Supplemental Material: Extra cost of erasure due to quantum lifetime broadening

Joe Dunlop,[1, *] Federico Cerisola,[1] Juliette Monsel,[2] Sofia Sevitz,[3] Jorge Tabanera-Bravo,[4]
Jonathan Dexter,[5] Federico Fedele,[6] Natalia Ares,[6] and Janet Anders[1, 3, †]
[1] *Physics and Astronomy, University of Exeter, Exeter EX4 4QL, United Kingdom*
[2] *Department of Microtechnology and Nanoscience (MC2),*
Chalmers University of Technology, S-412 96 Göteborg, Sweden
[3] *Institut für Physik, Potsdam University, 14476 Potsdam, Germany*
[4] *Mathematical Biophysics Group, Max Planck Institute for Multidisciplinary Sciences, Göttingen 37077, Germany*
[5] *Dept. of Materials, University of Oxford, Oxford OX1 3PH, United Kingdom*
[6] *Department of Engineering Science, University of Oxford,*
Parks Road, Oxford, OX1 3PJ, United Kingdom
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ELECTRON TUNNELING RATE EQUATION AND STEADY-STATE OCCUPATION

We first briefly note that, while the paper exclusively discusses a single-level quantum dot, all results should hold to a very good approximation for an arbitrary $N-$level dot, provided that the spacing of energy levels is much larger than the source-drain potential difference or broadening energy scales. In that case, at most one energy level (say, the j^{th}) lies within the bias window and is partially occupied, with the all lower levels fully occupied and all higher levels completely empty at all times. In that case, the bit is encoded in the occupation of the j^{th} level, and all others can be neglected.

We will treat the state of the dot as being described by a tunneling rate equation, which arises as the second-order master equation for the quantum dot Hamiltonian, where we neglect the superposition of different electronic states due to the particle-superselection rule; see [1, 2] for details. We assume that the electronic occupation of the source and drain electrodes are described by the Fermi-Dirac distributions $f_{S/D}(\varepsilon) = \frac{1}{1+\exp\left(\beta_{S/D}(\varepsilon-\mu_{S/D})\right)}$. The tunnelling rates between the quantum dot and electrodes are denoted Γ_S and Γ_D, and assumed to be constant. Then, at a given energy ε, the electron currents Γ in and out of the dot depend on the dot occupation p and the electrode occupations f_S and f_D are as follows:

$$\begin{aligned}
\Gamma_S^{\text{in}} &= \Gamma_S f_S(\varepsilon)\,(1-p(\varepsilon)),\\
\Gamma_S^{\text{out}} &= \Gamma_S (1-f_S(\varepsilon))\,p(\varepsilon),\\
\Gamma_D^{\text{in}} &= \Gamma_D f_D(\varepsilon)\,(1-p(\varepsilon)),\\
\Gamma_D^{\text{out}} &= \Gamma_D (1-f_D(\varepsilon))\,p(\varepsilon).
\end{aligned} \tag{S1}$$

The steady-state occupation p_0 is achieved when the total currents in and out are balanced:

$$\begin{aligned}
&\Gamma_S^{\text{in}} + \Gamma_D^{\text{in}} = \Gamma_S^{\text{out}} + \Gamma_D^{\text{out}}\\
&\qquad\implies p_0(\varepsilon) = \tfrac{\Gamma_S}{\Gamma_S+\Gamma_D} f_S(\varepsilon) + \tfrac{\Gamma_D}{\Gamma_S+\Gamma_D} f_D(\varepsilon).
\end{aligned} \tag{S2}$$

Introducing the shorthand $\gamma_{S/D} = \frac{\Gamma_{S/D}}{\Gamma_S+\Gamma_D}$, we arrive at Eq. (1) in the main text, noting that $\gamma_S + \gamma_D = 1$. In fast-tunnelling regimes, the energy level of the quantum dot is described by a distribution g about the chemical potential μ, due to lifetime broadening. Then the overall (broadened) occupation of the dot is given by averaging the unbroadened occupation $p_0(\varepsilon)$, weighted by the probability $g(\varepsilon\,|\,\mu)$ that the dot's energy level takes the value ε given that the chemical potential is μ:

$$p(\mu) = \int_{-\infty}^{\infty} g(\varepsilon\,|\,\mu) p_0(\varepsilon) d\varepsilon. \tag{S3}$$

If we additionally assume that the shape of the broadening distribution g is independent of the dot's chemical potential, such that we can write $g(\varepsilon\,|\,\mu) = g(\varepsilon-\mu)$, then the above leads to Eq. (2) (this assumption is revisited in Eqs (S36) onwards). In this case, the broadened occupation $p(\mu)$ is equivalent to the *cross-correlation* $(g \star p_0)(\mu)$, a product of functions closely related to convolution:

$$p(\mu) = g \star p_0(\mu) = \int_{-\infty}^{\infty} g(\varepsilon\, -\, \mu) p_0(\varepsilon) d\varepsilon. \tag{S4}$$

