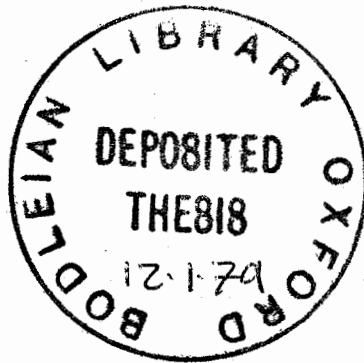


INTUITIONISTIC SET THEORY

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(i)

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ABSTRACT

We describe the formal system of *higher-order intuitionistic logic* with power types and (impredicative) comprehension which provides the basis for our "set theory"; this is adapted from the system of FOURMAN (D.Phil. Thesis, Oxford 1974), and such theories are equivalent to the notion of a *topos*. We interpret the basic concepts of *sets*, *relations* and *functions*; we consider relations of *equality*, *apartness* and *order*, and develop some of the intuitionistic theory of *complete Heyting algebras* (cHa's). We give the semantical definitions which make *sheaves* over a cHa into a model for the formal system. We give a unified description of the Dedekind reals and Baire space as the spaces of models of *geometric propositional theories*, from which follows the usual characterisation of them as they appear in sheaf models.

We investigate some notions from *topology*, adapting classical definitions and proofs where possible, and using sheaf models as counter-examples, especially for the Cauchy and Dedekind reals. Topologies are given by "open" sets as "closed" ones are unsuitable; strong forms of the basic *separation principles* arise in the presence of an apartness relation.

Well-founded relations are those satisfying the principle of induction, and *well-orderings* act as the unique representatives of their "ranks". This class of well-orderings has good closure properties, but they need not be linearly ordered nor have *Cantor normal forms*; the *Hartogs' number* of an infinite set forms a regular limit well-ordering. We consider some notions of *cardinality*, in particular the many possible notions of *finiteness*. For sets with apartness different ones arise; we characterise strongly the sense in which real polynomials have "finitely many" roots.

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PART ONE

HIGHER-ORDER LOGIC

INTRODUCTION

The notion of *set* to be investigated in this thesis corresponds to that expressed in the definition of a *topos*, since topoi are equivalent to formal systems of *intuitionistic higher-order logic* (for example FOURMAN [3]). Such systems allow the formation of *power* and *product* types, and include full (impredicative) *comprehension* axioms; power types are essential if we are to define within the system the central notion of a *complete Heyting algebra*. The system we use is adapted from that of FOURMAN [3] (see also FOURMAN and SCOTT [7]), and extends the "pure" system by the inclusion of a type of *natural numbers*, with suitable axioms; it also contains an *existence predicate* and *description operator* since these arise naturally in our principal models — *sheaves* over complete Heyting algebras.

In fact, we will not be very much concerned with formal systems in themselves, but rather with the distinctions and difficulties that arise in various parts of (set-based) mathematics when we are limited to intuitionistic reasoning; the result is part of what BISHOP [1] calls "affirmative mathematics". Definitions and proofs will of course never be strictly formal, but expressed rather in a loose, "mathematical" way.

We will also not be concerned with *Zermelo-Frankel* style set theories such as have been studied by, for example, *Feferman, Friedman, Myhill* and *Powell*. We considered some of the problems of working in such a theory in GRAYSON [9], which also includes the construction of sheaf models for the theory; but it has turned out that, for the problems that interest us, the type theory is a more natural setting.

We now give a brief summary of this first part:

Chapter I contains the description of the formal system and the interpretation of and notation for the basic concepts of sets, relations and functions within it; a *set* is simply an object of power type. We discuss briefly natural numbers, integers, rationals and (finite) sequences, as well as various *choice principles*; these are not included in the basic system (another difference from traditional intuitionism, or BISHOP [1]), and we use them rarely since they can fail to hold in sheaf models. *Apartness*

is the first example of a "strong" intuitionistic concept; many notions have stronger "duals" appropriate in the presence of an apartness. For example, sets with apartness form a *separated category* in the sense that each set of morphisms has an apartness.

In Chapter II we investigate various kinds of ordering. *Separated* orders have a natural apartness associated with them, and they coincide with the *pseudo-orders* already studied in the intuitionistic literature; a slight strengthening of this notion, which holds in the real line, is that each pair of elements has a maximum and minimum. *Complete Heyting algebras* (cHa's) are lattices suitable as sets of "truth-values" for the models of Chapter III. The property of *Kripke's Schema* provides a strong notion of "non-triviality" for a cHa, and the additional condition for *properness* acts as a constraint on its "logic".

In Chapter III we describe various constructions of *sheaves* over a cHa and give the semantical definitions which make them into a model for the formal system; such interpretations are a natural extension of *topological* models for intuitionistic first-order logic and analysis. The final section deals with some terms and formulae which are *absolute* for these interpretations.

In Chapter IV we construct some familiar spaces (power set of the natural numbers, Baire and Cantor spaces, Dedekind reals, complex numbers) as the spaces of models of *geometric propositional theories*, with the natural "finite information" topologies. Since the theories are absolute we obtain uniformly from this description the usual characterisation of these structures as they appear in topological sheaf models.

CHAPTER I
FUNDAMENTALS

I.1 THE FORMAL SYSTEM

The formal system of *higher-order intuitionistic logic* presented here is basically that of FOURMAN [3] and FOURMAN and SCOTT [7] to whom we refer for further elucidation. We leave a certain measure of freedom in the choice of ground types and constant terms, the only constraint being the inclusion of N , the type of *natural numbers*, and the constant 0 of type N .

The unfamiliar components of the language are the *existence predicate* (E), the *relation of equivalence* (\equiv), and the *description operator* (I). We may 'translate' them as follows:

$E\tau$: *the term τ denotes something*

$\sigma \equiv \tau$: *σ, τ are interchangeable salva veritate*

$Ix \in \alpha . \varphi$: *the object of type α such that φ .*

Terms then are not supposed necessarily to denote anything, which makes for a very simple theory of descriptions. When we quantify, of course, we do so over existing objects: thus we 'translate' the quantifiers as follows:

$\forall x \in \alpha . \varphi$: *for every existing object of type α , φ*

$\exists x \in \alpha . \varphi$: *an object of type α exists such that φ .*

We will introduce a *relation of equality* ($=$) by defining $(\sigma = \tau)$ iff $(E\sigma \wedge E\tau \wedge \sigma \equiv \tau)$, and 'translate' it by:

$\sigma = \tau$: *σ, τ both denote, and denote the same object.*

We could alternatively take $=$ as primitive in our language and define E, \equiv from it.

The logical axioms of the system are the natural ones for this interpretation. The remaining axioms govern natural numbers and n -tuples, and allow us to form arbitrary subsets by comprehension.

1. *The Language*

We now describe the language of the system, giving inductive definitions of the *types*, *terms* and *formulae*.

<u>TYPES</u> (α, β, \dots)	
N, ...	ground types including N
$P(\alpha)$	power types
$(\alpha_0 \times \dots \times \alpha_{n-1})$	product types
<u>TERMS</u> (σ, τ, \dots)	
0, ...	constant terms of various types including 0 of type N
x, y, z, \dots	variables of every type
$S(\tau)$	successor terms
$\langle \sigma_0, \dots, \sigma_{n-1} \rangle$	tuple terms
$\exists x \in \alpha. \varphi$	description terms
<u>FORMULAE</u> (φ, ψ, \dots)	
$E\tau, \sigma \equiv \tau, \sigma \in \tau$	atoms
$\left. \begin{array}{l} \varphi \wedge \psi, \varphi \vee \psi \\ \varphi \rightarrow \psi, \neg \varphi \end{array} \right\}$	propositional connectives
$\left. \begin{array}{l} \forall x \in \alpha. \varphi \\ \exists x \in \alpha. \varphi \end{array} \right\}$	quantifiers

NOTE: We will use $(\sigma = \tau)$ to abbreviate $(E\sigma \wedge E\tau \wedge \sigma \equiv \tau)$

2. *Assignment of types*

We now define by induction which terms and formulae are *well-formed* (w-f), and at the same time we *assign a type* to each w-f term:

<u>Term</u>	<u>Proviso</u>	<u>Type</u>
$\left. \begin{array}{l} \text{constants} \\ \text{variables} \end{array} \right\}$	none	as given
$S(\tau)$	τ w-f of type N	N
$\langle \sigma_0, \dots, \sigma_{n-1} \rangle$	σ_i w-f of type α_i	$(\alpha_0 \times \dots \times \alpha_n)$
$\exists x \in \alpha. \varphi$	φ w-f, x of type α	α
<u>Formula</u>	<u>Proviso</u>	
$E\tau$	τ w-f	
$\sigma \equiv \tau$	σ, τ w-f of same type	
$\sigma \in \tau$	τ w-f of type $P(\alpha)$ where σ w-f of type α	
$\left. \begin{array}{l} \varphi \wedge \psi, \varphi \vee \psi \\ \varphi \rightarrow \psi, \neg \varphi \end{array} \right\}$	φ, ψ w-f	
$\left. \begin{array}{l} \forall x \in \alpha. \varphi \\ \exists x \in \alpha. \varphi \end{array} \right\}$	φ w-f, x of type α .	

3. *Axioms and Rules (for well-formed formulae)*Propositional Connectives

The usual intuitionistic axioms govern $\wedge, \vee, \rightarrow, \neg$ together with the rules

$$(MP) \quad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

$$(Sub) \quad \frac{\varphi}{\varphi[\tau/x]}, \quad \varphi[\tau/x] \text{ being the result of substituting } \tau \text{ for } x \text{ at all free occurrences in } \varphi.$$

Quantifiers

$$(\forall) \quad \forall x \in \alpha. \varphi \wedge Ex \rightarrow \varphi$$

$$(\exists) \quad \varphi \wedge Ex \rightarrow \exists x \in \alpha. \varphi$$

$$(\forall^+) \quad \frac{\varphi \wedge Ex \rightarrow \psi}{\varphi \rightarrow \forall x \in \alpha. \psi}$$

$$(\exists^+) \quad \frac{\psi \wedge Ex \rightarrow \varphi}{\exists x \in \alpha. \psi \rightarrow \varphi}$$

Here x is to be of type α , and is not to occur free in φ in $(\forall^+), (\exists^+)$.

Equality

$$(\equiv) \quad \varphi[y/x] \wedge y \equiv z \rightarrow \varphi[z/x]$$

$$(E) \quad \forall x \in \alpha (x \equiv y \leftrightarrow x \equiv z) \rightarrow y \equiv z$$

$$(I) \quad \forall y \in \alpha [y \equiv \exists x \in \alpha. \varphi \leftrightarrow \forall x \in \alpha (\varphi \leftrightarrow x \equiv y)]$$

Natural Numbers

$$(OE) \quad E(0)$$

$$(O\equiv) \quad \neg (0 \equiv Sx)$$

$$(S) \quad x = y \leftrightarrow Sx = Sy$$

$$(N) \quad [0 \in y \wedge \forall x \in \mathbb{N} (x \in y \rightarrow Sx \in y)] \rightarrow \forall x \in \mathbb{N} . x \in y$$

Higher Types

$$(Comp) \quad E(\exists y \in P(\alpha) . \forall x \in \alpha (x \in y \leftrightarrow \varphi)) \quad (\text{for } y \text{ not free in } \varphi)$$

$$(\epsilon) \quad x \in y \rightarrow Ex \wedge Ey$$

$$(\langle \rangle E) \quad \forall x \in (\alpha_0 \times \dots \times \alpha_{n-1}) \exists x_0 \in \alpha_0, \dots, x_{n-1} \in \alpha_{n-1} . x = \langle x_0, \dots, x_{n-1} \rangle$$

$$(\langle \rangle =) \quad \langle x_0, \dots, x_{n-1} \rangle = \langle y_0, \dots, y_{n-1} \rangle \leftrightarrow (x_0 = y_0 \wedge \dots \wedge x_{n-1} = y_{n-1})$$

4. *Some remarks and notation*Abstractions

We introduce *abstraction terms* $\{x \in \alpha \mid \varphi\}$ to abbreviate $\exists y \in P(\alpha) . \forall x \in \alpha (x \in y \leftrightarrow \varphi)$. By the axiom (Comp) all such terms *exist*, and we have *extensionality* for each $P(\alpha)$:

$$\forall x, y \in P(\alpha) [x = y \leftrightarrow \forall z \in \alpha (z \in x \leftrightarrow z \in y)] .$$

We also use each *type* α as a *constant* of type $P(\alpha)$ to abbreviate

$\{x \in \alpha \mid x \equiv x\}$. If $\tau_0, \dots, \tau_{n-1}$ are all of type α , $\{\tau_0, \dots, \tau_{n-1}\}$ is $\{x \in \alpha \mid x = \tau_0 \vee \dots \vee x = \tau_{n-1}\}$; and if τ is of type α , and x_0, \dots, x_{n-1} of types $\alpha_0, \dots, \alpha_{n-1}$, $\{\tau \mid \varphi(x_0, \dots, x_{n-1})\}$ is $\{x \in \alpha \mid \exists x_0 \in \alpha_0, \dots, x_{n-1} \in \alpha_{n-1} \cdot x = \tau \wedge \varphi\}$.

Restricted Quantifiers, Abstractions, Descriptions

If τ is of type $P(\alpha)$ and x is not free in τ , we write

$\forall x \in \tau. \varphi$	for	$\forall x \in \alpha (x \in \tau \rightarrow \varphi)$
$\exists x \in \tau. \varphi$	for	$\exists x \in \alpha (x \in \tau \wedge \varphi)$
$\{x \in \tau \mid \varphi\}$	for	$\{x \in \alpha \mid x \in \tau \wedge \varphi\}$
$\text{I}x \in \tau. \varphi$	for	$\text{I}x \in \alpha (x \in \tau \wedge \varphi)$

Restrictions:

If τ is of type α we write $(\tau \upharpoonright \varphi)$ for $\text{I}x \in \alpha (x \equiv \tau \wedge \varphi)$, the *restriction of τ to φ* . Thus $\varphi \rightarrow (\tau \upharpoonright \varphi) \equiv \tau$ and $E(\tau \upharpoonright \varphi) \leftrightarrow (\varphi \wedge E\tau)$, so $(\tau \upharpoonright \varphi)$ exists only when φ holds, and then coincides with τ .

Projections:

If τ is of type $(\alpha_0 \times \dots \times \alpha_{n-1})$ and $i < n$, we write $\Pi_i^n(\tau)$ for $\text{I}y \in \alpha_i \exists y_0 \in \alpha_0, \dots, y_{n-1} \in \alpha_{n-1} [y \equiv y_i \wedge \tau \equiv \langle y_0, \dots, y_{n-1} \rangle]$. The axioms $(\langle \rangle E)$ and $(\langle \rangle \equiv)$ then guarantee that $\tau \equiv \langle \Pi_0^n(\tau), \dots, \Pi_{n-1}^n(\tau) \rangle$; $\Pi_i^n(\tau)$ is called the *i-th projection of τ* .

Connectives:

We write $\sigma \notin \tau$ for $\neg(\sigma \in \tau)$,

$$\bigwedge_{i < n} \varphi_i \text{ for } (\varphi_0 \wedge \dots \wedge \varphi_{n-1}), \quad \bigvee_{i < n} \varphi_i \text{ for } (\varphi_0 \vee \dots \vee \varphi_{n-1}).$$

5. *The law of excluded middle*

The law of excluded middle (*LEM*) is the axiom schema $(\varphi \vee \neg \varphi)$, which is notoriously not provable in our system (see III.2.4). A formula is said to hold *classically* iff it is provable on the assumption of *LEM*. There are many classical theorems whose truth would imply *LEM*, and we often use such facts to show that assertions are not provable in our system.

A weaker principle, the scheme $(\neg \varphi \vee \neg \neg \varphi)$, is also not provable: JOHNSTONE [14] has given a list of theorems equivalent to this principle, and we add to this in II.1.7.

I.2 SETS, RELATIONS AND FUNCTIONS

In this section we introduce expressions in words and symbols for the basic objects of our study of mathematics within the formal system of I.1. Thus we can translate ordinary mathematical language into the formal one. After this we will abandon the formal system almost entirely and proceed with definitions and proofs relying on an intuitive grasp of the constraints it imposes.

1. *Sets*

Definition: A *set* is an object whose type is a power-type. The *relation of inclusion* (\subseteq) between sets of the same type is given by $A \subseteq B$ iff $\forall a \in A. a \in B$.

If A is a set of type $P(\alpha)$, its *power-set* $P(A)$ is the set $\{X \in P(\alpha) \mid X \subseteq A\}$. The operations on $P(A)$ of *intersection*, *union* and *difference* (\cap, \cup, \setminus) are defined as usual.

If $X \subseteq A$, it is *inhabited* if $\exists a \in A (a \in X)$, and *detachable* (in A) iff $\forall a \in A (a \in X \vee a \notin X)$. We denote the set of detachable subsets of A by $P'(A)$.

If A is a set, its *empty* subset \emptyset_A is the set $\{a \in A \mid \neg Ea\}$. A subset X of A is a *singleton* iff $\forall a, b \in X (a = b)$; we denote the set of singleton subsets of A by \tilde{A} .

Remarks: Clearly a formula $a \in A$ or $A \subseteq B$ can only be *well-formed* if A, B are *sets*. Very often we introduce variables with an hypothesis such as "let $a \in A$ " or "if $A \subseteq B$ ", which determines the type of one variable in terms of that of the other, supposed already known. We also say that variables a, b, \dots are to *range over* a set A to mean that where they exist they are to belong to A ; so for example we may write $\forall a. \varphi$ to mean $\forall a \in A. \varphi$. This is just the standard mathematical practice of reserving some style of variable for elements of a domain.

2. *Relations*

Definition: If A_0, \dots, A_{n-1} are sets of types $P(\alpha_0), \dots, P(\alpha_{n-1})$, their *product* $(A_0 \times \dots \times A_{n-1})$ is the set

$$\{x \in (\alpha_0 \times \dots \times \alpha_{n-1}) \mid \bigwedge_{i < n} \Pi_i^n(x) \in A_i\}.$$

A *relation* on the sets A_0, \dots, A_{n-1} is a subset of their product. An *n-ary relation* on a set A is a subset of the n -fold product of A with itself, written A^n . The notions of *transitive*, *reflexive*, *irreflexive*, *symmetric*, *antisymmetric* and *asymmetric* binary relations are defined as usual.

Notation: If R is a relation we may write $R(a_0, \dots, a_{n-1})$ for $\langle a_0, \dots, a_{n-1} \rangle \in R$. If R is binary we may also write $a_0 R a_1$, and sometimes use the restricted quantifiers $\forall a_0 R a_1, \exists a_0 R a_1$, for given a_1 , in the usual way. We refer to unary relations as *predicates*.

3. Equalities

Definition: An *equality* on a set is a reflexive, symmetric, transitive binary relation. If \approx is an equality on A , it is *discrete* iff $\forall a, b \in A (a \approx b \vee \neg a \approx b)$. A subset X of A is *extensional with respect to \approx* iff $\forall a, b \in A (a \in X \wedge a \approx b \rightarrow b \in X)$.

If $\approx_0, \dots, \approx_{n-1}$ are equalities on A_0, \dots, A_{n-1} , their *product* is the equality \approx on $(A_0 \times \dots \times A_{n-1})$ such that

$$x \approx y \leftrightarrow \bigwedge_{i < n} \Pi_i^n(x) \approx_i \Pi_i^n(y).$$

If \approx is an equality on A and $a \in A$, the *equality class* of a is the set $\{b \in A \mid b \approx a\}$, written $[a]$. The *quotient* of A by \approx is $\{[a] \mid a \in A\}$ written (A/\approx) .

The *standard equality* on a set A is the relation determined by the symbol $=$.

Remarks: We "equip" a set with an equality when we do not wish to distinguish between elements which have some characteristic in common. In the presence of an equality subsets are supposed (unless otherwise indicated) to be extensional, that is, not to distinguish equal elements.

On the other hand we may wish to retain the identity of the elements with respect to the *standard* equality, and for this reason we refrain in general from taking quotients and reducing all equalities to the standard one. One may only do so "without loss of information" when the equality has a *choice operation* (I.2.5).

4. Functions

Definition: If \approx_A, \approx_B are equalities on A, B , F is a *partial function from A to B* ($F: A \xrightarrow{p} B$) iff it is an extensional subset of $(A \times B)$ such that

$$\forall a \in A \forall b, b' \in B (\langle a, b \rangle \in F \wedge \langle a, b' \rangle \in F \rightarrow b \approx_B b').$$

F is *total* ($F: A \rightarrow B$) iff

$$\forall a \in A \exists b \in B. \langle a, b \rangle \in F$$

F is *one-one* ($F: A \xrightarrow{p} B$) iff

$$\forall a, a' \in A \forall b \in B (\langle a, b \rangle \in F \wedge \langle a', b \rangle \in F \rightarrow a \approx_A a')$$

F is onto ($F: A \xrightarrow{p} B$) iff

$$\forall b \in B \exists a \in A. \langle a, b \rangle \in F.$$

The *domain* and *range* of F ($\text{dom} F, \text{rge} F$) are defined as usual, and so are $F(X), (F \upharpoonright X), F^{-1}(Y)$, for $X \subseteq A, Y \subseteq B$.

If $F: A \xrightarrow{p} B$ and $G: B \xrightarrow{p} C$, their *composition* ($G \circ F$) is the function $\{\langle a, c \rangle \mid \exists b \in B (\langle a, b \rangle \in F \wedge \langle b, c \rangle \in G)\}$. We form functions by λ -*abstraction* by putting

$$\lambda a \in A. \tau = \{\langle a, \tau \rangle \mid a \in A\}$$

The *identity function* relative to an equality \approx on A is just the set \approx , written also id_A .

Notation: We write $A \rightarrow B$ for $\exists F: A \rightarrow B$, and similarly for the other kinds of arrow. We also use the restricted quantifiers $\forall F: A \rightarrow B$ etc., as usual.

Remarks: Some care is needed in introducing terms formed by *functional application* when the equalities in question are not standard, since for example no proper sense can be given to the formula $b = F(a)$, for $F: A \xrightarrow{p} B$. We can introduce them however by giving the following interpretations to certain atomic formulae:

$$\begin{aligned} E(F(a)) & : \exists b \in B. \langle a, b \rangle \in F \\ F(a) \in Y & : \exists b \in Y. \langle a, b \rangle \in F \\ b \approx_B F(a) & : \langle a, b \rangle \in F. \end{aligned}$$

5. Operations

Definition: F is a (*partial*) *operation* from A to B ($F: A \xrightarrow{(p)} B$) iff it is a (*partial*) function with respect to their standard equalities.

F is *extensional* with respect to equalities \approx_A, \approx_B ($F: A \xrightarrow{(p)} B$) iff

$$\forall a, a' \in A (a \approx_A a' \rightarrow F(a) \approx_B F(a')).$$

If \approx is an equality on A , a *choice operation* for \approx is an $F: A \xrightarrow{0} A$ such that

$$\forall a, a' \in A [a \approx F(a) \wedge (a \approx a' \rightarrow F(a) = F(a'))].$$

Remarks: We only use the word "operation", of course, when there are *non-standard* equalities present and we need to distinguish the two kinds of function. This only occurs in IX.3 and X.2.

A choice operation for an equality serves to "choose" a representative from each equality class. When a choice operation exists we may pass freely between elements and equality classes, and between functions and

operations; in such a case there is therefore no loss in taking the quotient. The existence of choice operations is obviously equivalent to the full *axiom of choice*, discussed briefly in I.3.4.

6. *Disjoint unions*

Definition: The *disjoint union* $(A+B)$ of sets A, B is defined as follows:

We define $i: A \rightarrow (P(A) \times P(B))$ and $j: B \rightarrow (P(A) \times P(B))$ by $i(a) = \langle \{a\}, \emptyset_B \rangle$ and $j(b) = \langle \emptyset_A, \{b\} \rangle$, and put $(A+B) = (\text{rge}(i) \cup \text{rge}(j))$.

If \approx_A, \approx_B are equalities on A, B , their *disjoint union* is the equality \approx on $(A+B)$ such that

$$x \approx y \leftrightarrow [\exists a, a' \in A (i(a) = x \wedge i(a') = y \wedge a \approx_A a') \\ \vee \exists b, b' \in B (j(b) = x \wedge j(b') = y \wedge b \approx_B b')]$$

Remarks: The disjoint union serves to bring together two sets of different type as subsets of one set; we can also use it to "extend" a given set. We generally "identify" each element of A or B with its image under i or j , and i, j do not appear explicitly.

7. *Families of objects*

Definition: A *family of objects indexed by a set* I is a function with domain I ; we use the notation $\{F_i : i \in I\}$. If $\{F_i : i \in I\}$ is a family of sets, its *product*

$$\left(\prod_{i \in I} F_i \right) \text{ is } \left\{ f : I \rightarrow \bigcup_{i \in I} F_i \mid \forall i \in I. f(i) \in F_i \right\},$$

and its *disjoint union*

$$\left(\bigoplus_{i \in I} F_i \right) \text{ is } \left\{ \langle i, x \rangle \in I \times \bigcup_{i \in I} F_i \mid x \in F_i \right\}.$$

Note: If $\{F_i : i \in I\}$ is a family of objects, all the F_i 's must be of the *same type*, by definition. Thus the term $\bigcup_{i \in I} F_i$ is meaningful when they are all sets. If all the F_i 's are the *same* set A , their product is a "power" of A , written A^I , which is the set of functions from I to A .

8. *Structures*

Definition: A (*higher-order*) *structure* on a set A is a sequence (A_0, \dots, A_{n-1}) where each A_i is an element of a set obtained from A by forming power-sets and products. We define the notion of *structures for a given language*, or of a *certain type*, in the usual way. The preceding definition allows the formation of *families of structures*.

9. *Propositions - the power-set of $\{0\}$*

When we are concerned only with the *truth* of formulae (rather than with their *meaning*), we refer to them as *propositions* and use φ, ψ, \dots to vary over them. $P(\{0\})$ acts within the system as *the set of propositions* when we identify φ with $\{0 \uparrow \varphi\}$, and the subset X of $\{0\}$ with the proposition $(0 \in X)$. Then the *equivalence* (*conjunction, disjunction*) of propositions corresponds to the *equality* (*intersection, union*) of subsets. We carry this confusion to its logical conclusion and use φ, ψ, \dots to vary over propositions and $P(\{0\})$ indiscriminately.

A proposition φ is *decidable* iff $(\varphi \vee \neg \varphi)$, and *stable* iff $(\neg \neg \varphi \rightarrow \varphi)$. \top is the *true* proposition $(\varphi \rightarrow \varphi)$, and \perp is the *false* one $(\varphi \wedge \neg \varphi)$.

Classically then all propositions are decidable and stable, and in fact \top, \perp are the only propositions. Further, *LEM* is equivalent to the decidability (discreteness), or stability, of equality on $P(\{0\})$.

I.3 THE NATURAL NUMBERS AND RELATED STRUCTURES

We now give a very brief account of structure on the *natural numbers*, *integers* and *rationals*, and on *sets of (finite) sequences*. The intuitionistic theory is too well-known for us to need to say more. We also discuss *choice principles*.

1. *The natural numbers*

The induction axiom (N) justifies definitions by recursion on \mathbf{N} in the usual way (see also IX.1). We define then $[0] = \emptyset_{\mathbf{N}}$ and $[Sn] = [n] \cup \{n\}$, and put $m < n$ iff $m \in [n]$. The arithmetic sum, product and exponent are given the usual recursive definitions, and all equalities between (existing) terms are decidable.

We (generally) reserve i, j, k, m, n, \dots to range over \mathbf{N} .

2. *The integers and rationals*

We suppose the set of integers \mathbf{Z} and its order and arithmetic to be constructed in some standard way from \mathbf{N} , and similarly for the rationals \mathbf{Q} . We treat \mathbf{N} as a subset of \mathbf{Z} , and \mathbf{Z} as one of \mathbf{Q} . We use (generally) the variables p, q, r, \dots to range over \mathbf{Q} .

3. *Sequences*

Definition: An *infinite sequence* of elements of a set A is a function $f: \mathbf{N} \rightarrow A$.

A *finite sequence* is a function $f: \mathbf{N} \xrightarrow{p} A$ such that $\text{dom}(f) = [n]$ for some n , called the *length* of the sequence.

Notation: As in I.2.7 we use the notation $\{a_n : n \in \mathbf{N}\}$ for an infinite sequence from A ; similarly $\{a_{m,n} : m, n \in \mathbf{N}\}$ for a "double" sequence.

We denote the set of finite sequences from A by $\text{Seq}(A)$, the length of a sequence f by $\text{lh}(f)$ and the value $f(i)$ by $(f)_i$. The *concatenation* $(f * g)$ of two sequences is then the sequence $(f_0, \dots, f_{\text{lh}f-1}, g_0, \dots, g_{\text{lh}g-1})$ where, for $a_0, \dots, a_{n-1} \in A$, (a_0, \dots, a_{n-1}) denotes the obvious sequence.

4. *Choice principles*

The following principle of *finite choice* is easily justified by induction on n : if $\forall i < n \exists a \in A. \varphi(i, a)$, then $\exists f: [n] \rightarrow A (\forall i < n. \varphi(i, f(i)))$:

The principle of *countable choice (CAC)* extends this as follows:

if $\forall i \in \mathbb{N} \exists a \in A . \varphi(i, a)$, then $\exists f : \mathbb{N} \rightarrow A (\forall i \in \mathbb{N} . \varphi(i, f(i)))$. This principle fails to hold in some models (VIII.2) even when A is \mathbb{N} itself, so we do not include it in our basic system.

In VI.2 we prove some results in topology making use of the even stronger principle of *dependent choices* (DC), that if $\forall a \in A \exists b \in A . \varphi(a, b)$ then $\forall a \in A \exists f : \mathbb{N} \rightarrow A [f(0) = a \wedge \forall n \in \mathbb{N} . \varphi(f(n), f(Sn))]$. These results are included as "best approximations" to classical results.

Generally we do not wish to assume such principles, which makes our theory rather different at some points from that of traditional intuitionism, or that of BISHOP [1]. This point is related to questions of *continuity in parameters*, discussed in FOURMAN and HYLAND [6] and HYLAND [12] and [13].

The full *axiom of choice* (AC) asserts that if $\forall a \in A \exists b \in B . \varphi(a, b)$, then $\exists f : A \rightarrow B \forall a \in A . \varphi(a, f(a))$. As is well-known, this form implies LEM : given any proposition ψ , put $A = \{\{0, 1 \mid \psi\}, \{0 \mid \psi, 1\}\}$, $B = \{0, 1\}$ and $\varphi(a, b)$ iff $b \in a$; then if f is as given by AC , we have

$$\begin{array}{ll} f(\{0, 1 \mid \psi\}) = f(\{0 \mid \psi, 1\}) & \text{or not} \\ \text{so } \{0, 1 \mid \psi\} = \{0 \mid \psi, 1\} & \text{or not} \\ \text{so } (\psi \vee \neg \psi). & \end{array}$$

For this reason we obviously do not want AC as part of our system. There are however classical equivalents of AC (e.g. *Zorn's Lemma*) which do not imply LEM and indeed are valid in the models of Chapter III when AC is assumed "externally" (see GRAYSON [9] for details); on the other hand without LEM these principles do not have the usual classical applications. As is well-known the principles CAC and DC also hold in the models over *zero-dimensional* spaces, when they are assumed externally.

I.4 APARTNESS

The notion of an *apartness relation* is the first distinctively intuitionistic one we introduce; we refer to HEYTING [11] for a traditional exposition. Classically apartness just collapses to inequality, but intuitionistically it is much stronger. We "equip" a set with an apartness when we wish to *distinguish* its elements in some *positive* way. This occurs especially in the theories of *order* (II.1.4) and *topology* (VI.1.3), where apartness may be defined in terms of some more "primitive" structure. We use the tag "separated" to indicate the presence of an apartness in such cases.

We close the chapter with a digression on *separated categories* as an example of how notions appropriate in the presence of an equality very often have a stranger "dual" appropriate to an apartness.

