

MOTIVIC POISSON SUMMATION

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ABSTRACT. We develop a "motivic integration" version of the Poisson summation formula for function fields, with values in the Grothendieck ring of definable exponential sums. We also study division algebras over the function field, and obtain relations among the motivic Fourier transforms of a test function at different completions. We use these to prove, in a special case, a motivic version of a theorem of [DKV].

1. INTRODUCTION

The first order theory of valued fields associated with number theory has received a great deal of attention in the past half-century. A region of mystery remains around local fields of positive characteristic, but by and large local fields and associated geometric structures, are decidable and accessible to model-theoretic tools; in the hands of Denef this has been useful in the study of p -adic integration, later leading to the motivic integration of Kontsevich, Denef-Loeser and others. By Feferman-Vaught methods ([3], 6.2), one can similarly understand products of local fields or of their rings of integers; the underlying rings of the adèles are thus decidable; but without access to the discrete global field embeded in the adèles, this permits rather limited contact with the global geometry. No known decidable theory captures such a discrete embedding. The closest approach is [16], that can be understood as the theory of the non-archimedean adèles over \mathbb{Q}^a , with an embedded copy of \mathbb{Q}^a ; but as the authors make clear, it works precisely because of the cut-and-paste property, i.e. the absence of any global constraints. Every global field or adelic construction whose first-order theory is understood at all, is known to be undecidable.

Based on this evidence, one might guess that the line of decidability, for fields associated with number theory, coincides with the local/global distinction. The history of number theory, however, shows no such line at all; adelic methods are no less geometric than local ones, and for two hundred years have consistently decided relevant problems. We are not able to resolve the tension between these different conclusions in the present paper, but we try to reduce it a little. We study function fields and their associated adèles. We embed the function field only piecewise, as an Ind-definable object, and do not permit quantification over it. But in this setting we are able to interpret the Poisson summation formula motivically, leading to connections between Denef-Loeser motivic integrals over distinct local fields.

The term 'motivic' in this paper is used in its sense in the context motivic integration, to say that numbers are replaced by elements of the Grothendieck ring of varieties, and closely associated rings. We discuss such rings in §2 and §3; in particular we define the ring of exponential sums K_e , and a localization $K_e[Gr^{-1}]$, allowing division by certain classes of group varieties. In §4 we recall the motivic Fourier transform, in the very simple context of test functions that we need.

In §5, we define motivic global test functions, and the motivic analogue δ^K of the functional summing a test function over rational points of a function field. §6 sets up a first order context

useful for “everywhere-local” definitions. In particular we will be interested in *integral conjugacy classes* in a division algebra D over a function field $\mathbb{f}(t)$. For each place v we define a subring R_v of D ; we say two elements are locally integrally conjugate at v if they are conjugate by some element of R_v^* (possibly after base change), and integrally conjugate if they are locally integrally conjugate at every place v . Such sets are conveniently defined in our setting; their $\mathbb{f}(t)$ -rational points forms a constructible set over \mathbb{f} .

In §8 we compute explicitly the constructible set of rational points in an integral conjugacy class. We restrict attention to division algebras of prime degree n . Let $c \in D$ and let E be the subfield of D consisting of elements commuting with c . Then $E = \mathbb{f}(C)$ for some curve C . Let O_c be the integral conjugacy class of c . The value of $\delta^K(O_c)$ closely related to a certain Rosenlicht generalized Jacobian of the curve C , with ramification data connected with the integral structure.

More precisely, the value we obtain is connected with (a generalized) $Pic^0(C)$. Now when $C(\mathbb{f})$ has no rational point, the functor $Pic^0(C)$ is not represented by the Jacobian; in fact the functor $k \mapsto Pic^0(C \times_{\mathbb{f}} k)$ is not representable by a variety at all ([15]). Nevertheless after some discussion of quotients in 2.2, we manage to associate to $Pic^0(C)$ a class in a Grothendieck ring, treating it directly as an adelic quotient $T(\mathcal{O}) \backslash T(\mathbb{A}) / T(k(t))$. The class becomes equal to the class of the Jacobian over any field extension with a point of C . At all events, $\delta^K(O_c)$ is expressed in terms mentioning only a commutative subalgebra of D .

We test our method on a problem involving division rings. We consider two division algebras D, \dot{D} over $\mathbb{f}(t)$ associated with two distinct elements of a given cyclic Galois group of prime order n over \mathbb{f} . (§7). We work with a quotient K of $K_e[Gr^{-1}]$ appropriate for studying D or \dot{D} as a division algebra; namely, we factor out the class $[\epsilon_L]$ of a certain zero-dimensional variety ϵ_L that has no rational points in any field \mathbb{f}' such that D is a division ring over \mathbb{f}' . Consider local motivic test functions ϕ on D over $\mathbb{f}((t))$ that are invariant under conjugation by D^* . We will explain below how to match such functions on D^* with their homologues on \dot{D} (Definition 11.3); the conjugacy classes can be identified.

Theorem 1.1. *Let $\phi, \dot{\phi}$ be matching local motivic test functions on D, \dot{D} over $\mathbb{f}((t))$, with values in $K_e[Gr^{-1}]$. Then their Fourier transforms $\mathfrak{F}\phi, \mathfrak{F}\dot{\phi}$ also match.*

This is closely related (when specialized to finite $\mathbb{f} = \mathbb{F}_q$ and to numerically valued test functions) to results of [13]. In [13] the irreducible representations of each division algebra are shown to correspond to certain irreducible representations of GL_n ; as a consequence they correspond bijectively between two cyclic division algebras of the same dimension. Equivalently, the multiplicative convolution algebras of conjugation invariant test functions on the two algebras are canonically isomorphic. Such results are purely local, but appear to be very difficult to prove using local methods in high dimensions (in degree two this is done by Jacquet-Langlands.) See Appendix 3 for a relation between the multiplicative and additive convolution algebras.

The proof of Theorem 1.1 proceeds by expressing the local Fourier transform of ϕ at the place 0 in terms of the local Fourier transform at the place 1, and some global terms, measuring rational points on integral conjugacy classes (§ 9). The global term was shown to depend only on commutative subalgebras of D, \dot{D} ; these are canonically isomorphic. The matching of Fourier transforms over $\mathbb{f}((t))$ is thus reduced to the same question over $\mathbb{f}((t-1))$, where it is evident, since over $\mathbb{f}((t-1))$ the two algebras are isomorphic.

2. SOME GROTHENDIECK RING OPERATIONS

Here T may be any theory. We say “definable” for “definable by a quantifier-free formula in the language of T .” This shorthand is acceptable notationally since our main application is to

$T = ACF$, a theory with quantifier-elimination. In reality the quantifier elimination provides little help, since it is not reflected in the Grothendieck ring, and much of our effort is directed at staying with quantifier-free formulas; see Remark 2.20. Similarly we will assume that any definable function f is piecewise given by terms.

Let T_\forall be the universal part of T ; so that $A \models T_\forall$ iff A embeds into some $M \models T$.

Let $K(T)$ be the Grothendieck ring of definable sets, and let K be any $K(T)$ -algebra.

For any $A \models T_\forall$, we have a Grothendieck ring $K_A := K(T_A)$. So the Grothendieck ring is really a functor from models of T_\forall and embeddings among them, to the category of rings. If necessary we will denote the class of a definable set X in K_A by $[X]_A$. But usually we write $[X]$ for $[X]_A$, and write $[X] = [Y] \in K_A$ to express: $[X]_A = [Y]_A$. We also write K_b for K_A if b generates A .

This point of view is essential in discussing definable functions into K (cf. [11]). Let $f : X \rightarrow Y$ be a definable function, X, Y definable sets. We view the map

$$y \mapsto [f^{-1}(y)]$$

as a function on Y , and let $Fn(X, K)$ denote the family of all such functions. But it must be interpreted as follows: for any $A \models T_\forall$, we obtain a function $Y(A) \rightarrow K_A$, namely $b \mapsto [f^{-1}(b)]$.

We have the presheaf-like property:

2.1. *If $[X_b] = [X'_b] \in K_b$ for any $b \in Y$, then $[X] = [X'] \in K$.*

See [11].

Since we will consider various localizations and quotients of $K(T)$, it will be useful to discuss $K(T)$ -algebras in general.

Consider functors $A \mapsto R_A$ from models of T_\forall , together with natural transformations $K(T) \rightarrow R$. Given $A \models T_\forall$ and a T_A -definable set X , the image of $[X]_A$ in R_A is denoted $[X]_A^R$. We assume that 2.1 holds for R , i.e. if $f : X \rightarrow Y$ is T_A -definable, and if $[X_b]_b^R = [X'_b]_b^R$ for any $b \in Y$, then $[X]_A^R = [X']_A^R$. Call such functors Grothendieck algebras over T .

2.2. Localizing by a definable family. Let \mathcal{N} be an Ind-definable family of definable sets. Assume \mathcal{N} is closed under products. Then $\{[X] : X \in \mathcal{N}\}$ is a multiplicative subset of the Grothendieck ring.

Let K be a Grothendieck algebra for T . To define the localization of K by \mathcal{N} , consider the family of all Grothendieck algebras R , such that if $A \models T_\forall$ and $X \in \mathcal{N}(A)$ then $[X]_A^R$ is invertible in R_A . We let $K[\mathcal{N}^{-1}]$ be the universal object of \mathcal{C} .

It is natural to assume that each set in \mathcal{N} has a definable element. In this case, in any finite structure \mathfrak{f} , the number of points of $X \in \mathcal{N}$ is positive; so any zeta function defined on K factors through the localization $K[\mathcal{N}^{-1}]$. In the application the elements of \mathcal{N} will have a distinguished element 1, and indeed will be essentially classes of definable groups.

In practice we will only use the following consequence of the existence of inverses:

2.3. Let $(A_y : y \in Y) \subseteq \mathcal{N}$, let $(X_y), (X'_y)$ be two families of definable sets, and assume:

$$\begin{aligned} [A_y][X_y] &= [X_y]^2, & [A_y][X'_y] &= [X'_y]^2 \\ [X_y]^2 &= [X_y][X'_y] & &= [X'_y]^2 \end{aligned}$$

for $y \in Y$. Then

$$\sum_{y \in Y} [X_y] = \sum_{y \in Y} [X'_y].$$

Here is a proof that the relation holds if division by $[A_y]$ is possible, i.e. elements $e_y = [X_y]/[A_y]$, $e'_y = [X'_y]/[A_y]$ exist with $e_y[A_y] = [X_y]$, $e'_y[A_y] = [X'_y]$. Then $e_y = e_y^2 = e_y e'_y = (e'_y)^2 = e'_y$. So

$$\sum_{y \in Y} [X_y] = \sum_{y \in Y} e_y [A_y] = \sum_{y \in Y} e'_y [A_y] = \sum_{y \in Y} [X'_y]$$

Remark 2.4. *When A is a group, the relation $[A][X] = [X]^2$ is typical of principal homogeneous spaces X . For two torsors X, X' , the relation $[X][X'] = [X]^2 = [X']^2$ holds if X, X' generate the same subgroup of the Galois cohomology group, so that over any field extension, one represents the zero class iff the other does.*

Note that localization (even by families) commutes with quotients. Also note that $K[N^{-1}] = K[N^{-1}][GN^{-1}]$ canonically.

2.5. Representable quotients. Recall the notion of “piecewise definable” or Ind-definable from Appendix 1. In terms of saturated models M , an Ind-definable set is a union $\cup_{i \in I} X_i(M)$ of definable sets $X_i(M)$; where I is an index set, small compared to M .

Let E be an Ind-definable equivalence relation on an Ind-definable set V

Define a *weakly representative set* for (V, E) to be a definable set Y such that for some definable X and $f : X \rightarrow V$ and surjective $g : X \rightarrow Y$, every element of V is E -equivalent to some element $f(x)$, and $g(x) = g(x')$ iff $f(x)Ef(x')$.

We require that g is surjective only in the geometric sense, i.e. in models of T . For $b \in Y$ we let V_b be the E -equivalence class of $f(a)$, for any a with $g(a) = b$.

If Y, Y' are two weakly representative sets for (V, E) , then Y, Y' are definably isomorphic; moreover the isomorphism $y \mapsto y'$ is such that $V_y = V_{y'}$. We write $Y = V/E$.

If V is a definable group and E is the equivalence relation corresponding to a normal subgroup, then Y carries a definable group structure.

Lemma 2.6. *(V, E) is weakly representable iff*

- (1) *There exists a definable X and $f : X \rightarrow V$ such that any element of V is E -equivalent to some element of $f(X)$.*
- (2) *For any such f, X , the equivalence relation $f^{-1}E$ is a definable relation on X .*

Proof. Clear. □

We say that (V, E) admits a *set of unique representatives* if above one can choose $X = Y, g = Id_X$. Only in this case can we be sure of a bijection $V(f)/E \rightarrow Y(f)$.

More generally, let S be a subset of K , closed under multiplication. We say that (V, E) is S -representable if it is weakly representable by (X, Y, f, g) as above, and for some definable set Z with $[Z] \in S$, whenever (U, Y, f_1, g_1) is another weak representation of (V, E) , $X_b = g^{-1}(b)$, $U_b = g_1^{-1}(b)$, we have: $[U_b][Z] = [U_b][X_b] \in K(T_b)$ for any $b \in Y$.¹

The case we have in mind is with Z a definable group, and X_b a Z -torsor. Assume X_b becomes trivial over any point of U_b . Then we have an isomorphism $U_b \times Z \rightarrow U_b \times X_b$, over U_b .

We assign to (V, E) the class $[X]/[Z]$ in the localization $K[S^{-1}]$, and denote it by $[V : E]$. If E is the orbit equivalence relation of the action of a group H on V , we also write $[V : H]$.

If (V, E) is S -representable, then in particular (V, E) is weakly represented, so V/E is defined as well as $[V : E]$. But the image of $[V/E]$ in $K[S^{-1}]$ is not necessarily equal to $[V : E]$.

¹The weaker statement $[U \times Z] = [U \times_Y X]$ appears to suffice for our purposes.

Lemma 2.7. $[V : E]$ is well-defined. I.e. if (X', Y', Z', f', g', h') is another S -representation, then $[X]/[Z] = [X']/[Z']$ in $K[S^{-1}]$.

If $V = V_1 \dot{\cup} V_2$, $E_i = E|_{V_i}$, and (V_i, E_i) is S -representable, then so is (V, E) , and $[V : E] = [V_1 : E_1] + [V_2 : E_2]$.

Proof. $[X' \times Z] = [X' \times_Y X] = [X \times_Y X'] = [X \times Z']$ so dividing by $[Z \times Z']$ we find: $[X']/[Z'] = [X]/[Z]$.

If (V, E) is S -representable via (X, Y, f, g, Z) , then for any $Z' \in S$ it is also S -representable via $(X \times Z', f, g, Z \times Z')$. Hence by taking common denominators we may assume (V_i, E_i) is S -representable via the same denominator Z . In this case the statement on addition is immediate. \square

If $S \subset K$ is not closed under multiplication, we let $\langle S \rangle$ be the set of products of elements of S , and define S -representable to mean $\langle S \rangle$ -representable.

Lemma 2.8. Let Z be a definable group. Assume (V, E) is weakly representable, and let $Y = V/E$. Assume: for any $b \in Y$, there exists an $f(b)$ -definable Z -torsor R_b such that there exists a definable function $f_b : R_b \rightarrow V_b$, and for any f' with $f(b) \leq f'$ and $V_b(f') \neq \emptyset$, we have $R_b(f') \neq \emptyset$.

Then (V, E) is $[Z]$ -representable.

Proof. Say (V, E) is weakly representable via (X, f, g, Y) . Let $X_b = g^{-1}(b)$. By assumption, for any $b \in Y$ there exists an $f(b)$ -definable Z -torsor R_b , $f_b : R_b \rightarrow V_b$, and an $f(b)$ -definable map $h_b : X_b \rightarrow R_b$. By compactness and glueing, we can take (R_b, f_b, h_b) to be uniformly definable.

Let (U, Y, f_1, g_1) be another weak representation of (V, E) , $X_b = g^{-1}(b)$, $U_b = g_1^{-1}(b)$. Then $[U_b][Z] = [U_b][X_b] \in K(T_b)$. Indeed for any $c \in U_b$, using the point $h_b(c)$ we find a bijection $j_b : Z \rightarrow X_b$; these can be glued to give a bijection $U_b \times Z \rightarrow U_b \times X_b$, over the identity on U_b . \square

Let V is Ind-definable, E an Ind-definable equivalence relation on V , and X a definable set. By a definable function $X \rightarrow V/E$ we mean is definable relation $F \subset X \times V$ whose projection p to X is surjective and 1-1 modulo E (i.e. if $(x, v), (x, v') \in F$ then $(v, v') \in E$.) In this case if $x \in X$, we let $f(x)$ be the E -class of v for any v with $(x, v) \in F$.

We say that E is definable-in-definable families if for any definable $U \subset V$, the restriction of E to U is definable. In this case the quotient V/E is Ind-definable.

Remark 2.9. Let $f : X \rightarrow Y$ be a definable map between definable sets,. Let E be the equivalence relation $f(x) = f(y)$. If $[X : E]$ is defined, write $f_*[X] = [X : E]$. For a given structure A we may be interested in $f(X(A))$, but within the Grothendieck ring of quantifier-free formulas, or of formulas up to T -equivalence when A is not a model of T , we have no direct way to describe it. The class $f_*[X]$, when defined, offers a substitute.

2.10. Essentially representable sets. Let T^* be a universal theory containing the set T_V of universal consequences of T . An Ind-definable set (of T) is called *formally empty* if it is the union of definable sets U such that $T^* \models U = \emptyset$. A structure f' is said to be *negligible* if $W(f') \neq \emptyset$ for some formally empty W , or equivalently if $f' \not\models T^*$.

Let $K^* = K/I$, where $I = \{[X] \in K : T^* \models X = \emptyset\}$.

Let X be an Ind-definable set; write $X = \cup_i X_i$ with X_i definable. We say that X is T^* -*limited* if for some finite $I_0 \subseteq I$, letting $X_0 = \cup_{i \in I_0} X_i$, we have for all j , $T^* \models X_j \subseteq X_0$. We write $T^* \models X = X_0$ for short. In this case let $[X]$ be the image of X_0 in K^* . The definition clearly does not depend on the choice of representation $\cup_i X_i$ or on the finite set I_0 .

We say that V/E is (T^*, S) -representable if there exists an Ind-definable $V' \subseteq V$ and a formally empty Ind-definable V'' with $V = V' \cup V''$, and such that V'/E is S -representable. The image of $[V' : E]$ in $K^*[S^{-1}]$ is then well-defined, and denoted $[V : E]$.

A further variant of Lemma 2.8 will be useful.

Lemma 2.11. *Let E an Ind-definable equivalence relation on the Ind-definable set V . Let Z be a definable group acting freely on a T^* -limited set X , and $f : X \rightarrow V/E$ an Ind-definable function whose fibers are Z -orbits. Assume: for any $\mathfrak{f}' \geq \mathfrak{f}$ and any \mathfrak{f}' -definable Z -orbit U , $\cup_{c \in X(\mathfrak{f}')} f(c) \subseteq \cup_{c \in V(\mathfrak{f}')} cE$, where cE is the E -class of c ; with equality if $\mathfrak{f}' \models T^*$. Then V/E is (T^*, Z) -representable, and $[V : E] = [X]/[Z]$.*

Proof. We may express X as a direct limit of definable sets X_i . We have $X_i \subseteq ZX_i \subseteq X_{i'}$ for some $i' \geq i$. Replacing X_i by ZX_i , we may assume the X_i are Z -sets. Since X is T^* -limited, for some X_0 we have $X_i \setminus X_0$ formally empty, for all $i \geq 0$. Let $f_0 = f|_{X_0}$. Then for any $\mathfrak{f}' \geq \mathfrak{f}$ and any \mathfrak{f}' -definable Z -orbit $U \subseteq X_0$, $\cup_{c \in X(\mathfrak{f}')} f_0(c) \subseteq \cup_{c \in V(\mathfrak{f}')} cE$. Moreover if $\mathfrak{f}' \models T^*$, $c \in V(\mathfrak{f}')$ and $y \in cE$, then $y \in f(d)$ for some $d \in X(\mathfrak{f}')$. Since $X_i \setminus X_0$ is formally empty and $\mathfrak{f}' \models T^*$, we have $d \notin (X_i \setminus X_0)$ so $d \in X_0(\mathfrak{f}')$. This shows that the hypotheses hold for X_0, f_0 . Since $[X] = [X_0]$ by definition, we are reduced to this case. We may thus assume that X is a definable set.

For any $c \in X$, there exists $b \in \mathfrak{f}(c)$ such that $b \in f(c)$. By compactness, there exists a definable function $g : X \rightarrow V$ with $g(c) \in f(c)$.

We have $g(x)Eg(x')$ iff $f(x) = f(x')$ iff $x, x' \in X'$ are Z -conjugate. Let $V' = g(X)$; then V' is a definable subset of V , V'/E is weakly representable, and $V'/E \cong X/Z$.

For $b \in V$, if $\mathfrak{f}(b) \models T^*$ then $b \in f(c)$ for some $c \in X(\mathfrak{f}(b))$, so that $bEg(c)$. By compactness there exists $V'' \subseteq V$ such that $vEg(v)$ for $v \in V''$, and $V \setminus V''$ is formally empty.

The proof is now completed as in the 2nd paragraph of the proof of Lemma 2.8. \square

Remark 2.12. If, in Lemma 2.11, Z does not act freely, but the stabilizer of each point in X is one of finitely many groups $H_i \leq Z$, $Z_i = Z/H_i$, then V/E is essentially $\prod_{i=1}^k Z_i$ -representable, and $[V : E] = \sum_{i=1}^k [X_i]/[Z_i]$, where X_i is the union of the f -classes that are Z_i -orbits.

2.13. Absolute elements and invariant functions. A set S of elements of K is called *absolute* if whenever $s, s' \in S$ are defined and distinct over \mathfrak{f}' , and $\mathfrak{f}' \leq \mathfrak{f}''$, then the images of s, s' in $K_{\mathfrak{f}''}$ remain distinct. A function $\phi : E \rightarrow K$ is said to take absolute values on E if $\{\phi(e) : e \in E\}$ is absolute. For instance, $\{0, 1, [k], [k^2], \dots\}$ forms an absolute set in $K(ACF)$.

Let G be a definable group, with a definable action on a definable set D , fibered over ObG . Let $\phi : D \rightarrow K$ be a definable function.

Definition 2.14. ϕ is G -invariant if for $c \in D, g \in G$ we have $\phi(c) = \phi(gc) \in K_{c,g}$. We say ϕ is strongly G -invariant if for any such c, g we have $\phi(c) = \phi(gc) \in K_{c,gc}$.

The same definition could be made for a groupoid G . In this case for each object $a \in ObG$ we have a set D_a , and for each pair $a, b \in ObG$ we have a definable function $Mor_G(a, b) \times D_a \rightarrow D_b$, such that the obvious associativity relations hold. Given a family (ϕ_a) of functions $D_a \rightarrow K$, we have a definition of (strong) invariance as above. We will only use the case of two objects, with two corresponding definable division algebras D, \dot{D} ; we will have $G = Aut(D)$ as a division algebra, $\dot{G} = Aut(\dot{D})$, and also $M = Iso(D, \dot{D})$. A pair $(\phi, \dot{\phi})$ of functions on D, \dot{D} is then said to be (strongly) *matching* if it is invariant under the groupoid.

Lemma 2.15. *Let $\phi : D \rightarrow K$ be a definable function taking absolute values. If ϕ is invariant, then it is strongly invariant.*

Proof. Clear. \square

While G -invariance depends on the group or groupoid action, the notion of strong G -invariance depends only on the equivalence relation conj_G of G -conjugacy on D . For an equivalence relation E on D , say $\phi : D \rightarrow K$ is E -invariant if $\phi(c) = \phi(c') \in K_{c,c'}$ whenever $(c, c') \in E$. Then strong G -invariance is the same as conj_G -invariance. A still stronger notion is *descent to D/E* , i.e. existence of a definable $\bar{\phi} : D/E \rightarrow K$ with $\phi(d) = \bar{\phi}(d/E) \in K_d$. We have however:

Lemma 2.16. *Let E be an equivalence relation on D . Assume $\phi : D \rightarrow K$ is E -invariant. Let \mathcal{F} be the set of equivalence classes of E , and let $\tilde{K} = K[\mathcal{F}^{-1}]$ be the localization, § ?? . Let $\tilde{\phi} : D \rightarrow K'$ be the induced map. Then $\tilde{\phi}$ descends to D/E .*

Proof. For an equivalence class y of E , define $\bar{\phi}(y) = [y]^{-1} \sum_{d \in y} \phi(d)$. If $d \in y$, then by E -invariance we have $\phi(d') = \phi(d) \in K_{d,d'}$ for any $d' \in y$, so $\sum_{d' \in y} \phi(d') = [y]\phi(d) \in K_d$. It follows that $\phi(d) = \bar{\phi}(y) \in \tilde{K}_d$ for any $d \in y$. \square

2.17. Proof by cases. Let T be a theory of fields, with base field \mathfrak{f} ; let \mathfrak{l} be a finite Galois extension of this base field; so $\mathfrak{l} = L(\mathfrak{f})$ for some commutative definable algebra L . Note that for any field extension \mathfrak{f}' of \mathfrak{f} , \mathfrak{f}' contains a copy of \mathfrak{l} over \mathfrak{f} iff $L(\mathfrak{f}')$ is not a field. At all events $L(\mathfrak{f})$ has points - it is a $\dim(L)$ -dimensional extension of \mathfrak{f} - and should not be convused with the isomorphic copy \mathfrak{l} .

We explain how an identity in the semi-group can be proved by cases, according to whether L splits or not in extension fields of \mathfrak{f} . Similar considerations apply to definable finite sets in any theory.

Let I_L be a normal basis for \mathfrak{l} over \mathfrak{f} . Then I_L can be viewed as a finite definable set, so it has a class $[I_L] \in K(T)$. If I'_L is another normal basis, there exists a bijection $f : I_L \rightarrow I'_L$ left invariant by the Galois action, and hence definable; so $[I_L] = [I'_L]$. Thus the class $[I_L]$ depends only on L . Moreover $\epsilon_L := \frac{1}{n}[I_L]$ is an idempotent in $K[\frac{1}{n}]$.

Let K be any $K_+(T)$ -semi-algebra, such that:

- (1) n has a multiplicative inverse.
- (2) $\epsilon_L = 0$.