ACCOUNTING FOR WORK

Suppose that the dot's chemical potential is driven as $\mu(t)$, and that its occupation varies as $p(t)$. We take the rate at which work is done on the dot by the driving field to be given by [3]:

$$\tfrac{dW}{dt} = p(t) \tfrac{d\mu}{dt}. \tag{S5}$$

If the driving is much slower than the dot occupation's equilibration rate (i.e. the tunnelling rates), then the occupation is given by the instantaneous steady state $p(\mu(t))$ as given in Eq. (2). In this quasistatic regime, the explicit time-dependence can be eliminated from Eq. (S5), which integrates to:

$$W = \int_{\mu_i}^{\mu_f} p(\mu) d\mu, \tag{S6}$$

where μ_i and μ_f are the initial and final values of the dot's chemical potential. In the other extreme, when the chemical potential is suddenly *quenched* from μ_i to μ_f over a timescale much quicker than the tunnelling rates, then the occupation undergoes negligible evolution during the process. If the occupation is initially in the steady state $p(\mu_i)$, then the work done on the dot during the quench is:

$$W = \int_{\mu_i}^{\mu_f} p(\mu_i) d\mu = (\mu_f - \mu_i) p(\mu_i). \tag{S7}$$

Erasure to Zero—Starting in the configuration $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2})$, the dot level is quasistatically raised to some $\mu_{\text{hi}} \gg \mu_S$, for which the steady state occupation is vanishingly small: without lifetime broadening, $p(\mu_{\text{hi}}) \sim e^{-\beta(\mu_{\text{hi}} - \mu_S)}$. The new configuration is $(\mu = \mu_{\text{hi}}, p = p(\mu_{\text{hi}}))$, and by Eq. (S6), the work cost of raising the energy level is $\int_{\mu_{\frac{1}{2}}}^{\mu_{\text{hi}}} p(\mu) d\mu$. Then, the level is quenched back to $\mu_{\frac{1}{2}}$, so that the final configuration is $(\mu = \mu_{\frac{1}{2}}, p = p(\mu_{\text{hi}}))$. The (negative) work done on the dot during the quench is $-(\mu_{\text{hi}} - \mu_{\frac{1}{2}}) p(\mu_{\text{hi}})$, by Eq.(S7). Overall, in the limit as $\mu_{\text{hi}} \to +\infty$, the operation takes $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2}) \mapsto (\mu = \mu_{\frac{1}{2}}, p = 0)$, at a total work cost given by:

$$W^0 = \lim_{\mu_{\text{hi}} \to \infty} \left[\int_{\mu_{\frac{1}{2}}}^{\mu_{\text{hi}}} p(\mu) d\mu - (\mu_{\text{hi}} - \mu_{\frac{1}{2}}) p(\mu_{\text{hi}}) \right] = \int_{\mu_{\frac{1}{2}}}^{\infty} p(\mu) d\mu, \tag{S8}$$

assuming that $p(\mu)$ decays sufficiently fast.

Erasure to One—Again, starting at $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2})$, the dot level is slowly lowered to $\mu_{\text{lo}} \ll \mu_D$ so that the new configuration is $(\mu = \mu_{\text{lo}}, p = p(\mu_{\text{lo}}))$ with $p(\mu_{\text{lo}}) \approx 1$; during which the (negative) work done on the dot is $W = -\int_{\mu_{\text{lo}}}^{\mu_{\frac{1}{2}}} p(\mu) d\mu$. Then, the energy level is quickly quenched back to $\mu_{\frac{1}{2}}$ at a work cost $(\mu_{\frac{1}{2}} - \mu_{\text{lo}}) p(\mu_{\text{lo}})$. Taking the limit as $\mu_{\text{lo}} \to -\infty$, the protocol maps $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2}) \mapsto (\mu = \mu_{\frac{1}{2}}, p = 1)$, at a work cost given by:

$$W^1 = \lim_{\mu_{\text{lo}} \to -\infty} \left[(\mu_{\frac{1}{2}} - \mu_{\text{lo}}) p(\mu_{\text{lo}}) - \int_{\mu_{\text{lo}}}^{\mu_{\frac{1}{2}}} p(\mu) d\mu \right] = \int_{-\infty}^{\mu_{\frac{1}{2}}} (1 - p(\mu))\, d\mu. \tag{S9}$$

Thermodynamic reversibility—Both the erasure-to-zero and erasure-to-one protocols are reversible in the following sense. If the initial configuration was $(\mu = \mu_{\frac{1}{2}}, p = 0)$ and μ was quenched to μ_{hi} and then gradually lowered back to $\mu_{\frac{1}{2}}$ (the time-reverse of the erasure-to-zero protocol), then the configuration would be mapped to $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2})$

at a work cost $-W^0$. Therefore, if the erasure-to-zero protocol was applied to $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2})$ and *immediately* followed by its time-reverse, then the configuration would be unchanged, and the total work cost would vanish. The same is true for erasure to one. The caveat is that neither $(\mu = \mu_{\frac{1}{2}}, p = 0)$ nor $(\mu = \mu_{\frac{1}{2}}, p = 1)$ are steady-state configurations, and if μ were held fixed following the erasure protocols, the dot population would begin to irreversibly equilibrate back towards $\frac{1}{2}$.

