

Models of systemic risk in financial markets



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Abstract

This thesis studies systemic risk in financial markets and how it emerges through dynamical and structural amplification mechanisms.

In part (1) I study the dynamics and control of Basel leverage cycles. For this I develop a simple model of a financial system consisting of leveraged banks and an unleveraged fundamentalist investor (fund). Banks trade a risky asset with the fund and rely on historical information to estimate their portfolio risk. This risk estimate determines the banks' leverage limit. I show that these simple ingredients can lead to endogenous, irregular oscillations, which I call Basel leverage cycles.

I then proceed to evaluate alternative regulatory capital requirements based on their impact on endogenous risk. I find that in the microprudential limit, when the bank is small and exogenous volatility is high, the optimal policy is simply given by a Value-at-Risk constraint. However, when the bank is large, the optimal policy is constant leverage.

In part (2) I study contagion in financial networks for two examples. First, I study how intra-institutional linkages can amplify financial contagion when financial institutions are active in multiple over-the-counter markets. In particular, spillover within a diversified financial institution allows for contagion from one over-the-counter market to another. Using recent methods for coupled networks I illustrate that under certain circumstances, the presence of intra-institutional spillover can lead to the amplification of small shocks to the extent that trading across all markets collapses abruptly.

Finally, I develop a simple model of social learning in the context of a financial network. I study how banks' portfolio decisions can synchronize if banks rely both on outside information and information from their social network to compute the expected payoff of an investment opportunity. In the same model, I propose a simple boundedly rational decision mechanism for endogenous network formation based on the information content of a bank's neighbors' decisions.

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Chapter 1

Introduction

Systemic risk in financial markets can be defined as risk that emerges through the actions and interactions of the system's components. As such, systemic risk is endogenous by definition. It is this understanding of systemic risk as an emergent phenomenon that I will adopt throughout this thesis.¹ This understanding of systemic risk has important implications for financial stability. Due to its endogenous nature systemic risk can arise through the amplification of relatively small exogenous shocks. A case in point is the financial crisis of 2007/2008. As Adrian and Shin (2008) show, the losses in the US subprime mortgage market during the financial crisis were on the same order of magnitude of a 1 percent loss in the US stock market. Such losses in the stock market are relatively likely – so one wonders why certain small shocks have no system wide implications while others can lead to widespread disruption in the financial system. In order to tackle this question one must invariably understand the endogenous component of systemic risk. Ignoring the potential of the financial system to amplify shocks, can lead to an underestimate of the likelihood of financial crises, highly inaccurate stress tests and ultimately poor policy responses.

Broadly speaking, I believe that in order to understand systemic amplification effects one has to understand the *dynamics* and the *structure* that constrains the dynamics in a financial system. This, first, involves building dynamic models of how

¹However, this definition of systemic risk is not unique and differs from other definitions that focus on the extent rather than the origin of risk, see for example GroupOfTen (2001)

agents learn about their environment, such as asset markets, and how they act, for example through portfolio adjustments, on the information they have learned. Here the emphasis is on “dynamic” – dynamic models allow us to capture feedback effects between agents, their actions and the environment. As mentioned above, these feedback effects can ultimately cause systemic risk by amplifying small fluctuations. Second, one has to understand the structure that constrains the interactions of agents. In a financial system these interaction constraints can often be modeled by networks; these networks can represent connections through commonly held assets, credit exposures or information pathways. The structure of these networks of interaction is important as they determine to a large extent how stress in one part of the financial system propagates to other parts.

In this thesis I make an attempt at contributing to our understanding of the aforementioned dynamic and structural aspects of systemic risk in financial markets. First, I investigate the impact of bank risk management on endogenous risk and analyze how different risk management policies might help to mitigate systemic risk. Second, I study how the structure of financial networks affects both the propagation of stress and information in the financial system. More specifically, my thesis consists of two main parts: (i) the dynamics and control of the leverage cycle and (ii) contagion in financial networks. In the following I will discuss each part in more detail.

In chapter 2 I present a simple agent-based model of a financial system composed of leveraged investors (banks) and an unleveraged investor (fund). All agents hold a portfolio of risky and risk free assets. The fund’s portfolio decision is based on a fundamentalist strategy that is perturbed by an exogenous noise process. Banks manage their risk using a Value-at-Risk constraint and compute their level of perceived risk based on historical observations of asset prices. The Value-at-Risk constraint implies that when perceived risk is low, leverage is high and vice versa, a phenomenon that

has been dubbed *pro-cyclical leverage*. I complement the agent-based model with a simplified model that allows the derivation of analytical results and gives a deeper understanding of the dynamics and the nature of the feedback loops and instabilities present in the model. I show numerically and analytically that pro-cyclical leverage management as prescribed in Basel II or III can lead to the endogenous generation of recurring bubbles and crashes in asset prices. These dynamics are driven by the banks' historical risk estimation and leverage adjustment. I refer to this phenomenon as the Basel leverage cycle.

In chapter 3 I further investigate leverage cycles in a dynamical model that is more sophisticated than the simple model developed in chapter 2. In particular, this model allows for time varying, clustered exogenous volatility. I achieve this by making the noise process that perturbs the fundamentalist's investment decision a GARCH process. Thus, in the limit when the bank is small and has no price impact, the returns of the risky assets are heavy tailed. My analysis shows that the emergence of leverage cycles crucially depends on the size of the bank relative to the fund. If the bank is very small its balance sheet adjustments will have no market impact breaking the feedback loop between perceived risk, balance sheet adjustments and asset prices. Second, the amplitude of leverage cycles, i.e. the extent of price crashes, increases as the banks' riskiness increases.² By increasing this parameter sufficiently (and thereby increasing leverage targets) it is always possible to globally destabilize the system. I can show analytically that, in the deterministic case, the system dynamics bifurcate as the bank's leverage evaluated at the system's fixed point is increased. In particular the dynamics change discontinuously from a stable fixed point to locally unstable and chaotic dynamics before destabilizing globally.

I then propose a criterion for rating macroprudential policies based on their ability

²In the model this corresponds to banks increasing their Value-at-Risk by increasing the VaR quantile.

to minimize risk for a given average leverage. I construct a one parameter family of leverage policies that allows me to vary from the procyclical policies of Basel II or III, in which leverage decreases when volatility increases, to countercyclical policies in which leverage increases when volatility increases. I find the best policy depends critically on two parameters: The relative size of the bank to the fund and the degree of clustering of the exogenous volatility. Basel II is optimal when exogenous volatility is clustered and the bank is small; in the opposite limit where the bank is large the optimal policy is close to constant leverage. This concludes the first part of my thesis.

As mentioned above, in the second part of my thesis I study how the network structure of the financial system affects contagion – both of stress and information – through the system. In chapter 4 I develop a model of the financial system in which intermediaries (banks for short) consist of business units specialized in trading a particular asset. Assets are intermediated from sellers to buyers via exogenously fixed trading networks. Trading networks are random networks defined by a degree distribution. The novelty in this model is that I allow for intra-institutional spillovers. The failure of one business unit exerts an externality on other business units of the same bank. I model this externality by assuming a probability of failure of the entire bank conditional on the failure of one of its business units. This *couples* trading networks for different assets. I then study the resilience of such a system to exogenous random shocks using recent analytical methods for the study of percolation on interdependent networks. I define resilience as the size of the intermediation network after a given exogenous shock. Under an exogenous shock of a given size a fraction of business units trading a particular asset randomly fail.

When there is only one type of asset the transition from a regime in which all banks intermediate to a regime in which the fraction of banks that intermediate vanishes, is continuous in the size of the exogenous shock. However, when there are

multiple types of assets, under certain conditions intermediation breaks down not only at smaller shock sizes, it happens *abruptly*. This illustrates how the interaction between contagion processes on different networks can significantly amplify the overall contagion process. In numerical simulations, I relax a number of the assumptions made about the structure of intermediation networks and the strength of the intra-institutional spillover, i.e. the spillover probability. The extent of the amplification depends heavily on the spillover probability, where a lower spillover probability reduces the amplification effect. Furthermore, similarity between the trading networks for different assets, as modeled by degree correlations and link overlap, also reduces the extent of the amplification effect.

Finally, in chapter 5 I study how learning on a social network can lead to contagious synchronization of investment decisions in a financial network. I model a simple financial system in which banks decide on a binary investment based on a private belief about the state of the world and a social belief formed from observing the investment decisions of their peers in an exogenous financial network. Observing a larger group of peers can convey more information; therefore we allow for the weight of the social belief to depend on the size of the bank's neighborhood in the network. Extending the standard model of Bayesian updating in social networks, I show numerically that the probability that banks synchronize their investment strategy on a state non-matching action critically depends on the weighting between private and social belief and the initial distribution of actions in the system. Furthermore, I show analytically that for the complete network, there always exists a non zero probability of synchronization on a state non-matching action. I extend the model to allow for endogenous network formation under which banks choose their neighbors in order to maximize their expected utility gain from an additional link. Under endogenous network formation, less informed banks are capable of actively making connections to

more informed banks, leading to a more effective diffusion of information about the state of the world. This ultimately reduces the likelihood that the system synchronizes on a state non matching action.

Part I

Dynamics and control of the leverage cycle

Chapter 2

The dynamics of the Basel leverage cycle

2.1 Introduction

Borrowing is essential to economic activity, but leverage is inextricably linked to risk. On a systemic level, the collective leveraging and deleveraging of financial institutions can lead to booms and busts in asset markets. The recent financial crisis is a case in point for the systemic consequences of the use of leverage.

Wide-spread deleveraging typically occurs when leveraged investors hit a constraint on their leverage. Such a constraint may arise in a number of ways. If the investor is using collateralized loans to fund its investments, it must maintain margin on its collateral. Alternatively, a regulator may impose a risk contingent capital adequacy ratio. Finally, internal risk management considerations may lead the investor to adopt a Value-at-Risk¹ (VaR) constraint. All of these cases effectively impose a risk contingent leverage constraint.

Now, suppose there is a negative shock in the asset market associated with an increase in volatility, and as a result the leverage constraint tightens and investors are forced to sell part of their assets.² As investors sell into falling markets they

A large part of the work presented in this chapter has been published in Aymanns and Farmer (2015), DOI: <http://dx.doi.org/10.1016/j.jedc.2014.09.015>.

¹In simple terms Value-at-Risk is a measure of how much the bank could lose with a given small probability.

²In principle banks can react in two ways to an increase in market risk: they can raise more

cause prices to fall further. This is the start of a simple positive feedback loop in which selling causes a depression in prices which causes further selling. In a similar way positive news about prices can lead to a decline in perceived risk. This leads to increased leverage which leads to further price increases. This dynamic is referred to as a leverage cycle, see e.g. Geanakoplos (2003) and Gennotte and Leland (1990). A similar dynamic has also been studied in Brunnermeier and Pedersen (2008), where the authors investigate the destabilizing feedback between funding liquidity and market liquidity. A further discussion on the destabilizing effects of margin can be found in Gorton and Metrick (2010).

It is widely believed that an important driver of leverage cycles lies in the risk management of leveraged investors,³ see for example Adrian and Shin (2008), Shin (2010) and Danielsson et al. (2001). Further supporting evidence for the impact of risk management on the leverage cycle can be found in Tasca and Battiston (2012), Adrian and Shin (2014) and Adrian et al. (2012). However, less is known about the impact of the parameters of risk management on the dynamical properties of leverage cycles. In order to improve our understanding of the anatomy of leverage cycles, we develop a dynamic computational model of leveraged investors (which we will call banks from now on) that invest in an asset market (which for convenience we call the stock market). Banks have a target leverage that depends on the banks' perceived risk. In this way, we are able to capture pro-cyclical leverage policies, which correspond to banks having a Value-at-Risk constraint. Banks are boundedly rational and rely on historical data to estimate the risk of their portfolio. Banks then adjust their target leverage based on changes in perceived risk.

capital or sell assets. In practice many banks tended to do the latter, see Adrian and Shin (2008). This is simply because selling assets can be much easier and faster than raising equity, in particular in times of increasing market risk.

³A number of additional drivers of leverage cycles are discussed in the literature. In particular short-termism, herding in financial markets and incentive distortions can play a role in the development of the leverage cycle, see for example Aikman et al. (2012), de Nicolo et al. (2012) and Gennaioli et al. (2012).

Our model differs from existing models of leverage cycles and pro-cyclical leverage in a number of ways. We develop a fully *dynamic* model of endogenous leverage cycles. This differs from Geanakoplos (2003) and Geanakoplos (2010) who show the existence of leverage cycles in a two period general equilibrium model. Our model shows how a leverage cycle can be a sustained endogenous phenomenon where each cycle sows the seeds for the subsequent leverage cycle. We refer to this phenomenon as the Basel II leverage cycle. While our model set-up is similar to Danielsson et al. (2004), we show that the endogenous dynamics induced by leverage management and historical risk estimation are richer than initially thought.

Banks in our model rely on historical data for the estimation of their portfolio risk, and are in this sense explicitly boundedly rational. This sets us apart from the model developed in Zigrand et al. (2010) who consider endogenous risk as an equilibrium concept in a financial market with rational VaR constrained investors. The fact that rational investors can correctly anticipate future volatility and correctly estimate portfolio adjustments needed to reach their leverage targets, means that they settle into a fixed point equilibrium. In contrast, as we show here, boundedly rational investors will under or overshoot, and the inherent instability of the leverage cycle induces oscillations. Thus while as shown in Geanakoplos (2010) bounded rationality is not essential for the existence of leverage cycles, the mismatch between actual market risk and perceived portfolio risk as well as the uncertainty about the extent of price impact have important behavioral consequences that increase the severity of the leverage cycle. The fact that pursuing a target leverage policy can be destabilizing in the case of imperfect knowledge about the extent of market impact has been originally pointed out in Caccioli et al. (2014) and underlies much of the analysis done in this chapter.

Another important point of comparison is the agent-based model of leveraged investors developed in Thurner et al. (2010) and Poledna et al. (2013), which is

also dynamic and boundedly rational. They studied leveraged value investors such as hedge funds that are subject to a leverage ceiling imposed by the lender. In their model the leverage actually used by investors varies dramatically based on investment opportunities and as a consequence the leverage ceiling is only occasionally reached. In this model the bank is a dummy agent with infinite capital whose only role is to provide credit to funds. In contrast, in the model developed here the banks are the key strategic agents. They always use full leverage, adjusting it as needed to match a regulatory target. Thus the model introduced here studies the underlying mechanism of the financial crisis of 2008, whereas the earlier model of Thurner et al. (2010) is more relevant to circumstances such as the near meltdown of Long Term Capital Management in 1998.

In contrast to most models in the agent-based literature, we also develop and study a simple reduced model of the full multi-agent financial model. Its simplicity makes it amenable to the tools of dynamical systems theory. This approach improves our understanding of why the system displays endogenous oscillations in the deterministic limit.

Given that volatility is persistent in time, reducing leverage when historical volatility is high is rational from the point of view of a solipsistic individual who is unconcerned about possible systemic effects. Thus a risk manager who does not think about the market impact of her institution, or that of other similar institutions, will naturally pursue a *pro-cyclical* leverage policy. By this we mean that such an investor will lower leverage when historical volatility is high. As we explicitly show here, from a dynamical systems point of view this is inherently destabilizing.

A regulator can potentially correct for this systemic risk by imposing a *countercyclical* policy, in which leverage is actually decreased when historical volatility is low. We will examine this option in Chapter 3.

It is worth commenting that while countercyclical policies and systemic risk are

discussed in Basel III, the usage of expected shortfall⁴ is also inherently pro-cyclical, for the same reason that VaR is inherently pro-cyclical (since it is also inversely related to perceived risk). Thus without an active policy to manage expected shortfall countercyclically, similar results to those found here should apply.

Before proceeding, we would like to mention that the multi-agent financial model presented here can be coupled to a macroeconomic model in order to study the spill-over effects of the leverage cycle from the financial sector to the real economy. In fact, an unpublished earlier version of this chapter studies exactly this phenomenon. In that paper, we couple the financial model developed here with a simple macroeconomy by allowing banks to give loans to firms in the real economy. Via this credit channel the financial leverage cycle affects the activity in the real economy. We observe that, provided that the banks' stock and loan portfolios are sufficiently linked, leverage cycles originating in the stock market spill over into the real economy causing cycles in credit provision and output. We have not included these results here because they are not essential for the main results and the macro model complicates the exposition.

The remainder of this chapter is organized as follows. In Section 2.2 we provide some anecdotal evidence of a leverage cycle preceding the financial crisis in 2008. In Section 2.3 we outline the full multi-asset, agent-based model of the financial system. In Section 2.4 we will study the dynamics of the full agent-based model. We will then develop a reduced model in Section 2.5 and conclude in Section 2.6.

2.2 Empirical motivation

The period leading up to the global financial crisis was marked by a consistent decline in market volatility coupled with an increase in asset prices. In analogy to the term coined by Ben Bernanke,⁵ we refer to this period as the great moderation. The

⁴ Expected shortfall is the loss above a given quantile of the return distribution (whereas VaR is the quantile itself).

⁵Bernanke's speech is available at:
<http://www.federalreserve.gov/BOARDDOCS/speechES/2004/20040220/default.htm>

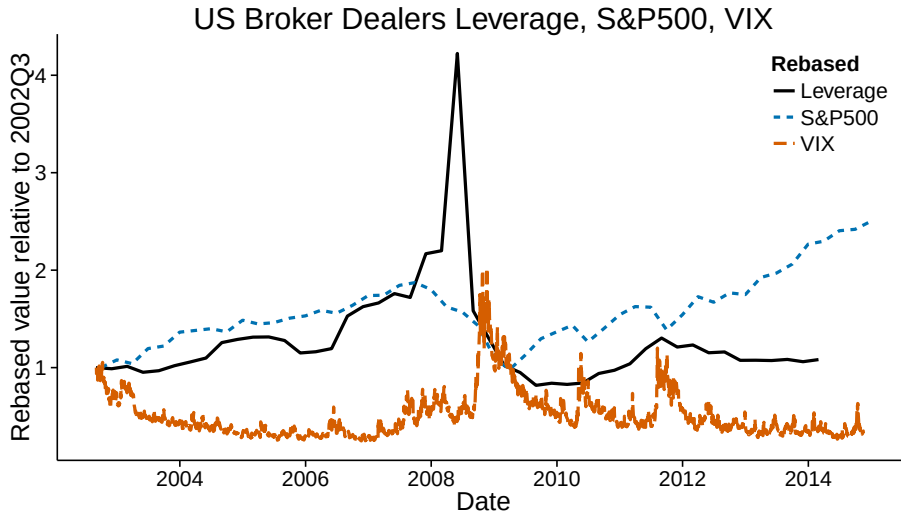


Figure 2.1: The leverage of US Broker-Dealers (solid black line) compared to the S&P500 index (dashed blue line) and the VIX S&P500 (red dash-dotted line). Leverage data is quarterly, S&P500 data is monthly and VIX data is daily; for further details see footnote 5.

period encompassing the great moderation and the subsequent global financial crisis starting in late 2007 is a case in point for the strong correlation between leverage, market volatility and asset prices. Detailed evidence over a longer time horizon for the link between asset prices and leverage for various types of financial institutions is provided by Adrian and Shin (2010). In this Section we focus only on the great moderation and the subsequent financial crisis.

During the great moderation perceived volatility, as measured by the VIX index of expected future volatility, declined consistently over several years, as shown by the dotted line in Figure 2.1. At the same time, in a near mirror image to volatility, asset prices (as measured by the S&P500 index) and leverage of financial institutions (as measured by the leverage of US security broker dealers) consistently increased.⁶ As financial institutions expanded their leverage their assets and liabilities grew corre-

⁶ It should be noted that US security broker dealers are a somewhat extreme example of leveraged financial institutions and are not representative for the behavior of commercial banks. Here we use their example to illustrate the stark correlation between leverage, volatility and asset prices in an anecdotal way. A more nuanced evaluation can be found in Adrian and Shin (2010). The data on US Security Broker Dealer Leverage, (defined as $\text{Assets}/(\text{Assets}-\text{Liabilities})$), is from US Federal Reserve Flow of Funds Data Package F.128 available at <http://www.federalreserve.gov/datadownload/>.

spondingly.

The great moderation came to a sudden and dramatic halt when the US subprime mortgage crisis began to unfold in late 2007 and the subsequent financial crisis sent asset prices into a downward spiral. As asset prices began to fall, market volatility increased. At the same time the leverage of financial institutions increased – in fact drastically so. This was due the fact that even relatively small drops in asset prices can massively increase the leverage of financial institutions that are already highly leveraged. Financial institutions responded quickly to increased market volatility and deleveraged by a factor of 2 in the span of one quarter.

This deleveraging likely had a drastic negative impact on asset prices. Of course, many other factors affected asset prices at that time. Nonetheless, the correlation observed between asset prices, volatility and leverage motivates the model that is developed here.

2.3 Multi-asset model of a financial system

We propose a simple model of a financial system consisting of leveraged investors, a fund investor, an outside lender to the leveraged investor⁷ and a regulator. In the following we will refer to these investors as banks. Banks are fundamentalists and distribute their assets across a range of tradable securities that provide an exogenous, stochastic dividend stream. For simplicity we refer to these securities as stocks. We focus our analysis on the dynamics induced by the leverage management and portfolio allocation of the banks. Therefore, we abstract from the influence other financial institutions may have on the market dynamics and subsume their activity in the fund investor.

Crucially, banks are boundedly rational and have imperfect information about the stock market. Therefore, banks must learn about the stock market by analyzing past

⁷We will not explicitly model this lender and assume that it always provides funds to the bank if requested by the bank. Thereby we effectively abstract from liquidity risk for the bank.

observations of market behavior. In particular, in order to manage their risk, banks estimate the covariance matrix of their portfolio from historical price movements. Banks then adjust their leverage based on their perception of portfolio risk. Due to the historical estimation banks' perceived portfolio risk may be drastically different from their actual risk. As we will demonstrate below, this mismatch between perception and reality of market conditions plays an important role in the dynamics of the model. In the following Sections we will outline the behavior of the banks and the fund investor more formally.

2.3.1 Bank

Accounting

The financial system is comprised of a set of banks indexed by $j \in \{0, \dots, N_b\}$. For ease of exposition we will drop this index wherever there is no risk of confusion. Banks are characterized by their balance sheet, their investment strategy, and potential regulatory requirements. Banks hold a portfolio of two types of assets: cash and shares in stocks. Cash is a non-interest bearing risk-free asset, while stocks are risky assets. More formally, the asset side of the bank balance sheet is given by

$$\begin{aligned}\mathcal{A}_t &= c_t + \mathbf{n}_t^T \mathbf{p}_t, \\ \mathbf{n}_t &= (n_{1,t}, \dots, n_{N_f,t})^T, \\ \mathbf{p}_t &= (p_{1,t}, \dots, p_{N_f,t})^T,\end{aligned}$$

where c_t is the cash investment, $\mathbf{n}_t$ the vector of stock ownership, $\mathbf{p}_t$ is vector of share prices, N_f is the number stocks. Finally, in this context T denotes the transpose of a vector. Note that the total number of shares of a given stock is normalized to 1 such that

$$\sum_{j=1}^{N_b} n_{ji,t} = 1 \quad \forall i, t,$$

where $n_{ji,t}$ is the fraction of asset i owned by bank j at time t . Also note that banks face a long only constraint, i.e. $n_{ji,t} \geq 0$. This implies that banks cannot short

an asset which they consider overpriced. This limits the extent of arbitrage that is possible in this setting. Mis-pricing may therefore persist longer than in the case where short selling is permitted. By definition the equity of the bank is given by

$$\mathcal{E}_t = \mathcal{A}_t - \mathcal{L}_t,$$

where $\mathcal{L}_t$ generically represents all liabilities of the bank at time t . These liabilities could be composed of deposits, interbank loans or other forms of short term debt. We assume for simplicity that the banks do not face funding restrictions, i.e. should a bank decide to increase its liabilities it always finds willing lenders. This is of course a strong simplifying assumption. If the bank faces exogenous constraints on its funding its ability to increase its leverage will be impaired. In this model, our focus is on the impact of risk management on the stability of the system. Therefore we explicitly try to exclude exogenous constraints if possible. Finally, we define the bank's leverage as follows:

$$\lambda_t = \frac{\mathcal{A}_t}{\mathcal{E}_t}.$$

Following LeBaron (2012) and Hommes and Wagener (2009), the budget constraint of the bank is given by

$$\begin{aligned} \mathcal{E}_t &= c_t + \mathbf{n}_t^T \mathbf{p}_t - \mathcal{L}_t \\ &= c_{t-1} + \mathbf{n}_{t-1}^T (\mathbf{p}_t + \boldsymbol{\pi}_{t-1}) - \mathcal{L}_{t-1}(1 + r_{\mathcal{L},t-1}) - d_{t-1}, \end{aligned}$$

where $\boldsymbol{\pi}_{t-1}$ is the $N_f \times 1$ vector of stock dividends, $r_{\mathcal{L},t-1}$ is a generic interest rate that the bank has to pay on its debts. Finally d_{t-1} represents a dividend paid to shareholders. We refer to the expression $\mathcal{L}_{t-1}r_{\mathcal{L},t-1} + d_{t-1}$ as the banks' funding cost.

For convenience we assume that each bank's income through dividends is equal to its funding cost, i.e.: $\mathbf{n}_{t-1}^T \boldsymbol{\pi}_{t-1} = \mathcal{L}_{t-1}r_{\mathcal{L},t-1} + d_{t-1}$. This amounts to assuming that the banks immediately pay all their dividend income as dividends and interest to their shareholders and lenders. However, dividend payments exclude income through

valuation gains from stock trading. This is a strong assumption but simplifies the decision problem of the bank significantly and allows for variable bank equity while ignoring the market for bank debt for now. In Section 2.5 we will simplify this assumption further and assume that the equity of banks is fixed.

This assumption implies that equity only changes via changes in the prices of the stocks in which the bank holds long positions. In particular, this assumption yields the following simplified budget constraint:

$$\begin{aligned}\mathcal{E}_t &= c_t + \mathbf{n}_t^T \mathbf{p}_t - \mathcal{L}_t \\ &= c_{t-1} + \mathbf{n}_{t-1}^T \mathbf{p}_t - \mathcal{L}_{t-1}.\end{aligned}\tag{2.1}$$

Expectation formation

In order to form expectations about future conditions of the stock market, banks rely entirely on the analysis of historical market data. In particular, banks use observations of past dividends and prices to compute the expected dividend price ratio, stock price return and variance of stock prices.

Banks are fundamentalist investors and as such base their investment decision on the expected dividend price ratio of a stock. The expected dividend price ratio is computed as an exponential moving average of past dividend price ratios. In particular we have for the dividend price ratio of stock i :

$$\hat{r}_{i,t+1} = (1 - \gamma)\hat{r}_{i,t} + \gamma \frac{\Pi_{i,t}}{p_{i,t}},\tag{2.2}$$

where $\Pi_{i,t}$ is the dividend paid by stock i at time t and $p_{i,t}$ is the corresponding stock price.⁸ We typically choose $\gamma \sim 0.1$ to ensure that estimated dividend price ratios are relatively slow moving.

In a similar fashion, banks estimate the covariance matrix of the stock price returns. Banks rely on the covariance matrix of stock prices for its risk management

⁸We initialize the expected dividend price ratio as $\hat{r}_{i,0} = \frac{\Pi_{i,0}}{p_{i,0}}$ such that initial expectations are consistent with the initial values chosen for the price and dividend processes. The initial values for price and dividend are given ad hoc numerical values. This choice does not affect the model dynamics.

as will be discussed in Section 2.3.1. The method used here is comparable to the RiskMetrics approach described in Longerstaey (1996) and Andersen et al. (2006). Consider the log return of the price of stock i : $x_{i,t} = \log \left(\frac{p_{i,t}}{p_{i,t-1}} \right)$ which are the components of the vector $\mathbf{x}_t$. First, we estimate the conditional sample mean of the return vector by

$$\hat{\boldsymbol{\mu}}_t = \delta \mathbf{x}_{t-1} + (1 - \delta) \hat{\boldsymbol{\mu}}_{t-1}, \quad (2.3)$$

where $\delta < 1$ determines the horizon of this exponential moving average.⁹ In the model the conditional sample mean of the stock price return is almost always zero. We estimate it anyway in order ensure unbiased estimates of the covariance matrix in all cases. We now estimate the conditional sample covariance matrix of returns by

$$\boldsymbol{\Sigma}_t = \delta (\mathbf{x}_{t-1} - \hat{\boldsymbol{\mu}}_t) (\mathbf{x}_{t-1} - \hat{\boldsymbol{\mu}}_t)^T + (1 - \delta) \boldsymbol{\Sigma}_{t-1}. \quad (2.4)$$

The bank then simply assumes that the covariance matrix in the next time step is the same as the current estimate of the covariance matrix, i.e.:

$$\boldsymbol{\Sigma}_{t+1} \approx \boldsymbol{\Sigma}_t. \quad (2.5)$$

Note the distinction between the two concepts of returns covered in this Section. On the one hand the bank uses price dividend ratios for its investment allocation. On the other hand it relies on stock price returns for its risk management as it has to protect itself against stock price devaluations.

Portfolio choice

Banks use a heuristic rule to compute their portfolio choice as a function of the expected risk-return relationship of the stocks. Banks compute the risk-return ratio as follows:

$$s_{i,t+1} = \hat{r}_{i,t+1} / \sigma_{i,t+1}, \quad (2.6)$$

⁹We initialize the conditional sample mean as $\hat{\boldsymbol{\mu}}_0 = 0$.

where $\sigma_{i,t+1}^2 = \Sigma_{ii,t+1}$ is the i th diagonal element of the covariance matrix and $\hat{r}_{i,t+1}$ is the expected dividend price ratio. Hence the bank's decision criterion is simply the Sharpe ratio of the stock. The portfolio weight $w_{i,t}$ of stock i is then given by the following rule:

$$w_{i,t} = (1 - w_c) \frac{\exp(\beta s_{i,t+1})}{\sum_j \exp(\beta s_{j,t+1})}, \quad (2.7)$$

where $w_c < 1$ is a fixed weight for the bank's cash reserve and $\beta > 0$ is an intensity of choice parameter. As $\beta \rightarrow 0$ the bank splits its portfolio more and more equally, effectively becoming insensitive to differences in stock return. Conversely as $\beta \rightarrow \infty$ the bank invests increasingly heavily into the stock with the highest return. In an alternative specification the bank's portfolio choice is determined by numerically optimizing a mean variance portfolio with "no short sale" constraints.¹⁰ We found that the key results of this chapter do not depend on the specific rule for the portfolio choice. Therefore we chose the more computationally efficient rule described above. In Appendix 2.A.3 we describe the alternative, optimizing portfolio choice specification and show that the results remain qualitatively unchanged.

Risk management

Risk management is an important component of any financial institution's activities. Here, we assume that banks use a Value-at-Risk (VaR) approach to control their exposure to the stock market. In particular, banks will try to ensure that their equity exceeds their Value-at-Risk in order to protect themselves against large losses on the stock market. This behavior may be due to a regulatory constraint as proposed in Zigrand et al. (2010), Adrian and Boyarchenko (2012) and Corsi et al. (2013). However, even in the absence of regulation, financial institutions are likely to control their Value-at-Risk. Rather than determining the existence of a VaR type risk management

¹⁰Note that the mean variance portfolio problem cannot be solved by the usual Lagrangian approach in the presence of "no short sale" constraints as required here. Therefore a numerical optimization method has to be used to determine the efficient portfolio.

approach, regulation can then be thought of as putting constraints on the parameters of the VaR model.

As before, the variables introduced in this Section apply to all banks and therefore carry an implicit index j that we drop for ease of exposition. For simplicity, we begin by assuming that banks believe that log stock returns are normally distributed with zero mean.¹¹ In this simple case we have for the per-dollar VaR

$$VaR_{a,t} = \sqrt{2}\sigma_{P,t} \text{erf}^{-1}(2a - 1), \quad (2.8)$$

where a is the VaR quantile and $\sigma_{P,t}$ is the per-dollar conditional estimate of the portfolio standard deviation at time t . The subscript “P” indicates that $\sigma_{P,t}$ refers to the bank’s entire portfolio as opposed to an individual stock. In the following we will also refer to the bank’s portfolio risk as perceived risk. This is different from the actual risk since it depends on the bank’s method of risk estimation. $\sigma_{P,t}$ is determined as follows:

$$\sigma_{P,t}^2 = \mathbf{w}_t^T \Sigma_t \mathbf{w}_t, \quad (2.9)$$

where $\mathbf{w}_t = (w_{1,t}, \dots, w_{N_f,t})^T$. Note that so far we have only defined VaR in terms of returns, i.e. as per-dollar VaR. In order to compute the nominal Value-at-Risk we need to multiply the per-dollar VaR by the total nominal amount of bank assets $\mathcal{A}_t$. The bank tries to ensure that its nominal VaR does not exceed its equity, hence:

$$VaR_{a,t} \mathcal{A}_t \leq \mathcal{E}_t. \quad (2.10)$$

For a bank that maximizes its return on equity this constraint will be binding. The bank then has the following target leverage:

$$\bar{\lambda}_t = VaR_{a,t}^{-1} = \left(\sqrt{2}\sigma_{P,t} \text{erf}^{-1}(2a - 1) \right)^{-1} = \alpha / \sigma_{P,t}, \quad (2.11)$$

¹¹Note that in the model we do find that the conditional sample mean of the stock price return is almost always zero making the zero mean assumption model consistent. Also note that banks invest into stocks to receive dividends rather than benefit from price movement which is why stocks are still an attractive investment despite the zero mean assumption on stock returns.

where we replaced the constant $(\sqrt{2} \operatorname{erf}^{-1}(2a - 1))^{-1}$ by α . Thus, from equation (2.11) we see that the bank's target leverage is inversely proportional to its perceived portfolio standard deviation.

While we derived equation (2.11) assuming normally distributed Gaussian returns, it is more general and includes both the Gaussian case as well as the maximally heavy-tailed symmetric return distribution with finite variance as we show in Appendix 2.A.1.

In a given time step the bank evaluates the difference to the desired target size of its balance sheet: $\Delta \mathcal{B}_t = \bar{\lambda}_t \mathcal{E}_t - \mathcal{A}_t$. If $\Delta \mathcal{B}_t > 0$ we assume that the bank simply raises the necessary funds in order to increase its assets and liabilities by the desired amount $\Delta \mathcal{B}_t$. If $\Delta \mathcal{B}_t < 0$ the bank will sell part of its assets in the next round in order to reduce its assets and liabilities by the desired amount. As mentioned above, we assume that the bank can always access the necessary funds from the external investor.

2.3.2 Fund investor

In order to control the market power of the banks, we introduce a fund investor. The fund investor can be thought of as representing a population of hedge or mutual funds in the market with fundamentalist trading patterns that are not fully modeled. This partial modeling introduces a random component in its portfolio allocation. In this model the fund investor effectively acts as market maker for the stock market.

Similar to banks the assets of the fund investor consist of a stock portfolio and a cash reserve: $A_{N,t} = c_t + \mathbf{n}_t^T \mathbf{p}_t$. The fund investor's stock portfolio weights $w_{i,t}$ are computed as follows:

$$w_{i,t+1} = (1 - w_c) \frac{v_{i,t+1}}{\sum_j v_{j,t+1}}, \quad (2.12)$$

where $v_{i,t}$ evolves according to a stochastic process and is not necessarily bounded between zero and one. Therefore we normalize $v_{i,t}$ and multiply by one minus the

required cash weight w_c to obtain the portfolio weight for stock i . Specifically, $v_{i,t}$ evolves according to the following process:

$$\frac{\Delta v_{i,t}}{v_{i,t}} = \underbrace{\rho(1/N_f - v_{i,t})}_{\text{Portfolio balance}} + \underbrace{\zeta(\hat{r}_{i,t+1} - \sum_j \hat{r}_{j,t+1}/N_f)}_{\text{Fundamentalist}} + \underbrace{\eta \xi_t}_{\text{Noise}}, \quad (2.13)$$

$$v_{i,t+1} = v_{i,t} + \Delta v_{i,t}.$$

$\rho < 1$ determines how quickly the portfolio returns to a balanced portfolio. ζ is a scaling parameter to ensure that contribution of the fundamentalist term is of comparable size to the other terms in the equation. $\hat{r}_{i,t+1}$ is the expected return and is computed in the same ways as for banks. The sum in the fundamentalist term is over all stocks and is simply the average stock return in the economy. This approximates the expected fundamental value of all stocks in the market, since we do not have an exogenously fixed fundamental value. Finally $\xi_t \sim \mathcal{N}(0, 1)$ is a standard Gaussian noise term while η determines the standard deviation of the random walk.

This specification of the fund investor's portfolio weights ensures that the weights do not diverge (portfolio balance term) and are economically sensible (fundamentalist term). In particular the second term ensures that the fund investor is a weak fundamentalist investing a larger fraction of its portfolio into stocks with higher than average return.

Due to this fundamentalist component and due the fact that the fund investor is unleveraged, it stabilizes the stock market. To see this consider the impact of the fundamentalist component of the weight update: If the price of a stock rises significantly, its dividend price ratio will decrease relative to other stocks. The fund investor will then shift its portfolio away from this stock. By divesting from this particular stock the fund investor will cause a decrease in the price of the stock. In fact, the fund investor will continue to shift its portfolio until the dividend price ratio of the stock is close to the average dividend price ratio. Assuming that dividends are slow moving with respect to price adjustments, this process leads to a mean reversion

in stock prices. The fact that the fund investor is not leveraged means that it acts as a counter-weight to the bank, which will actively manage its leverage, and thereby destabilizes the market as we will show in Section 2.4.

2.3.3 Dividend process

The dividend process follows a simple geometric random walk with drift

$$\frac{\Delta\pi_{i,t}}{\pi_{i,t}} = \mu + \varphi\xi_{i,t}, \quad (2.14)$$

where the drift term $\mu \approx 0$ and the standard deviation of the Gaussian noise term $\varphi \ll 1$. Under this specification we assume that all dividends follow independent geometric random walks with the same parameters. In a future version one could consider calibrating the parameters to actual dividend processes.

2.3.4 Stock market mechanism

In the following we propose a simple linear multi-asset market clearing mechanism that is based on the single asset markets developed in Hommes and Wagener (2009) and LeBaron (2012). The linearity of the approach allows efficient computation of prices and makes this approach scalable to potentially many assets.

First consider a single stock. As mentioned earlier the total supply of shares of a given stock is normalized to 1. In monetary units the supply of stocks, i.e. the number of shares in the market times their price, is therefore simply given by the market clearing price. The demand, in monetary units, for stock i is the sum over all banks' portfolio allocations to this stock, i.e.

$$\text{Demand}_{i,t+1} = \sum_{j \in \mathcal{I}} w_{ji,t+1} \mathcal{A}_{j,t+1}, \quad (2.15)$$

where $\mathcal{I}$ is the set of all investors in the stock market, i.e. in this case banks and the fund investor. $w_{ji,t+1}$ is investor j 's portfolio weight for stock i . Finally $\mathcal{A}_{j,t+1}$ is the value of investor j 's assets after market clearing.

In order to compute the market clearing price for stock i we equate demand (eq. (2.15)) and supply in monetary units. Recall that we assume that there is only one share, hence the supply in monetary units is simply $1 \times p_{i,t+1}$. We substitute into equation 2.15 the value for the investors' assets $\mathcal{A}_{j,t+1}$ taking into account any changes in assets due to additional leveraging or deleveraging. We then obtain:

$$p_{i,t+1} = \sum_{j \in \mathcal{I}} w_{ji,t+1} (c_{j,t} + \mathbf{n}_{j,t}^T \mathbf{p}_{t+1} + \Delta \mathcal{B}_{j,t}). \quad (2.16)$$

Note that the banks' demand depends on the price of the stock as it determines the wealth of the bank. Also recall that $\Delta \mathcal{B}_{j,t}$ is the overall change of the amount of the bank's asset. The portfolio weights then determine how this change is distributed across stocks. As discussed above, we assume that the investor chooses the portfolio weights prior to market clearing, i.e. the weights are not a function of the price computed in market clearing. Furthermore, banks compute their desired asset change prior to market clearing. Economically this corresponds to the case in which banks have to commit to their actions prior to receiving information about the market price. In this sense banks have bounded rationality during the market clearing process. This approximation linearizes the problem as it removes the price dependence of the bank's decision parameters. Expressed in matrix form we have:

$$\mathbf{p}_{t+1} = \mathbf{W}_{t+1} (\mathbf{N}_t \mathbf{p}_{t+1} + \mathbf{c}_t + \Delta \mathcal{B}_t), \quad (2.17)$$

where $\mathbf{W}$ is the $N_f \times |\mathcal{I}|$ matrix of stock portfolio weights, $\mathbf{N}$ is the $|\mathcal{I}| \times N_f$ stock ownership matrix, $\mathbf{c}_t$ is the vector of cash reserves and $\Delta \mathcal{B}_t$ is the vector of asset changes. Thus the market clearing vector of stock prices is given by:

$$\begin{aligned} \mathbf{p}_{t+1} &= \mathbf{U}_{t+1}^{-1} \mathbf{W}_{t+1} (\mathbf{c}_t + \Delta \mathcal{B}_t) \\ \mathbf{U}_{t+1} &= \mathbb{1} - \mathbf{W}_{t+1} \mathbf{N}_t \end{aligned} \quad (2.18)$$

Note that for the stock prices to be well defined two conditions have to be satisfied:

(1) There must exist a bank j such that $c_{j,t} + \Delta \mathcal{B}_{j,t} > 0$ and (2) the matrix $\mathbf{U}$ must be

invertible. Clearly condition (1) is always fulfilled since banks have to invest a non-zero amount into cash at every time. Further note that the portfolio weight matrix is substochastic, i.e. its columns sum to less than 1 (since the bank always invests a positive fraction into cash) while the ownership matrix is stochastic, i.e. its rows sum to 1. As a result their product is also substochastic. Therefore $\lim_{n \rightarrow \infty} (\mathbb{1} - \mathbf{U})^n \rightarrow 0$. Hence, the inverse of $\mathbf{U}$ exists (by Neumann series).

2.4 Leverage cycles in the multi-asset model

In this Section we will study the dynamics of the multi-asset model with one bank, three stocks and a fund investor. Below we will briefly comment on the choice of parameters for this Section. All parameters are summarized in Table 2.1. We will then proceed to discuss the results of the simulations.

2.4.1 Simulation set up

Banks: We run the simulation with only one bank since we are not explicitly studying the impact of heterogeneity among banks. In the homogeneous case we are considering here several banks could always be collected into one representative bank. The initial equity of the bank $\mathcal{E}_0$ sets the monetary scale of the simulation and has no impact on the actual dynamics of the simulation. We therefore just pick an arbitrary number. We choose an initial leverage λ_0 that is roughly of the order of magnitude of that of a real bank. The initial assets are then determined by the initial leverage and equity. We calibrate γ to introduce some momentum into the bank's portfolio weights.

We choose α to obtain roughly realistic levels of leverage in the simulation. Finally the value for the memory parameter in the exponential moving average to estimate the portfolio covariance matrix corresponds roughly to the value recommended in the original manual on the RiskMetrics approach, see Longestaey (1996).

Agent	Parameter		
	Notation	Description	Value
Bank	δ	Memory parameter in covariance estimation	0.1
	α	risk parameter	0.1
	$\mathcal{E}_0$	Initial bank equity	23
	λ_0	Initial bank leverage	5
	w_c	Cash weight	0.2
	γ	Memory parameter in return estimation	0.1
	β	Intensity of choice in portfolio allocation	0.1
	N_b	Number of banks	1
Fund investor	$A_{N,0}$	Initial fund investor assets	115
	ρ	Portfolio balance parameter	0.05
	ζ	Fundamentalist parameter	5
	η	Standard deviation Brownian motion	0.2
Dividend process	μ	Dividend drift term	10^{-5}
	ϕ	Standard deviation	0.05
	N_S	Number of stocks	3

Table 2.1: Overview of simulation parameters for full model. The most important parameters for dynamics of the model are δ and α . The initial conditions listed are only relevant in setting the relative sizes of the bank to the fund investor. This is important in order to stabilize the dynamics of the model.

Fund investor: We choose the size of the fund investor such that its market power is sufficient to stabilize the market dynamics somewhat while not dominating them. The remaining parameters are calibrated such that the fund investor on average neither loses nor gains money. In general we find that the investor with more stable portfolio weights tends to accumulate money over the course of the simulation. In order to counteract this tendency we increase the standard deviation η in the Gaussian noise term of the weight update rule until we find that on average neither agent accumulates equity.

Throughout this chapter we will take one time step to be roughly one quarter.

2.4.2 Leverage cycles

In order to demonstrate how leverage management under the Value-at-Risk constraint can affect the model dynamics, we will contrast two cases:

1. Passive case: The bank does not manage its leverage but invests into stocks according to its fundamentalist portfolio rule.¹² This means that the bank's leverage changes passively as stock prices change. If prices go up, the bank's leverage will go down and vice versa. In particular, a bank will not borrow more if its leverage goes down, nor will it sell assets when its leverage goes up.
2. Active case: The bank actively manages its leverage as outlined in Section 2.3.1 and invests into stocks according to their fundamentalist portfolio rule. This means that the bank has a target leverage that is determined by its perceived portfolio risk. If it is above its target leverage it will sell assets and repay part of its debt while it will borrow more and invest more if it is below its target leverage.

