Besov Functional Calculus for Two Commuting Operators

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Abstract

The subject of this dissertation is the construction of a functional calculus for functions of two variables and pairs of commuting operators.

Analytic Besov functions in one variable in the context of operator theory appeared in the works of Vladimir V. Peller, Steven White, and Pascal Vitse. More recently, Charles Batty, Aleander Gomilko, and Yuri Tomilov offered a novel and unifying approach to constructing a functional calculus for the generators of bounded semigroups on Hilbert spaces and generators of bounded holomorphic semigroups on Banach spaces. Our main aim is to extend the latter construction to the setting of two commuting operators.

We begin by providing an overview of the basic theory of one-parameter strongly continuous semigroups, and the theory of Besov functional calculus in one variable. We establish a new result concerning the spectral features of the one-dimensional Besov calculus, and show that compositions of certain functions are in the one-dimensional Besov class, $\mathcal{B}$.

We define and characterise a two-dimensional analogue of the class $\mathcal{B}$, denoted by $\mathcal{B}^2$, and show that it shares many desirable characteristics with its one-dimensional counterpart. We prove that the class $\mathcal{B}^2$ is a Banach algebra containing the two-dimensional Hille-Phillips algebra as its proper subspace, and discuss some of its topological properties. We obtain a number of results on spectral decompositions and provide useful approximation techniques. We show that all functions in our class enjoy representations given in terms of a partial duality, and prove a convergence lemma for sequences of functions in the class.

We then construct a two-dimensional functional calculus for pairs of commuting semigroup generators and functions in the class $\mathcal{B}^2$. We show that our construction yields a bounded algebra homomorphism. We demonstrate that the resulting calculus extends the two-dimensional Hille-Phillips calculus, that it is compatible with the joint holomorphic and half-plane calculi, and consistent with the Besov calculus for functions in one variable. Finally, we obtain a spectral mapping theorem, and establish that our calculus is essentially unique.

Contents

1 Introduction

1.1 Background

One of the key building blocks in the spectral theory of linear operators is the construction of a rich functional calculus for a relevant class of operators under investigation. As a first approximation, a functional calculus assigns meanings to expressions such as

$$A^\alpha, \quad e^{tA}, \quad \log A,$$

where A is a linear, in general unbounded, operator on some Banach space. To be slightly more precise: given a linear operator A of an appropriate type and a suitable class of functions $\mathcal{A}$, a functional calculus provides a mapping:

$$\mathcal{A} \ni f \mapsto f(A) \in L(X).$$

At the very least, it is required that certain properties of $f(A)$ come out as correlated in a natural manner with appropriate properties of the function f. These intuitive remarks will be made precise in the main text.

In the special case when A is a (not necessarily bounded) self-adjoint operator on a Hilbert space X, the existence of a functional calculus for A can be immediately deduced from a version of the Spectral Theorem. Indeed, by the Spectral Theorem (e.g. [24, Theorem 4.1]), there is a (localised) measure space (Ω, μ), a real-valued measurable function h on Ω, and a unitary operator $U : X \to L^2(\Omega, \mu)$ such that U maps $D(A)$ onto $D(M_h)$ and $A = U^* M_h U$. Here, M_h is the standard multiplication operator on $L^2(\Omega, \mu)$ associated with h. Denote by $\mathcal{A}^b$ the algebra of all bounded Borel-measurable complex-valued functions on $\sigma(A)$ and consider the map

$$\mathcal{A}^b \in f \mapsto f(A) = U^* M_{f \circ h} U \in L(X).$$

It can be seen, in particular, that such a mapping is an algebra homomorphism from $\mathcal{A}^b$ to $L(X)$, the set of bounded operators on X, and that if $\lambda \in \rho(A)$, the resolvent set of A, then the function $r_\lambda(t) = (\lambda - t)^{-1}$ applied to A yields the resolvent of A, i.e. $r_\lambda(A) = (\lambda - A)^{-1}$. With some care, this can be further extended to unbounded

Borel-measurable functions on $\sigma(A)$. Admittedly, the outlined approach has limited scope: since the procedure crucially relies on the Spectral Theorem, it does not admit a straightforward generalisation to situations where X is not a Hilbert space. Nonetheless, the functional calculus for self-adjoint operators represents an important prototype for more encompassing modern treatments.

Given a uniformly bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X with generator $-A$, one may wish to interpret $T(t) = e^{-tA}$ (by analogy with finite-dimensional deterministic systems, cf., for instance, [20, Chapter 1]), setting

$$f(A) = \int_0^\infty T(s)\,d\mu(s), \quad \text{where} \quad f(z) = \mathcal{L}\mu(z) = \int_0^\infty e^{-sz}\,d\mu(s), \quad \operatorname{Re} z > 0,$$

with μ in the space of bounded measures $M(\mathbb{R}_+)$. Here and subsequently, the integrals involving operators are in the strong operator topology. This yields a bounded homomorphism from $M(\mathbb{R}_+)$ to $L(X)$, which corresponds to the so-called *Hille-Phillips* (or HP-) functional calculus:

$$M(\mathbb{R}_+) \ni \mu \mapsto \mathcal{L}\mu(A) = \int_{\mathbb{R}_+} T(s)\,d\mu(s).$$

The HP-calculus relies on a fairly direct construction; nonetheless, it has proved to be an extremely useful concept in many areas of analysis.

Another example of a functional calculus is given by the so-called *Bochner-Phillips* calculus (cf. [46, Chapter 13]). Let $(\mu_t)_{t\geqslant 0}$ be a vaguely continuous semigroup of subprobability measures on $[0,\infty)$ and let $(T(t))_{t\geqslant 0}$ be a contraction C_0-semigroup. Then the subordinate semigroup $(T^f(t))_{t\geqslant 0}$ with respect to the subordinator $(\mu_t)_{t\geqslant 0}$ is given by the Bochner integral

$$T^f(t) = \int_0^\infty T(s)d\mu_t(s), \quad t \geqslant 0.$$

The superscript f in $T^f(t)$ refers to the Bernstein function f, which is given by the logarithm of the Laplace transform of μ_t, i.e.,

$$\mathcal{L}\mu_t(x) = \int_0^\infty e^{-sx}\,d\mu_t(s) = e^{-tf(x)}, \quad t, x \geqslant 0.$$

A classical result by Phillips ([43, Theorem 4.3]) states that the generator A^f of $(T^f(t))_{t\geqslant 0}$ is of the form

$$A^f u = au + bAu + \int_0^\infty (u - T(t)u)\,d\mu(t), \quad u \in D(A),$$

which corresponds naturally to the so-called Lévy-Khinchine representation of f on the half-line:

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, d\mu(t), \quad \lambda > 0. \tag{1.1.1}$$

In (1.1.1), $a, b \geqslant 0$, and μ is a measure on $(0, \infty)$ satisfying $\int_0^\infty \min(1, t) d\mu(t) < \infty$ (cf. [46, Theorem 3.2]).

In both of these approaches, the semigroup $(T(t))_{t \geqslant 0}$ is treated as a fundamental object, whereas the generator and its resolvent are viewed as in some sense derivative concepts.

One may, however, take an alternative route by focusing first and foremost on the resolvents. Suppose that a class of functions on a given set Ω has a reproducing kernel, so that for any f in the class

$$f(z) = \int_\Omega f(w) K(z, w)\, d\mu(w), \quad z \in \Omega,$$

for some measure μ. Knowing $K(A, w)$, one may attempt to define $f(A)$ by setting

$$f(A) = \int_\Omega f(w) K(A, w)\, d\mu(w).$$

With the right amount of luck, such a definition turns out to be meaningful and provides grounds for a rather powerful functional calculus. For instance, if Ω is a neighbourhood of $\sigma(A) \cup \{0\}$, and $f \in \mathrm{Hol}(\Omega)$, the algebra of functions holomorphic in Ω, then choosing an appropriate contour $\Gamma \subset \Omega$ one may set $K(z, w) = (w - z)^{-1}$, $K(A, w) = R(A, w)$, obtaining

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(w, A)\, dw.$$

This yields the classical *Riesz-Dunford* calculus. The idea can be extended to functions which have singularities at some points on the boundary of the spectrum. A drawback of this method is that, for unbounded A, it only produces bounded $f(A)$ in some restricted settings.

Following the seminal paper [34], one might proceed in this vein but restrict the class of operators under consideration. As it turns out, the natural class to look at is given by all those operators A having their spectrum in $S_\theta \cup \{0\}$, where $\theta \in (0, \pi)$ and $S_\theta = \{\lambda \in \mathbb{C} \backslash \{0\} : |\arg \lambda| < \theta\}$, and allowing resolvent estimate of the form

$$\|R(\lambda, A)\| \leqslant C/|\lambda|, \quad \lambda \in \mathbb{C} \backslash (S_\theta \cup \{0\}).$$

Any such operator is said to be *sectorial*; for a sectorial operator A, the infimum of all those θ for which A satisfies the above conditions is called the *sectorial angle* of A. In particular, generators of bounded C_0-semigroups are easily seen to be sectorial with sectorial angle less than or equal to $\pi/2$. Suppose A is sectorial with sectorial angle $\theta < \pi$. The point of departure is then to consider the class of holomorphic functions on S_θ decaying sufficiently fast at 0 and ∞, i.e.

$$H_0^\infty(S_\theta) = \left\{ f : S_\theta \to \mathbb{C} \text{ holomorphic and } |f(\lambda)| \leqslant C \frac{|\lambda|^\varepsilon}{1 + |\lambda|^{2\varepsilon}}, \text{ for some } \varepsilon > 0, C \geqslant 0 \right\}.$$

For functions in $H_0^\infty(S_\theta)$, one may then define $f(A)$ by using Cauchy's integral formula and setting

$$f(A) = \frac{1}{2\pi i} \int_{\partial S_\delta} f(z) R(z, A)\, dz,$$

for $\theta < \delta < \pi$. The integral can be shown not to depend on the choice of δ; the construction is then extended via a regularisation procedure to include more functions of interest. The calculus provides a closed operator $f(A)$ for any $f \in H^\infty(S_\theta)$ as long as A is injective; $f(A)$ may in general be unbounded. A systematic framework for such defined *holomorphic* calculus has been provided in [25]. It is worth noting that there are analogous approaches to functional calculi of strip-type and half-plane operators (cf. e.g. [25], [7]).

Once we have a functional calculus at our disposal, it becomes very natural to pose a number of questions. Of particular importance is the question whether, for a fixed function f, the mapping

$$\mathcal{M} : f \mapsto f(A)$$

preserves some relevant classes of operators, e.g. the class of generators of sectorially bounded holomorphic C_0-semigroups. The sectorial case, in particular, has drawn significant attention in connection with Bernstein functions. It is now known that Bernstein functions do preserve sectoriality and sectorial angles for operators with sectorial angle less than $\pi/2$; there exist several independent proofs of this result in the literature. The varied techniques used in these proofs use, e.g., the subclass of the so-called associated complete Bernstein functions in [23], the class of $\mathcal{NP}_+$-functions mapping the right half-plane and the right half-axis into themselves in [6], and the class of analytic Besov functions in the most recent work [4].

Whatever the means by which we define a functional calculus, it is customary to require that it should enjoy a number of desirable properties (cf. [4]). Ideally, we want it to be defined for wide classes of functions and operators, take values in $L(X)$ and yield sharp estimates for operator norms. So for instance, the norm-estimates obtainable via the HP-calculus are given in terms of the representing measure μ; and this is hardly ever optimal. In the case of the holomorphic calculus, of particular interest is the question of which operators enjoy a bounded H^∞-calculus, i.e. for which A does the following inequality

$$\|f(A)\| \leqslant C_A \|f\|_\infty,$$

hold for all functions in the admissible class and with constant C_A depending only on A. The existence of a bounded H^∞-calculus is related to bounds on the imaginary powers, and square function estimates (cf. [13], [26], [29], and [31]). In the Hilbert space case, the Boyadzhiev-deLaubenfels theorem ([10, Theorem 3.2]) states that every generator of a bounded C_0-group has bounded H^∞-calculus on vertical strips of heights exceeding the exponential type of the group. The analogue of this result for general Banach spaces under the assumption that the C_0-group is exponentially γ-bounded was originally obtained in [29, Theorem 6.8], and a new proof using transference techniques was given in [26, Theorem 6.5].

While there are known concrete examples of differential operators with bounded H^∞-calculus, it is usually quite difficult to verify that a given operator enjoys this property. At the very best, one can hope for positive results only in restricted settings. For instance, if $-A$ is the generator of a strongly continuous semigroup of contractions $(T(t))_{t\geqslant 0}$ on a Hilbert space, then the continuous version of von Neumann's inequality yields

$$\left\| \int_{\mathbb{R}_+} T(t)\, d\mu(t) \right\| \leqslant \|\mathcal{L}\mu\|_\infty, \quad \mu \in M(\mathbb{R}_+),$$

or equivalently

$$\|f(A)\| \leqslant \|f\|_{\infty,\mathbb{C}_+}, \quad f \in \mathcal{LM},$$

where $f = \mathcal{L}\mu$ for some $\mu \in M(\mathbb{R}_+)$. In general, however, it might be most reasonable to settle for identifying those classes of functions and semigroup generators for which it is possible to obtain non-trivial norm-estimates weaker than those stated in terms of the $\|\cdot\|_\infty$ norm. A class of functions that has recently received considerable attention in the context of operator theory is that of analytic Besov functions.

Analytic Besov functions appeared already in the work of Vladimir V. Peller [42], in which the author investigated power bounded operators on a Hilbert space, and obtained upper bounds for polynomials of such operators. Steven White in his PhD thesis extended Peller's methods and results to study polynomials in two commuting power bounded operators ([51, Chapter 4]), and obtained ([51, Section 5.5]) Besov estimate for generators of bounded semigroups on Hilbert spaces and

Schwartz functions. Pascale Vitse in [50] considered the Besov space $B^0_{\infty,1}(\mathbb{R})$, defined as consisting of holomorphic functions f on the complex right half-plane such that

$$\|f\|_B = \|f\|_\infty + \int_0^\infty \|f'(x + i\cdot)\|_\infty \, dx < \infty.$$

As shown in [4, Appendix], this space can be naturally identified with a closed subspace of $\mathcal{B}^0_{\infty,1}(\mathbb{R})$. Vitse studied the behaviour of $f(A)$, basing the construction of the functional calculus on the holomorphic sectorial calculus, and making use of the explicit representation

$$f(A) = \sum_{k\in\mathbb{Z}} (\mathcal{F}F * \mathcal{F}\phi_k)(A).$$

Here, $\mathcal{F}$ stands for the Fourier transform, F is a distribution with $\operatorname{supp}(F) \subset [0,\infty)$ such that $f = \mathcal{F}F$, ϕ_k is a triangular function with $\operatorname{supp}(\phi_k) \subset [2^{k-1}, 2^{k+1}]$ taking its maximal value at 2^k, $\sum_{k\in\mathbb{Z}} \phi_k = 1$, and A is a negative generator of a bounded holomorphic C_0-semigroup. Markus Haase put Vitse's results in a wider context of transference principles in [26]. In [26], Haase obtained functional calculus estimates for generators of C_0-semigroups and analytic Besov functions that are Laplace transforms of measures.

Building on work by Haase, Peller, Vitse, White, and others, Charles Batty, Alexander Gomilko, and Yuri Tomilov offered in [4] a novel and unifying approach to constructing a functional calculus for the generators of bounded semigroups on Hilbert and Banach spaces. We now sketch the construction from [4]; a more comprehensive survey of this approach is provided in Chapter 2.

Let $-A$ be the generator of a bounded C_0-semigroup on a Hilbert space X or the generator of a bounded holomorphic C_0-semigroup on a Banach space X. Then for all $x \in X$ and all $x^* \in X^*$, the weak resolvent $\langle (\cdot + A)^{-1}x, x^*\rangle$ belongs to the space $\mathcal{E}$ given by

$$\mathcal{E} = \left\{ g \in \operatorname{Hol}(\mathbb{C}_+) : \|g\|_{\mathcal{E}_0} = \sup_{\alpha>0} \alpha \int_{\mathbb{R}} |g'(\alpha + i\beta)| \, d\beta < \infty \right\}.$$

Consider the Banach algebra $\mathcal{B}$ given by

$$\mathcal{B} = \left\{ f \in \operatorname{Hol}(\mathbb{C}_+) : \|f\|_{\mathcal{B}_0} = \int_0^\infty \sup_{\beta\in\mathbb{R}} |f'(\alpha + i\beta)| \, d\alpha < \infty \right\},$$

with the norm $\|f\| = \|f\|_\infty + \|f\|_{\mathcal{B}_0}$. Then the following duality gives a natural pairing between $\mathcal{B}$ and $\mathcal{E}$:

$$\langle g, f\rangle_{\mathcal{B}} = \int_0^\infty \alpha \int_{-\infty}^\infty g'(\alpha - i\beta) f'(\alpha + i\beta) \, d\beta \, d\alpha, \quad f \in \mathcal{B}, g \in \mathcal{E}.$$

For A as above, one can define a mapping

$$\Phi_A : \mathcal{B} \to L(X, X^{**}),$$
$$\Phi_A(f) = f(A),$$

as a w^*-integral:

$$\langle f(A)x, x^* \rangle = f(\infty) + \frac{2}{\pi} \langle (\cdot + A)^{-1}x, f \rangle_{\mathcal{B}}, \quad x \in X,\ x^* \in X^*. \tag{1.1.2}$$

It then can be shown that (1.1.2) defines a bounded algebra homomorphism

$$\Phi_A : \mathcal{B} \to L(X),$$
$$\Phi_A(f) = f(A),$$

and that $\|f(A)\| \leqslant C_A \|f\|_{\mathcal{B}}$ holds for any $f \in \mathcal{B}$ with a constant C_A depending only on A. Our aim is to extend this construction to functions of two variables and pairs of commuting operators.

It is noteworthy that the Besov norm $\|\cdot\|_{\mathcal{B}}$ gives significantly better bounds for HP-functions than the standard HP-norm. For instance, let $-A$ be the generator of a bounded C_0-semigroup on a Hilbert space X, and let $V(A)$ be the Cayley transform $(A - I)(A + I)^{-1}$ of A. We have $V^n(A) = \chi^n(A)$, with $\chi(z) = (z - 1)(z + 1)^{-1}$. It can be shown (cf. [4, Section 5.5]) that the HP-norm of χ^n grows like $n^{1/2}$; whereas an elementary application of $\mathcal{B}$-calculus yields ([4, Corollary 5.9])

$$\|V(A)^n\| \leqslant 2K_A^2(3 + 2\log(2n)).$$

For some functional calculi, it is more or less straightforward to generalise them to a multivariable setting. Consider, for instance, a two-variable Bernstein function $f : [0, \infty)^2 \to [0, \infty)$ with the representation given by

$$f(\lambda) = a + b\lambda + \int_{\mathbb{R}_+^2 \setminus \{0\}} (1 - e^{-\lambda t})\, d\mu(t),$$

with $\lambda \in \mathbb{R}_+^2$, $a \geqslant 0$, $b \in \mathbb{R}_+^2$, and μ is a uniquely determined positive measure on $\mathbb{R}_+^2$ (cf. [37] and references therein). Then for a pair of commuting generators of uniformly bounded C_0-semigroups, $A = (A_1, A_2)$, we may obtain (as in [36])

$$f(A)x = ax + bAx + \int_{\mathbb{R}_+^2 \setminus \{0\}} (I - T(u))x\, d\mu(u),$$

with $T(u) = T_1(u_1)T_2(u_2)$, and study the properties of these operators. In particular, it is known that $-f(A)$ is a generator of a two-parameter bounded C_0-semigroup.

Similarly, we might define the joint holomorphic functional calculus for a pair of commuting sectorial operators A_1 and A_2 with sectorial angles θ_1 and θ_2, respectively, by considering first

$$H_0^\infty(S_{\mu_1} \times S_{\mu_2}) = \{f \in H^\infty(S_{\mu_1} \times S_{\mu_2}) : \exists_{s>0} \quad \psi^{-s} f \in H^\infty(S_{\mu_1} \times S_{\mu_2})\},$$

with

$$\psi(z_1, z_2) = \frac{z_1 z_2}{(1+z_1)^2(1+z_2)^2}.$$

Take $\delta_1 \in (\theta_1, \mu_2), \delta_2 \in (\theta_2, \mu_2)$. For $f \in H_0^\infty(S_{\mu_1} \times S_{\mu_2})$ we define

$$f(A_1, A_2) = -\frac{1}{4\pi^2} \int_{\partial S_{\delta_1} \times \partial S_{\delta_2}} f(\lambda_2, \lambda_2) R(\lambda_1, A_1) R(\lambda_2, A_2)\, d\lambda_1\, d\lambda_2.$$

The integral on the right-hand side can be shown to be independent of the choice of δ_1 and δ_2. Extension to a wider class of functions is then carried out in a very similar fashion to the one-dimensional case (see Chapter 6 below for details). The joint holomorphic functional calculus was studied in [30, Section 6], where the authors focus on sums of sectorial operators, in the spirit of the celebrated Dore-Venni theorem (cf. [17]).

Notwithstanding the fact that some multi-dimensional calculi can be set up in a rather straightforward manner, the progress in the area has been relatively slow. Many of the technical difficulties arise due to the fact that the underlying function theory in several variables is still in development and/or significantly more complicated than its one-dimensional counterpart. The present dissertation aims to fill this gap by adapting the approach presented in [4] and providing a construction of a functional calculus based on a suitable class of functions in two variables.

1.2 Overview of the thesis

We shall consider the function space $\mathcal{B}^2$, defined as the space of those bounded holomorphic functions f on $\mathbb{C}_+^2$ such that

$$\|f\|_{\mathcal{B}_0^2} + \|f\|_{1*} + \|f\|_{2*} < \infty,$$

where

$$\|f\|_{\mathcal{B}_0^2} = \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2 f(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2,$$

$$\|f\|_{1*} = \int_0^\infty \sup_{\substack{y_1\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f(x_1+iy_1, z_2)|\, dx_1,$$

$$\|f\|_{2*} = \int_0^\infty \sup_{\substack{y_2\in\mathbb{R}\\ z_1\in\mathbb{C}_+}} |D_2 f(z_1, x_2+iy_2)|\, dx_2 < \infty,$$

with $D_i = \frac{\partial}{\partial z_i}$ for $i = 1, 2$. We let

$$\mathcal{B}^2 = \{f \in H^\infty(\mathbb{C}_+^2) : \|f\|_{\mathcal{B}_0^2} + \|f\|_{1*} + \|f\|_{2*} < \infty\}.$$

We then endow $\mathcal{B}^2$ with the norm $\|f\|_{\mathcal{B}^2} = \|f\|_\infty + \|f\|_{1*} + \|f\|_{2*} + \|f\|_{\mathcal{B}_0^2}$. The crucial step will consist in pairing elements of $\mathcal{B}^2$ with functions in an auxiliary class $\mathcal{E}^2$ via a suitably constructed partial duality. More precisely, $\mathcal{E}^2$ is defined as the space of holmorphic functions g on $\mathbb{C}_+^2$ such that

$$\|g\|_{\mathcal{E}_0^2} = \sup_{x_1,x_2>0} x_1x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty |D_1D_2 g(x_1+iy_1, x_2+iy_2)|\, dy_1\, dy_2 < \infty.$$

The (partial) duality between $\mathcal{E}^2$ and $\mathcal{B}^2$ will be given by

$$\langle g, f\rangle_{\mathcal{B}^2} =$$

$$\int_0^\infty \int_0^\infty x_1x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty D_1D_2 g(x_1-iy_1, x_2-iy_2) D_1D_2 f(x_1+iy_1, x_2+iy_2) dy_1\, dy_2\, dx_1\, dx_2.$$

Let A_1, A_2 be closed commuting operators on a Banach space X, with $D(A_1) \cap D(A_2)$ dense in X. We shall assume that the spectra of A_1 and A_2 are contained in in $\overline{\mathbb{C}}_+$, and that the operators satisfy

$$\sup_{\alpha>0} \alpha \int_{\mathbb{R}} |\langle(\alpha - i\beta + A_i)^{-2}x, x^*\rangle|\, d\beta < \infty, \quad i = 1, 2, \quad (1.2.1)$$

$$\sup_{\alpha,\gamma>0} \alpha\gamma \int_{\mathbb{R}}\int_{\mathbb{R}} |\langle(\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}x, x^*\rangle|\, d\beta\, d\delta < \infty. \quad (1.2.2)$$

The inequality (1.2.1) is a version of the so-called Gomilko-Shi-Feng condition introduced in [22] and [48]; it ensures that weak resolvents of A_1 and A_2 belong to $\mathcal{E}^2$. As shown in [22] and [48], (1.2.1) also guarantees that $-A_1$ and $-A_2$ are generators of bounded C_0-semigroups.

We will then define

$$\begin{aligned}\langle f(A_1, A_2)x, x^*\rangle &= f(\infty,\infty)\langle x, x^*\rangle + \frac{2}{\pi}\langle\langle(\cdot + A_1)^{-1}x, x^*\rangle, f_1\rangle \\ &+ \frac{2}{\pi}\langle\langle(\cdot + A_2)^{-1}x, x^*\rangle, f_2\rangle + \frac{4}{\pi^2}\langle\langle(\cdot + A_1)^{-1}(\cdot + A_2)^{-1}x, x^*\rangle, f\rangle,\end{aligned} \quad (1.2.3)$$

with

$$f(\infty,\infty) = \lim_{\operatorname{Re} z_1, \operatorname{Re} z_2\to\infty} f(z_1, z_2), \quad f_1(z_1) = \lim_{\operatorname{Re} z_2\to\infty} f(z_1, z_2), \quad f_2(z_2) = \lim_{\operatorname{Re} z_2\to\infty} f(z_1, z_2).$$

We devote Chapter 2 to an overview of the basic theory of one-parameter strongly continuous semigroups, and the theory of Besov functional calculus in one variable. This will serve as a starting point for providing a corresponding framework for a two-dimensional analogue of the class $\mathcal{B}^1$ (cf. [4]). We base most of this discussion on [4] and [5].

In Proposition 2.3.17, we establish a new result showing that approximate λ-eigenvectors of A are approximate $f(\lambda)$-eigenvectors of $f(A)$. Before concluding Chapter 2, we show that compositions of certain functions are in the one-dimensional Besov class (Proposition 2.3.18), and that this allows one to consistently define operators on Hilbert spaces in two different ways (Proposition 2.3.19).

Chapter 3 of the present work is devoted to preliminary considerations concerning the class $\mathcal{B}^2$. We shall see that $\mathcal{B}^2$ shares many of its characteristics with its one-dimensional counterpart. In Section 3.1 we introduce the class, and define the subspace $\mathcal{B}^2_0$ which will play a key role in subsequent considerations. In Section 3.2 we prove that $\mathcal{B}^2$ is indeed a Banach space with respect to the norm $\|\cdot\|_{\mathcal{B}^2}$, and that the limiting functions $f_1(z_1) = \lim_{\operatorname{Re} z_1\to\infty} f(z_1, z_2)$ and $f_2(z_2) = \lim_{\operatorname{Re}\to\infty} f(z_1, z_2)$ are both in the one-dimensional Besov class, with $f_1(\infty) = f_2(\infty)$. We then establish in Section 3.3 that $\mathcal{B}^2$ is a Banach algebra. In Section 3.4 we introduce two alternative norms on $\mathcal{B}^2$, and show that they are both equivalent to $\|\cdot\|_{\mathcal{B}^2}$; we make extensive use of this fact throughout the thesis. We consider some classes of functions in $\mathcal{B}^2$ in Section 3.5; the results in this section are derived from the one-variable case. In Sections 3.6–3.8, we explain the relations between the classes $\mathcal{B}^2$ and $\mathcal{B}$, prove a two-variable analogue of Proposition 2.3.18 concerning compositions of functions, and provide further examples of functions in $\mathcal{B}^2$.

In Chapter 4, we prove a number of results on spectral decompositions, which we will need later on. We also provide approximation techniques for functions in $\mathcal{B}^2$ in terms of the partial duality with $\mathcal{E}^2$. Following auxiliary considerations in Section 4.1, we show in Section 4.2 that the union of suitably chosen spectral subspaces is dense in $\mathcal{B}_0^2$. We then introduce the class $\mathcal{E}^2$ and the partial duality $\langle\cdot,\cdot\rangle_{\mathcal{B}^2}$ in Section 4.3, and establish that any function $f \in \mathcal{B}^2$ admits an approximation in terms of $\langle\cdot,\cdot\rangle_{\mathcal{B}^2}$. In Section 4.4 we discuss topological properties of $\mathcal{B}^2$; we show that the subspace of Laplace transforms of measures is not norm-dense in $\mathcal{B}^2$, but it is dense in $\mathcal{B}^2$ in the topology of uniform convergence on compact subsets.

Chapter 5 extends the earlier discussion on functions in $\mathcal{B}^2$; it contains a characterisation of functions in $\mathcal{B}^2$ in terms of their representations, as well as a discussion of dual Banach spaces in the light of the partial duality, and a number of examples. Proposition 5.1.1 establishes that any $f \in \mathcal{B}^2$ can be represented in terms of the partial duality introduced in the preceding chapter. Section 5.2 covers a general method of constructing functions in $\mathcal{B}^2$; in Section 5.4 we characterise the behaviour of (two-parameter) semigroups of shifts acting on $\mathcal{B}^2$. After providing two negative results on dual spaces and density in Section 5.5, we close the chapter with Theorem 5.6.1 expressing a convergence lemma for sequences of functions in $\mathcal{B}^2$.

Chapter 6 encapsulates the core results on our two-dimensional functional calculus. We show that (1.2.3) defines a bounded algebra homomorphism

$$\Phi_{A_1,A_2} : \mathcal{B}^2 \to L(X), \quad \Phi_{A_1,A_2}(f) = f(A_1, A_2)$$

and that $\|f(A_1, A_2)\| \leqslant C_{A_1,A_2}\|f\|_{\mathcal{B}^2}$, with the constant C_{A_1,A_2} depending only on A_1 and A_2. We establish that the $\mathcal{B}^2$-calculus extends the two-dimensional Hille-Phillips calculus, that it is compatible with joint holomorphic and half-plane calculi, and consistent with the $\mathcal{B}^1$-calculus as defined in [4]. We shall obtain a spectral mapping theorem for the joint spectrum of A_1 and A_2. Finally, we will see that the $\mathcal{B}^2$-calculus is essentially unique.

1.3 Notation

We shall adopt the following notation throughout the dissertation. We denote Banach spaces by X and Y, and the dual space of X by X^*. The duality between $x \in X$ and $y \in X^*$ will be denoted by $\langle x, y \rangle$. For a Banach space X, $L(X)$ denotes the space of all bounded linear operators on X. The domain, spectrum and resolvent set of an (unbounded) operator A on X are denoted by $D(A)$, $\sigma(A)$ and $\rho(A)$, respectively. For $\lambda \in \rho(A)$, we let:

$$R(\lambda, A) = (\lambda - A)^{-1}.$$

Two linear operators, A_1 and A_2, are said to be commuting if for all $\lambda_1, \lambda_2 \in \rho(A_1) \cap \rho(A_2)$ we have

$$R(\lambda_1, A_1)R(\lambda_2, A_2) = R(\lambda_2, A_2)R(\lambda_1, A_1).$$

We introduce the following notational conventions.

$\mathbb{R}_+ = [0, \infty)$,

$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $\overline{\mathbb{C}}_+ = \{z \in \mathbb{C} : \operatorname{Re} z \geqslant 0\}$,

$S_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$ for $\theta \in (0, \pi)$; $S_0 = (0, \infty)$,

$\mathrm{R}_a = \{z \in \mathbb{C} : \operatorname{Re} z > a\}$,

$\operatorname{supp}(f)$ denotes the support of a function or distribution f on $\mathbb{R}$.

The Cartesian product of two sets, D and E, is denoted by $D \times E$, and $D^2 = D \times D$. For $f : \mathbb{C}_+ \to \mathbb{C}$, we write

$$f(\infty) = \lim_{\operatorname{Re} z \to \infty} f(z)$$

if this limit exists in $\mathbb{C}$. Similarly, for $f : \mathbb{C}_+^2 \to \mathbb{C}$, we write

$$f_1(z_1) = f(z_1, \infty) = \lim_{\operatorname{Re} z_2 \to \infty} f(z_1, z_2),$$
$$f_2(z_2) = f(\infty, z_2) = \lim_{\operatorname{Re} z_1 \to \infty} f(z_1, z_2),$$

and denote the partial complex derivatives $\frac{\partial f}{\partial z_1}$ and $\frac{\partial f}{\partial z_2}$ by $D_1 f$ and $D_2 f$, respectively.

For $a \in \overline{\mathbb{C}}_+$, we define functions on $\mathbb{C}_+$ by

$$e_a(z) = e^{-az}, \ r_a(z) = (z + a)^{-1}.$$

We use the following notation for spaces of functions or measures, and transforms, on $\mathbb{R}$ or $\mathbb{R}_+$:

$\mathcal{S}(\mathbb{R})$ denotes the Schwartz space on $\mathbb{R}$,

$\mathrm{BUC}(\mathbb{R})$ denotes the space of bounded, uniformly continuous, functions on $\mathbb{R}$, with the sup-norm,

$\mathrm{Hol}(\Omega)$ denotes the space of holomorphic functions on an open subset Ω of $\mathbb{C}_+$,

$M(\mathbb{R})$ denotes the Banach algebra of all bounded Borel measures on $\mathbb{R}$ under convolution, and $M(\mathbb{R}_+)$ denotes the corresponding algebra for $\mathbb{R}_+$. We shall identify $L^1(\mathbb{R}_+)$ with a subalgebra of $M(\mathbb{R}_+)$ in the usual way,

$\mathcal{L}$ denotes the Laplace transform applied to distributions, measures or functions on $\mathbb{R}_+$,

$\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}$. For $f \in L^1(\mathbb{R})$,

$$(\mathcal{F}f)(s) = \int_{\mathbb{R}} f(t)e^{-ist}\,dt.$$

We shall also consider $\mathcal{F}$ applied to measures and distributions on $\mathbb{R}$, and $\mathcal{F}^{-1}$ will be the inverse Fourier transform on $\mathbb{R}$.

We adopt analogous notation for spaces of functions or measures, and transforms on $\mathbb{R}^2$ or $\mathbb{R}^2_+$. In particular, we denote by $\mathcal{F}^{-1}_{(2)}$ the inverse Fourier transform on $\mathbb{R}^2$. We note that (see e.g. [49, Corollary 1.24]) if f and $\mathcal{F}_{(2)}f$ are both integrable, then the following holds a.e.

$$\mathcal{F}^{-1}_{(2)}f(s) = \frac{1}{4\pi^2}\mathcal{F}_{(2)}f(-s).$$

2 Preliminaries

2.1 C_0-semigroups

For a systematic treatment of C_0-semigroups and related topics see e.g. [2], [20], [15]. We begin with a definition of a strongly continuous function.

Definition 2.1.1. *A function $T : \mathbb{R}_+ \to L(X)$ is called* strongly continuous *if*

$$t \mapsto T(t)x$$

is continuous for all $x \in X$.

A C_0-semigroup can then be viewed as just a special type of a strongly continuous function, i.e.

Definition 2.1.2. *A strongly continuous semigroup, or a C_0-semigroup, is a strongly continuous function $T : \mathbb{R}_+ \to L(X)$, denoted by $(T(t))_{t \geqslant 0}$, such that*

$$\begin{aligned} T(t+s) &= T(t)T(s) \quad (t, s \geqslant 0), \\ T(0) &= I. \end{aligned}$$

To each C_0-semigroup we can associate an operator, called its *generator*. The generator of a C_0-semigroup is a linear, closed, but in general unbounded, operator defined only on a dense subspace $D(A)$ of the Banach space X.

Every norm-continuous semigroup $(T(t))_{t \geqslant 0}$ on a Banach space X can be characterised as an operator-valued exponential function (see [20, Theorem I.3.7]), i.e. there is an operator $A \in L(X)$ such that

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \quad t \geqslant 0, \tag{2.1.1}$$

where the convergence is understood to be in the operator norm. The notion of a C_0-semigroup generator provides us with an analogue of the bounded operator A appearing in (2.1.1).

Definition 2.1.3. *The generator A of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X is the operator*

$$Ax = \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x),$$

defined for every x in its domain

$$D(A) = \left\{ x \in X : \lim_{h\searrow 0} \frac{1}{h}(T(h)x - x) \text{ exists} \right\}.$$

Throughout the present section, we make the convention that A denotes the generator; when convenient, we take A to be the negative generator elsewhere in the dissertation. We believe that this slight abuse of notation should not cause any confusion.

The following lemma collects some basic properties of the C_0-semigroups and their generators.

Lemma 2.1.4 (Lemma II.1.3, [20])**.** *Let A be a closed operator that generates a C_0-semigroup $(T(t))_{t\geqslant 0}$. Then the following properties hold.*

1. *If $x \in D(A)$ then $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \quad \text{for all } t \geqslant 0.$$

2. *For $t \geqslant 0$ and $x \in X$ we have*

$$\int_0^t T(s)x\,ds \in D(A),$$
$$T(t)x - x = A\int_0^t T(s)x\,ds.$$

3. *For every $t \geqslant 0$ and $x \in D(A)$ we have*

$$T(t)x - x = \int_0^t T(s)Ax\,ds.$$

We have the following characterisation of C_0-semigroups which are norm-continuous on $\mathbb{R}_+$.

Theorem 2.1.5 (Theorem 3.1.10, [2])**.** *Let A be the generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$. The following assertions are equivalent:*

(i) *The operator A is bounded, i.e. $D(A) = X$.*

(ii) $\lim_{t\searrow 0} \|T(t) - I\| = 0$.

In that case, $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$*, for* $t \geqslant 0$.

Consider the abstract, i.e. Banach-space-valued, linear initial value problem:

$$\begin{cases} u'(t) = & Au(t), \quad (t \geqslant 0), \\ u(0) = & x, \quad x \in X, \end{cases}$$

where t represents time, u is a function with values in a Banach space X, $A : D(A) \subset X \to X$ a linear operator. This initial value problem is referred to as the *abstract Cauchy problem* (ACP) associated to $(A, D(A))$ and the initial value x. The importance of (ACP) stems from the fact that many equations of mathematical physics can be expressed in the above form, e.g. the wave equation or the heat equation (cf. [21]).

A function $u \in C^1(\mathbb{R}_+, X)$ such that $u(t) \in D(A)$ for each $t \geqslant 0$ and (ACP) holds is called a *a classical solution of* (ACP). A function $u \in C(\mathbb{R}_+, X)$ is called *a mild solution of* (ACP) if $\int_0^t u(s)ds \in D(A)$ for all $t \geqslant 0$ and

$$u(t) = A \int_0^t u(s)ds + x. \tag{2.1.2}$$

If now A generates a C_0-semigroup $(T(t))_{t \geqslant 0}$, then for each $x \in X$ the orbit map $u : t \to u(t) = T(t)x$ is known to be the unique mild solution to the associated (ACP).

In order to retrieve the C_0-semigroup $(T(t))_{t \geqslant 0}$ from its generator, a third object is needed, that is: the resolvent of the generator of $(T(t))_{t \geqslant 0}$.

Definition 2.1.6. *We call*

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is bijective}\}.$$

the resolvent set *of* A*. The holomorphic mapping*

$$R(\cdot, A) : \rho(A) \to L(X), \quad R(\lambda, A) = (\lambda - A)^{-1}$$

is called the resolvent *of* A*.*

We now briefly remark on the connection between resolvents and Laplace transforms of semigroups (cf. [2, Section 3.1]).

Definition 2.1.7. *Let* $\lambda_0 \in \mathbb{R}$ *and let* $R : (\lambda_0, \infty) \to L(X)$ *be a function. We say that* R *is a* Laplace transform *if there exists a strongly continuous function* $T : \mathbb{R}_+ \to L(X)$ *such that*

$$\operatorname{abs}(T) = \inf \left\{ \operatorname{Re}\lambda : \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} f(t)\, dt \text{ exists} \right\} \leqslant \lambda_0,$$

and

$$R(\lambda)x = \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} T(t)x\, dt \quad (\lambda > \lambda_0,\ x \in X).$$

It can be shown that C_0-semigroups are exactly those strongly continuous (operator-valued) functions whose Laplace transforms are resolvents (cf. [2, Section 3.1, Theorem 3.1.7]). The following proposition makes this fact precise, and shows some relations holding between the resolvent and the generator.

Proposition 2.1.8 (Theorems II.1.4 and II.1.10, [20]; Theorem 3.1.7, Definition 3.1.8, Proposition 3.1.9, [2])**.** *Let* A *be the generator of a* C_0*-semigroup* $(T(t))_{t\geqslant 0}$*. Then*

(1) *A is closed and densely defined, and there exist constants* $M, \omega \geqslant 0$ *such that* $\|T(t)\| \leqslant Me^{\omega t}$*, for* $t \in \mathbb{R}_+$*.*

(2) *Let*

$$\omega(T) = \inf \left\{ \omega \in \mathbb{R} : \sup_{t\geqslant 0} \|e^{-\omega t} T(t)\| < \infty \right\}.$$

Then for $\operatorname{Re}\lambda > \omega(T)$*,* λ *is in the resolvent set of* A*, and the following holds*

$$R(\lambda, A)x = \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} T(t)x\, dt, \quad x \in X. \tag{2.1.3}$$

Note that trivially $\operatorname{abs}(T) \leqslant \omega(T)$*.*

(3) $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$ *for all* $x \in X$*. Let* $(T(t))_{t\geqslant 0}$ *be a* C_0*-semigroup on* X *and let* A *be its generator. Then the following properties hold.*

(4) $R(\lambda, A)T(t) = T(t)R(\lambda, A)$ *for all* $\lambda \in \rho(A)$ *and* $t \geqslant 0$*.*

We note that property (4) in Proposition 2.1.8 is equivalent to (cf. e.g.[2, Proposition B.7])

$$\text{for all } x \in D(A) \text{ and } t \geqslant 0, \quad T(t)x \in D(A) \text{ and } A(T(t)x) = T(t)Ax.$$

The classical Hille-Yosida generation theorem provides necessary and sufficient conditions for a linear operator on a Banach space X to generate a C_0-semigroup.

Theorem 2.1.9 (Hille-Yosida). *Let $(A, D(A))$ be a linear operator on a Banach space X, $w \in \mathbb{R}, M \geqslant 1$. Then the following are equivalent.*

1. *A generates a C_0-semigroup $(T(t))_{t \geqslant 0}$ such that*

$$\|T(t)\| \leqslant Me^{wt} \quad \text{for all } t \geqslant 0.$$

2. *$(A, D(A))$ is closed, densely defined, and for every $\lambda > w$ we have $\lambda \in \rho(A)$ with*

$$\|[(\lambda - w)R(\lambda, A)]^n\| \leqslant M \quad \text{for all } n \in \mathbb{N}.$$

3. *$(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > w$ we have $\lambda \in \rho(A)$ with*

$$\|[(\lambda - w)R(\lambda, A)]^n\| \leqslant M \quad \text{for all } n \in \mathbb{N}.$$

The estimates appearing in Theorem 2.1.9 involve all powers of the resolvent; this limits the practical use of Theorem 2.1.9 in concrete applications. A result due to Gomilko (cf. [22]), Shi and Feng (cf. [48]), gives a sufficient condition on a general Banach space, involving only a single resolvent power. The condition turns out to be necessary on Hilbert spaces.

Theorem 2.1.10. *Let A be densely defined, closed operator on a Banach space X such that*

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leqslant 0\}.$$

Consider the following conditions:

(i) *For every $x \in X, x^* \in X^*$,*

$$\sup_{w>0} w \int_{\mathbb{R}} \|R(w + it, A)x\|^2 \, dt < \infty,$$
$$\sup_{w>0} w \int_{\mathbb{R}} \|R(w + it, A^*)x^*\|^2 \, dt < \infty.$$

(ii) *For every $x \in X, x^* \in X^*$,*

$$\sup_{w>0} w \int_{\mathbb{R}} |\langle R(w + it, A)^2 x, x^* \rangle|^2 \, dt < \infty.$$

(iii) *A generates a bounded C_0-semigroup.*

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). If X is a Hilbert space then the implication (iii) $\Longrightarrow$ (i) also holds.

A particularly important class of C_0-semigroups is constituted by the so-called holomorphic semigroups, i.e. semigroups that can be extended analytically.

Definition 2.1.11. *Let $\theta \in (0, \frac{\pi}{2}]$. A semigroup T on X is called holomorphic of angle θ if it has a holomorphic extension to S_θ which is bounded on $S_{\theta'} \cap \{z \in \mathbb{C} : |z| \leqslant 1\}$ for all $\theta' \in (0, \theta)$. A semigroup T is called a bounded holomorphic semigroup of angle θ' if T has a bounded holomorphic extension to $S_{\theta'}$ for each $\theta' \in (0, \theta)$.*

We will now introduce the key notion of resolvent commuting operators.

Definition 2.1.12. *Two closed operators A_1 and A_2 on a Banach space X are said to commute in the resolvent sense if $R(\lambda, A_1)R(\mu, A_2) = R(\mu, A_2)R(\lambda, A_1)$ for all $\lambda \in \rho(A_1)$ and $\mu \in \rho(A_2)$.*
The operators $A_1 + A_2$ and A_1A_2 are then understood with their natural domains

$$\begin{aligned} D(A_1 + A_2) &= D(A_1) \cap D(A_2), \\ D(A_1A_2) &= \{x \in D(A_2) : A_2x \in D(A_1)\}. \end{aligned}$$

Let A_1 and A_2 be the generators of C_0-semigroups $(T_1(t))_{t\geqslant 0}$ and $(T_2(t))_{t\geqslant 0}$, respectively. Then A_1 and A_2 commute in the resolvent sense if and only if the semigroups $(T_1(t))_{t\geqslant 0}$ and $(T_2(t))_{t\geqslant 0}$ commute (the right to left direction follows from (2.1.3); for the left to right direction see e.g. [16, Theorem 6.1.27]).

If A_1 and A_2 commute in the resolvent sense, then $\overline{A_1 + A_2}$, with the core $D(A_1) \cap D(A_2)$, generates the product semigroup $S(t) = T_1(t)T_2(t)$ (cf. [20, II.2.7]).

2.2 Sectorial operators

We begin by reviewing some definitions.

For $0 \leqslant \omega \leqslant \pi$ let:

$$S_\omega = \left\{ \begin{array}{lll} \{z \in \mathbb{C} : & z \neq 0, \text{ and } |\arg z| < \omega\} & \text{if } \omega \in (0, \pi], \\ (0, \infty) & & \text{if } \omega = 0. \end{array} \right\}$$

Definition 2.2.1. *Let $\omega \in [0, \pi)$. An operator A on X is called sectorial of angle ω, in short $A \in \mathrm{Sect}(\omega)$, if conditions (i) and (ii) are satisfied:*
(i) $\sigma(A) \subset \overline{S_\omega}$,

(ii) $M(A,\omega') = \sup\{\|\lambda R(\lambda, A)\| : \lambda \in \mathbb{C}\backslash\overline{S_{\omega'}}\} < \infty$ *for all* $\omega' \in (\omega, \pi)$.
In that case, the spectral angle (or sectoriality angle) of A *is defined by:*

$$\omega_A = \min\{0 \leqslant \omega < \pi | A \in \mathrm{Sect}(\omega)\}.$$

The set of sectorial operators on X *will be denoted by* Sect.

We note that the condition (i) in Definition 2.2.1 already implies that A is closed. Although in what follows we shall often assume that A is densely defined, we follow [25] in not making density of the domain a part of the definition of a sectorial operator.

