

# Markov semi-groups generated by elliptic operators with divergence-free drift

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## Abstract

In this paper we construct a conservative Markov semi-group with generator  $L = \Delta + b \cdot \nabla$  on  $\mathbb{R}^n$ , where  $b$  is a divergence-free vector field which belongs to  $L^2 \cap L^p$  with  $\frac{n}{2} < p$ . The research is motivated by the question of understanding the blow-up solutions of the fluid dynamic equations, which attracts a lot of attention in recent years.

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## 1 Introduction and the main result

In fluid dynamics, the velocity  $u(t, x)$  of fluid particles is described by, in the case of incompressible fluids, the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p, \quad \nabla \cdot u = 0, \quad (1.1)$$

in a domain of Euclidean space  $\mathbb{R}^3$ , subject to certain initial and boundary conditions. Here  $p(t, x)$  is the pressure which is uniquely determined by  $u(t, x)$  up to a constant at every  $t$ , and it solves the Poisson equation  $\Delta p = -\operatorname{div}(u \cdot \nabla u)$ . Hence  $p(t, x)$  is a non-linear and non-local term in the Navier-Stokes equations.

The first equation in (1.1) can be written as a parabolic type equation:

$$(\partial_t - \nu \Delta + u \cdot \nabla) u = -\nabla p, \quad (1.2)$$

which however possesses no much common features as (local) parabolic equations. But nevertheless the theory of parabolic equations is helpful in the analysis of the Navier-Stokes equations. It is a matter of fact that many quantities related to fluid flows such as the vorticity and the stress tensor fields also satisfy the same kind of parabolic evolution equations with the principal parabolic operator  $\partial_t - L$ , where  $L = \nu \Delta - u \cdot \nabla$ . The operator  $L$  is time non-homogeneous since  $u$  depends on  $t$ , and its formal adjoint  $L^* = \nu \Delta + u \cdot \nabla$  is the infinitesimal generator of a diffusion process, called Taylor's diffusion, which models the fluid flows in terms of Brownian particles. Taylor's diffusion solves formally the following stochastic differential equation

$$dX_t = u(t, X_t) + \sqrt{2\nu} dW_t, \quad (1.3)$$

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where  $W_t$  is the Brownian motion. Taylor's diffusion has been an important tool in the study of turbulent flows and in the development of numerical simulations to the solutions of the Navier-Stokes equations (such as vortex methods). For the Navier-Stokes equations, only global weak solutions have been constructed in general, and knowledge of weak solutions is still limited. There is a vast literature addressing the regularity of weak solutions, see e.g. [19, 24]. Leray's weak solution  $u(t, x)$  satisfies the energy balance equation, which implies that

$$u(t, x) \in L^2(0, T; H^1) \cap L^\infty(0, T; L^2). \quad (1.4)$$

For the most interesting case where dimension is three, this regularity only implies that  $u(t, x) \in L^2(0, T; L^6(\mathbb{R}^3)) \cap L^\infty(0, T; L^2(\mathbb{R}^3))$  and the classical parabolic regularity theory fails to apply.

Consider the following parabolic equation

$$\partial_t u(t, x) - \sum_{i,j=1}^n \partial_{x_i} (a_{ij}(t, x) \partial_{x_j} u(t, x)) + \sum_{i=1}^n b_i(t, x) \partial_{x_i} u(t, x) = 0 \quad (1.5)$$

on  $\mathbb{R}^n$ . A classical monograph on this class of parabolic equations is [10] by Ladyzhenskaya et al., in which the existence of a unique Hölder continuous weak solution  $u$  is proved under the conditions that  $a$  is uniform elliptic and  $b \in L^l(0, T; L^q(\mathbb{R}^n))$  with  $\frac{2}{l} + \frac{n}{q} \leq 1$ ,  $l \neq \infty$ . It is not known in general whether a Leray's weak solution has this regularity or not. Actually we know that when  $n = 3$ , (1.4) implies that  $\frac{2}{l} + \frac{3}{q} = \frac{3}{2}$ . This motivates us to consider the cases that  $\frac{2}{l} + \frac{n}{q} > 1$  together with the assumption that  $b$  is divergence-free.

If  $(a_{ij})$  and  $(b_i)$  are smooth, then there exists a unique fundamental solution  $\Gamma(t, x; \tau, \xi)$  associated with the Cauchy initial problem (1.5). In [17], the following Aronson type estimate in the time-inhomogeneous case with a super-critical drift  $b$  has been established, see also the related estimates in [1, 15, 18, 27].

**Theorem 1.** *Suppose  $a = (a_{ij})$  and  $b = (b_i)$  are smooth which satisfy that  $\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \frac{1}{\lambda} |\xi|^2$  and  $\operatorname{div} b = 0$ , and assume that  $b \in L^l(0, T; L^q(\mathbb{R}^n))$  for some  $n \geq 3$ ,  $l > 1$ ,  $q > \frac{n}{2}$  such that  $1 \leq \frac{2}{l} + \frac{n}{q} < 2$ . If  $\mu \equiv \frac{2}{2-\gamma+\frac{2}{q}} > 1$  with  $\gamma = \frac{2}{l} + \frac{n}{q}$ , then the fundamental solution has upper bound*