Lemma 2.18. *If $V(\mathfrak{f}') = 0$ for any \mathfrak{f}' with $L(\mathfrak{f}')$ a field, then $[V] = 0 \in K$.*

Proof. Assume the condition holds. Then whenever $b \in V$, $L(\mathfrak{f}(b))$ is not a field; hence $L(\mathfrak{f}(b)) \cong \mathfrak{f}(b)^n$; so $I_L \cong n$ over $\mathfrak{f}(b)$. (where n denotes a set of n definable points.) Hence $V \times I_L \cong V \times n$. Dividing by n we obtain $[V] = [V]\epsilon_L = 0$. \square

Typically K will be a $K(ACF_{\mathfrak{f}})$ -algebra, obtained as a localization of the Grothendieck ring $K(ACF_{\mathfrak{f}})$ or $K_e(ACF_{\mathfrak{f}})$, and factoring out ϵ_L .

Lemma 2.19. *Let K be any $K(ACF_{\mathfrak{f}})[n^{-1}]$ - algebra. Assume $[V_1] = [V_2]$ holds in $K_{\mathfrak{l}}$ and also in K/ϵ_L . Then $[V_1] = [V_2]$ in K .*

Proof. An element of $K_{\mathfrak{l}}$ is represented by an element X of K and a definable function $g : X \rightarrow I_L$. We saw that $[X] = [X][I_L]/n = [X]\epsilon$, i.e. $[X] \in K(1 - \epsilon)$. It is easy to see that $K_{\mathfrak{l}}$ is isomorphic to the ring $K\epsilon$ with unit ϵ , or equivalently to $K/(1 - \epsilon)$. Now if a class vanishes modulo ϵ and modulo $(1 - \epsilon)$, then it vanishes. \square

Remark 2.20. Let TF be the theory of perfect fields. Let $K(TF)$ be the Grothendieck ring of all formulas, including quantified ones, and including imaginary sorts. Let $K_c(PF)$ be the quotient of $K(TF)$ obtained by imposing a Cavalieri principle: if $f : X \rightarrow Y$ is a definable function, and each fiber is provably isomorphic to Z , then $[X] = [Z]$. Theorem 1.1

admits a considerably simpler proof if values are taken in this ring; to begin with, all issues regarding representability of quotients become superfluous. The extra effort required in using the quantifier-free Grothendieck ring is hopefully paid off in geometrically more precise answers.

3. THE GROTHENDIECK RING OF EXPONENTIAL SUMS

Let T be a theory of fields. In particular, T includes constants for a subfield F . It is possible to allow additional relations, but in the present paper we will not use them so one can take the language to be the language of F -algebras. The field sort is denoted k . We will assume that the models of T are perfect fields.

In some parts of the paper we will assume the existence of division algebras over $k(C)$, where C is a curve over k ; in particular k is not algebraically closed. Nonetheless constructions that do not require this assumptions are better carried out geometrically. Thus for instance we will define the Fourier transform of a test function, and the “sum over rational points” functional, over the theory ACF of algebraically closed fields. We define below a natural homomorphism from the Grothendieck ring of exponential sums over ACF, to the Grothendieck ring of exponential sums over T ; any equations holding at the ACF level will thus continue to hold.

We define the Grothendieck ring of exponential sums using generators and relations.

The generators are elements $[X, h]$ where X is a definable set, and $h : X \rightarrow k$ a definable function.

We write $\psi(c) = [\{c\}, Id]$; we think of ψ as an additive character, and think of $[X, h]$ as representing $\sum_{x \in X} \psi(h(x))$. We will impose the following relations:

$$(1) \quad [X, h][Y, g] = [X \times Y, h(x) + g(y)]; [0, 0] = 1$$

If X, Y are disjoint

$$(2) \quad [X, h] + [Y, g] = [X \cup Y, h \cup g]$$

if $g : X \rightarrow Y$ is a definable bijection,

$$(3) \quad [Y, h] = [X, h \circ g]$$

$$(4) \quad [k, Id] = 0$$

Let $K_{exp}(T)$ be the ring presented by the generators $[X, h]$ and relations (1)- (4). Define $\mathbb{L} = [k, 0]$. $K_{exp}(T)$ is naturally filtered by dimension: $F_d K_{exp}(T) = \{[X, h] : \dim(X) \leq d\}$. Let $K_e(T) = K_{exp}(T)[\mathbb{L}^{-1}]$.

Lemma 3.1. *Let $(u, x) \mapsto u + x$ be a definable action of $(k, +)$ on a definable set X , and let $h(t + x) = t + h(x)$. Then $[X, h] = 0$ in $K_e(T)$.*

Proof. For any $c \in k$, and X, h as above, we have

$$[X, h]\psi(c) = [X, h][\{c\}, Id] = [X \times \{c\}, h(x) + c] = [X, h]$$

The last equality uses (3), for the bijection $X \times \{c\} \rightarrow X$ given by the action of c . Thus $[X, h](\psi(c) - 1) = 0$. Summing over all c and using $\sum_{c \in k} \psi(c) = [k, Id] = 0$ we obtain $[X, h](0 - [k]) = -\mathbb{L}[X, h]$. Since \mathbb{L} is invertible in $K_e(T)$ the result follows. \square

It will sometimes be useful to consider the semiring with the same generators as $K_{exp}(T)$, relations (1)-(4), and in addition the relations $[X, h] = 0$ for $[X, h]$ as in 3.1. We will refer to these relations as 4’.

Remark 3.2. Let $K'_{exp}(T)$ be the ring presented by the generators $[X, h]$ and relations (1)-(3). Then (4) holds in $K'_{exp}(T)[(\psi(c) - c)^{-1}]$, for any definable element c of k . In particular if $\psi(1) - 1$ is inverted, relation (4) need not be explicitly imposed.

Proof. This follows from the computation $[X, h](\psi(c) - 1) = 0$ of the previous lemma. \square

Lemma 3.3. (1) If $k \models ACF_F$ then the conclusion of Lemma 3.1 follows from Equations 1-4, even without inverting $\psi(1) - 1$ or \mathbb{L} .
 (2) Every element of $K_{exp}(T)$ is represented in the form $[X, h]$ for some X, h .

Proof. (1) In this case, there exists (in the constructible category) a quotient Y of $X/(k, +)$, and by Hilbert's theorem 90, $X \rightarrow Y$ admits a constructible section f . So X is definably isomorphic as a $(k, +)$ -set to $Y \times k$. Thus $[X, h] \cong [Y, h \circ f] \times [k, x]$, and Lemma 3.1 follows from Equations 1 and the assumption $[k, Id] = 0$.

(2) The element -1 is represented by $[G_m, Id]$. Hence $[X, h] - [X', h'] = [X, h] + [G_m, Id][X', h']$. \square

If $h : Y \rightarrow k$ and $f : Y \rightarrow X$ are definable functions, and $n \in \mathbb{N}$, for $a \in X$ we obtain an element $L^{-n}[Y_a, h|Y_a]$ of $K_e(T_a)$. By a *definable function* $\theta : X \rightarrow K_e(T)$ we mean a function $a \mapsto \theta(a) \in K_{exp}(T_a)[\mathbb{L}^{-1}]$ of this form. Note in particular that this is not literally a function into $K_e(T)$. Given such a definable function θ on X , and $g : X \rightarrow k$, we write

$$(5) \quad \sum_{x \in X} \theta(x) \psi(g(x)) := [Y, h(y) + g(f(y))]$$

For $g = 0$ we obtain a definition of $\sum_{x \in X} \theta(x)$.

Let $A = Fn(X, K_e(T))$ be the set of definable functions $X \rightarrow K_e(T)$. Given $f, g \in A$ we can define $(f, g) = \sum_{x \in X} f(x)g(x)$.

Remark 3.4. For a generating a substructure of a model of T , let $F_a = Fn(X, K_e(T_a))$ be the set of T_a -definable functions $X \rightarrow K_e(T_a)$. If $g \in Fn(X, K_e(T))$ we obtain, for each a , a homomorphism $\chi_g : F_a \rightarrow K_e(T_a)$, namely $\chi_g(f) = \sum_{x \in X} g(x)f(x)$. Then if $f = f_t$ varies uniformly in some definable family, $\chi_g(f_t)$ is a definable function of t . It satisfies:

(*) $\chi_g(\sum_{y \in Y} \theta(y)f_y) = \sum_{y \in Y} \theta(y)\chi_g(f_y)$ where $(f_y : y \in Y)$ is a definable family of definable functions into $K_e(T)$, and $\theta \in Fn(Y, K_e(T))$.

(**) If $a = h(b)$ for a definable function h , we have a natural homomorphism $h^* : K_e(T_a) \rightarrow K_e(T_b)$, and by composition also $h^* : F_a \rightarrow F_b$. Then $\chi_g \circ h^* = h^* \circ \chi_g$.

Conversely, the χ_g are the only system of homomorphisms $F_a \rightarrow K_e(T_a)$, given uniformly in a , satisfying the above properties. For given such a system (χ) , define $g(a) = \chi(1_{\{a\}})$ where $1_{\{a\}}$ is the characteristic function of the element $a \in X$. Then from (*) it follows that $\chi(f) = \chi(\sum_{a \in X} f(a)1_{\{a\}}) = \sum_{a \in X} f(a)g(a) = \chi_g(f)$.

Note that (*) is an analogue of $K_e(T)$ -linearity, with finite additivity replaced by “motivically finite” additivity. In this sense the pairing (f, g) may be viewed as an isomorphism between A and its “motivic dual”.

3.5. Polynomial maps on semigroups. We prove a general lemma that will be used to extend the norm map, defined below, from the semiring to the ring $K_{exp}(T)$.

Let A be a commutative semi-group, B be an Abelian group, and $f : A \rightarrow B$ be a function. We say that f is a *polynomial map of degree 0* if f is constant. We say that f is *polynomial of degree $d + 1$* if $\phi : A^2 \rightarrow B$ is polynomial of degree d , where $\phi(x, z) = f(x + z) - f(x) - f(z)$.

Lemma 3.6. Let $f : A \rightarrow B$ be a polynomial map of any degree d . Let $a, b, c \in A$ and assume $a + c = b + c$. Then $f(a) = f(b)$.

Proof. For $d = 0$ this is clear. Assume it is true for polynomial maps of degree d , and f has degree $d + 1$. Let $\phi : A^2 \rightarrow B$, $\phi(x, z) = f(x + z) - f(x) - f(z)$. Then ϕ is polynomial of degree d . We have $(a, c) + (c, c) = (b, c) + (c, c)$ in the semigroup A^2 . Hence so $\phi(a, c) = \phi(b, c)$. In other words $f(a + c) - f(a) - f(c) = f(b + c) - f(b) - f(c)$. Since $a + c = b + c$, subtracting equal terms from this expression in the group B , we obtain $f(b) = f(a)$. \square

3.7. Irreducible definable sets. A definable set X is *irreducible* if $X \neq \emptyset$ and X contains no proper, nonempty definable sets. (Model theoretically one says that X *isolates a complete type*.) In ACF_F , of course, all irreducible sets are finite.

3.8. Hilbert 90. We will say that *Hilbert 90 holds for T* if any T_F -interpretable finite dimensional k -vector space admits a basis consisting of F -definable elements. This is true in ACF_F for any field F , field, by the vanishing of the first Galois cohomology of $GL_m(k)$, see [19] ch. 10, Prop. 3. On the other hand if Hilbert 90 holds for T , then any definable, finite dimensional k -algebra, as well as any definable finite dimensional module over such an algebra, are already T_F -definable, where F is the field of definable points. .

Lemma 3.9. *Let T be a theory of fields, and assume Hilbert 90 holds for T . Let \underline{R} be a definable finite dimensional algebra (associative, with 1). Then any definable \underline{R}^* -torsor has a definable point.*

Proof. If \underline{A} is a definable \underline{R} -module, let $\underline{A}^* = \{a \in \underline{A} : ra = 0 \rightarrow r = 0\}$. We say \underline{A} is *free on one generator* if this is the case for $\underline{A}(M)$, for some $M \models T$. If \underline{A} is free on one generator, then \underline{A}^* is an \underline{R}^* -torsor. Conversely, if \underline{B} is an \underline{R}^* -torsor, let \underline{A} be the quotient of $\underline{R} \times \underline{B}$ by the action of \underline{R}^* , $(x, y) \mapsto (xr, r^{-1}y)$. Then it is easy to define an \underline{R} -module structure on \underline{A} , making \underline{A} into a form of \underline{R} , with $\underline{A}^* = \underline{B}$. We thus have to show that if \underline{A} is a definable \underline{R} -module and \underline{A} is free on one generator c , then this generator can be chosen definable.

By the remark just above the lemma, \underline{R} and \underline{A} are ACF_F -definable. The non-generators of \underline{A} form a proper Zariski closed subset of \underline{A} . When F is infinite, $\underline{A}(F)$ is Zariski dense in F , and it suffices to choose any generator in $\underline{A}(F)$. When F is finite, we return to the connected algebraic group R^* , and use Lang's theorem instead. \square

Note also that if Hilbert 90 holds for T , then any definable $(k, +)$ -torsor H has a definable point, since H can be viewed as an affine line within a definable 2-dimensional k -space.

3.10. Norm map. In this paragraph let X be a *finite* definable set, and Def_X the category of definable sets over X , i.e. definable sets Y together with definable maps $Y \rightarrow X$; a morphism is a definable map $Y \rightarrow Y'$ commuting with the maps to X .

We have a functor $N : Def_X \rightarrow Def$, with $NY = \prod_{x \in X} Y_x$ the set of sections of $Y \rightarrow X$.

In case $T = ACF_F$ and X is irreducible, this is just the Weil restriction of scalars of Y_a from $F(a)$ to F (where $a \in X$.)

We will assume now that Hilbert 90 holds for T . Consider a triple (Y, f, h) , with $(Y, f) \in Def_X$ and $h : Y \rightarrow k$ a definable function. For $z \in NY$ define $Nh(z) = \sum_{x \in X} h(z(x))$; this is a finite sum taken in $(k, +)$. Let $N(Y, f, h) = [NY, Nh]$. Let $S = Fn(X, K_{exp}(T))$. We wish to show that N induces a map $N : S \rightarrow K_{exp}(T)$. In other words, we need:

Lemma 3.11. *Assume Hilbert 90 holds for T . Let (Y, f, h) represent a definable function $\Phi : X \rightarrow K_{exp}(T)$. Then $N(Y, f, h)$ depends only on Φ .*

Proof. First we let S^+ be the semiring of definable sections $X \rightarrow K_3^+(T)$, where K_3^+ is the semiring with generators and relations of (3) over X , with operations defined by Equations 1,2. It is easy to see from functoriality that (3) is respected by N , i.e. the norm of definably isomorphic pairs over X are definably isomorphic. We obtain a map $N : S^+ \rightarrow K_3(T)$.

We now verify that relation (4) is respected by N . In fact we will do more, and verify that the relations (4)' in Lemma 3.1 are respected too: assume k acts definably on each fiber $(X_a, h|_{X_a})$ of $Y \rightarrow X$, via a map $\rho : k \times Y \rightarrow Y$, with $f \circ \rho = f$ and $h(\rho(t, y)) = t + h(y)$. We will show that if $[Y, h] + [W, g] = [W', g']$ in S^+ , then $[NW, Ng] = [NW', Ng']$. To begin with we show that $[NY, Nh] = 0$. Let Nk be the group of maps $X \rightarrow k$, and for $\bar{t} \in Nk$ let $\phi\bar{t} = \sum_{x \in X} \bar{t}(x)$. We obtain by functoriality an action $N\rho : Nk \times NY \rightarrow NY$, with

$$(6) \quad Nh(N\rho(\bar{t}, \bar{y})) = \phi\bar{t} + Nh(\bar{y})$$

The k -vector space structure on k is inherited by Nk , and $\phi : Nk \rightarrow k$ is a surjective k -linear transformation. The kernel $\ker \phi$ is a k -vector space; so there exists a basis of $\ker \phi$ consisting of definable elements. Now using Hilbert 90, the torsor $\phi^{-1}(1)$ has a definable element \bar{t}_1 . Define an action of k on Nk by $(\alpha, \bar{y}) \mapsto N\rho(\alpha\bar{t}_1, \bar{y})$. Then the conditions of Lemma 3.1 are met, so $[NY, Nh] = 0$ in $K_{exp}(T)$.

Next we show, for Y as in the previous paragraph, not only that $[NY, Nh] = 0$ but also that if $[Y, h] + [W, g] = [W', g']$ in S^+ , then $[NW, Ng] = [NW', Ng']$. We can take W' to be the disjoint union of Y and W , and extend the action of k on Y to an action on W' , trivial on W . Let $N_i W'$ be the set of sections $s : X \rightarrow W'$, such that $|s^{-1}(Y)| = i$. Then NW' is the disjoint union of NW and of $N_i W'$ for $i \geq 1$. So it suffices to show that $N_i W' = 0 \in K_{exp}(T)$ for $i \geq 1$.

Let $[X]^i$ be the set of i -element subsets of X . For $w \in [X]^i$, consider the i -dimensional k -space k^w , and define $\phi_w : k^w \rightarrow k$ by $\phi_w(x) = \sum_{t \in w} x(t)$. Let B be the fiber product of all spaces k^w over k , via the maps ϕ_w . I.e. $B = \{(a_w) \in \prod_{w \in [X]^i} k^w : (\exists \alpha \in k)(\forall w \in [X]^i) \phi_w(a_w) = \alpha\}$. This is a k -space of dimension $(\binom{|X|}{i})(i-1) + 1$. We have a linear map $\phi : B \rightarrow k$, $\phi((a_w)) = \phi_w(a_w)$ (for any $w \in [X]^i$.) We have an action of B on $N_i W'$, as follows. We have a map $\psi : N_i W' \rightarrow [X]^i$, $\psi(s) = s^{-1}(Y)$. B will preserve the fibers of ψ ; on $\psi^{-1}(w)$, B will act via the w 'th coordinate, i.e. $(a_w) + s = s'$ with $s'(t) = a_{s^{-1}(Y)}(t) + s(t)$ if $s(t) \in Y$, $s'(t) = s(t)$ if $t \in W$. So Equation (6) holds. From here the proof is the same as in the paragraph following equation (6).

It follows that N induces a function on the semiring S_1^+ with generators 1-4'. It is easy to see that $N : S_1^+ \rightarrow K_{exp}(T)$ is polynomial of degree $|X|$. By Lemma 3.6 it induces a function on the image of S_1^+ in the associated ring S . This function is clearly multiplicative, $N(ab) = N(a)N(b)$. By Remark 3.3 this image equals S , so we obtain $N : S \rightarrow K_{exp}(T)$. \square

Inverting \mathbb{L}^{-1} , we find a norm map from the ring of definable maps $X \rightarrow K_e(T)$, into $K_e(T)$. We will also denote the norm of c by $\prod_{x \in X} c(x)$.

Let $Sym_n(X) = Y/Sym(n)$, where Y is the set of distinct n -tuples of X .² We also treat elements of $Sym_n(X)$ as n -element subsets of X .

Definition 3.12. Let $c : X \rightarrow K_{exp}(T)$ be a definable function. Define

$$\prod_{x \in X} (1 + c(x)t) = \sum_{n=0}^{\infty} b_n t^n$$

where $b_n = \sum_{s \in Sym_n(X)} \prod_{t \in s} c(t)$

By a definable function $X \rightarrow K_{exp}(T)[[t]]$ we mean one of the form $c(x) = \sum c_n(x)t^n$ with each c_n a definable function into $K_{exp}(T)$. Define $\prod_{x \in X} (1 + c(x)t)$ where $c : X \rightarrow K_{exp}(T)[[t]]$ is a definable function in the natural way. We have:

Lemma 3.13. If $f : Y \rightarrow X$ is definable and $b : Y \rightarrow K_{exp}(T)[[t]]$, define $a(x)$ by: $1 + ta(x) = \prod_{y \in Y_x} (1 + tb(y))$. Then $\prod_{x \in X} (1 + ta(x)) = \prod_{y \in Y} (1 + tb(y))$.

² $Sym_n(C)$ is a definable set in a possibly imaginary sort of T .

Proof. If we allow rational coefficients, this can be done using the analogous statement for sums, which is evident, and the isomorphism $\exp : tK_{exp}(T)_{\mathbb{Q}}[[t]] \rightarrow 1 + tK_{exp}(T)_{\mathbb{Q}}[[t]]$. Otherwise, a routine computation by coefficients. \square

3.14. Compatibility(1). Assume T admits quantifier-elimination (as may be achieved by Morleyzation.) Let T' be a theory extending T_{\forall} , possibly but not necessarily in a richer language. We assume any model of T' is definably closed as a substructure of a model of T . The generators $[X, h]$ of $K_{exp}(T)$ can be taken with X, h quantifier-free definable. As such they are also elements of $K_{exp}(T')$, and we define μ on generators by $[X, h]_T \mapsto [X, h]_{T'}$. Then equations 1-4 are respected by μ , and we obtain a ring homomorphism $\mu = \mu_{T/T'} : K_{exp}(T) \rightarrow K_{exp}(T')$.

Example 3.15. $T = ACF_F$ where F is a perfect field, $T' = Th(F)$, both in the language of F -algebras. This will interest us especially when F is finite or pseudo-finite.

When F is finite, $K_{exp}(T')$ is the group ring $\mathbb{Z}[(F, +)]$. Hence if we choose a homomorphism $\psi : (F, +) \rightarrow (\mathbb{C}, \times)$ we obtain by composition a nontrivial homomorphism $\mu_{\psi} : K_{exp}(T) \rightarrow \mathbb{C}$.

When F is pseudo-finite, $K(T')$ is related to virtual Chow motives, cf. [14].

The words “definable”, etc. continue to refer to T unless otherwise indicated.

If X is finite, we let $|X|$ be the number of points of X in a model of T .

Lemma 3.16. *Let X be an irreducible definable set. Let $M \models T, a \in X(M)$. Then $K_{exp}(T_a)$ is naturally isomorphic to the ring S of definable functions $X \rightarrow K_{exp}(T)$.*

Proof. Given a definable function $X \rightarrow K_{exp}(T)$, we obtain by evaluation at a an element of $K_{exp}(T_a)$. This gives a homomorphism $ev : S \rightarrow K_{exp}(T_a)$. Any definable set of T_a is T_a -definably isomorphic to one of the form $Y_a = f^{-1}(a)$, where Y is a definable set and $f : Y \rightarrow X$ a definable function. This shows that the evaluation map $S \rightarrow K_{exp}(T_a)$ is surjective. Moreover any definable function f on Y_a is the restriction of a definable function on Y . By irreducibility of X , if f restricts to a bijection $Y_a \rightarrow Y'_a$ then it must be a bijection $Y \rightarrow Y'$ over X . The required isomorphism follows already at the level of semirings, and hence extends to an isomorphism of rings. \square

We write $K_{exp}(T_X)$ for either of the rings in Lemma 3.16. If X is finite, we have the norm map $N_{T_X/T} : K_{exp}(T_X) \rightarrow K_{exp}(T)$. Let $T' = Th(F)$, as above.

Lemma 3.17. *Let X be a finite irreducible definable set. The composition $\mu_{T, Th(F)} \circ N_{T_X/T}$ is a ring homomorphism.*

Proof. Say $|X| = n$. Write $\mu = \mu_{T, Th(F)}$ and $N = N_{T_X/T}$. Since N is multiplicative and μ is a ring homomorphism, it is clear that $\mu \circ N$ is multiplicative. Let $a, b : X \rightarrow K_{exp}(T)$ be definable functions, and $c = a + b$. Then

$$N(c) = N(a) + N(b) + \sum_{m=1}^{n-1} \sum_{s \in \text{Sym}_m(X)} \prod_{x \in s} a(x) \prod_{x \in X \setminus s} b(x)$$

Now $\mu(\sum_{s \in \text{Sym}_m(X)} \prod_{x \in s} a(x) \prod_{x \in X \setminus s} b(x)) = 0$ simply because $\text{Sym}_m(X)(F) = \emptyset$ for $0 < m < |X|$. Hence $\mu N(c) = \mu N(a) + \mu N(b)$. \square

The ring homomorphism $\mu \circ N$ induces a ring homomorphism on the power series rings, $K_{exp}(T_X)[[t]] \rightarrow K_{exp}(T')[[t]]$, with $t \mapsto t^n$. This homomorphism coincides with $a \mapsto \mu(\prod_{x \in X} a(x))$. We note the corollary:

Lemma 3.18. *Let X be a finite irreducible definable set, $n = |X|$. Let $a_k : X \rightarrow K_{exp}(T)$ be a definable function, with $a_0 = 1$. Let $b_k(x) = \prod_{x \in X} a_k(x)$. Then*

$$\mu\left(\prod_{x \in X} \sum_{k=0}^{\infty} a_k(x) t^k\right) = \mu\left(\sum_{k=0}^{\infty} b_k(x) t^{nk}\right)$$

Proof. Follows from Lemma 3.17. Alternatively it can be proved directly, using the fact that there are no F -definable nontrivial partitions of X . \square

We mention in passing an additional, straightforward compatibility of motivic volumes with ultraproducts, in particular of formulas over pseudo-finite fields with corresponding motivic formulas over finite fields. In the next paragraph we compare the latter with classical adelic integration.

3.19. Compatibility(2). In this section we let $T = ACF_F$, with F a finite field. We wish to compare motivic adelic volumes to classical ones. For this purpose we do not need exponential sums, so let K be any algebra over the Grothendieck ring of T (typically, obtained by inverting the class of the affine line.)

While the classical treatment of adèles has a factor for each closed schematic point of the curve C , ours has a factor for each point of C in F^{alg} . Motivic volumes are nevertheless compatible with adelic volumes; for this the “Frobenius” Lemma 3.17 is essential.

let X be a scheme of finite type over F , and \mathcal{U} a scheme of finite type over X . For $x \in X(M)$ (where $M \models T$) let $a(x) = [\mathcal{U}_x] \in K_{exp}(F(x))$. This gives a definable function $X \rightarrow K_{exp}(F(x))$. On the other hand, let X_{closed} be the set of closed schematic points of X . For $v \in X_{closed}$, let F_v be the residue field, a finite extension of F , with q_v points. Let U_v be the fiber of \mathcal{U} above v , and $a(v) = |U_v(F_v)|$. Recall Definition 3.12.

Lemma 3.20.

$$\prod_{v \in X_{closed}} (1 + a(v) q_v^{-s}) = \mu_F \prod_{x \in X} (1 + a(x) t)_{t=p^{-s}}$$

Hence if $\prod_{v \in X_{closed}} (1 + a(v) q_v^{-s})$ converges absolutely to some $r \in \mathbb{C}$; then $\mu_F \prod_{x \in X} (1 + a(x) t)$ converges absolutely to r at $t = p^{-1}$.