If A is a closed operator and $(-\infty, 0) \subset \rho(A)$ and also $\sup_{t>0} \|t(t+A)^{-1}\| < \infty$, then the fact that A is sectorial follows from the von Neumann series expansion formula (see the proof of [25, Proposition 2.1.1a)]). Moreover, the inverse of an injective sectorial operator is sectorial. More specifically, one has the following proposition.

Proposition 2.2.2 (Proposition 2.1.1, [25]). *Let* A *be a closed operator on a Banach space* X.
(i) If $(-\infty, 0) \subset \rho(A)$ *and*

$$M(A) = \sup_{t>0} \|t(t + A)^{-1}\| < \infty,$$

then $M(A) \geqslant 1$ *and* $A \in \mathrm{Sect}(\pi - \arcsin(M(A)^{-1}))$.
(ii) If A *is injective and* $A \in \mathrm{Sect}(\omega)$ *for some* $\omega \in [0, \pi)$, *then* $A^{-1} \in \mathrm{Sect}(\omega)$ *and the fundamental identity:*

$$\lambda(\lambda + A^{-1})^{-1} = I - \frac{1}{\lambda}\left(\frac{1}{\lambda} + A\right)^{-1}$$

holds for all $\lambda \in \mathbb{C}\backslash\{0\}$ *such that* $\lambda^{-1} \in \rho(A)$.

If $-A$ generates a bounded semigroup $(T(t))_{t\geqslant 0}$, then by taking $M = \sup_{t\geqslant 0} \|T(t)\|$ one obtains:

$$\|(\lambda + A)^{-1}\| = \|R(\lambda, -A)\| \leqslant \frac{M}{\mathrm{Re}\,\lambda}. \quad \mathrm{Re}\,\lambda > 0.$$

This establishes that A is sectorial of angle $\pi/2$. Sectorial operators of angle strictly less than $\pi/2$ coincide with negative generators of bounded holomorphic semigroups ([20, Theorem 4.6]).

A family of operators $(A_i)_{i\in I}$ is called uniformly sectorial of angle $\omega \in [0, \pi)$ if $A_i \in \mathrm{Sect}(\omega)$ for each $i \in I$, and $\sup_{i\in I} M(A_i, \omega') < \infty$ for all $\omega' \in (\omega, \pi)$.

A uniformly sectorial sequence $(A_n)_{n\in\mathbb{N}}$ of angle ω is called a sectorial approximation for the operator A if for some (equivalently: for any, cf. [25, Proposition A.5.3]) $\lambda \in \mathbb{C}\backslash\overline{S}_\omega$ we have $\lambda \notin \sigma(A)$ and

$$R(\lambda, A_n) \to R(\lambda, A) \text{ in } L(X).$$

If $(A_n)_{n\in\mathbb{N}}$ is a sectorial approximation for A on S_ω, we write $A_n \to A\,(S_\omega)$ and speak of sectorial convergence. The following proposition collects some basic properties of sectorial approximations.

Proposition 2.2.3 (Proposition 2.1.3, [25])**.** *1. If $A_n \to A\,(S_\omega)$, and all A_n as well as A are injective, then $A_n^{-1} \to A^{-1}\,(S_\omega)$.*

2. If $A_n \to A\,(S_\omega)$ and $0 \in \rho(A)$, then $0 \in \rho(A_n)$ for large enough n.

3. If $A_n \to A\,(S_\omega)$, and $A \in L(X)$, then $A_n \in L(X)$ for large enough n, and $A_n \to A$ in norm.

4. If $(A_n)_{n\in\mathbb{N}} \subset L(X)$ is uniformly sectorial of angle ω, and if $A_n \to A$ in norm, then $A \in \mathrm{Sect}(\omega)$ and $A_n \to A\,(S_\omega)$.

5. If $A \in \mathrm{Sect}(\omega)$, then $(A+\varepsilon)_{\varepsilon>0}$ is a sectorial approximation for A on S_ω.

6. If $A \in \mathrm{Sect}(\omega)$ and $A_\epsilon = (A+\epsilon)(1+\epsilon A)^{-1}$ for $0 < \epsilon \leqslant 1$, then $(A_\epsilon)_{0<\epsilon\leqslant 1}$ is a sectorial approximation for A on S_ω.

Holomorphic functional calculus Let X be a Banach space, and let $\mathcal{M}$ be a commutative algebra with $\mathbb{1}$ and with a subalgebra $\mathcal{E} \subset \mathcal{M}$. Let Φ denote a homomorphism $\Phi : (e \to \Phi(e)) : \mathcal{E} \to L(X)$. Such defined triple $(\mathcal{E}, \mathcal{M}, \Phi)$ will be referred to as an *abstract functional calculus* over X. If $f \in \mathcal{M}$, then members of the set

$$\mathrm{Reg}(f) = \{e \in \mathcal{M} : ef \in \mathcal{M},\ \Phi(e) \text{ injective}\}$$

are called *regularisers* for f. Abstract functional calculus is said to be *proper* if $\mathrm{Reg}(\mathbb{1}) \neq \varnothing$. In the case of a proper abstract functional calculus $(\mathcal{E}, \mathcal{M}, \Phi)$ an extension procedure may be applied as follows. Take:

$$\mathcal{M}_r = \{f \in \mathcal{M} : \mathrm{Reg}(f) \neq \varnothing\}.$$

For $f \in \mathcal{M}_r$ define (abusing the notation):

$$\Phi(f) = \Phi(e)^{-1}\Phi(ef), \tag{2.2.1}$$

where $e \in \mathrm{Reg}(f)$ is an arbitrarily chosen regulariser for f. (2.2.1) can be easily seen to be independent of the chosen regulariser e. The new Φ can then be shown to extend the old one, i.e. we have

Theorem 2.2.4 (Lemma 1.2.1, [25])**.** *Let $(\mathcal{E}, \mathcal{M}, \phi)$ be a proper abstract functional calculus on a Banach space X. Then* (2.2.1) *defines a closed operator on X, and the map*

$$\Phi = (f \mapsto \Phi(f)) : \mathcal{M}_r \to \{\textit{closed operators on } X\}$$

extends the original mapping $\Phi : \mathcal{E} \to L(X)$. In general, the operator $\Phi(f)$ may be unbounded.

The holomorphic calculus for sectorial operators can now be defined in the following manner. Let $A \in \mathrm{Sect}(\omega)$, and for ϕ such that $\omega < \phi < \pi$ define

$$H_0^\infty(S_\phi) := \{f \in \mathcal{O}(S_\phi) : |f(z) \leqslant C \min(|z|^s, |z|^{-s}) \text{ for some } C, s > 0\},$$

where $\mathcal{O}(S_\phi)$ denotes the set of all holomorphic functions on the sector S_ϕ. Take

$$\tau(z) = \frac{z}{(1+z)^2},$$

and let

$$B(S_\phi) = \{f : S_\phi \to \mathbb{C} : \tau^n f \in H_0^\infty(S_\phi) \text{ for some } n \in \mathbb{N}\}.$$

For each $f \in H_0^\infty(S_\phi)$ we may now define

$$\Phi(f) = f(A) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-A)^{-1}\, dz, \tag{2.2.2}$$

where Γ is the downward oriented boundary of a sector S_{ω_0}, with $\omega < \omega_0 < \phi$. This definition is independent of the particular choice of ω_0, and

$$\Phi : H_0^\infty(S_\phi) \to L(X), \quad \Phi(f) = f(A),$$

is an algebra homomorphism, with $\Phi(\tau) = A(1+A)^{-2}$.

If A is injective, so is $\Phi(\tau)$. In that case, we have a proper abstract functional calculus for A, $(\mathcal{E}, \mathcal{M}, \Phi)$, with

$$\mathcal{E} = H_0^\infty(S_\phi), \quad \mathcal{M} = \mathcal{O}(S_\phi),$$

and the corresponding extended calculus is called the *holomorphic calculus* for A. Any function $f \in B(S_\phi)$ has a regulariser of the form τ^n for some $n \in \mathbb{N}$.

2.3 $\mathcal{B}^1$-calculus

The material in this section is based on the recent work [4] and [5].

Definition. We begin with the definition of the relevant class of analytic Besov functions. Let $\mathcal{B}^1$ be the space of those holomorphic functions f on $\mathbb{C}_+$ such that

$$\|f\|_{\mathcal{B}^1_0} = \int_0^\infty \sup_{y\in\mathbb{R}} |f'(x+iy)|\, dx < \infty. \tag{2.3.1}$$

It can be shown that functions in $\mathcal{B}^1$ enjoy the following properties.

Proposition 2.3.1 (Proposition 2.2, [4])**.** *Let $f \in \mathcal{B}^1$.*

1. *$f(\infty) = \lim_{\operatorname{Re} z\to\infty} f(z)$ exists in $\mathbb{C}$.*
2. *f is bounded, and $\|f\|_\infty \leqslant |f(\infty)| + \|f\|_{\mathcal{B}^1_0}$.*
3. *$f(is) = \lim_{x\to 0+} f(x+is)$ uniformly for $s \in \mathbb{R}$.*
4. *The extended function f is uniformly continuous on $\overline{\mathbb{C}}_+$, and so $f^b \in \mathrm{BUC}(\mathbb{R})$, where f^b denotes the boundary value of f.*
5. *If U is an open set containing the range of f, and h is a bounded holomorphic function with bounded derivative on U, then $h \circ f \in \mathcal{B}^1$.*
6. *If f is bounded away from 0, then $1/f \in \mathcal{B}^1$.*
7. *Assume that the range of f is contained in S_π. If $\beta > 1$, then $f^\beta(z) = f(z)^\beta \in \mathcal{B}^1$. If f is bounded away from 0, then $f^\beta \in \mathcal{B}^1$ for all $\beta \in \mathbb{R}$.*

A norm on $\mathcal{B}^1$ is then defined as follows:

$$\|f\|_{\mathcal{B}^1} = \|f\|_\infty + \|f\|_{\mathcal{B}^1_0} = \|f\|_\infty + \int_0^\infty \sup_{y\in\mathbb{R}} |f'(x+iy)|\, dx. \tag{2.3.2}$$

As observed in [4, Section 2.2], $\|\cdot\|_{\mathcal{B}^1}$ is equivalent to each norm of the form

$$|f(a)| + \|f\|_{\mathcal{B}^1_0} \qquad (a \in \overline{\mathbb{C}}_+ \cup \{\infty\}).$$

Furthermore, $\mathcal{B}^1$ is a Banach algebra and $\|\cdot\|_{\mathcal{B}^1}$ is an algebra norm ([4, Proposition 2.3(2)]).

Hille-Phillips algebra. A key thing to note is that $\mathcal{B}^1$ properly contains the Hille-Phillips algebra as a subalgebra. Let $\mathcal{LM}$ be the Hille-Phillips algebra, which is the subalgebra of $H^\infty(\mathbb{C}_+)$ consisting of Laplace transforms $\mathcal{L}\mu$ of measures $\mu \in M(\mathbb{R}_+)$. Let $m = \mathcal{L}\mu$ and $\|m\|_{\mathrm{HP}} = \|\mu\|$. Then

$$|m(z)| \leqslant |\mu|(\mathbb{R}_+), \quad m'(z) = -\int_{\mathbb{R}_+} te^{-tz}\, d\mu(t), \quad z \in \mathbb{C}_+,$$

and

$$\int_0^\infty \sup_{y>0} |m'(x+iy)|\, dx \leqslant \int_0^\infty \int_{\mathbb{R}_+} te^{-tx}\, d|\mu|(t)\, dx = |\mu|(0,\infty). \tag{2.3.3}$$

So $m \in \mathcal{B}^1$, $\|m\|_{\mathcal{B}^1} \leqslant 2\|m\|_{\mathrm{HP}}$ and $m(\infty) = \mu(\{0\})$. If μ is a positive measure, then equality holds in (2.3.3) and $\|m\|_{\mathcal{B}}^1 = \mu(\mathbb{R}_+)+\mu(0,\infty)$. The subspace $\mathcal{LM}$ is not closed in $\mathcal{B}^1$, and it is not dense in $\mathcal{B}^1$ in the norm-topology. However, $\mathcal{LM}$ is dense in $\mathcal{B}^1$ in the topology of uniform convergence on compact subsets of $\mathbb{C}_+$ by [4, Proposition 2.3].

Examples The following concrete functions are known to be members of $\mathcal{B}^1$ (the first three in virtue of belonging to $\mathcal{LM}$, cf. [4, Examples 2.12, Lemma 3.4, Lemma 3.5, Lemma 3.7]):

- $e_a(z) = e^{-az}$, for $a > 0$,
- $r_\lambda(z) = (\lambda + z)^{-1}$, for $\lambda \in \mathbb{C}_+$,
- $\eta(z) = (1 - e^{-z})/z$,
- $\exp(-t/(z+1))$, for $t > 0$,
- $((z-1)/(z+1))^n$, for $n \in \mathbb{N}$,
- Each function given by
$$h(z) = \frac{1}{\lambda + f(z^\alpha)^\beta},$$
with $\alpha \in (0,1), \beta \in (1, 1/\alpha]$ and an arbitrary Bernstein function f.

Subalgebra $\mathcal{G}$. Working directly with functions in $\mathcal{B}^1$ is not entirely straightforward. The setup in [4] involves subtle approximation techniques, utilising a certain subalgebra of $\mathcal{B}^1$ as its key ingredient. Let

$$\mathcal{G} = \left\{f \in H^\infty(\mathbb{C}_+) : \operatorname{supp}(\mathcal{F}^{-1} f^b) \text{ is a compact subset of } (0,\infty)\right\}$$
$$= \bigcup_{0<\varepsilon<\sigma<\infty} H^\infty[\varepsilon, \sigma].$$

Then, by [4, Lemma 2.5], $\mathcal{G}$ is a subalgebra of $\mathcal{B}^1$. Moreover, by [4, Proposition 2.10], the closure of $\mathcal{G}$ in $\mathcal{B}^1$ is

$$\overline{\mathcal{G}} = \mathcal{B}_0^1 = \left\{f \in \mathcal{B}^1 : f(\infty) = 0\right\}. \tag{2.3.4}$$

In other words, any function $f \in \mathcal{B}^1$ is the sum of a function in $\overline{\mathcal{G}}$ and a constant function.

Space $\mathcal{E}^1$. A partial duality between $\mathcal{B}^1$ and an auxiliary space, $\mathcal{E}^1$, is yet another building block of the approach in [4]. Let $\mathcal{E}^1$ be the space of holomorphic functions g on $\mathbb{C}_+$ such that

$$\|g\|_{\mathcal{E}_0^1} = \sup_{x>0} x \int_{-\infty}^{\infty} |g'(x+iy)|\, dy < \infty. \tag{2.3.5}$$

The constant functions are in $\mathcal{E}^1$, and $\|\cdot\|_{\mathcal{E}_0^1}$ is a seminorm which vanishes on the constant functions. A norm $\|\cdot\|_{\mathcal{E}^1}$ on $\mathcal{E}^1$ can be defined on $\mathcal{E}^1$ by setting

$$\|g\|_{\mathcal{E}^1} = |g(\infty)| + \|g\|_{\mathcal{E}_0^1}, \qquad g \in \mathcal{E}^1. \tag{2.3.6}$$

It can be seen (cf. [4, Section 2.5]) that $\mathcal{E}^1$ equipped with $\|\cdot\|_{\mathcal{E}^1}$ becomes a Banach space.

Partial duality. There is a (partial) duality between $\mathcal{E}^1$ and $\mathcal{B}^1$ given by

$$\langle g, f\rangle_{\mathcal{B}^1} = \int_0^{\infty} x \int_{-\infty}^{\infty} g'(x-iy) f'(x+iy)\, dy\, dx, \qquad g \in \mathcal{E}^1, \quad f \in \mathcal{B}^1. \tag{2.3.7}$$

The duality is only partial in the sense that $\mathcal{B}^1$ and $\mathcal{E}$ are not the dual or predual of each other with respect to this duality, and the constant functions in each space are annihilated in the duality.

Remark 2.3.2. The duality (2.3.7) induces a contractive map $\Psi_{\mathcal{B}^1} : \mathcal{E}^1 \to (\mathcal{B}_0^1)^*$, where $(\mathcal{B}_0^1)^*$ can be identified in the natural way with the space of functionals in $(\mathcal{B}^1)^*$ which annihilate the constant functions. Similarly, let $\mathcal{E}_0^1 = \{g \in \mathcal{E}^1 : g(\infty) = 0\}$, with the norm $\|\cdot\|_{\mathcal{E}^1}$ which coincides with $\|\cdot\|_{\mathcal{E}_0^1}$ on $\mathcal{E}_0^1$. Identify the dual $\mathcal{E}_0^{1*}$ with the space of linear functionals in $(\mathcal{E}^1)^*$ which annihilate the constant functions. Then (2.3.7) provides a contractive map $\Psi_{\mathcal{E}^1}$ from $\mathcal{B}^1$ to $(\mathcal{E}_0^1)^*$.

We have

Proposition 2.3.3 (Propositions 2.22 and 2.23, [4]). *1. The range of $\Psi_{\mathcal{B}^1}$ is not norm-dense in $(\mathcal{B}_0^1)^*$. 2. The range of $\Psi_{\mathcal{E}^1}$ is not norm-dense in $(\mathcal{E}_0^1)^*$*

Approximations. The importance of the partial duality stems from the fact that it provides means for approximating functions in $\mathcal{B}^1$; more specifically, any function in $\mathcal{B}^1$ can be weakly approximated (with respect to the partial duality) in the following way:

Lemma 2.3.4 (Lemma 2.19, [4])**.** *Let* $\mu \in M(\mathbb{R}_+)$ *satisfy*

$$\mu(\mathbb{R}_+) = 1, \qquad \int_{\mathbb{R}_+} t\, d|\mu|(t) < \infty.$$

Let $m = \mathcal{L}\mu \in \mathcal{LM}$*, and* $f \in \mathcal{B}^1$*. For* $\delta > 0$*, let*

$$m_\delta(z) = m(\delta z), \qquad f_\delta(z) = m_\delta(z) f(z) \qquad (z \in \mathbb{C}_+).$$

Then, for all $g \in \mathcal{E}^1$*,*

$$\lim_{\delta \to 0+} \langle g, f_\delta \rangle_{\mathcal{B}^1} = \langle g, f \rangle_{\mathcal{B}^1}.$$

Representations. With this at hand, one can show that any function in $\mathcal{B}^1$ can be represented in terms of the duality.

Proposition 2.3.5 (Proposition 2.20, [4])**.** *Let* $f \in \mathcal{B}^1$*,* $z \in \overline{\mathbb{C}}_+$ *and* $r_z(\lambda) = (\lambda + z)^{-1}$*. Then*

$$f(z) = f(\infty) + \frac{2}{\pi} \langle r_z, f \rangle_{\mathcal{B}^1}. \tag{2.3.8}$$

The following representations are variants of (2.3.8).

Proposition 2.3.6 (Propositions 2.4 and 2.5, [5])**.** *Let* $f \in \mathcal{B}^1_0$*, and* $z = x + iy \in \overline{\mathbb{C}}_+$*. Then*

$$\begin{aligned} f(z) &= -\frac{4}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \frac{\operatorname{Re} f'(\alpha + i\beta)}{(z + \alpha - i\beta)^2}\, d\beta\, d\alpha \\ &= -\frac{4i}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \frac{\operatorname{Im} f'(\alpha + i\beta)}{(z + \alpha - i\beta)^2}\, d\beta\, d\alpha \\ &= \frac{4}{\pi} \int_0^\infty \alpha (x + \alpha) \int_{\mathbb{R}} \frac{f''(\alpha + i\beta)}{(x + \alpha)^2 + (y - \beta)^2}\, d\beta\, d\alpha. \end{aligned}$$

$\mathcal{B}^1$**-calculus**. Let A be a closed operator on a Banach space X, with dense domain $D(A)$. We assume that the spectrum $\sigma(A)$ is contained in $\overline{\mathbb{C}}_+$ and

$$\sup_{\alpha > 0} \alpha \int_{\mathbb{R}} |\langle (\alpha + i\beta + A)^{-2} x, x^* \rangle|\, d\beta < \infty \tag{2.3.9}$$

for all $x \in X$ and $x^* \in X^*$. By the Closed Graph Theorem, there is a constant c such that

$$\frac{2}{\pi} \alpha \int_{\mathbb{R}} |\langle (\alpha + i\beta + A)^{-2} x, x^* \rangle|\, d\beta \leqslant c \|x\|\, \|x^*\| \tag{2.3.10}$$

for all $\alpha > 0$, $x \in X$ and $x^* \in X^*$. Note that (2.3.9) says precisely that the function

$$g_{x,x^*} : z \mapsto \langle (z+A)^{-1}x, x^* \rangle \tag{2.3.11}$$

belongs to $\mathcal{E}^1$. We let γ_A be the smallest value of c such that (2.3.10) holds, so

$$\gamma_A = \frac{2}{\pi} \sup\{\|g_{x,x^*}\|_{\mathcal{E}_0} : \|x\| = \|x^*\| = 1\}. \tag{2.3.12}$$

Assume that (2.3.9) holds, so that the functions g_{x,x^*} in (2.3.11) are in $\mathcal{E}^1$, and let $f \in \mathcal{B}^1$. We aim to define $f(A)$ by replacing z by A and r_z by $(z+A)^{-1}$ in the representation formula in Proposition 2.3.8.

Let $f \in \mathcal{B}^1$ and A be as above. Define

$$\begin{aligned} &\langle f(A)x, x^* \rangle && (2.3.13)\\ &= f(\infty)\langle x, x^* \rangle - \frac{2}{\pi}\int_0^\infty \alpha \int_{\mathbb{R}} \langle (\alpha - i\beta + A)^{-2}x, x^* \rangle f'(\alpha + i\beta)\, d\beta\, d\alpha \\ &= f(\infty)\langle x, x^* \rangle + \frac{2}{\pi}\langle g_{x,x^*}, f \rangle_{\mathcal{B}^1}, \end{aligned}$$

for all $x \in X$ and $x^* \in X^*$. Note that (2.3.13) defines a bounded linear mapping $f(A) : X \to X^{**}$, and that the linear mapping

$$\Phi_A : \mathcal{B}^1 \to \mathcal{L}(X, X^{**}), \qquad f \mapsto f(A),$$

is bounded, with

$$\|f(A)\| \leqslant |f(\infty)| + \gamma_A \|f\|_{\mathcal{B}_0^1} \leqslant \gamma_A \|f\|_{\mathcal{B}^1}. \tag{2.3.14}$$

In order to show that this is, in fact, a suitable definition, it is best to proceed in steps, building-up on the preceding considerations.

Step 1. A direct calculation allows one to establish that (2.3.13) is consistent with the Hille-Phillips functional calculus, i.e. we obtain

Lemma 2.3.7 (Lemma 4.2, [4])**.** *Let $f \in \mathcal{LM}$ and $f(A)$ be as defined as in* (2.3.13)*. Then $f(A)$ coincides with the operator $x \mapsto \int_{\mathbb{R}_+} T(t)x\, d\mu(t)$ as defined in the Hille-Phillips functional calculus. In particular, $f(A) \in L(X)$.*

Consequently, for any f in the Hille-Phillips algebra, $f(A) : X \to X^{**}$ turns out to be a bounded operator on X.

Step 2. The next step is a version of Lemma 2.3.4 for operators. Let

$$\eta(z) = \frac{1 - e^{-z}}{z}, \qquad \eta_\delta(z) = \eta(\delta z), \quad \delta > 0.$$

Lemma 2.3.8 (Lemma 4.3, [4]). *Let $f \in \mathcal{B}^1_0$, and assume that $f\eta_\delta \in \mathcal{LM}$ for each $\delta > 0$. Then $\lim_{\delta\to 0+}(f\eta_\delta)(A)x$ exists in X for every $x \in X$. Moreover $f(A)x = \lim_{\delta\to 0+}(f\eta_\delta)(A)x$ for all $x \in X$, and thus $f(A) \in L(X)$.*

Since, by [4, Lemma 2.13], $f\eta_\delta \in \mathcal{LM}$, for any $f \in \mathcal{G}$, Lemma 2.3.8 implies that $f(A) \in L(X)$ whenever $f \in \mathcal{G}$.

Step 3. In the final step, one obtains the main result showing that the map Φ_A has the essential properties of a bounded functional calculus.

Theorem 2.3.9 (Theorem 4.4, [4]). *Under the assumptions* (2.3.9)*, the map $\Phi_A : f \mapsto f(A)$ is a bounded algebra homomorphism from $\mathcal{B}^1$ into $L(X)$, which extends the Hille-Phillips calculus. Moreover, $\|\Phi_A\| \leqslant \gamma_A$.*

Theorem 2.3.9 essentially follows from Lemma 2.3.8 and the fact that $\mathcal{G}$ is norm-dense in $\mathcal{B}^1_0$.

Necessity and uniqueness. Somewhat surprisingly, such obtained functional calculus turns out to be optimal. To make this remark precise, let us say that a *(bounded) $\mathcal{B}^1$-calculus* for A is a bounded algebra homomorphism $\Phi : \mathcal{B}^1 \to L(X)$ such that $\Phi(r_z) = (z + A)^{-1}$ for all $z \in \mathbb{C}_+$.

Remark 2.3.10. If $\Phi(r_z) = (z + A)^{-1}$ for some $z \in \mathbb{C}_+$, then the same equation holds for all $z \in \mathbb{C}_+$. Moreover, the existence of Φ, and the fact that $\|r_z\|_{\mathcal{B}^1_0} = 1/\operatorname{Re} z$, imply that $\|(z + A)^{-1}\| \leqslant \|\Phi\|/\operatorname{Re} z$. Hence A is sectorial of angle at most $\pi/2$.

The initially imposed resolvent assumption on A can then be shown to be a necessary condition for the existence of a $\mathcal{B}^1$-calculus.

Theorem 2.3.11 (Theorem 6.1, [5]). *Let A be an operator on X with $\sigma(A) \subseteq \overline{\mathbb{C}}_+$, and suppose that there exists a B^1-calculus for A. Then the resolvent assumption* (2.3.9) *holds.*

We say that A admits a $\mathcal{B}^1$-calculus if there exists such bounded homomorphism Φ and/or if A satisfies (2.3.9). The following result shows that the calculus given in (2.3.13) is unique.

Theorem 2.3.12 (Theorem 6.2, [5]). *Let A be an operator on X with $\sigma(A) \subseteq \overline{\mathbb{C}}_+$. If A admits $\mathcal{B}^1$-calculus Φ, then Φ is unique, in the sense that the following holds*

$$\langle \Phi(f)x, x^* \rangle = f(\infty) + \frac{2}{\pi} \langle g_{x,x^*}, f \rangle_{\mathcal{B}^1}$$

for all $f \in \mathcal{B}^1$, $x \in X$ and $x^* \in X^*$. Here $g_{x,x^*}(z) = \langle (z + A)^{-1}x, x^* \rangle$, $g_{x,x^*} \in \mathcal{E}^1$.

Spectral features. The statement below shows that the $\mathcal{B}^1$-calculus possesses the standard spectral features.

Theorem 2.3.13 (Theorem 4.17, [4]). *Let A be a densely defined operator on a Banach space X such that $\sigma(A) \subset \overline{\mathbb{C}}_+$ and (2.3.9) holds. Let $f \in \mathcal{B}^1$ and $\lambda \in \mathbb{C}$.*

- *If $x \in D(A)$ and $Ax = \lambda x$, then $f(A)x = f(\lambda)x$.*
- *If $x^* \in D(A^*)$ and $A^*x^* = \lambda x^*$, then $f(A)^*x^* = f(\lambda)x^*$.*
- *If $\lambda \in \sigma(A)$ then $f(\lambda) \in \sigma(f(A))$.*
- *If $A \in \mathrm{Sect}(\pi/2-)$, then $\sigma(f(A)) \cup \{f(\infty)\} = f(\sigma(A)) \cup \{f(\infty)\}$, where $\mathrm{Sect}(\pi/2-) = \bigcup_{\theta\in[0,\pi/2)} \mathrm{Sect}(\theta)$.*

Convergence Lemma. We now present a Convergence Lemma formulated in the context of $\mathcal{B}^1$.

Lemma 2.3.14 (Lemma 8.1, [5]). *Let $(f_n)_{n\geqslant 1} \subset \mathcal{B}^1$ be such that $\sup_{n\geqslant 1} \|f_n\|_{\mathcal{B}^1} < \infty$. Assume that for every $z \in \mathbb{C}_+$ there exists*

$$f_0(z) = \lim_{n\to\infty} f_n(z) \in \mathbb{C}, \tag{2.3.15}$$

and for every $r > 0$ one has

$$\lim_{\delta\to 0+} \int_0^\delta \sup_{|\beta|\leqslant r} |f_n'(\alpha + i\beta)|\, d\alpha = 0, \tag{2.3.16}$$

uniformly in n. Let $g \in \mathcal{B}^1$ with $\lim_{|z|\to\infty} g(z) = 0$, and let $g_n(z) = f_n(z)g(z), n \geqslant 0$. Then $f_0 \in \mathcal{B}^1$, $g_n \in \mathcal{B}^1_0, n \geqslant 0$, and

$$\lim_{n\to\infty} \|g_n - g_0\|_{\mathcal{B}^1} = 0.$$

From Lemma 2.3.14, one can then obtain an immediate corollary applicable to operators.

Corollary 2.3.15 (Corollary 8.3, [5]). *Let A be an operator which admits $\mathcal{B}^1$-calculus, and let f_n and f_0 be as in Lemma 2.3.14. Then $f_n(A) \to f_0(A)$ in the strong operator topology. If $A \in L(X)$, then the convergence is in operator-norm.*

By applying Lemma 2.3.14 to the function $t \mapsto f(tA)$, one gets

Theorem 2.3.16 (Theorem 8.4, [5]). *Let A be an operator which admits $\mathcal{B}^1$-calculus, and let $f \in \mathcal{B}^1$. Then $t \mapsto f(tA)$ is continuous from $\mathbb{R}_+$ to $L(X)$ with the strong operator topology. If $A \in L(X)$, then the map is continuous with respect to the norm topology on $L(X)$.*

Fine structure of $\sigma(f(A))$. Suppose that A admits $\mathcal{B}^1$-calculus. It has been established in [4, Theorem 4.17] that, for any $f \in \mathcal{B}$,

$$\lambda \in \sigma(A) \implies f(\lambda) \in \sigma(f(A)). \tag{2.3.17}$$

This result does not, however, immediately imply anything about the fine structure of $\sigma(f(A))$. As already mentioned in [4, Remark 4.18], it is of some interest to investigate whether there exists a connection between the eigenvalues of A and the eigenvalues of $f(A)$. The following proposition is a new result that makes this connection explicit.

Proposition 2.3.17. *Suppose that A admits $\mathcal{B}^1$-calculus. Then approximate λ-eigenvectors of A are approximate $f(\lambda)$-eigenvectors of $f(A)$.*

Proof. If A admits $\mathcal{B}^1$-calculus, then, by [5, Theorem 6.1], A satisfies

$$\sup_{\alpha>0} \alpha \int_{\mathbb{R}} |\langle(\alpha + i\beta + A)^{-2}x, x^*\rangle| \, d\beta < \infty.$$

In particular, $-A$ generates a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$. We shall mainly follow the approach used in [38, p. 20]. For each $t \geqslant 0$, T extends canonically to a bounded operator on $l^\infty(X)$ given by

$$\hat{T}(t)x = (T(t)x_n)_{n\in\mathbb{N}},$$

for any $x = (x_n)_{n\in\mathbb{N}} \in l^\infty(X)$; here $l^\infty(X)$ denotes the space of bounded sequences with elements in X. Let $l^\infty_T(X)$ denote the subspace

$$\{(x_n)_{n\in\mathbb{N}} \in l^\infty(X) : \lim_{t\to 0} \|T(t)x_n - x_n\| = 0, \text{ uniformly in } n\}.$$

Let $(\tilde{T}(t))_{t\geqslant 0}$ denote the restriction of $(\hat{T}(t))_{t\geqslant 0}$ to $l^\infty_T(X)$. It is immediate to see that $(\tilde{T}(t))_{t\geqslant 0}$ is a strongly continuous semigroup; let $-\tilde{A}$ be its generator.
Consider the quotient space $E^T = l^\infty_T(X)/\mathcal{N}$, with $\mathcal{N} = (c_0(X) \cap l^\infty_T(X))$ equipped with the quotient norm and note that

$$|||(x_n)_{n\in\mathbb{N}} + \mathcal{N}||| = \limsup_{n\to\infty} \|x_n\|.$$

Define $(T_E(t))_{t\geqslant 0}$ on E^T by

$$T_E(t)((x_n)_{n\in\mathbb{N}} + \mathcal{N}) = (T(t)(x_n)_{n\in\mathbb{N}}) + \mathcal{N},$$

and denote its generator by $-A_E$. Take the mapping

$$\Psi_{A_E} : \mathcal{B}^1 \to E^T,$$
$$f \mapsto q \circ \widehat{f(A)},$$

where $\widehat{f(A)}$ is the canonical extension of $f(A)$ to $l^\infty_T(X)$, and q denotes the quotient map $q(x) = x+\mathcal{N}$. This is well defined, since $\widehat{f(A)}$ leaves $c_0(X)\cap l^\infty_T(X)$ invariant. We have that $\|q\circ\widehat{f(A)}\| \leqslant \|f(A)\|$, so in particular Ψ_{A_E} is bounded, and, by construction, for $f, g \in \mathcal{B}^1$,

$$\Psi_{A_E}(fg) = q(\widehat{(fg)(A)}) = q(\widehat{f(A)}\widehat{g(A)}) = \Psi_{A_E}(f)\Psi_{A_E}(g).$$

Fix any $z \in \mathbb{C}_+$. Then, by [38, p.78], $z \in \rho(A) = \rho(A_E)$. We have, again by [38, p. 78],

$$\Psi_{A_E}(r_z)q(x) = q(((z + A)^{-1}x_n)_n) = (z + A_E)^{-1}q(x).$$

for any $x = (x_n)_{n\in\mathbb{N}} \in l^\infty(X)$. This implies that Ψ_{A_E} is a bounded algebra homomorphism that acts on r_z in the expected fashion. Thus, A_E admits the $\mathcal{B}^1$-calculus.
We want to show that any approximate λ-eigenvector of A is an $f(\lambda)$-eigenvector of $f(A)$. Suppose that $\lambda \in \sigma(A)$, and that we have

$$(\xi_n)_{n\in\mathbb{N}} \subset D(A), \quad \|\xi_n\| = 1, n \in \mathbb{N}, \quad \|(\lambda - A)\xi_n\| \to_{n\to\infty} 0.$$

Define a vector in $l^\infty_T(X)$

$$\hat{\xi} = (\xi_n)_{n\in\mathbb{N}} + \mathcal{N}.$$

Then it can be shown, by [38, p. 78], that $\hat{\xi} \in E^T$, and that $\lambda \in \sigma(A_E)$, with $A_E\hat{\xi} = \lambda\hat{\xi}$. By [4, Theorem 4.17], $f(A_E)\hat{\xi} = f(\lambda)\hat{\xi}$, i.e.

$$(f(A_E) - f(\lambda))\hat{\xi} = ((f(A_E) - f(\lambda))\xi_n)_{n\in\mathbb{N}} + \mathcal{N},$$

with

$$\|\!|(f(A_E) - f(\lambda))\xi_n\|\!| = \limsup_{n\to\infty} \|(f(A_E) - f(\lambda))\xi_n\| = 0.$$

But this implies that there is $n_k \in \mathbb{N}$ such that for all $n > n_k$,

$$\|(f(A_E) - f(\lambda))\xi_n\| < 1/k.$$

Hence, $(\xi_n)_{n\in\mathbb{N}}$ is an approximate $f(\lambda)$-eigenvector for $f(A)$. □

Compositions.

Proposition 2.3.18. *Let $f \in \mathcal{B}^1$, and $h : \mathbb{C}_+ \to \mathbb{C}_+$ such that h is holomorphic, maps $\mathbb{R}_+$ to $\mathbb{R}_+$, is monotonic increasing on $\mathbb{R}_+$, and*

$$\operatorname{Re} h(z) \geqslant c_1 h(\operatorname{Re} z), \quad |h'(z)| \leqslant c_2 h'(\operatorname{Re} z),$$

for some strictly positive constants c_1 and c_2. If $g(z) = f(h(z))$, then $g \in \mathcal{B}^1$.

Proof. Note first that, by the maximum principle for f',

$$x \mapsto \sup_{\beta \in \mathbb{R}} |f'(x + i\beta)|$$

is non-increasing on $(0, \infty)$. Calculating directly

$$\begin{aligned}
\int_0^\infty \sup_{\beta \in \mathbb{R}} |g'(\alpha + i\beta)| \, d\alpha &= \int_0^\infty \sup_{\beta \in \mathbb{R}} |h'(\alpha + i\beta) f'(h(\alpha + i\beta))| \, d\alpha \\
&\leqslant \int_0^\infty \sup_{\beta, \delta \in \mathbb{R}} c_2 |h'(\alpha + i\beta) f'(\operatorname{Re} h(\alpha + i\beta) + i \operatorname{Im} h(\alpha + i\delta))| \, d\alpha \\
&\leqslant \int_0^\infty \sup_{\delta \in \mathbb{R}} c_2 |h'(\alpha) f'(c_1 h(\alpha) + i\delta)| \, d\alpha \\
&= c_2 \int_{c_1 h(0)}^{c_1 h(\infty)} \sup_{\delta \in \mathbb{R}} |f'(u + i\delta)| \, du < \infty.
\end{aligned}$$

□

The class of functions satisfying the assumptions of Proposition 2.3.18 includes all Bernstein functions (cf. [4, Section 3.5]). Given the above result, one can now consider some questions concerning functional calculi. Suppose we are working with a Hilbert space X. We have the following result.

Proposition 2.3.19. *Let X be a Hilbert space. Suppose that $-A$ generates a bounded C_0-semigroup on X, $f \in \mathcal{B}^1$, and h is as in the preceding proposition. Then if $h(A)$ is defined in the half-plane calculus and $-h(A)$ generates a bounded C_0-semigroup,*

$$f(h(A)) = (f \circ h)(A),$$

where both sides are given by the $\mathcal{B}^1$-calculus.

Here, half-plane calculus is understood in the sense of [7] (cf. also [24, Section 8]).

Proof. Since $h(A)$ is a negative semigroup generator on a Hilbert space, $h(A)$ satisfies the condition (i) in Theorem 2.1.10. Hence, by [4, Proposition 4.11 and Remarks 4.12], $f(h(A))$ exists and

$$f(h(A)) = \lim_{a \to 0+} f(h(A) + a),$$

where the left-hand side is given by the $\mathcal{B}^1$-calculus, the right-hand side limit exists in the operator norm, and each of the operators $f(h(A)+a)$ is defined by the half-plane calculus (as before, half-plane calculus is understood in the sense of [7], cf. also [24, Section 8])). On the other hand,

$$(f \circ h)(A) = \lim_{a \to 0+} (f_a \circ h)(A),$$

by convergence lemma for the half-plane calculus (since $f \circ h \in \mathcal{B}^1$, and A is of the half-plane type, $(f \circ h)(A)$ exists in the extended half-plane calculus and agrees with its $\mathcal{B}^1$ incarnation by the compatibility result [4, Proposition 4.11]). So it is then enough to show that, for each $a > 0$,

$$f(h(A) + a) = (f_a \circ h)(A).$$

Since $(f_a \circ h) \in \mathcal{B}^1$, it is bounded, and hence $\tau = r_1^2$ can serve as its regulariser. We have

$$\begin{aligned}
\tau(A)(f_a(h(A)) &= \frac{1}{2\pi i} \int_{\operatorname{Re}\lambda = \eta} f(\lambda)\tau(A)R(\lambda, (h(A) + a))d\lambda \\
&= \frac{1}{2\pi i} \int_{\operatorname{Re}\lambda = \eta} f(\lambda) \left(\frac{1}{(1+z)^2(\lambda - (h(z) + a))} \right)(A)\, d\lambda \\
&= \frac{1}{(2\pi i)^2} \int_{\operatorname{Re}\lambda = \eta} \int_{\operatorname{Re} z = \eta'} f(\lambda) \frac{1}{(1+z)^2(\lambda - (h(z) + a))} R(z, A)\, dz\, d\lambda \\
&= \frac{1}{2\pi} \int_{\operatorname{Re} z = \eta'} \frac{f(h(z) + a)}{(1+z)^2(\lambda - (h(z) + a))} R(z, A)\, dz \\
&= ((f_a \circ h) \cdot \tau)(A).
\end{aligned}$$

Injectivity of $\tau(A)$ implies $f_a(h(A)) = \tau(A)^{-1}(f_a(h) \cdot \tau)(A)$, and the conclusion follows from the definition of the extended half-plane calculus. □

3 The class $\mathcal{B}^2$ – preliminaries

In the paper [50], Pascale Vitse introduced and studied a functional calculus for negative generators of bounded holomorphic C_0-semigroups. In his PhD thesis, Steven White extended Peller's methods and results to study polynomials in two commuting power bounded operators ([51, Chapter 4]), and obtained ([51, Section 5.5]) a Besov estimate for generators of bounded semigroups on Hilbert spaces and Schwartz functions. Charles Batty, Alexander Gomilko, and Yuri Tomilov in [4] and [5] have recently refined and generalised Vitse's construction, relaxing the assumptions imposed on the admissible class of operators. The class of functions lying at the heart of those approaches is that of the so-called analytic Besov functions and is denoted by $\mathcal{B}^1$ (see Section 2.3 in Chapter 2 for a discussion).

Our primary aim in Chapter 3 is to obtain and describe a two-dimensional analogue of the class $\mathcal{B}^1$. We shall see that the class $\mathcal{B}^2$, as defined below, shares many characteristics with its one-dimensional counterpart. In Section 3.1, we write down the definitions and establish some elementary facts about our class $\mathcal{B}^2$. With this in place, we characterise Banach space properties of $\mathcal{B}^2$ in Section 3.2. We prove that $\mathcal{B}^2$ is indeed a Banach space with respect to the norm $\|\cdot\|_{\mathcal{B}^2}$ and that, for any $f \in \mathcal{B}^2$, the limiting functions $f_1(z_1) = \lim_{\operatorname{Re} z_2 \to \infty} f(z_1, z_2)$ and $f_2(z_2) = \lim_{\operatorname{Re} z_1 \to \infty} f(z_1, z_2)$ are both in $\mathcal{B}^1$, with $f_1(\infty) = f_2(\infty)$. As shown in Section 3.3, $\mathcal{B}^2$ is a Banach algebra, and a subalgebra of $H^\infty(\mathbb{C}_+^2)$. In Section 3.4 we establish the equivalence of some other norms $\mathcal{B}^2$. We provide a brief discussion of the relations between $\mathcal{B}^1$ and $\mathcal{B}^2$ in Section 3.5, and conclude the present chapter with examples of functions in $\mathcal{B}^2$ in Sections 3.6–3.8.

3.1 The space $H^\infty(\mathbb{C}_+^2)$

Throughout, we will be considering complex functions of two (complex) variables. We begin by surveying the known characterisations and basic properties of the space $H^\infty(\mathbb{C}_+^2)$. For a parallel discussion concerning functions of one variable, see [4] and references therein, and our Section 2.3.

We shall consider the space of bounded holomorphic functions on $\mathbb{C}_+^2$, equipped with the supremum norm and denoted by $H^\infty(\mathbb{C}_+^2)$. It is a standard result (cf. e.g. [49, Theorem 3.24], [47, p. 112]) that each $f \in H^\infty(\mathbb{C}_+^2)$ has a boundary function, $f^b \in L^\infty(\mathbb{R}^2)$, given by the Fatou relation

$$f^b(s_1, s_2) = f(is_1, is_2) = \lim_{t \to 0^+} f(t + is_1, t + is_2) \quad \text{a.e.} \tag{3.1.1}$$

The map $f \mapsto \tilde{f}$, where

$$\tilde{f}(w_1, w_2) = f\left(\frac{1 - w_1}{1 + w_1}, \frac{1 - w_2}{1 + w_2}\right)$$

is an isometric isomorphism of $H^\infty(\mathbb{C}_+^2)$ onto $H^\infty(\mathbb{D}^2)$; here $\mathbb{D}$ is the unit disc in $\mathbb{C}$. The existence of boundary functions in the $H^\infty(\mathbb{D}^2)$ case is proved e.g. in [44, Theorem 2.3.1], and the corresponding result for $H^\infty(\mathbb{C}_+^2)$ can be deduced from this.

The original f can then be recovered from f^b via the Poisson-formula (cf. [49, Chapter 3, Section 5]; see also [47, Section 2])

$$f(x_1 + iy_1, x_2 + iy_2) = \iint_{\mathbb{R}^2} f^b(y_1 - s_1, y_2 - s_2) p(s_1, s_2, x_1, x_2) ds_1\, ds_2, \tag{3.1.2}$$

where $x_1, x_2 > 0, y_1, y_2 \in \mathbb{R}$, with the two-dimensional Poisson kernel given by (cf. [49, p. 115])

$$p(s_1, s_2, x_1, x_2) = \frac{x_1 x_2}{\pi^2 (s_1^2 + x_1^2)(s_2^2 + x_2^2)}, \quad x_1, x_2 > 0, s_1, s_2 \in \mathbb{R}.$$

Moreover (cf. [47, Corollary 3.3]),

$$\|f^b\|_{L^\infty(\mathbb{R}^2)} = \|f\|_\infty, \tag{3.1.3}$$

and we have the following.

Theorem 3.1.1 (Theorems 2.2 and 2.4, [33])**.** *The mapping $f \mapsto f^b$ maps the space $H^\infty(\mathbb{C}_+^2)$ onto the space $H^\infty(\mathbb{R}_+^2)$ of those functions in $L^\infty(\mathbb{R}^2)$ for which the inverse distributional Fourier transform has support in $\mathbb{R}_+^2$, i.e.* $\operatorname{supp}(\mathcal{F}_{(2)}^{-1})(f^b) \subset \mathbb{R}_+^2$.

Theorem 3.1.1 is a special instance of higher-dimensional generalizations of the Paley–Wiener Theorem (see [33, Section 2] for a comprehensive discussion).

If $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ is a two-dimensional multiindex, and D^α is the corresponding higher-order partial derivative, then there is an absolute constant, C_α, such that

$$|D^\alpha f(z_1, z_2)| \leqslant \frac{C_\alpha \|f\|_\infty}{(\operatorname{Re} z_1)^{\alpha_1} (\operatorname{Re} z_2)^{\alpha_2}}, \quad (\alpha_1, \alpha_2) \neq (0, 0). \tag{3.1.4}$$

Given the Poisson formula (3.1.2), the estimate (3.1.4) can be justified by differentiating through the integral sign in (3.1.2) and estimating the appropriate norms. The inequality (3.1.4) can alternatively be deduced from the one-variable estimate [4, Lemma 2.1(3)] applied to functions $f(z_1, \cdot)$ and $D_2^k f(\cdot, z_2)$.