$$\Gamma(t, x; \tau, \xi) \leq \begin{cases} \frac{C_1}{(t-\tau)^{n/2}} \exp\left(-\frac{1}{C_2} \left(\frac{|x-\xi|^2}{t-\tau}\right)\right) & \frac{|x-\xi|^{\mu-2}}{(t-\tau)^{\mu-\nu-1}} < 1 \\ \frac{C_1}{(t-\tau)^{n/2}} \exp\left(-\frac{1}{C_2} \left(\frac{|x-\xi|^\mu}{(t-\tau)^\nu}\right)^{\frac{1}{\mu-1}}\right) & \frac{|x-\xi|^{\mu-2}}{(t-\tau)^{\mu-\nu-1}} \geq 1 \end{cases} \quad (1.6)$$

where  $\nu = \frac{2-\gamma}{2-\gamma+\frac{2}{q}}$ ,  $\Lambda = \|b\|_{L^l(0, T; L^q(\mathbb{R}^n))}$ ,  $C_1 = C_1(l, q, n, \lambda)$ ,  $C_2 = C_2(l, q, n, \lambda, \Lambda)$ . If  $\mu = 1$ , so that  $q = \infty$ , then

$$\Gamma(t, x; \tau, \xi) \leq \frac{C_1}{(t-\tau)^{n/2}} \exp\left(-\frac{(C_1 \Lambda (t-\tau)^\nu - |x-\xi|)^2}{4C_1(t-\tau)}\right). \quad (1.7)$$

The upper bound in Theorem 1 implies that  $\Gamma(t, x; \tau, \xi)$  decays exponentially in space variables, which yields the pre-compactness of the family of probability measures defined by  $\Gamma$ , in the sense that, the family of finite dimensional distributions

$$\prod_{i=1}^n \Gamma(t_i, x_i; t_{i-1}, x_{i-1}) dx_0 \cdots dx_n$$

for fixed  $s \leq t_0 < t_1 < \cdots < t_n$  is relatively compact under the topology of weak convergence in measure. The relative compactness allows us to construct  $\Gamma(t, x; \tau, \xi)$  for Borel measurable  $a$  and  $b$  which satisfy that  $\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \frac{1}{\lambda} |\xi|^2$ ,  $\operatorname{div} b = 0$ , and  $b \in L^l(0, T; L^q(\mathbb{R}^n))$  with  $\frac{2}{l} + \frac{n}{q} \in [1, 2)$ , but the convergence is too weak to ensure the Chapman–Kolmogorov equation, i.e.

$$\Gamma(t, x; \tau, \xi) = \int_{\mathbb{R}^n} \Gamma(t, x; s, y) \Gamma(s, y; \tau, \xi) dy.$$

Therefore, in this paper, we consider the time-homogeneous parabolic equation

$$\partial_t u(t, x) - \sum_{i,j=1}^n \partial_{x_i} (a_{ij}(x) \partial_{x_j} u(t, x)) + \sum_{i=1}^n b_i(x) \partial_{x_i} u(t, x) = 0 \quad (1.8)$$

where  $(a, b)$  satisfies that there exists a constant  $\lambda > 0$  such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \frac{1}{\lambda} |\xi|^2 \quad \text{E}$$

and  $b$  is divergence-free

$$\sum_{i=1}^n \partial_{x_i} b_i(x) = 0. \quad \text{S}$$

It worth noting that without the divergence-free condition (S), (1.8) may not have a solution for super-critical  $b$  (see for example [7, Example 1] and [28]).

The existence of weak solutions to (1.8) under mild conditions on  $b$  is easy to prove, while the uniqueness is not trivial. Here we prove the uniqueness using the conservativeness implied by Theorem 1. To be more specific, we construct a unique Markov semi-group  $(P_t)_{t \geq 0}$  for  $b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ ,  $q > \frac{n}{2}$  and show that it has kernel  $\Gamma$ , which satisfies the Chapman–Kolmogorov equation. Moreover, the semi-group is conservative, i.e.

$$P_t 1(x) = \int_{\mathbb{R}^n} \Gamma(t, x, y) dy = 1 \quad (1.9)$$

for a.e.  $x \in \mathbb{R}^n$ . Notice that in this case the corresponding bi-linear form

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^n} [\langle \nabla u, a \cdot \nabla v \rangle + (b \cdot \nabla u) v] dx \quad (1.10)$$

is not sectorial in general in the sense defined in [4, 14].

We are now in a position to state the main result of this paper.

**Theorem 2.** Suppose conditions (E), (S) hold and  $b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$  for  $q > \frac{n}{2}$ . There is a unique conservative Markov semi-group  $(P_t)_{t \geq 0}$  on  $L^2(\mathbb{R}^n)$  associated with operator  $\operatorname{div}(a \cdot \nabla) - b \cdot \nabla$  and it has transition probability kernel  $\Gamma(t, x, y)$  for  $t > 0$ ,  $x, y \in \mathbb{R}^n$ . Moreover, the uniqueness of weak solutions holds for the Cauchy initial problem to (1.8) and the solution is given by

$$u(t, x) = \int_{\mathbb{R}^n} \Gamma(t, x, y) u_0(y) dy \quad (1.11)$$

for any initial data  $u_0 \in L^2(\mathbb{R}^n)$ .

Here we assume that  $b$  is divergence-free and  $b \in L^q(\mathbb{R}^n)$  for  $q > \frac{n}{2}$  are primarily for proving the conservativeness of the semi-group through Theorem 1. The conservativeness is crucial for proving the uniqueness (see also [21]). When the dimension  $n = 3$ , the condition of Theorem 2 is satisfied if  $b \in L^2(\mathbb{R}^3)$ .

