditions $0.01 \leq \bar{f} < 0.3$ and the black curves to $0.3 \leq \bar{f} \leq 0.5$ with a step size of $\bar{f}_{\text{step}} = 0.01$. The order of the Taylor approximation is 20 and it is evaluated at $t_* = 10^{-3}$.

Recall that as $\bar{f} \to 0$, the solution approximates the cone solution, which satisfies $\dot{\omega}_{\text{cone}} \equiv 0$.

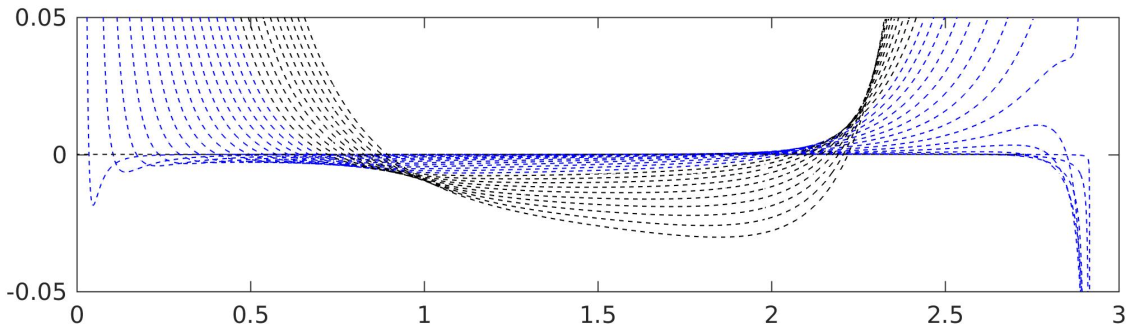

Figure 4.1.3: $\mathbb{H}P^{1.8}$-example: Long time behaviour of $\dot{\omega}$

In the $\mathbb{H}P^{1.8}$-example, for small $\bar{f} > 0$, the first root of $\dot{\omega}_{\bar{f}}$ already occurs as the solution 'oscillates' towards the cone solution, whereas in the $\mathbb{H}P^2$-example the roots of $\dot{\omega}_{\bar{f}}$ occur along the cone solution, one of which is before $\tau_{\bar{f}} > 0$, the other thereafter. Böhm [Böh98] could study these roots via the linearisation along the cone solution. However, the actual Einstein metric on $\mathbb{H}P^2\#\overline{\mathbb{H}P}^2$ corresponds to $\bar{f}^* \approx 1.08$.

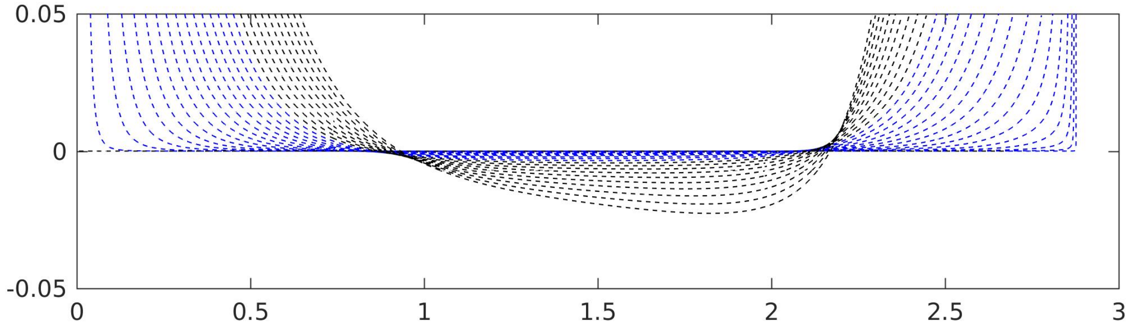

Figure 4.1.4: $\mathbb{H}P^2$-example: Long time behaviour of $\dot{\omega}$

In comparison to figure 4.1.4 above, figure 4.1.5 below has a different resolution which shows that indeed two roots occur as claimed.

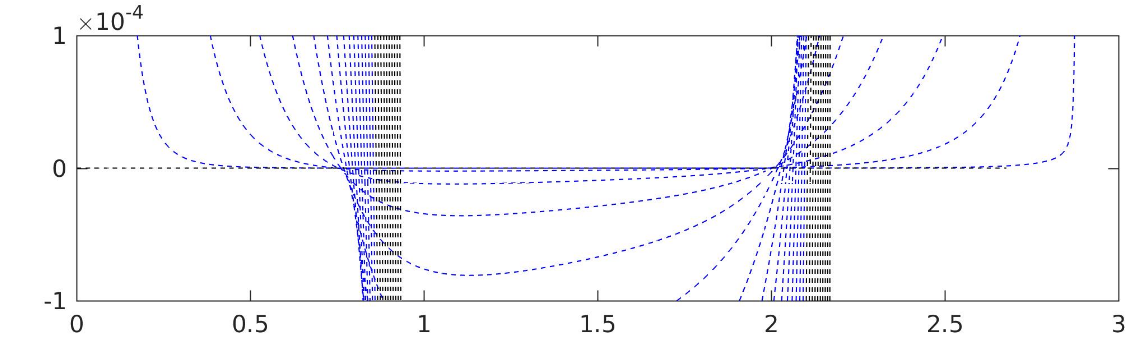

Figure 4.1.5: $\mathbb{H}P^2$-example: Long time behaviour of $\dot{\omega}$

As m increases, the qualitative behaviour of $\dot{\omega}_{\bar{f}}$ changes at approximately $m = 2.6$ and for $\bar{f} > 0$ sufficiently small, see figure 4.1.6. Along these solutions $\omega_{\bar{f}}$ only has one critical point, which has to appear after the maximal volume orbit, according to figures 4.1.1 and 4.1.2.

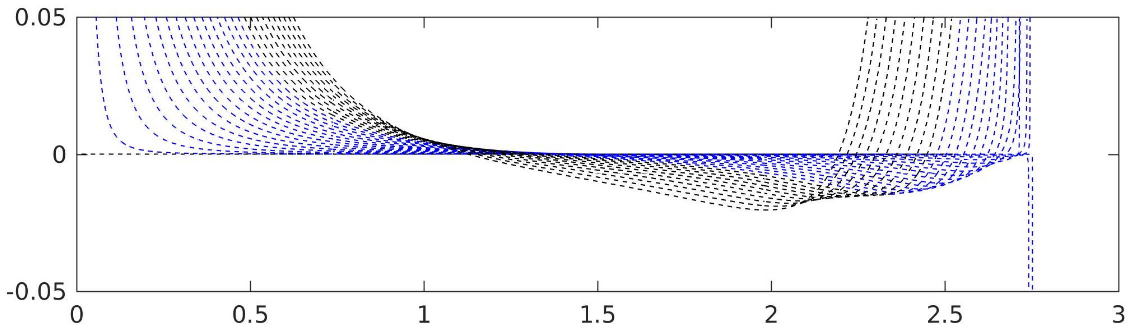

Figure 4.1.6: $\mathbb{H}P^{2.6}$-example: Long time behaviour of $\dot{\omega}$

A qualitatively similar picture is observed for higher values of m. Figure 4.1.7 shows the plot for $m = 3$.

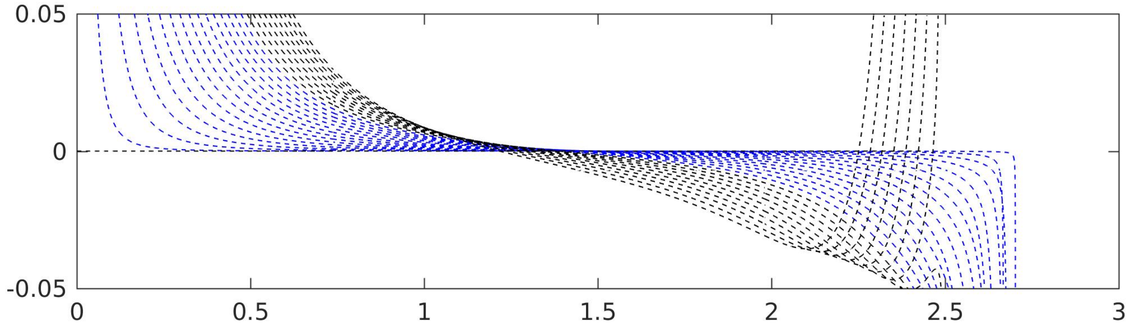

Figure 4.1.7: $\mathbb{H}P^3$-example: Long time behaviour of $\dot{\omega}$

The trajectory with $\bar{f}_* \approx 0.21$ corresponds to an Einstein metric on $\mathbb{H}P^3 \# \overline{\mathbb{H}P}^3$ and in particular it occurs before the qualitative behaviour of $\dot{\omega}_{\bar{f}}$ changes at $\bar{f}_{\text{jump}} \approx 0.36$. For $\bar{f} \in (\bar{f}_{\text{jump}}, \bar{f}^*)$ the trajectories have two critical points, but only the first occurs before $\tau_{\bar{f}}$. Remarkably, the qualitative behaviour changes again precisely when the second Einstein metric is observed at $\bar{f}^* \approx 1.08$. This is illustrated by the black and green curves in figure 4.1.8. Notice that $\dot{\omega}_{\bar{f}}$ has at most one critical point for $\bar{f} > \bar{f}^*$, which appears after $\tau_{\bar{f}}$, and for $\bar{f} \geq 1.35$ critical points of $\omega_{\bar{f}}$ do not occur at all. This is shown by the yellow curves in figure 4.1.8.

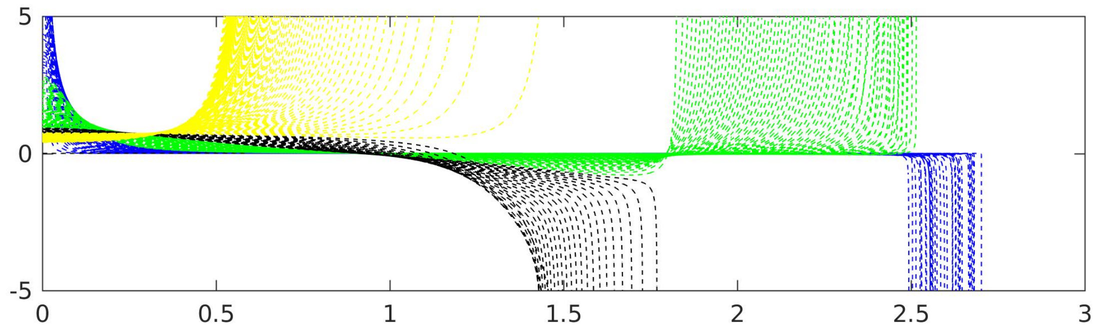

Figure 4.1.8: $\mathbb{H}P^3$-example: Long time behaviour of $\dot{\omega}$

The overall behaviour in the $\mathbb{H}P^2$-example is very similar, only the blue curves, which correspond to the smallest set of parameters of $\bar{f} > 0$, tend to positive infinity as well. In particular the Einstein metric, which is observed at $\bar{f}^* \approx 1.06$, occurs when also the qualitative behaviour changes.

Recall from figure 4.1.2 that in all $\mathbb{H}P^m$-examples, $m \geq 2$, along the trajectory with $\bar{f} = 1$ there exists a critical point of $\omega_{\bar{f}=1}$ before $\tau_{\bar{f}=1}$, which induces a symmetric solution for some $\bar{f}^* > 1$. However, in the $\mathbb{H}P^2$-example, the existence of this solution has been established via the limit $\bar{f} \to 0$, which is impossible in higher dimension. On the other hand, figure 4.1.9 plots the functions $t \mapsto \omega_{\bar{f}=1}(t)$ for $m \in \{0.6, 0.8, \ldots, 5\}$, where the green curves correspond to $m \in \{0.6, 0.8, \ldots, 2\}$, and illustrates that the qualitative behaviour of $\dot{\omega}_{\bar{f}=1}$ does not depend on the dimensional parameter m.

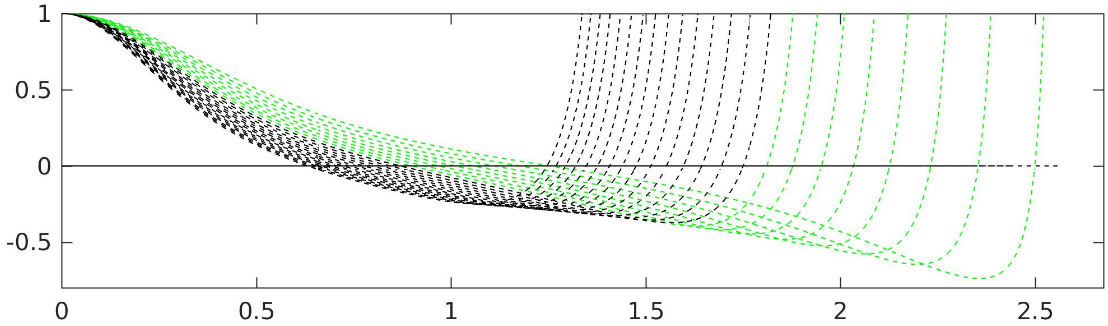

Figure 4.1.9: $\mathbb{H}P^m$-example, $0.6 \leq m \leq 5$: Qualitative behaviour $\dot{\omega}_{\bar{f}=1}$

If the qualitative behaviour of $\dot{\omega}_{\bar{f}=1}$ could be studied analytically and a stability statement under variation of m could be shown, then the existence of Einstein metrics on $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$, for $m \geq 3$, may be deduced from Böhm's [Böh98] symmetric Einstein metric on $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$.

4.1.2 The F^m-example

The analysis of the matching condition in the F^m-examples gives a result similar to the $\mathbb{H}P^m$-examples. Figures 4.1.10 and 4.1.11 provide analogous plots to figures 4.1.1 and 4.1.2, in particular, except for the geometric data from table 3.1.1, all technical data of the plots agree.

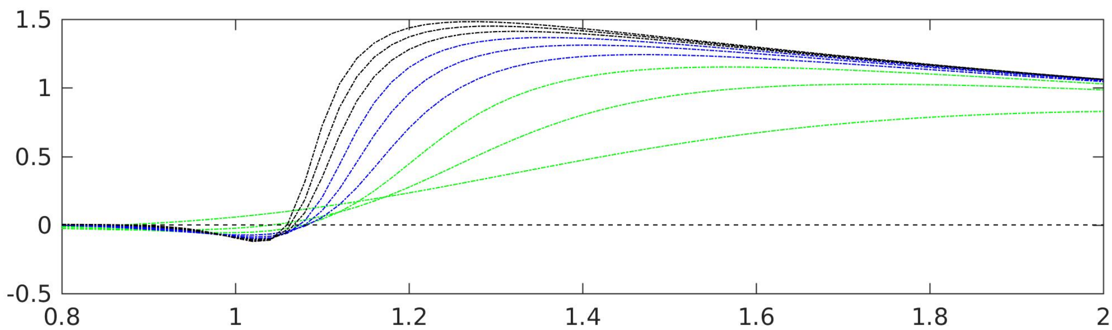

Figure 4.1.10: F^m-example: Matching conditions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$

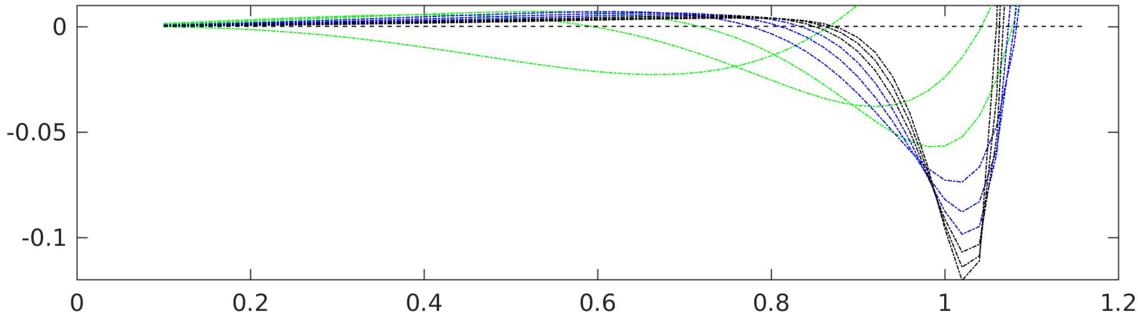

Figure 4.1.11: F^m-example: Matching conditions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$

The values $\bar{f}_*$ such that $\dot{\omega}_{\bar{f}_*}(\tau_{\bar{f}_*}) = 0$, which correspond to symmetric Einstein metrics, agree with the results of Böhm [Böh98], which suggests to conjecture:

Conjecture 4.1.4 *Symmetric cohomogeneity one Einstein metrics exist in all F^m-examples, $m \geq 2$. The metric is unique if $m = 2$ and otherwise there are exactly two.*

Moreover, the number of critical points of $\omega_{\bar{f}}$ which occur before the maximal volume orbit satisfies

$$\lim_{\bar{f}\to 0} \mathcal{C}(F^2, \tau_{\bar{f}}) = 1 \quad \textit{and} \quad \lim_{\bar{f}\to 0} \mathcal{C}(F^m, \tau_{\bar{f}}) = 0$$

for $m \geq 3$ and $\lim_{\bar{f}\to\infty} \mathcal{C}(F^m, \tau_{\bar{f}}) = 0$ for $m \geq 2$.

However, the analytic methods from the $\mathbb{H}P^2$-example to reconstruct the critical point of $\omega_{\bar{f}}$, which occurs in the limit as $\bar{f} \to 0$, via the linearisation of the Einstein equation along the cone solution do not carry over.

Figures 4.1.12 and 4.1.13 study the long time behaviour of $\dot{\omega}_{\bar{f}}$ analogously to figures 4.1.4 and 4.1.5. In particular these figures have the same technical data.

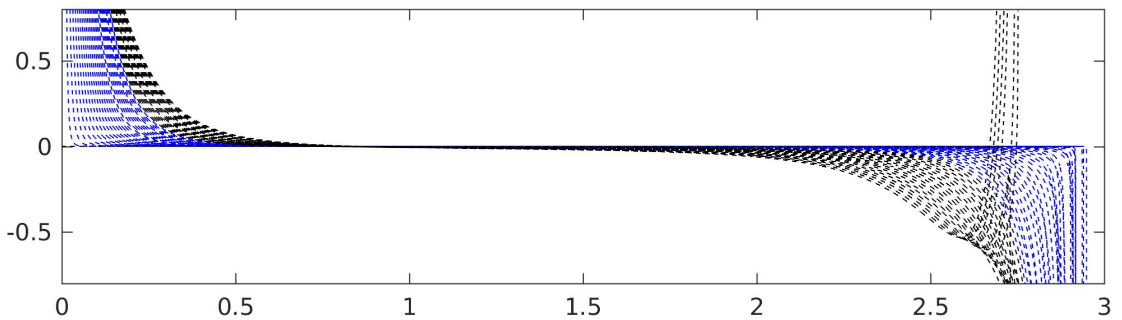

Figure 4.1.12: F^2-example: Long time behaviour of $\dot{\omega}$

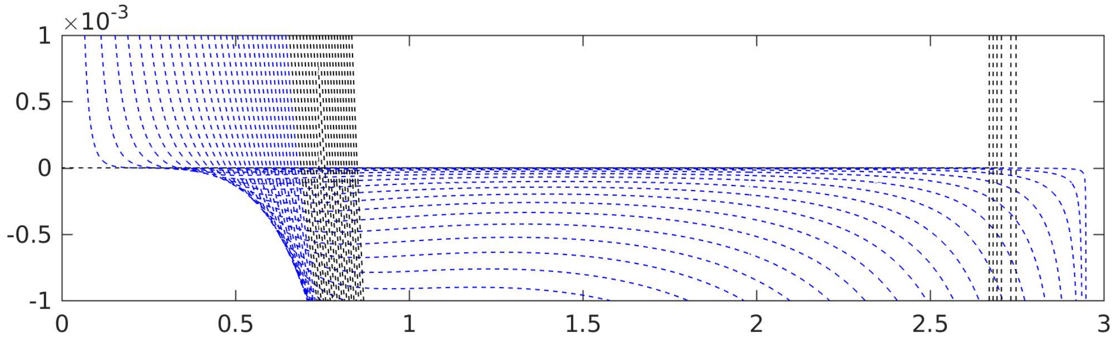

Figure 4.1.13: F^2-example: Long time behaviour of $\dot{\omega}$

Notice that the long time behaviour of $\dot{\omega}_{\bar{f}}$ in the above figures 4.1.12 and 4.1.13 is qualitatively different from the long time behaviour in the $\mathbb{H}P^2$-example observed in figures 4.1.4 and 4.1.5 but much more reminiscent of the long time behaviour of $\dot{\omega}_{\bar{f}}$ in the $\mathbb{H}P^3$-example in figure 4.1.8.

Therefore it seems that the numerically detected Einstein metrics on $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$, for $m \geq 3$, and $F^m \# \overline{F}^m$, for $m \geq 2$, are solutions which occur far away from the cone solutions and hence are in an entirely non-linear regime of the Einstein equations. Section 4.2 provides a different numerical argument in this direction.

4.1.3 The CaP^2-example

In contrast to the previous examples, the numerical analysis of the matching condition in the CaP^2-example indicates that there is no symmetric cohomogeneity one Einstein metric on $CaP^2 \# \overline{CaP}^2$. The corresponding plot is given in figure 4.1.14.

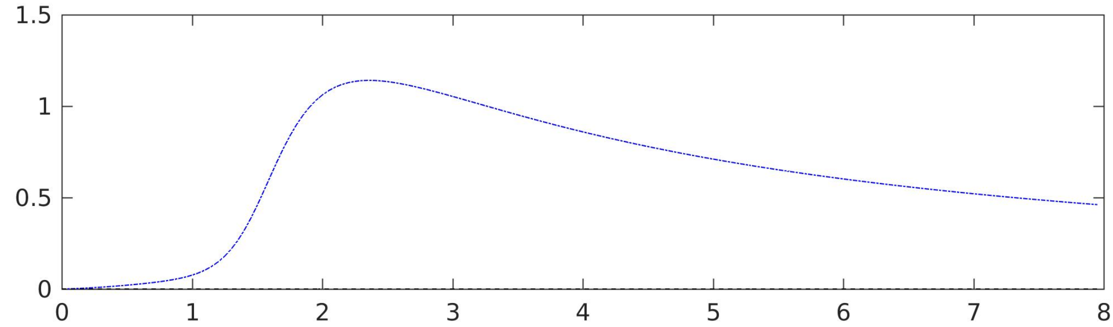

Figure 4.1.14: CaP^2-example: Matching conditions $\bar{f} \mapsto \dot{\omega}_{\bar{f}}(\tau_{\bar{f}})$

This is an agreement with the results in [Böh98] and [DHW13], where no Einstein metric could be detected either.

Recall that according to a theorem of Böhm [Böh98, Theorem 3.2] any non-symmetric Einstein solution induces a symmetric Einstein solution. Therefore one may conjecture:

Conjecture 4.1.5 *The Hopf fibrations do not induce a cohomogeneity one Einstein metric on* $CaP^2\#\overline{CaP}^2$.

A brief analysis of the long time behaviour of $\dot{\omega}_{\bar{f}}$ in figure 4.1.15 shows that for small values of $\bar{f} > 0$ a critical point does exist but it only occurs after the trajectory has passed the maximal volume orbit.

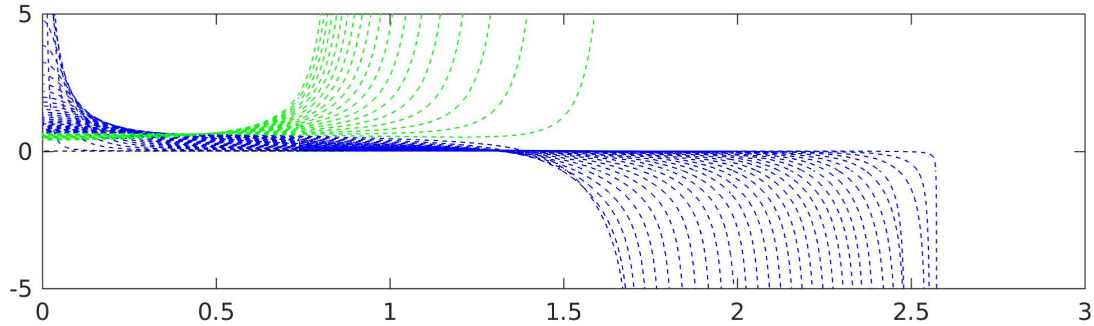

Figure 4.1.15: CaP^2 : Long time behaviour of $\dot{\omega}$

In figure 4.1.15, the blue curves have initial conditions $\bar{f} \in \{0.01, 0.06, \ldots, 1.51\}$ and the green curves proceed with the same step size up to $\bar{f}_{\max} = 2.46$. The Taylor expansion, which has been used to obtain an approximate solution to the Einstein equation, is of order $O(t^{20})$ and it has been evaluated at $t_* = 10^{-3}$.

4.2 Numerical analysis of the Ricci flat trajectories

The key simplification of Böhm's construction of Einstein metrics on products of spheres in section 3.5 is to consider the *Ricci flat* trajectory in the phase space defined by the rescaling with the inverse of the mean curvature. As section 3.4.2.1 explains, the two summands Ricci flat equation reduces to a planar ODE for the variables $(\overline{X}, \overline{Y})$, which is given in (3.19). In this coordinate system, the trajectory emanates from $(\frac{1}{d_1} - \frac{1}{n}, \frac{1}{d_1} - \frac{1}{nc_1})$, where (c_1, c_2) is the associated first cone solution, and finally converges to the origin.

Then due to the construction in section 3.5, any intersection of this trajectory with the $\overline{X}$-axis corresponds to a critical point of ω at a time before the maximal volume orbit is reached, and thus gives rise to a symmetric cohomogeneity one Einstein metric of positive scalar curvature.

Remark 4.2.1 This procedure corresponds to zooming in on the Einstein equation as follows:

Let $\mu > 0$ and suppose that $\mathrm{Ric}(g_1) = \lambda g_1$. Then the metric $g_2 = \mu g_1$ satisfies $\mathrm{Ric}(g_2) = \frac{\lambda}{\mu} g_2$. In particular, g_2 is also Einstein but its Einstein constant tends to zero as $\mu \to \infty$. This corresponds to letting the volume $\mathrm{Vol}(Q)$ of the singular orbit Q tend to zero.

Moreover, due to section 3.5, in the $\frac{1}{\mathrm{tr}(L)}$-rescaled coordinates, one obtains a well defined limit. The limiting trajectory is a solution to the Ricci flat equation and describes how the solutions to the Einstein equation with positive Einstein constant and small $\mathrm{Vol}(Q) > 0$ approach the base point of the cone solution.

The other extremal behaviour as $\mathrm{Vol}(Q) \to 0$ is described by the first cone solution $(f_1(t), f_2(t)) = (c_1 \sin(t), c_2 \sin(t))$ for $t \in (0, \pi)$. By studying its linearisation, Böhm [Böh98] could prove the existence of an Einstein metric on $\mathbb{H}P^2 \# \overline{\mathbb{H}P}^2$. However, section 4.1 suggests that this is the only example where the linearisation can be used to give a rigorous construction.

Therefore, this section discusses a numerical analysis of the rescaled Ricci flat equations of section 3.4.2.1 and the plots in figures 4.2.1 and 4.2.2 show the graph of the Ricci flat trajectory in the $\overline{X}$-$\overline{Y}$-plane in the $\mathbb{H}P^m$- and F^m-examples of table 3.1.1. The diagram on the left considers $m = 2$ and the diagram on the right considers $m = 3$ in both cases. The numerical construction of the plots is explained at the end of this section.

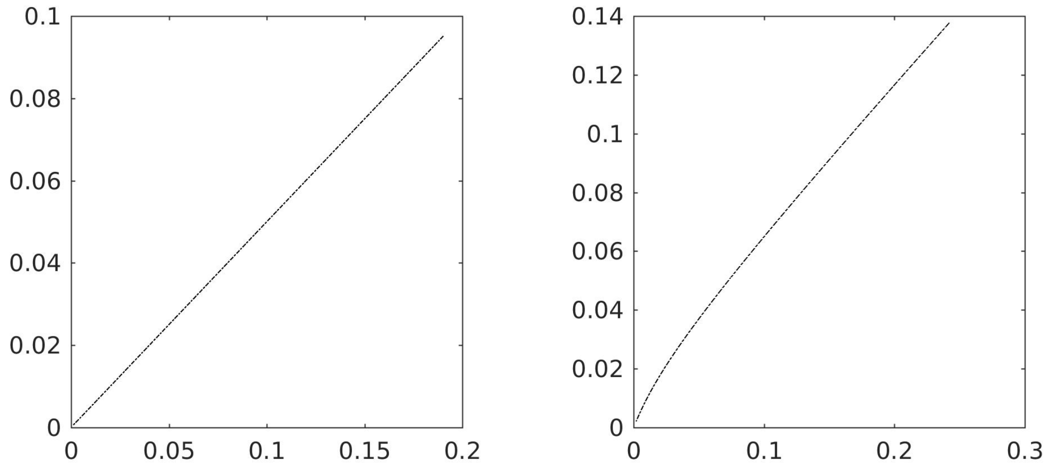

Figure 4.2.1: Ricci flat trajectories on $\mathbb{H}P^m \setminus \{ \text{ point } \}$ for $m = 2, 3$

However, in none of the cases a trajectory with $\overline{X} = 0$ has been found and thus no critical point of ω. Due to section 3.5, this suggests that the symmetric Einstein metrics on the connected sums $\mathbb{H}P^m \# \overline{\mathbb{H}P}^m$ and in the F^m-examples, which have been observed in the previous section, cannot be seen in this limiting procedure either and are hence entirely non-linear solutions to the Einstein equations.

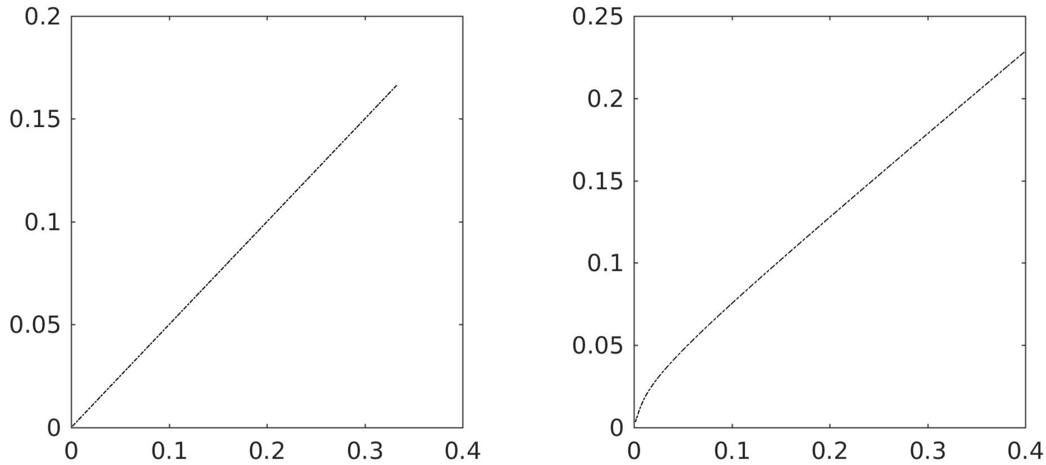

Figure 4.2.2: Ricci flat trajectories on $F^m \setminus \{$ point $\}$ for $m = 2, 3$

On the other hand, in the smallest dimensional cases, the Ricci flat trajectories are represented by straight lines. This numerical analysis is in fact the origin of theorem 3.4.10, which rigorously establishes this observation and gives new parametrisations of Ricci flat metrics of special holonomy due to Bryant-Salamon [BS89] and Gibbons-Page-Pope [GPP90].

Furthermore, the plots show that a complete Ricci flat metric should also exist in the F^3-example. However, this case is not yet settled analytically since it is neither covered by proposition 3.2.9 nor theorem 3.4.10.

The plots in figures 4.2.1 and 4.2.2 are constructed with the following numerical scheme:

Analogously to proposition 4.1.1, a Taylor expansion for the variables f_1, f_2, and $\dot{f}_1/f_1$, $\dot{f}_2/f_2$ in the Ricci flat case has been calculated up to order $O(t^{20})$ with the Symbolic Math Toolbox in MATLAB. The algorithm is the same as in section 4.1 and is based on the scheme in Eschenburg-Wang's work [EW00]. The Taylor expansions have then been converted into Taylor expansions for the rescaled variables X_i, Y_i, of (3.4) and were evaluated at $t_* = 10^{-3}$ and with $\bar{f} = f_2(0) = 10^{-2}$.