We wish to emphasize that the passive case is manifestly unrealistic, and is merely a reference point for comparison. Over the long run the leverage of such a bank would

¹²This implies $\Delta \mathcal{B}_t = 0$ throughout for the passive case.

make a random walk, and after a sufficiently long time would eventually become so large that a small shock would cause the bank to default. Nonetheless, this provides an interesting point of comparison because such a bank does not have to do any trading to adjust its leverage, and therefore has no systemic impact. Thus in a sense this represents a “pure” case. Since we are only simulating for a short period there is not enough time for the leverage to eventually reach large values.

In the following we will document the qualitative behavior of the model by studying exemplary time series. In the subsequent Sections we will study the model in more detail and develop an intuition for the drivers of the dynamics induced by leverage management.

In Figure 2.2 we compare two time series for the price of an individual stock generated by the model starting from the same initial conditions. The top panel shows case 1 in which the bank does not manage its leverage. This means that the bank does not have a target leverage and its leverage simply changes as the value of its assets changes: if the value of the assets appreciates the leverage of the bank decreases and vice versa. In contrast to the bank that actively manages its leverage, we have for the passive bank’s change in assets through leveraging or deleveraging $\Delta\mathcal{B}_t = 0$ throughout. Therefore, in the top panel the price movements are solely driven by the bank and fund investors portfolio adjustment.

The bottom panel corresponds to case 2 in which the bank actively manages its leverage. Recall that under active leverage management the bank reacts to decreases in perceived risk by increasing its target leverage. Similarly, it decreases its target leverage when perceived risk increases. Furthermore, the bank then adjusts the size of its asset portfolio in order to reach its desired target leverage by an amount $\Delta\mathcal{B}_t$.

By simple visual comparison the difference in the two price time series is apparent. In the case of active leverage management the price dynamics are characterized by recurring gradual increases in price followed by rapid and drastic collapses in price.

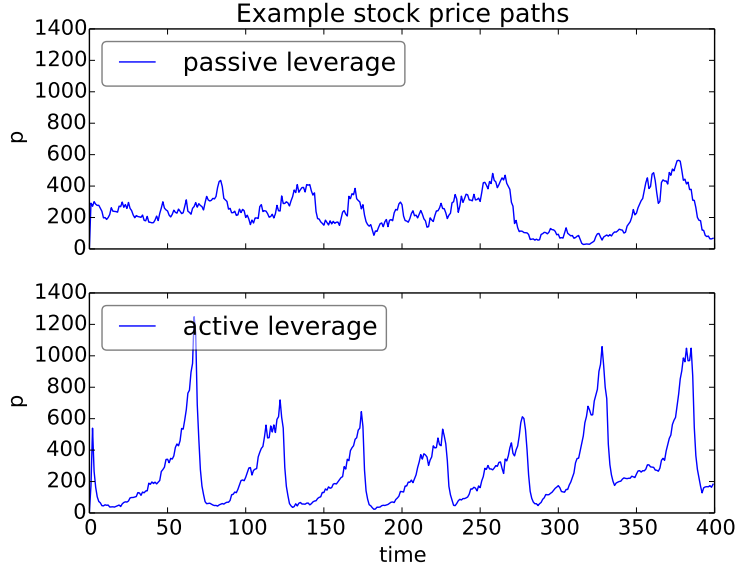


Figure 2.2: Comparison of two exemplary time series for the price of an individual stock. Top: Bank does not manage its leverage. Bottom: Bank actively manages its leverage. Clearly the variation in the active case is much larger than in the passive case. When the bank manages its leverage prices undergo recurring patterns of relatively gradual increase in price followed by drastic price crashes. When the bank does not actively manage its leverage, prices are driven exclusively by the portfolio adjustment of the bank to changing dividend price ratios and the activity of the fund investor.

The price dynamics in the passive case are driven by the bank's portfolio choice, the underlying driving dividend process and the behavior of the fund investor. The extent of the stock price variation in the passive case is small compared to the active case. Most importantly however, the fluctuations in the active case display a very clear recurring pattern while the variation in the passive case does not.

In Figure 2.3 we expand our analysis to a wider set of model outputs. As before, we are running the model with one bank, one fund investor and three stocks. We plot the following time series from top to bottom: (1) stock dividends, (2) dividend price ratio, (3) bank portfolio weights, (4) stock prices, (5) variance of stock returns, (6) leverage, (7) equity of bank and fund investor. A few points are worth noting:

Firstly, fluctuations in stock prices correlate strongly with fluctuations in leverage and fluctuations in leverage are anti-correlated to fluctuations in perceived portfolio

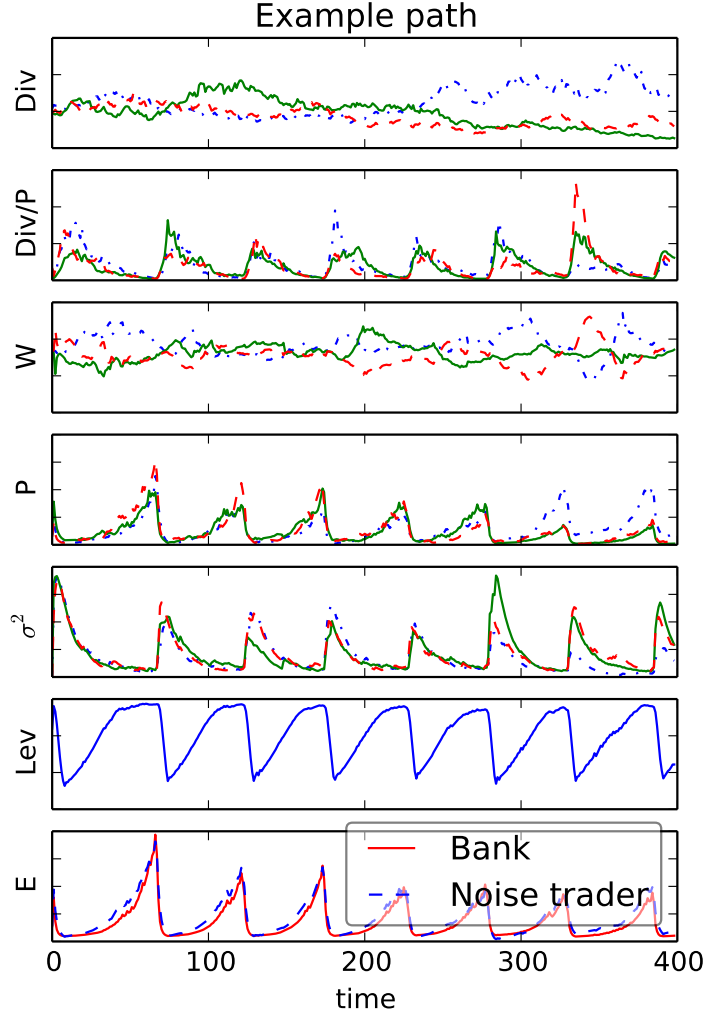


Figure 2.3: Exemplary time series of the model with a bank that actively manages its leverage, a fund investor and three stocks. Time series from top to bottom: (1) stock dividends, (2) dividend price ratio, (3) bank portfolio weights, (4) stock prices, (5) perceived risk of stock returns, (6) leverage, (7) equity of bank and fund investor. As mentioned before, the leverage management leads to recurring patterns of gradual price increases and drastic price crashes. Clearly the leverage is inversely related to the perceived portfolio risk and strongly positively related to the level of the stock market. Price crashes occur typically when perceived risk is very low and leverage is very high, i.e. when banks perceive the world as very tranquil they are in fact at the highest risk. Note that leverage cycles affect all endogenous variables in the system, in particular leading to large variations in bank and fund investor equity.

risk. In fact, Eq. (2.11) implies that perceived portfolio risk drives the changes in leverage. Interestingly, perceived risk is lowest just before a crash, i.e. when the actual market risk is highest.

Secondly, stock prices are strongly cross correlated. While differences in the evolution of the stock's dividend process affect the stock prices, prices are modulated by the changes in leverage. This makes sense since changes in leverage determine the overall level of investment of the bank and thereby affect all stock prices.

Thirdly, fluctuations in stock prices affect the equity of all investors in the market, including the fund investor. Investors gain equity as the stock market rises and lose a substantial part of their wealth during a crash. While not shown here, in general these crashes can cause the bankruptcy of the investors.

Finally, note that the variation in leverage, stock prices and equity significantly exceeds the underlying variation in the dividend process. This indicates that the leverage dynamics dominate the effect of the portfolio rebalancing. Taken together, figures 2.2 and 2.3 strongly suggest that the recurring fluctuations observed in leverage, stock prices and equity result from the bank's leverage management.

We refer to the property of leverage management leading to the observed dynamics as pro-cyclical leverage. Furthermore, we refer to the fluctuations in leverage and stock prices as Basel leverage cycles. This definition of pro-cyclical leverage extends the initial definition in Adrian and Shin (2008). While Adrian and Shin (2008) refer to pro-cyclical leverage as the destabilizing effect of Value-at-Risk constraints in financial markets, Figures 2.2 and 2.3 illustrate that leverage management can lead to persistent fluctuations. Put in the language of dynamical systems, Adrian and Shin (2008) describe an unstable feedback loop. Here, we demonstrate that the dynamics induced by VaR constraints are richer and not necessarily fully unstable but potentially cyclical.

2.5 Understanding leverage cycles – a constant equity model

In the previous Section we demonstrated the existence of leverage cycles in a relatively rich agent-based model of a financial sector. The banks' behavior is driven by two motives: adequately distributing their portfolio across assets based on their risk-return ratio and managing their risk by controlling their leverage. In order to better understand the drivers of the observed leverage cycles it is useful to consider simplified versions of the model. In this Section we will consider a deterministic two dimensional model with one asset and one investor with constant equity. This model can be thought of as a minimal model of leverage cycles. In Chapter 3 we will consider a more complete dynamical system with one asset, one investor and a fund investor whose behavior can be deterministic or stochastic.

2.5.1 Model description

In the following we reduce the agent-based model outlined in Section 2.3 to a very simple two dimensional model, which has the advantage that it reproduces the essence of the leverage cycle and gives clear insight into its essential features. To do this we assume a single asset and a single investor, which means that we can drop the indices i and j in Eq. (2.16), and that the portfolio weight $w_t = 1$ and the fraction of the stock owned by the investor is $n_t = 1$. Furthermore, we assume that the equity of the investor is constant over time. While this is a strong assumption and a departure from the original model, it is underpinned by empirical research done by Adrian and Shin (2008). In fact, Adrian and Shin (2008) show that investors respond to a change in assets by changing their leverage and keeping their equity fixed. We therefore consider it justified for the purposes of this Section.

If the investor always maintains her target leverage then by definition $\bar{\lambda}(t) = \mathcal{A}_t/E$, where E is the equity of the investor. With these assumptions Eq. (2.16)

reduces to $p_t = \mathcal{A}_t$, which can be written

$$p(t) = \bar{\lambda}(t)E. \quad (2.19)$$

As before, the estimated variance is given by Eq. (2.4) and the target leverage by Eq. (2.11),

$$\begin{aligned} \sigma^2(t+1) &= (1-\delta)\sigma^2(t) + \delta \left(\log \left(\frac{p(t)}{p(t-1)} \right) \right)^2, \\ \bar{\lambda}(t) &= \alpha/\sigma(t). \end{aligned} \quad (2.20)$$

Eliminating prices from the system of equations we obtain

$$\sigma^2(t+1) = (1-\delta)\sigma^2(t) + \frac{\delta}{4} \left(\log \left(\frac{\sigma^2(t-1)}{\sigma^2(t)} \right) \right)^2. \quad (2.21)$$

We can re-write this as a two-dimensional deterministic dynamical system by writing $z_1(t) = \sigma^2(t)$ and $z_2(t) = \sigma^2(t-1)$, which gives

$$\begin{aligned} z_1(t+1) &= (1-\delta)z_1(t) + \frac{\delta}{4} \left(\log \left(\frac{z_2(t)}{z_1(t)} \right) \right)^2, \\ z_2(t+1) &= z_1(t). \end{aligned} \quad (2.22)$$

In vector form this can be written $\mathbf{z}(t) = g(\mathbf{z}(t-1))$, where $\mathbf{z}(t) = (z_1(t), z_2(t))^T$. While this simple model will be useful to understand the destabilizing effects of procyclical leverage it has two important shortcomings. Firstly, the assumption that equity is fixed ignores an important aspect of leverage. Namely that if leverage is high, equity responds more drastically to shocks in assets. Therefore a bank with high leverage is inherently more risky than a bank with low leverage. This is not captured in this simple model. Secondly, the behavior of the model is independent of the riskiness of the bank α . This means that the model is effectively insensitive to the level of leverage the bank takes.

Intuition about system dynamics

We now develop an intuition for the dynamics of the system defined in Eq. ((2.22)). The left panel of Figure 2.4 clearly illustrates that the basic features of the leverage

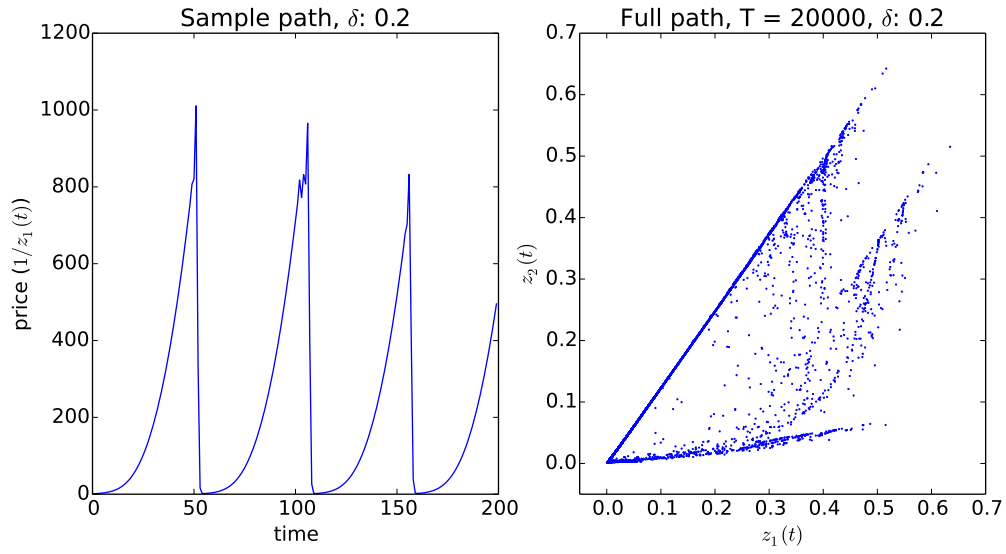


Figure 2.4: Exemplary time series of the two dimensional model with one asset, one investor and constant equity. Left: We plot an extract of the price time series for 200 time steps. As in the full model, the simple model displays clear leverage cycles (recall that in the simple model prices are simply proportional to leverage). Right: We plot the evolution of the dynamical system for 20,000 time steps in a phase plot, i.e. on the x-axis we plot z_1 and on the y-axis we plot z_2 . A state in the upper right corner of the figure moves along the straight nearly diagonal line towards the origin, then destabilizes close to the origin and quickly returns to the straight line via a chaotic trajectory (often lingering briefly on the curve at the bottom of the figure). The part of the trajectory on the straight line corresponds to the gradual increase in prices and leverage while the part of the trajectory below the straight line corresponds to the drastic price crash.

cycle are preserved in this simple model: repeating patterns of gradual price increase followed by drastic price collapses. In the right panel of Figure 2.4 we plot the evolution of the system spanned in the z_1, z_2 space for a long run with $T = 20,000$. The evolution of the system can be described follows:

Suppose we start in the top right corner of the phase space (the figure on the right), or equivalently at a trough of the time series plot of the price (the figure on the left). This corresponds to the high volatility and low leverage regime, i.e. perceived risk is high (z_1) and was high in the previous time step (z_2). In the phase space the system then moves along down the nearly diagonal straight line towards the origin, corresponding to decreasing volatility and increasing leverage. In the time series plot the price increases during this period, i.e. this is the bubble phase (e.g. the great moderation).

At the origin the perceived risk is zero and the leverage is infinite. Thus the system destabilizes as it approaches the origin, and a small perturbation is sufficient for the system to be ejected from its stable path towards the right of the phase plot where perceived risk is higher. In the time series plot this corresponds to the initiation of a price crash. The system then moves very quickly through the space below the diagonal of the phase plot until converging onto the high volatility and low leverage regime, where the price reaches a low value and the crash is over. The leverage cycle restarts once the system has reached the high volatility and low leverage regime. The irregular nature of the time series and the fractal structure of the phase plot suggest that the system is not on a limit cycle, but rather a chaotic attractor (it is deterministic so these are the only two options).

From an economic perspective the approach towards the origin of the phase space along the straight line corresponds to the situation where perceived risk is low and continually decreasing. However, while perceived risk is low and decreasing, the system is actually moving towards a more and more unstable configuration in which

small initial perturbations in perceived risk can cause rapid price movements and consequentially rapid adjustments of perceived risk.

The mismatch between perceived portfolio risk and actual systemic risk is crucial in the understanding of the dynamics observed here. As the bank learns about market risk it relies exclusively on historical data which falsely suggest an increasingly tranquil market. Instead, by adjusting to the lower perceived risk by taking on further leverage, the bank maneuvers itself into regions of increased systemic risk.

A more formal analysis of the stability properties of the constant equity model presented in this Section can be found in Appendix 2.A.2. Our analysis suggests that the system has a hyperbolic fixed point. This implies that, as the system approaches the fixed point along the stable manifold it becomes increasingly susceptible to small perturbations. Ultimately the system leaves the stable manifold and follows the unstable manifold towards increasing z_1 . Our stability analysis is therefore consistent with the computed trajectory of the system that we show in Figure 2.4.

2.6 Conclusion

In this chapter we study the implications of historically based bank leverage management on the stability properties of the financial system. To this end we develop an agent-based model of a multi-asset financial system and study it computationally in its full form and analytically in a simplified version.

The main result of this chapter is that the dynamics resulting from bank leverage management are richer than the unstable feedback between leverage, risk and asset prices (see Adrian and Shin (2008)) that is usually referred to as pro-cyclical leverage. In particular, we show that bank leverage management can cause recurring patterns of stock price bubbles and crashes which occur in a chaotic regime of the system. We refer to these recurring bubbles and crashes as Basel leverage cycles.

These dynamics are robust to the exact model specification as long as they combine

three crucial ingredients. First, a reliance on historical returns for the estimation of perceived portfolio risk. Second, a pro-cyclical risk management that uses the historically estimated perceived portfolio risk. Finally, a sufficiently large leveraged investor such that its trading activity has a significant market impact.

We demonstrate this robustness by developing two models of different degrees of complexity that all share the above requirements. Reducing the dynamics to its core elements in a simplified model, we are able to analytically characterize the leverage cycles dynamics in the deterministic limit.

Having developed a suite of models that display endogenous leverage cycles, a logical next step is to study how a policy maker might intervene in order to control the severity of the leverage cycle. In Chapter 3 we develop a further reduced model of leverage cycles and investigate how different leverage policy can affect the extent of financial crises.

A possible extension to the models presented here would be the introduction of a heterogeneous population of banks. Intuitively one would expect that heterogeneity in the portfolio positions of banks, their riskiness and their estimation horizon can increase the stability of the system. An interesting question would then be under which circumstances bank behavior synchronizes. Then, under synchronized action we would expect to recover the dynamics observed for one representative bank.

2.A Appendix

2.A.1 Risk management under heavy tailed returns

In the following we will demonstrate that while we derived Eq. (2.11) assuming normally distributed Gaussian returns, it is more general and includes both the Gaussian case as well as the maximally heavy-tailed symmetric return distribution with finite variance. In the Gaussian case for a confidence interval $a = 0.99$ we obtain $\alpha \approx 0.42$.

Now consider the maximally heavy tailed return distribution with finite variance.¹³ Given an arbitrary distribution with finite variance σ^2 , we have according to Chebychev's inequality for the probability that the loss X exceeds a threshold $k\sigma$:

$$P(X > k\sigma) \leq \frac{1}{2k^2}, \quad (2.23)$$

where $k > 1$ is simply a scaling parameter. In the maximally heavy tailed case the upper bound is tight. For a confidence interval of $a = 0.99$ we have $P(X > k\sigma) = 0.01$. Thus $k \approx 7.07$ and $\alpha \approx 0.14$. Therefore from the perspective of the bank, changing its beliefs about how heavy-tailed the return distribution is, does not fundamentally change the relationship between the perceived risk and leverage. Instead it only scales down leverage; the bank effectively becomes a more cautious investor.

2.A.2 Linear stability analysis of constant equity model

We can formalize the analysis of the dynamics of the constant equity model presented in Section 2.5.1 by studying the linear stability properties of the model. First recall the definition of the constant equity model:

$$\begin{aligned} z_1(t+1) &= (1-\delta)z_1(t) + \frac{\delta}{4} \left(\log \left(\frac{z_2(t)}{z_1(t)} \right) \right)^2, \\ z_2(t+1) &= z_1(t). \end{aligned} \quad (2.24)$$

Note that the system we are studying has a fixed point at $z^* = (0, 0)$. This can be seen from the definition of the model above. At a fixed point we must have $z_1 = z_2$ since z_2 is simply the previous position of z_1 ; at the fixed point z_1 is fixed hence $z_1 = z_2$. Taking the limit along $z_1 = z_2$, we obtain $\lim_{(z_1, z_2) \rightarrow (0, 0)} z_2/z_1 = 1$ due to l'Hospital's rule. Then the logarithm vanishes and z_1 is constant. Hence $z^* = (0, 0)$ is a fixed point.

To compute the linear stability of the system we derive the Jacobian of the dynamical system and then study the value of the leading eigenvalue of the Jacobian in

¹³The exact shape of this distribution is not relevant. It is maximally heavy tailed simply in the sense that Chebychev's inequality becomes tight.

phase space. Recall the definition of the Jacobian: $J_{ij} = \partial g_i / \partial z_j$. Then:

$$J = \begin{pmatrix} 1 - \delta - \delta \log\left(\frac{z_2}{z_1}\right) / (2z_1) & \delta \log\left(\frac{z_2}{z_1}\right) / (2z_2) \\ 1 & 0 \end{pmatrix}. \quad (2.25)$$

The absolute value of the largest eigenvalue of the Jacobian evaluated at the system's fixed point determines the stability of the system. Eigenvalues with absolute value greater than 1 imply an unstable system while eigenvalues with absolute value less than 1 imply a stable system.

We can now diagonalize the Jacobian to obtain its eigenvalues. We find for the eigenvalues:

$$\lambda_{\pm} = \frac{q_1(z_1, z_2, \delta) \mp q_2(z_1, z_2, \delta)}{q_3(z_1, z_2)}, \quad (2.26)$$

where

$$\begin{aligned} q_1(z_1, z_2, \delta) &= -2\delta z_1 z_2 - \delta z_2 \log\left(\frac{z_2}{z_1}\right) + 2z_1 z_2, \\ q_2(z_1, z_2, \delta) &= \sqrt{8\delta z_1^2 z_2 \log\left(\frac{z_2}{z_1}\right) + \left(2\delta z_1 z_2 + \delta z_2 \log\left(\frac{z_2}{z_1}\right) - 2z_1 z_2\right)^2}, \\ q_3(z_1, z_2) &= 4z_1 z_2. \end{aligned} \quad (2.27)$$

When computing the eigenvalues at the fixed point $z^* = (0, 0)$ we must be careful along which path to take the limit $(z_1, z_2) \rightarrow 0$. In general the limit will depend on the path taken. From Figure 2.4 we note that the system approaches the origin on a straight line with slope $m > 1$.¹⁴ We therefore take the limit along the path $z_2 = mz_1$, where $m > 1$. Then we obtain for the eigenvalues:

$$\begin{aligned} \lim_{(z_1, mz_1) \rightarrow (0,0)} \lambda_- &= 1/m, \\ \lim_{(z_1, mz_1) \rightarrow (0,0)} \lambda_+ &= -\infty. \end{aligned} \quad (2.28)$$

The corresponding eigenvectors are:

$$\mathbf{e}_{\pm} = (\lambda_{\pm}, 1)^T. \quad (2.29)$$

Therefore $z^* = (0, 0)$ is a hyperbolic fixed point which is stable along $\mathbf{e}_-$ and infinitely unstable along $\mathbf{e}_+$. The stable manifold corresponds to the nearly diagonal straight

¹⁴It turns out that this slope is exactly $m = 1/(1 - \delta)$.

line $z_2 = mz_1$ along which we are descending towards the origin. The unstable manifold corresponds to $z_2 = 0$.

As the system approaches the fixed point along the stable manifold it becomes increasingly susceptible to small perturbations. Ultimately the system leaves the stable manifold and moves along the unstable manifold towards increasing z_1 . Our stability analysis is therefore consistent with the computed trajectory of the system that we show in Figure 2.4.

The fact that the fixed point is hyperbolic is interesting from an economic point of view. For an agent that “lives” within this system, the dynamics may appear stable for long periods of time. The natural response for an investor in such a system is to take on more leverage. However, as noted above, it is precisely this response that moves the system closer to the hyperbolic fixed point and therefore makes it more susceptible to jumping to the unstable manifold. A system characterized by a hyperbolic fixed point can therefore transition very rapidly and unexpectedly from stable dynamics to highly unstable dynamics.

2.A.3 Alternative portfolio allocation

In the main model we propose a simple portfolio allocation rule based on the Sharpe ratio of a particular stock. In the following we will discuss an alternative portfolio allocation approach based on portfolio optimization subject to a Value-at-Risk constraint. The bank chooses the vector of portfolio weights $\mathbf{w}$ in order to maximize:

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w} \cdot \hat{\mathbf{r}} \\ \text{subject to} \quad & \mathcal{E} - a\mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \geq 0, \\ & w_i \geq 0, \forall i \end{aligned} \tag{2.30}$$

where $\hat{\mathbf{r}}$ is the vector of expected stock returns, $\mathcal{E}$ is the bank’s equity, a is a constant and $\mathbf{\Sigma}$ is the bank’s estimated portfolio variance. This problem cannot be solved analytically due to the no-shorting constraints on the portfolio weights. In

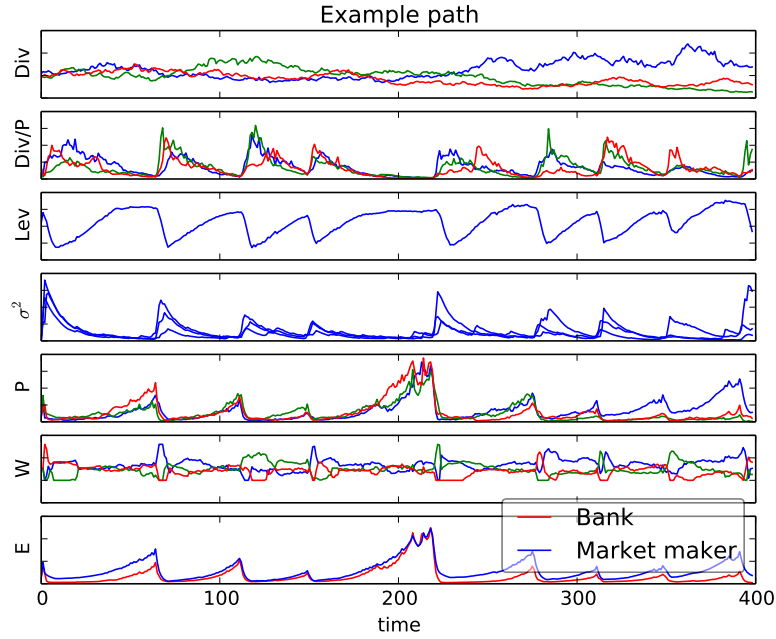


Figure 2.5: Exemplary time series of the model with a bank that actively manages its leverage. Time series from top to bottom: (1) dividends, (2) dividend price ratio, (3) leverage, (4) variance of stock prices, (5) stock prices, (6) bank portfolio weights, (7) equity of bank and fund investor.

the implementation of the simulation we use a numerical optimizer to compute the results.

In Figure 2.5 we show an example run for a simulation with all parameters as in Table 2.1 in Section 2.4.1. However instead of using the simple portfolio allocation based on the Sharpe ratio we solve the above optimization problem to compute the portfolio weights. The main dynamics, namely leverage cycles, remain unchanged under this alternative portfolio allocation scheme. Since our focus is on leverage cycles, we chose the simpler portfolio allocation method for computational efficiency. The simple portfolio allocation method runs approximately 20 times faster than the method based on numerical optimization.

Chapter 3

Macprudential policies for controlling the Basel leverage cycle

3.1 Introduction

In this chapter we extend the leverage cycle model developed in Chapter 2 and study alternative policies to control the Basel leverage cycle.

Similar to the previous chapter we focus on the side-effects of risk management as a driver of leverage cycles. A passive investor, i.e. an investor that never rebalances his or her portfolio, is *countercyclical* in the sense that falling prices drive leverage up and vice versa. In contrast, Adrian and Shin (2008) point out that many investors, such as commercial banks, use constant leverage targets, creating a positive feedback between the demand for an asset and its return. Since falling prices increase leverage, maintaining a constant target leverage causes investors to sell into a falling market and to buy into a rising market. Such behavior is inherently destabilizing: Higher (lower) demand leads to higher (lower) prices, that further increase (reduce) demand, and so on.

Adrian and Shin (2008) document even more destabilizing behavior by investors such as investment banks. These investors are actively *procyclical*, i.e. they lower leverage targets when prices fall and raise them when prices rise. This further amplifies the potentially destabilizing positive feedback between demand and returns.

Adrian and Shin argue that this can be due to regulatory risk management, since a risk neutral investor subject to a Value-at-Risk constraint increases her leverage when volatility is low and reduces it when volatility is high. In fact, volatility and prices cannot be disconnected. There is empirical evidence for a negative correlation between returns and volatility (Black, 1976; Christie, 1982; Nelson, 1991; Engle and Ng, 1993). This implies that when prices increase (decrease), volatility decreases (increases) and target leverage goes up (down), which results in leverage procyclicality. The fact that minimum capital requirements based on VaR are likely to result in procyclical behavior was also pointed out by Estrella (2004).

Since VaR risk management induces the procyclicality of leverage with respect to prices, in the following we will refer to a *procyclical leverage policy* as one for which banks are required to reduce their target leverage when volatility increases, and are allowed to increase it when volatility decreases. It is important to stress that this feedback can have significant macro-economic consequences. For instance Van den Heuvel (2002) investigates the effect of capital regulation on the transmission of monetary policy via bank lending, finding that leverage procyclicality can lead to an amplification of monetary policy.

In contrast to the procyclical case, we will refer to a *countercyclical leverage policy* as one for which banks can target a higher leverage if volatility is high, but are required to reduce their leverage when volatility is low. The rationale for such a policy is to counteract the potentially destabilizing positive feedback between demand and returns that occurs under procyclical policies. Here, however, we show that an actively countercyclical policy can also be destabilizing. In fact, under a policy that is countercyclical with respect to risk, we observe in our model that the relation between returns and volatility can be reversed, and that higher (lower) volatility is associated with higher (lower) prices. Therefore, if an investor buys when volatility increases, this can further increase volatility and prices, which raises leverage targets, etc., also

resulting in a positive feedback loop. The challenge for policy makers is to find a risk control policy that avoids the Scylla and Charybdis of excessively procyclical policies on one side or excessively countercyclical policies on the other. The aim of this chapter is to understand how the cyclicalities of leverage policies affects the properties of the financial system, and to seek the proper compromise between these two extremes.

It has to be stressed that the concept of cyclicalities we refer to in this chapter is with respect to risk, not with respect to the behavior of macroeconomic indicators. For example, Drehmann and Gambacorta (2012) provide counterfactual simulations showing how leverage policies that are countercyclical with respect to the difference between the credit-to-GDP ratio and its long-run average can help making the economy more stable. The focus of our chapter, however, is on the circumstances in which risk control can cause financial instability, and how to make an effective tradeoff between systemic vs. individual risk.

The consequences of procyclical leverage have now been studied by many authors.¹ The fact that feedback loops due to capital requirement constraints can lead to the amplification of shocks has been demonstrated, for example, by Zigrand et al. (2010), He and Krishnamurthy (2012), Thurner et al. (2010) and Adrian and Boyarchenko (2012, 2013). In Chapter 2 we go beyond this by showing how leverage constraints can lead to an endogenous cycle, i.e. one in which spontaneous oscillations occur. We call this the *Basel leverage cycle*.² While Adrian and Boyarchenko (2012, 2013) study this through a dynamic stochastic general equilibrium model, in Chapter 2 we consider a more stylized setting where investment decisions of households and unleveraged funds are not explicitly modeled. The latter model has the virtue of being very simple, and also of showing how under bounded rationality endogenous

¹See for example Danielsson et al. (2001), Danielsson et al. (2004), Adrian and Shin (2008), Shin (2010), Zigrand et al. (2010), Adrian et al. (2012), Tasca and Battiston (2012), Poledna et al. (2013), Adrian and Shin (2014) and Brummitt et al. (2014).

²However bear in mind that this can occur even without any regulatory policy constraint, due to prudent risk managers who limit the risk of individual institutions in isolation while failing to properly take systemic risk into account.

leverage cycles can occur even in a deterministic limit where there are no shocks. In this chapter we modify the model presented in Chapter 2 by adding a fundamental noise trader subject to exogenous noise with clustered volatility, which allows us to study the tradeoff between micro and macro prudential regulation. We also present a full stability analysis.

Our main contribution is the identification of optimal leverage policies. In this model the exponent of the relationship between perceived risk and target leverage is a free parameter. In this way it is possible to capture both procyclical and countercyclical leverage policies in a single model. The ability to interpolate between procyclical and countercyclical leverage policies in one model allows us to compare leverage policies of different “cyclicality” and study their effectiveness in controlling leverage cycles.

In order to do this it is necessary to choose a criterion for selecting the optimal policy. We believe a good policy is one that maximizes leverage at a given level of overall risk to the financial system. Maximizing leverage is desirable because this means that the capital of the financial system is put to full use in providing credit to the real economy. In fact, for reasons of convenience it is more feasible for us to minimize risk at a given leverage; this is essentially equivalent, since the average leverage can be adjusted to match any desired risk target (and the policy that is selected will be the same). We measure risk in terms of realized shortfall, i.e. the average of large losses to the financial system as a whole.

The main result of this chapter is that the optimal policy depends critically on two parameters: the relative size of the bank and the fundamentalist; and the amplitude of the exogenous noise. A procyclical leverage policy such as that of Basel II is optimal when the exogenous noise is high, the volatility is strongly clustered and the bank is small; in the opposite limit where these conditions are not met the optimal policy is closer to constant leverage.

3.2 A simple model of leverage cycles

As shown in the Section 2.2 of the previous chapter, there is at least anecdotal evidence for a strong correlation between changes in perceived risk, financial sector leverage and asset prices. This correlation motivates the question of causal dependency between these three variables: How do leveraged investors respond to changes in market prices? How are market prices affected by investors' portfolio adjustments? How does this potential feedback loop affect the overall dynamics of the financial system? Finally, what should regulators do to control this feedback loop, and how do they make an appropriate compromise between microprudential and macroprudential regulation?

3.2.1 Sketch of the model

We consider a financial system composed of a leveraged investor (called a *bank* for simplicity), an unleveraged fund investor, which we call the *fund*, and a passive outside lender that provides credit as required by the bank. The bank and the fund make a choice between investing in a risky asset whose price is determined endogenously vs. a risk free asset with fixed price, which we will call *cash*.

We focus on risk management by assuming the bank has a fixed preference for the relative weights of the risky asset and cash. The bank's risk management consists of two components: First, we assume that the bank estimates the future volatility of its investment in the risky asset by using an exponential moving average of historical returns. Second, the bank uses the estimated volatility to set its desired leverage. The target can be set either by internal risk management or by externally imposed regulatory constraints: The net result is the same. If the bank is below its desired leverage, it will borrow more and use the additional funds to expand its balance sheet. Conversely, if it is above its desired leverage, it will liquidate part of its investments and pay back part of its debt.

The fund is a proxy for the rest of the financial system, i.e. the part that does

not do leverage targeting. Leverage targeting creates inherently unstable dynamics, as it implies buying into rising markets and selling into falling markets. Thus it is necessary have at least one other investor who plays a stabilizing role. We model the fund as a weakly fundamentalist investor whose investment decisions are perturbed by exogenous random shocks reflecting information flow or decision processes outside the scope of this model. The exogenous random shocks display clustered volatility. The fund investor and the bank interact through the market for the risky asset and the market clearing price is determined by the investments of the fund and the bank. Because of the clustered volatility, in the absence of any systemic effects, the bank should adjust its leverage to maintain a constant Value at Risk. With systemic effects this becomes more complicated, which is what this model allows us to investigate.

In addition to its portfolio management decisions, we assume that the bank tries to maintain a constant target equity. This is consistent with the empirical observation that the equity of commercial and investment banks is roughly constant over time; see Adrian and Shin (2008). In order to conserve cash flow in our model, we assume that dividends paid out by the bank when the equity exceeds the target are invested in the fund, while new capital invested in the bank when the equity is below the target is withdrawn from the fund. This prevents all the wealth from accumulating with either the bank or the fund and makes the asymptotic dynamics stationary.

Figure 3.1 shows a diagrammatic representation of the model. The main driver of the dynamics is the feedback loop between changes in the price of the risky asset and balance sheet adjustments: The bank reacts to price changes to maintain its capital requirements under its perception of risk; similarly the fund invests as one expect from a fundamentalist, buying when prices are below value and selling when they are above. The balance sheet adjustments of the bank and the fund determine the price, which in turn feeds back to determine their decisions.

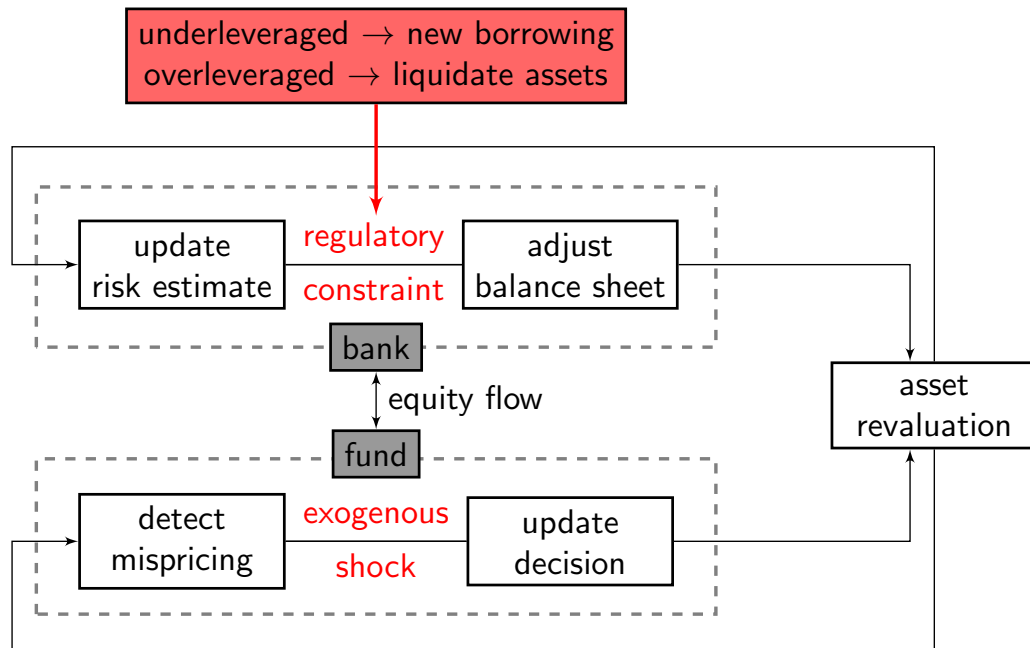


Figure 3.1: Diagrammatic representation of the model: The bank and the fund interact through price formation. The bank's demand for the risky asset depends on its estimated risk based on historical volatility and on its capital requirement. The demand of the fund consists of a mean reverting component that tends to push the price towards its fundamental value; in addition there is a random exogenous shock to the fund's demand that has clustered volatility. Price adjustments affect the bank's estimation of risk and the mean reverting behavior of the fund. The cash flow consistency in the model is enforced by equity flowing between the bank and the fund in equal amounts. The driver of the endogenous dynamics is the feedback loop between price changes, volatility and demand for the risky asset.

3.2.2 Leverage regulation:

The most important ingredient of our model is the fact that the bank is subject to a capital requirement policy that determines its investment strategy. The leverage ratio³ of the bank is defined as

$$\lambda(t) = \frac{\text{Total Assets}}{\text{Equity}}, \quad (3.1)$$

and the capital requirement policy implies a leverage constraint of the type $\lambda(t) \leq \bar{\lambda}(t)$, i.e. the bank is allowed a maximum leverage $\bar{\lambda}(t)$. Therefore we will use the term capital requirement/buffer policy and leverage policy synonymously throughout this paper. Conditional on the leverage constraint, it can be shown that the return on equity of the bank is maximized if $\lambda(t) = \bar{\lambda}(t)$ (see for instance Shin (2010)). We therefore assume that the bank always targets its maximum allowed leverage $\bar{\lambda}(t)$.⁴

We assume that $\bar{\lambda}(t)$ depends on the bank's estimate of the volatility of the risky asset, i.e. $\bar{\lambda}(t) = F(\sigma^2(t))$, where $\sigma^2(t)$ is the bank's perceived risk. Although nothing we do here depends on this, to gain intuition it is useful to compute the function F under the special case of a Value-at-Risk constraint with normally distributed returns. In this case the bank's target leverage is given by (see for example Zigrand et al. (2010)):

$$\bar{\lambda}(t) = F_{\text{VaR}}(\sigma^2(t)) = \frac{1}{\sigma(t)\Phi^{-1}(a)} \propto \frac{1}{\sigma(t)},$$

where Φ is the cumulative distribution of the standard normal, a is the VaR quantile, and σ the volatility of the risky asset. Under this specification the bank increases its

³We use this definition of leverage in analogy to the Tier 1 regulatory leverage ratio (Tier 1 capital over bank total assets). An alternative definition of the leverage ratio only considers risky assets in the numerator. Since in our model the bank holds the share of risky assets to total assets fixed, this alternative definition simply introduces a multiplicative constant into the leverage calculation and does not affect the qualitative outcome of our model.

⁴In reality banks usually keep more capital than required by regulation in order to reduce the cost of recapitalization or portfolio adjustments associated with violation of the minimal capital requirement. Using this perspective, Peura and Keppo (2006) explain the pattern of capital buffers observed in a sample of US commercial banks. However, note that our results remain valid even if we assume that banks hold more capital than required by the regulator. We only require that the resulting bank capital buffer responds to changes in perceived risk in a well defined way, i.e. in our model changes in the capital buffer are more important than the level of the capital buffer.

leverage when the volatility of the risky asset diminishes and decreases its leverage in the opposite case. Motivated by Adrian and Shin (2014), we classify leverage policies as follows:

Definition 1. A leverage policy $F(\sigma^2(t))$ is *procyclical* if $dF/d\sigma^2 < 0$ and *countercyclical* if $dF/d\sigma^2 > 0$.⁵

A class of leverage policies that allows us to interpolate between procyclical and countercyclical leverage policies is given by

$$\bar{\lambda}(t) = F_{(\alpha, \sigma_0^2, b)}(\sigma(t)) := \alpha(\sigma^2(t) + \sigma_0^2)^b, \quad (3.2)$$

where $\alpha > 0$, $\sigma_0^2 > 0$ and $b \in [-0.5, 0.5]$. We refer to α as the bank's riskiness. The larger α the larger the bank's target leverage for a given level of perceived risk $\sigma^2(t)$.⁶ For the special case where returns are normal with $b = -0.5$ and $\sigma_0^2 = 0$, α is linked to the quantile used to measure VaR by the inverse cumulative normal distribution. In the more general case where there are heavy tails or with other choices of parameters this correspondence is no longer valid. Nonetheless, the relationship between α and risk remains monotonic under any sensible risk measure, and one can simply think of α as a risk parameter and Equation ((3.2)) as a particular choice of F , corresponding to a volatility estimate based on historical standard deviation.

The parameter b is called the *cyclical parameter*, due to the fact that $F_{(\alpha, \sigma_0^2, b)}$ is procyclical for $b < 0$ and countercyclical for $b > 0$ (see Definition 1). For procyclical policies the leverage is inversely related to risk, i.e. leverage is low when risk is high and vice versa. For countercyclical policies the opposite is true; when risk is

⁵ This definition could be generalized for any risk measure Σ ; we use the standard deviation σ for simplicity.

⁶ Note that under standard Value-at-Risk the bank's leverage depends on the variance of its entire portfolio which in our model includes non risky cash holdings. Usually, the portfolio variance is computed as the inner product of the covariance matrix with the portfolio weights. In our case this implies that the portfolio variance is simply $\sigma^2(t)$ scaled by the bank's investment weight in the risky asset w_B . However, since we take w_B constant throughout, the resulting risk rescaling factor can be absorbed into α without loss of generality. Therefore, we make $F_{(\alpha, \sigma_0^2, b)}$ only a function of $\sigma(t)$.

high leverage is also high, see Figure 3.2. We explore countercyclical policies since naively one would expect that relaxing the leverage constraint in times of stress (high risk) can stabilize the system as it prevents banks from selling into falling markets. It is important to note that our definition of policy cyclicalities does not refer to macroeconomic measures such as the credit-to-GDP ratio or asset prices. Instead it is defined solely by the bank's response to changes in perceived risk. In this sense the countercyclical policies proposed in this model differ from the countercyclical capital buffer proposed by the Bank of England, see FPC (2014), which depends on the credit-to-GDP ratio.