In the past decades, there have been many researches on the semi-groups for non-sectorial bi-linear forms (1.10). The construction of the semi-group is studied in, for example, [7, 8, 12, 20, 21, 25]. The construction of the semi-groups in these papers all use approximation arguments of different types. In [21], Stannat shows the existence of sub-Markovian  $C_0$ -semi-group when

$b \in L^2_{loc}(\mathbb{R}^n)$ . Although  $b \in L^2_{loc}(\mathbb{R}^n)$  does not guarantee uniqueness, Stannat proves the equivalence between uniqueness, conservativeness and a characterization of the bi-linear form (see [21, Proposition 1.9]).

The proof of uniqueness and conservativeness is harder. One could prove them using the heat kernel estimates [3]. In [30], using the Aronson type heat kernel estimate, Zhikov constructed the unique approximation semi-group to

$$\partial_t u - \operatorname{div}((A + B) \cdot \nabla u) = 0, \quad (1.12)$$

for periodic  $B \in L^\infty(\mathbb{R}^n)$ ,  $\operatorname{div} B \in L^2_{loc}(\mathbb{R}^n)$  and  $\sup_{r \geq 1} \frac{1}{r^n} \|B\|_{L^n(B(0,r))}^n < \infty$ . Here  $A$  is symmetric matrix-valued and  $B$  is anti-symmetric matrix-valued. It is easy to see that such problems are equivalent to (1.8) with divergence-free  $b$  if we set  $a = A$  and  $b = -\operatorname{div} B$ . In [13], Liskevich and Sobol further proved the heat kernel estimate for the semi-groups under additional functional conditions on the bi-linear form using the idea developed in [2]. Another way to prove the conservative assumes growth conditions on the coefficient, analyze locally and then take limit to the whole space (see for example [5, 16]).

Conservativeness of stochastic processes is also studied extensively on manifolds. The closest conditions in literature to ensure the conservativeness (also called stochastic completeness) for unbounded  $b$  are those on the symmetric tensor  $\nabla^s b$  (Ricci curvature or Bakry-Émery condition) while  $\nabla \cdot b$  is the trace of  $\nabla^s b$ . Hence our condition impose a constrain on the “scalar curvature” of the operator.

In this paper, in addition to divergence-free, we only make weak integrability assumptions on  $b$ . For divergence-free  $b \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$  with  $q > \frac{n}{2}$ , we establish the existence of the Markov semi-group associated with parabolic equation (1.8) using approximation by smooth coefficients, which is inspired by the ideas from [29, 30]. Then, using the conservativeness implied by the new Aronson type estimate in Theorem 1, we proved the uniqueness of the Markov semi-group. Moreover, the unique semi-group gives the unique weak solution to (1.8) because we show that any weak solution is an approximation solution.

Once constructed the semi-group, a natural question is the construction of the corresponding diffusion process. Because the semi-group in Theorem 2 is only defined on  $L^2(\mathbb{R}^n)$ , without regularity results,  $\Gamma(t, x, y)$  is only unique a.e. on  $[0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ . Therefore, we can not define the diffusion process for all initial data  $x \in \mathbb{R}^n$ . By [21, Theorem 3.5] and [22, Chapter IV], Stannat constructed the diffusion in a weaker sense using the framework of generalized Dirichlet forms, i.e. the process is defined for any initial data  $x \in \mathbb{R}^n$  except on a zero capacity set. For more details, we refer to [21] and [22]. The solution has local singularities on a small set because of the local singularities of  $b$ . Therefore, if we further assume  $b$  to be more regular locally, then we can define the diffusion process for any initial data  $x \in \mathbb{R}^n$ . A classical one is the local Lipschitz condition. Then we have the following result by Theorem 2. As a criteria for the conservativeness of the diffusion processes, it is interesting by its own.

**Corollary 3.** *Let  $b$  be a divergence-free vector field in  $\mathbb{R}^n$  with  $n \geq 3$  that is locally Lipschitz and  $b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  where  $q > \frac{n}{2}$ , then the strong solution to*

$$dX_t = dB_t + b(X_t)dt, \quad X_0 = x$$

*is conservative.*

Here the local Lipschitz condition can be replaced by other conditions that implies the local existence and uniqueness of the strong solution for any initial data  $x \in \mathbb{R}^n$ , for example  $b \in L^p_{loc}(\mathbb{R}^n)$  for  $p > n$  (see [9]). Then we also have the same result as stated in Corollary 3. **It should be mentioned that for time-dependent super-critical  $b$  with upper bound on  $-\operatorname{div} b$ , Zhang and Zhao[28]**

recently proved the existence and weak uniqueness of the martingale solution  $\mathbb{P}_{s,x}$  for almost all  $(s,x)$  except on a null set. In the special case when  $\operatorname{div} b = 0$ , their result implies the conservative of the solution.

This paper is organized as follows. In section 2, we prove the existence of weak solution for  $b \in L^2(\mathbb{R}^n)$ . In section 3, we give the proof to Theorem 2 where the drift vector field  $b$  belongs to the function space  $L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ ,  $q > \frac{n}{2}$ .

## 2 Existence of weak solutions

Firstly, we show the existence of weak solutions to (1.8) when  $(a,b)$  satisfies conditions (E) and (S) and  $b \in L^2(\mathbb{R}^n)$ . A similar result was proved in [30] for (1.12) and here we borrow the same idea to deal with our case. Here we denote by  $\Gamma(t,x,\xi)$  the fundamental solution to (1.8).

