Free Curves on Varieties

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To my parents,
for their unceasing support.

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Abstract

In this thesis we study various ways in which every two general points on a variety can be connected by curves of a fixed genus, thus mimicking the notion of a rationally connected variety but for arbitrary genus. We assume the existence of a covering family of curves which dominates the product of a variety with itself either by allowing the curves in the family to vary in moduli, or by assuming the family is trivial for some fixed curve of genus g. A suitably free curve will be one with a large unobstructed deformation space, the images of whose deformations can join any number of points on a variety. We prove that, at least in characteristic zero, the existence of such a free curve of higher genus is equivalent to the variety being rationally connected. If one restricts to the case of genus one, similar results can be obtained even allowing the curves in the family to vary in moduli. In later chapters we study algebraic properties of such varieties and discuss attempts to prove the same rational connectedness result in positive characteristic.

Contents

Chapter 1

Introduction

Notation. A variety is an integral separated scheme of finite type over a field k. A curve will usually be integral but possibly singular unless otherwise specified whereas we will often emphasise the distinction by calling a possibly reducible curve a connected curve. We use the same notation for vector bundles and locally free sheaves without mention. If confusion arises, we distinguish between the stalk $\mathscr{E}_x$ at x of the locally free sheaf $\mathscr{E}$ on X and the fibre $\mathscr{E}(x) = \mathscr{E}_x/m_x$ above x of the vector bundle $\mathscr{E}$ by writing the latter as the tensor product of sheaves $\mathscr{E} \otimes k(x)$. We denote by $h^i(X, \mathscr{F})$ the integer $\dim H^i(X, \mathscr{F})$. A property of points of schemes will be general if it holds for every point in a dense open subset and very general if it holds for every point in the complement of a countable union of proper subschemes.

We give a brief historical account of the theory of rationally connected varieties, leaving the formal definitions till later chapters and then proceed to summarise the main results of this thesis.

Let k be an uncountable algebraically closed field. A smooth projective rationally connected variety, originally defined in [KMM92c] and [Cam92], is a variety such that through every two general points there passes the image of a rational curve. In the next few paragraphs we will try to justify why this notion is a much more natural, geometric and general notion than that of a projective space or a unirational variety (dominated by some projective space of the same dimension).

In characteristic zero a rationally connected variety is equivalent to the notion of a separably rationally connected variety, namely

in characteristic zero, a smooth projective variety is rationally connected if and only if there exists a morphism $f : \mathbb{P}^1 \to X$ such that $f^*\mathscr{T}_X$ is ample.

In characteristic p however one has to distinguish between these two notions (see the example in Remark 2.3.24). Deformations of a morphism $f : \mathbb{P}^1 \to X$ are controlled by the sheaf $f^*\mathscr{T}_X$ (see Theorem 2.1.1), hence studying positivity conditions of this bundle is intimately tied to deformation theory and the existence of many rational curves on X. A classical theorem of Grothendieck gives us easy numerical criteria for testing positivity statements for $f^*\mathscr{T}_X$ on $\mathbb{P}^1$.

Theorem 1.1. *([Gro57]) For k an algebraically closed field, every locally free sheaf $\mathscr{E}$ of rank n on $\mathbb{P}^1$ can be written as the direct sum of line bundles*

$$\mathscr{E} \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(a_n)$$

where $a_1 \geq \ldots \geq a_n$ are integers.

Corollary 1.2. *([Kol96] II.3.8) A locally free sheaf $\mathscr{E}$ of rank n on $\mathbb{P}^1$ is called semi-positive (respectively ample) if any of the following equivalent conditions are satisfied*

1. *$H^1(\mathbb{P}^1, \mathscr{E}(-1)) = 0$ (respectively $H^1(\mathbb{P}^1, \mathscr{E}(-2))$),*

2. *$\mathscr{E}$ (respectively $\mathscr{E}(-1)$) is generated by global sections,*

3. *$\mathscr{E} \cong \oplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$ (respectively ≥ 1) for all i,*

4. *$H^0(\mathbb{P}^1, \mathscr{E}) \to \mathscr{E} \otimes k(p)$ (respectively $H^0(\mathbb{P}^1, \mathscr{E}(-1)) \to \mathscr{E}(-1) \otimes k(p)$) is surjective for some $p \in \mathbb{P}^1$,*

5. *$\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ is nef (respectively ample) as a line bundle on $\mathbb{P}(\mathscr{E})$.*

A rationally connected surface is rational, since it has no global differential forms (see Conjecture 1.5) and such a surface is birational to $\mathbb{P}^2$ by the classification of surfaces. More generally in higher dimension, rationally connected varieties include the classes of rational and unirational varieties and an embarassing problem in the theory is that there is no proven example of a non-unirational rationally connected variety. The first result which greatly motivated the study of this theory (see [Cam92] and [KMM92b]) is that in characteristic zero

Fano varieties are rationally connected

(also in characteristic p but the result is that they are rationally chain connected). Rationally connected varieties have especially nice properties, often modelled on properties held by path connected spaces in topology. Note in particular the following crucial theorem of Graber, Harris and Starr in characteristic zero and by de Jong, Starr in positive characteristic, which we will make repeated use of throughout this thesis.

Theorem 1.3. *(Graber-Harris-Starr [GHS03], de Jong-Starr [dJS03]) Over k an algebraically closed field, let C be a smooth projective curve and $f : X \to C$ a proper flat morphism whose geometric generic fibre is normal and separably rationally connected. Then f has a section.*

Starting with an arbitrary smooth variety one can form the MRC fibration (see Theorem 2.2.17) over a normal and proper base such that the general fibre is rationally chain connected, or in other words that through every two points there is a chain of rational curves (i.e. a connected genus 0 curve), and maximal with respect to dimension. Smoothing of combs techniques (see Theorem 2.1.8) make the notions of rationally chain connected and rationally connected varieties equivalent in characteristic zero so in this case, the MRC fibration has rationally connected fibres. In positive characteristic, it is an open problem whether the classes of rationally chain connected and rationally connected varieties differ. A consequence of the above theorem in characteristic zero is the following (see [Deb03] Corollaire 3.1) fibration theorem

if $X \to Y$ is a morphism of smooth projective varieties such that the general fibre and Y are rationally connected, then so is X.

Another consequence is the following very useful result, which we shall frequently use. Note that it is in fact equivalent to the Graber-Harris-Starr Theorem in characteristic zero.

Corollary 1.4. *([Kol96] Proposition IV.5.6.3) Let X be a variety over an algebraically closed field k of characteristic zero and let $X \dashrightarrow R(X)$ be the MRC fibration. Then $R(X)$ is not uniruled.*

One of the big open conjectures in the area gives a numerical characterisation of when a variety can be rationally connected.

Conjecture 1.5. *(Mori-Mumford) A smooth projective variety X over an algebraically closed field of characteristic zero is uniruled if and only if*

$$H^0(X, \omega_X^{\otimes m}) = 0 \textit{ for all } m > 0$$

whereas it is rationally connected if and only if

$$H^0(X, (\Omega^1_X)^{\otimes m}) = 0 \textit{ for all } m > 0.$$

Note that if one assumes X to be uniruled or rationally connected then one direction of both conjectures follows from the statements in Theorem 2.3.23 (see [Deb01] Corollary 4.12), and that both directions of the conjectures are known in dimensions two and three. Finally, from the Graber-Harris-Starr Theorem one can conclude that the first conjecture implies the second (see [Deb03] Corollaire 3.5).

In this thesis we study various ways in which a variety can be connected by higher genus curves. We consider first varieties which admit a morphism from a family of irreducible curves of fixed arithmetic genus g whose product with itself dominates the product of the variety with itself and call these varieties "genus g connected", generalising the notion of there being a rational curve through two general points. If we assume that the family of curves is trivial, we define the notion of a C-connected variety, where there exists a family $C \times U \to X$ of a single smooth projective genus g curve C such that $C \times C \times U \to X \times X$ is dominant. Mori's important Bend and Break result allows us to produce rational curves going through a fixed point given that there exists a higher genus curve which deforms a lot keeping this point fixed.

Theorem 1.6. *(Bend and Break, [Deb01] 3.1) Let X be a projective variety over an algebraically closed field k, $f : C \to X$ a morphism from a smooth projective curve and $c \in C$. If $\dim_{[f]} \operatorname{Hom}(C, X; f|_{\{c\}}) \geq 1$, then there exists a rational curve on X through $f(c)$.*

In Proposition 2.2.16 we show that over any characteristic, if for any two general points of a smooth projective variety X with $\dim X \geq 3$ there passes the image of a morphism from a fixed curve C of genus g (coming from a suitable algebraic family), then X is uniruled or in other words there is a rational curve through a general point. In Chapter 2.2 we prove basic properties of such varieties and give examples but one quickly has to restrict to a less general setting.

A stronger condition than the aforementioned is a generalisation of the existence of a morphism from a curve which deforms a lot, as discussed for separably rationally connected varieties above. Namely, we introduce the existence of a free curve $f : C \to X$ first studied by Kollár [Kol96] where now C can be of any genus g. We call a morphism free if $f^*\mathscr{T}_X$ is globally generated as a vector bundle on C and also $H^1(C, f^*\mathscr{T}_X) = 0$. In the case of genus $g = 0$ one must distinguish between free and very free curves, geometrically meaning that $f : \mathbb{P}^1 \to X$ deforms so that its image covers all points in X or whether it can do so after fixing a point $x \in f(\mathbb{P}^1)$ respectively. In other words a variety with a free rational curve is uniruled whereas a variety with a very free curve is rationally connected. If $g \geq 1$ however, after defining an r-free curve to be one which deforms keeping any r points fixed, we show that the notions of the existence of a free (0-free) and very free (1-free) curve coincide and in fact are equivalent with the existence of a curve $f : C \to X$ such that $f^*\mathscr{T}_X$ is ample.

Theorem. *(Theorem 2.3.14) Let X be a smooth projective variety and C a smooth projective curve of genus $g \geq 1$ over an algebraically closed field k. Then for any $r \geq 0$, there exists an $f : C \to X$ which is r-free if and only if there exists a morphism $f' : C \to X$ such that $f'^*\mathscr{T}_X$ is ample.*

Work of Bogomolov-McQuillan (see Theorem 2.4.5) and more recently a proof of the same result by Kebekus-Solá Conde-Toma (see [BM01], [KSCT07]) on foliations which restrict to an ample bundle on a smooth curve sitting inside a complex variety X show that the leaves of such a foliation are not only algebraic but in fact have rationally connected closures (cf. [Miy87] Theorem 8.5 for a similar result on uniruledness). We reprove this result in the case of the foliation $\mathscr{F} = \mathscr{T}_X$ thus assuming the existence of a free curve $f : C \to X$, complementing the currently known connections between existence of curves with large deformation spaces and rationally connected varieties.

Theorem. *(Corollary 2.4.4) Let X be a smooth projective variety over an algebraically closed field of characteristic zero and let $f : C \to X$ be a smooth projective* *curve of genus $g \geq 1$ such that $f^*\mathscr{T}_X$ is globally generated and $H^1(C, f^*\mathscr{T}_X) = 0$. Then X is rationally connected.*

We give examples along the way and prove various useful properties of C-connected varieties and varieties admitting a free morphism from a curve of any genus g. Notably, in characteristic zero, this last class of varieties is the same as that of rationally connected varieties. One would hope that the existence of a higher genus free curve could on occasion be easier to prove, thus meaning our variety is rationally connected, however the author has not been able to find such a non-trivial example.

In the third chapter, we study the particular case of elliptically connected varieties (i.e. genus 1 connected varieties) where, even allowing a covering family of genus 1 curves to vary in moduli, one can in fact prove the following theorem.

Theorem. *(Theorem 3.3) Let X be a smooth projective variety over a field of characteristic zero. Then the following two statements are equivalent*

1. *There exists $\mathcal{C} \to U$ a flat projective family of irreducible genus 1 curves with a map $\mathcal{C} \to X$ such that $\mathcal{C} \times_U \mathcal{C} \to X \times X$ is dominant.*
2. *X is either rationally connected or a rationally connected fibration over a curve of genus 1.*

The final two chapters are concerned with attempts along the way to proving Corollary 2.4.4 in the case of positive characteristic. At this point we have not been able to prove that the existence of a higher genus free curve implies the existence of a very free rational curve (which would mean that X is separably rationally connected). In Chapter 4 however we work in this direction, establishing this result in dimensions two (with a short discussion about dimension three) and furthermore by studying algebraic implications of the existence of a free higher genus curve, such as the vanishing of pluricanonical forms and triviality of the Albanese variety. The final chapter discusses finiteness properties of the étale fundamental group of varieties that are rationally connected. We also give an example of a threefold in characteristic p whose MRC quotient is uniruled and which has infinite fundamental group.

Chapter 2

Curve connectedness

2.1 Moduli of curves and smoothing of combs

We start with some preliminaries by referencing results on the various moduli spaces we will be using. Many of the results on rationally connected varieties but also ones that we will be proving rely on being able to deform a nodal but connected curve into a smooth one, often preserving positivity of the normal bundle. This technique is known as smoothing of combs [Kol96] II.7 (or more generally links [Sta09]). None of the results in this section claim any originality.

For $X \to S$ a projective scheme over an arbitrary scheme S, we denote (see [Kol96] I) by $\mathrm{Hilb}_P(X/S) \to S$ the projective moduli space parametrising proper and flat subschemes of X with a fixed Hilbert polynomial. This is the prototype fine moduli space and so comes equipped with a universal family

$$\begin{tikzcd} \mathrm{Univ}_P(X/S) \arrow[r, hook] \arrow[d] & X \times_S \mathrm{Hilb}_P(X/S) \\ \mathrm{Hilb}_P(X/S). & \end{tikzcd}$$

There is a subfunctor $\mathbf{Hom}_S(X, Y)$ of the Hilbert functor which sends a scheme T to

$$\mathbf{Hom}_S(X,Y)(T) = \{T\text{-morphisms} : X \times_S T \to Y \times_S T\}$$

which if X and Y are projective S-schemes, one can prove is represented by an open subscheme

$$\mathrm{Hom}_S(X, Y) \subset \mathrm{Hilb}(X \times_S Y/S).$$

The theory is simpler in the case where X is a curve, and to illustrate this situation better, we specialise to the case of morphisms from a smooth projective curve C over a field k to a smooth projective variety X. Let $\mathscr{I}_B$ be the ideal sheaf of a closed finite subscheme of C with $g : B \to X$. One can construct $\mathrm{Hom}(C, X, g)$ the moduli space of maps $f : C \to X$ whose image passes through the image of g, namely that $f|_B = g$ (see [Kol96] II.1.5). The local behaviour of this moduli space was first described by Mori in his seminal paper [Mor79] and we give a general relative version of the main theorem below (see also [AK03]).