Proof. Any $v \in X_{closed}$ corresponds to a finite definable subset X_v of X , a Galois orbit. We have $\mu_F \prod_{x \in X_v} (1 + a(x) t) = 1 + \mu_F \prod_{x \in X_v} a(x) t^{deg(v)}$, by Lemma 3.18.

Claim 1. We have an equality of formal series:

$$\mu_F \prod_{x \in X} (1 + a(x) t) = \prod_{v \in X_{closed}} \mu_F \prod_{x \in X_v} (1 + a(x) t)$$

Proof. If X is finite, then $X = \cup_v X_v$, and the claim is a special case of Lemma 3.13, even without applying μ_F . In general, consider the coefficient b_N of t^N in $\prod_{x \in X} (1 + a(x) t)$. By Definition 3.12 we have $\mu_F b_N = \sum_{s \in Sym_n(X)(F)} \mu_F \prod_{u \in s} a(u)$. This is a finite sum, so for a sufficiently large finite definable $X' \subset X$, the coefficient of t^N in the finite product $\mu_F \prod_{x \in X'} (1 + a(x) t)$ is the same. Take X' so large that X'' contains no finite definable set of size $\leq N$. Let $X'' = X \setminus X'$. On the right hand side, the product decomposes into a product over X'_{closed} and a product over X''_{closed} ; the latter has no nonconstant terms of degree $\leq N$; so on the right hand side too the t^N term for X and for X' is the same. The claim follows. \square

On the other hand $\mu_F \prod_{x \in X_v} (1 + a(x) t) = 1 + \mu_F \prod_{x \in X_v} a(x) t^{deg(v)} = 1 + a(v) t^{deg(v)}$, so $(\mu_F \prod_{x \in X_v} (1 + a(x) t))_{t=p^{-s}} = 1 + a(v) q_v^{-s}$. The lemma follows. \square

The above lemma (slightly generalized) will permit the comparison of classical Tamagawa measures (with convergence factors) to motivic ones.

A similar comparison is valid for K_e . Let \mathbf{F} be a finite field $GF(q)$, together with a choice of character $\psi : (F, +) \rightarrow \mathbb{C}^*$.

Define $\mu_{\mathbf{F}}([X, h]) = \sum_{x \in X(F)} \psi(h(x))$. Then equations 1-4' are respected by $\mu_{\mathbf{F}}$; so a homomorphism $\mu_{\mathbf{F}} : K_{exp}(T) \rightarrow \mathbb{C}$. We have $\mu_{\mathbf{F}}(\mathbb{L}) = q$, and hence $\mu_{\mathbf{F}}$ extends to a ring homomorphism $K_{exp}(T)(\mathbb{L}^{-1}) \rightarrow \mathbb{C}$. We also write $\mu_{\mathbf{F}}$ for the natural extension to $K_{exp}(T)[[t]] \rightarrow \mathbb{C}[[t]]$. We have:

Lemma 3.21. *Let X be a finite irreducible definable set. Let $a : X \rightarrow K_{exp}(T)$ be a definable function. Let x_0 be a point of $X(F')$, where F' is a finite field extending F , and assume F' is generated by x_0 over F . Let $\psi'(x') = \psi(\text{tr}_{F'/F}(x'))$. Then $\mu_{\mathbf{F}}(\prod_{x \in X} a(x)) = \mu_{F'}a(x_0)$.*

The proof is similar to the case of counting.

4. LOCAL TEST FUNCTIONS AND INTEGRATION ON LINEAR SPACES

Let $k \models T$. Let K be a discrete valued field, with residue field k . In any valued field, \mathcal{O} will always denote the valuation ring, \mathfrak{M} the maximal ideal. A *parameter* is an element t of minimal positive valuation 1. Let

$$K(N; M) = \{x : \text{val}(x) \geq -N\} / \{x : \text{val}(x) \geq M\} = t^{-N}\mathcal{O}/t^M\mathcal{O}$$

We have an isomorphism $\alpha_{N,M} : k^{N+M} \rightarrow K(N; M)$,

$$x = (x_{-N}, \dots, x_M) \mapsto \sum x_i t^i$$

By a *test function of level (N, M)* we mean a function $\phi : K(N; M) \rightarrow K_e(T)$, such that $\phi^t := \phi \circ \alpha_{N,M}$ is a definable function $\phi^t : k^{N+M} \rightarrow K_e(T)$. We view ϕ as a function on K whose support is contained in $t^{-N}\mathcal{O}$, and invariant under $t^M\mathcal{O}$. Let $\mathcal{S}(K; N, M)$ be the set of test functions of level (N, M) , and let $\mathcal{S}(K) = \cup_{N, M \in \mathbb{N}} \mathcal{S}(K; N, M)$.

We define $\int^t : \mathcal{S}(K; N, M) \rightarrow K_e(T)$ by

$$\int^t \phi = \mathbb{L}^{-M} \sum_{x \in k^{N+M}} \phi^t(x)$$

If t' also has valuation 1, then for each integer r , $t' = \sum_{i=1}^r a_i t^i \pmod{t^{r+1}}$, for some (uniquely defined) $a_1, \dots, a_r \in k$. We call t, t' *definably equivalent* if for each r , each a_i is T -definable.

Lemma 4.1. *If t, t' are definably equivalent parameters, then $\int^t, \int^{t'}$ agree on $\mathcal{S}(K; N, M)$.*

Proof. $(\phi^t)^{-1}\phi^{t'}$ is definable and hence induces the identity on $K_e(T)$. \square

Hence \int^t depends only on the definable equivalence class of t . We will later describe a canonical choice of a definable equivalence class of parameters. We thus cease to write t in the superscript, letting $\int = \int^t$.

Lemma 4.2. (1) *Let $i : K(N, M) \rightarrow K(N+1, M)$ be the natural inclusion. It induces $i_* : \mathcal{S}(K; N, M) \rightarrow \mathcal{S}(K; N+1, M)$ (extension by 0.) We have $\int \circ i_* = \int$.*
 (2) *Let $\pi : K(N, M+1) \rightarrow K(N, M)$ be the natural projection. It induces $\pi^* : \mathcal{S}(K; N, M) \rightarrow \mathcal{S}(K; N, M+1)$ (composition). The image of π^* is the set of elements ϕ of $\mathcal{S}(K; N, M+1)$ invariant under $\ker(\pi)$. We have $\int \circ \pi^* = \int$.*

Proof. Clear. Regarding the image of π^* , note that π has a definable section. In general it is clear that if E is an equivalence relation on Y with a definable section $Y/E \rightarrow Y$, then an E -invariant definable map on Y arises from a definable map on Y/E . \square

We view both i_* and π^* as inclusion maps, and let $\mathcal{S}(K) = \cup_{N,M \in \mathbb{N}} \mathcal{S}(K; N, M)$. We view the elements of $\mathcal{S}(K)$ as (“smooth, bounded”) functions on K ; the integral as a function on $\mathcal{S}(K)$.

The same goes for families. If Y is a definable subset of k^n , $\alpha_{N,M}$ yields a notion of definable function on $Y \times K(N; M)$; and given such a definable function $\phi(y, x)$ we can integrate with respect to the $K(N; M)$ variables to obtain a definable function $Y \rightarrow K_e(T)$.

Let $K^m(N, M) = K(N, M)^m$. Define $\mathcal{S}(K^m; N, M)$, $\mathcal{S}(K^m) = \cup_{M,N} \mathcal{S}(K^m; M, N)$, and $\int : \mathcal{S}(K^m) \rightarrow K_e(T)$ in a similar way. If U is a vector space defined over K with a distinguished basis, we identify U with K^m , and in this way define $\mathcal{S}(U; N, M) = \mathcal{S}(K^m; N, M)$ etc.

Remark 4.3. In fact a definable element of $GL_m(K)$ induces an isomorphism $\mathcal{S}(U) \rightarrow \mathcal{S}(U)$ (not in general preserving levels), and using this system of isomorphisms it is possible to define $\mathcal{S}(U)$ for any U having a definable basis (but not a distinguished one); but we will not need this.

4.4. Finitely many valued fields. Assume given finitely many discrete valued fields $(K_i : i \in S)$, each with a parameter t_i and residue field k . We write K_S for the ring $\prod_{i \in S} K_i$. Let

$$K_S(N, M) = \prod_{i \in S} K_i(N, M)$$

We define $\mathcal{S}(K_S^m)$ and $\int : \mathcal{S}(K_S^m) \rightarrow K_e(T)$ in the same way as for one field above.

In practice the K_i will extend (F, v_i) where F is a fixed field, and v_i are valuations of F . We will also have a vector space U over F of dimension m , with an F -basis, and write $\mathcal{S}(U; K_S) = \mathcal{S}(\prod_{i \in S} K_i^m)$ using the basis for the identification.

Remark 4.5. Classically, the map $\mathcal{S}(K_1) \otimes \dots \otimes \mathcal{S}(K_n) \rightarrow \mathcal{S}(K_1 \times \dots \times K_n)$ is surjective, but this will not be the case for us. Nor is the image of this homomorphism preserved under the “sum of rational points” maps of §5.4.

4.6. Fourier transform. Let k, K, t, N, M be as above. We also fix a nonzero linear map $r : K \rightarrow k$, vanishing on $t^{-M}\mathcal{O}$ for some M . The dual of \mathcal{O} with respect to r is $\mathcal{O}^\perp = \{x : (\forall y \in \mathcal{O}) r(xy) = 0\}$; it contains $t^M\mathcal{O}$, and is an \mathcal{O} -module, so it must have the form $t^\nu\mathcal{O}$ for some $\nu \in \mathbb{Z}$. We assume ν is even.

We define the local Fourier transform of $\phi \in \mathcal{S}(K; N, M)$ by

$$(7) \quad \mathcal{F}(\phi)(x) = \mathbb{L}^{-\nu/2} \int_y \phi(y) \psi(r(xy))$$

It is clear that $\mathcal{F}(\phi)$ is invariant under $\{x : \text{val}(x) \geq N + M\}$. Using Lemma 3.1 one sees that $\mathcal{F}(\phi)$ is bounded. Hence $\mathcal{F} : \mathcal{S}(K) \rightarrow \mathcal{S}(K)$.

The inversion formula is easily proved.

On K^n we define $\mathcal{F} : \mathcal{S}(K^n) \rightarrow \mathcal{S}(K^n)$ by the same equation 7, with xy interpreted as the standard dot product.

More generally, given finitely many valued fields K_1, \dots, K_n , each with a parameter t_i and a linear map $r_i : K_i \rightarrow k$, we define for $\phi \in \mathcal{S}(K_1 \times \dots \times K_n)$

$$\mathcal{F}(\phi)(x_1, \dots, x_n) = \int_y \phi(y_1, \dots, y_n) \psi(\sum r_i(x_i y_i))$$

so that the Fourier transform can be computed one variable at a time.

4.7. Comparison with [11]. Let K be a discrete valued field, with distinguished subfield $F \leq \text{res}(K)$ and a distinguished parameter t . Assume $\text{char}(F) = 0$. Then for test functions ϕ both the above theory and the integration theory of [11] apply. We explain the connection. Let $i(\phi)$ denote the integral in the sense of [11]. Let T denote the theory of the residue field of K (including constants for F), and let T' be the theory of K (including the T -structure on the residue field and a constants for t .) Let j denote the “rational points” functor of [11], chapter 10, towards the theory T' . In general ji takes values in $K^+(T) \otimes K^+(\Gamma)/I_{sp}$. Let $K_{fin}^+(\Gamma)$ be the subsemiring of $K^+(\Gamma)$ represented by finite definable sets. Let $L = [G_a] \in K^+(T)$. We have a “weighted counting” homomorphism $K_{fin}^+(\Gamma) \rightarrow \mathbb{N}[L^{-1}] \leq K(T)[L^{-1}]$, where $L = [G_a]$; namely, the point $n \in \mathbb{Z} \leq \Gamma$ is assigned the value $[G_m(k)]L^{-n}$; sum over finite sets. This induces $h : K^+(T) \otimes K_{fin}^+(\Gamma) \rightarrow K(T)[L^{-1}]$ respecting the restriction of the congruence I_{sp} . (In fact h is an isomorphism between the image of $K^+(T) \otimes K_{fin}^+(\Gamma)$ in $K^+(T) \otimes K^+(\Gamma)/I_{sp}$, and $K^+(T)$.)

Lemma 4.8. (1) Suppose ϕ takes values in $K^+(T)$. Then $ji(\phi)$ lies in $K^+(T) \otimes K_{fin}^+(\Gamma)/I_{sp}$, and $hji(\phi) = \int \phi$.

Proof. The maps h, j, i and \int are completely additive over the residue field, i.e. if W is definable over T and $\phi = \sum_{w \in W} \phi_w$ then $hji(\phi) = \sum_{w \in W} hji(\phi_w)$, and similarly for $\int \phi$. Hence the statement on $ji(\phi)$ and the equality $hji(\phi) = \int \phi$ reduce to the case of characteristic functions of a point in $K(n, m)$; this is an immediate computation. \square

This immediately gives a change of variable formula, and the ability to integrate over varieties. Unfortunately it only applies in characteristic 0 so in general we are obliged to do this from scratch.

5. GLOBAL THEORY

We continue with the theory of fields T , containing constants for a field F .

Let C be a smooth, projective, absolutely irreducible curve over F . Let $K = k(C)$ be the function field. We will define global test functions and construct a Fourier transform operator, and a “sum over rational points” functional, on them.

We also fix a nonzero 1-form ω on C , defined over F . We assume ω can be chosen in such a way that every zero or pole of ω has even multiplicity. For $g = 0$ one can choose ω with one double pole, and no zeros; for $g = 1$ one can choose ω regular, with no zeros. When $g > 1$ such a form may not exist over F ; the next paragraph contains a proof that it can always be found over a finite extension. Various remedies are possible when one wants to remain over F , including the introduction of a formal square root of the affine line; since we will not require this case we will not enter into this discussion.

We have $\deg(\omega) = 2g - 2$, where g is the genus of C . Assume $g > 1$, and let $Jac(C)_{(2g-2)}$ be the divisor classes of degree $2g - 2$, a torsor of the Jacobian $Jac(C)$. Consider the map $f : C^g \rightarrow Jac(C)_{(2g-2)}$ given by $(c_1, \dots, c_{g-1}, c_g) \mapsto 2(c_1 + \dots + c_{g-2} - c_{g-1}) + 4c_g$. By subtracting $f(c_0)$ for some $c_0 \in C^g$ one obtains a map into the Jacobian, and if the image has dimension $h < g$ then the Jacobian is easily seen to have dimension h , a contradiction. Hence the image $f(C^g)$ has dimension g . Now $\dim(Jac(C)_{(2g-2)}) = \dim(Jac(C)) = g$. Since C^g is complete, the image is closed, so $f(C^g) = (Jac(C)_{(2g-2)})$. So we can find (c_1, \dots, c_g) such that $2(c_1 + \dots + c_{g-2} - c_{g-1}) + 4c_g$ represents the canonical class. Then there exists a form ω such that $\text{div}(\omega) = 2(c_1 + \dots + c_{g-2} - c_{g-1}) + 4c_g$.

For simplicity we assume that T contains ACF_F . This is not a serious restriction since the language may be larger than that of F -algebras, and may in particular include a predicate for a subfield K .

If v is a valuation on $F(C)$, the residue field of v is a finite extension F_v of F . The assumption that $T \models ACF_F$ is used to conclude that F_v is a subfield of k ; this simplifies the notation. It is not however contained in F , so a direct translation of the classical theory would lead to integrals with values in $K_e(T_{F_v})$ rather than $K_e(T_F)$. It is not clear how to multiply elements of $K_e(T_{F_v})$ for distinct v . However adelic integration with our “redundant” definition of the adeles involves taking products over several conjugate representatives of the same valuation; this means that the integral factors through the norm map, hence does belong to $K_e(T_F)$, and further products over distinct valuations make sense. Similarly if the language contains a predicate for a subfield K , the integrals of quantities defined over K will themselves be over K .

Our “sum over rational points” is actually a sum over $k(C)$, not $F(C)$, including notably $F^{alg}(C)$ -rational points. This is necessary to allow uniformity in definable families, e.g. Lemma 5.6. Nevertheless we show compatibility with the classical sum, via μ_F . And the Poisson summation formula holds motivically (in the sense of motivic integration), before μ_F is applied.

5.1. $k(C)$ as an Ind- definable field.

Let L_F^g be the language including the following symbols, *all viewed as relation symbols*.

1) A sort K intended to denote $k(C)$. On K , the language of F -algebras, made relational, i.e. with a relation symbol for the zero-set of each polynomial over F ; A unary predicate symbol for $k \subseteq K$.

2) Relation symbols $V_n \subseteq C(k) \times K$. Intended meaning: $V_n(\alpha, f)$ if $\text{ord}_\alpha(f) = n$, i.e. the order of vanishing of f at α equals n .

4) $\mathfrak{t} \subseteq V_1$ intended to pick out a parameter t_c for K_{v_c} , uniformly in c .

Let $F \leq k$. We impose the natural L_F^g structure on $k(C)$ (with auxiliary sort k^{alg} , and K interpreted as $k(C)$.)

5) $W = \{(\alpha, f, \gamma) \in C(k) \times k(C) \times k : \gamma = \text{res}_\alpha(f\omega)\}$.

According to Lemma 6.23, with this structure, $k(C)$ is piecewise definable over k .

5.2. Global test functions. For each $u \in C(k)$ let $k(C)_u$ be the valued field $k(C)$, with valuation corresponding to the point u , i.e. $\text{val}(f) > 0$ iff $f(u) = 0$. We could take the completion but for our immediate purposes it is not important since we really use only the vector spaces $t^{-m}\mathcal{O}_u/t^m\mathcal{O}_u$, where \mathcal{O}_u denotes the valuation ring and t is a parameter. Let \mathbb{A} be the restricted product of fields K_u relative to the rings \mathcal{O}_u . (Classically one takes only algebraic u , and only one copy for each conjugacy class; this would suffice to tell apart our test functions.)

A *global test function* on \mathbb{A}^n is given by a finite $S \subset C(k)$ and an element ϕ of $\mathcal{S}((\prod_{u \in S} k(C)_u)^n)$, with the understanding that if S' is disjoint from S , then (S, ϕ) is identified with $(S \cup S', \phi \otimes 1_{\mathcal{O}_{S'}})$. Here $\mathcal{O}_{S'} = \prod_{u \in S'} \mathcal{O}_u$.

A *definable subset* of \mathbb{A}^n is defined similarly, so that the characteristic function of a definable subset is a test function.

The form ω on C provides us with a linear map $k(C)_u \rightarrow k$, namely $f \mapsto \text{res}_u(f\omega)$. It is this map that we use in 4.6, to obtain a local Fourier transform $\mathcal{F}_u : \mathcal{S}(k(C)_u) \rightarrow \mathcal{S}(k(C)_u)$. If S is a finite subset of C , we define $r_S(f) = \sum_{u \in S} \text{res}_u(f\omega)$.

5.3. Fourier transform. The Fourier transform of a global test function on \mathbb{A} is defined by choosing a representative (S, ϕ) for the global test function such that ω is regular and nonzero

outside S , and letting

$$\mathcal{F}((S, \phi)) = (S, \mathcal{F}(\phi))$$

the latter \mathcal{F} being the semi-local Fourier transform defined at the end of 4.6. It is easy to check that this is well-defined.

We can also define $\mathcal{F}((S, \phi) = \phi'$ directly, $\phi' = \Lambda^{1-g} \int \phi(r_S(xy))\phi(x)dx$.

5.4. Summation over rational points. Let $u \in C(k)$. View $k(C)_u$ as an piecewise-definable valued field in T , with distinguished parameter $t = t_u$. For $f \in k(C)$, we write f_u for f viewed as an element of $k(C)_u$.

Fix a definable global test function ϕ , represented as (S, ϕ_S) for some S . We will define $\phi(f)$ for $f \in k(C)$.

Let $k(C)_S = \{f \in k(C) : (\forall u \notin S)(v_u(f) \geq 0)\}$. For $f \notin k(C)_S$ we define $\phi(f) = 0$. This is forced by the definition of global test functions, since if $v_u(f) < 0$, then ϕ is also represented by $(S \cup \{u\}, \phi_S \otimes 1_{\mathcal{O}_u})$ and $1_{\mathcal{O}_u}$ vanishes at f .

For $f \in k(C)_S$, let $f_S = (f_u : u \in S) \in \prod_{u \in S} k(C)_u$. Define $\phi(f) = \phi_S(f_S)$.

It is clear that $\phi(f)$ does not depend on the choice of S .

If f is definable, then $\phi(f) \in K_e(T)$. In general if f, ϕ are F' -definable then $\phi(f) \in K_e(T_{F'})$.

Let Y be a limited subset of $k(C)$. Then $y \mapsto \phi(y)$ is clearly a definable function $Y \rightarrow K_e(T)$. We thus have an element $\sum_{y \in Y} \phi(y) \in K_e(T)$. (cf. Equation (5)).

Let m be an integer such that ϕ_S is supported on $\prod_{u \in S} t_u^{-m} \mathcal{O}_u$. Let

$$Y_0 = \{f \in k(C)_S : (\forall u \in S) v_u(f) \geq -m\}$$

Then Y_0 is a limited subset of $k(C)$. We have $\phi(f) = 0$ for any $f \in k(C)_S \setminus Y_0$, hence for any $f \in k(C) \setminus Y_0$.

Define $\delta^K(\phi) = \sum_{y \in k(C)} \phi(y) = \sum_{y \in Y_0} \phi(y)$.

If $\phi = \phi(y, x)$ depends on other variables $x = (x_1, \dots, x_n)$, we define $\sum_{y \in k(C)} \phi(y, x)$ in the same way, and denote it $\delta_y^K \phi$.

Lemma 5.5. *Let ϕ be a test function in $n + 1$ variables x_1, \dots, x_n, y . Then $\delta_y^K \phi = \sum_{y \in k(C)} \phi(x, y)$ is a test function in n variables.*

Proof. Definability is clear. By assumption, $\phi_S(x, y)$ is invariant under C_S^{n+1} for some congruence subgroup C_S . From this it is clear that $\delta_y^K \phi$ is C_S^n -invariant. Since ϕ_S is supported on D^{n+1} for some bounded set D , it is clear that ϕ' is supported on D^n . \square

Call a global test function *simple* if it is represented by (S, ϕ_S) , where ϕ_S is the characteristic function of a single coset of $t_v^m \mathcal{O}_v$, for some m .

Lemma 5.6. (1) *Let W be a definable set of T , and let ϕ_a be a global test function, defined uniformly in $a \in W$. Let $\chi : W \rightarrow K_e(T)$ be a definable function. Then $\sum_{a \in W} \chi(a) \phi_a$ is a global test function.*

$$(2) \sum_{x \in k(C)} \sum_{a \in W} \phi_a(x) = \sum_{a \in W} \sum_{x \in k(C)} \phi_a(x).$$

$$(3) \int_x \sum_{a \in W} \chi(a) \phi_a(x) = \sum_{a \in W} \chi(a) \int_x \phi_a(x)$$

$$(4) \mathcal{F} \sum_{a \in W} \phi_a = \sum_{a \in W} \mathcal{F} \phi_a$$

(5) *Any global test function can be expressed as in (1).*

Proof. (1)-(4) are clear. As for (5), the test function is represented by (S, ϕ_S) for some S ; and for some m , ϕ_S factors through a function ϕ' on $\prod_{u \in S} t_u^{-m} \mathcal{O}_u / t_u^m \mathcal{O}_u$, pulled back to $\prod_{u \in S} t_u^{-m} \mathcal{O}_u$ and extended by 0 to $\prod_{u \in S} K_u$. Identify $t_u^{-m} \mathcal{O}_u / t_u^m \mathcal{O}_u$ with k^{2m} via the basis t_{m-1}, \dots, t_{-m} .

Let $W = (k^{2m})^S$. For $a \in W$ let ϕ_a be the simple test function concentrating on a , and $\chi(a) = \phi'(a)$. Then clearly $\phi = \sum_{a \in W} \chi(a) \phi_a$. \square

Lemma 5.7. *Assume F is a finite field, and fix a nontrivial character ψ of F . Let μ_ψ denote $\mu_{T/Th(F)}$ composed with $\psi : \mathbb{Z}[(F, +)] \rightarrow \mathbb{C}$. Let ϕ be a definable global test function in n variables. Then $\mu_\psi(\sum_{x \in k(C)^n} \phi(x)) = \sum_{x \in F(C)^n} \mu_\psi \phi(x)$.*

Proof. By opening up the definitions. The Ind-definable sum $\sum_{x \in k(C)^n}$ reduces to a certain definable sum $\sum_{y \in Y_a}$ (with limited Y_a depending on ϕ_a .) Now the union $Y = \cup_a Y_a$ is still limited, and we may write $\sum_{x \in k(C)^n} = \sum_{y \in Y}$ for any of the test functions in question. In general μ_ψ commutes with definable sums $\sum_{y \in Y}$. \square

Lemma 5.8. *Let $a \in F(C)$, and $\phi'(x) = \phi(x + a)$. Then $\delta^K \mathcal{F}(\phi') = \delta^K \mathcal{F}(\phi)$.*

Proof. We may assume $a\omega$ is holomorphic at u for $u \notin S$. Then $\mathcal{F}(\phi') = \psi(r_s(ab))\mathcal{F}(\phi)$. In particular for any $b \in k(C)_S$ we have $\mathcal{F}(\phi')(b) = \psi(r_s(ab))\mathcal{F}(\phi)(b)$. But the sum of residues $r_s(ab) = \sum_{u \in F(C)} \text{res}_u(ab\omega) = 0$. \square

5.9. Poisson summation formula.

Lemma 5.10. *Let $\phi(x)$ be a definable global test function, and $\psi(y) = \mathcal{F}\phi(x)$. Then $\delta^K \psi = \delta^K \phi$.*

Proof. In view of Lemma 5.6 we may assume ϕ is simple. We compute using representatives in $\prod_{u \in S} k(C)_u$, where S is a large enough finite set. Say ϕ is represented by (S, ϕ_S) with ϕ_S the characteristic function of $W = a + \prod_{u \in S} t_u^{m_u} \mathcal{O}_u$, $a \in \prod_{u \in S} k(C)_u$. Let D be the divisor on C supported at S , with multiplicity m_u at u . Then $L(D) := \{f \in k(C) : (\forall u \in C)(v_u(f) \geq m_u)\}$ is a finite-dimensional subspace of $k(C)$, defined over F . Let $D' = \text{div}(\omega) - D$ be the dual divisor. We write $k(C)_S$ for the image in $\prod_{u \in S} k(C)_u$ of $\{a \in k(C) : (\forall u \notin S)(v_u(a) \geq 0)\}$.