**Definition 4.** A function  $u \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n))$  is a weak solution to (1.8) corresponding to  $(a,b)$  and initial data  $u_0$  if

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} u(t,x) \partial_t \varphi(t,x) dx dt - \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u(t,x), \nabla \varphi(t,x) \rangle dx dt \\ & - \int_0^T \int_{\mathbb{R}^n} \langle b(x), \nabla u(t,x) \rangle \varphi(t,x) dx dt = - \int_{\mathbb{R}^n} u_0(x) \varphi(0,x) dx \end{aligned}$$

for any  $\varphi \in C_0^\infty([0,T] \times \mathbb{R}^n)$ .

When  $b \in L^q(\mathbb{R}^n)$  with  $q \geq n$ , for any initial data  $u_0 \in L^2(\mathbb{R}^n)$ , there exists a unique weak solution  $u$  satisfying that  $\partial_t u \in L^2(0,T;H^{-1}(\mathbb{R}^n))$  and  $u \in C([0,T],L^2(\mathbb{R}^n))$ . Moreover, it satisfies the energy identity

$$\frac{1}{2} \|u(T)\|_2^2 + \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u(t,x), \nabla u(t,x) \rangle dx dt = \frac{1}{2} \|u_0\|_2^2. \quad (2.1)$$

For more details, we refer to [10]. When  $n \geq 3$ ,  $b \in L^2(\mathbb{R}^n)$  we have the following results assuming that  $b$  is divergence-free.

**Proposition 5.** Suppose conditions (E) and (S) are satisfied and  $b \in L^2(\mathbb{R}^n)$ , there exists a weak solution to (1.8) with initial data  $u_0 \in L^2(\mathbb{R}^n)$ .

*Proof.* Denote by  $u_k$  the weak solution corresponding to  $(a,b_k)$  with the same initial data  $u_0$  as in Definition 4, where  $b_k \in C_0^\infty(\mathbb{R}^n)$  are divergence-free and  $b_k \rightarrow b$  in  $L^2(\mathbb{R}^n)$ . Then  $\{u_k\}$  is uniformly bounded in  $L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n))$  and has a sub-sequence which converges weakly to some  $u$ . This weak convergence allows us to take  $k \rightarrow \infty$  for

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} u_k(t,x) \partial_t \varphi(t,x) dx dt - \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u_k(t,x), \nabla \varphi(t,x) \rangle dx dt \\ & - \int_0^T \int_{\mathbb{R}^n} \langle b_k(x), \nabla u_k(t,x) \rangle \varphi(t,x) dx dt = - \int_{\mathbb{R}^n} u_0(x) \varphi(0,x) dx \end{aligned}$$

to obtain that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} u(t,x) \partial_t \varphi(t,x) dx dt - \int_0^T \int_{\mathbb{R}^n} \langle a(x) \cdot \nabla u(t,x), \nabla \varphi(t,x) \rangle dx dt \\ & - \int_0^T \int_{\mathbb{R}^n} \langle b(x), \nabla u(t,x) \rangle \varphi(t,x) dx dt = - \int_{\mathbb{R}^n} u_0(x) \varphi(0,x) dx. \end{aligned}$$

Hence the limit  $u$  is a weak solution to the corresponding problem with data  $(a,b)$ .  $\square$

We call the weak solution constructed in this way an approximation solution. Next, we show that every weak solution is an approximation solution in a weaker sense. This result follows from a similar argument in [27].

**Proposition 6.** *Suppose  $b \in L^2(\mathbb{R}^n)$  and  $b_k \in C_0^\infty(\mathbb{R}^n)$  are divergence-free such that  $b_k \rightarrow b$  in  $L^2(\mathbb{R}^n)$ . Let  $u$  and  $\{u_k\}$  be the weak solutions to (1.8) on  $[0, T] \times \mathbb{R}^n$  with the same initial data  $u_0$ , the same diffusion coefficient  $a$ , and drifts  $b$  and  $\{b_k\}$  respectively. Then  $u$  is the  $L^\infty(0, T; L^1(\mathbb{R}^n))$  limit of functions  $\{u_k\}$ .*

*Proof.* Consider the Cauchy problem

$$\partial_t u_k - \operatorname{div}(a \cdot \nabla u_k) + b_k \cdot \nabla u_k = 0$$

with initial data  $u_k(x, 0) = u_0(x)$ . Clearly  $u_k - u$  is a weak solution to

$$\partial_t(u_k - u) - \operatorname{div}(a \cdot \nabla(u_k - u)) + b_k \cdot \nabla(u_k - u) = (b - b_k) \cdot \nabla u$$

with 0 as the initial value. By assumption,  $\|(b - b_k) \cdot \nabla u\|_{L^2(0, T; L^1(\mathbb{R}^n))} \rightarrow 0$  as  $k \rightarrow \infty$ . Since  $b_k \in C_0^\infty(\mathbb{R}^n)$ , we have a representation given by

$$(u_k - u)(t, x) = \int_0^t \int_{\mathbb{R}^n} \Gamma_k(t - \tau, x, \xi) (b - b_k) \cdot \nabla u(\xi, \tau) d\xi d\tau,$$

where  $\Gamma_k$  is the fundamental solution corresponding to  $b_k$ . Then  $\Gamma_k^*(t, \xi, x) := \Gamma_k(t, x, \xi)$  is the fundamental solution to  $(\partial_t - L_k^*)u = 0$ , which is of the same form as the original equation (1.8) up to a sign on the drift. Hence

$$\int_{\mathbb{R}^n} \Gamma_k(t - \tau, x, \xi) dx = 1 \tag{2.2}$$

for any fixed  $(t - \tau, \xi)$ . This implies that

$$\int_{\mathbb{R}^n} |u_k - u|(t, x) dx \leq \int_0^t \int_{\mathbb{R}^n} |b - b_k| |\nabla u| d\xi d\tau \rightarrow 0$$

and the proof is done.  $\square$

The proposition above implies that all weak solutions are approximation solutions. Here the divergence-free condition is the key to have the dual operator being conservative to obtain (2.2).