Theorem 2.1.1. *([Mor79] Section 1) Let $\mathcal{C} \to S$ be a flat projective curve without embedded points and $X \to S$ a smooth quasi-projective scheme, where S is irreducible. Let $B/S \subset \mathcal{C}/S$ be a closed subscheme finite and flat over S. Assume that $\mathcal{C}/S$ is smooth along B and let $g : B/S \to X/S$ be a morphism. Let $s \in S$ be a point and $f_s \in \mathrm{Hom}(\mathcal{C}_s, X_s, g_s)(k(s))$ a morphism. Then*

1. *The Zariski tangent space of $\mathrm{Hom}(\mathcal{C}_s, X_s, g_s)$ at $[f_s]$ is given by*

$$\mathscr{T}_{\mathrm{Hom}(\mathcal{C}_s, X_s, g_s),[f_s]} \cong H^0(\mathcal{C}_s, f_s^*\mathscr{T}_{X_s} \otimes \mathscr{I}_{B_s})$$

2. *The dimension of every irreducible component of $\mathrm{Hom}(\mathcal{C}, X, g)$ at $[f_s]$ is at least*

$$\chi(\mathcal{C}_s, f_s^*\mathscr{T}_{X_s} \otimes \mathscr{I}_{B_s}) + \dim_s S$$

3. *If* $H^1(\mathcal{C}_s, f_s^*\mathscr{T}_{X_s} \otimes \mathscr{I}_{B_s}) = 0$ *then* $\mathrm{Hom}_S(\mathcal{C}, X, g)$ *is smooth over* S *at* $[f_s]$ *and the above dimension bound is exact.*

Proof. The proof is contained in [Mor79] or [Kol96] II.1.7. A point needs to be made about part (3) though (see [Deb01] 2.11). Since $h^1(\mathcal{C}_s, f_s^*\mathscr{T}_{X_s} \otimes \mathscr{I}_{B_s}) = 0$, part (2) implies that the dimension of $\mathrm{Hom}_S(\mathcal{C}, X, g)$ at $[f_s]$ is at least $h^0(\mathcal{C}_s, f_s^*\mathscr{T}_{X_s} \otimes \mathscr{I}_{B_s}) + \dim_s S$. From part (1), the dimension of the tangent space of $\mathrm{Hom}(\mathcal{C}_s, X_s, g_s) \to k(s)$ at $[f_s]$ is $h^0(\mathcal{C}_s, f_s^*\mathscr{T}_{X_s} \otimes \mathscr{I}_{B_s})$ hence this fibre is smooth at $[f_s]$ and every irreducible component of $\mathrm{Hom}_S(\mathcal{C}, X, g)$ passing through $[f_s]$ has dimension precisely the given lower bound. The final part [Kol96] II.1.7.3 implies that the morphism $\mathrm{Hom}_S(\mathcal{C}, X, g) \to S$ is smooth at $[f_s]$. $\square$

We now restrict the above again to the case where S is the spectrum of an algebraically closed field k and fix some notation of the following evaluation morphisms to be used in later sections

$$\begin{aligned}
F : C \times \mathrm{Hom}(C, X, g) &\rightarrow X \\
F^{(2)} : C \times C \times \mathrm{Hom}(C, X, g) &\rightarrow X \times X \\
\phi(p, f) : H^0(C, f^*\mathscr{T}_X \otimes \mathscr{I}_B) &\rightarrow f^*\mathscr{T}_X \otimes k(p) \\
\phi^{(2)}(p, q, f) : H^0(C, f^*\mathscr{T}_X \otimes \mathscr{I}_B) &\rightarrow f^*\mathscr{T}_X \otimes k(p) \oplus f^*\mathscr{T}_X \otimes k(q).
\end{aligned}$$

For a morphism $f : X \to Y$ and $x \in X$ denote the map on tangent spaces by $df(x) : \mathscr{T}_X \otimes k(x) \to \mathscr{T}_Y \otimes k(f(x))$.

We also introduce the coarse moduli space corresponding to the stack of stable genus g curves constructed over an arbitrary field k in [AK03] and give the general version of the algebraic space which is a coarse moduli space for stable curves over a fixed Noetherian base S.

The following is essentially the definition of Deligne and Mumford [DM69] but where one allows arbitrary base field.

Definition 2.1.2. Let X be a scheme over a field k, P a smooth zero dimensional k-scheme and $t : P \to X$ a k-morphism. A stable curve over X which is defined over k with base point $t(P)$ is a triple (C, f, σ) where

1. C is a proper and connected curve defined over k having only nodes,
2. $\sigma : P \to C$ is a k-embedding into the smooth locus of C,
3. $f : C \to X$ is a k-morphism such that $t = f \circ \sigma$ and
4. C has only finitely many automorphisms that commute with f and σ.

Theorem 2.1.3. *([AK03] Theorem 40) Let X be a projective scheme over a field k with a fixed ample divisor H, P a smooth zero dimensional k-scheme and $t : P \to X$ a k-morphism. Fix integers g and d. There is a projective k-scheme $\overline{\mathcal{M}}_g(X, d, t)$ which is a coarse moduli space for all genus g stable curves (C, f, σ) with base point $t(P)$ such that $f_*[C].H = d$.*

If one allows arbitrary base S then the following definition generalises the above.

Definition 2.1.4. Fix a Noetherian scheme S. Let X be a scheme over S, let $P \to S$ and $Q \to S$ be finite, flat morphisms and let $t : P \to X$ be a morphism. A stable curve over X with base point $t(P)$ and marking Q is a quadruple (C, f, σ, ρ) where

1. $C \to U$ is a flat, proper family of connected curves with only nodes over an S-scheme U,
2. $f : C \to X \times_S U$ is a U-morphism,
3. $\sigma : P \times_S U \to C$ and $\rho : Q \times_S U \to C$ are embeddings into the smooth locus of $C \to U$ with disjoint images,

4. $f \circ \sigma$ is the morphism $P \times_S U \to X \times_S U$ induced by t, and

5. for every point $\operatorname{Spec} k \to U$, C_k has only finitely many automorphisms commuting with f_k and fixing the closed points of $\sigma_k(P_k)$ and $\rho_k(Q_k)$.

Theorem 2.1.5. *([AK03] Theorem 50) Fix a Noetherian base scheme S. Let X be a projective scheme over S with an ample divisor H. Let $P \to S$ and $Q \to S$ be finite and flat morphisms of relative embedding dimension at most 1 and let $t : P \to X$ be a morphism. Fix integers g and d. Then there is a separated algebraic space $\overline{\mathcal{M}}_{g,Q}(X, d, t)$ of finite type over S which is a coarse moduli space for all genus g stable curves (C, f, σ, ρ) with base point $t(P)$ and marking Q such that $\deg_C f^*H = d$. Moreover, if $P \to S$ and $Q \to S$ are smooth, then $\overline{\mathcal{M}}_{g,Q}(X, d, t)$ is projective over S.*

Next, on to smoothing of morphisms of curves. We draw once more from Kollár's book [Kol96] (the original references being [KMM92a], [KMM92b], [KMM92c]). The idea here is that we start with a reducible curve C and a morphism $f : C \to X$ and study conditions that will ensure that we can obtain f as a limit of a family of morphisms from smooth curves. To be precise, one makes the following definition.

Definition 2.1.6. ([Kol96] II.7.1) Let $f : C \to X$ be a morphism from a curve to a variety X over a field k. A smoothing of f is a diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[d, "q"] & X \\ T & \end{tikzcd}$$

where $0 \in T$ is a connected smooth pointed curve such that

1. q is flat and proper over T,

2. q is smooth over $T \setminus \{0\}$

3. and $[F_{q^{-1}(0)} : q^{-1}(0) \to X] \cong [f : C \to X]$.

There is a similar definition where one fixes $x_1, \ldots, x_k$ distinct points in C and we require that every curve in the family $q : Y \to T$ when mapped to X via F passes through $f(x_1), \ldots, f(x_k)$.

The usual constructions one encounters with rationally connected varieties involves reducible curves which are constructed as chains of rational curves, or trees of rational curves, or more generally combs with an arbitrary base curve as handle as in the following definition.

Definition 2.1.7. ([Kol96] II.7.7) A comb with n teeth is a connected and reduced curve C with irreducible components D (the handle) and $C_1, \ldots, C_n$ (the teeth) with the following properties

1. D is smooth and every C_i is isomorphic to $\mathbb{P}^1$,

2. the only singularities of C are ordinary nodes and

3. every C_i intersects D in a single point x_i and $x_i \neq x_j$ for $i \neq j$ and there are no other intersections between the irreducible components.

Starting with an irreducible curve D, if one can attach free rational curves C_i (see Definition 2.3.4) to form a comb $C = D + \sum C_i$, then even after fixing a finite number of points on D there is a subcomb of C which is smoothable keeping these points fixed. The precise result is the following, which also gives bounds on the number of teeth C_i kept in the process.

Theorem 2.1.8. *([Kol96] II.7.9) Let $C = D + \sum_{1 \leq i \leq m} C_i$ be a comb and $p_1, \ldots, p_k \in D$ points different from the nodes x_i. Let $f : C \to X$ a morphism such that X is smooth along $f(C)$ and $f^* \mathscr{T}_X|_{C_i}$ is semi-positive for each i. Then there is a subcomb $C' = D + \sum' C_i$ with m' teeth such that $f' := f|_{C'} : C' \to X$ is smoothable fixing*

$f'(p_1), \ldots, f'(p_k)$ and m' satisfies both of the following inequalities

$$\begin{aligned} m' &\geq m - h^1(D, (f|_D)^* \mathscr{T}_X(-\textstyle\sum p_i)), \text{ and} \\ m' &\geq m - (K_X.D) - k \dim X + \dim X \cdot \chi(\mathscr{O}_D) - \dim_{[f|_D]} \operatorname{Hom}(D, X, f|_{\{p_1,\ldots,p_k\}}). \end{aligned}$$

Assuming moreover that the C_i are very free (i.e. that $f^*\mathscr{T}_X|_{C_i}$ is ample, see Definition 2.3.4) and that $f^*\mathscr{T}_X|_D$ satisfies certain positivity conditions, one proves furthermore that there is a subcomb as above whose smoothings are free over the fixed points. We will not prove either of these results, nor will we elaborate on how exactly they are used in the proof of Theorem 2.3.23 for example, since these are well established in the literature. Even though we have not given the formal definition of a separably rationally connected variety yet, one beautiful implication of the above constructions which we will use later is the following theorem, which should be thought of as an addendum to Theorem 1.3.

Theorem 2.1.9. *([KMM92c] 2.13) Let X be a projective variety over an algebraically closed field k and $f : X \to C$ a morphism onto a smooth curve. Assume that f has a section $\sigma : C \to X$ and let $c_1, \ldots, c_k \in C$ be closed points such that $X_i := f^{-1}(c_i)$ are smooth and separably rationally connected. Then picking arbitrary points $p_i \in X_i$, we can find a section $s = s_{p_1,\ldots,p_k} : C \to X$ such that $s(c_i) = p_i$.*

In combination with the Graber-Harris-Starr Theorem, this can be thought of as saying that assuming we have a rationally connected fibration $f : X \to C$ over a curve, then we can find a section passing through any finite number of prescribed points in smooth fibres of f.

2.2 Covering families

We start with the Rigidity Lemma of Mumford from [Mum70] (or [MFK94] Proposition 6.1, [KM98] Lemma 1.6). The underlying principle is that if one of the fibres of a proper flat family is contracted under a morphism to another variety, then every fibre is contracted. We will often subsume this result whenever we have a family of varieties mapping to another variety, thus often without assuming that one of the fibres does not get contracted so that the family is not trivial under the morphism. The proof claims no originality and uses the following lemma guaranteeing the existence of a section after a pullback.

Lemma 2.2.1. *([EGA] IV_4 17.16.2) Let $f : X \to S$ be a faithfully flat morphism which is locally of finite presentation. Then there exists a subscheme S' of X and a map $f|_{S'} = g : S' \to S$ faithfully flat and locally of finite presentation such that there exists a section $h : S' \to X \times_S S'$ to the projection coming from the fibre product.*

Lemma 2.2.2. *(Rigidity Lemma) Let $p : X \to B$ be a faithfully flat proper morphism of varieties over k and $f : X \to Y$ such that for some $b_0 \in B$, the image of the fibre $X_{b_0} = p^{-1}(b_0)$ under f is a point $y_0 \in Y$. Then for all $b \in B$, there exists a $y_b \in Y$ such that the image of X_b under f is y_b.*

Proof. From Lemma 2.2.1, we have a commutative diagram

$$\begin{tikzcd} X \times_B D \arrow[r, "pr_1"] \arrow[d, shift left, "pr_2"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \\ D \arrow[r, "e"] \arrow[u, shift left, bend left, "s"] & B & \end{tikzcd}$$

where D is a subvariety of X, e is faithfully flat and locally of finite presentation and s is a section of pr_2.

Define $\pi : D \to Y$ as $\pi = f \circ pr_1 \circ s$. If we can show that $f \circ pr_1 = \pi \circ pr_2$ then the result follows. From [Har77] exercise II.4.2, it is enough to show that these morphisms agree on an open subset. Let $U \subset Y$ be an affine open neighbourhood of y_0 and let $F = Y \setminus U$ and $G = p(f^{-1}(F))$. Since p is proper, G is closed in B. Furthermore, $b_0 \notin G$ since $f(X_{b_0}) = \{y_0\}$. Hence $V := B \setminus G$ is a non-empty open subset of B. Now for each $b \in V$ the variety X_b gets mapped by f into U and since X_b is complete and U is affine, its image is a point. Hence $f \circ pr_1$ and $\pi \circ pr_2$ agree on $pr_1^{-1}(p^{-1}(V))$ and so on all of X. $\square$

We now define various ways in which a variety can be covered by curves, generalising the classical notions of ruled, uniruled and rationally connected varieties. First, the basic objects of study are the following.

Definition 2.2.3. Let X be a variety of dimension n over a field k. We say that X is ruled (resp. uniruled, separably uniruled) if there is a scheme Y over k of dimension $n-1$ and a map $\mathbb{P}^1_Y \dashrightarrow X$ which is birational (resp. dominant, dominant and smooth at the generic point).

Definition 2.2.4. ([Kol96] IV.3.2) Let X be a variety over a field k. We say that X is rationally chain connected if there is a family of proper and connected algebraic curves $g : U \to Y$ over a variety Y whose geometric fibres have only rational components, and a map $u : U \to X$ such that

$$u^{(2)} : U \times_Y U \to X \times_k X$$

is dominant. We say that X is rationally connected if the geometric fibres of g are furthermore assumed to be irreducible rational curves. If we assume that $u^{(2)}$ is also smooth at the generic point then X is called separably rationally (chain) connected.

We can extend the definition of a uniruled variety to allow a covering family of morphisms from an arbitrary curve of higher genus as follows.

Definition 2.2.5. Let X be a variety over a field k and C a smooth projective curve over k. We say that X is C-ruled if there exists a variety Y over k of dimension $\dim Y = \dim X - 1$ and a dominant morphism $u : C \times Y \to X$.

We similarly extend the definition of rationally connected varieties to include curves of higher genus as follows.

Definition 2.2.6. We say that a variety X over a field k is connected by genus $g \geq 0$ curves (resp. chain connected by genus g curves) if there exists a proper flat morphism $\mathcal{C} \to Y$, for a variety Y, whose geometric fibres are irreducible genus g curves (resp. connected genus g curves) such that there is a morphism $u : \mathcal{C} \to X$ making the induced morphism $u^{(2)} : \mathcal{C} \times_Y \mathcal{C} \to X \times_k X$ dominant.

Definition 2.2.7. Keeping notation as in Definitions 2.2.5 and 2.2.6 we say that a variety X is separably (chain) connected by genus g curves if the morphism $\mathcal{C} \times_Y \mathcal{C} \to X \times_k X$ is also smooth at the generic point. One likewise defines a separably C-ruled variety.

A genus 0 connected variety is rationally connected. A variety which is connected by genus 1 curves will be called (with a slight abuse of notation) elliptically connected and similarly we have rationally and elliptically chain connected varieties. Even though we will often assume that a genus g connected variety is of dimension at least 2, note that we allow Y to be a point in the above definition. Consider for example that $\mathbb{P}^1$ is an elliptically connected variety.

Note that the notions of separably and non-separably genus g connected varieties coincide when k is of characteristic zero due to generic smoothness.

Remark 2.2.8. In the case of rationally connected varieties over an algebraically closed field, it is enough to consider trivial families $\mathbb{P}^1 \times Y \to X$ in our definition, as rational curves are all isomorphic to $\mathbb{P}^1$ over an algebraically closed field. Note that a non-constant family $u : E \times Y \to X$ where E is an elliptic curve will have the property that the image $u(E_y)$ for $y \in Y$ is isogenous to E.

