

# PROFINITE COMPLETIONS OF FREE-BY-FREE GROUPS CONTAIN EVERYTHING

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## Abstract

Given an arbitrary, finitely presented, residually finite group  $\Gamma$ , one can construct a finitely generated, residually finite, free-by-free group  $M_\Gamma = F_\infty \rtimes F_4$  and an embedding  $M_\Gamma \hookrightarrow (F_4 * \Gamma) \times F_4$  that induces an isomorphism of profinite completions. In particular, there is a free-by-free group whose profinite completion contains  $\hat{\Gamma}$  as a retract.

The finite quotients of a group  $\Gamma$  form a directed system and the *profinite completion* of  $\Gamma$  is the inverse limit of this system,  $\hat{\Gamma} := \varprojlim \Gamma/N$ . The natural map  $\Gamma \rightarrow \hat{\Gamma}$  is injective if  $\Gamma$  is residually finite, and two finitely generated groups  $\Gamma_1$  and  $\Gamma_2$  have the same set of finite images if and only if  $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$ . The purpose of this note is to demonstrate that the profinite completions of finitely generated, residually finite free-by-free groups contain, as retracts, the profinite completions of all subgroups of finitely presented groups.

**THEOREM A** *Given an arbitrary, finitely generated, recursively presented group  $\Gamma$  that is residually finite, one can construct a finitely generated, residually finite free-by-free group  $M_\Gamma = F_\infty \rtimes F_4$  and an embedding  $M_\Gamma \hookrightarrow (F_4 * \Gamma) \times F_4$  that induces an isomorphism of profinite completions.*

Note that  $D(\Gamma) := (F_4 * \Gamma) \times F_4$  is residually finite and the obvious retraction  $D(\Gamma) \rightarrow \Gamma$  induces a retraction  $\hat{M}_\Gamma \cong \widehat{D(\Gamma)} \rightarrow \hat{\Gamma}$ .

It follows from this theorem that the cohomological dimension of the profinite completion of a finitely generated, residually finite group of cohomological dimension 2 can be any positive integer or can be infinite (Section 2). And, despite being torsion-free itself, a free-by-free group can have all manner of torsion in its profinite completion (Section 3). The theorem also tells us that, with the possible (but unlikely) exception of certain free-by-free groups  $H$ , no statement of the following form can be valid for all pairs of finitely generated, residually finite groups  $\Gamma_1$  and  $\Gamma_2$ : ‘if  $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$  and  $\Gamma_1$  has a subgroup isomorphic to  $H$ , then  $\Gamma_2$  has a subgroup isomorphic to  $H$ .’ Furthermore, the theorem tells us that if a property  $\mathcal{P}$  is common to the subgroups of free-by-free groups but not to the subgroups of all finitely presented, residually finite groups, then  $\mathcal{P}$  is *not* a profinite invariant. Such properties include: being torsion-free; being locally indicable (i.e. every finitely generated subgroup maps onto  $\mathbb{Z}$ ); being left-orderable; all 2-generator subgroups being finitely presented (or coherent);

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all solvable (or amenable, or nilpotent) subgroups being finitely generated and abelian (of rank at most 2).

We shall see that Theorem A is a rather direct consequence of the following result, which is proved in [3] using celebrated theorems of Higman [4] and Baumslag, Dyer and Heller [1]. A group  $G$  is termed *acyclic* if  $H_i(G, \mathbb{Z}) = 0$  for all  $i \geq 1$ .

**THEOREM B ([3])** *There is a finitely presented acyclic group  $U$  such that:*

- (1)  *$U$  has no proper subgroups of finite index;*
- (2) *every finitely generated, recursively presented group can be embedded in  $U$ .*

## 1. The construction

The *fibre product* of a pair of epimorphisms  $p_i : G_i \twoheadrightarrow Q$  ( $i = 1, 2$ ) is the subgroup  $P = \{(g_1, g_2) \mid p_1(g_1) = p_2(g_2)\} < G_1 \times G_2$ . We need the following well-known lemma.

**LEMMA 1.1** *If  $G_1$  and  $G_2$  are finitely generated and  $Q$  is finitely presented, then  $P$  is finitely generated.*

*Proof.* For  $i = 1, 2$ , let  $S_i \subset G_i$  be a finite set that generates  $G_i$ . For each  $s \in S_1$  choose  $u_s \in G_2$  so that  $p_1(s) = p_2(u_s)$ . Similarly, for each  $t \in S_2$  choose  $v_t \in G_1$  so that  $p_2(t) = p_1(v_t)$ . As  $G_1$  is finitely generated and  $Q$  is finitely presented, there is a finite set  $R \subset G_1$  whose conjugates generate  $\ker p_1$ . It is easy to check  $P$  is generated by  $\{(s, u_s), (v_t, t), (r, 1) \mid r \in R, s \in S_1, t \in S_2\}$ .  $\square$

The following proposition originates in the work of Platonov and Tavgen [7]. They only considered the case  $p_1 = p_2$ , but the adaptation to the asymmetric case is straightforward [2].

**PROPOSITION 1.2** *For  $i = 1, 2$ , let  $p_i : G_i \twoheadrightarrow Q$  be an epimorphism of groups. If  $G_1$  and  $G_2$  are finitely generated and  $Q$  is finitely presented, with  $\hat{Q} = 1$  and  $H_2(Q, \mathbb{Z}) = 0$ , then the inclusion of the fibre product  $P \hookrightarrow G_1 \times G_2$  induces an isomorphism of profinite completions  $\hat{P} \cong \hat{G}_1 \times \hat{G}_2$ .*

We shall need the following refinement of Theorem B. I do not contend that there is any real significance to the integer 4 in this statement (and Theorem A); with sufficient effort, one might well be able to construct a 2-generator group  $U$  with the desired properties.