### 3 Uniqueness of the approximation semi-group and its kernel

In this section we prove our main result Theorem 2. The idea is to construct a unique approximation Markov semi-group corresponding to generator  $L = \operatorname{div}(a \cdot \nabla) - b \cdot \nabla$ . Since  $a$  is only Borel measurable, the generator  $L$  is not well defined as a differential operator. Hence we will construct  $L$  in the following, while we still use formal expression  $L = \operatorname{div}(a \cdot \nabla) - b \cdot \nabla$  for simplicity of notation. We start with the bi-linear form

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla v \rangle + (b \cdot \nabla u) v dx.$$

Naturally we consider the elliptic problem and its weak solutions, which is standard in literature.

**Definition 7.** Let  $(a, b)$  satisfies **(E)**, **(S)** and  $b \in L^2(\mathbb{R}^n)$ . For  $f \in L^2(\mathbb{R}^n)$ , if there exists a  $u \in H^1(\mathbb{R}^n)$  such that

$$\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla \varphi \rangle + (b \cdot \nabla u) \varphi + \alpha u \varphi \, dx = \int_{\mathbb{R}^n} f \varphi \, dx$$

for all  $\varphi \in C_0^\infty(\mathbb{R}^n)$ , we call  $u$  a weak solution to the elliptic problem  $(\alpha - L, f)$ , where  $\alpha \geq 0$ .

For  $b \in C_0^\infty(\mathbb{R}^n)$ , the bi-linear form is actually a Dirichlet form. We recall the following result on Dirichlet forms in [14, Chapter 1].

**Theorem 8.** Suppose  $(a, b)$  satisfies **(E)**, **(S)** and  $b \in C_0^\infty(\mathbb{R}^n)$ , then  $(\mathcal{E}, H^1(\mathbb{R}^n))$ , where

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla v \rangle + (b \cdot \nabla u) v \, dx$$

with  $u, v \in H^1(\mathbb{R}^n)$ , is a (non-symmetric) Dirichlet form. We still use  $L$  together with its domain  $D(L)$  to denote the generator associated with the Dirichlet form  $(\mathcal{E}, H^1(\mathbb{R}^n))$ . The resolvent  $R_\alpha = (\alpha - L)^{-1}$  for  $\alpha > 0$  is a bounded linear operator from  $L^2(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^n)$  with  $\|(\alpha - L)^{-1}\|_{L^2 \rightarrow L^2} \leq \alpha^{-1}$ , and it satisfies that

$$\mathcal{E}(R_\alpha f, v) + \alpha \int_{\mathbb{R}^n} (R_\alpha f) v \, dx = \int_{\mathbb{R}^n} f v \, dx. \quad (3.1)$$

Thus for  $b \in C_0^\infty(\mathbb{R}^n)$ ,  $\operatorname{div}(a \cdot \nabla) - b \cdot \nabla$  is understood as the generator  $L$  defined as in Theorem 8 above. Clearly, for any  $f \in L^2(\mathbb{R}^n)$ ,  $(\alpha - L)^{-1} f$  is the unique weak solution to  $(\alpha - L, f)$ . We can take  $v = (\alpha - L)^{-1} f$  and derive that

$$\|(\alpha - L)^{-1} f\|_{H^1} \leq \frac{1}{\min\{\lambda, \alpha\}} \|f\|_{L^2} \quad \text{and} \quad \|(\alpha - L)^{-1} f\|_{L^2} \leq \frac{1}{\alpha} \|f\|_{L^2} \quad (3.2)$$

for all  $\alpha > 0$  and  $f \in L^2(\mathbb{R}^n)$ . Next we prove the following estimate on  $R_\alpha$ , which follows from [30], plays an important role in proving our main result.

**Lemma 9.** Suppose  $b \in C_0^\infty(\mathbb{R}^n)$  and  $L$  as in Theorem 8, set  $u = (1 - L)^{-1} f$  for  $f \in C_0^\infty(\mathbb{R}^n)$ . Then for  $n \geq 3$ , we have

$$\int_{\mathbb{R}^n} [\ln(|x|^2 + e)]^{2\gamma} u^2(x) \, dx \leq C_0 \int_{\mathbb{R}^n} [\ln(|x|^2 + e)]^{2\gamma} f^2(x) \, dx$$

with sufficiently small positive  $\gamma$  and constant  $C_0$  depending only on  $n$ ,  $\lambda$ ,  $\gamma$  and  $\|b\|_{L^q(\mathbb{R}^n)}$  with  $q > \frac{n}{2}$ .