Lemma 2.2.9. *Let X be a genus g (chain) connected smooth projective variety over an algebraically closed field k. Then if $g' \geq 2g - 1$, X is also genus g' (chain) connected.*

Proof. Let $\mathcal{C}/Y \to X$ be a family making X a genus g (chain) connected variety with notation as in Definition 2.2.6. From Theorem 2.1.5 we have a projective algebraic space $Y' = \overline{\mathcal{M}}_{g',Y}(\mathcal{C}/Y, d)$ of finite type over Y parametrizing stable families of degree d curves of genus g' over $\mathcal{C} \to Y$. The condition $g' \geq 2g - 1$ coming from the Riemann-Hurwitz formula ensures that this moduli space is non-empty. From [ACG11] 12.9.2 there exists a normal scheme Z finite and surjective over Y' and a flat and proper family $\mathcal{X} \to Z$ of stable genus g' curves of degree d. Restricting to a suitable open subset $W \subset Z$ parametrizing irreducible curves we compose the family $\mathcal{X}|_W \to W$ with the evaluation morphism to X and the result follows. $\square$

An example of an elliptically connected variety over a non-algebraically closed field is given after the proof of Theorem 3.3.

Instead of assuming we have a possibly non-isotrivial family of curves dominating a variety X we may want to restrict to a single curve, keeping in mind Remark 2.2.8.

Definition 2.2.10. We say that a variety X over a field k is C-connected for a curve C if there exists a variety Y and a map $u : C \times Y \to X$ such that the induced map $u^{(2)} : C \times C \times Y \to X \times X$ is dominant. If $u^{(2)}$ is also smooth at the generic point, then we say that X is separably C-connected.

Obviously a C-connected variety is genus g connected where g is the genus of the curve C. Projective space is C-connected for every smooth projective curve C. An example of a C-connected variety which is not rationally connected is $C \times \mathbb{P}^n$ for C a smooth projective curve of genus greater than zero. To see that it is C-connected let $(c_1, x_1), (c_2, x_2)$ be any two points in $C \times \mathbb{P}^n$ and let $f : C \to \mathbb{P}^n$ a morphism which sends $c_i \mapsto x_i$. Considering the graph of f in $C \times \mathbb{P}^n$ we have found a curve isomorphic to C which goes through our two points. Using part (4) and (5) from Lemma 2.2.13 below, the result follows. More generally, examples can also be constructed from Proposition 2.2.15 below.

We begin by proving some basic properties of genus g connected and C-connected varieties.

Lemma 2.2.11. *Let X be a smooth projective variety over an algebraically closed field k. There exists an integer $g \geq 0$ such that X is connected by genus g curves.*

Proof. Let $d = \dim X$ and consider $\mathbb{P}^N$ as the space parametrizing hypersurfaces of high degree for some $N \gg d$. Let $T = \mathbb{P}^N \times \ldots \times \mathbb{P}^N$ be the product of $d-1$ copies of $\mathbb{P}^N$. Consider the family $\mathcal{C} \to T$ such that

$$\mathcal{C} = \{(x, H_1, \ldots, H_{d-1}) : x \in H_i \text{ for all } i = 1, \ldots, d-1\} \subset X \times T.$$

By Bertini's Theorem [Har77] II.8.18 and generic flatness we know that there is a non-empty locus $Y \subset T$ over which the fibres $H_1 \cap \ldots \cap H_{d-1}$ are smooth irreducible curves and the map to Y is flat. Finally, we replace $\mathcal{C} \to T$ with the irreducible component of $\mathcal{C}$ over Y which dominates X under the projection map onto the first factor. $\square$

Remark 2.2.12. When dealing with genus g connected (or C-connected) varieties, depending on what is to be proven, one can often assume they are projective by compactifying them. Namely, by Nagata's Compactification Theorem we have an open immersion of X into a proper variety $\overline{X}$. Now, by Chow's Lemma we have a proper birational morphism $\overline{X}' \to \overline{X}$ where $\overline{X}'$ is a projective variety over k. This gives a birational map $\overline{X} \dashrightarrow \overline{X}'$ and from Lemma 2.2.13 part (1) below we get that $\overline{X}'$ is also C-connected. If it is enough to show some statement for $\overline{X}'$, we can replace X by $\overline{X}'$ which is now projective.

The following are mostly straight-forward generalisations of various results in [Kol96] IV.3.

Lemma 2.2.13. *The following statements hold for a variety X over a field k and C a smooth projective curve.*

1. *If X is genus g connected and $X \dashrightarrow Y$ is a dominant rational map to a proper variety Y, then Y is also genus g connected. The same holds if X is C-connected.*

2. *If X is proper then it is C-connected if and only if there is a proper variety W, closed in $\mathrm{Hom}(C, X)$ such that $u^{(2)} : C \times C \times W \to X \times X$ is dominant.*

3. *If X is proper and C-connected over an algebraically closed field, then for arbitrary points $x_1, x_2 \in X$ there exists a morphism $C \to X$ whose image contains x_1, x_2.*

4. *If X is defined over a field k and K/k an extension of fields, then $X_K := X \times_k K$ is C-connected if and only if X_k is.*

5. *A variety X over an uncountable algebraically closed field is C-connected if and only if for all very general $x_1, x_2 \in X$ there exists a morphism $C \to X$ which passes through x_1, x_2.*

6. *A variety X over an uncountable algebraically closed field is genus g connected if and only if for all very general $x_1, x_2 \in X$ there exists a smooth irreducible genus g curve containing them.*

7. *Being rationally connected is closed under connected finite étale covers for proper varieties.*

8. *Being elliptically connected is closed under connected finite étale covers for proper varieties.*

Proof. To prove (1), let $u : \mathcal{C}/M \to X$ be a family making X genus g connected and denote by $u' : \mathcal{C}/M \dashrightarrow Y$ the composition. Restricting u' to the generic fibre $\mathcal{C}_{k(M)}$ we have a rational map $\phi : \mathcal{C}_{k(M)} \dashrightarrow Y$. Since Y is proper, by a corollary to the valuative criterion of properness (see [EGA] II Proposition 7.4.9) we can extend ϕ to a morphism $\phi : \mathcal{C}_{k(M)} \to Y$. By spreading out to an open subset $M' \subseteq M$ (see [EGA] IV_3 8.10.5 for properness and 11.2.6 for flatness of the family) we obtain a family $\mathcal{C}|_{M'} \to M'$ which makes Y also genus g connected.

For (2), consider $\mathrm{Hom}(C, X) = \cup R_i$ the decomposition into irreducible components. One direction of the statement is obvious, whereas for the other let $C \times W \to W$ be a family which makes X a C-connected variety. If $u_i : C \times R_i \to X$ the evaluation morphism, then for some i there is a morphism $h : W \to R_i$ such that $h(w) = [C_w \to X]$ for general $w \in W$. This implies that $u_i^{(2)} : C \times C \times R_i \to X \times X$ is also dominant. Statement (3) follows from (2) by noting that since the composition of proper morphisms is proper, we have that W is proper over k, and so $C \times C \times W \to X \times X$ is also proper. Now the result follows from the fact that a proper dominant morphism is surjective.

For the next statement (4), we may compactify X as in Remark 2.2.12 and also assume that it is projective. One direction is simply a pullback under the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$. For the other, if X_K is C_K-connected then from (2) there is a positive integer d such that the evaluation morphism

$$\mathrm{ev}_K^d : C_K \times C_K \times \mathrm{Hom}_d(C_K, X_K) \to X_K \times X_K$$

is dominant. Because of the universal property of the Hom scheme, we have that $\mathrm{Hom}(C, X) \times_k K = \mathrm{Hom}(C_K, X_K)$ and $(\mathrm{ev}^d)_K = \mathrm{ev}_K^d$ so $ev^d : C \times C \times \mathrm{Hom}(C, X) \to X \times X$ is also dominant.

For (5) and (6), again we may assume X is projective. If through every two very general points there passes the image of C under some morphism, then the map $u^{(2)} : C \times C \times \mathrm{Hom}(C, X) \to X \times X$ is dominant. Since $\mathrm{Hom}(C, X)$ has at most countably many irreducible components and X is the union of uncountably many proper subvarieties, the restriction of $u^{(2)}$ to at least one of the components R_i must be dominant, which proves (5). Similarly for (6), after fixing an ample line bundle, let

$$\mathcal{M}_{g,0}(X) = \bigsqcup_{n \geq 0} \mathcal{M}_{g,0}(X, d) \subset \bigsqcup_{n \geq 0} \overline{\mathcal{M}}_{g,0}(X, d)$$

be the subspace of smooth curves of the moduli space from Theorem 2.1.5, where we have taken P, Q to be trivial. By taking a one point marking for Q we have a family $\mathcal{M}_{g,1}(X) \to \mathcal{M}_{g,0}(X)$ with an evaluation map to X. Since for every two very general points in X we can find a smooth irreducible curve of genus g passing through them, the map

$$\begin{tikzcd} \mathcal{M}_{g,1}(X) \times_{\mathcal{M}_{g,0}(X)} \mathcal{M}_{g,1}(X) \arrow[r] \arrow[d] & X \times X \\ \mathcal{M}_{g,0}(X) & \end{tikzcd}$$

must be dominant. Arguing now with this family as we did for (5), the result follows.

For (7) and (8) we may assume that X is projective from Chow's Lemma. For (7) we include the proof from [Deb01] 4.4.(5). Namely let $X' \to X$ be a connected finite étale cover. Since X is proper, from (2) we know that there exists a d such that $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathrm{Hom}_d(\mathbb{P}^1, X) \to X \times X$ is dominant. Since $\mathbb{P}^1$ is étale simply connected, every morphism $\mathbb{P}^1 \to X$ lifts to a morphism $\mathbb{P}^1 \to X'$ of the same degree, so the morphism $\mathrm{Hom}_d(\mathbb{P}^1, X') \to \mathrm{Hom}_d(\mathbb{P}^1, X)$ given by composing with $X' \to X$ is surjective and so

(7) follows. For (8), let $\mathcal{C} \to U$ be a family which makes X elliptically connected and let $X' \to X$ be a connected finite étale cover. Consider the pullback diagram and $\mathcal{C}' \to U' \to U$ the Stein factorisation

$$\begin{tikzcd}
 & \mathcal{C}' = \mathcal{C} \times_X X' \arrow[dl] \arrow[d] \arrow[r] & X' \arrow[d] \\
U' \arrow[dr] & \mathcal{C} \arrow[r] \arrow[d] & X \\
 & U. &
\end{tikzcd}$$

After possibly restricting U' to the open subset of curves in $\mathcal{C}'$ which are irreducible, the family $\mathcal{C}' \to U'$ makes X' elliptically connected. $\square$

Remark 2.2.14. Note from part (1) above that being genus g connected or C-connected is a birational invariant.

Proposition 2.2.15. *Let X be a smooth projective variety over an algebraically closed field k and $f : X \to C$ a flat morphism to a smooth projective curve whose geometric generic fibre is separably rationally connected. Then X is C-connected.*

Proof. From Theorem 1.3, there is a section $\sigma : C \to X$ to f. Now from Theorem 2.1.9 we can find a section to f passing through any two points in different fibres over C, hence we can find a copy of C passing through two very general points. The result now follows from Lemma 2.2.13 parts (5) and (6) above after possibly passing to an uncountable extension K/k. $\square$

Proposition 2.2.16. *Let X be a C-connected variety of dimension at least 3 over an algebraically closed field k. Then X is uniruled.*

Proof. We can compactify X (as in Remark 2.2.12) and assume it is projective. Let $u : C \times Y \to X$ be a family such that $u^{(2)} : C \times C \times Y \to X \times X$ is dominant.

By dimension considerations, we have

$$\dim Y + 2 \geq 2 \dim X$$

from which $\dim Y \geq 2 \dim X - 2$ and so if $\dim X \geq 3$ we obtain $\dim Y \geq 4$. Now, pick general points $x \in X, c \in X$ and denote by $Z \subset Y$ the locus of curves $u_z : C_z \to X$ such that $x = u_z(c)$ for all $z \in Z$. We have that $\dim Z \geq \dim Y - (\dim X - 1) - 1$ and so for $\dim X \geq 3$, $\dim Z \geq 1$. From the Bend and Break Lemma 1.6 we obtain a rational curve through every general point. After possibly an extension to an uncountable algebraically closed field this implies that X is uniruled (see [Deb01] Remark 4.2(5)). $\square$

We should note that if $\dim X = 2$ and $\dim Y \geq 3$ then the same method proves X is uniruled. If C is rational the result is obvious in all dimensions, whereas if C is elliptic, it holds in dimensions at least 2 from Proposition 3.2.

For a normal and proper variety X over a field k, a rationally chain connected fibration is a rational map $\pi : X \dashrightarrow Z$ such that the restriction to a dense open is a proper morphism $\pi^0 : X^0 \to Z$ such that a very general fibre is rationally chain connected and $\pi^0_* \mathscr{O}_{X^0} = \mathscr{O}_Z$. After throwing away the bad locus, we can assume that Z is a normal variety and by Nagata's compactification we can also assume it is proper over k. In fact there is a construction giving a universal fibration given by the following theorem.

Theorem 2.2.17. *([KMM92c]) Let X be a normal and proper variety over a field k. Then there exists a rationally chain connected fibration $\pi : X \dashrightarrow R(X)$ which is maximal (called the MRC fibration), in the sense that if $\pi' : X \dashrightarrow Z$ is another RCC fibration then there is a unique rational map $\tau : Z \dashrightarrow R(X)$ such that $\pi = \pi' \circ \tau$.*

In characteristic zero after blowing up the indeterminacy locus (which is of codimension at least 2 - see [Deb01] 1.39) we get a birational morphism $X' \to X$ with a dominant morphism $X' \to R(X)$ such that the general fibre is rationally chain connected, and by generic smoothness we can assume that the general fibre is in fact smooth. By the theorem below, by smoothing chains of rational curves into rational curves ([Deb01] 4.6-7) we can in fact assume that the general fibre of the MRC fibration is rationally connected. Note that in positive characteristic we can have families where the general fibre is not even normal.

Theorem 2.2.18. *([KMM92c] 2.1) Let X be a smooth proper variety over an algebraically closed field of characteristic zero, then X is rationally chain connected if and only if X is rationally connected.*

Denoting $R^0(X) = X$, $R^1(X) = R(X)$, $R^2(X) = R(R(X))$ and so on, we can take successive MRC quotients and obtain a tower of MRC fibrations

$$X \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X).$$

Since X is finite dimensional, this tower must eventually stabilise. In characteristic zero, we in fact have $R(X) = \ldots = R^n(X)$ (see Corollary 2.2.20). In positive characteristic it can be shown that the tower has length greater than one - see the example given in Example 5.7.

Proposition 2.2.19. *Let X be a normal and proper C-connected variety of dimension $\dim X \geq 3$ over an algebraically closed field where C is a smooth projective curve. Then the tower $X \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X)$ of MRC quotients terminates in either a point or a curve.*

Proof. Let $C \times Y \to X$ be a family which makes X a C-connected variety. The map $\pi^0 : X \dashrightarrow R(X)$ is dominant. From Lemma 2.2.13 part (1) it follows that $R(X)$ is also C-connected. Using the same method we obtain that $R^i(X)$ are all C-connected. Hence from Proposition 2.2.16 and the comment following its proof, if $\dim R^i(X) \geq 2$, it is uniruled. This implies that $R^{i+1}(X)$ must have dimension strictly less than $R^i(X)$ and so the result follows. $\square$

Note that if k is algebraically closed of characteristic zero then we know from the Graber-Harris-Starr Theorem 1.4 that the MRC quotient $R^1(X)$ is not uniruled, so if X is C-connected as in Proposition 2.2.19, $R^1(X)$ must be either a curve or a point and one does not need to resort to taking successive quotients. Also, again in characteristic zero, from Theorem 1.3 it follows that the composition of two rationally connected fibrations is a rationally connected fibration, so in combination with Theorem 2.2.18 we obtain the following corollary.