3.2 The Banach space $\mathcal{B}^2$

Let $f : \mathbb{C}_+^2 \to \mathbb{C}$ be any bounded holomorphic function. Define

$$\|f\|_{\mathcal{B}_0^2} = \iint\limits_{\mathbb{R}_+^2} \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)|\, dx_1\, dx_2.$$

We will be interested in functions f for which $\|f\|_{\mathcal{B}_0^2}$ is finite. Note that, since f is assumed to be holomorphic, one has

$$\|f\|_{\mathcal{B}_0^2} = 0 \Leftrightarrow D_1 D_2 f = 0 \Leftrightarrow f(z_1, z_2) = g_1(z_1) + g_2(z_2),$$

where g_1 and g_2 are some (bounded) holomorphic functions on $\mathbb{C}_+$. It is not clear whether the space of all bounded functions f such that $\|f\|_{\mathcal{B}_0^2} < \infty$ is an algebra. In order to ensure that we obtain an algebra, let

$$\|f\|_{1*} = \int_0^\infty \sup_{\substack{y_1 \in \mathbb{R} \\ z_2 \in \mathbb{C}_+}} |D_1 f(x_1 + iy_1, z_2)|\, dx_1,$$

$$\|f\|_{2*} = \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 f(z_1, x_2 + iy_2)|\, dx_2.$$

We define $\mathcal{B}^2$ to be the space of those bounded holomorphic functions f on $\mathbb{C}_+^2$ such that

$$\begin{aligned} \|f\|_{\mathcal{B}_0^2} + \|f\|_{1*} + \|f\|_{2*} &= \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)|\, dx_1\, dx_2 \\ &+ \int_0^\infty \sup_{\substack{y_1 \in \mathbb{R} \\ z_2 \in \mathbb{C}_+}} |D_1 f(x_1 + iy_1, z_2)|\, dx_1 \\ &+ \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 f(z_1, x_2 + iy_2)|\, dx_2 < \infty. \end{aligned}$$

In other words,

$$\mathcal{B}^2 = \{f \in H^\infty(\mathbb{C}_+^2) : \|f\|_{\mathcal{B}_0^2} + \|f\|_{1*} + \|f\|_{2*} < \infty\},$$

and we endow $\mathcal{B}^2$ with the norm

$$\|f\|_{\mathcal{B}^2} = \|f\|_{\mathcal{B}_0^2} + \|f\|_{1*} + \|f\|_{2*} + \|f\|_\infty.$$

Proposition 3.2.1. *$\mathcal{B}^2$ with the norm $\|\cdot\|_{\mathcal{B}^2}$ is a Banach space.*

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{B}^2}$. Then, in particular, it is Cauchy with respect to $\|\cdot\|_\infty$. Hence, it converges uniformly on $\mathbb{C}_+^2$ to some f. By Weierstrass's limit theorem, f is holomorphic. It is now enough to show that $\|f\|_{\mathcal{B}^2} < \infty$ and that $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to $\|\cdot\|_{\mathcal{B}^2}$. We have

$$\|f\|_\infty = \lim_{n\to\infty} \|f_n\|_\infty.$$

Moreover, for $x + iy \in \mathbb{C}_+$ and $z \in \mathbb{C}_+$,

$$|D_1 f(x+iy, z_2)| = \lim_{n\to\infty} |D_1 f_n(x+iy, z_2)| \leqslant \liminf_{n\to\infty} \sup_{\substack{s\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(x+is, z_2)|.$$

From this and Fatou's lemma, it follows that

$$\begin{aligned}\int_0^\infty \sup_{\substack{y\in\mathbb{R},\\ z_2\in\mathbb{C}_+}} |D_1 f(x+iy, z_2)|\, dx &\leqslant \int_0^\infty \liminf_{n\to\infty} \sup_{\substack{s\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(x+is, z_2)|\, dx \\ &\leqslant \liminf_{n\to\infty} \int_0^\infty \sup_{\substack{s\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(x+is, z_2)|\, dx,\end{aligned}$$

and similarly for $D_2 f$. Let $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C}_+$. Then

$$\begin{aligned}|D_1 D_2 f(x_1+iy_1, x_2+iy_2)| &= \lim_{n\to\infty} |D_1 D_2 f_n(x_1+iy_1, x_2+iy_2)| \\ &\leqslant \liminf_{n\to\infty} \sup_{s_1,s_2\in\mathbb{R}} |D_1 f_n(x_1+is_1, x_2+is_2)|.\end{aligned}$$

Hence, applying Fatou's lemma once more,

$$\begin{aligned}&\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2 \\ &\qquad \leqslant \int_0^\infty \int_0^\infty \liminf_{n\to\infty} \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f_n(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2 \\ &\qquad \leqslant \liminf_{n\to\infty} \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f_n(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2.\end{aligned}$$

All in all,

$$\begin{aligned}\|f\|_{\mathcal{B}^2} &\leqslant \liminf_{n\to\infty} \|f_n\|_\infty \\ &+ \liminf_{n\to\infty} \int_0^\infty \sup_{\substack{y_1\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(x_1+iy_1, z_2)|\, dx_1 \\ &+ \liminf_{n\to\infty} \int_0^\infty \sup_{\substack{y_2\in\mathbb{R}\\ z_1\in\mathbb{C}_+}} |D_1 f_n(z_1, x_2+iy_2)|\, dx_2 \\ &+ \liminf_{n\to\infty} \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 f_n(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2 \\ &\leqslant \liminf_{n\to\infty} \|f_n\|_{\mathcal{B}^2} < \infty.\end{aligned}$$

Hence, $f \in \mathcal{B}^2$ and $\|f\|_{\mathcal{B}^2} < \infty$. Replacing f_n by $f_n - f_m$ with fixed $m \in \mathbb{N}$, and then letting $n \to \infty$, one obtains upper bounds for $\|f - f_m\|_{\mathcal{B}^2}$, i.e.

$$\|f - f_m\|_{\mathcal{B}^2} \leqslant \liminf_{n\to\infty} \|f_n - f_m\|_{\mathcal{B}^2} \to 0 \quad \text{as } m \to \infty,$$

by the Cauchy condition, which establishes that $(f_n)_{n\in\mathbb{N}}$ converges to f with respect to $\|\cdot\|_{\mathcal{B}^2}$. □

We note the following elementary properties of functions $f \in \mathcal{B}^2$.

Proposition 3.2.2. *Let $f \in \mathcal{B}^2$. Then the limits*

$$f(z_1, \infty) = \lim_{\operatorname{Re} z_2\to\infty} f(z_1, z_2),$$

$$f(\infty, z_2) = \lim_{\operatorname{Re} z_1\to\infty} f(z_1, z_2),$$

exist, so that the functions $f_1(z) = f(z, \infty)$, $f_2(z) = f(\infty, z)$ are well-defined and satisfy

$$\int_0^\infty \sup_{y\in\mathbb{R}} |f_j'(x+iy)|\, dx < \infty, \quad j = 1, 2.$$

Furthermore, the limits $f_j(\infty) = \lim_{\operatorname{Re} z\to\infty} f_j(z)$ exist in $\mathbb{C}$ for $j = 1, 2$ and are equal, i.e.

$$f_1(\infty) = f_2(\infty).$$

Moreover, the limit $\lim_{\operatorname{Re} z_1, \operatorname{Re} z_2\to\infty} f(z_1, z_2)$ exists and we have

$$\lim_{\substack{\operatorname{Re} z_1\to\infty\\ \operatorname{Re} z_2\to\infty}} f(z_1, z_2) = f_1(\infty) = f_2(\infty).$$

Proof. Without loss of generality, fix $z_1 \in \mathbb{C}_+$. Since

$$\int_0^\infty \sup_{y_2 \in \mathbb{R}} |D_2 f(z_1, x_2 + iy_2)|\, dx_2 \leqslant \sup_{z \in \mathbb{C}_+} \int_0^\infty \sup_{y_2 \in \mathbb{R}} |D_2 f(z, x_2 + iy_2)|\, dx_2 \leqslant \|f\|_{2*},$$

$f(z_1, \cdot) \in \mathcal{B}^1$ and Proposition 2.3.1 implies that $f_1(z_1)$ exists. Let $(z_{2,n})_{n\in\mathbb{N}} \subset \mathbb{C}_+$ be any sequence such that $\operatorname{Re} z_{2,n} \to \infty$ as $n \to \infty$. Then $(f(\cdot, z_{2,n}))_{n\in\mathbb{N}}$ is a (locally) bounded family of holomorphic functions such that $\lim_{n\to\infty} f(z, z_{2,n})$ exists for each $z \in \mathbb{C}_+$. Then, by Vitali's theorem [45, p. 44], $(f(\cdot, z_{2,n}))_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{C}_+$ to a holomorphic function. This establishes analyticity of $f_1(z)$. Now, one has

$$\|f_1\|_\infty \leqslant \liminf_{n\to\infty} \|f(\cdot, z_{2,n})\|_\infty.$$

Moreover, for $x + iy \in \mathbb{C}_+$,

$$|f_1'(x+iy)| = \lim_{n\to\infty} |D_1 f(x+iy, z_{2,n})| \leqslant \liminf_{n\to\infty} \sup_{s\in\mathbb{R}} |D_1 f(x+is, z_{2,n})|.$$

Fatou's lemma then yields

$$\begin{aligned} \int_0^\infty \sup_{y\in\mathbb{R}} |f_1'(x+iy)|\, dx &\leqslant \int_0^\infty \liminf_{n\to\infty} \sup_{y\in\mathbb{R}} |D_1 f(x+iy, z_{2,n})|\, dx \\ &\leqslant \liminf_{n\to\infty} \int_0^\infty \sup_{y\in\mathbb{R}} |D_1 f(x+iy, z_{2,n})|\, dx \\ &\leqslant \sup_{z_2\in\mathbb{C}_+} \int_0^\infty \sup_{y\in\mathbb{R}} |D_1 f(x+iy, z_2)|\, dx \leqslant \|f\|_{1*} < \infty. \end{aligned}$$

Since we showed

$$\int_0^\infty \sup_{y\in\mathbb{R}} |f_j'(x+iy)|\, dx < \infty, \quad j = 1, 2,$$

so that $f_j \in \mathcal{B}^1$, Proposition 2.3.1 then implies that $f_1(\infty)$ and $f_2(\infty)$ exist. Moreover, for fixed $z_1, z_2 \in \mathbb{C}_+$,

$$\begin{aligned} |f_1(\infty) - f_2(\infty)| &\leqslant |f_1(\infty) - f(z_1, \infty)| + |f(z_1,\infty) - f(z_1, z_2)| \\ &\quad + |f(z_1, z_2) - f_2(\infty, z_2)| + |f(\infty, z_2) - f_2(\infty)|. \end{aligned}$$

Each of the functions $f_1(\cdot), f_2(\cdot), f(\cdot, z_2), f(z_1, \cdot)$ is in the class $\mathcal{B}^1$, so that we have

$$|f_j(\infty) - f_j(z_1)| \leqslant \int_{\operatorname{Re} z_1}^{\infty} \sup_{s \in \mathbb{R}} |f_j'(t+is)|\, dt, \quad j = 1, 2,$$

$$|f(z_1, \infty) - f(z_1, z_2)| \leqslant \int_{\operatorname{Re} z_2}^{\infty} \sup_{s \in \mathbb{R}} |D_2 f(z_1, t+is)|\, dt,$$

$$|f(z_1, z_2) - f_2(\infty, z_2)| \leqslant \int_{\operatorname{Re} z_1}^{\infty} \sup_{s \in \mathbb{R}} |D_1 f(t+is, z_2)|\, dt.$$

Letting $\operatorname{Re} z_1 \to \infty$ and then $\operatorname{Re} z_2 \to \infty$ yields the conclusion that $f_1(\infty) = f_2(\infty)$. As for the final statement, we have

$$\begin{aligned} |f(z_1, z_2) - f_1(\infty)| &\leqslant |f(z_1, z_2) - f(z_1, \infty)| + |f(z_1, \infty) - f_1(\infty)| \\ &\leqslant \int_{\operatorname{Re} z_2}^{\infty} \sup_{s \in \mathbb{R}} |D_2 f(z_1, t+is)|\, dt + \int_{\operatorname{Re} z_1}^{\infty} \sup_{s \in \mathbb{R}} |f_1'(t+is)|\, dt \\ &\leqslant \int_{\operatorname{Re} z_2}^{\infty} \sup_{\substack{s \in \mathbb{R}, \\ z \in \mathbb{C}_+}} |D_2 f(z, t+is)|\, dt + \int_{\operatorname{Re} z_1}^{\infty} \sup_{s \in \mathbb{R}} |f_1'(t+is)|\, dt. \end{aligned}$$

□

Remark 3.2.3. Similarly to the one-dimensional case, the boundedness assumption can be dropped in the definition of $\mathcal{B}^2$. For consider $(f(\cdot, z_{2,n}))_{n \in \mathbb{N}}$ as in the proof of Proposition 3.2.2. Let us first show that this is a (locally) bounded family of holomorphic functions. Each function in the sequence is in $\mathcal{B}^1$, hence is bounded. Then we have that, for each $n \in \mathbb{N}$ and any $x > 0$,

$$\begin{aligned} |f(\infty, z_{2,n})| &\leqslant |f(x, z_{2,n})| + \int_x^{\infty} |D_1 f_n(t, z_{2,n})|\, dt. \\ &\leqslant |f(x, z_{2,n})| + \sup_{z_2 \in \mathbb{C}_+} \int_x^{\infty} |D_1 f(t, z_2)|\, dt, \end{aligned}$$

But the sequence $(f(x, z_{2,n}))_{n \in \mathbb{N}}$ converges for any particular choice of $x > 0$, hence is bounded. So the right-hand side is bounded by some constant and thus, for some $c > 0$

$$\sup_{m \in \mathbb{N}} |f(\infty, z_{2,m})| < c.$$

Now, for any $z \in \mathbb{C}_+$ and any $n \in \mathbb{N}$,

$$
\begin{aligned}
|f(z, z_{2,n})| &\leqslant |f(\infty, z_{2,n})| + |f(z, z_{2,n}) - f(\infty, z_{2,n})| \\
&\leqslant |f(\infty, z_{2,n})| + \int_x^\infty \sup_{s\in\mathbb{R}} |D_1 f(t+is, z_{2,n})|\, dt \\
&\leqslant \sup_{m\in\mathbb{N}} |f(\infty, z_{2,m})| + \sup_{z_2\in\mathbb{C}_+} \int_0^\infty \sup_{s\in\mathbb{R}} |D_1 f(t+is, z_2)|\, dt,
\end{aligned}
$$

which shows that $(f(\cdot, z_{2,n}))_{n\in\mathbb{N}}$ is indeed a locally bounded family. Take $f \in \mathcal{B}^2$. We now have by Proposition 3.2.2 that f_1 is well-defined and belongs to $\mathcal{B}^1$ and in particular f_1 is a bounded holomorphic function, and $f_1(\infty)$ exists in $\mathbb{C}$ by Proposition 2.3.1. Finally,

$$
\begin{aligned}
|f_1(\infty) - f(z_1, z_2)| &\leqslant |f_1(\infty) - f_1(z_1)| + |f(z_1, \infty) - f(z_1, z_2)| \\
&\leqslant 2\|f_1\|_\infty + \sup_{z_1\in\mathbb{C}_+} \int_0^\infty \sup_{s\in\mathbb{R}} |D_2 f(z_1, t+is)|\, dt \leqslant 2\|f_1\|_\infty + \|f\|_{2*}.
\end{aligned}
$$

So then it follows that f is bounded as required.

We shall need the following auxiliary result.

Proposition 3.2.4. *If $f \in \mathcal{B}^2$, then f is uniformly continuous on $\mathbb{C}_+^2$. Consequently, f extends to a uniformly continuous function $\overline{f}$ on $\overline{\mathbb{C}_+^2}$, and has a boundary function*

$$
f^b(s_1, s_2) = \lim_{\substack{z_1, z_2 \in \mathbb{C}_+ \\ (z_1, z_2)\to(is_1, is_2)}} f(z_1, z_2),
$$

where the limit is uniform in (s_1, s_2).

Proof. Let $f : \mathbb{C}_+^2 \to \mathbb{C}$ be holomorphic, and let

$$
\begin{aligned}
h_1(x) &= \sup_{\substack{y\in\mathbb{R} \\ z\in\mathbb{C}_+}} |D_1 f(x+iy, z)|, \quad x \in \mathbb{R}_+, \\
h_2(x) &= \sup_{\substack{y\in\mathbb{R} \\ z\in\mathbb{C}_+}} |D_2 f(z, x+iy)|, \quad x \in \mathbb{R}_+.
\end{aligned}
$$

Assume that

$$
\int_0^\infty h_i(x)\, dx < \infty, \quad i = 1, 2. \tag{3.2.1}
$$

Note that (3.2.1) is satisfied if $f \in \mathcal{B}^2$, since then we have both $\|f\|_{1*} < \infty$ and $\|f\|_{2*} < \infty$. We show that f is uniformly continuous on $\mathbb{C}_+^2$.

Let $\delta > 0$ and take $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{C}^2$ such that $|z - w| < \delta$. We want to estimate

$$|f(z_1, z_2) - f(w_1, w_2)|.$$

We have

$$|f(z_1, z_2) - f(w_1, w_2)| \leqslant |f(z_1, z_2) - f(w_1, z_2)| + |f(w_1, z_2) - f(w_2, w_2)|.$$

Consider first

$$|f(z_1, z_2) - f(w_1, z_2)|$$

with $|z_1 - w_1| < \delta$. Let $z_1 = x_1 + iy_1, w_1 = u_1 + iv_1 \in \mathbb{C}_+$. Let γ be a path in $\mathbb{C}_+$ from z_1 to w_1 of the following form

- either γ is one line-segment from z_1 to w_1, with gradient in $[-1, 1]$, or
- γ is the union of two line-segments, γ_1 with gradient 1 and γ_2 with gradient -1.

Each line segment is of length at most $\sqrt{2}\delta$. Now

$$f(w_1, z_2) - f(z_1, z_2) = \int_\gamma D_1 f(\xi, z_2)\, d\xi.$$

On each line-segment γ_i, $i = 1, 2$,

$$\left|\int_{\gamma_i} D_1 f(\xi, z_2)\, d\xi\right| \leqslant \int_{I_i} \sqrt{2} \sup_{z \in \mathbb{C}_+} |D_1 f(\xi(x), z)|\, d\xi$$

$$\leqslant \sqrt{2} \int_{I_i} \sup_{\substack{y \in \mathbb{R} \\ z \in \mathbb{C}_+}} |D_1 f(x + iy, z)|\, dx.$$

Here $I_i = \{\operatorname{Re} \xi : \xi \in \gamma_i\}$, which is an interval of length at most δ. Hence,

$$\left|\int_\gamma D_1 f(\xi, z_2)\, d\xi\right| \leqslant 2\sqrt{2} \int_I \sup_{\substack{y \in \mathbb{R} \\ z \in \mathbb{C}_+}} |D_1 f(x + iy, z)|\, dx = 2\sqrt{2} \int_I h_1(x)\, dx.$$

where I is also an interval of length at most δ. Let $\varepsilon > 0$. Since $h_1 \in L^1(\mathbb{R})$, there exists $\delta_1 > 0$ such that

$$\int_I h_1(x)\, dx < \varepsilon,$$

whenever I is an interval (or a measurable set) in $(0, \infty)$ of length (or measure) at most δ_1. Thus, δ_1 depends only on ε and f, i.e. δ_1 is independent of (z_1, z_2) and (w_1, w_2). Consequently,

$$|f(z_1, z_2) - f(w_1, z_2)| \leqslant 2\sqrt{2}\, \varepsilon, \quad \text{whenever } |z_1 - w_1| < \delta_1.$$

Similarly, we can show that there exists $\delta_2 > 0$ such that

$$|f(w_1, z_2) - f(w_1, w_2)| \leqslant 2\sqrt{2}\,\varepsilon, \quad \text{whenever } |z_2 - w_2| < \delta_2.$$

Hence,

$$|f(z_1, z_2) - f(w_1, w_2)| \leqslant 4\sqrt{2}\varepsilon, \quad \text{whenever } |(z_1, z_2) - (w_1, w_2)| < \min(\delta_1, \delta_2),$$

as required. □

We will often consider the subspace

$$\mathcal{B}_0^2 = \{f \in \mathcal{B}^2 : f(z_1, \infty) = f(\infty, z_2) = 0\}.$$

As we shall see below, $\|\cdot\|_{\mathcal{B}^2}$ and $\|\cdot\|_{\mathcal{B}_0^2}$ are equivalent norms on $\mathcal{B}_0^2$.

3.3 $\mathcal{B}^2$ is a Banach algebra

Take any $f, g \in \mathcal{B}^2$; we want the pointwise product fg to be in $\mathcal{B}^2$ with $\|fg\|_{\mathcal{B}^2} \leqslant \|f\|_{\mathcal{B}^2}\|g\|_{\mathcal{B}^2}$. We have, for example

$$\begin{aligned}
&\int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2(f(z_1, x_2 + iy_2)g(z_1, x_2 + iy_2))|\, dx_2 \\
&\qquad \leqslant \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |f(z_1, x_2 + iy_2) D_2 g(z_1, x_2 + iy_2)|\, dx_2 \\
&\qquad + \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |g(z_1, x_2 + iy_2) D_2 f(z_1, x_2 + iy_2)|\, dx_2.
\end{aligned}$$

Since f and g are both bounded, the right-hand side is then bounded from above by

$$\|f\|_\infty \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 g(z_1, x_2 + iy_2)|\, dx_2 + \|g\|_\infty \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 f(z_1, x_2 + iy_2)|\, dx_2,$$

which is finite by assumption, and similarly for the factor with only D_1 appearing in it. Now, as for the part of $\|\cdot\|_{\mathcal{B}^2}$ involving the mixed derivative, we have

$$\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 \left(f(x_1+iy_1, x_2+iy_2) g(x_1+iy_1, x_2+iy_2) \right)| \, dx_1 \, dx_2$$

$$\leqslant \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |g(x_1+iy_1, x_2+iy_2) D_1 D_2 f(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2$$

$$+ \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |f(x_1+iy_1, x_2+iy_2) D_1 D_2 g(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2$$

$$+ \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_2 f(x_1+iy_1, x_2+iy_2) D_1 g(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2$$

$$+ \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 f(x_1+iy_1, x_2+iy_2) D_2 g(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2.$$

The first two factors appearing on the right-hand side are dominated by

$$\|g\|_\infty \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2,$$

and

$$\|f\|_\infty \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 g(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2,$$

respectively, hence they are finite by the assumption that $f, g \in \mathcal{B}^2$. As for the remaining pair, we have for instance

$$\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_2 f(x_1+iy_1, x_2+iy_2) D_1 g(x_1+iy_1, x_2+iy_2)| \, dx_1 \, dx_2$$

$$\leqslant \int_0^\infty \sup_{\substack{y_2\in\mathbb{R},\\ z_1\in\mathbb{C}_+}} |D_2 f(z_1, x_2+iy_2)| \, dx_2 \int_0^\infty \sup_{\substack{y_1\in\mathbb{R},\\ z_2\in\mathbb{C}_+}} |D_1 g(x_1+iy_1, z_2)| \, dx_1 \leqslant \|f\|_{2*} \|g\|_{1*}.$$

Putting all this together, we have shown that

$$\|fg\|_{\mathcal{B}^2} \leqslant \|f\|_\infty \left(\int_0^\infty \sup_{\substack{y_1\in\mathbb{R} \\ z_2\in\mathbb{C}_+}} |D_1 g(x_1+iy_1, z_2)|\, dx_1 + \int_0^\infty \sup_{\substack{y_2\in\mathbb{R} \\ z_1\in\mathbb{C}_+}} |D_2 g(z_1, x_2+iy_2)|\, dx_2 \right.$$

$$\left. + \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 g(x_1+iy_1, x_2+iy_2)|\, dx_1, dx_2 \right)$$

$$+ \|g\|_\infty \left(\int_0^\infty \sup_{\substack{y_1\in\mathbb{R} \\ z_2\in\mathbb{C}_+}} |D_1 f(x_1+iy_1, z_2)|\, dx_1 + \int_0^\infty \sup_{\substack{y_2\in\mathbb{R} \\ z_1\in\mathbb{C}_+}} |D_2 f(z_1, x_2+iy_2)|\, dx_2 \right.$$

$$\left. + \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f(x_1+iy_1, x_2+iy_2)|\, dx_1 dx_2 \right)$$

$$+ \int_0^\infty \sup_{\substack{y_2\in\mathbb{R} \\ z_1\in\mathbb{C}_+}} |D_2 f(z_1, x_2+iy_2)|\, dx_2 \int_0^\infty \sup_{\substack{y_1\in\mathbb{R} \\ z_2\in\mathbb{C}_+}} |D_1 g(x_1+iy_1, z_2)|\, dx_1$$

$$+ \int_0^\infty \sup_{\substack{y_2\in\mathbb{R} \\ z_1\in\mathbb{C}_+}} |D_2 g(z_1, x_2+iy_2)|\, dx_2 \int_0^\infty \sup_{\substack{y_1\in\mathbb{R} \\ z_2\in\mathbb{C}_+}} |D_1 f(x_1+iy_1, z_2)|\, dx_1 + \|f\|_\infty \|g\|_\infty$$

$$\leqslant \|f\|_{\mathcal{B}^2} \|g\|_{\mathcal{B}^2}.$$

3.4 Equivalence of norms

It will sometimes be convenient to consider alternative norms on $\mathcal{B}^2$. In the present section, we define two such norms and establish that they are both equivalent to $\|\cdot\|_{\mathcal{B}^2}$.

Lemma 3.4.1. *Let $f \in \mathcal{B}^2$. Define*

$$|||f|||_{\mathcal{B}^2} = \|f\|_\infty + \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2$$

$$+ \sup_{z_1\in\mathbb{C}_+} \int_0^\infty \sup_{y_2\in\mathbb{R}} |D_2 f(z_1, x_2+iy_2)|\, dx_2$$

$$+ \sup_{z_2\in\mathbb{C}_+} \int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1 f(x_1+iy_1, z_2)|\, dx_1.$$

Then $\|\cdot\|_{\mathcal{B}^2}$ and $|||\cdot|||_{\mathcal{B}^2}$ are equivalent norms on $\mathcal{B}^2$.

Proof. Clearly, $|||\cdot|||_{\mathcal{B}^2} \leqslant \|\cdot\|_{\mathcal{B}^2}$, so it is enough to show that the reverse inequality holds with some constant, c, i.e. that

$$\|\cdot\|_{\mathcal{B}^2} \leqslant c|||\cdot|||_{\mathcal{B}^2}.$$

Consider two points $z_2, z_2' \in \mathbb{C}_+$ and the quantity

$$D_1 f(x+iy, z_2') - D_1 f(x+iy, z_2).$$

This is equal to the integral of $D_1 D_2 f$ along the line-segment from z_2 to z_2', that is

$$D_1 f(x+iy, z_2') - D_1 f(x+iy, z_2) = \int_{z_2}^{z_2'} D_2 D_1 f(x+iy, z)\, dz.$$

First, assume that the imaginary parts of z_2 and z_2' are equal, so the line-segment is horizontal. Then the integral is given by

$$\int_{\operatorname{Re} z_2}^{\operatorname{Re} z_2'} D_2 D_1 f(x+iy, t+i\operatorname{Im} z_2)\, dt,$$

and we have

$$\begin{aligned}\int_{\operatorname{Re} z_2}^{\operatorname{Re} z_2'} |D_2 D_1 f(x+iy, t+i\operatorname{Im} z_2)|\, dt &\leqslant \int_0^\infty |D_1 D_2 f(x+iy, t+i\operatorname{Im} z_2)| dt \\ &\leqslant \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 f(x+iy_1, x_2+iy_2)|\, dx_2.\end{aligned}$$

Next, assume that the gradient of the line from $[z_1, z_2]$ lies between -1 and 1, i.e. that $z_2 = x_2 + iy_2$ and $z_2' = (x_2+\varepsilon) + i(y_2+\varepsilon')$ with $|\frac{\varepsilon'}{\varepsilon}| \leqslant 1$. Then

$$\begin{aligned}|D_1 f(x+iy, z_1) - D_1 f(x+iy, z_2)| &= \left|\int_{z_2}^{z_2'} D_2 D_1 f(x+iy, z)\, dz\right| \\ &= \left|\int_{x_2}^{x_2+\varepsilon} \left(1+i\frac{\varepsilon'}{\epsilon}\right) D_2 D_1 f\left(x+iy, t+i\left(\frac{\varepsilon'}{\varepsilon}(t-x_2)+y_2\right)\right)\, dt\right| \\ &\leqslant \left(1+\frac{\varepsilon'^2}{\varepsilon^2}\right)^{1/2} \int_{x_2}^{x_2+\varepsilon} \left|D_2 D_1 f\left(x+iy, t+i\left(\frac{\varepsilon'}{\varepsilon}(t-x_2)+y_2\right)\right)\right|\, dt \\ &\leqslant \sqrt{2}\int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 f(x+iy_1, x_2+iy_2)|\, dx_2,\end{aligned}$$

where the last inequality follows from the assumption on ε' and ε.

In the general case (where the gradient may be infinite), one can argue by going from z_2 to z_2' by two line segments, as in the proof of [4, Proposition 2.2]. More explicitly, integrating D_2D_1f with respect to the second variable along the line segment from $z_2 = x_2 + iy_2$ to $x_2 + \varepsilon + i\operatorname{Im} z_2'$, and then the horizontal segment $[x_2 + \varepsilon + i\operatorname{Im} z_2', z_2']$. Choosing ε so that $|\varepsilon^{-1}(\operatorname{Im} z_2' - \operatorname{Im} z_2)| \leqslant 1$, we obtain as before

$$\left|\int_{z_2}^{x_2+\varepsilon+i\operatorname{Im} z_2'} |D_2D_1f(x+iy,z)|\,dz\right| \leqslant \sqrt{2}\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x+iy_1,x_2+iy_2)|\,dx_2.$$

We can then bound the integral over the horizontal segment by

$$\begin{aligned}\left|\int_{x_2+\varepsilon+i\operatorname{Im} z_2'}^{z_2'} D_2D_1f(x+iy,z)\,dz\right| &\leqslant \int_{x_2+\varepsilon}^{\operatorname{Re} z_2'} |D_2D_1f(x+iy,t+i\operatorname{Im} z_2)|\,dt \\ &\leqslant \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x+iy_1,x_2+iy_2)|\,dx_2.\end{aligned}$$

Altogether,

$$\int_{z_2}^{z_2'} |D_2D_1f(x+iy,z)|\,dz \leqslant (1+\sqrt{2})\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x+iy_1,x_2+iy_2)|\,dx_2.$$

It then follows that, for arbitrary x_1, y_1, z_2' and fixed z_2, that

$$|D_1f(x_1+iy_1,z_2')| \leqslant |D_1f(x_1+iy_1,z_2)| + (1+\sqrt{2})\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1,x_2+iy_2)|\,dx_2.$$

So now, taking sup over y_1, z_2' and integrating with respect to x_1,

$$\begin{aligned}\int_0^\infty \sup_{\substack{y_1\in\mathbb{R}\\ z_2'\in\mathbb{C}_+}} |D_1f(x_1+iy_1,z_2')|\,dx_1 &\leqslant \int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1f(x_1+iy_1,z_2)|\,dx_1 \\ &\quad + (1+\sqrt{2})\int_0^\infty\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1,x_2+iy_2)|\,dx_2\,dx_1 \\ &\leqslant \sup_{z\in\mathbb{C}_+}\int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1f(x_1+iy_1,z)|\,dx_1 \\ &\quad + (1+\sqrt{2})\int_0^\infty\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1,x_2+iy_2)|\,dx_2\,dx_1.\end{aligned} \tag{3.4.1}$$

□

Remark 3.4.2. Inspection of the above proof reveals that there exists yet another equivalent norm on $\mathcal{B}^2$, given by

$$\|f\|_{\hat{\mathcal{B}}^2} = \|f\|_\infty + \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2$$

$$+ \int_0^\infty \sup_{y_2\in\mathbb{R}} |D_2f(z_1, x_2+iy_2)|\, dx_2$$

$$+ \int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1f(x_1+iy_1, z_2)|\, dx_1,$$

with arbitrary $z_1, z_2 \in \mathbb{C}_+ \cup \{\infty\}$. More specifically, from the first inequality in (3.4.1) we obtain for arbitrary fixed $z_2 \in \mathbb{C}_+ \cup \{\infty\}$,

$$\int_0^\infty \sup_{\substack{y_1\in\mathbb{R}\\ z_2'\in\mathbb{C}_+}} |D_1f(x_1+iy_1, z_2')|\, dx_1 \leqslant \int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1f(x_1+iy_1, z_2)|\, dx_1$$

$$+ (1+\sqrt{2}) \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1, x_2+iy_2)|\, dx_2\, dx_1,$$

and similarly for an arbitrary fixed $z_1 \in \mathbb{C}_+ \cup \{\infty\}$

$$\int_0^\infty \sup_{\substack{y_2\in\mathbb{R}\\ z_1'\in\mathbb{C}_+}} |D_2f(z_1', x_2+iy_2)|\, dx_2 \leqslant \int_0^\infty \sup_{y_2\in\mathbb{R}} |D_2f(z_1, x_2+iy_2)|\, dx_2$$

$$+ (1+\sqrt{2}) \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1, x_2+iy_2)|\, dx_2\, dx_1,$$

so that

$$\|\cdot\|_{\mathcal{B}^2} \leqslant (2+2\sqrt{2})\|\cdot\|_{\hat{\mathcal{B}}^2}.$$

Since $\|\cdot\|_{\hat{\mathcal{B}}^2} \leqslant |||\cdot|||_{\mathcal{B}^2}$ trivially, the three norms are equivalent. When required, we shall use $\|\cdot\|_{\mathcal{B}^2}$, $|||\cdot|||_{\mathcal{B}^2}$, $\|\cdot\|_{\hat{\mathcal{B}}^2}$ interchangeably.

Remark 3.4.3. Note that

$$|f(\infty,\infty) - f(z_1,z_2)| \leqslant |f(\infty,\infty) - f(z_1,\infty)| + |f(z_1,\infty) - f(z_1,z_2)|$$

$$\leqslant \int_{\operatorname{Re} z_1}^\infty |D_1f(x_1 + i\operatorname{Im} z_1, \infty)|\, dx_1 + \int_{\operatorname{Re} z_2}^\infty |D_2f(z_1, x_2 + i\operatorname{Im} z_2)|\, dx_2$$

$$\leqslant \int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1f(x_1+iy_1,\infty)|\, dx_1 + \int_0^\infty \sup_{y_2\in\mathbb{R}} |D_2f(z_1, x_2+iy_2)|\, dx_2.$$

Since, by (3.4.1),

$$\int_0^\infty \sup_{y_2\in\mathbb{R}} |D_2 f_2(z_1, x_2+iy_2)|\, dx_2 \leqslant \|f_2\|_{\mathcal{B}_0^1} + (1+\sqrt{2})\|f\|_{\mathcal{B}_0^2},$$

we get, for any $z_1, z_2 \in \mathbb{C}_+$,

$$|f(z_1,z_2)| \leqslant |f(\infty,\infty)| + \|f_1\|_{\mathcal{B}_0^1} + \|f_2\|_{\mathcal{B}_0^1} + (1+\sqrt{2})\|f\|_{\mathcal{B}_0^2}.$$

Consequently, another equivalent norm on $\mathcal{B}^2$ is given by

$$|f(\infty,\infty)| + \|f_1\|_{\mathcal{B}_0^1} + \|f_2\|_{\mathcal{B}_0^1} + \|f\|_{\mathcal{B}_0^2}.$$

More precisely, taking $\|\cdot\|_{\hat{\mathcal{B}}^2}$ with $z_1 = z_2 = \infty$, we have e.g.

$$\|f\|_{\hat{\mathcal{B}}^2} \leqslant (2+2\sqrt{2})(|f(\infty,\infty)| + \|f_1\|_{\mathcal{B}_0^1} + \|f_2\|_{\mathcal{B}_0^1} + \|f\|_{\mathcal{B}_0^2}).$$

Observe that if $f \in \mathcal{B}_0^2$, then

$$f_1(z_1) = f_2(z_2) = f(\infty,\infty) = 0, \quad z_1, z_2 \in \mathbb{C}_+.$$

As a result, we have for any $f \in \mathcal{B}_0^2$

$$|f(\infty,\infty)| + \|f_1\|_{\mathcal{B}_0^1} + \|f_2\|_{\mathcal{B}_0^2} + \|f\|_{\mathcal{B}_0^2} = \|f\|_{\mathcal{B}_0^2}.$$

We thus obtain as an immediate corollary

Corollary 3.4.4. *Let f be holomorphic on $\mathbb{C}_+^2$ and satisfy*

$$\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2 f(x_1+iy_1, x_2+iy_2)\, dx_1\, dx_2 < \infty,$$

and $f(\infty, z_2) = f(z_1,\infty) = 0$. Then $f \in \mathcal{B}_0^2$. Moreover, on the subspace $\mathcal{B}_0^2$, the two norms $\|\cdot\|_{\mathcal{B}^2}$ and $\|\cdot\|_{\mathcal{B}_0^2}$ are equivalent.

We note the following proposition which is a direct analogue of [4, Proposition 2.3] (cf. the discussion in Section 2.3).

Proposition 3.4.5. *1. The closed unit ball U of $\mathcal{B}^2$ is compact in the topology of uniform convergence on compact subsets of $\mathbb{C}_+^2$.*
2. For $f \in \mathcal{B}^2$, the spectrum of f in $\mathcal{B}^2$ is the closure of the range of f and the spectral radius of f in $\mathcal{B}^2$ is $\|f\|_\infty$.

Proof. 1. By Montel's theorem (e.g. [39, Proposition 6, p. 8]) the closed unit ball of $H^\infty(\mathbb{C}_+^2)$ is compact in the topology of uniform convergence on compact sets. Thus, as in the proof of [4, Proposition 2.3], it suffices to consider a sequence $(f_n)_{n\in\mathbb{N}}$ in U which converges to a holomorphic function f uniformly on compact sets and to show that $f \in U$. One has

$$\|f\|_\infty \leqslant \liminf_{n\to\infty} \|f_n\|_\infty,$$

$$|D_1 f(x_1 + iy_1, z_2)| = \lim_{n\to\infty} |D_1 f_n(x_1 + iy_1, z_2)| \leqslant \liminf_{n\to\infty} \sup_{s_1\in\mathbb{R}} |D_1 f_n(x_1 + is_1, z_2)|,$$

$$|D_2 f(z_1, x_2 + iy_2)| = \lim_{n\to\infty} |D_2 f_n(z_1, x_2 + iy_2)| \leqslant \liminf_{n\to\infty} \sup_{s_2\in\mathbb{R}} |D_1 f_n(z_1, x_2 + is_2)|,$$

and also

$$|D_1 D_2 f(x + iy_1, x_2 + iy_2)| = \lim_{n\to\infty} |D_1 D_2 f_n(x_1 + iy_1, x_2 + iy_2)|$$

$$\leqslant \liminf_{n\to\infty} \sup_{s_1,s_2\in\mathbb{R}} |D_1 D_2 f_n(x_1 + iy_1, x_2 + iy_2)|.$$

It then follows from Fatou's Lemma that

$$\int_0^\infty \sup_{\substack{y\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f(x + iy, z_2)| dx_1 \leqslant \int_0^\infty \liminf_{n\to\infty} \sup_{\substack{s\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(x_1 + is, z_2)| dx_1$$

$$\leqslant \liminf_{n\to\infty} \int_0^\infty \sup_{\substack{s\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(x_1 + is, z_2)| dx_1,$$

and similarly for $D_2 f$. Also,

$$\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| dx_1\, dx_2$$

$$\leqslant \int_0^\infty \int_0^\infty \liminf_{n\to\infty} \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f_n(x_1 + iy_1, x_2 + iy_2)| dx_1\, dx_2$$

$$\leqslant \liminf_{n\to\infty} \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f_n(x_1 + iy_1, x_2 + iy_2)| dx_1\, dx_2.$$

Thus

$$\|f\|_{\mathcal{B}^2} \leqslant \liminf_{n\to\infty} \|f_n\|_{\mathcal{B}^2} \leqslant 1.$$

2. It is clear that, for any $f \in \mathcal{B}^2$, the range of f is contained in the spectrum of f in $\mathcal{B}^2$, and since f is holomorphic, so is the closure of the range. It remains to show

that $\lambda \notin \sigma(f)$ whenever λ is not in the closure of the range of f, i.e. that if $f \in \mathcal{B}^2$ and bounded away from 0, then $1/f \in \mathcal{B}^2$. Indeed, since

$$D_1 D_2 \frac{1}{f(z_1, z_2)} = \frac{2D_2 f(z_1, z_2) D_1 f(z_1, z_2) - f(z_1, z_2) D_1 D_2 f(z_1, z_2)}{f(z_1, z_2)^3},$$

and f is bounded away from 0, we have with constant $c = \sup_{z_1, z_2 \in \mathbb{C}_+} |f(z_1, z_2)|^{-3}$

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} \left| D_1 D_2 \frac{1}{f(x_1 + iy_1, x_2 + iy_2)} \right| dx_1\, dx_2 \\
&\leqslant 2c \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 f(x_1 + iy_1, x_2 + iy_2) D_2 f(x_1 + iy_1, x_2 + iy_2)|\, dx_1, dx_2 \\
&+ c \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |f(x_1 + iy_1, x_2 + iy_2) D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)|\, dx_1, dx_2 \\
&\leqslant 2c \int_0^\infty \int_0^\infty \sup_{\substack{y_1 \in \mathbb{R} \\ z_2 \in \mathbb{C}_+}} |D_1 f(x_1 + iy_1, z_2)| \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 f(z_1, x_2 + iy_2)|\, dx_1, dx_2 \\
&+ c\|f\|_\infty \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)|\, dx_1, dx_2 \\
&\leqslant 2c\|f\|_{1*}\|f\|_{2*} + c\|f\|_\infty \|f\|_{B_0^2} \\
&\leqslant 2c\|f\|_{\mathcal{B}^2}^2.
\end{aligned}$$

Finally, by [4, Proposition 2.2(6)], $1/f(\cdot, z_2), 1/f(z_1, \cdot) \in \mathcal{B}^1$ for some (and hence for any) $z_1, z_2 \in \mathbb{C}_+$, and thus, by the equivalence of norms and the above,

$$\begin{aligned}
\|1/f\|_{\mathcal{B}^2} &= \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 \frac{1}{f(x_1 + iy_1, x_2 + iy_2)}|\, dx_1\, dx_2 \\
&+ \int_0^\infty \sup_{\substack{y_1 \in \mathbb{R} \\ z_2 \in \mathbb{C}_+}} |D_1 \frac{1}{f(x_1 + iy_1, z_2)}|\, dx_1 \\
&+ \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 \frac{1}{f(z_1, x_2 + iy_2)}|\, dx_2 \\
&\leqslant C \left(\|1/f(\cdot, z_2)\|_{\mathcal{B}^1} + \|1/f(z_1, \cdot)\|_{\mathcal{B}^1} + \|f\|_{\mathcal{B}^2}^2 \right),
\end{aligned}$$

for arbitrary $z_1, z_2 \in \mathbb{C}_+$ and some constant $C > 0$. □

3.5 Relations between the classes $\mathcal{B}^2$ and $\mathcal{B}^1$

Note that if $f, g \in \mathcal{B}^1$, then the functions

$$(f \otimes \mathbb{1})(z_1, z_2) = f(z_1),$$

$$(\mathbb{1} \otimes g)(z_1, z_2) = g(z_2),$$

are in $\mathcal{B}^2$. Since $\mathcal{B}^2$ is an algebra, the function

$$(f \otimes g) = (f \otimes \mathbb{1})(\mathbb{1} \otimes g) : (z_1, z_2) \mapsto f(z_1)g(z_2)$$

is also in $\mathcal{B}^2$. The maps

$$g \mapsto (g \otimes \mathbb{1}),$$

$$g \mapsto (\mathbb{1} \otimes g),$$

are isometric algebra homomorphism of $\mathcal{B}^1$ onto subalgebras M_1 and M_2, respectively:

$$\begin{aligned} M_1 &= \{g \otimes \mathbb{1} : g \in \mathcal{B}^1\}, \\ M_2 &= \{\mathbb{1} \otimes g : g \in \mathcal{B}^1\}. \end{aligned} \tag{3.5.1}$$

Indeed, we have for instance

$$\|g \otimes \mathbb{1}\|_{\mathcal{B}^2} = \|g \otimes \mathbb{1}\|_\infty + \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 (g \otimes \mathbb{1})(x_1 + iy_1, x_2 + iy_2)| \, dx_1, dx_2$$

$$+ \int_0^\infty \sup_{\substack{y_2 \in \mathbb{R} \\ z_1 \in \mathbb{C}_+}} |D_2 (g \otimes \mathbb{1})(x_1 + iy_1, x_2 + iy_2)| \, dx_2 + \int_0^\infty \sup_{\substack{y_1 \in \mathbb{R} \\ z_2 \in \mathbb{C}_+}} |D_1 (g \otimes \mathbb{1})(x_1 + iy_1, x_2 + iy_2)| \, dx_1$$

$$= \|g\|_\infty + \int_0^\infty \sup_{y_1 \in \mathbb{R}} |D_1 g(x_1 + iy_1)| \, dx_1 = \|g\|_{\mathcal{B}^1}.$$

Moreover, for $i = 1, 2$, there exists a natural contractive projection of $\mathcal{B}^2$ onto M_i, denoted by P_i, and given by

$$P_1(f(z_1, z_2)) = f_1(z_1),$$

$$P_2(f(z_1, z_2)) = f_2(z_2).$$

As a direct consequence of Proposition 3.2.2, we have that $P_1 P_2 = P_2 P_1$, i.e. the projections commute, and $P_1 P_2(f(z_1, z_2)) = P_2 P_1(f(z_1, z_2)) = f(\infty, \infty)$. We can now define

$$P_{1,2} = P_1 + P_2 - P_1 P_2.$$

Note that $P_{1,2}$ is a projection of $\mathcal{B}^2$ onto $M_1 + M_2$ with kernel $\mathcal{B}^2_0$.

Finally, if $f \in \mathcal{B}^2$ then for fixed $x_2 + iy_2' \in \mathbb{C}_+$,

$$\int_{\mathbb{R}_+} \sup_{y_1 \in \mathbb{R}} |D_1 D_2 f(x_1 + iy_1, x_2 + iy_2')| \, dx_1 \leqslant \int_{\mathbb{R}_+} \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| \, dx_1, \tag{3.5.2}$$

where the right-hand side is finite for a.e. $x_2 \in \mathbb{R}_+$. Consequently, for a.e. $z_2 \in \mathbb{C}_+$, the function $D_2 f(\cdot, z_2)$ is in the class $\mathcal{B}^1$. Analogously, for a.e. $z_1 \in \mathbb{C}_+$, $D_1 f(z_1, \cdot) \in \mathcal{B}^1$.

Remark 3.5.1. It is not clear to us whether the class $\mathcal{B}^2$ can be naturally identified with any class of Besov functions, in the way that the class $\mathcal{B}^1$ is identified with $\mathcal{B}^{0+}_{\infty,1}(\mathbb{R})$ in [4, Appendix].

3.6 Basic examples

Laplace transforms of measures. A particularly important class of functions $f \in \mathcal{B}^2$ is the Hille-Phillips algebra, $\mathcal{LM}_{(2)}$, consisting of Laplace transforms $\mathcal{L}_{(2)}\mu$ of measures $\mu \in M(\mathbb{R}^2_+)$. We establish below that each $f \in \mathcal{LM}_{(2)}$ is already an element of $\mathcal{B}^2$. We will see in Proposition 4.4.2 that $\mathcal{LM}_{(2)}$ is not norm dense in $\mathcal{B}^2$. Nonetheless, functions in $\mathcal{LM}_{(2)}$ will be a key ingredient in certain approximation arguments involving partial duality.