1. Apertness relations

Definition: An *apartness relation* on a set A is a binary relation $\#$ such that

$$(i) \quad \forall a, b \in A (a = b \leftrightarrow \neg a \# b)$$

$$(ii) \quad \forall a, b \in A (a \# b \rightarrow b \# a)$$

$$(iii) \quad \forall a, b, c \in A (a \# c \rightarrow a \# b \vee b \# c).$$

A subset X of A is *strongly extensional with respect to* $\#$ iff $\forall a, b \in A (a \in X \rightarrow b \in X \vee a \# b)$.

If $\#_0, \dots, \#_{n-1}$ are apartness relations on A_0, \dots, A_{n-1} their *product* is the apartness $\#$ on $(A_0 \times \dots \times A_{n-1})$ such that

$$x \# y \leftrightarrow \bigvee_{i < n} \Pi_i^n(x) \#_i \Pi_i^n(y).$$

Remarks: As with equalities, in the presence of an apartness subsets are often supposed to be strictly extensional, that is, to "respect" the apartness. Note that condition (iii) for apartness is precisely that $\#$ be strongly extensional with respect to its product with itself (on A^2).

Condition (i) implies that (standard) equality on A is a *stable* relation. Thus by I.2.9 there can be no apartness on $P(\{0\})$ unless *LEM* holds. On the other hand, if equality is *decidable*, its *negation* determines an apartness, obviously, and then all subsets are strongly extensional. Fourman has given a simple argument to show that apartness relations are never unique.

2. *Apartness of functions*

If $\#_A, \#_B$ are apartness relations on A, B , we normally take functions $f: A \rightarrow B$ to be *strongly extensional* in the sense that

$$\forall a, a' \in A (f(a) \#_B f(a') \rightarrow a \#_A a').$$

This is easily seen to be equivalent to the strong extensionality of the relation $(f(a) \#_B b)$ as a subset of $(A \times B)$. The requirement that f itself be strongly extensional as a subset of $(A \times B)$ is far too strong; for example this condition applied to the *identity* function on A would be equivalent to

$$\forall a, a' \in A (a = a' \vee a \#_A a').$$

We will also be interested in functions which are *strongly one-one* ($f: A \twoheadrightarrow B$), that is,

$$\forall a, a' \in A (a \#_A a' \rightarrow f(a) \#_B f(a'))$$

Theorem (Scott): The category $\text{Set}^\#$ of sets with apartness relations and strongly extensional functions is cartesian closed.

Proof: To avoid describing the machinery of category theory we confine ourselves to the definitions of the apartness relations involved. Products have been defined above (I.4.1). Given $\#_A, \#_B$ on A, B and $f, g: A \rightarrow B$, we put

$$f \# g \leftrightarrow \exists a \in A. f(a) \#_B g(a).$$

Suppose that $f, g, h: A \rightarrow B$ and $f(a) \#_B g(a)$ for some $a \in A$. Then $(f(a) \#_B h(a) \vee h(a) \#_B g(a))$, so $(f \# h \vee h \# g)$, thus verifying condition (iii) for apartness. The other conditions are easily checked, and the necessary categorical properties are proved as for sets. \square

3. *Separated categories*

Elementary category theory is generally "constructive" in the sense that, for example, arguments by contradiction occur rarely. The category $\text{Set}^\#$ however suggests a line of development for a more overtly intuitionistic theory, which allows us to sharpen the statement of the preceding theorem.

Definition: A *separated category* is a category in which each set of morphisms $\text{Hom}(A, B)$ is equipped with an apartness such that the operation of *composition* of morphisms is strongly extensional.

Theorem: $\text{Set}^\#$ is a separated category and is cartesian closed in the strong sense that the operations of forming *pairs*, *products* and

exponential adjoints of morphisms are all strongly extensional.

Proof: With apartness of morphisms as before, let $f, f' : A \rightarrow B$, $g, g' : B \rightarrow C$, and $(g \circ f) \# (g' \circ f')$. Then for some $a \in A$, $g(f(a)) \#_C g'(f'(a))$, so $[g(f(a)) \#_C g'(f(a)) \vee g'(f(a)) \#_C g'(f'(a))]$. In the first case $g \# g'$, and in the second $f(a) \#_B f'(a)$ as g' is strongly extensional, so $f \# f'$. Thus $\text{Set}^\#$ is separated, and the remaining conditions may be checked in a similar way. \square

Remarks: It is not hard to devise other "strong" notions suitable in a separated category. For example we may define f to be a *strong monomorphism* by the condition $g \# g' \rightarrow (f \circ g) \# (f \circ g')$. It turns out in $\text{Set}^\#$ that strong monomorphisms are exactly the *strongly one-one* functions.

CHAPTER II

ORDERINGSII.1 ORDER RELATIONS

We define here the various kinds of *order relation* that will feature in later work. Many distinctions arise that do not exist classically, and some of these have already been investigated by the intuitionists (see HEYTING [11]). To show that such distinctions are genuine we define structures based on some given formula φ , and show that the identity of two notions would imply $(\varphi \vee \neg\varphi)$, or less strongly $(\neg\varphi \vee \neg\neg\varphi)$. (See I.1.5.)

We consider *order topologies* in VI.1, and *well-orderings* in IX.2.

1. Partial orders

Definition: An *inclusive partial order* (ipo) is a transitive, reflexive binary relation. If \leq is an ipo on the set A , its *associated equality* \approx is defined by

$$a \approx b \leftrightarrow (a \leq b \wedge b \leq a).$$

The *closed intervals* for $a, b \in A$ are given by

$$\begin{aligned} [a, \infty] &= \{c \in A \mid a \leq c\} \\ [-\infty, b] &= \{c \in A \mid c \leq b\} \\ [a, b] &= ([-\infty, b] \cap [a, \infty]). \end{aligned}$$

An *interval* is a subset X of A such that $\forall a, b \in X. [a, b] \subseteq X$.

Definition: A *strong partial order* (spo) is a transitive, irreflexive binary relation. If $<$ is an spo on A , the *open intervals* for $a, b \in A$ are given by

$$\begin{aligned} (a, \infty) &= \{c \in A \mid a < c\} \\ (-\infty, b) &= \{c \in A \mid c < b\} \\ (a, b) &= ((-\infty, b) \cap (a, \infty)). \end{aligned}$$

The spo $<$ is *dense* iff $\forall a, b \in A. (a < b \rightarrow \exists c \in (a, b))$.

Remarks: *Classically* we may pass between inclusive and strong partial orders (when the associated equality is *standard*) by the equivalences

$$\begin{aligned} \text{(i)} \quad a < b &\leftrightarrow (a \leq b \wedge \neg a = b) \\ \text{(ii)} \quad a \leq b &\leftrightarrow (a < b \vee a = b) \end{aligned}$$

But, in passing from an ipo to an spo and back again, we recover the original ipo only if equality is *discrete*. So, as with equality and apartness, we must separate these two notions which are interchangeable classically.

Example: On any power-set $P(A)$ we have the ipo of *inclusion* (\subseteq). We also have an spo $\not\subseteq$ given by

$$X \not\subseteq Y \leftrightarrow (X \subseteq Y \wedge \exists a \in (Y \setminus X)).$$

Classically these two are related by the equivalences above, but either of these (in $P(\{0\})$) would imply *LEM*:

(i) For any proposition φ (I.2.9), $\neg\neg(\varphi \vee \neg\varphi)$ so $(\perp \subseteq (\varphi \vee \neg\varphi) \wedge \perp \not\subseteq (\varphi \vee \neg\varphi))$ and (i) gives $\perp \not\subseteq (\varphi \vee \neg\varphi)$, whence $(\varphi \vee \neg\varphi)$.

(ii) For any proposition φ , $(\varphi \subseteq T)$ so (ii) gives $(\varphi \not\subseteq T \vee \varphi = T)$, whence $(\varphi \vee \neg\varphi)$.

2. Complete lattices

Definition: If \leq is an ipo on a set A and $X \subseteq A$, a *supremum* for X is an $a \in A$ such that

$$\forall b \in A (a \leq b \leftrightarrow \forall c \in X. a \leq b).$$

An *infimum* for X similarly is an $a \in A$ such that

$$\forall b \in A (b \leq a \leftrightarrow \forall c \in X. b \leq c).$$

Definition: A *complete lattice* is an ipo in which every subset has a supremum and infimum, and whose associated equality is standard (i.e. the relation is *antisymmetric*).

Notation: In a complete lattice suprema and infima are *unique*, and we use $\vee, \vee, \wedge, \wedge$ for these. We also write T for $(\vee A)$, the "top" element, and \perp for $(\wedge A)$, the "bottom" one.

Example: Every power-set $P(A)$ is a complete lattice under inclusion, with the operations of \vee and \wedge given by \cup and \cap .

3. Filters

Definition: If (A, \leq) is a complete lattice, a *filter* is a subset Δ of A such that $T \in \Delta$ and

$$\forall a, b \in A ((a \wedge b) \in \Delta \leftrightarrow (a \in \Delta \wedge b \in \Delta)).$$

Δ is a *superfilter* iff also

$$\forall X \subseteq A ((\vee X) \in \Delta \leftrightarrow \exists a \in X. a \in \Delta).$$

The *space of superfilters* $\Delta(A)$ consists of the set of superfilters with the *topology* of open sets $U_a = \{\Delta \mid a \in \Delta\}$ for $a \in A$. (See also IV.1.)

4. *Separated orders*

Any $\text{spo} <$ determines a natural notion of "distinctness" by the relation $(a < b \vee b < a)$; spo 's for which this is an *apartness* will turn out to coincide with the *pseudo-orders* familiar from the intuitionistic literature (e.g. HEYTING [11]), and we will assume without proof the well-known properties of these. Classically they are always *simple* (II.1.6), that is, they satisfy the *law of trichotomy*; we will see in II.1.7 that this assertion implies *LEM*.

Definition: A *separated order* is a $\text{spo} <$ such that the relation $\#$ defined by

$$a \# b \leftrightarrow (a < b \vee b < a)$$

is an apartness; this is called its *associated apartness*. The *associated ipo* is the relation \leq defined by

$$a \leq b \leftrightarrow \forall c < a . c < b.$$

Proposition: An $\text{spo} <$ is separated iff it satisfies

$$(i) \quad \forall a, b \in A (\neg a < b \wedge \neg b < a \rightarrow a = b), \text{ and}$$

$$(ii) \quad \forall a, b, c \in A (a < c \rightarrow a < b \vee b < c).$$

Proof: Let $<$ be separated, and $\#$ its associated apartness; then (i) is just condition (i) for apartness. For (ii), let $a < c$, so $a \# c$. Then as $\#$ is an apartness $(a \# b \vee b \# c)$, so by definition

$$(a < b \vee b < a \vee b < c \vee c < b)$$

and transitivity of $<$ gives $(a < b \vee b < c)$.

Conversely, we must prove condition (iii) for apartness from assumption (ii): if $a \# c$, without loss $a < c$. Then $(a < b \vee b < c)$, so $(a \# b \vee b \# c)$. □

Remarks: This proposition identifies separated with pseudo-orders, and we may then quote the following well-known results for them:

(i) The associated $\text{ipo} \leq$ satisfies

$$\forall a, b \in A (a \leq b \leftrightarrow \neg (a > b) \leftrightarrow \forall c > b . c > a).$$

(ii) $<$ is strongly extensional with respect to $\#$.

(iii) $\forall a, b \in A (\forall c \in A (c < a \leftrightarrow c < b) \rightarrow a = b)$.

The point of defining them as we have done is to highlight the logical relation with apartness. In VI.1 we do the same for topology and establish a nice connection via order topologies.

5. *Linear orders*

We now strengthen slightly the notion of a separated order by adding a property which holds of the order on the *real line* (see IV.2) and is equivalent to the associated ipo forming a *lattice*. It is still an open question as to whether any stronger "linearity" condition is valid in the reals. We will see in II.1.7 that these orders do not need to be simple.

Definition: A *linear order* is a separated spo with operations of *Max* and *Min* such that

$$\forall a, b, c \in A \quad (c > \text{Max}(a, b) \leftrightarrow c > a \wedge c > b)$$

and
$$\forall a, b, c \in A \quad (c < \text{Min}(a, b) \leftrightarrow c < a \wedge c < b).$$

Proposition: A separated spo is linear iff its associated ipo has suprema and infima for every pair of elements (i.e. it is a *lattice*)

Proof: Put $(a \vee b) = \text{Max}(a, b)$ and $(a \wedge b) = \text{Min}(a, b)$. □

6. *Simple orders*

Definition: A *simple spo* is one satisfying the *law of trichotomy*, that is,

$$\forall a, b \in A \quad (a = b \vee a < b \vee b < a) .$$

Examples: The usual orders on the natural numbers, integers and rationals are simple.

7. *Relation of separated, linear and simple orders*

The implications *Simple* \Rightarrow *Linear* \Rightarrow *Separated* are obvious, and the converses hold classically. That they fail intuitionistically follows from the

Proposition:

(i) The implication *Separated* \Rightarrow *Linear* is equivalent to the logical law $(\neg \varphi \vee \neg \neg \varphi)$ (see I.1.5).

(ii) The implication *Linear* \Rightarrow *Simple* implies *LEM* .

Proof: (i) Let $<$ be separated and suppose that $(\neg \varphi \vee \neg \neg \varphi)$ holds for all φ . Then given $a, b \in A$, $(\neg a < b \vee \neg \neg a < b)$; if $\neg (a < b)$, put $\text{Max}(a, b)$ equal to a , and if $\neg \neg (a < b)$, put it equal to b . A similar argument supplies $\text{Min}(a, b)$.

Conversely, for any proposition φ consider an spo on $(\{0, 1, 2\} / \approx)$, where $0 \approx 2$ iff φ , $1 \approx 2$ iff $\neg \varphi$,

$$[0] < [1] \text{ iff } [0] < [2] \text{ iff } \varphi$$

and
$$[1] < [0] \text{ iff } [1] < [2] \text{ iff } \neg \varphi .$$

This is *separated* and in fact every pair of elements has a *maximum*. But $\text{Min}([0],[1])$ is $[0]$ iff $\neg \neg \varphi$, it is $[1]$ iff $\neg \varphi$, and it is $[2]$ iff $(\neg \varphi \wedge \neg \neg \varphi)$. So the existence of $\text{Min}([0],[1])$ implies $(\neg \varphi \vee \neg \neg \varphi)$.

(ii) Consider an *spo* on $(\{0,1\}/\approx)$, where $0 \approx 1$ iff $\neg \neg \varphi$, and $[0] < [1]$ iff φ . This is *linear*, but *simple* only if $(\varphi \vee \neg \neg \varphi)$.

□

II.2 COMPLETE HEYTING ALGEBRAS

The use of *Heyting algebras* (*implicational lattices*, *pseudo-Boolean algebras*) to model intuitionistic propositional calculus is well-known (see RASIOWA and SIKORSKI [21]); for the interpretation of *quantifiers* in III.2 we need a *complete* lattice. We sketch those parts of the (classical) theory that we will need, and refer to FOURMAN and SCOTT [7] for an exposition in more detail. We also investigate some of the distinctive problems of the intuitionistic theory. These are concerned with the embedding of the *set of propositions* $P(\{0\})$ (see I.2.9) in a complete Heyting algebra, and are essentially trivial classically. Apart from the interesting distinctions which arise here, we need such properties for results in III.3; it was that need that first led to this investigation.

1. *Complete Heyting algebras*

Definition: A *complete Heyting algebra* (cHa) is a complete lattice (Ω, \leq) satisfying the *distributive law*: $\forall p \in \Omega \forall A \subseteq \Omega. p \wedge \bigvee A = \bigvee \{p \wedge q \mid q \in A\}$.

An *isomorphism* of cHa's Ω, Ω' is a function $F: \Omega \rightarrow \Omega'$ such that

$$\forall p, q \in \Omega. F(p) \leq' F(q) \leftrightarrow p \leq q.$$

Notation: We use Ω, Ω', \dots to denote cHa's and p, p', \dots to range over them. We write $(p \rightarrow q)$ for $\bigvee \{r \mid p \wedge r \leq q\}$, and $\neg p$ for $(p \rightarrow \perp)$.

2. *J-operators*

As examples of cHa's we have the lattices of *open* subsets of a *topological space* (see V.1). A well-known *representation theorem* tells us that all cHa's arise as *quotients* of these by *J-operators*, where we make the following

Definition: A *J-operator* on a cHa Ω is a function $J: \Omega \rightarrow \Omega$ such that

$$\forall p \in \Omega. p \leq Jp = J(Jp)$$

and

$$\forall p, q \in \Omega. J(p \wedge q) = (Jp \wedge Jq).$$

The *quotient* of Ω by J , written (Ω/J) , is the cHa formed by the *range* (or *fixed points*) of J , under the same ipo.

Examples: For each $p \in \Omega$ the functions $J_p = \lambda q. (p \wedge q)$, $J^p = \lambda q. (p \rightarrow q)$, $B_p = \lambda q. ((q \rightarrow p) \rightarrow p)$ are all J-operators. Since $B_\perp = \lambda q. (\neg \neg q)$, we write $(\neg \neg)$ for B_\perp .

For more information about J-operators see FOURMAN and SCOTT [7], whose results we shall use freely.

3. Positive elements - Kripke's schema

The notion of a *positive* element of a cHa is introduced to correspond in the general case to the property, in the topological case, of an open set being *inhabited*, that is, of being different from the bottom element \perp in a "positive" way. Kripke's Schema is then a "positive" condition of non-triviality for a cHa, and is classically equivalent to $T \neq \perp$. That this fails intuitionistically will follow from II.2.7.

Definition: Let Ω be a cHa, and φ any proposition. We put $\hat{\varphi} = V\{T \uparrow \varphi\}$, and say that $p \in \Omega$ is *positive* ($\text{Pos}(p)$) iff

$$\forall \varphi \in P(\{0\}). p \leq \hat{\varphi} \rightarrow \varphi \quad (\text{see I.2.9}).$$

Kripke's Schema (KS) is the condition $\text{Pos}(T)$.

Proposition: (i) The function $\lambda \varphi. \hat{\varphi}$ into a cHa Ω preserves \wedge and V , that is

$$\forall \varphi, \psi \in P(\{0\}). (\varphi \wedge \psi)^\wedge = (\hat{\varphi} \wedge \hat{\psi})$$

and

$$\forall A \subseteq P(\{0\}). (UA)^\wedge = V\{\hat{\varphi} \mid \varphi \in A\}.$$

(ii) If X is a topological space and $U \in \mathcal{O}(X)$, $\text{Pos}(U)$ iff $\exists x \in U$; so $\mathcal{O}(X)$ satisfies *KS* iff X is inhabited.

(iii) A quotient (Ω/J) satisfies *KS* iff

$$\forall \varphi \in P(\{0\}) (J\hat{\varphi} = T \rightarrow \varphi).$$

Proof: (i) is trivial. For (ii) note that for $U \in \mathcal{O}(X)$, $U \subseteq (\exists x \in U)^\wedge$, so that if $\text{Pos}(U)$, $(\exists x \in U)$. *Conversely* if $(\exists x \in U)$ and $U \subseteq \hat{\varphi}$, φ must hold since $\hat{\varphi} = \{y \in X \mid \varphi\}$.

(iii) We use the subscript "J" for operations in (Ω/J) . Then

$$\hat{\varphi}^J = V_J\{T_J \mid \varphi\} = J(V\{T \mid \varphi\}) = J(\varphi),$$

and the result follows. □

4. Equivalents of Kripke's Schema

We now prove a theorem stating the equivalence of *KS* to other assertions about a cHa. The first of these accounts for the name "Kripke's Schema" since it means that every proposition is equivalent to one in the language of the cHa. The second says that $P(\{0\})$ is *embedded monomorphically* in it.

Theorem: A cHa Ω satisfies *KS* iff either of the following conditions holds:

(i) $\forall \varphi \in P(\{0\}) \exists p, q \in \Omega \ (p \leq q \leftrightarrow \varphi)$

(ii) $(\lambda \varphi . \hat{\varphi})$ is one-one.

Proof: (i) If *KS* holds then, for every φ , $(\varphi \leftrightarrow T \leq \hat{\varphi})$. Conversely if (i) holds and φ is given, let $(\varphi \leftrightarrow p \leq q)$, so that $(p \wedge \hat{\varphi}) \leq q$. Then if $T = \hat{\varphi}$, $p = (p \wedge \hat{\varphi}) \leq q$; so φ must hold.

(ii) If *KS* holds and $\hat{\varphi} = \hat{\psi}$, then

$$\varphi \text{ iff } (T = \hat{\varphi}) \text{ iff } (T = \hat{\psi}) \text{ iff } \psi.$$

Conversely assuming (ii), since $\hat{T} = T$, if $T = \hat{\varphi}$, $\hat{\varphi} = \hat{T}$; so φ must hold. \square

5. *Properness*

The notion of a *proper* cHa arises when we add to *K.S.* some natural conditions on the predicate *Pos*, which certainly hold in the topological case, and are always true classically since we can take cases on whether an element p is \perp or not; that this fails intuitionistically will follow from II.2.7. In the theorem below we give some conditions equivalent to properness, which should make its import clearer: the first is a useful tool for later work; the second characterises *Pos* as the unique predicate satisfying certain natural conditions; the last makes the function $(\lambda \varphi . \hat{\varphi})$ on *isomorphic embedding* of the set of propositions. Thus a proper cHa includes all the complexities of $P(\{0\})$; this is why a complete *Boolean* algebra cannot be proper unless *LEM* holds (II.2.7).

Definition: A cHa Ω is *proper* iff it satisfies *KS* and
 (*) $\forall p \in \Omega . p \leq (\text{pos}(p))^\wedge$.

Theorem: A cHa Ω is proper iff any one of the following conditions holds:

(i) *KS* and $\forall p, q \in \Omega \ [(Pos(p) \rightarrow p \leq q) \rightarrow p \leq q]$

(ii) There is a subset X of Ω such that

(a) $T \in X$

(b) $\forall A \subseteq \Omega \ ((\forall A) \in X \rightarrow \exists p \in (A \cap X))$

(c) $\forall p, q \in \Omega \ [(p \in X \rightarrow p \leq q) \rightarrow p \leq q]$

(iii) $(\lambda \varphi . \hat{\varphi})$ is one-one and preserves \wedge , that is,

$$\forall A \subseteq P(\{0\}) . (\wedge A)^\wedge = \wedge \{\hat{\varphi} \mid \varphi \in A\} .$$

Proof: (i) If Ω is proper and $(Pos(p) \rightarrow p \leq q)$ then $p = (p \wedge (Pos(p))^\wedge) \leq q$. Conversely since $(Pos(p) \rightarrow p \leq (Pos(p))^\wedge)$, (i) gives $p \leq (Pos(p))^\wedge$.

(ii) Assuming (i) we show that Pos itself satisfies the conditions of (ii), for which we have just to check (b): Let $\text{Pos}(VA)$. For each $p \in A$, $p \leq (\text{Pos}(p))^\wedge$, so $(VA) \leq (\exists p \in A. \text{Pos}(p))^\wedge$, whence $\exists p \in A. \text{Pos}(p)$, by definition of $\text{Pos}(VA)$.

Conversely if X satisfies just (b) and (c) we can show that $X = \text{Pos}$: if $p \in X$ and $p \leq \hat{\varphi}$, $\hat{\varphi} \in X$ so φ holds by (b); if $\text{Pos}(p)$, since $(p \in X \rightarrow p \leq (\text{Pos}(p))^\wedge)$, (c) gives $p \leq (\text{Pos}(p))^\wedge$, whence $p \in X$. The result follows.

(iii) Let $A \subseteq P(\{0\})$. Then $(\cap A)^\wedge \leq \Lambda\{\hat{\varphi} \mid \varphi \in A\}$ obviously, and we prove the converse inclusion *on the assumption* that $\text{Pos}(\Lambda\{\hat{\varphi} \mid \varphi \in A\})$, making use of (i) again: we have $\forall \varphi \in A. \text{Pos}(\hat{\varphi})$, so each $\varphi \in A$ is *true*, whence $(\cap A)^\wedge = T$.

Conversely if (iii) holds, let $p \in \Omega$ and put $A = \{\varphi \mid p \leq \hat{\varphi}\}$. Then $(\cap A)^\wedge$ iff $\text{Pos}(p)$, while clearly $p \leq \Lambda\{\hat{\varphi} \mid \varphi \in A\}$, whence $p \leq (\text{Pos}(p))^\wedge$. \square

6. A representation theorem

We prove now a representation theorem for proper cHa's as quotients of inhabited topological spaces. These are not the only kinds of quotient which are proper since (Ω/J^P) , for example, is proper just in case Ω is proper and $\text{Pos}(p)$. We need first a

Lemma: (i) The condition (*) always holds for the open subsets of a topological space.

(ii) If J is a J-operator on Ω such that

$$\forall \varphi \in P(\{0\}). J(\hat{\varphi}) = \hat{\varphi},$$

then (Ω/J) is proper iff Ω is proper.

Proof: (i) As in the proof of Proposition II.2.3(ii), for each open U , $U \subseteq (\exists x \in U)^\wedge = (\text{Pos}(U))^\wedge$.

(iii) If J satisfies this condition, it preserves the embedding $(\lambda\varphi.\hat{\varphi})$, and the result follows by II.2.5(iii). \square

Theorem: A cHa is proper iff it is the quotient of an inhabited topological space by a J-operator J such that

$$\forall \varphi \in P(\{0\}). J(\hat{\varphi}) = \hat{\varphi}.$$

Proof: One direction is immediate from the Lemma. Let Ω be proper and let $X = \{p \in \Omega \mid \text{Pos}(p)\}$ with the topology of *downwards-closed* subsets. For $p \in \Omega$ put $[p] = \{q \leq p \mid \text{Pos}(q)\}$, and for $U \in \mathcal{O}(X)$ put $J(U) = [VU]$. By *KS* $T \in X$, so X is *inhabited*, and for any φ

$$J(\hat{\varphi}) = [V\{p \in \Omega \mid \text{Pos}(p) \wedge \varphi\}] = [\hat{\varphi}] = \hat{\varphi} .$$

Finally $(\lambda p. [p])$ is an *isomorphism* of Ω with $(\mathcal{O}(X)/J)$: $[p] \subseteq [q]$ we prove $p \leq q$ on the assumption that $\text{Pos}(p)$, making use of Theorem II.2.5(i); but if $\text{Pos}(p)$, $p \in [p]$, so $p \in [q]$ as required. \square

7. Complete Boolean algebras

Definition: A *complete Boolean algebra* (cBa) is a cHa Ω such that $\forall p \in \Omega. (p \vee \neg p) = T$.

Examples: Classically all power-set algebras are cBa's, but this would obviously imply *LEM* if true of $P(\{0\})$ for example. A result of *John Cartmell* (see FOURMAN and SCOTT [7]) identifies the cBa's as the quotients of cHa's by *J*-operators of the form B_p . The following result shows that we cannot expect cBa's to be proper, and the most immediate constructions of cBa's need not even satisfy *KS*. We construct a cBa satisfying *KS* in II.2.8.

Proposition: (i) The existence of a proper cBa implies *LEM*.

(ii) A quotient $\mathcal{O}(X)/(\neg \neg)$ can only satisfy *KS* if *LEM* holds.

Proof: (i) Let Ω be a proper cBa and φ be given. For $p \in \Omega$, if $(p \wedge \hat{\varphi}) = \perp$ and $\text{Pos}(p)$, $\neg \varphi$ must hold. Thus by properness, $\neg(\hat{\varphi}) = (\neg \varphi)^\wedge$. But then $(\varphi \vee \neg \varphi)^\wedge = (\hat{\varphi} \vee \neg(\hat{\varphi})) = T$ as Ω is Boolean; *KS* gives then $(\varphi \vee \neg \varphi)$.

(ii) Let $\mathcal{O}(X)/(\neg \neg)$ satisfy *KS* and φ be given: $\hat{\varphi}$ as evaluated in this cBa is $\{x \in X \mid \neg \neg \varphi\}$; so $(\varphi \vee \neg \varphi)^\wedge = T$, and *KS* gives $(\varphi \vee \neg \varphi)$. \square

8. A cBa satisfying *KS*

The preceding proposition shows that without *LEM* it not quite trivial to construct *any* cBa satisfying *KS*. We do so as follows:

Let Ω be $P(P(\{0\}))$ under inclusion, and consider (Ω/B_p) where $p = \{\{0\}\}$. To show that this cBa satisfies *KS* we use Proposition II.2.3(iii):

Suppose that $B_p(\hat{\varphi}) = T$,

that is $(\hat{\varphi} \rightarrow \{\{0\}\}) \subseteq \{\{0\}\}$.

Now $(\hat{\varphi} \rightarrow \{\{0\}\}) = \{\psi \mid \varphi \rightarrow \psi\}$

but $(\varphi \rightarrow \varphi)$ always, so $\varphi = \{0\}$, or, in other words: φ holds, since $\{0\}$ represents the *true* proposition. \square

9. *Anti-Boolean cHa's*

Definition: A cHa Ω is *anti-Boolean* iff

$$\bigwedge \{(p \vee \neg p) \mid p \in \Omega\} = \perp .$$

Remarks: Anti-Boolean cHa's have a positively "anti-classical" logic in a sense that will become precise in III.2. As *examples* we have the open sets of most decent topological spaces (e.g. the reals).

CHAPTER III

SHEAF MODELSIII.1 SHEAVES

In this section we describe those parts of the theory of *sheaves* that will be needed later on. What we call a "sheaf" is usually known as a "stack", but this notion is essentially equivalent to the normal sense of "sheaf"; for details we refer to FOURMAN and SCOTT [7]. The form we use is more suitable for giving models in III.2, as the "partial sections" of a sheaf are to correspond to the "partial elements" of the logic.

1. *Pre-sheaves*

Definition: A *pre-sheaf* \mathbf{S} over a *cha* Ω consists of a set S with functions of *extent* and *restriction*, $E: S \rightarrow \Omega$ and

$$\uparrow: S \times \Omega \rightarrow S$$

satisfying, for all $a \in S$, $p, q \in \Omega$,

$$(a \uparrow E a) = a, \quad E(a \uparrow p) = (E a \wedge p), \quad (a \uparrow p) \uparrow q = (a \uparrow p \wedge q).$$

The *equivalence evaluation* on \mathbf{S} is defined by

$$\llbracket a \equiv b \rrbracket = \bigvee \{ p \in \Omega \mid (a \uparrow p) = (b \uparrow p) \}$$

A subset A of S is *compatible* iff

$$\forall a, b \in A. (a \uparrow E b) = (b \uparrow E a)$$

An element a of S is *global* iff $E a = T$.

2. *Sheaves*

Definition: A *sheaf* is a pre-sheaf \mathbf{S} in which every compatible subset A has a *unique join*, that is, an element $\bigvee A$ of S such that

$$E(\bigvee A) = \bigvee \{ E a \mid a \in A \} \quad \text{and} \quad \forall a \in A. (\bigvee A) \uparrow E a = a$$

Remarks: This *join* condition allows the *evaluation of terms* in a sheaf (especially, descriptions; see III.2.2). A pre-sheaf may have potential elements "missing", but we can "complete" it by forming its *sheafification*, which is the least sheaf into which it can be embedded (see FOURMAN and SCOTT [7]). We treat Ω as "fixed".

3. *Subsheaves*

Definition: A *subsheaf* of a sheaf \mathbf{S} is a subset which forms a sheaf under the operations of \mathbf{S} , that is, it is closed under *joins* and *restrictions*.

If $A \subseteq S$ the *subsheaf generated by A* is the least one containing A, that is

$$\{(b \uparrow \vee \{\llbracket a \equiv b \rrbracket \mid a \in A\}) \mid b \in S\}$$

A *generates S* iff the subsheaf generated by it is S.

Remarks: We make tacit use here and elsewhere of the properties of the equivalence evaluation, for which we refer to FOURMAN and SCOTT [7]. We note that a pre-sheaf *generates* its *sheafification* in the sense above.