Case 1. There exists $a' \in k(C)_S \cap W$.

Since $k(C) \cap W$ is a torsor for $L(D)$, by Hilbert 90, $F(C) \cap (a + \prod_{u \in S} t_u^{m_u} \mathcal{O}_u) \neq \emptyset$. So we may take $a' \in F(C)$. By Lemma 5.8 we may translate by a' ; so we may assume $a' = 0$. In this case by direct computation we see that $\mathcal{F}(\phi) = \mathbb{L}^{\deg(D)+1-g} \phi'$ where ϕ' is the simple global test function concentrating on D' . The equality asserted in the lemma is precisely Riemann-Roch.

Case 2. $k(C)_S \cap W = \emptyset$. In this case we have $\delta^K \phi = 0$ and we have to show that $\delta^K \mathcal{F}\phi = 0$.

Let $A = \{x \in \prod_{u \in S} k(C)_u : (\forall u \in S)(v_u(x) \geq m_u)\}$. Using Riemann-Roch, $A/(k(C)_S \cap A)$ is finite dimensional. We use the form $r_S(xy)$ on $\prod_{u \in S} k(C)_u$. By definition of the Fourier transform and Lemma 3.1, $\mathcal{F}\phi$ is supported on A^\perp . For $y \in A^\perp$, $r_S(xy)$ takes a constant value $\rho(y)$ for all $x \in W$. This ρ is a linear map on A^\perp and in particular on $B = A^\perp \cap k(C)_S$. We have $\delta^K \mathcal{F}\phi = \text{vol}(W) \sum_{y \in B} \psi \rho(y)$. If ρ vanishes on B then $W \subseteq B^\perp = A + (k(C)_S)^\perp$ (as one obtains by factoring out to reduce to a finite dimensional situation.) But $k(C)_S$ is self-dual for r , since the sum of residues equals zero. Thus $W \subseteq A + k(C)_S$, contradicting the case definition. Thus ρ is not constant on B ; but then by Lemma 3.1 we have $\sum_{y \in B} \psi \rho(y) = 0$, so $\delta^K \mathcal{F}\phi = 0$. \square

Lemma 5.11. *Let $\phi(x, z)$ be a definable global test function, of several variables, and $\psi(y, z) = \mathcal{F}_x \phi(x, z)$. Then $\delta_y^K \psi = \delta_x^K \phi$.*

Proof. We take a representative ϕ_S of ϕ , with S sufficiently large, as usual. Since ϕ_S is smooth and of bounded support, we can view it as a function on a finite-dimensional k -space; the same goes for $\delta_y^K \psi$ and $\delta_x^K \phi$. To show that such functions into $K_e(T)$ are equal it suffices to show that

at any value b of z in any model of T , we have $\delta_y^K \psi(y, b) = \delta_x^K \phi(y, b)$. This is just Lemma 5.10 applied in T_b . \square

5.12. Characterization of δ^K . We remark that another proof of the Poisson summation formula is possible, using a self-dual *characterization* of δ^K among definable distributions.

The rational points functional clearly enjoys the following properties, where the equalities (1,2) take place in $K_{exp}(T_a)$:

- (1) $\delta^K(\psi(r(ax))\phi(x)) = \delta^K(\phi)$ for $a \in K$.
- (2) $\delta^K(\phi(ax)) = \delta^K(\phi)$ for $a \in K$.
- (3) $\delta^K(1_{\mathcal{O}}) = \mathbb{L} = [G_a]$

This is in fact a characterization of δ^K among definable distributions, provided that we invert $\psi(c) - 1$ for every $0 \neq c \in k$ uniformly, and that r is chosen so that K is self-dual for $r(xy)$. We sketch the proof.

Property (1) implies that δ^K concentrates on K -points. This uses the fact that $K^\perp = K$ and the invertibility of $\psi(c) - 1$ for $c \neq 0$. Property (3), along with (2), implies that $\delta^K(\phi) = 1$ for ϕ concentrated near 0, and with $\phi(0) = 1$. Using (2) again we obtain this for any rational point.

Properties (1),(2) are exchanged by the Fourier transform, while (3) is left invariant. Thus $\delta^K \circ \mathcal{F}$ enjoys the same properties, giving (under the stronger assumptions) another proof that $\delta^K \circ \mathcal{F} = \delta^K$.

6. THEORY OF VALUED FIELDS OVER A CURVE

6.1. Valued fields with a field of representatives. We begin with the local ingredient of the logical theory we will use. We take a three-sorted language of valued fields, with a sort VF for the valued field, a sort Γ for the value group, and a sort $\text{res}(VF)$ for the residue field. We take the usual language of valued fields, including, after Delon, a binary function symbol $\text{res}(\frac{x}{y})$; defined to be 0 when $\text{val}(x) < \text{val}(y)$, and otherwise the residue of $\frac{x}{y}$. In addition, the value group has a distinguished element $1 > 0$; and there is a function symbol $i : \text{res}(VF) \rightarrow VF$ for a section of res . Thus $k := i(\text{res}(VF))$ is a distinguished subfield $k \subset \mathcal{O}$, and the residue map is bijective on k .

The theory T_{loc} asserts that K is an algebraically closed field, val is a valuation, with valuation ring \mathcal{O} and maximal ideal \mathcal{M} , and $\mathcal{M} \oplus k = \mathcal{O}$.

Quantifier elimination for pairs of valued fields much more complex than ours is known; see [9], who attributes the case of T_{loc} to Delon. However the quantifier elimination in [9] takes to be basic formulas such as

$$(\exists x_1, x_2 \in k)(\text{val}(y - x_1 y_1 - x_2 y_2) > \text{val}(y_3))$$

asserting that y is closer to the vector space $y_1 k + y_2 k$ than y_3 to 0. The language we use allows only to take the coefficients of actual members of this vector space, not of nearby points. It seems simplest to give a direct proof of QE.

Lemma 6.2. *Let A be a subfield of $M \models T_{loc}$, closed under $i \circ \text{res}$. Then $A, k(M)$ are linearly disjoint over $k_A := A \cap k(M)$.*

Proof. We show by induction on n that if $a_1, \dots, a_n \in A$ are linearly dependent over $k(M)$, they are linearly dependent over k_A . If some $a_i = 0$ this is clear; this covers the case $n = 1$. Assume the statement holds below n , and let $a_1, \dots, a_n \in A$ be linearly dependent over $k(M)$. Reordering, we may assume $\text{val}(a_i) \geq \text{val}(a_n)$ for $i \leq n$. Dividing by a_n we may assume $a_n = 1$ and $\text{val}(a_i) \geq 0$ for each i . Let $b_i = i(\text{res}(a_i))$; then $b_i \in k_A$. Performing the column operation

$a_j \mapsto (a_j - b_j a_n)$, we may replace a_j by $a_j - b_j$ for $j < n$, so we may assume $\text{res}(a_j) = 0$ for $j < n$. Now for some $c_i \in k(M)$ we have $\sum_{i=1}^n c_i a_i = 0$, or $c_n = -\sum_{i=1}^{n-1} c_i a_i$. So $\text{val}(c_n) > 0$. But $c_n \in k$; so $c_n = 0$. Thus $\sum_{i=1}^{n-1} c_i a_i = 0$, and by induction c_1, \dots, c_{n-1} are linearly dependent over k_A . \square

Lemma 6.3. *T_{loc} admits quantifier elimination. k, Γ are stably embedded and strongly orthogonal. Their induced structure is the field and ordered group structure, respectively.*

Proof. Let \mathbb{U}, \mathbb{U}' be saturated. Write $k = k(\mathbb{U}), k' = k(\mathbb{U}')$

Claim . Let $f_k : k \rightarrow k'$ and $f_\Gamma : \Gamma(\mathbb{U}) \rightarrow \Gamma(\mathbb{U}')$ be given isomorphisms. Also let $f : A \rightarrow A'$ be an isomorphism between small subrings of \mathbb{U}, \mathbb{U}' , such that:

- i) f is a partial isomorphism of valued fields, carrying k to k' .
- ii) (Compatibility) $f(x) = f_k(x)$ for $x \in k_A$, and $\text{val} \circ f = f_\Gamma \circ \text{val}$ on A .
- iii) If $0 \neq a, b \in A$, $\text{val}(a) = \text{val}(b)$ then $i \circ \text{res}(a/b) \in A$ (hence the same holds for A').

Then $f_k \cup f_\Gamma \cup f$ extends to an isomorphism $\mathbb{U} \rightarrow \mathbb{U}'$.

Proof. Note in (iii) that $i \circ \text{res}(a/b)$ is the unique element c of k_A such that $\text{val}(a - bc) > \text{val}(a)$; so $f(c) = i \circ \text{res}(f(a)/f(b))$.

It suffices to show that f extends to a partial isomorphism with (i-iii) on a subring containing A and a given element c of \mathbb{U} . If this subring is not small, we can always replace it by a small subring still satisfying (i)-(iii), by Löwenheim-Skolem.

0) f extends to the field generated by A : it clearly extends to a valued field isomorphism, commuting with f_Γ . Any ratio b/b' of elements of the field of fractions has the form a/a' for some $a, a' \in A$, so (since $\text{res}(x/y)$ was taken to be a function of two variables) $\text{res}(b/b') = \text{res}(a/a')$ and commutativity with f_k is clear too. Hence we may assume A is a field.

1) By Lemma 6.2, k, A are linearly disjoint over $k \cap A$, and k', A' are linearly disjoint over $k' \cap A'$. Hence there exists a (unique) isomorphism $f_{kA} : kA \rightarrow k'A'$. Recall ([9], 1.1.2) that kA is separated over k in the sense of [1]: any finite dimensional k -subspace of kA has a basis c_1, \dots, c_n , such that $\text{val}(\sum_i a_i c_i) = \min \text{val}(a_i c_i)$ for any $a_1, \dots, a_n \in k$. It follows that $f_\Gamma \text{val}(x) \leq \text{val}(f_{kA}(x))$ for $x \in kA \setminus (0)$. By symmetry the other inequality holds. Hence f_{kA} is a valued field isomorphism compatible with f_Γ . Note that $\text{res}(kA) = \text{res}(k)$.

2) Extend f_{kA} to the field generated by kA using (0), and then to a valued field isomorphism $f_2 : \text{acl}(kA) \rightarrow \text{acl}(k'A')$. Note that (i), (iii) holds trivially.

3) Let $c \in \mathbb{U}$. If for some $a \in \text{acl}(kA)$ one has $\gamma = \text{val}(c - a) \notin \text{val}(A) = \text{val}(\text{acl}(kA))$, let $c' \in \mathbb{U}'$ be any element with $\text{val}(c' - f(a)) = f_\Gamma(\gamma)$. Then f_2 extends to $f_3 : \text{acl}(kA)(c) \rightarrow \text{acl}(k'A')(c')$ uniquely with $c \mapsto c'$. Indeed any element of $\text{acl}(kA)(c)$ is a product of elements of the form $c - d$, $d \in A$, and for such elements the data forces $\text{val}(c - d) = \min(\text{val}(d), \gamma)$; and similarly on the A' -side.

4) Otherwise, $kA(c)$ is an immediate extension of $\text{acl}(kA)$. Note that $\Gamma(\text{acl}(kA)) = \Gamma(A)$ and this is a small subset of Γ . Let b be the set of $\gamma \in \Gamma(A)$ such that the ball $B(\gamma) = B_\gamma(c)$ contains a point of $\text{acl}(kA)$; in this case $B(\gamma)$ is defined over $\text{acl}(kA)$, so $f_3(B(\gamma))$ is a ball over $\text{acl}(i(k)A')$. Using saturation, let c' be any point of $\cap_{\gamma \in b} f_3(B(\gamma))$. Then as in (3), f_2 extends uniquely to an isomorphism on $\text{acl}(kA)(c)$ with $c \mapsto c'$. Then f' satisfies the conditions for f . \square

To prove quantifier elimination, we need to extend a partial isomorphism on small substructures. Let $f : A \rightarrow A'$ be an isomorphism between small subrings of \mathbb{U}, \mathbb{U}' . By (0) of the Claim we may take A, A' to be fields. Find f_Γ and f_k compatible with f . Then the Claim implies that f extends to an isomorphism $\mathbb{U} \rightarrow \mathbb{U}'$.

Taking $\mathbb{U} = \mathbb{U}'$, the Claim gives the stable embeddedness (cf. [4], Appendix.) Since any automorphism of Γ and any automorphism of k lift, the statements on strong orthogonality and the induced structure on Γ and k follow. \square

If $A \leq M \models T$, $A(c)$ denotes the smallest substructure of M containing A, c . By *transcendence degree* of B/A we mean the transcendence degree of $B \cap \text{VF}$ over $A \cap \text{VF}$.

Remark 6.4. In general, for $c \in \text{VF}$, $A(c)$ can have infinite transcendence degree over A . (Let $\text{val}(t) > 0$ and consider $c = \sum a_i t^i$, with $a_i \in k$. Then each $a_i \in \mathbb{Q}(t, c)$.)

Recall that $\text{RV} = \text{VF}/(1 + \mathcal{M})$, $\text{rv} : \text{VF} \rightarrow \text{RV}$ and $\text{val}_{\text{rv}} : \text{RV} \rightarrow \Gamma$ are the natural maps, and RES is the subset of RV consisting of points whose image in Γ is definable. RES is a strict Ind-definable set, i.e. a union of definable sets.

Lemma 6.5. Let γ be a definable point of Γ , $V_\gamma = \text{val}_{\text{rv}}^{-1}(\gamma)$. Then there exists a definable map $g : V_\gamma \rightarrow \text{VF}$, with $\text{rv} \circ g = \text{id}$.

Proof. For some m , and some definable $c \in \text{VF}$, we have $m\gamma = \text{val}(c)$. Recall that k is embedded in \mathcal{O} . Let $A(\gamma) = \{y \in \text{VF} : \text{val}(y) = \gamma\}$, and $B(\gamma, c) = \{y \in A(\gamma) : y^m \in \text{ck}^*\}$. Then rv is injective on $B(\gamma, c)$; if $\text{rv}(y) = \text{rv}(y')$ then $y/y' \in k^*$, and $\text{rv}(y/y') = \text{res}(y/y') = 1$, so $y = y'$. Thus the restriction of rv to $B(\gamma, c)$ defines a bijection $r : B(\gamma, c) \rightarrow V_\gamma$, whose inverse is a section $V_\gamma \rightarrow B(\gamma, c) \subseteq A(\gamma)$. \square

Thus not only k but also RES admit a section into VF :

Lemma 6.6. RV is stably embedded, with the same induced structure as from ACVF .

Proof. This can be seen by extending a given automorphism f_{rv} of $\text{RV}(\mathbb{U})$, as in Lemma 6.3. In Step 1, note that when $\text{val}(\sum_i a_i c_i) = \min \text{val}(a_i c_i)$, it follows that $\text{rv}(\sum_i a_i c_i) = \sum_{i \in I_{\min}} \text{rv}(a_i c_i)$; where I_{\min} is the set of indices i such that $\text{val}(a_i c_i)$ is minimal, and where addition is defined naturally on elements of rv by $ex + ey = e(x + y)$, where $x, y, x + y \in k^*$ and $e \in \text{rv}$. In (2) we choose an extension to the algebraic closure compatible with the given isomorphism on rv ; this uses the stable embeddedness and known induced structure of RV in ACVF . Step (3) is the same, noting that the data determines $\text{rv}(c - d)$ too. Step (4) is identical; note that RV does not grow in immediate extensions. \square

6.7. Structure of definable sets. We take a further look at the structure of definable sets and raise some questions; the material here will not be needed for the main theorem, where only smooth functions will be used.

Definition 6.8. Let $T = T_{\text{loc}, A}$. Let $X \subseteq \text{VF}^n \times \Gamma^l$ be a definable set. A normal form for X is an expression

$$X = g(X^*)$$

where $\tilde{X} \subseteq \text{VF}^{n+m} \times \Gamma^l$ is an ACVF_A -definable set, and g an ACVF_A -definable function on \tilde{X} , and $X^* = \tilde{X} \cap (\text{VF}^n \times k^m) \times \Gamma^l$, such that:

(N) $g : X^* \rightarrow X$ is bijective.

We will say that X is normal via \tilde{X} , or via g .

Lemma 6.9. (1) If $X \subseteq \text{VF}^n \times \Gamma^l$ is the disjoint union of two definable sets X_1, X_2 , each having a normal form, then X has a normal form.

(2) Say $D \subseteq \text{VF}^n$ is weakly normal via $\tilde{D} \subseteq \text{VF}^{n+m+l}$ if \tilde{D} is ACVF -definable, and the projection $\pi' : \text{VF}^{n+m+l} \rightarrow \text{VF}^n$ induces a bijection $\tilde{D} \cap (\text{VF}^n \times k^m \times \text{VF}^l) \rightarrow D$. Then a weakly normal set D has a normal form, with g a coordinate projection.

(3) If D has a normal form, then it has one with g a coordinate projection.

Proof. (1) By adding 0's we may assume X_i is normal via $\pi : \tilde{X}_i \rightarrow X_i$ with $\tilde{X}_i \text{VF}^{n+m} \times \Gamma^l$. Let $\tilde{X}'_i = \tilde{X}_i \times \{i\}$ (with $i \in k$) and $\tilde{X} = \tilde{X}'_1 \cup \tilde{X}'_2 \subseteq \text{VF}^{n+(m+1)} \times \Gamma^l$. Then X is normal via $\tilde{X} \rightarrow X$.

(2) Let D' be a quantifier free formula equivalent in $ACVF_A$ to the projection of D^* to VF^{n+m} . Then D is normal via D' ; and if g is a function on D' , then g is $ACVF_A$ -definable iff $g \circ \pi''$ is $ACVF_A$ -definable.

(3) Say $X = g(X^*)$, as in Definition 6.8. Let $X'' = \{(g(x), x) : x \in \tilde{X}\}$. Then X has weakly normal form via X'' . By (2) it has normal form using a projection. \square

Lemma 6.10. (1) Every definable subset of $\text{VF}^n \times \Gamma^j$ has a normal form.

(2) if X is a definable set and $f : X \rightarrow \text{VF}$ a definable function, there exists \tilde{X} with X normal via $n : \tilde{X} \rightarrow X$, and an $ACVF_A$ -definable function F on \tilde{X} such that F agrees with $f \circ n$ on $\tilde{X} \cap X^*$.

Proof. We add to the language a new function symbol $D(x, y)$, defined to be $\frac{x}{y}$ when $y \neq 0$ and 0 when $y = 0$. Also replace $\text{res}(\frac{x}{y})$ by a unary function R , defined to be $i \circ \text{res}$ on \mathcal{O} , and elsewhere 0. Thus $\text{res}(\frac{x}{y})$ is now a composed term, $R(D(x, y))$.

To prove (1), let ϕ be a formula in variables $\text{VF}^n \times \Gamma^j$. The Γ coordinates play no role, and we will ignore them to simplify notation.

If ϕ does not involve R at all, it is already an ACVF formula. Otherwise, R occurs in some term in ϕ , and the innermost occurrence must have the form $R \circ t$, with t a rational function (more precisely, a term using $+$, \cdot , D .)

Let $\phi'(x, y)$ be the formula obtained from ϕ by replacing this instance of $R \circ t$ by y , and adding a conjunct:

$$(t \notin \mathcal{O} \implies y = 0) \& (t \in \mathcal{O} \implies \text{val}(y - t) > 0)$$

Then $\phi'(x, y) \& (y \in k)$ projects bijectively to ϕ . By induction, there exists an ACVF formula $\phi''(x, y, y')$ such that the projection $(\phi''(x, y, y') \& y' \in k^l) \rightarrow \phi'$ is bijective.

Then the projection $(\phi'' \& (y, y') \in k^{l+1}) \rightarrow \phi$ is also bijective, and shows that ϕ is normal.

For (2), apply (1) to the graph of f . We obtain a partition of f into sets Y_i , and ACVF-definable $\tilde{Y}_i \subseteq \text{VF}^n \times \text{VF} \times \text{VF}^m$, such that $\tilde{Y}_i \cap (\text{VF}^n \times \text{VF} \times k^m)$ projects injectively onto Y_i . Let $U_i = \{(x, z) \in \text{VF}^n \times \text{VF}^m : (\exists! y)(x, y, z) \in \tilde{Y}_i\}$. Then U_i is $ACVF_A$ -definable, and $\tilde{Y}_i \cap (\text{VF}^n \times \text{VF} \times k^m)$ is contained in the pullback \tilde{U}_i of U_i . Replacing \tilde{Y}_i by $\tilde{Y}_i \cap \tilde{U}_i$, we may assume the projection $\tilde{Y}_i \rightarrow \text{VF}^n \times \text{VF}^m$ is injective. Let \tilde{X}_i be the image of this projection. Then the projection $n_i : \tilde{X}_i \cap (\text{VF}^n \times k^m) \rightarrow X$ is injective; let X_i be the image. The composition $f \circ n_i$ is $ACVF_A$ -definable, since it is the section of the injective map $\tilde{Y}_i \rightarrow \tilde{X}_i$. \square

Corollary 6.11. Any definable set X admits a definable map $\xi : X \rightarrow \text{res}(\text{VF})^*$, whose fibers are definable by valued field formulas; $\xi^{-1}(c)$ is $ACVF_{F(i(c))}$ -definable.

Proof. Let $g : \tilde{X} \rightarrow X$ be a normal form for X , with $\tilde{X} \subseteq X \times \text{VF}^m$. Let $\pi : \tilde{X} \rightarrow \text{VF}^m$ be the projection, and let $h : X \rightarrow X^*$ be the inverse of the injective map $g|_{X^*}$. Let $\xi(x) = \text{res}\pi h(x)$. Then $\xi^{-1}(c) = g(\pi^{-1}(i(c)))$. \square

Definition 6.12. A definable X has VF-dimension $\leq n$ if there exists a definable $f : X \rightarrow \text{VF}^n$ whose fibers are internal to RV.

Unlike the case of pure Henselian fields of residue characteristic zero, the Zariski closure of $X \subseteq \text{VF}^m$ can have larger VF-dimension than X ; for instance k is Zariski dense, of VF-dimension zero.

We will use the valuation topology on VF, the discrete topology on RV, and the product topology on products.

Lemma 6.13. *Let X be a definable subset of VF^n . Then the boundary of X has dimension $< n$.*

Proof. We show that outside of a set of dimension $< n$, every point of X is an interior point. An RV^* -union of sets of dimension $< n$ still has dimension $< n$, so we may fiber over RV^* . By Corollary 6.11, we can take X to be defined by a valued field formula; but then the statement follows from the known one for ACVF. The same applies to the complement of X , so almost every point is interior either in X or in the complement. \square

6.14. Integration. In this subsection, we assume residue characteristic zero. We discuss an integration theory for more general sets than test functions. These results will not be required for the proof of Theorem 1.1.

Recall the measure-preserving bijections of [11]. These are definable bijections between subsets $X \subseteq \text{VF}^n \times \text{RV}^m$. We repeat the definition here, with the difference that we explicitly allow any bijection on the “discrete” RV sort.

Let RV^* denote $(\text{RV} \cup \Gamma)^m$. (For some unspecified m .)

Definition 6.15. *Fix n . An elementary admissible transformations of \mathbf{T}_A is a \mathbf{T}_A -definable function of one of the following types:*

(1) *Maps*

$$(x_1, \dots, x_n, y_1, \dots, y_l) \mapsto (x_1, \dots, x_{i-1}, x_i + a, x_{i+1}, \dots, x_n, y_1, \dots, y_l)$$

with $a = a(x_1, \dots, x_{i-1}, y_1, \dots, y_l) : \text{VF}^{i-1} \times \text{RV}^ \rightarrow \text{VF}$ an A -definable function of the coordinates y, x_1, \dots, x_{i-1} .*

(2) *Maps $(x_1, \dots, x_n, y_1, \dots, y_l) \mapsto (x_1, \dots, x_n, y_1, \dots, y_l, b(x, y))$ where $b : \text{VF}^n \times \text{RV}^* \rightarrow \text{RV}^*$ is any definable function.*

A function generated by elementary admissible transformations over A will be called an *admissible transformation* over A .

Recall also the category $\text{RV}_{\Gamma\text{-vol}}[n, \cdot]$ of definable subsets X of RV^* , along with definable functions $f : X \rightarrow \text{RV}^n$, and the lift Λ to the category of definable subsets of $\text{VF}^* \times \text{RV}^*$ of dimension $\leq n$. Morphisms are definable maps. The lift of (X, f) is just $X \times_{f, \text{rv}} (\text{VF}^*)^n$. The \cdot indicates that no bound is placed on the dimension “discrete” RV^* -component of $\text{VF}^* \times \text{RV}^*$, or correspondingly on the fibers of f . We will omit this symbol, as well as the letters referring to Γ -indexed volume since only such volumes will be considered, and write $\text{CRV}[\leq n]$ for $\text{RV}_{\Gamma\text{-vol}}[n, \cdot]$. $\text{CRV}[\leq n]$ is the direct sum of $\text{CRV}[k]$ over $k \leq n$. (cf. Def. 5.21; see also Theorem 8.29.) Here we will only consider volumes indexed by Γ .

Lemma 6.16. *Every definable $X \subseteq \text{VF}^n \times \text{RV}^*$ is in admissible bijection with the lift λU of some object U of $\text{CRV}[\leq n]$.*

Proof. Let $\xi : X \rightarrow \text{res}(\text{VF})^l$ be as in Corollary 6.11. By [11], for each $c \in \xi(X)$, there exists an $\text{ACVF}_{A(i(c))}$ -definable object $Y(c)$ of $\text{CRV}[\leq n]$ $f_c : X(c) \rightarrow \Lambda Y(c)$. Note that $Y(c)$ is actually $A_{\text{RV}}(c)$ -definable, by stable embeddedness. Putting these together (as in [11] Lemma 2.3), we obtain an object Y of $\text{CRV}[\leq n]$, and an admissible bijection $X \rightarrow U$. \square

Let E_μ be the equivalence relation on definable sets generated by the following steps:

- 1) If there exists an admissible $f : X \rightarrow Y$, then $(X, Y) \in E_\mu$.
- 2) Let X, Y differ only by a set of VF-dimension $< n$. Then $(X, Y) \in E_\mu$.

