



The Sure-Thing Principle[☆]

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ABSTRACT

The Sure-Thing Principle famously appears in Savage's axiomatization of Subjective Expected Utility. Yet Savage introduces it only as an informal, overarching dominance condition motivating his separability postulate P2 and his state-independence postulate P3. Once these axioms are introduced, by and large, he does not discuss the principle any more. In this note, we pick up the analysis of the Sure-Thing Principle where Savage left it. In particular, we show that each of P2 and P3 is equivalent to a dominance condition; that they strengthen in different directions a common, basic dominance axiom; and that they can be explicitly combined in a unified dominance condition that is a candidate formal statement for the Sure-Thing Principle. Based on elementary proofs, our results shed light on some of the most fundamental properties of rational choice under uncertainty. In particular they imply, as corollaries, potential simplifications for Savage's and the Anscombe-Aumann axiomatizations of Subjective Expected Utility. Most surprisingly perhaps, they reveal that in Savage's axiomatization, P3 can be weakened to a natural strengthening of so-called Obvious Dominance.

1. Introduction

1.1. Motivation

The Sure-Thing Principle (STP) famously appears in Savage's axiomatization of Subjective Expected Utility (SEU; Savage, 1954, 1972). In a section simply entitled “The Sure-Thing Principle”, Savage introduces it as follows (1954, p. 21): “A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter to himself, he asks whether he would buy if he knew that the Democratic candidate were going to win, and decides that he would. Similarly, he considers whether he would buy if he knew that the Republican candidate were going to win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy, even though he does not know which event obtains ... [E]xcept possibly for the assumption of simple ordering, I know of no other ... principle governing decisions that finds such ready acceptance.”

Savage does not directly employ the STP in his analysis, however. He leaves it as an informal, overarching dominance (also known as contingent—or case-by-case—reasoning) condition motivating several of his axioms. Most importantly, he uses it to motivate his separability postulate P2 and his state-independence postulate P3 (the formal statements of which we will recall shortly).¹ Savage comments (1954, p. 22): “The sure-thing principle cannot appropriately be accepted as a postulate in the sense that P1 [Savage's weak order, or “simple ordering”, axiom] is because it would introduce new undefined technical terms referring to knowledge and possibility that would render it mathematically useless without still more postulates governing these terms. It will be preferable to regard the principle as a loose one that suggests certain formal postulates well articulated with P1”. Once introduced, P2 and P3 supersede the STP in Savage's analysis. By and large (qualifications to follow), this is where Savage and the larger literature have left the principle, even more so, the investigation of the kinship of P2 and P3.

We pick up the analysis of the STP where Savage left it. We show that even without enriching his framework in any way (with primitive

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¹ Savage also uses the STP to motivate his postulate P7, that has a specific bite only when infinitely-valued options (known as “general acts”) come into play. (On the relationships between P7 and the other Savage axioms in that case, see Hartmann, 2020; Frahm and Hartmann, 2023.) As detailed in Section 2.1, we here focus on finitely-valued options (the so-called “simple acts”).