*Proof.* Let  $\psi = \gamma \psi_0$ ,  $\psi_0 = \ln \ln(|x|^2 + e)$ , for  $\gamma > 0$ , and consider the operator  $L_\psi = e^\psi L e^{-\psi}$ . For  $v = e^\psi u$ , we have  $L_\psi v - v = g = e^\psi f$  and

$$\int_{\mathbb{R}^n} -\langle \nabla(e^\psi v), a \cdot \nabla(e^{-\psi} v) \rangle - b \cdot \nabla(e^{-\psi} v) e^\psi v - v^2 \, dx = \int_{\mathbb{R}^n} g v \, dx.$$

It follows, together with **(E)** and **(S)**, that

$$\int_{\mathbb{R}^n} \lambda |\nabla v|^2 - \frac{1}{\lambda} \gamma^2 |\nabla \psi_0|^2 v^2 - \gamma (b \cdot \nabla \psi_0) v^2 + v^2 \, dx \leq - \int_{\mathbb{R}^n} g v \, dx.$$

Notice that

$$|\nabla \psi_0| \leq \frac{2|x|}{(|x|^2 + e) \ln(|x|^2 + e)},$$

which is bounded. Hence we have

$$\begin{aligned} \int_{\mathbb{R}^n} (b \cdot \nabla \psi_0) v^2 dx &\leq C \|b\|_{L^q} \|\nabla \psi_0\|_{L^\infty} \|v\|_{L^2}^{1-\theta} \|\nabla v\|_{L^2}^{1+\theta} \\ &\leq C \|b\|_{L^q} \|\nabla \psi_0\|_{L^\infty} C(\theta) (\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) \end{aligned}$$

where  $\theta = \frac{n}{q} - 1$  and  $C$  depends on  $n, q$ . Now we can take  $\gamma$  small enough such that  $\|v\|_{L^2} \leq C_0 \|g\|_{L^2}$  and the proof is complete.  $\square$

Given divergence-free  $b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  and a sequence of smooth functions  $b_k \rightarrow b$  in  $L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , Lemma 9 implies that for each fixed  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $L^2$ -norm of the approximate sequence  $\{(1 - L_k)^{-1} f\}$  will be uniformly concentrated on some balls. Now we can prove the compactness of  $\{(1 - L_k)^{-1} f\}$ .

**Lemma 10.** *Given divergence-free  $b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , smooth approximations  $b_k \rightarrow b$  in space  $L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , and  $f \in L^2(\mathbb{R}^n)$ , the sequence  $\{(1 - L_k)^{-1} f\}$  is strongly compact in  $L^2(\mathbb{R}^n)$  and weakly compact in  $H^1(\mathbb{R}^n)$ .*

*Proof.* By  $\|(1 - L_k)^{-1} f\|_{H^1} \leq \frac{1}{\min\{\lambda, 1\}} \|f\|_{L^2}$ , we have that the sequence  $\{(1 - L_k)^{-1} f\}$  is weakly compact in  $H^1(\mathbb{R}^n)$ . To prove the strong compactness in  $L^2(\mathbb{R}^n)$ , recall that we have proved  $\|(1 - L_k)^{-1}\|_{L^2 \rightarrow L^2} \leq 1$  for all  $k$ . Since the convergence of bounded linear operators is determined by its convergence on a dense subset (see Theorem 6 in [11, Ch15]), it is sufficient to establish the compactness of  $\{(1 - L_k)^{-1} f\}$  for  $f$  in a dense subset of  $L^2(\mathbb{R}^n)$ . For  $f \in C_0^\infty(\mathbb{R}^n)$ , by Lemma 9 and inequality  $\|(1 - L_k)^{-1} f\|_{H^1} \leq \frac{1}{\min\{\lambda, 1\}} \|f\|_{L^2}$ , the compactness of  $\{(1 - L_k)^{-1} f\}$  in  $L^2(\mathbb{R}^n)$  follows from the Fréchet–Kolmogorov theorem [26, Chapter X, Section 1].  $\square$

Lemma 10 above allows us to take  $k \rightarrow \infty$  and define the generator  $L$  for singular  $b$  as the limit of  $L_k$ .

**Lemma 11.** *Given  $L_k$  defined as in Theorem 8 corresponding to  $b_k$  which converges to  $b$  in  $L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , after a possible selection of a sub-sequence (denoted as  $L_k$  again), there exists a closed operator  $L$  defined on a dense subset of  $L^2(\mathbb{R}^n)$  such that  $\|(\alpha - L)^{-1}\|_{L^2 \rightarrow L^2} \leq \alpha^{-1}$  for all  $\alpha > 0$  and*

$$(\alpha - L_k)^{-1} f \rightarrow (\alpha - L)^{-1} f \quad \text{in } L^2(\mathbb{R}^n)$$

for all  $\alpha > 0$  and  $f \in L^2(\mathbb{R}^n)$ .

*Proof.* We first consider the case when  $\alpha = 1$ . By Theorem 6 in [11, Ch15], convergence of bounded linear operators  $\{(1 - L_k)^{-1}\}_{k=1}^\infty$  is determined by its convergence on a dense subset of  $L^2(\mathbb{R}^n)$ . We apply Lemma 10 to  $f$  in a countable dense subset of  $L^2(\mathbb{R}^n)$ . Then using the diagonal argument, we can find a sub-sequence of  $\{(1 - L_k)^{-1}\}_{k=1}^\infty$  that converges strongly. We still denote the sub-sequence as  $(1 - L_k)^{-1}$  and denote its limit as  $S$ , i.e.

$$(1 - L_k)^{-1} f \rightarrow S f$$

strongly in  $L^2(\mathbb{R}^n)$  for  $f \in L^2(\mathbb{R}^n)$ .