Consider now $m = \mathcal{L}_{(2)}\mu$, with $\mu \in M(\mathbb{R}^2_+)$. Then

$$D_1 m(z_1, z_2) = -\iint_{\mathbb{R}^2_+} t_1 e^{-t_1 z_1 - t_2 z_2} d\mu(t_1, t_2), \quad z_1, z_2 \in \mathbb{C}_+,$$

$$D_2 m(z_1, z_2) = -\iint_{\mathbb{R}^2_+} t_2 e^{-t_1 z_1 - t_2 z_2} d\mu(t_1, t_2), \quad z_1, z_2 \in \mathbb{C}_+,$$

$$D_1 D_2 m(z_1, z_2) = \iint_{\mathbb{R}^2_+} t_1 t_2 e^{-t_1 z_1 - t_2 z_2} d\mu(t_1, t_2), \quad z_1, z_2 \in \mathbb{C}_+.$$

Hence,

$$\sup_{z_2 \in \mathbb{C}_+} \int_0^\infty \sup_{y_1 > 0} |D_1 m(x + iy, z_2)| dx_1 \leqslant \int_0^\infty \iint_{\mathbb{R}^2_+} t_1 e^{-t_1 x_1 - t_2 x_2} d|\mu|(t_1, t_2) dx_1$$

$$\leqslant \iint_{\mathbb{R}^2_+} e^{-t_2 x_2} d|\mu|(t_1, t_2) \leqslant |\mu|(\mathbb{R}_+ \times \mathbb{R}_+),$$

and similarly for the other variable. Also,

$$\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2m(x_1+iy_1,x_2+iy_2)|\,dx_1\,dx_2$$

$$\leqslant \int_0^\infty \int_0^\infty \iint_{\mathbb{R}_+^2} t_1t_2e^{-t_1x_1-t_2x_2}d|\mu|dx_1dx_2 \leqslant |\mu|(\mathbb{R}_+^2),$$

so that $m \in \mathcal{B}^2$.

Remark 3.6.1. We note that if $f \in \mathcal{LM}_{(2)}$ with $f = \mathcal{L}_{(2)}\mu$, then

$$f_1 = \mathcal{L}_{(1)}\mu_1,$$
$$f_2 = \mathcal{L}_{(1)}\mu_2,$$

where

$$\mu_1(E) = \mu(E \times \{0\}),$$
$$\mu_2(E) = \mu(\{0\} \times E).$$

In particular, if $f \in \mathcal{LM}_{(2)} \cap \mathcal{B}_0^2$, with $f = \mathcal{L}_{(2)}\mu$, so that $f_1 = f_2 = 0$, then

$$\mu(\mathbb{R}_+ \times \{0\}) = \mu(\{0\} \times \mathbb{R}) = 0.$$

Holomorphic extensions to the left. For fixed $\omega, \omega_1, \omega_2 > 0$, we let

$$R_{-\omega} = \{z \in \mathbb{C} : \operatorname{Re} z > -\omega\}, \qquad H^\infty_{\omega_1,\omega_2} = H^\infty(R_{-\omega_1} \times R_{-\omega_2}),$$

$$\|f\|_{H^\infty_{\omega_1,\omega_2}} = \sup_{\substack{z_1\in R_{-\omega_1}\\ z_2\in R_{-\omega_2}}} |f(z_1,z_2)|, \qquad \|f\|_\infty = \sup_{z_1,z_2\in\mathbb{C}_+} |f(z_1,z_2)|.$$

Proposition 3.6.2. *Let $f \in H^\infty_{\omega_1,\omega_2}$, where $\omega_1, \omega_2 > 0$. Then $D_1D_2f \in \mathcal{B}^2$.*

Proof. By (3.1.4) applied to $f(z_1-\omega_1, z_2-\omega_2)$, we have for $x_i+iy_1 \in \mathbb{C}_+$ with $i = 1, 2$,

$$|D_1D_2f(x_1+iy_1,x_2+iy_2)| \leqslant \frac{C_{1,1}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_1+x_1)(\omega_2+x_2)},$$

$$|D_1^2D_2f(x_1+iy_1,x_2+iy_2)| \leqslant \frac{C_{2,1}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_1+x_1)^2(\omega_2+x_2)},$$

$$|D_1D_2^2f(x_1+iy_1,x_2+iy_2)| \leqslant \frac{C_{1,2}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_1+x_1)(\omega_2+x_2)^2},$$

$$|D_1^2D_2^2f(x_1+iy_1,x_2+iy_2)| \leqslant \frac{C_{2,2}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_1+x_1)^2(\omega_2+x_2)^2}.$$

Hence,

$$\|D_1D_2f\|_{\mathcal{B}^2} \leqslant \frac{C_{1,1}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{\omega_1\omega_2} + \int_0^\infty \frac{C_{2,1}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_1+x_1)^2\omega_2}\,dx_1 + \int_0^\infty \frac{C_{1,2}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_2+x_2)^2\omega_1}\,dx_2$$
$$+ \int_0^\infty\int_0^\infty \frac{C_{2,2}\|f\|_{H^\infty_{\omega_1,\omega_2}}}{(\omega_1+x_1)^2(\omega_2+x_2)^2}\,dx_1\,dx_2 < \infty.$$

□

3.7 Compositions

We now show that there is an analogue of Proposition 2.3.18 for functions in two variables; we consider 'Bernstein-like' functions, h_1 and h_2, characterised as in the aforementioned proposition and acting separately on each variable. The composition

$$g(z_1,z_2) = f(h_1(z_1),h_2(z_2))$$

then turns out to be in $\mathcal{B}^2$ for any $f \in \mathcal{B}^2$, as stated below.

Proposition 3.7.1. *Let $f \in \mathcal{B}^2$, $i = 1,2$, and $h_i : \mathbb{C}_+ \to \mathbb{C}_+$ such that h_i is holomorphic, maps $\mathbb{R}_+$ to $\mathbb{R}_+$, is monotone on $\mathbb{R}_+$, and*

$$\operatorname{Re} h_i(z) \geqslant c_i h_i(\operatorname{Re} z), \quad |h_i'(z)| \leqslant c|h_i'(\operatorname{Re} z)|,$$

for some constants c_1, c_2, $c \geqslant 0$. If $g(z_1,z_2) = f(h_1(z_1),h_2(z_2))$, then $g \in \mathcal{B}^2$.

Proof. First, fix $z_2 \in \mathbb{C}_+$. We have

$$\int_0^\infty \sup_{\beta\in\mathbb{R}} |D_1g(\alpha+i\beta,z_2)|\,d\alpha = \int_0^\infty \sup_{\beta\in\mathbb{R}} |h_1'(\alpha+i\beta)D_1f(h_1(\alpha+i\beta),h_2(z_2))|\,d\alpha$$
$$= \int_0^\infty \sup_{\beta\in\mathbb{R}} |h_1'(\alpha+i\beta)D_1f(\operatorname{Re} h(\alpha+i\beta) + i\operatorname{Im} h_1(\alpha+i\beta),h_2(z_2))|\,d\alpha$$
$$\leqslant \int_0^\infty \sup_{\beta\in\mathbb{R}} c|h_1'(\alpha)D_1f(ch(\alpha)+i\beta,h_2(z_2))|\,d\alpha$$
$$= c\int_{c_1h_1(0)}^{c_1h_1(\infty)} \sup_{\beta\in\mathbb{R}} |D_1f(u+i\beta,h_2(z_2))|\,du < \infty.$$

Analogously, we can show that, for fixed $z_1 \in \mathbb{C}_+$,

$$\int_0^\infty \sup_{\beta\in\mathbb{R}} |D_2 g(z_1, \alpha + i\beta)|\, d\alpha < \infty.$$

Also,

$$\begin{aligned}
\|g\|_{\mathcal{B}_0^2} &= \int_0^\infty \int_0^\infty \sup_{\beta\in\mathbb{R}} \sup_{\delta\in\mathbb{R}} |D_1 D_2 g(\alpha + i\beta, \gamma + i\delta)|\, d\alpha\, d\gamma \\
&= \int_0^\infty \int_0^\infty \sup_{\beta\in\mathbb{R}} \sup_{\delta\in\mathbb{R}} |h_1'(\alpha + i\beta) h_2'(\gamma + i\delta) D_1 D_2 f(h_1(\alpha + i\beta), h_2(\gamma + \delta))|\, d\alpha\, d\gamma \\
&\leqslant \int_0^\infty \int_0^\infty \sup_{\beta\in\mathbb{R}} \sup_{\delta\in\mathbb{R}} c^2 |h_1'(\alpha) h_2'(\gamma) D_1 D_2 f(c_1 h_1(\alpha) + i\beta, c_2 h_2(\gamma) + i\delta)|\, d\alpha\, d\gamma \\
&= c^2 \int_{c_2 h_2(0)}^{c_2 h_2(\infty)} \int_{c_1 h_1(0)}^{c_1 h_1(\infty)} \sup_{\beta\in\mathbb{R}} \sup_{\delta\in\mathbb{R}} |D_1 D_2 f(u_1 + i\beta, u_2 + i\delta)|\, du_1\, du_2 < \infty.
\end{aligned}$$

We have established that $\|g\|_{\hat{\mathcal{B}}^2} < \infty$ (cf. Remark 3.4.2); since $\|\cdot\|_{\hat{\mathcal{B}}^2}$ and $\|\cdot\|_{\mathcal{B}^2}$ are equivalent norms on $\mathcal{B}^2$, this implies that $\|g\|_{\mathcal{B}^2} < \infty$. □

3.8 Further examples

We shall first establish a certain useful result concerning a method of generating members of $\mathcal{B}^2$ out of functions in the class $\mathcal{B}^1$.

Lemma 3.8.1. *Let $g(z_1, z_2) = f(z_1 + z_2)$. Then $f \in \mathcal{B}^1$ if and only if $g \in \mathcal{B}^2$.*

Proof. Suppose that $f \in \mathcal{B}^1$. Remark 3.4.3 together with the fact that f is bounded and holomorphic if and only if g is imply that we only have to check whether $\|g\|_{\mathcal{B}_0^2} < \infty$. We have

$$\begin{aligned}\|g\|_{\mathcal{B}_0^2} &= \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2 g(x_1+iy_1, x_2+iy_2)| dx_1\, dx_2 \\ &= \int_0^\infty \int_0^\infty \sup_{y\in\mathbb{R}} |f''(x_1+x_2+iy)| dx_1\, dx_2 \\ &= \int_0^\infty \int_{x_2}^\infty \sup_{y\in\mathbb{R}} |f''(t+iy)|\, dt\, dx_2 = \int_0^\infty \int_0^t \sup_{y\in\mathbb{R}} |f''(t+iy)|\, dx_2\, dt \\ &= \int_0^\infty t \sup_{y\in\mathbb{R}} |f''(t+iy)|\, dt,\end{aligned}$$

so that $g \in \mathcal{B}^2$ if and only if

$$t \mapsto t \cdot \sup_{y\in\mathbb{R}} |f''(t+iy)| \in L^1(0,\infty).$$

Since $f \in H^\infty(\mathbb{C}_+)$, Cauchy's formula yields (cf. [4, Lemma 2.1(4)])

$$\sup_{y\in\mathbb{R}} |f''(2t+iy)| \leqslant \frac{\sup_{y\in\mathbb{R}} |f'(t+iy)|}{t}, \quad t > 0.$$

Hence,

$$4^{-1} \int_0^\infty t \cdot \sup_{y\in\mathbb{R}} |f''(t+iy)|\, dt \leqslant \int_0^\infty \sup_{y\in\mathbb{R}} |f'(t+iy)|\, dt \leqslant \|f\|_{\mathcal{B}_0}.$$

Conversely, if $g \in \mathcal{B}^2$, then $\|f\|_\infty = \|g\|_\infty$, and

$$\begin{aligned}\|f\|_{\mathcal{B}_0^1} = \int_0^\infty \sup_{y\in\mathbb{R}} |f'(t+iy)|\, dt &= \lim_{n\to\infty} \int_0^\infty \sup_{y\in\mathbb{R}} |f'(t+1/n+iy)|\, dt \\ &\leqslant \int_0^\infty \sup_{\substack{y_1\in\mathbb{R},\\ z_2\in\mathbb{C}_+}} |D_1 g(t+iy_1, z_2)|\, dx_1 \leqslant \|g\|_{\mathcal{B}^2}.\end{aligned}$$

□

Example 3.8.2. Consider

$$\tilde{f}_t(z_1,z_2) = \exp(-t/(z_1+z_2+1)), \quad t > 0, z_1, z_2 \in \mathbb{C}_+.$$

It was shown in [4, Lemma 3.4] that the function $\exp(-t/(z+1))$ is in $\mathcal{B}^1$ for any $t > 0$ and satisfies

$$\|\exp(-t/(z+1))\|_{\mathcal{B}^1} \leqslant \frac{1+2t}{1+t} + e^{-1}\log(t+1).$$

Consequently, by Lemma 3.8.1, $\tilde{f}_t(z_1, z_2) \in \mathcal{B}^2$ for any $t > 0$. It would be desirable to obtain explicit bounds on the $\|\cdot\|_{\mathcal{B}_0^2}$ norm of $\tilde{f}_t$.

We have

$$D_1 D_2 \tilde{f}_t(z_1, z_2) = \frac{t^2 \exp\left(-\frac{t}{z_1+z_2+1}\right)}{(z_1+z_2+1)^4} - \frac{2t \exp\left(-\frac{t}{z_1+z_2+1}\right)}{(z_1+z_2+1)^3} = \frac{t \exp\left(-\frac{t}{z_1+z_2+1}\right)(t - 2(z_1+z_2+1))}{(z_1+z_2+1)^4}.$$

Hence

$$\begin{aligned} |D_1 D_2 \tilde{f}_t(z_1, z_2)| &= \left|\frac{t(t - 2(x_1+iy_1+x_2+iy_2+1))}{(x_1+iy_1+x_2+iy_2+1)^4}\right| \exp\left(-\frac{t}{x_1+iy_1+x_2+iy_2+1}\right) \\ &= \left|\frac{t(t - 2(x_1+iy_1+x_2+iy_2+1))}{(x_1+iy_1+x_2+iy_2+1)^4}\right| \exp\left(-\frac{t(x_1+x_2+1)}{(x_1+x_2+1)^2+(y_1+y_2)^2}\right). \end{aligned}$$

Applying triangle inequality we see that the right-hand side is less than or equal to

$$\begin{aligned} &\left(\frac{t^2}{((x_1+x_2+1)^2+(y_1+y_2)^2)^2} + \frac{2t}{((x_1+x_2+1)^2+(y_1+y_2)^2)^{3/2}}\right) \\ &\times \exp\left(-\frac{t(x_1+x_2+1)}{(x_1+x_2+1)^2+(y_1+y_2)^2}\right). \end{aligned}$$

We shall consider each term in this sum separately. Let

$$g_{t,s}(r) = \frac{1}{(s^2+r)^2} \exp\left(-\frac{ts}{s^2+r}\right).$$

Then

$$\max_{r \geqslant 0} g_{t,s}(r) = \begin{cases} g_{t,s}(1/2(-2s^2+st)) = 1/(es^2t^2), & t > 2s \\ g_{t,s}(0) = e^{-(t/s)}/s^4, & 0 \leqslant t \leqslant 2s. \end{cases} \tag{3.8.1}$$

Our ultimate aim is to provide an upper bound for

$$\int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 \tilde{f}_t(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2.$$

We split the area of integration to make use of (3.8.1), thus obtaining

$$\begin{aligned} \int_0^t \int_0^{t-x_1} \frac{e^{-1}t^{-2}}{(x_1+x_2+1)^2}\, dx_2\, dx_1 &\leqslant \int_0^t \int_0^t \frac{e^{-1}t^{-2}}{(x_1+x_2+1)^2}\, dx_2\, dx_1 \\ &= \frac{2\log(t+1) - \log(2t+1)}{et^2}. \end{aligned}$$

Moreover

$$\int_0^t \int_{t-x_1}^{\infty} \frac{\exp\left(-t/(x_1+x_2+1)\right)}{(x_1+x_2+1)^4}\, dx_2\, dx_1 + \int_0^{\infty} \int_t^{\infty} \frac{\exp\left(-t/(x_1+x_2+1)\right)}{(x_1+x_2+1)^4}\, dx_1\, dx_2$$

$$\leqslant \int_0^{\infty} \int_0^{\infty} \frac{\exp\left(-t/(x_1+x_2+1)\right)}{(x_1+x_2+1)^4}\, dx_1\, dx_2 \leqslant \frac{e^{-t}\left(e^t(t-2)+t+2\right)}{t^3}.$$

Similarly, let

$$g_{t,s}(r) = \frac{2}{(s^2+r)^{3/2}} \exp\left(-ts/(s^2+r^2)\right).$$

Then

$$\max_{r\geqslant 0} g_{t,s}(r) = \begin{cases} g_{t,s}(1/3(-3s^2+2st)) = \frac{3\sqrt{\frac{3}{2}}}{2e^{-3/2}(st)^{3/2}}, & t > 3s/2 \\ g_{t,s}(0) = \exp\left(-t/s\right)/s^{-3}, & 0 \leqslant t \leqslant 3s/2. \end{cases}$$

We have

$$\int_0^{3t/2} \int_0^{3t/2-x_1} \frac{e^{-3/2}t^{-3/2}}{(x_1+x_2+1)^{3/2}}\, dx_2\, dx_1 \leqslant \int_0^{3t/2} \int_0^{3t/2} \frac{e^{-1}t^{-3/2}}{(x_1+x_2+1)^{3/2}}\, dx_1\, dx_2.$$

$$= \int_0^{3t/2} \left(\frac{2e^{-1}t^{-3/2}}{\sqrt{x_2+1}} - \frac{2e^{-1}t^{-3/2}}{\sqrt{x_2+3t/2+1}} \right) dx_2$$

$$= 4e^{-1}t^{-3/2} \left(\sqrt{6t/2+1} + 2\sqrt{3t/2+1} + 1 \right).$$

Moreover,

$$\int_0^{\infty} \int_0^{\infty} \exp\left(-t/(x_1+x_2+1)\right)/(x_1+x_2+1)^3\, dx_1\, dx_2 \leqslant \frac{t+e^{-t}-1}{t^2}.$$

All in all,

$$\|\tilde{f}_t\|_{\mathcal{B}_0^2} = \int_0^{\infty} \int_0^{\infty} \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 \tilde{f}_t(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2$$

$$\lesssim \frac{2\log(t+1) - \log(2t+1)}{e} + \frac{e^{-t}\left(e^t(t-2)+t+2\right)}{t}$$

$$+ \frac{4(\sqrt{6t/2+1} + 2\sqrt{3t/2+1} + 1)}{et^{3/2}} + \frac{t+e^{-t}-1}{t}.$$

Example 3.8.3. Consider the family of functions

$$\tilde{g}_t(z_1, z_2) = \exp\left(-\frac{t}{(z_1+1)(z_2+1)}\right), \quad t > 0,\ z_1, z_2 \in \mathbb{C}_+.$$

While the form of $\tilde{g}_t$ is not completely dissimilar from that of the functions $\tilde{f}_t$ considered in the previous example, Lemma 3.8.1 does not by itself imply that $\tilde{g}_t \in \mathcal{B}^2$. It is nonetheless true that $\tilde{g}_t \in \mathcal{B}^2$. We have for example

$$|D_1 \tilde{g}_t(z_1, z_2)| = \frac{t \exp\left(-\frac{t}{(z_1+1)(z_2+1)}\right)}{(z_1+1)^2(z_2+1)}.$$

Hence, by [4, Lemma 3.4] and for $z_2 = 1$,

$$\int_0^\infty \sup_{y_1 \in \mathbb{R}} |D_1 \tilde{g}_t(x_1 + iy_1, z_2)|\, dx_1 \leqslant \frac{1+t}{1+t/2} + e^{-1}\log(1+t/2).$$

Moreover,

$$|D_1 D_2 \tilde{g}_t(z_1, z_2)| = \frac{|t|\,|t - (z_1+1)(z_2+1)|\, e^{-\operatorname{Re}\left(\frac{t}{z_1 z_2 + z_1 + z_2 + 1}\right)}}{|z_1+1|^3\,|z_2+1|^3}.$$

Hence

$$\int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 \tilde{g}_t(x_1 + iy_1, x_2 + iy_2)|\, dx_1\, dx_2$$

$$\leqslant \int_0^\infty \int_0^\infty \frac{t^2}{(x_1+1)^3\,(x_2+1)^3}\, dx_1\, dx_2 + \int_0^\infty \int_0^\infty \frac{t}{(x_1+1)^2\,(x_2+1)^2}\, dx_1\, dx_2 = \frac{t^2}{4} + t.$$

Summarising,

$$\|\tilde{g}_t(z_1, z_2)\|_{\mathcal{B}^2} \leqslant \frac{2+2t}{1+t/2} + 2e^{-1}\log(1+t/2) + \frac{t^2}{4} + t.$$

4 Spectral subspaces and a partial duality

In this chapter we present certain results related to spectral subspaces. The theory of spectral subspaces, as introduced by Arveson, is typically developed in the abstract context of integrable representations. In most places, however, we restrict our attention to the special case of C_0-groups.

We begin by presenting some preliminary results on spectral decompositions in Section 4.1. In Section 4.2, we characterise the spectral subspaces of $\mathcal{B}^2$; we then identify (cf. Proposition 4.2.4) a dense subspace of $\mathcal{B}_0^2$ which will play a key role in the construction of our functional calculus. In Section 4.3, we introduce the auxiliary class $\mathcal{E}^2$ and define a (partial) duality between $\mathcal{B}^2$ and $\mathcal{E}^2$. We then establish that each function in $\mathcal{B}^2$ can be approximated in terms of that duality by functions with compact spectrum separated from zero. We consider some topological properties of $\mathcal{B}^2$ in Section 4.4; we show, in particular, that $\mathcal{LM}_{(2)}$ is not dense in $\mathcal{B}^2$ in the norm topology but that it is dense in $\mathcal{B}^2$ in the topology of uniform convergence on compact subsets.

4.1 Spectral decompositions

For any closed subset $I \subset \mathbb{R}_+^2$, define $H^\infty(I)$ as

$$H^\infty(I) = \{f \in H^\infty(\mathbb{C}_+^2) : \operatorname{supp}(\mathcal{F}_{(2)}^{-1} f^b) \subset I\},$$

with $\mathcal{F}_{(2)}$ denoting the two-dimensional Fourier transform. Since shifts of the Fourier transform correspond to multiplication by exponential functions, one has

$$|f(x_1 + iy_1, x_2 + iy_2)| \leqslant e^{-\sigma_1 x_1 - \sigma_2 x_2} \|f\|_\infty, \quad f \in H^\infty([\sigma_1, \infty) \times [\sigma_2, \infty)). \tag{4.1.1}$$

We say that an entire function $f : \mathbb{C}^2 \to \mathbb{C}$ is of exponential type at most $\tau = (\tau_1, \tau_2)$, where $\tau_1, \tau_2 > 0$, if for all $\varepsilon > 0$ there is a constant A_ε such that

$$|f(z_1, z_2)| \leqslant A_\varepsilon e^{(\tau_1+\varepsilon)|z_1| + (\tau_2+\varepsilon)|z_1|}, \quad z_1, z_2 \in \mathbb{C}. \tag{4.1.2}$$

The Paley-Wiener-Schwartz theorem [27, Theorem 7.3.1, p. 181] implies, in particular, that any function which is a Fourier transform of a distribution with a support contained in a convex compact subset of $\mathbb{R}^2$ is an entire function of exponential type. Furthermore, if $f \in H^\infty([0,\sigma_1] \times [0,\sigma_2])$, then, since $f^b \in L^\infty(\mathbb{R}^2)$, applying the multivariable version of Bernstein's inequality (cf. [11, Theorem 2.2]) yields

$$\|D_i f\|_\infty \leqslant \sigma_i \|f\|_\infty, \quad i = 1, 2. \tag{4.1.3}$$

Since the property of being a function of exponential type (at most τ) is preserved under taking partial derivatives by [11, Theorem 2.2], iterating (4.1.3) and applying it to $D_1 D_2 f$ gives

$$|D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| \leqslant \sigma_1 \sigma_2 \sup_{s_1, s_2 \in \mathbb{R}} |f(x_1 + is_1, x_2 + is_2)|. \tag{4.1.4}$$

We shall consider the spectral subspaces $H^\infty(I)$, where I is a compact subset of $\mathbb{R}_2$. In Proposition 4.2.6 we show analogues of the Bohr inequalities for these spaces. First we show that these spaces are contained in $\mathcal{B}^2$.

Lemma 4.1.1. *Let $f \in H^\infty([\varepsilon,\sigma]^2)$, $0 < \varepsilon < \sigma$. Then $f \in \mathcal{B}^2$.*

Proof. Note first that by (4.1.1), $\lim_{\operatorname{Re} z_1 \to \infty} f(z_1, z_2) = \lim_{\operatorname{Re} z_2 \to \infty} f(z_1, z_2) = 0$. In view of Corollary 3.4.4, it then suffices to check whether f satisfies the condition $\|f\|_{\mathcal{B}_0^2} < \infty$. Let $x_1, x_2 > 0$, $y_1, y_2 \in \mathbb{R}^2$. Then (4.1.1) gives

$$\sup_{s_1, s_2 \in \mathbb{R}} |f(x_1 + is_1, x_2 + is_2)| \leqslant e^{-\varepsilon(x_1 + x_2)} \|f\|_\infty,$$

and, by Bernstein's inequality (4.1.4),

$$|D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| \leqslant \sigma^2 \sup_{s_1, s_2 \in \mathbb{R}} |f(x_1 + is_1, x_2 + is_2)| \leqslant \sigma^2 e^{-\varepsilon(x_1 + x_2)} \|f\|_\infty.$$

On the other hand,

$$|D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| \leqslant \frac{\|f\|_\infty}{x_1 x_2}.$$

Thus,

$$|D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| \leqslant \|f\|_\infty \min\left((x_1 x_2)^{-1}, e^{-\varepsilon(x_1 + x_2)}\right) \leqslant \frac{2\|f\|_\infty}{x_1 x_2 + e^{\varepsilon(x_1 + x_2)}/\sigma^2}. \tag{4.1.5}$$

$$\int_0^\infty \int_0^\infty \frac{dx_1\, dx_2}{x_1 x_2 + e^{\varepsilon(x_1 + x_2)}/\sigma^2} = \int_0^\infty \int_0^\infty \frac{ds\, dt}{st + \varepsilon^2 e^{(s+t)}/\sigma^2}. \tag{4.1.6}$$

Fix $s \in \mathbb{R}_+$. Then

$$
\begin{aligned}
\int_0^\infty \frac{dt}{st+\varepsilon^2 e^{(s+t)}/\sigma^2} &\leqslant \int_0^1 \frac{dt}{st+\varepsilon^2 e^{s}/\sigma^2} + \int_1^\infty \frac{dt}{s+\varepsilon^2 e^{(s+t)}/\sigma^2} \\
&\leqslant s^{-1}\log\left(1+\frac{\sigma^2}{\varepsilon^2}se^{-s}\right) + s^{-1}\log\left(1+\frac{\sigma^2}{e\varepsilon^2}se^{-s}\right) \qquad (4.1.7) \\
&\leqslant 2s^{-1}\log\left(1+\frac{\sigma^2}{\varepsilon^2}se^{-s}\right).
\end{aligned}
$$

Hence,

$$
\int_0^\infty \int_0^\infty \frac{dx_1\,dx_2}{x_1x_2+e^{\varepsilon(x_1+x_2)}/\sigma^2} \leqslant 2\int_0^\infty \frac{\log\left(1+\frac{\sigma^2}{\varepsilon^2}se^{-s}\right)}{s}\,ds. \qquad (4.1.8)
$$

Denote σ^2/ε^2 by c and pick $a > e-1$. First,

$$
\int_0^1 \frac{\log(1+cse^{-s})}{s}\,ds \leqslant \int_0^1 \frac{\log(1+cs)}{s}\,ds \leqslant \mathrm{Li}_2(-c) < \infty, \qquad (4.1.9)
$$

where Li_2 denotes the dilogarithm function. Next, noting that $\frac{d}{dx}(\log x)^2 = 2\frac{\log x}{x}$,

$$
\begin{aligned}
&\int_1^{\log(c+a)} \frac{\log(1+cse^{-s})}{s}\,ds \leqslant 2\int_1^{\log(c+a)} \frac{\log(1+cs)}{1+s}\,ds \\
&\leqslant 2\left(\int_1^{\log(c+a)} \frac{\log(1+s)}{1+s}\,ds + \int_1^{\log(c+a)} \frac{\log c}{1+s}\,ds\right) \\
&= \log^2(1+\log(c+a)) - \log^2(2) + 2\log(c)\left(\log(1+\log(c+a)) - \log(2)\right). \qquad (4.1.10)
\end{aligned}
$$

Finally, by the fact that $\log(1+x) \leqslant x$,

$$
\int_{\log(c+a)}^\infty \frac{\log(1+cse^{-s})}{s}\,ds \leqslant \int_{\log(c+a)}^\infty ce^{-s}\,ds = \frac{c}{c+a} \leqslant 1. \qquad (4.1.11)
$$

Hence, by (4.1.5) and (4.1.8),

$$\begin{aligned}\|f\|_{\mathcal{B}_0^2} &= \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1 D_2 f(x_1+iy_1, x_2+iy_2)|\, dx_1\, dx_2 \\ &\leqslant 2\|f\|_\infty \int_0^\infty \int_0^\infty \frac{dx_1\, dx_2}{x_1x_2 + e^{\varepsilon(x_1+x_2)}/\sigma^2} \\ &\leqslant 4\|f\|_\infty \int_0^\infty \frac{\log\left(1+\frac{\sigma^2}{\varepsilon^2} s e^{-s}\right)}{s}\, ds.\end{aligned}$$

Since (4.1.9)–(4.1.11) jointly imply

$$\int_0^\infty \frac{\log\left(1+\frac{\sigma^2}{\varepsilon^2} s e^{-s}\right)}{s}\, ds < \infty,$$

we obtain that $\|f\|_{\mathcal{B}_0^2} < \infty$, as required. □

4.2 Spectral subspaces

Let $(T_1(t))_{t\in\mathbb{R}}$ and $(T_2(t))_{t\in\mathbb{R}}$ be commuting bounded C_0-groups on a Banach space X, with generators A_1 and A_2, respectively, so that $\sigma(A_j) \subset i\mathbb{R}$, for $j = 1, 2$. Denote

$$T(t_1, t_2) = T_1(t_1)T_2(t_2), \quad t_1, t_2 \in \mathbb{R}.$$

For $x \in X$, define the spectrum $\mathrm{sp}_T(x)$ of x as

$$\mathrm{sp}_T(x) = \mathrm{supp}\ \mathcal{F}_{(2)}(T(\cdot,\cdot)x).$$

For a closed subset $I \subset \mathbb{R}^2$, define the spectral subspace $X_T(I)$ to be

$$X_T(I) = \{x \in X : \mathrm{sp}_T(x) \subset I\}.$$

As described in [41, p. 299 and Lemma 8.1.2], any two-parameter C_0-group of bounded invertible isometries can be viewed as an integrable representation of $\mathbb{R}^2$ (corresponding to α in the notation of [41]). Moreover, one can define spectral subspaces (denoted by $M^\alpha(I)$ in [41]) in terms of annihilators of certain subspaces of the dual space (cf. [41, p. 300]); the two definitions are known to coincide (cf. [40]).

Proposition 4.2.1. *Let $(T(t_1, t_2))_{t_1,t_2\in\mathbb{R}}$ be as above and assume that the range of A_1A_2 is dense in X and that $\sigma(A_i) \subset i[0, \infty)$ for $i = 1, 2$. Then the set*

$$\bigcup\{X_T(I) : I \text{ compact}, I \subset (0,\infty)^2\}$$

is dense in X.

Proof. Reasoning as in the proof of [4, Proposition 2.8], take $f \in \mathcal{S}(\mathbb{R}^2)$ with compact support and identically 1 in a neighbourhood of $(0,0)$. Define

$$g_a(t_1,t_2) = a^2 f(at_1,at_2) - a^{-2} f(t_1/a,t_2/a), \quad t_1,t_2 \in \mathbb{R}, \quad a > 0.$$

Let $y \in D(A_1A_2)$, take $x = A_1A_2y$, and set

$$x_a = \int_{\mathbb{R}}\int_{\mathbb{R}} g_a(t_1,t_2)T(t_1,t_2)x\,dt_1\,dt_2.$$

Since $(T_i(t))_{t\in\mathbb{R}}$ is a C_0-group for $i = 1,2$,

$$\lim_{a\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} a^2 f(at_1,at_2)T(t_1,t_2)x\,dt_1\,dt_2 - x$$
$$= \lim_{a\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t_1,t_2)(T_1(t_1/a)T_2(t_2/a)x - x)\,dt_1\,dt_2 = 0. \quad (4.2.1)$$

Integrating by parts twice

$$\begin{aligned}
&\lim_{a\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} a^{-2} f(t_1/a,t_2/a)T(t_1,t_2)x\,dt_1\,dt_2 \\
&= \lim_{a\to\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t_1,t_2)T(at_1,at_2)A_1A_2y\,dt_1\,dt_2 \\
&= \lim_{a\to\infty} a^{-2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t_1,t_2)D_1D_2T(at_1,at_2)y\,dt_1\,dt_2 \\
&= \lim_{a\to\infty} a^{-2}\int_{\mathbb{R}}\int_{\mathbb{R}} D_1D_2f(t_1,t_2)\cdot T(at_1,at_2)y\,dt_1\,dt_2 = 0.
\end{aligned}$$

Hence, $\lim_{a\to\infty} x_a = x$. By [40, Lemma 2.2.1],

$$\mathrm{sp}_T(x_a) \subset \mathrm{supp}\,(\mathcal{F}_{(2)}g_a) \cap \mathrm{sp}_T(x).$$

Note that $\mathrm{supp}(\mathcal{F}_{(2)}g_a)$ is a compact subset of $\mathbb{R}^2\backslash(0,0)$. Since (cf. [41, Proposition 8.1.5])

$$X_T(-i\sigma(A_1)\times -i\sigma(A_1)) = X_{T_1}(-i\sigma(A_1)) \cap X_{T_2}(-i\sigma(A_2)),$$

and (cf. [2, Remark 4.6.2, Lemma 4.6.8], [15, Lemma 8.1.7])

$$X_{T_1}(-i\sigma(A_1)) = X_{T_2}(-i\sigma(A_2)) = X,$$

one has that $x \in X_T(-i\sigma(A_1)\times -i\sigma(A_1))$, which implies $\mathrm{sp}_T(x) \subset (-i\sigma(A_1)\times -i\sigma(A_1))$. Altogether,

$$\mathrm{sp}_T(x_a) \subset \mathrm{supp}\,(\mathcal{F}_{(2)}g_a) \cap (-i\sigma(A_1)\times -i\sigma(A_1)),$$

so the conclusion follows from the assumption. □

We shall make extensive use of products of shift semigroups on $\mathcal{B}^2$. The next propositions concern some basic properties of these.

Proposition 4.2.2. *Let*

$$(T_{\mathcal{B}^2}(a,b)f)(z_1,z_2) = f(z_1+a, z_2+b), \quad f \in \mathcal{B}^2,\ a,b \in \overline{\mathbb{C}}_+,\ z_1,z_2 \in \mathbb{C}_+.$$

Then for each $f \in \mathcal{B}^2$,

$$\|T_{\mathcal{B}^2}(a,b)f\|_{\mathcal{B}^2} \leqslant \|f\|_{\mathcal{B}^2}, \qquad \lim_{\substack{a,b\in\overline{\mathbb{C}}_+\\ a,b\to 0}} \|T_{\mathcal{B}^2}(a,b)f - f\|_{\mathcal{B}^2} = 0.$$

Moreover, the families $(T_{\mathcal{B}^2}(a,0))_{a\in\mathbb{C}_+}$ *and* $(T_{\mathcal{B}^2}(0,b))_{b\in\mathbb{C}_+}$ *are both holomorphic* C_0*-semigroups of contractions on* $\mathcal{B}^2$*. Furthermore,*

$$G(t_1,t_2) = T_{\mathcal{B}^2}(-it_1,-it_2), \quad t_1,t_2 \in \mathbb{R},$$

form a representation of $\mathbb{R}^2$ *by isometries on* $\mathcal{B}^2$.

Proof. It is easily seen that, for each $a,b \in \mathbb{C}_+$, $T_{\mathcal{B}^2}(a,b)$ is a contraction on $\mathcal{B}^2$. Since f is uniformly continuous on $\overline{\mathbb{C}}_+ \times \overline{\mathbb{C}}_+$, $\|T_{\mathcal{B}^2}(a,b)f - f\|_\infty \to 0$ as $a,b \to 0$.
Applying (3.1.4) to $D_1D_2^2f$ and $D_1^2D_2f$ we obtain

$$\begin{aligned}
&|D_1D_2f(z_1+a,z_2+b) - D_1D_2f(z_1,z_2)| \\
&\qquad\leqslant |D_1D_2f(z_1+a,z_2+b) - D_1D_2f(z_1+a,z_2)| + |D_1D_2f(z_1+a,z_2) - D_1D_2f(z_1,z_2)| \\
&\qquad\leqslant \frac{C_{(1,2)}|b|\|f\|_\infty}{\operatorname{Re} z_1(\operatorname{Re} z_2)^2} + \frac{C_{(2,1)}|a|\|f\|_\infty}{(\operatorname{Re} z_1)^2 \operatorname{Re} z_2}.
\end{aligned}$$

Hence,

$$\lim_{a,b\to 0} \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1+a, x_2+iy_2+b) - D_1D_2f(x_1+iy_1, x_2+iy_2)| = 0.$$

Now,

$$\begin{aligned}
&\sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1+a, x_2+iy_2+b) - D_1D_2f(x_1+iy_1, x_2+iy_2)| \\
&\qquad\leqslant \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1+a, x_2+iy_2+b)| + \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1, x_2+iy_2)|.
\end{aligned}$$

Since $f \in \mathcal{B}^2$, the functions on the right-hand side are integrable, i.e.

$$\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1+a, x_2+iy_2+b)|\,dx_1\,dx_2$$

$$= \int_{\operatorname{Re} a}^\infty \int_{\operatorname{Re} b}^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1, x_2+iy_2)|\,dx_1\,dx_2$$

$$\leqslant \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1, x_2+iy_2)|\,dx_1\,dx_2 \leqslant \|f\|_{\mathcal{B}^2} < \infty.$$

Moreover, since (cf. Proposition 3.2.2) $f_i \in \mathcal{B}^1$, $i = 1, 2$, it follows from the boundedness of f_i'' on $\{z : \operatorname{Re} z > 0\}$ that

$$\lim_{a,b\to 0} \sup_{y_1\in\mathbb{R}} |f_1'(x_1+iy_1+a) - f_1'(x_1+iy_1)| = 0,$$

$$\lim_{a,b\to 0} \sup_{y_2\in\mathbb{R}} |f_2'(x_2+iy_2+b) - f_2'(x_2+iy_2)| = 0.$$

By the maximum principle for f_1' on $\{z : \operatorname{Re} z \leqslant x_1\}$,

$$\sup_{y_1\in\mathbb{R}} |f_1'(x_1+iy_1+a) - f_1'(x_1+iy_1)| \leqslant \sup_{y_1\in\mathbb{R}} |f_1'(x_1+iy_1+a)| + \sup_{y_1\in\mathbb{R}} |f_1'(x_1+iy_1)|$$

$$\leqslant 2\sup_{y_1\in\mathbb{R}} |f_1'(x_1+iy_1)|,$$

and this function is integrable on $\mathbb{R}_+$, since $f_1 \in \mathcal{B}^1$. Analogously,

$$\sup_{y_2\in\mathbb{R}} |f_2'(x_2+iy_2+a) - f_2'(x_2+iy_2)| \leqslant 2\sup_{y_2\in\mathbb{R}} |f_2'(x_2+iy_2)|,$$

where the right-hand side is integrable since $f_2 \in \mathcal{B}^1$. By the dominated convergence theorem (cf. Remark 3.4.2 for the definition of $\|\cdot\|_{\hat{\mathcal{B}}^2}$; here we take $z_1 = z_2 = \infty$),

$$\lim_{a,b\to 0} \|T_{\mathcal{B}^2}(a,b)f - f\|_{\hat{\mathcal{B}}^2}$$

$$= \lim_{a,b\to 0} \int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2f(x_1+iy_1+a, x_2+iy_2+b) - D_1D_2f(x_1+iy_1, x_2+iy_2)|\,dx_1\,dx_2$$

$$+ \lim_{a,b\to 0} \int_0^\infty \sup_{y_1\in\mathbb{R}} |f_1'(x_1+iy_1+b) - f_1'(x_1+iy_1)|\,dx_1$$

$$+ \lim_{a,b\to 0} \int_0^\infty \sup_{y_2\in\mathbb{R}} |f_2'(x_2+iy_2+a) - f_1'(x_2+iy_2)|\,dx_2 = 0.$$

Remark 3.4.2 then yields

$$\lim_{a,b\to 0} \|T_{\mathcal{B}^2}(a,b)f - f\|_{\mathcal{B}^2} = 0.$$

Consider now the family of functionals

$$f \mapsto f(z_1, z_2), \quad z_1, z_2 \in \mathbb{C}_+.$$

Such functionals form a separating subspace of $(\mathcal{B}^2)^*$. Both

$$a \mapsto (T_{\mathcal{B}^2}(a,0)f)(z_1,z_2) = f(z_1 + a, z_2)$$

and

$$b \mapsto (T_{\mathcal{B}^2}(0,b)f)(z_1,z_2) = f(z_1, z_2 + b)$$

are holomorphic on $\mathbb{C}_+^2$. Since $T_{\mathcal{B}^2}(a,0)$ and $T_{\mathcal{B}^2}(0,b)$ are contractions, it follows immediately from a theorem of Arendt and Nikolski ([2, Theorem A.7]) that $T_{\mathcal{B}^2}(a,0)$ and $T_{\mathcal{B}^2}(0,b)$ are holomorphic on $\mathbb{C}_+^2$. □

We shall need the following auxiliary result.

Proposition 4.2.3. *Let $-A_{\mathcal{B}^2,1}$ and $-A_{\mathcal{B}^2,2}$ be the generators of the C_0-semigroups $(T_{\mathcal{B}^2}(t,0))_{t\geqslant 0}$ and $(T_{\mathcal{B}^2}(0,t))_{t\geqslant 0}$, respectively. Then*

$$\begin{aligned} D(A_{\mathcal{B}^2,1}) &= \{f \in \mathcal{B}^2 : D_1 f \in \mathcal{B}^2\}, \quad A_{\mathcal{B}^2,1} f = -D_1 f, \\ D(A_{\mathcal{B}^2,2}) &= \{f \in \mathcal{B}^2 : D_2 f \in \mathcal{B}^2\}, \quad A_{\mathcal{B}^2,2} f = -D_2 f. \end{aligned} \tag{4.2.2}$$

Moreover, the generators of the C_0-groups $(T_{\mathcal{B}^2}(-is,0))_{s\in\mathbb{R}}$ and $(T_{\mathcal{B}^2}(0,-is))_{s\in\mathbb{R}}$ are $-iA_{\mathcal{B}^2,1}$ and $-iA_{\mathcal{B}^2,2}$, respectively. Finally, $\sigma(A_{\mathcal{B}^2,1}) = \sigma(A_{\mathcal{B}^2,2}) = \mathbb{R}_+$.

Proof. If $f \in D(A_{\mathcal{B}^2,1})$, then trivially $A_{\mathcal{B}^2,1} f = -D_1 f$. Similarly, if $f \in D(A_{\mathcal{B}^2,2})$, then $A_{\mathcal{B}^2,2} f = -D_2 f$. Conversely, if $f, D_1 f \in \mathcal{B}^2$, then

$$t_1(T_{\mathcal{B}^2}(t,0)f - f) = t^{-1} \int_0^t T_{\mathcal{B}}(s,0) D_1 f \, ds, \quad t \geqslant 0.$$

Since this formula holds pointwise and the integral is a $\mathcal{B}^2$-valued integral with continuous integrand, letting $t \to 0+$ yields $f \in D(A_{\mathcal{B}^2,1})$. This together with analogous considerations in the case when $f, D_2 f \in \mathcal{B}^2$ establishes (4.2.2). It then follows (cf. [2, Corollary 3.9.9 and 3.9.10]) that the C_0-groups $(T_{\mathcal{B}^2}(-is,0))_{s\in\mathbb{R}}$ and $(T_{\mathcal{B}^2}(0,-is))_{s\in\mathbb{R}}$ have $-iA_{\mathcal{B}^2,1}$ and $-iA_{\mathcal{B}^2,2}$ as their respective generators. Finally, in view of Section 3.5, for any $a \in \mathbb{R}_+$, $e^{-az_1} \otimes \mathbb{1}, \mathbb{1} \otimes e^{-az_2} \in \mathcal{B}^2$. The functions $e^{-az_1} \otimes \mathbb{1}$ and $\mathbb{1} \otimes e^{-az_2}$ are easily seen to be eigenvectors of $A_{\mathcal{B}^2,1}$ and $A_{\mathcal{B}^2,2}$, respectively, corresponding to the eigenvalue a, hence $\mathbb{R}_+ \subseteq \sigma(A_{\mathcal{B}^2,1}) \cap \sigma(A_{\mathcal{B}^2,2})$. □

The following proposition establishes that there is a one-to-one correspondence between the closed subspaces $H^\infty(I)$ from Lemma 4.1.1 and the subspaces of the two-parameter group of vertical shifts arising from Proposition 4.2.2.

Proposition 4.2.4. *Let G be the C_0-group given by*

$$G(t_1, t_2) = T_{\mathcal{B}^2}(-it_1, -it_2), \quad t_1, t_2 \in \mathbb{R},$$

and let I be a compact subset of $(0, \infty)^2$. Then $\mathcal{B}^2_G(I) = H^\infty(I)$.

The proof proceeds in a similar fashion to the proof of [4, Lemma 2.9].

Proof. Let $(S(t_1, t_2))_{t_1,t_2 \in \mathbb{R}}$ be the two-parameter group of shifts on $\mathrm{BUC}(\mathbb{R}^2)$:

$$(S(t_1, t_2)f)(s_1, s_2) = f(s_1 - t_1, s_2 - t_2).$$

We have that

$$(\mathcal{F}_{(2)}(S(\cdot, \cdot)f(s_1, s_2))(t_1, t_2) = 4\pi^2(\mathcal{F}^{-1}_{(2)} f)(s_1, s_2)e^{-i(s_1t_1+s_2t_2)}.$$

Let $K : \mathcal{B}^2 \to \mathrm{BUC}(\mathbb{R}^2)$ be given by $Kf = f^b$. By (3.1.3), K is a bounded linear map. Then

$$S(t_1, t_2)K = KG(t_1, t_2).$$

Hence, for $f \in \mathcal{B}^2$,

$$\mathrm{sp}_S(f^b) = \mathrm{supp}\, \mathcal{F}_{(2)}(S(\cdot, \cdot)Kf) = \mathrm{supp}(K\mathcal{F}_{(2)}G(\cdot, \cdot)f) = \mathrm{sp}_G(f). \tag{4.2.3}$$

This implies $\mathcal{B}^2_G(I) \subset H^\infty(I)$. Conversely, if I is compact, then $H^\infty(I) \subset \mathcal{B}^2$ by Lemma 4.1.1, and then (4.2.3) implies that $H^\infty(I) \subset B^2_G(I)$. □

The next proposition identifies a dense subspace of $\mathcal{B}^2_0$ which will play a crucial role in the construction of the functional calculus.