The obtained set of data has been used as initial conditions for the ODE (3.5), which has been solved with the MATLAB ODE solver ode45. The simulation has been stopped once the numerical error in the conservation laws (3.8) increased significantly, up to the order of 10^{-3}.

Due to the preserved quantities in (3.8), the Ricci flat trajectory is completely determined by X_1 and Y_1, or equivalently by $\overline{X} = X_1 - \frac{1}{n}$ and $\overline{Y} = Y_1 - \frac{1}{nc_1}$, where (c_1, c_2) is the associated first cone solution, and which have been plotted on the x- and y-axes, respectively.

Chapter 5

Growth of the soliton potential

This chapter extends the general theory of cohomogeneity one Ricci solitons by introducing a growth estimate on the soliton potential.

Recall that if the Ricci soliton metric is complete, then in the steady case the soliton potential has linear growth at infinity [BDW15] whereas in the expanding case it has quadratic growth at infinity [BDGW15a].

However, in the construction problem for Ricci solitons, completeness cannot be assumed a priori. If the metric on the cohomogeneity one manifold remains in diagonal form and the shape operator of the principal orbit is positive definite, which is the case in all currently known examples, then theorem 5.1.1 guarantees *immediate linear growth* of the soliton potential with a *prescribed growth rate* provided that a condition on the second derivative of the soliton potential at the singular orbit is satisfied.

The theorem will be applied in section 5.2 to construct new complete Ricci solitons.

5.1 An estimate on the soliton potential

This section introduces and states the main growth estimate on the soliton potential. The proof follows in section 5.1.2. Firstly, the set-up and notation of chapter 2 of cohomogeneity one Ricci solitons with a singular orbit will briefly be recalled.

As every non-trivial steady or expanding Ricci soliton is non-compact, consider a Riemannian manifold (M^{n+1}, g) and a compact Lie group G which acts isometrically on M so that the orbit space M/G is isometric to $[0, T)$ for some $T > 0$ and denote by $K \subset H \subset G$ the isotropy groups of the principal and singular orbit, respectively. A choice of a unit speed geodesic on M then induces a G-equivariant diffeomorphism $\Phi\colon [0, T) \times G/K \to M_0$ onto the union of all principal orbits M_0 and the pullback metric is of the form $dt^2 + g_t$ for a 1-parameter family of metrics g_t on the principal orbit $P = G/K$.

The shape operator L_t of the hypersurface $\Phi(\{t\} \times P)$ will be regarded as a g_t-symmetric endomorphism of TP which satisfies $\dot{g}_t = 2g_tL_t$ and r_t is the Ricci endomorphism corresponding to g_t. Suppose that u is a G-invariant function on M. It follows from results in [DW11] that, due to the existence of a singular orbit, the gradient Ricci soliton equations for (M, g, u) reduce to the ODE system

$$- \operatorname{tr}(\dot{L}_t) - \operatorname{tr}(L_t^2) + \ddot{u} + \frac{\varepsilon}{2} = 0, \tag{2.3}$$

$$-\dot{L}_t - (-\dot{u} + \operatorname{tr}(L_t))L_t + r_t + \frac{\varepsilon}{2}\mathbb{I} = 0. \tag{2.4}$$

Conversely, if the metric is at least C^2-regular and the soliton potential is of class C^3, then a solution to these equations with $\dot{g}_t = 2g_tL_t$ induces the structure of a smooth gradient Ricci soliton on M. Furthermore, any C^3-regular gradient Ricci soliton has to satisfy the conservation law

$$\ddot{u} + (-\dot{u} + \operatorname{tr}(L))\dot{u} = C + \varepsilon u \tag{2.5}$$

for some constant $C \in \mathbb{R}$. After changing u by an additive constant, one may assume that $u(0) = 0$ and then the constant C of the conservation law appears as follows in the smoothness conditions of the soliton potential at the singular orbit:

$$u(0) = 0, \quad \dot{u}(0) = 0, \quad \ddot{u}(0) = \frac{C}{d_S + 1}, \tag{2.8}$$

where d_S denotes the dimension of the collapsing sphere H/K.

From now on restrict to the case of steady and expanding Ricci solitons, $\varepsilon \geq 0$. If the Ricci soliton is complete, there necessarily holds $C < 0$ but even along a possibly incomplete solution the soliton potential is strictly decreasing and concave (provided $L_t \neq 0$ if $\varepsilon = 0$). If the Ricci soliton is *complete* then the soliton potential is also necessarily concave in the steady case because the mean curvature of the principal orbit is then always positive and due to proposition 2.4.1 the soliton potential grows linearly at infinity. In the expanding case one has quadratic growth of the soliton potential at infinity according to proposition 2.4.2.

However, when investigating the Ricci soliton equations in order to construct new examples, the above estimates are *not* available a priori because the trajectory may not correspond to a complete Ricci soliton.

Therefore it is essential that theorem 5.1.1 does *not* assume that the Ricci soliton is complete. Furthermore, the growth rate of the soliton potential can also be prescribed for small times $t > 0$ when the existence of a solution to the Ricci soliton equations is a priori only guaranteed by short time existence results, e.g. [Buz11]. In contrast, propositions 2.4.1 and 2.4.2 only give estimates on the asymptotics for large times.

Theorem 5.1.1 *Let (M^{n+1}, g, u) be a steady or expanding gradient Ricci soliton which is of cohomogeneity one under the isometric action of the compact Lie group G. Suppose $\Phi\colon (0,T) \times P \to M_0$ is an isometric parametrisation of the union of all principal orbits M_0 and the pullback metric is given by $dt^2 + g_t$. Assume that there is a singular orbit G/H at $t = 0$ and that the isotropy representation of $P = G/K$ is monotypic, that is, it decomposes into pairwise inequivalent Ad_K-invariant irreducible summands.*

Fix $t_* \in [0,T)$ and $c_* > 0$ and suppose that the Killing form B of $\mathfrak{g}$ satisfies $-\operatorname{tr}_{g_{t_*}}(B_{|\mathfrak{p}}) < c_*$ and that the shape operator L_t of the hypersurface $P_t = \Phi(\{t\} \times P)$ is positive definite for all $t \geq t_*$.

Then for all $c > 0$ and for all $\tau \in (t_*, T)$ there exists a constant $C_0 \leq 0$ depending only on c_*, c, $\tau - t_*$ such that $-\dot{u}(t) \geq c$ on $[\tau, T)$, provided that $\ddot{u}(0) < C_0$.

Notice that in applications to constructions of Ricci solitons the initial condition $\ddot{u}(0) < C_0$ can simply be imposed.

Remark 5.1.2 The assumption of a monotypic isotropy representation can be relaxed. It suffices that the metric and the Ricci tensor can be diagonalised with respect to the *same* decomposition of the isotropy representation for all t as in (5.4).

It suffices to assume that the shape operator is positive definite on $[t_*, \tau)$ and moreover non-vanishing if $\varepsilon = 0$. Indeed, the proof in section 5.1.2 then shows that $-\dot{u}(t) \geq c$ holds at $t = \tau$ and the concavity of the soliton potential in proposition 2.3.1 ensures that this estimate is preserved for $t \geq \tau$.

The theorem even applies to classes of manifolds which are just foliated by hypersurfaces. These hypersurfaces may not be homogeneous but a similar structure theory has to apply. In particular the metric on the hypersurfaces has to satisfy a decomposition similar to (5.4) so that the Ricci soliton equations have the same form as if they came from a cohomogeneity one manifold.

The assumption of positive definiteness of the shape operator is natural in the sense that it holds in all currently known examples.

In the applications in section 5.2, the growth estimate will be applied in such a way to ensure that the shape operator indeed remains positive definite along the entire trajectory. This automatically implies completeness of the metric due to the following argument. The author thanks C. Böhm and B. Wilking for pointing this out.

Proposition 5.1.3 *Let $\varepsilon \geq 0$ and $(g_t, L_t, u)_{t\in(0,T)}$ be a maximal solution to the cohomogeneity one Ricci soliton equations corresponding to a smooth manifold with a singular orbit Q at $t = 0$, metric $g_{M\setminus Q} = dt^2 + g_t$ and $\ddot{u}(0) \leq 0$. Suppose that the shape operator L_t of the principal hypersurface is positive definite for all $t > 0$.*

Then $T = \infty$ and the manifold is complete.

Proof: In the steady case proposition 2.3.1 gives the bound $\mathrm{tr}(L) \leq \frac{n}{t}$ and hence the shape operator remains bounded along the flow.

In the expanding case, due to the concavity of the soliton potential, the differential inequality

$$\frac{d}{dt}\,\mathrm{tr}(L) \leq \frac{\varepsilon}{2} - \mathrm{tr}(L^2) \leq \frac{\varepsilon}{2} - \frac{1}{n}\,\mathrm{tr}(L)^2$$

again shows that $\mathrm{tr}(L)$ and hence L are bounded. Moreover, both arguments carry over to the Einstein case.

The metric satisfies the *linear* ODE $\frac{d}{dt}g_t = 2g_tL_t$ and hence cannot blow up in finite time either.

In the soliton case, an estimate on the potential is also required. One may assume that $u(0) = 0$. Recall that the conservation law $\ddot{u} + (-\dot{u} + \mathrm{tr}(L))\dot{u} = (d_S + 1)\ddot{u}(0) + \varepsilon u$ needs to be satisfied, where d_S is the dimension of the collapsing sphere.

In the steady case, it immediately follows that $\dot{u}^2 \leq -(d_S + 1)\ddot{u}(0)$ since $\dot{u}, \ddot{u} \leq 0$. In the expanding case, concavity of the soliton potential implies $-u(t) \leq -\dot{u}(t) \cdot t$. It follows that

$$\frac{d}{dt}(-\dot{u}) \leq -(d_S + 1)\ddot{u}(0) + \varepsilon(-\dot{u}) \cdot t - (-\dot{u})^2$$

and therefore $-\dot{u} \geq 0$ cannot blow up in finite time. $\square$

Proposition 5.1.4 *Complete cohomogeneity one steady Ricci solitons with a singular orbit and positive definite shape operator have non-negative Ricci curvature.*

Proof: This follows from the formulae for the Ricci curvature of a cohomogeneity one manifold in section 2.1, the cohomogeneity one Ricci soliton equations (2.2) - (2.4) and proposition 2.4.1. $\square$

5.1.1 Preparation for the proof: Scalar curvature of homogeneous spaces

The proof of theorem 5.1.1 involves an analysis of the scalar curvature $s_t = \mathrm{tr}(r_t)$ of the principal hypersurface P_t. This relies on the structure theory for homogeneous spaces, cf. [WZ86], [Bes87], [PS97], [Böh05], applied to the principal orbit $P = G/K$. This section revisits the relevant formulae.

Due to the compactness of G, there exists a biinvariant inner product b on the Lie algebra $\mathfrak{g}$ of G. Furthermore, $\mathfrak{g}$ decomposes into Ad_K-invariant, b-orthogonal summands $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and G-invariant metrics on G/K correspond to Ad_K-invariant inner products on $\mathfrak{p}$.

Let $B(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ be the Killing form of $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$ an Ad_K-invariant inner product on $\mathfrak{p}$ and e_α an $\langle \cdot, \cdot \rangle$-orthonormal basis of $\mathfrak{p}$. Then the scalar curvature s of G/K with respect to $\langle \cdot, \cdot \rangle$ is given by

$$s = -\frac{1}{4} \sum_{\alpha,\beta} \langle [e_\alpha, e_\beta]_{\mathfrak{p}}, [e_\alpha, e_\beta]_{\mathfrak{p}} \rangle - \frac{1}{2} \sum_{\alpha} B(e_\alpha, e_\alpha),$$

where $[\cdot, \cdot]_{\mathfrak{p}}$ is the projection of the Lie bracket onto $\mathfrak{p}$. By further decomposing $\mathfrak{p}$, this description can be refined. Suppose that

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_m \tag{5.1}$$

is a decomposition of $\mathfrak{p}$ into b-orthogonal, Ad_K-irreducible summands. In fact, for any Ad_K-invariant inner product on $\mathfrak{p}$ there exists such a decomposition which moreover satisfies

$$\langle \cdot, \cdot \rangle = x_1 b_{|\mathfrak{p}_1} \perp \ldots \perp x_m b_{|\mathfrak{p}_m}$$

for some $x_1, \ldots, x_m > 0$. This decomposition may not be unique.

Suppose that the $\langle \cdot, \cdot \rangle$-orthonormal basis $\{e_\alpha\}$ is adapted to the decomposition (5.1). That is, $e_\alpha \in \mathfrak{p}_i$ for some i and there holds $\alpha < \beta$ if $i < j$. Then the coefficients defined by $A_{\alpha\beta}^\gamma = b([e_\alpha, e_\beta], e_\gamma)$ satisfy $[e_\alpha, e_\beta]_{\mathfrak{p}} = \sum_\gamma A_{\alpha\beta}^\gamma e_\gamma$ and the structure constants of the decomposition are given by $[ijk] = \sum (A_{\alpha\beta}^\gamma)^2$, where the sum is taken over all indices α, β, γ such that $e_\alpha \in \mathfrak{p}_i$, $e_\beta \in \mathfrak{p}_j$ and $e_\gamma \in \mathfrak{p}_k$. Notice that the structure constants $[ijk]$ depend on the decomposition of $\mathfrak{p}$ but are independent of the choice of the orthonormal basis. Furthermore, $A_{\alpha\beta}^\gamma$ is skew-symmetric in all three indices and hence $[ijk]$ is symmetric in all three indices.

Let $d_i = \dim \mathfrak{p}_i$ and $B_{|\mathfrak{p}_i} = -b_i b_{|\mathfrak{p}_i}$. Then it follows that $b_i \geq 0$ with $b_i = 0$ if and only if $\mathfrak{p}_i \subset \mathfrak{z}(\mathfrak{g})$ and $[ijk] \geq 0$ with $[ijk] = 0$ if and only if $b([\mathfrak{p}_i, \mathfrak{p}_j], \mathfrak{p}_k) = 0$.

Furthermore, the scalar curvature of G/K with respect to the inner product $\langle\cdot,\cdot\rangle$ can now be written as

$$s = -\frac{1}{4}\sum_{i,j,k}[ijk]\frac{x_k}{x_i x_j} + \frac{1}{2}\sum_{i=1}^{m}\frac{d_i b_i}{x_i}. \tag{5.2}$$

More generally, the Ricci operator is given by

$$r_{|\mathfrak{p}_i} = \left\{\frac{1}{2}\frac{b_i}{x_i} - \frac{1}{2d_i}\sum_{j,k=1}^{m}[ijk]\frac{x_k}{x_i x_j} + \frac{1}{4d_i}\sum_{j,k=1}^{m}[ijk]\frac{x_i}{x_j x_k}\right\}\mathrm{id}_{\mathfrak{p}_i}. \tag{5.3}$$

Remark 5.1.5 Wang-Ziller [WZ86] have shown the following relation between the structure constants. For any $1 \leq i \leq m$ there holds

$$\sum_{j,k}[ijk] = d_i(b_i - 2c_i),$$

where the constants $c_i \geq 0$ describe the Casimir operator $\mathcal{C}_{\mathfrak{p}_i, b_{\mathfrak{k}}}$. By definition one has $\mathcal{C}_{\mathfrak{p}_i, b_{\mathfrak{k}}} = -\sum_i \mathrm{ad}(z_i)\circ\mathrm{ad}(z_i)$ for any b-orthonormal basis $\{z_i\}$ of $\mathfrak{k}$. However, due to the irreducibility condition it satisfies $\mathcal{C}_{\mathfrak{p}_i, b_{\mathfrak{k}}} = c_i\,\mathrm{id}_{\mathfrak{p}_i}$. Moreover, one has $c_i = 0$ if and only if $\mathfrak{p}_i$ is contained in the subspace $\mathfrak{p}_0$ which satisfies $\mathrm{Ad}_{K|\mathfrak{p}_0} = \mathrm{id}_{|\mathfrak{p}_0}$.

5.1.2 Proof of the growth estimate theorem 5.1.1

The assumption of a monotypic isotropy representation in theorem 5.1.1 guarantees that the *same* decomposition (5.1) can be chosen to diagonalise the metric on the hypersurface for all t. More generally, it is enough to assume that the metric is given by

$$g_t = f_1^2(t)b_{|\mathfrak{p}_1} + \ldots + f_m^2(t)b_{|\mathfrak{p}_m} \tag{5.4}$$

for all t. Notice that then the Ricci operator (5.3) also diagonalises with respect to the same decomposition. Moreover, the shape operator satisfies

$$L = \mathrm{diag}\left(\frac{\dot{f}_1}{f_1}\mathrm{id}_{\mathfrak{p}_1}, \ldots, \frac{\dot{f}_m}{f_m}\mathrm{id}_{\mathfrak{p}_m}\right)$$

since $\dot{g}_t = 2g_t L_t$.

One may assume $f_i > 0$ and $u(0) = 0$. Then the shape operator L is positive definite if and only if $\dot{f}_i > 0$ for all i and the smoothness conditions of the soliton potential are given by (2.8). In particular, the constant C in the conservation law (2.5) is given by $C = (d_S + 1)\ddot{u}(0)$.

Observe that the Ricci soliton equations (2.3), (2.4) imply

$$\ddot{u} = \mathrm{tr}(L^2) - (-\dot{u} + \mathrm{tr}(L))\,\mathrm{tr}(L) + \mathrm{tr}(r) + (n-1)\frac{\varepsilon}{2}.$$

Combining this with the conservation law (2.5), it follows that on any C^3-regular cohomogeneity one Ricci soliton there holds

$$2\ddot{u} = (d_S + 1)\ddot{u}(0) + \varepsilon u + \dot{u}^2 + \mathrm{tr}(L^2) - (\mathrm{tr}(L))^2 + \mathrm{tr}(r) + (n-1)\frac{\varepsilon}{2}. \tag{5.5}$$

In order to estimate the quantities in this equation, firstly recall that in the steady and expanding case, $\varepsilon \geq 0$, the soliton potential is negative for all $t > 0$ according to proposition 2.3.1. Secondly, it follows from the structure of the scalar curvature of P_t in (5.2) that $\mathrm{tr}(r_t) \leq \frac{1}{2} \sum_i \frac{d_i b_i}{f_i^2(t)}$. Thirdly, observe that as long as the shape operator L is positive definite one has $\dot{f}_i > 0$ and hence

$$\mathrm{tr}(r_t) \leq \frac{1}{2} \sum_i \frac{d_i b_i}{f_i^2(t_*)} = -\frac{1}{2} \mathrm{tr}_{g_{t_*}}(B_{|\mathfrak{p}})$$

for all $t \geq t_*$. Positive definiteness of L also implies that $\mathrm{tr}(L^2) - (\mathrm{tr}(L))^2 \leq 0$. Let $a > 0$ and suppose that $(d_S + 1)\ddot{u}(0) + \frac{c_*}{2} + (n-1)\frac{\varepsilon}{2} < -a$. Then it follows that

$$\ddot{u}(t) \leq -a + \frac{1}{2}\dot{u}^2(t)$$

for all $t \geq t_*$. It is useful to note that despite the singularity of (5.5) at $t = 0$, if the assumption $-\mathrm{tr}_{g_{t_*}}(B_{|\mathfrak{p}}) < c_*$ already holds at $t_* = 0$, then the above estimate on $\ddot{u}$ also holds at $t = 0$.

If $-\dot{u}(t_*) \geq c$ then the claim follows from the concavity of the soliton potential due to proposition 2.3.1. Otherwise, one can assume that $a > \frac{c^2}{2}$ and the claim then follows from the comparison principle for ODEs and remark 5.1.6 below. $\square$

Remark 5.1.6 Let $a > 0$. The initial value problem

$$y' = -a + \frac{y^2}{2} \quad \text{and} \quad y(s_*) = y_*$$

is solved by $y(s) = \sqrt{2a}\tanh(\sqrt{\frac{a}{2}}(s_* - s) + \mathrm{arctanh}(\frac{y_*}{\sqrt{2a}}))$ if $-a + \frac{y_*^2}{2} < 0$.

Fix $c > 0$, $s_0 > s_*$ and suppose that $0 \leq -y_* \leq c$. Then there exists an $a_0 \geq 0$ depending only on c, $s_0 - s_*$ such that for all $a > a_0$ the solution to the above initial value problem satisfies $-y(s) \geq c$ for all $s > s_0$.

This argument generalises an observation of Appleton [App18] for the soliton potential of steady Ricci solitons on complex line bundles over Fano Kähler Einstein manifolds.

5.2 Applications

In this section, theorem 5.1.1 will be used to construct new examples of complete steady and expanding Ricci solitons in the following geometric situations: the Dancer-Wang set-up [DW11], where previously only explicit Kähler Ricci solitons were known, the examples associated to Lü-Page-Pope [LPP04], who described explicit Ricci flat metrics, and the two summands case of chapter 3, where now all low dimensional examples are covered.

5.2.1 Revision of the two summands case

This section revisits the construction of complete steady and expanding Ricci solitons in the two summands case, which was considered in sections 3.1 and 3.2, and the main theorem 3.1.1 is extended to a larger dimension range.

Theorem 5.2.1 *On* $CaP^2 \setminus \{$ *point* $\}$, $\mathbb{H}P^{m+1} \setminus \{$ *point* $\}$ *for* $m \geq 1$ *and on the vector bundle associated to* $(G, H, K) = (Sp(m+1), Sp(1) \times Sp(m), U(1) \times Sp(m))$ *for* $m \geq 1$, *there exist a* 1*-parameter family of non-homothetic complete steady and a* 2*-parameter family of non-homothetic complete expanding Ricci solitons.*

Recall that the two summands case describes the setting in which the isotropy representation of $P = G/K$ decomposes into two inequivalent Ad_K-invariant irreducible summands $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$. Suppose that b is a biinvariant metric on G which induces the metric of constant curvature 1 on H/K. If $\mathfrak{p}_1$ and $\mathfrak{p}_2$ correspond to the tangent spaces of the collapsing sphere H/K and the singular orbit G/H, respectively, then metric on the principal orbit is given by

$$g_t = f_1(t)^2 b_{|\mathfrak{p}_1} + f_2(t)^2 b_{|\mathfrak{p}_2}$$

and the Ricci soliton equations take the form

$$\begin{aligned}
\ddot{u} &= \sum_{i=1}^{2} d_i \frac{\ddot{f}_i}{f_i} - \frac{\varepsilon}{2}, \\
\frac{d}{dt}\frac{\dot{f}_1}{f_1} &= -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}_1}{f_1} + \frac{\varepsilon}{2} + \frac{A_1}{d_1}\frac{1}{f_1^2} + \frac{A_3}{d_1}\frac{f_1^2}{f_2^4}, \\
\frac{d}{dt}\frac{\dot{f}_2}{f_2} &= -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}_2}{f_2} + \frac{\varepsilon}{2} + \frac{A_2}{d_2}\frac{1}{f_2^2} - 2\frac{A_3}{d_2}\frac{f_1^2}{f_2^4},
\end{aligned}$$

where $d_i = \dim \mathfrak{p}_i$ and the constants $A_i \geq 0$ are determined by the Riemannian submersion $(G/K, b_{|\mathfrak{p}}) \to (G/H, b_{|\mathfrak{p}_2})$, see section 3.1. In the following,

$$A_2,\ A_3 > 0 \ \text{ and } \ A_1 = d_1(d_1 - 1) \geq 0$$

will be assumed.

In order to smoothly extend the metric and the soliton potential over the singular orbit, the smoothness conditions

$$\begin{aligned} f_1(0) = 0,\ \dot{f}_1(0) = 1 \ \text{ and } \ f_2(0) = \bar{f} > 0,\ \dot{f}_2(0) = 0, \\ u(0) = 0,\ \dot{u}(0) = 0,\ (d_1 + 1)\ddot{u}(0) = C \leq 0 \end{aligned} \tag{5.6}$$

from (2.8) and (3.2) are imposed. Higher regularity of the solution then follows from elliptic regularity, cf. [Buz11]. In particular, any solution is uniquely determined by fixing $\bar{f} > 0$ and $C \leq 0$ and if $C = 0$ the trajectory is automatically Einstein. Recall that according to results of Chen [Che09a] any *complete* steady and expanding Ricci soliton with $u(0) = 0$ necessarily satisfies $C \leq 0$.

Fix $\varepsilon \geq 0$. Then $(d_1 + 1)\frac{\ddot{f}_2(0)}{f_2(0)} = \frac{\varepsilon}{2} + \frac{A_2}{d_2}\frac{1}{\bar{f}^2} > 0$ implies that $\dot{f}_2(t) > 0$ holds for sufficiently small $t > 0$. A key step in the proof of completeness of the corresponding Ricci soliton metrics is to show that $\dot{f}_2(t) > 0$ holds in fact for all $t > 0$. Since $\dot{f}_1 > 0$ is immediate, this also implies that the shape operator $L = \operatorname{diag}\left(\frac{\dot{f}_1}{f_1}\operatorname{id}_{d_1}, \frac{\dot{f}_2}{f_2}\operatorname{id}_{d_2}\right)$ is positive definite. In order to prove that $\dot{f}_2 > 0$ is preserved if $\varepsilon \geq 0$, it suffices to show that

$$\omega = \frac{f_1}{f_2} < \sqrt{\frac{A_2}{2A_3}}$$

is preserved if $\varepsilon \geq 0$. To this end, observe that ω satisfies

$$\begin{aligned} \frac{d}{dt}\omega &= \omega \left\{ \frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2} \right\}, \\ \frac{d^2}{dt^2}\omega &= \dot{\omega} \left\{ \frac{\dot{f}_1}{f_1} - \frac{\dot{f}_2}{f_2} - (-\dot{u} + \operatorname{tr}(L)) \right\} + \frac{\omega}{f_1^2} f(\omega), \end{aligned} \tag{5.7}$$

with $f(\omega)$ as in (3.14), and therefore $\omega(0) = 0$ and $\dot{\omega}(0) = \bar{f}^{-1}$.

According to proposition 3.4.6, $f(\omega)$ has two positive roots $\omega_1 < \omega_2$ if and only if the discriminant D of (3.17) is positive. In this case, there holds $\omega_1^2 < \frac{A_2}{4A_3}$ and $\omega_2^2 < \frac{A_2}{2A_3}$, and in particular it follows that $\dot{f}_2 > 0$ as long as $\omega < \omega_2$.

Furthermore, the geometric condition $A_1 = d_1(d_1 - 1)$ implies that $\omega_1 = 0$ if and only if $d_1 = 1$, which turns the S^1-bundle construction into a special case.

Remark 5.2.2 According to section 3.4, along any solution with $L_t > 0$, the asymptotic behaviour of the metric necessarily requires $D \geq 0$ for completeness, as this is the condition for the associated cone solution to be real valued.

Notice that the examples induced by the Hopf fibrations in example 3.4.5 indeed satisfy $D \geq 0$.

Lemma 5.2.3 *Suppose that $d_1 > 1$, $A_1 = d_1(d_1 - 1)$ and $D \geq 0$. As long as $\dot{f}_2 > 0$ and $\omega < \omega_2$, the following estimate holds*

$$\ddot{\omega} \leq \frac{d_1 - 1}{\omega} \left(\frac{1}{\bar{f}^2} - \dot{\omega}^2 \right) + \dot{\omega}(\dot{u} - n\frac{\dot{f}_2}{f_2}).$$

In particular, if furthermore $C = 0$ or $C < 0$ and $\varepsilon \geq 0$, then $\dot{\omega} \geq \frac{1}{\bar{f}}$ implies that $\ddot{\omega} \leq 0$, and thus the bound $\dot{\omega}(t) \leq \frac{1}{\bar{f}}$ holds for $t \geq 0$ and as long as $\omega(t) < \omega_2$.

Proof: The function $f(\omega) = \frac{2d_1+d_2}{d_1 d_2} A_3(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)$ has a local maximum at the origin and in particular one has $f(\omega) \leq f(0) = \frac{A_1}{d_1} = d_1 - 1$ if $0 \leq \omega \leq \omega_2$. Since f_2 is increasing there holds $f_2(t) \geq f_2(0) = \bar{f}$ for $t \geq 0$. Then first claim follows from

$$\ddot{\omega} = \frac{1}{\omega} \left(\frac{f(\omega)}{f_2^2} - (d_1 - 1)\dot{\omega}^2 \right) + \dot{\omega}(\dot{u} - n\frac{\dot{f}_2}{f_2}).$$

If $\varepsilon \geq 0$ and $C \leq 0$, the soliton potential either vanishes identically or is strictly decreasing by proposition 2.3.1. In particular, there holds $\dot{u} \leq 0$ and this completes the proof. $\square$

Theorem 5.2.1 now follows from proposition 5.2.4 below, which moreover produces complete steady and expanding Ricci solitons if $D \geq 0$ and $d_1 > 1$ and hence covers all known geometric examples. The case of a collapsing circle $d_1 = 1$ will be considered in proposition 5.2.7.

Proposition 5.2.4 *Let $d_1 > 1$, $A_1 = d_1(d_1 - 1)$ such that $D > 0$ and $\varepsilon \geq 0$. For any $\bar{f}_0 > 0$ there exists a constant $C_0 \leq 0$ such that for any $C < C_0$ and any $\bar{f} > \bar{f}_0$ the steady and expanding two summands Ricci soliton metrics with initial condition* (5.6) *are complete.*

In particular, these give rise to a 1-parameter family of complete steady and a 2-parameter family of complete expanding Ricci solitons.

Proof: Completeness follows from proposition 5.1.3 once the invariance of the set

$$\left\{ \dot{f}_2 > 0 \ \text{ and } \ 0 < \omega < \omega_2 \right\} \tag{5.8}$$

under the Ricci soliton ODE has been established for all $t > 0$.

Observe that it directly follows from the two summands Ricci soliton ODE that $\omega < \omega_2$ implies $\dot{f}_2 > 0$. It therefore remains to show that there are initial conditions as claimed so that ω cannot exceed ω_2 if $\dot{f}_2 > 0$.

Recall that $A_1 > 0$ implies that $\omega_1 > 0$. Suppose there exist t_*, $\delta > 0$ such that $\omega < \omega_2$ on $[0, t_*]$, $\omega \leq \omega_1$ on $(t_* - \delta, t_*)$ and $\omega > \omega_1$ on $(t_*, t_* + \delta)$. Set

$$t^* = \sup \left\{ \ t \geq t_* \mid \omega(s) \in [\omega_1, \omega_2] \ \text{ for all } \ s \in (t_*, t) \ \right\}.$$

Then $\omega(t_*) = \omega_1$ and the monotonicity of f_2 implies that $f_1(t_*) \geq \omega_1 \bar{f}$. It follows a posteriori that one can assume $\dot{\omega}(t_*) > 0$.

As long as $\omega_1 \leq \omega \leq \omega_2$, the ODE (5.7) implies that

$$\frac{d}{dt}\dot{\omega} \leq \dot{\omega} \left\{ \dot{u} - (d_1 - 1)\frac{\dot{f}_1}{f_1} - (d_2 + 1)\frac{\dot{f}_2}{f_2} \right\}$$

and with lemma 5.2.3 it follows that

$$\dot{\omega}(t) \leq \dot{\omega}(t_*)\frac{f_1^{d_1-1}(t_*) f_2^{d_2+1}(t_*)}{f_1^{d_1-1}(t) f_2^{d_2+1}(t)} \exp(u(t) - u(t_*)) \leq \dot{\omega}(t_*) \leq \frac{1}{\bar{f}}$$

for all $t \in (t_*, t^*)$. Notice in particular that $\omega < \omega_2$ holds on the interval $(t_*, t_* + T)$, where $T = \bar{f}_0(\omega_2 - \omega_1)$.

Furthermore, due to the monotonicity of f_1, f_2 one has

$$-\operatorname{tr}_{g_t}(B_{|\mathfrak{p}}) = \sum_{i=1}^{2} \frac{A_i}{f_i^2(t)} \leq \sum_{i=1}^{2} \frac{A_i}{f_i^2(t_*)} \leq \frac{A_1}{\omega_1^2 \bar{f}^2} + \frac{A_2}{\bar{f}^2} < \infty \tag{5.9}$$

for $t \geq t_*$ and as long as $\omega(t) < \omega_2$.

Now, according to theorem 5.1.1, the following holds: For any $c > 0$, any $\tau \in (0, T)$ and any $\bar{f}_0 > 0$ there exists a constant $C_0 \leq 0$ such that along any two summands Ricci soliton trajectory whose initial conditions (5.6) satisfy $C < C_0$ and $\bar{f} > \bar{f}_0$ the soliton potential obeys the estimate

$$u(t) \leq u(\tau + t_*) - c(t - (\tau + t_*))$$

for $t \geq \tau + t_*$ and as long as $\omega(t) < \omega_2$. (In fact, due to the concavity properties of u in proposition 2.3.1, the estimate actually holds for all $t \geq \tau + t_*$.)