Finally, the parameter σ_0^2 bounds the leverage policy from above for $b < 0$ such that $F_{(\alpha, \sigma_0^2, b)} \leq \alpha(\sigma_0^2)^b$ and from below for $b > 0$ such that $F_{(\alpha, \sigma_0^2, b)} \geq \alpha(\sigma_0^2)^b$. We introduce σ_0^2 primarily to regularize the system such that it does not have an infinite leverage at its fixed point. An infinite leverage at the fixed point would arise for $\sigma_0^2 = 0$ since at the fixed point prices are constant implying zero perceived risk ($\sigma^2 = 0$). We avoid this singularity by introducing σ_0^2 . For the remainder of this paper we will restrict our analysis to leverage policies of the class $F_{(\alpha, \sigma_0^2, b)}$. We illustrate the three corner cases of the above class of leverage policies with $\sigma_0^2 > 0$ in Figure 3.2.

3.2.3 Asset price dynamics:

The bank's target leverage $\bar{\lambda}(t)$ at time t defines a target portfolio value $\bar{A}_B(t) = \bar{\lambda}(t)E_B(t)$, where $E_B(t)$ is the equity of the bank. The difference between the target portfolio and the current portfolio then determines the magnitude of the change of the balance sheet required for the bank to achieve its target leverage. We denote this change of the balance sheet by $\Delta B(t)$:

- If $\Delta B(t) > 0$, the bank will borrow $\Delta B(t)$ and invest the corresponding amount into the risky and the risk free asset according to the bank's (fixed) portfolio weights.

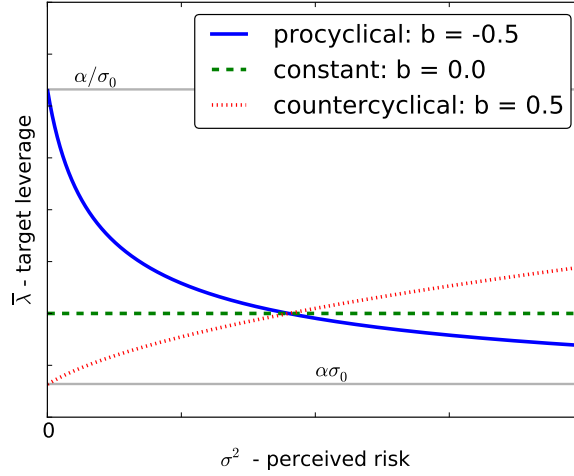


Figure 3.2: Illustration of target leverage as a function of perceived risk based on Equation (3.2) with $\sigma_0^2 > 0$. Continuous blue line: procyclical policy with $b = -0.5$. Dashed green line: constant leverage policy with $b = 0$. Dotted red line: countercyclical policy with $b = 0.5$. Continuous grey lines show relation to σ_0 .

- If $\Delta B(t) < 0$, the bank will liquidate part of its portfolio in order to pay back $\Delta B(t)$ of its liabilities.

The evolution of the fund's portfolio weight in the risky asset depends on the asset's price relative to a constant fundamental value μ , and also on random innovations. The fund investor therefore combines two economic mechanisms: (1) The constant fundamental value means that the price of the risky asset is ultimately anchored on the performance of unmodeled macro-economic conditions, which we assume are effectively constant over the length of one run of our model. (2) We allow random innovations in the portfolio weight that reflect exogenous shocks.

We assume that the fund invests a fraction $w_F(t)$ of its total assets in the risky asset, and that the time evolution of $w_F(t)$ follows a mean reverting process with a GARCH(1,1) noise term. Thus, the fundamentalist investor provides a source of time varying exogenous volatility to the model. It is important that the exogenous volatility is time varying as this motivates the need for microprudential risk control: To minimize risk, the bank must estimate the expected future volatility and adjust

its leverage accordingly.

Given the aggregate demand of the bank and the fund, and assuming for simplicity that there is a supply of exactly one unit of the risky asset that is infinitely divisible, the price of the risky asset is determined through market clearing by equating demand and supply.

3.2.4 Time evolution

The model evolves in discrete time-steps of length τ . We make this a free parameter so that the model has well-defined dynamics in the continuum limit $\tau \rightarrow 0$, which is useful for calibration. At each time-step the bank and the fund update their balance sheets as follows:

- The bank updates its historically-based estimate of future volatility and computes its new target leverage accordingly. Volatility estimation is done using an exponential moving average with an approach similar to RiskMetrics (see Longerstaey (1996));
- The bank pays dividends or raises capital to reach its target equity $\bar{E}$;
- The bank determines how many shares of the risky asset it needs to trade to reach its target leverage;
- At the same time, the fundamentalist fund submits its demand for the risky asset;
- The market clearing price for the risky asset is computed and trades occur.⁷

⁷ It is important to note that the decision concerning equity and investment adjustments are taken before the current trading price of the risky asset is revealed. We therefore assume that the bank uses the price of the previous time step as a proxy for the expected trading price, and acts accordingly. This assumption of myopic expectations marks a significant departure of our model from the general equilibrium setting of Adrian and Boyarchenko (2012) and Adrian and Boyarchenko (2013), but it is common in the literature on heterogeneous agents in economics (see for instance Hommes (2006)).

3.2.5 The model as a dynamical system

The dynamics of our model can be described as an iterated map for the state variable $x(t)$, defined as

$$x(t) = [\sigma^2(t), w_F(t), p(t), n(t), L_B(t), p'(t)]^T, \quad (3.3)$$

where σ is the historical estimation of the volatility of the risky asset; w_F is the fraction of wealth invested by the fund in the risky asset; p the current price of the risky asset; n the share of the risky asset owned by the bank; L_B the liabilities of the bank; and p' is the lagged price of the asset, i.e. the price at the previous time step. A detailed derivation of the model is presented in Appendix 3.A. Here we simply present the model and provide some basic intuition. Let us introduce the following definitions:

Bank assets	$A_B(t) = p(t)n(t)/w_B,$	
Target leverage	$\bar{\lambda}(t) = \alpha(\sigma^2(t) + \sigma_0^2)^b,$	
Balance sheet adjustment	$\Delta B(t) = \tau\theta(\bar{\lambda}(t)(A_B(t) - L_B(t)) - A_B(t)),$	(3.4)
Equity redistribution	$\kappa_B(t) = -\kappa_F(t) = \tau\eta(\bar{E} - (A_B(t) - L_B(t))),$	
Bank cash	$c_B(t) = (1 - w_B)n(t)p(t)/w_B + \kappa_B(t),$	
Fund cash	$c_F(t) = (1 - w_F(t))(1 - n(t))p(t)/w_F(t) + \kappa_F(t).$	

The parameters θ and η determine how aggressive the bank is in reaching its targets for leverage and equity (i.e. the bank aims at reaching the targets on time horizons of the order $1/\theta$ and $1/\eta$).

The model can be written as a dynamical system in the form

$$x(t + \tau) = g(x(t)) \quad (3.5)$$

where the function g is the following 6-dimensional map:

$$\sigma^2(t + \tau) = (1 - \tau\delta)\sigma^2(t) + \tau\delta \left(\log \left[\frac{p(t)}{p'(t)} \right] \frac{t_{\text{VaR}}}{\tau} \right)^2, \quad (3.6a)$$

$$w_F(t + \tau) = w_F(t) + \frac{w_F(t)}{p(t)} [\tau\rho(\mu - p(t)) + \sqrt{\tau}s\xi(t)], \quad (3.6b)$$

$$p(t + \tau) = \frac{w_B(c_B(t) + \Delta B(t)) + w_F(t + \tau)c_F(t)}{1 - w_B n(t) - (1 - n(t))w_F(t + \tau)}, \quad (3.6c)$$

$$n(t + \tau) = \frac{w_B(n(t)p(t + \tau) + c_B(t) + \Delta B(t))}{p(t + \tau)}, \quad (3.6d)$$

$$L_B(t + \tau) = L_B(t) + \Delta B(t), \quad (3.6e)$$

$$p'(t + \tau) = p(t). \quad (3.6f)$$

Each of these equations can be understood as follows:

- (a) The expected volatility σ^2 of the risky asset is updated through an exponential moving average. The parameter $\tau\delta \in (0, 1)$ defines the length of the time-window over which the historical estimation is performed, while the parameter t_{VaR} represents the time-horizon used by the bank in the calculation of VaR.
- (b) The adjustment of the fund's risky asset portfolio weight w_F drives the price towards the fundamental value μ , with an adjustment rate $\tau\rho \in (0, 1)$. The demand of the fund also depends on exogenous noise, which is assumed to be a normal random variable $\xi(t)$ with amplitude $s(t) \geq 0$. The amplitude varies in time so that the variable $\chi(t) = s(t)\xi(t)$ follows a GARCH(1,1) process. The factors of τ guarantee the correct scaling as $\tau \rightarrow 0$.
- (c) The market clears. $c_F(t)$ and $c_B(t)$ are the amount of cash held respectively by the fund and the bank.
- (d) The bank ownership of the risky asset $n(t + 1)$ adjusts according to market clearing.
- (e) Bank liabilities are updated to account for the change $\Delta B(t)$ in the asset side of the balance sheet.

	Notation	Description	Default	Unit
Bank	τ	Time step	0.1	year
	t_{VaR}	Horizon for VaR calculation	0.1	year
	δ	Memory parameter	0.5	1/year
	α	Risk parameter	0.075 (v)	1
	b	Cyclical parameter	-0.5 (v)	1
	$\bar{E}$	Bank's equity target	2.27 (v)	\$
	σ_0^2	Risk offset	10^{-6}	1
	w_B	Bank's weight for risky asset	0.3 (v)	1
	θ	Balance sheet adjustment speed	9.5 (v)	1/year
	η	Equity redistribution speed	10	1/year
Fund	μ	Fundamental value	25	\$
	ρ	Mean reversion	0.1	1/year
GARCH	a_0	Baseline return variance	10^{-3}	1
	a_1	Error autoregressive term	0.016	1
	b_1	Variance autoregressive term	0.87	1

Table 3.1: Overview of parameters for the numerical model solution. Values marked with (v) indicates that they are subject to change from their default values.

- (f) The lagged price variable $p'(t)$ is required to complete the state vector, and make the map a first order dynamical system of the form given in Equation (3.5).

3.3 Examples of leverage cycles

In order to explore the dynamical behavior of the model we solve it numerically. For now we will only consider leverage policies that are procyclical; in particular we choose $b = -0.5$ throughout this section, corresponding to the case of risk management under VaR.

3.3.1 Model calibration

While this model is too stylized to be fully calibrated, approximate values for some key parameters can be obtained. This then allows us to test the realism of some of the properties of the model. In the following we will briefly discuss the choice and effect of these key parameters, including the timescale of the risk estimation, the balance

sheet adjustment speeds and bank riskiness. A full list of parameters is provided in Table 3.1.

Timescale parameters

The parameter δ sets the timescale for the exponential moving average used to estimate volatility, and is the main determinant of the overall timescale of the dynamics. The contribution to the moving average of a squared return $y(t)$ observed at time t is $y(t + \Delta t) = (1 - \tau\delta)^{\Delta t/\tau} y(t)$ at time $t + \Delta t$. We define the typical time t_δ such that $y(t + t_\delta)/y(t) = 1/e$. Thus $t_\delta = -\tau/\log[1 - \tau\delta] \approx 1/\delta$ for $\tau\delta \ll 1$. According to the RiskMetrics approach Longerstaey (1996), a typical timescale is $t_\delta \approx 2$ years. For convenience we choose a time step of $\tau = 0.1$ years; because the model is well-defined in the continuum limit, this choice does not affect the behavior of the model.⁸ With this choice of τ , taking $\delta = 0.5 \text{ year}^{-1}$ gives $t_\delta \approx 2$ years.

Another important time-parameter that affects the dynamics of our model is t_{VaR} , the time horizon over which returns are computed for regulatory purposes. In practice, the timescale for the regulatory capital requirements varies depending on the liquidity of the asset portfolio and ranges from days to years. A good rule of thumb is to choose t_{VaR} roughly equal to the time needed to unwind the portfolio. We assume $t_{\text{VaR}} = \tau = 0.1$ years, i.e. a little more than a month.

The parameters θ and η define how aggressive the bank is in reaching its target for leverage and equity. Our default assumption is that the bank tries to meet its target on a timescale of about one time step of the dynamics, and so unless otherwise stated, in the following we set $\theta = 9.5 \text{ year}^{-1}$ and $\eta = 10 \text{ year}^{-1}$. This ensures that the bank's realized leverage is always close to its target. We will vary the parameter θ and discuss how it affects the stability of the dynamics in Section 3.4.3.

⁸ We have carefully constructed the dynamical system so that it reaches a continuum limit as $\tau \rightarrow 0$ in the deterministic case. For computational efficiency we chose τ to be the largest possible value with behavior similar to that in the continuum limit.

Bank size, leverage and exogenous volatility

The relative size of the bank vs. the fund is an important parameter affecting the stability of the system. As the bank becomes larger relative to the fund, its market impact increases, which can drive endogenous instabilities. As can be seen from equation (3.23) in the appendix, the parameters $(\bar{E}, w_b, \mu \text{ and } \sigma_0)$ all jointly determine the fraction of the assets owned by the bank. Note that the numerical values chosen for the target equity $\bar{E}$ and the fundamental price μ are arbitrary – ultimately only their ratio determines the dynamics. We have chosen values for the above parameters so that the relative size of the bank to the fund is approximately 0.27. Thus the bank makes up roughly 1/3 of the market for the risky asset. While this may be too much for the banking sector alone, it should be noted that the bank represents all investors with leverage targets, and the set of institutions with a comparable Value-at-Risk based leverage constraint⁹ is larger than the banking sector.

The other key parameter affecting the stability of the model is the bank riskiness α . Increasing α increases both the bank's market power and its default risk. Note that α is also related to t_{VaR} by the fact that, all else equal, increasing the timescale over which the risk is measured corresponds to taking more risk. (Increasing α usually increases leverage, though as discussed in footnote 14 this is not always true). The bank's portfolio weight w_B for the risky asset has a similar effect to the bank's equity target $\bar{E}$.¹⁰ Increasing any of these parameters increases the bank's market power.

In our calibration we choose a particular level of bank riskiness α and the relative size of the bank and the fund to match two basic properties of the run up and the subsequent collapse of leverage and asset prices during the global financial crisis in 2008/2009. First, we seek a peak to trough ratio in the price of the risky asset of roughly 2. Second, we target a period of oscillation of roughly ten years. Matching

⁹In principle the constraint can be either imposed by a regulator, creditors or internal risk management.

¹⁰ In fact, it can be shown the the critical leverage is inversely proportional to w_B .

these calibration targets comes at the price of achieving realistic levels of bank leverage in our simulations. Given our choice for α we obtain levels of bank leverage of around 6. This is below typically observed levels of leverage of around 20. The fact that we cannot calibrate our model to match several calibration targets simultaneously is a clear weakness of the model, but is not surprising given its simplicity.

Finally, we pick parameters for the fund GARCH process a_0 , a_1 and b_1 in order to achieve a randomly perturbed asset price path that still follows a leverage cycle roughly as observed in Figure 2.1; i.e. we allow for random perturbations but ensure that endogenous volatility due to the bank's leverage adjustment dominates over exogenous volatility due to the fund's investment perturbations.

3.3.2 Overview of model dynamics

We now build some intuition about the model dynamics. First, consider the extreme case where $\bar{E} \rightarrow 0$, i.e. where the market power of the bank is negligible so that the price dynamics are dominated by the fund. This is the purely *microprudential* case where the bank's actions have no significant effect on the market and the only source of volatility is exogenous. For a stochastic fund, i.e. $s > 0$, we expect the price to perform a mean reverting random walk around the fundamental price μ .¹¹ In the deterministic case, i.e. $s = 0$, the fund updates its portfolio weight until $p(t) = \mu$, i.e. until the price has converged to the fundamental price and the system settles to a fixed point.

When $\bar{E}$ is large enough such that the bank has a significant impact on the price process, the dynamics are less straightforward. We refer to this scenario as the *macro-*

¹¹While we do not provide a formal proof for this statement, we clearly observe this being the case in numerical simulations. We introduce the mean reversion in prices to ensure that the fund plays an actively stabilizing role in the dynamics and thereby counteracts the potentially destabilizing impact of the bank's behavior. The assumption of a fixed fundamental value of the risky asset implies that we assume that the financial system is anchored in a constant real economy. We concede that the real economy may change over our modeling horizon, however this simplifying assumption allows us to focus on the dynamics introduced by the financial sector while abstracting from any dynamics originating in the real economy.

prudential case. Suppose, for example, that there is a negative shock in the investment of the fund. This negative shock will lead to an increase in the perceived risk $\sigma^2(t)$. Under a procyclical leverage policy an increase in perceived risk causes a decline in the bank's leverage constraint. As a consequence the bank will have to deleverage in the time step following the negative shock, i.e. $\Delta B(t) < 0$. If the bank decreases its position and it has non-negligible market impact, the price will drop for $\Delta B(t) < 0$ ceteris paribus. This is clear from Equation (3.20) in the appendix. Thus an initial negative shock can be amplified by the bank's deleveraging response. This destabilizing feedback loop is a key ingredient for what is to come and distinguishes risk management in the macroprudential case from the microprudential case. In the macroprudential case the bank's risk management affects the system's state and introduces endogenous volatility on top of exogenous volatility.

In the following we will investigate the following four scenarios:

- (i) Deterministic, microprudential: $\bar{E} = 10^{-5}$ and $s = 0$,
- (ii) Deterministic, macroprudential: $\bar{E} = 2.27$ and $s = 0$,
- (iii) Stochastic, microprudential: $\bar{E} = 10^{-5}$ and $s > 0$.
- (iv) Stochastic, macroprudential: $\bar{E} = 2.27$ and $s > 0$,

Unless otherwise stated all parameters are as specified in Table 3.1. The first two cases are for the deterministic limit with $s = 0$, which is useful to gain intuition. The last two cases are with more realistic levels of exogenous noise. We summarize our results for scenarios (i) and (ii) in Figure 3.3 and for scenarios (iii) and (iv) in Figure 3.4.

The microprudential scenarios (i) and (iii) behave as expected: In the deterministic limit the system simply settles into a fixed point with prices equal to fundamental values. When there is exogenous noise the system makes excursions away from the

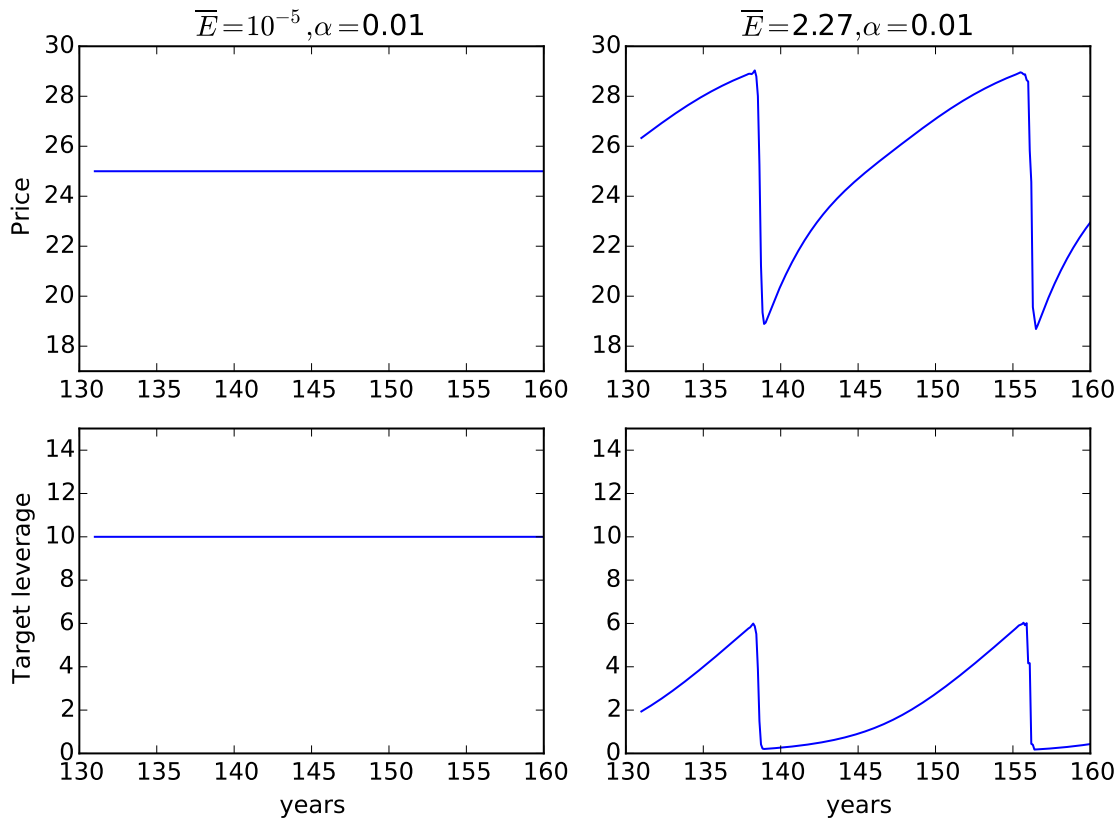


Figure 3.3: Time series of price and leverage in the deterministic case. Left panel: scenario (i) – microprudential, the fund dominates the bank ($\bar{E} = 10^{-5}$), i.e. the bank has no significant market impact. In this case the system goes to a fixed point equilibrium where the leverage and price of the risky asset remain constant. Right panel: scenario (i) – macroprudential, the bank has significant market impact ($\bar{E} = 2.27$). In this case the bank’s risk management leads to persistent oscillations in leverage and price of the risky asset with a time period of roughly 15 years.

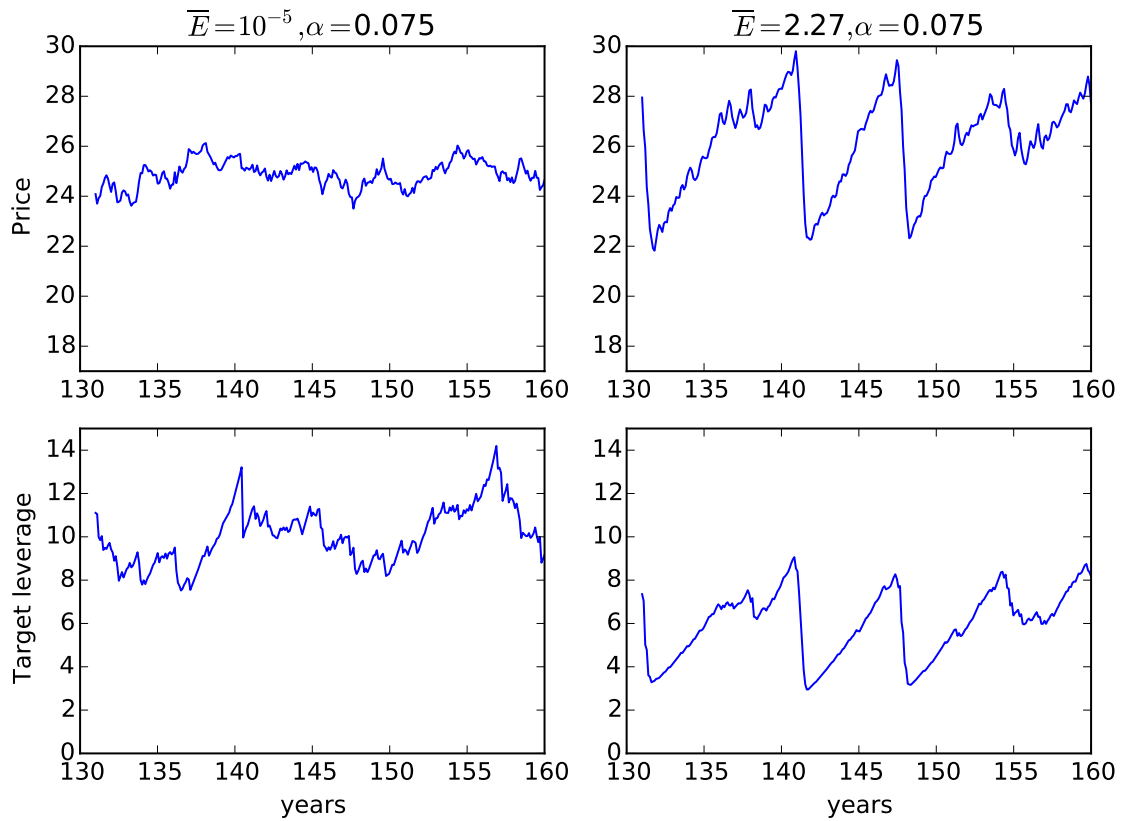


Figure 3.4: Time series of price and leverage in the stochastic case. Left panel: scenario (iii) – microprudential, the fund dominates the bank ($\bar{E} = 10^{-5}$), i.e. the bank has no significant market impact. In this case the price is driven by the fund’s trading activity and performs a mean reverting random walk around the fundamental value $\mu = 25$. Right panel: scenario (iv) – macroprudential, the bank has significant market impact ($\bar{E} = 2.27$). In this case the bank’s risk management leads to irregular oscillations in leverage and price of the risky asset that are similar to the deterministic case.

fixed point but never drifts far away from it, and the dynamics remain relatively simple.

In contrast the macroprudential scenarios (ii) and (iv) display large oscillations both in leverage and price. We refer to this oscillation as the *Basel leverage cycle*. Surprisingly, the oscillations occur even in the deterministic limit, i.e. without any external shocks. During the cycle the price and leverage slowly rise and then suddenly fall, with a period of about $\Delta t \approx 15$ years in the deterministic case. In the stochastic case we observe a period of about $\Delta t \approx 10$ years. This is roughly on the order of magnitude of the period of the Great Moderation and the subsequent financial crisis. Note that the period of oscillation depends heavily on the risk estimation horizon t_δ which allows us to roughly calibrate the model to the observed leverage cycle in Fig. 2.1.¹²

The oscillations have the following economic interpretation: Suppose we begin at about $t = 140$ years in the left panel of Figure 3.3, with leverage low, perceived risk high, and prices low but increasing. The perceived risk slowly decreases as the memory of the past crisis fades. From a mechanical point of view this is due to the smoothing action of the exponential moving average. As the moving average is updated on each timestep, σ^2 decreases; this causes the leverage to increase, and the bank buys more shares to meet its increased leverage target. The change in price is lower than the current historical average, so on the next step the volatility σ^2 drops, driving the leverage higher. As the leverage becomes very high the system becomes increasingly fragile and the target leverage becomes very sensitive to small changes in perceived risk.

¹² In fact, the period is roughly proportional to $1/t_\delta$. The period becomes large for very low values of $\delta\tau$ and then declines as the risk estimation horizon is increased – for the range of $\tau\delta$ the variation of the period ranges over roughly two orders of magnitude. The period also depends on the values of η , t_{VaR} , θ , w_B and α . As η or t_{VaR} are increased the period increases. As θ , α and w_B are increased, i.e. as the system moves to a more unstable regime, the period declines. However, for these parameters the variation in the period over the parameter range is only on roughly one order of magnitude. The period of oscillation is robust to changes in w_F .

As the price increases further its deviation from the fundamental value increases. Thus, the fund will decrease its investment into the risky asset. At first, the price impact of the trading of the fund and bank are in opposite directions and lead to small net changes in prices. However, I conjecture that as the mis-pricing increases, there will be a critical price at which the price impact of the fund will be sufficiently large to lead to an *increase* in perceived risk.

This increase in perceived risk then ends a sustained period of decreasing perceived risk, rising prices and leverage. The increase in perceived risk triggers a large downward adjustment of target leverage since target leverage is high and thus very sensitive to even small changes in perceived risk. The adjustment in target leverage leads to asset sales from the bank. This in turn leads to the sharp crash in asset price which increases perceived risk further.

The downward crash ultimately comes to an end by the increasingly heavy investment of the fundamentalist fund as the price drops below its fundamental value. After the crash volatility is high and leverage is low, and the cycle repeats itself. While we do not explore the exact mechanics of the cycle further, one might be able to determine a critical deviation of the price from its fundamental value at which the fund's downward price pressure becomes sufficiently large to trigger a price crash. This could then allow a further characterization of the determinants of the period of the leverage cycle, in particular it should corroborate the dependence on the risk estimation horizon t_δ and should suggest a dependence on the fund's mean reversion parameter ρ .

The fragility that drives the crashes comes from the fact that at high levels of leverage a small increase in risk is sufficient to cause a drastic tightening of the leverage constraint. This intuition can be made precise by comparing the derivative of the leverage policy for high vs. low leverage; for convenience we take $\sigma_0^2 \ll 1$.¹³

¹³ For $b < 0$ the parameter σ_0 imposes a cap on the target leverage; larger values for σ_0^2 would make this unrealistically low.

The result is that

$$\frac{dF_{(\alpha, \sigma_0^2, -0.5)}}{d\sigma^2(t)}(\sigma^2(t)) = \begin{cases} -0.5/\sigma_0 \ll 0 & , \text{ for } \sigma^2(t) \rightarrow 0 \wedge \sigma_0^2 \ll 1, \\ 0 & , \text{ for } \sigma^2(t) \rightarrow \infty \wedge \sigma_0^2 \ll 1. \end{cases}$$

In the high leverage limit, i.e. when the perceived risk is small, the sensitivity of the leverage target F to variations in risk tends to infinity as σ_0 vanishes. In contrast the sensitivity is zero in the opposite limit where leverage is low and perceived risk is large. Thus increasing leverage of the banking system has a two-fold destabilizing effect: It can make the dynamics unstable and lead to chaos (as we will show below), but it also makes it more sensitive to shocks, which can result in sudden deleveraging.

In summary, depending on the choice of parameters, the model either goes to a fixed point (scenario (ii)) or shows chaotic irregular cycles (scenarios (i) and (iii)). As expected the dynamics become more complicated when noise is added, but the essence of the Basel leverage cycle persists even in the zero noise limit.

3.4 Determinants of model stability

3.4.1 Deterministic case

In the deterministic case the standard tools of linear stability analysis can be used to characterize the boundary between the fixed point equilibrium and leverage cycles. In this section we will use this to characterize the behavior of the system as the risk parameter α and the cyclical parameter b are varied. We begin by studying the deterministic case, where we can compute things analytically, and then present numerical results for the stochastic case. The details of the stability analysis are presented in Appendix 3.A.

The system has a unique fixed point equilibrium x^* , given by

$$\begin{aligned} x^* &= (\sigma^{2*}, w_F^*, p^*, n^*, L_B^*, p'^*) \\ &= (0, w_F(0), \mu, \frac{1}{\mu} \alpha \sigma_0^{2b} \overline{E} w_B, (\alpha \sigma_0^{2b} - 1) \overline{E}, \mu). \end{aligned} \tag{3.7}$$

This corresponds to a leverage λ^* and relative size of bank to fund $R_c(x^*)$, given by

$$\lambda^* = \alpha \sigma_0^{2b},$$

$$R(x^*) = \frac{A_B^*}{A_F^*} = \frac{\lambda^* E_B^*}{(1 - n^*) p^* / w_F^*}. \quad (3.8)$$

At the equilibrium x^* the price is constant at its fundamental value and the bank is at its target leverage. The stability of the equilibrium depends on the parameters. Regime (i) observed in the numerical simulations of the previous section corresponds to the stable case. In this case, regardless of initial conditions, the system will asymptotically settle into the fixed point x^* . In contrast, when the fixed point x^* is unstable there are two possibilities. One is that there is a leverage cycle, in which the dynamics are locally unstable but exist on a chaotic attractor that is globally stable; the other is that the system is globally unstable, in which case the price either becomes infinite or goes to zero.

In Figure 3.5 we show the results of varying the risk parameter α and the cyclicity parameter b . The risk parameter α provides the natural way to vary the risk of the bank, but the realized risk for a given α depends on parameters due to other factors such as changes in volatility. For diagnostic purposes leverage is a better measure.¹⁴ Figure 3.5 shows each of the three regimes, corresponding to the stable equilibrium, leverage cycles or global instability, as a function of the leverage and the cyclicity parameter b . The boundary where the fixed point equilibrium becomes unstable is computed analytically based on the leverage λ_c^* where the modulus of the leading eigenvalue is one. The boundary for globally unstable behavior is more difficult to compute as it requires numerical simulation.

This diagram reveals several interesting results. As expected, for low leverage the system is stable and for higher leverage it is unstable. Somewhat surprisingly, the critical leverage λ_c^* is independent of b , and consequently the size of the regime

¹⁴ While α tends to increase leverage, when the risk policy is procyclical the behavior is not always monotonic. This is because increasing α tends to increase volatility, but increasing volatility drives the target leverage down, so the two effects compete with each other.

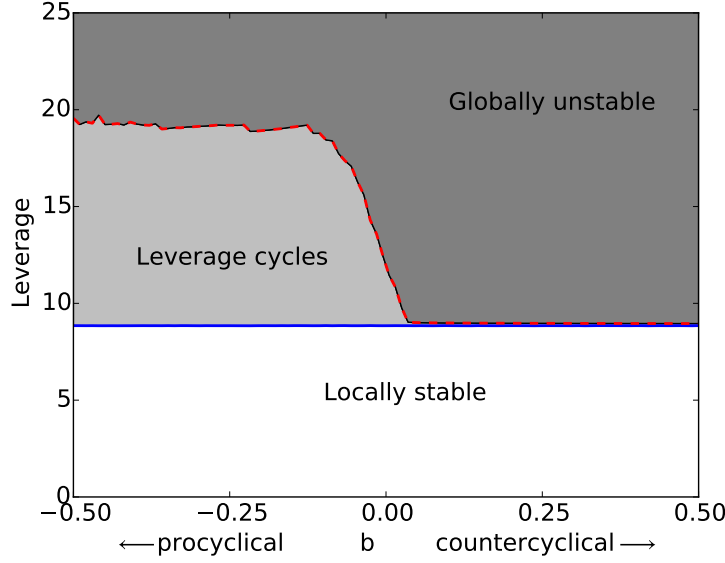


Figure 3.5: A bifurcation diagram showing the three regimes in the deterministic case. The risk parameter α and the cyclicity parameter b are varied while holding the other parameters constant at the value in Table 1. The white region corresponds to a stable fixed point equilibrium, the light gray region to leverage cycles and the dark gray region to global instability. The blue line corresponds to the critical leverage λ_c^* in Equation (3.8) at the critical value α_c where the fixed point becomes unstable.

with the stable equilibrium is unaffected by whether the risk control is procyclical or countercyclical. Note however that the choice of b does affect the system dynamics beyond the size of the stable regime. In particular, in the procyclical regime there is a substantial area of parameter space with leverage cycles. For the countercyclical regime, in contrast, there is only a small regime with leverage cycles. Throughout most of the parameter range the system makes a direct transition from the stable fixed point equilibrium to global instability. The instability is not surprising: In the countercyclical regime there is an unstable feedback loop in which increasing leverage drives increasing prices and increasing volatility, which further increases the leverage. Thus for high leverage there are unstable regimes for both pro- and countercyclical behavior, but the instability is even worse in the countercyclical regime.

3.4.2 Stability when there is exogenous noise

In the case where there is exogenous noise we can only measure the stability numerically. This is done by computing the largest Lyapunov exponent of the dynamics. The Lyapunov exponents are a generalization of eigenvalues that apply to trajectories that are more complicated than fixed points. The leading Lyapunov exponent measures the average rate at which the separation between two nearby points changes in time – when the dynamics are locally stable nearby points converge exponentially and the leading Lyapunov exponent is negative, and when they are locally unstable nearby points diverge exponentially and the leading Lyapunov exponent is positive. The Lyapunov exponent is a property of a trajectory, but for dissipative systems such as ours, it is also a property of the attractor.

Formally, we compute the largest Lyapunov in the usual way, i.e.: (1) Simulate the system for a number of iterations until it has settled on its attractor. (2) Perturb the system along one of its dimensions by a small amount $d_0 = 10^{-8}$. This yields a perturbed trajectory $x'(t)$ with $x'_0(t) \rightarrow x_0(t) + d_0$. (3) Iterate the reference and perturbed trajectories by one time step and compute the distance $d_1(t+1) = |x(t+1) - x'(t+1)|$. (4) Compute the rate of diversion of the reference and perturbed trajectories $\log(d_1(t+1)/d_0)$. (5) Rescale all components of the perturbed vector such that $d_0 = |x(t+1) - x'(t+1)|$. (6) Repeat for T_{LE} times. The largest Lyapunov exponent is then simply

$$LE = \frac{1}{T_{LE}} \sum_t^{T_{LE}} \log \left(\frac{d_1(t)}{d_0} \right). \quad (3.9)$$

A negative Lyapunov exponent implies a fixed point, and a positive Lyapunov exponent implies a chaotic attractor. As expected, in the deterministic case we observe that leverage cycles have a positive leading Lyapunov exponent, confirming that the dynamics are chaotic.

It is also possible to compute Lyapunov exponents for stochastic dynamics. To

understand the basic idea of how this is done, imagine two realizations of the dynamics with the same sequence of random shocks, but starting at slightly different initial conditions, see Crutchfield et al. (1982). Because the random noise is the same in both cases, it is possible to follow two infinitesimally separated points and measure the rate at which they separate. If the leading Lyapunov exponent is positive this means that the dynamics will strongly amplify the noise. Strictly speaking we compute the pathwise Lyapunov exponent for a given sequence of random shocks. We compute the Lyapunov exponent for 300 independent sequences of random shocks and then average. The results in Figure 3.6 are obtained using the average pathwise Lyapunov exponent. The standard deviation around the average Lyapunov exponent is small and typically on the order of 10% of the average value.¹⁵

We compare the stability for the stochastic and deterministic cases in Figure 3.6. This is done for the procyclical case only, since the chaotic regime disappears for the countercyclical case and Lyapunov exponents can no longer be calculated. In the stochastic case the critical leverage is computed as the time average of the target leverage observed in the simulation at which the system's Lyapunov exponent becomes positive. Interestingly, the critical leverage in the stochastic case first starts below the deterministic critical leverage and then approaches it as b is increased. This indicates that for strongly procyclical leverage policies noise destabilizes the system.

The most interesting conclusion from comparing the stochastic and deterministic cases is that when the dynamics are strongly procyclical (i.e. for $-0.5 < b < -0.2$) the noise significantly lowers the stability threshold. In contrast, for larger values of $b > -0.2$ there is little difference in the stability threshold in the two cases. This indicates that the dynamics becomes more stable when the risk policy is close to

¹⁵I estimated the uncertainty in the critical leverage due to the uncertainty in the average Lyapunov exponent by computing the critical leverage at which the average Lyapunov exponent shifted by one standard deviation down (up) becomes positive. This yields confidence bounds for the critical leverage. The bounds are very tight around the critical leverage found for the average Lyapunov exponent.

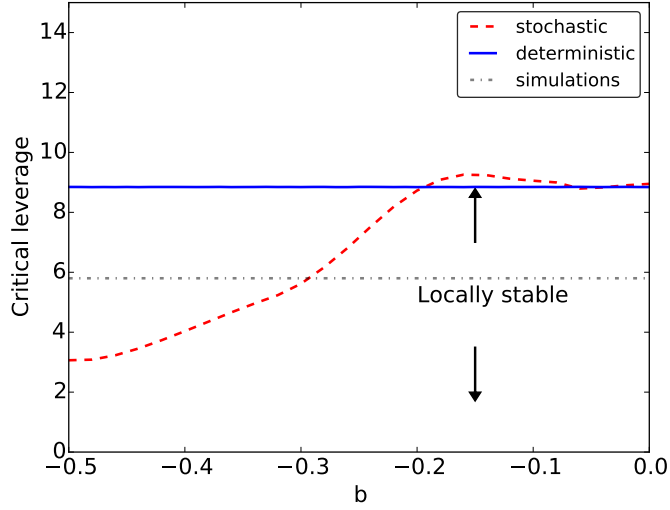


Figure 3.6: A comparison of stability when the dynamics are deterministic vs. stochastic for the procyclical region ($b < 0$). As in the previous figure, the critical leverage λ_c^* for the deterministic case is shown as a blue line. The dashed red line shows the parameter value where the dynamics become unstable as measured by the leading Lyapunov exponent; note the transition to chaos occurs at a much lower leverage.

constant leverage. This, together with the fact that in the countercyclical regime the system goes straight from stability to global instability, suggests that intermediate values of cyclicity (nearer to constant leverage) are likely to be most stable.

3.4.3 Slower adjustment leads to greater stability

The bank's balance sheet adjustment speed θ has a strong effect on stability with interesting regulatory implications. Intuitively, decreasing the adjustment speed should make the system more stable. To take an extreme case, in the limit $\tau\theta \rightarrow 0$, the bank would hold its balance sheet constant regardless of changes in perceived risk. This would eliminate the feedback loop between asset prices, perceived risk and investment. Even when $\theta > 0$, decreasing the adjustment speed should have a stabilizing effect.¹⁶

¹⁶We have considered the case where the bank increases its leverage quicker than it decreases it. We have done this introducing an asymmetry in the parameter θ that controls the speed of leverage adjustment, i.e. introducing a parameter θ_+ for the speed of levering up and a parameter θ_- for the speed of deleveraging. By allowing such asymmetric specification, we find that the dynamics

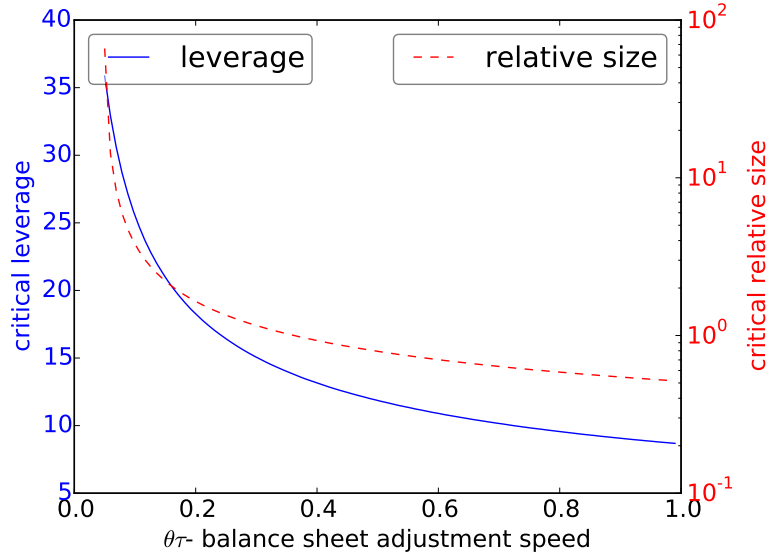


Figure 3.7: Critical leverage λ_c^* (solid blue line, left vertical axis) and the critical value of the relative size of the bank to the fund $R_c(x^*)$ (dashed red line, right vertical axis) as a function of the balance sheet adjustment speed $\theta\tau$. Other parameters are as in Table 1. The stability of the financial system can be dramatically improved by lowering the adjustment speed.

To test this we study how the critical leverage λ_c^* and critical relative size $R_c(x^*)$ depend on the adjustment speed $\theta\tau$ (we vary θ and hold τ constant). The relationship is shown in Figure 3.7, where the critical leverage is shown on the left vertical axis and the critical relative size on the right vertical axis. As expected, both the critical leverage (left axis, continuous line) and critical relative size of the bank (right axis, dashed line), decrease dramatically as $\theta\tau$ increases. This suggests that it is possible to dramatically improve the stability of the financial system if institutions adjust to their leverage targets slowly. Similarly, this illustrates the dangers of mark-to-market accounting, which can cause balance-sheet adjustments to be too rapid .

becomes more stable as θ_- is reduced. The qualitative behavior of the system, namely the existence of stable, locally unstable and globally unstable regimes, is preserved.

3.5 Leverage policies

Is it possible to dampen the Basel leverage cycle observed in the previous section by choosing an alternative risk management policy? If yes, is there an optimal policy under which financial stability is maximized? We tackle these questions by studying how the cyclical parameter of the bank's risk management policy affects the realized losses in the banking sector for a given level of leverage. In the following, we will first define our objective criterion and the corresponding optimization problem. We will then proceed to study optimal leverage policies.

3.5.1 Objective criterion

As mentioned above, we explore the family of leverage policies that can be parametrized through the relation

$$\bar{\lambda}(t) = F_{(\alpha, \sigma_0^2, b)} = \alpha(\sigma^2(t) + \sigma_0^2)^b, \quad (3.10)$$

where $b \in [-0.5, 0.5]$. We evaluate the impact of a particular choice of b on the financial stability of the system by computing the realized shortfall of the bank's equity for a given simulation. The realized shortfall measures the average tail loss of the bank equity beyond a certain quantile, i.e. it picks up large market crashes. Within our model realized shortfall provides an adequate proxy for financial stability as it allows us to quantify the extent to which the positive feedback between prices, risk and leverage amplifies exogenous volatility in general and in particular in the tail of the loss distribution. It is this amplification of volatility that leads to financial instability within our model and that we seek to minimize via an appropriate leverage policy.¹⁷

Technically, to compute the realized shortfall we simulate the system for $T = 5000$

¹⁷We concede that the concept of financial stability is much broader in practice and cannot be reduced to such a simple measure as realized shortfall. As an example, a more careful treatment in a richer model with heterogeneous agents would take into account the extent of contagion from one financial institution to another.

time steps for a given parameter configuration and record the changes of bank equity due to asset price fluctuations. The change in the bank's equity due to fluctuations in the price of the risky asset at time $t + 1$ is $\Delta E(t) = n(t)\Delta p(t)$, where $\Delta p(t) = (p(t + 1) - p(t))$. We then define the equity return as

$$\ell(t) = \log \left(\frac{E(t) + \Delta E(t)}{E(t)} \right). \quad (3.11)$$

Note that this captures both the leverage of the bank and the market return of the risky asset since

$$\frac{\Delta E(t + 1)}{E(t)} = \frac{n(t)p(t)}{E(t)} \frac{\Delta p(t)}{p(t)} = \lambda(t)w_B r(t),$$

where $r(t)$ is the market return on the risky asset. As expected, leverage amplifies the gains and losses. Then the realized shortfall at a confidence level q over a time horizon T is defined as

$$\text{RS}_q = -\frac{1}{qT} \sum_{t=0}^T \ell(t) \Theta(\ell_q - \ell(t)), \quad (3.12)$$

where $\Theta(x) = 1$ if $x > 0$ and zero otherwise, and ℓ_q is defined through the relation $\sum_{t=0}^T \Theta(\ell_q - \ell(t)) = qT$.¹⁸ In the following we will take $q = 0.05$. For a visual representation of realized shortfall, we refer the reader to Figure 3.8.