conditional preferences, knowledge operators, or anything else that would be additional to the traditional assumptions recapitulated in Section 2.1), more can be said than is currently known about the STP, and the kinship between P2 and P3 can be further confirmed. In particular, we show that each of P2 and P3 is equivalent to a dominance condition (Proposition 1); that they can be explicitly combined in a unified dominance condition that is a candidate formal statement for the full STP (Proposition 2); and that they strengthen in different directions a common, basic dominance axiom (Proposition 3). These results imply, as corollaries, potential simplifications for Savage’s and, incidentally, the Anscombe-Aumann (Anscombe and Aumann, 1963) axiomatizations of SEU (Corollaries 2 and 3). In particular (Corollary 3), and this may be our most surprising result, we show that in Savage’s axiomatization of SEU, P3 can be weakened to so-called Obvious Dominance (Li, 2017), suitably enriched with a strict clause. Obvious Dominance (further discussed before Corollary 3) is an extremely minimal rationality condition that has recently attracted considerable attention in various areas of economics, starting with mechanism design (via the notion of “obvious strategy-proofness”). It has been introduced and motivated as a dramatic weakening of stronger dominance conditions such as the one to which, taken alone, P3 proves equivalent—hence the surprising nature of the result referred to above. As we shall see, that result further underscores the importance of taking into account the overlap between P2 and P3, which is a key feature of our investigation of the STP. The first of the above results (Proposition 1) can essentially be found in Sec. 2.7 of Savage, 1954, and it is manifestly known in the literature. Still, even this result of ours improves on Savage’s in minor ways (see after Proposition 1 for details), and our other results (Propositions 2 and 3; Corollaries 2 and 3) are entirely new, to the best of our knowledge. Based on elementary proofs, together they shed light on the internal structure of and interplay between arguably the two most fundamental properties of rational choice under uncertainty. The modesty of the objective notwithstanding, no comparably comprehensive analysis of the STP (as originally envisaged by Savage) seems available in the current literature.

1.2. Literature review

The function of this subsection is to acknowledge important pre-existing work on the STP, broadly construed, while also emphasizing anew that this preexisting literature does not study the questions on which our paper focuses.

The STP may be the most scrutinized property in the literature if, as virtually all current decision theorists, one equates it with P2.² This is because P2 is the property generalized in most Non-Expected Utility models (the most famous generalization, the so-called “co-monotonic STP”, is presented and discussed in Schmeidler, 1989; Gilboa, 1987; Chew and Wakker, 1996). But the STP is rarely discussed if, like Savage, one understands it as uniting P2 and P3.³ Inspired especially by the second series of remarks by Savage previously quoted, some have analyzed the STP using tools from epistemic logic (see, e.g., Samet, 2022 and the references therein; also Chew and Wang, 2022 for a recent proposal to combine the STP, thus approached, with Obvious Dominance). But here as well, the focus is on P2. Similarly with Ghirardato, 2002 (and the relevant parts of the dynamic consistency literature to which this reference belongs), where primitive notions of conditional preferences

² For philosophical discussions of the STP, thus understood, from the point of view of Causal Decision Theory and the like, see for instance Jeffrey, 1982; Pearl, 2016.

³ It even seems that Savage discovered the principle starting from the P3, rather than the P2 side. The example in the 1950 letter to Samuelson which historians (see Moscatti, 2016, p. 230) take to contain the first occurrence of the principle is, upon close inspection, under the scope of P3, not P2. (This follows from Proposition 1.b.) Contrast the more famous early occurrence in Friedman and Savage, 1952 (p. 468), that is arguably under the scope of P2.

shed light on the nature of the STP—understood as P2. Valuable discussions of the STP that touch upon its potential difference from P2, yet not its relationship with P3, include Grant et al., 2000, Dietrich et al. 2021, Esponda and Vespa, 2021, and, in passing, Fleurbaey, 2010 (see its fn. 9). While evidently without the benefit of the more recent advances above, Sec. 2.7 of Savage, 1954 seems to remain the most complete analysis to date. We defer discussing exactly how we improve on Savage’s own analysis until after we state our first main result (see p. 7). For now, we only wish to reiterate that the questions underlying our other main results and their corollaries—viz. how P2 and P3 can be combined with one another, on the one hand, and decomposed in terms of a common and a distinctive part, on the other hand—have not hitherto been considered in the literature, to the best of our knowledge.

2. Analysis

2.1. Preliminaries

Let S be a state space, Σ a σ -algebra on S , and X a set of consequences. Elements of Σ are called events. For any event $E \in \Sigma$, \bar{E} denotes its complement. Given $E \in \Sigma$, we refer to a finite Σ -measurable partition $\{E_1, \dots, E_n\}$ of E simply as a partition of E . Acts are Σ -measurable mappings from S to X . More specifically, throughout this paper, by “acts” we mean simple (finite-valued) acts, the set of which we denote by F .^{4,5} With the usual abuse of notation, X also denotes the set of constant acts.