The monotonicity of f_1, f_2 and u and the a priori bound on $\dot{\omega}$ in lemma 5.2.3 then imply that for $t \in [\tau + t_*, t^*]$ there holds

$$\begin{aligned}
\omega(t) &\leq \omega(\tau + t_*) + \dot{\omega}(t_*) \int_{\tau + t_*}^{t} \exp(u(s) - u(t_*))ds \\
&\leq \omega(\tau + t_*) + \frac{\dot{\omega}(t_*)}{c} \\
&\leq \omega_1 + \frac{1}{\bar{f}_0}\left(\tau + \frac{1}{c}\right) < \omega_2,
\end{aligned}$$

if $c > 0$ is chosen large enough and $\tau > 0$ small enough. In particular, it follows that $t^* = \infty$ unless $\omega(t^*) = \omega_1$.

However, observe that the choices of c, $\tau > 0$ and hence $C_0 \leq 0$ depend on $\bar{f}_0 > 0$ but not on t_*. This implies that the corresponding trajectories with $C < C_0$ indeed satisfy $\omega(t) < \omega_2$ for all $t \geq 0$, which completes the proof. $\square$

Remark 5.2.5 (a) According to proposition 3.2.5, one can actually choose $C_0 = 0$ in proposition 5.2.4 if $d_1 > 1$ and $(d_1 + 1)A_2^2 > 4d_1 d_2 (2d_1 + d_2) A_3$.

(b) The asymptotics of the Ricci soliton metrics can be deduced as in sections 3.4.4 and 3.4.5. Indeed, the arguments carry over since $L_t > 0$ and $\omega \leq \omega_2 < \frac{A_2}{2A_3}$ hold on the constructed metrics.

In the steady case, any complete solution must in fact have positive definite shape operator.

Proposition 5.2.6 *Let $d_1 \geq 1$, $A_1 \geq 0$ and $\varepsilon = 0$ and let $(f_1(t), f_2(t), u(t))_{t \in [0,T)}$ be a maximal solution to the two summands Ricci solution ODE with initial condition* (5.6)*. Suppose there exists $t_0 > 0$ such that $\dot{f}_2(t_0) < 0$.*

Then $T < \infty$ and the associated metric is incomplete.

Proof: Recall that due to lemma 3.2.3 there holds $\dot{f}_2(t) < 0$ for all $t \geq t_0$, which implies $\dot{\omega}(t) > 0$ for all $t \geq t_0$. Hence there exists a constant $c_1 > 0$ such that

$$\frac{d}{dt}\frac{\dot{f}_1}{f_1} \geq -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}_1}{f_1} + c_1.$$

Hence, by comparison, there exists a constant $c_2 > 0$ such that

$$\frac{\dot{f}_1}{f_1} \geq c_1 \frac{e^u}{f_1^{d_1} f_2^{d_2}} + c_2,$$

which implies

$$\frac{d}{dt}\operatorname{tr}(L) = \ddot{u} - \operatorname{tr}(L^2) \leq -c_2^2 < 0$$

due to proposition 2.3.1.

However if $T = \infty$ this contradicts the estimate $\operatorname{tr}(L) \to 0$ as $t \to \infty$ which holds on any complete steady cohomogeneity one Ricci soliton with a singular orbit according to proposition 2.4.1. □

Notice that condition (5.9) cannot hold at $t = 0$ if $A_1 > 1$ because $f_1(0) = 0$ implies that $\operatorname{tr}_{g_t}(B_{|\mathfrak{p}})$ diverges as $t \to 0$.

However, if $A_1 = 0$ one can actually choose $t_* = 0$ and (5.9) holds initially. As $A_1 = d_1(d_1 - 1)$ holds in geometric applications, the proof can hence be simplified if $d_1 = 1$. In this case, the two summands system can be used to construct Ricci soliton metrics on complex line bundles over Fano Kähler Einstein manifolds, see section 3.3.

In the case of steady Ricci solitons, the following argument is due to Appleton [App18], who moreover has found a non-collapsed Ricci soliton in this family. Stolarski [Sto17] independently describes the steady Ricci solitons. Proposition 5.2.7 extends their results to the expanding case.

Proposition 5.2.7 *Let $d_1 = 1$, $A_1 = 0$ and $\varepsilon \geq 0$. For any $\bar{f}_0 > 0$ there exists a constant $C_0 \leq 0$ such that for all $C < C_0$ and $\bar{f} > \bar{f}_0$ the steady and expanding Ricci soliton trajectories with initial condition* (5.6) *give rise to families of complete Ricci soliton metrics on complex line bundles over Fano Kähler Einstein manifolds.*

In particular, these families include examples of non-Kähler type.

Proof: Recall that $A_1 = 0$ implies that $\omega_1 = 0$ and $\omega_2 = \frac{A_2}{A_3}\frac{d_1}{2d_1+d_2}$. As in the proof of proposition 5.2.4, it suffices to show that (5.8) is preserved.

Clearly, $\omega < \omega_2$ still implies $\dot{f}_2 > 0$ and as $\omega(0) = 0$ the condition $\omega < \omega_2$ holds initially. Moreover, since $d_1 = 1$ and $u(0) = 0$, one concludes as in the proof of proposition 5.2.4 that

$$\dot{\omega}(t) \leq \dot{\omega}(0)\frac{f_2^{d_2+1}(0)}{f_2^{d_2+1}(t)}\exp(u(t)) \leq \dot{\omega}(0) = \frac{1}{\bar{f}}$$

as long as $\omega(t) < \omega_2$. In particular, this condition is satisfied on $[0, \omega_2 \bar{f})$.

Furthermore, $-\operatorname{tr}_{g_t}(B_{|\mathfrak{p}}) = \sum_{i=1}^{2}\frac{A_i}{f_i^2(t)} = \frac{A_2}{f_2^2(t)} \leq \frac{A_2}{\bar{f}^2} < \infty$ also holds as long as $\omega(t) < \omega_2$. Thus, theorem 5.1.1 implies the following:

Let $\bar{f}_0 > 0$, $c > 0$ and $\tau \in (0, \omega_2 \bar{f}_0)$. Then there exists a constant $C_0 \leq 0$ such that along any two summands Ricci soliton trajectory with $d_1 = 1$, $A_1 = 0$ and $C < C_0$, $\bar{f} > \bar{f}_0$ in (5.6), the soliton potential satisfies $u(t) \leq u(\tau) - c(t - \tau)$ for $t \geq \tau$ and as long as $\omega(t) < \omega_2$.

Thus, one concludes that

$$\omega(t) \leq \omega(\tau) + \dot{\omega}(0) \int_\tau^t \exp(u(s))ds \leq \frac{1}{\bar{f}} \left(\tau + \frac{1}{c} \right) < \omega_2$$

for all $t \geq \tau$ if $c > 0$ is chosen large enough and $\tau > 0$ sufficiently small. In particular, in this case, $\omega(t) < \omega_2$ holds for all $t \geq 0$. $\square$

Remark 5.2.8 It follows from proposition 3.3.1 that proposition 5.2.7 holds with $C_0 = 0$ provided $A_2^2 > 2d_2(d_2 + 2)A_3$.

5.2.2 Generalisations of the circle bundle case

In this section two generalisations of proposition 5.2.7 will be discussed.

5.2.2.1 The Dancer-Wang set-up

Extending earlier work of Cao [Cao96] and Feldman-Ilmanen-Knopf [FIK03], Dancer-Wang [DW11] have found explicit families of Kähler Ricci solitons on certain complex line bundles over *products* of Fano Kähler Einstein manifolds. Their construction assumes that the Euler class of the line bundle is a rational linear combination of the first Chern classes of the base manifolds. However, the Kähler condition imposes strong restrictions on the coefficients. This section establishes that in fact on any of these line bundles there exist continuous families of steady and expanding Ricci solitons:

Theorem 5.2.9 *Let L be the total space of a complex line bundle over a product of Fano Kähler Einstein manifolds with $m \geq 1$ factors whose Euler class is a rational linear combination of the first Chern classes of the base manifolds. Then there exist an m-parameter family of complete non-trivial steady and an $(m+1)$-parameter family of complete non-trivial expanding gradient Ricci solitons on L.*

The precise geometric set-up for the construction of the Ricci soliton metrics is the following:

Let $\{(V_i, J_i, h_i)\}_{i=1,\ldots,m}$ be Fano Kähler Einstein manifolds of real dimension d_i. Then the first Chern classes are $c_1(V_i, J_i) = p_i \rho_i$ for positive integers p_i and indivisible classes $\rho_i \in H^2(V_i, \mathbb{Z})$. Suppose that the Kähler metrics h_i are normalised such that $\mathrm{Ric}(h_i) = p_i h_i$ and let η_i be the associated Kähler form. Choose $q_1, \dots, q_m \in \mathbb{Z} \setminus \{0\}$ and consider the principal circle bundle $\pi \colon P \to B$ with Euler class $\sum_{i=1}^m q_i \pi_i^* \rho_i$. Notice that on P there exists a principal connection θ with curvature $\Omega = \sum_{i=1}^m q_i \pi_i^* \eta_i$.

Then the Ricci soliton equations on $I \times P$ with respect to the Riemannian metric $g = dt^2 + g_t$, where $g_t = f^2(t)\theta \otimes \theta + \sum_{i=1}^m g_i^2(t) \pi_i^* h_i$, are given by

$$\ddot{u} = \frac{\ddot{f}}{f} + \sum_{i=1}^m d_i \frac{\ddot{g}_i}{g_i} - \frac{\varepsilon}{2}, \tag{5.10}$$

$$\frac{d}{dt}\frac{\dot{f}}{f} = -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}}{f} + \frac{\varepsilon}{2} + \sum_{i=1}^m \frac{d_i q_i^2}{4}\frac{f^2}{g_i^4}, \tag{5.11}$$

$$\frac{d}{dt}\frac{\dot{g}_i}{g_i} = -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{g}_i}{g_i} + \frac{\varepsilon}{2} + \frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4}, \tag{5.12}$$

for $i = 1, \dots, m$, and where $L = \mathrm{diag}\left(\frac{\dot{f}}{f}, \frac{\dot{g}_1}{g_1}\mathbb{I}_{d_1}, \dots, \frac{\dot{g}_m}{g_m}\mathbb{I}_{d_m}\right)$ is the shape operator of the principal hypersurface. A solution to these equations on $[0, \infty) \times P$ induces a complete Ricci soliton metric on the associated complex line bundle if the smoothness conditions

$$f(0) = 0,\ \dot{f}(0) = 1 \text{ and } g_i(0) > 0,\ \dot{g}_i(0) = 0, \qquad u(0) = \dot{u}(0) = 0,\ 2\ddot{u}(0) = C \tag{5.13}$$

are imposed, where C is the constant in the conservation law (2.5). Recall that if $\varepsilon \geq 0$ then $C \leq 0$ is a necessary condition to obtain a complete solution.

The metric on the associated complex line bundle is Kähler if $\frac{d}{dt} g_i^2 = -q_i f$ for $i = 1, \dots, m$, which necessarily requires $q_i < 0$, cf. [WW98].

Fix $\varepsilon \geq 0$ and observe that $2\frac{\ddot{g}_i(0)}{g_i(0)} = \frac{\varepsilon}{2} + \frac{p_i}{g_i^2(0)} > 0$ implies that $\dot{g}_i(t) > 0$ for sufficiently small $t > 0$.

Suppose that $C_0 \geq 1$ is a constant to be specified later and $\omega_i = \frac{f}{g_i}$ and $Q_{ij} = \frac{g_i}{g_j}$ are bounded by

$$\omega_i^2 \leq \frac{1}{mC_0^2}\frac{4}{(d_i + 2)q_i^2} \min_{k \in \{1, \dots, m\}} \{p_k\} \text{ and } Q_{ij} \leq C_0 \tag{5.14}$$

for all $i, j = 1, \dots, m$. Notice that the condition on ω_i is automatically satisfied initially as $\omega_i(0) = 0$ and that it implies

$$\frac{p_i}{g_i^2} - \frac{q_i^2}{2}\frac{f^2}{g_i^4} \geq \frac{d_i p_i}{d_i + 2}\frac{1}{g_i^2}. \tag{5.15}$$

In particular, as long as (5.15) holds one has $\dot{g}_i > 0$ for $i = 1, \dots, m$. The condition on Q_{ij} can be satisfied initially by simply choosing $C_0 \geq 1$ sufficiently large.

Remark 5.2.10 Notice that the a priori bound (5.14) also implies that all Q_{ij} are bounded from below.

If $m = 1$ one can choose $C_0 = 1$ and the condition $\omega^2 < \frac{4p}{(d+2)q^2}$ is the same as in proposition 5.2.7.

It will be shown that for a suitable choice of initial conditions in (5.13) the bounds in (5.14) are preserved along the corresponding steady and expanding Ricci soliton trajectories.

Observe that $\omega_i = \frac{f}{g_i}$ satisfies the equations

$$\dot{\omega}_i = \omega_i \left\{ \frac{\dot{f}}{f} - \frac{\dot{g}_i}{g_i} \right\} \quad \text{and}$$
$$\ddot{\omega}_i = \dot{\omega}_i \left\{ \dot{u} - (d_i + 1)\frac{\dot{g}_i}{g_i} - \sum_{j \neq i} d_j \frac{\dot{g}_j}{g_j} \right\} + \frac{\omega_i}{g_i^2} \left\{ \frac{q_i^2}{2}\omega_i^2 - p_i + \sum_{j=1}^{m} \frac{d_j q_j^2}{4} \frac{f^2}{g_j^2} \frac{g_i^2}{g_j^2} \right\}.$$

Remark 5.2.11 Any critical point of ω_i along a solution which satisfies (5.14) is a local maximum. Indeed, one has

$$\begin{aligned} \frac{d_i + 2}{4} q_i^2 \omega_i^2 - p_i + \sum_{j \neq i} \frac{d_j q_j^2}{4} \frac{f^2}{g_j^2} \frac{g_i^2}{g_j^2} &\leq \frac{d_i + 2}{4} q_i^2 \omega_i^2 - p_i + \frac{m-1}{m} \min \{p_k\} \\ &\leq \left(\frac{1}{m C_0^2} + \frac{m-1}{m} \right) \min \{p_k\} - p_i \leq 0 \end{aligned}$$

and equality implies $m = 1$, $C_0 = 1$ and equality in (5.14), in which case ω_1 is maximal by assumption. Therefore $\dot{\omega}_i$ can change its sign at most once. In particular, if the solution is defined for all times $t \geq 0$, then ω_i converges as $t \to \infty$.

Remark 5.2.12 An integral formula for $\dot{\omega}$ can be obtained from

$$\frac{d}{dt} \left\{ \dot{\omega}_i e^{-u} g_i^{d_i+1} \prod_{j \neq i} g_j^{d_j} \right\} = \omega_i e^{-u} g_i^{d_i - 1} \prod_{j \neq i} g_j^{d_j} \left\{ \frac{q_i^2}{2}\omega_i^2 - p_i + \sum_{j=1}^{m} \frac{d_j q_j^2}{4} \omega_j^2 Q_{ij}^2 \right\}.$$

Lemma 5.2.13 *Let $\varepsilon \geq 0$ and fix ε_0, $\bar{g} > 0$. Then there exists a constant $C_1 \leq 0$ such that along any Ricci soliton trajectory with $C < C_1$ and $g_i(0) > \bar{g}$ for all $i = 1, \ldots, m$ in* (5.13) *the estimate $\omega_i < \varepsilon_0$ is preserved for all $i = 1, \ldots, m$ as long as* (5.14) *holds.*

Proof: Observe that as long as assumption (5.14) is satisfied the second term in the expression for $\ddot{\omega}_i$ is nonpositive. Since $\dot{\omega}_i(0) = \frac{1}{g_i(0)}$, this implies as before that

$$\dot{\omega}_i(t) \leq \frac{g_i^{d_i}(0) \prod_{j \neq i} g_j^{d_j}(0)}{g_i^{d_i+1}(t) \prod_{j \neq i} g_j^{d_j}(t)} \exp(u(t)) \leq \frac{1}{g_i(0)} < \frac{1}{\bar{g}}$$

for $t \geq 0$ and as long as (5.14) holds. Observe that the bound on ω_i in (5.14) clearly holds on an interval $[0, T]$, where $T > 0$ only depends on $\bar{g} > 0$ and the bound itself.

In order to apply theorem 5.1.1, note that the Ricci soliton ODE and (5.15) imply positive definiteness of the shape operator, and that

$$-\operatorname{tr}_{g_t}(B_{|\mathfrak{p}}) = \sum_{i=1}^{m} \frac{d_i p_i}{g_i^2(t)} \leq \sum_{i=1}^{m} \frac{d_i p_i}{g_i^2(0)} \leq \overline{C}(d_i, p_i, \bar{g}) < \infty$$

is uniformly bounded in terms of $\bar{g} > 0$ and geometric or topological data. Hence, for all $c > 0$ and all $\tau \in (0, T)$ there exists a constant $C_1 \leq 0$ such that for all $C < C_1$ one has $-\dot{u}(t) \geq c$ for all $t > \tau$ and as long as (5.14) holds. This implies that

$$\omega_i(t) \leq \omega_i(\tau) + \dot{\omega}_i(0) \int_{\tau}^{t} \exp(u(s)) ds \leq \frac{1}{g_i(0)} \left(\tau + \frac{1}{c} \right)$$

and hence for sufficiently small $\tau > 0$ and sufficiently large $c > 0$, the condition $\omega_i < \varepsilon_0$ is preserved along the Ricci soliton trajectory. □

Now the existence of an a priori bound for the quotient $Q_{ij} = \frac{g_i}{g_j}$ will be established. Observe that

$$\dot{Q}_{ij} = Q_{ij} \left\{ \frac{\dot{g}_i}{g_i} - \frac{\dot{g}_j}{g_j} \right\} \quad \text{and}$$

$$\ddot{Q}_{ij} = \dot{Q}_{ij} \left\{ \dot{u} - \operatorname{tr}(L) + \frac{\dot{g}_i}{g_i} - \frac{\dot{g}_j}{g_j} \right\} + Q_{ij} \left\{ \frac{p_i}{g_i^2} - \frac{q_i^2}{2} \frac{f^2}{g_i^4} - \left(\frac{p_j}{g_j^2} - \frac{q_j^2}{2} \frac{f^2}{g_j^4} \right) \right\}.$$

It is easy to check that the initial conditions

$$\dot{Q}_{ij}(0) = 0 \quad \text{and} \quad \ddot{Q}_{ij}(0) = \frac{3}{2} Q_{ij}(0) \left\{ \frac{p_i}{g_i^2(0)} - \frac{p_j}{g_j^2(0)} \right\}$$

hold.

As a first step, an a priori estimate on $\dot{Q}_{ij}$ will be derived:

Lemma 5.2.14 *As long as* (5.15) *holds, one has the estimate*

$$\ddot{Q}_{ij} \leq \dot{Q}_{ij} \left\{ \dot{u} - \operatorname{tr}(L) + \frac{\dot{g}_i}{g_i} - \frac{\dot{g}_j}{g_j} \right\} + Q_{ij} \frac{p_i}{g_i^2}$$
$$= \dot{Q}_{ij} \left\{ \dot{u} - \sum_{k \neq i,j} d_k \frac{\dot{g}_k}{g_k} - (d_i + d_j) \frac{\dot{g}_j}{g_j} \right\} + \frac{1}{Q_{ij}} \left\{ \frac{p_i}{g_j^2} - (d_i - 1) \dot{Q}_{ij}^2 \right\}.$$

In particular, if furthermore $C = 0$ or $C < 0$ and $\varepsilon \geq 0$, then $\dot{Q}_{ij}(t_0) > \sqrt{\frac{p_i}{d_i - 1} \frac{1}{g_j^2(0)}}$ implies $\ddot{Q}_{ij}(t_0) < 0$ and hence the a priori bound

$$\dot{Q}_{ij}(t) \leq \max \left\{ \sqrt{\frac{p_i}{d_i - 1} \frac{1}{g_j^2(0)}} \right\}.$$

Proof: The first part is a straightforward calculation. For the second part recall that (5.15) implies that $\dot{g}_i > 0$ for $i = 1, \ldots, m$ and that for steady and expanding Ricci solitons the soliton potential is decreasing according to proposition 2.3.1. □

Lemma 5.2.15 *Suppose that $\varepsilon \geq 0$ and set $C_0 = \max\left\{Q_{ij}(0), \sqrt{\frac{d_j+2}{d_j}\frac{p_i}{p_j}}\right\} + 1$. Then for every $\bar{g} > 0$ there exists a constant $C_1 \leq 0$ such that along any Ricci soliton trajectory with $C < C_1$ and $g_i(0) > \bar{g}$ for $i = 1, \ldots, m$ in (5.13) the a priori bound (5.14) is preserved.*

Proof: Observe that, due to (5.14), the equation for $\ddot{Q}_{ij}$ and (5.15) imply that

$$\ddot{Q}_{ij} \leq \dot{Q}_{ij}\left\{\dot{u} - \mathrm{tr}(L) + \frac{\dot{g}_i}{g_i} - \frac{\dot{g}_j}{g_j}\right\} + \frac{Q_{ij}}{g_i^2}\left\{p_i - \frac{d_j p_j}{d_j+2}Q_{ij}^2\right\}.$$

Now suppose that $Q_{ij}(t) \geq \sqrt{\frac{d_j+2}{d_j}\frac{p_i}{p_j}}$ for $t > t_0$ and $t_0 \geq 0$ is minimal with that property. It follows a posteriori that $\dot{Q}_{ij}(t_0) > 0$ can be assumed and then

$$\dot{Q}_{ij}(t) \leq \dot{Q}_{ij}(t_0)\frac{\left(g_i^{d_i-1}g_j^{d_j+1}\prod_{k\neq i,j}g_k^{d_k}\right)(t_0)}{\left(g_i^{d_i-1}g_j^{d_j+1}\prod_{k\neq i,j}g_k^{d_k}\right)(t)}\exp(u(t)-u(t_0)) \leq \sqrt{\frac{p_i}{d_i-1}\frac{1}{g_j^2(0)}}$$

holds for $t \geq t_0$. Notice in particular that the bound on Q_{ij} in (5.14) is still satisfied on $[t_0, t_0+T)$, where $T = \bar{g}\cdot\max\{\frac{d_i-1}{p_i}\}^{-1/2} > 0$ can be chosen due to the definition of C_0.

Set $\varepsilon_0 = C_0$ in lemma 5.2.13. As the proof shows, for any $c > 0$ and any $\tau_0 > 0$ there exists a constant $C' \leq 0$ such that along any Ricci soliton trajectory with $C < C'$ and $g_i(0) > \bar{g}$ for all $i = 1, \ldots, m$ in (5.13) one has $-\dot{u}(t) \geq c$ for all $t > \tau_0$ as long as (5.14) holds. This implies that $\omega_i \leq C_0$ for a sufficiently small choice of $\tau_0 > 0$ and a sufficiently large choice of $c > 0$.

Recall that due to (5.14) the shape operator is positive definite and that the bound $-\mathrm{tr}_{g_t}(B_{|\mathfrak{p}}) \leq \sum_{i=1}^m \frac{d_i p_i}{g_i^2(0)}$ was established in the proof of lemma 5.2.13 for $t \geq 0$.

Fix $\tau_0 < T$ and apply theorem 5.1.1 with $c_* = \sum_{i=1}^m \frac{d_i p_i}{\bar{g}^2}$, $c > 0$, $t_* = t_0$ and $\tau = \tau_0 + t_0$. This yields a constant $C'' \leq 0$ such that along any Ricci soliton trajectory with $C < C''$ and $g_i(0) > \bar{g}$ for all $i = 1, \ldots, m$ in (5.13) one has $-\dot{u}(t) \geq c$ for all $t > \tau$ and as long as (5.14) holds. Then

$$\begin{aligned}
Q_{ij}(t) &\leq Q_{ij}(\tau) + \dot{Q}_{ij}(t_0)\int_\tau^t \exp(u(s)-u(t_0))ds \\
&\leq Q_{ij}(\tau) + \frac{\dot{Q}_{ij}(t_0)}{c} \\
&\leq Q_{ij}(t_0) + \sqrt{\frac{p_i}{d_i-1}\frac{1}{g_j^2(0)}}\left(\tau - t_0 + \frac{1}{c}\right) < C_0
\end{aligned}$$

for a sufficiently small choice of $\tau - t_0 = \tau_0 > 0$ and a sufficiently large choice of $c > 0$.

Recall that the constants $C', C'' \leq 0$ in theorem 5.1.1, for which the linear growth rate $-\dot{u} \geq c$ is guaranteed if $\ddot{u}(0) < C', C''$, respectively, only depend on $c_*, c > 0$ and the difference $\tau - t_* = \tau_0 > 0$, but neither on t_* nor τ. Hence there is a uniform choice of $\tau_0 \in (0, T)$ and $c > 0$ which implies that (5.14) is preserved along the Ricci soliton trajectory provided that $\ddot{u}(0) < C \leq C_1 := \min\{C', C''\} \leq 0$ and $g_i(0) > \bar{g}$ for all $i = 1, \ldots, m$. □

Completeness of the Ricci soliton metrics now follows from lemma 5.2.15 and proposition 5.2.16 below.

Proposition 5.2.16 *Let $\varepsilon \geq 0$ and consider a maximal Einstein or Ricci soliton trajectory in the Dancer-Wang set-up with initial conditions* (5.13) *and $\ddot{u}(0) \leq 0$. Suppose that $\frac{f^2}{g_i^2} < \frac{2p_i}{q_i^2}$ holds along the entire trajectory if $q_i \neq 0$.*

Then this trajectory is defined for all $t \geq 0$ and corresponds to a complete Einstein or Ricci soliton metric.

Proof: It is immediate that the shape operator remains positive definite along the trajectories and hence proposition 5.1.3 yields the claim. □

In the case of steady Ricci solitons, positive definiteness of the shape operator is actually a necessary condition for completeness.

Proposition 5.2.17 *Let $\varepsilon = 0$ and suppose that $(f(t), g_i(t), u(t))_{t \in [0,T)}$ is a maximal solution to the Ricci soliton equations* (5.10) - (5.12) *which satisfies the initial conditions* (5.13). *Suppose that $T = \infty$.*

Then there holds $\dot{f}(t)$, $\dot{g}_i(t) > 0$ for all $t > 0$.

Proof: It is clear that $\dot{f} > 0$ is preserved. Suppose that there exists $i_0 \in \{1, \ldots, m\}$ and $t_0 > 0$ such that $g_{i_0}(t_0) < 0$. Then an argument analogous to the proof of proposition 3.2.3 shows that $g_{i_0}(t) < 0$ for all $t \geq t_0$ and the proof of proposition 5.2.6 applied to $\dot{f}/f$ yields a contradiction: A comparison argument shows that $\dot{f}/f$ is bounded away from zero, which implies that the mean curvature decreases with definite speed. On the other hand, this is impossible as the mean curvature must tend to zero at infinity according to proposition 2.4.1. □

Remark 5.2.18 An alternative proof of proposition 5.2.16 in the spirit of remark 2.3.2 is also available: Set $\mathcal{L} = \frac{1}{-\dot{u}+\mathrm{tr}(L)}$ and consider the rescaling $\frac{d}{ds} = \mathcal{L} \cdot \frac{d}{dt}$. Then the Einstein and Ricci soliton loci of the variables

$$X_0 = \mathcal{L} \cdot \frac{\dot{f}}{f}, \ X_i = \mathcal{L} \cdot \frac{\dot{g}_i}{g_i} \ \text{ and } \ Y_0 = \frac{\mathcal{L}}{f}, \ Y_i = \frac{\mathcal{L}}{g_i} \ \text{ and } \ \omega_i = \frac{Y_i}{Y_0},$$

for $i = 1, \dots, m$, describe bounded regions in phase space as $\varepsilon \geq 0$, $\omega_i^2 < \frac{2p_i}{q_i^2}$ and the conservation law (2.5) in the form (2.6) yields

$$\sum_{i=0}^{m} d_i X_i^2 + \sum_{i=1}^{m} d_i p_i Y_i^2 - \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \omega_i^2 Y_i^2 + (n-1)\frac{\varepsilon}{2}\mathcal{L}^2 \leq 1.$$

Furthermore, the associated Ricci soliton ODE

$$\begin{aligned}
\frac{d}{ds}X_0 &= X_0 \left(\sum_{j=0}^{m} d_j X_j^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - 1 \right) + \frac{\varepsilon}{2}\mathcal{L}^2 + \sum_{j=1}^{m} \frac{d_j q_j^2}{4} \omega_j^2 Y_j^2, \\
\frac{d}{ds}X_i &= X_i \left(\sum_{j=0}^{m} d_j X_j^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - 1 \right) + \frac{\varepsilon}{2}\mathcal{L}^2 + p_i Y_i^2 - \frac{q_i^2}{2}\omega_i^2 Y_i^2, \quad && \text{for } \ i = 1, \dots, m, \\
\frac{d}{ds}Y_i &= Y_i \left(\sum_{j=0}^{m} d_j X_j^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - X_i \right), && \text{for } \ i = 0, \dots, m, \\
\frac{d}{ds}\mathcal{L} &= \mathcal{L} \left(\sum_{j=0}^{m} d_j X_j^2 - \frac{\varepsilon}{2}\mathcal{L}^2 \right), \\
\frac{d}{ds}\omega_i &= \omega_i \left(X_0 - X_i \right), && \text{for } \ i = 1, \dots, m,
\end{aligned}$$

has a polynomial right hand side, which implies long time existence. Completeness of the metric then follows as in proposition 3.2.9 and corollary 3.2.12 since $\omega_i^2 < \frac{2p_i}{q_i^2}$ implies $X_i \geq 0$ for $i = 1, \dots, m$.

Proposition 5.2.19 *The steady Ricci soliton metrics with* (5.14) *are complete and if furthermore* $\sum_{i=1}^{m} d_i + 2 < m\left(\min\{d_j\} + 2\right) C_0^2$ *it follows that*

$$-\dot{u}(t) \to \sqrt{-C} \quad \text{and} \quad \frac{g_i^2(t)}{t} \to \frac{2p_i}{\sqrt{-C}} \quad \text{and} \quad f(t) \text{ converges}$$

as $t \to \infty$.

Proof: In the steady case f and g_i satisfy the ODE system

$$\begin{aligned}
\ddot{f} &= -(-\dot{u} + \mathrm{tr}(L))\dot{f} - \frac{\dot{f}^2}{f} + \sum_{i=1}^{m} \frac{d_i q_i^2}{4}\frac{\omega_i^4}{f}, \\
\ddot{g}_i &= -(-\dot{u} + \mathrm{tr}(L))\dot{g}_i - \frac{\dot{g}_i^2}{g_i} + \frac{p_i - \frac{q_i^2}{2}\omega_i^2}{g_i}.
\end{aligned}$$

Due to (5.14) and remark 5.2.11, the limit $\omega_{i,\infty} = \lim_{t\to\infty} \omega_i(t)$ exists for all i. Moreover, the shape operator remains positive definite along the trajectories. It hence follows from proposition 2.4.1 that $\dot f/f$, $\dot g_i/g_i \to 0$ and $-\dot u \to \sqrt{-C}$ as $t \to \infty$. If $\omega_{i,\infty} > 0$ for some i, an application of lemma 3.4.18 as in the proof of proposition 3.4.19 yields

$$\left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right)\omega_{i,\infty}^2 = \sum_{j=1}^m \frac{d_j q_j^2}{4}\omega_{j,\infty}^4,$$

and in particular $\omega_{i,\infty} > 0$ for all i. Notice that this implies that

$$\sum_{j=1}^m d_j \omega_{j,\infty}^2 \left\{ p_j - \left(\sum_{i=1}^m d_i + 2\right)\frac{q_j^2}{4}\omega_{j,\infty}^2 \right\} = 0.$$

In the domain $\{ \ \omega_{j,\infty}^2 \leq \frac{4p_j}{\left(\sum_{i=1}^m d_i + 2\right)q_j^2} \mid j = 1, \ldots, m \ \}$ the unique solution to this equation is $\omega_{j,\infty}^2 = \frac{4p_j}{\left(\sum_{i=1}^m d_i + 2\right)q_j^2}$ for all j. However, by assumption, the functions ω_i satisfy an even stronger bound. Thus $\omega_{i,\infty} = 0$ holds for those the trajectories.

Then lemma 3.4.18 applied to g_i yields the asymptotic behaviour of all g_i. It remains to prove that f is bounded.

As $\dot\omega_i$ changes its sign at most once due to remark 5.2.11, there exists $t_0 > 0$ such that $\dot\omega_i(t) < 0$ for $t \geq t_0$. Hence, for all $\delta > 0$ there exists $t_0 > 0$ such that

$$\ddot\omega_i \leq -(\sqrt{-C} + \delta)\dot\omega_i - \left\{\frac{\sqrt{-C}}{2} - \delta\right\}\frac{\omega_i}{t}$$

for $t \geq t_0$.