Corollary 6.17. *Every definable $X \subseteq \text{VF}^n \times \text{RV}^*$ is E_μ -equivalent to a lift ΛU of some object U of $\text{CRV}[n]$.*

Proof. The lift of objects of $\text{RV}[k]$ for $k < n$ has VF-dimension $< n$, and can be discarded. \square

Let $K_+(T_A)$ be the semi-group of E_μ -classes of T_A -definable subsets of $\text{VF}^n \times \text{RV}^*$.

Corollary 6.18. *There exists an isomorphism $K_+(T_A) \rightarrow K_+(\text{CRV}[n])/I$, for some congruence I .*

Question 6.19. *Determine generators for I . Do the generators of I_{sp} of [11] suffice?*

Let Res denote the residue field.

Lemma 6.20. *If X, Y are definable subsets of Res^n , and $([X], [Y]) \in I$, X, Y defined over a finitely generated domain $R \subseteq A$, then for some $0 \neq d \in R$, for all homomorphisms $h : R[d^{-1}] \rightarrow F_q$ into a finite field, the F_q -varieties $X_h = X \otimes_R F_q, Y_h = Y \otimes_R F_q$ have the same number of points in F_q .*

Proof. Otherwise, embed R into the ultraproduct of the F_q ; interpret $X(F_q)$ as the volume of ΛX ; the fact that $([\Lambda X], [\Lambda Y]) \in E_\mu$ implies that (for some d) for all such h , $\Lambda X_h, \Lambda Y_h$ have the same measure. Here Λ and E_μ are interpreted in $F_q((t))^a$ with the natural section $F_q \rightarrow F_q((t))^a$. \square

Assume $\text{val}(A) = \mathbb{Z}$. By Lemma 6.13, any definable subset of VF^n has a well-defined Kontsevich motivic integral, with values in the completion $A[[L^{-1}]]$ where A is the Grothendieck ring of varieties over A_{res} , and L is the class of the affine line. This integral is linear on constructible functions. Moreover the power series obtained represent rational functions, by the proof of Denef-Loeser.

6.21. Valued fields over a curve. Let \mathfrak{f} be a field, and C a smooth curve over \mathfrak{f} .

We describe here a first-order theory $\mathbf{T} = \text{ACVF}_{C, \mathfrak{f}}$ convenient for adelic work. It has the following sorts.

k - an algebraically closed field with a distinguished field of constants \mathfrak{f} . k is endowed with the language of \mathfrak{f} -algebras.

$C(k)$ (when C has a distinguished point, or genus ≥ 2 , we can take $C \leq \mathbb{P}^n$ so a special sort is not necessary, but we take one nonetheless.)

Γ - an ordered Abelian group, with distinguished element $1 > 0$.

VF . This sort comes with a map $\text{VF} \rightarrow C(k)$; the fibers are denoted VF_x . Each VF_x comes with valuation ring \mathcal{O}_x , a surjective homomorphism $\text{res}_x : \mathcal{O}_x \rightarrow k$, and a ring embedding $i_x : k \rightarrow \mathcal{O}_x$, such that $\text{res}_x \circ i_x = \text{Id}_k$. Also, a map $v_x : \text{VF}_x \setminus \{0\} \rightarrow \Gamma$, denoting a valuation with valuation ring \mathcal{O}_x . For any variety V over \mathfrak{f} , we obtain using i_x a variety over VF_x ; let $V(\text{VF}) = \bigcup_{x \in C(k)} V(\text{VF}_x)$, a set fibered over $C(k)$.

We identify k with its image $i_x(k)$.

As a final element of structure, we have a function $c : C(k) \rightarrow C(\text{VF})$, such that $c(x) \in C(\text{VF}_x)$; and for any $f \in k(C)$, $\text{val}(c(x)) = \text{ord}_x(f) \cdot 1$.

Technically, the above depends on a specific chart for C as an abstract algebraic variety over \mathfrak{f} . We take C to be a complete curve. If C is given as a union of open subvarieties C_i embedded in n_i -dimensional affine space, $i = 1, \dots, m$, then c is by n_i -tuples c_i^j of functions $c_i^j : C(k) \rightarrow \text{VF}$;

the theory will state the natural compatibilities, and up to obvious bi-interpretation will not depend on the chart.

We note that Γ serves as the value group of each of the valued fields VF_x . This will be important later, for instance when considering divisors on C of degree 0. The identification of the various value groups

Lemma 6.22. *\mathbf{T} is complete. k and Γ are stably embedded. The induced structure on k is the \mathfrak{f} -algebra structure. The structure on Γ is the ordered group structure, with distinguished element 1.*

Moreover \mathbf{T} has quantifier-elimination.

Proof. Let \mathbb{U}, \mathbb{U}' be saturated models of \mathbf{T} of the same cardinality. We wish to show that \mathbb{U}, \mathbb{U}' are isomorphic. Since $k \cong k'$ and $\Gamma \cong \Gamma'$ we may assume they are equal. In particular $C(k) = C(k')$. Now for any $x \in C(k)$ we have VF_x and VF'_x . The structure $(VF_x, k, \Gamma, c(x))$ is a saturated model of T_{loc} , and so is $(VF'_x, k, \Gamma, c'(x))$. Moreover by quantifier-elimination, $tp_{T_{loc}}(c(x)/k, \Gamma) = tp_{T_{loc}}(c'(x)/k, \Gamma)$, since the valued fields $k(c(x)), k(c'(x))$ are both isomorphic to $k(C)$ with the valuation corresponding to the point x . Hence by stable embeddedness of k, Γ there exists an isomorphism $f_x : VF_x \rightarrow VF'_x$, with $f_x(c(x)) = c'(x)$. Putting together these isomorphisms we obtain an isomorphism $f : \mathbb{U} \rightarrow \mathbb{U}'$.

The same proof shows that any automorphism of k, Γ extends to an automorphism of \mathbf{T} . Hence they are stably embedded and their induced structure is as stated. In the same way we can extend partial isomorphisms, hence quantifier elimination. \square

See Appendix 1 for the notion of “piecewise definable”.

Lemma 6.23. *$k(C)$ is piecewise definable over k .*

Proof. For \mathbb{P}^1 this is completely elementary. The elements of $k(\mathbb{P}^1)$ can be identified with pairs (f, g) of polynomials, relatively prime, and with g monic or $g = 0, f = 1$. For any given bound on the degrees, this is clearly a constructible set. Given a pair f, g of polynomials of degree $\leq n$, the valuation at 0 of f/g is bounded between $-n$ and n , and each of the possible values is constructible. Moreover $PGL_2(k)$ acts constructibly on the polynomials of degree $\leq n$ and on the set of valuations, and for $\phi \in PGL_2(k)$, the valuation of f/g at $c = \phi^{-1}(0)$ equals the valuation of f^ϕ/g^ϕ at 0.

For other curves, the proof is more easily carried out using quantifier elimination for T^{loc} . We show that when $k \models ACF_{\mathfrak{f}}$, the structure $k(C)$ in the language $L_{\mathfrak{f}}^g$ is piecewise definable over k . Since the induced structure on k from $ACVF_{C;\mathfrak{f}}$ is the \mathfrak{f} -algebra structure, it suffices to piecewise interpret $k(C)$ in a model of $ACVF_{C;\mathfrak{f}}$, provided that the interpreted copy of $k(C)$ is contained in $dcl_M(k)$.

The field $k(C)$ as an \mathfrak{f} -algebra was treated in Lemma 10.1 (2); moreover the proof there shows that $k(C)$ as a differential algebra, i.e. with the additional ternary relation $df = hdg$, is also piecewise definable over k .

We have to show that on each limited subset Y of $k(C)$, and each n , the relation V_n restricted to $Y \times C(k)$ is definable. We may take C to be embedded in \mathbb{P}^m . For some d , each element f of Y is a quotient of two homogeneous polynomials of degree $d = d(Y)$. Now $V_n(f, \alpha)$ holds iff $ord_\alpha(f) = n$ iff $val(f(c(\alpha))) = n \cdot 1$. Hence $V_n \cap (Y \cap C(k))$ is definable in M , over \mathfrak{f} . By quantifier-elimination in $ACVF_{C;\mathfrak{f}}$ and the fact that the induced structure on k from $ACVF_{C;\mathfrak{f}}$ is the \mathfrak{f} -algebra structure, V_n is definable.

We now define \mathbf{t} . Let $t_0 \in F(C)$ be non-constant. For any $\alpha \in C(k^a)$ such that $t_0 : C \rightarrow \mathbb{P}^1$ is unramified at α , we let $t_\alpha = t_0 - t_0(\alpha)$. There are finitely many values of α where t_0 is ramified; there we make some choice of parameter t_α , in a Galois invariant way. Let $\mathbf{t} = \{(\alpha, t_\alpha) : \alpha \in C(k^a)\}$.

Given again a limited subset Y , we need to compute $\text{res}_a(f\omega)$, uniformly in $f \in Y$ and $\alpha \in C(k)$. Using the definable parameter t_α and the differential structure, we find the unique $g \in C(k)$ with $f\omega = gdt_\alpha$. Now g lies in some limited definable set Y' . For some n , for each $y \in Y'$, we have $v_\alpha(g - (c_n(y)t_\alpha^{-n} + \dots + c_1(y)t_\alpha^{-1})) \geq 0$ for some (unique) $c_n(y), \dots, c_1(y) \in k$. Then $\text{res}_a(f\omega) = c_1(g)$. \square

6.24. Adelic definable sets. Let C be a curve over a field \mathbb{f} , $F = \mathbb{f}(C)$. We continue working with the theory $ACVF_{C;\mathbb{f}}$; definability relates to this theory. For a subset of k^n , it is equivalent to $ACF_{\mathbb{f}}$ -definability, while for a subset of VF_v^n (with v a point of C) it is equivalent to $T_{\mathbb{f}(v)}^{\text{loc}}$ -definability. These theories admit quantifier-elimination; when we use a quantified formula, we mean the quantifier-free equivalent. The witness is not assumed to exist rationally over F .

Example 6.25. We will define algebras D_v with subrings R_v for each v , and consider R_v^* -conjugacy classes. On $D_v(F)$ -points, this will not be the same as $R_v^*(F)$ -conjugacy. Moreover, if \tilde{D}_v, \tilde{R}_v is another such pair, we define a point x of $D_v(F)$ to be conjugate to a point \tilde{x} of $\tilde{D}_v(F)$ if there exists an isomorphism $h : D_v \rightarrow \tilde{D}_v$ with $h(R_v) = \tilde{R}_v$ and $h(x) = \tilde{x}$. We are interested only in the case where $D_v(F), \tilde{D}_v(F)$ are non-isomorphic division algebras; so such an isomorphism never exists rationally; the notion of integral conjugacy across forms will nevertheless be useful.

Let X be a definable set, V a variety over F . By a *definable function* $f : X \rightarrow \prod_{v \in C} V(VF_v)$ we mean a definable function f on $X \times C$, such that $f(x, v) \in V(VF_v)$. We view f as a function into the product. We will only consider the case that $X \subseteq \text{dcl}(k)$.

Let ϕ be a formula in the language of $ACVF_{C;\mathbb{f}}$ enriched with additional unary function symbols $\xi_1, \dots, \xi_n : C \rightarrow VF$. Then ϕ defines a subset $Z = \phi(\prod_{v \in C} VF_v^m) := \{z \in \prod_{v \in C} VF_v^m : \phi(z)\}$. Note that Z is *definable-in-definable-families*, i.e. $f^{-1}(Z)$ is a definable subset of X whenever $f : X \rightarrow \prod_{v \in C} VF_v^m$ is definable. Similarly if Z is defined by an infinite disjunction of formulas ϕ_k , then Z is *Ind-definable-in-definable-families*. This extends to subsets of $\prod_{v \in C} V(VF_v)$ where V is any variety over F .

Assume V is an affine variety, or at any rate that a subset $V(\mathcal{O}_v)$ of integers is given in some way for each v , and $V(\mathcal{O}_v)$ is definable uniformly in v . Then $\prod_v V(\mathcal{O}_v)$ is definable in the above sense (by the formula $(\forall v \in C)(\xi(v) \in \mathcal{O}_v)$), and so if $f : X \rightarrow \prod_v V(VF_v)$ is definable then $f^{-1}(\prod_v V(\mathcal{O}_v))$ is a definable subset of X .

Example 6.26. Let D_v, R_v be as in Example 6.25, and write $R^* = \prod_v R_v$. Let $a \in D(\mathbb{A})$ be definable. The R^* -orbit O of a is then definable; for $x \in D$,

$$x \in O \iff (\forall v)(\exists u \in R_v^*)(uau^{-1} = x)$$

In particular, $O(K) := O \cap D(K)$ is Ind-definable.

If R_v is a bounded subset of D_v for all v , then $O(K)$ is a limited subset of K ; hence by Lemma 6.22 it is definable over k .

In practice we will have R_v bounded only for all $v \neq v_0$; in this case $O(K)$ is not limited, but is nearly so, and we will still be able to assign to $O(K)$ a class in an appropriate quotient of the Grothendieck ring.

Of course one can have a definable R^* -orbit with no definable element, and the same considerations hold.

Example 6.27. Let K_v^h be the Henselization within VF_v of $K = k(C)$. This is an Ind-definable set with parameter v : In the notation of § 6.21, an element of K_v^h has the form $h(a, c(v))$ for some definable function h , and an n -tuple a from k .

Note that any definable function $f : X \rightarrow \prod_v VF_v$ has image contained in $\prod_v K_v^h$.

The adelic points of $\Pi_v V(VF_v)$ are defined as the union over all m of the set $\cup_{|w|=m} \Pi_{v \in w} V(VF_v) \times \Pi_{v \notin w} V(\mathcal{O}_v)$. We let $V(\mathbb{A})$ be the set of adelic points of $\Pi_{v \in C(k)} V(K_v^h)$. For any subset Y of $C(k)$ we also let $V(\mathbb{A}_Y)$ be the set of adelic points of $\Pi_{v \in Y} V(K_v^h)$. We saw that any definable function on X into the adelic points maps into $V(\mathbb{A})$.

Let $f : X \rightarrow \Pi_{v \in C} V(VF_v)$ be a definable function. If $f(x) \in V(\mathbb{A})$ for any x we say that $f : X \rightarrow V(\mathbb{A})$. By compactness, this implies the existence of a fixed N , such that for each x , for some $w(x) \subseteq V$ of size N , $f(x) \in \Pi_{v \in w(x)} V(K_v^h) \times \Pi_{v \notin w} V(\mathcal{O}_v)$; moreover by stable embeddedness of k , if $X = W(k)$ is a constructible set over k , the map $x \mapsto w(x)$ can be taken to be constructible.

We will be interested in $G(\mathbb{A})/H(\mathbb{A})$ for certain congruence subgroups H ; since K is dense in K_v^h , for our purposes K_v^h and K could be used interchangeably, and we discuss K_v^h only to clarify the link with the classical presentation.

When H is a definable subgroup of a group G , and $X \subseteq G$, we write X/H for the image of X in G/H .

Example 6.28. Let G be a group scheme over F , and $H_v = G(\mathcal{O}_v)$; or more generally assume H_v is a uniformly $ACVF_v$ -definable subgroup of G . Write $H = \Pi_v H_v$. Given a finite subset s of C , let $(G/H)_s = \Pi_{v \in s} G(K_v^h)/H_v$; in our examples H_v will be open in G , and we will have $G(K_v^h)/H_v = G(K)/H_v$. At all events, $G(K)$ or $G(K_v)$ are Ind-definable sets; write $G(K) = \cup_n G[n]$, with $G[n]$ definable. Let $(G/H)_s[n] = \Pi_{v \in s} G[n]/H_v$. Then $(G/H)_s = \cup_n (G/H)_s[n]$ is Ind-definable.

When $s \subset s'$ we have a map $(G/H)_s \rightarrow (G/H)_{s'}$, mapping $a \mapsto a'$ where $a'(v) = H_v/H_v$ for $v \in s' \setminus s$. Consider the direct limit of $(G/H)_s$ over all finite subsets s of C . The directed set here is the set $P_\omega C$ of finite subsets of C ; this is itself Ind-definable, limit of the definable sets $P_{\leq n} C$ of all $\leq n$ -element subsets of C . Since $(P_\omega C)(M)$ depends on the model M , the direct limit $\lim_{s \in P_\omega(C)} (G/H)_s$ is not, as presented, an Ind-definable set. However, for fixed n , the disjoint union $\sqcup_{s \in P_{\leq n}(C)} (G/H)_s[n]$ is a definable set. There is a natural isomorphism $\lim_{s \in P_\omega C} (G/H)_s = \lim_n \sqcup_{s \in P_{\leq n} C} (G/H)_s[n]$. The latter is Ind-definable, and we denote it $(G/H)(\mathbb{A})$.

Note that when G is Abelian, $(G/H)(\mathbb{A})$ is an Ind-definable group, but is not in general an Ind-(definable group). A basic case: $G = G_m, H = G_m(\mathcal{O})$; in this case $(G/H)(\mathbb{A})$ is the group of cycles on C .

Example 6.29. Let $V = T$ be a multiplicative torus. We have a uniformly definable homomorphism $T(VF_v) \rightarrow X_*(T) \otimes \Gamma$; taking sums we obtain a homomorphism $\text{val}_T : T(\mathbb{A}) \rightarrow X_*(T) \otimes \Gamma$; the kernel is denoted $T(\mathbb{A})^0$. Then $T(\mathbb{A})^0$ is Ind-definable.

Example 6.30. Let C be a curve, S a nonempty finite definable subset. Let $I = \Pi_{v \in C} G_m(K_v^h)$, $T_S = \Pi_{v \in S} G_m(K_v^h) \times \Pi_{v \notin S} G_m(\mathcal{O}_v)$. We also have a diagonal embedding of K^* into I . Then I/KT_S is weakly representable if and only if for any $a, b \in S$, the image of $a - b$ in the Jacobian of C is a torsion point. In particular, if $|S| = 1$ then I/KT_S is weakly representable.

Indeed for checking weak representability we may change the base so that C has a rational point 0, in fact we can take $0 \in S$. In this case the Jacobian J can be identified with $I^0/K^* \Pi_{v \in C} G_m(\mathcal{O}_v)$ where I^0 is the group of ideles of degree 0. We have a natural homomorphism $J \rightarrow I/KT_S$. It is surjective, since $I_0 K_0 = I$. The kernel generated by the points $a - b$ with $a, b \in S$. If one of these points is not torsion, then condition (2) of Lemma 2.6 will not hold. If all points $a - b$ are torsion then the kernel is finite, and (1,2) of Lemma 2.6 are clear.

Example 6.31. The double-coset equivalence relation $(R \cap T)xT(K) = (R \cap T)yT(K)$ is Ind-definable. In this case for definable $t, t' : C \rightarrow T(\mathbb{A})$ we have tEt' iff $(\exists k \in T(K))(t^{-1}tk \in$

$(R \cap T))$ which is Ind-definable. The formula inside the quantifier is definable: $(\forall v \in C)(y \in (R \cap T)_v)$.

Let $\mathcal{T} = T(\mathbb{A})/(R \cap T)$ as in Example 6.28. It follows that the embedding of $T(K)$ in \mathcal{T} is Ind-definable.

When $T = G_m$, and $(R \cap T) = G_m(\mathcal{O}_v)$, $(R \cap T) \backslash T(\mathbb{A})^0/T(K)$ is the Jacobian of C . With more general $(R \cap T)$ one obtains Rosenlicht generalized Jacobians of C . If $T = R_{C'/C}G_m$ is obtained from a cover C' of C by reduction of scalars of G_m , this is a generalized Jacobian of C' .

7. DIVISION RINGS

7.1. Adelic structure of cyclic division rings. \mathfrak{f} is a perfect field. \mathbf{k} is a model of $ACF_{\mathfrak{f}}$. We denote $F = \mathfrak{f}(t)$, $K = \mathbf{k}(t)$. This section is purely algebraic, and adeles (or repartitions), when they are mentioned, are treated essentially classically.

When only one valuation is involved, we denote the valuation ring by \mathcal{O} . When many valuations v are involved, we denote the Henselization of F as a valued field by F_v , the valuation ring by \mathcal{O}_v , and let $\mathcal{O} = \prod_v \mathcal{O}_v$.

Let L be a commutative semi-simple algebra defined over \mathfrak{f} , of dimension n , with $\text{Aut}_{\mathfrak{f}}(L)$ a cyclic group; let g be a generator. Let

$$D_{g,t} = L[s]/(sa = g(a)s, s^n = t)$$

We view $D_{g,t}$ as an $ACF_{\mathfrak{f}(t)}$ -definable algebra. For most of the discussion, we fix g and denote $D = D_{g,t}$. Eventually we will compare D to another form $\dot{D} = D_{\dot{g},t}$ with \dot{g} another generator of $\text{Aut}(L)$.

We are mainly interested in the case that $L(\mathfrak{f})$ is a cyclic Galois field extension of \mathfrak{f} ; in this case we let $\mathfrak{l} = L(\mathfrak{f})$, see § 2.17. As we will see, $D(F)$ is then a division algebra over $F = \mathfrak{f}(t)$.

Let $d_1, \dots, d_n \in L(\mathfrak{f})$ be an \mathfrak{f} -basis for $L(\mathfrak{f})$. Then $(d_i s^j : 1 \leq i \leq n, j \in \mathbb{N})$ is a basis for D over K .

Let v_0, v_∞ be the valuations of $\mathbf{k}(t)$ with $v_0(t) > 0$, $v_\infty(t) < 0$, and let v_1 be the valuation of $F = \mathfrak{f}(t)$ over \mathfrak{f} with $v_1(t-1) > 0$.

Let $N = R^* \cap D^*(K)$. Note that \mathbf{L} is a normal subgroup of N , (indeed N is the normalizer of \mathbf{L} in D^* .)

Over $\mathfrak{l}(t)$ there exists a representation of D on L ; namely L acts diagonally, while s is the product of the permutation corresponding to g with a diagonal matrix $(1, \dots, 1, t)$. This defines an isomorphism of D with M_n over \mathfrak{l} .

For any valuation v on $\mathfrak{f}(t)$ with $v(t) = 0$, we define a subring R_v of D :

7.2. For $v(t) = 0$, for $x_{ij} \in \mathbf{k}(t)_v$, $i, j = 0, \dots, n-1$ we have:

$$\sum x_{ij} d_j s^i \in R_v \iff \bigwedge_{i,j=0}^{n-1} x_{ij} \in \mathcal{O}_v$$

When $v(t) \neq 0$, we let $R_v = D$. On two occasions we will refer to the ring defined by the same formula 7.2 at $v = 0$; but in this case we will denote it S_0 .

The family of rings is uniformly definable in the theory $ACVF_{\mathfrak{f},C}$ of § 6. We refer to this choice of subring, for each valuation v , as the adelic structure.

We write $\mathcal{O}_{\mathbb{A}}$ for $\prod_v \mathcal{O}_v$, and let $R = \prod_v R_v$. Denote the $n \times n$ matrix algebra by M_n . Let $\mathbf{M}_v = \mathcal{M}_v R_v$. Let Z be the center of D^* . Write $\mathbf{L} = L(\mathbf{k})$.

Lemma 7.3. (1) For $v(t) = 0$, $(D, L, R_v) \cong_{\mathfrak{l}(t)} (M_n, L, M_n(\mathcal{O}_v))$. This characterizes R_v uniquely up to conjugacy by Lemma 11.7 (3).

(2) $R(K) = \mathbf{L}[s, s^{-1}]$. If $x \in D(F)$ and $x \in ZR_v^*$ for each v with $v(t) = 0$, then $x \in Z(F)R(F)$.

When $L(\mathfrak{f})$ has no 0-divisors, we have $L[s]^*(\mathfrak{f}) = L^*(\mathfrak{f})$, and $R(F)^* = \mathbf{L}[s, s^{-1}]^*(\mathfrak{f}) = L^*(\mathfrak{f})s^{\mathbb{Z}} = N$.

- (3) a) Let v be such that $v(t) = 0$. Let F' be a Henselian valued field extension of (F, v) with residue field \mathfrak{f}' . Assume (*): $D_{g, \bar{t}}(\mathfrak{f}) \cong M_n(\mathfrak{f})$. Then $(D, R_v) \cong (M_n, M_n(\mathcal{O}))$ by an ACVF $_{F'}$ -definable isomorphism.
 b) If $v(t-1) > 0$, or if \mathfrak{f}' has a unique (and Galois) field extension of order n , then (*) holds.
- (4) Fix v , and denote by \bar{t} the image of t in R_v/\mathbf{M}_v . If $v(t) = 0$, then $R_v/\mathbf{M}_v \cong D_{g, \bar{t}}$. Assume $v(t) > 0$, and $L(\mathfrak{f})$ is a field. Then $D(F_v)$ is a valued division ring, and $s^{\mathbb{Z}}U = D(F_v)^*$ where $U = \{x : v(x) = 0\}$.
- (5) Assume $v(t) = 0$, n prime. Let $\tau \in F_v$ be a uniformizer, i.e. $v(F_v) = \mathbb{Z}v(\tau)$. If $D_{g, \bar{t}}(\mathfrak{f})$ is a division ring, then any nonzero element of $D(F_v)$ can be written as $z + \tau^m y$, where $z \in F_v, m \in \mathbb{Z}, y \in R_v$ and the residue $y + \mathbf{M}_v$ of y is regular semisimple.
- (6) Assume $L(\mathfrak{f})$ has no zero divisors. Then every left ideal of $\mathbf{L}[s]$ is principal.

Proof. (1) Over \mathbf{l} we have a definable basis e_1, \dots, e_n of L consisting of idempotents, such that $g(e_i) = e_j$ (where $j = i+1 \pmod n$.) The change of basis from (d_i) to (e_i) is effected by a matrix in $GL_n(\mathbf{l})$ and so it does not effect the adelic structure. We have an \mathbf{l} -definable isomorphism $D \rightarrow GL_n$, mapping e_i to the standard matrix e_{ii} (with 1 at (i, i) and zeroes elsewhere), and mapping s to the product of the cyclic permutation matrix, with the diagonal matrix $(1, 1, \dots, t)$. This is an isomorphism between D and GL_n ; when $v(t) = 0$, it is straightforward to verify that the ring R_v defined in (7.2) maps to $GL_n(\mathcal{O})$.

(2) Let $x = x_{ij}d_js^i \in D(k(t))$, and assume $x \in R_v$ for all $v \neq 0, \infty$. Then $v(x_{ij}) \geq 0$ for all such v , so $x_{ij} \in k[t, t^{-1}]$. Hence $\sum x_{ij}d_js^i \in L[s, s^{-1}]$.