Corollary 2.2.20. *Let X be a smooth projective variety over k an algebraically closed field of characteristic zero and assume that the tower of MRC quotients $X \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X)$ terminates with $R^n(X)$ being a point. Then X is rationally connected.*

Thus if X is C-connected with $\dim X \geq 3$ over an algebraically closed field of characteristic zero where C is a smooth projective curve, we either have that X is rationally connected or that it is a rationally connected fibration over a curve. From Proposition 2.2.15 the converse holds too. However we expect that this is not the case in positive characteristic.

Remark 2.2.21. As observed in [Occ06] Remark 4, over an algebraically closed field of characteristic zero, if the MRC quotient of a smooth projective variety X is a curve C, then the MRC fibration extends to the whole variety and coincides with the Albanese map. To see this, let $\pi : X \dashrightarrow C$ be the MRC fibration. We know that the Albanese map contracts all rational curves so we have a commutative diagram

$$\begin{tikzcd} X \arrow[d, dashed] \arrow[dr] & \\ C \arrow[r, dashed] & B \end{tikzcd}$$

where $B \subset \operatorname{Alb} X$ is the image of the Albanese morphism. Hence B must be a curve, and so the fibres of the Albanese morphism $X \to \mathrm{Alb}\, X$ are connected (see [Uen75] 9.19). We conclude that B is isomorphic to C.

2.3 Free morphisms

We work with varieties over an algebraically closed field k of arbitrary characteristic. In this section we define ways in which a morphism from a curve C to a variety X can deform enough to give a large family of morphisms from C so as to cover X.

A notion studied extensively by Hartshorne [Har70] is that of a (local complete intersection) subvariety Y in a smooth projective variety X such that the normal bundle $\mathscr{N}_{Y/X}$ is ample. In the case of curves we have the following lemma.

Lemma 2.3.1. *Let X be a smooth projective variety over an algebraically closed field k. Then for some $g \geq 0$ there exists a curve $C \subset X$ of genus g such that $\mathscr{N}_{C/X}$ is ample.*

Proof. See [Har70] III.4. By taking successive ample hyperplane sections $H_1 \cap \ldots \cap H_r$ given by Bertini's Theorem we obtain a curve C whose normal bundle decomposes as

$$\mathscr{N}_{C/X} = \oplus_i \mathscr{N}_{H_i/X}|_C.$$

Since each of the summands is ample, so is $\mathscr{N}_{C/X}$. □

Remark 2.3.2. In [Ott12], Ottem defines an ample closed subscheme $Y \subset X$ of codimension r to be one where the exceptional divisor $\mathscr{O}(E)$ of the blowup $\mathrm{Bl}_Y X$ of X along Y is an $(r-1)$-ample line bundle in the sense that for every coherent sheaf $\mathscr{F}$ there is an integer $m_0 > 0$ such that $H^i(X, \mathscr{F} \otimes \mathscr{O}(E)^m) = 0$ for all $m > m_0$ and $i > r-1$. In the case where Y is a Cartier divisor, this notion coincides with the usual notion of Y being an ample divisor. One can then prove that if Y is a local complete intersection subscheme of X which is ample, then the normal bundle $\mathscr{N}_{Y/X}$ is an ample bundle.

We impose the following stronger positivity condition than that of Hartshorne above.

Definition 2.3.3. ([Kol96] II.3.1) Let C be a smooth proper curve and X a smooth variety over a field k. Let $f : C \to X$ be a morphism and $B \subset C$ a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. The morphism f is called free over g if it is non-constant and one of the following two equivalent conditions is satisfied:

1. for every $p \in C$ we have $H^1(C, f^*\mathscr{T}_X \otimes \mathscr{I}_B(-p)) = 0$ or,
2. $H^1(C, f^*\mathscr{T}_X \otimes \mathscr{I}_B) = 0$ and $f^*\mathscr{T}_X \otimes \mathscr{I}_B$ is generated by global sections.

That the two conditions are equivalent follows by taking cohomology of the short exact sequence

$$0 \to f^*\mathscr{T}_X \otimes \mathscr{I}_B(-p) \to f^*\mathscr{T}_X \otimes \mathscr{I}_B \to (f^*\mathscr{T}_X \otimes \mathscr{I}_B)_p \to 0.$$

Definition 2.3.4. We say that a curve $f : C \to X$ is r-free if for all effective divisors D of degree $r \geq 0$, $H^1(C, f^*\mathscr{T}_X \otimes \mathscr{O}_C(-D)) = 0$ and $f^*\mathscr{T}_X \otimes \mathscr{O}_C(-D)$ is generated by global sections. A 0-free curve is called free whereas a 1-free curve is called very free.

Remark 2.3.5. The condition of r-freeness makes formal the notion that the curve C deforms in X while keeping any general r points fixed.

The following lemma follows immediately from Lemma A.4.

Lemma 2.3.6. *If $f : C \to X$ is an r-free curve then f is r'-free for all $r' \leq r$.*

In the case of $C = \mathbb{P}^1$, Grothendieck's Theorem 1.1 gives us that $f^*\mathscr{T}_X = \oplus_{i=1}^n \mathscr{O}_{\mathbb{P}^1}(a_i)$ with $a_1 \leq \ldots \leq a_n$. From Corollary 1.2 it follows that for $r \geq 0$ the condition that

$H^1(\mathbb{P}^1, f^*\mathscr{T}_X(-r-1)) = 0$ is equivalent to that of global generation of $f^*\mathscr{T}_X(-r)$ (see also [Deb01] Remark 4.6). We deduce that $f : \mathbb{P}^1 \to X$ is r-free if and only if $a_1 \geq r$.

Remark 2.3.7. By Riemann-Roch, if $f : C \to X$ is a free genus g curve on X of dimension n and $\deg f^*\mathscr{T}_X = -C.K_X = d$, then $h^0(f^*\mathscr{T}_X) = d + n(1-g)$.

We also give a relative version of the above definitions, which will be used in combination with the main deformation Theorem 2.1.1 given above. Note that these have already been discussed in [KSCT07].

Definition 2.3.8. Let $\mathcal{C} \to Y$ be a flat proper family of curves where Y is a variety over an algebraically closed field, and let X be a smooth Y-variety. Let $B \subset \mathcal{C}$ be a closed subscheme, finite and flat over Y and $g : B \to X$ a Y-morphism. We say that a Y-morphism $f : \mathcal{C} \to X$ is relatively free over g if $f|_B = g$, $f^*\mathscr{T}_{X/Y} \otimes \mathscr{I}_B$ is globally generated and $H^1(\mathcal{C}, f^*\mathscr{T}_{X/Y} \otimes \mathscr{I}_B) = 0$.

Proposition 2.3.9. *([KSCT07] Proposition 14, [Kol96] II.3.5–3.7) Let $f : \mathcal{C} \to X$ as above be a morphism which is free over $f|_B$ and assume that $\dim Y = 1$ and that $\mathcal{C} \to Y$ is surjective. Then*

1. *The connected component $\mathrm{Hom}_{[f]}(\mathcal{C}/Y, X/Y, f|_B) \subset \mathrm{Hom}(\mathcal{C}/Y, X/Y, f|_B)$ containing f is smooth at $[f]$.*

2. *The evaluation morphism $\mathcal{C} \times \mathrm{Hom}_{[f]}(\mathcal{C}/Y, X/Y, f|_B) \to X$ dominates X.*

3. *If $Z \subset X$ is any subset of codimension at least 2, and $[f'] \in \mathrm{Hom}_{[f]}(\mathcal{C}/Y, X/Y, f|_B)$ a general point, then $(f')^{-1}(Z) \subset B$.*

Remark 2.3.10. We should remark at this point that there do not exist complete intersection curves of large enough degree which are free on a general smooth hypersurface. For example, let X be a degree d smooth hypersurface in $\mathbb{P}^n$. Assume $d \leq n$ since otherwise X will be of general type or Calabi-Yau and will not have any free curves. Let Y_i be $n-2$ suitably general hypersurfaces in $\mathbb{P}^n$ all of degree e and let $C = X \cap_{i=1}^{n-2} Y_i$ be the resulting curve. The degree of C is de^{n-2} and we have a short exact sequence

$$0 \to \mathscr{T}_C \to \mathscr{T}_X|_C \to \mathscr{N}_{C/X} \to 0$$

where the normal bundle is

$$\mathscr{N}_{C/X} = \oplus_{i=1}^{n-2} \mathscr{O}_{\mathbb{P}^n}(Y_i)|_C = \oplus_{i=1}^{n-2} \mathscr{O}_{\mathbb{P}^n}(e)|_C.$$

By adjunction, since C is a complete intersection curve, we compute

$$\deg \mathscr{T}_C = -\deg \omega_C = -d(-n-1+d+\sum_{i=1}^{n-2} e).$$

Even setting $e = 1$ to make $\deg \mathscr{T}_C$ as large as possible, and taking into account that $\deg \mathscr{N}_{C/X} = e(n-2)$, we see that $\deg \mathscr{T}_X|_C = \deg \mathscr{T}_C + \deg \mathscr{N}_{C/X}$ is not going to be positive for large values of d and n. Positivity of the degree of $\mathscr{T}_X|_C$ would be necessary for any ampleness conditions.

The following result gives that a general deformation of a free curve is an embedding. We will often use this implicitly throughout the remainder of this section.

Theorem 2.3.11. *([Kol96] II.1.8) Let $f : C \to X$ be a morphism from a smooth projective curve to a smooth variety over k. Let $B \subset C$ be a subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$.*

1. *If $h^1(C, f^*\mathscr{T}_X \otimes \mathscr{I}_B(-2p) \leq \dim X - 2$ for every $p \in C$, then a general deformation of f over g is an immersion on $C \setminus B$. Furthermore, if g is an immersion, then a general deformation of f over g is an immersion on C.*

2. *If $h^1(C, f^*\mathscr{T}_X \otimes \mathscr{I}_B(-p-q) \leq \dim X - 3$ for every $p, q \in C$, then a general deformation of f over g is an embedding on $C \setminus B$. Furthermore, if g is an*

embedding, then a general deformation of f over g is an embedding on C.

Hence if the dimension of X is at least 3, a general deformation of a 2-free morphism is an embedding into X. We will see (Theorem 2.3.14) that in fact this holds for any free morphism too. We also have that being free over g is an open property by the following result.

Proposition 2.3.12. *([Kol96] II.3.2) Let $\mathcal{C} \to S$ be a proper flat family of curves and $X \to S$ a smooth morphism. Let $B/S \subset \mathcal{C}/S$ be a closed subscheme, flat over S. Let $F : \mathcal{C} \to X$ be a morphism over S and $G = F|_B$. Assume that for some $s \in S$ the induced morphism $F_s : \mathcal{C}_s \to X_s$ is free over G_s. Then there is an open neighbourhood $U \subset S$ of s such that $F_t : \mathcal{C}_t \to X_t$ is free over G_t for every $t \in U$.*

There is another type of positive curve one can consider for a smooth projective variety X, namely $f : C \to X$ such that $f^*\mathscr{T}_X$ is ample. Note that such a curve automatically has $\mathscr{N}_{C/X}$ ample as in Lemma 2.3.1 since the quotient of an ample bundle is ample in the following short exact sequence

$$0 \to \mathscr{T}_C \to f^*\mathscr{T}_X \to \mathscr{N}_{C/X} \to 0.$$

We will prove that the existence of a curve of genus $g \geq 1$ such that $f^*\mathscr{T}_X$ is ample is in fact equivalent to the existence of a free curve of the same genus.

Proposition 2.3.13. *Let X be a smooth projective variety over an algebraically closed field k and $f : C \to X$ a morphism from a smooth projective curve such that $f^*\mathscr{T}_X$ is ample. Then for any closed subscheme $B \subset C$ with ideal sheaf $\mathscr{I}_B$, there exists a morphism $f' : C \to X$ which is free over $\mathscr{I}_B$.*

Proof. If the characteristic of k is positive, then the result follows from Proposition A.9 by pulling back by Frobenius as we will see as a special case in the following. Now, assume the characteristic of k is 0, denote by H_X an ample divisor on X and consider the flat proper models $\mathcal{X}, \mathcal{C}, \mathcal{B}, H_{\mathcal{X}}$ of X, C, B, H_X respectively over the algebra R which is $\mathbb{Z}$ adjoined all the coefficients defining C, X etc. and all the morphisms between them, as in [Kol96] II.5.10. Denote by g_R the morphism $\mathcal{B} \to \mathcal{X}$. From the construction, the geometric generic fibre of $\mathcal{X}$ is X and after shrinking $\operatorname{Spec} R$ we can assume $\mathcal{C}$ is smooth and the same for $\mathcal{X}$ over the image of $f_R : \mathcal{C} \to \mathcal{X}$. Note also that after further suitably shrinking $\operatorname{Spec} R$ we can assume $f_R^* \mathscr{T}_{\mathcal{X}}$ is a locally free relatively ample bundle on $\mathcal{C}$ over R and the same for $H_{\mathcal{X}}$ on $\mathcal{X}$. Now, for any closed fibre $\mathcal{X}_p$ over a maximal p in R, the reduction of $f_p : \mathcal{C}_p \to \mathcal{X}_p$ will also have $f_p^* \mathscr{T}_{\mathcal{X}_p}$ ample and from Proposition A.9 we have (after pulling back by Frobenius) that there exists a morphism $f'_p : \mathcal{C}_p \to \mathcal{X}_p$ which is free over $\mathscr{I}_{\mathcal{B}_p}$. We can perform this operation for every closed point p in $\operatorname{Spec} R$ and note that from the proof of Proposition A.9 there is a global bound d on the intersection number $(f'_{p*}\mathcal{C}_p).H_{\mathcal{X}_p}$ depending on the genus g of C and the length of B. Considering now the relative moduli space $\operatorname{Hom}_R(\mathcal{C}, \mathcal{X}, g_R)$ and its subset $\rho : \operatorname{Hom}_R^{\text{free}, \leq d}(\mathcal{C}, \mathcal{X}, g_R) \to \operatorname{Spec} R$ of Proposition 2.3.9 containing curves which are free over g_R and of relative $H_{\mathcal{X}}$-degree at most d, we see that ρ has non-trivial fibres over the closed points, but these are dense in $\operatorname{Spec} R$. We also know that $\operatorname{Hom}_R^{\text{free}, \leq d}(\mathcal{C}, \mathcal{X}, g_R)$ is of finite type over R so by Chevalley's Theorem, the image of ρ is constructible and since it is dense, it must contain the generic point (see [Har77] ex. II.3.18) hence the morphism ρ must be surjective. Hence there exists a suitably free curve over the generic fibre which means a morphism $C \to X$ which is free over $\mathscr{I}_B$. We conclude that the curve $f'_p : C_p \to X_p$ lifts to a curve which is free over $\mathscr{I}_B$ in characteristic zero. $\square$

A closer look at the proof of Proposition A.9 shows that proving global generation and vanishing of first cohomology is just a matter of degrees, hence if $f : C \to X$ a curve such that $f^* \mathscr{T}_X$ is ample, then from the above proposition there is a morphism $f' : C \to X$ such that f' is r-free for any $r \geq 0$. Conversely from Lemma 2.3.6, if C is r-free for some $r \geq 0$ then it is free and from Proposition A.3 it follows that $f^* \mathscr{T}_X$

is ample. The above results and discussion lead to the following.