As in [App18, Lemma 6.5], it follows that for all $\delta > 0$ there exist c_0, $t_0 > 0$ such that $\omega_i(t) \leq c_0 t^{-1/2+\delta}$ for $t \geq t_0$. Hence there exist constants $c_1, c_2 > 0$ such that $\ddot f \leq -c_1 \dot f + \frac{c_2}{t^{2-\delta}}$ for $t \geq t_0$. This implies that f is bounded and, as $\dot f > 0$, thus converges. □

Remark 5.2.20 If $m = 1$, the long time analysis of ω in remark 5.2.11 can be improved as the bound $\omega^2 < \frac{4p}{q^2(d+2)}$ from (5.14) and remark 5.2.10 is now optimal in the following sense:

Observe that

$$\ddot\omega = \dot\omega\left\{\dot u - (d+1)\frac{\dot g}{g}\right\} + \frac{\omega}{g^2}\left\{\frac{(d+2)q^2}{4}\omega^2 - p\right\}$$

implies that critical points of ω are local maxima if $\omega^2 < \frac{4p}{q^2(d+2)}$ and local minimal if $\omega^2 > \frac{4p}{q^2(d+2)}$. In particular, as $\omega(0) = 0$, $\dot\omega$ can change its sign at most once and there exists $\lim_{t\to\infty}\omega(t) \in [0,\infty]$. This observation is due to Appleton [App18] who, by exploiting the fact that the S^1-bundle set-up does not yield complete Ricci flat metrics if $q > p$, could prove that in this case the steady Ricci soliton trajectory with

$$2\ddot{u}(0) = \inf \left\{ C \;\middle|\; \begin{array}{l} \text{the steady Ricci soliton metric} \\ \text{with } 2\ddot{u}(0) = C \text{ in (5.13) is complete} \end{array} \right\}$$

gives rise to a complete Ricci soliton with $\omega_\infty = \frac{4p}{(d+2)q^2}$. Appleton also shows that if $q > p$ any trajectory with $\omega > \frac{4p}{(d+2)q^2}$ blows up in finite time.

In particular, Appleton constructs a 4-dimensional, non-collapsed, non-Kähler steady Ricci soliton.

Remark 5.2.21 The Kähler condition $\frac{d}{dt}g_i^2 = -q_i f$ for the metric gives rise to the preserved locus of trajectories satisfying

$$\frac{\dot{g}_i}{g_i} = -\frac{q_i}{2}\frac{f}{g_i^2} \quad \text{and} \quad \left(-\dot{u} + \mathrm{tr}(L) + \frac{\dot{f}}{f}\right)\frac{\dot{g}_i}{g_i} = \frac{\varepsilon}{2} + \frac{p_i}{g_i^2}.$$

Due to the initial conditions of f and g_i, the condition $-q_i > 0$ is necessary for any Kähler metric to exist and then the shape operator is positive definite. In the steady case, proposition 2.4.1 furthermore implies $\frac{d}{dt}g_i^2 \to \frac{2p_i}{\sqrt{-C}}$ as $t \to \infty$ and thus

$$f(t) \to -\frac{2p_i}{\sqrt{-C}q_i} \quad \text{and} \quad \frac{g_i^2(t)}{t} \to \frac{2p_i}{\sqrt{-C}}$$

as $t \to \infty$.

In particular, $\frac{p_i}{q_i} < 0$ must be independent of i for any Kähler metric to exist.

Proposition 5.2.22 *The expanding Ricci soliton metrics with* (5.14) *are complete and satisfy*

$$\frac{-\dot{u}(t)}{t} \to \frac{\varepsilon}{2} \quad \textit{and} \quad \frac{L_t}{t} \to \mathbb{I}_n$$

as $t \to \infty$.

Proof: The asymptotics of the expanding Ricci solitons can be deduced as in section 3.4.5. Indeed, the argument in proposition 3.4.21 applies to the variables X_i, Y_i, ω_i and $\mathcal{L}$ of remark 5.2.18: Due to proposition 2.4.2 and the positivity of the shape operator, the variables X_i, Y_i and $\mathcal{L}$ tend to zero, but it follows from

$$\begin{aligned} \frac{d}{ds}\frac{X_0}{\mathcal{L}^2} &= \left(-\sum_{i=0}^{m} d_i X_i^2 - 1\right)\frac{X_0}{\mathcal{L}^2} + \sum_{i=1}^{m}\frac{d_i q_i^2}{4}\omega_i^2\frac{Y_i^2}{\mathcal{L}^2} + \frac{\varepsilon}{2}(X_0+1), \\ \frac{d}{ds}\frac{X_i}{\mathcal{L}^2} &= \left(-\sum_{i=0}^{m} d_i X_i^2 - 1\right)\frac{X_i}{\mathcal{L}^2} + \left(p_i - \frac{q_i^2}{2}\omega_i^2\right)\frac{Y_i^2}{\mathcal{L}^2} + \frac{\varepsilon}{2}(X_i+1) \end{aligned}$$

and the convergence of $\frac{Y_i}{\mathcal{L}} = \frac{1}{g_i}$ and ω_i, see remark 5.2.11, that

$$\frac{X_0}{\mathcal{L}^2} \to \Lambda_0 = \sum_{i=1}^{m} \frac{d_i q_i^2}{4} \omega_{i,\infty}^2 y_{i,\infty}^2 + \frac{\varepsilon}{2} \quad \text{and} \quad \frac{X_i}{\mathcal{L}^2} \to \Lambda_i = \left(p_i - \frac{q_i^2}{2}\omega_{i,\infty}^2\right) y_{i,\infty}^2 + \frac{\varepsilon}{2}$$

as $s \to \infty$, where $y_{i,\infty} = \lim_{s\to\infty} \frac{Y_i}{\mathcal{L}}$ and $\omega_{i,\infty} = \lim_{s\to\infty} \frac{Y_i}{Y_0} \leq \frac{2p_i}{q_i^2}$.

However, since $Y_i^{'} = Y_i \left(\sum_{i=0}^{m} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2 - X_i\right)$ and $\mathcal{L}^{'} = \mathcal{L}\left(\sum_{i=0}^{m} d_i X_i^2 - \frac{\varepsilon}{2}\mathcal{L}^2\right)$, L'Hôpital's rule implies $y_{i,\infty} = 0$ for $i = 0, \ldots, m$ by comparing the limits of $\frac{Y_i}{\mathcal{L}}$ and $\frac{Y_i{'}}{\mathcal{L}'}$ as in the proof of proposition 3.4.21.

The asymptotic behaviour of the soliton potential and the shape operator then follow as in [DW09a]. □

5.2.2.2 The Lü-Page-Pope set-up

Let L be the total space of a complex line bundle over a single Fano Kähler Einstein manifold whose Euler class is a rational multiple of the first Chern class of the base, equipped with the same geometry as in section 5.2.2.1. Lü-Page-Pope [LPP04] have found Ricci flat metrics on $L \times S^m$, for $m > 1$, of warped product type by exhibiting an explicit formula. All of their examples also support complete steady and expanding Ricci solitons.

Theorem 5.2.23 *There exist continuous families of complete non-trivial steady and expanding gradient Ricci solitons on any product $L \times S^m$ of a sphere S^m, $m > 1$, with the total space L of a complex line bundle over a Fano Kähler Einstein manifold M whose Euler class is a rational multiple of the first Chern class of M.*

In fact, Lü-Page-Pope already remark that in their construction the factor S^m can be replaced by any Einstein manifold (N^{d_2}, h_2) of positive Ricci curvature. Suppose that $\mathrm{Ric}_N = (d_2 - 1)h_2$. Then, with the same notation for the metric on the S^1-bundle as in section 5.2.2.1, the metric on the principal hypersurface is given by the warped product

$$g_t = f(t)^2 \theta \otimes \theta + g_1(t)^2 \pi^* h_1 + g_2(t)^2 h_2$$

and the Ricci soliton equations are given by the ODE system

$$
\begin{aligned}
\ddot{u} &= \frac{\ddot{f}}{f} + \sum_{i=1}^{2} \frac{\ddot{g}_i}{g_i} - \frac{\varepsilon}{2}, \\
\frac{d}{dt}\frac{\dot{f}}{f} &= -\frac{\dot{f}}{f}(-\dot{u} + \mathrm{tr}(L)) + \frac{\varepsilon}{2} + d_1 \frac{q_1^2}{4}\frac{f^2}{g_1^4}, \\
\frac{d}{dt}\frac{\dot{g}_1}{g_1} &= -\frac{\dot{g}_1}{g_1}(-\dot{u} + \mathrm{tr}(L)) + \frac{\varepsilon}{2} + \frac{p_1}{g_1^2} - \frac{q_1^2}{2}\frac{f^2}{g_1^4}, \\
\frac{d}{dt}\frac{\dot{g}_2}{g_2} &= -\frac{\dot{g}_2}{g_2}(-\dot{u} + \mathrm{tr}(L)) + \frac{\varepsilon}{2} + \frac{d_2 - 1}{g_2^2}.
\end{aligned}
$$

This ansatz gives rise to smooth Ricci soliton metrics provided the initial conditions

$$
\begin{gathered}
f(0) = 0,\ \dot{f}(0) = 1 \ \text{ and } \ g_i(0) > 0,\ \dot{g}_i(0) = 0, \\
u(0) = \dot{u}(0) = 0,\ 2\ddot{u}(0) = C
\end{gathered} \tag{5.16}
$$

are satisfied, where C is the constant in the conservation law (2.5).

Notice that this is a special case of the Dancer-Wang set-up in section 5.2.2.1 if one chooses $p_2 = d_2 - 1$ and $q_2 = 0$. However, this allows a simplified argument, which shall briefly be sketched:

Fix $\varepsilon \geq 0$. As in section 5.2.2.1 one has $\ddot{g}_1(0) > 0$ and the assumption $d_2 > 1$ implies that also $2\frac{\ddot{g}_2(0)}{g_2(0)} = \frac{\varepsilon}{2} + \frac{d_2-1}{g_2^2(0)} > 0$. In particular it follows that $\dot{g}_i(t) > 0$ for $i = 1, 2$ and sufficiently small $t > 0$. Furthermore it is clear from the ODE system that $\dot{f} > 0$ and $\dot{g}_2 > 0$ are preserved. Due to proposition 5.2.16, in order to establish completeness, it suffices to show that the set

$$
\left\{ \ \dot{g}_1 > 0 \ \text{ and } \ \frac{f^2}{g_1^2} < \frac{4p_1}{(d_1+2)q_1^2} \ \right\}
$$

is preserved by the Ricci soliton ODE if $\varepsilon \geq 0$. This is now straightforward:

Consider the quotient $\omega_1 = \frac{f}{g_1}$. It satisfies the ODE

$$
\ddot{\omega}_1 = \dot{\omega}_1 \cdot \frac{d}{dt}\left\{u - \ln g_1^{d_1+1} g_2^{d_2}\right\} + \frac{\omega_1}{g_1^2}\left\{\frac{(d_1+2)q_1^2}{4}\omega_1^2 - p_1\right\}.
$$

In particular, as long as $\omega_1^2(t) < \frac{4p_1}{(d_1+2)q_1^2}$, one has

$$
\dot{\omega}_1(t) \leq \frac{(g_1^{d_1} g_2^{d_2})(0)}{(g_1^{d_1+1} g_2^{d_2})(t)} \exp(u(t)) \leq \frac{1}{g_1(0)}
$$

as $u(0) = 0$ and $\dot{\omega}_1(0) = \frac{1}{g_1(0)}$. Then proposition 5.1.3 yields theorem 5.2.23:

Proposition 5.2.24 *Let $d_2 > 1$ and $\varepsilon \geq 0$. Then for any $\bar{g} > 0$ there exists a constant $C_0 \leq 0$ such that the steady and expanding Ricci soliton trajectories in the Lü-Page-Pope set-up with $C < C_0$ and $g_i(0) > \bar{g}$ for $i = 1, 2$ in* (5.16) *and are complete.*

Remark 5.2.25 It is easy to check that the proof of proposition 5.2.24 also works if $\varepsilon > 0$ and $d_2 = 1$. Moreover, in this case, it is sufficient to assume $g_2(0) > 0$.

Chapter 6

Remarks on related Geometries

In this chapter two geometric set-ups, which extend those from the previous chapters in different directions, will be discussed:

Section 6.1 extends the S^1-bundle case of section 5.2.2.1 to torus bundles and investigates the associated Ricci soliton equation. Section 6.2 introduces the m-quasi-Einstein equation and discusses its relationship to the Ricci soliton equation.

6.1 Ricci solitons from torus bundles

This section considers the cohomogeneity one Ricci soliton equation on manifolds that are foliated by r-torus bundles over products of Fano Kähler Einstein manifolds, thus generalising the set-up of section 5.2.2. The key difference between the S^1-bundle case and the torus bundle case is that the diagonal ansatz for the metric is not preserved by the Ricci soliton ODE.

Chen [Che11] has found Einstein metrics of positive, zero and negative scalar curvature in the case of 2-torus bundles by adapting the explicit ansatz of the S^1-bundle construction, cf. [BB82], [WW98].

This approach is in line with the construction of Kähler Ricci solitons due to Cao-Koiso [Cao96], [Koi90], Feldman-Ilmanen-Knopf [FIK03] and Dancer-Wang [DW11], who moreover consider a specific ansatz for the soliton potential.

However, in this section it will be shown that this ansatz does not generalise to the case of r-torus bundles for $r \geq 2$. On the other hand, originating from Chen's explicit formulae, a new invariant curvature condition will be described.

6.1.1 Geometry of principal torus bundles

This section briefly reviews a natural geometric structure on torus bundles [WZ90], [LW17] which will be used to describe the geometry of the principal orbit.

Let $\{(V_i, J_i, h_i)\}_{i=1,\ldots,m}$ be Fano Kähler Einstein manifolds of real dimension d_i. Principal r-torus bundles $\pi\colon P \to B$ with structure group $T^r = S^1 \times \ldots \times S^1$ over the base $B = V_1 \times \ldots \times V_m$ are classified by characteristic classes $\chi_1, \ldots, \chi_r \in H^2(B, \mathbb{Z})$. Each class χ_α comes from the Euler class of the circle bundle $P/T^{r-1} \to B$, where T^{r-1} is considered to be the subtorus with the α-th S^1-factor deleted.

According to a theorem of Kobayashi [Kob61], each V_i is simply connected and hence $H^2(V_i, \mathbb{Z})$ is torsion free. Therefore, the first Chern classes are $c_1(V_i, J_i) = p_i \rho_i$ for positive integers p_i and indivisible classes $\rho_i \in H^2(V_i, \mathbb{Z})$. Suppose that the Kähler metrics h_i are normalised such that $\mathrm{Ric}(h_i) = p_i h_i$, let η_i be the Kähler form associated to h_i and choose $Q \in \mathbb{Z}^{r \times m}$. Consider the principal circle bundle $\pi\colon P_\alpha \to B$ with Euler class $e(P_\alpha) = \chi_\alpha = \sum_{i=1}^m q_{\alpha i} \pi_i^* \rho_i$. On P_α there exists a principal connection θ_α with curvature $\Omega_\alpha = \sum_{i=1}^m q_{\alpha i} \pi_i^* \eta_i$. Then the principal r-torus bundle $\pi\colon P \to B$ associated to $\chi_1, \ldots, \chi_r$ is the pullback bundle

$$\begin{array}{ccc} P & \longrightarrow & P_1 \times \ldots \times P_m \\ \downarrow & & \downarrow \pi_1 \times \ldots \times \pi_r \\ B & \xrightarrow{\Delta} & B \times \ldots \times B, \end{array}$$

where Δ is the diagonal map.

Let $e_1, \ldots, e_r$ be a basis for the Lie algebra $\mathfrak{t}$ of T^r which is induced by the decomposition $T^r = S^1 \times \ldots \times S^1$ and let θ be the principal connection on P given by the pullback of $\theta_1 \times \ldots \times \theta_r$. Then its curvature is $\Omega = d\theta = \sum_{\alpha=1}^r \sum_{i=1}^m q_{\alpha i} \pi^* \eta_i \otimes e_\alpha$.

Choose a left invariant (and hence bi-invariant) metric $\langle e_\alpha, e_\beta \rangle = f_{\alpha\beta}$ on the torus T^r and consider the bundle metric

$$g = \sum_{\alpha,\beta=1}^r f_{\alpha\beta} \theta^\alpha \otimes \theta^\beta + \sum_{i=1}^m g_i^2 \pi_i^* h_i,$$

for $g_1, \ldots, g_m > 0$. This choice turns $\pi\colon P \to B$ into a Riemannian submersion with totally geodesic fibres, where the metric on B is the product metric $\sum_{i=1}^m g_i^2 \pi_i^* h_i$.

The O'Neill calculus for Riemannian submersions then provides a formula for the Ricci curvature:

$$\begin{aligned} \mathrm{Ric}(g) = & \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=1}^r \sum_{i=1}^m \frac{d_i}{4} q_{\alpha_2 i} q_{\beta_2 i} \frac{f_{\alpha_1\alpha_2} f_{\beta_1\beta_2}}{g_i^4} \theta^{\alpha_1} \otimes \theta^{\beta_1} \\ & + \sum_{i=1}^m \left\{ p_i - \frac{1}{2} \sum_{\alpha,\beta=1}^r q_{\alpha i} q_{\beta i} \frac{f_{\alpha\beta}}{g_i^2} \right\} \pi_i^* h_i. \end{aligned}$$

Notation 6.1.1 *For an $r \times m$-matrix Q let $Q_{\bullet i}$ denote the i-th column of Q.*

Furthermore, let $F = (f_{\alpha\beta})$ be the metric on the torus, $\tilde{e}_\alpha$ the fundamental vector field on P generated by e_α and $\mathcal{H}V_i$ the distribution on P of horizontal lifts of vector fields on V_i. Then the non-vanishing components of the Ricci endomorphism are

$$r(\tilde{e}_\alpha) = \sum_{\beta=1}^{r}\sum_{i=1}^{m}(FQ_{\bullet i}Q_{\bullet i}^T)_{\alpha\beta}\tilde{e}_\beta \quad \text{and} \quad r_{|\mathcal{H}V_i} = \left(\frac{p_i}{g_i^2} - \frac{1}{2g_i^4}(Q^TFQ)_{ii}\right)\mathrm{id}\,.$$

6.1.2 Ricci solitons foliated by torus bundles

Let $I \subset \mathbb{R}$ be an open interval and consider the manifold $M_0 = I \times P$ with the Riemannian metric $g = dt^2 + g_t$, where

$$g_t = \sum_{\alpha,\beta=1}^{r} f_{\alpha\beta}(t)\theta^\alpha \otimes \theta^\beta + \sum_{i=1}^{m} g_i^2(t)\pi_i^* h_i$$

is a smooth 1-parameter family of Riemannian metrics on the principal orbit P. Recall that the shape operator L_t of the hypersurface $\{t\} \times P$ satisfies $\dot{g}_t = 2g_tL_t$. Therefore it is given by

$$L_t(\tilde{e}_\alpha) = \sum_{\beta=1}^{r}\frac{1}{2}(\dot{F}F^{-1})_{\alpha\beta}\tilde{e}_\beta \quad \text{and} \quad L_{t|\mathcal{H}V_i} = \frac{\dot{g}_i}{g_i}\,\mathrm{id}\,.$$

Then the associated Ricci soliton equations (2.3) and (2.4) take the form

$$\begin{aligned}
\ddot{u} &= \frac{1}{2}\operatorname{tr}\left(\ddot{F}F^{-1} - \frac{1}{2}(\dot{F}F^{-1})^2\right) + \sum_{i=1}^{m} d_i\frac{\ddot{g}_i}{g_i} - \frac{\varepsilon}{2},\\
\frac{d}{dt}\frac{1}{2}\dot{F}F^{-1} &= -(-\dot{u} + \operatorname{tr}(L))\,\frac{1}{2}\dot{F}F^{-1} + \frac{\varepsilon}{2}\mathbb{I}_{r\times r} + \sum_{i=1}^{m}\frac{d_i}{4g_i^4}FQ_{\bullet i}Q_{\bullet i}^T,\\
\frac{d}{dt}\frac{\dot{g}_i}{g_i} &= -(-\dot{u} + \operatorname{tr}(L))\,\frac{\dot{g}_i}{g_i} + \frac{\varepsilon}{2} + \frac{p_i}{g_i^2} - \frac{1}{2g_i^4}Q_{\bullet i}^TFQ_{\bullet i},
\end{aligned}$$

for $i = 1, \ldots, m$.

Example 6.1.2 In the case of circle bundles, $r = 1$, one recovers the Ricci soliton equations from section 5.2.2.1 by setting $F = f^2$ and $Q = (q_i) \in \mathbb{Z}^{1\times m}$.

Remark 6.1.3 Notice that the term $FQ_{\bullet i}Q_{\bullet i}^T$ is non-diagonal. Hence, even though the matrix $\dot{F}F^{-1}$ may be diagonalised at any fixed time t_0, the diagonal form is not preserved in the case of r-torus bundles, for $r \geq 2$.

Let $\varepsilon \geq 0$. It follows from

$$\frac{d}{dt}\dot{F} = -(-\dot{u} + \mathrm{tr}(L))\dot{F} + \varepsilon F + \sum_{i=1}^{m} \frac{d_i}{2g_i^4} F Q_{\bullet i} Q_{\bullet i}^T F + \dot{F} F^{-1} \dot{F}$$

that simultaneous positive definiteness of F and $\dot{F}$ is preserved along the Ricci soliton ODE, if it holds initially.

However, the non-commutativity of matrix multiplication is an obstacle to carry out an analysis as in section 5.2.2.1 to prove positive definiteness of the shape operator. On the other hand, the proof of proposition 3.2.3 carries over to the torus bundle case and yields:

Proposition 6.1.4 *Let $\varepsilon \geq 0$, $i \in \{1, \ldots, m\}$ and $Q \in \mathbb{R}^{r \times m}$ such that the i-th column of Q is non-zero, and F, $\dot{F}$ are positive definite.*

Suppose that there exist $t_0 < t_1$ such that $\dot{g}_i(t_0) > 0$ and $\dot{g}_i(t_1) < 0$. Then $\dot{g}_i(t) < 0$ for all $t \geq t_1$.

Remark 6.1.5 Building up on the previous examples in [Cao96], [FIK03], [Che11], [DW11], two different compactifications of the cohomogeneity one manifold $I \times P$ with singular orbit at $t = 0$ will be discussed.

Case 1. The collapse of a circle in the torus bundle gives rise to the initial condition

$$F(0) = \mathrm{diag}\left(0, \bar{f}_2, \ldots, \bar{f}_r\right)$$

for $\bar{f}_2, \ldots, \bar{f}_r > 0$. The functions $f_{\alpha\beta}(t)$ are required to be even in t at $t = 0$ and the condition $\frac{d}{dt}\big|_{t=0} \sqrt{f_{11}(t)} = 1$ ensures that the collapsing circle becomes round. Similarly, the functions g_i also need to be even but with $g_i(0) = \bar{g}_i > 0$.

Case 2. By choosing $V_1 = \mathbb{C}P^{n_1}$ and $Q_{\bullet 1} = (-1, 0, \ldots, 0)^T$, one can simultaneously collapse a circle and the first base factor. Then as in [DW11] the conditions $g_1(0) = 0$ and $\dot{g}_1^2(0) = \frac{1}{2}$ are sufficient to ensure smoothness of the metric at the singular orbit, provided that otherwise the same conditions as in *case 1* hold.

The smoothness condition for the soliton potential is given in (2.8). With the normalisation $u(0) = 0$ one has $\dot{u}(0) = 0$ and $(d_S + 1)\ddot{u}(0) = C$, where C is the constant in the conservation law (2.5) and d_S is the dimension of the collapsing sphere. That is, $d_S = 1$ in *case 1* and $d_S = d_1 + 1$ in *case 2.*

By adopting the convention that choosing $d_1 = 0$ in *case 2* is allowed, *case 1* becomes a special case of *case 2.*

With these initial conditions it follows that $\mathrm{tr}(L) = \frac{1+d_1}{t} + O(t)$ and hence

$$(1+d_1)\frac{\ddot{g}_i(0)}{g_i(0)} = \frac{\varepsilon}{2} + \frac{p_i}{g_i^2(0)} - \frac{1}{2g_i^4(0)}\sum_{\alpha=2}^{r} q_{\alpha i}^2 f_{\alpha\alpha}(0)$$

for $i = 2, \dots, m$, whereas $\ddot{g}_1(0) = 0$. Furthermore notice that $\dot{F}(t) = 2t\ddot{F}(0) + O(t^3)$, $\lim_{t\to 0} t^2 F^{-1}(t) = E$, where $E = \mathrm{diag}(1, 0, \dots, 0)$, and $\lim_{t\to 0} \frac{F(t)Q_{\bullet 1}}{2g_1^2(t)} = -\ddot{F}_{\bullet 1}(0)$ imply

$$(1+d_1)\ddot{F}(0) = \varepsilon F(0) + 2d_1 \ddot{F}_{\bullet 1}(0)\ddot{F}_{\bullet 1}^T(0) + \sum_{i=2}^{m} \frac{d_i}{4g_i^4(0)} F(0)Q_{\bullet i}Q_{\bullet i}^T F(0) + 4\ddot{F}(0)E\ddot{F}(0).$$

Thus $\ddot{F}(0)$ is positive semi-definite if $\varepsilon \geq 0$ and since $F(0) = \mathrm{diag}\left(0, \bar{f}_2, \dots, \bar{f}_r\right)$ and $\ddot{f}_{11}(0) = 1$, it is positive definite if $\varepsilon > 0$. To obtain a sufficient condition for positive definiteness if $\varepsilon = 0$, observe that $F(0)Q_{\bullet i} = (0, \bar{f}_2 q_{2i}, \dots, \bar{f}_r q_{ri})^T$. Hence, for example, in the case of a 2-torus bundle over a single Fano Kähler Einstein manifold, $m = 2$ and $d_1 = 0$, positive definiteness of $\ddot{F}(0)$ follows if $q_{22} \neq 0$.

In particular, in these cases, $\dot{F}(t)$ is positive definite for sufficiently small $t > 0$.

6.1.3 The classical coordinate change, explicit solutions and invariant curvature conditions

In the existing literature, the following coordinate change has been used extensively. It goes back to Bérard-Bergery [BB82] who used it to integrate the Einstein equations on manifolds foliated by S^1-bundles over a single Fano Kähler Einstein manifold. Set

$$\alpha = \det F \quad \text{and} \quad \beta_i(s) = g_i^2(t) \quad \text{and} \quad \phi(s) = u(t) \quad \text{and} \quad \frac{d}{ds} = \frac{1}{\sqrt{\alpha}}\frac{d}{dt}. \tag{6.1}$$

Differentiation with respect to the time parameter s will be denoted by a prime $'$.

Set $v = \prod_{i=1}^m \beta_i^{d_i/2}$ and observe that $\alpha' = \alpha\,\mathrm{tr}(F'F^{-1})$. Then the Ricci soliton equations (2.3) and (2.4) yield

$$\begin{aligned}\frac{\alpha''}{2} &= \frac{\alpha}{2}\sum_{i=1}^{m} d_i \left\{\frac{1}{2}\left(\frac{\beta_i'}{\beta_i}\right)^2 - \frac{\beta_i''}{\beta_i}\right\} + \alpha\phi'' - \frac{\alpha'}{2}\left(-\phi' + (\log v)'\right) + \frac{\varepsilon}{2} \\ &\quad + \frac{(\alpha')^2}{4\alpha} - \frac{\alpha}{4}\mathrm{tr}((F'F^{-1})^2),\end{aligned} \tag{6.2}$$

$$\frac{\alpha}{2}\frac{d}{ds}F'F^{-1} = -\left(\alpha\left(-\phi' + (\log v)'\right) + \alpha'\right)\frac{1}{2}F'F^{-1} + \frac{\varepsilon}{2}\mathbb{I}_{r\times r} + \sum_{i=1}^{m}\frac{d_i}{4\beta_i^2}FQ_{\bullet i}Q_{\bullet i}^T, \tag{6.3}$$

$$\frac{\alpha}{2}\frac{d}{ds}\frac{\beta_i'}{\beta_i} = -\left(\alpha\left(-\phi' + (\log v)'\right) + \alpha'\right)\frac{1}{2}\frac{\beta_i'}{\beta_i} + \frac{\varepsilon}{2} + \frac{p_i}{\beta_i} - \frac{Q_{\bullet i}^T F Q_{\bullet i}}{2\beta_i^2}. \tag{6.4}$$

Remark 6.1.6 Notice that the term in the second line of (6.2) takes a simple form if the torus fibre is of small dimension. Indeed, there holds

$$(\alpha')^2 - \alpha^2 \operatorname{tr}((F^{'}F^{-1})^2) = \begin{cases} 0 & \text{if } r = 1, \\ 2\alpha \det F^{'} & \text{if } r = 2. \end{cases}$$

Furthermore, by taking the trace of (6.3) and then inserting (6.4) it follows that

$$\begin{aligned} \frac{\alpha''}{2} &= -\left(-\phi^{'} + (\log v)^{'}\right)\frac{\alpha'}{2} + \frac{\varepsilon}{2}\cdot r + \sum_{i=1}^{m} \frac{d_i}{4\beta_i^2} Q_{\bullet i}^T F Q_{\bullet i} \\ &= -\left(\alpha' + \frac{\alpha}{2}\left(-\phi^{'} + (\log v)^{'}\right)\right)(\log v)^{'} - \frac{\alpha}{2}(\log v)^{''} + \frac{\alpha'}{2}\phi^{'} \\ &\quad + \left\{r + \sum_{i=1}^{m}\frac{d_i}{2}\right\}\frac{\varepsilon}{2} + \sum_{i=1}^{m}\frac{d_i}{2}\frac{p_i}{\beta_i}. \end{aligned} \tag{6.5}$$

In the case of S^1-bundles, Dancer-Wang [DW11] have been able to integrate the Ricci soliton equations by considering the ansatz $\phi'' = 0$. This is based on the fact that in this case the conservation law (2.5), which takes the form

$$\alpha\phi'' + \alpha'\phi^{'} + \alpha\phi'\left(-\phi' + (\log v)'\right) = C + \varepsilon\phi, \tag{6.6}$$

can be integrated explicitly. Indeed, with $\phi(s) = a_1(s + a_0)$ and $a_1 \neq 0$ one obtains

$$\alpha' + \alpha(-a_1 + (\log v)^{'}) = \varepsilon s + E^*, \tag{6.7}$$

where $E^* = \frac{C}{a_1} + \varepsilon a_0$, and this implies

$$\alpha = v^{-1}e^{a_1 s}\int(\varepsilon s + E^*)e^{-a_1 s} v ds.$$

Furthermore, since $a_1 \neq 0$, by taking the derivative of the conservation law (6.6) it follows that

$$\alpha'' + \alpha'(-\phi^{'} + (\log v)') + \alpha(\log v)^{''} = \varepsilon. \tag{6.8}$$

Inserting (6.8) into (6.2) yields

$$\sum_{i=1}^{m} d_i \frac{\beta_i''}{\beta_i} = (\operatorname{tr}(F^{'}F^{-1}))^2 - \operatorname{tr}((F^{'}F^{-1})^2).$$

Hence, if also all β_i are assumed to be linear, then the restriction of the shape operator to the torus fibre cannot be positive definite unless the torus is a circle. However, note that the smoothness conditions 6.1.5 often require the restriction of the shape operator to the torus fibre to be positive definite close to the singular orbit. For example, this applies to the expanding case or, if the singular orbit is a non-trivial circle bundle over a single Fano Kähler Einstein manifold, also to the steady case. Therefore, in these examples, it is impossible to construct non-trivial Ricci solitons which are foliated by r-torus bundles, $r \geq 2$, by setting $\phi^{''} = \beta_1^{''} = \ldots = \beta_m^{''} = 0$.

On the other hand, in the case of circle bundles, the steady and expanding Ricci solitons of Dancer-Wang [DW11], which generalise the examples of Cao [Cao96] and Feldman-Ilmanen-Knopf [FIK03], all satisfy these conditions. In their construction the linearity of all β_i is in fact quite natural as it implies that the curvature tensor is invariant under the natural complex structure. This idea goes to back to the examples of Wang-Wang [WW98] in the Einstein case.