Let $x \in D(F)$ and suppose for each v with $v(t) = 0$ we have $x = z_v r_v$ with $z_v \in Z, r_v \in R_v^*$. Let $a = \det(x) \in F$. Then $v(a) = v(z_v^n)$ for each $v \neq 0, 1$; so $v(a)$ is divisible by n in $val F_v$, for each such v , and we can find $b \in F$ with $nv(b) = v(a)$ for each such v . Dividing by b we may assume $v(a) = 0$ for $v \neq 0, 1$. So $v(z_v) = 0$, so $z_v \in \mathcal{O}_v^*$ for each such v ; but then $x \in R_v^*$.

When $L(\mathfrak{f})$ has no 0-divisors, we have $L[s]^*(\mathfrak{f}) = L^*(\mathfrak{f})$, and $L[s, s^{-1}]^*(\mathfrak{f}) = L^*(\mathfrak{f})s^{\mathbb{Z}}$: it suffices to see that if $f, g \in L[s](\mathfrak{f})$ and $fg = s^n$ then $f, g \in Ls^{\mathbb{N}}$; this is clear by viewing f, g as non-commuting polynomials in s ; the product of a non-monomial with a polynomial is always non-monomial, by considering lowest and highest terms.

(3) (b) Note that $D_{g, 1} = M_n$ over any field, with the integral structure (7.2) coinciding with $M_n(\mathcal{O}_v)$.

a) Since $D_{g, \bar{t}}(\mathfrak{f}) \cong M_n(\mathfrak{f})$, we have $\bar{t} = N_{L(\mathfrak{f})/\mathfrak{f}}(\mathbf{c})$ for some $\mathbf{c} \in L(\mathfrak{f})^*$. (Proof: find an element s' normalizing L such that conjugation by s' has the effect of g on L , and such that $(s')^n = 1$; such an element exists in the matrix ring. It follows that $\mathbf{c} = s^{-1}s'$ centralizes L , so $s^{-1}s' \in L$; and we compute $N(\mathbf{c}) = s^n = \bar{t}$.) Let $h(X) = X^n + \dots \pm \bar{t}$ be the minimal polynomial of \mathbf{c} over \mathfrak{f} . Since \mathfrak{f} is perfect, h must have some nonzero monomials of degree strictly between 0 and n . So $h'(\mathbf{c}) \neq 0$. Lift h to $H(X) = X^n + \dots \pm t \in \mathcal{O}[X]$. Using Hensel's lemma find $c \in L(\mathfrak{f}_v)$ with $H(c) = 0$. This is an invertible element of R_v , and $t = N_{L(F_v)/F_v}(c)$. Dividing by c we reduce to the case of $D_{g, 1}$.

If $v(t-1) = 1$, i.e. $\bar{t} = 1$, we may take $\mathbf{c} = 1$. If the residue field has a unique (and Galois) extension of order n , then the norm map from this extension to \mathfrak{f}' is surjective.

(4) Clearly $L(F_v) = LF_v$ is a field, namely an unramified extension of F_v ; we view it as a valued field. Any nonzero element of $a \in D(F_v)$ may be written uniquely as $\sum_{i=0}^{n-1} u_i s^i$ with $u_i \in LF_v$; define $v(a) = \min_i \{v(u_i) + \frac{i}{n} \text{val}(t)\}$. Note that this minimum is attained at a unique i , since the summands are distinct elements of $\mathbb{Q}\text{val}(t)$, even modulo $\mathbb{Z}\text{val}(t)$. From this it

follows that $v(ab) = v(a) + v(b)$. In particular $D(F_v)$ has no zero-divisors; since D is finite-dimensional over the center, it follows that $D(F_v)$ is a division ring. We have $s^{\mathbb{Z}}U = D(F_v)^*$ where $U = \{x : v(x) = 0\}$.

(5) Note first that if \mathbf{D} is an n -dimensional division ring over a perfect field \mathfrak{f} (or any field \mathfrak{f} if $n \neq \text{char}(\mathfrak{f})$), then any nonzero element \bar{a} of E is either central or regular semisimple. Indeed over $\mathfrak{f}^{\text{alg}}$ there is an isomorphism $\alpha : \mathbf{D} \rightarrow GL_n$, and the set s of eigenvalues of $\alpha(\bar{a})$ does not depend on the choice of α . Since n is prime, these eigenvalues are all equal or all distinct. If all are equal, say to γ , then $\gamma \in \mathfrak{f}$ and $\bar{a} - \gamma$ is non-invertible, hence equal to 0.

Note also that $\mathbf{M}_v(F_v) = \tau R_v(F_v)$. For $x \in D(F_v)$, let $v(x)$ be the unique m with $\tau^{-m}x \in R_v \setminus \tau R_v$. Since $R_v(F_v)/\tau R_v(F_v)$ is a division ring, we have $v(xy) = 0$ whenever $v(x) = v(y) = 0$, and it follows that $v(xy) = v(x) + v(y)$ in general.

Let $a \in D(F_v)$. If a is central there is nothing to prove. Otherwise $ab \neq ba$ for some $b \in D(F_v)$. For any central z we have $ab - ba = (a - z)b - b(a - z)$, so $v(ab - ba) \geq \min(v((a - z)b), v(b(a - z))) = v(a - z) + v(b)$, or $v(a - z) \leq v(ab - ba) - v(b)$. Thus there exists a central z with $v(a - z)$ maximal; subtracting z , we may assume it is zero, i.e. $v(a) \geq v(a - z)$ for any central z . Multiplying by $\tau^{-v(a)}$ we may further assume that $v(a) = 0$. Now $v(a - z) \leq 0$ for any central $z \in R_v$; as central elements of $\mathbf{D} = R_v/\mathbf{M}_v$ lift to central elements of R_v , the residue \bar{a} of a cannot be central. By the first paragraph we have \bar{a} regular semisimple.

(6) The proof of the Euclidean algorithm for the commutative polynomial ring works equally well for the twisted polynomial ring $\mathfrak{l}[s]$, and so does the proof that every left ideal in a Euclidean ring is principal. \square

Let $\mathcal{O}' = \prod_{v \neq \infty} \mathcal{O}_v$, let \mathbb{A}'_F be the adeles over F without the factor at $v = \infty$, and let $R' = S_0 \times \prod_{v \neq 0, \infty} R_v$.

Let $R'(F)$ be the set of elements of $D(F)$, whose images in D_v fall into R_v for every v , and whose image in D_0 falls into S_0 . The rings R_v , $R'(F)$ have the property that left invertible elements are right invertible, and their invertible elements are denoted $(R_v)^*$, $(R')^*$.

Lemma 7.4. *Assume $L(\mathfrak{f})$ is a field. Then the natural map $D^*(F) \rightarrow (R')^* \setminus D^*(\mathbb{A}')$ is surjective. In particular, $D^*(F) \rightarrow R^* \setminus D^*(\mathbb{A})$ is surjective.*

Proof. To prove surjectivity, let $c \in D(\mathbb{A}'_F)$. Multiplying by a nonzero element of $\mathfrak{f}[t]$, we may assume $c \in R' \cap D(\mathbb{A}'_F)^*$. Let $I = R'c \cap D(F)$. This is an ideal in $R' \cap D(F)$. By Lemma 7.3 (6) we have $I = R'(F)d$ for some $d \in D(F)$. So $R'c \cap D(F) = (R' \cap D(F))d$. Using $c \in D(\mathbb{A}'_F)^*$ we see that $\det(c) \neq 0$ and $v(\det(c)) > 0$ for at most finitely many v , so $I \neq (0)$, indeed $I \cap \mathfrak{f}[t] \neq (0)$; thus $d \neq 0$. We have $R'cd^{-1} \cap D(F) = R' \cap D(F)$. Thus $cd^{-1} \in R'$, and $1 \in R'cd^{-1}$ so $cd^{-1} \in (R')^*$. \square

We aim to show that the class in a localized Grothendieck ring of the set of rational points on a given integral conjugacy class, does not depend on the form of the division ring. At this point we prove a special case: if one of these sets is nonempty, so is the other.

When c is a regular semi-simple element, let T_c denote the centralizer of c .

Lemma 7.5. *Let $D = D_{g,t}$, $\dot{D} = D_{\dot{g},t}$, where \dot{g} is another generator of $\text{Aut}(L)$. Fix a place v and let $c \in D(F_v)$ be a regular semisimple element.*

There exists $\dot{c} \in \dot{D}(F_v)$ such that (D, R_v, c) and $(\dot{D}, \dot{R}_v, \dot{c})$ are $k(t)_v$ -isomorphic. (We then say that c, \dot{c} match integrally at v .)

Moreover, there exists a definable isomorphism $i : T_c \rightarrow T_{\dot{c}}$, mapping $T_c \cap R_v$ to $T_{\dot{c}} \cap \dot{R}_v$.

Proof. Note:

(#) the class of $D_{\dot{g},\bar{t}}$ is a multiple of the class of $D_{g,\bar{t}}$ in the Brauer group of \mathfrak{f} .

If $D_{g,\bar{t}} \cong M_n$ then by Lemma 7.3 (1,3) we have $(D, R_v) \cong (M_n, M_n(\mathcal{O}))$. Moreover by (#), $D_{\dot{g},\bar{t}} \cong M_n$ and the statement is clear.

Assume therefore that $D_{g,\bar{t}}$ is a division ring. Then so is $D_{\dot{g},\bar{t}}$. Moreover the same conjugacy classes are represented in $D_{g,\bar{t}}, D_{\dot{g},\bar{t}}$. The reason is that a field extension \mathfrak{f}' of \mathfrak{f} embeds in $D_{g,\bar{t}}$ iff $[\mathfrak{f}' : \mathfrak{f}] = n$ and $D_{g,\bar{t}}$ splits over \mathfrak{f}' ([8], Theorem 4.8, p. 221); it is the same for $D_{\dot{g},\bar{t}}$; but by (#) it is clear that $D_{g,\bar{t}}$ splits over \mathfrak{f}' iff $D_{\dot{g},\bar{t}}$ does.

If $D(F_v)$ is a division ring, then by the same argument so is $\dot{D}(F_v)$, and they represent the same classes. In this case there exists $\dot{c} \in \dot{D}(F_v)$ with $(D, c), (\dot{D}, \dot{c})$ isomorphic over $k(t)$. In view of the definition of R_v this suffices in case $v(t) \neq 0$. Assume now that $v(t) = 0$.

By Lemma 7.3 (5) we may assume $c \in R_v$ and c has regular semisimple residue \bar{c} . By the above, there exists $'\bar{c} \in D_{\dot{g},\bar{t}}$ conjugate to \bar{c} . Lift $'\bar{c}$ to some $'c \in \dot{R}_v(F_v)$, with $'c$ lying in some unramified field extension of F_v ; this extension must be isomorphic to $F_v(c)$, and so there exists an element \dot{c} in this extension, with residue $'\bar{c}$, and such that \dot{c}, c satisfy the same minimal polynomial over F_v . Over $k(t)_v$ we have $R_v \cong M_n(\mathcal{O}) \cong \dot{R}_v$, and moving the question to $M_n(\mathcal{O})$, the two conjugate elements c, \dot{c} with regular semi-simple residues are clearly $GL_n(\mathcal{O})$ -conjugate. (A K -basis of eigenvectors v_i for c , with $v_i \in \mathcal{O}^n \setminus \mathcal{M}\mathcal{O}^n$, is an \mathcal{O} -basis for \mathcal{O}^n by Nakayama.)

The “moreover” follows from the main statement: any two $k(t)_v$ -isomorphisms $(D, c) \rightarrow (\dot{D}, \dot{c})$ differ by a conjugation of D by an element of T_c ; such a conjugation induces the identity on $T_{\dot{c}}$; hence all such isomorphisms induce the same isomorphism $i : T_c \rightarrow T_{\dot{c}}$, and i is definable. Since some $k(t)_v$ -isomorphism $(D, c) \rightarrow (\dot{D}, \dot{c})$ respects the integral structure this must be true for i . \square

Remark 7.6. The analogue of Lemma 7.5 is not true for S_0^* in place of R_0^* -conjugacy or matching. Indeed the element $s \in D$ does not match any element of \dot{D} in this sense. The result does become true if one uses the group $s^{\mathbb{Z}}S_0^*$ (an extension of order n), and replaces $k((t))$ by $k((t))^{alg}$ in the definition of matching.

Lemma 7.7. *Let $D = D_{g,t}, \dot{D} = D_{\dot{g},t}$, where \dot{g} is another generator of $Aut(L)$. Let $c \in D(F)$. Then there exists $\dot{c} \in \dot{D}(F)$ such that for any v , (D, R_v, c) and $(\dot{D}, \dot{R}_v, \dot{c})$ are $k(t)_v$ -isomorphic. (We will say that c, \dot{c} match adelically.)*

Moreover, there exists a definable isomorphism $T_c \rightarrow T_{\dot{c}}$, preserving adelic structure, i.e. mapping $T_c \cap R_v$ to $T_{\dot{c}} \cap \dot{R}_v$ for each v .

Proof. If $L(\mathfrak{f})$ is not a field, then D, \dot{D} are definably adelically isomorphic over F , so the statement is clear. Assume $L(\mathfrak{f})$ is a field.

Choose $\dot{b} \in \dot{D}(F)$ with $(D, c) \cong_{ACF_{k(t)}} (\dot{D}, \dot{b})$. By Lemma 7.5, for any v , there exists $\dot{c}_v \in \dot{D}(F_v)$ with $(D, R_v, c) \cong_{ACVF_{k(t)_v}} (\dot{D}, \dot{R}_v, \dot{c}_v)$. In particular, whenever $v(t) = 0$ and $c \in R_v$, we have $\dot{c} \in \dot{R}_v$, so $\dot{c} \in \dot{D}(\mathbb{A})$, and \dot{c}, \dot{b} are $\dot{D}(\mathbb{A})$ -conjugate.

By Lemma 7.4, they are $\dot{D}^*(F)\dot{R}^*$ -conjugate. So we can find b which is $\dot{D}^*(F)$ -conjugate to \dot{b} and for each v , $G(R_v)$ -conjugate to \dot{c} .

The “moreover” is as in the proof of the corresponding statement in Lemma 7.5. \square

8. RATIONAL POINTS IN INTEGRAL CONJUGACY CLASSES

In this section we define integral conjugacy classes O , and compute the class in the appropriate Grothendieck ring of $O(K) = O \cap D(K)$. The formula obtained will be independent from the form of the division ring. We begin with a discussion of local conjugacy classes. Let $\mathfrak{f}, k, F = \mathfrak{f}(t), K = k(t), L, L(\mathfrak{f}), D = L[s]$ be as in the previous section.

We will use a Grothendieck ring associated with $ACF_{\mathfrak{f}}$. Recall ϵ_L from § 2.17. Let K_0 be any $K_+(ACF_{\mathfrak{f}})$ -semi-algebra, such that $\epsilon_L = 0$; this can be achieved by passing to the quotient

K^* of K associated to the theory T^* whose models are fields $\mathfrak{f}' \geq \mathfrak{f}$ such that $\mathfrak{f}'(t)$ has no zero divisors (cf. § 2.10). Let Gr be the multiplicative monoid generated by the class $[\mathcal{T} : T(K)]$ for any torus T , and also by all classes of group varieties; this includes especially n and $[\mathbf{L}^*]$. Let $K = K_0[Gr^{-1}]$ (cf. § 2.2).

If \mathfrak{f}' is an extension field of \mathfrak{f} , we will write $K_{\mathfrak{f}'} = K(ACF_{\mathfrak{f}'}) \otimes_{K(ACF_{\mathfrak{f}})} K$. We also write K_c for $K_{\mathfrak{f}(c)}$. All classes $[X]$ in this section refer to K or some K_c , as specified.

Definition 8.1. *Let ϕ be a local test function on D_v . ϕ is R_v^* -invariant if for any $\mathfrak{f}' \models T^*$, for $c \in D(\mathfrak{f}'(t)_v)$ and $d \in R_v^*(\mathfrak{f}'(t)_v)$ we have $\phi(c) = \phi(d c) \in K_{\mathfrak{f}'}$.*

Note that any ϕ whose domain is formally empty counts as invariant.

It would also be possible to define R_v^* -invariance of local test functions, in terms of the action of R_v^* on the subquotients $K(N, M)^n = t^{-N} k[[t]]^n / t^M k[[t]]^n$; see the proof of Lemma 8.5 (c).

Lemma 8.2. *Let E be a finite dimensional algebra (with 1) over a perfect field F . If $c, c' \in E(F)$ and c, c' are $E^*(F^{alg})$ -conjugate, then c, c' are $E^*(F)$ -conjugate. The same is true for the loop ring $E[[x]]$, or in general for any pro-finite dimensional algebra.*

Proof. Let $X = \{a \in E^* : aca^{-1} = c'\}$, viewed as a pro-algebraic variety. We have to show that $X(F) \neq \emptyset$. If $E = \varprojlim E_n$ then $X = \varprojlim X_n$ with X_n defined in the same way; so the question reduces to the finite-dimensional case.

Note that X is coset of $T_c = \{a \in E^* : aca^{-1} = c\}$. Now T_c is the group of units of the ring $\bar{T}_c = \{a \in E : ac = ca\}$. By Lemma 3.9, X has a rational point. \square

Definition 8.3. *Let $c \in D(\mathfrak{f}'(t)_v)$, $\dot{c} \in \dot{D}(\mathfrak{f}'(t)_v)$. We say that c, \dot{c} are in matching conjugacy classes if $(D, R_v, c), (\dot{D}, \dot{R}_v, \dot{c})$ are isomorphic over $k(t)_v$.*

Let $\phi, \dot{\phi}$ be R_v^ -invariant functions on D, \dot{D} respectively. We say that $\phi, \dot{\phi}$ match if for any $\mathfrak{f}' \models T^*$, $\phi(c) = \dot{\phi}(\dot{c}) \in K_{\mathfrak{f}'}$ whenever $c \in D(\mathfrak{f}'(t)_v)$, $\dot{c} \in \dot{D}(\mathfrak{f}'(t)_v)$ are in matching conjugacy classes.*

This definition works thanks to Lemma 8.4, Lemma 8.5 (b) below.

Lemma 8.4. *For $c, c' \in D(\mathfrak{f}')$, $\mathfrak{f}' \models T^*$, we have $(D, R_v, c) \cong (D, R_v, c')$ over $k(t)$ iff c, c' are $R_v^*(\mathfrak{f}')$ -conjugate.*

Proof. For $v(t) \neq 0$ there is nothing to show, since $R_v = D$. Assume $v(t) = 0$.

Note that over $k(t)$, R_v is a matrix ring over a valuation ring, Lemma 7.3 (1). The stabilizer of R_v in D_v^* is ZR_v^* , where Z is the center of D^* . Now if $g \in M_n(k(t))$ and over K^{alg} we have $g \in ZGL_n(\mathcal{O})$, then $g \in ZGL_n(\mathcal{O}_K)$, as one easily sees by considering the valuation of matrix coefficients. Hence $ZR_v^*(K^{alg}) \cap M_n(K) = Z(K)R_v^*(K)$. One can also check directly that the $GL_n(K)$ -stabilizer of $M_n(\mathcal{O}_K)$ is $ZM_n(\mathcal{O}_K)$.

So c, c' are $ZR_v^*(k(t))$ -conjugate, and hence also $R_v^*(k(t))$ -conjugate. $R_v^*(k(t))$ may be viewed as the projective limit of $R_v^*(k(t))/(1 + \mathcal{M}_\alpha)$ over the various congruence ideals \mathcal{M}_α of R_v . By Lemma 8.2, c, c' are $R_v^*(\mathfrak{f}'(t))$ -conjugate. \square

In order to match anything, ϕ must be strongly R_v^* -invariant; but by Lemma 8.5 (b) this is automatic.

Lemma 8.5. (a) *Let $a, a' \in D(\mathfrak{f}(t)_v)$ and assume a', a' are R_v^* -conjugate over $k(t)_v$. Then they are conjugate by an element of $R_v^*((\mathfrak{f}(t)_v))$.*

(b) *Let $\phi : D(K_v) \rightarrow K$ be R_v^* -invariant (Definition 8.1). Then ϕ is strongly R_v^* -invariant (Definition 2.14). In other words, if $\mathfrak{f}' \models T^*$, for $c \in D(\mathfrak{f}'(t)_v)$ and $c' \in R_v^*(\mathfrak{f}'(t)_v)$ we have $\phi(c) = \phi(c') \in K_{\mathfrak{f}'}$ provided that c, c' are R_v^* -conjugate over $k(t)_v$.*

(c) Let $\phi, \dot{\phi}$ be R_v^* - (resp. \dot{R}_v^* -) invariant local test functions on D_v, \dot{D}_v respectively. To test them for matching, it suffices to show that $\phi(c) = \dot{\phi}(\dot{c}) \in K_{F'}$ for matching $c, c' \in D(F')$, $\mathfrak{f}' \models T^*$, $F' = \mathfrak{f}'(t)$. Also, if W is a proper subvariety of D , it suffices to consider $c, c' \in D(F') \setminus W$.

Proof. (a) If $v(t) \neq 0$, this is immediate from Lemma 8.2. Otherwise, multiplying by a scalar element we may assume $a, a' \in R_v(\mathfrak{f}(t)_v)$. Then the statement follows from the profinite part of Lemma 8.2.

(b) Immediate from (a) and the definitions.

(c) We prove the case $v(t) \neq 0$; the case $v(t) = 0$ is similar but easier, and will not be needed. Take $v(t) > 0$ and $\mathfrak{f}' = \mathfrak{f}$ to simplify notation; write $F = \mathfrak{f}(t)$.

Consider the ring defined by the formula 7.2 at $v(t) > 0$; we will refer to it as S_0 . Multiplying an element of D by a sufficiently large power of t will put it in S_0 , so it suffices to consider D_0^* -conjugacy on elements of S_0 .

S_0^* -conjugacy is a finer relation than D_0^* -conjugacy, so ϕ is S_0^* -invariant, and similarly $\dot{\phi}$. As in (b), ϕ is strongly S_0^* -invariant. On the other hand ϕ is invariant under additive $t^m S_0$ translations for large enough m , and in fact descends to a function $\bar{\phi}$ on the quotient $S[m] = S_0/t^m S_0$. It follows that $\bar{\phi}$ is $S_0/t^m S_0$ -translation invariant; using Lemma 8.2 for the ring $S_0/t^m S_0$ as in (a,b), we see that $\bar{\phi}$ is strongly S_0^* -invariant. By Lemma 2.16, $\bar{\phi}$ descends to a function ψ on the set Y of conjugacy classes of $S_0/t^m S_0$. (We use here that $S[m]^*$ and its quotient subgroups have invertible classes in $K = K[Gr^{-1}]$.) Let \dot{Y} be the corresponding construction for \dot{D} .

Let E be the equivalence relation of D^* -conjugacy, restricted to $S_0(F_v)$. Then E respects $t^m S_0$ -translation, so descends to $S_0(F_v)/t^m S_0(F_v)$. Being coarser than S_0^* -conjugacy, it induces an equivalence relation \bar{E} on the image Y_{F_v} of $S_0(F_v)$ in Y . Matching induces a bijection $Y_{F_v}/E \rightarrow \dot{Y}_{F_v}/\bar{E}$. Now ψ and the corresponding function $\dot{\psi}$ are E -invariant (resp. \bar{E} -invariant), and the question is whether the functions they induce on $Y_{F_v}/E, \dot{Y}_{F_v}/\bar{E}$ correspond. But $S_0(F_v)/t^m S_0(F_v) = S_0(F)/t^m S_0(F)$, so every element of Y_{F_v}, \dot{Y}_{F_v} can be represented by an F -rational point (outside a given subvariety W); hence it suffices to check invariance at these points. \square

8.6. Global conjugacy classes. By the *integral conjugacy class* of a regular semi-simple element $c \in D(K)$ we mean the set O of all elements $d \in D(K)$ such that for any place v , (D_v, R_v, c) and (D_v, R_v, d) are isomorphic over K .

Note that for almost all v , we have $c \in R_v$, and so $d \in R_v$, both with regular semisimple reduction.

Let $\mathbf{L} = L(k)$. Let $N_{L/k}$ be the norm map on L . Let L_1 be the kernel of the norm map $\mathbf{L} \rightarrow k$:

$$L_1 = \{a \in \mathbf{L}^* : \prod_{i=1}^n g^i(a) = 1\}$$

By Hilbert 90 $x \mapsto g(x)/x$ is a surjective map $\mathbf{L}^* \rightarrow L_1$; there exists a constructible section of this function; in particular $[L^*] = [L_1][G_m]$. In this expression, L^* and L_1 are viewed as $ACF_{\mathfrak{f}}$ -definable sets; to avoid confusion, we will use the notation $\mathbf{L} = L(k)$ when interested in k -points. Similarly $\mathbf{L}_1 = L_1(k)$.

Let $N = R^* \cap D^*(K)$, and let \mathbf{N} be the image of N in $PD = D^*/Z$. By Lemma 7.3 (2), we have $N = R^* \cap D^*(K) = L(k)[s, s^{-1}]^* = L(k)^* s^{\mathbb{Z}}$. We view $\mathbf{N} = N/t^{\mathbb{Z}}$ as a subgroup of PD , containing $\mathbf{L} := L(k)^*/k^*$ as a subgroup of index n . Note that we have definable representatives $1, s, \dots, s^{n-1}$ for the cosets of $L(k)^*/k^*$ in \mathbf{N} . Thus in the Grothendieck ring $K(ACF_{\mathfrak{f}})$ we have $[\mathbf{L}] = n[L^*/G_m] = n[L_1]$.

While N is not a limited set, it is easy to describe a limited set N' of elements of N , such that N, N' have the same image in PD . So \mathbf{N} can be seen as a constructible set over \mathfrak{f} .

Let $CN = \cup_{b \in N} T_b$, where T_b is the centralizer of b in D^* . The definition of CN is geometric, i.e. if $c \in D(F')$ and c commutes with some $n \in N(K^{alg})$, then c commutes with some $n' \in N(F')$. Indeed if $n = as^i$ with $a \in \mathbf{L}$, then c commutes with any element xs^i where a' lies in the linear space $\{x \in L : cxs^i = xs^i c\}$, intersected with the Zariski open set $\det(x) \neq 0$.