Proposition 4.2.5. *The set $\mathcal{G}^2 = \bigcup_{0<\varepsilon<\delta} H^\infty([\varepsilon, \delta]^2)$ is dense in $\mathcal{B}^2_0$.*

Proof. Consider $G(t_1, t_2)$ restricted to $\mathcal{B}^2_0$ and take $f \in \mathcal{B}^2_0$. By (3.1.4) one has

$$|D_1D_2f(z_1, z_2)| \leqslant \frac{C_{(1,2)}\|f\|_\infty}{\mathrm{Re}\, z_1 \, \mathrm{Re}\, z_2}.$$

Hence,

$$\lim_{t,s\to\infty} \|D_1D_2f(x_1 + t + i\cdot, x_2 + s + i\cdot)\|_\infty = 0,$$

for every $x_1, x_2 > 0$. Then, since $f \in \mathcal{B}_0^2$,

$$\lim_{t,s\to\infty} \|T_{\mathcal{B}^2}(t,s)f\|_{\mathcal{B}_0^2} = \lim_{t,s\to\infty} \int_0^\infty \int_0^\infty \|D_1 D_2 f(x+t+i\cdot, y+s+i\cdot)\|_\infty \,dx\,dy$$

$$= \lim_{t,s\to\infty} \int_t^\infty \int_s^\infty \|D_1 D_2 f(x+i\cdot, y+i\cdot)\|_\infty \,dx\,dy = 0.$$

By Corollary 3.4.4,

$$\lim_{t,s\to\infty} \|T_{\mathcal{B}^2}(t,s)f\|_{\mathcal{B}^2} = 0.$$

Take

$$f = \lim_{t,s\to\infty} (f - T_2(s)f - (T_1(t)f - T_1(t)T_2(s)f) = \lim_{t,s\to\infty} A_1 \int_0^t T_1(t_1)(f - T_2(s)f)\,dt_1$$

$$= -\lim_{t,s\to\infty} A_1 \int_0^t T_1(t_1) A_2 \int_0^s T_2(t_2) f\,dt_2\,dt_1 = -\lim_{t,s\to\infty} A_1 A_2 \int_0^t \int_0^s T(t_1,t_2) f\,dt_2\,dt_1.$$

This shows that the range of $A_1 A_2$ is dense in $\mathcal{B}_0^2$, so it now suffices to apply Proposition 4.2.1 to $G(t_1, t_2)$. □

We will need a two-dimensional version of the one-dimensional Bohr and Bernstein inequalities ([4, (2.6) and (2.7)]; the required two-dimensional version of Bernstein inequalities is already covered by (4.1.3) and (4.1.4) above). We want to obtain similar results for spaces of the form $H^\infty([\varepsilon_1, \varepsilon_2]^2)$, where $0 < \varepsilon_1 < \varepsilon_2$. Note that $[\varepsilon_1, \varepsilon_2]^2$ is contained in the quarter-annulus $\Omega = \{s \in \mathbb{R}_+^2 : \varepsilon_1 \leqslant |s| \leqslant \sqrt{2}\varepsilon_2\}$. In [3, Lemma 2.1], it is shown that, for any $k \in \mathbb{N}$, the norm

$$\max_{|\alpha|=k} \|\partial^\alpha f\|_\infty, \qquad f \in H^\infty[\Omega],$$

where the maximum is taken over all multi-indices of length k, is equivalent to the sup-norm $\|f\|_\infty$ on $H^\infty[\Omega]$. The following result shows that, on the smaller space $H^\infty([\varepsilon_1, \varepsilon_2]^2)$, the sup-norm is equivalent to each individual norm $\|\partial^\alpha f\|_\infty$.

Proposition 4.2.6. *Let α be a multi-index, and let $0 < \varepsilon_1 < \varepsilon_2$. Then the norm $\|\partial^\alpha f\|_\infty$ is equivalent to the norm $\|f\|_\infty$ on the space $H^\infty([\varepsilon_1, \varepsilon_2]^2)$.*

Proof. We give the proof in the case when $\alpha = (1,0)$, so $\partial^\alpha = D_1$. An identical argument covers the case when $\partial^\alpha = D_2$. Since (3.1.2) implies that if $f \in H^\infty([\varepsilon_1, \varepsilon_2]^2)$, then the functions $D_i f$, for $i = 1, 2$, also have distributional inverse Fourier transforms with support in $[\varepsilon_1, \varepsilon_2]^2$, the general result follows by an easy induction on the length of α.

Let X be the space $\mathrm{BUC}(\mathbb{R}^2)$, with the sup-norm, and let $S(t_1, t_2)$ be the 2-parameter group of vertical shifts as in the proof of Proposition 4.2.4. Let $S_1(t_1) = S(t_1, 0)$ and $S_2(t_2) = S(0, t_2)$, which define commuting C_0-groups on X. Let A_1 be the negative generator of $S_1(t)$. Then

$$D(A_1) = \{g \in X : \partial_1 g \text{ is defined on } \mathbb{R}^2 \text{ and } \partial_1 g \in X\}, \qquad A_1 g = \partial_1 g,$$

where $\partial_1 g$ denotes the derivative of g with respect to the first variable t_1.

Let $f \in H^\infty([\varepsilon_1, \varepsilon_2]^2)$. Then (4.1.3) shows that $\|D_1 f\|_\infty \leqslant \varepsilon_2 \|f\|_\infty$. For the lower bound, note that f extends to an entire function of exponential type (cf. Section 4.1) on $\mathbb{C}^2$ (also denoted by f) and that $D_1 f \in H^\infty([\varepsilon_1, \varepsilon_2]^2)$, since $(D_1 f)^b = -i\partial_1(f^b)$ and the support of the inverse Fourier transform of this is contained in the support for f^b. Moreover

$$f^b \in X_S([\varepsilon_1, \varepsilon_2]^2) \subseteq X_S([\varepsilon_1, \varepsilon_2] \times \mathbb{R}) = X_{S_1}([\varepsilon_1, \varepsilon_2]) =: Y,$$

by [40, Lemma 2.4.8]. By the theory of the Arveson spectrum for C_0-groups (see [40, Section 2.4]), Y is a closed subspace of X contained in $D(A_1)$, the generator of the C_0-group $S_1(t_1)|_Y$ is the restriction $\widetilde{A}_1$ of A_1 to Y, $\widetilde{A}_1$ is a bounded operator on Y and $\sigma(\widetilde{A}_1) \subseteq [\varepsilon_1, \varepsilon_2]$ (in fact, equality holds).

Now $f^b = (\widetilde{A}_1)^{-1}(\widetilde{A}_1 f^b)$, so

$$\|f^b\|_\infty \leqslant C' \|\widetilde{A}_1 f^b\|_\infty = C' \|D_1 f\|_\infty,$$

where $C' = \|(\widetilde{A}_1)^{-1}\|$, giving the lower bound.

Hence there exists a unique $g \in Y$ such that $\widetilde{A}_1 g = (D_1 f)^b = \widetilde{A}_1(f^b)$, and $\|g\|_\infty \leqslant C' \|(D_1 f)^b\|_\infty$, where C' is the norm of the inverse of $\widetilde{A}_1$. Thus $g - f^b \in \operatorname{Ker} A_1 = X_{S_1}(0)$. In addition $g - f^b \in X_{S_1}([\varepsilon_1, \varepsilon_2])$, so

$$g - f^b \in X_{S_1}(\{0\}) \cap X_{S_1}([\varepsilon_1, \varepsilon_2]) = X_{S_1}(\varnothing) = \{0\},$$

by [15, Lemma 8.9]. Hence

$$\|f\|_{H^\infty(\mathbb{C}^2_+)} = \|f^b\|_X = \|g\|_X \leqslant C' \|(D_1 f)^b)\|_X = C' \|D_1 f\|_{H^\infty(\mathbb{C}^2_+)},$$

giving the required estimate. It follows from (4.1.3) that $D_1 f \in \mathcal{B}^2$, and then $A_1 f^b = -i(D_1 f)^b$. Hence we have that

$$c\|f\|_\infty \leqslant \|D_1 f\|_\infty \leqslant C\|f\|_\infty,$$

where C is the norm of the restriction of A_1 to $X_{S_1}([\varepsilon_1, \varepsilon_2])$, and c^{-1} is the norm of the inverse of that operator. □

4.3 Class $\mathcal{E}^2$ and a partial duality

Define $\mathcal{E}^2$ to be the space of those holomorphic functions g on $\mathbb{C}_+^2$ such that

$$\|g\|_{\mathcal{E}_0^2} = \sup_{x_1,x_2>0} x_1x_2 \int_0^\infty \int_0^\infty |D_1D_2g(x_1+iy_1,x_2+iy_2)|\,dy_1\,dy_2 < \infty. \tag{4.3.1}$$

Consider now the (partial) duality between $\mathcal{E}^2$ and $\mathcal{B}^2$ given by

$$\langle g,f\rangle_{\mathcal{B}^2} =$$

$$\int_{-\infty}^\infty \int_{-\infty}^\infty x_1x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty D_1D_2g(x_1-iy_1,x_2-iy_2)D_1D_2f(x_1+iy_1,x_2+iy_2)\,dy_1\,dy_2\,dx_1\,dx_2. \tag{4.3.2}$$

It is easily seen that the partial duality is bounded. Calculating directly

$$\left|\int_0^\infty \int_0^\infty x_1x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty D_1D_2g(x_1-iy_1,x_2-iy_2)D_1D_2f(x_1+iy_1,x_2+iy_2)dy_1\,dy_2\,dx_1\,dx_2\right|$$

$$\leqslant \int_0^\infty \int_0^\infty x_1x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty (|D_1D_2g(x_1-iy_1,x_2-iy_2)|)$$

$$\times \left(\sup_{\beta_1,\beta_2\in\mathbb{R}} |D_1D_2f(x_1+i\beta_1,x_2+i\beta_2)|\right) dy_1\,dy_2\,dx_1\,dx_2$$

$$\leqslant \int_0^\infty \int_0^\infty \|g\|_{\mathcal{E}_0^2} \left(\sup_{\beta_1,\beta_2\in\mathbb{R}} |D_1D_2f(x_1+i\beta_1,x_2+i\beta_2)|\right) dx_1\,dx_2$$

$$\leqslant \|g\|_{\mathcal{E}_0^2}\|f\|_{\mathcal{B}_0^2}.$$

Consider now

$$\eta(z_1,z_2) = \frac{1-e^{-z_1}}{z_1}\frac{1-e^{-z_2}}{z_2}. \tag{4.3.3}$$

Then, as in [4, Example 2.12(4)], η is the double Laplace transform of Lebesgue measure on $[0,1]^2$. Hence, as a Laplace transform of a bounded measure, $\eta \in \mathcal{B}^2$. The following proposition shows how to approximate functions $f \in \mathcal{B}^2$ by functions in $\mathcal{LM}_{(2)}$ in certain senses, which enable us to extend results from $\mathcal{LM}_{(2)}$ to $\mathcal{B}^2$ (see Propositions 4.4.3, 5.1.1 and 6.1.5); the proof makes substantial use of Proposition 4.2.6.

Proposition 4.3.1. *Let $\eta(z_1,z_2)$ be as above. For $\delta > 0$ and any $f \in \mathcal{B}^2$, let*

$$_\delta\eta(z_1,z_2) = \eta(\delta z_1,\delta z_2), \quad _\delta f = {}_\delta\eta f.$$

Then, for each $g \in \mathcal{E}^2$,

$$\lim_{\delta\to 0+} \langle g, {}_\delta f\rangle_{\mathcal{B}^2} = \langle g, f\rangle_{\mathcal{B}^2}.$$

Proof. We may assume $f(z_1,\infty) = f(\infty, z_2) = 0$, so $f \in \overline{\mathcal{G}}_2$. Since (by direct calculation) $\|\eta\|_{\mathcal{B}^2} = \|{}_\delta\eta\|_{\mathcal{B}^2}$, the set $\{{}_\delta\eta : 0 < \delta < 1\}$ is bounded in $\mathcal{B}^2$. Consider $f \in H^\infty([\varepsilon_1,\varepsilon_2]^2)$ for $0 < \varepsilon_1 < \varepsilon_2$. In view of (4.1.4), $D_1 f, D_2 f, D_1D_2 f \in \mathcal{B}^2$. Moreover, the function f extends to an entire function of an exponential type (cf. Section 4.1) on $\mathbb{C}^2$ (also denoted by f). For fixed $x_1, x_2 > 0$, let:

$$f_{x_1,x_2}(y_1,y_2) := f(x_1+iy_1, x_2+iy_2).$$

It follows from (3.1.2) that if $f \in H^\infty(I)$, then the functions f_{x_1,x_2}, and $D_i f_{x_1,x_2}$, for $i = 1,2$, also have distributional inverse Fourier transforms with support in I. Applying Proposition 4.2.6 to f_{x_1,x_2} and $D_i f_{x_1,x_2}$, $i = 1,2$, and using (3.1.3) we obtain

$$\sup_{s_1,s_2\in\mathbb{R}} |f(x_1+is_1, x_2+is_2)| \leqslant c \sup_{s_1,s_2\in\mathbb{R}} |D_1D_2 f(x_1+is_1, x_2+is_2)|, \tag{4.3.4}$$

$$\sup_{s_1,s_2\in\mathbb{R}} |D_j f(x_1+is_1, x_2+is_2)| \leqslant c \sup_{s_1,s_2\in\mathbb{R}} |D_1D_2 f(x_1+is_1, x_2+is_2)|, \quad j = 1,2 \tag{4.3.5}$$

with some constant $c > 0$. These two inequalities imply in turn the following (with $z_j = x_j + iy_j$, $j = 1,2$)

$$|f(z_1,z_2)| \leqslant c \sup_{s_1,s_2\in\mathbb{R}} |D_1D_2 f(x_1+is_1, x_2+is_2)|, \tag{4.3.6}$$

$$|D_j f(z_1,z_2)| \leqslant c \sup_{s_1,s_2\in\mathbb{R}} |D_1D_2 f(x_1+is_1, x_2+is_2)|, \quad j = 1,2. \tag{4.3.7}$$

Now,

$$\begin{aligned}|D_1D_2 f - D_1D_2\, {}_\delta f| &\leqslant |f D_1D_2\, {}_\delta\eta| + |D_1 f D_2\, {}_\delta\eta| + |D_2 f D_1\, {}_\delta\eta| + |(1-{}_\delta\eta) D_1D_2 f| \\ &\leqslant (|f| + |D_1 f| + |D_2 f| + |D_1D_2 f|)\, \gamma_\delta,\end{aligned}$$

with

$$\gamma_\delta = |1 - {}_\delta\eta| + |D_1\, {}_\delta\eta| + |D_2\, {}_\delta\eta| + |D_1D_2\, {}_\delta\eta|.$$

Hence,

$$
\begin{aligned}
&|\langle g, f - {}_\delta f\rangle_{\mathcal{B}^2}| \\
&\quad \leqslant \left| \int_0^\infty \int_0^\infty x_1 x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty D_1 D_2 g(x_1 - iy_1, x_2 - iy_2) \right. \\
&\qquad \left. \times [D_1 D_2 f - D_1 D_2\, {}_\delta f](x_1 + iy_1, x_2 + iy_2) dy_1\, dy_2\, dx_1\, dx_2 \right| \\
&\quad \leqslant \Big(\int_0^\infty \int_0^\infty x_1 x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty |D_1 D_2 g(x_1 - iy_1, x_2 - iy_2)| \\
&\qquad \times [(|f| + |D_1 f| + |D_2 f| + |D_1 D_2 f|)\, \gamma_\delta](x_1 + iy_1, x_2 + iy_2) dy_1\, dy_2\, dx_1\, dx_2 \Big).
\end{aligned}
$$

Given (4.3.6) and (4.3.7),

$$
\begin{aligned}
&|\langle g, f - {}_\delta f\rangle_{\mathcal{B}^2}| \\
&\quad \leqslant C \int_0^\infty \int_0^\infty x_1 x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty |D_1 D_2 g(x_1 - iy_1, x_2 - iy_2)| \\
&\qquad \times \sup_{s_1, s_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + is_1, x_2 + is_2)| \times \gamma_\delta(x_1 + iy_1, x_2 + iy_2) dy_1\, dy_2\, dx_1\, dx_2,
\end{aligned}
$$

with some constant C. Since $f \in \mathcal{B}^2$, $g \in \mathcal{E}^2$, and

$$\lim_{\delta \to 0+} \gamma(z_1, z_2) = 0, \quad z_1, z_2 \in \mathbb{C}_+.$$

From these properties, the absolute convergence of the repeated integral (4.3.2), and the dominated convergence theorem we obtain

$$\lim_{\delta \to 0} \langle g, f - {}_\delta f\rangle_{\mathcal{B}^2} = 0.$$

□

4.4 Some topological properties

We shall need the following auxiliary result.

Proposition 4.4.1. *Let $f \in H^\infty([0,\sigma]^2)$, $g \in \mathcal{LM}_{(2)}$, with $\mathcal{F}_{(2)}^{-1} g^b \in L^2([0,\delta]^2)$, where $\sigma, \delta > 0$. Then $fg \in \mathcal{LM}_{(2)}$.*

As before (cf. (3.1.1)), f^b and g^b denote the boundary functions of f and g, respectively.

Proof. Since $f \in H^\infty([0,\sigma]^2)$, $\mathcal{F}_{(2)}^{-1} f^b$ is a distribution on $\mathbb{R}^2$ with support in $[0,\sigma]^2$. Then

$$\operatorname{supp}(\mathcal{F}_{(2)}^{-1} f^b * \mathcal{F}_{(2)}^{-1} g^b) \subset [0,\sigma+\delta]^2.$$

Here $\psi = \mathcal{F}_{(2)}^{-1} f^b * \mathcal{F}_{(2)}^{-1} g^b$ is understood as a convolution of distributions; it is well-defined since the support of each of the distributions is a subset of $[0,\sigma+\delta]^2$. Consequently, ψ is itself a distribution on $\mathbb{R}^2$ with compact support, and $\mathcal{F}_{(2)}\psi = f^b g^b$. Since f^b is bounded and $g^b \in L^2(\mathbb{R}^2)$ by Plancherel, $f^b g^b \in L^2(\mathbb{R}^2)$. Applying Plancherel once more, $\psi \in L^2(\mathbb{R}^2)$, and $\mathcal{L}\psi = fg$. Since ψ has support in $[0,\sigma+\delta]^2$, $\psi \in L^1(\mathbb{R}_+^2)$, and hence $fg \in \mathcal{LM}_{(2)}$. □

Proposition 4.4.2. *Consider $\mathcal{B}^2$ with its norm topology. Then $\mathcal{LM}_{(2)}$ is not dense in $\mathcal{B}^2$.*

Proof. For each measure $\mu \in M(\mathbb{R}^2)$, $\mathcal{F}_{(2)}^{-1}\mu$ is weakly almost periodic ([18, Corollary 4.2.4]), and weakly almost periodic functions form a proper closed subspace of $\mathrm{BUC}(\mathbb{R}^2)$ ([19, Theorem 5.3, 12.1]). Much like in the one-dimensional case, it can be seen that the entire functions of exponential type on $\mathbb{C}^2$ (cf. Section 4.1 and (4.1.2) above) are dense in $\mathrm{BUC}(\mathbb{R}^2)$. For let $f \in \mathrm{BUC}(\mathbb{R}^2)$. Define

$$K(t_1,t_2) = (2\pi^{-1})^2 (t_1^{-1}\sin(t_1/2))^2 (t_2^{-1}\sin(t_2/2))^2.$$

Let $\tau_1, \tau_2 > 0$ and consider the function

$$G(z_1,z_2) = \tau_1\tau_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\tau_1(t_1-z_1),\tau_2(t_2-z_2)) f(t_1,t_2)\,dt_1\,dt_2.$$

Then, as in the proof of [9, Theorem 12.11.1], K is entire of exponential type $\tau = (1,1)$, whereas G is entire of exponential type $\tau = (\tau_1,\tau_2)$. Also,

$$|G(x_1,x_2) - f(x_1,x_2)|$$
$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t_1,t_2)\,dt_1\,dt_2 \sup_{\alpha_1,\alpha_2\in\mathbb{R}} |f(\alpha_1+t_1/\tau_1,\alpha_2+t_2/\tau_2) - f(\alpha_1,\alpha_2)|,$$

where the right-hand side tends to 0 as $\tau_1,\tau_2 \to \infty$, since f is assumed to be in $\mathrm{BUC}(\mathbb{R}^2)$. Hence, there exist an entire function G of exponential type and $\delta > 0$ such that, for all $\mu \in M(\mathbb{R}^2)$,

$$\|G_{\mathbb{R}^2} - \mathcal{F}_{(2)}^{-1}\mu\|_\infty \geqslant \delta,$$

where $G_{\mathbb{R}^2}$ is the restriction of G to $\mathbb{R}^2$. If $\tau = (\tau_1,\tau_2)$ is the exponential type of G, take (σ,σ) with $\sigma \geqslant \tau_i$ for $i = 1,2$, so that $\mathcal{F}_{(2)}^{-1}G \subset ([-\sigma,\sigma]^2)$. Let

$$g(z_1,z_2) = e^{-2i\sigma(z_1+z_2)} G(z_1,z_2),$$

so that $\operatorname{supp}(\mathcal{F}^{-1}_{(2)}g) \subset ([\sigma, 3\sigma]^2)$. For any $\mu \in M(\mathbb{R}^2)$, let $\mu_\sigma(A_1 \times A_2) = \mu(A_1 + 2\sigma, A_2 + 2\sigma)$ for any Borel subsets A_1, A_2 of $\mathbb{R}$. Then

$$\|g_{\mathbb{R}^2} - \mathcal{F}^{-1}_{(2)}\mu\|_{L^\infty(\mathbb{R}^2)} = \|G_{\mathbb{R}^2} - \mathcal{F}^{-1}_{(2)}\mu_\sigma\|_{L^\infty(\mathbb{R}^2)} \geqslant \delta.$$

Letting $f(z_1, z_2) = g(-iz_1, -iz_2)$, one has $f \in H^\infty([\sigma, 3\sigma]^2) \subset \mathcal{B}^2$ and $\|f - m\|_{\mathcal{B}^2} \geqslant \|f - m\|_\infty \geqslant \delta$ for any $m \in \mathcal{LM}_{(2)}$. $\square$

Proposition 4.4.3. *The space $\mathcal{LM}_{(2)}$ is dense in $\mathcal{B}^2$ in the topology of uniform convergence on compact subsets of $\mathbb{C}^2_+$.*

Proof. Let η, ${}_\delta\eta$ be as in Proposition 4.3.1. For $f \in \mathcal{G}^2$, let ${}_\delta f(z_1, z_2) = f(z_1, z_2)\,{}_\delta\eta(z_2, z_2)$. Then ${}_\delta\eta(z_2, z_2) \to 1$ and hence ${}_\delta f(z_1, z_2) \to f(z_1, z_2)$ uniformly on compact subsets, as $\delta \to 0+$. By Proposition 4.4.1, ${}_\delta f \in \mathcal{LM}_{(2)}$. By Proposition 4.2.5, $\mathcal{G}^2$ is dense in $\mathcal{B}^2_0$.

By [4, Lemma 2.14], $\mathcal{LM}_{(1)}$ is dense in in $\mathcal{B}^1$. Furthermore, the embedding $f \mapsto f \otimes \mathbb{1}$ (respectively: the embedding $f \mapsto \mathbb{1} \otimes f$) is continuous from $\mathcal{B}^1$ to $\mathcal{B}^2$ (cf. Section 3.5) and maps $\mathcal{LM}_{(1)}$ to $\mathcal{LM}_{(2)}$, i.e. $\mu \mapsto \mu \otimes \delta_0$ (respectively: $\mu \mapsto \delta_0 \otimes \mu$)). So the conclusion then follows from Proposition 3.2.2.

Since $\mathbb{1} \otimes \mathcal{L}\mu, \mathcal{L}\mu \otimes \mathbb{1} \in \mathcal{LM}_{(2)}$, for any $\mu \in M(\mathbb{R}_+)$, the result follows from the one-dimensional version in [4, Lemma 2.14]. $\square$

5 The class $\mathcal{B}^2$ – further properties

In the present chapter we collect some important further properties of functions in the class $\mathcal{B}^2$, many of which are later employed in our construction of the functional calculus. In Section 5.1 we discuss representations of functions in $\mathcal{B}^2$. We have seen in Chapter 4 (cf. Proposition 4.4.2) that the subspace $\mathcal{LM}_{(2)}$ is not dense in $\mathcal{B}^2$. We have also shown that this is, in a sense, merely a technical inconvenience as we can approximate any function in $\mathcal{B}^2$ with functions in $\mathcal{LM}_{(2)}$ via the partial duality with $\mathcal{E}^2$. In the present context, in order to obtain that each $f \in \mathcal{B}^2$ enjoys the particular representation given in Proposition 5.1.1, we will first show the result holds for arbitrary functions in $\mathcal{LM}_{(2)}$, and secondly, extend it to the whole space using Proposition 4.3.1. In Section 5.4 we explore the behaviour of the two-parameter semigroup of shifts on $\mathcal{B}^2$ with respect to the partial duality. While the result expressed in Proposition 5.4.1 is not particularly interesting on its own, it then allows us to prove some topological results on the dual space of $\mathcal{E}_0^2$ in Section 5.5. In Section 5.6 we obtain a convergence lemma for sequences of functions f_n in the class $\mathcal{B}^2$.

5.1 Representations

The next proposition establishes that any function in $\mathcal{B}^2$ can be represented in terms of the partial duality introduced in Section 4.3.

Proposition 5.1.1. *Let $f \in \mathcal{B}^2$, $\mathsf{z} = (z_1, z_2) \in \overline{\mathbb{C}}_+ \times \overline{\mathbb{C}}_+$, and take*

$$r_{\mathsf{z}}(\lambda_1, \lambda_2) = (\lambda_1 + z_1)^{-1}(\lambda_2 + z_2)^{-1},$$

$$r_{z_1}(\lambda_1) = (\lambda_1 + z_1)^{-1},$$
$$r_{z_2}(\lambda_2) = (\lambda_2 + z_2)^{-1}.$$

Then

$$f(z_1, z_2) = f(\infty, \infty) + \frac{2}{\pi}\langle r_{z_1}, f_1\rangle_{\mathcal{B}^1} + \frac{2}{\pi}\langle r_{z_2}, f_2\rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\langle r_{\mathsf{z}}, f\rangle_{\mathcal{B}^2}. \tag{5.1.1}$$

Proof. Let $f = \mathcal{L}\mu \in \mathcal{LM}_{(2)} \cap \mathcal{B}_0^2$. Then (cf. Remark 3.6.1) $\mu(\{0\} \times \mathbb{R}_+) = \mu(\mathbb{R}_+ \times \{0\}) = 0$, so we may consider μ as a measure on $(0,\infty)^2$. Now

$$\langle r_{\mathsf{z}}, f\rangle_{\mathcal{B}^2} =$$

$$\int_0^\infty \int_0^\infty x_1 x_2 \int_{-\infty}^\infty \int_{-\infty}^\infty D_1 D_2 r_{\mathsf{z}}(x_1 - iy_1, x_2 - iy_2) D_1 D_2 f(x_1 + iy_1, x_2 + iy_2) dy_1\, dy_2\, dx_1\, dx_2$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^\infty \int_0^\infty \frac{x_1}{(x_1 - iy_1 + z_1)^2} \frac{x_2}{(x_2 - iy_2 + z_2)^2} \times$$

$$\times \int_{(0,\infty)^2} t_1 t_2 e^{-(x_1+iy_1)t_1} e^{-(x_2+iy_2)t_2} d\mu(t_1,t_2)\, dx_1\, dx_2\, dy_1\, dy_2$$

$$= \int_{(0,\infty)^2} \int_0^\infty \int_0^\infty x_1 t_1 x_2 t_2 e^{-x_1 t_1} e^{-x_2 t_2} \times$$

$$\times \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e^{-iy_1 t_1}}{(x_1 - iy_1 + z_1)^2} \frac{e^{-iy_2 t_2}}{(x_2 - iy_2 + z_2)^2} dy_1\, dy_2\, dx_1\, dx_2\, d\mu(t_1,t_2)$$

$$= 4\pi^2 \int_{(0,\infty)^2} \int_0^\infty \int_0^\infty \left(x_1 t_1^2 e^{-2x_1 t_1} e^{-z_1 t_1}\right) \left(x_2 t_2^2 e^{-2x_2 t_2} e^{-z_2 t_2}\right) dx_1\, dx_2\, d\mu(t_1,t_2)$$

$$= \frac{\pi^2}{4} \int_{(0,\infty)^2} e^{-t_1 z_1 - t_2 z_2}\, d\mu(t_1,t_2) = \frac{\pi^2}{4} f(z_1,z_2),$$

where we have used that $y \mapsto (x - iy + z)^2$ is the inverse Fourier transform (cf. Section 1.3) of $t \mapsto 2\pi t e^{-xt} e^{-zt}$ on $\mathbb{R}_+$, taken to be 0 on $(-\infty, 0)$.

This establishes that (5.1.1) holds for all functions in $\mathcal{LM}_{(2)} \cap \mathcal{B}_0^2$. Consider $f \in \mathcal{G}^2$, so with $f_1(z_1) = f_2(z_2) = 0$. Take

$$\eta(z_1,z_2) = \frac{1 - e^{-z_1}}{z_1} \frac{1 - e^{-z_2}}{z_2}.$$

Let $\delta > 0$. By Proposition 4.4.1 and [4, Example 2.12(4)],

$${}_\delta f(z_1,z_2) = f(z_1,z_2)\eta(\delta z_1, \delta z_2)$$

is in $\mathcal{LM}_{(2)}$; and since ${}_\delta f(z_1,\infty) = {}_\delta f(\infty, z_2) = 0$, we have ${}_\delta f \in \mathcal{B}_0^2$, so that ${}_\delta f$ satisfies (5.1.1). Hence, by Proposition 4.3.1,

$$f(z_1,z_2) = \lim_{\delta\to 0} {}_\delta f(z_1,z_2) = \lim_{\delta\to 0} \frac{4}{\pi^2}\langle r_{\mathsf{z}}, {}_\delta f\rangle = \frac{4}{\pi^2}\langle r_{\mathsf{z}}, f\rangle.$$

The formula then extends to $\mathcal{B}_0^2$ by continuity in view of Proposition 4.2.5, and then to $\mathcal{B}^2$ by adding arbitrary functions $f_1(z_1), f_2(z_2) \in \mathcal{B}^1$ such that $f_1(\infty) = f_2(\infty) = 0$, and adding an arbitrary constant corresponding to $f(\infty,\infty)$. This is consistent, since we established (cf. Section (3.5)) that

$$P_{1,2}f(z_1,z_2) = f_1(z_1) + f_2(z_2) - f(\infty,\infty),$$
$$f = (I - P_{1,2}f) + P_{1,2}f.$$

□

The following proposition is a two-dimensional version of [5, Proposition 2.2].

Proposition 5.1.2. *Let $f \in \mathcal{B}_0^2$, $g \in \mathcal{E}^2$, and suppose that for every $z_2 \in \mathbb{C}_+$ one has $D_2 g(\cdot, z_2) \in H^1(\mathbb{C}_+)$, and that for every $z_1 \in \mathbb{C}_+$ one also has $g(z_1,\cdot) \in H^1(\mathbb{C}_+)$. Then*

$$\langle g, f\rangle_{\mathcal{B}^2} = \frac{1}{16}\int_{\mathbb{R}}\int_{\mathbb{R}} g(-iy_1,-iy_2)f(iy_1,iy_2)\,dy_2\,dy_1. \tag{5.1.2}$$

Proof. We have

$$\int_0^\infty\int_0^\infty \alpha\gamma\int_{\mathbb{R}}\int_{\mathbb{R}} D_1D_2f(\alpha+i\beta,\gamma+i\delta)D_1D_2g(\alpha-i\beta,\gamma-i\delta)d\beta\,d\delta\,d\alpha\,d\gamma$$
$$= \int_0^\infty \gamma\int_{\mathbb{R}}\left(\int_0^\infty \alpha\int_{\mathbb{R}} D_1D_2f(\alpha+i\beta,\gamma+i\delta)D_1D_2g(\alpha-i\beta,\gamma-i\delta)\,d\beta\,d\alpha\right)d\delta\,d\gamma.$$

Now, for any $z_2 \in \mathbb{C}_+$, $D_2 g(\cdot,z_2) \in H^1(\mathbb{C}_+)$ and, for almost any $z_2 \in \mathbb{C}_+$, $D_2 f(\cdot,z_2) \in \mathcal{B}^1$ (cf. Section 3.5). Hence, applying Green's formula to

$$F(x,y) = D_2f(x+iy,z_2)D_2g(x-iy,z_2)$$

and using [5, Proposition 2.2], we obtain

$$\frac{1}{4}\int_{\mathbb{R}} D_2g(-iy_1,z_2)D_2f(iy_1,z_2)\,dy_1 = \int_0^\infty \alpha\int_{\mathbb{R}} D_1D_2g(\alpha-i\beta,z_2)D_1D_2f(\alpha+i\beta,z_2)\,d\beta\,d\alpha.$$

Let us fix $y_1 \in \mathbb{R}$ and consider

$$\int_{\mathbb{R}} g(-iy_1,-iy_2)f(iy_1,iy_2)\,dy_2.$$

Since, for any $z_1 \in \mathbb{C}_+$, $f(z_1,\cdot) \in \mathcal{B}_0^1$ and $g(z_1,\cdot) \in H^1(\mathbb{C}_+)$, we obtain by applying [5, Proposition 2.2] once more,

$$\frac{1}{4}\int_{\mathbb{R}} g(-iy_1,-iy_2)f(iy_1,iy_2)\,dy_2 = \int_0^\infty \gamma\int_{\mathbb{R}} D_2g(-y_1,\gamma-\delta)D_2f(y_1,\gamma+i\delta)\,d\delta\,d\gamma.$$

Hence,

$$\int_0^\infty \int_0^\infty \alpha\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} D_1 D_2 f(\alpha + i\beta, \gamma + i\delta) D_1 D_2 g(\alpha - i\beta, \gamma - i\delta) d\beta\, d\delta\, d\alpha\, d\gamma$$

$$= \int_0^\infty \gamma \int_{\mathbb{R}} \left(\int_0^\infty \alpha \int_{\mathbb{R}} D_1 D_2 f(\alpha + i\beta, \gamma + i\delta) D_1 D_2 g(\alpha - i\beta, \gamma - i\delta)\, d\beta\, d\alpha \right) d\delta\, d\gamma$$

$$= \int_0^\infty \gamma \int_{\mathbb{R}} \left(\frac{1}{4} \int_{\mathbb{R}} D_2 g(-iy_1, \gamma - i\delta) D_2 f(iy_1, \gamma + i\delta)\, dy_1 \right) d\delta\, d\gamma$$

$$= \frac{1}{4} \int_{\mathbb{R}} \left(\int_0^\infty \gamma \int_{\mathbb{R}} D_2 g(-iy_1, \gamma - i\delta) D_2 f(iy_1, \gamma + i\delta)\, d\delta\, d\gamma \right) dy_1$$

$$= \frac{1}{16} \int_{\mathbb{R}} \int_{\mathbb{R}} g(-iy_1, -iy_2) f(iy_1, iy_2)\, dy_2\, dy_1.$$

□

5.2 Constructing functions in $\mathcal{B}^2$

The present section serves to introduce certain methods of obtaining functions in $\mathcal{B}^2$. We first define an auxiliary class $\mathcal{W}^2$ of measurable functions on $\mathbb{C}^2_+$. We then study the behaviour of a certain linear operator on $\mathcal{W}^2$, and establish that any function in $\mathcal{B}^2_0$ can be constructed from a suitably chosen function in $\mathcal{W}^2$. We conclude this section by providing a procedure for approximating functions in $\mathcal{B}^2$.

Let $\mathcal{W}^2$ be the space of all equivalence classes of measurable functions $g : \mathbb{C}^2_+ \to \mathbb{C}$ such that

$$\|g\|_{\mathcal{W}^2} = \int_0^\infty \int_0^\infty \operatorname*{ess\,sup}_{\beta,\delta \in \mathbb{R}} |g(\alpha + i\beta, \gamma + i\delta)|\, d\alpha\, d\gamma < \infty, \tag{5.2.1}$$

with the norm given by (5.2.1). Clearly,

$$\{D_1 D_2 f : f \in \mathcal{B}^2\} \subset \mathcal{W}^2 \cap \mathrm{Hol}(\mathbb{C}^2_+).$$

For $g \in \mathcal{W}^2$, let

$$(Q_{(2)} g)(z_1, z_2) = \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \alpha\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g(\alpha + i\beta, \gamma + i\delta)}{(z_1 + \alpha - i\beta)^2 (z_2 + \gamma - i\delta)^2}\, d\beta\, d\delta\, d\alpha\, d\gamma. \tag{5.2.2}$$

If $f \in \mathcal{B}^2$, the reproducing formula (5.1.1) for f can now be written as

$$f(z_1, z_2) = f(\infty, \infty) + Q_{(1)}(f_1)(z_1) + Q_{(1)}(f_2)(z_2) + Q_{(2)}(f)(z_1, z_2).$$

where (cf. [5, Section 3])

$$(Q_{(1)}h)(z) = -\frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \frac{h(\alpha + i\beta)}{(z + \alpha - i\beta)^2} \, d\beta \, d\alpha,$$

for any $h \in \mathcal{W}^1$, i.e. the space of all (equivalence classes) of measurable functions $h : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\int_0^\infty \operatorname*{ess\,sup}_{\beta \in \mathbb{R}} |h(\alpha + i\beta)| \, d\alpha < \infty.$$

We obtain the following analogue of [5, Proposition 3.1(1)].

Proposition 5.2.1. *Let $\varphi \in L^\infty(\mathbb{R}^2)$, and $\alpha, \gamma > 0$. Let*

$$G_{\alpha,\gamma,\varphi}(z_1, z_2) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\varphi(\beta, \delta)}{(z_1 + \alpha - i\beta)^2 (z_2 + \gamma - i\delta)^2} \, d\beta \, d\delta, \quad z_1, z_2 \in \mathbb{C}_+.$$

Then $G_{\alpha,\gamma,\varphi} \in \mathcal{B}_0^2$, and

$$\|G_{\alpha,\gamma,\varphi}\|_{\mathcal{B}_0^2} \leqslant \left(\frac{16}{\alpha\gamma} \right) \|\varphi\|_{L^\infty},$$

$$\|G_{\alpha,\gamma,\varphi}\|_{\mathcal{B}^2} \leqslant \left(\frac{24 + \pi^2}{\alpha\gamma} \right) \|\varphi\|_{L^\infty}.$$

Proof. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and note that $G_{\alpha,\gamma,\varphi}(z_1, \infty) = G_{\alpha,\gamma,\varphi}(\infty, z_2) = 0$. We have

$$D_1 G_{\alpha,\gamma,\varphi}(z_1, z_2) = -2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\varphi(\beta, \delta)}{(z_1 + i\alpha - \beta)^3 (z_2 + i\gamma - \delta)^2} \, d\beta \, d\delta,$$

$$D_2 G_{\alpha,\gamma,\varphi}(z_1, z_2) = -2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\varphi(\beta, \delta)}{(z_1 + i\alpha - \beta)^2 (z_2 + i\gamma - \delta)^3} \, d\beta \, d\delta,$$

$$D_1 D_2 G_{\alpha,\gamma,\varphi}(z_1, z_2) = 4 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\varphi(\beta, \delta)}{(z_1 + i\alpha - \beta)^3 (z_2 + i\gamma - \delta)^3} \, d\beta \, d\delta.$$

Hence,

$$\begin{aligned} |D_1 G_{\alpha,\gamma,\varphi}(z_1, z_2)| &\leqslant 2\|\varphi\|_{L^\infty} \int_{\mathbb{R}} \frac{d\beta}{((x_1 + \alpha)^2 + (\beta - y_1)^2)^{3/2}} \int_{\mathbb{R}} \frac{d\delta}{(x_2 + \gamma)^2 + (\delta - y_2)^2} \\ &\leqslant 2 \frac{\|\varphi\|_{L^\infty}}{(x_1 + \alpha)^2 (x_2 + \gamma)} \int_{\mathbb{R}} \frac{d\tau_1}{(1 + \tau_1^2)^{3/2}} \int_{\mathbb{R}} \frac{d\tau_2}{(1 + \tau_2^2)} \\ &\leqslant \frac{4\pi \|\varphi\|_{L^\infty}}{(x_1 + \alpha)^2 (x_2 + \gamma)}. \end{aligned}$$

Consequently,

$$\int_0^\infty \sup_{\substack{y_1\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1G_{\alpha,\gamma,\varphi}(x_1+iy_1,z_2)|\,dx_1 \leqslant \frac{4\|\varphi\|_{L^\infty}}{\gamma}\int_0^\infty \frac{dx_1}{(x_1+\alpha)^2} = \frac{4\|\varphi\|_{L^\infty}}{\alpha\gamma}.$$

Similarly,

$$\begin{aligned}|D_1D_2G_{\alpha,\gamma,\varphi}(z_1,z_2)| &\leqslant 4\|\varphi\|_{L^\infty}\int_{\mathbb{R}}\frac{d\beta}{((x_1+\alpha)^2+(\beta-y_1)^2)^{3/2}}\int_{\mathbb{R}}\frac{d\delta}{((x_2+\gamma)^2+(\delta-y_2)^2)^{3/2}}\\ &\leqslant 4\frac{\|\varphi\|_{L^\infty}}{(x_1+\alpha)^2(x_2+\gamma)}\int_{\mathbb{R}}\frac{d\tau_1}{(1+\tau_1^2)^{3/2}}\int_{\mathbb{R}}\frac{d\tau_2}{(1+\tau_2^2)^{3/2}}\\ &\leqslant \frac{16\|\varphi\|_{L^\infty}}{(x_1+\alpha)^2(x_2+\gamma)^2}.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_0^\infty\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}}|D_1D_2G_{\alpha,\gamma,\varphi}(x_1+iy_1,x_1+iy_2)|\,dx_1\,dx_2 &\leqslant 16\|\varphi\|_{L^\infty}\int_0^\infty\int_0^\infty\frac{dx_1dx_2}{(x_1+\alpha)^2(x_2+\gamma)^2}\\ &= \frac{16}{\alpha\gamma}\|\varphi\|_{L^\infty}.\end{aligned}$$

Finally,

$$\|G_{\alpha,\gamma,\varphi}\|_\infty \leqslant \|\varphi\|_{L^\infty}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{d\beta}{\alpha^2+\beta^2}\frac{d\delta}{\gamma^2+\delta^2} = \frac{\pi^2}{\alpha\gamma}\|\varphi\|_{L^\infty}.$$

All in all,

$$\|G_{\alpha,\gamma,\varphi}\|_{\mathcal{B}^2_0} \leqslant \left(\frac{16}{\alpha\gamma}\right)\|\varphi\|_{L^\infty},$$

$$\|G_{\alpha,\gamma,\varphi}\|_{\mathcal{B}^2} \leqslant \left(\frac{24+\pi^2}{\alpha\gamma}\right)\|\varphi\|_{L^\infty}.$$

□

We turn to the operator $Q_{(2)}$ as defined in (5.2.2) above. We shall now establish that $Q_{(2)}$ is bounded, and that it maps the whole of $\mathcal{W}^2$ into $\mathcal{B}^2_0$.