By combining (6.8) with (6.5) and (6.7) one obtains

$$(\varepsilon s + E^*) \sum_{i=1}^{m} \frac{d_i}{2} \frac{\beta_i^{'}}{\beta_i} = \sum_{i=1}^{m} \frac{d_i p_i}{\beta_i} + \left\{ 2(r-1) + \sum_{i=1}^{m} d_i \right\} \cdot \frac{\varepsilon}{2},$$

and thus a necessary condition for $\beta_1, \ldots, \beta_m$, which, if $\varepsilon = 0$, does not depend on the number of torus factors.

In the special case of a single base factor, $m = 1$, and for $\varepsilon = 0$ it follows that

$$\beta_1^{'} = \frac{2p_1}{E^*},$$

and hence β_1 must be linear in the case of steady Ricci solitons.

On the other hand, if $\varepsilon \neq 0$ one obtains

$$\beta_1(s) = -\frac{2p_1}{b_1 \varepsilon} + \overline{C} \cdot (\varepsilon s + E^*)^{b_1},$$

where $b_1 = \frac{2(r-1)+d_1}{d_1}$ and $\overline{C}$ is a constant of integration. Notice that in the case of S^1-bundles β_1 is hence linear regardless of the sign of $\varepsilon \in \mathbb{R}$.

The above discussion is summarised in the following proposition:

Proposition 6.1.7 *Let (F, β_i, ϕ) be a solution to the Ricci soliton equations* (6.2) - (6.4) *on a manifold foliated by an r-torus bundle over an m-product of Fano Kähler Einstein manifolds. Suppose that the soliton potential ϕ is linear and non-constant.*

1. *In case of a single base factor, $m = 1$, solutions to the steady Ricci soliton equation must satisfy $\beta_1^{''} = 0$.*

2. If in fact all β_i are linear, then the shape operator of the torus fibre $\frac{1}{2}\dot{F}F^{-1}$ cannot be positive definite unless the torus bundle is a circle bundle.

In particular, if there is a singular orbit as in remark 6.1.5, solutions to the expanding Ricci soliton equation only exist on circle bundles and steady Ricci solitons do not exist in the case of 2-torus bundles if the singular orbit is a non-trivial circle bundle over a single Fano Kähler Einstein manifold, that is $r = 2$, $m = 2$, $d_1 = 0$ and $q_{22} \neq 0$ in remark 6.1.5.

Remark 6.1.8 Due to (6.1), the conditions $\phi'' = 0$ and $\beta_i'' = 0$ are equivalent to saying that the soliton potential u and $\frac{d}{dt}g_i^2$ are proportional to $\sqrt{\det F}$.

Recall that Dancer-Wang [DW11] indeed give examples of non-trivial Kähler Ricci solitons with $\phi'' = \beta_1'' = \ldots = \beta_m'' = 0$ in the case of circle bundles and, in this sense, proposition 6.1.7 is sharp in the case of expanding Ricci solitons. Conjecturally, steady Ricci solitons with $\phi'' = \beta_1'' = \ldots = \beta_m'' = 0$ only exist in the case of circle bundles too.

Notice furthermore that proposition 6.1.7 already rules out many 2-torus bundles over a single Fano Kähler Einstein manifold but in contrast, in this situation, Chen [Che11] has found explicit Einstein metrics from the ansatz $\beta = \kappa s$.

Following Chen's approach, set $m = 1$ and $\beta = \kappa s$. Then the Ricci soliton equation and the conservation law yield the ODE system

$$\begin{aligned}
\alpha'' &= -\frac{d}{s}\alpha' - \frac{\frac{d}{2}(\frac{d}{2}-1)}{s^2}\alpha + \frac{dp}{\kappa}\frac{1}{s} + (2r+d)\frac{\varepsilon}{2} + \phi'\left\{\frac{d}{2}\frac{\alpha}{s} + \alpha'\right\}, \\
\alpha\phi'' + \alpha'\phi' - \alpha(\phi')^2 + \alpha\phi'\frac{d}{2}\frac{1}{s} &= C + \varepsilon\phi.
\end{aligned}$$

The corresponding initial conditions are

$$\begin{aligned}
\alpha(s_1) = 0 \quad &\text{and} \quad \alpha'(s_1) = 2\prod_{\alpha=2}^{r}\sqrt{\bar{f}_\alpha}, \\
\phi(s_1) = 0 \quad &\text{and} \quad \phi'(s_1) = \frac{C}{d_S + 1},
\end{aligned}$$

where d_S is the dimension of the collapsing sphere and $s_1 > 0$ corresponds to the time $t = 0$. Therefore, in the general Ricci soliton case, this is a singular initial value problem. However, in the Einstein case, $\phi \equiv 0$, the first equation can be integrated explicitly:

$$\alpha = \frac{4(2r+d)}{(d+2)(d+4)}\frac{\varepsilon}{2}s^2 + \frac{4p}{\kappa(d+2)}s + c_1 s^{1-d/2} + c_2 s^{-d/2}.$$

Inserting this into the Ricci soliton equation yields

$$Q^T F Q = \frac{4p}{d+2}\kappa s + c_2\kappa^2 s^{-d/2} - \frac{2r-4}{d+4}\varepsilon\kappa^2 s^2$$

and thus

$$\frac{Q^T F Q}{2g^2} - \frac{2p}{d+2} + \frac{\varepsilon}{2}\frac{2r-4}{d+4}g^2 = c_2\kappa s^{-(d/2+1)}.$$

This can be expressed invariantly as follows: There exists a constant $\overline{C} \in \mathbb{R}$ such that

$$\frac{Q^T F Q}{2g^2} - \frac{2p}{d+2} + \frac{\varepsilon}{2}\frac{2r-4}{d+4}g^2 = \overline{C} g^{-(d+2)}. \tag{6.9}$$

Proof: Notice that

$$2g\dot{g} = \frac{d}{dt}g^2 = \sqrt{\alpha}\frac{d}{ds}\beta = \kappa\sqrt{\alpha} = \kappa\sqrt{\det F}.$$

Then the claim follows from the observation that, by differentiating the right hand side of (6.9), the logarithmic derivative of the left hand side is $\frac{d}{dt}\log g^{-(d+2)}$. □

Observe that in the construction of steady and expanding Ricci solitons ($\varepsilon \geq 0$) from circle bundles ($r = 1$) one necessarily has to have $\overline{C} < 0$ in order to satisfy (6.9). On the other hand, Chen [Che11] has shown that there are Einstein metrics of non-positive scalar curvature with $\overline{C} = 0$ in the case of 2-torus bundles $r = 2$.

In the general r-torus bundle case, these trajectories correspond to the following preserved curvature condition:

Proposition 6.1.9 *Suppose there exists $t_0 \in \mathbb{R}$ such that*

$$\frac{Q^T F Q}{2g^2} = \frac{2p}{d+2} \quad \textit{and} \quad Q^T \left(\frac{\dot{F}F^{-1}}{2}\right)^k F Q = \left(\frac{\dot{g}}{g}\right)^k Q^T F Q \tag{6.10}$$

for all $k \in \mathbb{N}$ at t_0.

Then (6.10) *is preserved by the flow of the Ricci soliton equation on any r-torus bundle over a single Fano manifold.*

Furthermore, if $\varepsilon \geq 0$ and the trajectory induces a smooth metric with a singular orbit at $t = 0$ whose shape operator becomes positive definite for $t > 0$, then the shape operator remains positive definite and the metric is complete.

Proof: It is straightforward to check that

$$\begin{aligned}
\frac{d}{dt}\, Q^T \left(\frac{\dot F F^{-1}}{2}\right)^k FQ \;&=\; Q^T \left(\frac{\dot F F^{-1}}{2}\right)^k \dot F Q \,+ \\
&+\; k\cdot \left\{ -(-\dot u + \operatorname{tr}(L)) Q^T \left(\frac{\dot F F^{-1}}{2}\right)^k FQ + \frac{\varepsilon}{2} Q^T \left(\frac{\dot F F^{-1}}{2}\right)^{k-1} FQ \right\} \\
&+\; \frac{d}{4g^4} \sum_{l=0}^{k-1} Q^T \left(\frac{\dot F F^{-1}}{2}\right)^l FQQ^T \left(\frac{\dot F F^{-1}}{2}\right)^{k-l-1} FQ, \\
\frac{d}{dt}\left(\frac{\dot g}{g}\right)^k Q^T FQ \;&=\; \left(\frac{\dot g}{g}\right)^k Q^T \dot F Q \;+ \\
&+\; k\cdot \left\{ -(-\dot u + \operatorname{tr}(L)) \left(\frac{\dot g}{g}\right)^k Q^T FQ + \left\{\frac{\varepsilon}{2} + \frac{p}{g^2} - \frac{Q^T FQ}{2g^4}\right\}\left(\frac{\dot g}{g}\right)^{k-1} Q^T FQ \right\}.
\end{aligned}$$

By assumption there holds

$$Q^T \dot F Q = 2\cdot Q^T \frac{\dot F F^{-1}}{2} FQ = 2\cdot \frac{\dot g}{g} Q^T FQ$$

at t_0 and hence also

$$\begin{aligned}
&\frac{d}{dt}\, Q^T \left\{ \left(\frac{\dot F F^{-1}}{2}\right)^k - \left(\frac{\dot g}{g}\right)^k \mathbb{I}_{r\times r} \right\} FQ \\
&\quad=\; 2\cdot \left\{ Q^T \left(\frac{\dot F F^{-1}}{2}\right)^{k+1} FQ - \left(\frac{\dot g}{g}\right)^{k+1} Q^T FQ \right\} \\
&\qquad + k\cdot \left\{ \frac{d}{4g^4}\left(\frac{\dot g}{g}\right)^{k-1} \left(Q^T FQ\right)^2 - \frac{dp}{d+2}\left(\frac{\dot g}{g}\right)^{k-1} \frac{Q^T FQ}{g^2} \right\} \;=\; 0, \\
&\frac{d}{dt}\,\frac{Q^T FQ}{2g^2} \;=\; \frac{1}{2g^2}\left(Q^T \dot F Q - 2\frac{\dot g}{g} Q^T FQ \right) \;=\; 0.
\end{aligned}$$

Therefore, (6.10) is preserved and any such trajectory satisfies $\frac{p}{g^2} - \frac{Q^T FQ}{2g^4} = \frac{dp}{d+2}\cdot\frac{1}{g^2}$. If $\varepsilon \geq 0$, this implies that $\dot g > 0$ is preserved along the flow and completeness of the metric then follows from remark 6.1.3 and proposition 5.1.3. $\square$

6.2 Quasi-Einstein Metrics

This section considers the m-quasi-Einstein equation

$$\operatorname{Ric} + \operatorname{Hess} u - \frac{1}{m} du \otimes du + \frac{\varepsilon}{2} g = 0$$

for a positive real m. It is motivated by the study of Einstein warped products and recovers the Ricci soliton equation in the formal limit $m = \infty$.

After a brief motivation and literature review, the methods from chapter 3 will be extended to the m-quasi-Einstein case.

6.2.1 Geometry of the m-Bakry-Émery Ricci tensor

Let (M^n, g) be a Riemannian manifold with $\mathrm{Ric} \geq (n-1)\kappa$. If M^n_κ denotes the simply connected model space of constant sectional curvature κ, then the Bishop-Gromov inequality asserts that

$$\frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}_{M^n_\kappa}(B(p,r))}$$

is non-increasing in $r > 0$, and if furthermore $\kappa > 0$, then according to Myers' theorem M is compact with $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{\kappa}}$.

However, if the volume is distorted by a function $u \in C^\infty(M)$, the m-Bakry-Émery Ricci tensor is appropriate quantity to consider.

Definition 6.2.1 *Let $(M^n, g, e^{-u} d\,\mathrm{Vol}_M)$ be a smooth metric measure space and let $m \in (0, \infty]$. Then*

$$\mathrm{Ric}_m = \mathrm{Ric} + \mathrm{Hess}\, u - \frac{1}{m} du \otimes du$$

is the associated m-Bakry-Émery Ricci tensor.

Let $B(p,r)$ denote the metric ball of radius $r > 0$ around $p \in M$ and let $\mathrm{Vol}_u(B(p,r))$ be its volume with respect to the measure $e^{-u} d\,\mathrm{Vol}_M$. Suppose that $\mathrm{Ric}_m \geq (n+m-1)\kappa$. Then due to results of Qian [Qia97]

$$\frac{\mathrm{Vol}_u(B(p,r))}{\mathrm{Vol}_{M^{n+m}_\kappa}(B(p,r))}$$

is non-increasing in $r > 0$, and if furthermore $\kappa > 0$, then M is compact with $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{\kappa}}$. Furthermore, the comparison geometry of the Bakry-Émery Ricci tensor has been extended by Lott [Lot03] and Wei-Wylie [WW09].

In particular, by assuming a lower bound on Ric_m as above, the smooth metric measure space $(M^n, g, e^{-u} d\,\mathrm{Vol}_M)$ behaves as if it was an $(m+n)$-dimensional space. Examples of this phenomenon naturally arise in the context of measured Gromov-Hausdorff convergence of Riemannian manifolds:

Definition 6.2.2 ([Lot07]) *An ε-approximation between two compact metric spaces (X_1, d_1) and (X_2, d_2) is a (not necessarily continuous) map $f\colon X_1 \to X_2$ such that for all $x_1, \tilde{x}_1 \in X_1$ there holds $|d_2(f(x_1), f(\tilde{x}_1)) - d_1(x_1, \tilde{x}_1)| \leq \varepsilon$ and for all $x_2 \in X_2$ there exists $x_1 \in X_1$ such that $d_2(f(x_1), x_2) \leq \varepsilon$.*

A sequence $\{(X_i, d_i, \nu_i)\}_{i\in\mathbb{N}}$ of compact metric measure spaces converges to (X, d, ν) in the measured Gromov-Hausdorff topology if there are measurable ε_i-approximations $f_i\colon X_i \to X$ with $\lim_{i\to\infty} \varepsilon_i = 0$ and $\lim_{i\to\infty}(f_i)_*\nu_i = \nu$ in the weak-$*$ topology on the space of probability measures on X.

Remark 6.2.3 The space of probability measures on a compact metric space is compact with respect to the weak-$*$ topology.

Theorem 6.2.4 ([LV09]) *Let $m \in (0, \infty)$ and suppose that $(M^n, g, e^{-u} d\operatorname{Vol}_M)$ is a compact smooth metric measure space that is a measured Gromov-Hausdorff limit of Riemannian manifolds with non-negative Ricci curvature and dimension at most $m + n$. Then there holds $\operatorname{Ric}_m(M) \geq 0$.*

Conversely, any compact smooth metric measure space $(M^n, g, e^{-u} d\operatorname{Vol}_M)$ with $\operatorname{Ric}_m(M) \geq 0$ for some integer $m \geq 2$ arises as a measured Gromov-Hausdorff limit of Riemannian manifolds with nonnegative Ricci curvature and dimension $m + n$.

Lott-Villani's proof actually uses a more general notion of non-negative N-Ricci curvature for measured length spaces, which is preserved under measured Gromov-Hausdorff convergence, and their characterisation that a smooth metric measure space $(M^n, g, e^{-u} d\operatorname{Vol}_M)$ with compact M has non-negative $(m+n)$-Ricci curvature if and only if $\operatorname{Ric}_m(M) \geq 0$.

Remark 6.2.5 Independently, Sturm [Stu06a, Stu06b] has developed a theory of Ricci curvature for non-smooth metric measure spaces too.

The converse then follows from a warped product construction of Lott [Lot03]. Consider $M \times S^m$ with the metrics $g_i = g_M + i^{-2} e^{-2\frac{u}{m}} g_{S^m}$ and let $p\colon M \times S^m \to M$ be the natural projection. Then the pushforward measure $p_*\left(\frac{d\operatorname{Vol}_{M_i}}{\operatorname{Vol}(M_i)}\right)$ agrees with the probability measure $e^{-u} d\operatorname{Vol}_M$ and p is an ε-approximation. Furthermore, the vertical part of the Ricci tensor is given by $\operatorname{Ric}^{S^m} - i^{-2} e^{-2\frac{u}{m}} \left\{\frac{\Delta u}{m} + \frac{|\nabla u|^2}{m^2}\right\} \cdot g_{S^m}$ and hence becomes positive as i increases, whereas the mixed terms vanish. The claim now follows from Lott-Villani's convergence results for metric measure spaces and the fact that horizontal part of the Ricci tensor is given by $\operatorname{Ric}_m \geq 0$.

Remark 6.2.6 The fact that the horizontal part of the Ricci tensor of the warped product $(M \times N, g_M + e^{-2\frac{u}{m}} g_N)$ is given by the m-Bakry-Émery Ricci tensor of M will reappear in the characterisation 6.2.9 of m-quasi-Einstein metrics.

6.2.2 An introduction to m-quasi-Einstein metrics

The m-Bakry-Émery Ricci tensor also naturally occurs in the context of warped product Einstein manifolds, where it has led to the notion of *m-quasi-Einstein metrics* or $(\lambda, n+m)$-Einstein metrics in the terminology of He-Petersen-Wylie.

Definition 6.2.7 *Let (M, g) be a Riemannian manifold of dimension n, $u \in C^\infty(M)$ a smooth function and $m \in (0, \infty]$. Then $(M, g, e^{-u} d\,\mathrm{Vol}_M)$ is called* m-quasi-Einstein manifold *if*

$$\mathrm{Ric} + \mathrm{Hess}\, u - \frac{1}{m} du \otimes du + \frac{\varepsilon}{2} g = 0. \tag{6.11}$$

The sum $m + n$ is called effective dimension *and $-\frac{\varepsilon}{2}$ is the* quasi-Einstein constant.

In analogy to the Ricci soliton equation, which one recovers by setting $m = \infty$, the the contracted second Bianchi identity $dR = \mathrm{tr}_{12} \nabla\, \mathrm{Ric}$ also implies a conservation law for finite $m < \infty$.

Lemma 6.2.8 *Let $(M^n, g, e^{-u} d\,\mathrm{Vol}_M)$ be a connected m-quasi-Einstein manifold of effective dimension $m + n < \infty$ and quasi-Einstein constant $-\frac{\varepsilon}{2} \in \mathbb{R}$. Then there exists a constant $\mu \in \mathbb{R}$, called* characteristic constant, *such that*

$$\Delta u - |\nabla u|^2 + m\mu e^{2u/m} + m\frac{\varepsilon}{2} = 0. \tag{6.12}$$

The following observation due to Kim-Kim [KK03] explains the relation between m-quasi-Einstein manifolds and Einstein warped products:

Proposition 6.2.9 ([KK03]) *Let $m > 1$ be an integer and let $(M^n, g, e^{-u} d\,\mathrm{Vol}_M)$ be an m-quasi-Einstein manifold of effective dimension $m + n \in \mathbb{N}$, with quasi-Einstein constant $-\frac{\varepsilon}{2} \in \mathbb{R}$ and characteristic constant $\mu \in \mathbb{R}$. Suppose that (N, h) is Einstein with $\mathrm{Ric}_h = \mu h$ and of dimension $\dim N = m$. Then the warped product manifold*

$$(M \times N, g + e^{-2u/m} h) \tag{6.13}$$

is Einstein with Einstein constant $-\frac{\varepsilon}{2}$.

Conversely, if the warped product (6.13) is Einstein with Einstein constant $-\frac{\varepsilon}{2}$, then the base manifold M is quasi-Einstein with quasi-Einstein constant $-\frac{\varepsilon}{2}$ and effective dimension $m + n$, (N, h) is Einstein and if $\mathrm{Ric}_h = \mu h$ then (6.12) holds.

As an immediate application, Kim-Kim [KK03] have shown that any Einstein warped product $M \times_u N$ with compact base M and nonpositive scalar curvature has constant warping function u.

Case-Shu-Wei [CSW11] have shown that any compact *Kähler* m-quasi Einstein metric for finite $m < \infty$ is Einstein. More generally, any complete simply-connected Kähler m-quasi-Einstein manifold $(M^n, g, e^{-u} d\,\mathrm{Vol}_M)$ with $m < \infty$ is a Riemannian product $M = M_1 \times M_2$, where M_1 is an $(n-2)$-dimensional Einstein manifold, M_2 a 2-dimensional quasi-Einstein manifold, and u can be considered as a function on M_2. Furthermore any compact 2-dimensional quasi-Einstein metric is trivial, i.e. Einstein. In contrast, recall that all *known* non-trivial compact Ricci solitons are Kähler.

As in the Ricci soliton case, compact m-quasi-Einstein metrics with constant scalar curvature are trivial [CSW11]. However, the rigidity properties of homogeneous Ricci solitons [PW09a] do not carry over. In particular, for any $m > 0$, He-Petersen-Wylie have constructed left-invariant m-quasi-Einstein metrics on 4-dimensional solvable Lie groups which are not rigid, i.e. they are neither Einstein nor is their universal cover a product of Einstein manifolds [HPW14]. They also provide a detailed study of homogeneous m-quasi-Einstein metrics, in particular with regard to their relation to homogeneous Ricci solitons, in [HPW15b].

Continuing with the theme of obtaining structure theorems for Einstein warped products from the m-quasi-Einstein equation, He-Petersen-Wylie have proven the following uniqueness result [HPW15a], see also [HPW15c]: Let (M, g_M) be a complete Riemannian manifold, and fix constants λ and m. Then, up to isometry, there is at most one warped product Einstein metric with Einstein constant λ on $M \times_u F$, where (F^m, g_F) is a simply connected space of constant curvature.

He-Petersen-Wylie have also obtained classification results for m-quasi-Einstein metrics with harmonic Weyl tensor [HPW12] or in dimension 3 [HPW14]: Fix $m > 1$. Then any complete, simply connected m-quasi-Einstein manifold with harmonic Weyl tensor W is rigid. If furthermore $W(\nabla u, \cdot, \cdot, \nabla u) = 0$, then $g = dt^2 + \psi^2(t) g_N$, for an Einstein metric g_N, and $u = u(t)$. In dimension 3, any m-quasi-Einstein metric with constant scalar curvature is a quotient of S^3, $S^2 \times \mathbb{R}$, $\mathbb{R}^3$, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{H}^3$.

Hall [Hal13] has constructed m-quasi-Einstein metrics on the total space of certain complex vector bundles over products of Fano Kähler Einstein manifolds by adapting the explicit constructions of Einstein metrics and Kähler Ricci solitons due to Wang-Wang [WW98] and Dancer-Wang [DW11], respectively. In particular, he uses the

S^1-bundle set up of section 5.2.2.1. Clearly, due to the rigidity result of Case-Shu-Wei [CSW11], only the non-Kähler Einstein metrics of Wang-Wang have m-quasi-Einstein analogues, and the case of a single base factor, as in section 5.2.2.2, is due to Lü-Page-Pope [LPP04]. Remarkably, the Lü-Page-Pope metrics are conformally Kähler and the associated Kähler class is a multiple of the first Chern class as shown by Batat-Hall-Jizany-Murphy [BHJM15]. Maschler [Mas11] has pointed out that, under suitable assumptions for computational simplicity, these are the only conformally Kähler m-quasi-Einstein metrics in complex dimension greater or equal three.

Remark 6.2.10 The m-quasi-Einstein equation with $m < 0$ also has a geometric interpretation. Chen [Che09b] has observed that it gives rise to conformally Einstein products:

Let (M^n, g_M) and (N^m, g_N) be two connected Riemannian manifolds of positive dimensions with $m + n > 2$ and $u \in C^\infty(M)$. Set $\rho = \exp\left(\frac{u}{m+n-2}\right)$.

Then $(M \times N, \rho^{-2}(g_M + g_N))$ is Einstein if and only if (N, g_N) is Einstein and (M, g_M) is $\left(-\frac{1}{m+n-2}\right)$-quasi-Einstein with respect to u and its quasi-Einstein constant is the Einstein constant of (N, g_N).

Remark 6.2.11 Let $(M \times N, g_M + f^2 g_N, u)$ be a warped product Ricci soliton. Due to a result of Lemes de Sousa-Pina [LdSP17], if f is non-constant, then (N^m, g_N) is necessarily Einstein and the soliton potential u restricts to the base manifold M^n.

An approach to studying Ricci soliton warped products, which is reminisent of the study of the m-quasi-Einstein equation, has been introduced by Feitosa-Freitas Filho-Gomes [FFFG17]: Let $m > 1$ and suppose $u \in C^\infty(M)$.

If $(M \times N, g_M + f^2 g_N, u, \frac{\varepsilon}{2})$ is a gradient Ricci soliton, then

$$\mathrm{Ric}_M + \mathrm{Hess}_M\, u + \frac{\varepsilon}{2} g_M - \frac{m}{f} \mathrm{Hess}_M\, f = 0 \tag{6.14}$$

and (N, g_N) is Einstein, $\mathrm{Ric}_N = \mu g_N$, where

$$\mu = f\Delta f + (m-1)|\nabla f|^2 - \frac{\varepsilon}{2} f^2 - f \nabla u(f). \tag{6.15}$$

Conversely, if (6.14), (6.19) and Hamilton's conservation law

$$\Delta u - |\nabla u|^2 + \frac{m}{f} \nabla u(f) = C + \varepsilon u$$

are satisfied on (M, g_M), and (N, g_N) is Einstein with Einstein constant μ, then the warped product $(M \times N, g_M + f^2 g_N, u, \frac{\varepsilon}{2})$ is a gradient Ricci soliton.

Remark 6.2.12 By setting $u = 0$ and $f = e^{-\tilde{u}/m}$ in (6.14), one recovers the m-quasi-Einstein equation (6.11).

In analogy to Kim-Kim's rigidity result for warped product Einstein metrics with non-positive scalar curvature, Feitosa-Freitas Filho-Gomes [FFFG17] use the above characterisation to prove that any complete steady or expanding warped product Ricci soliton with at least two dimensional fibre, and whose warping function attains both a maximum and a minimum, must be a Riemannian product.

6.2.3 Cohomogeneity one quasi-Einstein metrics

This section considers m-quasi-Einstein manifolds of cohomogeneity one. It studies the initial value problem at a singular orbit following Eschenburg-Wang [EW00] and Buzano [Buz11] and analyses the two summands system using the ideas of chapter 3.

The formulae for the Ricci curvature of a cohomogeneity one manifold in section 2.1 yield that the m-quasi-Einstein equation takes the form

$$-(\delta^{\nabla^t} L_t)^\flat - d(\mathrm{tr}(L_t)) = 0, \tag{6.16}$$

$$-\mathrm{tr}(\dot{L}_t) - \mathrm{tr}(L_t^2) + \ddot{u} - \frac{1}{m}\dot{u}^2 + \frac{\varepsilon}{2} = 0, \tag{6.17}$$

$$-\dot{L}_t - (-\dot{u} + \mathrm{tr}(L_t))L_t + r_t + \frac{\varepsilon}{2}\mathbb{I} = 0, \tag{6.18}$$

and the conservation law (6.12) is given by

$$\ddot{u} + (-\dot{u} + \mathrm{tr}(L))\dot{u} + m\mu e^{2u/m} + m\frac{\varepsilon}{2} = 0. \tag{6.19}$$

Remark 6.2.13 Notice that for $f = e^{-u/m}$ the conservation law is equivalent to

$$\frac{d}{dt}\frac{\dot{f}}{f} = -\left(m\frac{\dot{f}}{f} + \mathrm{tr}(L)\right)\frac{\dot{f}}{f} + \frac{\mu}{f^2} + \frac{\varepsilon}{2},$$

and hence *is* the Einstein equation for the added factor in Kim-Kim's [KK03] warped product construction 6.2.9.

The existence of a singular orbit implies dependences between the quasi-Einstein equations (6.16) - (6.18) and the conservation law (6.19), which in fact reduce this system to (6.18) and (6.19). This generalises an observation due to Back [Bac86] in the Einstein case, see also [EW00], and the proof is based on the calculations in the Ricci soliton case [DW11, Proposition 3.19].

Proposition 6.2.14 *Let M be a connected manifold and g a C^2-Riemannian metric on M. Suppose that G is a compact Lie group which acts isometrically and with cohomogeneity one on (M, g) and that the action has a singular orbit. Let $u \in C^3(M)$ be G-invariant.*

Then (6.18) *implies* (6.16) *and if the conservation law* (6.19) *is satisfied, then* (6.17) *holds as well.*

Proof: The fact that (6.18) implies (6.16) follows as in the Ricci soliton case, cf. [DW11, Proposition 3.19], as the equations are identical.

Let v_t be the relative volume of the principal orbit (P, g_t). Then it follows that $\frac{d}{dt}v = \operatorname{tr}(L)v$ and due to [DW11, Formula (3.16)] there holds

$$\frac{d}{dt}\left(v^2\left(\operatorname{Ric}(N,N)+\frac{\varepsilon}{2}\right)\right)+v^2\left(2\dot{u}\operatorname{tr}(L^2)+\frac{d}{dt}\left(\dot{u}\operatorname{tr}(L)\right)\right)=0.$$

By combining this with $\operatorname{Ric}(N,N) = -\operatorname{tr}(\dot{L}) - \operatorname{tr}(L^2)$ and the conservation law (6.19), one obtains

$$\frac{d}{dt}\left(v^2\left(\operatorname{Ric}(N,N)+\ddot{u}-\frac{1}{m}\dot{u}^2+\frac{\varepsilon}{2}\right)\right)=2\dot{u}v^2\left(\operatorname{Ric}(N,N)+\ddot{u}-\frac{1}{m}\dot{u}^2+\frac{\varepsilon}{2}\right).$$

Therefore $v^2\left(\operatorname{Ric}(N,N)+\ddot{u}-\frac{1}{m}\dot{u}^2+\frac{\varepsilon}{2}\right)$ is a multiple of e^{2u} which vanishes at the singular orbit, and thus vanishes identically. □

The contracted second Bianchi identity has already been used implicitly in the proof of proposition 6.2.14. It is also useful to show that the m-quasi Einstein equation is a quasi-linear elliptic system for the metric and the potential, which will imply strong regularity properties in proposition 6.2.16.

Lemma 6.2.15 *Let (M^n, g) be a Riemannian manifold of dimension n.*

(i) Any 1-form $\omega \in \Omega^1(M)$ satisfies the identities

$$\begin{aligned}\operatorname{tr}_{12}\nabla(\omega\otimes\omega) &= (\operatorname{tr}\nabla\omega)\omega+\nabla_{\omega^\#}\omega,\\ 2\operatorname{tr}_{12}\nabla\delta^*\omega &= \Delta\omega+\nabla(\operatorname{tr}\nabla\omega)+\operatorname{Ric}(\cdot,\omega^\#),\end{aligned}$$

where δ^* is the symmetrised covariant differential.