3.5.2 Optimization problem

Clearly, the level of realized shortfall of the bank's equity depends on the bank's leverage (since equity losses are proportional to bank leverage). Furthermore, the bank's market impact depends on its relative size to the fund. This in turn affects whether there is significant amplification of exogenous volatility. Thus, in order to make a fair comparison between different leverage policies, these two quantities need be held fixed. Essentially, the idea is to investigate the counterfactual scenario under which the bank's leverage policy has been modified while all structural properties of the financial system remain unchanged.

¹⁸In practice, the realized shortfall is the average loss beyond the q -quantile ℓ_q of the loss distribution, that is empirically computed as the average loss over the worst $q \times T$ time observations.

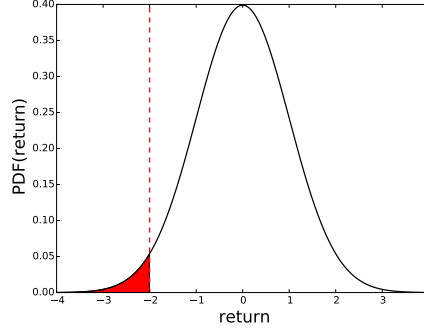


Figure 3.8: Visual representation of realized shortfall. The solid black lines represents a hypothetical return distribution. The red vertical dashed line is drawn in correspondence of the q -quantile of the distribution. The realized shortfall is the average of the distribution in the red region to the left of the vertical line.

Similar to realized shortfall, the bank's leverage and relative size are computed ex post, i.e. after simulating the model for T time steps. In particular, we have for the bank leverage $\hat{\lambda} = \frac{1}{T} \sum_{t=0}^T \bar{\lambda}(t)$ and for the relative size $\hat{R} = \frac{1}{T} \sum_{t=0}^T R(t)$. For a benchmark scenario, we choose levels of leverage and relative size consistent with the previous section, i.e. $\hat{R} \approx 0.27$ and $\hat{\lambda} \approx 5.8$.¹⁹ We can now define the optimal leverage policy b^* as

$$\begin{aligned} & \underset{b \in [-0.5, 0.5]}{\text{minimize}} && \text{RS}_q(b), \\ & \text{subject to} && \hat{R} = 0.27, \\ & && \hat{\lambda} = 5.8. \end{aligned} \tag{3.13}$$

3.5.3 Optimal policies

We present our results for the optimal leverage policy in Figure 3.9 for different parameter scenarios.²⁰ We find that the optimal cyclical parameter depends crucially on the bank's relative size and on the strength of the exogenous volatility process due to the fund investment. As we will outline below, depending the choice of parameters, the optimal policy ranges from constant leverage to Value-at-Risk. In the following

¹⁹For a given choice of cyclical parameter b , we vary α and $\bar{E}$ to match the target for $\hat{R}$ and $\hat{\lambda}$. All other parameters are as in Table 3.1 unless otherwise specified.

²⁰The observations only extend up to $b \approx 0.1$ as for larger values of b there exists no model solution with the required output targets for relative size and leverage.

we will discuss these scenarios in more detail.

First consider the benchmark scenario when the bank is large ($\hat{R} = 0.27$) and the exogenous volatility process is mildly clustered (GARCH parameters: $a_0 = 0.001$, $a_1 = 0.016$, $b_1 = 0.874$). In this case the bank has a large market impact and we find that the realized shortfall of bank equity reaches a minimum at $b^* = 0$, blue circles in Figure 3.9. This corresponds to the case where the bank holds constant leverage.

In the following we will put this result into context by considering two illustrative scenarios. First note that when exogenous volatility is clustered (time changing), there exists a trade off between adjusting leverage to changing levels of *exogenous volatility* and the leverage adjustment itself which leads to increased *endogenous volatility*. Thus, the optimal cyclical parameter will depend on the relative size of these effects. As we see in the benchmark case, if the bank is large and exogenous volatility clustering is small, it is optimal to hold leverage constant. However, as the size of the bank is decreased, the relative size of these two effects changes. In a scenario with a smaller bank ($\hat{R} = 0.1$) but the same exogenous volatility process (red squares in Figure 3.9), we find that realized shortfall is minimized at an intermediate cyclical parameter, i.e. $b^* \approx -0.2$.

The relative effect size of exogenous versus endogenous volatility can be further increased by making the bank even smaller and increasing the clustering in the exogenous volatility²¹ (green diamonds in Figure 3.9). Now, the destabilizing effect of bank leverage adjustment is negligible. Intuition therefore suggests that it should be optimal for the bank to follow a plain microprudential Value-at-Risk constraint capital policy – i.e. we expect $b^* = -0.5$ for the optimal cyclical parameter. As can be seen in Figure 3.9, the corresponding curve for realized shortfall indeed reaches a minimum at $b = -0.5$.²²

²¹In particular this corresponds to the following configuration: GARCH parameters: $a_0 = 0.001$, $a_1 = 0.04$, $b_1 = 0.95$, output targets: $\hat{R} = 10^{-5}$ and $\hat{\lambda} = 5.8$.

²²We have replicated the above analysis with different parameter configurations, including a setting in which average leverage, relative size and average volatility are closer to levels observed in

The sensitivity analysis above underlines that the optimal choice depends on the relative magnitudes of endogenous vs. exogenous volatility. Of course it is difficult to back out these latent properties from observed time series of asset returns. Therefore, any quantitative statement about an optimal cyclicity parameter comes with considerable uncertainty attached. Nevertheless the qualitative conclusion from our analysis remains valid despite this uncertainty: bank risk management amplifies exogenous volatility and can lead to substantial, unintended, endogenous volatility. This effect has to be considered when designing effective macroprudential risk management policies.

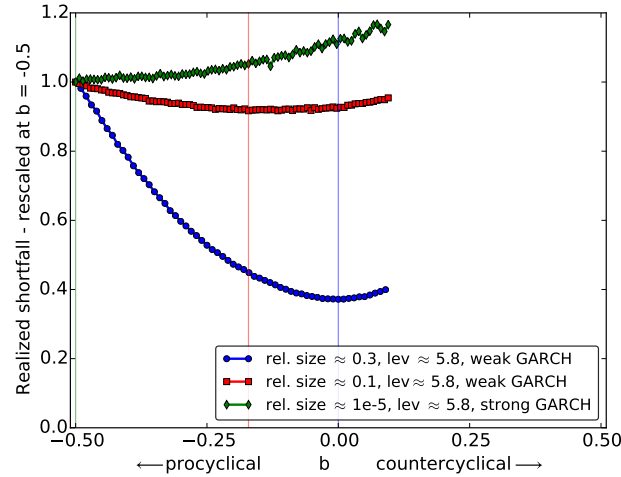


Figure 3.9: Numerical evaluation of the realized shortfall $RS(b)$ for different values of cyclicity b to determine the optimal policy. **Blue circles:** $RS(b)$ for GARCH parameters are as in Table 3.1 ($a_0 = 0.001$, $a_1 = 0.016$, $b_1 = 0.874$), α and $\bar{E}$ adjusted for each b to match output targets for leverage ($\hat{\lambda} = 5.8$) and relative size ($\hat{R} = 0.27$). **Red squares:** $RS(b)$ for GARCH parameters are as in Table 3.1 ($a_0 = 0.001$, $a_1 = 0.016$, $b_1 = 0.874$), α and $\bar{E}$ adjusted for each b to match output targets for leverage ($\hat{\lambda} = 5.8$) but smaller relative size ($\hat{R} = 0.1$). **Green diamonds:** $RS(b)$ for strong GARCH parameters ($a_0 = 0.001$, $a_1 = 0.04$, $b_1 = 0.95$), α and $\bar{E}$ adjusted for each b to match output targets for leverage ($\hat{\lambda} = 5.8$) but much smaller relative size ($\hat{R} = 10^{-5}$). All values for $RS(b)$ have been rescaled at $b = -0.5$. Observations only extend up to $b \approx 0.1$ as for larger values of b there exists no model solution with the required output targets for relative size and leverage.

reality. We achieve this at the expense of matching the period and peak to trough ratio of the leverage cycle as observed prior and following the crisis in 2007/20008. In line with the above sensitivity analysis, the optimal cyclicity parameter is roughly $b^* \approx -0.2$ – i.e. at some intermediate level between the optimal policy in the absence of endogenous volatility and the dominance of endogenous volatility.

3.6 Conclusion

In this chapter we have investigated the effect of different risk-based leverage policies on financial stability. To this end we have considered a modification of the dynamical model of leverage cycles introduced in the previous chapter.

The model is a simple agent-based model that includes a leveraged bank, a fundamentalist unleveraged fund and an outside lender, and can be described in terms of a discrete dynamical system defined by six recursion equations. We have roughly calibrated the model to match basic features of the S&P500 trajectory prior and following the 2007/2008 crash. The model can be simulated in a stochastic or a deterministic regime.

We consider a class of risk based leverage policies that interpolates between procyclical and countercyclical policies. Under a procyclical policy the bank decreases its leverage when perceived risk is high. In contrast, under a countercyclical policy the bank is allowed to increase its leverage when perceived risk is high.²³

Based on this specification we study the stability of the model for different values of bank leverage and policy cyclicity. We find three different regimes: (1) For low leverage the system is stable and settles into a fixed point. (2) As leverage is increased, leverage cycles emerge; however, for countercyclical policies ($b > 0$) this region in parameter space is very small. (3) As leverage is increased further, the system becomes globally unstable. The behavior of the critical leverage at which cycles emerge, depends on whether the system is deterministic or stochastic. In the deterministic case, the critical leverage is constant for all values of the cyclicity parameter b . In the stochastic case, however, the critical leverage increases with b .

In our model both procyclical and countercyclical policies are associated with

²³It is important to note that our definition of policy cyclicity does not refer to the level of macroeconomic indicator such as the credit-to-GDP ratio or the level asset prices but is defined solely by the bank's response to changes in perceived risk. As such countercyclical policies proposed in this model differ from alternative policies such as the Basel III countercyclical capital buffer, see for example FPC (2014).

positive feedback loops that can amplify market fluctuations. When leverage is procyclical, banks reduce their balance sheets when perceived volatility is high, pushing prices down and increasing volatility even further, potentially triggering a crash. When leverage is countercyclical, instead, high volatility leads to higher leverage. This increases demand for assets, which pushes up prices and volatility. This positive feedback loop can be destabilizing in a similar way that procyclical leverage is destabilizing.

Our main contribution is the evaluation of different leverage policies based on the realized bank equity shortfall for a given average leverage and bank size. This yields our central result: the optimal policy is procyclical in perceived risk but the precise value of the cyclical parameter depends crucially on two factors: the bank's size relative to that of the fundamentalist, and on the amplitude of exogenous volatility introduced by the fund investor.

In the microprudential limit when the bank is small and exogenous volatility is high, the optimal policy is simply given by Basel II, i.e. Value-at-Risk ($b = -0.5$). As the bank increases the size of its balance-sheet (either through increasing its equity or its leverage) the optimal policy becomes less procyclical. In the limit when the bank is very large, the optimal policy is constant leverage, $b = 0$. The fact that the degree of cyclicity of the optimal policy depends on the size and the leverage of the banking sector suggests that regulation should be flexible to adapt to changes in the financial environment.

Our paper clearly illustrates the interplay between exogenous and endogenous volatility: the microprudential response to exogenous volatility can itself cause endogenous volatility which may end up dominating over the exogenous volatility. This insight is crucial for the effective design of macroprudential policies. Such policies must critically evaluate the potential endogenous amplification effects any risk management policy may have.

Clearly, in this paper we have abstracted from a number of aspects that a more realistic model should incorporate. For instance, by abstracting from the bank's portfolio allocation problem, i.e. by holding the bank's portfolio weights fixed, we explicitly exclude the effect of flight to quality from the asset price dynamics. Moreover, our assumption that the bank can always access new funding from an outside lender neglects funding liquidity risk. Finally, by considering only a representative leveraged investor, we abstract from important heterogeneity in the financial system. These simplifications have been taken in order to isolate the interaction between asset prices and leverage, which was the focus of this paper. We leave the task of incorporating all of the above aspects into our model as a challenge for future work.

3.A Detailed description of the model

In the following we will describe the modeling details of the assets, the bank, the fund and the market. The model is set in discrete time indexed by $t = \{\tau, 2\tau, \dots, T\tau\}$, where τ is the length of a time-step. We leave the length of the time-step as a free parameter in order to allow us to calibrate the model.

3.A.1 Assets

Let $p(t)$ be the price of the risky asset at time t . The risky asset can be thought of as any traded security such as stocks, asset backed securities or bonds. We assume that there is exactly one unit of the risky asset which is infinitely divisible. The return on the price of the risky asset is $r(t) = \log[p(t)/p(t - \tau)]$. We denote the fraction of the risky asset held by the bank by $n(t) \in [0, 1]$. Since only the bank and the fund can invest in the risky asset, the fraction of the risky asset held by the fund is simply $1 - n(t)$. The risk free asset is analogous to cash. The price of the risk free asset is constant and equal to one.

3.A.2 Agents

There are two representative agents. The first is a bank, denoted B, and the second is a fund, denoted F.

3.A.3 Bank

Balance sheet Assume the bank divides its assets $A_B(t)$ in a fixed ratio w_B between the risky asset and cash $c_B(t)$, so that the bank owns $n(t)$ shares of the risky asset with price $p(t)$. The relevant accounting relations are:

$$\text{Risky investment} = n(t)p(t) = w_B A_B(t),$$

$$\text{Risk free investment} = c_B(t) = (1 - w_B)A_B(t),$$

$$\text{Total assets} = A_B(t) = c_B(t) + n(t)p(t).$$

The bank's liabilities L_B have a maturity of one time step and are freely rolled over or expanded. There is no limit to the reduction in L_B ; in principle the bank could pay back its entire liabilities in one time step.

The bank adjusts its equity toward a fixed target $\bar{E}$. This guarantees that neither the bank nor the fund asymptotically accumulates all the wealth and makes the long-term dynamics stationary, with only a small effect on the short term dynamics. The dividends paid out by the bank are invested in the fund and new capital invested in the bank comes from the fund. If the bank deviates from its equity target $\bar{E}$ it either pays out dividends or attracts new capital from outside investors at a rate η to adjust its equity closer to the target, so that its equity changes by

$$\kappa_B(t) = \tau\eta(\bar{E} - E_B(t)). \quad (3.14)$$

Taking both the changes in price and the active adjustments in equity into account, the bank's equity at time $t + \tau$ is

$$E_B(t + \tau) = n(t)p(t + \tau) + c_B(t) - L_B(t) + \kappa_B(t), \quad (3.15)$$

and the bank's leverage is

$$\lambda(t + \tau) = \frac{\text{Total Assets}}{\text{Equity}} = \frac{n(t)p(t + \tau)/w_B}{E_B(t + \tau)}. \quad (3.16)$$

We assume the bank enforces risk control through a target leverage $\bar{\lambda}(t)$, corresponding to a target portfolio value $\bar{A}_B(t)(t) = \bar{\lambda}(t)E_B(t)$.

Estimation of perceived risk The bank relies on historical data to estimate the perceived variance of the risky asset $\sigma^2(t)$. To do so the bank computes an exponential moving average of squared returns of the risky asset. This approach is similar to the RiskMetrics approach, see Longerstaey (1996). In particular

$$\begin{aligned} \sigma^2(t + \tau) &= (1 - \tau\delta)\sigma^2(t) + \tau\delta r^2(t) \\ &= (1 - \tau\delta)\sigma^2(t) + \tau\delta \left(\log \left[\frac{p(t)}{p(t - \tau)} \right] \frac{t_{\text{VaR}}}{\tau} \right)^2, \end{aligned} \quad (3.17)$$

where the term t_{VaR}/τ rescales the return over one time-step τ to the return over the horizon t_{VaR} used in the computation of the capital requirement. The parameter $\tau\delta \in (0, 1)$ implicitly defines the length of the time window over which the historical estimation is performed. We define the typical time t_δ as the time at which an observation made at $t - t_\delta$ has decayed to $1/e$ of its original contribution to the exponential moving average. Thus $t_\delta = -\tau/\log[1 - \tau\delta] \approx 1/\delta$ for $\tau\delta \ll 1$.

3.A.4 Fund investor

The fund investor represents the rest of the financial system, and plays the role of a fundamentalist noise trader. Since the fund is not leveraged its assets $A_F(t)$ are equal to its equity, i.e. $E_F(t) = A_F(t)$. Just as for the bank, the fund invests $w_F(t)$ of its assets in the risky asset and $1 - w_F(t)$ in cash; a key difference is that the fund adjusts its portfolio weight $w_F(t)$ whereas the bank's weight is fixed. The relevant

accounting relations are

$$\text{Risky investment} = (1 - n(t))p(t) = w_F(t)A_F(t),$$

$$\text{Risk free investment} = c_F(t) = (1 - w_F(t))A_F(t),$$

$$\text{Total assets} = A_F(t) = c_F(t) + (1 - n(t))p(t),$$

and the fund's equity is

$$E_F(t + \tau) = (1 - n(t))p(t + \tau) + c_F(t) + \kappa_F(t). \quad (3.18)$$

The fund's cash flow $\kappa_F := -\kappa_B$ mirrors the dividend payments or capital injections of the bank.

We have already explained the motivation for the fund's demand function in the main text. Here, we simply note that we rescale the deviation of the price of the risky asset to the fundamental by the current price of the risky asset in order to make portfolio weight adjustments independent of the scale of the price of the risky asset. Otherwise, the portfolio weight would likely exceed its natural bounds, i.e. $w_F \in [0, 1]$. In order to introduce heteroskedasticity we make s^2 time varying according to a simple GARCH(1,1) process of the form

$$\begin{aligned} s^2(t) &= a_0 + a_1\chi^2(t-1) + b_1s^2(t-1), \\ \chi(t) &= s(t)\xi(t). \end{aligned} \quad (3.19)$$

When the parameters a_1 and b_1 are zero the returns $r(t)$ of the risky asset are normally distributed as $n(t) \rightarrow 0$, i.e. as the fund dominates the market, and the price process will resemble a mean reverting random walk with constant volatility.

3.A.5 Market mechanism

The price of the risky asset is determined by market clearing. For this we construct the demand functions for the bank and fund (D_B and D_F respectively) as follows:

$$\begin{aligned} D_B(t + \tau) &= \frac{1}{p(t + \tau)} w_B A_B(t + \tau) \\ &= \frac{1}{p(t + \tau)} w_B (n(t)p(t + \tau) + c_B(t) + \Delta B(t)), \\ D_F(t + \tau) &= \frac{1}{p(t + \tau)} w_F(t + \tau) A_B(t + \tau) \\ &= \frac{1}{p(t + \tau)} w_F(t + \tau) ((1 - n(t))p(t + \tau) + c_F(t)). \end{aligned}$$

Recall that there is a supply of exactly one unit of the risky asset that is infinitely divisible. We can then compute the market clearing price by equating demand and supply $1 = D_B(t + \tau) + D_F(t + \tau)$. Solving for the market clearing price we obtain

$$p(t + \tau) = \frac{w_B(c_B(t) + \Delta B(t)) + w_F(t + \tau)c_F(t)}{1 - w_B n(t) - w_F(t + \tau)(1 - n(t))}. \quad (3.20)$$

Given the new price we can compute the fraction of the risky asset owned by the bank as follows:

$$n(t + \tau) = \frac{1}{p(t + \tau)} w_B (n(t)p(t + \tau) + c_B(t) + \Delta B(t)). \quad (3.21)$$

3.A.6 Finding the fixed point

We begin by considering the conditions for a fixed point of the $g(\cdot)$ as defined in Equation (3.6).²⁴

1. The price is at the noise trader's fundamental value:

$$p^* = \mu \implies w_F(t + \tau) = w_F(t).$$

²⁴For the deterministic system it is simple to derive a set of differential equations for the continuous-time limit. We have checked that the qualitative behavior of the system in continuous time is the same as that of the discrete system in this case. For simplicity, and for consistency with Section 3.5 where numerical simulations for the discrete stochastic case are considered, we present here results for the discrete dynamical system.

2. The bank's perceived risk is 0:

$$\sigma^{2*} = 0 \wedge p(t) = p(t - \tau) = \mu \implies \sigma^2(t) = \sigma^2(t + \tau).$$

3. The bank is at its target leverage consistent with $\sigma^{2*} = 0$:

$$\lambda^* = \frac{A^*}{A^* - L^*} = \bar{\lambda}(t) = \alpha(\sigma_0^2)^b \implies \Delta B(t) = 0.$$

4. The bank is at its target equity:

$$E^* = A^* - L^* = \bar{E} \implies \Delta E_B(t) = 0.$$

5. The bank's ownership of the risky asset is consistent with the price, leverage target and equity target at the fixed point:

$$n^* = \lambda^* E^* w_B / \mu.$$

The fixed point is therefore:

$$\begin{aligned} x^* &= (\sigma^{2*}, w_F^*, p^*, n^*, L^*, p'^*) \\ &= (0, w_F(0), \mu, \frac{1}{\mu} \alpha \sigma_0^{2b} \bar{E} w_B, (\alpha \sigma_0^{2b} - 1) \bar{E}, \mu), \end{aligned} \tag{3.22}$$

where we picked $w_F^* = w_F(t = 0)$, the initial value of the fund's investment weight, since at $p^* = \mu$ any w_F will remain unchanged. Since w_F^* is not specified by the fixed point condition, there is essentially a set of fixed points for $w_F \in [0, 1]$. As such it is useful to interpret w_F^* as a parameter of the model determined by an appropriate initial condition. We choose $w_F(0) = 0.5$ throughout.

We can distinguish two cases for the existence of the fixed point. In the first case $\sigma_0^2 = 0$. In this case, provided that $b < 0$, the fixed point has an infinity in n^* . This fixed point has no economic meaning. Therefore, we will restrict our analysis to the case where $\sigma_0^2 > 0$ in which the fixed point is well defined.

In the case studies in Figure 3.3 and Figure 3.4 in Section 3.3 we saw that the properties of the system dynamics depended heavily on the relative proportions of the fund versus the bank as this determines the impact of the bank on the price of the risky asset. Therefore, before moving on we define the relative size of the bank

to the fund at the fixed point as:

$$\begin{aligned}
R(x^*) &= \frac{\text{Total assets bank}}{\text{Total assets fund}} \\
&= \frac{\lambda^* E^*}{(1 - n^*) p^* / w_F^*} = \left(\frac{\mu}{\bar{E}} \frac{1}{\alpha \sigma_0^{2b} w_F(0)} - \frac{w_B}{w_F(0)} \right)^{-1}
\end{aligned} \tag{3.23}$$

Clearly, as the equity of the bank goes up, its size relative to the noise trader will increase. Similarly if the bank risk parameter α or the risk off set σ_0^2 is increased, the bank's leverage at the fixed point will increase whereby its size relative to the fund will increase.

3.A.7 Existence of critical leverage and bank riskiness

In order to assess the stability of the fixed point we compute the Jacobian matrix $J_{ij} = \partial g_i / \partial x_j$. We then evaluate the Jacobian at the fixed point x^* and compute the eigenvalues e_i of the corresponding matrix. In this particular case the eigenvalues cannot be found analytically. Instead, we compute the eigenvalues numerically using the parameters specified in Table 3.1. With the help of the eigenvalues we can distinguish between local stability and instability of the fixed point. If the absolute value of the largest eigenvalue $|e_+| > 1$ the system exhibits chaotic oscillations, while it is locally stable if $|e_+| < 1$. We assess the global stability of the system via numerical iteration of the map in Equation (3.6).

Now, suppose we increase the bank risk parameter α and study how the eigenvalues of the Jacobian change while keeping all other model parameters constant. It is clear from Equation (3.23) and Equation (3.7), that as we increase α we will increase both the bank's leverage and relative size to the fund at the fixed point. We therefore expect that firstly the bank's market impact increases and secondly that the bank becomes more fragile due to its increased leverage. Thus overall, we expect that if we increase α sufficiently, we should observe a transition from the fixed point dynamics to leverage cycles. We summarize the evolution of the two largest eigenvalues of the Jacobian in the complex plane in Figure 3.10. The eigenvalues start out at a point

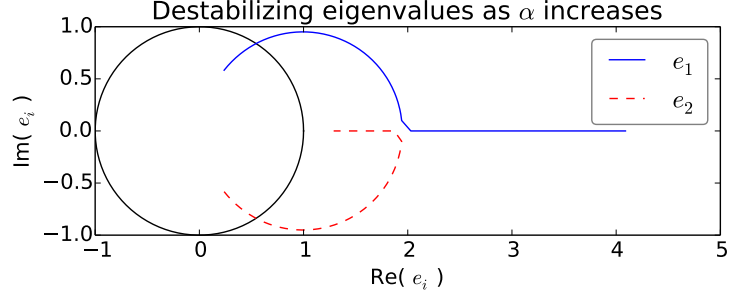


Figure 3.10: Numerical evaluation of the destabilizing eigenvalues (specific results will depend on parameter choice) for different values of α . The destabilizing eigenvalues are the two largest eigenvalues that first cross the unit circle from within.

within the unit circle on the complex plane (i.e. $|e_i| < 1$). Then as α is increased the magnitude of the eigenvalues increases. The critical bank riskiness α_c at which the eigenvalues cross the unit circle, corresponds to the point at which leverage cycles emerge. Since we keep all other parameters constant, this critical bank riskiness also corresponds to a critical leverage and a critical relative size of the bank to the fund. In particular

$$\begin{aligned} \lambda_c^* &= \alpha_c \sigma_0^{2b}, \\ R_c(x^*) &= \frac{\lambda_c^* E^*}{(1 - n^*) p^* / w_F^*}. \end{aligned} \tag{3.24}$$

Part II

Contagion in financial networks

Chapter 4

Contagion in Coupled Financial Networks

4.1 Introduction

In June 2007, at the start of the financial crisis, two funds owned and operated by Bear Stearns got into trouble. The funds were trading in collateralized debt obligations (CDOs) which had been hit hard due to the emerging crisis in subprime mortgages. Following the failure of these two funds Bear Stearns' overall position deteriorated due to reasons ranging from investor law suits to funding issues. Less than a year later Bear Stearns effectively failed and was acquired by JP Morgan Chase with the a loan from the Federal Reserve. This anecdote is an example of what we call *intra-institutional spillover*: Bear Stearns – a relatively well diversified securities broker – failed due to the prior failure of two of its business units.

In this chapter we argue that intra-institutional spillover is relevant beyond the example of Bear Stearns. To see this, note that portfolio diversification implies that large financial intermediaries trade in a wide range of different types of assets, usually through specialized business units or subsidiaries. These business units are not independent of each other but tend to have strong *intra-institutional* ties, for example in the form of intra-group loans or ownership (in the case of subsidiaries). The impor-

tance of intra-institutional ties – particularly in the form of internal capital markets – has been widely studied in the context of monetary transmission and corporate finance.¹ Acharya et al. (2013) empirically study the role of conduits, subsidiaries of larger financial intermediaries specialized in trading certain types of asset backed securities, in the global financial crisis. The authors show that conduits allowed banks to hold less capital while retaining similar levels of risk.² Despite this research, the impact of the complex structure of interlinkages between subsidiaries of different financial institutions on financial stability is less understood. We will illustrate in this chapter that intra-institutional links establish connections between markets for different, independently traded assets and therefore permit the contagion of stress from one asset market to another. As we will show in this chapter, this form of contagion may have dramatic consequences.

In this chapter we develop a simple model of multi-asset trading in which N banks intermediate K types of assets from a finite group of sellers to a finite group of buyers. Trading relationships between banks are exogenously fixed and assets trade over-the-counter. Assets can be thought of, for example, as credit default swaps which are traded over-the-counter.³ We model banks as being comprised of K *business units* specialized in trading assets of type $\mu \in \{1, \dots, K\}$ only. Each business unit faces an operational cost that has to be covered by profits from asset intermediation. Thus, a business unit that does not intermediate assets ceases to operate and fails. In other words, a business unit fails if “liquidity” in the asset that it is trading dries up. A

¹See Campello (2002), Cetorelli and Goldberg (2012), Gertner et al. (1994), Houston et al. (1997), amongst many others.

²They also provide evidence for one of the central assumptions for our model: financial intermediaries sponsor a range of conduits each specializing in a certain class of asset backed securities. From the author’s analysis a class of asset backed securities can be characterized by three variables: asset origin (e.g. US), asset rating (e.g. AAA) and underlying type (e.g. residential mortgages).

³Indeed, empirical research on the structure of CDS OTC markets suggests that a majority of the notional is concentrated in trades between intermediaries (or dealers) and between intermediaries and ultimate buyers and sellers of CDS, see for example Peltonen et al. (2014). Cont and Minca (2015) also stress the importance of intermediation chains in the CDS market.

key assumption in our model is that the failure of one business unit exerts a negative externality on other business units belonging to the same bank. This spill-over can be motivated for example with losses due to asset fire-sales (e.g. Shleifer and Vishny (1992), Allen and Gale (1994)), increased refinancing costs of the bank because of reduced profitability (a functioning business unit is by definition profitable), or dead-weight losses in the form of legal fees. By considering intra-institutional spill-overs from one asset market to another, our model introduces a channel of contagion that is somewhat related to contagion due to overlapping portfolios where stress in one asset market can spread to other markets via bank portfolio adjustment, see for example Caccioli et al. (2014). It should be noted that many other channels of contagion have been studied including roll-over risk, counterparty risk or information contagion.⁴ Throughout this paper we will abstract from these other channels of contagion and focus exclusively on the intra-institutional spill-over.

Buyers have a higher valuation of assets of type μ than sellers which implies gains from trade and intermediation yields a positive total surplus. This surplus can be divided amongst sellers, intermediaries, and buyers. The fraction of the total surplus that each agent receives depends on her bargaining power and other market specifics. For the sake of our model, any division of surplus is acceptable that allows all agents involved in the intermediation of an asset to survive. For simplicity we assume that each agent involved in the intermediation of an asset receives an equal share of the total surplus realized by intermediating this asset. Since business units face fixed costs, only those business units do not default that are connected in a *feasible* trading network, i.e. one which contains at least one buyer and one seller. We are thus able to characterize the state of our system simply by the number of banks in a feasible trading network. To find this number, we employ the method of generating func-

⁴For a more thorough discussion of the different forms of contagion, see for example Bandt et al. (2009).

tions, first applied to networks by Wilf (1990) and widely used in various applications of graph theory. In deriving our results we rely heavily on methods developed by Buldyrev et al. (2010) for the study of percolation on multi-layer networks.

Throughout this paper, we are interested in the resilience of feasible trading networks to exogenous shocks. Under an exogenous shock a certain fraction of business units trading a particular asset fail randomly. We then measure resilience as the number of banks in the feasible trading network following an exogenous shock of a particular size. In Section 4.3.1 we present a number of analytical results in the limit of large, tree-like trading networks. We first show that the feasible trading network always corresponds to the giant component of the network. Second, we study the intermediation of a single asset, i.e. the case $K = 1$. We show that the size of the largest feasible intermediation network continuously declines with the size of the exogenous shock until it finally vanishes entirely when the exogenous shock becomes too large. It is important to note that the feasible trading network is not only reduced by the exogenous shock itself, but also by the contagion that follows. To see this, imagine a feasible trading network with two components that are connected by a single bank only. If buyers and sellers are both in one of the components and the exogenous shock hits the bank connecting the two components, by definition only the component with buyers and sellers is feasible. All banks in the other component fail due to contagion and as a consequence of the exogenous shock without being initially affected.

The main contribution of our paper is to study the resilience of coupled financial networks. In Section 4.3.2 we study the case in which banks, comprised of two business units each, intermediate two types of assets, i.e. $K = 2$. The default of one business unit of a bank exerts an externality on the bank's other business units. In our analytical application we assume that the externality is such that the default of one business unit leads to the default of the entire bank, i.e. of all other business

units. This spillover within a bank fundamentally changes the contagion process. In the single-asset case, an exogenous shock hits a business unit and leads to its default. This can trigger contagion, as described above. In the single-asset case, the process ends here. In the multi-asset case, the default of bank i 's business unit in layer μ triggers the default of business units in another layer $\nu \neq \mu$ and a new round of contagion ensues. It is now possible that contagion in layer ν leads to the default of a bank j , which was not affected by the contagion process in layer μ . This further fragments the feasible trading network. It is this reinforcement of contagion between different asset types that leads to additional fragility. This result is robust for many network structures. Furthermore, for some network structures, the size of the largest feasible intermediation network is no longer continuously decreasing with the size of the exogenous shock, but rather exhibits a sudden, discontinuous decrease once a critical shock size is reached. When the collapse of intermediation is discontinuous, it also occurs at a smaller exogenous shock relative to the single-asset case. Our analysis thus shows that there is a qualitative difference in the resilience of single-asset trading networks and coupled trading networks.

In order to obtain the above analytical results, we make a number of simplifying assumptions. First, we restrict the number of assets to $K = 2$. Second, we assume that the size of the externality is large, i.e. that the probability of intra-bank spillover is one. Third, we assume that the different trading networks are uncorrelated (i.e. their degrees are uncorrelated). Fourth, we assume that there is no overlap (i.e. there are no edges in common) between the trading networks of different types. And fifth, we assume that networks are tree like. This excludes core-periphery networks which have been found to be prevalent in the financial system. We relax these limitations one by one in Section 4.5.

Numerically we find, first, that the system of coupled trading networks becomes increasingly unstable with an increasing number of assets K . Second, we find that the system of coupled trading networks is less resilient to exogenous shocks, the larger the size of the externality q . Third, we find that correlated coupled trading networks are more stable than uncorrelated trading networks. However, this effect is reversed as soon as one uncorrelated trading network is added. The system of three coupled trading networks of which two are highly correlated and only the third one is uncorrelated is less stable than the system of two coupled trading networks that are uncorrelated. We, fourth, find that the system of coupled trading networks with less overlap is less resilient to exogenous shocks. In the case of perfect overlap, we recover the results for the single-asset trading system. And finally, we find that core-periphery networks are very resilient when the overlap between the different network layers is large, and extremely fragile when the overlap is very small.

4.2 Model

We consider a simple model of a multi-asset over-the-counter market in which market participants consist of sub-units of financial institutions called business units. Each business unit specializes in the trade of one particular asset. Business units within a financial institution are linked by a “failure externality” such that the failure of one business unit within a financial institution may lead to the failure of other business units within the same institution.

There is a countable set of financial institutions $N = \{1, \dots, n\}$, called banks for short, and two periods, initial and final.⁵ There are $K \geq 1$ assets that are traded over-the-counter, $\mu = \{1, \dots, K\}$. All assets mature in the final period and have the same return r . Assets could be thought of as derivatives traded in an over-the-counter

⁵For our analytical work we will later on let the number of banks tend to infinity.

market. Each bank is subdivided into K business units each trading in a particular asset μ .

In one concrete interpretation, the banks' business units can be thought of as off-balance sheet vehicles sponsored by the bank similar to the conduits studied in Acharya et al. (2013). Typically, the sponsoring bank provides guarantees for the conduit's outstanding commercial paper. This allows the conduit to borrow at low rates. Via these guarantees the bank effectively provides an emergency line of credit to the conduit that it can tap into when it cannot roll over its commercial paper. While conduits and the sponsoring bank may not be subject to a common solvency constraint, a conduit's survival is to a large extent contingent on the sponsor's ability to provide credit guarantees. This is particularly true in times of market stress.

The sponsor's ability to provide credit guarantees will depend on a number of factors including its funding liquidity, solvency position or cash flow. Throughout this paper we will assume that the sponsor's overall health and thereby its ability to provide credit guarantees is a function of the state of its sponsored conduits. Therefore, the failure of one conduit can impair the sponsor's ability to support other conduits. Via this implicit funding channel the failure of one conduit can lead to the failure of other conduits. Thus the failure of one conduit exerts an externality on the sponsor overall and its constituent conduits. It is precisely this intra-institutional spillover that we will study throughout this chapter. In the following we will model this spillover via a spillover probability such that a larger externality corresponds to a larger spillover probability.

In the initial period, a finite number of banks in the subset $E_\mu \subseteq N$ receive a random endowment of asset μ . We assume that the number of sellers is much less than the total number of banks, i.e. $|E_\mu| \ll |N|$ and does not scale with $|N|$. In

the final period, a finite subset $D_\mu \subseteq N$ of banks face inelastic demand for asset μ from outside investors, where again the number of buyers is much less than the total number of banks, i.e. $|D_\mu| \ll |N|$ and does not scale with $|N|$. Banks with an endowment of a particular asset do not necessarily face demand for that asset from outside investors. This motivates intermediation of the assets.⁶

Over-the-counter asset trading is relationship based on the level of business units. A set of fixed networks of trading relationships between bank business units allows banks to trade assets at the end of the initial period.⁷ These trading relationships may differ from asset to asset.⁸ While bank A might have a trading relationship with bank B for asset μ the same relationship might not exist for asset μ' . Therefore, we consider a network of trading relationships for each asset μ . First consider the trading network for a single asset μ only.

Let $\mathcal{G}'_\mu$ denote a random multigraph⁹ with n vertices. The vertex i in $\mathcal{G}_\mu$ represents the business unit of bank i operating in the market for asset μ . Later we will consider multigraphs in the limit as $n \rightarrow \infty$. The edges of the graph correspond to the trading relationships between different business units.

We define the random multigraph following Britton et al. (2007) and Amini et al. (2013). Let $(k_i)_1^n = (k_i^{(n)})_1^n$ be a sequence of non-negative integers such that $\sum_{i=1}^n k_i$ is even. Then the random multigraph with degree sequence $(k_i)_1^n$ is given by the config-

⁶As pointed out in the introduction, empirical evidence of over-the-counter markets indeed suggests that a large fraction of the volume traded in over-the-counter markets is traded by intermediaries, see for example Peltonen et al. (2014). In addition, theoretical models have predicted intermediation in over-the-counter markets early on, see for example Amihud and Mendelson (1980).

⁷To motivate that not all banks can trade with each other we could introduce, for example, costs of establishing and maintaining trading relationships.

⁸This is a reasonable assumption since assets are traded in different business units of the bank. To the extent that over-the-counter markets are based on personal relationships, the trading relationships will vary from business unit to business unit. Of course, trading relationships may also be highly correlated for different assets. We will deal with this case numerically in Section 4.5.

⁹A multigraph explicitly allows for multiple edges and self-loops.

uration model and denoted by $\mathcal{G}'_\mu(n, (k_i)_1^n)$.¹⁰ A detailed review of the configuration model can be found in Jackson et al. (2008) and Newman (2010). In order to ensure that the configuration model yields a uniform distribution over random graphs with a given degree distribution we need to condition on the multigraph being a simple graph. This (among other convenient properties) is achieved by assuming the following regularity conditions on the degree sequence $(k_i)_1^n$, cf. Molloy and Reed (1998) and Britton et al. (2007):

Assumption 1. *For each n , $(k_i)_1^n = (k_i^{(n)})_1^n$ is a sequence of non-negative integers such that $\sum_{i=1}^n k_i$ is even and, for some probability distribution over degrees $(P(k))_{k=0}^\infty$ independent of n , and with $n_j := \#\{i : k_i = j\}$,*

1. $n_j/n \rightarrow P(j)$ for every $j \geq 0$ as $n \rightarrow \infty$,
2. $E(k) = \lambda := \sum_k kP(k) \in (0, \infty)$,
3. $\sum_{i=1}^n k_i/n \rightarrow \lambda$ as $n \rightarrow \infty$,
4. $P(2) < 1$,
5. $P(1) > 0$,
6. $\sum_i k_i^2 = O(n)$.

Clearly we have that $\sum_i k_i^2 = \sum_j n_j j^2$. This, together with condition (6) in Assumption 1 implies that the asymptotic degree distribution has finite variance, $\sum_k P(k)k^2 < \infty$, cf. Britton et al. (2007). We denote the resulting simple graph by $\mathcal{G}_\mu(n, (k_i)_1^n)$. These assumptions will allow us to make a branching process approximation and use generating functions in deriving some analytical results in Section 4.3.

¹⁰ A random instance of the configuration model can be generated as follows. Generate n nodes where each node i has k_i link “stubs” emanating from it. Then pick two link stubs uniformly at random and form a full link. Repeat this process until all link stubs have been matched.

Furthermore, we assume that the degree distribution permits a single large connected component, the so called giant component. First, we define the giant component as

Definition 4.1. *A giant component $\mathcal{G}_\mu^*$ is a component of $\mathcal{G}_\mu(n, (k_i)_1^n)$ that contains $n^* \propto n$ nodes as $n \rightarrow \infty$, i.e. its size is proportional to the size of the network.*

Molloy and Reed (1998) and Janson (2009) show that there exists a unique giant component if the following condition is satisfied which we will assume to be true throughout this chapter.

Assumption 2. *For a given degree distribution $(P(k))_{k=0}^\infty$, $\sum_k k(k-2)P(k) > 0$.*

Before proceeding, we introduce the following definitions for a particular instance of the random graph $\mathcal{G}_\mu(n, (k_i)_1^n)$. The set N of all banks and the set of all trading relationships $A_\mu = \{A_{ij;\mu} | i, j \in N\} \subseteq N \times N$ (also called *links*) for asset μ form a trading network denoted by $\mathcal{G}_\mu(n, A_\mu)$, which we refer to as the network *layer* μ . Layer μ is represented by its adjacency matrix $\mathbf{A}_\mu$ which takes values in $\{0, 1\}$. Trading relationships are bilateral, i.e. $\mathbf{A}_{ij;\mu} = \mathbf{A}_{ji;\mu}$. We refer to the set of all trading partners of bank i for asset μ as the bank's neighborhood $\mathcal{N}_{i;\mu}$, i.e. $\{\mathcal{N}_{i;\mu} : j \in N | A_{ij;\mu} = 1\}$. The size of the neighborhood (or degree) $k_\mu(i)$ of bank i in layer μ is given by the number of links bank i has in that layer: $k_{i;\mu}(\mathbf{A}) = \sum_{ij} \mathbf{A}_{ij;\mu}$.

A trading path $i \hookrightarrow c$ from bank i to c in layer μ is an alternating sequence of nodes and links $i, A_{ij;\mu}, j, A_{jk;\mu}, \dots, b, A_{bc;\mu}, c$ with $A_{lm;\mu} = 1$ for all consecutive l and m in $i \hookrightarrow c$. A trading network $\mathcal{G}_\mu$ is connected if there exists a trading path between any two nodes in the network. Finally, a component of a trading network is a nonempty subnetwork $\mathcal{G}'_\mu = \mathcal{G}'(N', A'_\mu)$ such that $\emptyset \neq N' \subset N$, $A'_\mu \subset A_\mu$ and: (i) $\mathcal{G}'_\mu$ is connected, and (ii) for $i \in N'$, and $A_{ij;\mu} \in A_\mu$, then $j \in N'$ and $A_{ij;\mu} \in A'_\mu$.

Recall, that being able to trade one asset does not necessarily imply being able to trade the other. Thus, $\mathbf{A}_\mu$ is not necessarily equal to $\mathbf{A}_\nu$, $\mu, \nu \in \{1, \dots, K\}, \mu \neq \nu$. The set of all network layers, together with the convention that node i in layer μ is uniquely mapped to node i in layer ν for $\mu, \nu \in \{1, \dots, K\}$, is denoted the multi-layer network $\mathcal{G}$.¹¹

For intermediation to take place there has to be at least one buyer and one seller connected by a network of bilateral trades. We denote a network in which intermediation can take place as a *feasible* intermediation network.

Definition 4.2. *A network for some asset μ , $\mathcal{G}_\mu$, with node set N_μ is a feasible intermediation network if it (i) is connected and (ii) contains both sellers and buyers, i.e. $E_\mu, D_\mu \neq \emptyset$.*

Bargaining between banks determines the division of total surplus, i.e. the difference between the valuation of the buyers and sellers. In our model we do not define a specific bargaining process and trading protocol. Rather, we are interested in under which conditions intermediation is possible, when it becomes impossible and how this transition depends on exogenous shocks to the network. We assume that each intermediary has equal bargaining power and that the total surplus is thus split equally amongst all banks (i.e. seller, intermediaries, and buyer). Formally, let S_μ be the total surplus from trading asset μ . Further let $n_\mu = |N_\mu|$ be the size of the feasible intermediation network $\mathcal{G}_\mu$. Then, the individual surplus for a business unit of bank i dedicated to trading asset μ is simply:

$$S_{i;\mu} = \begin{cases} S_\mu/n_\mu & , \text{ if } i \in N_\mu, \\ 0 & , \text{ otherwise.} \end{cases}$$

¹¹Technically, we can introduce a set of nodes $N = \{i_\mu, j_\mu, \dots, n_\mu\}$ for $\mu \in \{1, \dots, K\}$ corresponding to the business units of bank i . Since banks are identified uniquely across layers in our model, however, we can drop the index for the layer and simply write $i, j, \dots, n$.