Given any  $f \in L^2(\mathbb{R}^n)$ , since  $\{(1 - L_k)^{-1} f\}$  is weakly compact in  $H^1$ , it also converges to  $S f$  weakly in  $H^1$ . It is easy to see that the limit  $S f$  is a weak solution to  $(1 - L, f)$ . Since  $S$  is bounded linear operator from  $L^2(\mathbb{R}^n)$  to itself, we can define its adjoint operator  $S^*$  by  $\langle S f, g \rangle = \langle f, S^* g \rangle$  for all  $f, g \in L^2(\mathbb{R}^n)$ . We already know that  $\lim_{k \rightarrow \infty} \langle (1 - L_k)^{-1} f, g \rangle = \langle S f, g \rangle$  for all  $f, g \in L^2(\mathbb{R}^n)$  and  $\langle (1 - L_k)^{-1} f, g \rangle = \langle f, (1 - L_k^*)^{-1} g \rangle$ . Hence we can see that  $S^* g$  is a weak solution to  $(1 - L^*, g)$ . Proposition 13 below implies that both  $S$  and  $S^*$  have  $K(S) = K(S^*) = 0$  and hence have dense

range in  $L^2(\mathbb{R}^n)$  by  $K(S^*) = R(S)^\perp$ . Now we can define  $L = 1 - S^{-1}$ , which has dense domain  $D(L)$  and  $D(L) \subset H^1$ . Since  $S^{-1}$  is a closed operator,  $L$  is also closed.

Clearly, for each  $u \in D(L)$ , it is the weak solution to  $(-L, -Lu)$ . Hence for any  $\alpha > 0$ ,  $(\alpha - L)$  is also a closed operator and we can define  $(\alpha - L)^{-1}f$  to be the weak solution to  $(\alpha - L, f)$  for  $f$  in the range of  $(\alpha - L)$ , i.e.  $f \in R(\alpha - L)$ . Notice that for  $(\alpha - L)$  and its dual operator  $(\alpha - L^*)$ , Proposition 13 implies that  $K(\alpha - L^*) = 0$ . By the closed range theorem, we have that  $R(\alpha - L) = K(\alpha - L^*)^\perp = L^2(\mathbb{R}^n)$ . Hence the resolvent operator  $(\alpha - L)^{-1}$  is well defined for any  $\alpha > 0$ . Finally we prove that  $(\alpha - L_k)^{-1}f \rightarrow (\alpha - L)^{-1}f$  for any  $\alpha > 0$  and  $f \in L^2(\mathbb{R}^n)$ . As shown in Theorem 1.3 in [6, Ch.8], we can derive

$$\begin{aligned} & (\alpha - L_k)^{-1} - (\alpha - L)^{-1} \\ &= (1 + (\alpha - 1)(\alpha - L_k)^{-1}) ((1 - L_k)^{-1} - (1 - L)^{-1}) (1 + (\alpha - 1)(\alpha - L)^{-1}), \end{aligned}$$

from the resolvent equation and  $(1 - L_k)^{-1}f \rightarrow (1 - L)^{-1}f$  implies that  $(\alpha - L_k)^{-1}f \rightarrow (\alpha - L)^{-1}f$  for any  $\alpha > 0$ . Since estimates (3.2) are true for  $\{(\alpha - L_k)^{-1}f\}$ , it is also true for the limit  $(\alpha - L)^{-1}f$ .  $\square$

*Remark 12.* Given  $f \in L^2(\mathbb{R}^n)$ , we have that  $Sf$  is the limit of  $\{(1 - L_k)^{-1}f\}$  weakly in  $H^1(\mathbb{R}^n)$  and it is easy to check that  $Sf$  is a weak solution to  $(1 - L, f)$ . Next we show that for  $b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ , there is a unique  $S$  defined as in last Theorem. The uniqueness of  $S$  implies that the definition of  $L$  is independent of the choice of the convergent sub-sequence.

**Proposition 13.** Suppose  $(a, b)$  satisfies conditions (E) and (S). For any  $f \in L^2$ , there exists a unique weak solution  $u \in H^1$  to the elliptic problem  $(\alpha - L, f)$  for  $n \geq 3$ ,  $b \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  and  $\alpha > 0$ .

*Proof.* The existence of weak solution can be obtained using the same approximation argument as in Lemma 11. To prove the uniqueness, we consider a weak solution  $u$  to  $(\alpha - L, 0)$ . Here we can take the test function to be  $h = \bar{u}\varphi$  with  $\bar{u} = u \wedge N \vee (-N)$  and  $\varphi \in C_0^\infty$ . Given any  $r > 0$ , there is  $\varphi_r$  satisfying

$$\varphi_r = \begin{cases} 1 & |x| \leq \frac{r}{2}, \\ 0 & |x| \geq r, \end{cases} \quad |\nabla \varphi| \leq \frac{4}{r}.$$

Then we have

$$\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla(\bar{u}\varphi_r) \rangle + b \cdot \nabla u(\bar{u}\varphi_r) + \alpha u(\bar{u}\varphi_r) dx = 0.$$

Because  $\bar{u}\varphi_r \rightarrow \bar{u}$  in  $H^1(\mathbb{R}^n)$  and a.e., we can take  $r \rightarrow \infty$  to obtain

$$\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla(\bar{u}) \rangle + b \cdot \nabla u(\bar{u}) + \alpha u(\bar{u}) dx = 0.$$

Next we consider the second term in the equation above. Since  $\int_{\mathbb{R}^n} b \cdot \nabla \bar{u} \bar{u} dx = 0$ , we have

$$\begin{aligned} \int_{\mathbb{R}^n} b \cdot \nabla u \bar{u} dx &= \int_{\mathbb{R}^n} b \cdot (\nabla u - \nabla \bar{u}) \bar{u} dx \\ &= N \int_{\{u > N\}} b \cdot (\nabla u - \nabla \bar{u}) dx - N \int_{\{u < -N\}} b \cdot (\nabla u - \nabla \bar{u}) dx = 0 \end{aligned}$$

Now we can take  $N \rightarrow \infty$  to obtain

$$\int_{\mathbb{R}^n} \langle \nabla u, a \cdot \nabla u \rangle + \alpha u^2 dx = 0$$

and conclude that  $u = 0$ .  $\square$

Finally, to prove the representation (1.11), we also need the convergence of the fundamental solutions.