Theorem 2.3.14. *Let X be a smooth projective variety and C a smooth projective curve of genus $g \geq 1$ over an algebraically closed field k. Then for any $r \geq 0$, there exists an $f : C \to X$ which is r-free if and only if there exists a morphism $f' : C \to X$ such that $f'^*\mathscr{T}_X$ is ample.*

Note that the above theorem is not true if C was a rational curve, namely there exist varieties which have a free rational curve (uniruled varieties) which do not have a very free rational curve (rationally connected varieties) - see Theorem 2.3.23.

Remark 2.3.15. Assume $g : \mathbb{P}^1 \to X$ is such that $g^*\mathscr{T}_X$ is ample and compose with a separable finite morphism $f : C \to \mathbb{P}^1 \to X$ where C is smooth of genus g. Since $C \to \mathbb{P}^1$ is finite, the bundle $f^*\mathscr{T}_X$ is also ample, hence from the above, the existence of a very free rational curve implies the existence of a very free curve of any genus g. Moreover, if $\dim X \geq 3$ we can assume from Theorem 2.3.11 that f has deformations which are embeddings. The case where $\dim X = 2$ follows from the fact that separably rationally connected surfaces are rational.

Lemma 2.3.16. *Let X be a smooth variety over an algebraically closed field k and $f : C \to X$ a free morphism from a smooth projective curve over k. Then the image of C misses only finitely many divisors in X.*

Proof. From Proposition 2.3.25 below, we know that X is separably C-connected, so there exists a variety Y such that $u : C \times Y \to X$ and $u^{(2)} : C \times C \times Y \to X \times X$ are dominant. Let U be an open subset in the image of u. Since the algebraically equivalent cycles which are the images of C_y under u are numerically equivalent, the images of deformations of f, and hence the image of f, may not intersect with only the complement of U. The complement though is a proper closed subset of X and so contains a finite number of irreducible closed subsets so a finite number of divisors. $\square$

Lemma 2.3.17. *Let X be a smooth variety over an algebraically closed field k with $D \subset X$ a divisor and $f : C \to X$ a free morphism from a smooth projective curve over k and $p \in C$. Then there exists a deformation $f' : C \to X$ with $f'(p) \notin D$.*

Proof. By semicontinuity let $U \subset \operatorname{Hom}(C, X)$ be a connected open neighbourhood of $[f]$ such that $H^1(C, f_t^* \mathscr{T}_X) = 0$ for all $[f_t] \in U$. From Theorem 2.1.1 it follows that the dimension of U is $h^0(C, f^* \mathscr{T}_X)$. Denote by $\mathscr{I}_p$ the ideal sheaf on C of the closed subscheme with unique point p. Since f is free, we have $H^1(C, f_t^* \mathscr{T}_X \otimes \mathscr{I}_p) = 0$ for all $[f_t] \in U$ and so by fixing a point $x \in X$ such that $p \mapsto x$, we have

$$\begin{aligned} \dim(\operatorname{Hom}(C, X; p \mapsto x) \cap U) &= h^0(C, f^* \mathscr{T}_X \otimes \mathscr{I}_p) \\ &= h^0(C, f^* \mathscr{T}_X) - \dim X \\ &= \dim U - \dim X. \end{aligned}$$

Next, denote by

$$V = \{[f_t] \in U \mid f_t(p) \in D\} = \bigcup_{x \in D} \{[f_t] \in U \mid f_t(p) = x\}$$

the subspace of all morphisms in U which send p to a point in the divisor D. It follows that

$$\begin{aligned} \operatorname{codim}(V, U) &\geq \dim U - \dim V \\ &= h^0(C, f^* \mathscr{T}_X) - (h^0(C, f^* \mathscr{T}_X) - \dim X + \dim X - 1) \\ &= 1 \end{aligned}$$

and hence there exists an $[f'] \in U \setminus V$ such that $f'(p) \notin D$. $\square$

Remark 2.3.18. Note that this does not hold for C-connected varieties since it may be that $H^1(C, f^* \mathscr{T}_X \otimes \mathscr{I}_p) \neq 0$ so when we fix a point in the source and target, the dimension of the space of morphisms does not necessarily decrease by $\dim X$. Consider for example $C \to C \times \mathbb{P}^n$.

A similar proof leads to the following result, stating that a general deformation of a free curve will miss a codimension two and above subset.

Lemma 2.3.19. *([Kol96] II.3.7) Let X be a smooth projective variety over an algebraically closed field k and $f : C \to X$ a morphism from a smooth projective curve which is free over $g : B \to X$. Let $Z \subset X$ be a subscheme of codimension at least 2. Then $f'(C \setminus B) \cap Z = \emptyset$ for a general deformation f' of f over g.*

Proposition 2.3.20. *Let X be a smooth variety over an algebraically closed field k and $f : C \to X$ a smooth projective curve which is free over $B \subset C$ a closed subscheme with ideal sheaf $\mathscr{I}_B$. Let $g : X \dashrightarrow Y$ be a rational map to a smooth proper variety Y. Then it follows that $f' := g \circ f : C \dashrightarrow Y$ can be deformed to a morphism free over B if either of the following two conditions holds*

1. *the characteristic of k is 0 or*
2. *the characteristic of k is positive and g is generically smooth.*

Proof. First note that in any characteristic we can deform $f : C \to X$ so that it misses the codimension 2 exceptional locus of g (from Lemma 2.3.19) so we can assume that the composition $g \circ f : C \dashrightarrow Y$ is in fact a morphism. We also have an exact sequence of sheaves on X

$$0 \to \mathscr{T}_{X/Y} \to \mathscr{T}_X \to g^* \mathscr{T}_Y$$

and applying f^* and tensoring with $\mathscr{I}_B$ we obtain the following exact sequence

$$0 \to f^* \mathscr{T}_{X/Y} \otimes \mathscr{I}_B \to f^* \mathscr{T}_X \otimes \mathscr{I}_B \to (g \circ f)^* \mathscr{T}_Y \otimes \mathscr{I}_B. \tag{2.1}$$

If k has characteristic zero then by generic smoothness (see [Har77] III.10) we may assume that g is a smooth morphism and the above sequences are exact on the right. We can now apply Lemma A.5 to conclude that $(g \circ f)^* \mathscr{T}_Y \otimes \mathscr{I}_B$ is globally generated and has $H^1(C, (g \circ f)^* \mathscr{T}_Y \otimes \mathscr{I}_B) = 0$. Assume now that the characteristic of k is positive and that $g : X \dashrightarrow Y$ is generically smooth. There exists an open $U \subset X$ such that $g|_U : U \to Y$ is smooth, and since f is free, we can find a deformation whose image misses the complement of U but for a finite number of points. Hence we have an exact sequence

$$f^* \mathscr{T}_X \to (g \circ f)^* \mathscr{T}_Y \to \mathscr{Q} \to 0$$

where $\mathscr{Q}$ is a torsion sheaf supported on this finite set of points. This implies that $f^* \mathscr{T}_X \to (g \circ f)^* \mathscr{T}_Y$ has full rank (i.e. its image has the same rank as the target). The result now follows from Lemma A.5. $\square$

Proposition 2.3.21. *([Kol96] II.3.5) Let C be a smooth proper curve, $p, q \in C$ and X a smooth variety and $[f] \in \operatorname{Hom}(C, X, g)$ a smooth point. Using notation of the evaluation morphisms from the beginning of this section we have*

1. *if $\phi(p, f)$ is surjective then F is smooth at $(p, [f])$. The converse holds if $H^0(C, \mathscr{T}_C \otimes \mathscr{I}_B) \to \mathscr{T}_C \otimes k(p)$ is surjective.*
2. *If $\phi^{(2)}(p, q, f)$ is surjective then $F^{(2)}$ is smooth at $(p, q, [f])$. The converse holds if $H^0(C, \mathscr{T}_C \otimes \mathscr{I}_B) \to \mathscr{T}_C \otimes k(p) \oplus \mathscr{T}_C \otimes k(q)$ is surjective.*

Proof. From [Kol96] Proposition II.3.4, we have that

$$\begin{aligned} dF(p, [f]) &= df(p) + \phi(p, f) \\ df^{(2)}(p, q, [f]) &= (df(p) + \phi(p, f), df(q) + \phi(q, f)). \end{aligned}$$

Since these are all vector spaces, if $\phi(p, f)$ is surjective, so is $dF(p, [f])$ and so F is smooth at $(p, [f])$ and the same holds for $dF^{(2)}(p, q, [f])$.

Assume now that F is smooth at $(p, [f])$. Since $dF(p, [f])$ and $df(p)$ are surjective (p is a smooth point on C), we have that $\phi(p, f)$ is surjective as a map to

$$f^* \mathscr{T}_X \otimes k(p) / \operatorname{im} (\mathscr{T}_C \otimes k(p) \to f^* \mathscr{T}_X \otimes k(p)) .$$

From the assumption, we have that $\mathscr{T}_C \otimes k(p)$ is in the image of $\phi(p, f)$ and so that $\phi(p, f)$ is surjective. The proof for $F^{(2)}$ is similar. $\square$

Corollary 2.3.22. *If C is smooth and $f : C \to X$ is free over $g : B \to X$, then $F : C \times \operatorname{Hom}(C, X, g) \to X$ is smooth along $(C \setminus B) \times [f]$.*

We summarise the known results for rationally connected varieties in the following theorem which is essentially a converse to the above corollary. The proofs are mostly from [KMM92c] and elaborated on in [Deb01] for example.

Theorem 2.3.23. *Let X be a smooth projective variety over an algebraically closed field k. If the characteristic of k is zero then the following are equivalent*

1. *There exists a very free morphism $f : \mathbb{P}^1 \to X$.*
2. *X is $\mathbb{P}^1$-connected.*
3. *X is genus 0 connected.*
4. *X is genus 0 chain connected.*

If the characteristic of k is positive, then $(1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$. In any characteristic, if we furthermore assume separability in $(2), (3)$ and (4) then all statements are equivalent to (1).

Similarly, if k is algebraically closed of any characteristic, X is separably uniruled if and only if there exists a free rational curve $f : \mathbb{P}^1 \to X$.

Remark 2.3.24. An example of a smooth projective variety in positive characteristic p which is rationally connected but does not have any free rational curves (hence no very free rational curves either) is given by the Fermat hypersurface $x_0^d + x_1^d + \ldots + x_N^d$ where $N \geq 3$ and $d = p^r + 1$ (see [Deb01] 4.4).

In the case of higher genus curves though there exist genus g connected varieties which do not have a free or very free curve for all $g \geq 1$, for example consider $E \times \mathbb{P}^1$ where E is an elliptic curve. We will prove later in Theorem 3.3 that $E \times \mathbb{P}^1$ is elliptically connected yet it is not possible that there exists a morphism $f : C \to E \times \mathbb{P}^1$ from a curve C such that $f^*\mathscr{T}_{E\times\mathbb{P}^1}$ is ample since this bundle is isomorphic to $\mathscr{O}_C \oplus \mathscr{O}_C(2)$ which has a non-ample quotient $\mathscr{O}_C$. One can however prove the following proposition.

Proposition 2.3.25. *Let X be a smooth variety over an algebraically closed field and $f : C \to X$ a very free morphism for some smooth projective curve C. Then X is separably C-connected.*

Proof. Let $[f] \in Y \subset \mathrm{Hom}(C, X)$ be an open and smooth neighbourhood with cycle map $u : C \times Y \to X$. We first show that the evaluation map

$$\phi^{(2)}(p, q, f) : H^0(C, f^*\mathscr{T}_X) \to f^*\mathscr{T}_X \otimes k(p) \oplus f^*\mathscr{T}_X \otimes k(q)$$

is surjective for $p \neq q$ general points in C. Consider the following exact sequences of sheaves

$$0 \to f^*\mathscr{T}_X(-p-q) \to f^*\mathscr{T}_X \to (f^*\mathscr{T}_X \otimes k(p)) \oplus (f^*\mathscr{T}_X \otimes k(q)) \to 0$$
$$0 \to f^*\mathscr{T}_X(-p-q) \to f^*\mathscr{T}_X(-p) \to f^*\mathscr{T}_X(-p) \otimes k(q) \to 0$$

and note that by taking the long exact sequence in cohomology of the first, to show that $\phi^{(2)}(p, q, f)$ is surjective, it suffices to show that $H^1(C, f^*\mathscr{T}_X(-p-q)) = 0$.

Since f is very free we have from the second sequence that $H^0(C, f^*\mathscr{T}_X(-p)) \to f^*\mathscr{T}_X(-p) \otimes k(q)$ is surjective and also that $H^1(C, f^*\mathscr{T}_X(-p)) = 0$ from which it follows that $H^1(C, f^*\mathscr{T}_X(-p-q)) = 0$.

Since $\phi^{(2)}(p, q, f)$ is surjective, it follows from Proposition 2.3.21 that $u^{(2)} : C \times C \times Y \to X \times X$ is smooth at $(p, q, [f])$. We conclude that X is separably C-connected and thus also separably connected by genus g curves. □

Of course, if the genus of C is at least 1, from Theorem 2.3.14 it follows that the same result holds if there exists a free morphism and also one such that $f^*\mathscr{T}_X$ is ample.

2.4 Proving uniruledness and rational connectedness

In this section we prove that the existence of a free curve of genus at least 1 in characteristic zero implies the existence of a very free rational curve.

Proposition 2.4.1. *Let X be a smooth variety over an algebraically closed field k and $f : C \to X$ a free morphism for C a smooth projective curve of genus g. Then X is uniruled.*

Proof. Like in the proof of Proposition 2.2.16, we can assume that X is projective (note that by Lemma 2.3.19 we can find a free deformation of f which misses the exceptional locus of a birational map). The case of genus zero follows by definition so we may assume $g \geq 1$. From the Bend and Break Lemma 1.6, if we show that $\dim_{[f]} \operatorname{Hom}(C, X; f|_{\{c\}}) \geq 1$, then there will be a rational curve in X passing through $f(c)$. This condition is equivalent to $-K_X.C - g \dim X \geq 1$. Since f is free, Riemann-Roch gives

$$\begin{aligned} h^0(C, f^*\mathscr{T}_X) &= \deg f^*\mathscr{T}_X + (1-g)\dim X \\ &= -K_X.C + (1-g)\dim X \end{aligned}$$

and since $f^*\mathscr{T}_X$ is globally generated, we have $h^0(C, f^*\mathscr{T}_X) \geq d = \dim X$. In fact, consider d linearly independent sections $f_1, \ldots, f_d \in H^0(C, f^*\mathscr{T}_X)$ which generate f^*T_X at a point $p \in C$ and consider the short exact sequence given by these sections

$$0 \to \mathscr{O}_C^{\oplus d} \to f^*\mathscr{T}_X \to \mathscr{Q} \to 0,$$

where $\mathscr{Q}$ is a torsion sheaf. Pick a point q in the support of $\mathscr{Q}$ (such a point exists since $f^*\mathscr{T}_X$ is a non-trivial bundle since it has vanishing first cohomology) and consider the stalks for the above exact sequence at q. We deduce that $h^0(C, f^*\mathscr{T}_X) > d$. Hence we obtain $-K_X.C + (1-g)\dim X > \dim X$ from which $-K_X.C - g \dim X \geq 1$ and so Bend and Break applies. $\square$

From Lemma 2.3.6 we obtain that the same result holds if f was very free or such that $f^*\mathscr{T}_X$ was ample. Note also the above result follows (at least if $\dim X \geq 3$) as a combination of Theorem 2.3.14, Proposition 2.3.25 and Proposition 2.2.16, but the above proof is simpler.