Beside gaining a surplus from trading a business unit also faces a fixed cost. Formally we have:

Assumption 3. *Each business unit faces a fixed cost $c_F > 0$ to fund its operations and, at the initial time, gains sufficient surplus from trading¹² to cover its fixed cost, i.e.: $S_{i;\mu} > c_F, \forall i, \mu$.*

This implies that a business unit trading asset μ must be part of a feasible intermediation network for this asset in order to cover its fixed costs. If a business unit cannot cover its fixed costs it becomes illiquid and is said to fail. We indicate failure of business unit μ of bank i by the binary variable $\sigma_{i;\mu} \in \{0, 1\}$ which takes the value $\sigma_{i;\mu} = 1$ if business unit μ of bank i has failed and $\sigma_{i;\mu} = 0$ otherwise. This, together with Assumption 3 implies that

$$\sigma_{i;\mu} = \begin{cases} 0 & , \text{ if } i \in N_\mu, \\ 1 & , \text{ otherwise.} \end{cases}$$

where as above N_μ is the set of all banks in the feasible intermediation network for asset μ . The failure of one business unit may lead to the failure of further business units in a given bank due to a failure related externality, i.e. intra-institutional spillover. We indicate the failure of the entire bank i by the binary variable $\Sigma_i \in \{0, 1\}$, which takes value $\Sigma_i = 1$ if bank i has failed and $\Sigma_i = 0$ otherwise. We assume that the probability of the entire bank failing conditional on the failure of at least one of its business units can be modeled as follows.

Assumption 4. *The failure probability of bank i conditional on the failure of at least one of its business units is given by a Bernoulli distribution such that*

$$P(\Sigma_i = 1 | \sum_{\nu} \sigma_{i;\nu} > 0) = q_i.$$

¹²Any nodes for which this might not be case could simply be removed from the network prior to the initial period and only the remaining sub-network would be studied. Also note that the assumptions about the homogeneity of fixed costs and distribution of surplus imply that either $S_{i;\mu} > c_F$ for all nodes or none.

There are different possible motivations for the externality in Assumption 4. The failure of a viable business unit of bank i , for example, makes the bank less profitable and thus increases the refinancing costs for the bank's other business units.¹³ The externality can be further motivated by deadweight losses, e.g. in the form of legal fees, or losses due to fire-sales when a business unit fails. If the externality is sufficiently large the bank as a whole may become unprofitable. In this case the bank will cease intermediation across all layers. When solving the model analytically, we choose the simplest possible form of the externality: $q_i = 1$ for all banks i , i.e. we assume the externality is homogeneous across banks and assets.

Note that this contagion process differs from overlapping portfolio contagion, cf. Caccioli et al. (2014). In the setting of overlapping portfolios, banks invest in multiple assets and are subject to a solvency constraint that takes into account the value of the entire portfolio. Thereby gains in one asset can offset losses from another. In our model of intra-institutional spillover this “cross-subsidy” between different asset markets is not possible. The failure of a business unit in one market can lead to an immediate failure of the entire institution in all other markets regardless of the state of the other markets. In that sense, the percolation process considered here may be more “violent” than the process considered for overlapping portfolios.

4.3 Resilience of intermediation: analytical results

As mentioned above, we study resilience of intermediation networks to shocks. We measure resilience by the fraction of banks in the feasible intermediation network after a given exogenous shock. The exogenous shock to the intermediation network comes in the form of the failure of a certain fraction of business units trading a particular

¹³As mentioned in the introduction, a real world example of an intra-bank spill over of stress is the case of Bear Stearns in June 2007. Then, two of Bear Stearns funds trading in collateralized debt obligations (CDO) got into trouble due to stress in the market for CDOs. The initial failure of these funds ultimately lead to the failure of the entire institution. For a more detailed description of the events see for example <http://www.nytimes.com/2007/06/23/business/23bond.html>.

asset. In this section we will consider the large network limit as $n \rightarrow \infty$. We will begin by considering the special case of an intermediation network with a single asset μ (i.e. $K = 1$) to introduce notation and intuition. We will then proceed to study multi-asset intermediation networks ($K > 1$).

4.3.1 Single asset intermediation networks

First, consider an arbitrary network $\mathcal{G}_\mu(n, (k_i)_1^n)$ satisfying the conditions laid out in the previous section. As before we have the set N_μ of business units overall and sets of sellers E_μ and buyers D_μ . Also, as before let $n = |N_\mu|$. Note that by Assumption 2, $\mathcal{G}_\mu$ has a giant component to start with.

Now suppose business units in market μ fail uniformly at random with probability $1 - x$ due to an exogenous shock. Then $1 - x$ is also the expected fraction of business units that fail due to the exogenous shock. In the following we will simply refer to the term $1 - x$ as the exogenous shock. We denote the resulting network by $\mathcal{G}_\mu(x)$. Following the removal of these nodes, the network may fragment into a number of disconnected components $\{\mathcal{G}_\mu^l(x)\}$ of sizes $\{n^l(x)\}$. The size of these disconnected components will vary and depending on the size of the exogenous shock the network will still have a unique giant component. The giant component is particularly important in our analysis, as becomes apparent from the following proposition.

Proposition 4.1. *In the limit of large networks ($n \rightarrow \infty$), the giant component remains the only feasible intermediation network after fragmentation of the original network.*

Proof see Appendix 4.A.

Studying the resilience of an intermediation network to exogenous shocks can be complicated, even for a single asset. The importance of Proposition 4.1 is that it reduces this task to finding the giant component of the intermediation network. This is a well known problem in network theory and has been solved analytically using a variety of mathematical techniques for a large range of different network topologies, for example for Erdős-Rényi networks see seminal papers by Erdős and Rényi (1960) and Bollobás (1984).

One such technique is the generating function method which, under certain conditions, allows the computation of the size of a giant component for random graphs with a given degree distribution. In particular, the generating function method can be applied in the limit as $n \rightarrow \infty$ and if the networks are locally tree-like, i.e. short cycles are rare. In this chapter this condition is ensured by Assumption 1, cf. Britton et al. (2007). Generating functions have only recently been used to study economic problems.¹⁴

In the following we proceed in two steps. For the convenience of the reader, we first introduce generating functions as a methodology to find the size of a giant component in a network with a given degree distribution. Then, we show how the degree distribution changes once exogenous shocks are introduced.

¹⁴Campbell (2013) studies a word-of-mouth model in which consumers are connected in a social network using generating function. Gai and Kapadia (2009) also use generating functions to study financial contagion in interbank networks.

Generating Functions

The generating functions method has been extensively used to study the spread of epidemic diseases (see, for example, Newman (2002)) and the stability of critical infrastructure (see, for example, Buldyrev et al. (2010)). A key distinguishing feature of the generating function method is that it considers random collections, so called “ensembles”, of large locally tree-like networks (i.e. networks without short cycles) that share the same degree probability distribution rather than a specific instance of a network. Therefore, when computing a giant component using generating functions we compute it for an average network in the ensemble of networks specified by a given degree distribution. As mentioned above, for the analytical results in this paper we will consider intermediation networks amenable to the generating function method, i.e. networks following Assumptions 1, 2 and in the limit as $n \rightarrow \infty$. Before proceeding, it is useful to consider a simple example of a degree distribution.

Example - random matching: Suppose two intermediaries B_1 and B_2 randomly chosen out of the population of N_μ intermediaries meet with fixed probability p . Assume for simplicity that intermediaries B_1 and B_2 form a link with probability 1 if they meet. A given intermediary therefore establishes a link to any other of the $n - 1$ intermediaries with probability p . It is a well known result (see for example Newman (2003)) that the probability that a given intermediary has k links to other intermediaries is given by

$$P_\mu(k) = \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

The probabilities $\{P_\mu(k)\}_{0 \leq k \leq \infty}$ are then referred to as the degree distribution of the network $\mathcal{G}_\mu$ generated by the above outlined random matching process. This type of network is also referred to as an Erdős-Rényi network.

For a network $\mathcal{G}_\mu$ with a given degree distribution the generating function is defined as follows.

Definition 4.3. *The generating function of a network $\mathcal{G}_\mu$ with the degree distribution $\{P_\mu(k)\}$ is given by*

$$G_\mu(z) = \sum_{k=0}^{\infty} P_\mu(k) z^k. \quad (4.1)$$

We are interested in the probability that a randomly selected node belongs to the giant component of the network $\mathcal{G}_\mu$. This probability implicitly defines the fraction of nodes that belong to the giant component and thus its size. First, suppose we pick an intermediary in the network $\mathcal{G}_\mu$ with k trading partners. We then follow one of its trading relationships to another intermediary. Let f_μ denote the probability that such a randomly selected trading relationship does not lead to a giant component, i.e. that it leads to a finite-sized component. One can find that (see, for example, Newman et al. (2001)) the probability f_μ satisfies the following self-consistency relationship:

$$f_\mu = H_\mu(f_\mu), \quad (4.2)$$

where

$$H_\mu(z) = \frac{\sum_{k=0}^{\infty} P_\mu(k) k z^{k-1}}{\sum_{k=0}^{\infty} k P_\mu(k)}.$$

This can be seen as follows. The (unnormalized) probability of reaching an intermediary with degree k is $P_\mu(k)k$, i.e. the probability of the intermediary having degree k multiplied by the degree because it is k times more likely to reach an intermediary of degree k than an intermediary of degree 1. Given that we arrived on one link, there are $k - 1$ remaining links on a node with degree k . By definition, each link does not lead to the giant component with probability f_μ . Thus, the probability that none of the links lead to the giant component is simply f_μ^{k-1} . This hinges on the assumption that the probabilities that two neighbors do not lead to the giant component are independent. This assumption is justified since short cycles are rare for the networks

under consideration here. Taking into account all possible degrees we obtain

$$P(\text{Link does not lead to giant component}) = f_\mu = \sum_{k=0}^{\infty} P_\mu(k) k f_\mu^{k-1} / Z, \quad (4.3)$$

where Z is a normalization constant. Clearly, this yields a self consistency relationship for f_μ . In order to find Z , consider the special case where $f_\mu = 1$. Then,

$$f_\mu = 1 = \sum_{k=0}^{\infty} P_\mu(k) k / Z. \quad (4.4)$$

Clearly $Z = \sum_{k=0}^{\infty} P_\mu(k) k$. We thus have:

$$f_\mu = \frac{\sum_{k=0}^{\infty} P_\mu(k) k f_\mu^{k-1}}{\sum_{k=0}^{\infty} P_\mu(k) k}. \quad (4.5)$$

Note that $\sum_{k=0}^{\infty} P_\mu(k) k = E(k)$ is simply the average degree of the network $\mathcal{G}_\mu$. The probability that a randomly selected bank with k links has no trading relationship that leads to a giant component is f_μ^k . This is simply the probability that this bank does not belong to a giant component. When averaging over all banks in the network we find that the probability that a bank does belong to a giant component is

$$g_\mu(f_\mu) = 1 - \sum_{k=0}^{\infty} P_\mu(k) f_\mu^k = 1 - G_\mu(f_\mu). \quad (4.6)$$

Resilience to exogenous shocks

Now return to the initial scenario, where a random fraction $1 - x$ of an initially intact network $\mathcal{G}_\mu$ fails due to an exogenous shock. Following Newman (2002) one finds that after a random removal of $1 - x$ nodes the updated degree distribution is

$$P'_\mu(k) = \sum_{k_0=k}^{\infty} P_\mu(k_0) \binom{k_0}{k} (1-x)^{k_0-k} x^k, \quad (4.7)$$

where the term $\binom{k_0}{k} (1-x)^{k_0-k} x^k$ gives the probability that $k_0 - k$ neighbors were deleted in the random removal.¹⁵ Therefore the new generating function after the

¹⁵This of course only works since under our assumptions the failures of two neighbors of a particular node due to an exogenous shock are independent.

removal is:

$$\begin{aligned}
G_\mu(z, x) &= \sum_{k=0}^{\infty} P'_\mu(k) z^k = \sum_{k=0}^{\infty} \sum_{k_0=k}^{\infty} P_\mu(k_0) \binom{k_0}{k} (1-x)^{k_0-k} (zx)^k \\
&= \sum_{k_0=0}^{\infty} P_\mu(k_0) \sum_{k=0}^{k_0} \binom{k_0}{k} (1-x)^{k_0-k} (zx)^k \\
&= \sum_{k_0=0}^{\infty} P_\mu(k_0) (1-x+zx)^{k_0} = G_\mu(1-x+zx)
\end{aligned} \tag{4.8}$$

The probability g_μ that a randomly chosen bank belongs to the giant component, after removing a fraction $1-x$ of banks from the network is thus given as

$$g_\mu(f_\mu, x) = 1 - G_\mu(f_\mu, x) = 1 - \sum_{k=0}^{\infty} P_\mu(k) (1-x+f_\mu x)^k, \tag{4.9}$$

where f_μ satisfies the recursive relationship (4.2) with the new argument:

$$f_\mu(x) = H_\mu(f_\mu, x) = \frac{\sum_{k=0}^{\infty} P_\mu(k) k (1-x+f_\mu x)^{k-1}}{E(k)}. \tag{4.10}$$

We are now in the position to formulate a first result on the resilience of an intermediation network of a single asset subject to random exogenous shocks (See also Campbell (2013)).

Proposition 4.2. *Let $\{P_\mu(k)\}$ be the degree distribution of a random intermediation network $\mathcal{G}_\mu$. Suppose that Assumptions 1 and 2 hold. Then, in the limit as $n \rightarrow \infty$ there exists a critical exogenous shock of size $1-x_c$ at which the size of the feasible intermediation network vanishes, i.e. $\exists x_c \in (0, 1]$ such that:*

1. $g_\mu(f_\mu, x) = 0 \ \forall x \in (0, x_c]$,
2. $g_\mu(f_\mu, x) \geq 0 \ \forall x \in [x_c, 1]$,
3. $g_\mu(f_\mu, x_c^-) = g_\mu(f_\mu, x_c^+)$,

where

$$x_c = \frac{1}{E(k^2)/E(k) - 1}.$$

Proof see Appendix 4.A.

Proposition 4.2 implies that, as the size of the exogenous shock on an intermediation network is increased, the size of the remaining feasible network monotonically decreases until it vanishes. Therefore, the transition from the intermediation to the no-intermediation regime is *continuous*. Intermediation vanishes because there is no longer a feasible intermediation network: due to the exogenous shock, the network becomes fragmented to the extent that sellers can no longer find an intermediation path to potential buyers. Also note that for networks with very broad degree distributions the critical exogenous shock $1 - x_c$ vanishes. In particular, for degree distributions with infinite second moment $E(k^2)$ but finite first moment the critical fraction $1 - x_c$ vanishes, i.e. $E(k^2) \rightarrow \infty$ implies $x_c \rightarrow 0$ and $g_\mu(f_\mu, x) > 0 \forall x > 0$. This is the case, for example, for some networks with power law degree distributions.

4.3.2 Two-asset intermediation networks

In the following we will consider how the resilience of an intermediation network is affected as we extend our model to two assets with an externality between different business units. Assets A and B are traded by a set of N_A and N_B business units with $|N_A| = |N_B|$ in intermediation networks $\mathcal{G}_A$ and $\mathcal{G}_B$ with degree distributions $\{P_A(k)\}$ and $\{P_B(k)\}$, respectively. Before proceeding we make the following technical assumption.

Assumption 5. $g_\mu(f_\mu, x)$ is concave in x for $\mu \in \{A, B\}$.

This implies that the size of the feasible intermediation network declines faster as shocks become larger.

Furthermore, for analytical convenience we make two simplifying assumptions, which we will relax in Section 4.5. First, we assume that the networks $\mathcal{G}_A$ and $\mathcal{G}_B$

are uncorrelated. This implies that the degree sequences of the two networks are uncorrelated, i.e. the degree of a business unit trading asset μ does not contain information about the degree of the business unit trading asset ν within the same bank. While this appears to be a strong assumption, we conjecture that for multiple assets $K > 2$ it is sufficient for one of the intermediation networks to be uncorrelated with the other networks to qualitatively recover the results for the uncorrelated two asset case, which would make this assumption somewhat more benign. This conjecture is supported by numerical experiments in Section 4.5. Second, we study the case when the size of the externality exerted by a defaulting business unit of bank i on another business unit of the same bank is large, i.e. when $q = 1$.

We show in this section that within our model portfolio diversification—banks trading multiple assets—leads to systemic instability. When the probability of the externality from one business unit of a bank to another business unit is positive, trading multiple assets by the same bank makes the intermediation network less resilient to exogenous shocks. This result has a simple intuition in our model framework. Suppose that, as in the single asset case, a fraction $1 - x$ of business units in network $\mathcal{G}_A$ fail due to an exogenous shock. The network fragments as a consequence and only those business units survive that remain in the feasible intermediation network of asset A . For a single asset the contagion process stops here. However, a failed business unit trading asset A exerts an externality on another business unit in the same institution. Because we assume that $q = 1$, the other business unit trading asset B , fails as well. The failed business units in network $\mathcal{G}_B$ fragment this trading network and lead to further failures of business units that are no longer connected to the feasible intermediation network. The process reverts to $\mathcal{G}_A$ in the next step. Thus, the result of a potentially small exogenous shock in one network can be amplified through *intra-bank spillovers*. In the case of a smaller intra-bank externality,

i.e. $q < 1$, the bank's business units are less likely to fail conditional on the failure of one of the bank's business units failure. The cascading effect outlined above will be less pronounced, but present nonetheless. We study this in more detail in Section 4.5.

In order to determine the extent of the cascade resulting from an initial shock we will closely follow the methodology proposed by Buldyrev et al. (2010). Consider the following iterative procedure (see Fig. 4.1 for an illustration). Denote the fraction of surviving nodes after spillover (or equivalently an exogenous shock) in network A by ψ_n and network B by ϕ_n at cascade step n . In the first step an exogenous shock causes the failure of a fraction $1 - x$ of business units in network $\mathcal{G}_A$, i.e. a fraction $\psi_1 = x$ survives. The new feasible intermediation network for asset A is determined by the probability that a bank has survived the initial attack times the probability that it belongs to the giant component, i.e. $xg_A(x)$ (Note that we dropped the term f_A in $g_A(\cdot)$ for readability). Next, spillover from asset A to asset B ensues and because of $q = 1$ we have for the fraction of banks that survive after spillover $\phi_1 = xg_A(x)$. The new feasible intermediation network for asset B is then given by $g_B(\phi_1)$. In the second step spillovers from asset B to asset A ensue, i.e. $\psi_2 = xg_B(\phi_1)$. The feasible intermediation network for asset A is now given by $xg_A(\psi_2)$. The second step is concluded when spillover from asset A to B ensues and $\phi_2 = xg_A(\psi_2)$. Again, the new feasible intermediation network for asset B is given by $g_B(\phi_2)$. Note that this cascade implies that, whenever there is spillover, more business units will fail in the multi-layer case than in the single layer case. In fact, evaluating the size of the feasible intermediation network for different cascade steps provides a quick way to estimate a lower bound on how much spillover amplifies the initial shock compared to the single layer case.

Generalizing this cascading process to n steps the fraction of nodes surviving after spillover from asset B to asset A is given as $\psi_n = xg_B(\phi_{n-1})$, which yields a feasible

intermediation network for asset A given by $xg_A(\psi_n)$ after fragmentation. Similarly, after the subsequent spillover from asset A to asset B we have $\phi_n = xg_A(\psi_n)$ and obtain a new feasible intermediation network for asset B determined by $g_B(\phi_n)$. Substituting the expression for ϕ_{n-1} in to ψ_n we obtain the following iterative equation:

$$\psi_n = xg_B(xg_A(\psi_{n-1})).$$

We seek the fixed point of this iterative equation, in particular we want to find y^* such that:

$$y^* = F(x, y^*) = xg_B(xg_A(y^*)). \quad (4.11)$$

The largest fixed point to the above allows the computation of the size of the feasible intermediation network at the end of the cascade of failures triggered by the initial exogenous shock of size $1 - x$. The size of the giant component in network A is then $xg_A(y^*)$; the size of the giant component in network B can be computed analogously. Studying the fixed point equation (4.11) yields the following Proposition.

Proposition 4.3. *Suppose there exists a range of exogenous shocks $1 - x$ for which the largest solution y^* to $y = F(y, x)$ is non zero. Suppose that Assumptions 1, 2 and 5 hold. Then, in the limit as $n \rightarrow \infty$ there exists an exogenous shock $1 - x_c$ such that asset intermediation breaks down for both assets. The transition from intermediation to the break down of intermediation is abrupt and discontinuous in the size of the exogenous shock, i.e. $\exists x_c$ such that:*

1. $F(y^*, x) = 0 \ \forall x \in [0, x_c),$
2. $F(y^*, x) > 0 \ \forall x \in [x_c, 1],$
3. $F(y^*, x)$ is discontinuous at $x = x_c$, i.e $F(y^*, x_c^-) = 0 \neq F(y^*, x_c) > 0.$

Proof see Appendix 4.A.

Proposition 4.3 tells us that intermediation networks with two assets are susceptible to sudden collapse following an initial, potentially small exogenous shock. As in the single asset case, intermediation collapses when the networks have been fragmented to the extent that sellers can no longer find intermediation paths to potential buyers. Hence, the market freezes. However, in contrast to the single asset case, the network fragmentation is not exclusively the result of an initial exogenous shock, but occurs due to the cascading feedback between the two asset intermediation networks. It is this cascading feedback that leads to the sudden collapse of intermediation at the critical exogenous shock size $1 - x_c$. It is also worth noting that whenever there is a cascade of failures as outlined above, the size of the surviving intermediation network in the multi-layer case is always smaller than in the single layer case – even for shocks smaller than the critical exogenous shock.

It can be shown that the destabilizing feedback between the two intermediation networks not only makes the collapse of intermediation more abrupt, it also decreases the overall resilience of the system, i.e. it reduces the critical size of the exogenous shock $1 - x_c$ at which intermediation breaks down relative to the corresponding thresholds if there was no spillover between the two intermediation networks $\mathcal{G}_A$ and $\mathcal{G}_B$. In other words, when there is an externality, intermediation in the multilayer network breaks down for values of $1 - x_c$ at which intermediation in the single-layer case does not break down. We summarize this result in the following Proposition.

Proposition 4.4. *Suppose there exists a range of exogenous shocks $1 - x$ for which the largest solution y^* to $y = F(y, x)$ is non zero. Suppose that Assumptions 1, 2 and 5 hold and $n \rightarrow \infty$. Let $1 - x_{c,A}$ and $1 - x_{c,B}$ be the critical exogenous shocks for networks A and B respectively at which intermediation vanishes if there is no externality between business units. Furthermore, let $1 - x_c$ be the critical exogenous shock for two asset intermediation networks in the presence of an externality. Then the presence of an externality leads to an earlier break down of intermediation, i.e.*

$$x_{c,A}, x_{c,B} < x_c.$$

Proof see Appendix 4.A.

Our stylized model of financial intermediation suggests that stress in different asset markets can be severely amplified via intra-institution spillover: stress from one asset spreads to another via institutions that trade in both assets. This cross-asset amplification can lead to an abrupt break down of trading across all assets and suggests that coupled asset markets may be significantly less resilient to exogenous shocks compared to a benchmark without cross-asset amplification. These findings contribute to and are broadly consistent with the literature on multi-layer financial networks and contagion via overlapping investment portfolios, see for example Caccioli et al. (2013).

4.4 The Resilience of Different Network Structures

In this section we study the resilience of two different example network structures to exogenous shocks. In particular, we study a network with a relatively homogeneous degree distribution, i.e. an Erdős-Rényi network, and a scale-free network with a relatively heterogeneous degree distribution both partially inspired by Buldyrev et al. (2010).

4.4.1 Erdős-Rényi network

Recall that an Erdős-Rényi network (ER) network can be easily generated through a random matching process between the banks willing to trade a particular asset, see Section 4.3.1. Now, consider two such networks A and B composed of the set of N_A and N_B banks with $n = |N_A| = |N_B|$. The two networks have degree distribution

$$P_\mu(k) = \binom{n-1}{k} p_\mu^k (1-p_\mu)^{n-k-1}, \quad \forall \mu \in \{A, B\},$$

p_μ is the connection probability of two randomly selected banks. In the limit $n \rightarrow \infty$ with $z = np_\mu = \text{const.}$, the corresponding generating functions become:

$$G_\mu(u) = H_\mu(u) = e^{z_\mu(u-1)},$$

where z_μ is the average degree of a bank in network $\mu \in \{A, B\}$, cf. Newman (2002). After an exogenous shock removing a fraction $1-x$ of nodes, the generating functions become

$$G_\mu(u, x) = H_\mu(u, x) = e^{z_\mu x(u-1)},$$

As before, assume that the spillover probability is $q = 1$. We therefore have:

$$g_\mu(y) = 1 - G_\mu(y) = 1 - \exp[z_\mu y(f_\mu(y) - 1)],$$

$$f_\mu(y) = H_\mu(y) = \exp[z_\mu y(f_\mu(y) - 1)],$$

$$F(y, x) = xg_B(xg_A(y)).$$

We seek the largest solution y^* to $y = F(y, x)$, where y^* corresponds to the probability that a bank has survived the failure cascade. We can then compute the final size of the feasible intermediation network A as follows:

$$S(x) = y^*(x)g_A(y^*(x)).$$

In Figure 4.4 we solve $S(x)$ numerically for two single asset networks A and B with $z_A = 3$ and $z_B = 4$, respectively, and for the coupled two asset intermediation network with $q = 1$. Both the results from Proposition 4.3 and 4.4 are evident: in the two asset case with spillover intermediation breaks down abruptly at a significantly smaller exogenous shock.

4.4.2 Scale free networks

Now consider a scale free network that emerges for example via a preferential attachment network formation process, see Barabási and Albert (1999). The two intermediation networks A and B have degree distributions $P_A(k)$ and $P_B(k)$ such that $P_\mu(k) = C_\mu k^{-\alpha_\mu}$, $\alpha_\mu \in (2, 3]$ for $k > 1$. Note that for this explicit calculation we relax some of the conditions in Assumption 1, in particular we consider the case where the second moment of the degree distribution diverges.¹⁶ As before $n = |N_A| = |N_B|$. The constant that normalizes the degree distribution is $C_\mu = 1/(\zeta(\alpha_\mu) - 1)$, where $\zeta(\cdot)$ is the Riemann zeta function. Also define

$$y = F(y, x) = x g_B(x g_A(y)),$$

for $0 < y \leq 1$ and with

$$\begin{aligned} g_\mu(y) &= 1 - G_\mu(y) = 1 - \sum_k P_\mu(k)(y f_\mu(y) + 1 - y)^k, \\ f_\mu(y) &= H_\mu(f_\mu(y), y) = \frac{\sum_k P_\mu(k) k (y f_\mu(y) + 1 - y)^{k-1}}{E_\mu(k)}, \end{aligned}$$

for $\mu \in \{A, B\}$. First make the substitution $z = y f(y) + 1 - y$, i.e. $f = (z - 1)/y + 1$. Then:

$$f_\mu(y) = (z - 1)/y + 1 = \frac{1}{E_\mu(k)z} \sum_{k=2}^{\infty} C_\mu \frac{z^k}{k^{\alpha_\mu-1}} = \frac{1}{z(\text{Li}_{\alpha_\mu-1}(1) - 1)} (\text{Li}_{\alpha_\mu-1}(z) - z),$$

where $\text{Li}_s(z)$ is the polylogarithmic function defined by:

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s},$$

where s is complex number and z is a complex number with $|z| < 1$, which is clearly valid here. In the following we will only consider real s and z . The intersection

¹⁶This choice of parameters implies that nodes are linked to at least two other nodes. This is important since it makes it more likely that the network is initially fully connected. Furthermore, the range of α_μ implies that the distribution has a well defined first moment but a diverging second moment. For $\alpha_\mu \leq 2$ the first moment diverges. For $\alpha_\mu > 3$ the second moment becomes finite. Also the estimated tail exponents for many empirically observed networks fall into this range.

between $f_\mu(z)$ and $(z-1)/y+1$ on the interval $y \in (0, 1]$ and $z \in [0, 1)$ determines z for a given y . Again making use of the substitution $z = yf(y) + 1 - y$ we can write

$$g_\mu(z) = 1 - \sum_{k=2}^{\infty} P_\mu(k) z^k = 1 - C_\mu (\text{Li}_{\alpha_\mu}(z) - z).$$

As before, we seek the largest solution y^* to $y = F(y, x)$, where y^* corresponds to the probability that a node has survived the failure cascade. Again we can compute the final size of the feasible intermediation network A as follows:

$$S(x) = y^*(x)g_A(y^*(x)).$$

In Figure 4.5 we solve $S(x)$ numerically for a scale-free single asset intermediation network with exponent $\alpha = 3$, a multi-asset intermediation network composed of two networks with exponent $\alpha = 3$ and spillover $q = 1$ and an Erdős-Rényi network with the same average degree as the scale-free networks. Note that in the single asset scale-free network intermediation technically never breaks down since $\alpha \leq 3$ and thus $E_\mu(k^2) \rightarrow \infty$ as noted in Proposition 4.2. However, in the multi-asset scale-free network intermediation does break down after a sufficiently large exogenous shock.

Interestingly, scale-free networks are typically known to be more resilient to random attack than Erdős-Rényi networks (see Newman (2003)). This relationship is reversed in the example presented here. The intuition behind this finding is as follows: in a scale-free intermediation network there are a small number of very highly connected banks and a large number of banks with only a small number of trading relationships. Thus, an exogenous shock is most likely to hit a bank with few trading relationships. Due to its peripheral role in the intermediation network, the failure of a low degree bank will only lead to a small fragmentation of the overall network. Hence, a scale-free network for a single asset is relatively resilient against a random exogenous shock. However, as soon as intra-institutional spillover is added, things look differently. Now, a bank with relatively low degree in the intermediation network

for asset A may have very high degree in intermediation network B . Thus, an exogenous shock in one asset market may be greatly amplified through the presence of banks with very heterogeneous numbers of trading relationships for different assets.

4.5 Resilience of intermediation: numerical results

In our model we abstract from a number of important features of realistic financial systems. In the following, we study resilience of intermediation networks numerically and relax a number of these assumptions. In particular, we study (i) the case of more than two asset markets; (ii) the impact of reducing the spillover probability q ; (iii) the effect of correlated and overlapping intermediation networks; and finally (iv) the resilience of intermediation networks with a stylized core-periphery topology. Thus, these numerical experiments allows us to test our results on a more general set of intermediation networks and externalities.

For our analysis we generate K intermediation networks each with n nodes.¹⁷ The K intermediation networks are generated in order to match the desired correlation and overlap structure. We ex-ante fix the size of the externality between the banks' business units by creating "*spillover links*" between business units with probability q . These spillover links are used to determine whenever the default of one business unit leads to the default of other business units. This yields the multi-asset intermediation network $\mathcal{G}$. The protocol for numerical computation of the resilience of intermediation networks follows the iterative procedure outlined in section 4.3.2. First, layer $\mu = 1$ is subject to a random exogenous shock of size $1 - x$. In the next step we evaluate which business units in layer $\mu = 1$ belong to the largest connected component (i.e. the giant component) using a simple breadth-first search algorithm (see for example Skiena (1998)). Essentially, we replace the function $g_\mu(\cdot)$ for the size of the giant

¹⁷We use the C++ Snap library, see Leskovec and Sosič (2014).

component by the breadth-first search algorithm. As before, all banks found not to be in the giant component fail. We then compute the spillover to the remaining asset intermediation networks and evaluate the corresponding giant components. We repeat the process until convergence. We perform this protocol for different values of the initial exogenous shock and with ten different random seeds. At the end of the failure cascade we measure the total size of the intermediation network on business unit level, i.e. we define

$$S = \text{Final fraction of nodes in feasible network} = \frac{\sum_{\mu} |N_{\mu}|}{nK}, \quad (4.12)$$

where N_{μ} is the set of business units in the feasible intermediation network for asset μ when the failure cascade has converged while n is the size of the network before the initial shock hits. All numerical results are averages over ten runs. The standard deviation within this sample is typically of the order of 10^{-6} . Error bars are therefore not shown on the plots. In the following we discuss each of the four numerical scenarios separately.

4.5.1 Multi-asset intermediation

In order to generalize our results for the resilience of intermediation networks with multiple types of assets, we investigate networks with $K = 2$, $K = 3$ and $K = 5$ different assets. All intermediation networks are uncorrelated Erdős-Rényi networks with mean degree $z = 5$ and $n = 20000$ banks. As before, we take a uniform spillover probability of $q = 1.0$. We present the results of the simulations together with the analytical predictions for a single asset network and a network with two assets in Figure 4.6. For $K = 2$ the numerical results are well in line with our analytical predictions. Interestingly, as we increase the number of assets in the intermediation network, the network becomes less and less resilient. The size of the feasible intermediation network declines faster for a given exogenous shock $1 - x$ and intermediation breaks down earlier for larger K . Our simulations suggest the following:

Numerical result 1. *Consider two uncorrelated multi-asset intermediation networks with K and K' assets respectively and $q = 1$. Let $1 - x_c$ and $1 - x'_c$ denote the respective critical exogenous shocks at which intermediation breaks down. Then, the multi-asset intermediation network that includes more assets is less resilient to exogenous shocks, i.e.:*

$$x_c > x'_c, \text{ if } K > K'.$$

Thus, as banks diversify their business across many assets, the fragility of the system and thus the probability of catastrophic collapse of intermediation increases.

4.5.2 Reducing the spillover probability

So far we have only considered the case of $q = 1$, i.e. the externality from the failure of one business units always leads to the failure of the entire bank. Thus, multi-asset intermediation networks are ex-ante fragile. We test the sensitivity of our results for a set of different values for $q = \{0.1, 0.3, 0.5, 0.8, 0.9\}$. As before we run simulations with intermediation networks that are uncorrelated Erdős-Rényi networks with mean degree $z = 5$ and $n = 20000$ banks. Intuitively, for $q = 0$ we should recover the dynamics for a single asset network in the network for the asset that faces the initial exogenous shock, while the network corresponding to the “unshocked” network should remain entirely intact. Therefore we expect that $\lim_{q \rightarrow 0} S \rightarrow 0.5$.

We summarize our results in Figure 4.7. As expected, as the spillover probability decreases, networks become more resilient. In fact in the case of an Erdős-Rényi network we observe that for sufficiently low spillover probability $q < 0.8$, the abrupt collapse of intermediation disappears and the network dynamics resemble more the analytical single asset case than the analytical multi-asset case with $q = 1$. This finding can be summarized as:

Numerical result 2. *Consider two uncorrelated multi-asset intermediation networks with spillover probabilities q and q' respectively. Let $1 - x_c$ and $1 - x'_c$ denote the respective critical exogenous shocks at which intermediation breaks down. Then, the multi-asset intermediation network that has a higher spillover probability is less resilient to exogenous shocks, i.e.:*

$$x_c > x'_c, \text{ if } q > q'.$$

This result shows the importance of intra-institutional safe guards to minimize the spillover of stress from one business unit to the other. Such measures can include ring fencing (for the current discussion about ring-fencing in the UK, see Bank of England (2014)) or requirements on sufficient cash buffers in order to absorb adverse shocks from other business units.

4.5.3 Correlated and overlapping intermediation networks

Often trading networks for different assets are correlated and overlapping. Network correlation and network overlap are related but slightly different concepts. Before proceeding we define these concepts more formally.

Definition 4.4. *Consider the degree distributions of two networks $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$: $\{P_\mu(k)\}$ and $\{P_\nu(k)\}$ respectively. Then the correlation of networks $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ is simply the Pearson correlation of the degrees, i.e.:*

$$C_{\mu\nu} = \frac{\sum_{i=1}^n (k_{i\mu} - z_\mu)(k_{i\nu} - z_\nu)}{\sqrt{\sum_{i=1}^n (k_{i\mu} - z_\mu)^2 (k_{i\nu} - z_\nu)^2}}, \quad (4.13)$$

where z_μ and z_ν are the average degrees of networks $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ respectively.

Definition 4.5. *Consider the degree distributions of two networks $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$: $\{P_\mu(k)\}$ and $\{P_\nu(k)\}$ respectively. Then the overlap o of networks $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ is the fraction of edges that are present in $\mathcal{G}_\mu$ that are also present in $\mathcal{G}_\nu$.*

Real financial networks are likely to be both correlated and overlapping since the network formation processes in each layer are unlikely to be independent.¹⁸ Correlation arises for example when a bank that is large overall is likely to have many trading relationships for all its business units, i.e. a large degree in layer μ is likely to be associated with a large degree in layer ν . Overlap arises when it is more likely that a bank establishes a trading relationship with a bank that it already trades with than a bank it has no relationship with at all. In the following we consider Erdős-Rényi intermediation networks with at least two assets, mean degree $z = 5$, $n = 20000$ banks and spillover probability $q = 1$. In particular, we study the following cases: (i) full correlation between intermediation networks for two assets A and B, $C_{AB} = 1$, (ii) varying degree of overlap $o = \{0.1, 0.5, 0.7, 1.0\}$ between intermediation networks for two assets A and B.¹⁹

We present our results for the full correlation case in Figure 4.8. We find that in the case of Erdős-Rényi intermediation networks a full degree correlation preserves the qualitative dynamics of the failure process, i.e. there is still an abrupt collapse of intermediation at some critical exogenous shock $1 - x_c$. However, the correlation increases the critical shock size. Thus, correlated intermediation networks are more stable than uncorrelated intermediation networks. Similar findings were also found by Buldyrev et al. (2011). Interestingly though, the addition of one uncorrelated layer to an intermediation network of two correlated networks leads to a significant increase in fragility. In fact, our simulations show that a network with $K = 3$ assets of which two are fully correlated and one uncorrelated is more fragile than an uncorrelated

¹⁸In Germany, Roukny et al. (2014) find that, over a period of 12 years, the German interbank network was positively correlated with the market for credit default swaps.

¹⁹We generate two fully correlated networks by first generating one random network in the usual way. We then create a copy of the first network and randomly rewire the links within the copied network while preserving the degree of each node. We generate two overlapping networks in a similar fashion. First we create one random network as well as a copy of it. Then we randomly rewire a fraction o of the spillover links.

network with $K = 2$. This suggests that, while correlations across intermediation networks increase system stability, this increase itself is not robust as it is lost simply by adding an additional uncorrelated intermediation network. Given that many banks trade in many assets, i.e. $K \gg 1$ it is likely that a network for at least one asset is uncorrelated. This suggests the following generalized statement:

Numerical result 3. *Consider the degree distributions of three networks $\mathcal{G}_\mu$, $\mathcal{G}_\nu$ and $\mathcal{G}_\rho$: $\{P_\mu(k)\}$, $\{P_\nu(k)\}$ and $\{P_\rho(k)\}$ respectively. Further assume that the network correlations are $C_{\mu\nu} = 1$, $C_{\mu\rho} = 0$ and $C_{\nu\rho} = 0$ respectively. Let $\mathcal{G}_{\mu\nu}$ be the two asset intermediation network formed by $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$. $\mathcal{G}_{\mu\rho}$ and $\mathcal{G}_{\nu\rho}$ are similarly defined. The critical exogenous shocks are denoted by $1 - x_{c,\mu\nu}$, $1 - x_{c,\mu\rho}$ and $1 - x_{c,\nu\rho}$ respectively. Then the networks fragility takes the following order:*

$$x_{c,\mu\rho} < x_{c,\mu\nu} < x_{c,\nu\rho}.$$

We summarize the results for the case of varying link of overlap in Figure 4.9. For small levels of network overlap, i.e. $o = 0.1$, the numerical solution is in good agreement with the analytical solution for the case without overlap. As the network overlap increases, the critical exogenous shock at which intermediation breaks down increases. Furthermore, the collapse of intermediation at the critical shock level becomes less abrupt as the overlap increases. For a complete overlap $o = 1$, we recover the analytical solution for a single asset network. These findings suggest the following generalized statement:

Numerical result 4. *Consider two multi-asset intermediation networks with overlap o and o' respectively. Let $1 - x_c$ and $1 - x'_c$ denote the respective critical exogenous shocks at which intermediation breaks down. Then, the multi-asset intermediation network that has less overlap is less resilient to exogenous shocks, i.e.:*

$$x_c > x'_c, \text{ if } o < o'.$$

This result can be understood as follows. In the limit of a perfect overlap $o = 1$, the networks $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ are identical. In this case the multi-asset amplification effect is not present. The reason for this is intuitive. Denote by B_μ the set of nodes in network $\mathcal{G}_\mu$ that failed (i.e. are no longer part of the giant component) after an initial shock to $\mathcal{G}_\mu$. The failure of the nodes in B_μ cause the failure of an identical set B_ν in network $\mathcal{G}_\nu$. However, since the networks are identical, all nodes B_ν lie outside the giant component. Only nodes that are outside the giant component in network $\mathcal{G}_\mu$ but inside the giant component in $\mathcal{G}_\nu$ can cause further failures in network $\mathcal{G}_\nu$. Therefore, there are no further failures in the case of perfect overlap $o = 1$.

One can now apply this intuition naively to the case of $o < 1$ to obtain an approximate analytical expression. Suppose $\mathcal{G}_\mu$ is subject to an exogenous shock $1 - x$. Then

$$\begin{aligned} \psi_1 &= x, \\ \phi_1 &= x g_\mu(\psi_1), \end{aligned} \tag{4.14}$$

$$\psi_2 = x(g_\nu(\phi_1)(1 - o) + o),$$

where the last equation can be obtained as follows. Fragmentation in network $\mathcal{G}_\mu$ leads to the failure of $1 - \phi_1$ nodes in network $\mathcal{G}_\mu$ and $\mathcal{G}_\nu$ by dependency links. However, by the same logic as above for full overlap, a fraction of nodes in the failed set B_ν will already be outside the giant component and will therefore not cause any further failures. We conjecture that this fraction is well approximated by the overlap fraction o . Then the remaining fraction $1 - o$ will cause failures as usual. The combined effect is then simply $g_\nu(\phi_1)(1 - o) + o$. The modified system of equations at cascade

convergence is then

$$\begin{aligned} z &= x(g_\nu(y)(1 - o) + o), \\ y &= x(g_\mu(z)(1 - o) + o). \end{aligned} \tag{4.15}$$

This approach approximates numerical simulations relatively well for small or large levels of overlap but relatively poorly for intermediate levels of overlap, see Figure 4.10.

4.5.4 Core-periphery networks

Many empirical studies of financial networks have found a core-periphery structure, see for example Craig and Von Peter (2014). In a stylized core-periphery network there exists a core of nodes which are fully connected. Peripheral nodes have a single connection to a node in the core and are not connected among each other. We consider two-asset intermediation networks composed of such stylized core-periphery networks with a core of 10% of the total number of banks in the network. As in the previous section, we consider networks with different level of network overlap $o = \{0.0, 0.1, 0.3, 0.7, 1.0\}$. We summarize our results in Figure 4.11.

First note that there is never a critical exogenous shock at which intermediation breaks down entirely. As the shock size increases the size of the feasible intermediation network decreases but $S > 0$ for all $x > 0$. The intuition for this observation is as follows. Suppose first that a bank in the periphery fails. A bank in the periphery never causes any additional failures since it cannot fragment the network. Suppose now a bank in the core fails. The failure of a core node can only lead to the failure of peripheral nodes. The core cannot fragment since it is fully connected. Hence, there can never be a cascade of failures that leads to the disappearance of the giant component in the network. Hence there is no critical exogenous shock. As such core-periphery networks appear very robust to random shocks.

However, while there is never a complete breakdown of intermediation, there is still

amplification of initial shocks via the cascades of failures induced by the intra-bank spillovers. As the network overlap decreases, the extent of this amplification becomes large, leading to a drastic reduction in the size of the feasible intermediation network for even relatively small exogenous shocks. In this light, core periphery networks appear significantly less stable than for example Erdős-Rényi networks provided that the network overlap is sufficiently small.

4.6 Conclusion

Intra-institutional links can be induced by an externality when the failure of a business unit within a financial intermediary causes adverse consequences for other business units in the same intermediary. We capture such intra-institutional linkages in a simple model of the financial system in which banks trade different types of assets through specialized business units. If one of the business units defaults it increases the refinancing cost of other business units within the same bank, e.g. via losses through asset fire-sales or deadweight losses due to legal fees. This increase can trigger default of the other business units. When there is considerable stress in the financial system, already a small increase in a business unit's refinancing cost can lead to its default. Our analysis is thus particularly important during times of large stress, such as during the global financial crisis.

We obtain closed-form analytical solutions for the resilience of a system of coupled trading networks. We show that in the case of a single asset the size of the largest feasible trading system, i.e. the part of the network where intermediation can take place, continuously declines with the size of the exogenous shock. However, for some network structures, when there are at least two coupled trading systems, the size of the largest feasible trading system experiences a sudden collapse once the size of the exogenous shock exceeds a certain threshold. When applied to different network

topologies, we show that heterogeneous coupled networks (i.e. scale-free) are *less* resilient than homogeneous coupled networks (i.e. Erdős-Rényi), which is in stark contrast to the case of a single asset. Our paper thus contributes to the literature on financial stability, and in particular to the literature on the stability of different network structures.

We relax all crucial assumptions, necessary to obtain closed-form analytical solutions, in a numerical extension. Our main result, that intermediation breaks down suddenly and abruptly in coupled trading systems, is stronger for larger coupling strengths, i.e. when the externality exerted by a defaulting business unit on other business units is larger. Our result is also stronger when the trading networks for different assets are uncorrelated. Should they be correlated, however, the addition of a single uncorrelated trading network leads to a sudden and abrupt breakdown of intermediation, for an even smaller exogenous shock. Finally, we find that coupled core-periphery networks are more resilient to large exogenous shocks than coupled Erdős-Rényi networks, but are *less* resilient at smaller shock sizes.