Lemma 8.7. *Let O be an integral conjugacy class. If $O \cap CN \neq \emptyset$ then $O \subset CN$.*

Proof. Let $c \in O \cap CN$, and let $c' \in O$, with say $c, c' \in D(F')$. Then $c' = rcr^{-1}$ for some $r \in D^*(F')$, and for each v we have $rT_c = r_v T_c$ for some $r_v \in R_v$. Since $c \in CN$, there exists $n \in N$ with $n \in T_c$. So $rn r^{-1} = r_v n r_v^{-1}$. It follows that $rn r^{-1} \in R_v$ for each v (since $N \subseteq R_v$.) So $rn r^{-1} \in \cap_v R_v = N$. Thus rcr^{-1} commutes with an element of N , so $c' \in CN$. \square

Let $D = D_{g,t}$, $\dot{D} = D_{\dot{g},t}$; define $\dot{N} = \mathbf{L}\dot{s}^{\mathbb{Z}}$, and $\dot{CN} = \cup_{b \in \dot{N} \setminus \mathbf{L}} T_b(K)$.

Lemma 8.8. *There exists a definable bijection $CN \rightarrow \dot{CN}$, mapping integral conjugacy classes to matching integral conjugacy classes.*

Proof. Over \mathbf{l} we have a bijection $h : D \rightarrow \dot{D}$ preserving adelic structure, i.e. preserving R_v for each v . In particular as noted above we have $h(N) = \dot{N}$, and h maps integral conjugacy classes to matching ones. Moreover $h(\mathbf{L}) = \mathbf{L}$ since e.g. \mathbf{L} is the subspace generated by $N \cap (1 + N)$. $h|_{\mathbf{L}}$ must be a Galois automorphism; by pre-composing with conjugation by a power of s we may assume that $h|_{\mathbf{L}} = Id_{\mathbf{L}}$. It remains to consider $CN \setminus \mathbf{L}$.

If $a \in CN \setminus \mathbf{L}$ then there exists some element $bs^i \in T_a \cap \mathbf{L}s^i$, $i \neq 0$. We may take $i = 1$, since if $b_1 \in \mathbf{L}$ is any element with $g^i(b_1)/b_1 = g(b)/b$ then $b_1 s \in T_{bs^i}$ (and using Hilbert 90). Hence $CN \setminus \mathbf{L} = \cup_{b \in \mathbf{L}^*} (T_{bs}(K))$, and similarly on the \dot{D} side.

Given $b \in \mathbf{L}^*$, the elements $bs \in D^*$, $b\dot{s} \in \dot{D}^*$ have the characteristic polynomial $X^n - N_{\mathbf{L}/\mathbf{k}}(b)t$. If $b's$ has the same characteristic polynomial as bs , then $b' = cb$ for some $c \in L_1$. In this case by Hilbert 90 we have $a^{-1}g(a) = c$ for some $c \in \mathbf{L}$; so conjugation by a takes bs to $b's$ while preserving integral conjugacy. This shows that the integral conjugacy class of bs is precisely $L_1 bs$. It follows that $h(L_1 bs) = L_1 b\dot{s}$.

In particular $h(L_1 s) = L_1 \dot{s}$; we may assume $h(s) = \dot{s}$. So $h(bs) = h(b)\dot{s} = b\dot{s}$, for $b \in L^*$.

The isomorphism $h|_{T_{bs}}$ takes bs to $b\dot{s}$, and as such, is determined up to conjugation by the centralizer of bs . Hence the restriction $h|_{T_{bs}}$ is determined uniquely, given b ; we denote it $\alpha_b : T_{bs} \rightarrow T_{b\dot{s}}$. So α_b is $\mathbf{f}(b)$ -definable.

Note that if $k \in \mathbf{k}^*$ then $T_{kbs} = T_{bs}$, $T_{b\dot{s}} = T_{kb\dot{s}}$ and $\alpha_{kb} = \alpha_b$.

Note that α_b takes any element of T_{bs} to an element of the matching integral conjugacy class (since this is true of h .)

Also, two centralizers $T_{b_1 s}, T_{b_2 s}$ are disjoint or equal; they are equal when $b_1 s, b_2 s$ commute, i.e. when $b_1/b_2 = g(b_1)/g(b_2)$; equivalently when $b_1/b_2 \in \mathbf{k}^*$.

Thus we may define $\alpha : \cup_{b \in \mathbf{L}^*} T_{bs} \rightarrow \cup_{b \in \mathbf{L}^*} T_{b\dot{s}}$ by $\alpha(x) = \alpha_b(x)$ when $x \in T_{bs}$. It is clearly a bijection, and for any two matching integral conjugacy classes O, \dot{O} , it restricts to a bijection $\cup_{b \in \mathbf{L}^*} (O \cap T_{bs}(K)) \rightarrow \cup_{b \in \mathbf{L}^*} (\dot{O} \cap T_{b\dot{s}}(K))$. \square

Let O be an integral conjugacy class of regular semi-simple elements. O is a $\mathbf{T}_f = ACVF_{C,f}$ -Ind-definable subset of $D(K)$ (§6.21.) By Lemma 6.22, O is also Ind-definable for ACF_f . O is T^* -limited, since R_v is bounded for any v with $v(t) = 0$, and R_v/Z is T^* -limited for $v(t) \geq 0$, where Z is the center of D^* .

We define the canonical torus with adelic structure T associated to $O(K)$ as follows. For any $c \in O(K)$ let $T_c = \{a \in D^* : ac = ca\}$. If $c, c' \in O \cap D(K)$ there exists $d \in D(K^{alg})$ with $dcd^{-1} = c'$; such a d gives an isomorphism $ad_d : T_c \rightarrow T_{c'}$, preserving adelic structure,

$ad_d(x) = dxd^{-1}$; but ad_d does not depend on the choice of d , so we can write $f_{c,c'} = ad_d$. Clearly $f_{c',c''}f_{c,c'} = f_{c,c''}$. We can factor out this system to obtain a torus T , with adelic structure; given any $c \in O$ we have an isomorphism $f_c : T \rightarrow T_c$; and T is definable over the field of definition of O .

Let $\mathcal{T} = (Z(\mathbb{A})(R \cap T)) \backslash T(\mathbb{A})$. This is an Ind-definable group. The diagonal embedding $T(K) \rightarrow T(\mathbb{A})$ induces a homomorphism $T(K) \rightarrow \mathcal{T}$. The image of $T(K)$ in \mathcal{T} is Ind-definable, and hence so is the corresponding coset equivalence relation E on \mathcal{T} .

Proposition 8.9. *Assume $O(K) \cap CN = \emptyset$, and let $c \in O(K)$. Then $\mathcal{T}/T(K)$ is (T^*, nL_1) -representable, and we have*

$$[O(K)] = n[L_1][\mathcal{T} : T(K)]$$

in K_c .

Proof. Fix $c \in O(K)$. We identify T with T_c . We will verify the conditions of Lemma 2.11, with V, X, Z of that lemma corresponding here to $\mathcal{T}, O(K), \mathbf{N}$, respectively, and E being the $T(K)$ -coset equivalence relation.

We identify \mathbf{L}^*/k^* with L_1 , and $\mathbf{N} = N/t^{\mathbb{Z}}$ with the semi-direct product of L_1 with $s^{\mathbb{Z}}/s^{n\mathbb{Z}}$. N acts on $O(K)$ by conjugation inducing an action of \mathbf{N} ; the latter action is free because of the assumption: $O \cap CN = \emptyset$.

Define $f : O(K) \rightarrow \mathcal{T}/T(K)$ as follows. Let $d \in O(K)$. Then $d = aca^{-1} = bcb^{-1}$ for some $a \in D^*(K), b \in R^*$. We have $b^{-1}a \in T(\mathbb{A})$. If also $d = a'ca'^{-1} = b'cb'^{-1}$ with $a' \in D^*(K), b' \in R^*$, then $a^{-1}a' \in T_c(K)$ and $b^{-1}b' \in (T_c \cap R) = (R \cap T)$, so $b^{-1}a, b'^{-1}a'$ have the same class in $(R \cap T) \backslash T(\mathbb{A})/T(K)$. and in particular in $((R \cap T)Z) \backslash T(\mathbb{A})/T(K)$. Let $f(d)$ denote this class. The graph of f pulls back to an Ind-definable subset of $O(K) \times \mathcal{T}$. So f is Ind-definable.

If $f(d) = f(d')$, then for some $a, a' \in D^*(K), b, b' \in R^*$ we have $b^{-1}a = b'^{-1}a'$ away from 0, and $d = aca^{-1} = bcb^{-1}$, $d' = a'ca'^{-1} = b'cb'^{-1}$. So $b'b^{-1} = a'a^{-1} \in D^*(K) \cap Z(\mathbb{A})R^*$. By Lemma 7.3 (2), we have $a'a^{-1} \in ZN$. it follows that $ZN a = ZN a'$ so d, d' are \mathbf{N} -conjugate. So the fibers of f are \mathbf{N} -orbits.

We now show: for any $f' \geq f$ and any f' -definable \mathbf{N} -orbit U , $\cup_{d \in O(K)(f')} f(d) \subseteq \cup_{d \in \mathcal{T}(f')} dT(K)$.

Let $d \in O(K)(f')$. We have to show that $f(d)$ contains a point of $\mathcal{T}(f')$. Indeed by Lemma 8.2 there exists $a \in D^*(f')$ with $aca^{-1} = d$. By definition of $O(K)$ there exists $b \in R^*$ with $bcb^{-1} = d$; $b(R \cap T)$ is uniquely determined. Thus $(R \cap T)b^{-1}a \in (R \cap T) \backslash T(\mathbb{A}) = \mathcal{T}$ is determined, and $Z_0(R \cap T)b^{-1}a \in \mathcal{T}(f')$.

Now assume $f' \models T^*$; so $L(f')$ is a field. Let $Z_0(R \cap T)e \in \mathcal{T}(f')$, $e \in T(\mathbb{A})$. We have $(R \cap T)e \in \mathcal{T} = (R \cap T) \backslash T(\mathbb{A}) \leq R^* \backslash D^*(\mathbb{A})$. By Lemma 7.4 there exists $a \in D^*(f'(t))$ with $R^*a = R^*e$; so $b^{-1}a = e$ for some $b \in R^*$. Since $e \in T(\mathbb{A})$ we have $aca^{-1} = bcb^{-1}$. Let $d = aca^{-1}$. Then $d \in O(K)$ and $f(d) = (R \cap T)eT(K)$. \square

Remark 8.10. It follows from the proposition, in particular, that there exists a definable W with $\mathcal{T} \setminus W$ formally empty, and such that E is definable on W . It is not the case that E is definable-in-definable-families. It is possible, but unnecessary for our purposes, to modify $(R \cap T)$ by using a definable subgroup H_0 at 0 such that H_0 is bounded modulo the center, but $H_0(f'((t)))$ contains $D(f'((t)))$ for any $f' \models T^*$. One can take $H_0 = s^{\mathbb{Z}}ZS_0^*$. This has no effect on classes in K since $\epsilon_L = 0 \in K$, but yields an equivalence relation that is definable-in-definable-families.

If D, \dot{D} are two forms of M_n with the same adelic structure (§7.1) it is possible to match their integral conjugacy classes; see Lemma 11.3 for the matching of R_v^* -classes of D to those

of \dot{D} , for any v . We say that c, \dot{c} match if their R_v^* -conjugacy classes match for any v . Write \dot{O} for the \dot{R}^* -class corresponding to O .

Corollary 8.11. *Assume $O(K)$ has an F -rational point. Then $[O(K)] = [\dot{O}(K)]$.*

Proof. By Lemma 7.7, $\dot{O}(K)$ also has an F -rational point. So Proposition 8.9 is valid in K , and by either this or Lemma 8.8 we obtain $[O(K)] = [\dot{O}(K)]$. \square

Remark 8.12. Let $\tilde{J} = \mathcal{T}^0/T(K)$; it is an algebraic group over \mathfrak{f} . The relation between $[\mathcal{T}^0 : T(K)]$ and $[\tilde{J}]$ is very close. The interesting case is that the torus T does not split, and we have $T = R_{C'/C}G_m$ for a certain curve C' over $C = \mathbb{P}^1$; i.e. T is obtained by restriction of scalars from $\mathfrak{f}(C')$ to $\mathfrak{f}(C)$. The adelic constructions commute with restriction of scalars, and \tilde{J} can be identified with a certain Rosenlicht generalized Jacobian of C' . For any field \mathfrak{f}' such that $C'(\mathfrak{f}') \neq \emptyset$, it can be shown that there exists a rational section $\tilde{J} \rightarrow \mathcal{T}^0$, and therefore $[\mathcal{T}^0 : T(K)] = [\tilde{J}]$, so we simply have the class of an algebraic group. In general while \mathcal{T}^0, \tilde{J} and the exact sequence $T(K) \rightarrow \mathcal{T}^0 \rightarrow \tilde{J}$ are all defined over \mathfrak{f} , no section exists, so the quotient group cannot quite be identified with the algebraic group; cf. [15].

Remark 8.13. on a group theoretic level, we are using a special case of the bijection

$$(R \cap K) \backslash (RS \cap KT) / (S \cap T) \cong (R \cap S) \backslash (RK \cap ST) / (K \cap T)$$

valid for any subgroups R, S, K, T of a group G . The bijection maps the $(R \cap K), (S \cap T)$ -double coset of $rs = kt$ to the $(R \cap S), (K \cap T)$ -double coset of $r^{-1}k = st^{-1}$. In our case we take $S = T$ and have $RK = G$, yielding $T \cap R \backslash T / T \cap K$ on the right.

Remark 8.14. Let O be an R^* -conjugacy class, defined over \mathfrak{f} . Given an element $c \in O$, we defined above a torus T_c , a Jacobian J_c , and a map $z_c : O \rightarrow J_c$; the construction depended on c . However the triple (T_c, J_c, z_c) descends to (T, U, z) where U is an \mathfrak{f} -definable torsor of an Abelian variety J over \mathfrak{f} , and $z : O \rightarrow U$. For any $c \in O$ we have a c -definable isomorphism $(T_c, J_c, z_c) \rightarrow (T, U, z)$.

Proof. Let O be an R^* -conjugacy class, defined over \mathfrak{f} . We define the canonical torus with adelic structure T associated to $O \cap D(K)$ as follows. For any $c \in O \cap D(K)$ let $T_c = \{a \in D^* : ac = ca\}$. If $c, c' \in O \cap D(K)$ there exists $d \in D(K^{alg})$ with $dcd^{-1} = c'$; such a d gives an isomorphism $ad_d : T_c \rightarrow T_{c'}$, preserving adelic structure, $ad_d(x) = dxd^{-1}$; but ad_d does not depend on the choice of d , so we can write $f_{c,c'} = ad_d$. Clearly $f_{c',c''}f_{c,c'} = f_{c,c''}$. We can factor out this system to obtain a torus T , with adelic structure; given any $c \in O$ we have an isomorphism $f_c : T \rightarrow T_c$; and T is definable over any field of definition for O . This induces an isomorphism $f_c : J \rightarrow J_c$, where $J = (R \cap T) \backslash T(\mathbb{A}) / T(K)$.

For any $c, c' \in O \cap D(K)$ we obtain an element $j(c, c')$ of J : pick $a \in R, b \in D(K)^*$ with $ac = bc = c'$; let $t = a^{-1}b$; then $t \in T_c(\mathbb{A})$; let $j(c, c') = f_c^{-1}(t)$.

If $c_1, c_2, c_3 \in O \cap D(K)$, let $a_i \in R, b_i \in K, a_1c_1 = b_1c_1 = c_2, a_2c_2 = b_2c_2 = c_3$. Let $a_3 = a_2a_1, b_3 = b_2b_1$; so $a_3c_1 = b_3c_1 = c_3$. Let $t_i = a_i^{-1}b_i$. Then $j(c_1, c_3) = f_{c_1}(t_3) = f_{c_1}(a_1^{-1}a_2^{-1}b_2b_1) = f_{c_1}((a_1^{-1}a_2^{-1}b_2a_1)(a_1^{-1}b_1))$. Now $f_{c_1}((a_1^{-1}a_2^{-1}b_2a_1)) = f_{c_2}(a_2^{-1}b_2) = f_{c_2}(t_2) = j(c_2, c_3)$; and $f_{c_1}(a_1^{-1}b_1) = f_{c_1}(t_1) = j(c_1, c_2)$. Thus $j(c_1, c_3) = j(c_2, c_3)j(c_1, c_2)$. It follows that there exists a J -torsor U , defined over \mathfrak{f} , and a map $z : O \rightarrow U$, with $z(c_1, c_2) + z(c_1) = z(c_2)$. \square

Question 8.15. Does the equation of Proposition 8.9, descend to K ?

8.16. δ_K is geometric. Consider global functions ϕ given by a uniformly definable family (ϕ_v) . We assume that ϕ_v has bounded support for all v , contained in R_v for almost all v ; but not necessarily that ϕ_v is locally constant. Assume ϕ_v is R_v^* -invariant. Recall that $K = K[Gr^{-1}]$.

Proposition 8.17. *Let $\phi, \dot{\phi}$ be matching definable global functions as above. Then $\delta^K(\phi) = \delta^K(\dot{\phi}) \in K$.*

Proof. The support of ϕ is a limited subset X of $D(K)$; the equivalence relation E of integral conjugacy is definable on X . Similarly, let \dot{X} be the support of $\dot{\phi}$, and let \dot{E} be integral conjugacy. Since $\phi, \dot{\phi}$ match, we can identify the quotients $X/E, \dot{X}/\dot{E}$; so we have quotient maps $\pi : X \rightarrow Y, \dot{\pi} : \dot{X} \rightarrow Y$. By fibering over Y (2.1) we can reduce to the case that Y is a point, i.e. X, \dot{X} form a single integral conjugacy class. If this class is central, the statement is clear. If it is not regular semi-simple, then $[X] = [\dot{X}] = 0 \in K$ since $X(\mathfrak{f}) = \emptyset$ whenever $L(\mathfrak{f})$ is a field. So we assume X, \dot{X} are integral conjugacy classes of regular semi-simple elements.

Since $K[\mathcal{N}^{-1}] = K[\mathcal{N}^{-1}][\mathcal{N}^{-1}]$, replacing K by $K[Gr^{-1}]$ we may assume $K = K[Gr^{-1}]$.

For any $c \in X'$, let $\mathfrak{f}' = \mathfrak{f}(c)$.

Claim . $\delta^K(\phi) = \delta^K(\dot{\phi}) \in K_{\mathfrak{f}'}$.

Indeed there exists $\dot{c} \in \dot{X}(\mathfrak{f}')$ (Lemma 7.5). By strong invariance of $\phi, \dot{\phi}$ we have $\delta^K(\phi) = \phi(c)[X]$, and $\delta^K(\dot{\phi}) = \dot{\phi}(\dot{c})[\dot{X}]$, and by strong matching we have $\phi(c) = \dot{\phi}(\dot{c}) \in K_{\mathfrak{f}'}$. By Lemma 8.11, $[X] = [\dot{X}] \in K(\mathfrak{f}')$. So $\delta^K(\phi) = \delta^K(\dot{\phi}) \in K_{\mathfrak{f}'}$.

Let $U = \delta^K(\phi), \dot{U} = \delta^K(\dot{\phi})$. Since $[U] = [\dot{U}] \in K(\mathfrak{f}(c))$, for any $c \in X$, summing over $c \in X$ we obtain:

$$[\dot{U}][X] = [U][X]$$

Similarly, since $[X] = [\dot{X}] \in K(\mathfrak{f}(c))$ for any $c \in X$, we have:

$$[X]^2 = [\dot{X}][X]$$

and by symmetry, $[\dot{X}]^2 = [X][\dot{X}]$.

Let $[A] = [L_1][\mathcal{T} : T(K)]$. Then $[A] \in Gr$.

If $X \cap CN = \emptyset$, then by Proposition 8.9 we have the relations of 2.3. Thus $[X] = [\dot{X}] \in K$.

Otherwise, by Lemma 8.7, $X \subseteq CN$. So $[X] = [\dot{X}] \in K$. by Lemma 8.8. \square

Remark 8.18. *The proof of Lemma 8.17 does not require subtraction, and goes through for the Grothendieck semiring.*

9. AN EXPRESSION FOR THE FOURIER TRANSFORM, AND PROOF OF THEOREM 1.1

We can now express the Fourier transform at $\mathfrak{f}((t))$ in terms of the Fourier transform at $\mathfrak{f}((t-1))$ (where D splits) and δ^K . This will lead to a proof of Theorem 1.1.

For a finite set of places w , let $R_w = \prod_{v \in w} R_v$. If $\phi = (\phi_v)_{v \in S}$ is a family of local test functions on a set S of places, and $\phi' = (\phi'_v)_{v \in S'}$ is a family of local test functions on a disjoint set S' of places, we write $\phi \frown \phi'$ for their conjunction on $S \cup S'$.

We will write K_0, K_1 for K_{v_0}, K_{v_1} ; here v_1 is the valuation with $v_1(t-1) > 0$.

Let O be an integral conjugacy class. Since O is T^* -limited, there exist bounded definable sets B_0, B_∞ such that if $\mathfrak{f}' \models T^*$ then $O_v \subseteq B_v$ for $v(t) \neq 0$. Recall that $[O]$ is defined to be the class of $O' := B_0 \cap O \cap B_\infty$. More generally, let $\delta_O^K(\phi) = \sum \{\phi(a) : a \in O'(K)\} \in K$; this clearly does not depend on the choice of B_0, B_∞ . Also let $\delta_c^K = \delta_{O(c)}^K$, where $O(c)$ is the integral conjugacy class of c . Since $O(c)$ is already a limited set, $\delta_c^K(\phi)$ is defined even if ϕ does not have bounded support.

Let ϕ_v^{st} be the characteristic function of R_v . We call this the standard test function at v . Let $\phi_{v;n}(x) = \phi_v(ex)$ where e is any element of $k(t)$ with $v(e) = n$; for instance $c = (t - \alpha)^n$ if

$v(t - \alpha) > 0$. This function clearly does not depend on the choice of c ; we call such functions semi-standard.

Let $\phi_1^{c,m}$ be the characteristic function of $Ad_{R_1}(c) + (t - 1)^m R_1$.

A collection $(\phi_v)_v$ is semi-standard if ϕ_v is a semi-standard test function for each v , and standard almost everywhere.

Lemma 9.1. *For $v \neq 1$ let $Y_v \subset D$ be a bounded $ACVF_{\mathfrak{f}(t)_v}$ -definable set, with $Y_v = R_v$ for all v outside some $ACF_{\mathfrak{f}}$ -definable finite subset of \mathbb{P}^1 . Let $c \in D(\mathfrak{f}(t))$ with $c \in Y_v$ for $v \neq 1$. Then there exists a bounded, R_1^* -invariant $\mathfrak{f}(t)_{v_1} = \mathfrak{f}((t - 1))$ -definable neighborhood Y_1 of c such that if $y \in Y = D(\mathfrak{k}(t)) \cap \cap_{v \neq 1} Y_v$ then y, c are R_1^* -conjugate.*

Proof. Let $U_m = c + (t - 1)^m R_1$; it is a bounded, $\mathfrak{f}(t)_{v_1} = \mathfrak{f}((t - 1))$ -definable, neighborhood of c . So is $V_m = ad_{R_1^*}(U_m) = \{x : (\exists y \in R_1^*)(xyx^{-1} \in U_m)\}$; moreover V_m is R_1^* -invariant. Let $Y^0 = D(\mathfrak{k}(t)) \cap \cap_{v \neq 1} Y_v \cap V_0$. Then Y^0 is a limited subset of $D(\mathfrak{k}(t))$.

Let $R_{1_y}^* = Ad_{R_1}(y)$ be the R_1^* -conjugacy class of y . Being the image of a bounded set defined by weak inequalities under a continuous definable map, $Ad_{R_1}(y)$ is a closed and bounded subset of $D(K_1)$ in the valuation topology.

Claim . For any $y \in Y^0$ there exists $m = m(y)$ such that if $u \in U_m$ and u, y are R_1^* -conjugate then u, c are R_1^* -conjugate.

Proof. Fix $y \in Y^0$. If $y \in Ad_{R_1}(c)$ then $m = 0$ will do, since if u, y are R_1^* -conjugate then so are u, c . If $y \notin Ad_{R_1}(c)$, then since $Ad_{R_1}(y)$ is closed, and $c \notin Ad_{R_1}(y)$, some neighborhood U_m of c is disjoint from $Ad_{R_1}(y)$. In this case no $u \in U_m$ is R_1^* -conjugate to y . \square

Now Y^0 is a limited (so $ACF_{\mathfrak{f}}$ -definable) set, and so by compactness, for some m , for any $y \in Y^0$, if $u \in U_m$ and u, y are R_1^* -conjugate then u, c are R_1^* -conjugate. Let $Y_1 = ad_{R_1^*}(U_m)$. This is a bounded, R_1 -invariant $\mathfrak{f}((t - 1))$ -definable neighborhood of c . Define Y as above. If $y \in Y$ then $y \in Y^0$, and y is R_1^* -conjugate to some $u \in U_m$. But then u, c and hence y, c are also R_1^* -conjugate, as required. \square

Lemma 9.2. *Let ϕ_0 be a local test function on D over $K_0 = K_{v_0}$, and let $c \in D(F)$.*

Let $(\theta_v : v \neq 0, 1)$ be a semi-standard collection, with $\theta_v(c) = 1$. Let $\phi_1 = \phi_1^{m,c}$, $\theta' = \mathfrak{F}^{-1}\theta$, $\phi'_1 = \mathfrak{F}^{-1}\phi_1$

Then if m is sufficiently large, we have

$$\delta_c^K \mathfrak{F}\phi_0 = \delta^K(\phi_0 \frown \phi'_1 \frown \theta')$$

If ϕ_0 is R_0^ -invariant, and $c \notin CN$, we have:*

$$\mathcal{F}\phi_0(c) = (n[L_1][\mathcal{T}_c : T(K)])^{-1} \delta^K(\phi_0 \frown \phi'_1 \frown \theta')$$

Proof. By Lemma 9.1 we have $\delta^K(\phi_0 \frown \phi_1 \frown \theta) = \delta_c^K(\phi_0 \frown \phi_1 \frown \theta)$. Now ϕ_1, θ take the value 1 on δ_c^K . So $\delta^K(\phi_0 \frown \phi_1 \frown \theta) = \delta_c^K(\phi_0)$. This gives:

$$\delta_c^K(\phi_0) = \delta^K(\phi_0 \frown \phi_1 \frown \theta)$$

Applying this formula to $\phi'_0 := \mathfrak{F}\phi_0$, we find:

$$\delta_c^K(\phi'_0) = \delta^K(\phi'_0 \frown \phi_1 \frown \theta)$$

By Poisson summation, $\delta^K(\phi_0 \frown \phi'_1 \frown \theta') = \delta^K(\phi'_0 \frown \phi_1 \frown \theta)$. The first formula follows.