4. Power-sheaves

To correspond to the formation of *power types* we need:

Definition: A *predicate* on a pre-sheaf S is a function $\Phi: S \rightarrow \Omega$ such that

$$\forall a \in S \forall p \in \Omega. \Phi(a \uparrow p) = (\Phi(a) \wedge p).$$

The *power-sheaf* $P(S)$ of a pre-sheaf S consists of $\{\langle \Phi, p \rangle \mid \Phi \text{ a predicate on } S \text{ and } \forall a \in S. \Phi(a) \leq p\}$, with $E(\langle \Phi, p \rangle) = p$

and $\langle \langle \Phi, p \rangle \uparrow q \rangle = \langle (\Phi \uparrow q), (p \wedge q) \rangle$

where $\forall a \in S. (\Phi \uparrow q)(a) = (\Phi(a) \wedge q)$.

Remarks: The power-sheaf of a pre-sheaf is always a sheaf, and is generated by its *global* elements since $\langle \Phi, p \rangle = (\langle \Phi, T \rangle \uparrow p)$; for this reason we tend to ignore the *extent* component of an element and just work with the *predicate* component.

Predicates Φ correspond to *subsheaves* S' by the equations

$$S' = \{(a \uparrow \Phi(a)) \mid a \in S\}$$

and

$$\Phi(a) = \vee \{p \mid (a \uparrow p) \in S'\}$$

If A generates S' we can define a predicate Φ on S' by specifying just the values $\Phi(a)$ for $a \in A$. In conformity with the notation of III.2, we often write $\llbracket a \in \Phi \rrbracket$ instead of $\Phi(a)$.

5. Product sheaves

To correspond to the formation of *product types* we need

Definition: The *product* $(S_0 \times \dots \times S_{n-1})$ of sheaves S_0, \dots, S_{n-1} consists of

$$\{\langle a_0, \dots, a_{n-1} \rangle \in S_0 \times \dots \times S_{n-1} \mid E a_0 = \dots = E a_{n-1}\}$$

with $E(\langle a_0, \dots, a_{n-1} \rangle) = E a_0$

and $\langle \langle a_0, \dots, a_{n-1} \rangle \uparrow p \rangle = \langle (a_0 \uparrow p), \dots, (a_{n-1} \uparrow p) \rangle$.

Remarks: A product of sheaves is always a sheaf. It is sometimes convenient to treat an arbitrary element

$$\langle a_0, \dots, a_{n-1} \rangle \text{ of } (S_0 \times \dots \times S_{n-1}) \text{ as the element}$$

$$\langle a_0 \uparrow \left(\bigwedge_{i < n} E a_i \right), \dots, a_{n-1} \uparrow \left(\bigwedge_{i < n} E a_i \right) \rangle$$

of the product sheaf.

6. Sheaf morphisms

To make sheaves over a cHa into a *category* we need:

Definition: A *sheaf morphism* between sheaves \mathbf{S}, \mathbf{S}' is a function $F: \mathbf{S} \rightarrow \mathbf{S}'$ such that

$$\forall a \in \mathbf{S} \forall p \in \Omega \left(F(a) \uparrow p = F(a \uparrow p) \wedge E a = E F(a) \right).$$

Remarks: As a subset of $(\mathbf{S} \times \mathbf{S}')$ a sheaf morphism is always a *subsheaf* of $(\mathbf{S} \times \mathbf{S}')$. Sheaf morphisms will correspond to *operations* in the interpretation of III.2.

7. Locally constant sheaves

Every *set* "appears" as a sheaf over any cHa according to the following

Definition: For a set A and cHa Ω the *locally constant A -valued sheaf over Ω* is the sheafification \hat{A} of the following pre-sheaf \mathbf{S} :

$$\mathbf{S} = (\{ \langle a, p \rangle \mid a \in A, p \in \Omega \} / \approx)$$

where

$$\langle a, p \rangle \approx \langle b, q \rangle \leftrightarrow [p = q \wedge (a = b \vee p = \perp)],$$

with

$$E([\langle a, p \rangle]) = p$$

and

$$([\langle a, p \rangle] \uparrow q) = [\langle a, p \wedge q \rangle].$$

Remarks: This rather complicated-looking definition is designed to embed A as the *global* elements of \mathbf{S} , while allowing for the requirement that elements with extent \perp must be equal. We write \hat{a} for $[\langle a, (Ea)^\wedge \rangle]$, for a ranging over A ; then the elements $\{\hat{a} \mid a \in A\}$ generate \hat{A} . So, for example, we can define a predicate or sheaf morphism on \hat{A} by specifying its values for these elements, and indeed *any* such specification will determine one.

We have for all $a, b \in A$, $[[\hat{a} = \hat{b}]] = (a = b)^\wedge$, but we can only deduce in general that

$$[[\hat{a} = \hat{b}]] = T \text{ iff } a = b$$

in case Ω satisfies *KS*. If A is *discrete*, however, this will hold for any Ω , obviously.

8. *The sheaf \hat{N}*

In order to model the natural numbers and their terms in III.2., we define the *successor* sheaf morphism $S: \hat{N} \rightarrow \hat{N}$ by specifying

$$S(\hat{n}) = (S_n)^\wedge, \text{ for each } n \in \mathbf{N}.$$

9. *The sheaf $\underline{\Omega}$*

We make a cHa Ω into a "sheaf over itself", $\underline{\Omega}$, consisting of

$$\{\langle p, q \rangle \mid p, q \in \Omega \wedge p \leq q\}$$

with $E(\langle p, q \rangle) = q$

and $(\langle p, q \rangle \upharpoonright r) = \langle (p \wedge r), (q \wedge r) \rangle$.

Clearly $\underline{\Omega}$ is isomorphic to a sheaf $P(S)$ where S consists of the restrictions of a single global element. Thus in the interpretations of III.2, $\underline{\Omega}$ will correspond to the *set of propositions*, and *global propositions* to elements of Ω .

10. *Sheaves of partial morphisms*

All the specific sheaves that we will consider will be essentially subsheaves of sheaves formed by the following important construction (for example \hat{A} corresponds to the case where $\Omega' = P(A)$):

Definition: A *partial morphism* between cHa's Ω', Ω is a function $\Phi: \Omega' \rightarrow \Omega$ such that

$$\forall p', q' \in \Omega'. \Phi(p' \wedge q') = (\Phi(p') \wedge \Phi(q'))$$

and $\forall A \subseteq \Omega'. \Phi(VA) = V\{\Phi(p') \mid p' \in A\}$.

The *sheaf of partial morphisms* $M(\Omega', \Omega)$ is the sheaf over Ω consisting of the set of partial morphisms with $E(\Phi) = \Phi(T)$

and $(\Phi \upharpoonright p)(p') = (\Phi(p') \wedge p)$.

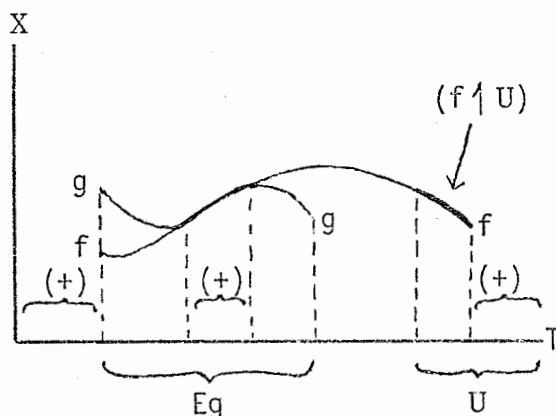
Remarks: See IV.1.6 for further discussion of this construction in terms of the sheaf of partial models for a geometric propositional theory. We consider the case where Ω is topological in the next subsection.

11. *Sheaves of partial continuous functions*

For the definitions of the topological concepts used here we refer to V.1; we describe the present construction as a special case of a *bundle* in V.2.7. The beauty of this construction is the splendid *picture* we can give of the elements and their valuations.

Definition: If X, T are topological spaces, the *sheaf of partial continuous functions* from T to X , $\mathbf{C}(T, X)$, consists of the set of functions $f: T \xrightarrow[p]{\text{cts}} X$ with $E(f) = \text{dom}(f)$ and $(f \upharpoonright U) = (f \upharpoonright U)$.

A picture of $\mathbf{C}(T, X)$



The union of the parts marked (+) is the value

$$\llbracket f \equiv g \rrbracket = \{t \mid f(t) \equiv g(t)\},$$

i.e. where f, g "agree".

The following is a corollary of IV.1.5 and 6:

Theorem: For any space T and *cHa* Ω' the sheaves $\mathbf{M}(\Omega', \mathbf{0}(T))$ and $\mathbf{C}(T, \Delta(\Omega'))$ are isomorphic, where $\Delta(\Omega')$ is the *space of superfilters* defined in II.1.3. □

Remarks: This result is intended to account for the name "partial morphisms" as corresponding to *partial* functions. A *cHa-morphism* in the usual sense is a ϕ such that $\phi(T) = T$; these correspond to *total* functions.

The *locally constant* sheaf \hat{X} over $\mathbf{0}(T)$ is embedded in $\mathbf{C}(T, X)$ as the elements which are literally "locally constant"; it is generated by the *constant functions* $\hat{x} = (\lambda t. x)$ for $x \in X$ (by abuse of notation).

III.2 INTERPRETATIONS

We define now the notion of an *interpretation in sheaves* for the formal language of Chapter I: terms are evaluated as elements of sheaves and formulae as elements of the cHa. The axioms and rules are then *sound* and (classically) *complete* for these interpretations (see FOURMAN [3]). Such interpretations are a natural extension of the *topological models* for intuitionistic propositional calculus (RASIOWA and SIKORSKI [21]) and analysis (SCOTT [24], MOSCHOVAKIS [17], van DALEN [2])(see also IV.2).

The constructions given in III.1 make sheaves over a cHa into a *topos* with Ω as subobject classifier, so the interpretations given here are a special case of *interpretations in topoi* (FOURMAN [3] and [4]). Many other formal systems have been proposed for this general context, their main differences being the omission of the existence predicate and the consequent necessary restrictions on the rule (MP) (e.g. OSIUS [19]). The present system certainly seems more natural for the concrete case of sheaves.

Grothendieck topoi generalise our sheaf topoi by taking sheaves over an arbitrary *site*; but for the types over \mathbf{N} (III.3.1) there is no real gain in generality.

1. *Interpretations*

Definition: An *interpretation over a cHa* Ω for the formal language of I.1 assigns

- (i) to each type α a sheaf S_α over Ω
- (ii) to each constant of type α an element of S_α
in such a way that

$$S_{p(\alpha)} = P(S_\alpha) \quad \text{and} \quad S_{(\alpha_0 \times \dots \times \alpha_{n-1})} = S_{\alpha_0} \times \dots \times S_{\alpha_{n-1}}$$

A *standard interpretation* assigns $\hat{\mathbf{N}}$ to \mathbf{N} and $\hat{0}$ to 0.

Definition: The *language* of an interpretation is the expansion of our original language by the addition of

- (i) for each type α and each $a \in S_\alpha$ a constant a of type α .
- (ii) for each $p \in \Omega$ a propositional constant p .

2. *Evaluation of terms and formulae*

Definition: We now define by induction on their construction the *evaluation* of the closed (well-formed) terms and formulae in the language of a standard interpretation:

if τ is a term of type α , $\llbracket \tau \rrbracket \in S_\alpha$
and if φ is a formula, $\llbracket \varphi \rrbracket \in \Omega$.

TERMS

Constants in the original language: as given by the interpretation, in particular $\llbracket 0 \rrbracket = \hat{0}$.

$$\llbracket a \rrbracket = a, \text{ for } a \in S_\alpha$$

$$\llbracket S\tau \rrbracket = S(\llbracket \tau \rrbracket), \text{ as defined in III.1.8}$$

$$\llbracket \langle \sigma_0, \dots, \sigma_{n-1} \rangle \rrbracket = \langle \llbracket \sigma_0 \rrbracket, \dots, \llbracket \sigma_{n-1} \rrbracket \rrbracket$$

$$\llbracket \exists x \in \alpha. \varphi \rrbracket = V\{(a \uparrow \llbracket \forall x \in \alpha (\varphi \leftrightarrow a \equiv x) \rrbracket \mid a \in S_\alpha)\},$$

that is, the *join* of this compatible set.

FORMULAE

$$\llbracket p \rrbracket = p, \text{ for } p \in \Omega$$

$$\llbracket E\tau \rrbracket = E(\llbracket \tau \rrbracket)$$

$$\llbracket \sigma \equiv \tau \rrbracket = \llbracket \llbracket \sigma \rrbracket \equiv \llbracket \tau \rrbracket \rrbracket \quad \left. \begin{array}{l} \text{these being evaluated} \\ \text{in the appropriate sheaf} \end{array} \right\}$$

$$\llbracket \sigma \in \tau \rrbracket = \pi_0^2(\llbracket \tau \rrbracket)(\llbracket \sigma \rrbracket)$$

$$\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \wedge \llbracket \psi \rrbracket$$

$$\llbracket \varphi \vee \psi \rrbracket = \llbracket \varphi \rrbracket \vee \llbracket \psi \rrbracket$$

$$\llbracket \varphi \rightarrow \psi \rrbracket = \llbracket \varphi \rrbracket \rightarrow \llbracket \psi \rrbracket$$

$$\llbracket \neg \varphi \rrbracket = \neg \llbracket \varphi \rrbracket$$

$$\llbracket \forall x \in \alpha. \varphi \rrbracket = \wedge \{ \llbracket E a \rightarrow \varphi [a/x] \rrbracket \mid a \in S_\alpha \}$$

$$\llbracket \exists x \in \alpha. \varphi \rrbracket = V \{ \llbracket E a \wedge \varphi [a/x] \rrbracket \mid a \in S_\alpha \}.$$

3. *Validity*

Definition: A formula φ in the language of a standard interpretation with free variables x_0, \dots, x_{n-1} of types $\alpha_0, \dots, \alpha_{n-1}$ is *valid* (in the interpretation) iff

$$\forall a_0 \in S_{\alpha_0}, \dots, a_{n-1} \in S_{\alpha_{n-1}}. \llbracket \varphi [a_0/x_0, \dots, a_{n-1}/x_{n-1}] \rrbracket = T.$$

Theorem (FOURMAN and SCOTT [7]): all the axioms of I.1.3 are valid in every standard interpretation, and all the rules preserve validity. \square

If some closed formula φ is valid in all standard interpretations over a cHa Ω , we say that φ *holds in sheaves over* Ω ; we may write then

$\text{Sh}(\Omega) \models \omega$, and, if Ω is $\mathbf{O}(T)$ for some topological space T , we may write $\text{Sh}(T)$ for $\text{Sh}(\Omega)$. We distinguish terms and formulae as interpreted in $\text{Sh}(\Omega)$ from those of the "outside world" by using the words "internal" and "external" respectively.

4. *The law of excluded middle*

Internal (global) propositions just correspond to elements of Ω (III.1.9), so that if Ω is *Boolean* (II.2.7), $\text{Sh}(\Omega) \models \text{LEM}$; while if Ω is *anti-Boolean* (II.2.9), $\text{Sh}(\Omega) \models \neg \text{LEM}$. In this latter case then all (classical) theorems which imply *LEM* are automatically falsified.

Thus both *LEM* and its negation are consistent relative to our system; we will just occasionally have to use classical arguments in proving a relative consistency result, but the consistency of *LEM* guarantees the result relative to the original system.

5. *Sets, relations and functions*

We now describe how some of the expressions introduced in I.2 appear "internally" in an interpretation. We will of necessity assume some facility in manipulating the evaluations of terms and formulae; as a good exercise in this we suggest the evaluations of *topological* concepts in *bundles* in the chapters (V - VIII) on topology, which we state mostly without proof.

Sets are just *sheaves* (subsheaves of some \mathbf{S}_α) and *subsets* are *subsheaves*, or *predicates* (III.1.4). If Φ, Ψ are predicates on a sheaf \mathbf{S}

$$\begin{aligned} \llbracket \Phi \subseteq \Psi \rrbracket &= \wedge \{ (\Phi(a) \rightarrow \Psi(a)) \mid a \in S \} \\ &= \vee \{ p \mid \forall a \in S. (p \wedge \Phi(a)) \leq \Psi(a) \} \end{aligned}$$

and $\llbracket \Phi \cap \Psi \rrbracket(a) = (\Phi(a) \wedge \Psi(a))$, for each $a \in S$, and so on. Any formula φ with free variable x of type α , say, determines a predicate on \mathbf{S}_α through the evaluation of $\llbracket \text{Ea} \wedge \varphi[a/x] \rrbracket$ for $a \in \mathbf{S}_\alpha$, and this will be the evaluation of $\llbracket \{x \in a \mid \varphi\} \rrbracket$.

Relations on sheaves $\mathbf{S}_0, \dots, \mathbf{S}_{n-1}$ are predicates on their product, and conditions (reflexivity, transitivity, etc.) are placed on them by requiring the appropriate formulae to be valid. Thus *equalities* and *functions* are predicates related in suitable ways; in particular if Φ acts as a *total operation* from \mathbf{S}_α to \mathbf{S}_β , it corresponds to the *sheaf morphism* F from \mathbf{S}_α to \mathbf{S}_β such that $\forall a \in S. F(a) = \llbracket \{x \in \beta. \langle a, x \rangle \in \Phi\} \rrbracket$.

III.3 ABSOLUTENESS

This section represents an attempted account of those terms and formulae which are *absolute* in the sense that in any interpretation their evaluations are the "locally constant" ones for their external interpretations. This turns out to work most satisfactorily over *proper* cHa's, but enough is true in the arbitrary case for the applications in this section and IV.2. We have included these results for the sake of their applications, but they seem rather *ad hoc*; we feel that there must be some more elegant and general statement possible.

1. *The hierarchy of locally constant sheaves*

We now confine ourselves to the language of *types over N*, that is, with \mathbf{N} as the sole ground type. We want to define a *sublanguage* of this whose terms and formulae are all *absolute*, and we start with the *atomic* cases. We regard " Ω " as "fixed", as usual.

Definition: By induction on the types over \mathbf{N} we define for each term τ a term $\hat{\tau}$ so that if τ is of type α , $\hat{\tau} \in S_\alpha$ in any standard interpretation. (Note that this is *fixed*, \mathbf{N} being the only ground type.)

- (i) If τ is of type \mathbf{N} , $\hat{\tau}$ is as given in III.1.7.
- (ii) If τ is of type $P(\alpha)$, $\hat{\tau} = \langle \Phi, (E\tau)^\wedge \rangle$ where Φ corresponds to the subsheaf of S_α generated by $\{\hat{x} \mid x \in \tau\}$.
- (iii) If τ is of type $(\alpha_0 \times \dots \times \alpha_{n-1})$,

$$\hat{\tau} = \langle (\Pi_0^n(\tau))^\wedge, \dots, (\Pi_{n-1}^n(\tau))^\wedge \rangle.$$

Remark: If τ is of power type, $\hat{\tau}$ corresponds to a sheaf isomorphic to the *locally constant τ -valued sheaf* (III.1.7), as will follow from the next lemma. This definition just makes each such $\hat{\tau}$ (and its elements) act internally as an object of the right type.

2. *Embedding Lemma*

Lemma: If the cHa Ω is *proper*, for all terms σ, τ of appropriate types, in the standard interpretation

$$\llbracket \hat{\sigma} = \hat{\tau} \rrbracket = (\sigma = \tau)^\wedge \quad \text{and} \quad \llbracket \hat{\sigma} \in \hat{\tau} \rrbracket = (\sigma \in \tau)^\wedge.$$

Proof: We prove the two statements simultaneously by induction on types using the results of II.2, the crucial case being (=) for σ, τ of power type:

By *extensionality* (I.1.4), for σ, τ of type $P(\alpha)$,

$$\begin{aligned}
 \llbracket \hat{\sigma} = \hat{\tau} \rrbracket &= \llbracket E\hat{\sigma} \wedge E\hat{\tau} \wedge \forall x \in \alpha (x \in \hat{\sigma} \leftrightarrow x \in \hat{\tau}) \rrbracket \\
 &= \llbracket E\hat{\sigma} \wedge E\hat{\tau} \rrbracket \wedge \wedge \{ \llbracket \hat{x} \in \hat{\tau} \rrbracket \mid x \in \sigma \} \wedge \wedge \{ \llbracket \hat{x} \in \hat{\sigma} \rrbracket \mid x \in \tau \} \\
 &\quad \text{(by the definition of } \hat{\sigma}, \hat{\tau} \text{)} \\
 &= (E\sigma \wedge E\tau)^\wedge \wedge \wedge \{ (x \in \tau)^\wedge \mid x \in \sigma \} \wedge \wedge \{ (x \in \sigma)^\wedge \mid x \in \tau \} \\
 &\quad \text{(by induction hypothesis)} \\
 &= (E\sigma \wedge E\tau \wedge \forall x \in \sigma. x \in \tau \wedge \forall x \in \tau. x \in \sigma)^\wedge \\
 &= (\sigma = \tau)^\wedge \quad \text{(by Theorem II.2.5(iii))} \quad \square
 \end{aligned}$$

Remarks: As Ω satisfies *KS* we can deduce that $\llbracket \hat{\sigma} = \hat{\tau} \rrbracket = T$ iff $\sigma = \tau$, and $\llbracket \hat{\sigma} \in \hat{\tau} \rrbracket = T$ iff $\sigma \in \tau$; conversely these forms imply *KS* since for any φ , $\llbracket \hat{\varphi} = \hat{T} \rrbracket = (\varphi = T)^\wedge = \hat{\varphi}$. ~~We can also deduce the (*) condition for properness from the conclusion of the lemma, in the form of Theorem II.2.5(iii): Let $A \subseteq P(\{0\})$, so that~~

$$\begin{aligned}
 \wedge \{ \hat{\varphi} \mid \varphi \in A \} &= \llbracket \forall \varphi \in \hat{A}. \varphi \rrbracket \\
 &= \llbracket (\cap A)^\wedge = \hat{T} \rrbracket = (\cap A = T)^\wedge = (\cap A)^\wedge,
 \end{aligned}$$

~~when the conclusion of the lemma holds. So the hypothesis of (*) really is necessary to prove this (classically evident) proposition (see also the proof of III.3.4).~~

3. *Some absolute terms and formulae*

We now define classes of terms and formulae which are to be construed in the language of types over \mathbf{N} according to the definitions of Chapter I. We use the word "some" advisedly in the title of this subsection; the list we give is rather *ad hoc*, and is not claimed to be exhaustive.

TERMS

Variables for types over \mathbf{N} ;

$\mathbf{N}, 0, S(\tau), [\tau]$;

$\langle \sigma_0, \dots, \sigma_{n-1} \rangle, (\sigma_0 \times \dots \times \sigma_{n-1}), \Pi_i^n(\tau)$;

$\text{Seq}(\tau), \text{Lh}(\tau), \tau(i), (\sigma_0, \dots, \sigma_{n-1})$, (as in I.3.3);

$(\sigma \cup \tau), \{ \tau \mid \varphi(y_0, \dots, y_{n-1}) \wedge \bigwedge_{j < n} y_j \in \tau_j \}, \exists x \in \tau. \varphi$.

FORMULAE

$$\varepsilon\tau, \sigma = \tau, \sigma \in \tau,$$

$$\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \neg\varphi$$

$$\forall x \in \tau. \varphi \quad \exists x \in \tau. \varphi$$

4. *Absoluteness theorem*

Definition: a term τ and formula φ with free variables among x_0, \dots, x_{m-1} are *absolute* (over Ω) iff in the standard interpretation (over Ω)

$$\llbracket \tau(\hat{x}_0, \dots, \hat{x}_{m-1}) \rrbracket = (\tau(x_0, \dots, x_{m-1}))^\wedge$$

and $\llbracket \varphi(\hat{x}_0, \dots, \hat{x}_{m-1}) \rrbracket = (\varphi(x_0, \dots, x_{m-1}))^\wedge$.

Theorem: All the terms and formulae of the above classes are absolute over any *proper* cHa.

Proof: Among the *terms* the only difficult case is

$$\{\tau \mid \varphi(y_0, \dots, y_{n-1}) \wedge \bigwedge_{j < n} y_j \in \tau_j\} :$$

The induction hypothesis applied to $\varphi, \tau, \tau_0, \dots, \tau_{n-1}$ shows that the evaluation of this term is the sheaf generated by the elements (suppressing the free variables)

$$\{\hat{\tau} \mid \varphi(y_0, \dots, y_{n-1}) \wedge \bigwedge_{j < n} y_j \in \tau_j\}$$

since these are the only ones with *positive extent*; but this is

$$(\{\tau \mid \varphi(y_0, \dots, y_{n-1}) \wedge \bigwedge_{j < n} y_j \in \tau_j\})^\wedge$$

by definition.

For the *atomic* formulae we use the Embedding lemma, and for \wedge, \vee, \exists we use the fact that $(\lambda\varphi. \hat{\varphi})$ *preserves* \wedge and \vee (proposition II.2.3(i)). For \rightarrow we need *KS* in the form that $(\lambda\varphi. \hat{\varphi})$ is *one-one* (Theorem II.2.4(iii)), and for \forall we need $(*)$ as in the Embedding lemma to ensure that $(\lambda\varphi. \hat{\varphi})$ *preserves* \wedge (Theorem II.2.5(iii)). \square

Remarks: The sense of "absoluteness" is thus that (in a standard interpretation) an *absolute term* is evaluated as the locally constant object for its external interpretation, and an *absolute formula* is valid iff it is true externally.

5. *Locally connected cHa's*

The present facts are well-known, but we include them for use in Chapter VIII, and to show how they fit into the general context.

Definition: An element p of a cHa Ω is *connected* iff

$$\forall q \in \Omega [p \leq (q \vee \neg q) \rightarrow p \leq q \vee p \leq \neg q]$$

Ω is *locally connected* iff

$$\forall q \in \Omega . q \leq \bigvee \{p \leq q \mid p \text{ is connected}\}.$$

Remark: This definition corresponds in the topological case to the *weak connectedness* of VII.2.

Proposition: If Ω is locally connected, the Absoluteness theorem holds when we extend the formation of terms by adding: $(\tau^\sigma) \uparrow$ (τ is discrete).

Proof: Since *discreteness* is obviously absolute, we suppose that τ is discrete, so that $\hat{\tau}$ is also discrete internally, and show that (τ^σ) is *generated* by the elements $\{\hat{f} \mid f: \sigma \rightarrow \tau\}$:

Let $q \in \Omega$ be *connected* and Φ an object such that

$$q \leq \llbracket \Phi: \sigma \rightarrow \tau \rrbracket . \text{ Then for } x \in \sigma, y \in \tau,$$

$$q \leq \llbracket \Phi(\hat{x}) = \hat{y} \vee \neg \Phi(\hat{x}) = \hat{y} \rrbracket \quad (\text{by discreteness})$$

$$\text{so} \quad q \leq \llbracket \Phi(\hat{x}) = \hat{y} \rrbracket \quad \text{or} \quad q \leq \llbracket \neg \Phi(\hat{x}) = \hat{y} \rrbracket .$$

$$\text{Thus} \quad \llbracket \lambda x \in \sigma . (\exists y \in \tau . q \leq \llbracket \Phi(\hat{x}) = \hat{y} \rrbracket) \rrbracket : \sigma \rightarrow \tau ,$$

$$\text{and denoting this by } f, \quad q \leq \llbracket \hat{f} = \Phi \rrbracket . \quad \square$$

6. *Integers and rationals*

Any standard construction of the sets of integers and rationals is *absolute*, so that \mathbf{Z}, \mathbf{Q} appear internally as $\hat{\mathbf{Z}}, \hat{\mathbf{Q}}$; also all the *arithmetic* and *order relations* on \mathbf{N}, \mathbf{Z} and \mathbf{Q} are absolute. It will be clear by inspection that the *geometric theories* introduced in IV.2 are absolute too (over proper cHa's).

It turns out in fact for these cases that the assumption that the cHa be *proper* is not needed, essentially because at each stage everything is *decidable*. We have been unable so far however to determine satisfactory general classes of terms and formulae which are absolute over arbitrary cHa's. If one thinks classically, of course, or confines oneself to topological cHa's, there is no problem.

CHAPTER IV
GEOMETRIC THEORIES

IV.1 MODELS OF GEOMETRIC PROPOSITIONAL THEORIES

We define here what we regard as a fundamental mathematical construction, the *space of models* of a *geometric propositional theory*, which serves to unify the constructions given in IV.2 of Baire and Cantor spaces, the Dedekind reals and the complex numbers. We also describe the *internal models* of a *locally constant* theory over a cHa as a slight generalisation of a *sheaf of partial morphisms*; every cHa gives rise naturally to a theory for which these two sheaves actually coincide.

We use the word "theory" here although we give no *formal system of deduction* for the "sequents" involved. It is quite possible to do so, and interesting relations are then obtained between *completeness theorems* and familiar *mathematical* properties of the spaces of models (see VIII.2); these results are due to FOURMAN [5].

Historical precedence must be given for the basic idea here to *André Joyal*, though we discovered it more or less independently in attempting to unify the obviously similar well-known sheaf constructions for the cases considered in IV.2; this was something we "predicted" in the Conclusion of GRAYSON [9]. Geometric (predicate) logic has many applications, for example the work of *Kock*, *Wraith* and *Reyes* in algebra; see also REYES [22].

1. *Languages and theories*

Definition: A *conjunctive propositional language* is a structure $A = (A, \perp, \top)$ where $\perp, \top \in A$, the set of *atoms*. The *formulae* of A are $\text{Seq}(A)$, denoted by F . A *sequent* of A is a pair $\langle \delta, \Gamma \rangle$ where $\delta \in F$ and $\Gamma \subseteq F$.

Notation: We use γ, δ to range over F and Γ, Γ' to range over $P(F)$. We write $(\gamma \wedge \delta)$ for $(\gamma * \delta)$ and $(\delta \vdash \forall \Gamma)$ for $\langle \delta, \Gamma \rangle$; also $(\gamma \vee \delta)$ for $V\{\gamma, \delta\}$.

Definition: A *geometric propositional theory* (in the language A) is a set Θ of sequents of A .

Remarks: We have expressed these definitions in terms of sets and elements so that in IV.1.3 it will be clear what it means to interpret such things in sheaves. However, as our notation and nomenclature is intended

to suggest, we think of *formulae* as representing *propositions* on which there is an operation of *conjunction*; and a *sequent* $(\delta \vdash \vee \Gamma)$ expresses that δ *implies the disjunction of* Γ . \perp, T are the *false, true* propositions.

2. Spaces of models

With the intuitive motivation given above in mind, we make the following natural

Definition: A *model* of a (geometric propositional) theory Θ is a subset Δ of F such that

$$\perp \notin \Delta, T \in \Delta, \forall \gamma, \delta \in F ((\gamma \wedge \delta) \in \Delta \leftrightarrow \gamma \in \Delta \wedge \delta \in \Delta),$$

and

$$\forall (\delta \vdash \vee \Gamma) \in \Theta (\delta \in \Delta \rightarrow \exists \gamma \in (\Delta \cap \Gamma)).$$

Definition: The *space of models* of a theory Θ is the set of models $\text{Mod}(\Theta)$ with the *topology* of open sets $U_\Gamma = \{\Delta \mid \exists \gamma \in (\Delta \cap \Gamma)\}$ for $\Gamma \subseteq F$.

Remarks: For the definitions of topological concepts see V.1. We could have equivalently defined the topology on $\text{Mod}(\Theta)$ by giving it the *basis* of sets $U_\gamma = \{\Delta \mid \gamma \in \Delta\}$ for $\gamma \in F$. This is the *finite information topology* since membership of an open set can be asserted on the basis of a *finite* amount of *information* about a model.

3. Sheaves of partial models

In order to give a good concrete representation of internal models in an interpretation we define for each theory and cHa a special sheaf as follows:

Definition: If Θ is a theory and Ω a cHa, a *partial model of* Θ *over* Ω is a function $\Phi: F \rightarrow \Omega$ such that

$$\Phi(\perp) = \perp, \forall \gamma \in F. \Phi(\gamma) \leq \Phi(T), \forall \gamma, \delta \in F. \Phi(\gamma \wedge \delta) = (\Phi(\gamma) \wedge \Phi(\delta)),$$

and

$$\forall (\delta \vdash \vee \Gamma) \in \Theta. \Phi \delta \leq \vee \Phi(\Gamma).$$

Definition: The *sheaf of partial models of* Θ *over* Ω consists of the set of partial models

with
$$E(\Phi) = \Phi(T)$$

and
$$(\Phi \uparrow p)(\gamma) = (\Phi(\gamma) \wedge p)$$

We denote this sheaf by $M(\Theta, \Omega)$.