**Proposition 14.** *Given a sequence of probability measures  $\{P_n\}$  on  $\mathbb{R}^n$  which have densities  $\{f_n\}$  uniformly bounded from above by a continuous function  $h$ . Suppose  $h$  satisfies*

$$\lim_{R \rightarrow \infty} \int_{B(0,R)^c} h(x) dx = 0,$$

*where  $B(0,R)$  is the open ball in  $\mathbb{R}^n$  centered at 0 with radius  $R$ . Then  $\{P_n\}$  is weakly compact in the space of probability measure. Suppose we take a convergent sub-sequence, then its limit  $P$  has density  $f$  which is also bounded from above by  $h$ .*

*Proof.* It is easy to see that  $\{P_n\}$  is tight, which implies that it is weakly compact by Prohorov's theorem. So we just need to show that  $P$  has density  $f$  which is bounded by  $h$ . Firstly, we show that  $P$  is absolutely continuous with respect to the Lebesgue measure  $m$ . Suppose  $A \subset \mathbb{R}^n$  such that  $m(A) = 0$ , then there is a decreasing sequence of open sets  $\{O_i\}$  containing  $A$  such that  $\lim_{i \rightarrow \infty} m(O_i) = 0$ . Therefore  $\lim_{i \rightarrow \infty} P_n(O_i) \rightarrow 0$  uniformly for all  $P_n$ . By the Portmanteau theorem [23, Theorem 1.1.1], we have  $P(O_i) \leq \limsup_{n \rightarrow \infty} P_n(O_i)$ , which implies that  $\lim_{i \rightarrow \infty} P(O_i) = 0$  and hence  $P(A) = 0$ . So  $P$  has a density  $f$  by Radon–Nikodym's theorem.

Next we show that this  $f$  is bounded by  $h$ . If not, we can find a bounded set  $A$  such that  $m(A) > 0$  and  $f > h$  a.e. on  $A$ . Since  $h$  is continuous, we can find an open set  $O$  small enough such that it contains  $A$  and  $P(O) > \int_O h \geq P_n(O)$  for all  $n$ . Clearly this contradicts to that  $P_n \rightarrow P$  weakly in measure.  $\square$

Now we are in a position to complete the proof of Theorem 2.

*Proof.* By the fundamental approximation theorem of semi-groups in [6, Cp 9, Theorem 2.16], the convergence of resolvents proved in Lemma 11 implies that  $e^{tL_k} \rightarrow e^{tL}$  as bounded linear operators from  $L^2(\mathbb{R}^n)$  to  $L^2(\mathbb{R}^n)$  and is uniform for  $t$  in any finite interval  $[0, T]$ . Moreover, we know that  $L$  is unique by Remark 12 and Proposition 13. Using Proposition 6, we know that any weak solution to (1.8) with initial condition  $u_0 \in L^2(\mathbb{R}^n)$  must equal to  $e^{tL}u_0$ . Therefore,  $e^{tL}$  is the unique semi-group which generates the unique weak solution.

Next, we show the existence of the unique fundamental solution corresponding to (1.8). We have proved above that  $u_k \rightarrow e^{tL}u_0$  in  $L^2(\mathbb{R}^n)$ , which implies that there is a subsequence, denoted as  $u_k$  again, converging a.e. to  $e^{tL}u_0$ . Let  $\Gamma_k(t, x, y)$  be the corresponding fundamental solution to  $(\partial_t - L_k)u = 0$ . Then

$$u_k(t, x) = \int_{\mathbb{R}^n} \Gamma_k(t, x, y) u_0(y) dy = e^{tL_k} u_0$$

for any  $u_0 \in L^2(\mathbb{R}^n)$  and  $k = 1, 2, \dots$ . By Theorem 1 and Proposition 14, we have that for each fixed  $(t, x)$  (and  $(t, y)$ ), the family of transition probabilities  $\{\Gamma_k(t, x, y) dy\}$  (and also the family  $\{\Gamma_k(t, x, y) dx\}$ ) is tight and hence converges weakly in measure to some  $\Gamma(t, x, y) dy$  which has the same upper bound as that of  $\Gamma_k(t, x, y)$ . Define

$$u(t, x) = \int_{\mathbb{R}^n} \Gamma(t, x, y) u_0(y) dy$$

for  $u_0 \in C_0^\infty(\mathbb{R}^n)$ , then for each fixed  $(t, x)$ ,  $u_k(t, x)$  has a subsequence that converges to  $u(t, x)$  by the weak convergence of measure. Now we proved that  $u = e^{tL}u_0$  in  $L^2(\mathbb{R}^n)$ . Since  $C_0^\infty(\mathbb{R}^n)$  is dense in  $L^2(\mathbb{R}^n)$ , we can extend it to  $L^2(\mathbb{R}^n)$  and conclude that operator  $e^{tL}$  has a kernel  $\Gamma(t, x, y)$ .  $\square$

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