Resetting the energy level to $\mu_{\frac{1}{2}}$ following erasure ensures that there is no average change in the dot's internal energy $U = \mu p$. In the erasure-to-zero operation $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2}) \mapsto (\mu = \mu_{\frac{1}{2}}, p = 0)$, the change in internal energy is $\Delta U = -\frac{1}{2}\mu_{\frac{1}{2}}$, and in erasure-to-one $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2}) \mapsto (\mu = \mu_{\frac{1}{2}}, p = 1)$, the change is $\Delta U = \frac{1}{2}\mu_{\frac{1}{2}}$. Averaging with equal weighting gives $\overline{\Delta U} = 0$. Moreover, if the quantum dot has additional energy levels outside the bias window, resetting μ means that there is no change in the internal energy associated with those levels.

Relation to Mean Absolute Deviation—Defining the average work cost $\overline{W} \equiv \frac{1}{2}\left(W^0 + W^1\right)$, we have:

$$\begin{aligned}
2\overline{W} &= \int_{\mu_{\frac{1}{2}}}^{\infty} p(\mu) d\mu + \int_{-\infty}^{\mu_{\frac{1}{2}}} (1 - p(\mu))\, d\mu \\
&= \Big[(\mu - \mu_{\frac{1}{2}}) p(\mu)\Big]_{\mu_{\frac{1}{2}}}^{\infty} - \int_{\mu_{\frac{1}{2}}}^{\infty} (\mu - \mu_{\frac{1}{2}}) \frac{dp}{d\mu}\, d\mu + \Big[(\mu - \mu_{\frac{1}{2}})(1 - p(\mu))\Big]_{-\infty}^{\mu_{\frac{1}{2}}} - \int_{-\infty}^{\mu_{\frac{1}{2}}} (\mu - \mu_{\frac{1}{2}}) \left(-\frac{dp}{d\mu}\right) d\mu \\
&= \int_{-\infty}^{\infty} |\mu - \mu_{\frac{1}{2}}| \left(-\frac{dp}{d\mu}\right) d\mu.
\end{aligned} \tag{S10}$$

If p_0 is as in Eq. (S2) and g is a well-defined probability density function, then $p = g \star p_0$ meets the definition of a complementary cumulative distribution function (i.e. monotone decreasing, $p \to 0$ as $\mu \to +\infty$, $p \to 1$ as $\mu \to -\infty$). The corresponding probability density function (PDF) is $p'(\mu) = -\frac{dp}{d\mu}$, with median equal to $\mu_{\frac{1}{2}}$. Equation (S10) is equivalent to the mean absolute deviation (MAD) of that distribution: in general for a PDF $f(x)$ with median m, the MAD is given by:

$$D(f) = \int_{-\infty}^{\infty} |x - m| f(x) dx \;\equiv\; \min_{y \in \mathbb{R}} \int_{-\infty}^{\infty} |x - y| f(x) dx. \tag{S11}$$

The property that the MAD is minimised about the median has a physical consequence: the mean cost of erasure[1] from an arbitrary initial steady state (p, μ) can never be lower than from $(\mu = \mu_{\frac{1}{2}}, p = \frac{1}{2})$, even though W^0 and W^1 might radically differ.

BOUNDING THE MEAN ABSOLUTE DEVIATION

We here state and prove two general properties of the MAD, which will later be applied to bound $\overline{W}$.

Lemma 1. Let f and g be probability density functions. Then the mean absolute deviation D of the cross-correlated function (called the "convolution" in the main text for short) $f \star g$ is bounded by:

$$\begin{aligned}
&\text{(i)} \qquad D(f \star g) \geq \max\{D(f), D(g)\} \\
&\text{(ii)} \qquad D(f \star g) \leq D(f) + D(g).
\end{aligned} \tag{S12}$$

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[1] Assuming equal weighting of W^0 and W^1.