Our primitive is a binary relation \succsim over F , interpreted as the preferences of a decision-maker among acts. Its asymmetric and symmetric parts are denoted \succ and \sim , respectively. In keeping with Savage’s own nomenclature, the axioms—Savage preferred to say: the postulates, hence his nomenclature—which he imposes on \succsim will be here referred to as the P-axioms. We always assume that \succsim is complete and transitive—in other words, that Savage’s axiom P1 holds.⁶

P1. \succsim is a weak order (i.e., complete and transitive).

For $f, g \in F$ and $E \in \Sigma$, fEg denotes the act resulting in f on E and g on \bar{E} . An event $E \in \Sigma$ is null if $fEh \sim gEh$ for all $f, g, h \in F$. Otherwise E is non-null. An event $E \in \Sigma$ is essential if neither it nor its complement is null. While all of the above is standard, such is not the case of the following notation. First, for any $E \in \Sigma$, $f \in F$, we write $f(E)$ if and only if f is constant over E , and we also let $f(E)$ denote the constant consequence of f on E . Second, for any $f, g, h \in F$, ternary partition $\{E_1, E_2, E_3\}$ of S , fE_1gE_2h denotes the act equal to f on E_1 , g on E_2 , and h on E_3 .

Next, we recall the statement of Savage’s axioms P2 and P3.

P2. For all $f, g, h, h' \in F$, $E \in \Sigma$, $fEh \succsim gEh \Leftrightarrow fEh' \succsim gEh'$.

P3. For all $x, y \in X$, non-null $E \in \Sigma$, $h \in F$, $x \succsim y \Leftrightarrow xEh \succsim yEh$.

P2 and P3 are well appreciated to be logically independent.⁷ Like P1, they are necessary conditions for SEU to hold, i.e., for the existence

⁴ Hence the fact that we do not discuss Savage’s P7, despite its being also motivated by the STP. P7 becomes relevant only when general (infinitely-valued) acts are considered.

⁵ Our main proofs are those of Props. 1 and 3, Lemmas 3, 4, and 6. Our proofs of Proposition 3.a and Lemma 6 hold for simple acts only. The other proofs hold for general acts.

⁶ This is evidently a major restriction. It is in line with Savage’s already quoted remark that the STP covers “certain formal postulates well articulated with P1” (1954, p. 22; emphasis added).

⁷ For instance, an SEU model with ordinarily state-dependent utilities satisfies P2 but not P3 (more details in, e.g., Baccelli, 2017, Sec. 2.3) and conversely, a Non-Expected Utility Probabilistically Sophisticated model (as defined in Machina and Schmeidler, 1992) satisfies P3 but not P2.

of a utility function $u : X \rightarrow \mathbb{R}$ and a probability measure P on (S, Σ) such that for all $f, g \in F$,

$$f \succcurlyeq g \Leftrightarrow \int_S u(f(s)) dP(s) \geq \int_S u(g(s)) dP(s).$$

Savage’s Theorem refers to the axiomatization of SEU by **P1**, **P2**, **P3** together with three axioms that can be left in the background of our analysis, viz. the comparative probability axiom **P4**, the non-triviality axiom **P5**, and the continuity axiom **P6** (see Savage, 1954 and, for a modern exposition, Thm. 10.1 in Gilboa, 2009). The Anscombe-Aumann Theorem refers to a popular alternative axiomatization of the SEU representation, set in a different analytical framework that is a suitably construed mixture space. The axiomatization is in terms of a non-triviality assumption, the von Neumann - Morgenstern axioms of expected utility under risk, and the standard State-Wise Dominance condition (see Anscombe and Aumann, 1963; Thm. 14.1 in Gilboa, 2009).