Proposition 5.2.2. *The operator $Q_{(2)}$ belongs to $L(\mathcal{W}^2, \mathcal{B}_0^2)$.*

Proof. $Q_{(2)}$ is clearly linear. If $x_1, x_2 > 0$, then

$$|(Q_{(2)}g)(x_1+iy_1, x_2+iy_2)|$$
$$\leqslant \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \sup_{\beta,\delta\in\mathbb{R}} |g(\alpha+i\beta,\gamma+i\delta)| \int_{\mathbb{R}} \frac{d\beta}{(x_1+\alpha)^2+(y_1-\beta)^2} \int_{\mathbb{R}} \frac{d\delta}{(x_2+\gamma)^2+(y_2-\delta)^2}\, d\alpha\, d\gamma$$
$$= 4\int_0^\infty \frac{\alpha}{x_1+\alpha} \int_0^\infty \frac{\gamma}{x_1+\gamma} \sup_{\beta,\delta\in\mathbb{R}} |g(\alpha+i\beta,\gamma+i\delta)| d\alpha\, d\gamma \leqslant 4\|g\|_{\mathcal{W}^2}.$$

Moreover, $(Q_{(2)}g)(\infty, z_2) = (Q_{(2)}g)(z_1, \infty) = (Q_{(2)}g)(\infty, \infty) = 0$, by the monotone convergence theorem. We also have

$$D_1(Q_{(2)}g)(x_1+iy_1, x_2+iy_2) = \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma D_1 G_{\alpha,\gamma,\varphi_{\alpha,\gamma,g}}(x_1+iy_1, x_2+iy_2)\, d\alpha\, d\gamma,$$

and

$$\sup_{y_1\in\mathbb{R}} |D_1(Q_{(2)}g)(x_1+iy_1, x_2+iy_2)| \leqslant \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \sup_{y_1\in\mathbb{R}} |D_1 G_{\alpha,\gamma,\varphi_{\alpha,\gamma,g}}(x_1+iy_1, x_2+iy_2)|\, d\alpha\, d\gamma,$$

so that for fixed $z_2 \in \mathbb{C}_+$

$$\begin{aligned}\int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1(Q_{(2)}g)(x_1+iy_1, z_2)|\, dx_1 &\leqslant \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \int_0^\infty \sup_{y_1\in\mathbb{R}} |D_1 G_{\alpha,\gamma,\varphi_{\alpha,\gamma,g}}(x_1+iy_1, z_2)|\, dx_1\, d\alpha\, d\gamma \\ &\leqslant \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \frac{4}{\alpha\gamma} \sup_{\beta,\delta\in\mathbb{R}} |g(\alpha+i\beta,\gamma+i\delta)| d\alpha\, d\gamma \\ &= \frac{16}{\pi^2}\|g\|_{\mathcal{W}^2}.\end{aligned}$$

Similarly, for arbitrary $z_1 \in \mathbb{C}_+$,

$$\int_0^\infty \sup_{y_2\in\mathbb{R}} |D_2(Q_{(2)}g)(z_1, x_2+iy_2)|\, dx_2 \leqslant \frac{16}{\pi^2}\|g\|_{\mathcal{W}^2}.$$

Furthermore,

$$D_1D_2(Q_{(2)}g)(x_1+iy_1, x_2+iy_2) = \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma D_1D_2 G_{\alpha,\gamma,\varphi_{\alpha,\gamma,g}}(x_1+iy_1, x_2+iy_2)\, d\alpha\, d\gamma,$$

and hence using the estimate from Proposition 5.2.1 with $\varphi(\beta,\delta) = g(\alpha+i\beta,\gamma+i\delta)$ for fixed $\alpha > 0$ and $\gamma > 0$

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2(Q_{(2)}g)(x_1+iy_1,x_2+iy_2)|\,dx_1\,dx_2 \\
&\quad \leqslant \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \alpha\gamma \int_0^\infty\int_0^\infty \sup_{y_1,y_2\in\mathbb{R}} |D_1D_2G_{\alpha,\gamma,\varphi_{\alpha,\gamma,g}}(x_1+iy_1,x_2+iy_2)|\,dx_1\,dx_2\,d\alpha\,d\gamma \\
&\quad \leqslant \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \alpha\gamma\frac{16}{\alpha\gamma}\sup_{\beta,\delta\in\mathbb{R}}|g(\alpha+i\beta,\gamma+i\delta)|d\alpha\,d\gamma \\
&\quad = \frac{64}{\pi^2}\|g\|_{\mathcal{W}^2}.
\end{aligned}$$

□

Consider now the cut-off operators V_n, $n \geqslant 1$, on $\mathcal{W}^2$ defined by

$$(V_ng)(z_1,z_2) = \begin{cases} g(z_1,z_2) & |\operatorname{Im} z_1|, |\operatorname{Im} z_2| \leqslant n, \\ 0 & \text{otherwise.}\end{cases}$$

We also consider the operators $Q_{(2)}V_n : \mathcal{W}^2 \to \mathcal{B}_0^2$, given by

$$(Q_{(2)}V_ng)(z_1,z_2) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \alpha\gamma\int_{-n}^n\int_{-n}^n \frac{g(\alpha+i\beta,\gamma+i\delta)}{(z_1+\alpha-i\beta)^2(z_2+\gamma-i\delta)^2}\,d\beta\,d\delta\,d\alpha\,d\gamma.$$

Proposition 5.2.3. *For every $g \in \mathcal{W}^2$, $z_1, z_2 \in \mathbb{C}_+$, and each $n \geqslant 2\max(|\operatorname{Im} z_1|, |\operatorname{Im} z_2|)$,*

$$|(Q_{(2)}V_ng)(z_1,z_2) - (Q_{(2)}g)(z_1,z_2)| \leqslant \frac{64}{\pi^2}S_n(g), \tag{5.2.3}$$

where

$$S_n(g) = \int_0^\infty\int_0^\infty \left(\frac{\alpha}{\alpha+n/2} + \frac{\gamma}{\gamma+n/2}\right)\sup_{\beta,\delta\in\mathbb{R}}|g(\alpha+i\beta,\gamma+i\delta)|\,d\alpha\,d\gamma \to 0, \quad n\to\infty.$$

Proof. Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Then

$$
\begin{aligned}
&|(Q_{(2)}V_n g)(z_1, z_2) - (Q_{(2)}g)(z_1, z_2)| \\
&\quad = \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \int_{|\delta|\geqslant n}\int_{|\beta|\geqslant n} \iint\limits_{[n,\infty)\times[0,\infty)\cup[0,\infty)\times[n,\infty)} \frac{g(\alpha+i\beta, \gamma+i\delta)}{(z_1+\alpha-i\beta)^2(z_2+\gamma-i\delta)^2}\, d\beta\, d\delta\, d\alpha\, d\gamma \\
&\quad \leqslant \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \sup_{s_1,s_2\in\mathbb{R}} |g(\alpha+is_1, \gamma+is_2)| \iint\limits_{[n,\infty)\times[0,\infty)\cup[0,\infty)\times[n,\infty)} \frac{d\delta}{|z_2+\gamma-i\delta|^2}\frac{d\beta}{|z_1+\alpha-i\beta|^2}.
\end{aligned}
$$

If $n > 2\max(|y_1|, |y_2|)$, we have

$$
\begin{aligned}
&\iint\limits_{[n,\infty)\times[0,\infty)\cup[0,\infty)\times[n,\infty)} \frac{d\delta}{|z_2+\gamma-i\delta|^2}\frac{d\beta}{|z_1+\alpha-i\beta|^2} \\
&\quad \leqslant 4\int_0^\infty \frac{d\beta}{(x_1+\alpha)^2+(\beta-|y_1|)^2}\int_n^\infty \frac{d\delta}{(x_2+\gamma)^2+(\delta-|y_2|)^2} \\
&\qquad + 4\int_n^\infty \frac{d\beta}{(x_1+\alpha)^2+(\beta-|y_1|)^2}\int_0^\infty \frac{d\delta}{(x_2+\gamma)^2+(\delta-|y_2|)^2} \\
&\quad \leqslant 4\left(\int_0^\infty \frac{d\beta}{\alpha^2+\beta^2}\int_{n/2}^\infty \frac{d\delta}{\gamma^2+\delta^2} + \int_{n/2}^\infty \frac{d\beta}{\alpha^2+\beta^2}\int_0^\infty \frac{d\delta}{\gamma^2+\delta^2}\right) \\
&\quad \leqslant 16\left(\int_0^\infty \frac{d\beta}{(\alpha+\beta)^2}\int_{n/2}^\infty \frac{d\delta}{(\gamma+\delta)^2} + \int_{n/2}^\infty \frac{d\beta}{(\alpha+\beta)^2}\int_0^\infty \frac{d\delta}{(\gamma+\delta)^2}\right) \\
&\quad = 16\left(\frac{1}{\alpha(\gamma+n/2)} + \frac{1}{(\alpha+n/2)\gamma}\right).
\end{aligned}
$$

It follows that (5.2.3) holds. Since $g \in \mathcal{W}^2$, the monotone convergence theorem implies that

$$
\frac{64}{\pi^2}\int_0^\infty \int_0^\infty \left(\frac{\alpha}{\alpha+n/2} + \frac{\gamma}{\gamma+n/2}\right) \sup_{\beta,\delta\in\mathbb{R}} |g(\alpha+i\beta, \gamma+i\delta)|\, d\alpha\, d\gamma \to 0
$$

as $n \to \infty$. □

Consider the operator $K_m^{(2)}$ on $\mathcal{W}^2$ defined by

$$(K_m^{(2)}g)(z_1,z_2) = \begin{cases} g(z_1,z_2), & 1/m \leqslant |\operatorname{Re} z_1|, |\operatorname{Re} z_2| \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, consider the operator K_m^{Δ} on $\mathcal{B}^2$ given by

$$K_m^{\Delta} f = Q_{(2)}(K_m^{(2)} D_1 D_2 f) + Q_{(1)}(K_m^{(1)} f_1') + Q_{(1)}(K_m^{(1)} f_2'),$$

where the operator $K_m^{(1)}$ on $\mathcal{W}^1$ is defined as follows (cf. [5, Section 3])

$$(K_m^{(1)}g)(z) = \begin{cases} g(z), & 1/m \leqslant |\operatorname{Re} z| \leqslant m, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} &(K_m^{\Delta} f)(z_1,z_2) = \\ &+ \frac{4}{\pi^2} \int_{1/m}^{m} \int_{1/m}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\alpha\gamma D_1 D_2 f(\alpha + i\beta, \gamma + i\delta)}{(z_1 + \alpha - i\beta)^2 (z_2 + \gamma - i\delta)^2} \, d\beta \, d\delta \, d\alpha \, d\gamma \\ &- \frac{2}{\pi} \int_{1/m}^{m} \int_{\mathbb{R}} \frac{\alpha D_1 f(\alpha + i\beta, \infty)}{(z_1 + \alpha - i\beta)^2} \, d\beta \, d\alpha \\ &- \frac{2}{\pi} \int_{1/m}^{m} \int_{\mathbb{R}} \frac{\gamma D_2 f(\infty, \gamma + i\delta)}{(z_2 + \gamma - i\delta)^2} \, d\delta \, d\gamma. \end{aligned} \tag{5.2.4}$$

We are now ready to show that, for any $f \in \mathcal{B}_0^2$, the sequence $(K_m^{\Delta} f)_{m \in \mathbb{N}}$ approximates f in the $\|\cdot\|_{\mathcal{B}^2}$ norm.

Proposition 5.2.4. *For every $f \in \mathcal{B}_0^2$, and every $m \geqslant 2$, the following holds:*

$$(K_m^{\Delta} f)(z_1,z_2) = 16 \int_{1/m}^{m} \int_{1/m}^{m} \alpha\gamma D_1^2 D_2^2 f(2\alpha + z_1, 2\gamma + z_2) \, d\alpha \, d\gamma.$$

Moreover,

$$\|K_m^{\Delta} f\|_{\mathcal{B}_0^2} \leqslant C \|f\|_{\infty} \log^2 m, \tag{5.2.5}$$

where $C > 0$ is a constant which does not depend on the choice of f, and

$$\|f - K_m^{\Delta} f\|_{\mathcal{B}_0^2} \leqslant \frac{64}{\pi^2} R_m(f), \tag{5.2.6}$$

where

$$R_m(f) = \left\{ \int_0^{1/m} + \int_m^{\infty} \right\} \left\{ \int_0^{1/m} + \int_m^{\infty} \right\} \sup_{\beta,\delta \in \mathbb{R}} |D_1 D_2 f(\alpha + i\beta, \gamma + i\delta)| \, d\alpha \, d\gamma \to 0, \quad m \to \infty.$$

Proof. Let $f \in \mathcal{B}_0^2$, and let $z_1, z_2 \in \mathbb{C}_+$, with $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Applying Cauchy's theorem twice we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{D_1D_2f(\alpha+i\beta,\gamma+i\delta)}{(z_1+\alpha-i\beta)^2(z_2+\gamma-i\delta)^2}\,d\beta\,d\delta \\
&\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{D_1D_2f(\alpha+i\beta,\gamma+i\delta)}{(2\alpha+z_1-(\alpha+i\beta))^2(2\gamma+z_2-(\gamma+i\delta))^2}\,d\beta\,d\delta \\
&\qquad = 4\pi^2 D_1^2D_2^2 f(2\alpha+z_1, 2\gamma+z_2),
\end{aligned}$$

so multiplying by $\alpha\gamma$ and integrating,

$$(K_m^\Delta f)(z_1,z_2) = 16\int_{1/m}^m\int_{1/m}^m \alpha\gamma D_1^2D_2^2 f(2\alpha+z_1,2\gamma+z_2)\,d\alpha\,d\gamma.$$

Moreover, applying (3.1.4) we obtain

$$\begin{aligned}
|(D_1D_2K_m^\Delta f)(z_1,z_2)| &\leqslant \frac{4}{\pi^2}\int_{1/m}^m\int_{1/m}^m \alpha\gamma\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|D_1D_2f(\alpha+i\beta,\gamma+i\delta)|}{|z_1+\alpha-i\beta|^3|z_2+\gamma-i\delta|^3}\,d\beta\,d\delta\,d\alpha\,d\gamma \\
&\leqslant \frac{4}{\pi^2}\int_{1/m}^m\int_{1/m}^m\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{C_{1,1}\|f\|_\infty}{|z_1+\alpha-i\beta|^3|z_2+\gamma-i\delta|^3}\,d\beta\,d\delta\,d\alpha\,d\gamma \\
&\leqslant \frac{16\,C_{1,1}\|f\|_\infty}{\pi^2}\int_{1/m}^m\int_{1/m}^m \frac{d\alpha\,d\gamma}{(x_1+\alpha)^2(x_2+\gamma)^2},
\end{aligned}$$

so that

$$\begin{aligned}
\|K_m^\Delta f\|_{\mathcal{B}_0^2} &\leqslant \frac{16\,C_{1,1}\|f\|_\infty}{\pi^2}\int_0^\infty\int_0^\infty\int_{1/m}^m\int_{1/m}^m \frac{d\alpha\,d\gamma\,dx_1\,dx_2}{(x_1+\alpha)^2(x_2+\gamma)^2} \\
&\leqslant \frac{16\,C_{1,1}\|f\|_\infty}{\pi^2}\int_{1/m}^m\int_{1/m}^m \frac{d\alpha\,d\gamma}{\alpha\gamma} \\
&\leqslant C\|f\|_\infty \log^2 m,
\end{aligned}$$

with $C = 64\,C_{1,1}\pi^{-2}$. Finally, if $f \in \mathcal{B}_0^2$, we have by Proposition 5.2.3

$$\|f - K_m^\Delta f\|_{\mathcal{B}_0^2} = \|Q_{(2)}(D_1D_2f - K_m^{(2)}D_1D_2f)\|_{\mathcal{B}_0^2} = \|D_1D_2f - K_m^{(2)}\|_{\mathcal{W}^2} \leqslant \left(\frac{64}{\pi^2}\right)R_m(f).$$

□

5.3 Density of $\mathcal{LM}_{(2)}$ in other topologies

In Proposition 4.4.3, we estabished that $\mathcal{LM}_{(2)}$ is dense in $\mathcal{B}^2$ in the topology of uniform convergence of compact subsets. We now show density in a slightly stronger sense; our discussion parallels that of [5, Section 4.1].

Proposition 5.3.1. *Let $g \in \mathcal{W}^2$, and assume that g is supported by $([a,\infty) \times [-ib, ib])^2 \subset \mathbb{C}_+^2$, with $a, b > 0$. Then $Q_{(2)}g \in \mathcal{LM}_{(2)}$, and*

$$\|Q_{(2)}g\|_{\mathrm{HP}} \leqslant \frac{16b^2}{\pi^2 a^2}\|g\|_{\mathcal{W}^2}.$$

Proof. Since

$$\frac{1}{(z+\alpha-i\beta)^2} = \int_0^\infty te^{-(z+\alpha-i\beta)t}\,dt, \qquad z \in \mathbb{C}_+,$$

we have $Q_{(2)}g = \mathcal{L}_{(2)}h$, where

$$h(t_1,t_2) = \frac{4t_1t_2}{\pi^2}\int_a^\infty\int_a^\infty \alpha\gamma e^{-\alpha t_1-\gamma t_2}\int_{-b}^b\int_{-b}^b g(\alpha+i\beta,\gamma+i\delta)e^{i(\beta t_1+\delta t_2)}\,d\beta\,d\delta\,d\alpha d\gamma.$$

Thus $Q_{(2)}g \in \mathcal{LM}_{(2)}$ and

$$\|Q_{(2)}g\|_= \int_0^\infty\int_0^\infty |h(t_1,t_2)|\,dt_1\,dt_2$$

$$\leqslant \frac{4}{\pi^2}\int_0^\infty\int_0^\infty t_1t_2\int_a^\infty\int_a^\infty \alpha\gamma e^{-\alpha t_1-\gamma t_2}\int_{-b}^b\int_{-b}^b |g(\alpha+i\beta,\gamma+i\delta)|\,d\beta\;d\delta\,d\alpha\,d\gamma\,dt_1\,dt_2$$

$$\leqslant \frac{16b^2}{\pi^2}\int_0^\infty\int_0^\infty t_1t_2\int_a^\infty\int_a^\infty \alpha\gamma e^{-\alpha t_1-\gamma t_2}\sup_{s_1,s_2\in\mathbb{R}}|g(\alpha+is_1,\gamma+is_2)|\,d\alpha\;d\gamma\,dt_1\,dt_2$$

$$= \frac{16b^2}{\pi^2}\int_a^\infty\int_a^\infty \alpha^{-1}\gamma^{-1}\sup_{s_1,s_2\in\mathbb{R}}|g(\alpha+is_1,\gamma+is_2)|\,d\alpha\,d\gamma \leqslant \frac{16b^2}{\pi^2a^2}\|g\|_{\mathcal{W}^2}.$$

□

For $f \in \mathcal{B}^2$ and $n > 0$, define $Q_n^\Delta f = Q_{(2)}V_nK_n^{(2)}D_1D_2f$, so that

$$(Q_n^\Delta f)(z_1,z_2) = \frac{4}{\pi^2}\int_{1/n}^n\int_{1/n}^n \alpha\gamma\int_{-n}^n\int_{-n}^n \frac{D_1D_2f(\alpha+i\beta,\gamma+i\delta)}{(z_1+\alpha-i\beta)^2(z_2+\gamma-i\delta)^2}\,d\beta\,d\delta\,d\alpha\,d\gamma, \qquad z_1,z_2 \in \mathbb{C}_+.$$

Proposition 5.3.2. *For every $f \in \mathcal{B}_0^2$,*

$$Q_n^\Delta f \in \mathcal{LM}_{(2)}, \quad n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} |(Q_n^{\Delta} f)(z_1, z_2) - f(z_1, z_2)| = 0,$$

uniformly for all $z_1, z_2 \in \overline{\mathbb{C}}_+$ *with* $|\operatorname{Im} z_1| + |\operatorname{Im} z_2| \leqslant c$, *for any* $c > 0$.

Proof. Since $D_1 D_2 f \in \mathcal{W}^2$, it follows from Proposition 5.3.1 that $Q_{(2)} V_n K_n^{(2)} D_1 D_2 f \in \mathcal{LM}_{(2)}$. By Proposition 5.2.3,

$$\sup_{|\operatorname{Im} z_1|+|\operatorname{Im} z_2|\leqslant c} |(Q_{(2)} K_n^{(2)} D_1 D_2 f)(z_1, z_2) - (Q_n^{\Delta} f)(z_1, z_2)|$$

$$\leqslant \frac{64}{\pi^2} S_n(K_n^{(2)} D_1 D_2 f) \leqslant \frac{64}{\pi^2} S_n(D_1 D_2 f), \quad n > c.$$

Note that $Q_{(2)} K_n^{(2)} D_1 D_2 f - f \in \mathcal{B}_0^2$; hence combining Remark 3.4.3 and (5.2.6) we obtain

$$\|Q_{(2)} K_n^{(2)} D_1 D_2 f - f\|_\infty \lesssim \|Q_{(2)} K_n^{(2)} D_1 D_2 f - f\|_{\mathcal{B}_0^2} \leqslant \frac{64}{\pi^2} R_n(f).$$

Consequently,

$$\sup_{|\operatorname{Im} z_1|+|\operatorname{Im} z_2|\leqslant c} |f(z_1, z_2) - (Q_n^{\Delta} f)(z_1, z_2)| \lesssim \frac{64}{\pi^2} \left(S_n(D_1 D_2 f) + R_n(f)\right), \quad n > c.$$

□

5.4 Semigroups on $\mathcal{B}^2$ and $\mathcal{E}^2$

We have seen in Section 4.2 that the two-parameter semigroup of shifts on $\mathcal{B}^2$

$$(T_{\mathcal{B}^2}(a, b) f)(z_1, z_2) = f(z_1 + a, z_2 + b), \quad a, b \in \mathbb{R}_+.$$

plays a distinguished role in considerations on spectral subspaces and related matters. We will now show that shifting f within the pairing $\langle g, f\rangle_{\mathcal{B}^2}$ yields the same result as shifting g.

Proposition 5.4.1. *Let* $f \in \mathcal{B}^2$, $a, b \in \mathbb{R}_+$ *and* $T_{\mathcal{B}^2}(a, b)$ *be as above. Define* $T_{\mathcal{E}^2}(a, b)$ *on* $\mathcal{E}^2$ *by*

$$(T_{\mathcal{E}^2}(a, b) g)(z_1, z_2) = g(a + z_1, b + z_2), \quad g \in \mathcal{E}^2.$$

Then

$$\langle T_{\mathcal{E}^2}(a, b) g, f\rangle_{\mathcal{B}^2} = \langle g, T_{\mathcal{B}^2}(a, b) f\rangle_{\mathcal{B}^2}, \quad a, b \in \mathbb{R}_+.$$

Proof. Let $f \in \mathcal{B}^2$, $x_1, x_2 > 0$, and let

$$h(z_1, z_2) = D_1 D_2 g(a + x_1 - z_1, b + x_2 - z_2) D_1 D_2 f(x_1 + z_1, x_2 + z_2)$$

for $-x_1 < \operatorname{Re} z_1 < x_1 + \operatorname{Re} a$, and $-x_2 < \operatorname{Re} z_2 < x_2 + \operatorname{Re} b$. Since $f \in \mathcal{B}^2$ and $g \in \mathcal{E}^2$, we have that

$$\iint_{\mathbb{R}^2} |D_1 D_2 g(a + x_1 - iy_1, b + x_2 - iy_2) D_1 D_2 f(x_1 + iy_1, x_2 + iy_2)| \, dy_1 \, dy_2$$

$$\leqslant \iint_{\mathbb{R}^2} |D_1 D_2 g(a + x_1 - iy_1, b + x_2 - iy_2)| \sup_{s_1, s_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + is_1, x_2 + is_2)| \, dy_1 \, dy_2$$

$$\leqslant \sup_{s_1, s_2 \in \mathbb{R}} |D_1 D_2 f(x_1 + is_1, x_2 + is_2)| \cdot \|g\|_{\mathcal{E}^2},$$

with the product in the last line being bounded for almost any choice of x_1 and x_2. Moreover, since

$$\int_0^\infty \int_0^\infty |D_1 D_2 g(a + x_1 - t_1 - iy_1, b + x_2 - t_2 - iy_2)| \, dy_1 \, dy_2 \leqslant \frac{\|g\|_{\mathcal{E}^2}}{(a + x_1 - t_1)(b + x_2 - t_2)},$$

$$|D_1 D_2 f(x_1 + t_1 + iy_1, x_2 + t_2 + iy_2)| \leqslant \frac{\|f\|_\infty}{(x_1 + t_1)(x_2 + t_2)},$$

we obtain

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_0^b \int_0^a h(t_1 + iy_1, t_2 + iy_2) \, dt_1 \, dt_2 \right| dy_1 \, dy_2$$

$$\leqslant \int_0^b \int_0^a \frac{\|g\|_{\mathcal{E}^2} \|f\|_\infty}{(x_1 + t_1)(a + x_1 - t_1)(x_2 + t_2)(b + x_2 - t_2)} \, dt_1 \, dt_2 < \infty.$$

Consider now a double contour integral around a product of two rectangles with vertices at $\pm iy_n$ and $a \pm iy_n$, and $\pm iy_n$ and $b \pm iy_n$, respectively. Taking suitable $y_n \to \infty$ and noting that, by Cauchy's theorem, the contribution from the horizontal parts of the contour tend to 0, we conclude that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} D_1D_2g(a+x_1-iy_1, b+x_2-y_1)D_1D_2f(x_1+iy_1, x_2+iy_2)\,dy_1\,dy_2$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} D_1D_2g(a+x_1-iy_1, x_2-y_1)D_1D_2f(x_1+iy_1, b+x_2+iy_2)\,dy_1\,dy_2$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} D_1D_2g(x_1-iy_1, b+x_2-y_1)D_1D_2f(a+x_1+iy_1, x_2+iy_2)\,dy_1\,dy_2$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} D_1D_2g(x_1-iy_1, x_2-y_1)D_1D_2f(a+x_1+iy_1, b+x_2+iy_2)\,dy_1\,dy_2.$$

Consequently, this shows that

$$\langle T_{\mathcal{E}^2}(a,b)g, f\rangle_{\mathcal{B}^2} = \langle T_{\mathcal{E}^2}(a,0)g, T_{\mathcal{B}^2}(0,b)f\rangle_{\mathcal{B}^2}$$

$$= \langle T_{\mathcal{E}^2}(0,b)g, T_{\mathcal{B}^2}(a,0)f\rangle_{\mathcal{B}^2} = \langle g, T_{\mathcal{B}^2}(a,b)f\rangle_{\mathcal{B}^2},$$

as required. □

If $z_1, z_2 \in \mathbb{C}_+$, $a, b \in \mathbb{R}_+$, we have two formulas for $f(z_1+a_1, z_2+a_2)$, obtained by either shifting the resolvent function or the function f itself. On the one hand, by Proposition 5.1.1,

$$f(z_1+a_1, z_2+a_2) = f(\infty,\infty) + \frac{2}{\pi}\langle r_{z_1+a_1}, f_1\rangle_{\mathcal{B}^1} + \frac{2}{\pi}\langle r_{z_2+a_2}, f_2\rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\langle r_{z_1+a_1, z_2+a_2}, f\rangle_{\mathcal{B}^2}.$$

On the other hand, once more by Proposition 5.1.1,

$$f(z_1+a_1, z_2+a_2) = f(\infty,\infty) + \frac{2}{\pi}\langle r_{z_1}, T_{\mathcal{B}^1}(a_1)f_1\rangle_{\mathcal{B}^1} + \frac{2}{\pi}\langle r_{z_2}, T_{\mathcal{B}^1}(a_2)f_2\rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\langle r_{z_1,z_2}, T_{\mathcal{B}^2}(a_1,a_2)f\rangle_{\mathcal{B}^2}.$$

Proposition 5.4.1 then allows us to establish that this does not lead to an inconsistency, i.e. that these two representations agree with each other.

Corollary 5.4.2. *Let* $f \in \mathcal{B}^2$*. If* $z_1, z_2, a, b \in \mathbb{C}_+$*, then*

$$f(z_1+a, z_2+b) = f(\infty,\infty) + \frac{2}{\pi}\langle r_{z_1+a}, f_1\rangle_{\mathcal{B}^1} + \frac{2}{\pi}\langle r_{z_2+b}, f_2\rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\langle r_{(z_1+a, z_2+b)}, f\rangle_{\mathcal{B}^2}$$

$$= f(\infty,\infty) + \frac{2}{\pi}\langle r_{z_1}, T_{\mathcal{B}^2}(a,b)f_1\rangle_{\mathcal{B}^1} + \frac{2}{\pi}\langle r_{z_2}, T_{\mathcal{B}^2}(a,b)f_2\rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\langle r_{\mathbf{z}}, T_{\mathcal{B}^2}(a,b)f\rangle_{\mathcal{B}^2}.$$

Proof. This follows immediately from (5.1.1), Proposition 5.4.1 and [4, Remark 2.21]. □

5.5 A closer look at the duality between $\mathcal{B}^2$ and $\mathcal{E}^2$

The partial duality defined in Section 4.3 induces a bounded linear mapping

$$\mathcal{E}^2 \ni g \mapsto \langle g, \cdot \rangle_{\mathcal{B}^2} \in (\mathcal{B}^2)^*.$$

By restricting the functional $\langle g, \cdot \rangle_{\mathcal{B}^2}$ to the subspace $\mathcal{B}^2_0$, we obtain a bounded linear mapping $\Psi_{\mathcal{B}^2}$ from $\mathcal{E}^2$ to $(\mathcal{B}^2_0)^*$, given by

$$(\Psi_{\mathcal{B}^2} g)(f) = \langle g, f \rangle_{\mathcal{B}^2}, \quad g \in \mathcal{E}^2, f \in \mathcal{B}^2_0.$$

We shall see that the range of $\Psi_{\mathcal{B}^2}$ is not norm-dense in $(\mathcal{B}^2_0)^*$. Namely, we have

Proposition 5.5.1. *The range of $\Psi_{\mathcal{B}^2}$ is not dense in $(\mathcal{B}^2_0)^*$.*

The proof is similar to the one variable case ([4, Proposition 2.22]).

Proof. Suppose that the range is dense in $(\mathcal{B}^2_0)^*$. Take $f \in \mathcal{G}^2$ and let ${}_\delta f$ be as in Proposition 4.3.1. Then Proposition 4.3.1 shows that $\lim_{\delta \to 0+} \psi({}_\delta f) = \psi(f)$ for all ψ in the range of $\Psi_{\mathcal{B}^2}$. Note that the functions ${}_\delta f$ form a bounded subset of $\mathcal{G}^2$, so convergence for all functionals ψ in the range of $\Psi_{\mathcal{B}^2}$ implies convergence for all functionals in the closure of the range; which, by assumption, implies convergence for all functionals in $(\mathcal{B}^2_0)^*$.

By Proposition 4.4.1, ${}_\delta f \in \mathcal{LM}_{(2)}$. This would imply that $\mathcal{LM}_{(2)}$ is weakly dense in $\mathcal{G}^2$, and thus in $\mathcal{B}^2_0$. By Mazur's theorem, $\mathcal{LM}_{(2)}$ would be norm-dense in $\mathcal{B}^2_0$. But this would contradict Proposition 4.4.2. Thus, the range of $\Psi_{\mathcal{B}^2}$ is not dense in $(\mathcal{B}^2_0)^*$. □

Let $\mathcal{E}^2_0 = \{g \in \mathcal{E} : g(z_1, \infty) = g(\infty, z_2) = 0\}$, with the norm $\| \cdot \|_{\mathcal{E}^2_0}$ which coincides with $\| \cdot \|_{\mathcal{E}^2}$ on $\mathcal{E}^2_0$. Identify the dual $(\mathcal{E}^2_0)^*$ with the space of linear functionals in $(\mathcal{E}^2)^*$ which annihilate the functions of one variable. Our duality then provides a contractive map $\Psi_{\mathcal{E}^2}$ from $\mathcal{B}^2$ to $(\mathcal{E}^2_0)^*$. The following is a two-dimensional analogue of [4, Proposition 2.23].

Proposition 5.5.2. *The range of $\Psi_{\mathcal{E}^2}$ is not norm dense in $(\mathcal{E}^2_0)^*$.*

Proof. Let $(T_{\mathcal{B}^2}(t_1, t_2))_{t_1, t_2 \geqslant 0}$ and $(T_{\mathcal{E}^2}(t_1, t_2))_{t_1, t_2 \geqslant 0}$ be as before. Suppose that the range of $\Psi_{\mathcal{E}^2}$ is norm-dense in $(\mathcal{E}^2_0)^*$. We obtain

$$\langle T_{\mathcal{E}^2}(t, 0) g, f \rangle_{\mathcal{B}^2} = \langle g, T_{\mathcal{B}^2}(t, 0) f \rangle_{\mathcal{B}^2} \to \langle g, f \rangle_{\mathcal{B}^2},$$

as $t \to 0+$, by Propositions 4.2.2 and 5.4.1. It then follows that

$$\lim_{t\to 0+} \psi(T_{\mathcal{E}^2} g) = \psi(g),$$

for all $\psi \in (\mathcal{E}_0^2)^*$ and $g \in \mathcal{E}_0$. By [15, Proposition 1.23], this implies that

$$\lim_{t\to 0+} \|T_{\mathcal{E}^2}(t,0)g - g\|_{\mathcal{E}^2} = 0, \tag{5.5.1}$$

for all $g \in \mathcal{E}_0$. Consider now

$$g(z_1, z_2) = \frac{1}{z_1 z_2}.$$

We have that

$$\begin{aligned}
\|g\|_{\mathcal{E}_0^2} &= \sup_{x_1,x_2>0} x_1 x_2 \int_0^\infty \int_0^\infty |D_1 D_2 g(x_1+iy_1, x_2+iy_2)|\, dy_1\, dy_2 \\
&= \sup_{x_1>0} x_1 \int_0^\infty \left|\frac{1}{(x_1+iy_1)^2}\right| dy_1 \cdot \sup_{x_2>0} x_2 \int_0^\infty \left|\frac{1}{(x_2+iy_2)^2}\right| dy_2 < \infty,
\end{aligned}$$

since $\frac{1}{z} \in \mathcal{E}^1$ by [4, Examples 2.17(1)], so that $g \in \mathcal{E}_0^2$. Estimating directly

$$\begin{aligned}
&\|T_{\mathcal{E}^2}(a,0)g - g\|_{\mathcal{E}_0^2} \\
&= \sup_{x_1,x_2>0} x_1 x_2 \int_0^\infty \int_0^\infty \left|\frac{1}{(x_1+iy_1)^2 (x_2+iy_2)^2} - \frac{1}{(x_2+iy_2)^2 (a+x_1+iy_1)^2}\right| dy_1\, dy_2 \\
&= \sup_{x_2>0} x_2 \int_0^\infty \left|\frac{1}{(x_2+iy_2)^2}\right| dy_2 \cdot \sup_{x_1>0} x_1 \int_0^\infty \left|\frac{1}{(x_1+iy_1)^2} - \frac{1}{(a+x_1+iy_1)^2}\right| dy_1 \\
&\geqslant \|r_0\|_{\mathcal{E}_0^1} \cdot \sup_{s>0} sJ(s;1) > 0,
\end{aligned}$$

where (as in the proof of [4, Lemma 2.18(3)])

$$J(x_1; a) = \int_0^\infty \left|\frac{1}{(x_1+iy_1)^2} - \frac{1}{(a+x_1+iy_1)^2}\right| dy_1,$$

so that $x_1 J(x_1; a) = sJ(s;1)$, with $s = x_1/a$. Hence, $(T_{\mathcal{E}^2}(t,0))_{t\geqslant 0}$ is not strongly continuous, which contradicts (5.5.1). □

5.6 Convergence lemma

In [5, Lemma 8.1], the authors obtain a convergence lemma for sequences of functions f_n in the Besov class $\mathcal{B}^1$. The result applies only to sequences of functions satisfying the following auxiliary condition: for any $r > 0$,

$$\lim_{\delta\to +0} \int_0^\delta \sup_{|\beta|<r} |f_n'(\alpha + i\beta)|\, d\alpha = 0 \tag{5.6.1}$$

uniformly in n. The condition (5.6.1), while perhaps somewhat artificial, appears to be crucial for the proof, and it is valid for examples (cf. [4, Examples 4.16], [5, Theorem 8.4]). The following result is intended to be viewed as a natural analogue of [5, Lemma 8.1]. Our assumptions (5.6.2) and (5.6.3) play the roles of [5, (8.5)] and [5, (8.6)], respectively.

Theorem 5.6.1. *Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{B}^2$ be such that $\sup_{n\geqslant 1} \|f_n\|_{\mathcal{B}^2} < \infty$. Assume that for all $z_1, z_2 \in \mathbb{C}_+$ there exists*

$$f_0(z_1, z_2) = \lim_{n\to\infty} f_n(z_1, z_2) \in \mathbb{C}, \tag{5.6.2}$$

and for every $r > 0$ one has

$$\lim_{\delta\to 0+} \iint\limits_{\substack{0<\alpha<\delta \\ or \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 D_2 f_n(\alpha + i\beta, \gamma + is)|\, d\alpha\, d\gamma = 0, \tag{5.6.3}$$

$$\lim_{\delta\to 0+} \iint\limits_{\substack{0<\alpha<\delta \\ or \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_n(\alpha + i\beta, \gamma + is)|\, d\alpha\, d\gamma = 0, \tag{5.6.4}$$

$$\lim_{\delta\to 0+} \iint\limits_{\substack{0<\alpha<\delta \\ or \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_2 f_n(\alpha + i\beta, \gamma + is)|\, d\alpha\, d\gamma = 0, \tag{5.6.5}$$

uniformly in n. Suppose further that $g \in \mathcal{B}^2_0$, and that for $i = 1, 2$,

$$D_i g \text{ is bounded }, \quad D_i g(z_1, z_2) \to 0, \quad g(z_1, z_2) \to 0 \quad \text{as } |z_1| + |z_2| \to \infty, \tag{5.6.6}$$

and that

$$G_{12}(\alpha) = \int_0^\infty \sup_{\beta, s\in\mathbb{R}} |D_2 g(\alpha + i\beta, \gamma + is)|\, d\gamma \to 0 \text{ as } \alpha \to \infty,$$

$$G_{21}(\alpha) = \int_0^\infty \sup_{\beta, s\in\mathbb{R}} |D_1 g(\alpha + i\beta, \gamma + is)|\, d\alpha \to 0 \text{ as } \gamma \to \infty.$$

Let $g_n(z_1, z_2) = g(z_1, z_2) f_n(z_1, z_2)$, $n \geqslant 0$. Then $f_0 \in \mathcal{B}^2$ and

$$\lim_{n\to\infty} \|g_n - g_0\|_{\mathcal{B}^2} = 0.$$

Proof. Note that $g \in \mathcal{B}^2_0$ with $\lim_{|z_1|+|z_2|\to\infty} |g(z_1, z_2)| = 0$, and for all $\lambda_1, \lambda_2 \in \mathbb{C}_+$ we have $\lim_{|z_1|\to\infty} |g(z_1, \lambda_2)| = 0$ and $\lim_{|z_2|\to\infty} |g(\lambda_1, z_2)| = 0$. Since bounded subsets of $\mathcal{B}^2$ are relatively compact in the topology of uniform convergence on compact sets (see Proposition 3.4.5), there is a subsequence of $(f_n)_{n\in\mathbb{N}}$ which converges to a function in $\mathcal{B}^2$ uniformly on compact subsets, and since that function must be f_0, $f_0 \in \mathcal{B}^2$. Assuming without loss of generality that $f_0 = 0$, we need to show that $\lim_{n\to\infty} \|g_n\|_{\mathcal{B}^2} = 0$. Consider

$$\int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha \\ \operatorname{Re} z_2=\gamma}} |D_1 D_2 g_n(z_1, z_2)| \, d\alpha \, d\gamma$$

$$\leqslant \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha \\ \operatorname{Re} z_2=\gamma}} |D_1 D_2 f_n(z_1, z_2) g(z_1, z_2)| \, d\alpha \, d\gamma$$

$$+ \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha \\ \operatorname{Re} z_2=\gamma}} |D_1 f_n(z_1, z_2) D_2 g(z_1, z_2)| \, d\alpha \, d\gamma$$

$$+ \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha \\ \operatorname{Re} z_2=\gamma}} |D_2 f_n(z_1, z_2) D_1 g(z_1, z_2)| \, d\alpha \, d\gamma$$

$$+ \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha \\ \operatorname{Re} z_2=\gamma}} |f_n(z_1, z_2) D_1 D_2 g(z_1, z_2)| \, d\alpha \, d\gamma$$

$$= I_n + J_n + K_n + L_n.$$

We shall estimate each of the four terms separately. Let $0 < \delta < r$. Beginning with I_n, we have

$$I_n = \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha \\ \operatorname{Re} z_2=\gamma}} |D_1 D_2 f_n(z_1, z_2) g(z_1, z_2)| \, d\alpha \, d\gamma$$

$$\leqslant \|g\|_\infty \sup_{R \in \mathbb{N}} \iint\limits_{\substack{0<\alpha<\delta \\ \text{or} \\ 0<\gamma<\delta}} \sup_{\substack{|\beta| \leqslant r \\ |s| \leqslant r}} |D_1 D_2 f_R(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma$$

$$+ \|g\|_\infty \iint\limits_{\substack{\delta \leqslant \alpha \leqslant r \\ \delta \leqslant \gamma \leqslant r}} \sup_{\substack{|\beta| \leqslant r \\ |s| \leqslant r}} |D_1 D_2 f_n(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma$$

$$+ \sup_{|z_1|+|z_2| \geqslant r} |g(z_1, z_2)| \iint\limits_{\substack{0 \leqslant \alpha \leqslant r \\ 0 \leqslant \gamma \leqslant r}} \sup_{\substack{|\beta| \geqslant r \\ \text{or} \\ |s| \geqslant r}} |D_1 D_2 f_n(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma$$

$$+ \sup_{|z_1|+|z_2|\geqslant r} |g(z_1,z_2)| \iint\limits_{\substack{\alpha\geqslant r\\ \text{or}\\ \gamma\geqslant r}} \sup_{\beta,s\in\mathbb{R}} |D_1D_2f_n(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$=: I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.$$

Note that

$$I_{n,3} + I_{n,4} \leqslant \sup_{|z_1|+|z_2|\geqslant r} |g(z_1,z_2)| \|f_n\|_{\mathcal{B}_0^2} \leqslant C \sup_{|z_1|+|z_2|\geqslant r} |g(z_1,z_2)|,$$

and that $I_{n,2} \to 0$ as $n \to \infty$, by Vitali's theorem [28, Corollary 1.2.6] as applied to D_1D_2f. Hence,

$$\limsup_{n\to\infty} I_n \leqslant C \sup_{|z_1|+|z_2|\geqslant r} |g(z_1,z_2)|$$

$$+ \|g\|_\infty \sup_{R\in\mathbb{N}} \iint\limits_{\substack{0<\alpha<\delta\\ \text{or}\\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r\\ |s|\leqslant r}} |D_1D_2f_R(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma.$$

Letting $\delta \to 0+$, we obtain from the assumption (5.6.3)

$$\limsup_{n\to\infty} I_n \leqslant C \sup_{|z_1|+|z_2|\geqslant r} |g(z_1,z_2)|.$$

By (5.6.6), letting $r \to \infty$ yields $\lim_{n\to\infty} I_n = 0$. Consider now

$$J_n = \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1=\alpha\\ \operatorname{Re} z_2=\gamma}} |D_1f_n(z_1,z_2)D_2g(z_1,z_2)|\,d\alpha\,d\gamma$$

$$= \iint\limits_{\substack{0<\alpha<\delta\\ \text{or}\\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r\\ |s|\leqslant r}} |D_1f_n(\alpha+i\beta,\gamma+is)D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$+ \iint\limits_{\substack{\delta<\alpha<r\\ \text{and}\\ \delta<\gamma<r}} \sup_{\substack{|\beta|\leqslant r\\ |s|\leqslant r}} |D_1f_n(\alpha+i\beta,\gamma+is)D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$+ \iint\limits_{\substack{\alpha>r\\ \gamma\in\mathbb{R}_+}} \sup_{\beta,s\in\mathbb{R}} |D_1f_n(\alpha+i\beta,\gamma+is)D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$+ \iint\limits_{\substack{\alpha\in\mathbb{R}_+ \\ \gamma>r}} \sup_{\beta,s\in\mathbb{R}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$+ \iint\limits_{\substack{\alpha<r \\ \text{and} \\ \gamma<r}} \sup_{\substack{\beta\in\mathbb{R} \\ |s|\geqslant r}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$+ \iint\limits_{\substack{\alpha<r \\ \text{and} \\ \gamma<r}} \sup_{\substack{|\beta|\geqslant r \\ s\in\mathbb{R}}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$=: J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4} + J_{n,5} + J_{n,6}.$$

We have

$$J_{n,1} = \iint\limits_{\substack{0<\alpha<\delta \\ \text{or} \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$\leqslant \|D_2 g\|_\infty \iint\limits_{\substack{0<\alpha<\delta \\ \text{or} \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_n(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$\leqslant \|D_2 g\|_\infty \sup_{R\in\mathbb{N}} \iint\limits_{\substack{0<\alpha<\delta \\ \text{or} \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_R(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma =: \tilde{J}_1.$$

As for $J_{n,2}$,

$$J_{n,2} = \iint\limits_{\substack{\delta<\alpha<r \\ \text{and} \\ \delta<\gamma<r}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$\leqslant \|D_2 g\|_\infty \iint\limits_{\substack{\delta<\alpha<r \\ \text{and} \\ \delta<\gamma<r}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_n(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma \to 0 \quad \text{as } n\to\infty,$$

Since $\|D_2 g\|_\infty < \infty$ by (5.6.6), the last line follows by Vitali's theorem
By (5.6.1)

$$J_{n,3} \leqslant G_{12}(r)\|f_n\|_{\mathcal{B}^2} \leqslant \tilde{J}_3 = C\cdot G_{12}(r) \to 0, \text{ as } r\to\infty.$$

Similarly,

$$J_{n,4} = \iint\limits_{\substack{\alpha\in\mathbb{R}_+\\ \gamma>r}} \sup_{\beta,s\in\mathbb{R}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$\leqslant \int\limits_{\alpha\in\mathbb{R}} \sup_{\substack{\beta\in\mathbb{R}_+\\ z_2\in\mathbb{C}_+}} |D_1 f_n(\alpha+i\beta,z_2)| G_{12,r}(\alpha)\,d\alpha,$$

where

$$G_{12,r}(\alpha) = \int\limits_{\gamma>r} \sup_{\beta,s\in\mathbb{R}_+} |D_2 g(\alpha+i\beta,\gamma+is)|\,d\gamma.$$

Consequently,

$$J_{n,4} \leqslant C \int\limits_{\gamma>r} \sup_{\beta,s\in\mathbb{R}_+} |D_2 g(\alpha+i\beta,\gamma+is)|\,d\gamma \leqslant C \int\limits_{\gamma>r} \sup_{\substack{z_1\in\mathbb{C}_+\\ s\in\mathbb{R}_+}} |D_2 g(z_1,\gamma+is)|\,d\gamma =: \tilde{J}_4.$$

Let us now turn to $J_{n,5}$.

$$J_{n,5} = \iint\limits_{\substack{\alpha<r\\ \text{and}\\ \gamma<r}} \sup_{\substack{\beta\in\mathbb{R}\\ |s|\geqslant r}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$\leqslant \int\limits_{\alpha<r} \sup_{\substack{\beta\in\mathbb{R}\\ z_2\in\mathbb{C}_+}} |D_1 f_n(\alpha+i\beta,z_2)| \int\limits_{\gamma<r} \sup_{\substack{z_1\in\mathbb{C}_+\\ |s|\geqslant r}} |D_2 g(z_1,\gamma+is)|\,d\alpha\,d\gamma.$$

Since $g \in \mathcal{B}^2$ and by (5.6.6)

$$\int\limits_{\gamma\in\mathbb{R}_+} \sup_{\substack{z_1\in\mathbb{C}_+\\ s\in\mathbb{R}}} |D_2 g(z_1,\gamma+is)|\,d\gamma < \infty, \quad \text{and} \quad \sup_{\substack{z_1\in\mathbb{C}_+\\ |s|\geqslant r}} |D_2 g(z_1,\gamma+is)| \searrow 0, \text{ as } r\to\infty.$$

Thus

$$J_{n,5} \leqslant \tilde{J}_5 = C\cdot \int\limits_{\gamma<r} \sup_{\substack{z_1\in\mathbb{C}_+\\ |s|\geqslant r}} |D_2 g(z_1,\gamma+is)|\,d\gamma \to 0 \text{ as } r\to\infty.$$

Finally, as for $J_{n,6}$,

$$J_{n,6} = \iint\limits_{\substack{\alpha<r \\ \text{and} \\ \gamma<r}} \sup_{\substack{|\beta|\geqslant r \\ s\in\mathbb{R}}} |D_1 f_n(\alpha+i\beta,\gamma+is) D_2 g(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma$$

$$\leqslant \int\limits_{\alpha<r} \sup_{\substack{|\beta|\geqslant r \\ z_2\in\mathbb{C}_+}} |D_1 f_n(\alpha+i\beta,z_2)| \int\limits_{\gamma<r} \sup_{\substack{s\in\mathbb{R} \\ \alpha'\in\mathbb{R}_+ \\ |\beta|>r}} |D_2 g(\alpha'+i\beta,\gamma+is)|\, d\gamma\, d\alpha.$$

$$\leqslant C\cdot \int_0^\infty \sup_{\substack{s\in\mathbb{R} \\ \alpha'\in\mathbb{R}_+ \\ |\beta|>r}} |D_2 g(\alpha'+i\beta,\gamma+is)|\, d\gamma =: \tilde{J}_6.$$

The last integral is finite for $r=0$ since $g\in\mathcal{B}^2$, decreasing in r and tending to 0 so $J_{n,6}\to 0$ as $r\to\infty$. Now

$$J_n \leqslant \tilde{J}_1 + J_{n,2} + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5 + \tilde{J}_6.$$

Taking first $n\to\infty$, we have $J_{n,2}\to 0$, hence

$$\limsup_{n\to\infty} J_n \leqslant \tilde{J}_1 + \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5 + \tilde{J}_6.$$

The bounds on all terms on the right-hand side depend on r; in addition, the bound on $\tilde{J}_1$ depends also on δ, i.e.

$$\tilde{J}_1 \leqslant \|D_2 g\|_\infty \sup_{R\in\mathbb{N}} \iint\limits_{\substack{0<\alpha<\delta \\ \text{or} \\ 0<\gamma<\delta}} \sup_{\substack{|\beta|\leqslant r \\ |s|\leqslant r}} |D_1 f_R(\alpha+i\beta,\gamma+is)|\, d\alpha\, d\gamma.$$

Letting $\delta\to 0$, we obtain by (5.6.5)

$$\limsup_{n\to\infty} J_n \leqslant \tilde{J}_3 + \tilde{J}_4 + \tilde{J}_5 + \tilde{J}_6.$$

Finally, taking $r\to\infty$, we have

$$\limsup_{n\to\infty} J_n = 0.$$

The exact same reasoning, with the roles of z_1 and z_2 swapped, and using (5.6.6) and (5.6.1), leads to the conclusion that $K_n\to 0$ as $n\to\infty$. As for L_n, let $\delta\in(0,1)$, $r>0$, and take $C_\infty = \sup_{n\in\mathbb{N}} \|f_n\|_\infty$. Let

$$S_\delta = (\delta, 1/\delta)\times(\delta, 1/\delta).$$

Then

$$
\begin{aligned}
L_n &= \int_0^\infty \int_0^\infty \sup_{\substack{\operatorname{Re} z_1 = \alpha \\ \operatorname{Re} z_2 = \gamma}} |f_n(z_1, z_2) D_1 D_2 g(z_1, z_2)| \, d\alpha \, d\gamma \\
&\leqslant \iint\limits_{\mathbb{R}^2_+ \setminus S_\delta} \sup_{\beta, s \in \mathbb{R}} |f_n(\alpha + i\beta, \gamma + is) D_1 D_2 g(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma \\
&+ \iint\limits_{S_\delta} \sup_{\substack{|\beta| \leqslant r \\ |s| \leqslant r}} |f_n(\alpha + i\beta, \gamma + is) D_1 D_2 g(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma \\
&+ \iint\limits_{S_\delta} \sup_{\substack{|\beta| \geqslant r \\ \text{or} \\ |s| \geqslant r}} |f_n(\alpha + i\beta, \gamma + is) D_1 D_2 g(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma \\
&=: L_{n,1} + L_{n,2} + L_{n,3}.
\end{aligned}
$$

Looking at $L_{n,1}$ first,

$$
\begin{aligned}
\limsup_{n\to\infty} L_{n,1} &= \limsup_{n\to\infty} \iint\limits_{\mathbb{R}^2_+ \setminus S_\delta} \sup_{\beta, s \in \mathbb{R}} |f_n(\alpha + i\beta, \gamma + is) D_1 D_2 g(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma \\
&\leqslant C_\infty \iint\limits_{\mathbb{R}^2_+ \setminus S_\delta} \sup_{\beta, s \in \mathbb{R}} |D_1 D_2 g(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma,
\end{aligned}
$$

which tends to 0 as $\delta \to \infty$, since $g \in \mathcal{B}^2$, so that

$$
\lim_{n\to\infty} L_{n,1} = 0.
$$

Now,

$$
\begin{aligned}
L_{n,2} &= \iint\limits_{S_\delta} \sup_{\substack{|\beta| \leqslant r \\ |s| \leqslant r}} |f_n(\alpha + i\beta, \gamma + is) D_1 D_2 g(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma \\
&\leqslant \sup_{\substack{\alpha, \gamma > \delta \\ \beta, s \in \mathbb{R}}} |D_1 D_2 g(\alpha + i\beta, \gamma + is)| \iint\limits_{S_\delta} \sup_{\substack{|\beta| \leqslant r \\ |s| \leqslant r}} |f_n(\alpha + i\beta, \gamma + is)| \, d\alpha \, d\gamma \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

where the last line follows from Vitali's theorem [28, Corollary 1.2.6] applied to f_n.