Theorem 2.4.2. *Let X be a smooth projective variety over an algebraically closed field k and $f : C \to X$ a free morphism from a smooth projective curve of genus $g \geq 1$. Then the tower of MRC fibrations terminates with a point.*

Proof. Applying Theorem 2.3.14 twice we can find a very free morphism $f' : C \to X$. Replacing f by f', it follows from Proposition 2.3.25 that X is separably C-connected. Following the argument as in Proposition 2.2.19, by showing that at each step of the tower of MRC fibrations $X \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X)$ the target $R^i(X)$ is uniruled, it must be that $R^n(X)$ is either a point or a curve C'. In the latter case, call the dominant composition map $\pi : X \dashrightarrow C'$ where we can assume that C' is a smooth projective curve. By invoking Lemma 2.3.19 it follows that we can assume that C misses the codimension two exceptional locus of π and from Lemma 2.3.17, for a point $p \in C$, we can deform f so that the image of p misses the inverse image under π of $\pi(f(p))$. Hence there is an at least 1 dimensional family of non-constant morphisms from C to C' and from de Franchis' Theorem 2.4.3 below (see also [Kol96] II.1.9.2) it follows that C' has genus 0 or 1. Assume that C' has genus 1. We have $\mathscr{T}_{C'} = \mathscr{O}_{C'}$ and $f^*\pi^*\mathscr{O}_{C'} \cong \mathscr{O}_C$ and we thus obtain $\mathscr{O}_C$ as a quotient of $f^*\mathscr{T}_X$ since

$$f^*\mathscr{T}_X \to f^*\pi^*\mathscr{O}_{C'} \to 0$$

is exact. Taking cohomology we see that since f is free, one obtains a contradiction since $H^1(C, \mathscr{O}_C) \neq 0$. Finally note that $C' = \mathbb{P}^1$ is ruled out since we have assumed that the tower of MRC fibrations is maximal. $\square$

Theorem 2.4.3. *(de Franchis' Theorem [ACG11] Theorem 8.27) Let $g \geq 1, g' \geq 2, d \geq 1$ be integers and let C be a smooth curve of genus g. Then the following are true.*

1. *Up to isomorphism, there are only finitely many genus g' curves C' such that there exists a non-constant morphism $C \to C'$.*
2. *Up to isomorphism, for any smooth genus g' curve C', there are only finitely many non-constant morphisms $C \to C'$.*
3. *Up to isomorphism, there are only finitely many genus 1 curves E such that there exists a degree d morphism $C \to E$.*
4. *For any smooth genus 1 curve E, up to conjugation with a translation of E, there are only finitely many degree d morphisms $C \to$.*

Applying Corollary 2.2.20 we conclude the following.

Corollary 2.4.4. *In the above setup, if k has characteristic zero then X is rationally connected.*

We will see in the next section that a straightforward application of Mori's Bend and Break Lemma 1.6 in combination with the Graber-Harris-Starr Theorem 1.3 leads to proving that every elliptically connected variety over an arbitrary algebraically closed field of characteristic zero is either rationally connected or a rationally connected fibration over an elliptic curve (see Theorem 3.3). Assuming the ampleness of the restriction of a foliation to a smooth curve in characteristic zero, results of this type have been demonstrated in the work of various people. A short summary follows (see [MP97] for an introduction to foliations).

Theorem 2.4.5. *([BM01] Theorem 0.1) Let X be a normal complex projective variety and $C \subset X$ a complete curve in the smooth locus of X. Assume that $\mathscr{F} \subset \mathscr{T}_X$ is a foliation regular along C and such that $\mathscr{F}|_C$ is ample. If $x \in C$ is any point, the leaf through x is algebraic and if $x \in C$ is general then the closure of the leaf is also rationally connected.*

This result was reproved in [KSCT07] using a simpler Mori-type technique of reduction to characteristic p, proving uniruledness of the MRC quotient and deriving a contradiction using the Graber-Harris-Starr Theorem [GHS03], in a manner similar to our results. See also [KSC06].

Theorem 2.4.6. *([BDPP04] Corollary 0.3) Let $X/\mathbb{C}$ be a projective manifold. If K_X is not pseudo-effective then X is uniruled.*

Using this theorem, Peternell proved a weaker version of Mumford's conjecture on numerical characterisation of rationally connected varieties (Conjecture 1.5) from which one can deduce the following theorem.

Theorem 2.4.7. *([Pet06] 5.4, 5.5) Let $X/\mathbb{C}$ be a projective manifold and $C \subset X$ a possibly singular curve. If $\mathscr{T}_X|_C$ is ample then X is rationally connected. If $\mathscr{T}_X|_C$ is nef and $-K_X.C > 0$ then X is uniruled.*

Using either of Theorems 2.4.7 or 2.4.5, a smooth projective variety X over an algebraically closed field of characteristic zero with a free genus g curve $f : C \to X$ such that $g \geq 1$ is automatically rationally connected since from Proposition A.3 we have that $f^*\mathscr{T}_X$ is ample. Similarly, from Proposition 2.3.13 and Corollary 2.4.4 we obtain the following.

Theorem 2.4.8. *Let X be a smooth projective variety over an algebraically closed field k of characteristic zero and assume that there exists a smooth curve $f : C \to X$ of genus g such that $f^*\mathscr{T}_X$ is ample. Then X is rationally connected.*

Remark 2.4.9. At this point we do not know whether the above statement holds in characteristic p or whether, still in positive characteristic, assuming that we have a free curve $f : C \to X$ of genus $g \geq 1$ implies that X is separably rationally connected or even rationally connected. It is tempting to hope that both statements are true though. Jason Starr informs us that his maximal free rational quotient (MFRC) [Sta06] gives a generically (on the source) smooth morphism $X \to R_f(X)$ over any algebraically closed field k, so if X contained a free rational curve $f : \mathbb{P}^1 \to X$, then $\dim R_f(X) < \dim X$. Hence, if $f : C \to X$ a free curve of genus $g \geq 1$ implied that we have a free rational curve $\mathbb{P}^1 \to X$ (we do not know how to show this), taking successive MFRC quotients and using Proposition 2.3.20 would reduce the tower of MFRC quotients to a point. This does not mean that X will necessarily be rationally connected, but since there is a free rational curve on X, it will at least be separably uniruled. Even though Bend and Break arguments give us the existence of many rational curves, the author does not know any techniques to construct free rational curves in positive characteristic.

Chapter 3

Elliptically connected varieties

Let k be an algebraically closed field of arbitrary characteristic. In this section we will study more carefully the cases of genus 1 connected (elliptically connected), E-connected varieties and free morphisms $E \to X$ where E is an elliptic curve.

Denoting RC and EC to mean rationally and elliptically connected respectively, it follows from Lemma 2.2.9 that we have the following inclusions of sets of varieties

$$\{\text{rational}\} \subsetneq \{\text{unirational}\} \subseteq \{\text{RC}\} \subsetneq \{\text{EC}\} \subsetneq \{\text{uniruled}\}.$$

It is an open problem whether there exists a non-unirational rationally connected variety but it is widely expected these do exist. An example of an elliptically but not rationally connected variety is given by $E \times \mathbb{P}^n$ as we saw in the discussion after Definition 2.2.10. A more substantial example of an elliptically connected variety which is not E-connected for some genus 1 curve is given at the end of this section. We begin with a Bend and Break lemma.

Lemma 3.1. *Let $f : E \to X$ be a morphism from a smooth genus 1 curve E to a smooth projective variety X over an algebraically closed field k and let $e \in E$. Assume that $-K_X.f_*E > 0$ (for example $-K_X$ ample). Then there exists a rational curve on*

X through $f(e)$.

Proof. Mori's Bend and Break Lemma 1.6 tells us that the conclusion of this lemma holds if the condition $\dim_{[f]} \operatorname{Hom}(E, X; f|_e) \geq 1$ holds. By [Deb01] 2.11 this is equivalent to

$$-K_X.f_*E - \dim X \geq 1.$$

From the projection formula, the integer $-K_X.f_*E$ is equal to $f^*(-K_X).E$. Now we can consider $[n] : E \to E$ the multiplication by n isogeny on the elliptic curve E and precompose $E \to X$ by this isogeny thus arbitrarily increasing $-K_X.f_*E$ the anti-canonical degree of E. $\square$

Proposition 3.2. *Let X be an elliptically connected smooth projective variety. Then*

1. *if X is a surface it is either rationally connected or ruled, in particular it is uniruled,*
2. *if $\dim X \geq 3$ then X is uniruled.*

Proof. Let $\mathcal{E} \to Y$ be a family of genus 1 curves which makes X elliptically connected. For (i), assume that X is a smooth elliptically connected surface. It suffices to work with a minimal model of X. Pick x a point on X and $f : C \to X$ a smooth irreducible genus 1 curve whose image passes through x. Since elliptic curves have trivial canonical bundle, the converse condition in Proposition 2.3.21 (1) is satisfied and it follows that $f^*\mathscr{T}_X$ is globally generated at x. This holds for all $x \in C$ so $f^*\mathscr{T}_X$ is a globally generated vector bundle. Now, we can assume C is smoothly immersed in X and consider the short exact sequence

$$0 \to \mathscr{T}_C \to \mathscr{T}_X|_C \to \mathscr{N}_{C/X} \to 0$$

where $\mathscr{T}_C = \mathscr{O}_C$ is trivial and $\mathscr{N}_{C/X}$ is the normal line bundle of C in X. Since $\mathscr{T}_X|_C$ and $\mathscr{T}_C$ are globally generated, it follows that $\deg \mathscr{N}_{C/X} > 0$. From the adjunction formula $2g_C - 2 = C.(C + K_X)$ and so $C.K_X = -C^2 = -\deg \mathscr{N}_{C/X} < 0$ from which it follows that K_X is not nef. From the classification of surfaces (see for example [Fri98] Theorem 10.4), this implies that X is either rational or ruled and hence uniruled. Alternatively, since $-C.K_X > 0$ we can apply Bend and Break as in Lemma 3.1 and produce a rational curve through every point of X.

For (ii), we apply Bend and Break as follows. Fix a general point $x \in X$. From Lemma 2.2.2, we know that no fibre of $\mathcal{E} \to Y$ is contracted in X. Hence, since $\mathcal{E} \times_Y \mathcal{E}$ dominates $X \times X$ we know that $\dim Y \geq 2 \dim X - 2$. Now, the subspace Z in Y of curves which go through x has dimension at least $\dim Y - (\dim X - 1)$ and so the subspace Z' in Z of curves of fixed j-invariant that go through x has dimension $\dim Z' \geq \dim Y - (\dim X - 1) - 1$ or is empty. Hence $\dim Z' \geq \dim X - 2 \geq 1$ since X has dimension at least 3. From Lemma 1.6 it now follows that there is a rational curve through x. After possibly an extension to an uncountable algebraically closed field this implies that X is uniruled (see [Deb01] Remark 4.2(5)). $\square$

Theorem 3.3. *Let X be a smooth projective variety over an algebraically closed field k of characteristic zero. Then X is elliptically connected if and only if it is rationally connected or a rationally connected fibration over an elliptic curve.*

Proof. Consider the MRC fibration $\pi : X \dashrightarrow R(X)$. From Corollary 1.4, we know that $R(X)$ is not uniruled. On the other hand, we know from Lemma 2.2.13 part (1) that $R(X)$ is elliptically connected since π is dominant. Since $R(X)$ is elliptically connected and not uniruled, it follows from Proposition 3.2 that it must be either of dimension 0 and thus X is rationally connected, or of dimension 1 and so an elliptic curve E by the Riemann-Hurwitz Theorem. By blowing up the indeterminacy locus of the rational map $X \dashrightarrow R(X)$ we find a birational model X' of X such that $X' \to E$

has rationally connected general fibre. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ \operatorname{Alb} X \arrow[r] & E \end{tikzcd}$$

where $X \to \operatorname{Alb} X$ is the Albanese morphism (which contracts all rational curves in X) and $\operatorname{Alb} X \to E$ comes from the universality of the Albanese variety. We obtain that $\operatorname{Alb} X$ is an elliptic curve E' isogenous to E and that the fibres of $X \to E'$ are rationally connected.

Conversely, we have seen that a rationally connected variety is elliptically connected in Lemma 2.2.9. If on the other hand X is a rationally connected fibration over an elliptic curve E then from Proposition 2.2.15 we know that it is E-connected. $\square$

Remark 3.4. In case k was of positive characteristic, using the same methods as in Lemma 2.2.19 we deduce that for an elliptically connected variety X, the tower of the MRC fibration $X \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X)$ terminates with $R^n(X)$ a point or a curve.

We now give an example of a way of constructing elliptically connected varieties over an arbitrary field that are not rationally connected. Families of such objects were recently constructed by Bjorn Poonen [Poo10] as counterexamples to the sufficiency of the Brauer-Manin obstruction applied to étale covers and we outline the construction in the following paragraph.

Let k be any field and let $P_\infty(x), P_0(x)$ be two relatively prime and separable degree 4 polynomials in $k[x]$. Consider the vector bundle $\mathscr{L} = \mathscr{O}(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and define the section

$$s := u^2\widetilde{P}_\infty(w,x) + v^2\widetilde{P}_0(w,x) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{L}^{\otimes 2})$$

where $\widetilde{P}_\infty, \widetilde{P}_0$ are the homogenisations and (u,v) and (w,x) are the coordinates of each of the two copies of $\mathbb{P}^1$ respectively. For $a \in k^*$, the equation $y^2 - az^2 = s$ defines a conic bundle $V \to \mathbb{P}^1 \times \mathbb{P}^1$. Composing with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, the fibre in V above $\infty \in \mathbb{P}^1$ is the Châtelet surface $V_\infty : y^2 - ax^2 = P_\infty(x)$. Now let $f : E \to \mathbb{P}^1$ be a dominant map from an elliptic curve such that f is étale over the branch locus of the first projection of the zero locus of s to $\mathbb{P}^1$ (i.e. the locus $Z \in \mathbb{P}^1 \times \mathbb{P}^1$ over which the fibres in V split into a nodal union of two lines). We have a pullback diagram

$$\begin{tikzcd}
X := V \times_{\mathbb{P}^1} E \arrow[r] \arrow[d, "\pi"] & V \arrow[d] \\
E \arrow[r, "f"] & \mathbb{P}^1
\end{tikzcd}$$

where now $X \to E$ is a fibration in Châtelet surfaces over an elliptic curve. In other words, after passing to an uncountable algebraically closed field from Lemma 2.2.13 parts (4) and (6) and using Theorem 3.3, X is an elliptically connected threefold which is not rationally connected.

Chapter 4

Towards a positive characteristic analogue

From Remark 2.4.9 and the work preceding it, we would like to demonstrate that the existence of a free higher genus curve implies the existence of a free rational curve in positive characteristic, something which holds in characteristic zero by combining Theorems 2.4.4 and 2.3.23. In this section and the next we will prove small results in this direction.

Lemma 4.1. *Let X be a smooth projective variety of dimension n over an algebraically closed field k with $f : C \to X$ a free morphism from a smooth projective curve of genus $g \geq 2$. Then*

$$C.K_X \leq -n(g-1).$$

If $g = 1$ then $C.K_X \leq -1$.