(ii) If $\omega \in \Omega^1(M)$ satisfies $\operatorname{Ric}+\delta^*\omega-\frac{1}{m}w\otimes\omega+\frac{\varepsilon}{2}=0$ *then also*

$$\Delta\omega+\operatorname{Ric}(\cdot,\omega^\#)+\frac{1}{m}\cdot\left\{d(|\omega|^2)-2\operatorname{tr}(\nabla\omega)\omega-2\nabla_{\omega^\#}\omega\right\}=0.$$

(iii) Any $u \in C^\infty(M)$ which satisfies the m-quasi-Einstein equation satisfies

$$\Delta(du) + \mathrm{Ric}(\cdot, \operatorname{grad} u) - \frac{2}{m}\,(\Delta u)\, du = 0.$$

Proof: (i) Extend an orthonormal basis $e_1, \ldots, e_n$ of T_pM and $X \in T_pM$ via parallel transport to local vector fields. Then there holds $\nabla_{e_i} e_j = \nabla_{e_i} X = 0$ and thus

$$\begin{aligned}
\mathrm{tr}_{12} \nabla(\omega \otimes \omega)(X) &= \sum_i \nabla_{e_i}(\omega \otimes \omega)(e_i, X) \\
&= \sum_i \nabla_{e_i}(\omega(e_i)) w(X) + \sum_i w(e_i) \nabla_{e_i}(w(X)) \\
&= (\mathrm{tr}\, \nabla\omega)\omega(X) + \nabla_{\omega^\#}\omega(X).
\end{aligned}$$

Furthermore, the Ricci identity $(\nabla^2\omega)(Y, X, Z) = (\nabla^2\omega)(X, Y, Z) + [R(X, Y)\omega]\,(Z)$, where $R(X, Y) = \nabla^2_{Y,X} - \nabla^2_{X,Y}$, implies that

$$\begin{aligned}
2\,\mathrm{tr}_{12}(\nabla\delta^*\omega)(X) &= 2\sum_i \nabla(\delta^*\omega)(e_i, e_i, X) = 2\sum_i \nabla_{e_i}\left((\delta^*(e_i, X)\right) \\
&= \sum_i \nabla_{e_i}\left\{(\nabla_{e_i}\omega)(X) + (\nabla_X\omega)(e_i)\right\} \\
&= \sum_i (\nabla^2\omega)(e_i, e_i, X) + \sum_i (\nabla^2\omega)(e_i, X, e_i) \\
&= (\Delta\omega)(X) + \sum_i (\nabla^2\omega)(X, e_i, e_i) + [R(X, e_i)\omega]\,(e_i) \\
&= (\Delta\omega)(X) + \nabla_X(\mathrm{tr}\,\nabla\omega) - \sum_i \omega(R(X, e_i)e_i) \\
&= (\Delta\omega)(X) + \nabla_X(\mathrm{tr}\,\nabla\omega) + \sum_i g(R(X, e_i)\omega^\#, e_i) \\
&= (\Delta\omega)(X) + \nabla_X(\mathrm{tr}\,\nabla\omega) + \mathrm{Ric}(X, \omega^\#).
\end{aligned}$$

(ii) Combining part (i) with $R + \mathrm{tr}\,\nabla\omega - \frac{1}{m}|\omega|^2 + n\frac{\varepsilon}{2} = 0$ and the contracted second Bianchi identity $dR = 2\,\mathrm{tr}_{12} \nabla \,\mathrm{Ric}$ yields

$$\begin{aligned}
-dR &= 2\,\mathrm{tr}_{12} \nabla \left[\delta^*\omega - \frac{1}{m}\omega \otimes \omega\right] \\
&= \Delta\omega + \nabla(\mathrm{tr}(\nabla\omega)) + \mathrm{Ric}(\cdot, \omega^\#) - \frac{2}{m}\left\{\mathrm{tr}(\nabla\omega)\omega + \nabla_{\omega^\#}\omega\right\} \\
&= \Delta\omega - dR + \frac{1}{m} d(|\omega|^2) + \mathrm{Ric}(\cdot, \omega^\#) - \frac{2}{m}\left\{\mathrm{tr}(\nabla\omega)\omega + \nabla_{\omega^\#}\omega\right\}.
\end{aligned}$$

(iii) Consider $\omega = du$ in part (ii) and observe that $d|\operatorname{grad} u|^2 = 2\,\mathrm{Hess}\, u(\operatorname{grad} u, \cdot)$ and $\nabla_{\operatorname{grad} u} du = \mathrm{Hess}\, u(\operatorname{grad} u, \cdot)$. □

Proposition 6.2.16 *Let M be a smooth manifold of dimension* $\dim M \geq 3$. *Suppose that a solution of the m-quasi-Einstein equation on M is given by a C^2-Riemannian metric g and $u \in C^3(M)$. Then g and u are real analytic in harmonic and geodesic normal coordinates.*

Proof: According to lemma 6.2.15, the m-quasi-Einstein equation and the contracted second Bianchi identity give rise to the PDE

$$\begin{aligned} \mathrm{Ric} + \mathrm{Hess}\, u - \frac{1}{m} du \otimes du + \frac{\varepsilon}{2} g = 0, \\ \Delta(du) + \mathrm{Ric}(\cdot, \mathrm{grad}\, u) - \frac{2}{m}(\Delta u)\, du = 0 \end{aligned}$$

for (g, u). Notice that the $\frac{1}{m}$-terms are of lower order and thus the principal symbol is the same as in the Ricci soliton case. Hence, (g, u) is a solution of a quasi-linear elliptic system and the regularity analysis in [DW11, Lemma 3.2] carries over without any changes, see also [DK81, Theorem 5.2]. $\square$

Similarly, the initial value problem at a singular orbit can be treated as in the gradient Ricci soliton case, which has been solved by Buzano [Buz11]. Indeed, due to proposition 6.2.14, it suffices to consider (6.18), (6.19) and the relation $\dot{g}_t = 2g_t L_t$. Then setting up an ODE system for (g_t, L_t, u) as in [Buz11], one observes that the $-\frac{1}{m}\dot{u}^2$-term simply disappears in the error terms that occur in Buzano's proof because it is of lower order. In particular, the construction of a formal power series solution is unchanged. Due to the real analyticity of m-quasi-Einstein metrics in proposition 6.2.16, a theorem of Malgrange [Mal74, Theorème 7.1] then yields a genuine solution. Alternatively, a Picard iteration may be applied as in [EW00].

One may also observe that the m-quasi-Einstein equation gives rise to an Einstein equation that is formally induced by a warped product due to Kim-Kim's [KK03] construction 6.2.9. If $m \notin \mathbb{N}$, then this construction cannot be realised geometrically as m corresponds to the dimension of the fibre. However, due to remark 6.2.13, the resulting equations have otherwise exactly the structure of a cohomogeneity one Einstein equation and hence Eschenburg-Wang's [EW00] ODE analysis applies.

Theorem 6.2.17 *Let G be a compact Lie group acting isometrically on a connected Riemannian manifold (M, g) and suppose there exists a singular orbit $Q = G/H$. Choose $q \in M$ such that $Q = G \cdot q$ and denote by $V = T_qM/T_qQ$ the normal space of Q at q. Then H acts linearly and orthogonally on V and a tubular neighbourhood of Q may be identified with its normal bundle $E = G \times_H V$. The principal orbits are*

$P = G/K = G \cdot v$ for any $v \in V \setminus \{0\}$. These can be identified with the sphere bundle of E (with respect to an H-invariant scalar product on V). Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_-$ be a decomposition of the Lie algebra of G where $\mathfrak{p}_-$ is an Ad_H-invariant complement of $\mathfrak{h} = \mathrm{Lie}(H)$.

Assume that V and $\mathfrak{p}_-$ have no common irreducible factors as K-representations.

Then for any $\varepsilon \in \mathbb{R}$, any $m \in (0, \infty]$, any G-invariant metric g_Q on Q and any shape operator $L\colon E \to \mathrm{Sym}^2(T^*Q)$ there exists a G-invariant m-quasi-Einstein metric on some open disc bundle of E.

Proof: The basic set-up is due to Eschenburg-Wang [EW00]: Observe that a smooth G-invariant symmetric bilinear form $a \in C^\infty(S^2TE)$ is determined by a smooth H-equivariant map

$$a\colon V \to \mathrm{Sym}^2(V \oplus \mathfrak{p}_-).$$

Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_+$ be a reductive decomposition of $\mathfrak{h}$ and set $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Since V and $\mathfrak{p}_-$ are assumed to have no common irreducible factors as K-representations, g_t and L_t split into $+$ and $-$ parts. Then there exist K-invariant endomorphisms $x(t), \eta(t) \in \mathrm{End}(\mathfrak{p})^K$ which preserve the splitting and such that the identities

$$g_t = t^2 x_+(t) \oplus x_-(t) \quad \text{and} \quad L_t = \left(\frac{1}{t} I_+ + \eta_+(t)\right) \oplus \eta_-(t)$$

hold. In particular, $x(t)$, $\eta(t)$ have to satisfy the initial conditions

$$x(0) = I \quad \text{and} \quad \eta_+(0) = 0 \quad \text{and} \; \eta_-(0) = L_1(v_0),$$

where the H-equivariant map $L_1\colon V \to \mathrm{Sym}^2(\mathfrak{p}_-)$ corresponds to the shape operator of Q via evaluation of L on the collapsing sphere, $v_0 \in S^k = H/K$.

Set $y = x\eta$. Then the m-quasi-Einstein equation (6.18), the conservation law (6.19) and $\dot{g}_t = 2g_tL_t$ yield the ODE system

$$\begin{aligned}
\dot{x} &= 2y, \\
\dot{y} &= \frac{1}{t^2} A(x) + \frac{1}{t} B(x, y) + C(x, y, t), \\
\frac{d^3}{dt^3} u &= \frac{1}{t^2} \widetilde{A}(\dot{u}) + \frac{1}{t} \widetilde{B}(\dot{u}, \ddot{u}) + \widetilde{C}(\dot{u}, \ddot{u}, t),
\end{aligned}$$

where

$$\begin{aligned}
A(x) &= (1-k)x_+ + x r_{\mathrm{sing}}, \\
B(x, y) &= -ky - \mathrm{tr}(x^{-1}y)x_+ + \dot{u}x_+, \\
C(x, y, t) &= 2yx^{-1}y + x r_{\mathrm{reg}} - \mathrm{tr}(x^{-1}y)y + \dot{u}y + \frac{\varepsilon}{2} x
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{A}(\dot{u}) &= k\dot{u} + (k-1)\dot{u}\,\mathrm{tr}(x^{-1}x_+) - \dot{u}\,\mathrm{tr}(r_{\mathrm{sing}}),\\
\widetilde{B}(\dot{u},\ddot{u}) =& -k\ddot{u} + k\dot{u}\,\mathrm{tr}(x^{-1}y) + \dot{u}\,\mathrm{tr}(x^{-1}y)\,\mathrm{tr}(x^{-1}x_+) - \mathrm{tr}(x^{-1}x_+)\dot{u}^2 + \frac{2}{m}k\dot{u},\\
\widetilde{C}(\dot{u},\ddot{u},t) =& -\mathrm{tr}(x^{-1}y)\ddot{u} - \mathrm{tr}(r_{\mathrm{reg}})\dot{u} + \mathrm{tr}(x^{-1}y)\,\mathrm{tr}(x^{-1}y)\dot{u} - \mathrm{tr}(x^{-1}y)\dot{u}^2,\\
& -(n+1)\frac{\varepsilon}{2}\dot{u} + 2\ddot{u}\dot{u} + \varepsilon\dot{u} + \frac{2}{m}\left\{\ddot{u} + \mathrm{tr}(x^{-1}y)\dot{u} - \dot{u}^2\right\}\dot{u},
\end{aligned}$$

and $k = \dim H/K$ is the dimension of the collapsing sphere.

The corresponding initial conditions

$$\begin{aligned}
A(x(0)) = 0 \quad &\text{and} \quad 2(dA)_{x(0)} \cdot y(0) + B(x(0), y(0)) = 0,\\
\widetilde{A}(\dot{u}(0)) = 0 \quad &\text{and} \quad (d\widetilde{A})_{\dot{u}(0)}\ddot{u}(0) + \widetilde{B}(\dot{u}(0), \ddot{u}(0)) = 0
\end{aligned}$$

follow easily from the $\frac{1}{t}$- and $\frac{1}{t^2}$-dependencies of the coefficients.

However, up to the additional $\frac{1}{m}$-terms, which are of lower order, the expressions agree with the Ricci soliton case covered by Buzano [Buz11]. In particular, only the error terms in [Buz11, Formulae (5.11) and (5.12)] are changed but the operators $\mathcal{L}$, $\widetilde{\mathcal{L}}$ of [Buz11, Formula (5.13)] agree in the Ricci soliton and m-quasi-Einstein case. Therefore the construction of a formal power series solution carries over without changes and due to the real analyticity of m-quasi-Einstein metrics in proposition 6.2.16, a theorem of Malgrange [Mal74, Theorème 7.1] then yields a genuine solution. Alternatively, a Picard iteration may be applied as in [EW00]. $\square$

Remark 6.2.18 The assumption that V and $\mathfrak{p}_-$ have no common irreducible factors as K-representations is primarily a technical simplification but as Eschenburg-Wang point out in [EW00, Remark 2.7] it is also natural in the context of the Kaluza-Klein construction.

The analysis of the two summands case in chapter 3 can easily be adapted to the m-quasi-Einstein case.

Recall from section 3.1 that in the two summands case the metric on the principal orbit is given by $g_t = f_1(t)^2 g_S + f_2(t)^2 g_Q$, and set $f_3 = e^{-u/m}$. Due to proposition 6.2.14, it suffices to consider (6.18) and (6.19), and thus the two summands m-quasi-

Einstein equations take the form

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\dot{f}_1}{f_1}\right) &= -\operatorname{tr}(\widehat{L})\frac{\dot{f}_1}{f_1} + \frac{\varepsilon}{2} + \frac{A_1}{d_1}\frac{1}{f_1^2} + \frac{A_3}{d_1}\frac{f_1^2}{f_2^4},\\
\frac{d}{dt}\left(\frac{\dot{f}_2}{f_2}\right) &= -\operatorname{tr}(\widehat{L})\frac{\dot{f}_2}{f_2} + \frac{\varepsilon}{2} + \frac{A_2}{d_2}\frac{1}{f_2^2} - 2\frac{A_3}{d_2}\frac{f_1^2}{f_2^4},\\
\frac{d}{dt}\left(\frac{\dot{f}_3}{f_3}\right) &= -\operatorname{tr}(\widehat{L})\frac{\dot{f}_3}{f_3} + \frac{\varepsilon}{2} + \frac{\mu}{f_3^2},
\end{aligned}$$

where $\widehat{L} = \operatorname{diag}\left(\frac{\dot{f}_1}{f_1}\mathbb{I}_{d_1}, \frac{\dot{f}_2}{f_2}\mathbb{I}_{d_2}, \frac{\dot{f}_3}{f_3}\mathbb{I}_m\right)$ corresponds to the shape operator in Kim-Kim's [KK03] warped product construction. Notice that $\widehat{L}$ is only well-defined if $m \in \mathbb{N}$, but its trace always is.

The metric can be smoothly extended over the singular orbit if the initial conditions

$$f_1(0) = 0,\ \dot{f}_1(0) = 1 \ \text{ and } \ f_2(0) = \bar{f} > 0,\ \dot{f}_2(0) = 0$$

are imposed. Clearly one may fix $u(0) = 0$ and then

$$f_3(0) = 1 \ \text{ and } \ \dot{f}_3(0) = 0 \ \text{ and } \ \ddot{f}_3(0) = \varepsilon + 2\mu$$

are the corresponding smoothness conditions for f_3.

Fix $\varepsilon \geq 0$ and $\mu > 0$. Then it follows that $\dot{f}_i(t) > 0$ for $i = 1, 2, 3$ and sufficiently small $t > 0$, and due to proposition 5.1.3, it suffices to show that $\dot{f}_i(t) > 0$ is preserved for $i = 1, 2, 3$ to obtain a complete metric.

However, $\dot{f}_1, \dot{f}_3 > 0$ is immediate and $\dot{f}_2 > 0$ follows if $\omega = \frac{f_1}{f_2} < \sqrt{\frac{A_2}{2A_3}}$ is preserved. As in section 3.2, consider the function

$$\mathcal{K} = \frac{1}{2}\omega^{2(d_1-1)}\left(\dot{f}_1 - f_1\frac{\dot{f}_2}{f_2}\right)^2 - \widehat{\mathcal{G}}(\omega),$$

with $\widehat{\mathcal{G}}$ as in (3.11), and notice that

$$\frac{d}{dt}\mathcal{K} = -(n-1)\frac{\dot{f}_2}{f_2}\omega^{2(d_1-1)}\left(\dot{f}_1 - f_1\frac{\dot{f}_2}{f_2}\right)^2.$$

In particular, $\mathcal{K}$ is non-increasing if $\dot{f}_2 > 0$ and hence the arguments in the proofs of propositions 3.2.5 and 3.3.1 carry over and show:

Theorem 6.2.19 *Let $d_1 \geq 1$, $A_1 = d_1(d_1-1)$ and $(d_1+1)A_2^2 > 4d_1d_2(2d_1+d_2)A_3 > 0$ and fix $m > 0$.*

Then the associated two summands ODE gives rise to a 1-parameter family of non-trivial, complete, non-homothetic m-Bakry-Émery Ricci flat metrics and a 2-parameter family of non-trivial, complete, non-homothetic m-quasi Einstein metrics with quasi-Einstein constant $-\frac{\varepsilon}{2} < 0$, all of which have positive characteristic constant.

Furthermore, there exists a 1-parameter family of complete, non-trivial m-quasi-Einstein metrics with quasi-Einstein constant $-\frac{\varepsilon}{2} < 0$ and vanishing characteristic constant.

Completeness of the corresponding Ricci soliton metrics has been established in proposition 3.2.9 and corollary 3.2.12, and the corresponding Ricci flat metrics and Einstein metrics of negative scalar curvature have been constructed by Böhm [Böh99].

Remark 6.2.20 Case [Cas10] has shown that any complete, non-trivial m-Bakry-Émery Ricci flat quasi-Einstein manifold has positive characteristic constant.

Notice that if $A_3 = 0$ and $m \in \mathbb{N}$, the above construction gives rise to a triply warped product Einstein metric.

Multiple warped product Einstein metrics of non-positive scalar curvature have been constructed by Böhm [Böh99] on $\mathbb{R}^{d_1+1} \times M_2 \times \ldots \times M_r$ if $d > 1$, for Einstein manifolds (M_i, g_i) of positive scalar curvature $\mu_i > 0$. (Böhm actually chose each (M_i, g_i) to be a non-flat, compact, isotropy irreducible homogeneous spaces, in order to obtain proper cohomogeneity one manifolds.) The corresponding steady and expanding Ricci solitons have been constructed by Dancer-Wang [DW09b, DW09a] who in joint work with Buzano and Gallaugher [BDGW15a, BDW15] have also settled the case $d_1 = 1$. These authors also describe the asymptotic behaviour of the metric at infinity.

Away from the singular orbit, on $(0, \infty) \times S^{d_1-1} \times M_2 \times \ldots \times M_r$, the metrics are of the form $dt^2 + \sum_{i=1}^r f_i^2(t) g_i$. Notice that the corresponding m-quasi-Einstein equations are

$$\frac{d}{dt}\frac{\dot{f}_i}{f_i} = -(-\dot{u} + \mathrm{tr}(L))\frac{\dot{f}_i}{f_i} + \frac{\varepsilon}{2} + \frac{\mu_i}{f_i^2} \quad \text{for} \quad i = 1, \ldots, r,$$

where $\mu_1 = d_1 - 1$. If $m < \infty$ then $f_{r+1} = e^{-u/m}$ satisfies an analogous equation, where $\mu_{r+1} > 0$ will be the characteristic constant of the induced m-quasi-Einstein metric.

As before, by imposing the initial conditions $f_1 = 0$, $\dot{f}_1 = 1$ and $f_i > 0$ for $i \geq 2$ at $t = 0$ and by requiring that $f_i(t)$ for $i \geq 2$ and $u(t)$ are even, one induces a smooth m-quasi-Einstein metric on the trivial $\mathbb{R}^{d_1+1}$-bundle over $M_2 \times \ldots \times M_r$.

Therefore, as $\dot{f}_i > 0$ is trivially preserved if $\varepsilon \geq 0$, proposition 5.1.3 yields the corresponding result for m-quasi-Einstein metrics and hence a unified proof of the works of Böhm [Böh99] and Buzano-Dancer-Gallaugher-Wang [DW09b,DW09a,BDGW15a, BDW15].

Theorem 6.2.21 *Let $M_2, \ldots, M_r$ be Einstein manifolds of positive scalar curvature. Let $d_1 \geq 1$ and $m \in (0, \infty]$.*

Then there is an $r-1$ parameter family of non-trivial, non-homothetic, complete, smooth m-Bakry-Émery Ricci flat quasi-Einstein manifolds and an r parameter family of non-trivial, non-homothetic, complete, smooth m-quasi-Einstein manifolds with quasi-Einstein constant $-\frac{\varepsilon}{2} < 0$ on $\mathbb{R}^{d_1+1} \times M_2 \times \ldots \times M_r$.

Chapter 7

Uniform Poincaré inequalities for differentials on Riemann surfaces

This chapter is based on joint work with Melanie Rupflin.

Let (M, g) be a compact Riemannian manifold and fix $p \in [1, \infty)$. Then there exists a constant $C > 0$ such that

$$\|u - \bar{u}\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

for all $u \in C^\infty(M)$, where $\bar{u} = \frac{1}{\mathrm{Vol}(M)} \int_M u$ is the average of u. However, the constant C does depend on the manifold, the metric and the L^p-norm. In the case $p = 2$ the optimal constant C is the inverse of the smallest positive eigenvalue of the Laplace operator of (M, g).

If (M, c) is a compact Riemann surface with complex structure c and genus $\gamma \geq 2$, there exists a unique hyperbolic metric g on M that is compatible with c. However, the constant C in the associated Poincaré inequality for functions on M still depends on the choice of the complex structure.

In contrast, Rupflin-Topping [RT15] proved a *uniform* L^1-Poincaré inequality for quadratic differentials on closed hyperbolic surfaces. That is, the associated Poincaré constant only depends on the genus of M but not on the complex structure.

In this chapter uniform L^p-Poincaré inequalities for k-differentials, $k \geq 2$, are proven. Examples show that there is no corresponding uniform Poincaré inequality for 1-differentials.

Let (M, c) be a compact Riemann surface of genus $\gamma \geq 2$, K its canonical bundle and g the hyperbolic metric compatible with c. Then $\Omega^0(M, K^k)$ and $H^0(M, K^k)$ are the spaces of smooth and holomorphic k-differentials, respectively.

Let z be a local complex coordinate on M. Then locally the metric is given by $g = \rho^2 dz d\bar{z}$ and any $\Psi \in \Omega^0(M, K^k)$ can be written as

$$\Psi = \psi dz^k,$$

where ψ is a smooth function. Furthermore, its antiholomorphic derivative $\bar{\partial}\Psi$ is

$$\bar{\partial}\Psi = \partial_{\bar{z}}\psi d\bar{z} \otimes dz^k$$

and its L^p-norm can be calculated via

$$||\psi dz^k||^p_{L^p(M,g)} = \int_M |\psi|^p |dz^k|^p dv_g = 2^{kp/2} \int_M |\psi|^p \rho^{2-pk} \frac{i}{2} dz \wedge d\bar{z}.$$

Notice that the associated L^2-inner product

$$\langle \phi dz^k, \psi dz^k \rangle_{L^2(M,g)} = \int_M \phi\bar{\psi} |dz^k|^2 dv_g$$

induces an L^2-orthogonal projection $P_g \colon \Omega^0(M, K^k) \to H^0(M, K^k)$.

The main theorem of this chapter is:

Theorem 7.0.1 *Let (M, c) be a closed Riemann surface of genus $\gamma \geq 2$, $p \in [1, \infty)$ and $k > 1$ a natural number. Then there exists a constant $C = C(\gamma, p, k) < \infty$ such that for any k-differential $\Psi \in \Omega^0(M, K^k)$ there holds*

$$||\Psi - P_g(\Psi)||_{L^p(M,g)} \leq C \cdot ||\bar{\partial}\Psi||_{L^p(M,g)},$$

where g is the hyperbolic metric compatible with the complex structure c on M.

Observe that the total derivative ∇ from the Poincaré inequality for functions has been replaced by the $\bar{\partial}$-operator. In particular, the associated kernels have changed from constant functions to holomorphic differentials.

Proposition 7.1.12 shows that the estimate cannot be uniform for any $p \in [1, \infty)$ if $k = 1$. Furthermore, the argument will show that Poincaré-Sobolev type estimates hold on a fixed hyperbolic Riemann surface (M, c, g) but these cannot be uniform either, see example 7.1.13.

The proof mainly follows [RT15], which is based on the structure theory for quadratic differentials on degenerating sequences of hyperbolic surfaces from [RTZ13]. Section 7.1 extends the main results to k-differentials and provides proofs unless the argument carries over without any changes. The proof of the main theorem 7.0.1 is carried out in section 7.2.

7.1 Deligne-Mumford compactness and k-differentials

In this section, the structure theory for degenerating sequences of compact hyperbolic Riemann surfaces is extended to the associated spaces of differentials. In particular, a splitting theorem for the space of holomorphic k-differentials identifies the subspaces that converge to the space of integrable holomorphic k-differentials on the limiting hyperbolic punctured surface. This is analogous to the case of quadratic differentials which is due to Rupflin-Topping-Zhu [RTZ13].

The space of integrable holomorphic k-differentials on a hyperbolic punctured surface has the following characterisation.

Theorem 7.1.1 *Suppose (Σ, h) is a hyperbolic punctured surface, $\Psi \in H^0(\Sigma, K^k)$ a holomorphic k-differential on Σ and $k \geq 1$. Then $\Psi \in L^1(\Sigma, h)$ if and only if at any puncture Ψ has a pole of order at most $k-1$.*

Proof: According to [Hum97, Proposition IV.4.2], for each puncture there is a neighbourhood $U \subset \Sigma$ which is isometric to the punctured hyperbolic disc $\hat{D}$ given by

$$\left(D_{e^{-\pi}}(0) \setminus \{0\}, \frac{1}{(|z| \log |z|)^2} dz^2 \right).$$

Thus, if the local representation $\Psi = \psi dz^k$ holds on U, then $|\Psi|_h = 2^{k/2} |\psi| (|z| \log |z|)^k$ and hence

$$||\Psi||_{L^1(U,h)} = 2^{k/2} \int_{\hat{D}} |\psi| (|z| \log |z|)^{k-2} dm(z).$$

Then $||\Psi||_{L^1(\Sigma,h)} < \infty$ implies that the function $|\psi|(|z| \log |z|)^{k-2}$ can have at most a simple pole, cf. [RTZ13, Lemma A.11], and therefore ψ can have a pole of order at most $k-1$. With these observations, the converse is also clear. $\square$

In the following, the space of integrable holomorphic k-differentials on a hyperbolic punctured surface (Σ, c, h) will be denoted by

$$\mathcal{H}ol_k(\Sigma, h) = H^0(\Sigma, K^k) \cap L^1(\Sigma, h).$$

Corollary 7.1.2 *Suppose that (Σ, c, h) is a complete hyperbolic surface of genus γ with m punctures.*

Then the space of integrable holomorphic k-differentials has dimension

$$\dim \mathcal{H}ol_k(\Sigma, h) = (2k-1)(\gamma-1) + (k-1)m$$

if $k, m \geq 1$ or if $m \geq 0$ and $k \geq 2$.

Proof: In view of theorem 7.1.1, consider the space of holomorphic sections of the k-th power of the canonical bundle on a compact Riemann surface of genus γ with poles of order at most $k-1$ at the punctures $z_1, \ldots, z_m$. The Riemann-Roch theorem applied to the divisor $D = \sum_{j=1}^{m}(k-1)z_j$ yields

$$\begin{aligned} h^0(K^k \otimes \mathcal{O}(D)) &= \deg(K^k \otimes \mathcal{O}(D)) + 1 - \gamma + h^0(K^{-k+1} \otimes \mathcal{O}(-D)) \\ &= \deg(K^k) + \deg(D) + 1 - \gamma + h^0(K^{-k+1} \otimes \mathcal{O}(-D)). \end{aligned}$$

Recall that $\deg(K) = 2(\gamma-1)$ and observe that $\deg D = (k-1)m$. Hence, under the assumptions in the theorem, one has

$$\deg(K^{-k+1} \otimes \mathcal{O}(-D)) = 2(1-k)(\gamma-1) - \deg D < 0,$$

which implies $h^0(K^{-k+1} \otimes \mathcal{O}(-D)) = 0$ and the claim. $\square$

Recall that a closed oriented hyperbolic surface M of genus $\gamma \geq 2$ can be decomposed into pairs of pants by a collection of $3(\gamma-1)$ disjoint simple closed geodesics, cf. [Hum97], but the dimension of the space of holomorphic 1-differentials on M satisfies $\dim H^0(M, K) = \gamma < 3(\gamma-1)$ since $\gamma \geq 2$. This is a key insight for the construction of sequences of 1-differentials that cannot satisfy a uniform Poincaré estimate for any $p \in [1, \infty)$, see proposition 7.1.12.

The compactness theorem due to Deligne and Mumford gives a precise description of how a sequence of closed hyperbolic Riemann surfaces of fixed genus $\gamma \geq 2$ pinches $1 \leq m \leq 3(\gamma-1)$ simple closed curves and thus degenerates to a hyperbolic punctured Riemann surface. Notice that $3(\gamma-1)$ is in fact the maximum number of simple closed geodesics of length $< 2\,\mathrm{arsinh}(1)$ on a hyperbolic Riemann surface of genus $\gamma \geq 2$, cf. [Hum97].

Theorem 7.1.3 (Deligne-Mumford compactness, [DM69]) *Let (M, g_i, c_i) be a sequence of closed hyperbolic Riemann surfaces of fixed genus and let $l(g_i)$ denote the length of the shortest simple closed geodesic on (M, g_i).*

1. *Suppose that $l(g_i) > \varepsilon > 0$ for all i. Then there is a hyperbolic Riemann surface (M, c, h) of the same genus such that, after passing to a subsequence, (M, c_i, g_i) converges to (M, c, h) in the following sense:*

   There exist diffeomorphisms $f_i \colon M \to M$ such that $f_i^* g_i \to h$ and $f_i^* c_i \to c$ smoothly on M as $i \to \infty$.

2. *Suppose that $l(g_i) \to 0$ as $i \to \infty$. Then there exists a hyperbolic punctured Riemann surface (Σ, c, h) such that, after passing to a subsequence, (M, c_i, g_i) degenerates to (Σ, c, h) in the following sense:*

   There exist pairwise disjoint, homotopically non-trivial, simple closed geodesics $\{\sigma_i^j\}_{j=1}^m$ on (M, c_i, g_i) of length $l(\sigma_i^j) \to 0$ as $i \to \infty$ as well as diffeomorphisms $f_i \colon \Sigma \to M \setminus \cup_{j=1}^m \sigma_i^j$ such that $f_i^* g_i \to h$ and $f_i^* c_i \to c$ smoothly locally on Σ as $i \to \infty$.

Remark 7.1.4 If the hyperbolic surface (Σ, h) is obtained from a sequence (M, c_i, g_i) of hyperbolic surfaces by collapsing m closed, disjoint, homotopically non-trivial curves, then corollary 7.1.2 implies that

$$\dim \mathcal{H}ol_k(\Sigma, h) = \dim H^0(M, K^k) - m.$$

For the proof observe that Σ is not connected if a separating curve is collapsed. However, in this case the genera of the two parts satisfy the relation $\gamma_1 + \gamma_2 = \gamma$. On the other hand, a collapse of a non-separating curve reduces the genus by one. In contrast, any such collapse increases the number of punctures by two.

The pinching of a simple closed geodesic on a hyperbolic surface can be studied very concretely. The Keen-Randol collar lemma identifies a cylindrical collar region around the curve and provides an explicit coordinate description of this neighbourhood. This is very useful to prove estimates on the degenerating parts of hyperbolic surfaces.

Lemma 7.1.5 (Keen-Randol Collar Lemma, [Ran79]) *Let (M, g) be a closed hyperbolic Riemann surface and let σ be a simple closed geodesic of length $l > 0$. Then there exists a neighbourhood $\mathcal{C}$ of σ in (M, g) which is isometric to the cylinder $\mathcal{C}(l) = ((-X(l), X(l)) \times S^1, \rho^2(ds^2 + d\theta^2))$, where*

$$X(l) = \frac{2\pi}{l}\left(\frac{\pi}{2} - \arctan\left(\sinh\left(\frac{l}{2}\right)\right)\right) \quad \textit{and} \quad \rho(s) = \frac{l}{2\pi \cos(\frac{l}{2\pi} s)}.$$

In particular, the geodesic σ corresponds to the circle $\{0\} \times S^1 \subset \mathcal{C}(l)$.

Moreover, for $0 < l < \operatorname{arsinh}(1)$ the δ-thin part of the collar, i.e. the set of points on the cylinder with injectivity radius strictly less than δ, is given by the subcylinder $((-X_\delta(l), X_\delta(l)) \times S^1 \subset \mathcal{C}(l)$ with

$$X_\delta(l) = \frac{2\pi}{l}\left(\frac{\pi}{2} - \arcsin\left(\frac{\sinh(l/2)}{\sinh(\delta)}\right)\right)$$

if $l/2 \leq \delta < \operatorname{arsinh}(1)$ and is empty otherwise.

Remark 7.1.6 The function ρ is of order δ at X_δ in the sense that there is a uniform constant $c < \infty$ such that

$$c^{-1}\delta \leq \rho(X_\delta(l)) \leq c\delta$$

for any $0 < l < 2\operatorname{arsinh}(1)$ and any $\delta \in (l/2, \operatorname{arsinh}(1))$, see [RT15, Formula (2.6)]. Furthermore, it is worth noting that

$$\int_{-X}^{X} \rho^{-\beta}(s)ds \sim l^{-\beta-1} \text{ for any } \beta > -1 \text{ and } \int_{-X}^{X} \rho(s)ds \sim |\log(l)|$$

as $l \to 0$.

Suppose that ψdz^k is the local coordinate representation of the holomorphic k-differential Ψ on a hyperbolic collar $(\mathcal{C}, g)$. Then the Laurent series

$$\psi(s,\theta) = \sum_{n=-\infty}^{\infty} b_n e^{ns} e^{in\theta}$$

induces an $L^2(\mathcal{C}, g)$-orthogonal decomposition of Ψ into the *principal part* $b_0(\Psi)dz^k$ and the *collar decay part* $\omega^\perp(\Psi) = \Psi - b_0(\Psi)dz^k$, whose name originates from the following lemma.