4. *Internal models of a locally constant theory*

Theorem: The internal models in $\text{Sh}(\Omega)$ of the theory $\hat{\Theta}$ in the language $\hat{\mathbf{A}}$ form a sheaf isomorphic to $\mathbf{M}(\Theta, \Omega)$.

Proof: By the Remarks of III.1.7, an internal model Δ of $\hat{\Theta}$ will be determined by the values $[[\hat{\gamma} \in \Delta]]$ for $\gamma \in \mathbf{F}$, and clearly the conditions of these values imposed by the definition of a *model* of $\hat{\Theta}$ correspond to those for a *partial model* of Θ over Ω . \square

Remark: The applications of this theorem in IV.2 will be to *absolute* theories Θ , so that $\hat{\Theta}$ really is the "right" theory when we construct Θ internally.

5. *The topological case*

In the case where Ω is *topological* we have an even more "concrete" description of internal models, in terms of the external *space of models*:

Theorem: If Θ is a theory and T a topological space, the sheaves $\mathbf{M}(\Theta, \mathbf{O}(T))$ and $\mathbf{C}(T, \text{Mod}(\Theta))$ are isomorphic.

Proof: The correspondence is established by

$$f(t) = \{\gamma \mid t \in \Phi(\gamma)\} \quad \text{for } t \in \Phi(T),$$

and

$$\Phi(\gamma) = f^{-1}(U_\gamma) \quad \text{for } \gamma \in \mathbf{F},$$

from which the necessary properties are obvious. \square

6. *The theory associated to a cHa*

Every cHa Ω gives rise naturally to an *associated language* \mathbf{A}_Ω and *theory* Θ_Ω as follows: \mathbf{A}_Ω is (Ω, \perp, T) , and Θ_Ω consists of

$$(p) \wedge (q) \vdash (p \wedge q), \quad (p \wedge q) \vdash (p) \wedge (q) \quad \text{for } p, q \in \Omega$$

and

$$(p) \vdash \forall \Gamma, \quad \text{for } p \in \Omega, \Gamma \subseteq \Omega \text{ such that } p \leq \bigvee \Gamma.$$

Then it is fairly obvious that $\mathbf{M}(\Omega', \Omega)$ and $\mathbf{M}(\Theta_{\Omega'}, \Omega)$ are *isomorphic* for any cHa's Ω, Ω' , and that $\Delta(\Omega')$ is *homeomorphic* to $\text{Mod}(\Theta_{\Omega'})$. (See II.1.3 and III.1.10).

In the *converse* direction caution is needed since it is *not* the case in general that $\mathbf{M}(\Theta, \Omega)$ is isomorphic to $\mathbf{M}(\mathbf{O}(\text{Mod}(\Theta)), \Omega)$. This assertion amounts to a *completeness theorem* for Θ in some natural sequent calculus; that it can fail for the cases considered in IV.2 will appear in VIII.2 and 3 (see FOURMAN [5], and FOURMAN and HYLAND [6].)

IV.2 APPLICATIONS

We now use the definitions and results of IV.1 to construct some familiar spaces as the *spaces of models of geometric theories*, and to describe them as they appear in sheaf interpretations. Spaces obtained in this way have a very simple "logic" — that of the *manipulation of finite information*. The *real numbers* we construct are *Dedekind cuts*; as will be seen in VIII.2, the logic of *Cauchy sequences* is too complex for such a description.

All the theories here are *absolute*, so that these structures appear internally as sheaves of partial models, by IV.1.4. The descriptions in the topological case given by IV.1.5 are familiar in the folk-lore of sheaf-theory; our contribution is to show how they arise *uniformly* from their constructions as *spaces of models*. These models essentially include the *topological models* of SCOTT [24], van DALEN [2], and MOSCHOVAKIS [17], since in the sheaf description the *global* elements generate the whole sheaf. The model for the reals over themselves provides very intuitive and pictorial counter-examples to classical properties of *order* and *apartness*; we consider *topological* properties in Chapter VIII.

1. *The power-set of \mathbf{N}*

We begin with an essentially trivial example, $P(\mathbf{N})$: The language has *atoms* $n \in X$, for $n \in \mathbf{N}$, plus \perp, \top ; the *theory* is just *empty*!

Models are subsets X of \mathbf{N} with a *basis* for the topology consisting of the sets $U_e = \{X \subseteq \mathbf{N} \mid e \subseteq X\}$ for e a (\mathcal{Q} -) finite subset of \mathbf{N} (see X.1).

2. *Baire space*

To construct Baire space \mathbf{B} , the set of infinite sequences from \mathbf{N} with the Baire topology, we use the following:

The language has *atoms* $u \subseteq \alpha$, for $u \in \text{Seq}(\mathbf{N})$, plus \perp, \top ; the *theory* is the union of

$$\{(u \subseteq \alpha \wedge u' \subseteq \alpha) \vdash \perp \mid \neg(u \subseteq u' \vee u' \subseteq u)\}$$

and

$$\{u \subseteq \alpha \vdash \forall \{u * (k) \subseteq \alpha \mid k \in \mathbf{N}\} \mid u \in \text{Seq}(\mathbf{N})\} .$$

Models are infinite sequences α from \mathbf{N} with *basis* the sets $V(u) = \{\alpha : \mathbf{N} \rightarrow \mathbf{N} \mid u \subseteq \alpha\}$ for $u \in \text{Seq}(\mathbf{N})$.

3. *Cantor space*

Cantor space \mathbf{C} is the space of infinite sequences from $\{0,1\}$ and is constructed similarly to \mathbf{B} :

Atoms $u \subseteq \alpha$, for $u \in \text{Seq}(\{0,1\})$, plus \perp, \top ; *theory* the union of

and $\{(u \subseteq \alpha \wedge u' \subseteq \alpha) \vdash \perp \mid \neg(u \subset u' \vee u' \subset u)\}$
 $\{u \subseteq \alpha \vdash (u * (0) \subseteq \alpha \vee u * (1) \subseteq \alpha) \mid u \in \text{Seq}(\{0,1\})\}$

Basis sets $V(u) = \{\alpha : \mathbb{N} \rightarrow \{0,1\} \mid u \subseteq \alpha\}$.

4. *The Dedekind reals*

The most complicated example is that of the Dedekind reals \mathbb{R} , where we use a theory first introduced by *Miles Tierney* :

The language has *atoms* $p \in L, p \in U$, for $p \in \mathbb{Q}$, plus \perp, \top ; the *theory* consists of

- (i) $\top \vdash \forall \{p \in L \wedge q \in U \mid p, q \in \mathbb{Q}\}$
- (ii) $(p \in L \wedge p \in U) \vdash \perp$ for all p
- (iii) $p \in L \vdash q \in L, q \in U \vdash p \in U$ for $q < p$
- (iv) $p \in L \vdash \forall \{q \in L \mid p < q\}$
and $p \in U \vdash \forall \{q \in U \mid q < p\}$ for all p
- (v) $\top \vdash (p \in L \vee q \in U)$ for $p < q$.

Models are pairs $\langle L, U \rangle$ of *inhabited* ((i)) subsets of \mathbb{Q} which are *disjoint* ((ii)), *open* ((iv)), *lower* (or *upper*) *cuts* ((iii)), and are *close together* ((v)). This is obviously classically equivalent to any standard classical definition of a *Dedekind cut in \mathbb{Q}* ; this "geometric" form is just what we need intuitionistically.

We use x, y, \dots to range over \mathbb{R} , with $x = \langle L_x, U_x \rangle$. We treat \mathbb{Q} as a subset of \mathbb{R} in the usual way.

5. *Order on \mathbb{R}*

We define an *spo* $<$ on \mathbb{R} by

$$x < y \leftrightarrow \exists p \in (U_x \cap L_y).$$

As is well-known from the intuitionistic literature this is a *dense linear order*; for example $\text{Max}(x, y)$ is $\langle (L_x \cup L_y), (U_x \cap U_y) \rangle$. The *order topology* for $<$ (V.1.1) coincides with the *finite information topology*, having as basis the *open intervals* $(p, q) = \{x \in \mathbb{R} \mid p < x < q\}$ for $p, q \in \mathbb{Q}$.

6. *The sheaf $\mathbf{C}(\mathbb{R}, \mathbb{R})$*

The "picture of $\mathbf{C}(T, X)$ " given in III.1.11 is of course really a picture of $\mathbf{C}(\mathbb{R}, \mathbb{R})$ (insofar as \mathbb{R} represents the "line"!), and this picture provides an excellent intuition about properties of the internal Dedekind reals in $\text{Sh}(\mathbb{R})$.

For example let ι be the *generic real* in $\mathbf{C}(\mathbb{R}, \mathbb{R})$, that is, $\iota = \text{id}_{\mathbb{R}}$, and consider its internal relation to the constant function $\hat{0}$:

$$\llbracket \hat{0} < \iota \rrbracket = (0, \infty)$$

and $\llbracket \iota < \hat{0} \rrbracket = (-\infty, 0)$

Thus $\llbracket \iota \neq \hat{0} \rrbracket = ((-\infty, 0) \cup (0, \infty)) \neq \top$,

while $\llbracket \iota = \hat{0} \rrbracket = \perp$.

Also $\llbracket \hat{0} \leq \iota \vee \iota \leq \hat{0} \rrbracket = ((-\infty, 0) \cup (0, \infty))$.

This provides elegant counter-examples to classical properties of *order* and *apartness* on \mathbb{R} , which are normally obtained by the device of "weak counter-examples" in standard intuitionistic treatments.

7. *The complex numbers*

It is an easy exercise to construct the complex numbers in this fashion, essentially as pairs from \mathbb{R} . Many interesting questions arise, especially for the model of the complex numbers over themselves, but we do not investigate them here (see ROUSSEAU [23], FOURMAN and HYLAND [6]).

PART TWO

TOPOLOGY

INTRODUCTION

In this part we investigate some of the basic notions of classical *topology* from our intuitionistic view-point. FREUDENTHAL [8] and TROELSTRA [26] have made investigations of this sort, but mostly for spaces with *non-topological* structure, such as *metric* spaces, and using *continuity principles*. Our programme is to rephrase *classical* definitions and proofs so as to preserve the classical theory as far as possible, and to use sheaf models as counter-examples otherwise, especially for properties of the Cauchy and Dedekind reals; of special interest, we consider, is the role that *apartness* is found to play. From the classical point of view then, no entirely novel concepts are introduced; classical results whose proofs require no material alteration are quoted without proof, our basic reference here being NAGATA [18].

We give now a brief summary of Part Two:

In Chapter V we define some of the fundamental concepts of topology, *topologies* being given as sets of *open* sets; the classical duality with *closed* sets is not valid since the complement of a closed set need not be open nor the union of two closed sets be closed. We construct a wide class of internal topological spaces over a space as *the sheaves of sections of bundles*; in each chapter following we interpret the various notions in bundles.

Chapter VI is concerned with the various conditions of *separation*, and here the existence of a suitably behaved *apartness* corresponds to a strong " T_1 " property. Sets with apartness are nicely embedded into these "separated" spaces via their *strongly discrete* topologies. We define some new notions of "closed" set which are sometimes suitable substitutes for the ordinary one in classical theorems. *Tychonoff's embedding theorem* holds for our notion of *completely regular* space and (assuming *DC*) we prove forms of *Urysohn's lemma* and *Tietze's extension theorem* for strong *normality* conditions; unfortunately only much weaker properties are provable for metric spaces.

The notion of *compact* space considered in Chapter VII is that each open cover has a finite subcover, which is not comparable with that of BISHOP [1] for metric spaces. We characterise compact subspaces using our new notions of "closed" set. We define two conditions of *connectedness* for which many of the usual results hold. *Sober* spaces have a rather different-looking definition from the classical one: that every superfilter on the opens is principal. We give some criteria for soberness and evaluate some *soberifications* over a space; the rationals need not be sober although classically every Hausdorff space is sober.

In Chapter VIII we discuss *metric* spaces briefly; the usual definition of *uniform* space must be strengthened to guarantee that the topology be separated, but still complete regularity seems not to follow. The *completion* of a metric space by *Cauchy filters* may be given a metric, and this "Dedekind" completion can (in the absence of *CAC*) be larger than the usual completion by *Cauchy sequences*. We consider compactness, connectedness and soberness properties for the Cauchy and Dedekind reals, and give a model due to FOURMAN and HYLAND [6] in which the unit interval is not compact. We conclude with a few remarks on Baire and Cantor spaces.

CHAPTER V

SPACESV.1 TOPOLOGICAL SPACES

Topologies as defined here are given by sets of "open" sets; we have seen in IV.1.2 how such things arise naturally on the models of geometric propositional theories. "Closed" sets are relatively complicated and, not surprisingly, we cannot recover the *open* sets from the *closed* ones as is done classically.

The sense of "finiteness" used throughout this part is that of *Q-finiteness* (X.1), that is, of being "finitely indexed".

1. *Topological spaces*

Definition: A *topological space* X has a set of *open subsets* $\mathbf{O}(X)$ such that

$$X \in \mathbf{O}(X) , \forall U, V \in \mathbf{O}(X) . (U \cap V) \in \mathbf{O}(X)$$

and

$$\forall \mathbf{U} \subseteq \mathbf{O}(X) . \cup \mathbf{U} \in \mathbf{O}(X) .$$

If $A \subseteq X$, its *interior* A° is $\cup \{U \in \mathbf{O}(X) \mid U \subseteq A\}$; A is a *neighbourhood* of x , for $x \in X$, iff $x \in A^\circ$. The notions of (*neighbourhood*) (*sub-*) *basis* for a topology are defined as usual.

Notation: We use x, y, \dots to range over X ; U, V, \dots over $\mathbf{O}(X)$ and $\mathbf{U}, \mathbf{V}, \dots$ over $P(\mathbf{O}(X))$. For $A \subseteq X$ we write $\neg A$ for $(X \setminus A)^\circ$.

Examples:

- (i) The *largest* topology on a set X is the *discrete* one, with $\mathbf{O}(X) = P(X)$.
- (ii) The *smallest* topology is a *trivial* one, with just the opens $\hat{\varphi} = \cup \{X \mid \varphi\} = \{x \in X \mid \varphi\}$, for each proposition φ . Classically of course this consists of X and \emptyset_X only.
- (iii) If $<$ is an spo on X , its *order* topology has as subbasis its open intervals (II.1.1).

2. *Closure*

Definition: If A is a subset of a space X , its *closure* A^- is $\{x \mid \forall U \exists x \exists y \in (U \cap A)\}$; A is *closed* iff $A = A^-$, and *dense* iff $A^- = X$.

Remarks: We may prove as usual that $A \subseteq A^-$ and $(A^-)^- = A^-$, and that complements of open sets are closed. Other classical properties fail however:

Proposition: In $C(\mathbb{R}, \mathbb{R})$

- (i) the union of two closed sets may not be closed, and
- (ii) the complement of a closed set may not be open.

Proof: (The reader may check the details directly or use the results of V.2 and VI.1).

- (i) Every *singleton* subset is closed, so let ι be the *generic real* (IV.2.6) and $f = (\lambda t \in \mathbb{R}. \text{Max}(0, t))$. Then $\llbracket f = \iota \vee f = \hat{0} \rrbracket = ((-\infty, 0) \cup (0, \infty))$ while $\llbracket f \in \{\hat{0}, \iota\}^- \rrbracket = \top$, so that $\{\hat{0}, \iota\}$ is not closed internally.
- (ii) Although $\{\hat{0}\}$ is closed internally and $\llbracket \iota \neq \hat{0} \rrbracket = \top$, $\llbracket \iota \in \neg \{\hat{0}\} \rrbracket = \llbracket \iota \neq \hat{0} \rrbracket \neq \top$, so that its complement is not open. □

3. *Topological cHa's*

The open sets of a space X form a cHa in the usual way; in particular, $\wedge U = (\cap U)^\circ$. As an example of a difference from the classical theory, let J be $\lambda U. (U^-)^\circ$; classically this is the same as $(\neg \neg)$, so that $(\mathcal{O}(X)/J)$ is *Boolean*. Intuitionistically we have:

$$J \text{ is a } J\text{-operator and } \forall \varphi \in P(\{0\}). J(\hat{\varphi}) = \hat{\varphi},$$

so that (if X is inhabited) $(\mathcal{O}(X)(J))$ is *proper* and is *not* Boolean without *LEM*. To check the second assertion observe that $\hat{\varphi}$ is always both *closed* and *open* since $(x \in (\hat{\varphi})^- \rightarrow \varphi)$.

4. *Subspaces, continuous functions, quotients*

Definition: If X is a space and $Y \subseteq X$, the *subspace topology* on Y has $\mathcal{O}(Y) = \{U \cap Y \mid U \in \mathcal{O}(X)\}$.

Definition: F is a (*partial*) *continuous function* between spaces X and Y ($F: X \xrightarrow[\text{(p)}]{\text{cts}} Y$) iff it is a (*partial*) function such that

$$\forall V \in \mathcal{O}(Y). F^{-1}(V) \in \mathcal{O}(X).$$

Y is a *quotient* of X iff $(X \xrightarrow{\text{cts}} Y)$ (see I.2.4).

Note: Partial continuous functions as we have defined them always have *open domains*; this is convenient for the sheaf constructions of III.1.11 and V.2.1.

5. Products

Classical definitions of the product of a family $\{X_i : i \in I\}$ of spaces (e.g. NAGATA [18], p.57) usually involve negative statements; for example, that a basic open is a product $(\prod_{i \in I} U_i)$ where each $U_i \in \mathcal{O}(X_i)$ and for at most a finite number of indices $U_i \neq X_i$. To obtain a satisfactory definition we use the natural "finite information" form:

Definition: If $\{X_i : i \in I\}$ is a family of spaces, the *product topology* on $(\prod_{i \in I} X_i)$ has as basis the sets

$$\left\{ f \in \prod_{i \in I} X_i \mid \bigwedge_{k < n} f(i_k) \in U_k \right\}$$

where

$$i_0, \dots, i_{n-1} \in I \text{ and } \bigwedge_{k < n} U_k \in \mathcal{O}(X_{i_k}).$$

6. Covers, refinements

Definition: If X is a space and \mathbf{A} is a set of (open) subsets of X , \mathbf{A} is an (*open*) *cover* iff $\cup \mathbf{A} = X$. If $\mathbf{A} \subseteq \mathcal{P}(X)$ and $B \subseteq X$,

$$S(B; \mathbf{A}) = \cup \{A \in \mathbf{A} \mid \exists x \in B \cap A\},$$

and for $x \in X$,

$$S(x; \mathbf{A}) = \cup \{A \in \mathbf{A} \mid x \in A\}.$$

If $\mathbf{A}, \mathbf{B} \subseteq \mathcal{P}(X)$, we say that

\mathbf{B} *refines* \mathbf{A} ($\mathbf{B} < \mathbf{A}$) iff $\forall B \in \mathbf{B} \exists A \in \mathbf{A}. B \subseteq A$

\mathbf{B} *star-refines* \mathbf{A} ($\mathbf{B} <^* \mathbf{A}$) iff $\forall B \in \mathbf{B} \exists A \in \mathbf{A}. S(B; \mathbf{B}) \subseteq A$.

7. Filters

Definition: If \mathbf{F} is a filter on $\mathcal{P}(X)$ (see II.1.3) and $x \in X$, we say that

\mathbf{F} *is proper* iff $\forall A \in \mathbf{F}. \exists y \in A$

\mathbf{F} *converges to* x iff $\forall U \ni x. U \in \mathbf{F}$

x *is a cluster point of* \mathbf{F} iff $\forall A \in \mathbf{F}. x \in A^-$

V.2 BUNDLES

Bundles of all sorts have been widely investigated, but we will be interested almost exclusively in the (internal) *topological* properties of their *sheaves of sections*. More discussion may be found in FOURMAN and SCOTT [7], where a *Representation Theorem* is proved, of which a corollary is that all internal spaces arise essentially as subspaces of those obtained from bundles. Thus we have a good "picture" of many internal spaces, which includes as a special case the *sheaves of partial functions* of III.1.11. Detailed evaluations in bundles are routine and tedious, so we leave them as an exercise for the reader.

1. *Sections of bundles*

Definition: If E, T are spaces, a *projection* of E into T is a function $\pi: E \xrightarrow{\text{cts}} T$. A *section* of a projection π is a partial continuous right inverse of it, that is, a function $a: T \xrightarrow[\text{p}]{\text{cts}} E$ such that

$$\forall t \in \text{dom}(a) . \pi(a(t)) = t.$$

A *bundle* over T is a structure (E, π) where π is a projection of E onto T such that each point x of E has a section a of π *through* it, that is,

$$\pi(x) \in \text{dom}(a) \wedge a(\pi(x)) = x.$$

(Thus a bundle has no "redundant" points with respect to its sections.)

To avoid triviality we always assume T to be *inhabited*.

Notation: For π a projection of E into T , we use x, y, \dots to range over E ; s, t, \dots over T ; U, U', \dots over $\mathcal{O}(T)$; W, W', \dots over $\mathcal{O}(E)$; a, b, \dots over \underline{E} the set of sections of π . For $E' \subseteq E$ and $A \subseteq T$, we write E'_A for $(E' \cap \pi^{-1}(A))$, and for $t \in T$, E'_t for $E'_{\{t\}}$. The set E_t is given the subspace topology and is called the *fibre* at t .

The following is sometimes useful:

Definition: A bundle (E, π) over T has a (topological) property *locally* iff T has a neighbourhood basis of subsets A such that the subspace E_A of E has the property.

2. *Sheaves of sections*

The sections of a bundle form a sheaf in a natural way (compare III.1.11, V.2.7):

Definition: The *sheaf of sections* \underline{E} of a bundle (E, π) over T is the sheaf over $\mathcal{O}(T)$ consisting of the set \underline{E} of sections of π

with $E(a) = \text{dom}(a)$

and $(a \uparrow U) = (a \downarrow U)$.

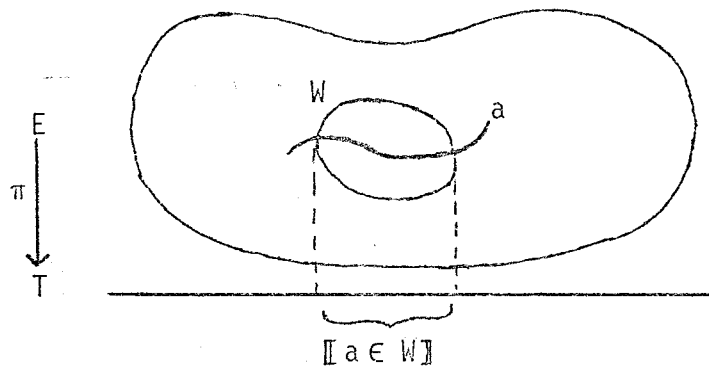
If $E' \subseteq E$ we define a *predicate* \underline{E}' on E by putting

$$\llbracket a \in \underline{E}' \rrbracket = (\{t \mid a \downarrow t\} \in E')^\circ.$$

3. Bundles as internal spaces

Definition: We make a sheaf of sections E into an *internal topological space* by defining $\mathcal{O}(E)$ to be the subsheaf of $\mathcal{P}(E)$ generated by the *global* elements \underline{W} for $W \in \mathcal{O}(E)$.

A picture of E



Some evaluations in E

- (i) $\llbracket a \in \underline{W} \rrbracket = a^{-1}(W)$, as W is open and a continuous.
- (ii) $\llbracket \underline{W} \subseteq \underline{W}' \rrbracket = U\{U \mid W_U \subseteq W'_U\}$, by the bundle condition.
- (iii) $\llbracket (\underline{W} \cap \underline{W}') \rrbracket = \llbracket \underline{W} \cap \underline{W}' \rrbracket = T$.
- (iv) For $W \subseteq \mathcal{O}(E)$ put $\llbracket \underline{W} \in \underline{W} \rrbracket = (W \in W)^\wedge$,
then $\llbracket (\underline{U} \underline{W}) = U(\underline{W}) \rrbracket = T$.
- (v) For Φ a predicate on $\mathcal{O}(E)$ put $\underline{W} = \{(W)_{\Phi(\underline{W})} \mid W \in \mathcal{O}(E)\}$,
then $\llbracket U\Phi = U(\underline{W}) \rrbracket = T$.

Corollary: From these evaluations we may deduce that we have indeed defined a topology on E , whose global elements are *exactly* the predicates \underline{W} for $W \in \mathcal{O}(E)$.

Examples:

- (i) E has the *discrete* topology iff π is a *local homeomorphism*, that is (E, π) is a "sheaf" in the original sense.
- (ii) E has the *trivial* topology iff $\mathcal{O}(E) = \{E_U \mid U \in \mathcal{O}(T)\}$.

4. *Evaluations for predicates on E - subbundles*

If ϕ is a predicate on E , its *range* $\text{rge}(\phi)$ is $\{a(t) \mid t \in \phi(a)\}$. Although ϕ is not *completely* determined by its range, many of its topological properties are, for example:

$$(i) \quad \text{rge}(\underline{\text{rge}(\phi)}) = \text{rge}(\phi)$$

$$(ii) \quad \llbracket \exists a \in \phi \rrbracket = \llbracket \exists a \in (\text{rge } \phi) \rrbracket = \pi(\text{rge } \phi)$$

$$(iii) \quad \text{rge}(\phi \cap W) = (\text{rge}(\phi) \cap W)$$

from which follows

$$(iv) \quad \llbracket \phi^- = (\text{rge } \phi)^- \rrbracket = \mathbf{T}.$$

The range of a predicate is clearly a *subbundle* in the sense of forming a bundle under the projection π . *Conversely* if $A \subseteq E$ is a subbundle, $A = \text{rge}(A)$, so that *every* subbundle arises in this way. Thus for properties such as those above, quantification over $\mathbf{P}(E)$ can be replaced by quantification over *subbundles*.

5. *Bundle morphisms*

To obtain a description of *partial continuous functions* between spaces arising from bundles we make a

Definition: If $(E, \pi), (E', \pi')$ are bundles over T , a *bundle morphism* between them is a function $f: E \xrightarrow[\text{p}]{\text{cts}} E'$ commuting with the projections, that is,

$$\forall x \in \text{dom}(f) . \pi(x) = \pi'(f(x)).$$

Before stating the next theorem we observe that in considering partial continuous functions it is sufficient to treat *global* objects which are *globally* continuous: a simple *logical* argument shows that these generate.

Theorem: Assuming that E' is T_0 (VI.1.1), the sheaf of internal partial continuous functions from E to E' has global elements corresponding exactly to bundle morphisms from (E, π) to (E', π') in a natural way.

Proof: The correspondence between bundle morphisms f and (partial) sheaf morphisms F (III.1.6), that are internally continuous, is established by

$$(+) \quad F(a) = (f \circ a), \text{ for } a \in E.$$

Then for $W' \in \mathcal{O}(E')$, $\llbracket F^{-1}(W') \rrbracket = \underline{f^{-1}(W')} \rrbracket = \mathbf{T}$,

from which the continuity conditions follow easily. The only tricky part is to show that, given F , (+) does define f properly, that is:

$$a(t) = b(t) \rightarrow F(a)(t) = F(b)(t):$$

Let $a(t) = b(t)$ and $F(a)(t) \in W' \in \mathcal{O}(E')$,

i.e. $t \in \llbracket a \in F^{-1}(W') \rrbracket$.

Then $b(t) = a(t) \in \text{rge}(F^{-1}(W')) \in \mathcal{O}(E)$, as F continuous, so $F(b)(t) \in W'$ also; and the result follows from the T_0 condition on E' . \square

6. Product bundles

Since an arbitrary family of bundles does not have a product bundle in a natural way, we give the definition only for *finite* families.

Definition: The *product* of bundles $(E_0, \pi_0), \dots, (E_{n-1}, \pi_{n-1})$ is the bundle (E, π)

with $E = \{ \langle x_0, \dots, x_{n-1} \rangle \mid \pi_0(x_0) = \dots = \pi_{n-1}(x_{n-1}) \}$

and $\pi(x) = \pi_0(\pi_0^n(x))$, for $x \in E$,

and E having its topology as a subspace of the product of E_0, \dots, E_{n-1} .

It is easy to see that E acts as the *internal product* of the spaces E_0, \dots, E_{n-1} , via the equation

$$a(t) \equiv \langle a_0(t), \dots, a_{n-1}(t) \rangle, \text{ for } a \in \underline{E}, a_i \in \underline{E}_i, t \in T.$$

We denote by (E^2, π^2) the product of (E, π) with itself.

7. The bundle X_T

Each external space X determines a natural bundle over any space T , which is essentially the sheaf $\mathbf{C}(T, X)$.

Definition: The bundle X_T is $((T \times X), \pi)$ where $(T \times X)$ has the *product* topology, and π is the *first projection* π_0^2 .

Theorem:

(i) The sheaves \underline{X}_T and $\mathbf{C}(T, X)$ are naturally isomorphic.

(ii) If θ is a geometric propositional theory, the topology on $\underline{\text{Mod}}(\theta)_T$ coincides (under the isomorphism of (i)) with the finite information topology on $\text{Mod}(\hat{\theta})$ (see IV.1).

Proof:

(i) The isomorphism is established by the equation $a(t) \equiv \langle t, f(t) \rangle$ for $a \in \underline{X}_T$, $f \in \mathbf{C}(T, X)$, $t \in T$.

(ii) The topology on $\underline{\text{Mod}}(\theta)_T$ has as a basis the predicates $(\underline{U} \times \underline{U}_\gamma)$, for $U \in \mathcal{O}(T)$, $\gamma \in F$. But this is the same as $(\hat{U} \cap \hat{U}_\gamma)$ as an internal open of $\text{Mod}(\hat{\theta})$, and the result follows. \square

Thus sheaves of partial continuous functions and internal spaces of models of (locally constant) theories are special cases of the bundle construction; note also that E is always a *subsheaf* of $C(T,E)$, though not a *subspace*.

For convenience we abuse our notation and write X_T for $C(T,X)$; and, for $f \in C(T,X)$, $W \in \mathcal{O}(T \times X)$, put $\llbracket f \in W \rrbracket = \{t \mid \langle t, f(t) \rangle \in W\}$. So, for example, the Dedekind reals in $\text{Sh}(\mathbb{R})$ appear as $\mathbb{R}_{\mathbb{R}}$, and we have an excellent "picture" of their *topology* via the topology of \mathbb{R}^2 (see V.2.3!).

Finally, if (E, π) is any bundle, bundle morphisms f from it to a bundle X_T "reduce" to functions $g: E \xrightarrow[\text{p}]{\text{cts}} X$ via the equation

$$f(x) = \langle \pi(x), g(x) \rangle, \text{ for } x \in E.$$

This will be useful later when considering internal *continuous real-valued functions* on a bundle.

CHAPTER VI

SEPARATIONVI.1 SEPARATION PRINCIPLES

We start our investigation with a consideration of possible forms of the classical T_0, T_1, T_2 conditions. As commonly stated these involve an hypothesis of *inequality* between points of the space, but this is obviously not suitable intuitionistically. One possibility is to *contrapose* them into conditions describing how various degrees of "sameness" or "closeness" make points *equal*. Stronger conditions arise when we replace *inequality* by *apartness* in some sense, and here a natural notion of *separated space* corresponds to the strong " T_1 " property; nice connections exist then with *separated orders* through their order topologies. TROELSTRA [26] has a notion of a "space with an apartness", but the conditions are not strong enough for a good theory. The conditions we use are a development of ones abstracted from the internal properties of a "separated" bundle (VI.1.6) by Mike Fourman.