Finally,

$$L_{n,3} = \iint\limits_{S_\delta} \sup_{\substack{|\beta|\geqslant r \\ \text{or} \\ |s|\geqslant r}} |f_n(\alpha+i\beta,\gamma+is)D_1D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$\leqslant C_\infty \iint\limits_{S_\delta} \sup_{\substack{|\beta|\geqslant r \\ \text{or} \\ |s|\geqslant r}} |D_1D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma \to 0 \qquad \text{as } r\to\infty,$$

Hence

$$\limsup_{n\to\infty} L_n \leqslant C_\infty \iint\limits_{\mathbb{R}^2_+\setminus S_\delta} \sup_{\beta,s\in\mathbb{R}} |D_1D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma$$

$$+ C_\infty \iint\limits_{S_\delta} \sup_{\substack{|\beta|\geqslant r \\ \text{or} \\ |s|\geqslant r}} |D_1D_2g(\alpha+i\beta,\gamma+is)\,d\alpha\,d\gamma.$$

Letting $r\to\infty$,

$$\limsup_{n\to\infty} L_n \leqslant C_\infty \iint\limits_{\mathbb{R}^2_+\setminus S_\delta} \sup_{\beta,s\in\mathbb{R}} |D_1D_2g(\alpha+i\beta,\gamma+is)|\,d\alpha\,d\gamma.$$

And letting $\delta\to 0$, we see that

$$\lim_{n\to\infty} L_n = 0.$$

We have shown that $\|g_n\|_{\mathcal{B}^2_0} \to 0$ as $n\to\infty$. Since $(g_n)_{n\in\mathbb{N}} \subset \mathcal{B}^2_0$, and by Remark 3.4.3 the norm $\|\cdot\|_{\mathcal{B}^2}$ is equivalent to $\|\cdot\|_{\mathcal{B}^2_0}$ on $\mathcal{B}^2_0$, we are done. □

In Theorem 6.5.1 we will obtain a convergence result for operators applying the above result with $g(z_1,z_2) = (1+z_1)^{-1}(1+z_2)^{-1}$.

6 The $\mathcal{B}^2$-calculus

6.1 Definitions and setup

Let A_1, A_2 be closed commuting operators on a Banach space X, with $D(A_1) \cap D(A_2)$ dense in X. We shall assume throughout this section that $\sigma(A_i) \subset \overline{\mathbb{C}}_+$, $i = 1, 2$, that both operators satisfy

$$\sup_{\alpha>0} \alpha \int_{\mathbb{R}} |\langle(\alpha - i\beta + A_i)^{-2}x, x^*\rangle| \, d\beta < \infty, \quad i = 1, 2, \tag{6.1.1}$$

for all $x \in X$ and $x^* \in X^*$ and that the following holds

$$\sup_{\alpha,\gamma>0} \alpha\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle(\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}x, x^*\rangle| \, d\beta \, d\delta < \infty, \tag{6.1.2}$$

for all $x \in X$ and $x^* \in X^*$. By the Closed Graph Theorem, there are constants c_1 and c_2 such that, for all $x \in X$ and $x^* \in X^*$

$$\frac{2}{\pi} \sup_{\alpha>0} \alpha \int_{\mathbb{R}} |\langle(\alpha - i\beta + A_i)^{-2}x, x^*\rangle| \, d\beta \leqslant c_i \|x\| \|x^*\|, \quad i = 1, 2. \tag{6.1.3}$$

Similarly, it follows from (6.1.2) that there exists a constant $c_{1,2}$ such that, for all $x \in X$ and $x^* \in X^*$

$$\frac{4}{\pi^2} \sup_{\alpha,\gamma>0} \alpha\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle(\alpha - i\beta + A_1)^{-2}(\gamma - i\delta A_2)^{-2}x, x^*\rangle| \, d\beta \, d\delta \leqslant c_{1,2} \|x\| \|x^*\|. \tag{6.1.4}$$

Note that (6.1.1) says that, for all $x \in X$ and $x^* \in X^*$

$$g^{(i)}_{x,x^*} : z \mapsto \langle(z_i + A_i)^{-1}x, x^*\rangle, \quad i = 1, 2, \tag{6.1.5}$$

belongs to the class $\mathcal{E}^1$, as defined in [4, Section 2.5]. Correspondingly, (6.1.2) states that, for all $x \in X$ and $x^* \in X^*$

$$g^{(1,2)}_{x,x^*} : (z_1, z_2) \mapsto \langle(z_1 + A_1)^{-1}(z_2 + A_2)^{-1}x, x^*\rangle \tag{6.1.6}$$

belongs to the class $\mathcal{E}^2$ in the sense of Chapter 2.

We let γ_{A_i} be the smallest value of c_i such that (6.1.3) holds, i.e.

$$\gamma_{A_i} = \frac{2}{\pi} \sup\{\|g^{(i)}_{x,x^*}\|_{\mathcal{E}^1_0} : \|x\| = \|x^*\| = 1\}.$$

Analogously, we define

$$\gamma_{A_1,A_2} = \frac{4}{\pi^2} \sup\{\|g^{(1,2)}_{x,x^*}\|_{\mathcal{E}^2_0} : \|x\| = \|x^*\| = 1\}.$$

Example 6.1.1. Suppose that $-A_1$ and $-A_2$ generate commuting bounded C_0-semigroups on a Hilbert space X. Then it follows from [4, Examples 4.1] that both operators automatically satisfy (6.1.1). By a similar argument we see that they have to satisfy (6.1.2) as well. First, by applying Plancherel's theorem in the Hilbert space $L^2(\mathbb{R}^2_+, X)$ with $\alpha, \gamma > 0$ we obtain for any $x \in X$

$$\int_0^\infty \int_0^\infty \|(\alpha - i\beta + A_1)^{-1}(\gamma - i\delta + A_2)^{-1}x\|^2 d\beta\, d\delta$$

$$= 4\pi^2 \int_0^\infty \int_0^\infty e^{-2\alpha t_1} e^{-2\gamma t_2} \|T_1(t_1)T_2(t_2)x\|^2\, dt_1\, dt_2 \leqslant \frac{\pi^2 K^2_{A_1} K^2_{A_2}}{\alpha\gamma} \|x\|^2,$$

where $K_{A_i} = \sup_{t \geqslant 0} \|T_i(t)\|$, $i = 1, 2$. Similarly, for $y \in X$ we have

$$\int_0^\infty \int_0^\infty \|(\alpha - i\beta + A_1^*)^{-1}(\gamma - i\delta + A_2^*)^{-1}y\|^2 d\beta\, d\delta$$

$$= 4\pi^2 \int_0^\infty \int_0^\infty e^{-2\alpha t_1} e^{-2\gamma t_2} \|T_1^*(t_1)T_2^*(t_2)y\|^2\, dt_1\, dt_2 \leqslant \frac{\pi^2 K^2_{A_1} K^2_{A_2}}{\alpha\gamma} \|y\|^2.$$

The Cauchy-Schwarz inequality then yields

$$\begin{aligned}
&\alpha\gamma \int_0^\infty \int_0^\infty |\langle(\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}x, x^*\rangle|\, d\beta\, d\delta \\
&= \alpha\gamma \int_0^\infty \int_0^\infty |\langle(\alpha - i\beta + A_1)^{-1}(\gamma - i\delta + A_2)^{-1}x, (\alpha + i\beta + A_1^*)^{-1}(\gamma + i\delta + A_2^*)^{-1}x^*\rangle|\, d\beta\, d\delta \\
&\leqslant \pi^2 K^2_{A_1} K^2_{A_2} \|x\|\|x^*\|,
\end{aligned} \tag{6.1.7}$$

where A_i^* is the Hilbert space adjoint of A_i, with $i = 1, 2$. Since the right-hand side of (6.1.7) does not depend on the choice α and γ, the conclusion follows, that is

$$\gamma_{A_i} \leqslant 2K^2_{A_i}, \quad \gamma_{A_1,A_2} \leqslant 4K^2_{A_1} K^2_{A_2}.$$

Example 6.1.2. Suppose now that both $-A_1$ and $-A_2$ generate sectorially bounded commuting holomorphic semigroups. Then it is easy to see (cf. [4, Examples 4.1(2)]) that both operators satisfy (6.1.1) and (6.1.2). For the sake of completeness, we give an explicit argument concerning (6.1.2) below. We have

$$\|(\alpha + i\beta + A_1)^{-1}\| \leqslant M_{A_1} |\alpha + i\beta|^{-1},$$

$$\|(\gamma + i\delta + A_2)^{-1}\| \leqslant M_{A_2} |\gamma + i\delta|^{-1},$$

where M_{A_1} and M_{A_2} are the sectoriality constants of A_1 and A_2, respectively, i.e. $M_{A_i} = \sup_{z \in \mathbb{C}_+} \|z(z + A_i)^{-1}\|$, $i = 1, 2$. Hence,

$$\begin{aligned} &\alpha\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle (\alpha - i\beta + A_1)^{-2} (\gamma - i\delta + A_2)^{-2} x, x^* \rangle| \, d\beta \, d\delta \\ &\leqslant M_{A_1}^2 M_{A_2}^2 \|x\| \|x^*\| \int_{\mathbb{R}} \frac{\gamma}{|\gamma - i\delta|^2} \, d\delta \int_{\mathbb{R}} \frac{\alpha}{|\alpha - i\beta|^2} \, d\beta \\ &\leqslant \pi^2 M_{A_1}^2 M_{A_2}^2 \|x\| \|x^*\|. \end{aligned} \tag{6.1.8}$$

This yields

$$\gamma_{A_i} \leqslant 2M_{A_i}^2, \quad \gamma_{A_1,A_2} \leqslant 4M_{A_1}^2 M_{A_2}^2.$$

We now state our definition of the $\mathcal{B}^2$-calculus.

Definition 6.1.3. *Let $f \in \mathcal{B}^2, x \in X, x^* \in X^*$. Let A_1, A_2 be closed commuting operators on a Banach space X, with $D(A_1) \cap D(A_2)$ dense in X with $\sigma(A_i) \subset \overline{\mathbb{C}}_+$, $i = 1, 2$, and assume that A_1 and A_2 satisfy* (6.1.1) *and* (6.1.2)*. Define*

$$\begin{aligned} &\langle f(A_1, A_2)x, x^* \rangle \\ &= f(\infty, \infty) \langle x, x^* \rangle \\ &- \frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \langle (\alpha - i\beta + A_1)^{-2} x, x^* \rangle f_1'(\alpha + i\beta) \, d\beta \, d\alpha \\ &- \frac{2}{\pi} \int_0^\infty \alpha \int_{\mathbb{R}} \langle (\alpha - i\beta + A_2)^{-2} x, x^* \rangle f_2'(\alpha + i\beta) \, d\beta \, d\alpha \\ &+ \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \alpha\gamma \int_{\mathbb{R}} \int_{\mathbb{R}} \langle (\alpha - i\beta + A_1)^{-2} (\gamma - i\delta + A_2)^{-2} x, x^* \rangle \times \\ &\quad \times D_1 D_2 f(\alpha + i\beta, \gamma + i\delta) \, d\beta \, d\delta \, d\alpha \, d\gamma, \end{aligned} \tag{6.1.9}$$

for all $x \in X$, $x^* \in X^*$. Here, f_1 and f_2 denote (cf. Proposition 3.2.2 above)

$$f_1(z_1) = \lim_{\operatorname{Re} z_2 \to \infty} f(z_1, z_2),$$
$$f_2(z_2) = \lim_{\operatorname{Re} z_1 \to \infty} f(z_1, z_2).$$

Note that using (6.1.5) and (6.1.6) we can write equivalently:

$$\langle f(A_1, A_2)x, x^* \rangle = f(\infty, \infty)\langle x, x^* \rangle + \frac{2}{\pi}\langle g^{(1)}_{x,x^*}, f_1 \rangle_{\mathcal{B}^1} + \frac{2}{\pi}\langle g^{(2)}_{x,x^*}, f_2 \rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\langle g^{(1,2)}_{x,x^*}, f \rangle_{\mathcal{B}^2}. \tag{6.1.10}$$

It is easily seen that this defines a bounded linear mapping $f(A_1, A_2) : X \to X^{**}$, and that the linear mapping

$$\Phi_{A_1,A_2}(f) : \mathcal{B}^2 \to \mathcal{L}(X, X^{**}),$$
$$f \mapsto f(A_1, A_2),$$

is bounded. More precisely,

$$\|f(A_1, A_2)\| \leqslant |f(\infty, \infty)| + \gamma_{A_1}\|f_1\|_{\mathcal{B}^1_0} + \gamma_{A_2}\|f_2\|_{\mathcal{B}^1_0} + \gamma_{A_1,A_2}\|f\|_{\mathcal{B}^2_0}$$
$$\leqslant (\gamma_{A_1} + \gamma_{A_2} + \gamma_{A_1,A_2})\|f\|_{\mathcal{B}^2}.$$

If A_1 and A_2 satisfy (6.1.1), then they are negative generators of C_0-semigroups $(T_1(t))_{t \geqslant 0}$ and $(T_2(t))_{t \geqslant 0}$, respectively (cf. Theorem 2.1.10 and the discussion in Section 2.3). In such case it is meaningful to consider the mapping

$$M(\mathbb{R}^2_+) \ni \mu \mapsto \int_{\mathbb{R}^2_+} T_1(t_1)T_2(t_2)d\mu(t_1, t_2), \tag{6.1.11}$$

where the integral is in the strong operator topology, to which we shall refer as the two-dimensional Hille-Phillips calculus. It is then immediate that each operator of the form as in (6.1.11) is bounded with norm less than $K_{A_1}K_{A_2}$. We will now establish a simple but important consistency result between (6.1.10) and the two-dimensional Hille-Phillips calculus as in (6.1.11).

Lemma 6.1.4. *Let $f \in \mathcal{LM}_2$, $f = \mathcal{L}_{(2)}\mu$, and $f(A_1, A_2)$ be defined as in* (6.1.10). *Then $f(A_1, A_2)$ coincides with the operator*

$$x \mapsto \int_{\mathbb{R}^2_+} T(t_1, t_2)x d\mu(t_1, t_2) = \int_{\mathbb{R}^2_+} T_1(t_1)T_2(t_2)x d\mu(t_1, t_2).$$

Proof. Suppose that $f \in \mathcal{B}_0^2$, $f = \mathcal{L}_{(2)}\mu$ for some $\mu \in M(\mathbb{R}_+^2)$. Let $f \in \mathcal{B}_0^2$, so that $\mu(\{0\} \times \mathbb{R}_+) = \mu(\mathbb{R}_+ \times \{0\}) = 0$, and we may thus consider μ as a measure on $(0,\infty)^2$. Then using (6.1.2), Fubini's theorem and (2.1.3)

$$\int_{(0,\infty)^2} \langle T(t_1,t_2)x, x^*\rangle \, d\mu(t) = \int_{(0,\infty)^2} \langle T_1(t_1)T_2(t_2)x, x^*\rangle \, d\mu(t)$$

$$= 4\int_{(0,\infty)^2} \int_0^\infty \alpha t_1^2 e^{-2\alpha t_1} \langle T_1(t_1)T_2(t_2)x, x^*\rangle \, d\alpha \, d\mu(t)$$

$$= 16\int_{(0,\infty)^2} \int_0^\infty \int_0^\infty \alpha t_1^2 e^{-2\alpha t_1} \gamma t_2^2 e^{-2\gamma t_2} \langle T_1(t_1)T_2(t_2)x, x^*\rangle \, d\alpha \, d\gamma \, d\mu(t)$$

$$= \frac{4}{\pi^2}\int_{(0,\infty)^2} \int_0^\infty \int_0^\infty \alpha t_1 e^{\alpha t_1} \beta t_2 e^{\beta t_2}$$

$$\times \int_{\mathbb{R}}\int_{\mathbb{R}} \langle (\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}x, x^*\rangle e^{-i\beta t_1} e^{-i\delta t_2} \, d\beta \, d\delta \, d\alpha \, d\gamma \, d\mu(t)$$

$$= \frac{4}{\pi^2}\int_0^\infty \int_0^\infty \alpha\gamma \int_{\mathbb{R}}\int_{\mathbb{R}} \langle (\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}x, x^*\rangle$$

$$\times \int_{(0,\infty)^2} t_1 t_2 e^{-(\alpha+i\beta)t_1} e^{-(\gamma+i\delta)t_2} d\mu(t) \, d\beta \, d\delta \, d\alpha \, d\gamma$$

$$= \frac{4}{\pi^2}\langle g_{x,x^*}^{(1,2)}, f\rangle.$$

The general case then follows by considering $f(z_1,z_2) - f(\infty,z_2) - f(z_1,\infty) + f(\infty,\infty)$ and noting that $f(z_1,\infty)$ and $f(\infty,z_2)$ are the Laplace transforms of measure μ restricted to $\mathbb{R}_+ \times \{0\}$ and $\{0\} \times \mathbb{R}_+$, respectively (with $f(\infty,\infty) = \mu(\{0,0\})$). □

As in Section 5.1, take

$$\eta(z_1,z_2) = \frac{1-e^{-z_1}}{z_1}\frac{1-e^{-z_2}}{z_2}, \quad {}_\delta\eta(z_1,z_2) = \eta(\delta z_1, \delta z_2). \tag{6.1.12}$$

The following is an analogue of [4, Lemma 4.3] and is an operator version of our Proposition 4.3.1.

Proposition 6.1.5. *Let $f \in \mathcal{B}_0^2$, and assume that $f\,{}_\delta\eta \in \mathcal{LM}_2$ for all $\delta > 0$. Then*

$$\lim_{\delta\to 0+} (f\,{}_\delta\eta)(A_1,A_2) = f(A_1,A_2)$$

in the strong operator topology. Hence, $f(A_1,A_2) \in L(X)$.

Proof. Take $r_1(z_1, z_2) = (1 + z_1)^{-1}(1 + z_2)^{-1}$, so $r_1 = \mathcal{L}_{(2)} e_{1,1}$, where $e_{1,1}(z_1, z_2) = e^{-z_1 - z_2}$, hence $r_1(A_1, A_2) = (1+A_1)^{-1}(1+A_2)^{-1}$. If $\delta > 0$, ${}_\delta\eta$ is the Laplace transform of $\delta^{-2}\chi_{[0,\delta]^2}(t_1, t_2)$. Note that $(1 - e^{-\delta z})(\delta z)^{-1}(1 + z)^{-1} = \mathcal{L}_{(1)}((\delta^{-1}\chi_{[0,\delta]} * e_1)(t))$, and that $(\delta^{-1}\chi_{[0,\delta]} * e_1)(t)$ converges to e_1 in $L^1(\mathbb{R}_+)$, due to the fact that the functions $\delta^{-1}\chi_{[0,\delta]}$ form an approximimate unit for $L^1(\mathbb{R}_+)$ as $\delta \to 0+$. Since r_1 and ${}_\delta\eta$ can be viewed as the tensor products $(1+z_1)^{-1} \otimes (1+z_2)^{-1}$ and $\frac{1-e^{-z_1}}{z_1} \otimes \frac{1-e^{-z_2}}{z_2}$, respectively, continuity of the mappings (cf. Section 3.5)

$$\mathcal{B}^2 \ni \mathbb{1} \otimes f \mapsto f \in \mathcal{B}^1,$$

$$\mathcal{B}^2 \ni f \otimes \mathbb{1} \mapsto f \in \mathcal{B}^1,$$

together with the fact that the Laplace transform is a bounded algebra homomorphism from $L^1(\mathbb{R}_+)$ to $\mathcal{B}^1$ implies

$$\| {}_\delta\eta r_1 - r_1 \|_{\mathcal{B}^2} \to 0, \quad \delta \to 0.$$

It then follows from Proposition 4.4.1 and Lemma 6.1.4 that

$$(f\, {}_\delta\eta r_1)(A_1, A_2) = (f\, {}_\delta\eta)(A_1, A_2) r_1(A_1, A_2) = (f\, {}_\delta\eta)(A_1, A_2)(1 + A_1)^{-1}(1 + A_2)^{-1}.$$

For $x \in D(A_1 A_2)$, we obtain

$$(f\, {}_\delta\eta)(A_1, A_2)x = (f\, {}_\delta\eta r_1)(A_1, A_2)(1+A_1)(1+A_2)x \to (f r_1)(A_1, A_2)(1+A_1)(1+A_2)x.$$

Moreover, for $\delta > 0$, $\|(f\, {}_\delta\eta)(A_1, A_2)\| \leqslant c\|f\|_{\mathcal{B}^2}$. Now, $D(A_1 A_2) = \operatorname{Ran}((1 + A_1)^{-1}(1 + A_2)^{-1})$; since the range of $(1 + A_i)^{-1}$, with $i = 1, 2$, is dense, $D(A_1 A_2)$ is dense in X. Hence, $\lim_{\delta \to 0+} (f\, {}_\delta\eta)(A_1, A_2)x$ exists for every $x \in X$. By Proposition 4.3.1,

$$\langle \lim_{\delta \to 0+} (f\, {}_\delta\eta)(A_1, A_2)x, x^* \rangle = \lim_{\delta \to 0+} \langle g^{(1,2)}_{x,x^*}, f\, {}_\delta\eta \rangle_{\mathcal{B}^2} = \langle f(A_1, A_2)x, x^* \rangle.$$

□

6.2 Main theorem

We now have all the ingredients necessary to prove our main result. In Theorem 6.2.1 we show that the map Φ_{A_1,A_2} has the essential properties of a bounded functional calculus. We shall subsequently refer to Φ_{A_1,A_2} as the $\mathcal{B}^2$-calculus for (A_1, A_2). We will show later (see Section 6.3) that the $\mathcal{B}^2$-calculus is essentially unique.

Theorem 6.2.1. *The map $\Phi_{A_1,A_2} : f \mapsto f(A_1, A_2)$ is a bounded algebra homomorphism from $\mathcal{B}^2$ to $L(X)$.*

Proof. Let ${}_\delta\eta$ be given by (6.1.12). If $f \in \mathcal{G}^2$, then $f \cdot {}_\delta\eta \in \mathcal{LM}_2$ by Proposition 4.4.1, so Proposition 6.1.5 applies to all such f. In particular Φ_{A_1,A_2} maps $\mathcal{G}^2$ into $L(X)$. Since, by Proposition 4.2.5, $\mathcal{G}^2$ is norm-dense in $\mathcal{B}_0^2$, it follows that $\Phi_{A_1,A_2}(f) \in L(X)$ whenever $f_1(z_1) = f_2(z_2) = 0$. Since $\Phi_{A_1,A_2}(f \otimes \mathbb{1}) = f(A_1)$ and $\Phi_{A_1,A_2}(\mathbb{1} \otimes f) = f(A_2)$ for any $f \in \mathcal{B}$, it follows from [4, Theorem 4.4] that Φ_{A_1,A_2} maps $\mathcal{B}^2$ into $L(X)$. Consider now $f, g \in \mathcal{G}^2$. Take $\delta, \delta' > 0$. We have

$$(f\,{}_\delta\eta g\,{}_{\delta'}\eta)(A_1, A_2) = (f\,{}_\delta\eta)(A_1, A_2)(g\,{}_{\delta'}\eta)(A_1, A_2).$$

Letting first $\delta' \to 0+$ and then $\delta \to 0+$, and using Proposition 6.1.5, we obtain $(fg)(A_1, A_2) = f(A_1, A_2)g(A_1, A_2)$. □

Example 6.2.2. If $f \in \mathcal{B}^2$, then by Lemma 3.8.1, $f^{(2)}(z_1, z_2) = f(z_1 + z_2)$ is in $\mathcal{B}^2$. Let A_1 and A_2 be sectorial commuting operators on X such that $\omega_{A_1} + \omega_{A_2} < \pi$, where ω_A is the spectral angle of A (Definition 2.2.1). In view of Lemma 3.8.1, we can now define, for any $f \in \mathcal{B}^1$, the operator $f^{(2)}(A_1, A_2)$ in accordance with the $\mathcal{B}^2$-calculus. It is well known that the operator $A_1 + A_2$, while necessarily closable, may in general fail to be closed. The operator $A_1 + A_2$ is guaranteed to be closed, for instance, when X is an UMD space and both A_1 and A_2 admit bounded imaginary powers, in virtue of the Dore-Venni theorem (see e.g. [25, Theorem 9.3.11]). That said, we may always consider the closure $\overline{A_1 + A_2}$ instead and define $f(\overline{A_1 + A_2})$ in the one-dimensional $\mathcal{B}^1$-calculus. One can then ask if the two definitions agree, i.e. whether the operator $f^{(2)}(A_1, A_2)$ defined in the $\mathcal{B}^2$-calculus is the same as $f(\overline{A_1 + A_2})$ defined in the $\mathcal{B}^1$-calculus. The next proposition shows that this is indeed the case.

Lemma 6.2.3. *Suppose that $f(z) \in \mathcal{B}^1$ and let $f^{(2)}(z_1, z_2) = f(z_1 + z_2)$. Then the operator $f^{(2)}(A_1, A_2)$ defined in the $\mathcal{B}^2$-calculus equals $f(\overline{A_1 + A_2})$.*

Proof. Assume first that $f = \mathcal{L}\mu$ for some $\mu \in M(\mathbb{R}_+)$. We then have for $z_1, z_2 \in \mathbb{C}_+$

$$f(z_1 + z_2) = \int_{\mathbb{R}_+} e^{-t(z_1+z_2)}\, d\mu(t) = \int_{\mathbb{R}_+^2} e^{-(t_1 z_1 + t_2 z_2)}\, d\tilde{\mu}(t_1, t_2),$$

with $\tilde{\mu}(E) = \mu(q^{-1}(E))$ for E in the Borel σ-algebra of $\mathbb{R}_+^2$ and $q : \mathbb{R}_+ \to \mathbb{R}_+^2$ given by $q(t) = (t, t)$. Then, in view of [4, Lemma 4.2] and Lemma 6.1.4, for $x \in D(A_1) \cap D(A_2)$,

$$f(\overline{A_1 + A_2})x = \int_{\mathbb{R}_+} e^{-t(A_1+A_2)} x\, d\mu(t) = \int_{\mathbb{R}_+^2} e^{-(t_1 A_1 + t_2 A_2)} x\, d\tilde{\mu}(t_1, t_2) = f^{(2)}(A_1, A_2)x,$$

so that the two definitions agree, by density of the domain and boundedness of the calculi. If $f(z) \in \mathcal{G}^1$, then $f\,{}_\delta g \in \mathcal{LM}_{(1)}$, with ${}_\delta g = \frac{1-e^{-\delta z}}{\delta z}$ by [4, Lemma 2.13], and $(f\,{}_\delta g)^{(2)}(z_1, z_2) \in \mathcal{LM}_{(2)}$, so that

$$f(\overline{A_1 + A_2})x = \lim_{\delta \to 0+} (f\,{}_\delta g)(\overline{A_1 + A_2})x = \lim_{\delta \to 0+} (f\,{}_\delta g)^{(2)}(A_1, A_2)x = f^{(2)}(A_1, A_2)x,$$

by [4, Lemma 4.3] and Proposition 6.1.5. Since $\mathcal{G}^1$ is dense in $\mathcal{B}_0^1$, the result holds for all $f \in \mathcal{B}_0^2$. In the general case, it suffices to consider $f(z) - f(\infty)$ and observe that

$$\begin{aligned} f(\overline{A_1 + A_2})x &= (f - f(\infty))(\overline{A_1 + A_2})x + f(\infty)(\overline{A_1 + A_2})x \\ &= (f - f(\infty))^{(2)}(A_1, A_2)x + f^{(2)}(\infty, \infty)x \\ &= f^{(2)}(A_1, A_2)x. \end{aligned}$$

□

6.3 Necessity and uniqueness

In the present section, we shall be concerned with the optimality of the $\mathcal{B}^2$-calculus. Firstly, we shall see that the resolvent assumptions (6.1.1) and (6.1.2) are optimal, and secondly, that the calculus is uniquely defined, so that it is necessarily given by the reproducing formula (6.1.10).

Let (A_1, A_2) be a pair of closed commuting operators on a Banach space X, with dense domains, and assume that $\sigma(A_1) \cup \sigma(A_2) \subseteq \overline{\mathbb{C}}_+$. An *abstract (bounded) $\mathcal{B}^2$-calculus* for (A_1, A_2) is a bounded algebra homomorphism

$$\Phi : \mathcal{B}^2 \to L(X), \tag{6.3.1}$$

such that

$$\Phi((\lambda_1 + z_1)^{-1} \otimes \mathbb{1}) = (\lambda_1 + A_1)^{-1}, \quad \Phi(\mathbb{1} \otimes (\lambda_2 + z_2)^{-1}) = (\lambda_2 + A_2)^{-1}. \tag{6.3.2}$$

We say that the pair (A_1, A_2) *admits* a $\mathcal{B}^2$-calculus in case there is a homomorphism Φ such that (6.3.1) and (6.3.2) hold. We have the following result concerning the necessity of our resolvent assumptions.

Theorem 6.3.1. *Let A_1 and A_2 be commuting operators on X with $\sigma(A_i) \subseteq \overline{\mathbb{C}}_+$ for $i = 1, 2$, and assume that the pair (A_1, A_2) admits a $\mathcal{B}^2$-calculus Φ. Then the resolvent assumptions* (6.1.1) *and* (6.1.2) *hold.*

Proof. We only provide an explicit argument for (6.1.2). As for (6.1.1), consider for instance the embedding of $\mathcal{B}^1$ in $\mathcal{B}^2$ given by $f \mapsto f \otimes 1$. Then $\Phi(f \otimes 1)$ is a $\mathcal{B}$-calculus for A_1, and [5, Theorem 6.1] yields that (6.1.1) holds for $i = 1$. The same reasoning with A_1 and $f \otimes 1$ replaced by A_2 and $1 \otimes f$, respectively, establishes that (6.1.1) holds for $i = 2$ as well.

Let φ be a continuous function on $\mathbb{R}^2$ with compact support, and let $K_{\alpha,\gamma,\varphi}$ be given by

$$K_{\alpha,\gamma,\varphi}(z_1, z_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\varphi(\beta,\delta)}{(z_1 + \alpha - i\beta)^2(z_2 + \gamma - i\delta)^2}\, d\beta\, d\delta, \quad z_1, z_2 \in \mathbb{C}_+. \tag{6.3.3}$$

We showed in Proposition 4.2.2 that the two-parameter semigroup

$$T_{\mathcal{B}^2}(a, b)f(z_1, z_2) = f(z_1 + a, z_2 + b),$$

is a product of strongly continuous semigroups on $\mathcal{B}^2$. Since the function

$$r_{\lambda_1,\lambda_2}(z_1, z_2) = (z_1 + \lambda_1)^{-1}(z_2 + \lambda_2)^{-1}$$

is a product of two functions in $\mathcal{B}^1$ (cf. [4, Examples 2.12(2)]), r_{λ_1,λ_2} is itself in $\mathcal{B}^2$ for all $\lambda_1, \lambda_2 \in \mathbb{C}_+$. Hence the function

$$\mathbb{R}^2 \ni (\beta, \delta) \mapsto r^2_{\alpha-i\beta,\gamma-i\delta} \tag{6.3.4}$$

belongs to $C(\mathbb{R}^2, \mathcal{B}^2_0)$ for every $\alpha, \gamma > 0$, and is bounded. Indeed, the continuity follows of (6.3.4) follows from the continuity of $\lambda \mapsto r_\lambda$ considered as a mapping from $\mathbb{R}$ to $\mathcal{B}^1$, the continuity of $f \mapsto f \otimes 1$(or $1 \otimes f$) viewed as a mapping from $\mathcal{B}^1$ to $\mathcal{B}^2$ (see Section 3.5) and then the continuity of multiplication in the Banach algebra $\mathcal{B}^2$ (see Section 3.3).

This establishes that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\beta,\delta) r^2_{\alpha-i\beta,\gamma-i\delta}\, d\beta\, d\delta$$

exists as a $\mathcal{B}^2$-valued Bochner integral. Since the point evaluations are continuous in the $\mathcal{B}^2$-norm, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\beta,\delta) r^2_{\alpha-i\beta,\gamma-i\delta}(z_1, z_2)\, d\beta\, d\delta = K_{\alpha,\gamma,\varphi}(z_1, z_2).$$

Note that φ is compactly supported and continuous, so by the Dominated Convergence Theorem $K_{\alpha,\gamma,\varphi}(z_1, \infty) = K_{\alpha,\gamma,\varphi}(\infty, z_2) = 0$ for any $z_1, z_2 \in \mathbb{C}_+$. We have

$$D_1 D_2 K_{\alpha,\gamma,\varphi}(z_1, z_2) = 4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\varphi(\beta,\delta)}{(z_1 + \alpha - i\beta)^3(z_2 + \gamma - i\delta)^3}\, d\beta\, d\delta, \quad z_1, z_2 \in \mathbb{C}_+.$$

Hence

$$
\begin{aligned}
&|D_1D_2K_{\alpha,\gamma,\varphi}(x_1+iy_1,x_2+iy_2)| \\
&\quad \leqslant 4\|\varphi\|_{L^\infty}\int_{-\infty}^{\infty}\frac{d\beta}{((x_1+\alpha)^2+(\beta-y_1)^2)^{3/2}}\int_{-\infty}^{\infty}\frac{d\delta}{((x_2+\gamma)^2+(\delta-y_2)^2)^{3/2}} \\
&\quad \leqslant \frac{4\|\varphi\|_{L^\infty}}{(x_1+\alpha)^2(x_2+\gamma)^2}\int_{-\infty}^{\infty}\frac{d\tau_1}{(1+\tau_1^2)^{3/2}}\int_{-\infty}^{\infty}\frac{d\tau_2}{(1+\tau_2^2)^{3/2}} \\
&\quad = \frac{16\|\varphi\|_{L^\infty}}{(x_1+\alpha)^2(x_2+\gamma)^2},
\end{aligned}
$$

so that

$$
\|K_{\alpha,\gamma,\varphi}\|_{\mathcal{B}_0^2} \leqslant \frac{\pi^2}{\alpha\gamma}\|\varphi\|_\infty. \tag{6.3.5}
$$

Moreover, we have

$$
\|K_{\alpha,\gamma,\varphi}\|_\infty \leqslant \|\varphi\|_\infty\int_{-\infty}^{\infty}\frac{d\beta}{\alpha^2+\beta^2}\int_{-\infty}^{\infty}\frac{d\gamma}{\gamma^2+\delta^2} = \frac{\pi^2}{\alpha\gamma}\|\varphi\|_\infty. \tag{6.3.6}
$$

If Φ is a $\mathcal{B}^2$-calculus for (A_1,A_2), then

$$
\Phi(K_{\alpha,\gamma,\varphi}) = \int_{-\infty}^{\infty}(A_1+\alpha-\beta)^{-2}(A_2+\gamma-\delta)^{-2}\varphi(\beta,\delta)\,d\beta\,d\delta,
$$

Hence, by the boundedness of the calculus,

$$
\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle(A_1+\alpha-i\beta)^{-2}(A_2+\gamma-\delta)^{-2}x,x^*\rangle\varphi(\beta,\delta)\,d\beta\,d\delta\right| \leqslant \|\Phi\|\|K_{\alpha,\gamma,\phi}\|_{\mathcal{B}^2}.
$$

for all $x\in X$, $x^*\in X^*$ with $\|x\|=\|x^*\|=1$. By (6.3.5) and (6.3.6),

$$
\begin{aligned}
&\alpha\gamma\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle(A_1+\alpha-i\beta)^{-2}(A_2+\gamma-\delta)^{-2}x,x^*\rangle\varphi(\beta,\delta)\,d\beta\,d\delta\right| \\
&\qquad\qquad \leqslant \|\Phi\|\alpha\gamma\|K_{\alpha,\gamma,\phi}\|_{\mathcal{B}^2} \leqslant (16+\pi^2)\|\Phi\|\|\varphi\|_{L^\infty}.
\end{aligned}
$$

Since the continuous functions with compact support are weak*-dense in $L^\infty(\mathbb{R}^2)$, it follows that (6.1.2) holds. □

Having proved the necessity of our resolvent assumptions, the following question addressing uniqueness of $\mathcal{B}^2$-calculus is natural. If commuting (bounded) negative semigroup generators A_1 and A_2 admit $\mathcal{B}^2$-calculus, then is the calculus given by

$$\begin{aligned}\langle \Phi(f)x, x^*\rangle &= f(\infty,\infty) + \frac{2}{\pi}\big\langle\langle(\cdot + A_1)^{-1}x, x^*\rangle, f_1\big\rangle_{\mathcal{B}^1} \\ &\quad + \frac{2}{\pi}\big\langle\langle(\cdot + A_2)^{-1}x, x^*\rangle, f_2\big\rangle_{\mathcal{B}^1} + \frac{4}{\pi^2}\big\langle\langle(\cdot + A_1)^{-1}(\cdot + A_2)^{-1}x, x^*\rangle, f\big\rangle_{\mathcal{B}^2}\end{aligned}$$

for all $x \in X$ and $x^* \in X^*$?

We show that this question has a positive answer.

Theorem 6.3.2. *Let A_1, A_2 be commuting operators on X with $\sigma(A_1) \cup \sigma(A_2) \subseteq \overline{\mathbb{C}}_+$. If the pair (A_1, A_2) admits a $\mathcal{B}^2$-calculus Φ, then Φ is unique.*

Proof. Note first that the functions $e_{a_1,a_2}(z,z_2) = e^{-a_1z_1-a_2z_2}$ for $a_1, a_2 > 0$ span a dense subspace of $L^1(\mathbb{R}_+^2)$ (this follows from the uniqueness of Laplace transforms for functions in $L^\infty(\mathbb{R}^2)$ considered as the dual of $L^1(\mathbb{R}^2)$ together with the Hahn-Banach theorem). Since $\mathcal{L}_{(2)}e_{a_1,a_2} = (z_1 + a_1)^{-1}(z_2 + a_2)^{-1}$, $\Phi(\mathcal{L}_{(2)}g)$ is uniquely determined for every $g \in L^1(\mathbb{R}_+^2)$. Let $f \in \mathcal{G}^2$ and $g \in L^2([0,\tau]^2) \subseteq L^1(\mathbb{R}_+^2)$. By Proposition 4.4.1, $f\mathcal{L}_{(2)}g$ is also the Laplace transform of a function in $L^1(\mathbb{R}_+^2)$. Thus $\Phi(\mathcal{L}_{(2)}g)$ and $\Phi(f)\Phi(\mathcal{L}_{(2)}g)$ are both determined uniquely. It follows that $\Phi(f)$ is uniquely determined on the range of $\Phi(\mathcal{L}_{(2)}g)$. In particular this implies that $\Phi(f)$ is uniquely determined on vectors of the form

$$\int_0^n \int_0^n e^{-t_1}e^{-t_2}T_1(t_1)T_2(t_2)x\,dt_1\,dt_2,$$

and hence on $(1+A_1)^{-1}(1+A_2)^{-1}x$. Here $(T_1(t))_{t\geqslant 0}$ and $(T_2(t))_{t\geqslant 0}$ are the semigroups generated by $-A_1$ and $-A_2$, respectively. Since A_1 and A_2 are densely defined and commuting, it follows that $\Phi(f)$ is uniquely determined for all $f \in \mathcal{G}^2$. By the density of $\mathcal{G}^2$ in $\mathcal{B}_0^2$, Φ is uniquely determined on $\mathcal{B}_0^2$. Recall that (cf. Section 3.5) $M_1 = \{g \otimes \mathbb{1} : g \in \mathcal{B}^1\} \subset \mathcal{B}^2$ and $M_2 = \{\mathbb{1} \otimes g : g \in \mathcal{B}^1\} \subset \mathcal{B}^2$. Hence $\Phi(f \otimes \mathbb{1})$ and $\Phi(\mathbb{1} \otimes f)$ are uniquely determined for all $f \in \mathcal{B}_0^1$ by Theorem 2.3.12. □

6.4 Spectral inclusion and mapping

In the one-dimensional case, as noted in [4, Section 4.5], given a semigroup generator $-A$ and a functional calculus Υ, the full spectral mapping theorem typically fails, and we are left with the inclusion $f(\sigma(A)) \subset \sigma(\Upsilon_A(f))$. In the present setting, the situation is further complicated by the existence of several types of joint spectra of families of commuting closed operators (see [35] for a discussion). We shall establish below that, for a particular notion of the joint spectrum, the spectral inclusion theorem for $\mathcal{B}^2$-calculus holds.

Let $-A_1$ and $-A_2$ generate bounded commuting C_0-semigroups $(T_1(t))_{t\geqslant 0}$ and $(T_2(t))_{t\geqslant 0}$, respectively, on a Banach space X. Denote by $\mathcal{A}$ the Banach algebra generated by all the operators $f(A_1, A_2)$, with $f \in \mathcal{B}^2$, and all their resolvents. Then $\mathcal{A}$ is an abelian subalgebra of $L(X)$ and contains $R(\lambda, A_i)$ for all $\lambda_i \in \rho(A_i)$ and $i = 1, 2$. Clearly, if $f \in \mathcal{B}^2$ then the spectrum of $f(A_1, A_2)$ in $\mathcal{A}$ coincides with its spectrum in $L(X)$. It can be shown (cf. [35, p. 1238]) that, for $i = 1, 2$, the Gelfand spectrum $\mathfrak{M}_{\mathcal{A}}$ of $\mathcal{A}$ splits into two subsets $\mathfrak{M}_i$ and $\mathfrak{U}_i$, and that there is a continuous function α_i on $\mathfrak{M}_i$ such that for every character χ of $\mathcal{A}$ we have

$$\chi(R(z, A_i)) = \begin{cases} (z - \alpha_i(\chi))^{-1} & \text{if } \chi \in \mathfrak{M}_i, \\ 0 & \text{if } \chi \in \mathfrak{U}_i, \end{cases}$$

for all $z \in \rho(A_i)$, and $\sigma(A_i) = \alpha_i(\mathfrak{M}_i)$. We define the joint spectrum $\sigma(A_1, A_2)$ of a pair of C_0-semigroup generators, A_1 and A_2 on X by

$$\sigma(A_1, A_2) = \{(\alpha_1(\chi), \alpha_2(\chi)) : \chi \in \mathfrak{M}_1 \cap \mathfrak{M}_2\}.$$

We are now ready to state the theorem.

Theorem 6.4.1. *Assume that $\sigma(A_i) \subset \overline{\mathbb{C}}_+$ for $i = 1, 2$ and that (6.1.1) and (6.1.2) hold. Let $f \in \mathcal{B}^2$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. If $(\lambda_1, \lambda_2) \in \sigma(A_1, A_2)$, then $f(\lambda_1, \lambda_2) \in \sigma(f(A_1, A_2))$.*

Proof. Suppose that $f \in \mathcal{B}^2_0$. We shall first assume that the resolvents of A_1 and A_2 are bounded on the left-half plane. By the resolvent identity, we have for $z \in \mathbb{C}_+$

$$\|(z + A_i)^{-2}(1 + A_i)^{-2}\| \leqslant C(1 + |z - 1|)^{-2}, \quad i = 1, 2. \tag{6.4.1}$$

From (6.1.10), (6.4.1), and the fact that f_i are identically zero for $i = 1, 2$, one may then obtain the following formula, with the integral absolutely convergent in the operator norm:

$$\begin{aligned} &f(A_1, A_2)(1 + A_1)^{-2}(1 + A_2)^{-2} \\ &= \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha\gamma(\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}(1 + A_1)^{-2}(1 + A_2)^{-2} \\ &\times D_1 D_2 f(\alpha + i\beta, \gamma + i\delta)\, d\alpha\, d\gamma\, d\beta\, d\delta. \end{aligned} \tag{6.4.2}$$

Let $(\lambda_1, \lambda_2) \in \sigma(A_1, A_2)$. Then, for $i = 1, 2$, $(1 + \lambda_i)^{-1} \in \sigma((1 + A_i)^{-1})$, so there exists a character χ in $\mathcal{A}$ such that $\chi((1 + A_i)^{-1}) = (1 + \lambda_i)^{-1}$, with $\alpha_i(\chi) = \lambda_i$. Since χ is multiplicative, applying it to both sides of (6.4.2) and using (5.1.1) we obtain

$$\chi(f(A_1, A_2))(1 + \lambda_1)^{-2}(1 + \lambda_2)^{-2}$$

$$= \frac{4}{\pi^2} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha\gamma(\alpha - i\beta + \lambda_1)^{-2}(\gamma - i\delta + \lambda_2)^{-2}(1 + \lambda_1)^{-2}(1 + \lambda_2)^{-2}$$

$$\times D_1 D_2 f(\alpha + i\beta, \gamma + i\delta)\, d\alpha\, d\gamma\, d\beta\, d\delta$$

$$= f(\lambda_1, \lambda_2)(1 + \lambda_1)^{-2}(1 + \lambda_2)^{-2}.$$

Thus $f(\lambda_1, \lambda_2) = f(\alpha_1(\chi), \alpha_2(\chi)) = \chi(f(A_1, A_2)) \in \sigma(f(A_1, A_2))$. If $f_1 \neq 0$ and $f_2 \neq 0$ with $f_1(\infty) = f_2(\infty) = 0$, then by [4, Theorem 4.17] (or repeating the above calculation with the representation formula [4, (4.20)] and the same choice of χ) $f_i(\lambda_i) = f_i(\alpha_i(\chi)) = \chi(f_i(A_i)) \in \sigma(f_i(A_i))$, $i = 1, 2$, hence

$$f(\lambda_1, \lambda_2) = (f(\lambda_1, \lambda_2) - f_1(\lambda_1) - f_2(\lambda_2) + f_1(\lambda_1) + f_2(\lambda_2))$$

$$= \chi((f - f_1 - f_2)(A_1, A_2)) + \chi(f_1(A_1)) + \gamma(f_2(A_2))$$

$$= \chi(f(A_1, A_2))) \in \sigma(f(A_1, A_2)).$$

For the general case, we proceed as follows. If $\varepsilon > 0$, $i = 1, 2$, then $\lambda_i + \varepsilon \in \sigma(A_i + \varepsilon)$. Applying the case above to $A_i + \varepsilon$, with the same choice of χ, we obtain $f(\lambda_1 + \varepsilon, \lambda_2 + \varepsilon) = \chi(f(A_1 + \varepsilon, A_2 + \varepsilon))$. By Proposition 4.2.2 and Proposition 5.4.1, $\|f(A_1 + \varepsilon, A_2 + \varepsilon) - f(A_1, A_2)\|_{L(X)} \to 0$ as $\varepsilon \to 0+$, which implies the conclusion. □

Remark 6.4.2. As seen in the proof, it is critical that we should be able to find a character χ such that its value at $R(\lambda, A_i)$ is just $(1 + \lambda_i)^{-1}$, with $\lambda_i \in \sigma(A_i)$, for $i = 1, 2$. Our choice of the notion of a joint spectrum is then dictated by this particular requirement. This may be a limiting feature of the Banach algebraic methods employed in the proof; it remains an open question whether similar spectral mapping theorems hold for other types of joint spectra.