Proof. We have a short exact sequence

$$0 \to \mathscr{T}_C \to f^* \mathscr{T}_X \to \mathscr{N}_{C/X} \to 0.$$

Since f is free by taking cohomology of the above short exact sequence we know that $H^0(C, \mathscr{N}_{C/X}) \to H^1(C, \mathscr{T}_C)$ is surjective and $H^1(C, \mathscr{N}_{C/X}) = 0$. If $g \geq 2$, by Serre duality, $h^1(C, \mathscr{T}_C) = h^0(C, \omega_C^{\otimes 2}) = 3g - 3$ and so $h^0(C, \mathscr{N}_{C/X}) \geq 3g - 3$, where g is the genus of C. By Riemann-Roch,

$$\begin{aligned}
\deg \mathscr{N}_{C/X} &= h^0(C, \mathscr{N}_{C/X}) - h^1(C, \mathscr{N}_{C/X}) + (n-1)(g-1) \\
&\geq (g-1)(n+2).
\end{aligned}$$

We now have that

$$\begin{aligned}
C.K_X &= \deg f^*\omega_X \\
&= \deg (f^* \wedge^n \mathscr{T}_X^*) \\
&= \deg \left(\mathscr{T}_C^* \otimes \wedge^{n-1} \mathscr{N}_{C/X}^*\right) \\
&= 2g - 2 - \deg \mathscr{N}_{C/X}
\end{aligned}$$

and so $C.K_X \leq -n(g-1)$. Finally, if $g = 1$ then $h^1(C, \mathscr{T}_C) = 1$ so $h^0(C, \mathscr{N}_{C/X}) \geq 1$ from which the result follows. □

Proposition 4.2. *Let X be a smooth projective surface over an algebraically closed field k with $f : C \to X$ a very free morphism from a smooth projective curve C. It follows that X is separably rationally connected.*

Proof. If C is of genus zero then X is separably rationally connected by definition. In general we only need to assume that the curve is free and from the above Lemma 4.1 we have, after picking a minimal model for X, that K_X is not nef and from the classification of surfaces ([Fri98] Theorem 10.4) this means that X is either a rational or ruled surface. If X was ruled it would admit a birational morphism to $\mathbb{P}^1 \times C'$. From Proposition 2.3.20 the free morphism $f : C \to X$ would give a free morphism $C \to \mathbb{P}^1 \times C'$ which would mean C' must be a rational curve and that X was birational to $\mathbb{P}^2$. In other words, the only possibility is that X is a rational surface and the result follows. $\square$

Remark 4.3. We would now like to prove a similar result in the case where the dimension of X is 3 and X is smooth. Of course, as above, the result follows in characteristic zero from work in previous chapters. Unfortunately, in the generality required by the minimal model program, there are still many open problems in the classification of 3-folds in positive characteristic. From the work of Keel [Kee99], the existence of contractions is only known where the target is an algebraic space. It is however expected that a statement of the following form is true:

"If K_X is not nef then X is a Fano fibration over a point, curve or surface".

From the main theorem in [Kol91], we can contract extremal rays in the cone of curves in arbitrary characteristic, assuming X is smooth (or more generally there are results in the singular direction in [Kee99]) and that it admits a free morphism from a curve, to obtain a Fano fibration over a curve, surface or point (i.e. X is Fano). In the case where there exists a conic fibration $X \to Y$ where Y is a smooth surface, Kollár proves that if the characteristic of k is not 2 then the general fibre is smooth. From Proposition 2.3.20 it follows that the composition morphism $C \to Y$ is free and so from the above proposition for the case of surfaces, Y is a rational surface. Hence X is a conic bundle over a rational surface so it is rational, hence separably rationally connected. If $X \to Y$ a Fano fibration over a curve, to the author's knowledge, it is not known whether the fibres of the del Pezzo surface fibration over Y obtained in this way must be smooth. Assuming for the time being that they were, they would be separably rationally connected, and from the deformation theory argument in Theorem 2.4.2 and de Franchis' Theorem 2.4.3, Y would be $\mathbb{P}^1$ and from the de Jong-Starr Theorem 1.3 we would obtain sections $\mathbb{P}^1 \to X$ from which we could assemble combs to be smoothed to very free rational curves in X, showing that X is separably rationally connected.

The following result is well known in the case of $\mathbb{P}^1$ (see [Deb01] 4.18) and easily extends to higher genus.

Proposition 4.4. *Let $f : C \to X$ be a very free morphism from a smooth projective curve C to a smooth projective variety X over an algebraically closed field k. Then for all positive integers m, p*

$$H^0(X, (\Omega^p_X)^{\otimes m}) = 0.$$

Proof. Since $f : C \to X$ is very free, from Proposition 2.3.25 there is a variety U such that $C \times U \to X$ makes X separably C-connected. Being very free is an open property from Proposition 2.3.12 so we can assume that the general morphism $f_u : C_u \to X$ for $u \in U$ is very free, and so $f_u^* \mathscr{T}_X$ is ample from Proposition A.3 (and by definition of a very free curve in the genus 0 case). We conclude that for a general point $x \in X$ there is a morphism $f_u : C_u \to X$ such that $f_u^* \mathscr{T}_X$ is ample and whose image passes through x. Hence since $f_u^* \Omega^1_X$ is negative, any section of $(\Omega^p_X)^{\otimes m}$ must vanish on the image $f(C_u)$ hence on a dense open subset of X, and so on X. $\square$

Proposition 4.5. *Let $f : C \to X$ be a very free morphism from a smooth projective curve C to a smooth projective variety X over an algebraically closed field k. It follows that the Albanese variety $\operatorname{Alb} X$ is trivial.*

Proof. Note that in characteristic zero, we have that

$$\dim \operatorname{Alb} X = \dim \operatorname{Pic}^0 X = \dim H^1(X, \mathscr{O}_X) = h^{0,1}.$$

Hodge duality gives that $h^{1,0} = h^{0,1}$ but more generally over any algebraically closed field we have (see [Igu55]) that $\dim \operatorname{Alb} X \leq h^{1,0} = h^0(X, \Omega^1_X)$. The result follows from Proposition 4.4. □

Remark 4.6. At this point the author expects that if $f : C \to X$ a very free curve, then the n-torsion in the Picard group $\mathrm{Pic}(X)[n]$ is trivial for all integers n.

Chapter 5

Fundamental groups

Following [SGA1] or [Sza09], we begin by briefly introducing the algebraic (étale) fundamental group for a scheme X. We then summarise what is known for étale fundamental groups of rationally and separably rationally connected varieties over an algebraically closed field, highlighting the differences between positive and 0 characteristic. Finally, a discussion follows of certain results one can obtain for C-connected varieties and varieties admitting a free morphism from a curve C.

Let X be a scheme and $\overline{x} : \operatorname{Spec} \overline{k} \to X$ a fixed geometric base point. Denote by FEt_X the category of finite étale coverings $\pi_Y : Y \to X$. Consider the fibre functor

$$F_{\overline{x}} : \mathrm{FEt}_X \to \mathrm{Set} \ \text{ such that } Y \mapsto \pi_Y^{-1}(\overline{x})$$

Call the arithmetic fundamental group of X, denoted by $\pi_1^{\mathrm{et}}(X, \overline{x})$, the group of natural automorphisms $\mathrm{Aut}(F_{\overline{x}})$ of the fibre functor $F_{\overline{x}}$. This is a profinite group when X is connected which is seen more clearly via the construction of $\pi_1^{\mathrm{et}}(X, \overline{x})$ as the automorphism group of the universal covering space which in this case is a pro-object in the category of schemes pro-representing the fibre functor. Up to continuous isomorphism of profinite groups, the étale fundamental group is independent of base points. With appropriate choices of base points, π_1^{et} is also covariantly functorial and we have the following useful result.

Lemma 5.1. *([Kol03] Lemma 11) Let $p : Y \to X$ be a dominant morphism of geometrically irreducible normal varieties over an algebraically closed field k. Let $\overline{y} \to Y$ be a geometric point of Y and $\overline{x} = p(\overline{y})$. Then the image of*

$$\pi_* : \pi_1^{\text{et}}(Y, \overline{y}) \to \pi_1^{\text{et}}(X, \overline{x})$$

is a finite index closed subgroup of $\pi_1^{\text{et}}(X, \overline{x})$.

The first result in the direction of rationally connected varieties is the following.

Proposition 5.2. *([Cam91], see also [Kol03] Corollary 12) Let X be a normal proper rationally connected variety over an algebraically closed field k and $\overline{x} : \operatorname{Spec} \overline{k} \to X$ a geometric point. Then $\pi_1^{\text{et}}(X, \overline{x})$ is finite.*

Proof. Since this result is independent of base points we leave them out of the notation. As X is proper and rationally connected, there is a morphism $u : \mathbb{P}^1 \times Y \to X$ such that $\mathbb{P}^1 \times \mathbb{P}^1 \times Y \to X \times X$ is surjective from Lemma 2.2.13 part (3). Hence for every point $x \in X$, restricting to the subset W' of W parametrising curves whose image passes through x, we have a dominant morphism $u' : W' \times \mathbb{P}^1 \to X$ such that $u'(W' \times \{(0:1)\}) = \{x\}$. From Lemma 5.1 the image of u' has finite index in $\pi_1^{\text{et}}(X)$, however

$$\begin{aligned} \pi_1^{\text{et}}(W \times \mathbb{P}^1) &= \pi_1^{\text{et}}(W) \times \pi_1^{\text{et}}(\mathbb{P}^1) \\ &= \pi_1^{\text{et}}(W \times \{(0:1)\}) \end{aligned}$$

which is mapped to the identity under $u'_* : \pi_1^{\text{et}}(W \times \mathbb{P}^1) \to \pi_1^{\text{et}}(X)$. The result follows.

□

There do exist however examples of rationally connected varieties in positive characteristic which are not simply connected. We will see such an example of Shioda (see [Shi74], [Shi77]) in the construction of the Godeaux surface in Example 5.7, a general type unirational surface in positive characteristic with fundamental group isomorphic to $\mathbb{Z}/5\mathbb{Z}$.

On the other hand, if we furthermore assume that a variety is smooth and separably rationally connected, then it is also simply connected by the following result.

Theorem 5.3. *([Cam91], Kollár [Deb03] Corollaire* 3.6*) Let X be a smooth proper separably rationally connected variety over an algebraically closed field k of arbitrary characteristic. Then X is simply connected.*

Proof. When k has characteristic zero, then X being separably rationally connected is equivalent to it being rationally connected from Theorem 2.3.23. Now, since there exists a very free rational curve on X, we have that $H^0(X, \Omega^n) = 0$ for all $n \geq 1$ from Lemma 4.4 and so by Hodge duality, $H^n(X, \mathscr{O}_X) = 0$ for all $n \geq 1$. Hence $\chi(X, \mathscr{O}_X) = 1$. For an arbitrary connected étale cover $\pi : \widetilde{X} \to X$, we know from Lemma 2.2.13 Part (7) that $\widetilde{X}$ is also rationally connected, hence $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = 1$. However $\chi(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = (\deg \pi)\chi(X, \mathscr{O}_X)$ so $\deg \pi = 1$ from which π must be an isomorphism and so X is algebraically simply connected.

If k is of positive characteristic, we refer to the slightly more intricate proof given by Kollár in [Deb03] Corollaire 3.6, which uses the de Jong-Starr Theorem 1.3. □

A statement in the generality of rationally chain connected varieties was given by Chambert-Loir, whose nice proof we will not recount.

Theorem 5.4. *([CL03]) Let X be a normal rationally chain connected variety over an algebraically closed field of characteristic $p \geq 0$.*

1. *The fundamental group $\pi_1^{\text{et}}(X, \overline{x})$ is finite.*

2. *If k is of characteristic zero then the topological fundamental group of X is also finite.*

3. *If X is proper and smooth, the order of $\pi_1^{\text{et}}(X, \overline{x})$ is prime to p.*

Of course if X is smooth and k is of characteristic zero, then from the above and Theorem 2.3.23 we recover Theorem 5.3.

Finally in the case of a higher genus curve C, the group $\pi_1^{\text{et}}(C, \overline{c})$ is of course not trivial, but the topology of a C-connected variety is reflected in the topology of C in the following sense.

Proposition 5.5. *Let X be a proper normal C-connected variety over an algebraically closed field k, where C is a smooth projective curve. It follows that $\pi_1^{\text{et}}(C, \overline{c}) \to \pi_1^{\text{et}}(X, \overline{x})$ has finite index.*

Proof. Let $\text{ev} : C \times U \to X$ be a family such that $\text{ev}^{(2)} : C \times C \times U \to X \times X$ is a dominant morphism. Let $x_0 \in X$ be a general point and $\overline{x}$ a geometric point lying over it. Define

$$U' = \{(u, c) : u \in U, c \in C, \text{ev}_u(c) = x_0\}$$

and denote by $\sigma : U' \to C \times U'$ the natural section to the projection onto the second factor that sends $u \in U'$ to (u, c) where $c \in C_u$, noting that σ is not the trivial section. Since X is C-connected, the map $\text{ev}' : C \times U' \to X$ is also dominant and in fact the composition $\text{ev}' \circ \sigma : U' \to X$ is constant with image x_0. The induced morphism on fundamental groups gives a diagram

$$\begin{tikzcd}
\pi_1^{\text{et}}(U', \overline{u}') \times \pi_1^{\text{et}}(C, \overline{c}) \arrow[r, "\text{ev}'_*"] \arrow[d, shift left, "\text{pr}_1"] & \pi_1^{\text{et}}(X, \overline{x}) \\
\pi_1^{\text{et}}(U', \overline{u}') \arrow[u, shift left, "\sigma_*"] &
\end{tikzcd}$$

and the image of $\mathrm{ev}'_* \circ \sigma_* : \pi_1^{\mathrm{et}}(U', \overline{u}') \to \pi_1^{\mathrm{et}}(X, \overline{x})$ is the identity. From [Kol03] Lemma 11 it follows that ev'_* has finite index and in fact this remains true if we quotient $\pi_1^{\mathrm{et}}(U', \overline{u}') \times \pi_1^{\mathrm{et}}(C, \overline{c})$ by the normal closure $\overline{\mathrm{im}\,\sigma_*}$ of the image of σ_*. It is now sufficient to prove that $\pi_1^{\mathrm{et}}(C, \overline{c})$ surjects onto the quotient

$$(\pi_1^{\mathrm{et}}(U', \overline{u}') \times \pi_1^{\mathrm{et}}(C, \overline{c}))/\overline{\mathrm{im}\,\sigma_*}.$$

This follows from the following trivial lemma in group theory. □

Lemma 5.6. *Let G, H be two groups, $\mathrm{pr}_1 : G \times H \to G$ the projection onto the first factor and $\sigma : G \to G \times H$ any section. It follows that H surjects onto the quotient of $G \times H$ by the normal closure of the image of σ.*

Proof. Write the section σ as $\sigma(g) = (g, f(g))$. Considering the inclusion $H \to G \times H$ and composing with the quotient map observe that the class of an element $[(g,h)]$ in the quotient $(G \times H)/\overline{\mathrm{im}\,\sigma}$ contains the element $(g,h)(g^{-1}, f(g^{-1})) = (1, h')$. □

Example 5.7. We now give an example of a variety in positive characteristic whose tower of MRC fibrations is of length two yet whose fundamental group is infinite.

Let X be the Fermat quintic surface $x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0$ in $\mathbb{P}^3_k$ over an algebraically closed field k of characteristic p. In [Shi74] and [Shi77] Shioda proves that if $p \neq 5$ and p is not congruent to 1 modulo 5, then X is a unirational general type surface and if we quotient by the action of the group G of 5-th roots of unity $x_i \mapsto \zeta^i x_i$, then we obtain a Godeaux surface which is again unirational but has algebraic fundamental group $\pi_1^{\mathrm{et}}(X/G, \overline{y}) \cong \mathbb{Z}/5\mathbb{Z}$.

Let C be a smooth 5 to 1 cover of $\mathbb{P}^1$, with defining affine equation of the form

$y^5 = f(x)$ where f is a general polynomial of high degree. We have an action of $G = \mathbb{Z}/5\mathbb{Z}$ on C which we can extend to the product $X \times C$ of the above Fermat quintic X with C. Projecting from the quotient onto the second factor we have a morphism

$$(X \times C)/G \to \mathbb{P}^1$$

where we have identified C/G with $\mathbb{P}^1$. The general fibre of this morphism is isomorphic to X. We have a short exact sequence

$$1 \to \pi_1^{\mathrm{et}}(X, \overline{x}) \times \pi_1^{\mathrm{et}}(C, \overline{c}) \to \pi_1^{\mathrm{et}}((X \times C)/G, \overline{z}) \to G \to 1.$$

Hence we have constructed an example of a smooth projective variety over an algebraically closed field of characteristic p whose fundamental group is infinite yet whose tower of MRC quotients terminates with a point.