Lemma 7.1.7 *Let $\Psi = \psi dz^k$ be a holomorphic k-differential on the collar $(\mathcal{C}(l), g)$ for $0 < l < \operatorname{arsinh}(1)$. Then there exist $\delta_0 > 0$ and $C < \infty$ such that for any $0 < \delta < \delta_0$*

$$||\omega^\perp(\Psi)dz^k||_{L^\infty(\delta-thin(\mathcal{C}(l)))} \leq C \cdot \delta^{-k} e^{-\pi/\delta} ||\Psi||_{L^2(\delta_0-thick(\mathcal{C}(l)))}$$

Proof: Following [RTZ13, Proof of Lemma 2.2], fix $0 < \delta_0 < \operatorname{arsinh}(1)$ with the property that $X(l) - X_{\delta_0}(l) \geq 1$ for all $0 < l < l_0 := 2\delta_0$. Notice that for $l \geq l_0$ the δ-thin part of $\mathcal{C}(l)$ is empty. Normalise $\omega^\perp(\Psi)dz^k$ so that $||\Psi||_{L^2(\delta_0-thick(\mathcal{C}(l)))} = 1$.

Then it follows as in [RTZ13, Proof of Lemma 2.2] that there exists a constant $c = c(k) < \infty$ such that for any $0 < \delta < \delta_0$ there holds

$$\begin{aligned} ||\omega^{\perp}(\Psi)dz^k||_{L^\infty(\delta-thin(\mathcal{C}(l)))} \leq \ & c \sum_{n\in\mathbb{Z}\setminus\{0\}} \sqrt{n}e^{-|n|X(l)} \sup_{s\in[-X_\delta(l),X_\delta(l)]} e^{ns}|dz^k|(s) \\ = \ & 2^{k/2}c \sum_{n\in\mathbb{Z}\setminus\{0\}} \sqrt{n}e^{-|n|X(l)} \sup_{s\in[-X_\delta(l),X_\delta(l)]} e^{ns}\rho^{-k}(s). \end{aligned}$$

Notice that the supremum can be estimated by $C(k)e^{|n|X_\delta}\rho^{-k}(X_\delta)$, which proves the claim as $X_\delta(l) = \frac{\pi^2}{l} - \frac{\pi}{\delta} + O(1)$ and ρ is of order δ at X_δ according to remark 7.1.6. $\square$

Remark 7.1.8 For any $k \in \mathbb{N}$, $\delta > 0$ and any closed surface M there exists a constant $C < \infty$ depending only on k, δ and the genus of M such that for every hyperbolic metric g on M and any $1 \leq p \leq \infty$ there holds

$$||\Psi||_{L^p(\delta-thick(M,g))} \leq C \cdot ||\Psi||_{L^1(M,g)}$$

for any holomorphic differential Ψ. This is analogous to [RTZ13, Lemma A.8].

Theorem 7.1.9 below is the main structure theorem for differentials on degenerating sequences of hyperbolic surfaces. The proof follows as in [RTZ13, Proofs of Lemma 2.4 and Theorem 2.6]. The main ingredients are the dimension formula of proposition 7.1.2 and the collar decay lemma 7.1.

Theorem 7.1.9 *Suppose that (M, c_i, g_i) is a sequence of hyperbolic surfaces that degenerates to the punctured hyperbolic surface (Σ, c, h) by collapsing the collars $\mathcal{C}_i^j$, $j = 1, \dots, m$, each corresponding to a simple closed geodesic of length $l_i^j \to 0$ as $i \to \infty$, as described in theorem 7.1.3. Let $k \geq 1$ and let*

$$W_i = \{\Psi \in H^0(M, K^k) \mid b_0(\Psi, \mathcal{C}_i^j)dz^k = 0 \ \text{ for all } \ j = 1, \dots, m\}$$

be the space of integrable holomorphic k-differentials with vanishing principal part on every collar. It has the following properties:

1. *There is a constant $C < \infty$ depending only on k and the genus of M such that for any $\delta > 0$ with δ-thin$(M, g_i) \subset \cup_{j=1}^m \mathcal{C}_i^j$ there holds*

$$||w||_{L^\infty(\delta-thin(M,g_i))} \leq C \cdot \delta^{-k} e^{-\pi/\delta} ||w||_{L^2(M,g_i)} \ \text{for all } w \in W_i.$$

2. *There exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ one has*

$$\dim W_i = (2k-1)(\gamma-1) - m.$$

3. *Suppose that the diffeomorphisms $f_i\colon \Sigma \to M \setminus \cup_{j=1}^m \sigma_i^j$ describe the convergence of M to Σ as in theorem 7.1.3.*

   (a) *The sequence W_i converges to $\mathcal{H}ol_k(\Sigma, h)$ in the following sense: For any $\Psi \in \mathcal{H}ol_k(\Sigma, h)$ there exist $\Psi_i \in W_i$ such that*

   $$f_i^*\Psi_i \to \Psi \text{ in } \Omega^0_{loc}(\Sigma, K^k) \text{ and } \lim_{i\to\infty} ||\Psi_i||_{L^p(M,g_i)} = ||\Psi||_{L^p(\Sigma,h)}$$

   for any $1 \le p \le \infty$.

   (b) *The corresponding L^2-orthogonal projections converge, that is, if $f_i^*\Psi_i \to \Psi$ locally in $L^1(\Sigma, h)$ as $i \to \infty$, then*

   $$f_i^*(P_{g_i}^{W_i}\Psi_i) \to P_h^{\mathcal{H}ol_k(\Sigma,h)}\Psi \text{ locally smoothly as } i \to \infty \text{ and moreover}$$
   $$\lim_{i\to\infty} ||P_{g_i}^{W_i}\Psi_i||_{L^p(M,g_i)} = ||P_h^{\mathcal{H}ol_k(\Sigma,h)}\Psi||_{L^p(\Sigma,h)}$$

   for any $1 \le p \le \infty$.

The degeneration of the collars can thus also be seen on the level of differentials. In fact, there are natural differentials that describe this process and which in particular concentrate on the collapsing collars.

Theorem 7.1.10 *Let (M, g) be a closed hyperbolic surface and $k \ge 1$. Then there exists a constant $C < \infty$ depending only on k and on the genus of M such that on any collar $\mathcal{C}$ that corresponds to a simple closed geodesic of length $0 < l \le 2\,\mathrm{arsinh}(1)$ there exist $\Omega \in H^0(M, K^k)$ with $||\Omega||_{L^2(M,g)} = 1$ and $b_0 \in \mathbb{R}$ which satisfy*

$$||\Omega||_{L^\infty(M\setminus\mathcal{C},g)} \le Cl^{1/2},$$
$$||\Omega - b_0 dz^k||_{L^\infty(\mathcal{C},g)} \le Cl^{1/2},$$
$$\big|1 - |b_0|\; ||dz^k||_{L^2(\mathcal{C},g)}\big| \le Cl^{1/2}.$$

Proof: This follows as in [RT15, Lemma 2.6]. The only changes in the proof are due to the k-dependence of the norms of the differentials. □

Remark 7.1.11 In fact, for a degenerating of a sequence of hyperbolic surfaces as in theorem 7.1.9, there exist $\Omega_i^1, \dots, \Omega_i^n \in W_i^\perp$ which concentrate on the different degenerating collars $\mathcal{C}_i^1, \dots, \mathcal{C}_i^n$ in the following sense:

The principal parts satisfy $b_0(\Omega_i^a, \mathcal{C}_i^b) \neq 0$ if and only if $a = b$. For each collar $\mathcal{C}_i^j$ the conclusion of theorem 7.1.10 is satisfied by Ω_i^j and the constant b_0 is the principal part $b_0(\Omega_i^j, \mathcal{C}_i^j) > 0$. Furthermore, the above estimates hold with an improved bound of $l^{3/2}$ and the elements are approximately L^2-orthogonal in the sense that $\langle \Omega_i^a, \Omega_i^b \rangle_{L^2(M,g_i)} \leq C(l_i^a)^{3/2}(l_i^b)^{3/2}$, where l_i^j denotes the length of the shortest simple closed geodesic on the respective collar. This can be shown as in [RT18, Lemma 4.5].

Proposition 7.1.12 *Fix $\gamma \geq 2$ and $1 \leq p < \infty$. Let $\mathcal{M}_\gamma(\varepsilon)$ denote the set of all hyperbolic surfaces (M,g) of genus γ and injectivity radius inj$(M,g) \geq \varepsilon > 0$. Then it follows that*

$$\sup_{(M,g)\in\mathcal{M}_\gamma(\varepsilon)} \sup_{\Psi\in H^0(M,K)} \frac{||\Psi - P_g(\Psi)||_{L^p(M,g)}}{||\bar{\partial}\Psi||_{L^p(M,g)}} \to \infty$$

as $\varepsilon \to 0$.

Proof: Recall that the number of geodesics which are necessary to decompose M into pairs of pants is $3(\gamma-1)$. Pick a Riemann surface such that all geodesics $\sigma_1, \ldots, \sigma_{3(\gamma-1)}$ in this decomposition are of the same length $l > 0$.

Furthermore, let $\Omega^1, \ldots, \Omega^\gamma$ be an $L^2(M,g)$-orthonormal basis of $H^0(M,K)$ and $\alpha \in C_c^\infty(-X(l), X(l))$ with $\alpha \equiv 1$ on $(-X(l)+1, X(l)-1)$ and with uniformly bounded derivative. Set

$$\Psi = \sum_{j=1}^{3(\gamma-1)} c_j \alpha(s_j) \mathbb{1}_{\mathcal{C}(\sigma_j)} dz_j,$$

where $c_j \in \mathbb{R}$ are constants to be chosen later and (s_j, θ_j), where $z_j = s_j + i\theta_j$ are the standard coordinates on the collar $\mathcal{C}(\sigma_j)$ as in lemma 7.1.5.

It follows that

$$\begin{aligned}
\langle \Psi, \Omega^i \rangle_{L^2(M,g)} &= \sum_{j=1}^{3(\gamma-1)} c_j \langle \alpha dz, \Omega^i \rangle_{L^2(\mathcal{C}(\sigma_j))} \\
&= \sum_{j=1}^{3(\gamma-1)} c_j \langle \alpha dz, b_0^j(\Omega^i) dz \rangle_{L^2(\mathcal{C}(\sigma_j))} \\
&= \int_{\mathcal{C}(l)} \alpha |dz|^2 dv_g \cdot \sum_{j=1}^{3(\gamma-1)} c_j b_0^j(\Omega^i)
\end{aligned}$$

and therefore there holds $P_g(\Psi) = 0$ if and only if $(c_1, \ldots c_{3(\gamma-1)})$ is a solution of the system of linear equations

$$\sum_{j=1}^{3(\gamma-1)} c_j b_0^j(\Omega^i) = 0,$$

where $i = 1, \dots, \gamma$. Notice that there exists a non-trivial solution since $\gamma < 3(\gamma - 1)$ for $\gamma \geq 2$ and one may assume $\max |c_j| = 1$.

By construction there holds

$$||\bar{\partial}\Psi||_{L^p} \leq C_p$$

for a constant $C_p > 0$ which is independent of the length $l > 0$ of the geodesics σ_j. On the other hand one has

$$||\Psi||_{L^1} \geq \int_{\mathcal{C}(l)} |\alpha| \cdot |dz| dv_g \geq c \int_{-X(l)+1}^{X(l)-1} \rho ds \geq c|\log(l)|,$$

which is unbounded as $l \to 0$. By the Gauss-Bonnet theorem the surfaces $(M, g) \in \mathcal{M}_\gamma$ have uniformly bounded area, which yields the claim for any $p \in [1, \infty)$. $\square$

Example 7.1.13 Let (M, g) be a compact hyperbolic Riemann surface and $\mathcal{C} = \mathcal{C}(l)$ be a collar of (M, g).

Suppose that $\alpha \colon [(-X(l), X(l)] \to \mathbb{R}$ is an odd function which piecewise linearly interpolates $\alpha(0) = \alpha(X) = 0$ and $\alpha(X/2) = 1$. On $\mathcal{C}$ let $\Psi = \alpha dz^k$ and otherwise set $\Psi \equiv 0$. Then it follows that

$$\begin{aligned} ||\Psi||^p_{L^p(M,g)} &\geq c \int_{\frac{X}{4} \leq |s| \leq \frac{3X}{4}} |dz|^p dv_g \geq c l^{1-kp}, \\ ||\bar{\partial}\Psi||_{L^1(M,g)} &\leq \frac{c}{X} \int_{-X}^{X} \rho^{2-(k+1)} ds \leq c l^{1-k} \end{aligned}$$

for $l > 0$ small enough. Furthermore Ψ satisfies $P_g(\Psi) = 0$ as for any holomorphic k-differential $\Omega \in H^0(M, K^k)$ there holds

$$\langle \Psi, \Omega \rangle_{L^2(M,g)} = \langle \Psi, \Omega \rangle_{L^2(\mathcal{C})} = b_0(\Omega) \int_{-X}^{X} \alpha(s) |dz^k|^2 dv_g = 0,$$

since α is odd with mean zero. Therefore one has

$$\frac{||\Psi - P_g(\Psi)||_{L^p(M,g)}}{||\bar{\partial}\Psi||_{L^1(M,g)}} \geq c l^{1/p-1} \to \infty$$

as $l \to 0$ if $p > 1$. Hence the Poincaré-Sobolev type estimate

$$||\Psi - P_g(\Psi)||_{L^p(M,g)} \leq C ||\bar{\partial}\Psi||_{L^1(M,g)}$$

cannot be uniform if $p > 1$.

7.2 Proof of the main theorem

The first step in the proof of the main theorem 7.0.1 is a local compactness result for functions. The proof extends the argument in [RT16, Lemma 2.3], where the L^1-case has been established.

Lemma 7.2.1 *Suppose that $\psi_i \in C^1(D, \mathbb{C})$ is a sequence of functions on the unit disc $D \subset \mathbb{C}$ such that*

$$||\psi_i||_{L^p} + ||\partial_{\bar{z}}\psi_i||_{L^p} \leq C$$

for some constant $C < \infty$. Suppose that $1 \leq p < 2$ and $1 \leq q < \frac{2p}{2-p}$ or $p \geq 2$ and $q \in [1, \infty)$.

Then there exists $\psi_\infty \in L^q(D_{1/2}, \mathbb{C})$ such that after passing to a subsequence there holds

$$\psi_i \to \psi_\infty$$

in $L^q(D_{1/2}, \mathbb{C})$. Moreover, if $||\partial_{\bar{z}}\psi_i||_{L^p} \to 0$ as $i \to \infty$ then ψ_∞ is holomorphic.

Proof: Notice that for $w \in D_{1/2}$ and $\varepsilon < 1/2$ there holds

$$\begin{aligned}\left(|\partial_{\bar{z}}\psi|\mathbb{1}_D * \frac{1}{|z|}\mathbb{1}_{D_\varepsilon(0)}\right)(w) &= \int_{\mathbb{R}^2} |\partial_{\bar{z}}\psi|(z)\mathbb{1}_D(z)\frac{1}{|w-z|}\mathbb{1}_{D_\varepsilon(0)}(w-z)dm(z) \\ &= \int_{D_\varepsilon(w)} \frac{|\partial_{\bar{z}}\psi|(z)}{|z-w|}dm(z).\end{aligned}$$

Then Young's convolution inequality shows that

$$\begin{aligned}\int_{D_{1/2}}\left(\int_{D_\varepsilon(w)}\frac{|\partial_{\bar{z}}\psi|(z)}{|z-w|}dm(z)\right)^q dm(w) &= ||\ |\partial_{\bar{z}}\psi|\mathbb{1}_D * \frac{1}{|z|}\mathbb{1}_{D_\varepsilon(0)}||^q_{L^q(D_{1/2})} \\ &\leq ||\ |\partial_{\bar{z}}\psi|\mathbb{1}_D * \frac{1}{|z|}\mathbb{1}_{D_\varepsilon(0)}||^q_{L^q(\mathbb{R}^2)} \\ &\leq ||\ |\partial_{\bar{z}}\psi|\mathbb{1}_D||^q_{L^p(\mathbb{R}^2)} \cdot ||\frac{1}{|z|}\mathbb{1}_{D_\varepsilon(0)}||^q_{L^r(\mathbb{R}^2)} \\ &= ||\partial_{\bar{z}}\psi||^q_{L^p(D)} \cdot ||\frac{1}{z}||^q_{L^r(D_\varepsilon)}\end{aligned}$$

if $1 \leq p, q, r \leq \infty$ and $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. For the second integral to exist, $r < 2$ is necessary. As in [RT16], one proves with the general Cauchy-formula that

$$|\psi(w) - \psi^\varepsilon(w)| \leq \frac{1}{2\pi}\int_{D_\varepsilon(w)}\frac{|\partial_{\bar{z}}\psi|(z)}{|z-w|}dm(z)$$

holds for any $\psi \in C^1(D, \mathbb{C})$, any ε-mollification ψ^ε which is computed with respect to a radially symmetric, sufficiently locally supported smoothing function, $w \in D_{1/2}$ and $\varepsilon \in (0, 1/2]$. In particular it follows that

$$||\psi - \psi^\varepsilon||_{L^q(D_{1/2})} \leq \frac{\varepsilon^{2-r}}{2-r}||\partial_{\bar{z}}\psi||_{L^p(D)}.$$

Notice that this estimate can in particular be established for $q < p^* = \frac{2p}{2-p}$. According to the Rellich-Kondrachov compactness theorem, for any open, bounded set $U \subset \mathbb{R}^2$ with C^1-boundary, the assumptions in the theorem imply that $W^{1,p}(U)$ is precompact in $L^q(U)$, cf. [Eva98, Section 5.7]. In particular, for fixed ε, the mollification ψ_i^ε is precompact in $L^q(D_{1/2})$. The result now follows by a diagonal sequence argument.

If $||\partial_{\bar{z}}\psi_i||_{L^p} \to 0$ as $i \to \infty$ then holomorphicity of the limit can be established exactly as in [RT16]. □

The main argument in the proof of theorem 7.0.1 is by contradiction: Suppose that there is a sequence of closed hyperbolic surfaces (M_i, c_i, g_i) of fixed genus $\gamma \geq 2$ and a sequence of smooth k-differentials Ψ_i on M_i such that

$$\frac{||\Psi_i - P_{g_i}(\Psi_i)||_{L^p(M_i,g_i)}}{||\bar{\partial}\Psi_i||_{L^p(M_i,g_i)}} \to \infty$$

as $i \to \infty$. In fact, one can assume that $M = M_i$, as all closed surfaces of fixed genus are diffeomorphic, and that Ψ_i satisfies

$$P_{g_i}(\Psi_i) = 0 \quad \text{and} \quad ||\Psi_i||_{L^p(M,g_i)} = 1 \quad \text{and} \quad ||\bar{\partial}\Psi_i||_{L^p(M,g_i)} \to 0 \tag{7.1}$$

by replacing Ψ_i with a suitable normalisation of $\Psi_i - P_{g_i}(\Psi_i)$.

Then the Deligne-Mumford compactness theorem 7.1.3 can be applied. A local argument in coordinates and the compactness lemma 7.2.1 for functions then yield as in [RT15, Lemma 2.1]:

Lemma 7.2.2 *Let (M, c_i, g_i) be a sequence of hyperbolic surfaces that converges to a possibly punctured hyperbolic surface (Σ, c, h) in the sense of theorem 7.1.3 and suppose that either $1 \leq p < 2$ and $1 \leq q < p^* = \frac{2p}{2-p}$ or $p \geq 2$ and $q \in [1, \infty)$. Then for any sequence of k-differentials $\Psi_i \in L^p(M, g_i)$ with*

$$||\Psi_i||_{L^p(M,g_i)} + ||\bar{\partial}\Psi_i||_{L^p(M,g_i)} \leq C < \infty$$

there exists a subsequence which converges locally in L^q to $\Psi_\infty \in L^q(\Sigma, h)$. That is, with respect to the diffeomorphisms $f_i \colon \Sigma \to M \setminus \cup_{j=1}^m \sigma_i^j$ as in theorem 7.1.3, one has $f_i^*\Psi_i \to \Psi_\infty$ in $L^q_{loc}(\Sigma, h)$.

Furthermore, if $||\bar{\partial}\Psi_i||_{L^p(M,g_i)} \to 0$, then Ψ_∞ is holomorphic.

In particular, lemma 7.2.2 applies to the sequence in (7.1) and moreover, by theorem 7.1.9, the holomorphic limit Ψ_∞ must satisfy $P_h^{\mathcal{H}ol_k(\Sigma,h)}(\Psi_\infty) = 0$. Hence, it vanishes identically, $\Psi_\infty \equiv 0$.

However, it will be shown that the L^p-norm is preserved, thus providing a contradiction. Due to [RTZ13, Lemma A.7], the conclusion holds on the δ-thick parts, i.e. for any fixed δ with $0 < \delta < \operatorname{arsinh}(1)$ one has

$$||\Psi_\infty||_{L^p(\Sigma^\delta,h)} = \lim_{i\to\infty} ||\Psi_i||_{L^p(M^\delta,g_i)}.$$

This is a consequence of the fact that the δ-thick part of $(\Sigma, f_i^* g_i)$ converges to (Σ^δ, h) in the sense that the measure of their symmetric difference tends to zero.

On the other hand, there holds $||\Psi_i||^p_{L^p(M^\delta,g_i)} = 1 - ||\Psi_i||^p_{L^p(\delta-thin(M),g_i)}$ and then lemma 7.2.3 below shows that

$$\sup_{\delta>0} ||\Psi_\infty||^p_{L^p(\Sigma^\delta,h)} = 1 - \inf_{\delta>0} \lim_{i\to\infty} ||\Psi_i||^p_{L^p(\delta-thin(M),g_i)} = 1,$$

which gives the claimed contradiction. It therefore remains to show that there is no L^p-norm concentration on the degenerating collars:

Lemma 7.2.3 *Let (M,g) be a hyperbolic surface of genus γ, $k \geq 2$ an integer and $1 \leq p < \infty$. Then there exists a constant $C = C(\gamma, k, p) < \infty$ such that for every $\delta > 0$ any L^p-integrable k-differential Ψ with $P_g(\Psi) = 0$ satisfies*

$$||\Psi||_{L^p(\delta-thin(M,g))} \leq C \cdot (||\bar{\partial}\Psi||_{L^p(M,g)} + \delta^{1/2}||\Psi||_{L^p(M,g)}).$$

Since the δ-thin part is contained in a union of collar regions for sufficiently small $\delta > 0$, lemma 7.2.3 will follow from explicit estimates on hyperbolic collars.

According to the Keen-Randol collar lemma 7.1.5, any collar region is isometric to a hyperbolic cylinder of the form $\mathcal{C}(l) = ((-X(l), X(l)) \times S^1, \rho^2(ds^2 + d\theta^2))$. With respect to the associated collar coordinates (s, θ), where $z = s + i\theta$, the angular mean value of a k-differential $\Psi = \psi dz^k$ on $\mathcal{C}(l)$ is

$$\alpha(s) = \frac{1}{2\pi} \int_{S^1} \psi(s,\theta) d\theta$$

for $s \in (-X(l), X(l))$.

As in [RT15], the proof of lemma 7.2.3 now follows in two steps:

Lemma 7.2.4 *Let (M,g) be a closed hyperbolic surface and $k \geq 1$. Then there exists a constant $c < \infty$ depending only on k and the genus of M such that for any collar region $\mathcal{C}$, any k-differential Ψ and any $\delta > 0$, $p \in [1,\infty)$ there holds*

$$||\Psi||_{L^p(\delta-thin(\mathcal{C}))} \leq c(\left(\int_{-X}^{X} |\alpha(s)|^p \rho^{2-kp}(s)ds\right)^{1/p} + ||\bar{\partial}\Psi||_{L^p(M,g)} + \delta^{1/2}||\Psi||_{L^p(M,g)}).$$

In order to control the integral of the mean values, one appeals to the following estimate.

Lemma 7.2.5 *Let (M,g) be a closed hyperbolic surface, $p \in [1,\infty)$ and $k \geq 2$. Then there exists a constant $c < \infty$ depending only on p,k and the genus of M such that any k-differential Ψ with $P_g(\Psi) = 0$ satisfies*

$$\int_{-X}^{X} |\alpha(s)|^p \, \rho^{2-kp}(s)ds \leq c\left(||\bar{\partial}\Psi||^p_{L^p(M,g)} + l||\Psi||^p_{L^1(M,g)}\right)$$

on each collar $\mathcal{C}$.

The combination of the two lemmas finishes the proof of lemma 7.2.3 as for $l > 2\delta$ the δ-thin part of the collar $\mathcal{C}(l)$ is empty.

Proof (of lemma 7.2.5): Recall from [RT15] that

$$A(s_0) := |\alpha(0) - \alpha(s_0)| \leq \frac{1}{\pi}\int_0^{s_0}\int_{S^1} |\partial_{\bar{z}}\psi| d\theta ds.$$

Hence there holds

$$\begin{aligned} A(s_0)^p &\leq c\left(\int_0^{s_0}\int_{S^1} |\bar{\partial}\Psi|_g \rho^{k-1}(s)dv_g\right)^p \\ &\leq c\left(\int_0^{s_0}\int_{S^1} |\bar{\partial}\Psi|^p_g \rho^{p(k-1)}(s)dv_g\right)\left(\text{Area}([0,s_0]\times S^1, g)\right)^{p-1} \end{aligned}$$

and since $\text{Area}([0,s_0]\times S^1, g) \leq 2\pi\rho(s_0)$ it follows with Fubini's theorem that

$$\begin{aligned} \int_{-X}^{X} A(s_0)^p \rho^{2-kp}(s_0)ds_0 &\leq c\int_{-X}^{X}\left(\int_0^{s_0}\int_{S^1} |\bar{\partial}\Psi|^p_g \rho^{p(k-1)}(s)dv_g\right)\rho^{1-p(k-1)}(s_0)ds_0 \\ &= c\int_{-X}^{X}\int_{S^1}\left(|\bar{\partial}\Psi|^p_g\rho^{p(k-1)}(s)\right)\left(\int_{|s|}^{X}\rho^{1-p(k-1)}(s_0)ds_0\right)dv_g \\ &\leq c\int_{-X}^{X}\int_{S^1} |\bar{\partial}\Psi|^p_g \rho^{p(k-1)+1-p(k-1)-1}dv_g = c||\bar{\partial}\Psi||^p_{L^p(\mathcal{C},g)}, \end{aligned}$$

where in the last estimate the monotonicity of ρ in $|s|$, the assumption $k \geq 2$ and the observation

$$\rho(s)\cdot(X(l) - |s|) \leq \rho(s)\cdot\left(\frac{\pi^2}{l} - |s|\right) = \frac{\frac{\pi}{2} - \frac{l|s|}{2\pi}}{\sin(\frac{\pi}{2} - \frac{l|s|}{2\pi})} \leq \sup_{x\in(0,\pi/2)}\frac{x}{\sin(x)} = \frac{\pi}{2}$$

from [RT15] were used. Since $||dz^k||_{L^p(\mathcal{C},g)} \sim l^{k+1/p-1}||dz^k||^2_{L^2(\mathcal{C},g)}$ as $l \to 0$, the claim follows from

$$\begin{aligned}\left(\int_{-X}^{X} |\alpha(s_0)|^p \rho^{2-kp}(s_0) ds_0\right)^{1/p} &\leq \left(\int_{-X}^{X} A(s_0)^p \rho^{2-kp}(s_0) ds_0\right)^{1/p} + |\alpha(0)|\ ||dz^k||_{L^p(\mathcal{C},g)} \\ &\leq c||\bar{\partial}\Psi||_{L^p(\mathcal{C},g)} + cl^{k+1/p-1}|\alpha(0)|\ ||dz^k||_{L^2(\mathcal{C},g)},\end{aligned}$$

provided that a sufficient estimate for the last term can be established. To this end, notice that

$$\alpha(0) \cdot \frac{||dz^k||^2_{L^2(\mathcal{C},g)}}{2^{k+1}\pi} = \int_{-X}^{X} (\alpha(0) - \alpha(s))\rho^{2(1-k)}(s) ds + \int_{-X}^{X} \alpha(s)\rho^{2(1-k)}(s) ds.$$

Due to the monotonicity of ρ, the first term can be handled as above

$$\begin{aligned}\int_{-X}^{X} \left(\int_{[0,s_0]\times S^1} |\partial_{\bar{z}}\psi|\rho^{2(1-k)}(s_0) d\theta ds\right) ds_0 &= \int_{0}^{X} \left(\int_{-s_0}^{s_0}\int_{S^1} |\partial_{\bar{z}}\psi|\rho^{2(1-k)}(s_0) d\theta ds\right) ds_0 \\ &\leq \int_0^X \rho^{1-k}(s_0) \left(\int_{-s_0}^{s_0}\int_{S_1} |\bar{\partial}\Psi|_g dv_g\right) ds_0 \\ &\leq \int_0^X \rho^{1-k}(s_0) \cdot ||\bar{\partial}\Psi||_{L^p(\mathcal{C},g)} \cdot \mathrm{Area}([-s_0, s_0] \times S^1, g)^{1-1/p} ds_0 \\ &\leq \int_0^X \rho^{2-k-1/p}(s_0) ds_0\ ||\bar{\partial}\Psi||_{L^p(\mathcal{C},g)} \leq cl^{1-k-1/p}||\bar{\partial}\Psi||_{L^p(\mathcal{C},g)}.\end{aligned}$$

In order to estimate the second term, choose a holomorphic k-differential Ω adapted to the collar in the sense of theorem 7.1.10. Then its principal part can be estimated by $|b_0| \geq cl^{k-1/2}$. Furthermore, $P_g(\Psi) = 0$ implies $\langle\Psi, \Omega\rangle_{L^2(M,g)} = 0$ and hence

$$\begin{aligned}cl^{k-1/2}\left|\int_{-X}^{X}\int_{S^1} \psi\rho^{2(1-k)} d\theta ds\right| \leq |b_0| \left|\int_{-X}^{X}\int_{S^1} \psi\rho^{2(1-k)} d\theta ds\right| &= \left|\langle\Psi, b_0 dz^k\rangle_{L^2(\mathcal{C},g)}\right| \\ &\leq \left|\langle\Psi, \Omega - b_0 dz^k\rangle_{L^2(\mathcal{C},g)}\right| + \left|\langle\Psi, \Omega\rangle_{L^2(M\setminus\mathcal{C},g)}\right| \\ &\leq \left(||\Omega - b_0 dz^k||_{L^\infty(\mathcal{C})} + ||\Omega||_{L^\infty(M\setminus\mathcal{C})}\right) ||\Psi||_{L^1(M)} \\ &\leq cl^{1/2}||\Psi||_{L^1}(M, g).\end{aligned}$$

In particular, this yields

$$\left|\int_{-X}^{X} \alpha(s)\rho^{2(1-k)}(s) ds\right| \leq cl^{1-k}||\Psi||_{L^1(M,g)},$$

which completes the proof. $\square$

Proof (of lemma 7.2.4): As in [RT15], the proof relies on the general Cauchy formula for C^1-functions. Given a k-differential $\Psi = \psi dz^k$ on a collar $(-X, X) \times S^1$, extend the function ψ periodically to $\mathbb{R}^2 \cong \mathbb{C}$. For $z_0 = (s_0, \theta_0) \in (-X_{\delta_0}, X_{\delta_0}) \times [0, 2\pi]$ and $b \in [0, 2\pi]$ consider the domain

$$\Omega_b(z_0) = \{z_0\} + [-\rho^{-1/2}(s_0), \rho^{-1/2}(s_0)] \times [-(\rho^{-1/2}(s_0) + b), \rho^{-1/2}(s_0) + b].$$

By setting

$$\begin{aligned} I_\Omega(z_0, b) &= \int_{\Omega_b(z_0)} \frac{\partial_{\bar{z}}\psi}{z - z_0} dz \wedge d\bar{z}, \\ I_H^\pm(z_0, b) &= \mp \int_{h^-(s_0)}^{h^+(s_0)} \frac{\psi(s, \theta_0 \pm (\rho^{-1/2}(s_0) + b))}{s - s_0 \pm i(\rho^{-1/2}(s_0) + b)} ds, \\ I_V^\pm(z_0, b) &= \pm i \int_{\theta_0 - (\rho^{-1/2}(s_0) + b)}^{\theta_0 + \rho^{-1/2}(s_0) + b} \frac{\psi(h^\pm(s_0), \theta)}{\pm\rho^{-1/2}(s_0) + i(\theta - \theta_0)} d\theta, \end{aligned}$$

where $h^\pm(s_0) = s_0 \pm \rho^{-1/2}(s_0)$, the Cauchy formula implies

$$2\pi i \psi(z_0) = (I_\Omega + I_V^+ + I_V^- + I_H^+ + I_H^-)(z_0, b)$$

for any $b \in [0, 2\pi]$. Then notice that

$$\begin{aligned} ||\Psi||^p_{L^p(\delta-\mathrm{thin}(\mathcal{C},g))} &= \int_{-X_\delta}^{X_\delta} \int_{S^1} |\psi|^p |dz^k|^p \rho^2 d\theta_0 ds_0 = \ 2^{kp/2} \int_{-X_\delta}^{X_\delta} \int_{S^1} |\psi|^p \rho^{2-kp} d\theta_0 ds_0 \\ &= \left(\frac{2^{k/2-1}}{\pi}\right)^p \int_{-X_\delta}^{X_\delta} \int_{S^1} \left| \fint_0^{2\pi} (I_\Omega + I_V^+ + I_V^- + I_H^+ + I_H^-)(z_0, b) db \right|^p \rho^{2-kp}(s_0) d\theta_0 ds_0 \end{aligned}$$

and thus

$$\begin{aligned} \frac{\pi}{2^{k/2-1}} ||\Psi||_{L^p(\delta-thin(\mathcal{C},g))} \leq \quad & \sup_{b \in [0,2\pi]} \left(\int_{-X_\delta}^{X_\delta} \int_{S^1} |I_\Omega(z_0, b)|^p \rho^{2-kp}(s_0) d\theta_0 ds_0 \right)^{1/p} \\ &+ \sum_{\bullet \in \{\pm\}} \sup_{b \in [0,2\pi]} \left(\int_{-X_\delta}^{X_\delta} \int_{S^1} |I_V^\bullet(z_0, b)|^p \rho^{2-kp}(s_0) d\theta_0 ds_0 \right)^{1/p} \\ &+ \sum_{\bullet \in \{\pm\}} \left(\int_{-X_\delta}^{X_\delta} \int_{S^1} \left| \fint_0^{2\pi} I_H^\bullet(z_0, b) db \right|^p \rho^{2-kp}(s_0) d\theta_0 ds_0 \right)^{1/p}. \end{aligned} \tag{7.2}$$

It will be shown that the first term can be bounded by $c||\bar{\partial}\Psi||_{L^p(2\delta-thin(\mathcal{C},g))}$ and all other terms by $c(\int_{-X_{2\delta}}^{X_{2\delta}} |\alpha(s)|^p \rho^{2-kp}(s) ds)^{1/p} + c\delta^{1/2} \cdot ||\Psi||_{L^p(2\delta-thin(\mathcal{C}))}$, for some constant $c < \infty$ independent of p, which will prove the claim.