1. *Weak separation*

Definition: In any space X we may define three relations of "sameness" or "closeness" as follows:

$$x \approx_0 y \leftrightarrow \forall U(x \in U \leftrightarrow y \in U)$$

$$x \approx_1 y \leftrightarrow \forall U(x \in U \rightarrow y \in U)$$

$$x \approx_2 y \leftrightarrow \forall U, V(x \in U \wedge y \in V \rightarrow \exists z \in U \cap V)$$

We then define, for $i=0, 1, 2$, that X is T_i iff

$$\forall x, y \in X (x \approx_i y \rightarrow x = y).$$

Remarks: The relation \approx_0 is really the "natural" equality on a space, and *open* and *closed* sets, and *continuous* functions are always *extensional* with respect to it. So we may without loss generally *assume* the condition T_0 . We quote the following simple facts that are proved much as in the classical case:

(i) $T_2 \Rightarrow T_1 \Rightarrow T_0$, and not conversely .

(ii) X is T_1 iff every singleton subset is closed.

(iii) X is T_2 iff every proper filter on $\mathbf{O}(X)$ converges to at most one point.

(iv) T_0, T_1 and T_2 spaces are closed under subspaces and products.

2. *Weak separation in bundles*

If (E, π) is a bundle over T and $a, b \in \underline{E}$,

$$[a \approx_i b] = (\{t \mid a(t) \approx_i b(t) \text{ in the fibre } E_t\})^\circ$$

for $i=0, 1, 2$; so in particular

(i) if each fibre is T_i , so is E , and

(ii) X_T is T_i iff X is T_i , for any space X .

3. *Strong separation*

Definition: In parallel with subsection 1 we may define in any space X two relations of "difference":

$$x \#_1 y \leftrightarrow \exists U (x \in U \wedge y \notin U)$$

$$x \#_2 y \leftrightarrow \exists U, V (x \in U \wedge y \in V \wedge (U \cap V) = \emptyset)$$

We then define, for $i=1, 2$, that X is $T_i^\#$ iff it is T_0 and

$$(*)_i \forall U \forall x, y \in X (x \in U \rightarrow (y \in U \vee y \#_i x)).$$

The meaning of these will become clearer after another

Definition: X is a *separated* space iff it has an *apartness* $\#$ which is *open*, that is,

$$\forall y \in X. \{x \mid x \# y\} \in \mathcal{O}(X),$$

and with respect to which every open is *strongly extensional*.

That this is a natural and fundamental notion of a "space with an apartness" will appear through the examples, and through the role it plays in the re-statement of classical results. The connection with the preceding definition is established by:

Theorem: X is separated iff it is $T_1^\#$; and the apartness relation is always $\#_1$.

Proof: Let X be separated with apartness $\#$. If $x \# y$, $x \in \{z \mid z \# y\} \in \mathcal{O}(X)$ but $y \notin \{z \mid z \# y\}$, so $x \#_1 y$; conversely if $x \in U$ and $y \notin U$, strong extensionality of U gives $x \# y$. Thus $\#$ and $\#_1$ coincide, and X must be $T_1^\#$.

Conversely, if X is $T_1^\#$, $\#_1$ is an apartness making X separated. We check first the conditions for apartness:

(i) For $x, y \in X$, $\neg(x \#_1 y) \leftrightarrow x \approx_0 y \leftrightarrow x = y$, as X is T_0 .

(ii) If $x \in U$ and $y \notin U$, $y \#_1 x$, by $(*)_1$.

(iii) If $x, y, z \in X$ and $x \in U$ and $y \notin U$, $(*)_1$ gives $(z \in U \vee z \#_1 x)$ whence $(z \#_1 y \vee z \#_1 x)$.

Openness of $\#_1$ is immediate, and $(*)_1$ is exactly the *strong extensionality* of every open. \square

Remarks: The fact that the existence of a suitably behaved apartness is as strong as the (classical) T_1 condition accounts for the absence of a " $T_0^\#$ " condition. The following are easily seen to hold:

(i) X is $T_2^\#$ iff it is $T_1^\#$ and $\#_1, \#_2$ coincide.

(ii) $T_2^\# \Rightarrow T_2$ and $T_1^\# \Rightarrow T_1$, and not conversely.

(iii) $T_1^\#$ and $T_2^\#$ spaces are closed under subspaces and products.

For the counter-example in (ii) observe that the *discrete* topology on $P(\{0\})$ is T_2 , but there can be no apartness on it without *LEM* (I.4.1).

4. Order topologies

We have good examples of separated spaces, including the Dedekind reals, through the following

Proposition: The order topology of a *separated* spo is separated and T_2 ; also if it is *dense*, the topology is $T_2^\#$. But the order topology of a *simple* order need not be $T_2^\#$ in general.

Proof: Let $<$ be a separated spo on A , and show that all opens are strongly extensional with respect to the *associated apartness* $\#$ by considering the basic open

$$\bigcap_{i < n} (a_i, b_i), \text{ for } a_i, b_i \in A:$$

If $\forall i < n. a_i < a < b_i$ and $b \in A$,

then $\forall i < n ((a_i < b \vee b < a) \wedge (b < b_i \vee a < b))$

so $(a \# b \vee \forall i < n. a_i < b < b_i)$.

For the T_2 property, let $a, b \in A$ and $a \approx_2 b$, and show $\neg(a \# b)$: if $a < b$, $a \approx_2 b$ gives $\exists c. a < c < b$; but then $a \in (-\infty, c)$ and $b \in (c, \infty)$, *contradicting* $a \approx_2 b$.

If $<$ is also *dense*, and $a < b$, let $a < c < b$, so that, as before, $a \#_2 b$; whence the space is $T_2^\#$.

Finally, for the *counter-example*, let A be the set $\{0, (\neg \varphi), 2\}$ for some given proposition φ , with its order inherited from \mathbf{N} . This is clearly *simple*, and $0 \#_2 2$; but if $0 \#_2 2$, there must be $c, d \in A$ such that $0 < c, d < 2$, and $(-\infty, c) \cap (d, \infty) = \emptyset$. Then

either $(c = 1 \vee d = 1)$, so φ holds,
or $(c = 2 \wedge d = 0)$, so $\neg \varphi$ holds. \square

(We prove a stronger property for the topologies of *dense linear orders* in VI.2.2.)

5. *Strongly discrete topologies*

Definition: If $\#$ is an apartness on a set X , its *strongly discrete topology* consists of all strongly extensional subsets of X .

This topology is obviously the *largest* separated topology on X with the given apartness. We will see in VIII.2.9 how a weak form of *Brouwer's continuity principle* implies that \mathbf{R}, \mathbf{B} and \mathbf{C} all have the strongly discrete topology.

The assignment of strongly discrete topologies embeds $\text{Set}^\#$ in the category $\text{Top}^\#$ of separated spaces and continuous functions in a way whose "faithfulness" is described by the following:

Proposition: (i) All continuous functions between separated spaces are strongly extensional.

(ii) All strongly extensional functions from a space with the strongly discrete topology are continuous.

Proof: (i) Let X, Y be separated and $f: X \xrightarrow{\text{cts}} Y$. If $x, x' \in X$ and $f(x) \# f(x')$, let $f(x) \in V \in \mathcal{O}(Y)$ and $f(x') \notin V$. Then $x \in f^{-1}(V) \in \mathcal{O}(X)$, while $x' \notin f^{-1}(V)$.

(ii) Now let X be strongly discrete and $f: X \rightarrow Y$ be strongly extensional; we show, for $V \in \mathcal{O}(Y)$, that $f^{-1}(V)$ is strongly extensional: let $x, x' \in X$ and $x \in f^{-1}(V)$, so $f(x) \in V$, whence $(f(x') \in V \vee f(x') \# f(x))$. Then $(x' \in f^{-1}(V) \vee x' \# x)$, as required. \square

6. *Strong separation in bundles*

If (E, π) is a bundle over T and $a, b \in \underline{E}$,

$$\llbracket a \#_1 b \rrbracket = \{t \in \text{dom}(a) \cap \text{dom}(b) \mid a(t) \in \neg \text{rge}(b)\}$$

with $\llbracket a \#_2 b \rrbracket = \{t \in \text{dom}(a) \cap \text{dom}(b) \mid a(t) \#_2 b(t)\}$,

by straightforward evaluations. We deduce then

(i) E is $T_1^\#$ iff for $W \in \mathcal{O}(E)$, $b \in \underline{E}$ and $t \in \text{dom}(b)$ if $b(t) \in W$ then $\forall x \in E_t (x \in W \vee x \in \neg \text{rge}(b))$.

(ii) E is $T_2^\#$ iff for $W \in \mathcal{O}(E)$, $t \in T$ and $y \in E_t$ if $y \in W$ then $\forall x \in E_t (x \in W \vee x \#_2 y)$.

Furthermore, for the case of the bundle X_T ,

(iii) if X_T is $T_1^\#$ so is X , but not conversely

(iv) X_T is $T_2^\#$ if X is $T_2^\#$.

The condition on (E, π) expressed in (ii) is precisely that the projection is "separated" in the sense of bundle theory; it was *Mike Fourman's* internal interpretation of this that led to the formulation of these strong separation principles.

For (iii) and (iv), if X_T is $T_1^\#$ (or $T_2^\#$), one proves that X is too by using the *constant functions* (III.1.11). For the *converse* in (iv), suppose $\langle t, y \rangle \in W \in \mathcal{O}(T \times X)$, and let $U \in \mathcal{O}(T)$, $V \in \mathcal{O}(X)$ and $\langle t, y \rangle \in (U \times V) \subseteq W$. Then for $x \in X$, $(x \in V \vee x \#_2 y)$ since X is $T_2^\#$, whence

$$(\langle t, x \rangle \in W \vee \langle t, x \rangle \#_2 \langle t, y \rangle)$$

and the result follows by (ii).

For the *counter-example* in (iii) let X be the following space: we adjoin a point θ to \mathcal{Q} with its order topology and give as open neighbourhoods of θ the *complements of finite subsets* of \mathcal{Q} . X is clearly $T_1^\#$, but not T_2 , since $\forall p \in \mathcal{Q} . p \approx_2 \theta$. Now in the bundle $X_{\mathcal{Q}}$ (considered as $\mathcal{C}(\mathcal{Q}, X)$ as in V.2.7) let f be the identity function ($\lambda p . p$), and let

$$g(p) = \begin{cases} \theta & \text{if } p = 0 \\ p & \text{otherwise} \end{cases}$$

Then, if $W = (\mathcal{Q} \times \mathcal{Q})$, $\llbracket f \in \underline{W} \rrbracket = T$

while $\llbracket g \in \underline{W} \rrbracket = (\mathcal{Q} \setminus \{0\})$ and $\llbracket g \#_1 f \rrbracket = \perp$.

The *continuity* of g follows from $0 \approx_2 \theta$.

7. *Apartness of points from sets*

We consider now some properties of subsets of a space which are most conveniently expressed in terms of a relation of "apartness" between points and subsets, that can be defined in any space but works most satisfactorily in *separated* ones. These notions are taken from TROELSTRA [26] who investigates them more fully, and we refer to him for most proofs; BISHOP [1] uses parallel notions in the context of metric spaces.

Definition: A point x of a space X is *apart* from a subset A of X ($x \# A$) iff $\exists U \exists x . (U \cap A) = \emptyset$. (Thus $\neg A$ as defined in V.1.1 is $\{x | x \# A\}$.)

We then define for $A, B \subseteq X$

A is *located* ($L(A)$) iff $\forall U \forall x \in U (\exists y \in (U \cap A) \vee x \# A)$

A is *well-contained* in B ($A \ll B$) iff $(\neg A \cup B^\circ) = X$.

Finally for $A, B \subseteq P(X)$

B *well-refines* A ($B \ll A$) iff $\forall B \in \mathcal{B} \exists A \in \mathcal{A}. B \ll A$.

Remarks: Classically, $x \# A$ iff $x \notin A^-$; all subsets are located; $A \ll B$ iff $A^- \subseteq B^\circ$; and well-refinements coincide with "cushioned" ones. It should be clear that intuitionistically our conditions are *stronger*, and it will turn out on many occasions that these stronger forms are exactly what are needed, for example in the results on *normal* spaces in VI.2. We summarise some of the properties of these notions in the following

Proposition: If X is separated and inhabited, for all $x, y \in X$, $A, B, C, D \subseteq X$, and $\varphi \in P(\{0\})$,

- (i) $x \# A \rightarrow (y \# A \vee x \# y)$
- (ii) $x \# A \leftrightarrow x \# A^-$
- (iii) $x \# (A \cup B) \leftrightarrow x \# A \wedge x \# B$
- (iv) $L(A) \leftrightarrow L(A^-)$
- (v) $L(A) \wedge L(B) \rightarrow L(A \cup B)$
- (vi) $L(\hat{\varphi}) \leftrightarrow (\varphi \vee \neg \varphi)$
- (vii) $A \ll B \wedge C \ll D \rightarrow (A \cup C)^- \ll (B \cap D)^\circ$.

Proof: Straightforward (see TROELSTRA [26]). For (vi), if $L(\hat{\varphi})$, let $x \in X$ (as X is inhabited); then $(\exists y \in \hat{\varphi} \vee x \# \hat{\varphi})$, whence $(\varphi \vee \neg \varphi)$. \square

8. *New notions of closure*

We now introduce two new properties classically equivalent to the ordinary one of being *closed*, and prove their distinctness; they do not appear to arise as the fixed points of *closure operations* as the ordinary one does. The more important of these notions ("w-closed") will turn out to be exactly the right substitute for the ordinary notion in matters relating to *finite open covers* (*normality* in VI.2 and *compactness* in VII.1).

Definition: If X is separated and $A \subseteq X$,

A is *a-closed* iff $\forall x \in X (\forall y \in A. x \# y \rightarrow x \# A)$

A is *w-closed* iff $\forall U (A \subseteq U \rightarrow A \ll U)$.

Proposition: For all $x \in X$ and $A, B \subseteq X$,

- (i) $\{x\}$ is w -closed
- (ii) if A, B are a -(w -)closed, so is $(A \cup B)$
- (iii) if $A \subseteq B \subseteq A^c$ and A is a -(w -)closed, so is B
- (iv) if A is w -closed, it is a -closed and located
- (v) w -closed $\not\Rightarrow$ closed $\not\Rightarrow$ a -closed $\not\Rightarrow$ w -closed.

Proof: (i) This is simply the *strong extensionality* of all opens in a separated space. (ii), (iii) follow easily from the previous proposition.

(iv) Let A be w -closed and $x \in X$. If $\forall y \in A. x \# y$, then $A \subseteq \neg\{x\}$, so $A \subset \neg\{x\}$; but $x \notin \neg\{x\}$, so $x \# A$. Next, if $x \in U$, we have $\forall y \in X (y \in U \vee y \# x)$, whence $A \subseteq ((\exists y \in U \cap A) \wedge U \neg\{x\})$; then as before $(x \in (\exists y \in U \cap A) \wedge x \# A)$, so that $(\exists y \in (U \cap A) \vee x \# A)$.

(v) Proposition V.1.2 combined with (i) and (ii) above shows that w -closed $\not\Rightarrow$ closed. Again in the sheaf $\mathbf{C}(\mathbb{R}, \mathbb{R})$, let ι be the generic real and $g = (\hat{0} \uparrow (-\infty, 0))$: $\{g\}$ is internally closed, but not a -closed since $E(g) \subseteq \llbracket \iota \# g \rrbracket$, while $0 \notin \llbracket \iota \# \{g\} \rrbracket$. Finally, every $\hat{\varphi}$ is a -closed, since $(\forall y \in \hat{\varphi}. x \# y \rightarrow \neg\varphi)$; but if it is w -closed, $(\varphi \vee \neg\varphi)$ by (iv) and Proposition VI.1.7(vi). \square

9. Some interpretation in bundles

If (E, π) is a bundle over T , $a \in \underline{E}$ and ϕ is a predicate on E ,

$$\llbracket a \# \phi \rrbracket = \llbracket a \# (\text{rge } \phi) \rrbracket = \{t \mid a(t) \# \text{rge}(\phi)\}.$$

Combining this with V.2.4 we see that all the properties of subsections 7 and 8 depend only on the *ranges* of predicates.

We evaluate further for *subbundles* A, B of E

- (i) $\llbracket L(A) \rrbracket$ is the interior of $\{t \mid \forall W \forall x \in W_t (\exists y \in (W \cap A)_t \vee x \# A)\}$
- (ii) for $U \in \mathbf{O}(T)$, $U \subseteq \llbracket A \subset B \rrbracket$ iff $A_U \subset B_U$ in the subspace E_U .
- (iii) for $U \in \mathbf{O}(T)$, $U \subseteq \llbracket A \text{ is } w\text{-closed} \rrbracket$ iff, for $W \in \mathbf{O}(E)$ and $U' \subseteq U$, if $A_{U'} \subseteq W$ then $U' \subset \llbracket A \subset W \rrbracket$.

VI.2 REGULAR, COMPLETELY REGULAR AND NORMAL SPACES

We give here suitably adapted versions of the stronger separation principles of classical topology, which are usually expressed using negations and closures. *Tychonoff's embedding theorem* seems to justify our definition of a *completely regular* space, and then the condition of being *separated regular* fits nicely between this and $T_2^{\#}$; the weaker regularity property is useful for *uniform* spaces (VIII.1.3). The situation with *normality* is rather more confused: conditions such as we give, which are strong enough at least to imply complete regularity, are not provable of metric spaces; those which are provable (e.g. STOUT [25]) are too weak to give anything like the classical results, it seems (see VIII.1.1).

1. *Regularity, complete regularity, normality*

Definition: A topological space X is

regular iff $\forall U \forall x \in U \exists V \ni x . \bar{V} \subseteq U$

separated regular iff $\forall U \forall x \in U \exists V \ni x . V \subseteq U$

completely regular iff $\forall U \forall x \in U \exists f: X \xrightarrow{\text{cts}} [0,1] (f(x) = 0 \wedge f^{-1}([0,1]) \subseteq U)$

A separated space X is

$normal_1$ iff $\forall A, B \subseteq X (A \subseteq B \rightarrow \exists C . A \subseteq C \subseteq B)$

$normal_2$ iff $\forall U, V (U \cup V = X \rightarrow \exists W . U \cup W = X = V \cup \bar{W})$

Remarks: The difference between these two notions of *normality* is very slight, $normal_1$ just being the specialisation of $normal_2$ to the cases where " U " is of the form " \bar{A} ". The former will appear as the natural condition for the proof of Urysohn's lemma; the latter is suitable for properties relating to *finite open covers*.

The following present no special difficulty:

(i) Regular $\Rightarrow T_2$

(ii) Completely regular \Rightarrow separated regular $\Rightarrow T_2^{\#}$.

(iii) Regular and completely regular spaces are closed under subspaces and products.

(iv) (Tychonoff's embedding theorem) Every completely regular space is homeomorphic to a subspace of a product of metric spaces. The converse also holds by (iii) and VIII.1.1.

To show how our notions fit together, we prove a

Proposition: A w -closed subspace of a $normal_2$ (or $normal_1$) space is also $normal_2$ (or $normal_1$).

Proof: Let X be normal₂, and $A \subseteq X$ be w -closed, and suppose that $A \subseteq (U \cup V)$. Then $A \subseteq (U \cup V)$, that is, $((U \cup \neg A) \cup V) = X$. So let $(U \cup \neg A) \cup W = X = V \cup \neg W$, whence $A \subseteq (U \cup W)$ and $A \subseteq (V \cup \neg W)$, as required. \square

2. Dense linear orders

As examples of *separated regular* spaces we have the order topologies of *dense linear orders*:

Let $(A, <)$ be a dense linear order, and observe that, since $(a, b) \cap (a', b') = (\text{Max}(a, a'), \text{Min}(b, b'))$, the open intervals form a *basis* for its order topology.

Now if $a < c < b$, let $a < a' < c < b' < b$, by *denseness*; then $c \in (a', b') \cap (a, b)$, as $<$ is *separated*, and the result follows.

3. Urysohn's Lemma

Assuming *DC* (I.3.4) we have the following

Lemma: X is normal₁ iff whenever $A \subseteq B \subseteq X$,
 $\exists f: X \xrightarrow{\text{cts}} [0, 1] (f(A) \subseteq \{0\} \wedge f^{-1}([0, 1]) \subseteq B)$.

Proof: We can follow the usual pattern of proof (e.g. NAGATA [18] p.75), but replacing " $A \subseteq B$ " by " $A \subseteq B$ " throughout:

Let X be normal₁ and $A \subseteq B \subseteq X$, and choose (according to *DC*), for successive values of n and for all $m \leq 2^n$, subsets $A(m/2^n)$ such that

- (i) $A(0) = A$, $A(1) = B$
- (ii) $A(m/2^n) = A(2m/2^{n+1})$
- (iii) $A(m/2^n) \subseteq A(2m+1/2^{n+1}) \subseteq A(m+1/2^n)$.

The required function f is then given by specifying that

$$\begin{aligned} p < f(x) & \text{ iff } \exists (m/2^n) > p (x \notin A(m/2^n) \vee m = 0) \\ q > f(x) & \text{ iff } \exists (m/2^n) < q (x \in A(m/2^n) \vee m = 2^n) \end{aligned}$$

for all $p, q \in \mathbb{Q}$, $x \in X$.

That $f(x) \in \mathbb{R}$ follows from condition (iii) above since then

$$\forall m < 2^n (f(x) < (m+1/2^n) \vee (m/2^n) < f(x));$$

the remaining properties are proved as usual.

Conversely, if $A \subseteq B \subseteq X$, and $f: X \xrightarrow{\text{cts}} [0, 1]$ is of the kind described in the lemma, we can put $C = \{x \mid f(x) < \frac{1}{2}\}$ and prove that $A \subseteq C \subseteq B$, since

$$\forall x \in X [(f(x) < 1 \vee f(x) > \frac{1}{2}) \wedge (f(x) > 0 \vee f(x) < \frac{1}{2})] \quad \square$$

As *corollaries* we may deduce (assuming *DC*):

- (i) $\text{Normal}_1 \Rightarrow$ completely regular.
- (ii) If X is normal_1 and $(\neg A \cup \neg B) = X$, then
 $\exists f: X \xrightarrow{\text{cts}} [0,1] (f(A) \subseteq \{0\} \wedge f(B) \subseteq \{1\})$.
- (iii) X is normal_2 iff whenever $(U \cup V) = X$
 $\exists f: X \xrightarrow{\text{cts}} [0,1] (f^{-1}((0,1]) \subseteq V \wedge f^{-1}([0,1)) \subseteq U)$.

4. *Tietze's extension theorem*

Assuming *DC* we have the following

Theorem: If F is a w -closed subset of a normal_1 space X , and $f: F \xrightarrow{\text{cts}} [-1,1]$, then f has a continuous extension $\varphi: X \xrightarrow{\text{cts}} [-1,1]$.

Proof: Again we follow NAGATA [18], (p.78) with a similar notation for comparison:

With the hypotheses of the theorem, we put

$$G = \{x \in F \mid f(x) < -\frac{1}{3}\}, \quad H = \{x \in F \mid f(x) > \frac{1}{3}\}.$$

Then $F \subseteq (\neg G \cup \neg H)$, so $F \subset (\neg G \cup \neg H)$;

but $G, H \subseteq F$, so $(\neg G \cup \neg H) = X$.

Now corollary (ii) of Urysohn's lemma gives $g: X \xrightarrow{\text{cts}} [0,1]$ such that $(g(G) \subseteq \{0\} \wedge g(H) \subseteq \{1\})$.

Then put $\varphi_0(x) = \frac{2}{3} \cdot (g(x) - \frac{1}{2})$

so that $\forall x \in X. |\varphi_0(x)| \leq (\frac{1}{3})$

and $\forall x \in F. |f(x) - \varphi_0(x)| \leq (\frac{2}{3})$.

The proof is completed by the usual iteration argument. \square

Remarks: This appears to be the best we can do in this general context; BISHOP [1] has a form for metric spaces which may well be more useful.

5. *Finite open covers*

To illustrate the use of the normal_2 condition we prove from it some facts about *finite open covers* familiar from classical treatments. We need first a

Lemma: If $\{U_i \mid i < n\}$ is an open cover of a normal_2 space X , then there is an open cover $\{W_i \mid i < n\}$ such that $\forall i < n. W_i \subset U_i$.

Proof: We choose successively W_0, \dots, W_{n-1} so that, for $i < n$, $\{W_0, \dots, W_{i-1}, U_i, \dots, U_{n-1}\}$ covers X , and $(\neg W_i \cup U_i) = X$. \square

Theorem: If X is separated and inhabited, then

- (i) X is normal_2 iff every finite open cover has a finite open well-refinement,
 (ii) if X is normal_2 every finite open cover has a finite open star-refinement.

Proof: (i) One direction is immediate from the lemma. For the converse, if $U \cup V = X$, let $\{W_i \mid i < n\} \subset \{U, V\}$; then without loss we may suppose that

$$W_0, \dots, W_{k-1} \subset V \text{ and } W_k, \dots, W_{n-1} \subset U.$$

Putting $W = \bigcup_{i < k} W_i$ we have $U \cup W = X = V \cup \neg W$

(ii) Let $\{U_i \mid i < n\}$ cover X and let $\{W_i \mid i < n\}$ be as in the lemma. For $a \in P'([n])$ (I.2.1, I.3.1) we put

$$U_a = \bigcap_{i \in a} U_i \cap \bigcap_{i \notin a} \neg W_i.$$

Then $\{U_a \mid a \in P'([n])\}$ is also a finite open cover of X (since $P'([n]) \twoheadrightarrow [2^n]$), Finally, for $B \in P'(P'([n]))$ we put

$$V_B = \bigcap_{a \in B} U_a \cap \bigcap_{a \notin B} \neg W_a \text{ and } V = \{V_B \mid B \in P'(P'([n]))\};$$

we show that this finite open cover of X star-refines the original cover $\{U_i \mid i < n\}$:

Given $B \in P'(P'([n]))$, *detachability* gives

either $\forall j < n \exists a \in B. j \notin a$

or $\exists j < n \forall a \in B. j \in a.$

In the first case, $\forall j < n. V_B \subseteq \neg W_j$, so $V_B = \emptyset$.

In the second case, if $j < n$ and $\forall a \in B. j \in a$, we can show that

$$S(V_B; V) \subseteq U_j:$$

Suppose $x \in (V_B \cap V_{B'})$, and let $a \in P'([n])$ and $x \in W_a$; then $a \in (B \cap B')$, since $(a \notin B \rightarrow V_B \subseteq \neg W_a)$.

Thus $V_{B'} \subseteq U_a$,

while $j \in a$, so $U_a \subseteq U_j$. □

(The classically valid converse to (ii) cannot be expected to hold — see the discussion for uniform spaces in VIII.1.3.)

6. *Regularity and normality in bundles*

If (E, π) is a bundle over T ,

(i) E is *regular* iff for $x \in W_0 \in \mathcal{O}(E)$
 $\exists U, W_1 [x \in (W_1)_U \wedge \forall a \in \underline{E}. \cup \{t \mid a(t) \in W_1\}^\circ \subseteq W_0]$

(ii) E is *separated regular* iff for $x \in W_0 \in \mathcal{O}(E)$
 $\exists U, W_1 [x \in (W_1)_U \wedge W_1 \subset W_0 \text{ in the subspace } E_U].$

(iii) E is *normal₁* iff for $W \in \mathcal{O}(E)$ and A a subbundle
 if $A_U \subset W$ in the subspace E_U
 then $U \subseteq \cup \{U' \mid \exists W'. A_{U'} \subset W' \subset W \text{ in the subspace } E_{U'}\}.$

(iv) E is *normal₂* iff for $W_0, W_1 \in \mathcal{O}(E)$
 if $E_U \subseteq (W_0 \cup W_1)$
 then $U \subseteq \cup \{U' \mid \exists W. E_{U'} \subseteq (W_0 \cup W) \cap (W_1 \cup W)\}$

Then we can deduce:

(v) If E is *locally* (separated) regular (or *normal₁* or *normal₂*),
 then E is also (separated) regular (or *normal₁* or *normal₂*).

(vi) X_T is (separated) regular iff X is (separated) regular.

7. *Complete regularity in bundles*

We use the analysis of continuous functions given in V.2.5 and 7 to interpret this condition in a bundle (E, π) :

E is *completely regular* iff for $a \in \underline{E}$ and $a(t) \in W \in \mathcal{O}(E)$
 $\exists U \exists t \exists f: E_U \xrightarrow{\text{cts}} [0,1] (\forall s \in U. f(a(s)) = 0 \wedge f^{-1}([0,1]) \subseteq W).$

The questions of whether the properties of complete regularity or normality are "preserved" in the passage from a space X to a bundle X_T , are still open; but the answers seem likely to be "no".

CHAPTER VII

COMPACTNESS, CONNECTEDNESS, SOBERNESSVII.1 COMPACT SPACES

The notion of *compactness* considered here is the familiar classical one relating to *open covers*; since it does not "provide points" in the space, it does not tie in with properties of being "complete" or "closed" as in classical topology. The condition of being *w-closed* arose first, as a substitute for the ordinary notion, in connection with compact subspaces. The incomparability of this notion of compactness with that of BISHOP [1] (and originally Brouwer) will appear in VIII.2 when we consider compactness properties in the reals. The work of FOURMAN [5] in this connection suggests that we consider a notion more applicable to open sets than to closed ones, but this has yet to be properly investigated.

1. *Compactness*

Definition: A space X is *compact* iff every open cover has a finite subcover; that is, if $\mathcal{U} \subseteq \mathcal{O}(X)$ covers X , then there are $U_0, \dots, U_{n-1} \in \mathcal{U}$ such that

$$\bigcup_{i < n} U_i = X.$$

X is *locally compact* iff it has a neighbourhood basis of compact subspaces.

Remark: Our definition of a "locally compact" space corresponds to what is usually called "properly locally compact"; *classically* the two coincide for T_2 spaces. This seems to be the right notion for applications (Remark VII.1.2 and VIII.3.1).

The following gives versions of well-known classical results, using some of our previous versions of the separation principles.

Proposition: (i) Compact $T_2^{\#} \Rightarrow \text{Normal}_2$.

(ii) Compact spaces are closed under quotients and finite products, but not infinite ones.

Proof: (i) Let X be compact $T_2^{\#}$, and show first that it is *separated regular*:

Let $x \in U \in \mathcal{O}(X)$, so that $\forall y \in X (y \#_2 x \vee y \in U)$;

thus, if $W = \{W \mid x \# W\}$, $(W \cup \{U\})$ covers X .

Then, if $\{W_0, \dots, W_{n-1}, U\}$ is a finite subcover,

put $V = \neg \left(\bigcup_{i < n} W_i \right)$, so that $x \in V \subset U$.

Next, to prove X $normal_2$, let $(U \cup V) = X$; thus $\{W \mid W \subset U \vee W \subset V\}$ covers X , as it is separated regular. Then a finite subcover is a finite open w -refinement of $\{U, V\}$, as required in VI.2.5.

(ii) Closure under quotients is standard, and the proof that the product of two compact spaces is compact presents no special difficulty. For the counter-example we refer to VIII.3 where we use a result of FOURMAN and HYLAND [6] to show that Cantor space need not be compact although it is the power $[2]^{\mathbb{N}}$ (where $[2]$ has the *discrete* topology.) \square

2. Compact subspaces

The results of this subsection show that the notion of a w -closed subset is exactly the right substitute for the ordinary notion for characterising the compact subspaces of a compact $T_2^{\#}$ space.

Lemma: Let A be a compact subspace of X . Then

- (i) if X is $T_2^{\#}$, A is a -closed, and is w -closed iff it is located;
- (ii) if X is separated regular, A is w -closed.

Proof: (i) Let X be $T_2^{\#}$ and A a compact subspace, and suppose that $\forall y \in A. y \neq x$: Thus $\{W \mid x \notin W\}$ covers A so let $\{W_0, \dots, W_{n-1}\}$ be a finite subcover. Then

$$A \subseteq \bigcup_{i < n} W_i$$

while $\forall i < n. x \notin W_i$; so

$$x \notin \bigcup_{i < n} W_i$$

whence $x \notin A$.

In view of Proposition VI.1.8(iv) we have only to show that, if A is located, it is w -closed:

Let $A \subseteq U$ and fix $x \in X$; since $\forall y \in A (y \neq_2 x \vee x \in U)$, $\{W \mid x \notin W\} \cup \{(x \in U)^\wedge\}$ covers A .