Our numerical analysis demonstrates that knowledge of the empirical structure of intermediation networks is crucial to evaluate whether real financial networks are susceptible to the abrupt break down of intermediation observed in our model. Therefore a logical next step to the theoretical analysis presented here is an empirical analysis of over-the-counter intermediation networks and their links via diversified institutions. A potentially promising dataset for such a study would be the data on conduits trading in asset backed securities presented and analyzed in Acharya et al. (2013).

4.A Appendix: Proofs

4.A.1 Overview of technical assumptions

Recall the following technical assumptions made throughout this paper.

- For each n , $(k_i)_1^n = (k_i^{(n)})_1^n$ is a sequence of non-negative integers such that $\sum_{i=1}^n k_i$ is even and, for some probability distribution over degrees $(P(k))_{k=0}^\infty$ independent of n , and with $n_j := \#\{i : k_i = j\}$,
 1. $n_j/n \rightarrow P(j)$ for every $j \geq 0$ as $n \rightarrow \infty$,
 2. $E(k) = \lambda := \sum_k kP(k) \in (0, \infty)$,
 3. $\sum_{i=1}^n k_i/n \rightarrow \lambda$ as $n \rightarrow \infty$,
 4. $P(2) < 1$,
 5. $P(1) > 0$,
 6. $\sum_i k_i^2 = O(n)$.
- For a given degree distribution $(P(k))_{k=0}^\infty$ we have $\sum_k k(k-2)P(k) > 0$.
- $g_\mu(f_\mu, x)$ is concave in x for $\mu \in \{A, B\}$.
- The probability of spillover is homogeneous across banks and assets and takes the value $q = 1$.

4.A.2 Preliminaries

Recall the following definitions:

$$g(f, x) = 1 - G(f, x) = 1 - \sum_k P(k)(xf + 1 - x)^k,$$

$$f = H(f, x) = \frac{\sum_k P(k)k(xf + 1 - x)^{k-1}}{E(k)},$$

where $1 - x$ is an exogenous shock leading to the random failure of $1 - x$ nodes.

4.A.3 Proposition 4.1

Proof. Proposition 4.1: An intermediation network is feasible if it is connected and contains at least one buyer and one seller. By definition any of the components remaining after an exogenous shock will be connected within. It remains to show that in the limit of large networks the probability of finding a seller and a buyer in a component that is not the giant component vanishes. Recall that the sellers and buyers constitute a subset of the original network and that their number does not scale with the size of the network. Without loss of generality, assume there is only a single buyer and a single seller, i.e. $|D_\mu| = |E_\mu| = 1$.²⁰ It is easy to see that the probability that a seller e and a buyer b are present in a particular component $\mathcal{G}_\mu^l(x)$ of size $n^l(x)$ is proportional to the square of the size of the component,

$$P(e \in N_\mu^l \wedge b \in N_\mu^l) \propto \left(\frac{n^l(x)}{n} \right)^2.$$

Thus,

$$\lim_{n \rightarrow \infty} P(e \wedge b \in N_\mu^l) \propto \left(\frac{n^l(x)}{n} \right)^2 \rightarrow \begin{cases} \text{const.} & , \text{ if } n^l(x) \propto n, \\ 0 & , \text{ otherwise.} \end{cases}$$

Since $n^l(x)$ can at most be proportional to n , only a giant component has a non-zero probability of containing a seller and a buyer and hence remains the only feasible intermediation network after fragmentation of the original network. $\square$

4.A.4 Proposition 4.2

This proof is adapted from Campbell (2013). First consider the following Lemma.

²⁰For $|D_\mu| = |E_\mu| > 1$ (but fixed, i.e. the number buyers and sellers does not scale with the number of banks) the probability of finding m sellers (buyers) in a given component will be given by a binomial distribution. The basic argument remains the same in this case.

Lemma 1. For fixed $x \in [0, 1]$, $H(f, x)$

1. has $f = 1$ as a trivial solution.
2. is continuous in f ,
3. is monotonically increasing in f ,
4. is convex in f ,

Proof. Lemma 1: $H(1, x) = 1$ can be checked by simple substitution. $H(f, x)$ is a polynomial in f with positive coefficients. Therefore it is continuous, increasing and convex. We can show the latter two explicitly by computing the first and second derivatives:

$$\begin{aligned}\frac{dH}{df} &= \frac{\sum_k P(k)k(k-1)(xf+1-x)^{k-2}}{E(k)}x \geq 0, \\ \frac{d^2H}{df^2} &= \frac{\sum_k P(k)k(k-1)(k-2)(xf+1-x)^{k-3}}{E(k)}x^2 \geq 0.\end{aligned}$$

□

Proof. Proposition 4.2: We illustrate the graphical intuition for this proof in Fig. 4.2. As seen in Lemma 1 $H(f, x)$ is convex in f and $H(1, x) = 1$ for a given x . Then there exists at most one more solution to $H(f, x) = f$ in $f \in [0, 1]$. $H(f, x)$ will intersect the diagonal in the interval $f \in [0, 1]$ if $dH/df(1, x) > 1$, otherwise $H(f, x)$ will stay above the diagonal for $f \in [0, 1]$. Note that the derivative at $f = 1$ is a function of x . The knife edge case occurs when $dH/df(1, x_c) = 1$, i.e when $H(f, x)$ just touches the diagonal at $f = 1$. Then

$$\begin{aligned}\frac{dH}{df}(1, x_c) &= \frac{\sum_k P(k)k(k-1)(x+1-x)^{k-2}}{E(k)}x_c = \frac{\sum_k P(k)k(k-1)}{E(k)}x_c = 1, \\ \implies x_c &= \frac{E(k)}{E(k^2) - E(k)} = \frac{1}{E(k^2)/E(k) - 1}.\end{aligned}$$

This implies that:

- $x \in [0, x_c]$: $f = 1$.

- $x \in (x_c, 1]$: $f < 1$.

Note that f is continuous at x_c since $\lim_{x \rightarrow x_c^+} f \rightarrow 1$. For $f = 1$: $g(1, x) = 0 \forall x$.

Thus, for $g(f, x)$ we have:

- $x \in [0, x_c]$: $g(f = 1, x) = 0$.
- $x \in (x_c, 1]$: $0 < g(f < 1, x) < 1$

Note that since f is continuous at x_c , so is $g(x, f)$. Clearly for finite $E(k)$, $\lim_{E(k^2) \rightarrow \infty} x_c \rightarrow 0$ and thus for degree distributions with infinite second moment $E(k^2)$, $x_c \rightarrow 0$ and $g(x) > 0 \forall x > 0$. □

4.A.5 Proposition 4.3

Preliminary Lemmas

Consider the following Lemmas.

Lemma 2. We have for $f(x)$ and $g(f, x)$:

1. $f(x) = H(f(x), x)$ is continuous, monotonically decreasing in x for $x \in [0, 1]$.
2. $g(f, x)$ is continuous and monotonically increasing in x for $x \in [0, 1]$.

Proof. Lemma 2:

1. $f(x) = H(f(x), x)$ is continuous, monotonically decreasing and convex in x for $x \in [0, 1]$.
 - (a) $f(x) = H(f(x), x)$ continuous: See proof of Proposition 4.2.
 - (b) $f(x)$ monotonically decreasing: Using the result from the proof of Proposition 4.2 we have for $x \in [0, x_c]$: $f = 1 \implies df/dx = 0$. Now consider

$x \in (x_c, 1]$: $f < 1$. In this case we derive for df/dx :

$$\begin{aligned} \frac{df}{dx} &= \frac{1}{E(k)} \underbrace{\sum_k P(k)k(k-1)(xf(x) + 1 - x)^{k-2}}_{dH/df \cdot 1/x} \left(f(x) + x \frac{df}{dx} - 1 \right), \\ \frac{df}{dx} &= \frac{dH}{df}(f, x) \frac{1}{x} \left(f(x) + x \frac{df}{dx} - 1 \right), \\ \frac{df}{dx} &= \frac{dH}{df}(f, x) \frac{1}{x} \left(1 - \frac{dH}{df}(f, x) \right)^{-1} (f(x) - 1). \end{aligned}$$

Note that for supercritical x the derivative of $H(f, x)$ with respect to f evaluated at the intersection with the diagonal is less than one, i.e. for $x \in (x_c, 1]$ $\frac{dH}{df}(f, x) < 1$, where $H(f, x) = f < 1$. This can be seen as follows. Clearly for there to exist a solution $f < 1$ to $f = H(f, x)$, $H(f, x)$ must cross the diagonal. But since $\frac{dH}{df}(1, x) > 1$ for $x \in (x_c, 1]$ and $H(1, x) = 1$, $H(f, x)$ must cross the diagonal from below when approaching the intersection from the right. This implies that $\frac{dH}{df}(f, x) < 1$ at the intersection. This together with $0 < x$, $f(x) < 1$ and $\frac{dH}{df}(f, x) > 0$ implies that $\frac{df}{dx} < 0$.

2. $g(f, x)$ is continuous and monotonically increasing in x for $x \in [0, 1]$.

(a) $g(f, x)$ is continuous: See proof Proposition 4.2.

(b) $g(f, x)$ is monotonically increasing:

$$\frac{dg}{dx} = - \sum_k P(k)k(f(x)x + 1 - x)^{k-1} \underbrace{\left(\frac{df}{dx}x + f(x) - 1 \right)}_{\leq 0} \geq 0.$$

□

Two-asset intermediation network

In the following we will take the number of assets to be $K = 2$. The two networks A and B have degree distributions $P_A(k)$ and $P_B(k)$ with $E_A(k^2) < \infty \wedge E_B(k^2) < \infty$

and $E_A(k) < \infty \wedge E_B(k) < \infty$. Furthermore, assume that the spillover probability is $q = 1$ throughout. Recall

$$y = F(y, x) = xg_B(xg_A(y)),$$

for $0 < x \leq 1$ and with

$$g_\mu(y) = 1 - G_\mu(y) = 1 - \sum_k P_\mu(k)(yf_\mu(y) + 1 - y)^k,$$

$$f_\mu(y) = H_\mu(f_\mu(y), y) = \frac{\sum_k P_\mu(k)k(yf_\mu(y) + 1 - y)^{k-1}}{E_\mu(k)},$$

and $g_\mu(y)$ concave for $\mu \in \{A, B\}$. First consider the following Lemma:

Lemma 3. For $y \in (0, 1]$

1. $F(y, x)$ is continuous in y ,
2. $F(y, x)$ is monotonically increasing in y ,
3. $F(y, x)$ is bounded from above: $F(y, x) < x$,
4. $F(y, x)$ is concave in y .
5. $\lim_{y \rightarrow 0} F(y, x) \rightarrow 0$,
6. $\lim_{y \rightarrow 0} \frac{\partial F(y, x)}{\partial y} \rightarrow 0$.

Proof. Lemma 3: For this proof we invoke results from Lemma 2. For $y \in (0, 1]$:

1. $F(y, x)$ is continuous: $g_\mu(y)$ is continuous as shown in Lemma 2. $F(y, x)$ is a function of $g_\mu(y)$ and therefore also continuous in y .
2. $F(y, x)$ is monotonically increasing: $g_\mu(y)$ is monotonically increasing as shown in Lemma 2. $F(y, x)$ is therefore also monotonically increasing in y .
3. $F(y, x)$ is bounded from above - $F(y, x) < 1$: Clearly $g_\mu(y)$ is bounded from above since $g_\mu(y) \leq 1$. Furthermore, $x \leq 1$. Therefore $F(y, x) = xg_B(xg_A(y)) <$

1. Also note that the above implies that $F(y, x)$ has a maximum at $y = 1$ which scales with x , i.e. as x is decreased the maximum of $F(x, y)$ decreases by at least the same amount.

4. $F(y, x)$ is concave in y :

$$\frac{\partial^2 F}{\partial y^2}(y, x) = x^2 \left(x \frac{d^2 g_A}{dy^2}(y) \left(\frac{dg_B}{dy}(y) \right)^2 + \frac{dg_A}{dy}(y) \frac{d^2 g_B}{dy^2}(y) \right)$$

Since $\frac{dg_\mu}{dy}(y) > 0$, $\frac{d^2 g_\mu}{dy^2}(y) < 0$ (by assumption) and $x > 0$ we must have $\frac{\partial^2 F}{\partial y^2} < 0$, i.e. $F(y, x)$ concave.

5. $\lim_{y \rightarrow 0} F(y, x) \rightarrow 0$: Since for $y < y_{c,\mu}$ we have $f_\mu(y) = 1$ and $g_\mu(y) = 0$, where $y_{c,\mu}$ is the percolation threshold for network μ at which the giant component vanishes as given in Proposition 4.1. In other words, there exists a critical $y_{c,\mu}$ at which the giant component in one of the intermediation networks vanishes (recall that we assume that $E(k) < \infty$, hence there always exists this critical $y_{c,\mu}$ by Proposition 4.2).

6. $\lim_{y \rightarrow 0} \frac{\partial F}{\partial y}(y, x) \rightarrow 0$:

$$\lim_{y \rightarrow 0} \frac{\partial F}{\partial y}(y, x) = x^2 \frac{dg_A}{dv}(v) \frac{dg_B}{dy}(y) \rightarrow 0.$$

Since for $y < y_{c,\mu}$ we have $f_\mu(y) = 1$ and $g_\mu(y) = 0$. Hence for $y < y_{c,\mu}$ we have $\frac{dg_\mu}{dy}(y) = 0$. In other words, since there exists a critical $y_{c,\mu}$ at which the giant component vanishes in one of the intermediation networks, there is a region for values of $y < y_{c,\mu}$ in which $F(x, y)$ is flat.

These observations show that, under the assumptions made here, $F(x, y)$ can be decomposed into two regions: (i) for small values of y ($y < y_{c,\mu}$) $F(x, y)$ vanishes ($F(x, y) = 0$) and is flat ($\partial F / \partial y = 0$). (ii) for larger values of y ($y > y_{c,\mu}$) $F(x, y)$ is strictly monotonically increasing and concave but bounded from above ($F(x, y) < 1$). □

Proof. Proposition 4.3: This proof invokes results from Lemma 3 and relies in particular on our observations of the shape of $F(x, y)$ in the interval $y \in [0, 1]$. We illustrate the graphical intuition for this proof in Fig. 4.3.

First note that $y = 0$ is a trivial solution to $y = F(y, x)$ for all x since $g_\mu(0) = 0$. Furthermore as shown in Lemma 3 there exists a region for sufficiently small y in which $F(y, x)$ is constant and equal to zero. As seen in Lemma 3, for all $y > y_{c,\mu}$ the function $F(y, x)$ is strictly increasing and concave provided $g_\mu(y)$ is concave. The fact that $F(x, y)$ is constant and flat close to $y = 0$ implies that in at least some of the interval $y \in [0, 1]$, $F(x, y)$ must lie below the diagonal. If for $y > y_{c,\mu}$ the function $F(x, y)$ increases sufficiently fast to cross the diagonal there will exist two solutions in addition to the trivial solution (since $F(x, y) < 1$ and hence cannot remain above the diagonal for the entire interval $y \in [0, 1]$).

In Proposition 4.3 we assume these nontrivial solutions exist. Since we are investigating cascades following a small exogenous shock we are only interested in the largest fixed point y^* of the map $y_n = F(y_{n-1}, x)$ with $y_0 = x$. This fixed point will be stable due to the concavity of $F(y, x)$ and because at y^* the slope of $F(x, y)$ is $\partial F / \partial y(y^*, x) < 1$.

Now consider how the largest fixed point y^* changes when the initial exogenous shock $1 - x$ is increased. Clearly, when x goes down, y^* goes down as well. This is because for a smaller value of x the curve $F(y, x)$ will have a smaller maximum value. This pushes the entire segment of the curve of $F(x, y)$ for $y > y_{c,\mu}$ downwards. Therefore $F(y, x)$ will intersect the diagonal at a smaller value. When both x and y^* decrease further the curve $F(y, x)$ will ultimately become tangent to the diagonal. This will correspond to some critical value x_c . At this point the largest solution y^* merges with the second largest on the diagonal.

If x is decreased further ($x < x_c$) both non trivial solutions vanish and only the trivial solution at $y = 0$ remains. In summary, if there exists some fixed point of

$F(x, y)$, y^* , and some critical exogenous shock $1 - x_c$ such that $F(x, y)$ is tangent to the diagonal ($\frac{\partial F}{\partial y}(y^*, x_c) = 1$), then there will be a region below x_c where only the trivial solution exists ($y^* = 0$) and a region above x_c where a non trivial solution $0 < y^* < 1$ exists.

Note that, since there exists some value $y_{c,\mu} > 0$ at which the derivative $\partial F / \partial y(y, x)$ vanishes, $F(x, y)$ must lie below the diagonal close to $y = 0$. Therefore, the non trivial solution must always be greater than zero, i.e. $y^* > 0$ for $x \geq x_c$. Therefore

$$\lim_{\epsilon \rightarrow 0} F(y^*, x_c - \epsilon) = 0 \neq F(y^*, x_c) > 0.$$

Hence $F(y, x)$ is discontinuous in x at $x = x_c$. From the above it also follows that, if there exists no $0 < y^* < 1$ such that at some $x = x_c > 0$, $\frac{\partial F}{\partial y}(y^*, x_c) = 1$, then only the trivial solution can exist and $F(y^*, x) = 0 \forall x < 1$. In this case a minimal disturbance of the network leads always to a complete collapse of the network. $\square$

Proof. Proposition 4.4: Consider the following cases:

- $x_{c,B} \leq x_c \leq x_{c,A}$: Then $F(y, x_c) = x_c g_B(x_c g_A(y)) < x_{c,A} \forall y$, since $g_i(y) < 1$. However, at a fixed point $F(y, x_c) = y$. But $g_A(y) = 0 \forall y < x_{c,A} \implies y^* = 0$. This contradicts $F(y^*, x_c) > 0$.
- $x_{c,A} \leq x_c \leq x_{c,B}$: Then $F(y, x_c) = x_c g_B(\underbrace{x_c g_A(y)}_{< x_{c,B}}) = 0 \forall x$. This contradicts $F(y^*, x_c) > 0$.
- $x_{c,B}, x_{c,A} \geq x_c$: Same argument as for the two cases above.

Therefore by contradiction the only remaining case is $x_{c,A}, x_{c,B} < x_c$. $\square$

4.B Appendix: Figures

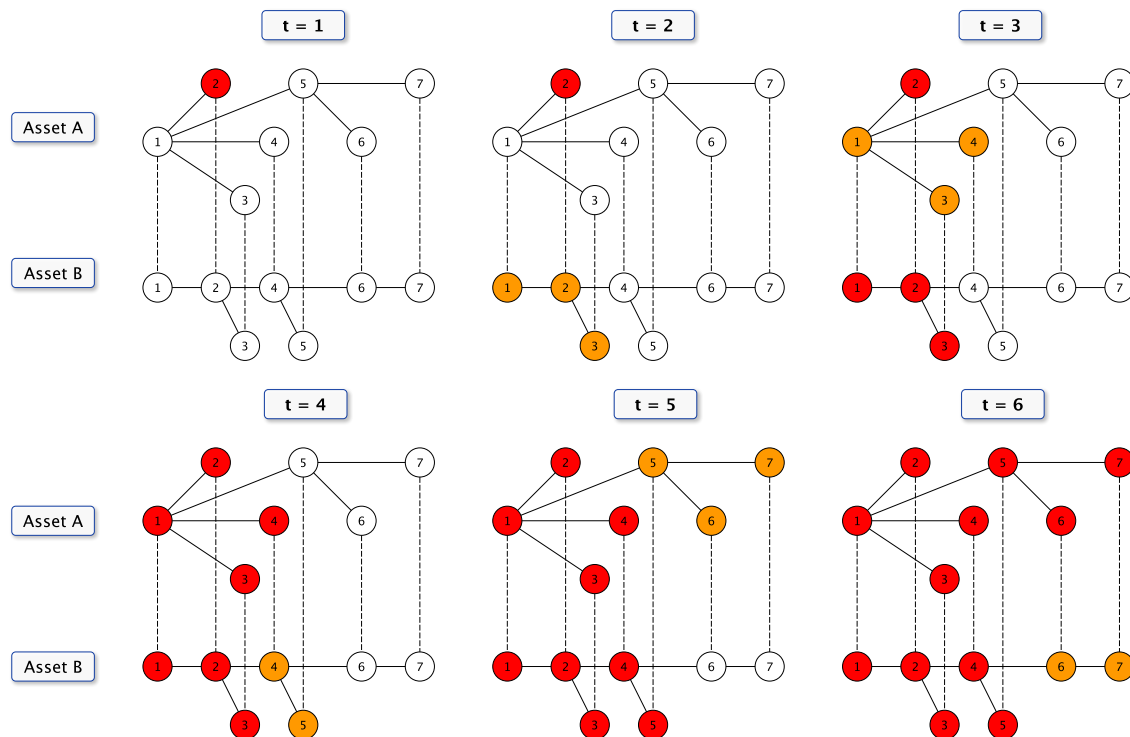


Figure 4.1: Illustration of failure cascades due to intra-bank spillover. We show stylized trading networks for two assets *A* and *B*. Nodes represent banks. Solid links between nodes are trading relationships. Dashed links represent spillover links due to externalities. An exogenous shock in the intermediation network for asset *A* leads the business unit of bank 2 in asset market *A* to fail. Spillover then causes a cascade of failures leading ultimately to the failure of all banks in the system.

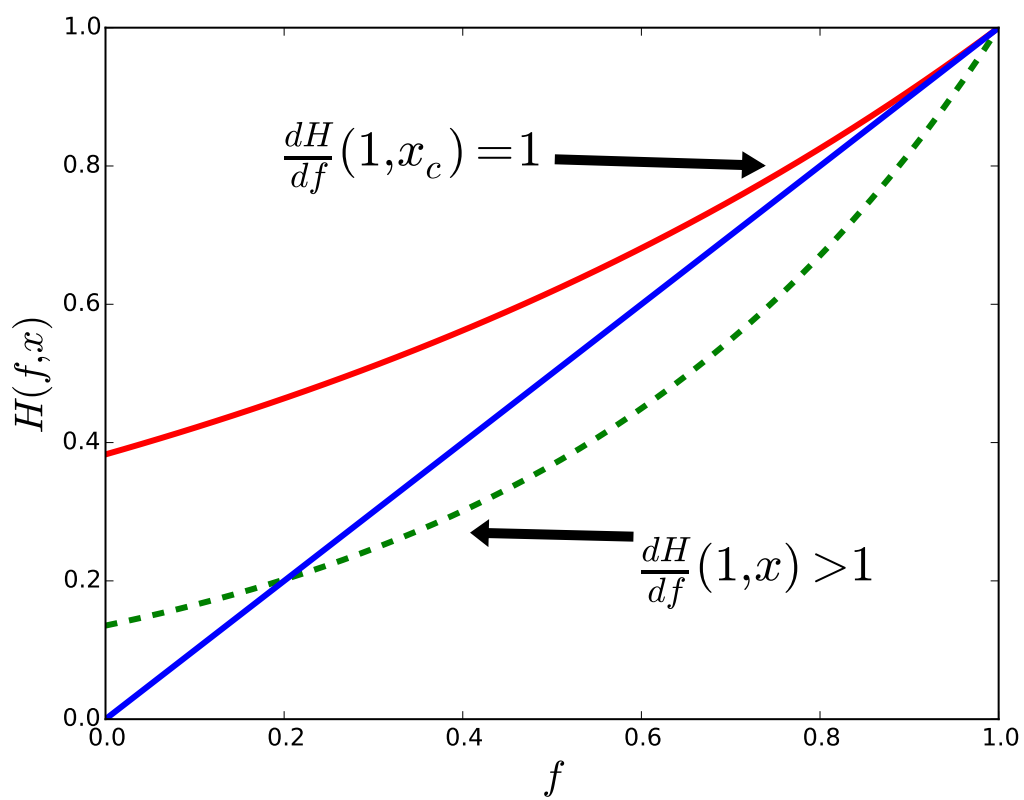


Figure 4.2: Graphical intuition for proof of Proposition 4.2.

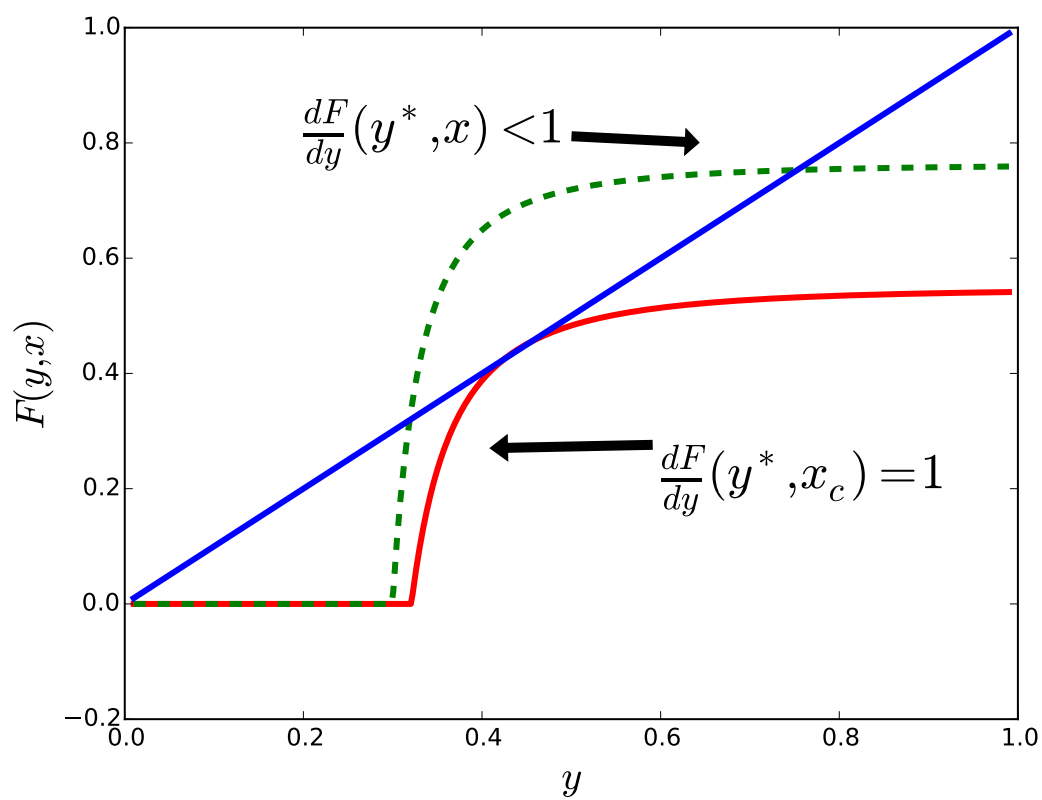


Figure 4.3: Graphical intuition for proof of Proposition 4.3.

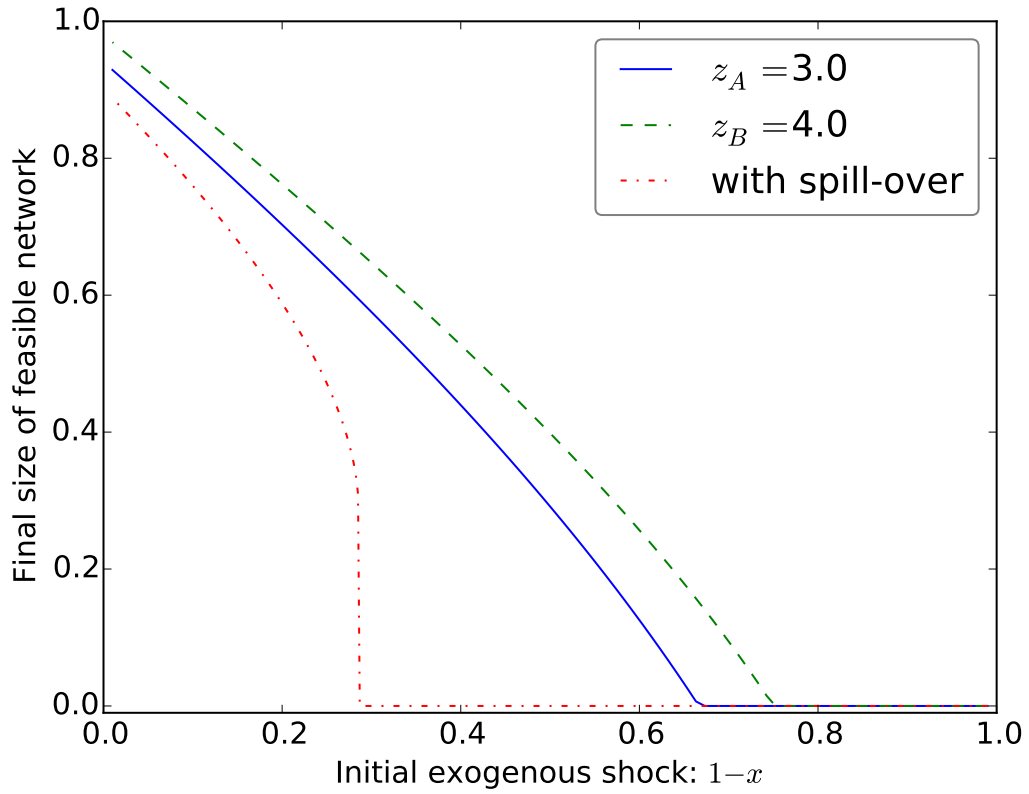


Figure 4.4: Analytical examples for single and two asset Erdős-Rényi intermediation networks. We consider two single asset networks A and B in isolation with average degrees $z_A = 3$ and $z_B = 4$. We plot the final size of the intermediation network after an exogenous shock $1 - x$ for the two isolated single asset networks (continuous blue, and dashed green lines respectively). We also consider the combined two-asset intermediation network in the presence of spillover (dash red line).

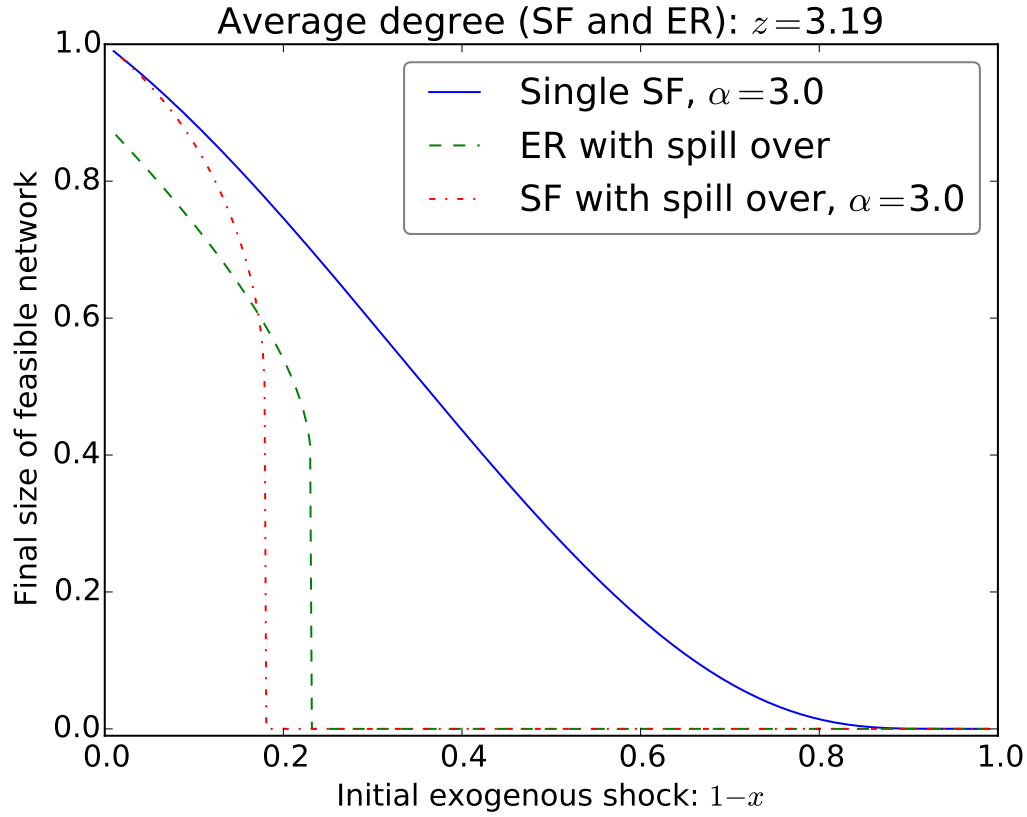


Figure 4.5: Analytical examples for single and two asset scale-free intermediation networks. We consider a single asset network with power law exponent $\alpha = 3$ and average degree $z = 3.19$. We plot the final size of the intermediation network after an exogenous shock $1 - x$ for the isolated single asset network (continuous blue line). We also consider the combined two-asset scale-free intermediation network in the presence of spillover (dash red line). We contrast these results with the results for a two asset Erdős-Rényi intermediation network with average degree $z = 3.19$ for both assets (dashed green line).

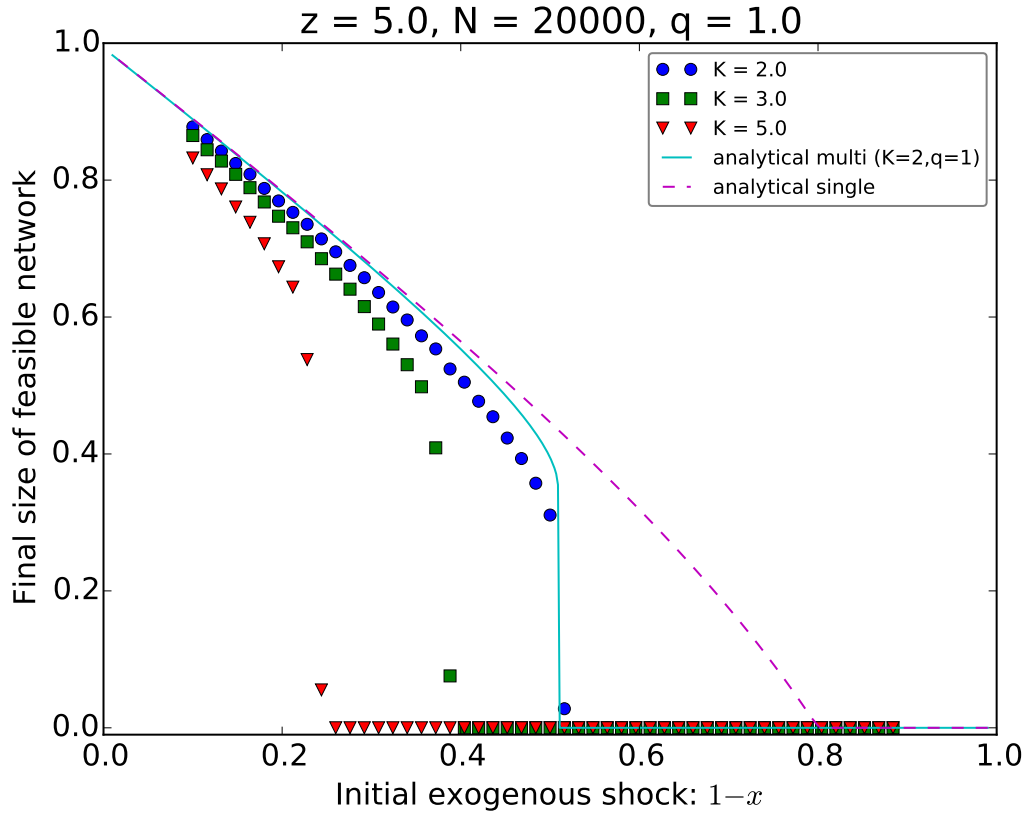


Figure 4.6: Numerical computation of the resilience of intermediation networks with multiple assets. We have: Number of assets $K = 2$, $K = 3$, $K = 5$. All intermediation networks are Erdős-Rényi with mean degree $z = 5$ and $n = 20000$ banks. We take a uniform spillover probability of $q = 1.0$. We compare the analytical solution for a single asset intermediation network (dashed purple line) and the analytical solution for a two asset intermediation network (continuous blue line) with the numerical solution for the respective number of assets (markers). Discrepancies between analytical and numerical solutions are likely due to finite size effects.

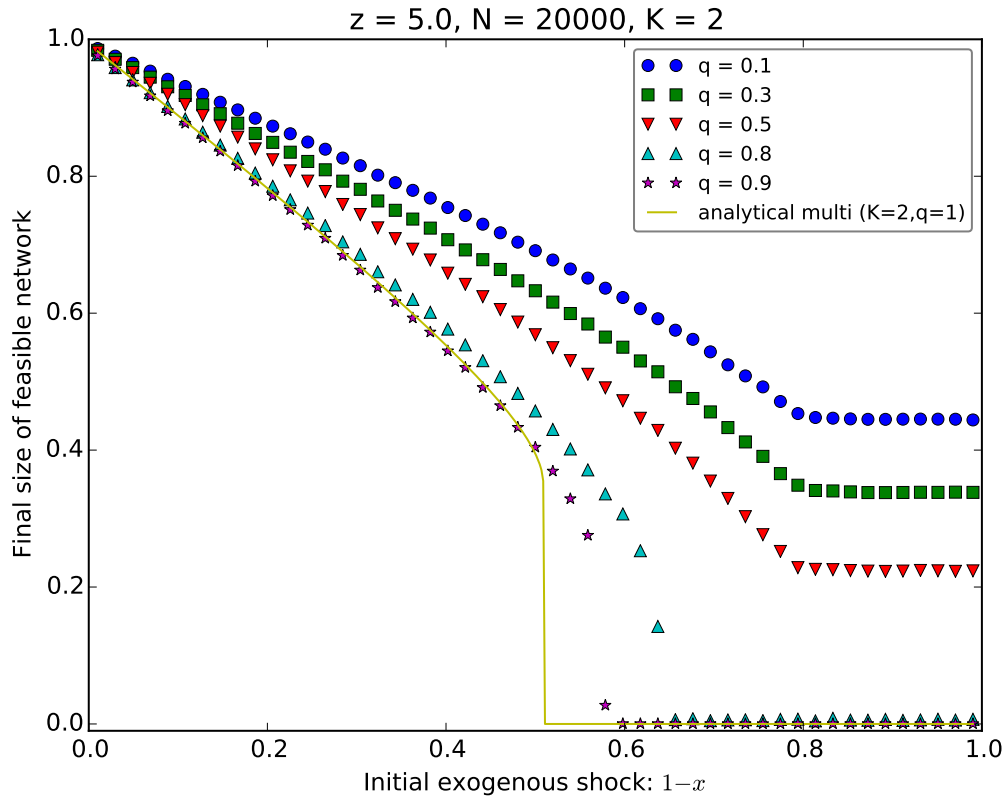


Figure 4.7: Numerical computation of the resilience of intermediation networks with multiple assets. Numerical computation of the resilience of intermediation networks with different spillover probabilities. We have: $q = \{0.1, 0.3, 0.5, 0.8, 0.9\}$. All intermediation networks are Erdős-Rényi with mean degree $z = 5$ and $n = 20000$ banks. We compare the analytical solution for a two asset intermediation network with $q = 1$ (continuous yellow line) with the numerical solution for the respective spillover probabilities (markers).

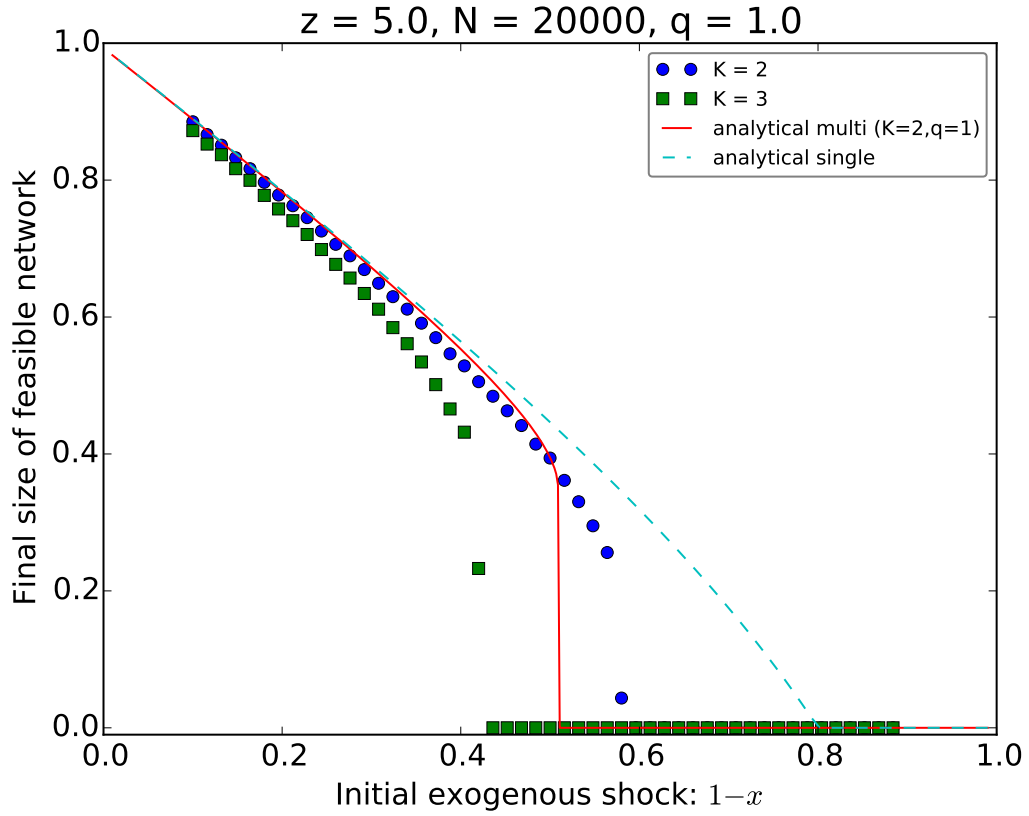


Figure 4.8: Numerical computation of the resilience of intermediation networks with different correlation structure. All intermediation networks are Erdős-Rényi with mean degree $z = 5$ and $n = 20000$ banks. We take a uniform spillover probability of $q = 1.0$. We compare the analytical solution for a single asset intermediation network (dashed blue line) and the analytical solution for two uncorrelated asset intermediation networks (continuous red line) with the numerical solution for the respective correlation structure (markers). Blue dots: Two fully correlated intermediation networks ($K = 2$). Green squares: Two fully correlated intermediation networks and one uncorrelated intermediation network $K = 3$.

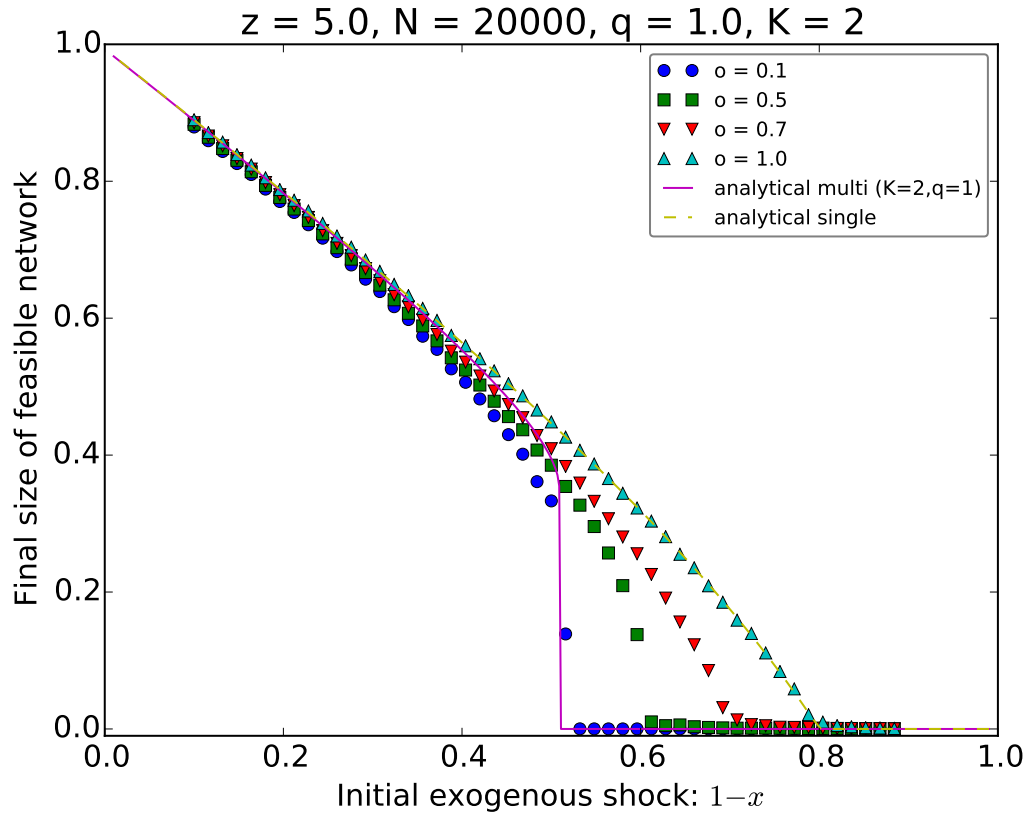


Figure 4.9: Numerical computation of the resilience of intermediation networks with multiple assets. All intermediation networks are Erdős-Rényi with mean degree $z = 5$ and $n = 20000$ banks. We take a uniform spillover probability of $q = 1.0$. We compare the analytical solution for a single asset intermediation network (dashed yellow line) and the analytical solution for two uncorrelated asset intermediation networks (continuous purple line) with the numerical solution for the respective network overlap $o = \{0.1, 0.5, 0.7, 1.0\}$ (markers).