Proof. Recall $f \star g = \int_{-\infty}^{\infty} f(y)g(y-x)dy = \int_{-\infty}^{\infty} f(y+x)g(y)dy$. Letting m_{fg} be the median of $f \star g$, we have:

$$\begin{aligned} D(f \star g) &= \int_{-\infty}^{\infty} |x - m_{fg}| \left(\int f(y+x)g(y)dy \right) dx \\ &= \int \left(\int |x - m_{fg}| f(y+x)dx \right) g(y)dy \\ &\geq \int D(f)g(y)dy \\ &= D(f). \end{aligned} \tag{S13}$$

Since $f \star g(x) = g \star f(-x)$, and since the mean absolute deviation of a function $h(x)$ is equal to that of $h(-x)$, then by the above reasoning we also have $D(f \star g) \geq D(g)$, which completes the proof of 1(i). On the other hand to show the upper bound,

$$\begin{aligned} D(f \star g) &= \int_{-\infty}^{\infty} |u - m_{fg}| f \star g(u)du \\ &\leq \int |u - (m_f - m_g)| f \star g(u)du \\ &= \int |u - (m_f - m_g)| \left(\int f(u+y)g(y)dy \right) du \\ &= \int \left(\int |u - (m_f - m_g)| f(u+y)du \right) g(y)dy \\ &= \int \left(\int |x - y - (m_f - m_g)| f(x)dx \right) g(y)dy \\ &\leq \iint \left(|x - m_f| + |y - m_g| \right) f(x)g(y)dxdy \\ &= \int \left(\int |x - m_f| f(x)dx \right) g(y)dy + \int \left(\int |y - m_g| g(y)dy \right) f(x)dx \\ &= \int D(f)g(y)dy + \int D(g)f(x)dx \\ &= D(f) + D(g), \end{aligned} \tag{S14}$$

where m_f and m_g are the medians of f and g respectively (unless otherwise specified, the integrals run from $-\infty$ to ∞). In the first line, we used Eq.(S11), and in the fifth line we used the substitution $x = u + y$. This completes the proof of 1(ii).

Lemma 2. Let f be a probability density function which is even about its median m_f, such that for all x, $f(m_f + x) = f(m_f - x)$. Similarly, let g be a probability density function which is even about its median m_g, where $m_g \leq m_f$. Let p_f and p_g be probabilities such that $p_f + p_g = 1$. Then the mean absolute deviation of the convex sum $p_f f + p_g g$ is bounded by:

$$\begin{aligned} &\text{(i)} \qquad D(p_f f + p_g g) \geq \max\left\{ p_f D(\mu_f) + p_g D(\mu_g),\ \ \min\{p_f, p_g\}(m_f - m_g) \right\} \\ &\text{(ii)} \qquad D(p_f f + p_g g) \leq p_f D(\mu_f) + p_g D(\mu_g) + \min\{p_f, p_g\}(m_f - m_g). \end{aligned} \tag{S15}$$

Proof. Let us denote $h(x) = p_f f(x) + p_g g(x)$; note that h is also a probability density function and its median $m_h \in [m_g, m_f]$. We will use the fact that the median minimises the mean absolute deviation, in that for all a,

$\int_{-\infty}^{\infty} |x-a| f(x)dx \geq \int_{-\infty}^{\infty} |x-m_f| f(x)dx$. Then the mean absolute deviation of h is bounded as follows:

$$\begin{aligned} D(h) &= \int_{-\infty}^{\infty} |x-m_h|(p_f f(x) + p_g g(x))dx \\ &= p_f \int |x-m_h| f(x)dx + p_g \int |x-m_h| g(x)dx \\ &\geq p_f \int |x-m_f| f(x)dx + p_g \int |x-m_g| g(x)dx \\ &= p_f D(f) + p_g D(g). \end{aligned} \tag{S16}$$

On the other hand, we also have:

$$\begin{aligned} \int_{-\infty}^{\infty} |x-m_h| f(x)dx &= \int_{m_h}^{\infty} |x-m_h| f(x)dx - \int_{-\infty}^{m_h} |x-m_h| f(x)dx + 2\int_{-\infty}^{m_h} |x-m_h| f(x)dx \\ &= \int_{-\infty}^{m_f} (x-m_h) f(x)dx + \int_{m_f}^{\infty} (x-m_h) f(x)dx + 2\int_{-\infty}^{m_h} |x-m_h| f(x)dx \\ &= \int_{0}^{\infty} (m_f-u-m_h) f(m_f-u)du + \int_{0}^{\infty} (m_f+u-m_h) f(m_f+u)du \\ &\qquad\qquad + 2\int_{-\infty}^{m_h} |x-m_h| f(x)dx \\ &= 2(m_f-m_h)\int_{0}^{\infty} f(m_f+u)du + 2\int_{-\infty}^{m_h} |x-m_h| f(x)dx \\ &= m_f - m_h + 2\int_{-\infty}^{m_h} |x-m_h| f(x)dx. \end{aligned} \tag{S17}$$