Remark 7.2.6 It follows from [RT15, Remark 2.8] that for $\delta_0 > 0$ sufficiently small, any $0 < l < 2\delta_0$ and any $0 < \delta \leq \delta_0$ there holds $\Omega(z_0, b) \subset (-X_{2\delta}, X_{2\delta}) \times \mathbb{R}$ for all $z_0 \in \delta$-*thin*$(\mathcal{C}(l))$ and moreover there is a constant $C = C(\delta_0) > 0$ such that

$$C^{-1}\rho(s) \leq \rho(s_0) \leq C\rho(s)$$

for all $s_0 \in (-X_{2\delta_0}, X_{2\delta_0})$ and $s \in (-X_{\delta_0}, X_{\delta_0})$.

To prove the bound on the first term, recall from [RT15] that

$$|I_\Omega(z_0,b)| \le 2\int_{z_0+[-2\pi,2\pi]^2} \frac{|\partial_{\bar z}\psi|}{|z-z_0|}d\theta ds + c\int_{h^-(s_0)}^{h^+(s_0)}\int_{S^1}|\partial_{\bar z}\psi|\log(\rho^{-1}(s))d\theta ds. \quad (7.3)$$

To estimate the first summand in (7.3) note that by remark 7.2.6 one has

$$\int_{-X_\delta}^{X_\delta}\int_{S^1}\left(\int_{z_0+[-2\pi,2\pi]^2}\frac{|\partial_{\bar z}\psi|}{|z-z_0|}d\theta ds\right)^p \rho^{2-kp}(s_0)d\theta_0 ds_0 \le$$
$$\le c^p \int_{-X_\delta}^{X_\delta}\int_{S^1}\left(\int_{z_0+[-2\pi,2\pi]^2}\frac{|\partial_{\bar z}\psi|\rho^{2/p-k}(s)}{|z-z_0|}d\theta ds\right)^p d\theta_0 ds_0,$$

which can be viewed as an L^p-norm of a convolution by considering $\frac{1}{|z|}\mathbb{1}_{[-2\pi,2\pi]^2}$ and $|\partial_{\bar z}\psi|\rho^{\frac{2}{p}-k}\mathbb{1}_{\delta-thin(\mathcal{C},g)} = |\bar\partial\Psi|_g\rho^{\frac{2}{p}+1}\mathbb{1}_{\delta-thin(\mathcal{C},g)}$. Hence Young's convolution inequality implies

$$\int_{-X_\delta}^{X_\delta}\int_{S^1}\left(\int_{z_0+[-2\pi,2\pi]^2}\frac{|\partial_{\bar z}\psi|}{|z-z_0|}d\theta ds\right)^p \rho^{2-kp}(s_0)d\theta_0 ds_0 \le$$
$$\le c^p \;||\; 1/|z| \;||^p_{L^1([-2\pi,2\pi]^2)} \cdot \int_{-X_\delta}^{X_\delta}\int_{S^1}|\bar\partial\Psi|_g^p\rho^{2+p}(s)d\theta ds$$
$$= c^p \;||\; 1/|z| \;||^p_{L^1([-2\pi,2\pi]^2)} \cdot \int_{-X_\delta}^{X_\delta}\int_{S^1}|\bar\partial\Psi|_g^p\rho^{p}(s)dv_g.$$

Notice that this term is actually a lot smaller than $||\bar\partial\Psi||^p_{L^p(2\delta-thin(\mathcal{C},g))}$ due to the extra factor of ρ^p.

Remark 7.2.7 It is useful to recall from [RT15, Remark 2.8] that there is $\delta_0 > 0$ and a constant $C_{\delta_0} < \infty$ such that for all $0 < \delta \le \delta_0$ and $0 < l < 2\delta_0$ both functions $h^\pm(s) = s \pm \rho(s)^{-1/2}(s)$ are invertible on $[-X_{\delta_0}(l), X_{\delta_0}(l)]$, the derivatives of their inverses are uniformly bounded and

$$((h^-)^{-1} - (h^+)^{-1})(s) \le C_{\delta_0}\rho^{-1/2}(s).$$

Also notice that $[(h^+)^{-1}(s), (h^-)^{-1}(s)] \subset [-X_{2\delta}(l), X_{2\delta}(l)]$.

With regard to the second term in (7.3), notice that $h^+(s_0) - h^-(s_0) = 2\rho(s_0)^{-1/2}$.

Then remark 7.2.6, Hölder's inequality and Fubini's theorem imply that

$$\begin{aligned}
&\int_{-X_\delta}^{X_\delta}\int_{S^1}\left(\int_{h^-(s_0)}^{h^+(s_0)}\int_{S^1}|\partial_{\bar z}\psi|\log(\rho^{-1}(s))d\theta ds\right)^p\rho^{2-kp}(s_0)d\theta_0 ds_0\\
&\leq(4\pi)^{p-1}\int_{-X_\delta}^{X_\delta}\int_{S^1}\left(\int_{h^-(s_0)}^{h^+(s_0)}\int_{S^1}|\partial_{\bar z}\psi|^p(\log(\rho^{-1}(s)))^p d\theta ds\right)\rho^{2-kp-\frac{p-1}{2}}(s_0)d\theta_0 ds_0\\
&\leq(4\pi)^{p-1}\int_{-X_{2\delta}}^{X_{2\delta}}\int_{S^1}|\partial_{\bar z}\psi|^p(\log(\rho^{-1}(s)))^p\cdot\left(\int_{(h^+)^{-1}(s)}^{(h^-)^{-1}(s)}\rho^{2-kp-\frac{p-1}{2}}(s_0)ds_0\right)d\theta ds\\
&\leq c^p\int_{-X_{2\delta}}^{X_{2\delta}}\int_{S^1}|\partial_{\bar z}\psi|^p(\log(\rho^{-1}(s)))^p\rho^{2-kp-\frac{p-1}{2}-\frac{1}{2}}(s)d\theta ds\\
&\leq c^p\int_{-X_{2\delta}}^{X_{2\delta}}\int_{S^1}|\partial_{\bar z}\psi|^p\rho^{2-p(k+1)}(s)d\theta ds=c^p\ ||\bar\partial\Psi||^p_{L^p(2\delta-thin(\mathcal{C},g))},
\end{aligned}$$

where $\log(x)\leq\sqrt{x}$ was used in the last inequality. This proves the bound for I_Ω.

The key idea to estimate the boundary integrals $I_H^\pm$ and $I_V^\pm$ is that $\frac{1}{z-z_0}$ is almost constant along line segments of the boundary of $\Omega_b(z_0)$. More precisely, one uses

$$\begin{aligned}
\sup_{z\in\{h^+(s_0)\}\times(\{\theta_0+2\pi k\}+[0,2\pi])}\left|\tfrac{1}{z-z_0}-\tfrac{1}{\rho^{-1/2}(s_0)+2\pi ik}\right| &\leq 2\pi\rho(s_0),\\
\sup_{z\in\{s\}\times(\{\theta_0+\rho^{-1/2}(s_0)\}+[0,2\pi])}\left|\tfrac{1}{z-z_0}-\tfrac{1}{s-s_0+i\rho^{-1/2}(s_0)}\right| &\leq 2\pi\rho(s_0)
\end{aligned}$$

to bound I_H^+ and I_V^+.

As carried out in [RT15], this leads to the estimate

$$\left|\fint_0^{2\pi}I_H^\pm(z_0,b)db\right|\leq\rho^{1/2}(s_0)\int_{h^-(s_0)}^{h^+(s_0)}|\alpha(s)|ds+\rho(s_0)\int_{h^-(s_0)}^{h^+(s_0)}\int_0^{2\pi}|\psi(s,\theta)|d\theta ds.$$

Therefore the $I_H^\pm$-term in (7.2) splits up into two pieces and the first term can be controlled by

$$\begin{aligned}
&\int_{-X_\delta}^{X_\delta}\left(\int_{h^-(s_0)}^{h^+(s_0)}|\alpha(s)|ds\right)^p\rho^{2-kp+\frac{p}{2}}(s_0)ds_0\\
&\qquad\leq 2^{p-1}\int_{-X_\delta}^{X_\delta}\int_{h^-(s_0)}^{h^+(s_0)}|\alpha(s)|^p\rho^{2-kp+\frac{p}{2}-\frac{p-1}{2}}(s_0)ds ds_0\\
&\qquad\leq 2^{p-1}\int_{-X_{2\delta}}^{X_{2\delta}}\left(\int_{(h^+)^{-1}(s)}^{(h^-)^{-1}(s)}\rho^{2-kp+1/2}(s_0)ds_0\right)|\alpha(s)|^p ds\\
&\qquad\leq c^p\int_{-X_{2\delta}}^{X_{2\delta}}|\alpha(s)|^p\rho^{2-kp}(s)ds,
\end{aligned}$$

where firstly Hölder's inequality, then Fubini's theorem and finally remarks 7.2.6 and 7.2.7 were used. Similarly, one finds that

$$\begin{aligned}
&\int_{-X_\delta}^{X_\delta} \left(\int_{h^-(s_0)}^{h^+(s_0)} \int_0^{2\pi} |\psi(s,\theta)| d\theta ds \right)^p \rho^{2-kp+p}(s_0) ds_0 \\
&\quad \leq \int_{-X_\delta}^{X_\delta} \left(\int_{h^-(s_0)}^{h^+(s_0)} \int_0^{2\pi} |\psi(s,\theta)|^p d\theta ds \right) \left(4\pi \rho^{-1/2}(s_0)\right)^{p-1} \rho^{2-kp+p}(s_0) ds_0 \\
&\quad \leq (4\pi)^{p-1} \int_{-X_{2\delta}}^{X_{2\delta}} \int_{(h^+)^{-1}(s)}^{(h^-)^{-1}(s)} \left(\int_0^{2\pi} |\psi(s,\theta)|^p d\theta \right) \rho^{2-kp+\frac{p}{2}+\frac{1}{2}}(s_0) ds_0 ds \\
&\quad \leq c^p \left(\sup_{s_0 \in [-X_{2\delta}, X_{2\delta}]} \rho^{p/2}(s_0) \right) \int_{-X_{2\delta}}^{X_{2\delta}} \int_0^{2\pi} |\psi(s,\theta)|^p \rho^{2-kp}(s) d\theta ds \\
&\quad \leq c^p \delta^{p/2} ||\Psi||^p_{L^p(2\delta-thin(\mathcal{C}))}.
\end{aligned}$$

Putting these estimates together, one obtains the bound

$$\begin{aligned}
\left(\int_{-X_\delta}^{X_\delta} \int_{S^1} \left| \fint_0^{2\pi} I_H^{\pm}(z_0,b) db \right|^p \rho^{2-kp}(s_0) ds_0 d\theta_0 \right)^{1/p} &\leq c \int_{-X_{2\delta}}^{X_{2\delta}} |\alpha(s)|^p \rho^{2-kp}(s) ds \\
&\quad + c\delta^{1/2} \cdot ||\Psi||_{L^p(2\delta-thin(\mathcal{C}))}
\end{aligned}$$

as claimed.

Finally, for the estimate of the $I_V^{\pm}$-term in (7.2), recall from [RT15] that

$$|I_V^+(z_0,b)| \leq c \cdot |\alpha(h^+(s_0))| + c\rho^{1/2}(s_0) \int_{S^1} |\psi(h^+(s_0),\theta)| d\theta$$

for each $b \in [0,2\pi]$. Then remarks 7.2.6 and 7.2.7 imply that

$$\begin{aligned}
&\left(\int_{-X_\delta}^{X_\delta} \int_{S^1} |I_V^+(z_0,b)|^p \rho^{2-kp}(s_0) d\theta_0 ds_0 \right)^{1/p} \\
&\leq \quad c \left(\int_{-X_\delta}^{X_\delta} \int_{S^1} |\alpha(h^+(s_0))|^p \rho^{2-kp}(s_0) d\theta_0 ds_0 \right)^{1/p} \\
&\quad + c \left(\int_{-X_\delta}^{X_\delta} \int_{S^1} \rho^{p/2+2-kp}(s_0) \left(\int_{S^1} |\psi(h^+(s_0),\theta)| d\theta \right)^p d\theta_0 ds_0 \right)^{1/p} \\
&\leq \quad c \left(\int_{-X_{2\delta}}^{X_{2\delta}} \int_{S^1} |\alpha(s)|^p \rho^{2-kp}(s) ds \right)^{1/p} \\
&\quad + c \sup_{s \in [-X_\delta, X_\delta]} (\rho(s))^{1/2} ||\Psi||_{L^p(2\delta-thin(\mathcal{C},g))} \\
&\leq \quad c \left(\int_{-X}^{X} |\alpha(s)|^p \rho^{2-kp}(s) ds \right)^{1/p} + c\delta^{1/2} ||\Psi||_{L^p(2\delta-thin(\mathcal{C},g))},
\end{aligned}$$

which completes the proof of lemma 7.2.4. □

Bibliography

[AIK11] Sigurd B. Angenent, James Isenberg, and Dan Knopf, *Formal matched asymptotics for degenerate Ricci flow neckpinches*, Nonlinearity **24** (2011), no. 8, 2265–2280.

[AIK15] ———, *Degenerate neckpinches in Ricci flow*, J. Reine Angew. Math. **709** (2015), 81–117.

[App18] Alexander Appleton, *A family of non-collapsed steady Ricci solitons in even dimensions greater or equal to four*, arXiv:1708.00161 (2018).

[Bac86] Allen Back, *Local Theory of Equivariant Einstein Metrics and Ricci Realizability on Kervaire Spheres*, available at www.math.cornell.edu/∼back/einstein.ps (1986).

[BB75] Lionel Bérard-Bergery, *Sur certaines fibrations d'espaces homogènes riemanniens*, Compositio Math. **30** (1975), 43–61.

[BB82] ———, *Sur de nouvelles variétés riemanniennes d'Einstein*, Institut Élie Cartan, 6, Inst. Élie Cartan, vol. 6, Univ. Nancy, Nancy, 1982, pp. 1–60.

[BDGW15a] Maria Buzano, Andrew S. Dancer, Michael Gallaugher, and McKenzie Wang, *Non-Kähler expanding Ricci solitons, Einstein metrics, and exotic cone structures*, Pacific J. Math. **273** (2015), no. 2, 369–394.

[BDGW15b] Maria Buzano, Andrew S. Dancer, Michael Gallaugher, and McKenzie Y. Wang, *A family of steady Ricci solitons and Ricci-flat metrics*, arXiv:1309.6140 (2015), includes numerical studies not published in [BDW15].

[BDW15] M. Buzano, A. S. Dancer, and M. Wang, *A family of steady Ricci solitons and Ricci-flat metrics*, Comm. Anal. Geom. **23** (2015), no. 3, 611–638.

[Bes87] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge · Band 10, Springer-Verlag, Berlin, 1987.

[BHJM15] Wafaa Batat, Stuart J. Hall, Ali Jizany, and Thomas Murphy, *Conformally Kähler geometry and quasi-Einstein metrics*, Münster J. Math. **8** (2015), no. 1, 211–228.

[Böh98] Christoph Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math. **134** (1998), no. 1, 145–176.

[Böh99] ———, *Non-compact cohomogeneity one Einstein manifolds*, Bull. Soc. Math. France **127** (1999), no. 1, 135–177.

[Böh05] ———, *Non-existence of homogeneous Einstein metrics*, Comment. Math. Helv. **80** (2005), no. 1, 123–146.

[Bre13] Simon Brendle, *Rotational symmetry of self-similar solutions to the Ricci flow*, Invent. Math. **194** (2013), no. 3, 731–764.

[Bre14] ———, *Rotational symmetry of Ricci solitons in higher dimensions*, J. Differential Geom. **97** (2014), no. 2, 191–214.

[Bry05] Robert L. Bryant, *Ricci flow solitons in dimension three with $SO(3)$-symmetries*, available at www.math.duke.edu/∼bryant/3DRotSymRicciSolitons.pdf, 2005.

[BS89] Robert L. Bryant and Simon M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), no. 3, 829–850.

[BS09] Simon Brendle and Richard Schoen, *Manifolds with $1/4$-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), no. 1, 287–307.

[Buz11] Maria Buzano, *Initial value problem for cohomogeneity one gradient Ricci solitons*, J. Geom. Phys. **61** (2011), no. 6, 1033–1044.

[BW08] Christoph Böhm and Burkhard Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) **167** (2008), no. 3, 1079–1097.

[Cao96] Huai-Dong Cao, *Existence of gradient Kähler-Ricci solitons*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, 1996, pp. 1–16.

[Cao97] ———, *Limits of solutions to the Kähler-Ricci flow*, J. Differential Geom. **45** (1997), no. 2, 257–272.

[Cao10] ———, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 1–38.

[Cas10] Jeffrey S. Case, *The nonexistence of quasi-Einstein metrics*, Pacific J. Math. **248** (2010), no. 2, 277–284.

[CC13] Huai-Dong Cao and Qiang Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. **162** (2013), no. 6, 1149–1169.

[CDS15a] Xiuxiong Chen, Simon Donaldson, and Song Sun, *Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. **28** (2015), no. 1, 183–197.

[CDS15b] ———, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than* 2π, J. Amer. Math. Soc. **28** (2015), no. 1, 199–234.

[CDS15c] ———, *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches* 2π *and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), no. 1, 235–278.

[Che09a] Bing-Long Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. **82** (2009), no. 2, 363–382.

[Che09b] Dezhong Chen, *Construction of Conformally Compact Einstein Manifolds*, arXiv:0908.1430 (2009).

[Che11] ———, *Examples of Einstein manifolds in odd dimensions*, Ann. Global Anal. Geom. **40** (2011), no. 3, 339–377.

[CS18] Jacob Cable and Hendrik Süß, *On the classification of Kähler-Ricci solitons on Gorenstein del Pezzo surfaces*, Eur. J. Math. **4** (2018), no. 1, 137–161.

[CSW11] Jeffrey Case, Yu-Jen Shu, and Guofang Wei, *Rigidity of quasi-Einstein metrics*, Differential Geom. Appl. **29** (2011), no. 1, 93–100.

[CV96] Thierry Chave and Galliano Valent, *On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties*, Nuclear Phys. B **478** (1996), no. 3, 758–778.

[Der12] Andrzej Derdziński, *Ricci solitons*, Wiad. Mat. **48** (2012), no. 1, 1–32.

[DHW13] Andrew S. Dancer, Stuart J. Hall, and McKenzie Y. Wang, *Cohomogeneity one shrinking Ricci solitons: an analytic and numerical study*, Asian J. Math. **17** (2013), no. 1, 33–61.

[DK81] Dennis M. DeTurck and Jerry L. Kazdan, *Some regularity theorems in Riemannian geometry*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 3, 249–260.

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[Don08] Simon K. Donaldson, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry*, Handbook of geometric analysis. No. 1, Adv. Lect. Math. (ALM), vol. 7, Int. Press, Somerville, MA, 2008, pp. 29–75.

[DS16] Ved Datar and Gábor Székelyhidi, *Kähler-Einstein metrics along the smooth continuity method*, Geom. Funct. Anal. **26** (2016), no. 4, 975–1010.

[DW09a] Andrew S. Dancer and McKenzie Y. Wang, *Non-Kähler expanding Ricci solitons*, Int. Math. Res. Not. IMRN (2009), no. 6, 1107–1133.

[DW09b] ———, *Some new examples of non-Kähler Ricci solitons*, Math. Res. Lett. **16** (2009), no. 2, 349–363.

[DW11] ———, *On Ricci solitons of cohomogeneity one*, Ann. Global Anal. Geom. **39** (2011), no. 3, 259–292.

[EMT11] Joerg Enders, Reto Müller, and Peter M. Topping, *On type-I singularities in Ricci flow*, Comm. Anal. Geom. **19** (2011), no. 5, 905–922.

[Eva98] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[EW00] J.-H. Eschenburg and McKenzie Y. Wang, *The initial value problem for cohomogeneity one Einstein metrics*, J. Geom. Anal. **10** (2000), no. 1, 109–137.

[FFFG17] F. E. S. Feitosa, A. A. Freitas Filho, and J. N. V. Gomes, *On the construction of gradient Ricci soliton warped product*, Nonlinear Anal. **161** (2017), 30–43.

[FH17] Lorenzo Foscolo and Mark Haskins, *New G_2-holonomy cones and exotic nearly Kähler structures on S^6 and $S^3 \times S^3$*, Ann. of Math. (2) **185** (2017), no. 1, 59–130.

[FIK03] Mikhail Feldman, Tom Ilmanen, and Dan Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geom. **65** (2003), no. 2, 169–209.

[FLGaRo11] Manuel Fernández-López and Eduardo Garcí a Rí o, *Rigidity of shrinking Ricci solitons*, Math. Z. **269** (2011), no. 1-2, 461–466.

[GHP03] G. W. Gibbons, Sean A. Hartnoll, and C. N. Pope, *Bohm and Einstein-Sasaki metrics, black holes, and cosmological event horizons*, Phys. Rev. D (3) **67** (2003), no. 8, 084024, 24.

[GK04] Andreas Gastel and Manfred Kronz, *A family of expanding Ricci solitons*, Variational problems in Riemannian geometry, Progr. Nonlinear Differential Equations Appl., vol. 59, Birkhäuser, Basel, 2004, pp. 81–93.

[GPP90] G. W. Gibbons, D. N. Page, and C. N. Pope, *Einstein metrics on S^3, $\mathbf{R}^3$ and $\mathbf{R}^4$ bundles*, Comm. Math. Phys. **127** (1990), no. 3, 529–553.

[Gua95] Daniel Zhuang-Dan Guan, *Quasi-Einstein metrics*, Internat. J. Math. **6** (1995), no. 3, 371–379.

[GZ08] Hui-Ling Gu and Xi-Ping Zhu, *The existence of type II singularities for the Ricci flow on S^{n+1}*, Comm. Anal. Geom. **16** (2008), no. 3, 467–494.

[Hal13] Stuart James Hall, *Quasi-Einstein metrics on hypersurface families*, J. Geom. Phys. **64** (2013), 83–90.

[Ham82] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[Ham88] ———, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262.

[Ham95] ———, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995, pp. 7–136.

[HPW12] Chenxu He, Peter Petersen, and William Wylie, *On the classification of warped product Einstein metrics*, Comm. Anal. Geom. **20** (2012), no. 2, 271–311.

[HPW14] ———, *Warped product Einstein metrics over spaces with constant scalar curvature*, Asian J. Math. **18** (2014), no. 1, 159–189.

[HPW15a] ———, *Uniqueness of warped product Einstein metrics and applications*, J. Geom. Anal. **25** (2015), no. 4, 2617–2644.

[HPW15b] ———, *Warped product Einstein metrics on homogeneous spaces and homogeneous Ricci solitons*, J. Reine Angew. Math. **707** (2015), 217–245.

[HPW15c] ———, *Warped product rigidity*, Asian J. Math. **19** (2015), no. 1, 135–170.

[Hum97] Christoph Hummel, *Gromov's compactness theorem for pseudo-holomorphic curves*, Progress in Mathematics, vol. 151, Birkhäuser Verlag, Basel, 1997.

[IKS18] James Isenberg, Dan Knopf, and Natasa Sesum, *Non-Kähler Ricci flow singularities that converge to Kähler-Ricci solitons*, arXiv:1703.02918 (2018).

[IS17] Nathan Ilten and Hendrik Süss, *K-stability for Fano manifolds with torus action of complexity 1*, Duke Math. J. **166** (2017), no. 1, 177–204.

[Ive93] Thomas Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. **3** (1993), no. 4, 301–307.

[Ive94] ______, *New examples of complete Ricci solitons*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 241–245.

[Jab14] Michael Jablonski, *Homogeneous Ricci solitons are algebraic*, Geom. Topol. **18** (2014), no. 4, 2477–2486.

[Jab15] ______, *Homogeneous Ricci solitons*, J. Reine Angew. Math. **699** (2015), 159–182.

[KK03] Dong-Soo Kim and Young Ho Kim, *Compact Einstein warped product spaces with nonpositive scalar curvature*, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2573–2576.

[Kle90] Bruce Alan Kleiner, *Riemannian four-manifolds with nonnegative curvature and continuous symmetry*, ProQuest LLC, Ann Arbor, MI, 1990, Thesis (Ph.D.)–University of California, Berkeley.

[Kob61] Shoshichi Kobayashi, *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. of Math. (2) **74** (1961), 570–574.

[Koi90] Norihito Koiso, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 327–337.

[Lau01] Jorge Lauret, *Ricci soliton homogeneous nilmanifolds*, Math. Ann. **319** (2001), no. 4, 715–733.

[LdSP17] Márcio Lemes de Sousa and Romildo Pina, *Gradient Ricci solitons with structure of warped product*, Results Math. **71** (2017), no. 3-4, 825–840.

[Li11] Chi Li, *On rotationally symmetric Kähler-Ricci solitons*, arXiv:1004.4049 (2011).

[Lot03] John Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv. **78** (2003), no. 4, 865–883.

[Lot07] ———, *Optimal transport and Ricci curvature for metric-measure spaces*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 229–257.

[LPP04] H. Lü, Don N. Page, and C. N. Pope, *New inhomogeneous Einstein metrics on sphere bundles over Einstein-Kähler manifolds*, Phys. Lett. B **593** (2004), no. 1-4, 218–226.

[LV09] John Lott and Cédric Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903–991.

[LW17] Peng Lu and Y. K. Wang, *Ancient solutions of the Ricci flow on bundles*, Adv. Math. **318** (2017), 411–456.

[Mal74] Bernard Malgrange, *Sur les points singuliers des équations différentielles*, Enseignement Math. (2) **20** (1974), 147–176.

[Mas11] Gideon Maschler, *Conformally Kähler base metrics for Einstein warped products*, Differential Geom. Appl. **29** (2011), no. 1, 85–92.

[Máx14] Davi Máximo, *On the blow-up of four-dimensional Ricci flow singularities*, J. Reine Angew. Math. **692** (2014), 153–171.

[MS13] Ovidiu Munteanu and Natasa Sesum, *On gradient Ricci solitons*, J. Geom. Anal. **23** (2013), no. 2, 539–561.

[Nab10] Aaron Naber, *Noncompact shrinking four solitons with nonnegative curvature*, J. Reine Angew. Math. **645** (2010), 125–153.

[Per02] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 (2002).

[Per03a] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math/0307245 (2003).

[Per03b] ———, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109 (2003).

[PP86] Don N. Page and C. N. Pope, *Einstein metrics on quaternionic line bundles*, Classical Quantum Gravity **3** (1986), no. 2, 249–259.

[PS97] Joon-Sik Park and Yusuke Sakane, *Invariant Einstein metrics on certain homogeneous spaces*, Tokyo J. Math. **20** (1997), no. 1, 51–61.

[PW09a] Peter Petersen and William Wylie, *On gradient Ricci solitons with symmetry*, Proc. Amer. Math. Soc. **137** (2009), no. 6, 2085–2092.

[PW09b] ______, *Rigidity of gradient Ricci solitons*, Pacific J. Math. **241** (2009), no. 2, 329–345.

[Qia97] Zhongmin Qian, *Estimates for weighted volumes and applications*, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 190, 235–242.

[Ran79] Burton Randol, *Cylinders in Riemann surfaces*, Comment. Math. Helv. **54** (1979), no. 1.

[RT15] Melanie Rupflin and Peter Topping, *A uniform Poincaré estimate for quadratic differentials on closed surfaces*, Calc. Var. Partial Differential Equations **53** (2015), no. 3-4, 587–604.

[RT16] Melanie Rupflin and Peter M. Topping, *Flowing maps to minimal surfaces*, Amer. J. Math. **138** (2016), no. 4, 1095–1115.

[RT18] ______, *Teichmüller harmonic map flow into nonpositively curved targets*, J. Differential Geom. **108** (2018), no. 1, 135–184.

[RTZ13] Melanie Rupflin, Peter M. Topping, and Miaomiao Zhu, *Asymptotics of the Teichmüller harmonic map flow*, Adv. Math. **244** (2013), 874–893.

[Ses06] Natasa Sesum, *Convergence of the Ricci flow toward a soliton*, Comm. Anal. Geom. **14** (2006), no. 2, 283–343.

[Son13] Jian Song, *Ricci flow and birational surgery*, arXiv:1304.2607 (2013).

[ST17] Jian Song and Gang Tian, *The Kähler-Ricci flow through singularities*, Invent. Math. **207** (2017), no. 2, 519–595.

[Sto17] Maxwell Stolarski, *Steady Ricci Solitons on Complex Line Bundles*, arXiv:1511.04087 (2017).

[Stu06a] Karl-Theodor Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131.

[Stu06b] ———, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177.

[Tia99] Gang Tian, *Kähler-Einstein manifolds of positive scalar curvature*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., vol. 6, Int. Press, Boston, MA, 1999, pp. 67–82.

[TZ00] Gang Tian and Xiaohua Zhu, *Uniqueness of Kähler-Ricci solitons*, Acta Math. **184** (2000), no. 2, 271–305.

[Win17a] Matthias Wink, *Cohomogeneity one Ricci Solitons from Hopf Fibrations*, arXiv:1706.09712 (2017).

[Win17b] ———, *Complete Ricci solitons via estimates on the soliton potential*, arXiv:1710.11108 (2017).

[Wu14] Haotian Wu, *On Type-II singularities in Ricci flow on* $\mathbb{R}^N$, Comm. Partial Differential Equations **39** (2014), no. 11, 2064–2090.

[WW98] Jun Wang and McKenzie Y. Wang, *Einstein metrics on* S^2*-bundles*, Math. Ann. **310** (1998), no. 3, 497–526.

[WW09] Guofang Wei and Will Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405.

[Wyl08] William Wylie, *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1803–1806.

[WZ86] McKenzie Y. Wang and Wolfgang Ziller, *Existence and nonexistence of homogeneous Einstein metrics*, Invent. Math. **84** (1986), no. 1, 177–194.

[WZ90] ———, *Einstein metrics on principal torus bundles*, J. Differential Geom. **31** (1990), no. 1, 215–248.

[WZ04] Xu-Jia Wang and Xiaohua Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188** (2004), no. 1, 87–103.

[Yan12] Bo Yang, *A characterization of noncompact Koiso-type solitons*, Internat. J. Math. **23** (2012), no. 5, 1250054, 13.

[Zil13] Wolfgang Ziller, *Group actions*, https://www.math.upenn.edu/∼wziller/math661/LectureNotesLee.pdf, 2013, Lecture notes for Math 260P, University of Pennsylvania, taken by Lee Kennard.