Let $\{W_0, \dots, W_{n-1}, (x \in U)^\wedge\}$ be a finite subcover, and put

$$V = \neg \left(\bigcup_{i < n} W_i \right).$$

Then assuming A *located* we have $(\exists y \in (V \cap A) \vee x \notin A)$; but if $y \in (V \cap A)$, $y \in (x \in U)^\wedge$, so $(x \in U \vee x \notin A)$ as required.

(ii) Let X be separated regular and A a compact subspace, and suppose that $A \subseteq U$:

Thus $\{W \mid W \subset U\}$ covers A , so let $\{W_0, \dots, W_{n-1}\}$ be a finite subcover; then $A \subseteq \bigcup_{i < n} W_i \subset U$. \square

Theorem: If X is compact T_2^{**} and $A \subseteq X$, A is compact iff it is w -closed.

Proof: One direction is immediate from (ii) of the lemma since X is separated regular by Proposition VII.1.1(i).

For the converse let A be w -closed and $A \subseteq \bigcup U$; then $A \subset \bigcup U$, that is, $(\bigcup U \setminus \{A\})$ covers X . But if $\{U_0, \dots, U_{n-1}, \neg A\}$ is a finite subcover of X , so is $\{U_0, \dots, U_{n-1}\}$ one of A . \square

Remark: It is not hard to see, using these ideas, that locally compact $T_2^{**} \Rightarrow$ Separated regular.

3. *Incompactness - divergence*

As noted in the introduction to this section, compactness in our sense does not "provide points", and we will see in VIII.2 that in a compact space not every proper filter need have a cluster point. To express the content of this classical result we introduce strong notions of "non-compactness" and "non-convergence" as follows:

Definition: A space X is *incompact* iff it has an open cover \mathbf{U} such that, if $\{U_0, \dots, U_{n-1}\}$ is any finite subset,

$$\exists x \in (X \setminus \bigcup_{i < n} U_i).$$

A proper filter \mathbf{F} on $P(X)$ is *divergent* iff $\forall x \in X \exists A \in \mathbf{F}. x \notin A$.

Theorem: X is incompact iff it has a divergent proper filter.

Proof: (This is really just the well-known classical proof.) If X is incompact through some cover \mathbf{U} , put

$$\mathbf{F} = \{A \subseteq X \mid \exists \{U_0, \dots, U_{n-1}\} \subseteq \mathbf{U}. A \supseteq (X \setminus \bigcup_{i < n} U_i)\};$$

then \mathbf{F} is a divergent proper filter.

Conversely, given such an \mathbf{F} , put $\mathbf{U} = \{\neg A \mid A \in \mathbf{F}\}$; \mathbf{U} covers by divergence, while for

$$\{A_0, \dots, A_{n-1}\} \subseteq \mathbf{F}, (X \setminus \bigcup_{i < n} \neg A_i) \supseteq \bigcap_{i < n} A_i,$$

which is inhabited by properness. \square

4. *Compactness in bundles*

If (E, π) is a bundle over T , E is *compact* iff for $W \subseteq \mathbf{O}(E)$ and $U \in \mathbf{O}(T)$, if $E_U \subseteq \bigcup W$ then

$$U \subseteq U' \mid \exists \{W_0, \dots, W_{n-1}\} \subseteq W. E_{U'} \subseteq \bigcup_{i < n} W_i.$$

Then we can deduce

- (i) If E is locally compact, E is compact.
- (ii) X_T is compact iff X is compact.

For (ii), let X be compact and suppose that $W \subseteq \mathcal{O}(T \times X)$ covers $(U \times X)$:

Fix $t \in U$, so that $\{V \in \mathcal{O}(X) \mid \exists U' \ni t \exists W \in W. (U' \times V) \subseteq W\}$ covers X (by definition of the product topology).

Then, if $\{V_0, \dots, V_{n-1}\}$ is a finite subcover, choose $U_0, \dots, U_{n-1}, W_0, \dots, W_{n-1}$ such that

$$\forall i < n (t \in U_i \wedge W_i \in W \wedge (U_i \times V_i) \subseteq W_i).$$

So if

$$U' = \bigcap_{i < n} U_i, \quad t \in U' \quad \text{and} \quad (U' \times X) \subseteq \bigcup_{i < n} W_i,$$

as required.

For the converse one uses constant functions in the usual way.

VII.2 CONNECTED SPACES

TRÖELSTRA [28] has considered notions of *connectedness* in intuitionistic topology, with applications to separable metric spaces and using continuity principles. The two notions we investigate are based on properties of *open* sets, the stronger one having a very attractive "geometric" form. Unfortunately (VIII.2.7) it may fail to hold of intervals of the real line when *CAC* does not hold. In fact we have not found so far any "advantage" in this notion over the weaker one (which *does* hold in the reals – VIII.2.3).

1. *Connectedness*

Definition: A space X is *connected* iff

$$\forall U, V [(U \cup V) = X \wedge \exists x \in U \wedge \exists x \in V \rightarrow \exists x \in (U \cap V)]$$

and *weakly connected* iff

$$\neg \exists U, V [(U \cup V) = X \wedge \exists x \in U \wedge \exists x \in V \wedge (U \cap V) = \emptyset]$$

Some properties: The following are easily verified as in the classical theory:

- (i) If $(U \cap V)$, $(U \cup V)$ are both (weakly) connected subspaces of X , then so are U and V .
- (ii) If $A \subseteq B \subseteq A^- \subseteq X$ and A is (weakly) connected, then so is B .
- (iii) If $A \subseteq P(X)$ is a set of (weakly) connected sets and $(\cap A)$ is inhabited, then $(\cup A)$ is (weakly) connected.
- (iv) For each $x \in X$, its (*weak*) *component* is $U\{A \subseteq X \mid x \in A \wedge A \text{ (weakly) connected}\}$, which is (weakly) connected and closed.
- (v) (Weakly) connected spaces are closed under quotients and products over discrete index sets.

2. *Order topologies*

If the order topology of a separated order $(A, <)$ is *connected* the order must be *dense*; for if $a, b \in A$ and $a < b$,

$$[(-\infty, b) \cup (a, \infty) = A \wedge a \in (-\infty, b) \wedge b \in (a, \infty)]$$

so connectedness gives $\exists c \in (-\infty, b) \cap (a, \infty)$.

But this seems to be all that can be salvaged of the classical connexion with *complete dense* orders; VIII.2.6 provides an example of a connected space which is not complete with respect to Dedekind cuts.

3. *Partitions - clopen sets*

We now give alternative characterisations of weakly connected spaces in terms of *finite partitions* and *clopen* subsets; as noted in V.1.3 every \hat{C} is both *closed* and *open*, so our notion of "clopen" set has to be much stronger than this.

Definition: A *finite partition* of a space X is a set of opens $\{U_0, \dots, U_{n-1}\}$ such that

$$X = \bigcup_{i < n} U_i \quad \text{and} \quad \forall i < j < n. U_i \cap U_j = \emptyset.$$

A subset A of X is *clopen* iff $A \cup \neg A = X$.

Proposition: X is weakly connected iff, for every finite partition $\{U_0, \dots, U_{n-1}\}$, $\exists i < n. U_i = X$ (assuming X inhabited).

Proof: One direction is obvious; conversely, if X is weakly connected and $\{U_0, \dots, U_{n-1}\}$ is any partition, let $x \in X$ (as X is inhabited) and suppose $x \in U_i$:

Put $U = U_i$ and $V = \bigcup_{j \neq i} U_j$; so that

$$[(U \cup V) = X \wedge \exists x \in U \wedge (U \cap V) = \emptyset]$$

whence $V = \emptyset$, and $U_i = X$. □

Since every element of a finite partition is clopen, we may deduce the following

Corollary: If X is inhabited, it is weakly connected iff every clopen subset is either X or \emptyset .

4. *Connectedness in bundles*

If (E, π) is a bundle over T

(i) E is *connected* iff for $W, W' \in \mathcal{O}(E)$ and $U \in \mathcal{O}(T)$

if $E_U \subseteq (W \cup W')$ and $U \subseteq (\pi(W) \cap \pi(W'))$

then $U \subseteq \pi(W \cap W')$.

(ii) E is *weakly connected* iff for $W, W' \in \mathcal{O}(E)$ and $U \in \mathcal{O}(T)$

if $E_U \subseteq (W \cup W')$ and $(W \cap W' \cap E_U) = \emptyset$

then $(U \cap \pi(W) \cap \pi(W')) = \emptyset$.

Then we can deduce:

(iii) If each fibre is (weakly) connected, so is E .

(iv) X_T is (weakly) connected iff X is.

VII.3 SOBER SPACES

Classically a *sober* space is one in which every irreducible closed set is the closure of a (unique) point. The definition we use is due to *Mike Fourman* who used it to give a *Representation Theorem* for internal sober spaces in sheaves over a sober space; see FOURMAN and SCOTT [7], where the more *algebraic* and *categorical* aspects are presented. We confine ourselves to *topological* questions mostly, where the theory is rather different from the classical.

1. *Soberness*

Definition: A T_0 space X is *space* iff for every super-filter Δ on $\mathbf{O}(X)$ there is a point x of X such that $\Delta = \Delta_x = \{U \in \mathbf{O}(X) \mid x \in U\}$.
(Note: The *uniqueness* of such a point is guaranteed by the condition T_0 .)

- Some properties:* (i) The space of models of a geometric propositional theory (IV.1) is sober; so also
(ii) the space of superfilters on a cHa is sober.
(iii) Sober spaces are closed under products.

For (i), let Θ be a theory and $\underline{\Delta}$ a superfilter on $\mathbf{O}(\text{Mod}(\Theta))$, and put $\Delta = \{\Upsilon \mid U_\Upsilon \in \underline{\Delta}\}$; then $\Delta \in \text{Mod}(\Theta)$, and for $U_\Gamma \in \mathbf{O}(\text{Mod}(\Theta))$

$$\Delta \in U_\Gamma \leftrightarrow \exists \Upsilon \in (\Delta \cap \Gamma) \leftrightarrow \exists \Upsilon \in \Gamma. U_\Upsilon \in \underline{\Delta} \leftrightarrow U_\Gamma \in \underline{\Delta}.$$

2. *Some conditions for soberness*

The following provide criteria for soberness which will be useful in VIII.1.6:

Lemma: If $x \in X$ and Δ is a superfilter on $\mathbf{O}(X)$,

- (i) if X is regular and Δ converges to x , then $\Delta = \Delta_x$;
(ii) if X is separated regular and x is a cluster point of Δ , then $\Delta = \Delta_x$.

Proof: (i) Let X be regular and Δ converge to x , so $\Delta_x \subseteq \Delta$; conversely suppose that $U \in \Delta$:

By regularity $U = U\{V \mid V^- \subseteq U\}$ so $\exists V \in \Delta. V^- \subseteq U$.

But if $V \in \Delta$, $x \in V^-$; so $x \in U$.

(ii) Let X be separated regular and x a cluster point of Δ ; so $\Delta \subseteq \Delta_x$ as in (i). Conversely let $x \in U$ so $(U \cup \neg\{x\}) = X$; but $x \notin \neg\{x\}$ so $\neg\{x\} \notin \Delta$, whence $U \in \Delta$. \square

Corollary: (i) If X is regular and every superfilter on $\mathcal{O}(X)$ converges, X is sober.

(ii) If X is separated regular and every superfilter on $\mathcal{O}(X)$ has a cluster point, X is sober.

3. Soberification

Every space can be "extended" to a sober one in the following way:

Definition: The *soberification* of a space X is the space $\text{Sob}(X)$ of superfilters on $\mathcal{O}(X)$.

By VIII.3.1 this space is always sober, and X is embedded in it by taking x to Δ_x ; it is in fact the *unique* sober space Y such that $\mathcal{O}(Y)$ is isomorphic to $\mathcal{O}(X)$ (up to homeomorphism). If X is a subspace of Y we can give a very simple condition for this isomorphism:

$$(*) \quad \forall U, V \in \mathcal{O}(Y) \quad ((U \cap X) \subseteq V \rightarrow U \subseteq V) .$$

The following rather subtle result will be very useful in VIII.2:

Lemma: If Y is a $T_2^\#$ space and X is a *dense* subspace such that Y has a neighbourhood basis of subsets A for which $(A \cap X)$ is *compact*, then $(*)$ holds.

Proof: With the conditions stated, let $U, V \in \mathcal{O}(Y)$ with $(U \cap X) \subseteq V$ and $y \in U$:

Let $y \in A^\circ$, $A \subseteq U$ and $(A \cap X)$ be compact.

For $x \in X$, $(x \#_2 y \vee x \in U)$; but if $x \in U$, $x \in V$, so that $(x \#_2 y \vee y \in V)$.

Thus $(\{W \in \mathcal{O}(Y) \mid y \# W\} \cup \{(y \in V)^\wedge\})$ covers X ,

and we have a finite subcover $\{W_0, \dots, W_{n-1}, (y \in V)^\wedge\}$ of $(A \cap X)$. Put

$$W = \bigcup_{i < n} W_i$$

so that $y \in (A^\circ \cap \neg W)$ whence *denseness* of X gives $\exists x \in (A^\circ \cap \neg W \cap X)$.

But then $\exists x \in (y \in V)^\wedge$ so $y \in V$ as required. \square

Remark: Since X and $\text{Sob}(X)$ have isomorphic cHa 's of open subsets, they *share* those topological properties which can be expressed entirely in terms of opens; these include *compact*, *(weakly) connected*, *normal₂* and *separated regular*.

4. *Soberification of bundles*

General questions about soberness in bundles are considered in FOURMAN and SCOTT [7]. We just note that if (E, π) is a bundle over T and E is sober then so is E , and concentrate on the special cases of the form X_T .

Lemma:

- (i) X_T is sober iff X is.
- (ii) $\text{Sob}(\hat{X}) \cong \text{Sob}(X_T) \cong (\text{Sob}(X))_T$, that is, the *internal* soberifications of \hat{X} and X_T coincide with the space obtained from the bundle $(\text{Sob}(X))_T$.

Proof: (i) The results of IV.1 combined with V.2.7 show that an internal superfilter on $\mathcal{O}(X_T)$ corresponds to an element of $\mathcal{C}(T, \text{Sob}(X))$, so that X being sober makes X_T sober also. For the converse use constant functions as usual.

(ii) By (i), $(\text{Sob}(X))_T$ is internally sober and we have only to check the condition (*) of VII.3.3 for \hat{X} regarded as a subspace of $(\text{Sob}(X))_T$; that is, for

$$W, W' \in \mathcal{O}(T \times \text{Sob}(X)), \mathbb{I}(W \cap \hat{X}) \subseteq W' \mathbb{I} \subseteq \mathbb{I}W \subseteq W' \mathbb{I}:$$

If $t \in \mathbb{I}(W \cap \hat{X}) \subseteq W' \mathbb{I}$, $(W_t \cap X) \subseteq W'_t$

so $W_t \subseteq W'_t$ and the result follows. \square

5. *Soberness of \mathcal{Q}*

Classically every T_2 space is sober; we use the results above to give a counter-example.

Theorem: In $\text{Sh}(\mathcal{Q})$ the rationals are not sober.

Proof: Over any space T the rationals appear as the subspace $\hat{\mathcal{Q}}$ of \mathcal{Q}_T (giving \mathcal{Q} its usual (order) topology, of course) by the results of III.3, so the immediately preceding lemma evaluates their internal soberification as $(\text{Sob}(\mathcal{Q}))_T$, which certainly includes \mathcal{Q}_T .

Now in $\mathcal{Q}_{\mathcal{Q}}$ we have a *generic* element $\iota = (\lambda p, p)$ which is nowhere locally constant so not an element of $\hat{\mathcal{Q}}$; thus $\hat{\mathcal{Q}}$ is not sober internally. \square

(See also VIII.2.6 for a similar application to the *Cauchy reals*.)

CHAPTER VIII
SPECIAL SPACES

VIII.1 METRIC AND UNIFORM SPACES

1. *Metrics*

We give here the standard "constructive" definition of a *metric space*, which has already been thoroughly investigated (e.g. BISHOP [1], TROELSTRA [26]). More general notions are possible – and interesting – for example via the specification of "open balls"; but we do not consider them.

Definition: A *metric space* is a structure (X, ρ) such that $\rho: X^2 \rightarrow [0, \infty)$ and

- (i) $\forall x, y \in X (\rho(x, y) = 0 \leftrightarrow x = y)$
- (ii) $\forall x, y \in X. \rho(x, y) = \rho(y, x)$
- (iii) $\forall x, y, z \in X. \rho(x, z) \leq (\rho(x, y) + \rho(y, z)).$

If (X, ρ) is a metric space, the *metric topology* on X has as basis the *open spheres*

$$S(x; \delta) = \{y \in X \mid \rho(x, y) < \delta\} \quad \text{for } x \in X, \delta > 0.$$

If $\mathcal{O}(X)$ is a topology on X , a metric ρ on X is *compatible* with it iff the metric topology coincides with $\mathcal{O}(X)$.

Remarks: Metric topologies are easily seen to be *completely regular*, for if $x \in U$ and U is open let $\delta > 0$ and $S(x; \delta) \subseteq U$; then the required continuous function is $\lambda y. \text{Min}(\delta, \rho(x, y))/\delta$. The Dedekind reals have the usual metric, of course, which is compatible with the order topology; we will see in VIII.2.7 that this space need not be normal₂. The most that we can prove seems to be the following, due essentially to STOUT [25]:

Proposition: If (X, ρ) is a metric space and $A, B \subseteq X$ are such that $A \subseteq \neg B$ and $B \subseteq \neg A$, then $\exists U, V [A \subseteq U \wedge B \subseteq V \wedge (U \cap V) = \emptyset]$.

Proof: With A, B as described we put

$$U = \{x \mid \exists \delta > 0 (S(x; \delta) \cap B = \emptyset \wedge \exists a \in A \cap S(x; \delta))\},$$

and define V similarly. □

Note: The hypotheses here are classically equivalent to $(A \cap B) = \emptyset$, when A, B are *closed*, so this is a "normality" property. Alternative versions may be given, by requiring for example that A, B be "metrically located" (BISHOP [1]).

2. *Metrics on bundles*

Since a metric is compatible with a given topology iff it is a continuous function and the open spheres form a basis, we can use the analysis of V.2.7 to describe metrics on bundles:

If (E, π) is a bundle over T , a metric on E compatible with its topology corresponds to a function $\sigma : E^2 \xrightarrow{\text{cts}} [0, \infty]$ which acts as a metric on each fibre of (E, π) and satisfies:

for $a \in \underline{E}$, $W \in \mathcal{O}(E)$, $t \in T$, if $a(t) \in W$
 then $\exists \delta > 0 \exists U \ni t \forall x \in E_U [\sigma(a(\pi(x)), x) < \delta \rightarrow x \in W]$.

Here (E^2, π^2) is the product of (E, π) with itself (V.2.6). The last condition above corresponds to the internal condition that

$$a \in \underline{W} \rightarrow \exists \delta > 0 . S(a; \delta) \subseteq \underline{W} .$$

In the special case of a bundle X_T , where X has the metric topology for a metric ρ , we may define such a function σ by putting

$$\sigma(\langle t, x \rangle, \langle t, y \rangle) = \rho(x, y).$$

Alternatively we may describe this internal metric as a sheaf morphism $\sigma : C(T, X)^2 \rightarrow C(T, [0, \infty))$ which "reads off" the distance "pointwise"; that is

$$\sigma(f, g)(t) = \rho(f(t), g(t)) \quad \text{for } t \in \text{dom}(f) \cap \text{dom}(g).$$

3. *Uniformities*

We define *uniformities* in very much the usual way via sets of "uniform covers", but an extra condition is needed even to make the topology separated, and this still seems to be inadequate for complete regularity. Is there some stronger "refinement" property which holds in metric and completely regular spaces (see examples below) and implies complete regularity of the uniform topology (assuming *DC* of course) ?

Definition: A *uniform space* is a structure (X, C) where C is a set of *covers* of X which includes $\{X\}$ and satisfies (with notation as in V.1.6):

$$(i) \quad \forall A \in C \forall B > A . B \in C$$

$$(ii) \quad \forall A, B \in C . \{A \cap B \mid A \in A, B \in B\} \in C$$

$$(iii) \quad \forall A \in C \exists B \in C . B <^* A$$

$$(iv) \quad \forall x, y \in X [\forall A \in C \exists A \in A (x \in A \wedge y \in A) \rightarrow x = y].$$

If (X, C) is a uniform space, the *uniform topology* on X has as neighbourhood basis the sets $S(x; A)$ for $x \in X, A \in C$, and the uniformity is *compatible* with a given topology iff these topologies coincide.

Understanding the relation " \subset " in the uniform topology we define finally that (X, \mathcal{C}) is a *separated uniform space* iff it is a uniform space satisfying also

$$(v) \quad \forall A \in \mathcal{C} \exists B \in \mathcal{C}. B \subset A.$$

Remarks: Classically condition (v) follows from the others, since $\forall Y \subseteq X \forall A \in \mathcal{C}. Y \subset S(Y; A)$; but this is equivalent to $\forall Y \subseteq X. L(Y)$, which implies *LEM* (see VI.1.7). We give a counter-example below using the following, which are easily proved:

- (i) (Separated) uniform topologies are (separated) regular.
- (ii) (Separated) uniform spaces are closed under subspaces and products.

4. *Examples*

- (i) The uniformity on $P(\{0\})$ containing *all* covers is compatible with the *discrete* topology; but it cannot be *separated* unless *LEM* holds, since $P(\{0\})$ cannot have an apartness.
- (ii) *Topological groups* may be given the usual uniformity.
- (iii) If (X, ρ) is a *metric space*, the *metric uniformity* consists of those covers with refinements of the form $U_\delta = \{S(x; \delta) \mid x \in X\}$ for $\delta > 0$. This is compatible with the metric topology and is *separated* since, for $\delta < \delta'$ and $x \in X$, $S(x; \delta) \subset S(x; \delta')$ whence $U_\delta \subset U_{\delta'}$.
- (iv) *Completely regular* spaces have compatible separated uniformities by Tychonoff's embedding theorem. The *converse* implication, that (separated) uniform topologies are completely regular, is proved classically again using the property $\forall Y \subseteq X \forall A \in \mathcal{C}. Y \subset S(Y; A)$ and there appears to be no way of getting round this.

5. *Uniform continuity*

Definition: If $(X, \mathcal{C}_X), (Y, \mathcal{C}_Y)$ are uniform spaces, a function $f: X \rightarrow Y$ is *uniformly continuous* iff $\forall A \in \mathcal{C}_Y. \{f^{-1}(A) \mid A \in A\} \in \mathcal{C}_X$.

Remarks: A uniformly continuous function is always continuous with respect to the uniform topologies.

The usual proof shows that a continuous function from a compact metric space is uniformly continuous, but this depends on the property of the uniformity that

$$\forall A \in \mathcal{C}. \{A \in A \mid \exists x \in A\} \in \mathcal{C}$$

which holds in metric uniformities but fails in general; we can not add this condition to the definition without losing closure under subspaces.

6. *Filter-completeness*

Definition: If (X, \mathcal{C}) is a uniform space, a *Cauchy filter* on it is a proper filter \mathcal{F} on $P(X)$ such that $\forall A \in \mathcal{C} \exists A \in (\mathcal{F} \cap \mathcal{A})$.

(X, \mathcal{C}) is *filter-complete* iff every Cauchy filter converges. If \mathcal{F}, \mathcal{G} are Cauchy filters on (X, \mathcal{C}) , they are *equal* iff $\forall A \in \mathcal{C} \exists A \in (\mathcal{F} \cap \mathcal{G} \cap \mathcal{A})$.

Remarks: These are the usual definitions and require little comment; the usual results go through, for example that a uniformly continuous function from a dense subspace of a uniform space to a filter-complete space has a unique uniformly continuous extension to the whole space. We have also a nice tie-up with some previous notions:

Proposition: Filter-complete uniform spaces are sober.

Proof: Let (X, \mathcal{C}) be filter-complete and Δ a superfilter on $\mathcal{O}(X)$, and put $\mathcal{F} = \{A \subseteq X \mid \exists U \in \Delta. U \subseteq A\}$. Since every uniform cover has a uniform open refinement by condition (iii), \mathcal{F} is Cauchy. Then \mathcal{F} converges, so Δ does too.

Now we can apply Corollary VII.3.2(i) to deduce soberness, since X is regular by VIII.1.3. □

7. *Filter completions*

We may prove exactly as in classical treatments (e.g. NAGATA [18], p.232) that the equality classes of Cauchy filters on a (separated) uniform space form a filter-complete (separated) uniform space into which the original space is unimorphically embedded as a dense subspace.

8. *Dedekind completions*

One may define as usual the condition on a metric space of being *complete* with respect to *Cauchy sequences*; but in the absence of *CAC* one naturally asks for a notion not involving the existence of sequences. We provide this as follows:

Definition: If (X, ρ) is a metric space, its *Dedekind completion* is obtained by giving a metric σ on the filter completion of its metric uniformity, by specifying for $p \in \mathbf{Q}, \mathcal{F}, \mathcal{G}$ Cauchy:

$$\sigma(\mathcal{F}, \mathcal{G}) < p \text{ iff } \exists q, r \exists x, y [S(x; q) \in \mathcal{F} \wedge S(y; r) \in \mathcal{G} \wedge p > (q+r + \rho(x, y))]$$

$$\sigma(\mathcal{F}, \mathcal{G}) > p \text{ iff } \exists q, r \exists x, y [S(x; q) \in \mathcal{F} \wedge S(y; r) \in \mathcal{G} \wedge p < (\rho(x, y) - (q+r))]$$

It is a purely routine matter to check that this does indeed define the required metric and that this extends the original metric under the embedding of the original space.

We call this the "Dedekind" completion since the Dedekind reals are obviously the Dedekind completion of the rationals. With DC one can always find a Cauchy *sequence* to correspond to any Cauchy *filter* in a metric space; but we will see in VIII.2.6 how completions by Cauchy sequences may diverge from Dedekind completions in suitable sheaf models.

9. *Totally bounded uniformities*

Definition: A uniform space is *totally bounded* iff every uniform cover has a finite subcover (not necessarily uniform).

Remarks: Classically a uniform space has a *compact* topology iff it is *totally bounded* and *filter-complete*, and this latter condition has been used as a notion of "compactness" for metric spaces (e.g. BISHOP [1]). We will see in VIII.2 that a metric space with compact topology need not be sober, and that a filter-complete totally bounded metric space need not even be normal_2 .

Examples:

- (i) The metric uniformity on the *unit interval* $[0,1]$ is totally bounded and complete.
- (ii) If X is a normal_2 space, the results of VI.2.5 show that the covers of X which have finite open refinements form a totally bounded separated uniformity compatible with the original topology.

10. *Uniformities in bundles*

We consider just the special bundles X_T over T : If C is a compatible uniformity on X , we can define a compatible internal uniformity C_T on X_T by specifying for $W \subseteq \mathbf{0}(T \times X)$ the evaluation

$$[W \in C_T] = U\{U \mid \exists A \in C. \{(U \times A) \mid A \in A\} < W\}.$$

If C is separated, complete or totally bounded, so is C_T .

VIII.2 CAUCHY AND DEDEKIND REALS

We now consider topological properties of the *Cauchy and Dedekind reals*, defining the Cauchy reals as a certain subspace of \mathbb{R} . These two spaces have many properties in common, but not, most notably, soberness. We use two particular sheaf models to provide the counter-examples promised in the preceding chapters. We also discuss briefly some consequences of "Brouwer's continuity principle".

1. *Cauchy reals*

Definition: A (Dedekind) real x is *Cauchy* iff there is some *Cauchy sequence* of rationals converging to it, i.e.

$$\exists F : \mathbb{N} \rightarrow \mathbb{Q} \forall n \exists k \forall m \geq k. |x - F(m)| < n^{-1}.$$

We denote the set of Cauchy reals by \mathbb{R}^C and for $A \subseteq \mathbb{R}$ write A^C for $(A \cap \mathbb{R}^C)$.

Remarks: Classically every real is Cauchy, and this would follow intuitionistically from *CAC*; it fails to hold in $\text{Sh}(\mathbb{R}^C)$ as will be seen in VIII.2.6.

2. *Separation*

Both \mathbb{R} and \mathbb{R}^C are metric spaces of course, and both are dense linear orders in the usual way; so they are both completely regular spaces. They may be normal_2 (VIII.2.6) or they may not (VIII.2.7); note that this latter model is not topological, and it is still an open problem to find a *space* over which \mathbb{R} is not normal_2 .

As an example of the notions of VI.1.8 we observe that in \mathbb{R} (and \mathbb{R}^C) closed bounded inhabited intervals are *w-closed*: for if $a \leq b$ and $[a, b] \leq U$, let $\delta > 0$ be such that $(a - \delta, b + \delta) \subseteq U$; then for $x \in \mathbb{R}$

either $(a - \delta) < x < (b + \delta)$ so $x \in U$
or $(x < a \vee b < x)$ so $x \notin [a, b]$.

This argument is easily generalised to the closed spheres of any *normed linear space*, in particular to finite products of \mathbb{R} (and \mathbb{R}^C).

3. *Connectedness*

Proposition: All intervals of \mathbb{R} (and \mathbb{R}^C) are weakly connected, and, assuming *CAC*, also connected.

Proof: We prove the two statements simultaneously pointing out the differences where they occur.

Let I be an interval of \mathbb{R} (or \mathbb{R}^C) and suppose that

$$[I \subset (U \cup V) \wedge \exists x \in (U \cap I) \wedge \exists x \in (V \cap I)] .$$

Let $p, q \in \mathbb{Q}$ with $[p \in (U \cap I) \wedge q \in (V \cap I)]$

and put $r = (p+q)/2$; then $r \in I$, so $r \in (U \cup V)$

Now for *weak connectedness* we will suppose also that $(I \cap U \cap V) = \emptyset$ so that just one of $r \in U, r \in V$ holds; for *connectedness* we need *CAC* to "choose between" the cases $r \in U, r \in V$ at each stage.

Thus we put $p_0 = p, q_0 = q$ and

$$(p_{n+1} = p_n \wedge q_{n+1} = (p_n + q_n)/2) \text{ if } (p_n + q_n)/2 \in V ,$$

$$(p_{n+1} = (p_n + q_n)/2 \wedge q_{n+1} = q_n) \text{ if } (p_n + q_n)/2 \in U .$$

Then $\{p_n : n \in \mathbb{N}\}, \{q_n : n \in \mathbb{N}\}$ are Cauchy sequences in $(U \cap I), (V \cap I)$ respectively, with the same (Cauchy) real x as their limit.

Now $x \in I$ being between p and q so $x \in U$, say, whence $q_n \in U$ for some n . But then $q_n \in (I \cap U \cap V)$, which will be the desired conclusion for *connectedness*, and contradict the hypotheses for *weak connectedness*. □

Remarks: By VII.2.4 connectedness of intervals will hold over any space when it is assumed externally (in particular, classically); VIII.2.7 gives a counter-example to the general case. Notice that this property for \mathbb{R}^C *implies* that for \mathbb{R} ; the converse is still open, but it seems unlikely.

4. Compactness

FOURMAN [5] shows that the compactness of closed intervals (equivalently of $[0,1]$) of \mathbb{R} is equivalent to a *completeness theorem* for the geometric theory defining \mathbb{R} (Chapter IV). From VII.1.4 we know that this property holds over any space when it is assumed externally (in particular, classically); VIII.2.7 gives a counter-example to the general case.

We denote the interval $[0,1]$ by I .

Notice that by Lemma VII.3.3 compactness of I^C implies that $\mathbb{R} = \text{Sob}(\mathbb{R}^C)$, whence I is compact also; the converse is still open, but it seems unlikely.