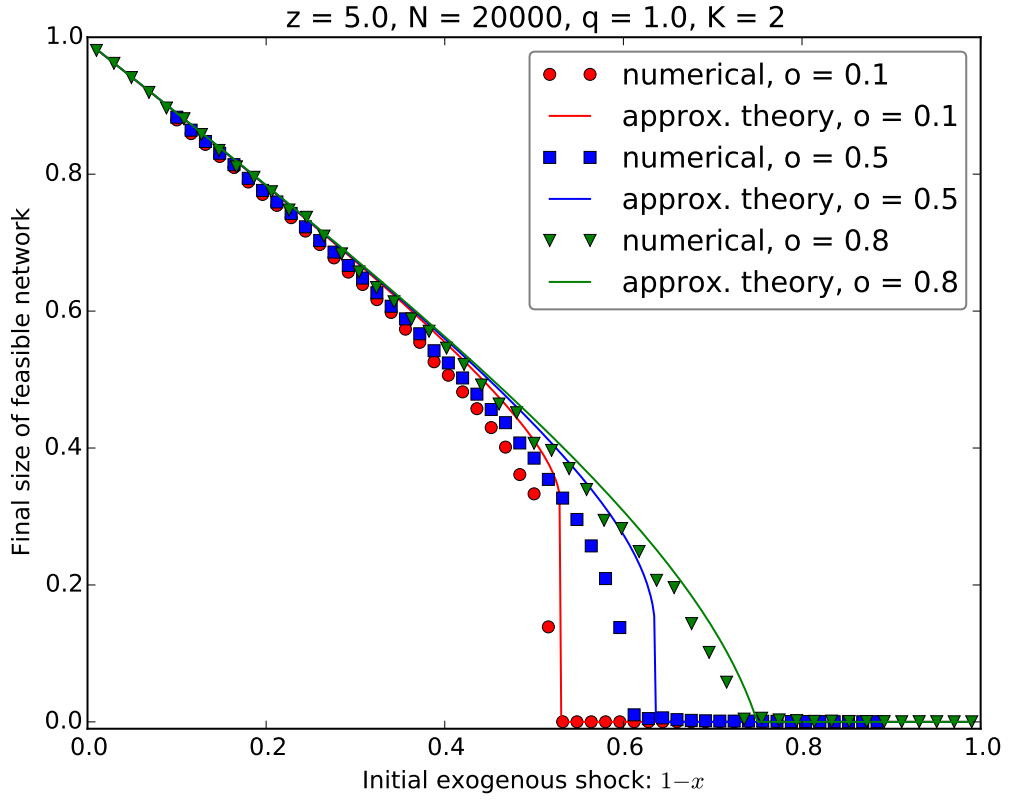


Figure 4.10: Numerical computation of the resilience of intermediation networks with multiple assets. All intermediation networks are Erdős-Rényi with mean degree $z = 5$ and $n = 20000$ banks. We take a uniform spillover probability of $q = 1.0$. We compare the theoretical approximation for a particular overlap with the numerical solution for the respective network overlap $o = \{0.1, 0.5, 0.7, 1.0\}$ (markers). The theoretical approximation works well in the limit of small and large overlap and performs relatively poorly for intermediate overlaps.

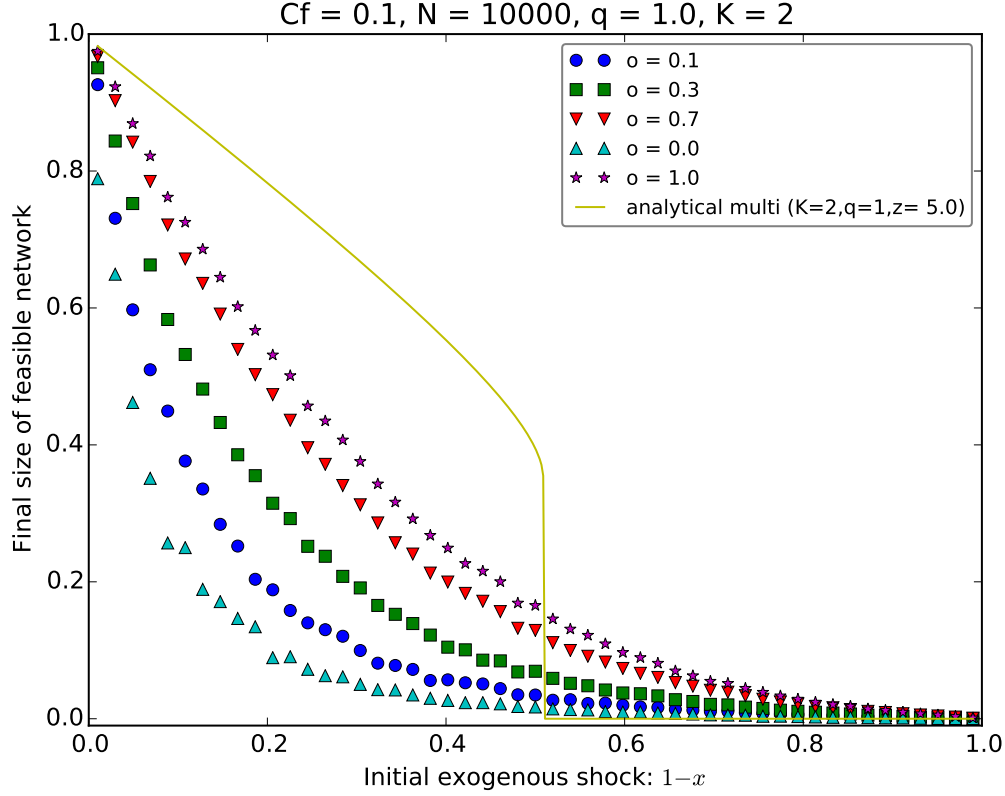


Figure 4.11: Numerical computation of the resilience of intermediation networks with multiple assets. All intermediation networks are stylized core periphery networks with a fully connected core that contains $C_f = 10\%$ of the network's nodes and $K = 2$. We compare multi-asset core periphery networks with different value of network overlap $o = \{0.0, 0.1, 0.3, 0.7, 1.0\}$ (markers). We also plot the analytical solution to the two asset Erdős-Rényi intermediation network with mean degree $z = 5$ and $q = 1$. Since the structure of ER and core-periphery networks is very different, direct stability comparisons between networks is hard. It should be noted however that the response of ER network to exogenous shocks is *concave* while the response of core-periphery networks is *convex*.

Chapter 5

Contagious Synchronization and Endogenous Network Formation in Financial Networks

5.1 Introduction

Information is key to a successful investment strategy. Investors therefore collect and aggregate information from a variety of different sources in order to make the best possible investment decision. In a financial market in which similar institutions invest in similar assets, observing the investment choices of others may allow an investor to infer information otherwise not available to her. In other words, investors have an incentive to learn from each other's actions. We refer to this process throughout this chapter as social learning.

While social learning may be effective in disseminating private information accessible to only a few investors, it also bears the risk of contagious synchronization – or herding. In such a scenario, rather than coordinating on a successful investment strategy, investors' flawed beliefs reinforce each other such that ultimately all

This work expands substantially on initial work done by Co-Pierre Georg published as working paper (see Georg (2014)). I modify and extend the model presented in Georg (2014), all results presented here are based on my own calculations and numerical simulations. A large part of the work presented here has been published in Aymanns and Georg (2015), DOI: <http://dx.doi.org/10.1016/j.jbankfin.2014.06.030>.

investors choose an unsuccessful strategy. This process of contagious synchronization can provide one possible contributing factor for excessive investment in certain securities, e.g. technology stocks during the dot-com bubble in the early 2000s. Rather than investing in costly due diligence on the fundamental value of the stocks, investors simply imitated their peers' investment decisions.

We develop a simple model of a financial market in which financial intermediaries, banks for short, take a binary investment decision based on a “private” and “social” belief about the state of the world. If a bank's investment is state-matching, the bank receives a positive payoff and zero otherwise. Banks receive an independent and informative private signal about the state of the world. They combine this private signal with a social signal, which is the average of their peers' action, in a form of boundedly rational Bayesian updating.¹ The set of actions a particular bank can observe is determined by a financial network. Within the network the bank can only observe the actions of its direct neighbors. This network can be thought of as a network of trading relationships in an over-the-counter market. At first, we take the financial network as exogenously given by a random Erdős-Rényi graph. In a second step, banks endogenously choose their neighbors based on the expected utility of establishing a link.

We obtain two sets of results, one for networks with exogenous network structure, and one for endogenously formed networks. First, we analyze different ways of weighting private and social belief. In particular, we compare an *equal weighting* scenario in which agents place equal weight on their private and social belief, with two scenarios where agents place more weight on the social belief when they have more

¹We refer to the bank's learning process as boundedly rational Bayesian updating as it combines both a traditional Bayesian component and a heuristic learning method. In particular, the agents use Bayesian updating to compute the posterior probability of the state of the world (private belief) based on the private signal they receive. Agents then combine this rational posterior in a weighted average with a simple average of its neighbors actions (social belief). The weighted average of private and social beliefs then yields a boundedly rational belief about the state of the world.

neighbors. In the *neighborhood size* scenario the social belief is weighted by the size of the neighborhood, i.e. the private signal has the same weight as the signal observed from each neighbor. Thus if there are many neighbors the social belief will dominate over the private belief. In the *relative neighborhood* scenario banks put more weight on the social belief when the neighborhood constitutes a larger share of the overall network. For completely uninformative private signals there is no difference between these weighting functions. For informative signals, however, the weighting function has an impact on the probability that agents synchronize their investment decisions on a state non-matching action, i.e. for the probability that choosing a state-non-matching action is contagious.² Here, contagion is very generally understood as the transmission of adverse effects from one agent to another. In this model contagion is more likely if agents place greater weight on their social belief. For example, the probability of contagion increases by an order of magnitude in the neighborhood size scenario compared to the equal weighting scenario. The likelihood of contagion also depends on the density of the underlying exogenous network structure.³

We show that contagious synchronization occurs even if private signals are informative and if agents are initialized with an action that is on average state matching. In the equal weighting scenario with a complete network we show this analytically. For different weighting scenarios and network structures, we show numerically that the probability of contagion depends non-monotonically on the density of the network. For small network densities ($\rho \lesssim 0.1$) the probability of contagion increases sharply and then decreases slowly for larger network densities. We confirm the robustness of our results by conducting 2,000 independent simulations where we observe the

²Such *informational cascades* are a well-documented empirical phenomenon. See, for example, Alevy et al. (2007), Bernhardt et al. (2006), Chang et al. (2000), Chiang and Zheng (2010), and Cipriani and Guarino (2014).

³For a more thorough discussion of the different forms of contagion, see for example Bandt et al. (2009).

average final action as a function of the average initial action with varying network densities.

Second, turning to the extension of endogenously formed networks, we show that endogenous link formation can significantly improve the speed of learning (i.e. the number of iterations needed for a consensus to form) and reduce the probability of contagious synchronization relative to Erdős-Rényi networks. When private signals are less informative but the cost of establishing a link remains the same, the additional net utility from forming a link is smaller and the endogenously formed network is less dense. This in turn can increase the probability of contagious synchronization. Heightened uncertainty about the state of the world, i.e. a less informative signal, does therefore not only directly increase the probability of contagious synchronization, but also indirectly because agents have less incentives to endogenously form links. If agents are heterogeneous in the “informativeness” of their private signals, i.e. if some agents receive stronger signals than others, we show that the resulting endogenous network structure is of a core-periphery type. This is interesting since the structure of real-world interbank markets is often of this particular type⁴, as for example Craig and Von Peter (2014) show. Naturally, these endogenously formed networks transfer information more effectively from highly informed agents to less informed agents than simple random networks.

This paper combines aspects from the literature on Bayesian and non-Bayesian learning on exogenous social networks. Similar to the models proposed in Acemoglu et al. (2011) and Gale and Kariv (2003)), agents receive a private signal drawn from a known distribution such that agents are able to form a private posterior belief based on the observed signal. However, the social belief is obtained via a boundedly

⁴Of course there are many mechanisms driving the formation of financial networks – learning considerations are likely to play only a minor role in the formation of a financial network.

rational learning rule inspired by DeMarzo et al. (2003). Under this learning rule, banks aggregate their neighbors' actions via a simple averaging mechanism rather than rigorously treating each observed action as a new signal in a Bayesian updating process. In an increasingly complex financial system, it appears justified to assume that banks use simple heuristics to aggregate large amounts of information from their environment.

A key difference to most of the existing literature on Bayesian learning in social networks is that we allow agents to endogenously form links based on the utility they expect from an improved social belief. Once links are formed agents learn about the state of the world and take their investment decisions. One example in the literature featuring endogenously formed social networks in a Bayesian learning setup is Acemoglu et al. (2014). Acemoglu et al. (2014) model endogenous network formation as a two stage game with a communication cost matrix where some agents (in a *social clique*) can communicate at low costs, while others communicate at high cost. The network formation mechanism we propose in our model is simpler and computationally tractable even for larger networks. This is because it effectively truncates infinite sequences in the computation of the expected marginal utility of an additional link. We then obtain a resulting endogenous network structure which is pairwise stable in the sense of Jackson and Wolinsky (1996).

The remainder of this paper is organized as follows. The next section develops the baseline model and presents the results in the limiting case of an exogenous network structure. Section 5.3 generalizes the model by allowing agents to endogenously form links in a first stage of the model. Section 4.6 concludes.

5.2 Contagious Synchronization with Fixed Network Structure

5.2.1 Model

Agents, actions and beliefs

There is a countable set of dates $t = 0, 1, \dots, T$ and a set of agents A $i = 1, \dots, N$ which represent financial institutions and are called banks for short. The state of the world θ can take two values: $\theta \in \{0, 1\}$. The probability that the world is in a particular state is $\Pr(\theta = 1) = \Pr(\theta = 0) = \frac{1}{2}$. At each point in time t each bank i chooses one of two investment strategies $x_t^i \in \{0, 1\}$ which yields a positive return if the state of the world is revealed and matches the investment strategy chosen, and nothing otherwise. Agents take an action by choosing a certain investment strategy. Taking an action and switching between actions is costless. For simplicity we assume that the utility of bank i from investing is given as:

$$u^i(x^i, \theta) = \begin{cases} 1 & \text{if } x^i = \theta \\ 0 & \text{else} \end{cases} \quad (5.1)$$

The state of the world is unknown ex-ante and revealed at time T . This setup captures a situation where the state of the world is revealed less often (e.g. quarterly) than banks take investment decisions (e.g. daily).

Banks can form interconnections in the form of trading relationships. The set of banks to which bank i is directly connected is denoted $K^i \subseteq A$. Bank i thus has $k^i = |K^i|$ direct connections called neighbors. This implements the notion of a network of banks $\mathbf{g}$ which is defined as the set of banks together with a set of undirected links $L = \cup_{i=1}^n \{(i, j) : j \in K^i\}$. A link is undirected since trading relationships are mutual and captured in the symmetric adjacency matrix g of the network. Whenever a bank i and j have a link, the corresponding entry $g^{ij} = 1$, otherwise $g^{ij} = 0$. For the remainder of this section, I assume that the network structure is exogenously

fixed and does not change over time. Furthermore, I assume that the network is an instance of an Erdős-Rényi random graph. Finally, I assume that banks monitor each other continuously when establishing a relationship and thus observe their respective actions.

At $t = 0$ there is no previous decision of agents. Thus, each bank decides on its action in autarky. Banks receive a signal about the state of the world and form a private belief. Based on this private belief they decide about their investment strategy $x_{t=0}^i$.⁵ The private signal received at time t is denoted $s_t^i \in \mathbb{R}$. Signals are independently generated according to a probability distribution that depends on the state of the world θ . Throughout this paper I will assume that s_t^i is drawn from a Gaussian distribution with mean and standard deviation (μ_0, σ_0) for $\theta = 0$ denoted by $f_0(s)$ and (μ_1, σ_1) for $\theta = 1$ denoted by $f_1(s)$, where $\mu_1 = 1 - \mu_0$.⁶ At each time $t \geq 0$ bank i receives a signal s_t^i and observes the $t - 1$ actions x_{t-1}^j of its neighbors $j \in K^i$. This protocol is repeated until the state of the world is revealed at time T or until the dynamics have converged.

Banks form a private belief at time t based on their privately observed signal s_t^i and a social belief based on the observed actions x_{t-1}^j their neighboring banks took in the previous period. In a related model Acemoglu et al. (2011) show that the bank maximizes its expected payoff if it chooses its action according to the following

⁵Alternatively one can initialize the system with a particular distribution of initial actions $\{x_{t=0}^i\}$

⁶I assume a Gaussian distribution for two reasons. Firstly, it simplifies all further calculations greatly. Secondly, it can be shown that in a sequential Bayesian learning model on a network learning occurs asymptotically for many network topologies when the private beliefs are unbounded, cf. Acemoglu et al. (2011). The private belief resulting from the signal structure is said to be unbounded if there exist some private beliefs which are arbitrary close to zero and some private beliefs which are arbitrary close to one. The private belief is unbounded if $\inf_s \frac{f_1(s)}{f_2(s)} = 0$ and $\sup_s \frac{f_1(s)}{f_2(s)} = \infty$. The Gaussian signal structure assumed here results in unbounded private beliefs. While in this chapter we do not consider the limit $N \rightarrow \infty$, we still take private beliefs to be unbounded to be in line with the literature on Bayesian learning in social networks.

condition:

$$x^i = \begin{cases} 1 & \text{if } \Pr(\theta = 1|s_t^i) + \Pr(\theta = 1|x_{t-1}^j, j \in K_{t-1}^i) > 1 \\ 0 & \text{if } \underbrace{\Pr(\theta = 1|s_t^i)}_{\text{private belief } p^i} + \underbrace{\Pr(\theta = 1|x_{t-1}^j, j \in K_{t-1}^i)}_{\text{social belief } q^i} < 1. \end{cases} \quad (5.2)$$

The first term on the right-hand side of Equation (5.2) is the private belief, the second term is the social belief, and the threshold is fixed at 1. This equation can be generalized heuristically by introducing weights on the private and social belief. In this generalized form, it can be written as:

$$x^i = \begin{cases} 1 & \text{if } t(p^i, q^i) > \frac{1}{2} \\ 0 & \text{if } t(p^i, q^i) < \frac{1}{2} \end{cases} \quad (5.3)$$

where $t(p^i, q^i)$ is a weighting function depending on the private and social belief. A simple weighting function

$$t(p^i, q^i) = \begin{cases} \frac{1}{2}(p^i + q^i) & \text{if } |K^i| > 0, \\ p^i & |K^i| = 0, \end{cases} \quad (5.4)$$

implements the model of Acemoglu et al. (2011) where agents place equal weight on their private and social belief (we denote this weighting function as the *equal weighting* scenario).

However, it is worthwhile considering alternative weighting functions that depend on the size of a bank's neighborhood. We consider two different scenarios for the weighting function beyond the equal weighting scenario: (i) The private signal and each observed action are equally weighted (called the *neighborhood size* scenario):

$$t(p^i, q^i) = \left(\frac{1}{k_{t-1}^i + 1} \right) p^i + \left(\frac{k_{t-1}^i}{k_{t-1}^i + 1} \right) q^i \quad (5.5)$$

And (ii) observed actions are weighted with the relative size of the neighborhood (called the *relative neighborhood* scenario):

$$t(p^i, q^i) = \left(1 - \frac{k_{t-1}^i}{N-1} \right) p^i + \left(\frac{k_{t-1}^i}{N-1} \right) q^i \quad (5.6)$$

The private belief of bank i is denoted $p^i = \Pr(\theta = 1|s^i)$ and can easily be obtained using Bayes' rule if the signal likelihoods are known:

$$p^i = \left(1 + \frac{f_0(s_t^i)}{f_1(s_t^i)}\right)^{-1} \quad (5.7)$$

where f_0 and f_1 are the Gaussian densities corresponding to $\theta = 0$ and $\theta = 1$ respectively. Bank i is assumed to form a social belief q^i by simply averaging over the actions of all neighbors $j \in K_{t-1}^i$:

$$q^i = \Pr(\theta = 1|K_t^i, x^j, j \in K_{t-1}^i) = 1/k_{t-1}^i \sum_{j \in K_{t-1}^i} x_{t-1}^j \quad (5.8)$$

Given these private and social beliefs, agents choose an action according to equation ((5.3)).

Averaging over the actions of neighbors is a special case of DeGroot (1974) who introduces a model where a population of N agents is endowed with initial opinions $p(0)$. Agents are connected to each other but with varying levels of trust, i.e. their interconnectedness is captured in a weighted directed $n \times n$ matrix T . A vector of beliefs p is updated such that $p(t) = Tp(t-1) = T^t p(0)$. DeMarzo et al. (2003) point out that this process is a boundedly rational approximation of a much more complicated inference problem where agents keep track of each bit of information to avoid a persuasion bias (effectively double-counting the same piece of information). Therefore, the model this paper develops is also boundedly rational.⁷

The combination of a heuristic rule for the social belief with a Bayesian private belief adds a further dimension of bounded rationality. In previous models of Bayesian learning on social networks, cf. Acemoglu et al. (2011), the social belief is also computed in a Bayesian framework. We depart from this approach and instead use the simple averaging rule outlined in Eq. (5.8). Therefore, one could refer to the

⁷This bounded rationality can be motivated analogously to DeMarzo et al. (2003) who argue that the amount of information agents have to keep track of increases exponentially with the number of agents and increasing time, making it computationally impossible to process all available information.

learning framework proposed here as a form of boundedly rational Bayesian updating.

Markov chain representation

The model outlined above can be represented as a discrete state space Markov chain. The state space S is the set of possible combinations of bank actions $\{x^i\}_{i=1}^N$. We label each of the 2^N states of the model by an index l .⁸ Furthermore, we write x_l^i for the action of bank i in state l . We denote the state of the system at time t by the random variable X_t . Now consider the transition probability from state l to state m . Note that all agents choose their actions independently, therefore we can write the transition probability as

$$\Pr(X_{t+1} = m \mid X_t = l) = \prod_{i=1}^N \Pr(x^i = x_m^i \mid X_t = l) = \prod_{i=1}^N \Pr(x^i = x_m^i \mid q^i(l)), \quad (5.9)$$

where we write for convenience $q^i(l)$ to indicate the social belief of bank i in state l . Based on the signal structure, the update of private beliefs, and the neighbors' action choice we can derive the probability that bank i chooses a particular action $\Pr(x^i = x_m^i \mid q^i(l))$ for the equal weighting scenario.⁹ To see this, we first derive the distribution of private beliefs. Recall that the private signal structure is given as:

$$\begin{aligned} f_0(s) &= \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(\frac{-(s - \mu_0)^2}{2\sigma_0^2}\right), \\ f_1(s) &= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(\frac{-(s - \mu_1)^2}{2\sigma_1^2}\right), \end{aligned} \quad (5.10)$$

where we assume that $\sigma_0 = \sigma_1 = \sigma$ and that $\mu_0 = 1 - \mu_1$. Furthermore, denote the probability distribution of agent i 's private belief p^i as $f_p(p^i)$. We can then state the following:

⁸Since the size of the state space grows as 2^N with the number of agents, explicit computation of the transition matrix of this Markov chain becomes prohibitive as the number of agents becomes large. Nevertheless the Markov chain formalism is useful to obtain some analytical results. Furthermore, as we will show below, for the complete network a Markov chain on a reduced state space can be defined that approximates the actual model dynamics very well.

⁹Similar transition probabilities can be derived for the other weighting scenarios but are omitted here.

Proposition 5.1. *For $\theta \in \{0, 1\}$ the probability that agent i chooses a particular action $x^i = \theta$, given a social belief $q^i = q$ and private belief p^i distributed according to $f_p(\cdot)$ and under the assumption that $\mu_0 = 1 - \mu_1$, is given by:*

$$\Pr(x^i = \theta \mid q^i = q) = \int_0^{1-q} f_p(p^i \mid \theta = 0) dp^i = \int_q^1 f_p(p^i \mid \theta = 1) dp^i, \quad (5.11)$$

where the distribution $f_p(p^i \mid \theta = 0)$ of agent i 's private belief is given as:

$$f_p(p^i \mid \theta = 0) = \frac{\sigma \exp\left(-\frac{((\mu_0 - \mu_1)^2 - 2\sigma^2 \log(\frac{1}{p} - 1))^2}{8\sigma^2(\mu_0 - \mu_1)^2}\right)}{\sqrt{2\pi}(1-p)p(\mu_0 - \mu_1)}, \quad (5.12)$$

and similarly

$$f_p(p^i \mid \theta = 1) = \frac{\sigma \exp\left(-\frac{((\mu_0 - \mu_1)^2 + 2\sigma^2 \log(\frac{1}{p} - 1))^2}{8\sigma^2(\mu_0 - \mu_1)^2}\right)}{\sqrt{2\pi}(1-p)p(\mu_0 - \mu_1)}, \quad (5.13)$$

Proof, see Appendix 5.C. ■

This fully defines the transition probability $\Pr(X_{t+1} = m \mid X_t = l) = P_{ml}$ and thereby the transition matrix P .

5.2.1.1 Relation to other social interaction models

Before proceeding to the model dynamics I will briefly put the model proposed in this chapter in relation to other social interaction models in the literature. Besides DeGroot (1974) and DeMarzo et al. (2003), our model also bears some resemblance to the models of Cont and Löwe (2010), Schelling (1971) and spin models in the Physics literature. In particular, these models consider the social interaction of agents characterized by a binary state variable. As in our model, the interaction typically depends on the state of the neighbors and can be perturbed by exogenous noise. In addition the dynamics of an agent's state variable can be subject to an external bias that pushes it towards one of the two states.

Usually it can be shown that depending on the level of exogenous noise, which can re-interpreted in a number of ways including heterogeneity in preferences, cf. Cont

and Löwe (2010), there exists a transition between an ordered state in which agents choose the same action and an unordered state in which agents do not synchronize. While this insight is clearly relevant for the herding dynamics we aim to study in this chapter, it cannot be translated one-to-one to our model. This is primarily due to fact that the agent’s dynamics follow different transition rules. For example, in Cont and Löwe (2010) agents draw a particular action from a Boltzmann distribution. As seen above, this is not the case in our model.

Furthermore, while an exogenous bias towards one state is explicitly considered in the Physics literature this is not typically done the case in the social interaction literature. In our model however, due to the informativeness of the agent’s private signal, the “noise” is biased towards the true state of the world which in turn biases the agent’s actions.

Finally, our primary objective here is to study the impact of network structure on the social interactions for a fixed and finite number of agents. We therefore only consider two scenarios for the agent’s private signal: a highly informative signal and a less informative signal. In order to establish whether our model exhibits a similar phase transition as observed in Cont and Löwe (2010), we would have to consider a continuum of signal structures potentially differing both in the informativeness $|\mu_0 - \mu_1|$ and variance σ^2 while taking the limit as $N \rightarrow \infty$. While this is an interesting avenue of research we will not consider it here.

5.2.2 Herding with Exogenous Network Structures

Our interest is to understand under which conditions agents in the model with an exogenously fixed network structure coordinate on a state non-matching action. Our analysis will proceed in three steps. First, we build some intuition by discussing limiting cases for the strength of the private signal. Second, we will study analytically the dynamics of the model for the example of a complete network in the equal weight-

ing scenario. Finally, we will analyze the model dynamics on random networks with different weighting scenarios via numerical simulation.

Limiting cases for the strength of the private signal

Let the state of the world be $\theta = 0$ and assume that $f_0 = mf_1$. For $m = 1$ the signal is completely uninformative. For $m > 1$ the signal is informative and more so the larger m is. In the equal weighting scenario, equation (5.2) together with equations (5.7) and (5.8) yields the following condition for $x^i = 1$:

$$\frac{1}{2}(1+m)^{-1} + \frac{1}{2} \frac{1}{k_{t-1}^i} \sum_{j \in K_{t-1}^i} x_{t-1}^j > \frac{1}{2} \Leftrightarrow \sum_{j \in K_{t-1}^i} x_{t-1}^j > k_{t-1}^i \left[1 - \frac{1}{(1+m)} \right] \quad (5.14)$$

For completely uninformative signals, $m = 1$, equation (5.14) implies that an agent will always follow the majority of her neighbors. For very informative signals, $m \gg 1$, equation (5.14) implies that an agent will only ignore her private signal if she receives a strong social signal. The required strength of the social signal increases with the precision of the private signal. Now consider the neighborhood size scenario. The equation analogous to (5.14) for this scenario reads:

$$\left(\frac{1}{k_{t-1}^i + 1} \right) (1+m)^{-1} + \left(\frac{k_{t-1}^i}{k_{t-1}^i + 1} \right) \frac{1}{k_{t-1}^i} \sum_{j \in K_{t-1}^i} x_{t-1}^j > \frac{1}{2} \quad (5.15)$$

$$\Leftrightarrow \sum_{j \in K_{t-1}^i} x_{t-1}^j > \frac{k_{t-1}^i}{2} + \left[\frac{1}{2} - \frac{1}{1+m} \right] \quad (5.16)$$

For completely uninformative signals $m = 1$ this condition reduces to the equal weighting scenario. For highly informative signals, $m \gg 1$, however, the agent is almost as willing to follow her neighbors as in the uninformative equal weighting scenario if $k_{t-1}^i \gg 1$. In the relative neighborhood scenario equation (5.14) reads:

$$\left(1 - \frac{k_{t-1}^i}{N-1} \right) (1+m)^{-1} + \left(\frac{k_{t-1}^i}{N-1} \right) \frac{1}{k_{t-1}^i} \sum_{j \in K_{t-1}^i} x_{t-1}^j > \frac{1}{2} \quad (5.17)$$

$$\Leftrightarrow \sum_{j \in K_{t-1}^i} x_{t-1}^j > (N-1) \left[\frac{1}{2} - \frac{1}{1+m} \right] + \frac{k_{t-1}^i}{1+m} \quad (5.18)$$

Again, for completely uninformative signals this equation reduces to the equal weighting case. For highly informative signals, $m \gg 1$, this equation reduces to $\sum_{j \in K_{t-1}^i} x_{t-1}^j > \frac{1}{2}(N-1)$ if the neighborhood is sufficiently small, i.e. $m \gg k_{t-1}^i$. Note that by construction, the central node in a star network will thus always follow the majority of the spokes and will ignore his private signal while spokes will almost always follow their private signal.

Dynamics on a complete network

Before studying the special case of the complete network of banks, consider a few general properties of the Markov chain $(X_t)_{t \geq 0}$ in the equal weighting scenario. In particular, consider the probability that bank i adopts action y given the state of the Markov chain is $X_t = l$. From equation (5.11) we know that

$$\Pr(x^i = y \mid q^i(l)) = \begin{cases} 0 & , \text{ if } q^i(l) = \tilde{y}, \text{ where } \tilde{y} = 1 \text{ if } y = 0 \text{ and } \tilde{y} = 0 \text{ if } y = 1. \\ 1 & , \text{ if } q^i(l) = y, \\ z & , \text{ where } 0 < z < 1, \text{ otherwise.} \end{cases} \quad (5.19)$$

Therefore a bank will adopt the action of its neighbors with probability one if all of its neighbors chose the same action in the previous time step, i.e. if there is consensus in its neighborhood. Eq. (5.19) implies that the Markov chain has at least two absorbing states,¹⁰ namely the two consensus states in which all banks choose the same action. The existence of further absorbing states or recurrence classes depends on the network structure.

To see this consider the example of a star network. In a star network there are two types of nodes, a single hub node and spoke nodes which are connected to the hub but not among each other. Suppose at time t all spoke nodes choose action $x = 1$ while the hub node chooses action $x = 0$. Since for each node in the network there is consensus in its neighborhood, it will choose its neighbor's action in the following

¹⁰An absorbing state is a state which the Markov chain never leaves.

time step. Therefore at time $t + 1$ all spoke nodes choose action $x = 0$ while the hub node chooses action $x = 1$. By the same argument the Markov chain will return to its state at time t in the following time step. Therefore, the two states outlined above form a recurrence class – the Markov chain returns to them with probability one.

In contrast to the star network, the complete network has only the consensus states as absorbing states. This follows immediately from equation (5.19). In the complete network with N nodes each node is connected to the remaining $N - 1$ nodes. Therefore for all banks i the social belief in state l satisfies

$$q^i(l) = y \text{ with } y \in \{0, 1\} \text{ iff } x_l^j = y \ \forall \ j.$$

Thus, $0 < \Pr(x^i = y \mid q^i(l)) < 1$ for all states l that are not consensus states. Therefore, in the complete network the only two absorbing states are the consensus states. This finding also implies that there always exists a positive, though often very small, probability that the banks coordinate on a state non-matching action unless the system is initialized in consensus on the state matching action.

For the special case of a complete network it is possible to write down a reduced state space of the Markov Chain. Since all banks are equivalent, it is a reasonable simplification to study the average action in the system rather than each bank's individual action. Then for a system of N banks the reduced state space S' is simply $l \in \{0, 1/N, 2/N, \dots, (N - 1)/N, 1\}$. The reduced state space scales linearly with the number of banks and therefore permits explicit evaluation of the transition matrix for non trivial numbers of banks. The transition probabilities on this state space are given by a simple binomial distribution with success probability equal to the probability of a bank choosing action $x = 1$ given an average state of the system l , i.e.

$$\Pr(m \mid l) = \binom{N}{Nl} \Pr(x = 1 \mid q = l)^{Nl} (1 - \Pr(x = 1 \mid q = l))^{N(1-l)}. \quad (5.20)$$

With the transition matrix P defined by the above relation, it is possible to explicitly compute the absorption probabilities of the two consensus states conditional on a

particular initial state of the system. These probabilities then tell us the probability that the system will coordinate on a state non-matching action depending on the system's initial state. The absorption probabilities can be computed using the standard machinery of Markov chains. Let Q be the transition matrix between transient states (i.e. non absorbing states) and let R be the transition matrix between transient states and absorbing states. Then the probability of ending up in the absorbing state indexed by a when starting from a transient state indexed by b is given by M_{ab} , where the matrix M is given by $M = (\mathbb{1} - Q)^{-1}R$.

To illustrate this method we compare the absorption probabilities computed from the transition matrix and the absorption probabilities estimated by numerical simulation for a network of $N = 40$ banks, all possible initial states and different signal informativeness. We summarize our results in Figure 5.1. The theoretical prediction and the numerical results are in agreement. Our results indicate that, in the complete network, the probability of choosing a state non-matching action depends critically on the initial state of the system. As the informativeness of the private signal increases, the probability of coordinating on a state non-matching action becomes very small for most initial states – even for initial states in which most banks start off with a state non-matching action. It should be noted though that even in the case of highly informative signals, the probability of coordinating on a state non-matching action does not vanish entirely as we have shown above.

Herdung in random networks - numerical simulations

The above exercises show that the impact of the weighting function on agents' strategies and model dynamics is easily understood in the case of completely uninformative and fully informative signals or very simple network structures. But what happens in the more realistic region of interim signal informativeness? Are densely connected random networks more conducive for agents to coordinate on state non-matching actions or sparse networks? And how does the fraction of nodes that coordinate on a

state non-matching action depend on the initial conditions? We address these questions in numerical simulations for two cases. In the case (I) of *informed agents* the distance between the mean of the two signals is $\mu_1 - \mu_0 = 0.6 - 0.4 = 0.2$ while in the case (U) of uninformed agents the distance between the mean of the two signals is $\mu_1 - \mu_0 = 0.51 - 0.49 = 0.02$. In both cases we use a standard deviation of $\sigma_{0,1} = \sqrt{0.1}$.¹¹ We conduct our simulations with $N = 100$ agents and update $T = 100$ times. To analyze the impact of the network structure on the probability of coordination on a state non-matching action, we vary the network density ρ of a random (Erdős-Rényi) graph within $\rho = [0.0, 0.95]$ in 20 steps. Each simulation is repeated $N_S = 1,000$ times to account for stochasticity. For all simulations we assume that the state of the world is $\theta = 0$. An overview of the parameters used can be found in Table 5.1.

Figure 5.2 shows the average final action of the system after $T = 100$ update steps for a random graph with varying densities in the informed and uninformed case for the equal weighting, neighborhood size, and relative neighborhood scenarios. In all cases, when the network is disconnected ($\rho = 0$), agents effectively act on the basis of their private signal only and the fraction of agents that choose a state non-matching action is close to the signal informativeness μ_0 . With increasing network density, social learning sets in and the fraction of agents with a state non-matching action is reduced. It should be noted that after $T = 100$ the system may not have converged, i.e. the system may not be in a consensus state. Therefore, the average action does not provide a good estimate of the probability of contagious synchronization. This will be addressed in a second step. In the following we will consider each weighting scenario in turn.

¹¹A larger standard deviation would make the signal less informative, without changing our results qualitatively.

Equal weighting (top row): With an informative signal, equal weighting and a network density $\rho > 0.1$ there is almost no agent that chooses a state non-matching action after $T = 100$ update steps. This suggests that as soon as the network is fully connected, the strong private signal in combination with social learning efficiently spreads “knowledge” of the state of the world. This also suggests that once the network is fully connected, adding further links does not improve the performance of the agents. In the uninformative case, social learning also improves the average action of banks but only slightly. As in the informed case, this improvement occurs for small network densities while increasing the network density further only has a marginal effect. This occurs even though in the uniformed case there remains ample room for improvement.

Neighborhood size (middle row): The results in the neighborhood size scenario resemble closely the equal weighting scenario. First, actions improve considerably with network density and then level off. However, in the informative case, the average action flattens off at a higher level than for the equal weighting scenario. This is because in the neighborhood size scenario a bank’s social belief has a stronger weight than the private belief for $k > 1$. Thus, we expect the system to be more sensitive to initial conditions. Then, the system becomes susceptible to “action-belief” lock in if the system happens to be initialized with a majority of banks choosing a state non-matching action. Since the social signal dominates, the system cannot recover from this poor initialization with the help of private signals.

Relative neighborhood (bottom row): Interestingly, in the relative neighborhood scenario results are different. First, in the case of an informative signal improvements due to social learning accumulate much slower as the network density is increased. This is because the weight on the social belief on average starts off at a much lower value than in the equal or neighborhood size scenarios. The weight then

gradually increases to 0.5 when $\rho \approx 0.5$. Not surprisingly, this is when the average action reaches values similar to the average action in the equal weighting case. Second, both in the informative and uninformative case, the effect of increasing network density is non-monotonic. If the network density is increased beyond a certain point, the average action deteriorates. In fact, the average action approaches levels observed in the neighborhood size scenario. As in the neighborhood size scenario, the reason for the system converging on a state non-matching action lies in the “action-belief” lock following unfavorable initial conditions. Given these results, one can interpret the relative neighborhood scenario as interpolating between the neighborhood size and equal weighting scenarios. These results also suggest that, once the network is fully connected, changing the network density only affects the average action via changes in the weights of the private and social beliefs; not via changes in the network structure.

As mentioned above, the initial conditions may be important determinants of the average action and the probability of contagious synchronization. In order to understand how exactly the probability of contagious synchronization depends on the initial conditions, Figure 5.3 shows the probability that a large fraction ($> 80\%$) of agents coordinate on a state non-matching action for three cases: (1) For a full sample of $N_S = 1,000$ simulations; (2) conditional on agents starting on average with a state matching action: $\hat{x} = \sum_i x_0^i / N < \frac{1}{2}$; (3) conditional on agents starting on average with a state non-matching action: $\hat{x} > \frac{1}{2}$.

Interestingly, the effect of increasing the network density on the likelihood of contagion is quite different to its effect on the average action. In fact, increasing the network density at first increases the probability of contagion in all scenarios while it decreased the average action. This suggests that indeed the system has not converged to a consensus state for many data points in Figure 5.2. Whether the effect of an

increased network density can be beneficial depends on the initial conditions and weighting scenario. In the equal and neighborhood size scenarios, if the system is initialized such that $\hat{x} < \frac{1}{2}$, the probability of contagion decreases for larger network densities. This is not the case if $\hat{x} > \frac{1}{2}$. Thus, learning from more neighbors can make positive outcomes more likely if the system starts off in a favorable state but it cannot reverse the dynamics if the system starts off in an initial state with $\hat{x} > \frac{1}{2}$.

As expected, the probability that agents coordinate on a state non-matching action drastically increases when agents initially start with a state non-matching action. However, less so in the equal weighting scenario when private signals are informative because agents place less weight on their social beliefs. A comparison of the left-center with the left-top panel in Figure 5.3 shows that the probability of contagion increases by a roughly an order of magnitude in the neighborhood size scenario compared to the equal weighting scenario. Again, for uninformative signals this effect is not present.

While Figure 5.3 shows the existence of contagion even for initial actions that are state matching, the relationship between average initial and average final action can be further quantified. In Figure 5.4 we plot the average final action versus the average initial action in a density plot. We conducted a total of $N_S = 1,000 \times 20$ simulations and show the resulting pair of average initial and average final action as a dot with the respective coordinates. The averages are taken over the 1000 replications for a given network density. We draw initial actions according to the private signal. The left side of Figure 5.4 shows the results for an informative signal. The mean of initial distributions is thus $\hat{x} < 0.5$, i.e. informative on average. In the right panel we show the same results for an uninformative signal where the mean of initial distributions is thus much closer to $\hat{x} = \frac{1}{2}$. For the equal weighting scenario (top), only very few simulations yield a final average action that is state non-matching. A similar picture can be seen for the relative neighborhood (bottom) scenario. In the neighborhood

size (center) scenario, however, a substantial number of simulations with an initially state matching average action yield a final state non-matching average action.

5.3 Contagious Synchronization in Endogenously Formed Networks

Banks form trading relationships endogenously. The decision whether or not two banks establish a trading relationship depends in reality on many factors. In the previous section we analyzed how one bank can learn about an underlying state of the world by observing the action of another bank to which it has established a relationship. In the following we will consider a network formation process in which the bank only considers the possible “learning” benefit from establishing a link. We abstract from other motives for establishing links. We compute the value of an additional link in three steps. First, we compute the probability that an agent chooses a state matching action, given her signal structure, private beliefs and neighbors’ actions. Given this probability, we compute, second, an agent’s expected utility conditional on her social belief which depends on her strategic choice to establish a link. Once an agent’s expected utility with and without a link is computed, we can use the concept of pairwise stable networks to determine the equilibrium network structure.

5.3.1 Agents’ Expected Utility

Recall that an agent i ’s utility u^i is given as:

$$u^i(x^i) = \begin{cases} 1 & \text{if } x^i = \theta \\ 0 & \text{if } x^i \neq \theta \end{cases} \quad (5.21)$$

The expected utility of agent i conditional on her social belief q^i is thus:

$$\bar{u}^i(x^i \mid q^i) = \Pr(x^i = \theta \mid q^i = q). \quad (5.22)$$

The value of a link is given by the marginal utility from establishing a link, which in turn depends on the change in the social belief q . An agent can thus influence her social belief by strategically choosing neighbors. The probability that a neighbor takes a state matching action depends in turn on the social belief that this neighbor forms about her neighbors, which leads to complex higher-order effects which we neglect in this paper. Rather, we assume that an agent i has constant beliefs about her neighbors' social beliefs q' . For simplicity we simply assume that the agent takes her neighbors' social belief as $q' = \frac{1}{2}$. This assumption also allows us to exploit the symmetry of the distribution the private belief, as will become apparent below. The expected utility of agent i conditional on a given q' and neighborhood K^i is given as:

$$\bar{u}^i(q', K^i) = \sum_{a \in Q^i} \Pr(q^i = a \mid q') \Pr(x^i = \theta \mid q^i = a), \quad (5.23)$$

where Q^i is the set of all possible values of the social belief of agent i . The first term on the right-hand side of equation (5.23) is the probability that agent i has a certain social belief given the social belief of her neighbors. The second term is the probability of choosing a state matching action given that social belief is given by equation (5.11). For a given size of the agent's neighborhood k^i , Q^i is simply $Q^i = \{n/k^i \mid n \in \mathbb{Z}, 0 \leq n \leq k^i\}$. The probability of a particular social belief can be computed by summing over the probabilities of combinations of actions chosen by the neighbors of agent i . Define the set of feasible action vectors of i 's neighbors conditional on agent i having a social belief $q^i = a$:

$$X^{ai} = \{\mathbf{x} \mid \sum_j x_j = a, x_j \in \{0, 1\}, j \in K_i\}. \quad (5.24)$$

i.e. X^{ai} is the set of all action vectors that are compatible with a social belief $q^i = a$. Then the probability of agent i having a social belief of $q^i = a$ given the social beliefs of all i 's neighbors, q' , is given as:

$$\Pr(q^i = a \mid q') = \sum_{\mathbf{y} \in X^{ai}} \prod_{j \in K^i} \Pr(x^j = y^j \mid q^j = q'). \quad (5.25)$$

Now define the probability that neighbor j chooses a state matching action as:

$$z^j = \Pr(x^j = \theta \mid q^j = q') = \int_0^{1-q'} f_p(p^j \mid \theta = 0) dp = \int_{q'}^1 f_p(p^j \mid \theta = 1) dp, \quad (5.26)$$

where the last equality is true since $q' = \frac{1}{2}$ and due to the symmetry of the distribution of the private belief around $p^i = \frac{1}{2}$. Therefore, z^j can be computed without knowledge of the state of the world θ . Note, that $f_p(p^j \mid \theta = 0)$ and $f_p(p^j \mid \theta = 1)$ depend on the signal structure of neighbor j .

We can write for the probability of agent i having a social belief of $q^i = a$ in equation (5.25):

$$\Pr(x^j = y^j \mid q^j = q') = \begin{cases} z^j & \text{if } y^j = \theta \\ 1 - z^j & \text{if } y^j \neq \theta \end{cases} \quad (5.27)$$

If $z^j = z \forall j$ the distribution in (5.25) would be a simple binomial distribution. In general, however, this is not the case and we need to resort to numerical methods to compute the equilibrium network structures.