In the penultimate line we used that $f(m_f+x) = f(m_f-x)$. In the final line we used that $\int_{m_f}^{\infty} f(x)dx = \frac{1}{2}$, by definition of the median. By a similar argument for g,

$$\int_{-\infty}^{\infty} |x-m_h| g(x)dx = m_h - m_g + 2\int_{m_h}^{\infty} |x-m_h| g(x)dx. \tag{S18}$$

Combining (S17) and (S18):

$$D(h) = p_f(m_f-m_h) + p_g(m_h-m_g) + 2\left(p_f\int_{-\infty}^{m_h} |x-m_h| f(x)dx + p_g\int_{m_h}^{\infty} |x-m_h| g(x)dx\right) \tag{S19}$$

Since the integral terms are always positive, it follows that:

$$\begin{aligned} D(h) &\geq p_f(m_f-m_h) + p_g(m_h-m_g) \\ &\geq \min\{p_f, p_g\}(m_f-m_g). \end{aligned} \tag{S20}$$

Taking inequalities (S16) and (S20) together, the claim 2(i) follows. Moreover, since $D(h) = \min_m \int_{-\infty}^{\infty} |x-m| h(x)dx$, then from (S19) we have:

$$\begin{aligned} D(h) &= \min_m \left\{ p_f(m_f-m) + p_g(m-m_g) + 2\left(p_f\int_{-\infty}^{m} |x-m| f(x)dx + p_g\int_{m}^{\infty} |x-m| g(x)dx\right)\right\} \\ &\leq p_f(m_f-m_g) + p_g(m_g-m_g) + 2p_f\int_{-\infty}^{m_g} |x-m_g| f(x)dx + 2p_g\int_{m_g}^{\infty} |x-m_g| g(x)dx \\ &= p_f(m_f-m_g) + p_g D(g) + 2p_f\int_{-\infty}^{m_g} |x-m_g| f(x)dx \\ &\leq p_f(m_f-m_g) + p_g D(g) + 2p_f\int_{-\infty}^{m_f} |x-m_f| f(x)dx \\ &= p_f(m_f-m_g) + p_g D(g) + p_f D(f). \end{aligned} \tag{S21}$$

In the penultimate line we used that $m_f \geq m_g$, and in the final line we used the assumption that f is even about m_f. If we instead take $m = m_f$, then by the same arguments,

$$D(h) \leq p_g(m_f - m_g) + p_g D(g) + p_f D(f). \tag{S22}$$

Combining (S21) and (S22), we certainly have $D(h) \leq \min\{p_f, p_g\}(m_f - m_g) + p_f D(f) + p_g D(g)$, which was the claimed bound 2(ii).

BOUNDING THE WORK COST OF ERASURE

We now apply Lemmas 1 and 2 to bound the work cost of erasure in the quantum dot device. Recall from Eq. (S10) that $\overline{W} = \frac{1}{2}D(p')$, where $p'(\mu) = -\frac{dp}{d\mu}$. Combining equations (S2) and (S4), it can be seen that:

$$\overline{W} = \frac{1}{2}D(g \star (\gamma_S f'_S + \gamma_D f'_D)), \tag{S23}$$

where $f'_\nu(\varepsilon) = -\frac{d}{d\mu} f_\nu(\varepsilon)$, for $\nu = S, D$; which can be explicitly written as:

$$f'_\nu(\varepsilon) = \frac{\beta_\nu e^{\beta_\nu(\varepsilon - \mu_\nu)}}{(1 + e^{\beta_\nu(\varepsilon - \mu_\nu)})^2}. \tag{S24}$$

f'_ν defines a probability density function with median equal to the chemical potential μ_ν. The mean absolute deviation is given by:

$$D(f'_\nu) = \frac{2 \ln 2}{\beta_\nu}. \tag{S25}$$

Returning to (S23), we can apply lemmas 1(i) followed by 2(i), using the fact that that $f'_\nu(\mu_\nu + \varepsilon) = f'_\nu(\mu_\nu - \varepsilon)$ for all ε:

$$\begin{aligned} \overline{W} &\geq \frac{1}{2} \max\left\{D(g),\ \ D(\gamma_S f'_S + \gamma_D f'_D)\right\} \\ &\geq \frac{1}{2} \max\left\{D(g),\ \ \gamma_S D(f'_S) + \gamma_D D(f'_D),\ \ \min\{\gamma_S, \gamma_D\}(\mu_S - \mu_D)\right\} \\ &= \max\left\{\frac{1}{2}D(g),\ \ \left(\tfrac{\gamma_S}{\beta_S} + \tfrac{\gamma_D}{\beta_D}\right)\ln 2,\ \ \frac{1}{2}\min\{\gamma_S, \gamma_D\}(\mu_S - \mu_D)\right\}. \end{aligned} \tag{S26}$$