2.2. Results

2.2.1. From the STP to P2 and P3, and back

In this subsection we explain how, inspired by Savage’s informal STP, one may see each of **P2** and **P3** as a dominance condition (Proposition 1). This clarifies how the STP supports both **P2** and **P3**. Conversely, we also explain how these dominance conditions help formalizing the STP (Proposition 2). This clarifies how **P2** and **P3** can harness the full power of the STP.

Recall the reasoning of Savage’s businessman quoted in the introduction. The core intuition is one of case-by-case reasoning. Now consider the following two axioms. They are dominance conditions—hence their being here labeled (like any other dominance condition to come) as D-axioms. They correspond to different ways of cashing out what it means to “kn[o]w” (Savage, 1954, p. 21) that an event occurs, and thus of sustaining case-by-case reasoning. In a nutshell, in the first case, knowing that an event obtains means suitably conditioning on it, while in the second case, it implies that uncertainty has been fully resolved.⁸

D2. For all $f, g, h \in F$, partition $\{E_1, \dots, E_n\}$ of S ,

- i. if $f E_i h \succcurlyeq g E_i h$ for all E_i , then $f \succcurlyeq g$;
- ii. if in addition $f E_i h \succ g E_i h$ for some E_i , then $f \succ g$.⁹

D3. For all $f, g \in F$, partition $\{E_1, \dots, E_n\}$ of S ,

- i. if $f(E_i) \succcurlyeq g(E_i)$ for all E_i , then $f \succcurlyeq g$;
- ii. if in addition $f(E_i) \succ g(E_i)$ for some non-null E_i , then $f \succ g$.

D3 is essentially Strong State-Wise Dominance, i.e., State-Wise Dominance (**D3.i**)—as in, say, the Anscombe-Aumann Theorem—enriched with a strict clause (**D3.ii**). Apart from the intended parallelism with **D2**, the main reason for the non-traditional phrasing adopted here is that a state-wise dominance property cannot be non-trivially enriched with a strict clause of the same kind when the state space is infinite and all states must be null—as is the case in, e.g., Savage’s Theorem. By contrast, as **D3** illustrates, a similarly inspired event-wise dominance property can be so enriched. **D2** on the other hand, despite being a familiar condition, seems to have no established name in the literature.¹⁰ The key point is that as the parallelism between the two axioms makes

⁸ As regards the second axiom, recall that for any $E \in \Sigma$, $f \in F$, we write $f(E)$ if and only if f is constant over E , and that we also let $f(E)$ denote the constant consequence of f on E .

⁹ In **D2.ii** there is no need to assume that E_i be non-null; this follows from $f E_i h \succ g E_i h$.

¹⁰ For some of the issues that arise in connection with infinitary variants of **D2**, see for instance Seidenfeld and Schervish, 1983.

transparent, **D2** is, like **D3**, a dominance or a monotonicity property. While the latter expresses *ex post* dominance, the former arguably expresses *interim* dominance. This is in the sense that it effectively applies case-by-case reasoning to partial, rather than full, resolutions of uncertainty.

Now, recalling that **P1** is assumed throughout our analysis, consider our first main results, viz. the equivalences in Proposition 1.^{11,12}

Proposition 1.

- a. **P2** holds if and only if **D2** holds;
- b. **P3** holds if and only if **D3** holds.

Proof. See the Appendix. \square

In light of Proposition 1, assuming **P2** or **D2**—respectively, **P3** or **D3**—in the statement of Savage’s Theorem is merely a “matter of taste” (Savage, 1954, p. 26). There could be principled methodological reasons to choose one way rather than the other, however. For instance, the dominance format **D2** and **D3** clearly is the one under which the required properties look most attractive from a normative point of view. On the other hand, one may speculate that to that format, Savage preferred the one that most closely corresponds to the role the properties play in the proof of the existence of the SEU representation—the initial format **P2** and **P3**. Incidentally, it is also the format under which the axiom are the easiest to test.