As mentioned above, in the one-dimensional case the spectral inclusion is the best we can generally hope for. However, provided that the operator involved satisfies some sharp resolvent estimates, e.g. is sectorial of angle less than $\pi/2$, one may expect the equality

$$f(\sigma(A)) \cup \{f(\infty)\} = \sigma(\Upsilon_A(f)) \cup \{f(\infty)\},$$

to hold for a given functional calculus Υ_A (cf. [4, Theorem 4.17)]). We will now verify that the $\mathcal{B}^2$-calculus does indeed satisfy a spectral mapping theorem of this type. Suppose that the Banach space operators A_1 and A_2 commute, and that $A_1, A_2 \in \mathrm{Sect}(\pi/2-)$, where $\mathrm{Sect}(\pi/2-) = \bigcup_{\theta\in[0,pi/2)} \mathrm{Sect}(\theta)$. Then the estimates

$$\|(\alpha - i\beta + A_i)^{-2}\| \leqslant M_{A_i}^2(\alpha^2+\beta^2)^{-1}, \quad i = 1, 2, \tag{6.4.3}$$

where $M_{A_i} = \sup_{z\in\mathbb{C}_+} \|z(z + A_i)^{-1}\|$, imply that the integrals in (6.4.4) are convergent in the operator norm

$$\begin{aligned} f(A_1, A_2) = f(\infty,\infty) &- \frac{2}{\pi}\int_0^\infty \alpha \int_{\mathbb{R}} (\alpha - i\beta + A_1)^{-2} f_1'(\alpha + i\beta) d\beta\, d\alpha \\ &- \frac{2}{\pi}\int_0^\infty \alpha \int_{\mathbb{R}} (\alpha - i\beta + A_2)^{-2} f_2'(\alpha + i\beta) d\beta\, d\alpha \\ &+ \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \alpha\gamma \int_{\mathbb{R}}\int_{\mathbb{R}} (\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2} \times \\ &\quad \times D_1D_2 f(\alpha + i\beta, \gamma + i\delta)\, d\beta\, d\delta\, d\alpha\, d\gamma, \end{aligned} \tag{6.4.4}$$

and the identity follows from (6.1.10). Reasoning as in Example 6.1.2 above, we have, for instance,

$$\begin{aligned} &\left\| \int_0^\infty\int_0^\infty \alpha\gamma \int_{\mathbb{R}}\int_{\mathbb{R}} (\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2} D_1D_2 f(\alpha + i\beta, \gamma + i\delta)\, d\beta\, d\delta\, d\alpha\, d\gamma \right\| \\ &\quad \leqslant M_{A_1}^2 M_{A_2}^2 \int_0^\infty\int_0^\infty \alpha\gamma \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|D_1D_2 f(\alpha + i\beta, \gamma + i\delta)|}{|\alpha + i\beta|^2|\gamma + i\delta|^2}\, d\beta\, d\delta\, d\alpha\, d\gamma \\ &\quad \leqslant M_{A_1}^2 M_{A_2}^2 \|f\|_{\mathcal{B}^2} \|1/(z_1z_2)\|_{\mathcal{E}^2} < \infty, \end{aligned}$$

and similarly for the other terms, which yields convergence in the operator norm. We can now obtain the desired result using (6.4.4); the proof remains almost identical to the proof of the final part of [4, Theorem 4.17(4)], with some obvious modifications.

Theorem 6.4.3. *If A_1 and A_2 are a pair of commuting operators on a Banach space X and $A_1, A_2 \in \mathrm{Sect}(\pi/2-)$, where Sect$(\pi/2-) = \bigcup_{\theta\in[0,\pi/2)} \mathrm{Sect}(\theta)$, then*

$$\sigma(f(A_1, A_2)) \cup \{f(\infty,\infty)\} = f(\sigma(A_1, A_2)) \cup f_1(\sigma(A_1)) \cup f_2(\sigma(A_2)) \cup \{f(\infty,\infty)\}.$$

Proof. Let $\mathcal{A}$ be as above, and let $f(\infty,\infty) = 0$. Let χ be any character of $\mathcal{A}$.

Applying χ to (6.4.4) gives

$$\begin{aligned}\chi(f(A_1,A_2)) = &-\frac{2}{\pi}\int_0^\infty \alpha\int_{\mathbb{R}} \chi((\alpha - i\beta + A_1)^{-2}) f_1'(\alpha + i\beta) d\beta\, d\alpha \\ &-\frac{2}{\pi}\int_0^\infty \alpha\int_{\mathbb{R}} \chi((\alpha - i\beta + A_2)^{-2}) f_2'(\alpha + i\beta) d\beta\, d\alpha \\ &+\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \alpha\gamma\int_{\mathbb{R}}\int_{\mathbb{R}} \chi((\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2})\times \\ &\times D_1 D_2 f(\alpha + i\beta, \gamma + i\delta)\, d\beta\, d\delta\, d\alpha\, d\gamma\end{aligned} \tag{6.4.5}$$

We consider three cases.

1. If $\chi \in \mathfrak{U}_1 \cap \mathfrak{U}_2$, then $\chi((z + A_1)^{-1}) = \chi((z + A_2)^{-1}) = 0$, for all $z \in \mathbb{C}_+$, and then $\chi(f(A_1, A_2)) = 0 = f(\infty, \infty)$.

2. Suppose that $\chi \in \mathfrak{M}_1 \cap \mathfrak{U}_2$. Then $\chi((z + A_2)^{-1}) = 0$, for all $z \in \mathbb{C}_+$. Also, there exists $\mu_1 \in \sigma(A_1)$ such that $\chi((z + A_1)^{-1}) = (z + \mu_1)^{-1}$. Consequently, equation (6.4.5) reduces to

$$\begin{aligned}\chi(f(A_1,A_2)) &= -\frac{2}{\pi}\int_0^\infty \alpha\int_{\mathbb{R}} \chi((\alpha - i\beta + A_1)^{-2}) f_1'(\alpha + i\beta) d\beta\, d\alpha \\ &= f_1(\mu_1) \in f_1(\sigma((A_1)),\end{aligned}$$

where the last equality follows from [4, (4.18)]. Respectively, if $\chi \in \mathfrak{U}_1 \cap \mathfrak{M}_2$, we obtain that $\chi(f(A_1, A_2)) \in f_2(\sigma(A_2))$.

3. Finally, suppose that $\chi \in \mathfrak{M}_1 \cap \mathfrak{M}_2$. Then there exist μ_1, μ_2 such that $(\mu_1, \mu_2) \in \sigma(A_1, A_2)$, with $\chi((z + A_1)^{-1}) = (z + \mu_1)^{-1}$ and $\chi((z + A_2)^{-1}) = (z + \mu_2)^{-1}$, so that

$$\chi(f(A_1, A_2)) = f(\mu_1, \mu_2) \in f(\sigma(A_1, A_2)).$$

□

Fine structure of the spectrum. Let $f \in \mathcal{B}^2$ and suppose that A_1 and A_2 admit $\mathcal{B}^2$-calculus. We have shown in Theorem 6.4.1 that there is a natural analogue of (2.3.17) which holds in the two-dimensional setting, i.e. we have

$$(\lambda_1, \lambda_2) \in \sigma(A_1, A_2) \implies f(\lambda_1, \lambda_2) \in \sigma(f(A_1, A_2)). \tag{6.4.6}$$

It is, perhaps, interesting to ask whether one can obtain some analogue of Proposition 2.3.17 in the present context. The answer is: yes, as long as we restrict ourselves to a suitable subset of the joint spectrum. For a pair of commuting closed operators A_1 and A_2, define the approximate joint spectrum, $\sigma_{ap}(A_1, A_2)$ as the set of all $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that there is a sequence of vectors $x_n \in D(A_1) \cap D(A_2)$ with

$$\|A_i x_n - \lambda_i x_n\| \to_{n\to\infty} 0, \quad \|x_n\| = 1, \quad i = 1, 2.$$

It is known (cf. [35, Theorem 1]) that the approximate joint spectrum behaves as expected, that is, we have $\sigma_{ap}(A_1, A_2) \subset \sigma(A_1, A_2)$. Proceeding essentially as before, we can now obtain

Proposition 6.4.4. *Suppose that A_1, A_2 are commuting operators such that they admit $\mathcal{B}^2$-calculus. Then approximate (λ_1, λ_2)-eigenvectors of (A_1, A_2) are approximate $f(\lambda_1, \lambda_2)$-eigenvectors of $f(A_1, A_2)$.*

Proof. The proof is similar to that of Proposition 2.3.17 but with the following modifications. Let $(T_1(t))_{t\geqslant 0}$ and $(T_2(t))_{t\geqslant 0}$ be the bounded C_0-semigroups generated by $-A_1$ and $-A_2$, respectively. Consider the space

$$\begin{aligned} l^\infty_{(T_1,T_2)}(X) =&\{(x_n)_{n\in\mathbb{N}} \in l^\infty(X) : \lim_{t\to 0} \|T_i(t)x_n - x_n\| = 0, \\ &\text{uniformly in } n \text{ for } i = 1, 2\}. \end{aligned}$$

This can be viewed as a closed invariant subspace of either $l^\infty_{T_1}(X)$ or as a closed invariant subspace of $l^\infty_{T_2}(X)$. Similarly as in the proof of Proposition 2.3.17, consider the space $E^{(T_1,T_2)} = l^\infty_{(T_1,T_2)}(X)/\mathcal{N}$, with $\mathcal{N} = (c_0(X) \cap l^\infty_{(T_1,T_2)}(X))$, equipped with the norm

$$|||(x_n)_{n\in\mathbb{N}} + \mathcal{N}||| = \limsup \|x_n\|.$$

For $i = 1, 2$, define $(T^i_E(t))_{t\geqslant 0}$ on $E^{(T_1,T_2)}$ by

$$T^i_E(t)((x_n)_{n\in\mathbb{N}} + \mathcal{N}) = (T_i(t)(x_n)_{n\in\mathbb{N}}) + \mathcal{N},$$

and denote its generator by $-A^i_E$. It now follows from Theorem 6.3.2 that the pair $A = (A^1_E, A^2_E)$ has $\mathcal{B}^2$-calculus. The only slightly non-trivial part in the present context is ensuring that 'there are enough' vectors in $D(A^1_E) \cap D(A^2_E)$ to allow the argument to go through. So suppose that $\lambda = (\lambda_1, \lambda_2) \in \sigma(A_1, A_2)$ and consider the corresponding approximate λ-eigenvector $(\xi_n)_{n\in\mathbb{N}}$. For $i = 1, 2$, we have ([38, p. 78]):

$$\|T_i(t)\xi_n - \xi_n\| \leqslant \|T_i(t)\xi_n - e^{\lambda_i t}\xi_n\| + |e^{\lambda_i t} - 1|$$

$$= \left\| \int_0^t e^{\lambda_i(t-s)} T_i(s)(\lambda_i - A_i)\xi_n \, ds \right\| + |e^{\lambda_i t} - 1|,$$

which converges uniformly to 0 as $n \to \infty$. Hence, it is meaningful to define a vector in $l^\infty_{(T_1,T_2)}(X)$ by

$$\hat{\xi} = (\xi_n)_{n\in\mathbb{N}} + \mathcal{N}.$$

Since $\lambda_1 \in \sigma_p(A^1_E)$ and $\lambda_2 \in \sigma_p(A^2_E)$, we obtain that $f(A^1_E, A^2_E)\hat{\xi} = f(\lambda_1, \lambda_2)\hat{\xi}$, and from this it follows as before that $(\hat{\xi}_n)_{n\in\mathbb{N}}$ is an approximate $f(\lambda_1, \lambda_2)$-eigenvector for $f(A_1, A_2)$.

□

6.5 Convergence lemma for operators

Theorem 6.5.1. *Let A_1 and A_2 be a pair of commuting operators which admit the $\mathcal{B}^2$-calculus, and let f_0 and f_n be as in Theorem 5.6.1. Then $f_n(A_1, A_2) \to f_0(A_1, A_2)$ in the strong operator topology. If $A_1, A_2 \in L(X)$, then the convergence is in operator norm.*

Proof. Let $g(z_1, z_2) = (1 + z_1)^{-1}(1 + z_2)^{-1}$. Then $D_1 g, D_2 g$, and $D_1 D_2 g$ are clearly all bounded; $D_1 g(z_1, z_2), D_2 g(z_1, z_2)$, and $D_1 D_2 g(z_1, z_2)$ all tend to 0 as $|z_1| + |z_2| \to \infty$; moreover (using the notation as in the statement of Theorem 5.6.1),

$$G_{12}(\alpha) = \int_0^\infty \sup_{\beta,\delta\in\mathbb{R}} |(1 + \alpha + i\beta)^{-1}(1 + \gamma + i\delta)^{-2}| \, d\gamma = (1 + \alpha)^{-1} \to 0, \text{ as } \alpha \to \infty,$$

$$G_{21}(\alpha) = \int_0^\infty \sup_{\beta,\delta\in\mathbb{R}} |(1 + \alpha + i\beta)^{-2}(1 + \gamma + i\delta)^{-1}| \, d\alpha = (1 + \gamma)^{-1} \to 0, \text{ as } \gamma \to \infty.$$

Thus, the first statement follows from Theorem 5.6.1, continuity of the $\mathcal{B}^2$-calculus, and Corollary 3.4.4 (i.e. the fact that the norms $\|\cdot\|_{\mathcal{B}^2}$ and $\|\cdot\|_{\mathcal{B}^2_0}$ are equivalent on $\mathcal{B}^2_0$). Since $D(A_1 A_2)$ is dense and $f_n(A_1, A_2) \in L(X)$,

$$g_n(A_1, A_2) = f_n(A_1, A_2)(1 + A_1)^{-1}(1 + A_2)^{-1} \to f_0(A_1 A_2)(1 + A_1)^{-1}(1 + A_2)^{-1}$$

in operator norm. □

6.6 Compatibility with the holomorphic calculus

We now proceed to show that the $\mathcal{B}^2$-calculus is compatible with two other functional calculi, namely the sectorial holomorphic calculus and the half-plane holomorphic functional calculus.

To keep the presentation concise, we assume throughout the remainder of this section that both A_1 and A_2 are injective Banach space operators.

Let $A_i \in \mathrm{Sect}(\theta_i)$, where $\theta_i \in [0, \pi/2)$, for $i = 1, 2$. Assume that A_1 and A_2 commute in the resolvent sense. For $\phi_i \in (\theta_i, \pi)$, we denote by $H^\infty(S_{\phi_1} \times S_{\phi_2})$ the Banach algebra of all bounded holomorphic scalar-valued functions on $S_{\phi_1} \times S_{\phi_2}$. Denote

$$\psi(z_1, z_2) = \frac{z_1 z_2}{(1 + z_1)^2 (1 + z_2)^2}. \tag{6.6.1}$$

Given $\phi_1, \phi_2 \in (0, \pi)$, we define a subspace of $H^\infty(S_{\phi_1} \times S_{\phi_2})$ as follows

$$H_0^\infty(S_{\phi_1} \times S_{\phi_2}) = \{f \in H^\infty(S_{\phi_1} \times S_{\phi_2}) : \exists_{s>0} \quad \psi^{-s} f \in H^\infty(S_{\phi_1} \times S_{\phi_2})\}.$$

For $f \in H_0^\infty(S_{\phi_1} \times S_{\phi_2})$ we define

$$f(A_1, A_2) = -\frac{1}{4\pi^2} \iint\limits_{\Gamma_{\theta_1'} \times \Gamma_{\theta_2'}} f(\lambda, \lambda') R(\lambda, A_1) R(\lambda', A_2) \, d\lambda \, d\lambda',$$

for $\theta_i' \in (\theta_i, \phi_i), i = 1, 2$. Here, $\Gamma_{\theta_i'}$ is the downward oriented boundary of the sector $S_{\theta_i'}$. This integral converges and is independent of the choice of $\theta^{\prime}_i$. To extend this to general $f \in H^\infty(S_{\phi_1} \times S_{\phi_2})$, we may observe that $f\psi$ belongs to $H_0^\infty(S_{\phi_1} \times S_{\phi_2})$, so that we may define

$$f(A_1, A_2) = \psi(A_1, A_2)^{-1} (f\psi)(A_1, A_2).$$

Let $f \mapsto \Psi_{A_1, A_2}(f)$ stand for the thus obtained (extended) holomorphic functional calculus. We note that, given (6.6.1), we have

$$\Psi_{A_1, A_2}(\psi) = A_1 (1 + A_1)^{-2} A_2 (1 + A_2)^{-2} = \Phi_{A_1, A_2}(\psi). \tag{6.6.2}$$

Interested readers should consult [32] for the properties of the joint holomorphic calculus. The following elementary lemma is a two-dimensional analogue of [25, Lemma 2.6.7]; it will be used subsequently in the proof of Proposition 6.7.3.

Lemma 6.6.1. *Let $A_i \in \mathrm{Sect}(\theta_i)$ for $i = 1, 2$ with commuting resolvents. Let $(A_{i,n})_{n \in \mathbb{N}}$ be a sectorial approximation of A_i, with $i = 1, 2$. Take $f \in H_0^\infty(S_{\theta_1} \times S_{\theta_2})$. Then $f(A_{1,n}, A_{2,n}) \to f(A_1, A_2)$ in norm as $n \to \infty$.*

Proof. We have

$$\lim_n f(A_{1,n}, A_{2,n}) = \lim_n \left(-\frac{1}{4\pi^2} \iint\limits_{\Gamma_{\theta_1} \times \Gamma_{\theta_2}} f(\lambda, \lambda') R(\lambda, A_{1,n}) R(\lambda', A_{2,n})\, d\lambda\, d\lambda' \right)$$

$$= -\frac{1}{4\pi^2} \iint\limits_{\Gamma_{\theta_1} \times \Gamma_{\theta_2}} f(\lambda, \lambda') \lim_n R(\lambda, A_{1,n}) R(\lambda', A_{2,n})\, d\lambda\, d\lambda'$$

$$= -\frac{1}{4\pi^2} \iint\limits_{\Gamma_{\theta_1} \times \Gamma_{\theta_2}} f(\lambda, \lambda') R(\lambda, A_1) R(\lambda', A_2)\, d\lambda\, d\lambda'$$

$$= f(A_1, A_2).$$

□

Since we assume that both A_1 and A_2 are injective, our class $\mathcal{B}^2$ is automatically contained in the domain of Ψ_{A_1,A_2}. We have the following compatibility result.

Proposition 6.6.2. *Let $f \in \mathcal{B}^2 \cap H_0^\infty(\mathbb{C}_+^2)$. If A_1 and A_2 are commuting injective operators on a Banach space X, and $A_1, A_2 \in \operatorname{Sect}(\pi/2-)$, then*

$$\Phi_{A_1,A_2}(f) = \Psi_{A_1,A_2}(f).$$

The proof of Proposition 6.6.2 resembles closely that of [4, Proposition 4.9], except for some necessary adjustments.

Proof. For $i = 1, 2$, let A_i be an injective operator in $\operatorname{Sect}(\theta_i)$, where $\theta_i \in [0, \pi/2)$. Then, for any $\theta_i' \in (\theta_i, \pi/2)$,

$$\|A_i(I + A_i)^{-1}(z_i - A_i)^{-1}\| = \|(z_i(z_i - A_i)^{-1} - 1)(1 + A_i)^{-1}\| \leqslant \frac{C_{\theta_i'}}{1 + |z_i|}, \quad z_i \notin S_{\pi - \theta_i'}.$$

Let $f \in \mathcal{B}^2$, and let $g = f(r_1 \otimes 1)(1 - r_1 \otimes 1)^{-2}(1 \otimes r_2)(1 - 1 \otimes r_2)^{-2}$, so that $g \in \mathcal{B}_0^2 \cap H_0^\infty(S_{\pi/2} \times S_{\pi/2})$. It then follows from the definition of Ψ_{A_1,A_2}, Proposition 5.1.1 together with the fact that $g \in \mathcal{B}_0^2$, the identities (6.6.1) and (6.6.2), and Fubini's theorem that

$$\begin{aligned}
&\Psi_{A_1,A_2}(\psi)\Psi_{A_1,A_2}(g)\\
&= -\Psi_{A_1,A_2}(\psi)\frac{1}{4\pi^2}\int_{\partial S_{\theta_1'}}\int_{\partial S_{\theta_2'}}(z_1-A_1)^{-1}(z_2-A_2)^{-1}g(z_1,z_2)\,dz_2\,dz_1\\
&= \frac{1}{\pi^4}\Psi_{A_1,A_2}(\psi)\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg(\int_{\partial S_{\theta_1'}}\int_{\partial S_{\theta_2'}}(z_1-A_1)^{-1}(z_2-A_2)^{-1}\\
&\quad\times(\alpha-i\beta+z_1)^{-2}(\gamma-i\delta+z_2)^{-2}\,dz_2\,dz_2\Bigg)\alpha\gamma D_1D_2g(\alpha+i\beta,\gamma+i\delta)\,d\beta\,d\delta\,d\alpha\,d\gamma.
\end{aligned}$$

Now, by applying the primary functional calculus for invertible sectorial operators first to $(z_1-A_1)^{-1}(\alpha-i\beta+z_1)^{-2}$ and then to $(z_2-A_2)^{-1}(\gamma-i\delta+z_2)^{-2}$ we obtain

$$\begin{aligned}
&\int_{\partial S_{\theta_1'}}\int_{\partial S_{\theta_2'}}(z_1-A_1)^{-1}(z_2-A_2)^{-1}(\alpha-i\beta+z_1)^{-2}(\gamma-i\delta+z_2)^{-2}\,dz_2\,dz_1\\
&\quad= 2\pi iA_1(1+A_1)^{-1}(\alpha-i\beta+A_1)^{-2}\int_{\partial S_{\theta_2'}}(\gamma-i\delta+z_2)^{-2}\,dz_2\\
&\quad= -4\pi^2(\alpha-i\beta+A_1)^{-2}(\gamma-i\delta+A_2)^{-2}.
\end{aligned}$$

Hence, it follows from (6.4.4) and (6.6.2) that

$$\begin{aligned}
&\Psi_{A_1,A_2}(\psi)\Psi_{A_1,A_2}(g)\\
&= \frac{4}{\pi^2}\Psi_{A_1,A_2}(\psi)\int_0^\infty\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\Bigg((\alpha-\beta+A_1)^{-2}(\gamma-\delta+A_2)^{-2}\\
&\quad\times\alpha\gamma D_1D_2g(\alpha+i\beta,\gamma+i\delta)\Bigg)\,d\beta\,d\delta\,d\alpha\,d\gamma\\
&= \Psi_{A_1,A_2}(\psi)\Phi_{A_1,A_2}(g).
\end{aligned}$$

Since A_1 and A_2 are both injective, it follows that $\Psi_{A_1,A_2}(g) = \Phi_{A_1,A_2}(g)$.Then it follows from (6.6.1) and (6.6.2) that $\Psi_{A_1,A_2}(f) = \Phi_{A_1,A_2}(f)$. □

6.7 Compatibility with the half-plane calculus

Following the notation and terminology of [8], we denote by

$$L_\omega = \{z \in \mathbb{C} : \operatorname{Re} z < \omega\}, \quad R_\omega = \{z \in \mathbb{C} : \operatorname{Re} z > \omega\},$$

the open left and right half-planes defined by the abscissa $\operatorname{Re} z = \omega$. An operator A on X is said to be *of half-plane type* $\omega \in (-\infty, \infty]$, $A \in HP(\omega)$, if $\sigma(A) \subseteq \overline{R_\omega}$ and, for every $\alpha < \omega$,

$$M_\alpha(A) = \sup\{\|R(z, A)\| : \operatorname{Re} z \leqslant \alpha\} < \infty.$$

An operator A is said to be *of half-plane type* if it is of half-plane type for some $\omega \in (-\infty, \infty]$. Let $\omega_1, \omega_2 \in (-1, \infty)$. Let

$$\psi(z_1, z_2) = (1 + z_1)(1 + z_2), \quad z_i \in R_{\omega_i},$$

and let

$$H_0^\infty(R_{\omega_1} \times R_{\omega_2}) = \{f \in Hol(R_{\omega_1} \times R_{\omega_2}) : \psi^s f \in H^\infty(R_{\omega_1} \times R_{\omega_2}) \text{ for some } s > 1\}.$$

Then $H_0^\infty(R_{\omega_1} \times R_{\omega_2})$ is an algebra.

By applying [8, Lemma 2.2] separately to each variable we obtain

Proposition 6.7.1. *Let $f \in H_0^\infty(R_{\omega_1} \times R_{\omega_2})$, and let $\omega_i < \delta_i$, $i = 1, 2$. Then*

$$f(a_1, a_2) = -\frac{1}{4\pi^2} \int_{\partial R_{\delta_1}} \int_{\partial R_{\delta_2}} \frac{f(z_1, z_2)}{(z_1 - a_1)(z_2 - a_2)} \, dz_1 \, dz_2, \quad (\delta_1 < \operatorname{Re} a_1, \delta_2 < \operatorname{Re} a_2).$$

Now, let A_i, for $i = 1, 2$, be resolvent commuting operators of half-plane type, with $\omega_i < \delta_i < s_0(A_i)$, where

$$s_0(A) = \max\{\omega : A \in \mathrm{HP}(\omega)\} = \sup\{\alpha : \sup_{\operatorname{Re} z \leqslant \alpha} \|R(z, A)\| < \infty\}.$$

For $f \in H_0^\infty(R_{\omega_1} \times R_{\omega_2})$ we let

$$f(A_1, A_2) = -\frac{1}{4\pi^2} \int_{\partial R_{\delta_1}} \int_{\partial R_{\delta_2}} f(z_1, z_2) R(z_1, A_1) R(z_2, A_2) \, dz_1 \, dz_2,$$

where $\omega_i < \delta_i$, for $i = 1, 2$. It is easy to see that the so-defined mapping

$$\Psi^*_{A_1, A_2} : H_0^\infty(R_{\omega_1} \times R_{\omega_2}) \to L(X)$$

is a homomorphism of algebras (cf. [8, Proposition 2.3]). For example, to establish multiplicativity, one can reason directly as in [12, VII.4.7]. Alternatively, supposing that $\delta_i < \delta_i' < s_0(A_i)$,

$f(A_1,A_2)g(A_1,A_2) =$

$$\frac{1}{16\pi^4}\int_{\partial R_{\delta_2}}\int_{\partial R_{\delta_1}}\int_{\partial R_{\delta_2'}}\int_{\partial R_{\delta_1'}} f(z_1,z_2)g(\xi_1,\xi_2)R(z_1,A_1)R(z_2,A_2)R(\xi_1,A_1)R(\xi_2,A_2)\,dz_1\,dz_2\,d\xi_1\,d\xi_2$$

$$= \frac{1}{8\pi^3 i}\int_{\partial R_{\delta_2}}\int_{\partial R_{\delta_2'}}\int_{\partial R_{\delta_1'}} f(z_1,z_2)g(z_1,\xi_2)R(z_1,A_1)R(z_2,A_2)R(\xi_2,A_2)dz_1\,dz_2\,d\xi_2$$

$$= -\frac{1}{4\pi^2}\int_{\partial R_{\delta_1'}}\int_{\partial R_{\delta_2'}} f(z_1,z_2)g(z_1,z_2)R(z_1,A_1)R(z_2,A_2)\,dz_1\,dz_2,$$

by repeated application of [8, Proposition 2.3(a)].

Now suppose that $f \in Hol(R_{\omega_1}\times R_{\omega_2})$ and $\psi^{-n}f \in H^\infty(R_{\omega_1}\times R_{\omega_2})$ for some $n \geqslant 0$. In this case, $\psi^{-(n+2)}f \in H_0^\infty(R_{\omega_1}\times R_{\omega_2})$ and one defines

$$f(A_1,A_2) = (I + A_1)^{n+2}(I + A_2)^{n+2}(\psi^{-(n+2)}f)(A_1,A_2)$$

with the appropriate domain. Standard arguments show that this operator is closed and independent of the choice of n.

In the remainder of the section we will prove that $\mathcal{B}^2$-calculus and the half-plane holomorphic calculus are compatible. We shall need the following auxiliary lemma, which is a two-dimensional analogue of [8, Theorem 3.1].

Lemma 6.7.2. *Let A_1, A_2 be densely defined resolvent commuting operators of half-plane type on X, and let $\omega_i < s_0(A_i)$ for $i = 1, 2$. Consider a net of functions*

$$(f_\iota)_{\iota\in I} \subseteq H^\infty(R_{\omega_1}\times R_{\omega_2}),$$

indexed by I, with the following properties:

1) $\sup_\iota \|f_\iota\|_{H^\infty(R_{\omega_1}\times R_{\omega_2})} < \infty$;

2) $f_\iota(A_1,A_2) \in L(X)$ *for all* ι, *and* $\sup_\iota \|f_\iota(A_1,A_2)\| < \infty$;

3) $f(z_1,z_2) = \lim_\iota f_\iota(z_1,z_2)$ *exists in* $\mathbb{C}$ *for all* $z_1 \in R_{\omega_1}$ *and* $z_2 \in R_{\omega_2}$.

Then $f \in H^\infty(R_{\omega_1}\times R_{\omega_2})$, $f(A_1,A_2) \in L(X)$, $f_\iota(A_1,A_2)x \to f(A_1,A_2)x$ for each $x \in X$, and $\|f(A_1,A_2)\| \leqslant \limsup_\iota \|f_\iota(A_1,A_2)\|$.

Proof. By Vitali's convergence theorem, [39, Proposition 7], f is holomorphic and the convergence $f_\iota \to f$ is uniform on compact subsets of $R_{\omega_1}\times R_{\omega_2}$. Condition 1) then implies that f is bounded. Let $\omega_i < \delta_i < s_0(A_i)$, for $i = 1, 2$. We then have (cf. the proof of [8, Theorem 3.1])

$$(f_\iota(z_1, z_2)(\mu_1 - z_1)^{-2}(\mu_2 - z_2)^{-2})(A_1, A_2) =$$

$$\lim_{n\to\infty} \frac{1}{4\pi^2} \int_{-n}^{n} \int_{-n}^{n} \frac{f_\iota(\delta_1 + is_1, \delta_2 + is_2)}{(\mu_1 - \delta_1 - is_1)(\mu_2 - \delta_2 - is_2)} R(\delta_1 + is_1, A_1) R(\delta_2 + is_2, A_2)\, ds_1\, ds_2,$$

where the limit is uniform in ι. Since the integrand converges uniformly on $[-n, n] \times [-n, n]$, this implies

$$(f_\iota(z_1, z_2)(\mu_1 - z_1)^{-2}(\mu_2 - z_2)^{-2})(A_1, A_2) \to (f(z_1, z_2)(\mu_1 - z_1)^{-2}(\mu_2 - z_2)^{-2})(A_1, A_2)$$

in norm. Hence, for $x \in D(A_1^2 A_2^2)$ (cf. Definition 2.1.12),

$$f_\iota(A_1, A_2)x = \left(\frac{f_\iota(z_1, z_2)}{(\mu_1 - z_1)^2(\mu_2 - z_2)^2} \right)(A_1, A_2)(\mu_1 - A_1)^2(\mu_2 - A_2)^2 x \to$$

$$\left(\frac{f(z_1, z_2)}{(\mu_1 - z_1)^2(\mu_2 - z_2)^2} \right)(A_1, A_2)(\mu_1 - A_1)^2(\mu_2 - A_2)^2 x = f(A_1, A_2)x.$$

By an argument as in the proof of Proposition 6.1.5, $D(A_1^2 A_2^2)$ is dense in X. Clearly, $\|f(A_1, A_2)x\| \leqslant \limsup_\iota \|f_\iota(A_1, A_2)x\| \|x\|$. Hence, $f(A_1, A_2)$ is bounded with $\|f(A_1, A_2)\| \leqslant \limsup_\iota \|f_\iota(A_1, A_2)\|$, and since $D(f(A_1, A_2))$ is dense, $f_\iota(A_1, A_2) \to f(A_1, A_2)$ strongly.

□

Proposition 6.7.3. *Let $f \in \mathcal{B}^2$, and assume that either of the following conditions holds:*

(C1) For some $\varepsilon > 0$, $\sigma(A_i) \subset R_\varepsilon$ and $\|R(z, A_i)\|$ is bounded for $z \in R_{-\varepsilon}$, for $i = 1, 2$;

(C2) There exist $\varepsilon > 0$, $n \geqslant 2$, and a holomorphic extension of f to $R_{-\varepsilon} \times R_{-\varepsilon}$ such that $\psi^{-n} f \in H^\infty(R_{-\varepsilon} \times R_{-\varepsilon})$.

Assume further that A_1 and A_2 are commuting operators such that $\sigma(A_1) \cup \sigma(A_2) \subseteq \overline{\mathbb{C}}_+$, and they satisfy both (6.1.1) *and* (6.1.2)*.*

Then

$$\Phi_{A_1,A_2}(f) = \Psi^*_{A_1,A_2}(f),$$

whenever $\Phi_{A_1,A_2}(f)$ is defined as a closed operator.

Proof. Assume first that (C1) holds.

a) Assume further that $|f(z_1, z_2)| \leqslant C(1+|z_1|)^{-2}(1+|z_2|)^{-2}$ for $z_1, z_2 \in \mathbb{C}_+$. Consider

$$\Psi^*_{A_1,A_2}(f) = -\frac{1}{4\pi^2}\int_{\partial R_\eta \times \partial R_\eta} (z_1 - A_1)^{-1}(z_2 - A_2)^{-1} f(z_1, z_2)\, dz_1\, dz_2,$$

with $0 < \eta < \varepsilon$. We obtain

$$\Psi^*_{A_1,A_2}(f) = \frac{-4}{\pi^2}\int_0^\infty \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} (\alpha - i\beta + A_1)^{-2}(\gamma - i\delta + A_2)^{-2}\alpha\gamma$$
$$\times D_1 D_2 f(\alpha + i\beta, \gamma + i\delta)\, d\beta\, d\delta\, d\alpha\, d\gamma = \Phi_{A_1,A_2}(f).$$

b) Assuming that A_1 and A_2 are both invertible, replace f by $g(z_1, z_2) = f(z_1, z_2) \cdot \frac{1}{(1+z_1)^2(1+z_2)^2}$. Then g decreases quadratically as $|z_i| \to \infty$, for $i = 1, 2$, and

$$(1 + A_1)^{-2}(1 + A_2)^{-2}\Psi^*_{A_1,A_2}(f) = \Psi^*_{A_1,A_2}(g) = \Phi(g) = (1 + A_1)^{-2}(1 + A_2)^{-2}\Phi(f).$$

c) Assume now that (C2) holds for some $\varepsilon > 0$. Assume initially that $f \in H^\infty(R^2_{-\varepsilon})$ and let $\eta \in (0, \varepsilon)$. Then the operators $A_1 + \eta$, $A_2 + \eta$ satisfy (C1) with ε replaced by η. We obtain from b) that

$$\Psi^*_{A_1+\eta,A_2+\eta}(f) = \Phi_{A_1+\eta,A_2+\eta}(f).$$

From Corollary 5.4.2 we get that $\Phi_{A_1+\eta,A_2+\eta}(f) = \Phi_{A_1,A_2}(f_\eta)$. Also, $\Psi^*_{A_1+\eta,A_2+\eta}(f) = \Psi^*_{A_1,A_2}(f_\eta)$ by applying the argument in the proof of [4, Proposition 4.10] separately to each variable for $f \in H^\infty_0(R^2_{-\varepsilon})$ and then extending to more general functions. Let $\eta \to 0+$. By Lemma 6.6.1, $\Phi_{A_1,A_2}(f_\eta) \to \Phi_{A_1,A_2}(f)$ in the operator norm. By Lemma 6.7.2, we have that $\Psi^*_{A_1,A_2}(f_\eta) \to \Psi^*_{A_1,A_2}(f)$ in the strong operator topology.
Finally, if (C2) holds and f is polynomially bounded in $R^2_{-\varepsilon}$, then we can apply the above case to $f(z_1, z_2)$ replaced by $\frac{f(z_1,z_2)}{(1+z_1)^{n_1}(1+z_2)^{n_2}}$ for sufficiently large n_1, n_2, and the result follows.

□

7 Epilogue

7.1 The class $\mathcal{E}^2$

As discussed in [4], there is an interesting relation linking the classes $\mathcal{B}^1$ and $\mathcal{E}^1$; namely, we have (cf. [4, Proposition 2.15]) that if $g \in \mathcal{E}^1$ and $g_a(z) = g(a+z), z \in \mathbb{C}_+, a > 0$, then $g_a \in \mathcal{B}^1$ for each $a > 0$. A direct analogue of this result fails for $\mathcal{B}^2$; since we do not impose any explicit restrictions on the first-order derivatives in our definition of $\mathcal{E}^2$, horizontally translated functions in $\mathcal{E}^2$ may not in general belong to $\mathcal{B}^2$. One may nonetheless ask whether a result in the vein of [4, Proposition 2.15] can be obtained by restricting attention to a certain subclass of $\mathcal{E}^2$. We will now discuss our attempt at obtaining such result.

For fixed integer d, we denote by $H^1(\mathbb{C}_+^d)$ the space of analytic functions $f : \mathbb{C}_+^d \to \mathbb{C}_+$ such that

$$\|f\|_{H^1} := \sup_{x>0} \int_{\mathbb{R}^d} |f(x+iy)|\, dy < \infty.$$

The notation $x > 0$ where $x \in \mathbb{R}^d$ means that $x_j > 0$ for each $j \in \{1, \ldots, d\}$.

Let $\mathcal{E}_+^2$ be the space of all functions g holomorphic on $\mathbb{C}_+^2$ such that

$$\begin{aligned} \|g\|_{\mathcal{E}_+^2} = \sup_{x_1,x_2>0} &\int_0^\infty \int_0^\infty |D_1D_2 g(x_1+iy_1, x_2+iy_2)|\, dy_1\, dy_2 \\ &+ \int_0^\infty \sup_{\substack{x_1>0 \\ z_2\in\mathbb{C}_+}} |D_1 g(x_1+iy_1, z_2)|\, dy_1 \\ &+ \int_0^\infty \sup_{\substack{z_1\in\mathbb{C}_+ \\ x_2>0}} |D_2 g(z_1, x_2+iy_2)|\, dy_2 < \infty. \end{aligned} \tag{7.1.1}$$

If $g \in \mathcal{E}_+^2$, the condition expressed in (7.1.1) implies that, for each $z_1, z_2 \in \mathbb{C}_+$ and $a, b > 0$, we have

$$D_1 g(\cdot + a, z_2), D_2 g(z_1, \cdot + b) \in H^1(\mathbb{C}_+),$$

$$D_1 D_2 g(\cdot + a, \cdot + b) \in H^1(\mathbb{C}_+^2).$$

Consequently, by [4, Proposition 2.4], $D_1 g(\cdot + a, z_2), D_2 g(z_1, \cdot + b) \in \mathcal{B}^1$. The following conjecture concerning functions in the class $H^1(\mathbb{C}_+^2)$ is far from new, but no proof or counterexample is known to us.

Conjecture 7.1.1. *If $f \in H^1(\mathbb{C}_+^2)$, then*

$$\int_0^\infty \int_0^\infty \frac{|\mathcal{F}_{(2)}^{-1} f(x_1, x_2)|}{x_1 x_2} \, dx_1 \, dx_2 < \infty. \tag{7.1.2}$$

Under the supposition that Conjecture 7.1.1 holds we can establish that horizontal translations of functions in $\mathcal{E}^2$ are in our class $\mathcal{B}^2$.

Proposition 7.1.2. *Let $g \in \mathcal{E}_+^2$, and let $g_{a,b}(z_2, z_2) = g(z_1 + a, z_2 + b)$, $z_1, z_2 \in \mathbb{C}_+$, $a, b > 0$. Suppose that Conjecture 7.1.1 holds. Then for each $a, b > 0$, $g_{a,b} \in \mathcal{B}^2$.*

Proof. We will first show that

$$\int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 g_{a,b}(x_1 + iy_1, x_2 + iy_2)| \, dx_1 \, dx_2 < \infty. \tag{7.1.3}$$

Note that, by the preceding remarks, $D_1 D_2 g_{a,b} \in H^1(\mathbb{C}_+^2)$, and so by [33, Theorem 2.1] its non-tangential boundary function, $h(t_1, t_2)$, exists as a function in $L^1(\mathbb{R}^2)$, and $\operatorname{supp} \mathcal{F}_{(2)}^{-1} h \subset \mathbb{R}_+^2$; we also have

$$D_1 D_2 g_{a,b}(z_1, z_2) = \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty e^{-(z_1 \xi_1 + z_2 \xi_2)} \mathcal{F}_{(2)}^{-1} h(\xi_1, \xi_2) \; d\xi_1 \, d\xi_2.$$

We can now estimate

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |D_1 D_2 g_{a,b}(x_1 + iy_1, x_2 + iy_2)| \, dx_1 \, dx_2 \\
&\quad \leqslant \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} |e^{-(z_1 \xi_1 + z_2 \xi_2)} \mathcal{F}_{(2)}^{-1} h(\xi_1, \xi_2)| \, d\xi_1 \, d\xi_2 \, dx_1 \, dx_2 \\
&\quad = \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(x_1 \xi_1 + z_2 \xi_2)} |\mathcal{F}_{(2)}^{-1} h(\xi_1, \xi_2)| \, dx_1 \, dx_2 \, d\xi_1 \, d\xi_2 \\
&\quad = \frac{1}{4\pi^2} \int_0^\infty \int_0^\infty \frac{|\mathcal{F}_{(2)}^{-1} h(\xi_1, \xi_2)|}{\xi_1 \xi_2} \, d\xi_1 \, d\xi_2 < \infty,
\end{aligned}$$

where the last inequality follows from (7.1.2).

Finally, since $D_1 g(\cdot + a, z_2), D_2 g(z_1, \cdot + b) \in \mathcal{B}^1$ for any $z_1, z_2 \in \mathbb{C}_+$, then by the equivalence of norms on $\mathcal{B}^2$ and the above,

$$
\begin{aligned}
\|g_{a,b}\|_{\mathcal{B}^2} = & \int_0^\infty \int_0^\infty \sup_{y_1, y_2 \in \mathbb{R}} D_1 D_2 g_{a,b}(x_1 + iy_1, x_2 + iy_2)\, dx_1\, dx_2 \\
& + \int_0^\infty \sup_{\substack{z_2 \in \mathbb{C}_+ \\ y_1 \in \mathbb{R}}} |D_1 g_{a,b}(x_1 + iy_1, z_2)|\, dx_1 \\
& + \int_0^\infty \sup_{\substack{z_1 \in \mathbb{C}_+ \\ y_2 \in \mathbb{R}}} |D_2 g_{a,b}(z_1, x_2 + iy_2)|\, dx_2 \\
& \leqslant C \sup_{z_1, z_2 \in \mathbb{C}_+} \left(\|D_1 D_2 g_{a,b}\|_{H^1_A(\mathbb{C}^2_+)} + \|D_1 g_{a,b}(\cdot, z_2)\|_{\mathcal{B}^1} + \|D_2 g_{a,b}(z_1, \cdot)\|_{\mathcal{B}^1} \right),
\end{aligned}
$$

for some $C > 0$. □

Remark 7.1.3. Assuming that Conjecture 7.1.1 holds, Proposition 7.1.2 and Proposition 3.2.2 jointly imply as an immediate corollary that if $g \in \mathcal{E}^2_+$, then the limit $g(\infty, \infty) := \lim_{\operatorname{Re} z_1, \operatorname{Re} z_2 \to \infty} g(z_1, z_2)$ exists.

Define a norm on $\mathcal{E}^2_+$ by setting

$$
|||g|||_{\mathcal{E}^2_+} := |g(\infty, \infty)| + \|g\|_{\mathcal{E}^2_0}, \quad g \in \mathcal{E}^2.
$$

If Conjecture 7.1.1 is true, then we can show that $\mathcal{E}^2_+$ equipped with the norm $|||\cdot|||_{\mathcal{E}^2_+}$ becomes a Banach space. To see this, let $(g_n)_{n\in\mathbb{N}} \subset \mathcal{E}^2_+$ be a Cauchy sequence with respect to $|||\cdot|||_{\mathcal{E}^2_+}$. Then $(g_n(\infty, \infty))_{n\in\mathbb{N}}$ is convergent in $\mathbb{C}$. Let $a > 0$, and $f_n(z_1, z_2) := g_n(a + z_1, a + z_2) - g_n(\infty, \infty)$. By Proposition 7.1.2, (f_n) is Cauchy in $\mathcal{B}^2_0$, and hence in $H^\infty(\mathbb{C}^2_+)$. Thus the limit $f(z_1, z_2) = \lim_{n\to\infty} f_n(z_1, z_2)$ for every $z_1, z_2 \in \mathbb{C}_+$, and f is holomorphic. Proceeding essentially as in the proof of Proposition 3.2.1, we have $|||f|||_{\mathcal{E}^2_+} \leqslant \liminf_{n\to\infty} |||f_n|||_{\mathcal{E}^2_+}$, and hence $|||f - f_n|||_{\mathcal{E}^2_+} \leqslant \sup_{m\geqslant n} |||f_m - f_n|||_{\mathcal{E}^2_+}$.

7.2 The case of n commuting operators

We have not so far considered the question whether the methods employed in the present work may be extended to cover the case of n commuting operators, where $n > 2$. We believe that this may not be an entirely straightforward task. When attempting to construct a functional calculus for n commuting operators along the lines of the present work, one faces the challenge of choosing an appropriate class of functions for which the calculus is to be defined. It is not obvious to us that the approach we rely on extends readily even to the case $n = 3$. Our considerations in Chapter 3, where we define the class $\mathcal{B}^2$, involve establishing a number of results concerning the limiting functions and equivalences of norms. Those results, however, seem to us to be rather specific to the two variable case and may not admit immediate generalisations to functions of more than two variables. The same is true but even more evident e.g. for Lemma 4.1.1 and Theorem 5.6.1, the proofs of which rely on estimates specific to our setting.

It is plausible that the situation is analogous to that of von Neumann's inequality which states that if T is a contraction on a Hilbert space and p is a polynomial in one variable with complex coefficients, then

$$\|p(T)\| \leqslant \sup_{|z| \leqslant 1} |p(z)|. \tag{7.2.1}$$

The inequality (7.2.1) is essentially equivalent to the fact that a sectorial operator $A \in \mathrm{Sect}(\pi/2)$ is m-accretive if and only if A has a bounded H^∞-calculus on $S_{\pi/2}$ with constant 1 (cf. [25, Theorem 7.1.7]). Ando [1] proved that (7.2.1) extends to the case of two commuting contractions, i.e. if S and T are commuting contractions on a Hilbert space and p is a polynomial in two variables with complex coefficients, we have

$$\|p(T, S)\| \leqslant \sup_{|z_1|, |z_2| \leqslant 1} |p(z_1, z_2)|. \tag{7.2.2}$$

However, the analogue of this for three commuting contractions is not true in general (cf. [14] and discussion therein).

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