5. *Soberness*

As \mathbf{R} is the space of models of a geometric theory (alternatively: as it is a filter-complete uniform space), it is sober; VIII.2.6 shows that \mathbf{R}^C need not be, but first we evaluate the *soberification* of \mathbf{R}^C in sheaves over a space T using the following notation:

$\hat{\mathbf{R}}^C$ denotes the locally constant \mathbf{R}^C -valued elements of \mathbf{R}_T

$(\mathbf{R}_T)^C$ denotes the Cauchy elements of \mathbf{R}_T

$(\mathbf{R}^C)_T$ denotes the \mathbf{R}^C -valued elements of \mathbf{R}_T .

Proposition: Over any space T , $\hat{\mathbf{R}}^C \subseteq (\mathbf{R}_T)^C \subseteq (\mathbf{R}^C)_T$ whence $\text{Sob}((\mathbf{R}_T)^C) \cong (\text{Sob}(\mathbf{R}^C))_T$.

Proof: If F is a Cauchy sequence converging to x , \hat{F} converges internally to \hat{x} , so $\hat{\mathbf{R}}^C \subseteq (\mathbf{R}_T)^C$.

If ϕ is an internal Cauchy sequence converging to $f \in \mathbf{R}_T$ at $t \in T$, we put

$$F_t(n) = \{ q \in \mathbf{Q} \mid (t \in \llbracket \hat{q} = \phi(\hat{n}) \rrbracket) \} \text{ for } n \in \mathbf{N}$$

so that F_t is a Cauchy sequence converging to $f(t)$, whence $(\mathbf{R}_T)^C \subseteq (\mathbf{R}^C)_T$.

Now the final assertion follows by VII.3.4. \square

Remarks: This shows that the property $\mathbf{R} = \text{Sob}(\mathbf{R}^C)$ holds over any space when it is assumed externally (in particular, classically); it fails for example in the model $K(\mathbf{I} \times \mathbf{I})$ (FOURMAN and HYLAND [6]).

6. *The model over \mathbf{R}^C*

We now bring together a lot of previous results to show how the model over \mathbf{R}^C provides counter-examples to various things.

Lemma: In $\text{Sh}(\mathbf{R}^C)$ all Cauchy reals are locally constant.

Proof: By VIII.2.3 \mathbf{R}^C is locally weakly connected, so that all rational sequences are locally constant (III.3.5); in particular then all Cauchy sequences converge to locally constant reals.

We list some consequences of this for the *internal* Cauchy reals over \mathbf{R}^C , assuming \mathbf{I}^C to be compact *externally* (in particular, *classically*):

- (a) \mathbf{R}^C is not sober, since the generic real $\iota = (\lambda t.t)$ is Cauchy-valued but nowhere locally constant.
- (b) \mathbf{I}^C is a compact metric space, but the superfilter determined by ι has no cluster point (by VII.3.2).

(c) $\mathbb{R}^{\mathbb{C}}$ has a connected order topology, but is not complete.

(d) From (a) $\mathbb{R} \neq \mathbb{R}^{\mathbb{C}}$, so *CAC* must fail.

Next we list for reference some of the topological properties of \mathbb{R} in this model as seen *classically*:

(a) \mathbb{R} is normal_2 since the bundle $\mathbb{R}_{\mathbb{R}}$ is locally normal, being a metric space (VI.2.6(v)): in fact, (in the absence of *CAC*) the stronger property expressed in *Urysohn's Lemma* (VI.2.3, corollary (iii)) also holds.

(b) Closed intervals are compact, and all intervals are connected.

(c) $\mathbb{R} = \text{Sob}(\mathbb{R}^{\mathbb{C}})$

7. The model over $K(\mathbb{R})^*$

We describe briefly here a model due to FOURMAN and HYLAND [6] and use it to provide more counter-examples.

Definition: An open subset U of a separated space T is *coperfect* iff

$$\forall V \in \mathcal{O}(T) \forall t \in T ((\forall \eta \neg \{t\} \subseteq \eta \rightarrow V \subseteq \eta).$$

This is easily seen to correspond classically to the usual definition as the *complement of a perfect closed set*; it appears to be the right notion constructively, certainly for the present applications. The coperfect opens form a cHa denoted by $K(T)$.

Next we adjoin a point $*$ to T and define $K(T)^*$ to be those $U \subseteq (T \cup \{*\})$ such that $(U \cap T) \in K(T)$ and $[(\exists t \in (U \cap T)) \rightarrow * \in U]$. (Thus all inhabited elements contain $*$.)

Lemma: Over $K(T)^*$ all Dedekind reals are locally constant.

Proof: If ϕ is an internal Dedekind real,

put $p < x < q$ iff $* \in [\hat{p} < \phi < \hat{q}]$.

Then $x \in \mathbb{R}$ and $E(\phi) \leq [\hat{x} = \phi]$ □

Proposition: Over $K(\mathbb{R})^*$ the Dedekind reals are not connected or normal_2 .

Proof: We define two internal opens Φ, Ψ of the Dedekind reals by specifying for $x \in \mathbb{R}$

$$[\hat{x} \in \Phi] = (-\infty, x) \cup \{*\}$$

$$\text{and } [\hat{x} \in \Psi] = (x, \infty) \cup \{*\}$$

Now in $K(\mathbb{R})$, $(-\infty, x) \vee (x, \infty) = T$ so that

$$\forall x \in \mathbb{R}. [\hat{x} \in \Phi \vee \hat{x} \in \Psi] = T$$

whence $(\Phi \cup \Psi)$ covers \mathbb{R} internally. Furthermore

$$\bigcup_{x \in \mathbb{R}} [\hat{x} \in \Phi] = T = \bigcup_{x \in \mathbb{R}} [\hat{x} \in \Psi]$$

so that both Φ and Ψ are inhabited.

But, for $x \in \mathbb{R}$, $[\hat{x} \in \Phi \wedge \hat{x} \in \Psi] = \{*\}$ so that $[\exists x \in \Phi \cap \Psi] \neq T$, and \mathbb{R} is not connected.

Finally, if Φ' is an internal open and $(\Phi \cup \Phi')$ covers \mathbb{R} , for any $x \in \mathbb{R}$, $(x+1) \in [\hat{x} \in \Phi']$ so $* \in [\hat{x} \in \Phi']$ whence $[\hat{x} \notin \Phi'] = \perp$. Thus $(\neg \Phi')$ is empty internally, so that $(\Psi \cup \neg \Phi')$ cannot cover \mathbb{R} , and \mathbb{R} is not normal₂. □

(Note: in this model of course $\mathbb{R} = \mathbb{R}^C$.)

An exactly similar argument shows that over $K(I)^*$ the interval I is also not normal₂, from which (by VII.1.1) it follows that I is not compact, although it is a filter-complete totally bounded metric space.

8. Irrationals

A Dedekind real is said to be *irrational* iff it is apart from every rational. This notion was used by Brouwer, and many classical proofs of "irrationality" may be made to yield this property (e.g. HEYTING [11], p.28). Mike Fourman observed that all irrationals are Cauchy, by an argument similar to that in VIII.2.3. In fact irrationality is exactly the condition needed to ensure that the standard algorithm for generating a *continued fraction* converging to a real continues *ad infinitum*. From this it follows that the subspace of irrationals in $[0,1]$ is homeomorphic to Baire space, in the usual way.

9. Brouwer's continuity principle

We do not consider here the widely discussed *principles of continuity of functions* on Baire space or the reals, but just mention one consequence of the form of "Brouwer's principle" used in TROELSTRA [27]:

From this form follows

$$(*) : \text{ if } \{A_n \mid n \in \mathbb{N}\} \text{ is any cover of } \mathbb{R} \\ \{(A_n)^\circ \mid n \in \mathbb{N}\} \text{ is also a cover.}$$

Now suppose that A is a *strongly extensional* subset of \mathbb{R} , and let $x \in A$; then $(A \cup \neg\{x\}) = \mathbb{R}$ and (*) implies $(A^\circ \cup \neg\{x\}) = \mathbb{R}$, whence $x \in A^\circ$ and A is *open*. Thus (*) implies that \mathbb{R} has the *strongly discrete topology*, defined in VI.1.5.

It is not hard to construct (*classically*) a counter-example to this property over any space; on the other hand this property holds over any separated space if it is assumed *externally*. The problem of finding a CHA over which it is provable is still open. (Similar statements hold for Baire and Cantor spaces.)

VIII.3 BAIRE AND CANTOR SPACES

1. *Baire space*

We constructed Baire space \mathbb{B} in IV.2.2 as the space of models of a geometric theory; FOURMAN [5] shows that the "completeness theorem" for this theory is equivalent to the following principle of *Bar Induction (BI)*:
If $X \subseteq \text{Seq}(\mathbb{N})$ satisfies

$$\forall \alpha \in \mathbb{B} \exists n. (\alpha(0), \dots, \alpha(n-1)) \in X,$$

$$\forall u \in X \forall v \in \text{Seq}(\mathbb{N}) [u \subseteq v \rightarrow v \in X],$$

and $\forall u \in \text{Seq}(\mathbb{N}) [\forall n. u^*(n) \in X \rightarrow u \in X],$

then $(\) \in X$ (or equivalently: $X = \text{Seq}(\mathbb{N})$).

This is easily seen to be equivalent to the usual "Bar Induction with monotonicity" using two subsets of $\text{Seq}(\mathbb{N})$.

We quote some results from FOURMAN and HYLAND [6] for which we were partly responsible:

If we assume *BI* externally then

(a) *BI* holds over any locally compact space

(b) *BI* holds over \mathbb{B}

(c) *BI* fails over \mathbb{C} and over $\text{Seq}(\mathbb{N})$ with a suitable topology.

Among topological consequences of *BI* we note that (assuming *CAC*) each basic open $V(a)$ of \mathbb{B} is *normal*₂ and *zero-dimensional*. We remark also that \mathbb{B} is a filter-complete metric space in the usual way, and that each basic open is *clopen* in the sense of VII.2.3.

2. *Cantor space*

Here the completeness theorem is equivalent to a form of the *Fan Theorem (FT)*:

If $X \subseteq \text{Seq}(\{0,1\})$ and $\forall \alpha \in \mathbb{C} \exists n. (\alpha(0), \dots, \alpha(n-1)) \in X$
then $\exists k \forall \alpha \in \mathbb{C} \exists n \leq k. (\alpha(0), \dots, \alpha(n-1)) \in X.$

This is easily seen to be equivalent to the *compactness* of \mathbb{C} . Thus *FT* holds over any space when it is assumed externally; it fails over $K(I)^*$ (VIII.2.7) since *it implies the compactness of I*:

We embed \mathbb{C} in I in the usual way, taking $\alpha \in \mathbb{C}$ to

$$\sum_{n \in \mathbb{N}} (\alpha(n) \cdot 2^{-(n+1)});$$

by \mathcal{F} this image is compact, so that (as for \mathbb{R}^C) \mathbb{I} is its soberification and thus compact also.

\mathbb{C} is of course also a totally bounded filter-complete metric space.

PART THREE

SET THEORY

INTRODUCTION

The title of this part refers to the fact that we look here at two topics which form the backbone of classical set theory, namely *well-ordering* and *cardinality*.

Well-founded relations as defined in Chapter IX are those satisfying the principle of *induction*, and *morphisms* between them are *rank-preserving* functions. *Well-orderings* are well-founded, transitive and *extensional* (elements with the same predecessors are equal) and they serve as unique representatives of the ranks of well-founded relations. They are *simply* ordered just in case the order is *decidable*, and one may construct over suitable cHa 's well-orderings with *incomparable* elements. We define successors, suprema, sums, products and exponents of well-orderings; without *LEM* *Cantor normal forms* may not exist. We define a notion of *regular* well-ordering and construct some as the *Hartogs' numbers* of infinite sets; the use of *equalities* and *operations* obviates the need for choice principles here.

In Chapter X we consider notions of *finiteness*, building on work of TROELSTRA [29] and MINIO [16], with special reference to their *closure properties*. One group of notions makes a set "finite" iff it is *enumerated* in some way by natural numbers; a second group places a *bound* on the number of its elements. We use as an example here the "finiteness" of the set $P(\{0\})$ of *propositions*, over various cHa 's. For subsets of sets with *apartness* one can make the obvious modifications to these notions; we define also a dual notion of *coboundedness* in order to express fully the classical theorem that real polynomials have "finitely many" roots.

The only cardinality *relation* between sets that we consider is one of *productivity* (taken essentially from GREENLEAF [10]) which is used to express classical theorems on "higher cardinality" such as Cantor's theorem. The general area of cardinality relations is still very murky, and nothing like the classical theory seems likely to emerge; we found the wealth of possibilities even at the lowest level, of "finiteness", both distracting and discouraging.

CHAPTER IX

ORDINALSIX.1 WELL-FOUNDED RELATIONS

Since *proofs by induction* and *definitions by recursion* are really what one wants from a notion of "well-founded" relation, the natural choice of definition is that the relation be "inductive"; other classically equivalent definitions are either too strong or too weak. Our notion also includes as a component a (non-standard) *equality*, which is essential for the results of IX.3.5.

1. *Well-foundedness*

Definition: A *well-founded (Wf) relation* A consists of a set A , an equality \approx_A and a binary relation $<_A$ such that

$$\forall X \subseteq A [\forall a \in A (\forall b <_A a . b \in X \rightarrow a \in X) \rightarrow X = A].$$

An element a of A is *minimal in* $X \subseteq A$ iff

$$(a \in X \wedge \neg \exists b \in X . b <_A a).$$

Remarks: The axiom (N) says precisely that the relation $(S, =, <)$ is well-founded on N .

One may prove (by induction on the first element) that a well-founded relation A can have no *infinite descending sequences*; in particular $<_A$ is *irreflexive* and *asymmetric*. On the other hand it is obvious that this condition is too weak to imply the induction principle.

The *least element principle* (that every inhabited subset has an element minimal in it) is far too strong a condition to be useful. For if $(A, <_A)$ is any non-trivial relation with $a <_A b$ say, put $X = \{(a \uparrow \varphi), b\}$ for any proposition φ , and $(\varphi \vee \neg \varphi)$ follows if either a or b is minimal in X .

2. *Transitive closures*

Definition: Let A be Wf. For $a \in A$ the set of *predecessors* P_a of a is $\{b \in A \mid b <_A a\}$; an extensional subset X of A is *transitive* (or an *initial segment*) $(Tr(X))$ iff $\forall a \in X . P_a \subseteq X$. If S is an initial segment we denote by S the Wf relation $(S, \approx, <)$ with relations inherited from A .

For $a \in A$ the *proper initial segment* S_a below a is the set $\bigcap \{X \subseteq A \mid \text{Tr}(X) \wedge P_a \subseteq X\}$. The *transitive closure* of $<_A$ is the relation $<_A^*$ defined by

$$a <_A^* b \text{ iff } a \in S_b.$$

We denote by A^* the Wf relation $(A, \approx_A, <_A^*)$.

Remarks: $<^*$ is transitive of course as a relation, and one may prove by induction that for $a \in A$

$$S_a = (P_a \cup \bigcup \{S_b \mid b <_A a\}).$$

3. *Recursion*

The usual inductive proof justifies the definition of functions (or operations) by *recursion* in a Wf relation; similarly for *simultaneous induction* and recursion.

4. *Morphisms - rank*

If A is Wf one may think intuitively of the "rank" of an element of A as being determined inductively in terms of the "ranks" of its predecessors. Our notion of a *morphism* between Wf relations is then of a "rank-preserving" function.

Definition: A *morphism* between Wf relations A, B is a function $f: A \rightarrow B$ such that

$$\forall a \in A. S_{f(a)} = f(S_a).$$

We write then $f: A \rightarrow B$, and also $f: A \rightarrowtail B$ (or $f: A \twoheadrightarrow B$) if f is *one-one* (or *onto*). If $(\exists f: A \twoheadrightarrow B)$ we write $A \approx B$; note that $A \approx B$ iff A^*, B^* are *isomorphic* as spo's (with equality), and in particular $A \approx A^*$ always holds.

IX.2 WELL-ORDERINGS

The definition we give of a *well-ordering* looks rather different from the usual one, although it is classically equivalent. The idea is to provide unique representative of the "ranks" of Wf relations; previous intuitionistic notions have always been too strong to provide a sufficiently large class for this purpose. Our notion corresponds to the notion of an *ordinal* in ZF set theory (a transitive set of transitive sets) that we considered in GRAYSON [9] under the influence of POWELL [20].

1. *Well-orderings*

Definition: A *well-ordering* (Wo) is a Wf relation A such that $<_A$ is *transitive* and *extensional*, that is

$$\forall a, b \in A [\forall c \in A (c <_A a \leftrightarrow c <_A b) \rightarrow a \approx_A b].$$

Remarks: The *extensionality* condition can be seen as making elements with the same "rank" *equal*; this will be the key to the *uniqueness* properties proved below. It is sometimes convenient to treat $<_A$ as "primitive" and \approx_A as defined from it by the extensionality condition; we may also define an *associated ipo* \leq_A by

$$a \leq_A b \text{ iff } \forall c <_A a. c <_A b$$

of which the associated equality is just \approx_A . The relation with the classical definition is described by the following

Proposition: If A is Wo, $<_A$ is simple in the sense that

$$\forall a, b \in A [a <_A b \vee a \approx_A b \vee b <_A a]$$

iff it is decidable; the simpleness of all Wo implies *LEM*.

Proof: If $<_A$ is simple it is clearly decidable. Conversely if $<_A$ is decidable we can prove by a *double induction* in A using the condition of *extensionality* that

$$\forall a, b \in A [(\neg a <_A b \wedge \neg b <_A a) \rightarrow a \approx_A b]$$

whence the result follows.

For a given proposition φ let $A = \{0, 1\}$ with $0 <_A 1$ iff φ , and $0 \approx_A 1$ iff $\neg\varphi$; then A is Wo but $<_A$ is simple iff $(\varphi \vee \neg\varphi)$. \square

Examples: Giving \mathbb{N} its usual order and equality, we obtain a W_0 which we denote by ω ; this is in fact the *transitive closure* of the W_f relation ($Sm=n$). The *proper initial segments* of ω are just the W_0 's $([n], =, <)$ for $n \in \mathbb{N}$, which we denote by \underline{n} .

Any initial segment of a W_0 is again a W_0 obviously. In the case of ω all its initial segments are *simple*, but in IX.2.7 we will construct in a suitable sheaf model an initial segment of ω which is *incomparable* with it, being neither ω itself nor \underline{n} for any $n \in \mathbb{N}$.

2. Morphisms in well-orderings

In order to describe the relation between W_f and W_0 we need a

Lemma: Let A be W_0 and B W_f ; then

(i) there is at most one morphism from B into A .

(ii) all morphisms from A into B are one-one.

Proof: (i) If $f, g: B \rightarrow A$ we prove by induction in B that $\forall b \in B. f(b) \approx_A g(b)$:

Suppose that $\forall x <_B b. f(x) \approx_A g(x)$, and let $y <_A f(b)$.

As f is a morphism there is an $x <_B b$ such that $f(x) \approx_A y$.

Then as g is a morphism, $g(x) <_A g(b)$, while $g(x) \approx_A y$.

Thus $f(b) \leq_A g(b)$ and conversely, so $f(b) \approx_A g(b)$.

(ii) If $f: A \rightarrow B$ we prove by induction in A that

$$\forall a, a' \in A (f(a) \approx_B f(a') \rightarrow a \approx_A a')$$

by extensionality in A as usual. □

As *corollaries* we may deduce that morphisms between W_0 's are unique and one-one, and in particular that morphisms *onto* are *isomorphisms*.

3. The rank theorem

We now define a construction on W_f relations and morphisms which may be called a "functor" from W_f into W_0 , in the language of category theory.

Definition: Let A be W_f and define by simultaneous recursion on a, b in A^* the relation \approx'_A on A by

$$a \approx'_A b \text{ iff } [(\forall x <^*_A a \exists y <^*_A b. x \approx y) \wedge (\forall y <^*_A b \exists x <^*_A a. x \approx y)]$$

Next define $a <'_A b$ iff $(\exists x <^*_A b. a \approx'_A x)$, and let $\rho(A)$ be the structure $(A, \approx'_A, <'_A)$.

If B is also Wf and $f: A \rightarrow B$ we take $\rho(f)$ to be

$$\{\langle a, b \rangle \in (A \times B) \mid \exists \langle x, y \rangle \in f. a \approx'_A x \wedge b \approx'_A y\}.$$

Finally ρ_A is the *identity* function on A with respect to \approx'_A (i.e. ρ_A is the same as \approx'_A as a set).

Remarks: The relation \approx'_A captures our intuition about "sameness of rank" among the elements of A , and \langle'_A then orders these "ranks". It is clear that if A is Wo, $\rho(A)$ is just the same as A .

Theorem: For any Wf A, B and $f: A \rightarrow B$ we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho_A} & \rho(A) \\ f \downarrow & & \downarrow \rho(f) \\ B & \xrightarrow{\rho_B} & \rho(B) \end{array}$$

with $\rho(A), \rho(B)$ both Wo.

Proof: It is easy to see by manipulating the definitions that \approx'_A is an equality extending \approx_A , and \langle'_A is a transitive relation extending \langle_A and extensional with respect to \approx'_A . For *induction* let $X \subseteq A$ and

$$\forall a \in A [(\forall b \langle'_A a. b \in X) \rightarrow a \in X]$$

and prove by induction on a in A^* that

$$\forall a \in A \forall b \in A (b \approx'_A a \rightarrow b \in X).$$

Thus $\rho(A)$ is Wo and, as \approx'_A extends \approx_A , ρ_A is a *function* for these equalities; to show $\rho_A: A \twoheadrightarrow \rho(A)$ observe that by definition of \langle'_A for $a \in A$

$$\{b \mid b \langle'_A a\} = \{b \mid \exists x \langle^*_A a. x \approx'_A b\} = \rho(S_a),$$

and ρ_A is obviously onto.

Finally we must show that $\rho(f): \rho(A) \rightarrow \rho(B)$ since commutativity is obvious (this illustrates the general technique of proof here).

By induction in A^* we prove

$$\forall a, b \in A [a \approx'_A b \rightarrow f(a) \approx'_B f(b)]:$$

Let $a \approx'_A b$ and $x \langle^*_B f(a)$, so as f is a morphism $x \approx_B f(y)$ for some $y \langle^*_A a$; then as $a \approx'_A b$; $y \approx'_A z \langle^*_A b$ for some z .

Then $f(y) \approx'_B f(z)$ by induction hypothesis,
 while $f(z) <^*_B f(b)$ as f is a morphism.

So $x \approx_B f(y) \approx'_B f(z) <^*_B f(b)$ whence $x <'_B f(b)$ as required.

Last of all, for the *morphism* condition on $\rho(f)$:

If $a <'_A b$ let $a \approx'_A x <^*_A b$,

so that $f(a) \approx'_B f(x) <^*_B f(b)$ whence $f(a) <'_B f(b)$;

conversely if $y <'_B f(b)$ let $y \approx'_B z <^*_B f(b)$,

so that $z \approx_B f(a)$ for some $a <^*_A b$, whence $y \approx'_B f(a)$. \square

Remarks: By the lemma of IX.2.2 ρ_A, ρ_B and $\rho(f)$ are the unique possible morphisms into $\rho(A), \rho(B)$, and $\rho(f)$ is *one-one*; furthermore if f is *onto* so is $\rho(f)$, whence $\rho(A) \approx \rho(B)$. Thus we have constructed for each Wf relation a W_o which serves to represent its "rank" and is unique up to (unique) isomorphism.

4. *Order on W_o*

We will be considering only W_o 's from now on, and we use the variables $\alpha, \beta, \gamma, \dots$ to stand for W_o 's A, B, C, \dots , so as to make them look more like the usual "ordinals" of set theory.

Definition: For $\alpha \in W_o$ and $a \in A$ (the domain of α) we denote by α_a the W_o $(S_a, \approx_A, <_A)$. Then for $\alpha, \beta \in W_o$ we define

$$\alpha < \beta \text{ iff } \exists b \in B. \alpha \approx \beta_b$$

$$\alpha \leq \beta \text{ iff } \forall a \in A. \alpha_a < \beta$$

A set X of W_o 's is *transitive* iff $\forall \alpha \in X \forall a \in A. \alpha_a \in X$.

Remarks: The order $<$ on W_o corresponds to the ordering of "ranks". We note the following

(i) $(\alpha < \beta \leq \gamma \rightarrow \alpha < \gamma)$, but not $(\alpha \leq \beta < \gamma \rightarrow \alpha < \gamma)$
 as we will see in IX.2.7.

(ii) $(\alpha \leq \beta \leftrightarrow \exists f: \alpha \rightarrow \beta)$ by IX.2.2, whence

(iii) $(\alpha \leq \beta \leq \alpha \rightarrow \alpha \approx \beta)$.

A *transitive* set of W_o 's acts as an "initial segment" in W_o ; the sense in which W_o is a "well-ordered class" is then expressed by the following

Proposition: If X is a transitive set of W_o 's, $(X, \approx, <)$ is W_o .

Proof: As \leq is the associated ipo to $<$ on X , (iii) above establishes extensionality. For induction let $Y \subseteq X$ and

$$\forall \alpha \in X [\forall \beta \in X (\beta < \alpha \rightarrow \beta \in Y) \rightarrow \alpha \in Y].$$

and prove by induction in α for $\alpha \in X$ that

$$\forall a \in A. \alpha_a \in Y. \quad \square$$

Note: If we denote this Wo by ξ it is clear that $\forall \alpha \in X. \alpha \approx \xi_\alpha$. In fact it is obvious in general that, for $\alpha \in Wo$ and $a, b \in A$, $\alpha_a < \alpha_b$ iff $a <_A b$, $\alpha_a \leq \alpha_b$ iff $a \leq_A b$, and $\alpha_a \approx \alpha_b$ iff $a \approx_A b$.

5. *Successors, suprema, limits*

Definition: If α is Wo we form its *successor* α^+ by adjoining a point to its "top" as usual.

If X is a set of Wo we form its *supremum* $Sup(X)$ as the Wo $(X', \approx, <)$ where $X' = \{\alpha_a \mid \alpha \in X, a \in A\}$.

Remarks: For successors we have for $\alpha, \beta \in Wo$

$$(i) \quad \beta < \alpha^+ \leftrightarrow (\beta \approx \alpha \vee \beta < \alpha), \text{ whence}$$

$$(ii) \quad \alpha^+ \leq \beta \leftrightarrow \alpha < \beta.$$

As will be seen in IX.2.7, $(\alpha < \beta \rightarrow \alpha^+ \approx \beta \vee \alpha^+ < \beta)$ is not valid.

For suprema we have for X a set of Wo and $\beta \in Wo$

$$(i) \quad \beta < Sup(X) \leftrightarrow \exists \alpha \in X. \beta < \alpha, \text{ whence}$$

$$(ii) \quad Sup(X) \leq \beta \leftrightarrow \forall \alpha \in X. \alpha \leq \beta.$$

We note further that each $Wo \alpha$ is obtained by forming successors and suprema of smaller ones since

$$\alpha \approx Sup\{(\alpha_a)^+ \mid a \in A\}.$$

As will be seen in IX.2.7 this closure under *suprema* fails for other notions.

Definition: A $Wo \alpha$ is a *limit* iff $\forall a \in A. \alpha_a^+ < \alpha$, and a *weak limit* iff $\forall a \in A \exists b \in A. a <_A b$.

Remarks: A limit is always a weak limit, but the converse implies LEM: for a given proposition φ let $A = \mathbb{N}$ with the usual order and equality except that $0 <_A 1$ iff φ , and $0 \approx_A 1$ iff $\neg \varphi$; then α is Wo and a weak limit, but $\alpha_0^+ < \alpha$ iff $(\varphi \vee \neg \varphi)$.

6. *Initial segments*

Definition: If α is Wo its *full successor* α^S is the Wo $(X, \approx, <)$ where X is the set of initial segments $(S, \approx_A, <_A)$ of α .

Remarks: By the results of IX.2.7 α^S is much "fuller" than α^+ , since for $\beta \in \text{Wo}$

$$\beta < \alpha^S \leftrightarrow \beta \leq \alpha,$$

while $\beta \leq \alpha \rightarrow (\beta \approx \alpha \vee \beta < \alpha)$ is not valid.

Notice that when α is \perp its initial segments are just subsets of $\{0\}$, and this gives a well-ordering of $P(\{0\})$ in which $\varphi < \psi$ iff $(\psi \wedge \neg \varphi)$, and equality is standard.

7. *Incomparable well-orderings*

One may easily construct examples to show that various classical assertions about the order on Wo imply *LEM*; e.g. for $(0 \approx \alpha \vee 0 < \alpha)$ we can take $A = \{0 \uparrow \varphi\}$ for a given proposition φ .

We establish here some stronger results which show how these assertions can actually be *falsified* in suitable sheaf models:

Let Ω be a cHa (e.g. $\mathbf{O}(\mathbf{R})$) containing elements p_0, p_1, \dots such that $\bigwedge_{n \in \mathbf{N}} p_n = \perp$ while

$$\forall n \in \mathbf{N} [p_n \geq p_{n+1} \wedge (\neg p_n) = \perp].$$

We let S be the initial segment of $\hat{\omega}$ in $\text{Sh}(\Omega)$ generated by the elements $(\hat{n} \uparrow p_n)$ for $n \in \mathbf{N}$, and denote this internal Wo by σ .

Then $[\hat{n} < \sigma] = p_n$ so the conditions on p_0, p_1, \dots guarantee the following internal statements:

- (i) $\forall n. \neg \hat{n} < \sigma$ (since $\neg p_n = \perp$)
 (ii) $\neg(\omega \leq \sigma)$ (since $\bigwedge_{n \in \mathbf{N}} p_n = \perp$)

From these follow

- (iii) $\neg(\omega < \sigma \vee \omega \approx \sigma \vee \sigma < \omega)$
 (iv) $\sigma \leq \omega < \omega^+$ but $\neg \sigma < \omega^+$
 (v) $\neg(\omega^+ \leq \sigma^+ \vee \sigma^+ \leq \omega^+)$
 (vi) σ is neither a weak limit nor a successor.
 (vii) $\text{Sup}\{\sigma^+, \omega^+\}$ contains incomparable elements.

We note finally that all the Wo's involved here are *simple*, so that the situation cannot be retrieved by imposing further "linearity" conditions; in particular (vii) shows that stronger notions cannot be closed under forming *suprema*.

IX.3 ORDINAL ARITHMETIC1. *Sums and products*

Sums and products of Wo's are defined exactly as usual and have the expected properties, the only exception being the "recursion equation" for *sum* which has to take the form

$$(\alpha + \beta) \approx \text{Sup}(\{(\alpha + \beta_b)^+ \mid b \in B\} \cup \{\alpha\})$$

since we cannot assume that $(\underline{0} \approx \beta \vee \underline{0} < \beta)$. The equation for *product* is the usual one

$$(\alpha \times \beta) \approx \text{Sup}(\{(\alpha \times \beta_b) + \alpha \mid b \in B\}).$$

(Of course we also cannot take cases on whether β is a limit or a successor.)

We denote by $\sum_{i < n} \alpha_i$ the sum $(\alpha_0 + \dots + \alpha_{n-1})$.

2. *Exponents*

More modification is need in the definition of the *exponent* (α^β) of Wo's α and β , since the usual form uses the set of functions from B to A which are $\underline{0}$ (i.e. *minimal* in A) at all except a finite number of arguments (compare the discussion of the product of topological spaces in V.1.5). Our definition uses instead the set of finite partial functions from B to A which are always "positive", with essentially the *lexicographic* ordering:

Definition: If α, β are Wo their *exponent* (α^β) is the Wo $(X, \approx_X, <_X)$ where X is the set of finite sequences $(\langle a_0, b_0 \rangle, \dots, \langle a_{n-1}, b_{n-1} \rangle)$ from $(A \times B)$ such that $[b_{n-1} <_B \dots <_B b_0 \wedge \forall i < n \exists a <_A a_i]$; and denoting such a sequence by $(\underline{a}, \underline{b})$,

$$(\underline{a}, \underline{b}) <_X (\underline{a}', \underline{b}') \text{ iff for some } i \leq \min(n, n')$$

$$[\forall j < i (a_j \approx_A a'_j \wedge b_j \approx_B b'_j) \wedge ((i = n < n') \vee (b_i <_B b'_i) \vee (b_i \approx_B b'_i \wedge a_i <_A a'_i))];$$

$$\text{finally } (\underline{a}, \underline{b}) \approx_X (\underline{a}', \underline{b}') \text{ iff } [n = n' \wedge \forall i < n (a_i \approx_A a'_i \wedge b_i \approx_B b'_i)]$$

Remarks: With this definition we have a "recursion equation" like that for sum

$$(\alpha^\beta) \approx \text{Sup}(\{\alpha^{\beta_b} \times \alpha \mid b \in B\} \cup \{1\}).$$

Notice that $\alpha^\beta > \underline{0}$ always holds, since the *empty* sequence is always *minimal* in the exponent.