Be that all as it may, two further comments on Proposition 1 are in order. First, **D2.i** is demonstrably equivalent to **D2**.¹³ Accordingly, in Proposition 1.a, **D2.i** could be stated equivalent to **P2**. By contrast, no similar analysis applies to **D3**, **D3.i**, and **P3**, since the second of these properties is strictly weaker than the first.¹⁴ One of our additional results in the Appendix (Lemma 3) identifies the weakening of **P3** to which **D3.i** is equivalent.

Second, it is instructive to compare the results in Proposition 1 to those in Sec. 2.7 of Savage, 1954.¹⁵ Savage proves that **P2** \Rightarrow **D2** (1954, Thm. 2) but, whatever the reason, he does not discuss the converse. We have not found the full equivalence elementarily proved in the literature; but it is undoubtedly a folk theorem. We merely extend Savage’s proof in the obvious way.¹⁶ On the other hand, Savage does

¹¹ Proposition 1, hence also Proposition 2, holds for general acts.

¹² Occasionally, one can find in the literature the claim that the STP, construed exclusively as **P2**, is “stronger” than State-Wise Dominance (see, e.g., Mongin and Pivato, 2016, p. 727). In light of Proposition 1 and the already noted logical independence of **P2** and **P3** (see fn. 7), that extends to **P2** and **D3.i** alone, these claims call for interpretation. They should not be understood as logical statements, but as informal comparisons of normative strength. A widely received view is indeed that while **P2** is—among the properties specific to uncertainty—the most contentious requirement of rationality, State-Wise Dominance is essentially uncontroversial. In light of Proposition 1, the latter claim is debatable inasmuch as one can debate that ordinal state-independence is a requirement of rationality (more on this in, e.g., Karni, 2008).

¹³ This follows from the fact that each of **D2.i** and **D2** proves equivalent to **P2**. Further, **D2** proves equivalent to the special case of **D2.i** referring to a binary partition.

¹⁴ For instance, Maxmin Expected Utility (Gilboa and Schmeidler, 1989) and Choquet Expected Utility (Schmeidler, 1989; Gilboa, 1987) satisfy **D3.i** but in general not **D3**.

¹⁵ Savage focuses on the strengthenings of **D2** and **D3** featuring conditional antecedents and consequents. These strengthenings are only apparent, however, for the conditional—as in Savage—and unconditional—as in our paper—conditions are demonstrably equivalent. (This follows from the fact that they each prove equivalent to **P2** and **P3**, respectively.) Accordingly, we ignore this difference in the clarifications to follow.

¹⁶ Contrast our simple proof and the more involved treatments in, e.g., Marschak, 1986 (in a generalization of Savage’s framework), LaValle, 1992 (over decision trees), or Zimper, 2008 (in the context of

prove the full equivalence $P3 \Leftrightarrow D3$ (1954, Thm. 3)—an equivalence of which the larger literature is clearly aware of.¹⁷ Yet Savage suggests, and we have not found it denied in the literature, that it holds only when $P2$ is assumed, while our result shows that $P2$ is not needed—an important aspect on which our next subsection will elaborate.

Meanwhile, Proposition 1 helps one discern how $P2$ and $P3$ can be explicitly combined in a unified condition expressing, as much as can be within Savage’s framework, the full STP—a simple question which we have not seen raised in the literature. While based on the original conditions $P2$ and $P3$, it is unclear how such a unification could be achieved, it is entirely transparent based on their equivalent dominance format $D2$ and $D3$. As registered in Proposition 2 below, the answer—simplified as much as possible—is given by $D4$, the hybrid dominance condition stated next. The condition is hybrid in the sense that, much like Savage with his initial businessman example, it leaves open the exact kind of dominance reasoning employed.

D4. For all $f, g, h \in F$, partition $\{E_1, \dots, E_n\}$ of S ,

- i. if for all E_i either $fE_ih \geq gE_ih$ or $f(E_i) \geq g(E_i)$, then $f \geq g$;
- ii. if in addition $f(E_i) > g(E_i)$ for some non-null E_i , then $f > g$.