In the above, we have identified the three independent energy scales which contribute to the work cost of erasure:

$$\begin{aligned} E_{\text{therm}} &= \left(\tfrac{\gamma_S}{\beta_S} + \tfrac{\gamma_D}{\beta_D}\right)\ln 2 \\ E_{\text{bias}} &= \frac{1}{2}\min\{\gamma_S, \gamma_D\}(\mu_S - \mu_D) \\ E_{\text{broad}} &= \frac{1}{2}D(g). \end{aligned} \tag{S27}$$

Eq. (S26) is not a tight bound, and so far we cannot rule out that in fact an optimal erasure protocol would require much *more* work that the right hand side of (S26) suggests. However, applying lemmas 1(ii) and 2(ii) to (S23), we can upper-bound the minimum work cost:

$$\begin{aligned} \overline{W} &\leq \frac{1}{2}\Big(D(g)\ \ +\ \ D(\gamma_S f'_S + \gamma_D f'_D)\Big) \\ &\leq \frac{1}{2}\Big(D(g)\ \ +\ \ \gamma_S D(f'_S) + \gamma_D D(f'_D)\ \ +\ \ \min\{\gamma_S, \gamma_D\}(\mu_S - \mu_D)\Big) \\ &= \frac{1}{2}D(g)\ \ +\ \ \left(\tfrac{\gamma_S}{\beta_S} + \tfrac{\gamma_D}{\beta_D}\right)\ln 2\ \ +\ \ \frac{1}{2}\min\{\gamma_S, \gamma_D\}(\mu_S - \mu_D). \end{aligned} \tag{S28}$$

Putting this together with (S26), we have the full bound:

$$\max\Big\{E_{\text{therm}}, E_{\text{bias}}, E_{\text{broad}}\Big\} \leq \overline{W} \leq E_{\text{therm}} + E_{\text{bias}} + E_{\text{broad}}. \tag{S29}$$

DEPENDENCE ON LIFETIME BROADENING DISTRIBUTION

We have established that the work cost of erasure is related to the mean absolute deviation of the lifetime broadening distribution, which in turn depends on the total tunnelling rate between quantum dot and electrodes. However, the exact relationship between $D(g)$ and Γ_{tot} is determined by the form of the broadening distribution, which in general depends on the details of the coupling between dot and electrodes. Here we will treat two common models: Lorentzian and Gaussian broadening.

Gaussian broadening—If lifetime broadening is described by a Gaussian $g(\varepsilon-\mu)=\frac{1}{\hbar\Gamma_{\text{tot}}\sqrt{2\pi}}\exp\Big(-\frac{1}{2}(\frac{\varepsilon-\mu}{\hbar\Gamma_{\text{tot}}})^2\Big)$, then the mean absolute deviation straightforwardly integrates as:

$$D(g)=\frac{1}{\hbar\Gamma_{\text{tot}}\sqrt{2\pi}}\int_{-\infty}^{\infty}|\varepsilon-\mu|\exp\left(-\frac{1}{2}\left(\frac{\varepsilon-\mu}{\hbar\Gamma_{\text{tot}}}\right)^2\right)d\varepsilon=\hbar\Gamma_{\text{tot}}\sqrt{\tfrac{2}{\pi}}. \tag{S30}$$

The lifetime broadening energy scale is then $E_{\text{broad}}=\frac{1}{2}D(g)=\frac{\hbar\Gamma_{\text{tot}}}{\sqrt{2\pi}}$.

Lorentzian broadening—If lifetime broadening is described by a Lorentzian distribution, then the work cost of perfect erasure to either $p=0$ or $p=1$ is unbounded, since the mean absolute deviation of a Lorentzian does not converge. However, we can derive an expression for work required to 'erase' to p within some small η-neighbourhood of 0 or 1.

Suppose that temperature and bias are negligible, and take $\mu_S=\mu_D=0$ without loss of generality; such that the unbroadened distribution is described by a Heaviside step function $p_0(\mu)=H(-\mu)$. Take lifetime broadening to be described by a Lorentzian:

$$g(\epsilon-\mu)=\frac{\hbar\Gamma_{\text{tot}}}{\pi\left[(\hbar\Gamma_{\text{tot}})^2+(\epsilon-\mu)^2\right]}. \tag{S31}$$

Then, using the symmetry of $g(\epsilon-\mu)$ about μ, we have:

$$\begin{aligned} p(\mu)=g\star p_0(\mu)&=\int_{-\infty}^{\infty}g(\varepsilon-\mu)H(-\varepsilon)d\varepsilon\\ &=\int_{-\infty}^{0}g(\varepsilon-\mu)d\varepsilon\\ &=\int_{\mu}^{\infty}g(\varepsilon)d\varepsilon \end{aligned} \tag{S32}$$

Let $0<\eta<1$, and let μ_η be such that $p(\mu_\eta)=\eta$. In fact μ_η is described by the quantile function for the Lorentzian:

$$\mu_\eta=\hbar\Gamma_{\text{tot}}\tan\big(\pi(\tfrac{1}{2}-\eta)\big). \tag{S33}$$