By Lemma 8.5, $\mathcal{F}\phi_0$ is strongly invariant. So $\mathcal{F}\phi_0(y) = \mathcal{F}\phi_0(c) \in K_y$, and $\delta_c^K(\mathcal{F}\phi_0) = \sum_{y \in O(c)} \mathcal{F}\phi_0(y) = [O(c)]\mathcal{F}\phi_0(c)$. Hence, $\mathcal{F}\phi_0(c) = [O_c]^{-1} \delta_c^K(\mathcal{F}\phi_0)$, and the lemma follows from the first formula and Proposition 8.9. \square

Note that θ' above is easily computed, and gives the same (absolute) value for D, \dot{D} . Since at 1 we have an isomorphism of D, \dot{D} preserving integral structure, the Fourier transform of ϕ_1 can be computed with respect to either ring, giving the same result ϕ'_1 . Finally, note that the global term $[\mathcal{T}_c : T(K)]$ is the same, via an explicit bijection, for adelicly matching c, c' (Lemma 7.7.)

We can now deduce a *proof of Theorem 1.1*. Let $\phi_0, \dot{\phi}_0$ be matching R_0^* -invariant local test functions at 0. We wish to show that $\mathcal{F}\phi_0, \mathcal{F}\dot{\phi}_0$ also match. By Lemma 8.5 (c), it suffices to consider rational points c, \dot{c} be of matching conjugacy classes O, \dot{O} of D, \dot{D} ; with $c \notin CN$ (and so $\dot{c} \notin C\dot{N}$.) By Lemma 7.7 there exists c' adelicly conjugate to c ; by invariance we have $\mathcal{F}\dot{\phi}_0(\dot{c}) = \mathcal{F}\dot{\phi}_0(c')$; so we may assume c, \dot{c} match at every place. In this case, Proposition 8.17 and the explicit formula of Lemma 9.2 shows that $\mathcal{F}\phi_0(c) = \mathcal{F}\dot{\phi}_0(\dot{c})$.

10. APPENDIX 1: IND-DEFINABLE SETS

We include here some standard definitions, largely lifted from the exposition in [4].

A structure N for a finite relational language L is *piecewise-definable* over another structure k if there exist definable L -structures N_i and definable L -embeddings $N_i \rightarrow N_{i+1}$ such that $\lim_i N_i$ is isomorphic to N . A *definable subset* of $\lim_i N_i$ is just a definable subset of some N_i . A *morphism* $f : N \rightarrow N'$ is an L -embedding such that for any definable $S \subseteq N$, $f(S)$ is a definable subset of N' , and $f|_S$ is a definable map $S \rightarrow f(S)$.

A piecewise definable set is an Ind-object over the category of definable sets with *injective* definable maps; we will not consider other Ind-objects in this paper, so will use the term "Ind-definable" synonymously with "piecewise definable".

Let C be the category of L -structures interpretable in k , with definable L -embeddings between them. Since all maps in C are injective, every object of $\text{Ind}C$ is strict. If $A \in \text{Ind}C$ is represented by a system $(A_i)_{i \in I}$, let $\phi(A) = \lim_i A_i$ (inductive limit of L -structures.) For a map $f : A \rightarrow B$ in $\text{Ind}C$, $\phi(f)$ is defined in the obvious way. Then ϕ is an equivalence of categories. Unlike the case of $\text{Pro}C$, there is no saturation requirement on k .

Lemma 10.1. (1) *If N is quantifier-free definable over L , and L is piecewise-definable over k , then N is piecewise-definable over k .*

(2) *Let k be a field, and let $L = k(b_1, \dots, b_n)$ be a finitely generated field extension of k . Then $(L, +, \cdot, b_1, \dots, b_n, k)$ is piecewise definable over k . (More precisely there exists a piecewise definable k -algebra L' and an isomorphism $\psi : L \rightarrow L'$ of k -algebras.)*

(3) *For any variety V over L , $V(L)$ can be viewed as piecewise-definable over k . (I.e. $\psi(V(L)) = V^\psi(L')$ is piecewise-definable over k .)*

Proof. (1) is clear. For (2), L is a finite extension of a purely transcendental extension $k(t) = k(t_1, \dots, t_n)$ of k . Clearly L is quantifier-free definable over $k(t)$. Hence by (1) it suffices to show that $k(t)$ is piecewise-definable over k . Indeed let S_n be the set of rational functions $f(t)/g(t)$ with $\deg(f), \deg(g) \leq n$, and let $+, \cdot$ be the graphs of addition and multiplication restricted to S_n^3 . Then $\lim_n S_n = k(t)$.

(3) Note that the k -algebra isomorphism ψ induces a map $V(L) \rightarrow V^\psi(L')$, also denoted ψ . (3) follows from (1) and (2). \square

Definition 10.2. *Let L, V be as in 10.1. A subset Y of $V(L)$ is called limited if for some isomorphism $\psi : L \rightarrow L'$ to a piecewise-definable field, $\psi(Y)$ is contained in a definable subset of the piecewise-definable set $V^\psi(L')$.*

Let us mention two further equivalent formulations, one geometric and one model-theoretic.

- (1) When $\text{tr.deg.}_k L = 1$, and when V comes with a projective embedding, one has the notion of a Weil height of a point of $V(L)$. Then a limited subset of $V(L)$ is a set of bounded height.
- (2) Let T be the ω -stable theory of pairs (k, K) of algebraically closed fields, with $k < K$. Assume $(k, K) \models T$. A subset Y of $V(K)$ is *limited* if it is k -internal, i.e. $Y \subseteq \text{dcl}(b, k)$ for some finite b . (In this case $Y \subseteq V(L)$ for some subfield L of K , finitely generated over k .)

11. APPENDIX 2: FORMS

Definition 11.1. Let T be a theory, D a definable set, and R_i a definable subset of D^{m_i} ($i = 1, \dots, m$). By a form of $(D, R_i)_i$ we mean a structure $(D', \dot{R}_i)_i$, with D' definable in T and \dot{R}_i a T -definable subset of D^{m_i} , such that for any $M \models T$ there exists a T_M -definable isomorphism $(D, R_i)_i \rightarrow (D', \dot{R}_i)_i$, i.e. a T_M -definable bijection $D \rightarrow D'$ carrying R_i to \dot{R}_i .

For instance, a torus is by definition a form of G_m^n , with respect to the theory ACF .

For the rest of the section we discuss forms for $ACVF$ or $ACVF_F$, where F is a valued field, with residue field \mathfrak{f} . Let \mathcal{O} denote the valuation ring, \mathfrak{M} the maximal ideal.

Let M_n (respectively $M_n(\mathcal{O})$) denote the ring of $n \times n$ (integral) matrices. Thus D is a form of $(M_n, M_n(\mathcal{O}))$ iff D is a form definable finite dimensional central simple algebra.

A form of $(M_n, M_n(\mathcal{O}))$ over F is a pair (D, R) , with D an ACF_F -definable algebra and R a definable subring, such that if $K \models ACF_F$ then there exists an ACF_K -definable isomorphism $D \rightarrow M_n$ carrying M_n to $M_n(\mathcal{O})$.

If V is a definable vector space, by a *lattice* we mean a definable \mathcal{O} -submodule Λ of V , such (V, Λ) is a form of (K^n, \mathcal{O}^n) (for $n = \dim(V)$).

Thus (D, R) is a form of $(M_n, M_n(\mathcal{O}))$ iff there exists a an ACF_K -definable D -module A of dimension n , and a definable lattice $\Lambda \leq A$, such that $R = \{r \in D : r\Lambda = \Lambda\}$. If A, Λ can be found over an unramified extension of F , we say that (D, R) is an *unramified* form. While it is mostly unramified forms that are of interest for us, much of the discussion can be carried out more generally.

Since $GL_n(\mathcal{O})$ leaves invariant the ideal $\mathfrak{M}M_n(\mathcal{O})$, any definable integral form R has a unique definable ideal M , such that (D, R, M) is a form of $(M_n, M_n(\mathcal{O}), \mathfrak{M}M_n(\mathcal{O}))$.

The trace map $\text{tr} \circ \phi$ does not depend on the choice of ϕ , so it is defined over F , and we denote it by tr . Similarly for \det . In particular we have a bilinear form $\text{tr}(xy)$ defined over F .

11.2. Characterizations of integral forms. For an $ACVF_F$ -definable ring R , we will say “definably compact” for “ $(R, +)$ is generically metastable”, i.e. for: “ $(R, +)$ admits a stably dominated translation invariant type” ([7].) For a subring of an algebra D , this is equivalent to: R is bounded, and definable by weak valuation inequalities. In this case, for some ϕ, θ , R is defined by a formula $\phi(x, a)$, with $a \in \theta(F)$, and for almost all local fields F' and $a' \in \theta(F')$, $\phi(x, a')$ defines a compact ring. Say R is “maximally definably compact” if it is definably compact and is contained in no bigger definably compact ring, even over K .

Given a definable lattice $\Lambda \leq D$, let $\Lambda^\perp = \{x : (\forall y \in \Lambda)(\text{tr}(xy) \in \mathcal{O})\}$. This is another definable lattice, freely generated as an \mathcal{O} -module by the dual basis to a basis for Λ . Say Λ is self-dual if $\Lambda^\perp = \Lambda$.

Let R be definably compact, $K \models ACVF_F$. Then R is contained in a conjugate \dot{R} of $GL_n(\mathcal{O})$ (defined over K). Any such conjugate is self-dual. If R is also self-dual, $\dot{R} \subseteq R$, so $R = \dot{R}$. Conversely, if R is maximally definably compact then $R = \dot{R}$, so R is self-dual.

Thus the following conditions on a definably compact subring R are equivalent: R is self-dual, R is a maximal definably compact, R is (eventually, i.e. in a model) conjugate to $M_n(\mathcal{O})$.

For any definable subring R of GL_n , let $N(R) = \{a \in GL_n : (\forall b \in R)(a^{-1}ba \in R)\}$ be the normalizer. This is a definable subgroup of GL_n containing the center Z . We call R *self-normalizing* if $N(R) = ZR^*$.

Lemma 11.3. *Let R be a self-normalizing subring of D , a form of M_n . Let (\dot{D}, \dot{R}) be a form of (D, R) . Let E be the definable equivalence relation of R^* -conjugacy on D , and similarly \dot{E} . Let $D/E, \dot{D}/\dot{E}$ be interpreted in ACVF. There exists a definable bijection $f : D/E \rightarrow \dot{D}/\dot{E}$.*

Proof. Let $\text{Hom}^*(A, B)$ denote the set of k -algebra isomorphisms $A \rightarrow B$. The $\text{Hom}^*(D, \dot{D})$ is a form of $\text{Hom}^*(M_n, M_n) = PGL_n$; in particular it is a definable set. Let $H = \{h \in \text{Hom}^*(D, \dot{D}) : h(R) = \dot{R}\}$. Then H is also a definable set, a torsor for $N(R)/Z$. Any $h \in H$ induces a bijection $D/E \rightarrow \dot{D}/\dot{E}$, which is h -definable. However this bijection does not depend on the choice of h , so it is definable. \square

Though we will use only forms of $M_n(\mathcal{O})$, we note in passing another, non-maximal, compact definable ring.

Example 11.4. Let I be the Iwahori algebra of $n \times n$ matrices from \mathcal{O} with superdiagonal entries in \mathcal{M} . I can be viewed as a definable subring of the matrix ring M_n . Then I is self-normalizing as an ACVF-definable ring, though $I(\mathbb{Q}_p)$ or $I(\mathbb{C}((t)))$ are not. First, if $a \in GL_n$ normalizes I , then it must normalize $M_n(\mathcal{O})$; the reason is that by considering elements of the form t^α for $\alpha \rightarrow 0$, one can approximate elements of $M_n(\mathcal{O})$ by elements of I . Since in $M_n(\mathcal{O}/\mathcal{M})$, the algebra of lower triangular matrices is self-normalizing, it follows that I is self-normalizing in GL_n , modulo the center.

Lemma 11.5. *Let D be a form of M_n over a nontrivially valued field F , and let $K \models ACVF_F$. Then there exists an $ACVF_F$ -definable subring R such that there exists an ACF_K -definable isomorphism $h : D \rightarrow M_n$ with $h(R) = M_n(\mathcal{O})$.*

Proof. Since $F^a \models ACVF_F$, there exists a finite $ACVF_F$ definable set W and for $w \in W$ an $F(w)$ -definable representation $V(w)$ of D , of dimension n . Given $w, w' \in W$, there exists a finite $F(w, w')$ -definable set Y_w and for $y \in Y$ an $F(w, w', y)$ -definable isomorphism g_y between the two representations. Moreover two isomorphisms $g_y, g_{y'} : V_w \rightarrow V_{w'}$ differ by a scalar $c(y, y')$. We may assume all $y \in Y$ have the same type over $F(w, w')$. But we can define a partial ordering on Y , $y \leq y'$ iff $\text{val}(c(y, y')) \geq 0$. Since Y is finite the partial ordering must be trivial, i.e. $\text{val}(c(y, y')) = 0$ for all y, y' . It follows that the maps g_y induce a unique isomorphism $g_{w, w'} : \mathcal{O}^* \backslash V_w \rightarrow \mathcal{O}^* \backslash V_{w'}$. (Alternatively by Hilbert 90, since $\text{Hom}_D(V_w, V_{w'})$ is a 1-dimensional vector space, there exists $G_{w, w'} : V_w \rightarrow V_{w'}$, defined over $F(w, w')$; let $g_{w, w'} : \mathcal{O}^* \backslash V_w \rightarrow \mathcal{O}^* \backslash V_{w'}$ be the induced map.)

Now we need some Galois cohomology, which is easiest to do from first principles. Let $C^m(W, \Gamma)$ be the set of definable maps $W^m \rightarrow \Gamma$. Define a coboundary map $d : C^m(W, \Gamma) \rightarrow C^{m+1}(W, \Gamma)$ in the usual way. Namely given $f \in C^m(W, \Gamma)$, let $df = F$ where $F(w_0, \dots, w_m) = \sum (-1)^i f(\text{omit } w_i)$. Since Γ is uniquely divisible, the cohomology groups $H^m(W, \Gamma)$, $m \geq 1$ are trivial: let $f \in C^m(W, \Gamma)$ and assume $df = 0$. Given $w \in W$, let $f_w \in C^{m-1}(W, \Gamma)$ be defined by $f_w(w_1, \dots, w_{m-1}) = -f(w, w_1, \dots, w_{m-1})$. Then formally the relation $df = 0$ gives $df_w = f$. Now f_w is not definable, but let F be the average of all f_w ; then F is definable and $dF = f$.

In particular, let $f(w, w', w'') = g_{w, w'}^{-1} g_{w', w''} g_{w, w''}$. Then $f(w, w', w'')$ is a scalar endomorphism of V_w , modulo \mathcal{O}^* , so it can be viewed as an element of Γ . We have $df = 0$ (this can be verified by fixing some w_0 , identifying $\mathcal{O}^* \backslash V_w$ with $\mathcal{O}^* \backslash V_{w_0}$ via $g_{w_0, w}$, and using the commutativity of Γ .) So $f = dF$ for some $F \in C^2(W, \Gamma)$. Replacing $g_{w, w'}$ by $F(w, w')^{-1} g_{w, w'}$ (i.e. by $g_{w, w'}$ composed by the endomorphism of division by a scalar with value $F(w, w')$), we may

assume $f = 0$, i.e. $g_{w,w''} = g_{w',w''}g_{w,w'}$. So we have a commuting system of isomorphisms between the integrally projectivized representations $\mathcal{O}^* \backslash V_w$.

Now we can find an $F(w)$ -definable lattice $\Lambda_w \in L(V_w)$. Let $\Lambda_w^* = \cap_{w' \in W} g_{w,w'}^{-1} L(V_{w'})$. Then Λ_w^* is also an $F(w)$ -definable lattice³; and $g_{w,w'}^{-1} \Lambda_{w'}^* = \Lambda_w^*$.

Let $R_w = \{r \in D : r\Lambda_w^* \subseteq \Lambda_w^*\}$. Then R_w does not depend on w as one sees using $g_{w,w'}$. Let $R = R_w$. Then R clearly satisfies the requirements. \square

Let T_n be the diagonal subalgebra of M_n . Consider diagonalizable algebraic subrings of D , i.e. definable subrings T such that (D, T) is a form of (M_n, T_n) . Then there exists a unique definable subring \mathcal{O}_T of T , such that (D, T, \mathcal{O}_T) is a form of $(M_n, T_n, T_n(\mathcal{O}))$. Indeed after base change, there exists an isomorphism $(D, T) \rightarrow (M_n, T_n)$ of pairs of rings; it is well-defined up to composition with an element of the Weyl group $\text{Sym}(n)$; since $\text{Sym}(n)$ respects $T_n(\mathcal{O})$, the pullback of $T_n(\mathcal{O})$ does not depend on the choice of isomorphism, and is definable. It is not necessarily the case that \mathcal{O}_T is the \mathcal{O} -module generated by $\mathcal{O}_{T(F)}$. Note that by Hilbert 90, T is determined by $T(F)$.

We call R a *definable integral form* for (D, T) if (D, T, R) is a form of $(M_n, T_n, M_n(\mathcal{O}))$. So $R \cap T = \mathcal{O}_T$.

Lemma 11.6. *Let D be a form of M_n over a nontrivially valued field F , and let $K \models \text{ACVF}_F$. Let T be defined over F , with (D, T) a form of (M_n, T_n) . Then there exists definable integral form for (D, T) .*

Proof. Same as Lemma 11.5. We take $V(w)$ to be graded by one-dimensional eigenspaces of T , and we choose Λ_w to be generated by T -eigenvectors. \square

In case $D \cong M_n$, we have $D \cong \text{End}(T)$; let $R_T = \text{End}_{\mathcal{O}} \mathcal{O}_T$; then (D, T, R_T) is a definable integral form for (D, T) .

Let T be a diagonalizable subring, with normalizer N . Let $\mathcal{D} = \mathcal{D}(D, T, F)$ be the set of definable integral forms of (D, T) over F . Let Z be the center of D^* . Let $X = \text{Hom}(G_m, T^*/Z)$ and $\Delta = X \otimes \Gamma$ (so Δ is parameterically isomorphic to $\Gamma^{\dim(T)-1}$.) Let \mathcal{D}_T^F be the definable integral forms for (D, T) , and $\bar{\mathcal{D}}_T^F = \mathcal{D}_T^F / N(F)^*$ be $\mathcal{D}_T(F)$ up to $D(F)^*$ -conjugacy. If the identity of F is clear we will omit the superscript.

If $K \models \text{ACVF}$ and R, \dot{R} are two definable integral forms for D (or for (D, T)), then R, \dot{R} are conjugate in $D(K)$ (respectively, in $R(K)Z(K)$.) We will use this in (1),(2) below.

In (4),(5) below we use the fact that D splits over the maximal unramified algebraic extension F^{unr} of F ; see [18] II 3.2, Corrolaire, and 3.3(c) ($H^1(F^{unr}, \text{PGL}^n) = 0$.)

In (2) below we assume residue char. 0; in each case what we really use is that $H^1(F, A) = 0$ for certain definable unipotent groups A . For ACVF, unlike ACF, this is not automatic even over perfect fields; but the instances we require may be true in char. p too.

Lemma 11.7. (1) $\bar{\mathcal{D}}_T^F$ has a canonical structure of torsor over $\Delta_{\text{def}}(F)/\Delta(F)$; where $\Delta_{\text{def}}(F)$ is the set of points of $\Delta(F^{\text{alg}})$ invariant under $\text{Aut}(F^{\text{alg}}/F)$, and $\Delta(F)$ is the set of points of Δ represented in F .

(2) (char. 0.) Assume there exists a definable B such that (D, T, B) is a form of (M_n, T_n, B_n) , with B_n the upper triangular matrices. Then there is a canonical retraction $\rho : \mathcal{D} \rightarrow \mathcal{D}_T$. In residue char. 0, any $R \in \mathcal{D}$, R and $\rho(R)$ are $D^*(F)$ -conjugate. Hence every definable integral form for D is $D(F)$ -conjugate to a definable integral form for (D, T) .

³By [6], a definable \mathcal{O} -submodule of n -space is a lattice iff the intersection with each one-dimensional subspace is a closed ball; this property is evidently preserved under finite intersections.

- (3) Let F'/F be an unramified field extension. Then the natural map $\bar{\mathcal{D}}_T^F \rightarrow \bar{\mathcal{D}}_T^{F'}$ is injective.

Proof. (1) Let $R \in \mathcal{D}_T$. Let $N \leq D^*$ be the normalizer of T (a definable group.) As observed above, any element of \mathcal{D}_T has the form $a^{-1}Ra$ for some $a \in N(K)$. But $NR = T^*R$, since the Weyl group is represented in $R(K)^*$. So we can take $a \in T(K)$. Now $a^{-1}Ra$ is by assumption $ACVF_F$ -definable, and the normalizer of R is R^*Z , so $N_T(R) := T \cap R = \mathcal{O}_T Z$, using the observation above that $T \cap R = \mathcal{O}_T$. hence $a\mathcal{O}_T Z$ is definable. Now $T/(\mathcal{O}_T Z) = \Delta$. The definable points of Δ are $\Delta(F^{alg})$. This gives a surjection $\Delta_{def}(F) \rightarrow \mathcal{D}_T$, and hence $\Delta_{def}(F)/\Delta(F) \rightarrow \bar{\mathcal{D}}_T$. It is easy to check injectivity.

- (2) Any element of \mathcal{D} has the form $aGL_n(\mathcal{O})a^{-1}$ for some $a \in GL_n(K)$. Since $GL_n = B_n GL_n(\mathcal{O})$, we can take $a \in B_n(K)$. Let $ss(x)$ be the semi-simple part of x . Then ss commutes with conjugation, hence gives a well-defined map on D ; it induces a homomorphism $B_n \rightarrow T_n$. This in turn induces a map $s : B_n/B_n(\mathcal{O})Z \rightarrow T_n/T_n(\mathcal{O})Z$.

It remains to show that $\rho(R), R$ are $D^*(F)$ -conjugate. Say $R = aGL_n(\mathcal{O})a^{-1}$ with $a \in B_n$; write $a = a_s a_u$ with $a_s \in T_n$ and $a_u \in U_n$, this being the strictly upper-triangular matrices. So $\rho(R) = a_s GL_n(\mathcal{O})a_s^{-1}$, hence $R = a_u \rho(R) a_u^{-1}$. Now $S = \{u \in U_n : u\rho(R)u^{-1} = \rho(R)\}$ is an $ACVF_F$ -definable subgroup of U_n . This group is geometrically connected. In characteristic 0 it is clear that $H^1(\text{Aut}(F^a/F), S) = 0$, so there exists a definable point.

- (3) Pick $R \in \mathcal{D}_T$. Then by change of scalars we can view R as an element of \mathcal{D}_T over F' . By (1), $\bar{\mathcal{D}}_T^F$ corresponds bijectively to a subgroup of $\Delta(F^{alg})/\Delta(F)$, while $\bar{\mathcal{D}}_T^{F'}$ corresponds bijectively to $\Delta(F^{alg})/\Delta(F')$. However these two groups are the same. \square

Remark 11.8. Let G_{ch} be a Chevalley group. Let G be a form of G_{ch} over F . Define an *integral form* of G to be an $ACVF_F$ -definable subgroup H of G , such that (G, H) is a form of $G_{ch}, G_{ch}(\mathcal{O})$. It seems that analogues of the above results should be true.

Above we used the fact that M_n has no outer automorphisms, which is not true for G_{ch} . However every outer automorphism of $G_{ch}(K)$ is an inner automorphism composed with a graph automorphism, and the graph automorphisms preserve $G_{ch}(\mathcal{O})$. This makes it possible to consider definable integral forms of G for a form G of G_{ch} , so that two forms are G -conjugate.

12. APPENDIX 3: MULTIPLICATIVE CONVOLUTION.

Our results on the stability of the Fourier transform have an immediate consequence for additive convolution: given two pairs of matching local test functions on D, \dot{D} , their additive convolutions also match.

This statement can be phrased without the intervention of additive characters, and may be valid for the Grothendieck ring $K[Gr^{-1}]$; our proof however shows it in $K_e[Gr^{-1}]$. Indeed the Fourier transform transposes the problem into a similar one using pointwise products, which is obvious.

Here we assume characteristic 0 in order to point out a relation between this additive result and the analogous multiplicative statement, as in [13].

We note that the “orbit method” isomorphism between convolution algebras of nilpotent groups and their Lie algebras ([2] Prop. 2.4) goes through for motivic convolution algebras.

In our setting, we have the division algebra D , with subring R , ideals M_n of elements of determinantal valuation $\geq n$. The exponential map defines a bijection $x \mapsto 1 + x + \dots$,

$A_n := M_1/M_n \rightarrow (1+M_1)/(1+M_n) =: G_n$. This induces a bijection \exp between D^* -conjugacy classes on the algebraic groups A_n and on G_n .

Lemma 12.1. *\exp induces an isomorphism of motivic rings $F_n(G_n)^{D^*} \rightarrow F_n(A_n)^{D^*}$.*

Proof. Let C_1, C_2 be two conjugacy classes, and let $c \in A_n$. Then we have to show that $A = \{(x_1, x_2) \in C_1 \times C_2 : x_1 + x_2 = c\}$ has the same class in the Grothendieck group as $B = \{(x_1, x_2) \in C_1 \times C_2 : \exp(x_1)\exp(x_2) = \exp(c)\}$.

Let $H(x_1, x_2) = (\exp(\text{ad}\phi(x, y))(x), \exp(\text{ad}\psi(x, y))(y))$, where $\phi(X, Y), \psi(X, Y)$ are the Lie polynomials from [2] Lemma. 2.5. By this lemma, $B = H^{-1}(A)$. We are thus done given:

Claim. Let $\phi(X, Y), \psi(X, Y)$ be Lie polynomials. Then the function $A_n^2 \rightarrow A_n^2$ defined by:

$$(x, y) \mapsto (\exp(\text{ad}\phi(x, y))(x), \exp(\text{ad}\psi(x, y))(y))$$

is bijective.

Proof. of Claim: if $H(u) = H(u')$, we show by induction on $k \leq n$ that $u \equiv u' \pmod{M_k}$. Given that $u = u' \pmod{M_k}$ we have $\phi(u) = \phi(u'), \psi(u) = \psi(u') \pmod{M_{k+1}}$, so $u = u' \pmod{M_{k+1}}$. \square

\square

In the classical case, given the isomorphism on U , the full multiplicative isomorphism can be obtained using character-theoretic methods. As the characters involved are uniformly definable, it seems likely that this can be done motivically too.

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