Proposition 2. $P2$ and $P3$ hold if and only if $D4$ holds.

Proof. Immediate from Proposition 1. \square

Since it combines $P2$ and $P3$, $D4$ covers patterns of dominance reasoning justified by neither axiom alone. For instance, for any $f, g, h \in F$, $E \in \Sigma$, $D4$ justifies concluding $f \geq g$ from $fEh \geq gEh$ and $f(\bar{E}_i) \geq g(\bar{E}_i)$ for each cell \bar{E}_i of some partition $\{\bar{E}_1, \dots, \bar{E}_n\}$ of \bar{E} —a particularly useful schema. Taken in isolation from one another, neither $P2$ nor $P3$ could support that conclusion. In terms of Savage’s initial businessman example, the above pattern of dominance reasoning could correspond to the following scenario. Assume that there is not only a Democrat and a Republican, but also an Independent candidate. For simplicity, further assume that the election of either candidate would determine exactly one consequence for the act of buying or not buying the property. Let E_1 (respectively: E_2 ; E_3) denote the event that the Republican (respectively: the Democrat; the Independent) candidate wins, and let f (respectively: g) denote the act of buying (respectively: not buying) the property. For instance because of the tax policy to which the Republican candidate would have committed, it might be that the businessman has the preference $f(E_1) \geq g(E_1)$. While for similar reasons he may well strictly prefer $g(E_2)$ to $f(E_2)$ but $f(E_3)$ to $g(E_3)$, it might also be that for some other business venture h , he has the preference $fE_2 \cup E_3h \geq gE_2 \cup E_3h$. From these two heterogeneous pieces of data, $D4$ justifies concluding $f \geq g$, which taken alone neither $P2$ nor $P3$ could.¹⁸ This also manifests, more generally, that $D4$ seems flexible enough to capture many possible explications of the businessman example motivating Savage’s introduction of the STP.

Finally, since it exactly corresponds to the conjunction of $P2$ and $P3$, $D4$ can replace these axioms in the statement of Savage’s Theorem. This is recorded in Corollary 1 below.

Corollary 1. In the statement of Savage’s Theorem, keeping all the other axioms, $P2$ and $P3$ can be conjointly replaced by $D4$.

Proof. Immediate from Proposition 2. \square

decision-making under risk). As these papers illustrate, and as recently emphasized among others by Li et al. (2023), the fact that separability is equivalent to a form of dominance or monotonicity has been known for a long time.

¹⁷ For instance, witness the different names (viz. “State-Wise Dominance”, “Monotonicity”, or “State-Independence”) which the literature has given to the last axiom in the Anscombe-Aumann characterization of SEU.

¹⁸ The models mentioned in fn. 7 could be invoked here once again to illustrate this point.

2.2.2. The STP: $P2$, $P3$, and their intersection

In this subsection we further investigate the internal structure of the STP, starting from the observation that $P2$ and $P3$ have an intersection, and that in that sense the STP has a basic core. Given our previous results, our main result in this section (Proposition 3) suggests that three distinct properties underpin the STP. To further highlight the difference with Savage’s own P-axioms or the equivalent dominance conditions stated as the D-axioms, we will label these more basic properties as the A-axioms.

Previously (after Proposition 1), we noted that to show the equivalence of $P3$ and $D3$, it is unnecessary to assume $P2$. To best appreciate why such is the case, one may observe that the weak implication of $P2$ stated next, $A1$, is also implied by $P3$ alone.

A1. For all $x, y \in X$, $E \in \Sigma$, $h, h' \in F$, $xEh \geq yEh \Leftrightarrow xEh' \geq yEh'$.