The work cost of raising μ from $\mu_{\frac{1}{2}}(=0)$ to μ_η while the dot occupation remains in the steady state, followed by resetting the energy level from μ_η to 0 at constant population $p=\eta$, is:

$$\begin{aligned} W^\eta&=\int_0^{\mu_\eta}p(\mu)d\mu+\int_{\mu_\eta}^{0}p(\mu_\eta)d\mu\\ &=[\mu p(\mu)]_0^{\mu_\eta}-\int_0^{\mu_\eta}\mu\frac{dp}{d\mu}d\mu-\eta\mu_\eta\\ &=\int_0^{\mu_\eta}\mu g(\mu)d\mu\\ &=\frac{\hbar\Gamma_{\text{tot}}}{2\pi}\left[\ln\big((\hbar\Gamma_{\text{tot}})^2+\mu^2\big)\right]_0^{\mu_\eta}, \end{aligned} \tag{S34}$$

where we have used that $\frac{dp}{d\mu} = -g(\mu)$ in the third line. Substituting our expression for μ_η:

$$\begin{aligned} W^\eta &= \frac{\hbar\Gamma_{\text{tot}}}{2\pi}\left[\ln\Big((\hbar\Gamma_{\text{tot}})^2 - \big(\hbar\Gamma_{\text{tot}}\tan\big(\pi(\tfrac{1}{2}-\eta)\big)\big)^2\Big) - \ln\big((\hbar\Gamma_{\text{tot}})^2\big)\right] \\ &= \frac{\hbar\Gamma_{\text{tot}}}{2\pi}\ln[1+\tan^2(\pi(\tfrac{1}{2}-\eta))] \\ &= \frac{\hbar\Gamma_{\text{tot}}}{2\pi}\ln[\sec^2(\pi(\tfrac{1}{2}-\eta))]. \end{aligned} \tag{S35}$$

So, while the energy cost of exact erasure diverges, for any given $\eta > 0$ the broadening-related work cost of erasing to within η of 0 or 1 is proportional to $\hbar\Gamma_{\text{tot}}$: the tunneling rates still determine the characteristic energy scale.

Energy-dependent broadening— We assumed that the quantum dot's lifetime broadening distribution does not change shape depending on its chemical potential μ but instead is only shifted, allowing us to write $g(\varepsilon|\mu) = g(\varepsilon-\mu)$. we found that $p(\mu) = g\star p_0(\mu)$ is a well-defined complementary cumulative distribution function (i.e. monotone-decreasing with $\lim_{\mu\to\infty} p(\mu) = 0$ and $\lim_{\mu\to-\infty} p(\mu) = 1$); and moreover that the mean absolute deviation $D(p) \geq D(g)$.

If we relax the assumption that $g(\varepsilon|\mu) = g(\varepsilon - \mu)$, and instead impose only that μ is the median of $g(\varepsilon|\mu)$, then $p(\mu)$ is given by Eq. (S3), and $D(g)$ is no longer a fixed quantity but depends on μ:

$$D(g)|_\mu = \int_{-\infty}^{\infty} |\varepsilon - \mu|\, g(\varepsilon|\mu)\, d\varepsilon. \tag{S36}$$

We might still hope to prove something like $D(p) \geq \min_y D(g)|_y$. However, this is not true at all. Let's consider the version of Eq. (S36) involving probability density functions $f(x), g(x|y)$:

$$h(y) = \int_{-\infty}^{\infty} f(x)\, g(x|y)\, dx. \tag{S37}$$

Here, $h(y)$ is not necessarily a well-defined probability density function, and $D(h)$ may vanish even if f and g are well-defined and have $D(f) > 0$ and $D(g)|_y > 0$, $\forall y$. In particular, $g(x|y)$ might be fine-tuned such that for all y, $g(x|y)$ and $f(x)$ have non-overlapping support. A pathological example is as follows:

$$\begin{aligned} f(x) &= \begin{cases} 1 \text{ for } -\frac{1}{2} < x < \frac{1}{2} \\ 0 \text{ elsewhere} \end{cases} \\ g(x|y) &= \tfrac{1-\eta}{2}\delta(x-(y+|y|+1)) + \tfrac{1-\eta}{2}\delta(x-(y-|y|-1)) + \eta\delta(x-y), \end{aligned} \tag{S38}$$

for some small $\eta > 0$. In this case y is always the median of $g(x|y)$, and $D(g)|_y = (1-\eta)(|y|+1) \geq 1-\varepsilon$. However, we have:

$$h(y) = \begin{cases} \eta \text{ for } -\frac{1}{2} < y < \frac{1}{2} \\ 0 \text{ elsewhere} \end{cases}, \tag{S39}$$

which is not a normalised probability density function since $\int h(y)dy = \eta$, and moreover has mean absolute deviation $D(h) = \frac{\eta}{4}$, vanishing in the limit of small η. While this example is clearly un-physical, it illustrates that nontrivial assumptions about the form of $g(\varepsilon|\mu)$ are necessary to lower-bound $D(p)$ in terms of $D(g)$.