**LEMMA 1.3** *There is a 4-generator group  $U$  with the properties described in Theorem B.*

*Proof.* Theorem B is proved on pages 20 and 21 of [3]. The construction of  $U$  begins with Higman's universal group  $U_0$ , which can be generated by two elements. A particular HNN extension  $U^\dagger$  of  $U_0$  is constructed and  $U$  is an amalgamated free product  $U^\dagger *_\mathbb{Z} B$  where  $B$  is any finitely presented acyclic group that has an element of infinite order  $\tau$  but no non-trivial finite quotients. The amalgamation identifies  $\tau$  with the stable letter of the HNN extension  $U^\dagger$ , so  $U$  is generated by  $U_0$  and  $B$ . We take  $B$  to be the 2-generator group constructed in [6].  $\square$

*Proof of Theorem A.* Let  $U$  be a 4-generator group that satisfies Theorem B. We fix an epimorphism  $\mu : F_4 \twoheadrightarrow U$ . Given a finitely generated, recursively presented group  $\Gamma$ , we fix an embedding  $\psi : \Gamma \hookrightarrow$

$U$  and extend this to an epimorphism  $\Psi: F_4 * \Gamma \rightarrow U$  that restricts to  $\mu$  on  $F_4$  and  $\psi$  on  $\Gamma$ . Consider the fibre product of  $\Psi$  and  $\mu$ ,

$$P < (F_4 * \Gamma) \times F_4.$$

Lemma 1.1 assures us that  $P$  is finitely generated and Proposition 1.2 tells us that the inclusion  $P \hookrightarrow (F_4 * \Gamma) \times F_4$  induces an isomorphism of profinite completions.

The restriction of  $\Psi$  to each conjugate of  $\Gamma$  is injective, so by the Kurosh subgroup theorem  $\ker \Psi$  is free. The projection from  $P$  to the second factor of  $(F_4 * \Gamma) \times F_4$  is onto and has kernel  $\ker \Psi$ . Thus,  $P$  is free-by-free; more precisely, it is of the form  $F_\infty \rtimes F_4$ . Define  $M_\Gamma = P$ .  $\square$

REMARK 1.4  $M_\Gamma$  is not finitely presented.

## 2. Cohomological dimension

The fibre products  $M_\Gamma$  that we are considering are free-by-free (and not free) and hence have cohomological dimension 2. But if  $\Gamma$  has cohomological dimension  $d$  then  $D(\Gamma) := (F_r * \Gamma) \times F_4$  has cohomological dimension  $d + 1$ . Thus, Theorem A yields pairs of finitely generated, residually finite groups that have the same profinite completion but have an arbitrary difference in their cohomological dimensions; for example, we can take  $\Gamma \cong \mathbb{Z}^d$ . Moreover,  $D(\mathbb{Z}^d)$  is good in the sense of Serre [8] and it retracts onto  $\mathbb{Z}^{d+1}$ , so  $\widehat{D(\mathbb{Z}^d)} \cong \widehat{M}_\Gamma$  also has cohomological dimension  $d + 1$ . Thus, Theorem A provides us with examples of groups of cohomological dimension 2 whose profinite completions have cohomological dimension  $d + 1$ , where  $d$  is arbitrary. One can also arrange for  $\widehat{M}_\Gamma$  to have infinite cohomological dimension (even if it is torsion free).

## 3. Torsion

In [5], Lubotzky used the congruence subgroup property to exhibit ‘as much torsion as one can wish’ in the profinite completion of certain torsion-free arithmetic groups. Theorem A shows that torsion is similarly unconstrained in the profinite completions of free-by-free groups, since  $\widehat{M}_\Gamma \cong \widehat{D(\Gamma)}$  retracts onto  $\widehat{\Gamma}$ . In this case, one can encode the torsion into  $\Gamma$  directly.

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## References

1. G. Baumslag, E. Dyer and A. Heller, The topology of discrete groups, *J. Pure Appl. Algebra* **16** no. 1 (1980), 1–47. [10.1016/0022-4049\(80\)90040-7](https://doi.org/10.1016/0022-4049(80)90040-7)

2. M. R. Bridson, The strong profinite genus of a finitely presented group can be infinite, *J. Eur. Math. Soc. (JEMS)* **18** no. 9 (2016), 1909–1918. [10.4171/jems/633](https://doi.org/10.4171/jems/633)
3. M. R. Bridson, The homology of groups, profinite completions, and echoes of Gilbert Baumslag, *Elementary Theory of Groups and Group Rings, and Related Topics*, 11–28, De Gruyter, Berlin, 2020.
4. G. Higman, Subgroups of finitely presented groups, *Proc. Royal Soc. Series A.* **262** (1961), 455–475.
5. A. Lubotzky, Torsion in profinite completions of torsion-free groups, *Quart. J. Math. Oxford (2)* **44** no. 3 (1993), 327–332. [10.1093/qmath/44.3.327](https://doi.org/10.1093/qmath/44.3.327)
6. A. Y. Ol’shanskii and M. V. Sapir, A 2-generated, 2-related group with no non-trivial finite quotients, *Algebra and Discrete Mathematics* **2** (2007), 111–114.
7. V. P. Platonov and O. I. Tavgen, Grothendieck’s problem on profinite completions and representations of groups, *K-Theory* **4** no. 1 (1990), 89–101. [10.1007/BF00534194](https://doi.org/10.1007/BF00534194)
8. J. -P. Serre, *Galois Cohomology*. Springer-Verlag, Berlin, 1997.