Appendix A

Preliminaries on ample vector bundles and Frobenius

We begin with a collection of some statements concerning positivity of vector bundles. Unless otherwise mentioned, assume that a variety X is over an algebraically closed field and if projective, $\mathscr{O}_X(1)$ is a fixed ample vector bundle.

Definition A.1. A locally free sheaf $\mathscr{E}$ on a scheme X is called ample if $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ has this property.

Ampleness was originally discussed in [Har71], [Har66], [Har70] where the definition was given as follows: $\mathscr{E}$ is ample if for every coherent sheaf $\mathscr{F}$ on X, there exists a positive integer n_0 such that for all $n > n_0$, the $\mathscr{O}_X$-module $\mathscr{F} \otimes S^n(\mathscr{E})$ is generated by global sections. This definition is equivalent with the one given above.

Lemma A.2. *Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic zero and $\mathscr{E}$ a locally free sheaf on C such that $H^1(C, \mathscr{E}) = 0$. It follows that $\mathscr{E}$ is ample.*

Proof. From [Har71] Theorem 2.4, it suffices to show that every non-trivial quotient locally free sheaf of $\mathscr{E}$ has positive degree. Let $\mathscr{E} \to \mathscr{E}' \to 0$ be a quotient. From the long exact sequence in cohomology we see that $H^1(C, \mathscr{E}')$ is also 0. From the Riemann-Roch formula $\chi(\mathscr{E}') = \deg \mathscr{E}' + \operatorname{rk} \mathscr{E}' \chi(\mathscr{O}_X)$ from which we obtain $h^0(C, \mathscr{E}') = \deg \mathscr{E}' + (\operatorname{rk} \mathscr{E}')(1-g)$ and since $g \geq 2$ we deduce that $\deg \mathscr{E}' > 0$. $\square$

More generally, over any characteristic if we further assume that our locally free sheaf is globally generated then the same result holds so long as the genus is at least 1.

Proposition A.3. *Let C be a smooth projective curve of genus $g \geq 1$ over an algebraically closed field k and $\mathscr{E}$ a globally generated locally free sheaf on C such that $H^1(C, \mathscr{E}) = 0$. Then $\mathscr{E}$ is ample.*

Proof. Since $\mathscr{E}$ is globally generated, there exists a positive integer n such that $\mathscr{O}_C^{\oplus n} \to \mathscr{E} \to 0$ is exact. This gives (see [Har77] ex. II.3.12) a closed immersion of the respective projective bundles $\mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}_C^{n-1}$. By projecting onto the first factor we have the following diagram

$$\begin{tikzcd}
\mathbb{P}(\mathscr{E}) \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & \mathbb{P}^{n-1} \times C \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathbb{P}^{n-1} \\
& C &
\end{tikzcd}$$

and from [Har77] II.5.12 we have $\mathrm{pr}_1^* \mathscr{O}_{\mathbb{P}^{n-1}}(1) = \mathscr{O}_{\mathbb{P}_C^{n-1}}(1)$. Also, since i is a closed immersion it follows that $i^* \mathscr{O}_{\mathbb{P}_C^{n-1}}(1) = \mathscr{O}_{\mathbb{P}_C^{n-1}}(1)|_{\mathbb{P}(\mathscr{E})} = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ which implies that $i^* \mathrm{pr}_1^* \mathscr{O}_{\mathbb{P}^{n-1}}(1) = \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. To show that $\mathscr{E}$ is an ample locally free sheaf on C it is enough to show that this invertible sheaf is ample. Since we know that $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ is ample though, it is sufficient to show that $i \circ \mathrm{pr}_1$ is a finite morphism. Since it is projective, we need only show that it is quasi-finite. Hence assuming that the fibre of $i \circ \mathrm{pr}_1$ over a general point $p \in \mathbb{P}^{n-1}$ is not finite, it must be the whole of C. We now embed this fibre $j : C \to \mathbb{P}(\mathscr{E})$ as a section to π and pull back the surjection $\pi^* \mathscr{E} \to \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ via j, obtaining $j^* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ as a quotient of $j^* \pi^* \mathscr{E} = \mathscr{E}$ (see [Har77] II.7.12). However $\mathrm{pr}_1 \circ i \circ j : C \to \mathbb{P}^{n-1}$ is a constant map so $j^* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathscr{O}_C$. Taking cohomology of the corresponding short exact sequence given by this quotient, we obtain a contradiction since $H^1(C,\mathscr{E}) = 0$ whereas $H^1(C,\mathscr{O}_C)$ is not trivial for $g \geq 1$. $\square$

In Proposition A.9 below we will prove that given an ample bundle on a curve in positive characteristic, then after pulling back by Frobenius, we can make this bundle be globally generated and have vanishing first cohomology.

Lemma A.4. *Let C be a smooth projective curve over an algebraically closed field k, $d \geq 0$ an integer and $\mathscr{E}$ a locally free sheaf on C. If $H^1(C,\mathscr{E}(-D)) = 0$ for all effective divisors D of fixed degree d then for $d' < d$ it follows that $H^1(C,\mathscr{E}(-D')) = 0$ and $\mathscr{E}(-D')$ is globally generated for all effective divisors D' of degree d'.*

Proof. The first result follows from the short exact sequence

$$0 \to \mathscr{E}(-D'-R) \to \mathscr{E}(-D') \to \mathscr{E}(-D')|_R \to 0$$

where R is an effective divisor of degree $d-d'$. For the second, let $p \in C$. From the first part we have $H^1(C,\mathscr{E}(-D'-p)) = 0$ since $D'+p$ is an effective divisor of degree $d'+1 \leq d$ so the following sequence is exact

$$0 \to H^0(C,\mathscr{E}(-D'-p)) \to H^0(C,\mathscr{E}(-D')) \to \mathscr{E}(-D') \otimes k(p) \to 0.$$

Hence $\mathscr{E}(-D')$ is globally generated at p and the result follows. $\square$

Lemma A.5. *Let C be a smooth projective curve and $\phi : \mathscr{E} \to \mathscr{F}$ a morphism between two locally free sheaves on C. Assume that the image of ϕ has full rank (i.e. that the image of ϕ has rank $\operatorname{rk}\mathscr{F}$), that $\mathscr{E}$ is globally generated and $H^1(C,\mathscr{E}) = 0$. Then $\mathscr{F}$ is also globally generated and $H^1(C,\mathscr{F}) = 0$.*

Proof. Let $\mathscr{F}'$ be the locally free sheaf associated to the image of ϕ. For a torsion sheaf $\mathscr{Q}$ and a locally free sheaf $\mathscr{K}$ we have the following short exact sequences of sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{Q} \to 0,$$
$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{F}' \to 0.$$

As $\mathscr{F}'$ is a quotient of $\mathscr{E}$ we have that it is globally generated and $H^1(C, \mathscr{F}') = 0$. From the cohomology of the first exact sequence above we obtain that $H^1(C, \mathscr{F}) = 0$ too. Now for $p \in C$, from the short exact sequence

$$0 \to \mathscr{F}'(-p) \to \mathscr{F}' \to \mathscr{F}' \otimes k(p) \to 0$$

we obtain that $H^1(C, \mathscr{F}'(-p)) = 0$ for any $p \in C$. Twisting the first short exact sequence above by $\mathscr{O}_C(-p)$ we similarly obtain that $H^1(C, \mathscr{F}(-p)) = 0$ and from $0 \to \mathscr{F}(-p) \to \mathscr{F} \to \mathscr{F} \otimes k(p) \to 0$ we see that $\mathscr{F}$ is globally generated at p. □

As a reference for the following see [BK05] and [Uen03]. Let k be an algebraically closed field of characteristic $p > 0$ and $F_X : X \to X$ be the absolute Frobenius morphism of a k-scheme X which leaves the topological space X unchanged but raises functions to the p-th power. More generally, consider an S-scheme X where S is a scheme over k. Pulling back the map $X \to S$ via F_S we obtain

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\
& X^{(1)} \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

where $F_{X/S} : X \to X^{(1)}$ is the relative Frobenius morphism which we get from the universality of the pull-back diagram. In the case where $S = \operatorname{Spec} k$ and X is a variety the relative Frobenius morphism $F_{X/k}$ raises variables of the equation of X to the p-th power, α is called the k-linear Frobenius morphism and raises the coefficients to the p-th power whereas F_X raises everything to the p-th power.

Remark A.6. Some comments on the various uses of Frobenius morphisms in the literature. In Hartshorne's original paper on ample vector bundles [Har66] as also in [BK05], the authors refer to the absolute Frobenius morphism, for which they prove the results of the following Lemma A.7. Note that these results do not hold for the k-linear Frobenius morphisms, which is the morphism most commonly used in Mori-type arguments (see [Deb01], [KSCT07]). In Hartshorne's book [Har77], α the k-linear Frobenius is proven to be finite of degree p, which means that pulling back locally free sheaves increases their degree by multiplying by p (not necessarily making it so that the pullback of the invertible sheaf $\mathscr{L}$ is $\mathscr{L}^{\otimes p}$ as in the case of absolute Frobenius). Also since α is a finite morphism, the pull-back of an ample bundle remains ample.

Lemma A.7. *Let X be a quasi-projective variety over a field of characteristic $p > 0$ and let $F = F_X$ denote the absolute Frobenius morphism. The following are true*

1. *The functor $F^* : \mathscr{E} \mapsto \mathscr{E}^{(1)}$ from coherent sheaves on X to coherent sheaves on X is the unique such functor such that*
   - *(a) F^* is additive and right exact,*
   - *(b) if $\mathscr{L}$ is an invertible sheaf then $F^*\mathscr{L} = \mathscr{L}^{\otimes p}$,*
   - *(c) if $\phi : \mathscr{L}_1 \to \mathscr{L}_2$ is a morphism of invertible sheaves then $F^*\phi : \mathscr{L}_1^{\otimes p} \to \mathscr{L}_2^{\otimes p}$ is $\phi^{\otimes p}$.*
2. *Assume X is either normal or proper, then for $\mathscr{E}$ a locally free sheaf on X we have that $\mathscr{E}$ is ample if and only if $\mathscr{E}^{(1)}$ is ample.*
3. *For $\mathscr{E}$ a locally free sheaf, $F_*F^*\mathscr{E} \cong \mathscr{E} \otimes_{\mathscr{O}_X} F_*\mathscr{O}_X$.*

Proof. The first and second parts are [Har66] Proposition 6.1 and 6.2 respectively. The final part follows from the projection formula. $\square$

A partial converse to Proposition A.3 in characteristic p is given by the following, using $\mathbb{Q}$-twisted vector bundles as in [Laz04] II.6.4.

Proposition A.8. *([KSCT07] Proposition 9) Let C be a smooth projective curve of genus g over a field of characteristic $p > 0$, $\mathscr{E}$ a locally free sheaf of rank r and $\delta \in \mathbb{Q}_+$. Assume $\mathscr{E}\langle -\delta \rangle$ is ample and $b_p(\delta) := p\delta - 2g + 1 \geq 0$. Then for $F : C^{(1)} \to C$ the k-linear Frobenius morphism and every subscheme $B \subset C^{(1)}$ of length smaller than or equal to $b_p(\delta)$ we have*

$$H^1(C^{(1)}, F^*\mathscr{E} \otimes \mathscr{I}_B) = 0$$

and $F^*\mathscr{E} \otimes \mathscr{I}_B$ is globally generated.

We prove the following different version of this result.

Proposition A.9. *Let C be a smooth projective curve of genus g over an algebraically closed field k of characteristic p and let $\mathscr{E}$ be an ample locally free sheaf on C. Let $B \subset C$ be a closed subscheme of length b and ideal sheaf $\mathscr{I}_B$. Then there exists a positive integer n such that $H^1(C^{(n)}, F_n^*\mathscr{E} \otimes \mathscr{I}_B) = 0$ and $F_n^*\mathscr{E} \otimes \mathscr{I}_B$ is globally generated on $C^{(n)}$ where $F_n : C^{(n)} \to C$ the n-fold composition of the k-linear Frobenius morphism.*

Proof. We proceed by induction. First, assume we can write $\mathscr{E}$ as an extension

$$0 \to \mathscr{M} \to \mathscr{E} \to \mathscr{Q} \to 0$$

where $\mathscr{M}$ is an ample line bundle. If $\mathscr{Q}$ is not torsion free, consider the saturation of $\mathscr{M}$ in $\mathscr{E}$ instead and take $\mathscr{Q}$ as that quotient. Since $\mathscr{E}$ is ample, so is its quotient $\mathscr{Q}$. Note also that the rank of $\mathscr{Q}$ is one less than that of $\mathscr{E}$ and that if we can prove the result for $\mathscr{Q}$ then we will have it for $\mathscr{E}$ too by considering cohomology of the appropriate exact sequences. We thus reduce to the case of $\mathscr{E} = \mathscr{L}$ an invertible sheaf of positive degree (since it is ample). We now know from Lemma A.7 that an invertible sheaf $\mathscr{L}$ pulls back under the n-fold composition of the linear Frobenius morphism to an invertible sheaf $F_n^*\mathscr{L}$ of degree $p^n \deg \mathscr{L}$. To show that $H^1(C^{(n)}, F_n^*\mathscr{L} \otimes \mathscr{I}_B) = 0$, it is equivalent by Serre duality to show that $\mathrm{Hom}_{C^{(n)}}(F_n^*\mathscr{L}, \mathscr{O}_{C^{(n)}}(B) \otimes \omega_{C^{(n)}}) = 0$. Since the invertible sheaf $\mathscr{O}_{C^{(n)}}(B) \otimes \omega_{C^{(n)}}$ has degree $b+2g-2$ and by picking n large enough, we can ensure $p^n \deg \mathscr{L} > b + 2g - 2$ from which we obtain $H^1(C^{(n)}, F_n^*\mathscr{L} \otimes \mathscr{I}_B) = 0$ and hence $H^1(C^{(n)}, F_n^*\mathscr{E} \otimes \mathscr{I}_B) = 0$ for a locally free sheaf of any rank.

To show that $F_n^*E \otimes \mathscr{I}_B$ is globally generated, pick a point $q \in C$. Then $\mathscr{I}_B \otimes \mathscr{I}_q$ has length $b + 1$ and from the discussion above $H^1(C^{(n)}, F_n^*E \otimes \mathscr{I}_B \otimes \mathscr{I}_q)$ vanishes when $p^n \deg L > b + 1 + 2g - 2$ so we can just pick n large enough to fit this condition. Now, by taking the long exact sequence in cohomology of

$$0 \to F_n^*E \otimes \mathscr{I}_B \otimes \mathscr{I}_q \to F_n^*E \otimes \mathscr{I}_B \to (F_n^*E \otimes \mathscr{I}_B) \otimes k(q) \to 0$$

we conclude that $F_n^*E \otimes \mathscr{I}_B$ is globally generated.

That $\mathscr{E}$ can not be written as an extension of $\mathscr{M}$ an ample line bundle and a quotient locally free sheaf $\mathscr{Q}$ is equivalent to $H^0(C, \mathscr{E} \otimes \mathscr{M}^{-1}) = 0$. However there exists a positive integer m and an ample line bundle $\mathscr{M}_{C^{(m)}}$ on $C^{(m)}$ for which $H^0(C^{(m)}, (F_m^*\mathscr{E}) \otimes \mathscr{M}_{C^{(m)}}^{-1}) \neq 0$ and we proceed as before with the sheaf $(F_m^*\mathscr{E})$. $\square$

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