WORK PENALTY FOR FINITE-SPEED ERASURE

Finally, we include a note about the additional work penalty which results when erasure is performed in finite time, rather than in the quasistatic limit.

A consequence of the rate equation model (S1) is that the time-evolution of the average dot occupation is described by

$$\frac{dp}{dt} = -\Gamma_{\text{tot}}\left[p(t) - p_{\text{ss}}(\mu(t))\right], \tag{S40}$$

where we have introduced a subscript to distinguish the actual occupation $p(t)$ from the steady state $p_{\rm ss}(\mu(t))$ as given in (S3). Now, consider that μ is ramped at a uniform rate $\dot{\mu}$ from $\mu_{\frac{1}{2}}$ up to $+\infty$, in order to perform erasure to zero. Since there is a one-to-one correspondence between t and μ, we can consider p to be a function of μ along that trajectory, and we can write $\frac{dp}{dt} = \dot{\mu}\frac{dp}{d\mu}$. Then (S40) becomes

$$\dot{\mu}\frac{dp}{d\mu} = -\Gamma_{\rm tot}\left[p(\mu) - p_{\rm ss}(\mu)\right]. \tag{S41}$$

We are interested in the work cost of erasure to zero, $W^0 = \int_{\mu_{\frac{1}{2}}}^{\infty} p d\mu$. But this is easy to find from (S41):

$$\begin{aligned} &\frac{\dot{\mu}}{\Gamma_{\rm tot}}\frac{dp}{d\mu} = -p(\mu) + p_{\rm ss}(\mu) \\ &\implies p(\mu) = p_{\rm ss}(\mu) - \frac{\dot{\mu}}{\Gamma_{\rm tot}}\frac{dp}{d\mu} \\ &\implies \int_{\mu_{\frac{1}{2}}}^{\infty} p(\mu) d\mu = \int_{\mu_{\frac{1}{2}}}^{\infty} p_{\rm ss}(\mu) d\mu - \frac{\dot{\mu}}{\Gamma_{\rm tot}}\int_{\mu_{\frac{1}{2}}}^{\infty}\frac{dp}{d\mu}d\mu \\ &\implies W^0 = W^0_{\rm QS} - \frac{\dot{\mu}}{\Gamma_{\rm tot}}\Delta p. \end{aligned} \tag{S42}$$

Here we identified that $\int_{\mu_{\frac{1}{2}}}^{\infty} p_{\rm ss}(\mu) d\mu$ is the quasistatic work cost $W^0_{\rm QS}$. Moreover, since we have erased from $p = \frac{1}{2}$ to $p = 0$, we have that $\Delta p = -\frac{1}{2}$, so that

$$W^0 = W^0_{\rm QS} + \frac{\dot{\mu}}{2\Gamma_{\rm tot}}, \tag{S43}$$

i.e. a work penalty of $\frac{\dot{\mu}}{2\Gamma_{\rm tot}}$ for finite-speed driving. We can follow a similar argument for erasure to one – in fact everything is the same up to the second line of (S42), but now we are integrating for $W^1 = \int_{-\infty}^{\mu_{\frac{1}{2}}}(1-p)d\mu$, so we get:

$$\begin{aligned} &\int_{-\infty}^{\mu_{\frac{1}{2}}}(1 - p(\mu))d\mu = \int_{-\infty}^{\mu_{\frac{1}{2}}}(1 - p_{\rm ss}(\mu))d\mu + \frac{\dot{\mu}}{\Gamma_{\rm tot}}\int_{-\infty}^{\mu_{\frac{1}{2}}}\frac{dp}{d\mu}d\mu \\ &\implies W^1 = W^1_{\rm QS} + \frac{\dot{\mu}}{\Gamma_{\rm tot}}\Delta p. \end{aligned} \tag{S44}$$

Here, $\Delta p = +\frac{1}{2}$ so in the end we have the same penalty $W^1 = W^1_{\rm QS} + \frac{\dot{\mu}}{2\Gamma_{\rm tot}}$, and therefore also for the average

$$\overline{W} = \overline{W}_{\rm QS} + \frac{\dot{\mu}}{2\Gamma_{\rm tot}}. \tag{S45}$$

This fits with the intuition that the finite-speed penalty should depend on the ratio of driving rate to tunneling rate (the population equilibration rate). Note, however, that this analysis has assumed uniform driving, leaving open the possibility of improvements with an adaptive strategy.

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[*] he/him j.dunlop@exeter.ac.uk

[†] she/her janet@qipc.org

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