3. *Cantor normal form*

Definition: If α, γ are W_0 , γ is expressed in *Cantor normal form (cnf)* to base α iff there are W_0 's $\beta_0, \dots, \beta_{n-1}, \alpha_0, \dots, \alpha_{n-1}$ such that

$$(\beta_{n-1} < \dots < \beta_0 \wedge \forall i < n (\alpha_i < \alpha \wedge \exists x \in A_i))$$

and

$$\gamma = \sum_{i < n} (\alpha^{\beta_i} \times \alpha_i).$$

Remarks: Notice first that the definition of the order of the exponents (α^β) has precisely the effect that for $(\underline{a}, \underline{b})$ in the domain and

$$\alpha_i = \alpha(a_i), \beta_i = \beta(b_i), \sum_{i < n} (\alpha^{\beta_i} \times \alpha_i) \approx (\alpha^\beta)_{(\underline{a}, \underline{b})}$$

and the left-hand side be in cnf.

I am grateful to *Dana Scott* for pointing out the next fact to me, that the existence of cnf's for all W_0 to base $\underline{0}$ or ω implies *LEM*: The argument applies in fact to any base α such that

$$[\alpha > \underline{0} \wedge \forall a \in A ((\exists b <_A a) \rightarrow \alpha_a > \underline{0})].$$

Let α be such a base and $\sum_{i < n} (\alpha^{\beta_i} \times \alpha_i) \approx \gamma$

be in cnf. Then

either $n = 0$ so $\gamma \approx \underline{0}$
or $n > 0$ so $\gamma \geq (\alpha^{\beta_0} \times \alpha_0)$.

Now $\alpha_0 < \alpha$ and $(\exists x \in A_0)$, so $\alpha_0 > \underline{0}$ by the condition on α ; but then $\gamma \geq (\alpha^{\beta_0} \times \alpha_0) \geq \alpha^{\beta_0} > \underline{0}$. Thus we have deduced $(\gamma \approx \underline{0} \vee \gamma > \underline{0})$ which implies *LEM* if assumed for all W_0 .

Observe finally that our first remarks now imply that for such a base α we cannot find for each γ a β such that $\gamma < \alpha^\beta$.

4. *Regular well-orderings*

Our definition of a *regular* W_0 is essentially the classical one except we now at last reap the benefits of having carried *equalities* along with us in the possibility of distinguishing *operations* from *functions*; this is necessary for the results on Hartogs' numbers in the following subsection.

Definition: A W_0 α is (*p*-) *regular* iff it has an operation Sup such that, for $a \in A$ and $f: Sa \xrightarrow{\text{ox}} A$ a (*partial*) *extensional operation*, $\text{Sup}(f) \in A$ and $\alpha_{\text{Sup}(f)} \approx \text{Sup}\{\alpha_{f(b)} \mid b \in \text{dom}(f)\}$.

α is a *strong limit* iff it has an operation $\text{Suc} : A \xrightarrow{0} A$ such that $\forall a \in A. \alpha_{\text{Suc}(a)} \approx (\alpha_a)^+$.

Remarks: Thus in a strong limit regular $\text{Wo } \alpha$ we can "choose representatives" for successors and suprema; in particular we can define *operations* of sum, product and exponent "inside" α according to the recursion equations of IX.3.1 and 2.

ω is a strong limit regular Wo , but is not necessarily p-regular (see X.2.3): we construct "arbitrarily large" p-regular Wo 's below.

5. Hartogs' numbers

POWELL [20] has considered various kinds of Hartogs' numbers as *ordinals* in ZF set theory. By using *equalities* and not taking quotients we can have *operations* to do our "choosing" for us, and so avoid the use of *choice principles* in proving regularity.

Definition: The *Hartogs' number* $H(X)$ of a set X is the Wo $(H(X), \approx, <)$ where $H(X)$ is the set of Wo 's $(A, \approx_A, <_A)$ with $A \subseteq X$, and $\approx, <$ are as in Wo . If X has an equality \approx_X we require each \approx_A to extend \approx_X and each A to be *extensional*.

Remarks: Note that by allowing the equalities on elements of $H(X)$ to "expand" we have ensured that

$$\beta < H(X) \text{ iff } (X \xrightarrow[p]{\text{p}} \beta), \text{ for any } \beta.$$

For our main theorem we need the notion from X.2.2 of a *P-infinite* set X , that is, having *at least two elements* and a *pairing operation* $[\lambda x, y \in X. (x, y)]$ into X .

Theorem:

(i) If X is P-infinite, $H(X)$ is a p-regular strong limit.

(ii) If β is a p-regular strong limit and $\alpha < \beta$, then

$$H(A) \leq \beta \text{ (and } \alpha < H(A) \text{ of course).}$$

Proof: (i) To define Suc we have just to "adjoin a point" to any $A \subseteq X$: fix $x_0, x_1 \in X$ such that $x_0 \neq x_1$ and put

$$A^+ = \{(x_0, a) \mid a \in A\} \cup \{(x_0, x_1)\}.$$

Next let $\alpha \in H(X)$ and $f : S_\alpha \xrightarrow[p]{0x} H(X)$, and put $\text{Sup}(f) = \beta$ where

$$B = \{(a, y) \mid a \in A \wedge y \in f(\alpha_a)\}$$

and $(a, y) <_B (a', y')$ iff $(f(\alpha_a))_y < (f(\alpha_{a'}))_{y'}$ and \approx_B similarly.

(ii) Without loss suppose that $\alpha = \beta_b$ for some $b \in B$ and define by recursion in $H(A)$ an extensional operation $\Phi: H(A) \xrightarrow{OX} B$ such that $\forall \gamma \in H(A). \gamma \approx \beta_{\Phi(\gamma)}$, as follows (Sup, Suc being the operations in β):

$$\Phi(\gamma) = \text{Sup}(\lambda a \in C . \text{Suc}(\Phi(\gamma_a)))$$

for $\gamma = (C, \approx_C, <_C) \in H(A)$; note that the partial operation on A used here is *extensional* since \approx_C extends \approx_A . \square

For constructions of P-infinite sets we refer to X.2.2; in particular then for every $Wo \alpha$ we can find larger p-regular strong limit, and if A is P-infinite, $H(A)$ is the *least* such larger than α . This case includes ω of course, and we denote $H(N)$ by ω_1 ; a sense in which ω_1 is of "higher cardinality" than ω will be given in X.2.3. Note that ω_1 is *very far* from being *simple*, since it includes for example all the initial segments of ω (see IX.2.7).

6. Topology of Wo

We conclude this chapter with a few remarks about the *order topology* of Wo 's :

- (a) The relation \approx_1 of VI.1.1 on a $Wo \alpha$ coincides with \approx_A .
- (b) If α is *simple*, α^+ has *compact* topology by the usual proof; this seems unlikely for ω_1^+ , but we do not have a counter-example yet.
- (c) One may define *normal* functions on regular strong limits and prove them continuous.

CHAPTER X

CARDINALSX.1 FINITENESS

We consider here just a few of the limitless possibilities for notions of *finiteness* that are available in intuitionistic mathematics. Previous investigations include TROELSTRA [29] and MINIO [16], to whom we refer for many results. We restrict our attention to some of the simpler ones, concentrating on *closure properties* and the "finiteness" of the set $P(\{0\})$ of *propositions*. The only *worthwhile* notion to emerge in the general context seems to be that of being "finitely indexed" (Q); we look at some others suitable in the presence of an *apartness* in X.3.

1. *Cardinality notions - notions of finiteness*

By a *cardinality* notion we mean a condition φ on *sets* invariant under *one-one correspondence*, that is

$$\varphi(X) \wedge (X \xrightarrow{\sim} Y) \rightarrow \varphi(Y).$$

(For the various notations for functions here we refer to I.2.4).

We propose then the following *minimal* conditions for a cardinality notion to be a notion of *finiteness*

- (i) $\forall n \in \mathbb{N}. \varphi([n])$
- (ii) $\varphi(X) \rightarrow \neg \forall n \in \mathbb{N} ([n] \xrightarrow{\sim} X)$,

that is, it should include all sets of predecessors of natural numbers but no "arbitrarily large" sets.

In future we introduce "class names" for cardinality notions, writing " $X \in F$ " for " $\varphi(X)$ " for example.

2. *Closure properties*

We now give a list of possible *closure properties* for a notion of finiteness F ; as we shall see (X.1.6) these are not all mutually compatible. We will take for granted the obvious implications between them (for example $(\text{Sing}) \wedge (\text{Sub}) \rightarrow (\text{Sing S})$).

$$(\text{Sub}) \quad [X \subseteq Y \in F \rightarrow X \in F]$$

$$(\text{Quot}) \quad [Y \in F \wedge (Y \twoheadrightarrow X) \rightarrow X \in F]$$

$$(\text{Sing}) \quad [Ex \rightarrow \{x\} \in F]$$

$$(\text{Sing S}) \quad [\{x\} \in F]$$

$$(\cup) \quad [(X \in F \wedge Y \in F \wedge (X \cap Y) = \emptyset) \rightarrow (X \cup Y) \in F]$$

$$(\cup Q) \quad [X \in F \wedge Y \in F \rightarrow (X \cup Y) \in F]$$

$$(U) \quad [(X \in F \wedge \forall A \in X (A \in F) \wedge \forall A, B \in X (A = B \vee (A \cap B) = \emptyset)) \rightarrow UX \in F]$$

$$(UQ) \quad [(X \in F \wedge \forall A \in X. A \in F) \rightarrow UX \in F]$$

$$(\text{Prod}) \quad [X \in F \wedge Y \in F \rightarrow (X \times Y) \in F]$$

$$(\text{Power}) \quad [X \in F \rightarrow (P(X) \cap F) \in F]$$

$$(\neg \neg) \quad [(X \in F \wedge \forall y \in Y \neg \neg y \in X) \rightarrow Y \in F]$$

$$(\sim) \quad [X \in F \rightarrow \tilde{X} \in F]$$

Remarks: We always think here of sets as *existing*. We use the tags "S" and "Q" as a mnemonic to indicate some relation to *subsets* and *quotients*.

We write " $F \models (P)$ " to denote that F *satisfies* the property (P). We will be interested below in notions that are the *least* ones satisfying certain properties, that is, they are included in any other notions with the properties.

3 *Enumeration notions*

We consider first four notions according to which a set is "finite" iff it is *enumerated* in some way by the predecessors of some natural number. MINIO [16] (following TROELSTRA [29]) has studied other similar notions and shown them to reduce to these four.

Definition: We define

$$X \in N \quad \text{iff} \quad \exists n \ ([n] \twoheadrightarrow X)$$

$$X \in Q \quad \text{iff} \quad \exists n \ ([n] \dashrightarrow X)$$

$$X \in S \quad \text{iff} \quad \exists n \ ([n] \twoheadrightarrow_p X)$$

$$X \in SQ \quad \text{iff} \quad \exists n \ ([n] \dashrightarrow_p X)$$

Relations between these:

The implications
$$\begin{array}{ccc} N & \longrightarrow & Q \\ \downarrow & & \downarrow \\ S & \longrightarrow & SQ \end{array}$$
 hold,

but no others without *LEM*, since in fact

- (i) if $X \in SQ$, $X \in S$ iff X is *discrete*
- (ii) if $X \in Q$, $X \in N$ iff X is *discrete*
- (iii) if $X \in SQ$, $X \in Q$ iff $\exists n \exists f: [n] \xrightarrow{p} X$, $\text{dom } f \in P'([n])$
- (iv) if $X \in S$, $X \in N$ iff $\exists n \exists f: [n] \xrightarrow{p} X$, $\text{dom } f \in P'([n])$

Closure properties:

We list next the closure properties from the list above which are satisfied by these notions; the proofs are all easy using the arithmetic of \mathbf{N} . That no stronger ones hold follows mostly from the facts about these being the *least* notions satisfying various properties; the other counter-examples are given in X.1.6 and 7. The proofs of the "least notion" properties just involve simple inductions in \mathbf{N} .

- (i) N is the least notion of finiteness
- (ii) $S, SQ \models (\text{Sub})$, and S is the least such notion of finiteness
- (iii) $Q, SQ \models (\text{Quot})$, and Q is the least such notion of finiteness
- (iv) $N \models [(\text{Sing}) \wedge (\cup)]$, and N is the least such cardinality notion
- (v) $S \models [(\text{Sing}) \wedge (\cup)]$, and S is the least such cardinality notion
- (vi) $Q \models [(\text{Sing}) \wedge (\cup Q)]$, and Q is the least such cardinality notion
- (vii) $SQ \models [(\text{Sing } S) \wedge (\cup Q)]$, and SQ is the least such cardinality notion
- (viii) $N \models (U)$, $Q \models (UQ)$
- (ix) $N, S, Q, SQ \models (\text{Prod})$
- (x) $N, Q \models (\text{Power})$

Remarks: On the basis of its closure properties (especially (UQ) , (Power)) Q is the most useful of these notions when we are not concerned that the enumeration should give *distinct* elements; one might describe a Q -finite set as "finitely indexed". This was the natural notion for *topology* in Part Two, and it is the right notion often for *algebra* too — for example, for *finitely-generated* substructures or for "finite" elements

of algebraic lattices generally. When distinctness is wanted there will usually be an *apartness* present: we consider this situation in X.3.

Property (vi) above identifies Q with the *Kuratowski* finiteness of KOCK, LECOUTURIER and MIKKELSEN [15], also studied in the topos setting by Acuña-Ortega, Johnstone and Linton. When the type \mathbb{N} is not included in the formal system (as for general topoi) one can take (iv) - (vii) as *defining* these notions; it is still far easier and more natural to prove closure properties etc. for these definitions in the "internal logic" of a topos than to reason "externally" about objects and morphisms.

4. *Boundedness*

The next two notions place a *bound* on enumerations of elements of a "finite" set. The weaker one is Brouwer's "bounded in number" (HEYTING [11]), and the stronger is a more "positive" version of this; one important difference between them will be in their application to $P(\{0\})$ in X.1.5. We consider a "dual" notion for sets with apartness in X.3.2. These two notions do not appear to be the least ones for any natural closure property, nor do they seem to be of much use in mathematics.

Definition: We define for $n \in \mathbb{N}$

$$B_n(X) \text{ iff } \forall f : [n+1] \rightarrow X \exists i < j \leq n. f(i) = f(j)$$

$$B'_n(X) \text{ iff } \neg ([n+1] \twoheadrightarrow X)$$

and then: $X \in B$ iff $\exists n. B_n(X)$, $X \in B'$ iff $\exists n. B'_n(X)$.

Some properties:

- (i) $SQ \rightarrow B \rightarrow B'$, but not conversely by X.1.5.
- (ii) $(X \in B' \wedge X \text{ discrete}) \rightarrow X \in B$.
- (iii) $B, B' \models (\text{Sub}), (\text{Quot}), (\text{Sing } S), (\cup Q)$, and (Prod) but not (U) by X.1.7.
- (iv) $B' \models (\neg \neg)$ whence $B' \models (\sim)$ and $\forall n. P([n]) \in B'$.

The proofs are all fairly easy except perhaps for $B \models (\text{Prod})$: if $B_m(X)$ and $B_n(Y)$ one proves $B_{m,n}(X \times Y)$ by "sorting" the values of any function $f : [m.n+1] \rightarrow (X \times Y)$ according to their first components.

For (iv) note that for $A \subseteq [n]$, $\neg \neg (A \in P'([n]))$.

5. *Finiteness of* $P(\{0\})$

From the preceding we see that $P(\{0\}) \in B'$, in fact $B'_2(P(\{0\}))$ so that (even for intuitionists) any "third" possibility (other than the *true* and *false* propositions) really is "excluded" after all!

The assertion $B_n(P(\{0\}))$ is equivalent to the logical schema

$$\left(\bigvee_{i < j \leq n} \varphi_i \leftrightarrow \varphi_j \right)$$

which is well-known not to be provable intuitionistically for any n ; thus $(B' \rightarrow B)$ fails. In sheaves over a cHa (global) propositions appear as elements of the cHa (III.2.4), and it is easy to see that if $B_n(P(\{0\}))$ holds over any space it does so externally. Thus for the following construction we have to assume LEM:

We give \mathbf{R} the topology consisting of the empty set and all complements of finite subsets, and denote it by Ω .

In $\text{Sh}(\Omega)$, $B_3(P(\{0\}))$ since at any point t of \mathbf{R} an element of Ω must look *either* empty, *or* like an interval, *or* like an interval with t missing.

But also in $\text{Sh}(\Omega)$, $P(\{0\})$ is not even *countable* in the sense that $\neg (N \xrightarrow{p} P(\{0\}))$; this is because the intersection of any sequence $\{U_n : n \in \mathbf{N}\}$ of non-empty elements of Ω is *dense* in \mathbf{R} (by the *Baire category theorem*), while if

$$t \in \bigcap_{n \in \mathbf{N}} U_n \text{ and } V = (\mathbf{R} \setminus \{t\}), \quad t \notin \bigcup_{n \in \mathbf{N}} (U \leftrightarrow V).$$

Thus $\text{Sh}(\Omega)$ provides a counter-example to $(B \rightarrow \text{SQ})$.

6. *The size of* $P(P(\{0\}))$

We show here that in $\text{Sh}(\mathbf{Q})$, $P(P(\{0\}))$ is *infinite* in the sense that $[N \xrightarrow{\quad} P(P(\{0\}))]$, so that we cannot include it in general in any notion of *finiteness*; thus also no notion of finiteness can be closed under the full power-set operation, nor therefore satisfy both (Sub) and (Power).

For the proof let $\left\{ \{q_{k,n} : k \in \mathbf{N}\} : n \in \mathbf{N} \right\}$ be a *disjoint* sequence of sequences from \mathbf{Q} each of which is *dense*. Then put $U_{k,n} = (\mathbf{Q} \setminus \{q_{k,n}\})$ and let a_n be the subsheaf of $\underline{\Omega}$ generated by the (global) elements $\{U_{k,n} \mid k \in \mathbf{N}\}$. By *disjointness*, for $m \neq n$, $(U_{j,m} \leftrightarrow U_{k,n}) = (U_{j,m} \cap U_{k,n})$ for all j, k ;

and then by *denseness*, for $m \neq n$,

$$\llbracket a_m = a_n \rrbracket \leq \bigwedge_{k \in \mathbb{N}} U_{k,n} = \perp.$$

Thus $(\lambda n. a_n)$ is (internally) a function: $\mathbb{N} \rightarrow P(P(\{0\}))$.

We note that a similar proof shows $\neg B(P(\{0\}))$ in $\text{Sh}(Q)$.

7. Closure under unions

Our final example is to show that in $\text{Sh}(\mathbb{R})$ the union of a singleton (so S -finite) set of N -finite sets need not even be B' -finite (so $S, SQ, B, B' \not\vdash (U)$):

Let A be the join in $P(\hat{\mathbb{N}})$ of the sections

$$\{[\hat{n}] \uparrow ((n+2)^{-1}, (n+1)^{-1}) \mid n \in \mathbb{N}\},$$

and let $X = \{A\}$ internally; so X is a singleton and where A exists it is N -finite being just $[\hat{n}]$ for some n . But also for each $n, 0 \in ([\hat{n}] \subseteq A)^\perp$, so $0 \notin \llbracket UX \in B' \rrbracket$.

8. Tarski-finiteness

Our last notion corresponds to a definition of "finite" set due to *Tarski*. (Recall that $A \subsetneq B$ iff $[A \subseteq B \wedge \exists x \in (B \setminus A)]$.)

Definition: We define

$$X \in T \text{ iff } (P(X), =, \subsetneq) \text{ is well-founded.}$$

Some properties:

- (i) $B' \rightarrow T \rightarrow \neg \neg B'$ (whence T is a notion of finiteness).
- (ii) $X \in T$ iff $(P(X), =, \supsetneq)$ is Wf.
- (iii) $T \models (\text{Sub}), (\text{Quot}), (\text{Sing } S), (\cup Q), (U)$
- (iv) T is the least notion satisfying (T):

$$[(\forall Y \subsetneq X. Y \in T) \rightarrow X \in T].$$

For (i) one proves $(B'_n(X) \rightarrow X \in T)$ by induction on n , and if $X \in T$, $(\forall Y \subseteq X. \neg \neg B'(Y))$ by induction in $P(X)$.

By (iii) T catches the set constructed in X.1.7, so that $(T \rightarrow B')$ fails; it is still open whether T satisfies (UQ) or (Prod) .

One may express the definitions of N, S, Q and SQ in a similar way to that of T : For example, for Q , define $A < B$ iff $(\exists x \in B. B = AU\{x\})$; then $X \in Q$ iff $(P(X), =, <)$ is Wf, essentially by the same argument as for X.1.3(vi).

Another interesting property of this notion is that any relation on a T-finite set, whose transitive closure is irreflexive, is *well-founded*; thus we get the induction principle "for free" in virtue of this "finiteness".

X.2 COUNTABLE AND INFINITE SETS1. *Countability*

MINIO [16] considers various notions according to which a set is "countable" iff it is enumerated in some way by \mathbb{N} , and the same sorts of distinctions arise as in X.1.3; we used the weakest of these, $(\mathbb{N} \overline{p} \gg X)$ in X.1.5. The most useful seems to be $(\mathbb{N} \dashrightarrow X)$, for much the same reasons as Q-finiteness in X.1.3.

2. *Infiniteness*

MINIO [16] also investigates similar notions of infinity, of which we used one, $(\mathbb{N} \gg X)$, in X.1.6. A much stronger notion (the equivalence implies AC classically!) is the following, which we found useful in IX.3.5:

Definition: We define

$$X \in P \text{ iff } [(X^2 \gg X) \wedge ([2] \gg X)]$$

that is, X has *at least two elements* and a *pairing* function, usually denoted by $\lambda x, y \in X. (x, y)$.

Some properties:

- (i) $X \in P \rightarrow (\mathbb{N} \gg X)$
- (ii) $\mathbb{N} \in P$
- (iii) $P \models (\text{Prod})$; $[X \in P \rightarrow P(X) \in P]$.
- (iv) If $([2] \gg X)$, $(X^{\mathbb{N}})$ belongs to P .

For (i) let $x_0, x_1 \in X$ and $x_0 \neq x_1$, and put $f(0) = (x_0, x_1)$ and $f(Sn) = (f(n), x_0)$; then $f: \mathbb{N} \gg X$. \mathbb{N} has its usual (diagonal) pairing function. In view of (iv) any set can be "extended" to a P -infinite one.

3. *Higher cardinality*

This brief excursion into questions of *higher cardinality* is the only point at which we consider cardinality *relations* between sets; we owe the idea here to GREENLEAF [10], where he suggests a relation of *productivity* as a strong measure of being "of higher cardinality than",

Definition: We define Y is $(p-)$ productive in X

$$(X \ll_{(p)} Y) \text{ iff } \forall f: X \xrightarrow{(p)} Y \exists y \in (Y \setminus \text{rge}(f)).$$

(If X, Y have equalities we modify this to

$$\forall f: X \xrightarrow{(p)} Y \exists y \forall x \in \text{dom}(f) . y \neq_Y f(x).)$$

Some properties:

- (i) $X \ll_p P(X)$ and $X \ll [2]^X$ by Cantor's theorem.
- (ii) $[X \in P \longrightarrow X \ll_p H(X)]$ since for $f: X \xrightarrow{(p)} H(X)$ we can form $\text{Sup}(f) \in H(X)$, and $\text{Suc}(\text{Sup}(f))$ does the job.
- (iii) $\forall n ([n] \ll [n+1])$ by induction in \mathbb{N} , but in $\text{Sh}(\mathbb{R})$, $\neg(\{0\} \ll_p \mathbb{N})$; so in particular ω is not a p -regular ω_0 .

For (iii) let $(\lambda m, n . (m, n))$ be the usual pairing function on \mathbb{N} ; put $U_n = U\{([m, n] + 2)^{-1}, [m, n]^{-1} \mid m \in \mathbb{N}\}$, and let a be the join in $\hat{\mathbb{N}}$ of $\{\hat{n} \uparrow U_n \mid n \in \mathbb{N}\}$.

Then map $f: \{\hat{0}\} \xrightarrow{(p)} \hat{\mathbb{N}}$ by $f(\hat{0}) \equiv a$,

so that $\forall n . 0 \in (\llbracket f(\hat{0}) = \hat{n} \rrbracket)^-$

whence $0 \notin \llbracket \exists n . n \in \text{rge}(f) \rrbracket$.

4. *The continuum hypothesis*

On this subject we just note that one may easily modify the usual classical proof to show that, in $\text{Sh}([2]^{P(P(\mathbb{N}))})$, $(\hat{\mathbb{N}} \ll_p (P(\mathbb{N}))^\wedge \ll_p P(\mathbb{N}))$ so that the continuum hypothesis fails in a strong way.

X.3 NOTIONS FOR SETS WITH APARTNESS

We investigate finally some cardinality notions applicable to the subsets of some fixed set A with an *apartness* $\#$. Note that as \mathbb{N} is *discrete* all functions from it into A are *strongly extensional* already.

1. *Finiteness*

Two new natural notions in this context arise by using the property of *strong one-one-ness* (I.4.2):

$$X \in N^\# \text{ iff } \exists n([n] \twoheadrightarrow X)$$

$$X \in S^\# \text{ iff } \exists n([n] \xrightarrow[p]{} X)$$

These then satisfy modified closure properties such as

$$(U\#) [(X \in F \wedge Y \in F \wedge \forall x \in X, y \in Y. x \# y) \rightarrow (X \cup Y) \in F].$$

One may then adapt the form of the *Bolzano-Weierstrass theorem* of HEYTING [11], p.48, to give:

If I is *compact* and $X \subseteq \mathbb{R}$ is *bounded* and *isolated* in the sense that each point of \mathbb{R} has a neighbourhood whose intersection with X is a *singleton*, then $X \in S^\#$.

Conversely, if $X \in N^\#$ it is bounded and isolated, but in $C(\mathbb{R}, \mathbb{R})$ even singletons may not be bounded (e.g. $\lambda x \# 0. x^{-1}$).

2. *Coboundedness*

The most interesting notion here is a "dual" of the notion B' which arose first in connection with the results of X.3.3 below. We treat again subsets X of some fixed set A with apartness, and read $C_n(X)$ as: for any $(n+1)$ apart elements of A one at least must belong to X . Thus X is a "large" subset of A .

Definition: We define for $n \in \mathbb{N}$

$$C_n(X) \text{ iff } \forall f: [n+1] \twoheadrightarrow A \exists i \leq n. f(i) \in X$$

and then $X \in C$ iff $\exists n. X \in C_n$.

Some properties:

(i) $[C_n(X) \rightarrow B'_n(A \setminus X)]$ since if $f: [n+1] \twoheadrightarrow A$, $\neg \neg (f: [n+1] \twoheadrightarrow A)$; whence $[X \in C \rightarrow (A \setminus X) \in B']$, and the converse is true classically of course.

(ii) $\forall x \in A. C_1(\{y \mid y \# x\})$, but in \mathbb{R} for example the complement of a singleton need not belong to C :

In $C(\mathbb{R}, \mathbb{R})$ let f be an "oscillating" function defined on $(0, \infty)$ which takes all values in $(0, 1)$ arbitrarily close to 0. Then for any $x \in (0, 1)$, $0 \in [\hat{x} \neq f]$ so that the internal complement of $\{f\}$ does not belong to C .

(iii) $[(C_m(X) \wedge C_n(Y)) \rightarrow C_{m+n}(X \cap Y)]$ so that C is closed under intersections; it is also obviously closed upwards under \subseteq , so that it forms a *filter* on $P(A)$.

3. *Non-roots of polynomials*

We now apply the preceding notion to characterise strongly the sense in which real polynomials have "finitely many roots"; this was something we worked out with *Dana Scott*.

Definition: Let A be any *field with apartness* (HEYTING [11], p.49). A *polynomial* of degree at most n with coefficients in A is determined by a sequence $\mathbf{p} = (p_0, \dots, p_n)$ from A , which determines a function $p: A \rightarrow A$ given as usual by

$$p(a) = \sum_{k \leq n} (a^k \cdot p_k).$$

A polynomial \mathbf{p} is *non-trivial* iff $\exists i \leq n. p_i \neq 0$; an element a of A is a *non-root* of \mathbf{p} iff $p(a) \neq 0$.

Theorem: If \mathbf{p} is a non-trivial polynomial of degree at most n and X is its set of non-roots, then $C_n(X)$.

Proof: Let $\mathbf{a} = (a_0, \dots, a_n) \in \text{Seq}(A)$ be such that $\forall i < j \leq n (a_i \neq a_j)$, and let M be the $(n+1)$ -by- $(n+1)$ matrix with (k, i) entry $(a_i)^k$. Then the *determinant* of M is

$$\prod_{i < j} (a_i - a_j) \neq 0,$$

so M has an *inverse* M^{-1} with entries $(b_{k,i})$ say.

Denoting by \mathbf{q} the result of the multiplication $(\mathbf{p} \cdot M)$ we have $(\mathbf{q} \cdot M^{-1}) = \mathbf{p}$, so that for each k ,

$$\sum_{i \leq n} (q_i \cdot b_{k,i}) = p_k.$$

Now as \mathbf{p} is *non-trivial*, $p_k \neq 0$ for some k , whence $q_i \neq 0$ for some i . But $q_i = p(a_i)$, so that a_i is a *non-root* of \mathbf{p} .

Remarks: It follows from this of course that the set of *roots* $\{a \mid p(a) = 0\}$ belongs to B'_n ; our aim here was to extract the maximum possible information from the usual classical proof of this result. To say more about these sets would involve one in the rather delicate question of the *existence* of roots (for example FOURMAN and HYLAND [6] and work of Kock, Wraith and Reyes).

CONCLUSION

As the reader will have found, any *originality* in this thesis consists in many small ideas rather than a few large ones. Some of these ideas originated in discussions with *Dana Scott*, *Mike Fourman* and *Martin Hyland*, and it is often hard to remember just who thought of what; so one can only repeat the acknowledgement of the debt and thanks that are owing to them. The origins of other ideas in the established literature are clearly recognisable, so we have refrained from trying to give exact credit there.

We hope that the reader will share our feeling that it is not only the things that *go wrong* in intuitionistic logic that are interesting, but also the ways in which they can be made to *go right*. In all areas there are endless possibilities for "going wrong", some of which we have worked on but omitted here because they are uninteresting (and take up a lot of space!). We have concentrated rather on the distinctions to be made between "reasonable" alternatives, such as weak and strong separation, connectedness, boundedness, limits etc..

There have been many small outstanding problems noted at various points: From III.3 there is the question of *absoluteness* over cHa's not known to be proper. From Part Two there are questions in *topology*, about normal and uniform spaces, or the problem of finding a model in which we can prove "Brouwer's principle". We should also like a characterisation of the topological properties which are *preserved* in the passage from a space X to a bundle X_T ; many of the conditions considered were found to be so preserved. Of course there are still vast areas of topology completely untouched, as well as all sorts of questions about *cardinality*.

A new direction is suggested by the *formal spaces* of FOURMAN [5]: In terms of the *geometric theories* of Chapter IV one thinks of a formula δ determining a "formal" open U_δ rather than a set of points (or models), and reads a sequent $(\delta \vdash \forall \Gamma)$ as " U_δ is covered by $\{U_\gamma \mid \gamma \in \Gamma\}$ ". Thus one interprets topological notions without reference to *points*. We have already made such interpretations in general cHa's in the notions of *positive*, *proper* and *connected*, and FOURMAN and HYLAND [6] contains a notion of a *locally compact* cHa.

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