The fact that z^j is independent of the state of the world and that we integrate over all possible values of the social belief q in Eq. (5.23) implies that the expected utility can be computed without knowledge of the state of world.¹²

5.3.2 The Network Formation Process

We now have all the necessary ingredients to compute the expected utility of an agent i given her neighborhood K^i and expectations about her neighbors' social belief q' . In the endogenous network formation process the agent i will seek to maximize her "network utility" u^i by changing her neighborhood while holding q' fixed. In this section we outline an algorithm for endogenous network formation that ensures a pairwise stable network in the sense of Jackson and Wolinsky (1996):

¹²This can also be justified intuitively: Under the symmetry of the signal structure in both states of the world the signal should carry the same amount of information. Thus changing the state of the world should not change an agent's expected utility which only depends on the agent's signal and his neighbors' signals.

Definition 5.1. A network defined by an adjacency matrix g is called pairwise stable if

(i) For all banks i and j directly connected by a link, $l^{ij} \in L$: $u^i(g) \geq u^i(g - l^{ij})$
and $u^j(g) \geq u^j(g - l^{ij})$

(ii) For all banks i and j not directly connected by a link, $l^{ij} \notin L$: $u^i(g + l^{ij}) < u^i(g)$
and $u^j(g + l^{ij}) < u^j(g)$

where the notation $g + l^{ij}$ denotes the network g with the added link l^{ij} and $g - l^{ij}$ the network with the link l^{ij} removed. When maintaining a link is costly, there will be some network density that depends on the cost $c > 0$ per link. In this model, the marginal utility of an additional link decreases with the number of links because the expected utility is bounded from above by 1 (the pay-off is 1 and the probability of choosing the correct action is less than, or equal to, 1).

The algorithm to ensure a pairwise stable equilibrium starts by choosing a random agent i from the set of agents N . Then, choose a second agent j from the set of agents $N \setminus K^i$ that are not yet neighbors of i . An agent is chosen with the following probability:

$$w^j = \exp(\beta E^j) / Z, \quad (5.28)$$

where $Z = \sum_k \exp(\beta E^k)$ is a normalization constant and $E^j = |\frac{1}{2} - \mu_0^j|$ is a proxy for agent j 's signal strength. For $\beta = 0$, i chooses the new agent with equal probability. While this makes it more likely that agent i considers forming a link with agent j when j has a higher signal strength, it does not imply that such a link is actually formed. This decision is solely based on the utility that both i and j obtain from establishing the link.

Now, let $K^i = K^i \cup j$, i.e. the neighborhood of agent i after adding agent

j , and similarly $K'^j = K^j \cup i$. The marginal expected utilities of adding j and i to the respective neighborhoods are then:

$$\begin{aligned}\Delta \bar{u}^i(q', K'^i, K^i) &= \bar{u}^i(q', K'^i) - \bar{u}^i(q', K^i) \\ \Delta \bar{u}^j(q', K'^j, K^j) &= \bar{u}^j(q', K'^j) - \bar{u}^j(q', K^j)\end{aligned}\tag{5.29}$$

Given the marginal utilities of agents i and j and their cost of maintaining link c^i and c^j the agents will form a link if $\Delta \bar{u}^i(q', K'^i, K^i) > c^i$ and $\Delta \bar{u}^j(q', K'^j, K^j) > c^j$. If $\Delta \bar{u}^i(q', K'^i, K^i) > c^i$ and $\Delta \bar{u}^j(q', K'^j, K^j) < c^j$, the algorithm selects the least informative agent in the neighborhood of j :

$$m = \underset{n \in K^j}{\operatorname{argmin}} E^n.\tag{5.30}$$

Now, define $K''^j = K'^j \setminus m$. If $\Delta \bar{u}^i(q', K'^i, K^i) > c^i$ and $\Delta \bar{u}^j(q', K''^j, K^j) > c^j$, form the link l^{ij} and remove the link l^{jm} (and similarly for the substitutions $i \rightarrow j$ and $j \rightarrow i$). Otherwise, don't form the link. If $\Delta \bar{u}^i(q', K'^i, K^i) < c^i$ and $\Delta \bar{u}^j(q', K'^j, K^j) < c^j$ repeat the previous step, i.e. consider removing the least informative neighbor and re-evaluate the utilities.

5.3.3 Equilibrium Networks

The endogenously formed network in an economy with identically informed agents and positive cost c of maintaining a link is a simple Erdős-Rényi network with a network density depending on the signal structure and link cost. To analyze more realistic situations, we use numerical simulations. In the following we assume that agents are heterogeneously informed about the underlying state of the world: a few “informed” agents have relatively precise signals and low costs of maintaining a link, while many “uninformed” agents have relatively imprecise signals and higher cost of maintaining a link. Table 5.2 summarizes the parameters we are using for the rest of this section.

In order to compare the dynamics on the endogenous networks to the Erdős-Rényi networks we run the following simulations. We first create 1,000 networks using the network formation algorithm described above. Then, we run the social learning algorithm described in Section 5.2 while holding the network structure constant throughout. The underlying assumption is that banks are updating their investment decisions faster than the network structure changes. This can be empirically corroborated by looking at the term structure of interbank lending. While 90% of the turnover in interbank markets is overnight, about 90% of exposures between banks stems from the term segments, i.e. longer term exposures, cf. Roukny et al. (2014).

We also run the social learning algorithm with an initialization bias in which we set the initial action of all agents to some predefined value. To assess the efficiency of the endogenous network formation, we compare the performance of the endogenously formed networks to the performance of Erdős-Rényi networks with the same average degree and same signal structure. All simulations in this section are conducted using the equal weighting scenario. An example of a resulting network structure can be found in Figure 5.5 and the cumulative degree distribution of an ensemble of 1000 endogenously formed networks is shown in Figure 5.6. The degree distribution of the endogenously formed networks has a much heavier tail than an Erdős-Rényi network with the same average degree. Also, the endogenous degree distribution appears bimodal. One peak corresponds to the uninformed nodes with small degree while the second peak corresponds to the informed nodes with high degree. Figure 5.7 shows the distribution of the final action for the 1,000 simulations conducted. A clear improvement over the Erdős-Rényi networks can be seen, highlighting the importance of core banks with more precise private signals. Since core banks are highly interconnected, there is a higher chance that they are in the neighborhood of a peripheral bank (as opposed to a peripheral bank being in the neighborhood of an-

other peripheral bank) which increases the precision of peripheral banks' social belief.

To further understand the difference between endogenously formed and Erdős-Rényi networks, we analyze the time it takes learning to converge. We assume the learning has converged at time t if:

$$\sum_i |x_i(t) - x_i(t + \Delta t)|/N < 0.05, \text{ where } \Delta t = 15. \quad (5.31)$$

It can be seen from Figure 5.8 that, except for very long convergence times, the system always converges faster in the endogenous network case than in the Erdős-Rényi case. Note, that this simulation was conducted without initialization bias, i.e. with average initial action of $\frac{1}{2}$. Finally, the probability of contagion as a function of an initialization bias, i.e. as a function of average initial action is shown in Figure 5.9. Again, the picture shows that the probability of contagion, i.e. the probability that more than 80% of agents coordinate on a state non-matching action is significantly smaller in the endogenous network case than in the case of a random Erdős-Rényi graph.

5.4 Conclusion

This paper develops a model of contagious synchronization of bank's investment strategies. Banks are connected in a financial network via trading relationships and learn from their neighbors' actions. The network structure is either an exogenously given Erdős-Rényi network or an endogenously formed network that heuristically maximizes each bank's expected "network utility" - i.e. the benefit from social learning. Banks receive a private signal about the state of the world and observe the average action of their counterparties – the social belief.

We compare three scenarios of weighting functions on the private and social belief. First, in the equal weighting scenario, agents place equal weights on their private and

social belief. Second, in the neighborhood scenario agents place proportionately more weight on the social signal when the size of the neighborhood increases. Third, in the relative neighborhood scenario agents place more weight on the social belief if their neighborhood constitutes a larger fraction of the overall network. Social learning can increase the probability of choosing a state matching action and thus agents' utility. When agents strategically choose their neighbors, they take the additional utility from learning from an additional bank into account, as well as the associated cost of establishing a link. Since, in this model the expected utility is bounded from above, the more neighbors a given agent has, the lower is the marginal utility from another link. Therefore the network converges to an equilibrium configuration.

We obtain two main results. First, in a complex financial system where agents cannot take the action of all their peers into account when taking an investment decision, the probability of contagious synchronization depends on two things: (i) the weighting between the private and social belief; and (ii) the density of the financial network. In the case of exogenous networks, as networks become more tightly connected, banks can exploit the “social wisdom” better and the probability of contagious synchronization declines when the bank's initial action is correct on average. When networks are endogenous and banks have heterogeneous signal strength, the endogenous network formation process converges to network structures that improve the spreading of information on the network, making contagious synchronization less likely relative to the Erdős-Rényi benchmark.

One drawback of the model is that there is no closed-form analytical solution for the utility a bank obtains through learning from a peer that takes into account higher order effects. This utility will depend on whether or not a neighboring bank chose a state matching or state non-matching action in the previous period and thus on the social belief of neighboring banks. In the former case, the benefit will be

positive, while in the latter case it will be negative. Agents have ex ante no way of knowing what type of action (state or non state matching) a neighboring bank selected until the state of the world is revealed ex post. Finding such a closed-form solution is beyond the scope of the present paper which focuses on the application in an agent-based model, but would provide a fruitful exercise for future research.

5.A Appendix: Tables

Variable	Description	Value	
		1 - (I)	2 - (U)
N	Number of agents	100	100
μ_0	Average signal for $\theta = 0$	0.4	0.49
μ_1	Average signal for $\theta = 1$	0.6	0.51
σ_0	Standard deviation of signal for $\theta = 0$	$\sqrt{0.1}$	$\sqrt{0.1}$
σ_1	Standard deviation of signal for $\theta = 1$	$\sqrt{0.1}$	$\sqrt{0.1}$
T	Number of iterations of updater	100	100
ρ	Density of ER network	$[0, 0.95]$	$[0, 0.95]$
N_S	# of simulations per param. config.	1000	1000

Table 5.1: Parameters for runs for the case of informed (I), uninformed (U).

Notation	Description	Value
N	Number of agents	30
N_I	Number of informed agents	4
N_U	Number of uninformed agents	26
μ_{0I}	Average signal for $\theta = 0$ for informed agents	0.3
μ_{1I}	Average signal for $\theta = 1$ for informed agents	0.7
σ_{0I}	Standard deviation of signal for $\theta = 0$ for informed agents	$\sqrt{0.1}$
σ_{1I}	Standard deviation of signal for $\theta = 1$ for informed agents	$\sqrt{0.1}$
μ_{0U}	Average signal for $\theta = 0$ for uninformed agents	0.4
μ_{1U}	Average signal for $\theta = 1$ for uninformed agents	0.7
σ_{0U}	Standard deviation of signal for $\theta = 0$ for uninformed agents	$\sqrt{0.1}$
σ_{1U}	Standard deviation of signal for $\theta = 1$ for uninformed agents	$\sqrt{0.1}$
c_I	Cost per link for informed agent	0
c_U	Cost per link for uninformed agent	0.1
T_C	Number of iterations of network algorithm	400
q'	Agent's belief of average action of neighbors of neighbors	0.5
β	Intensity of choice in agent selection	30
T	Number of iterations of action updater	100
n	Number of endogenously formed networks used for simulation	1000
N_S	Number of simulations per parameter configuration	100
ρ	Average density of ER networks	0.08

Table 5.2: Parameters for network formation and runs with endogenous networks.

5.B Appendix: Figures

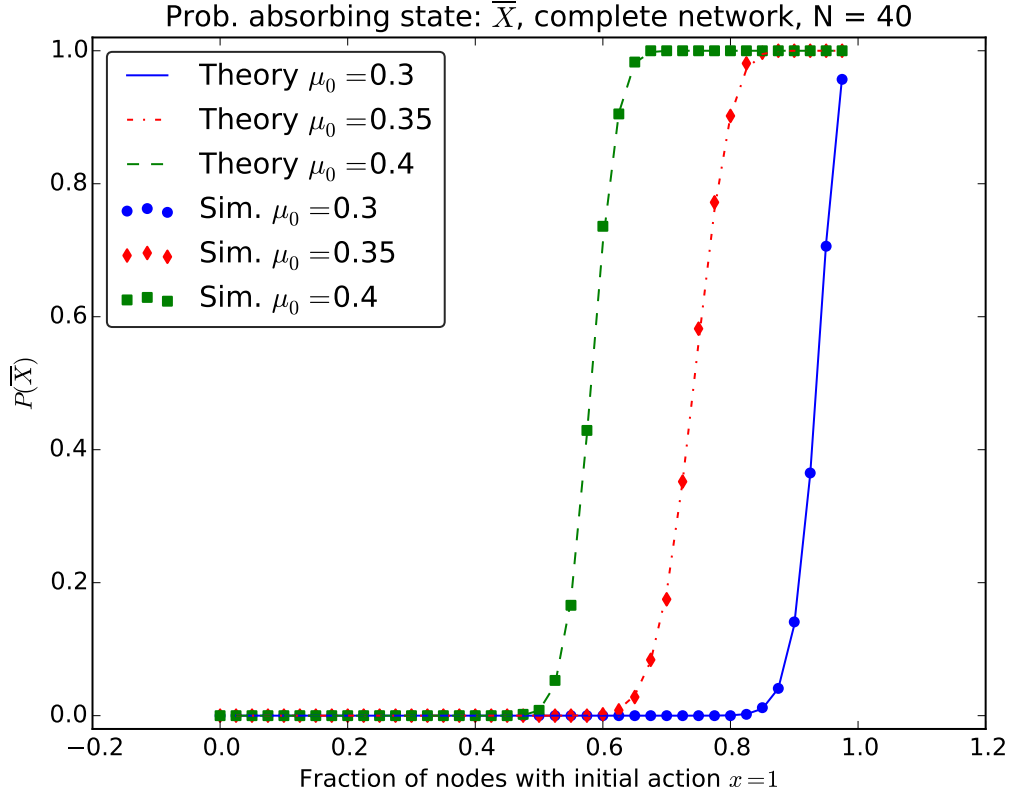


Figure 5.1: Probability of absorption for consensus state $\bar{X}$, i.e. state in which all banks choose action $x = 1$, as a function of initial state parameterized by the fraction of banks initially choosing action $x = 1$. We assume that $\mu_1 = 1 - \mu_0$. Markers: Numerically estimated absorption probabilities. Lines: Absorption probabilities predicted by reduced state space Markov chain representation. Note that theoretical absorption probabilities are computed for the same initial states as for numerical estimation; lines are drawn to interpolate between discrete observations for clarity only. Theory predicts the numerical results very well.

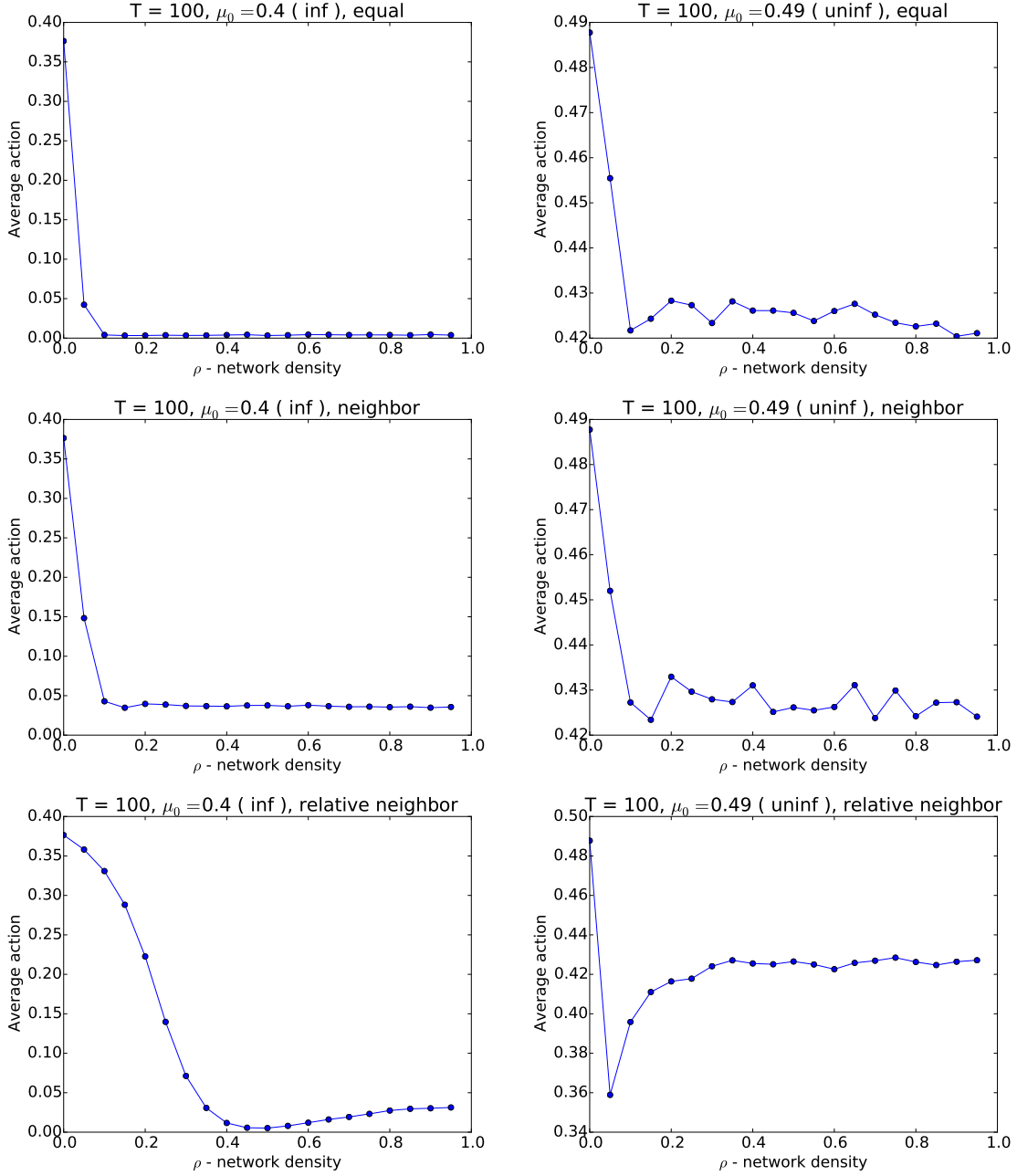


Figure 5.2: Average final action of agents as a function of the network density ρ of a random graph for the informed (left) and uninformed (right) case. Top: Equal weighting scenario; Center: Neighborhood size scenario; Bottom: relative neighborhood scenario.

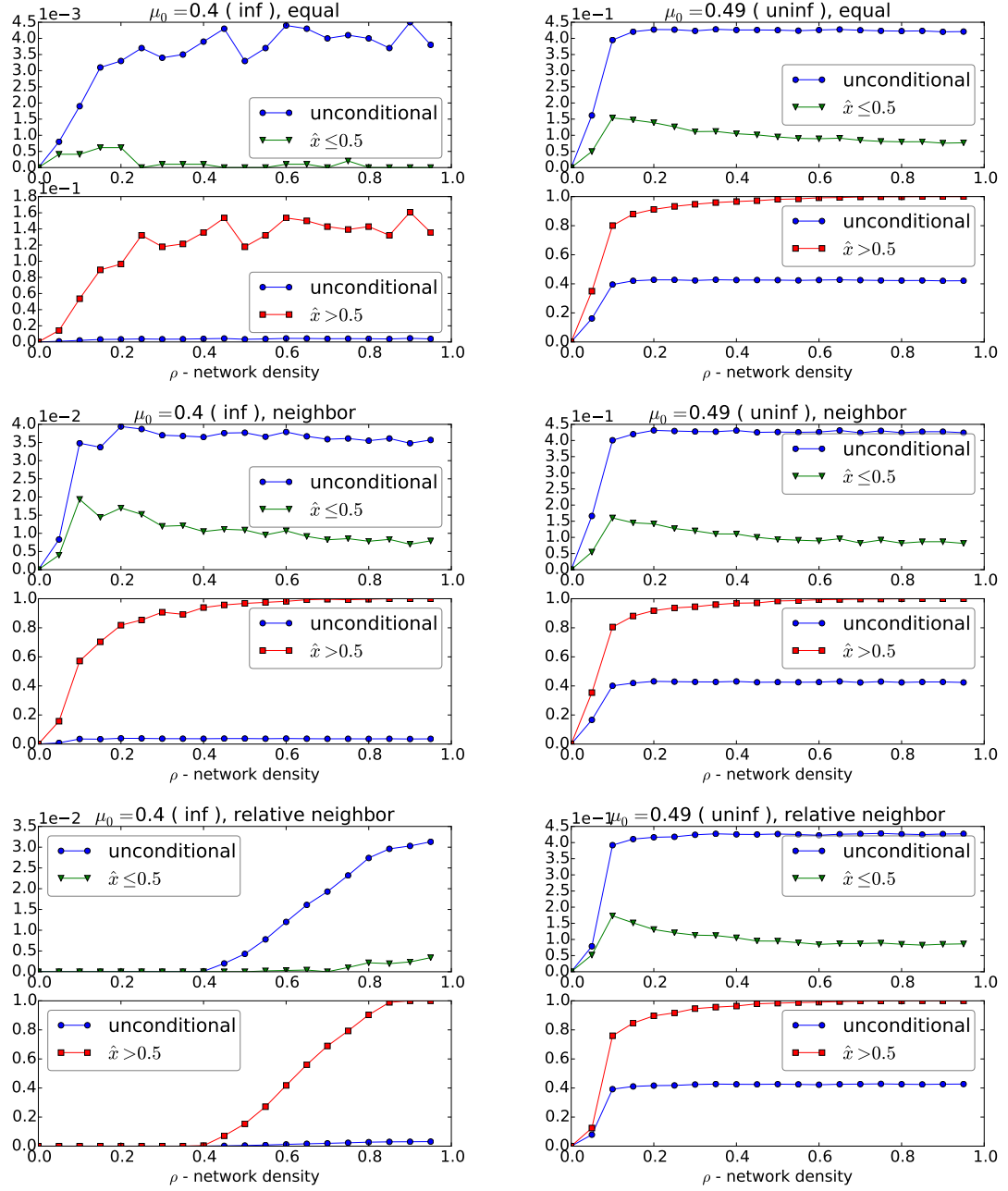


Figure 5.3: Fraction of simulations per parameter configuration N_S (1000) in which agents synchronize on the state non-matching action (more than 80% of agents choose the state non-matching action) as a function of network density ρ of a random graph for the informed (left) and uninformed (right) case. Top: Equal weighting scenario; Center: Neighborhood size scenario; Bottom: relative neighborhood scenario. We distinguish three cases: (1) unconditional: we compute the fraction based on the full sample S . (2) conditional $\hat{x} \leq \frac{1}{2}$: we compute the fraction based on the sub-set of simulations in which the average initial action $\hat{x} = \sum_i x^i(0)/N \leq \frac{1}{2}$, i.e. when the agents start with a state matching action. (3) conditional $\hat{x} > \frac{1}{2}$: we compute the fraction based on the sub-set of simulations when the agents start with a state non-matching action.

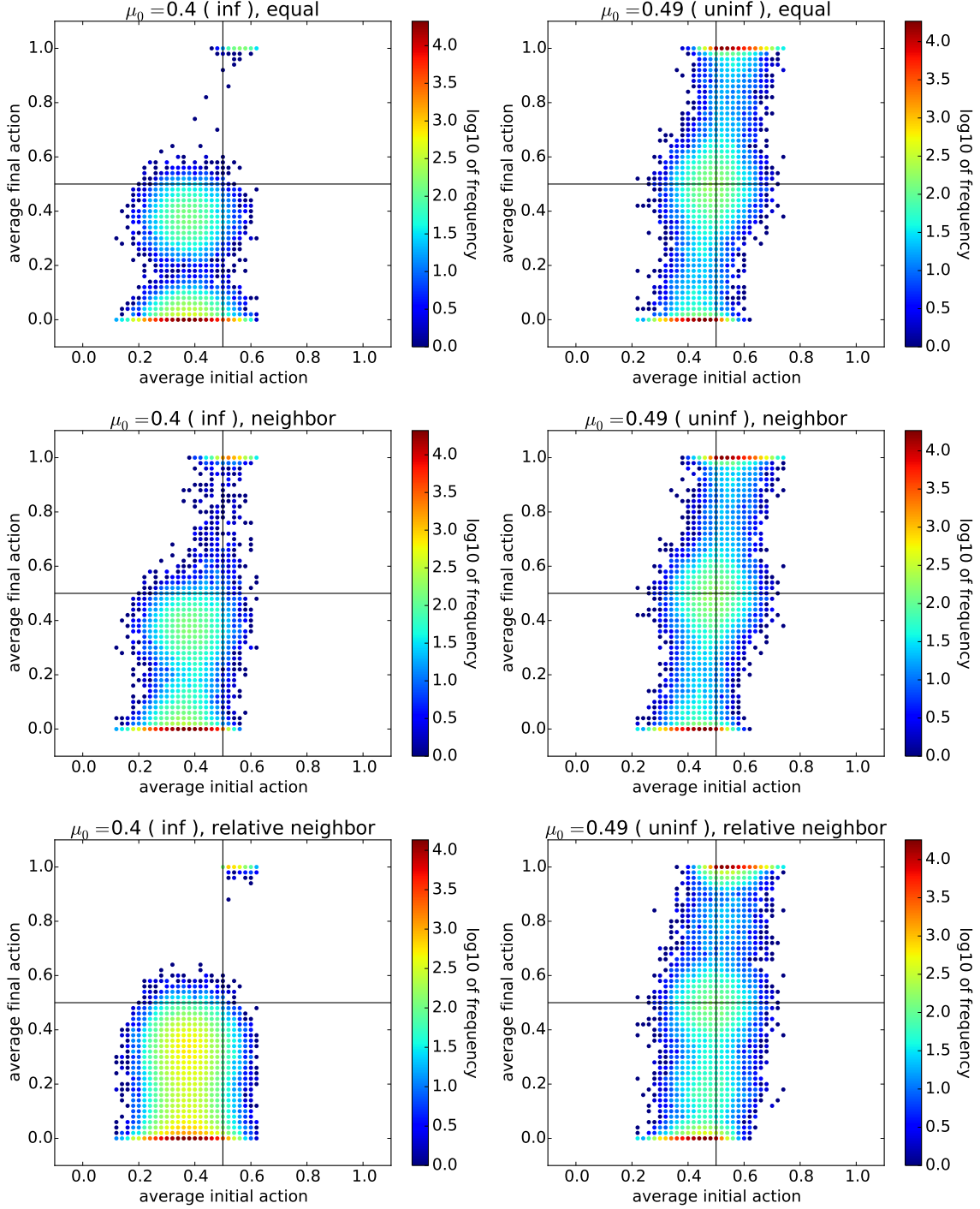


Figure 5.4: Average final action $\hat{x}_F = \sum_i x^i(T)/N$ versus the average initial action $\hat{x}_I = \sum_i x^i(0)/N$ for the informed (left) and uninformed (right) case. Top: Equal weighting scenario; Center: Neighborhood size scenario; Bottom: relative neighborhood scenario. Data points are averages over $N_S = 1000$ simulations and all network densities ρ (20 values equally distributed over the interval $[0, 0.95]$). The color code indicates the frequency with which a point occurs in the sample (total size 20×1000), the scale of the color code is logarithmic of base 10.

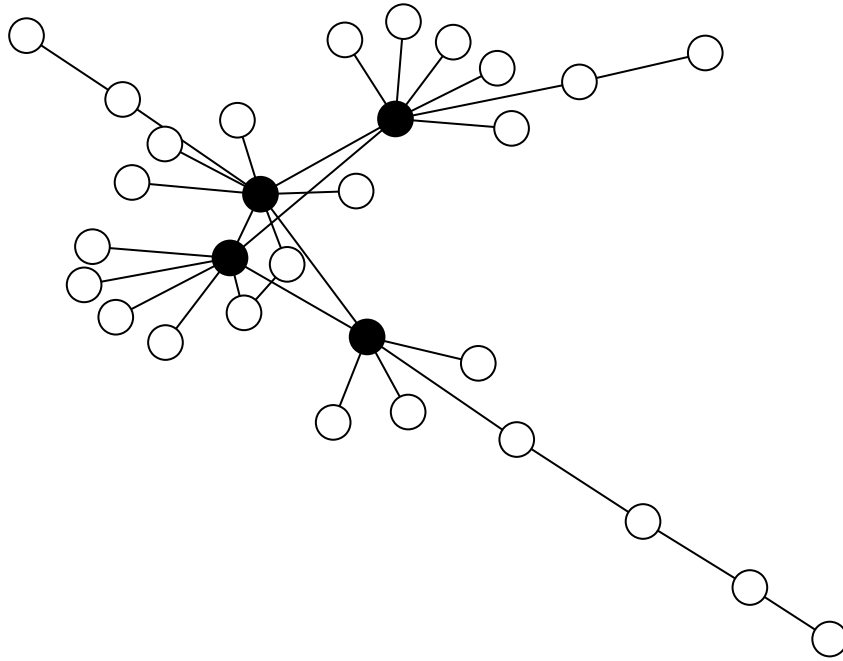


Figure 5.5: Example network from ensemble of $n = 1000$ endogenously formed networks. Black nodes are “informed” agents, while white nodes are “uninformed”.

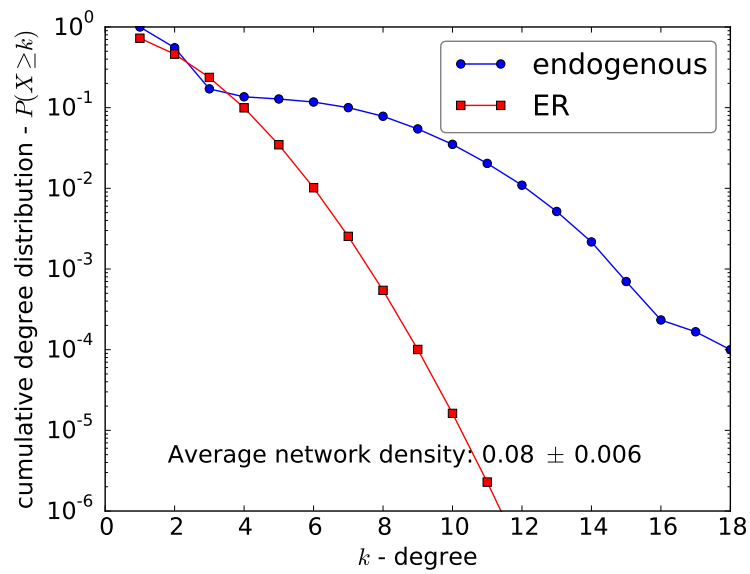


Figure 5.6: Degree distribution of ensemble of $n = 1000$ endogenously formed networks compared to ER networks with same average density.

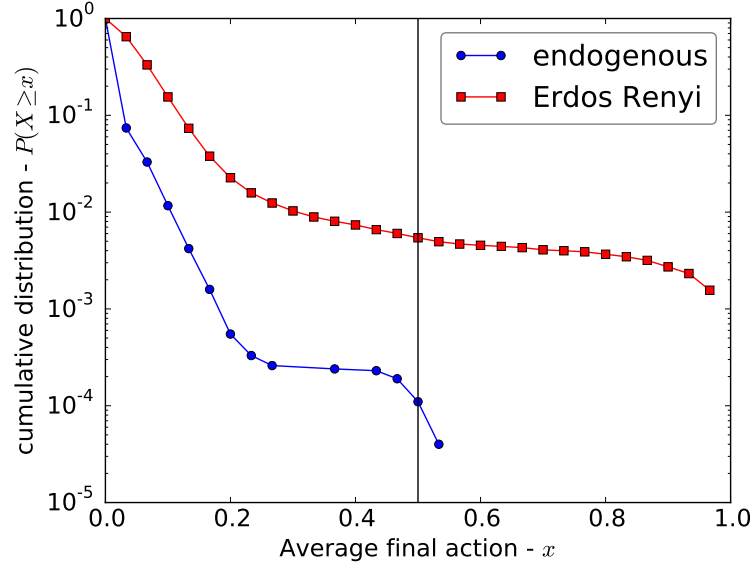


Figure 5.7: Distribution of final action in ER networks vs. endogenous networks. This is without bias, i.e. the initial action is random based on the private belief only.

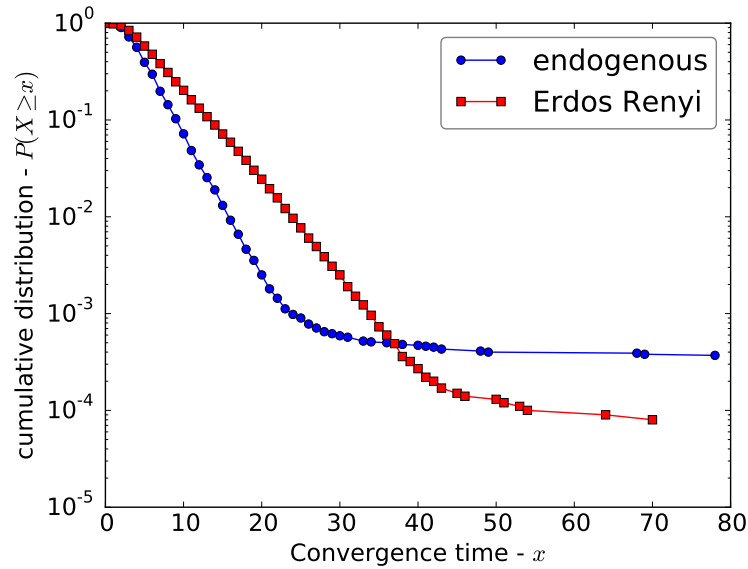


Figure 5.8: Distribution of convergence time ER networks vs. endogenous networks. This is without bias, i.e. the initial action is random based on the private belief only.

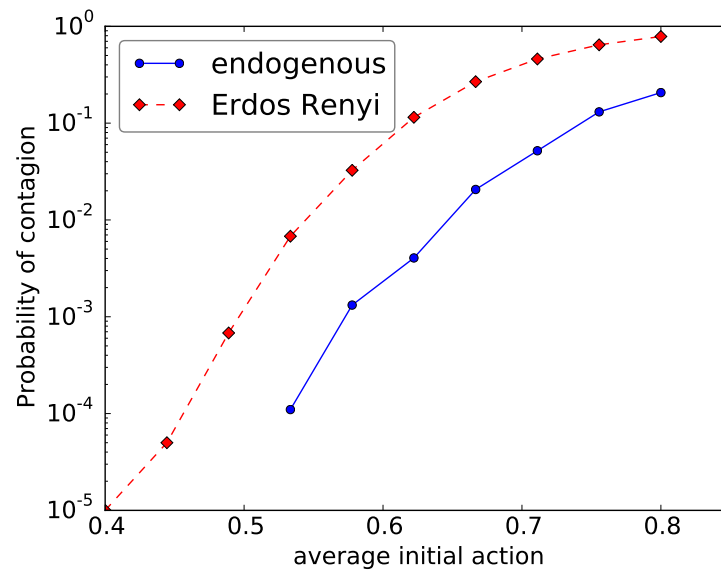


Figure 5.9: Probability of contagion vs. initialization bias. Missing values correspond to zero frequency.

5.C Appendix: Proofs

Proof of Proposition (5.1). We use the notation $f_P(p^i \mid \theta = 0)$ to indicate that the functional form of the probability distribution of the private belief p^i has been derived assuming that $\theta = 0$. Now, let $s(p^i)$ be the inverse of the private belief:

$$s(p^i) = \frac{\mu_0^2 - \mu_1^2 + 2\sigma^2 \log\left(\frac{1-p^i}{p^i}\right)}{2(\mu_0 - \mu_1)} \quad (5.32)$$

The distribution of the private belief can be computed as follows:

$$f_p(p^i) = \frac{\partial s(p^i)}{\partial p^i} f_s(s(p^i)). \quad (5.33)$$

where the probability density function for signal s is given as:

$$f_s(s) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(s - \mu_0)^2}{2\sigma^2}\right) \quad (5.34)$$

and the private belief $p^i(s)$ is given by Equation (5.7). Substituting in the expression for $s(p^i)$ and computing the partial derivative we obtain:

$$f_p(p^i \mid \theta = 0) = \frac{\sigma \exp\left(-\frac{((\mu_0 - \mu_1)^2 - 2\sigma^2 \log(\frac{1-p^i}{p^i}))^2}{8\sigma^2(\mu_0 - \mu_1)^2}\right)}{\sqrt{2\pi}(1-p)p(\mu_0 - \mu_1)}. \quad (5.35)$$

Assuming that $\theta = 1$ and using the corresponding density for the signal s , we obtain

$$f_p(p^i \mid \theta = 1) = \frac{\sigma \exp\left(-\frac{((\mu_0 - \mu_1)^2 + 2\sigma^2 \log(\frac{1-p^i}{p^i}))^2}{8\sigma^2(\mu_0 - \mu_1)^2}\right)}{\sqrt{2\pi}(1-p)p(\mu_0 - \mu_1)}. \quad (5.36)$$

Examples of $f_p(p^i \mid \theta = 0)$ are shown in Figure 5.10. We have $\mu_1 = 1 - \mu_0$ and $\sigma^2 = 0.1$. Note, that the majority of the probability density of the private belief is to the left of 0.5 in all cases. Therefore, the private signal tends to produce private beliefs that yield the state matching action. If we increase the signal informativeness $|\mu_0 - 0.5|$ the distribution becomes more skewed towards the actual state of the world. Hence the private belief becomes more informative.

Now that we have defined the pdf of the private belief we can compute the probability that the agent chooses $x^i = 0$ if the state of the world is $\theta = 0$ given some social belief $q = q^i$ as:

$$\Pr(x^i = 0 \mid q^i = q) = \int_0^{1-q} f_p(p^i \mid \theta = 0) dp^i, \quad (5.37)$$

where we use the notation $f_p(p^i \mid \theta = 0)$ to indicate that the functional form of f_p has been derived assuming that $\theta = 0$. Similarly, when the state of the world is $\theta = 1$ we have

$$\Pr(x^i = 1 \mid q^i = q) = \int_{1-q}^1 f_p(p^i \mid \theta = 1) dp^i. \quad (5.38)$$

Note that $f_p(p^i \mid \theta = 0)$ transforms into $f_p(p^i \mid \theta = 1)$ by reflection around $p^i = 0.5$ and vice versa, i.e. $f_p(p^i \mid \theta = 0) = f_p(1 - p^i \mid \theta = 1)$. We can introduce the change of variable $v^i = 1 - p^i$. Then

$$\int_0^{1-q} f_p(p^i \mid \theta = 0) dp^i = \int_q^1 f_p(1 - v^i \mid \theta = 0) dv^i = \int_q^1 f_p(p^i \mid \theta = 1) dp^i. \quad (5.39)$$

This implies that when $q = 0.5$ the probability of choosing a state matching action is the same irrespective of the state of the world. This result is important since, in Section 5.3, it allows us to compute the expected utility for an additional link irrespective of the underlying state of the world if we approximate an agent's neighbors' social belief by $q' = \frac{1}{2}$.

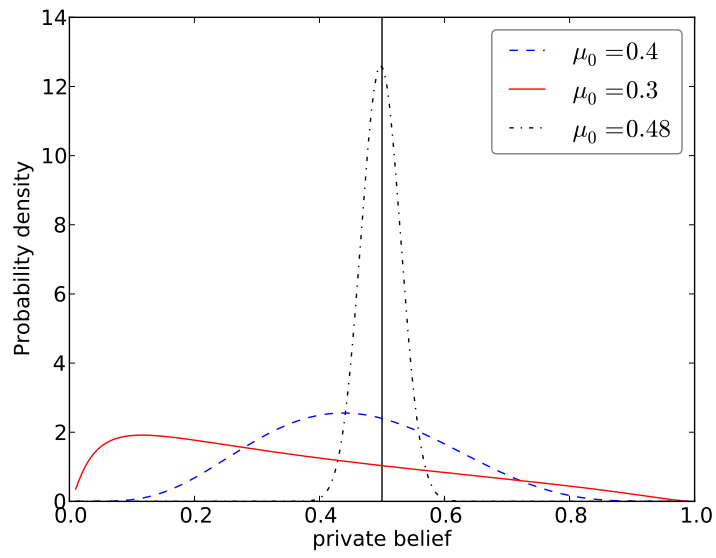


Figure 5.10: Probability density function of the private belief given that the state of the world is $\theta = 0$ $f_P(p \mid \theta = 0)$ for three values of $\mu_0 \in \{0.3, 0.4, 0.48\}$.

Chapter 6

Conclusion

In this thesis I have explored the notion of endogenous systemic risk from a theoretical perspective. For this, I have developed three models for the dynamics and structure of the interaction of agents in a financial system.

In order to study how learning and risk management can lead to systemic instabilities, I develop a simple model of a leveraged investor (bank for short) that uses historical returns on a risky asset to estimate her portfolio risk. The perceived risk then translates into a leverage requirement that the bank tries to reach through balance sheet adjustments. The price of the risky asset is determined by the bank's balance sheet adjustments as well as the investment decision of an unleveraged fundamentalist investor that is subject to exogenous random fluctuations. I show that, when the bank has sufficient market impact, the feedback between balance sheet adjustments, price impact and perceived risk can lead to endogenous irregular oscillations that I call Basel leverage cycles.

Using this model as a basis, I investigate whether certain leverage policies can mitigate the severity of the leverage cycle. In particular, I study risk based leverage policies that can be described by a cyclical parameter that interpolates between procyclical and countercyclical leverage policies. Procyclical policies require the bank to reduce its leverage when perceived risk is high while countercyclical policies allow the bank to increase its leverage in same situation. I show analytically and

numerically that both countercyclical and procyclical policies can be destabilizing. Taking the realized shortfall of the bank's equity as a measure of financial stability, I find that the optimal cyclical parameter depends critically on the relative effect sizes of the endogenous volatility due to the bank's balance sheet adjustments and the exogenous volatility due to the fundamentalist's investment perturbations. If exogenous volatility is weak and endogenous volatility strong, the leverage policy that maximizes financial stability is simply a constant leverage ratio. By contrast, if exogenous volatility is strong and endogenous volatility is weak, a Value-at-Risk constraint policy is optimal.

My theoretical analysis primarily yields qualitative results for the limiting cases in which volatility is either purely exogenous or purely endogenous. Of course, in reality volatility in financial markets is both driven by exogenous events as well as the endogenous interaction of the market participants. However, the split between these two effects is latent and it is therefore difficult to back out these properties from observed time series of asset returns. Thus, any quantitative statement about an optimal cyclical parameter comes with considerable uncertainty attached. Nevertheless the qualitative conclusion from my analysis remains valid despite this uncertainty: bank risk management amplifies exogenous volatility and can lead to substantial, unintended, endogenous volatility which has to be considered when designing effective macroprudential risk management policies.

In the second part of my thesis I focus on different types of contagion in financial networks. First, I study how intra-institutional linkages can amplify financial contagion when financial institutions are active in multiple over-the-counter markets. In particular, spillover within a diversified financial institution allows for contagion from one over-the-counter market to another. Under certain circumstances, the presence of intra-institutional spillover can lead to the amplification of small shocks to the extent that trading across all markets collapses abruptly. This is not the case

when there is only contagion within a market but not between markets. However, this finding depends critically on assumptions about the structure of the different over-the-counter markets. In particular, when the networks of trading relationships in the markets for different assets are similar, the amplification effect is weak. Therefore, while this chapter provides some interesting theoretical insight into multi-asset contagion processes, further empirical research on the structure of over-the-counter markets is necessary to determine whether the effect observed here is relevant in real financial networks.

Finally, I develop a simple model of social learning in the context of a financial network. Banks must make a binary investment choice and depending on their choice, as well as the state of the world, they either receive a positive payoff or nothing. Therefore, in order to maximize their expected utility, banks must learn about the current state of the world. They do this by observing an informative exogenous signal and the previous actions of their peers. The fact that banks respond to the actions of their neighbors permits an outcome in which all banks coordinate on a state non-matching action, despite having access to an informative exogenous signal. I refer to this outcome as contagious synchronization. Another contribution of this chapter is the development of a simple, boundedly rational, network formation mechanism based on the expected marginal utility of adding a particular neighbor. The network formation mechanism becomes computationally feasible by abstracting from higher order network interactions in the computation of marginal utility. I show that, provided banks form networks endogenously, contagious synchronization can become less likely.

One possible avenue for future research would be to better integrate dynamic and network models of systemic risk and thereby combine two essential elements of endogenous risk. As illustrated in this thesis, both dynamic and structural aspects of financial markets can lead to substantial levels of systemic risk. Yet, these two

aspects are not independent and an increased integration of these modeling aspects would certainly lead to more realistic models. Interesting advances in this area with a focus on fire sales have already been made in Cont and Wagalath (2014). Hence, a promising direction for future research would be to build on Cont and Wagalath (2014) and others, e.g. Caccioli et al. (2012), extend the leverage cycle model presented in chapter 3 to multiple assets and banks, and explicitly consider the effect of contagion across asset markets.

As mentioned at the outset, in this thesis I have studied endogenous risk primarily from a theoretical perspective. However, often a tighter link between the available data and models is required to be able to derive quantitative conclusions from a theory. This can be challenging, since in many cases the determining quantities are either latent (e.g. exogenous versus exogenous volatility) or very costly to observe (e.g. the structure of over-the-counter markets). Nevertheless, I believe that building theory, with the available data in mind, will be crucial in the advance of the study of endogenous systemic risk. For me, this will be both a challenge and guiding principle for my future work.

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