$A1$ weakens $P2$ by ranging only over acts with constant (instead of general, i.e. variable) non-common parts while it weakens $P3$ by focusing on only one event at a time (instead of reasoning across non-null events). In neither case is this supposed to isolate an uncontroversial implication of the stronger axiom; this is only to highlight that these stronger axioms do intersect, and that to this extent the STP has a basic core. Instructively, as further noted in the Appendix (Lemma 4), $A1$ proves equivalent to an especially minimal dominance property ($D1$), that is transparently a special case of $D2$ as well as, less transparently, $D3$.

The fact that $A1$ is an implication of both $P2$ and $P3$ ¹⁹ naturally raises the following question. How exactly is the former property to be strengthened to obtain either of the latter properties—or both, to reach the full STP? As an inspection of the three axioms makes clear, $P2$ strengthens the invariance in $A1$ by requiring that it also hold over acts with variable (non-constant) non-common parts, while $P3$ strengthens it by requiring, instead, that it also hold across different (non-null) events.²⁰ The properties characterizing these orthogonal strengthenings are introduced next.²¹

A2. For any $f, g, h, h' \in F$, $x, y \in X$, partition $\{E_1, E_2, E_3\}$ of S , if either $fE_1h > gE_1h$ and $xE_2h < yE_2h$, or $fE_1h < gE_1h$ and $xE_2h > yE_2h$, then $fE_1xE_2h \geq gE_1yE_2h \Leftrightarrow fE_1xE_2h' \geq gE_1yE_2h'$.

A3. For all $x, y \in X$, essential $E \in \Sigma$, $xEy \geq y \Leftrightarrow yEx \geq y$.

As will be seen shortly, $A2$ captures the part of $P2$ that $P3$ cannot deliver. The axiom considers a basic pattern of preference conflict over some event. Indeed, calling E the union of E_1 and E_2 , the axiom ranges over acts a, b such that $aE_1h > bE_1h$ but $aE_2h < bE_2h$ or the other way around—hence our referring to a “preference conflict”, viz. over E —with a and b constant on at least one of E_1 and E_2 —hence our calling the conflict pattern “basic”.²² What $A2$ states is that whatever the settlement of that preference conflict (i.e., whether it is a or b that over E the decision-maker prefers), it cannot depend on the common values assigned to a and b outside the event of interest (i.e., h, h' , etc. on $\bar{E} = E_3$). In that sense, $A2$ is a kind of trade-off consistency

¹⁹ One may further conjecture that $A1$ is the strongest property common to, and in that sense the common core of, $P2$ and $P3$.

²⁰ The latter fact may be seen most clearly by considering, instead of $P3$, the following logically equivalent property: For all $x, y \in X$, non-null $E, E' \in \Sigma$, $h, h' \in F$, $xEh \geq yEh \Leftrightarrow xE'h' \geq yE'h'$.

²¹ As regards the first axiom, recall that for any $f, g, h \in F$, ternary partition $\{E_1, E_2, E_3\}$ of S , we let fE_1gE_2h denote the act equal to f on E_1 , g on E_2 , and h on E_3 .

²² The example developed after $D4$ is readily adapted to illustrate exactly such a pattern.

axiom.²³ P3 being effectively, as D3 illustrates, a unanimity axiom, it has no bearing on such patterns of conflicting preferences. Conversely, A3 captures the part of P3 that P2 cannot deliver. It is an especially minimal ordinal state-independence axiom. The common part of the acts it considers is restricted to one of the two consequences involved in their non-common part (y , in the above notation). Against that background, A3 states only that consequences must be ordered in the same way across any essential event and its complement (viz., for any such event, $xEy \geq yEy$ if and only if $x\bar{E}y \geq y\bar{E}y$).²⁴ P2 has no bearing on such issues of cross-event consistency.

These axioms lead us to our next main results, that are the equivalences in Proposition 3.²⁵

Proposition 3.

- a. P2 holds if and only if A1 and A2 hold;
- b. P3 holds if and only if A1 and A3 hold.

Proof. See the Appendix. □

Proposition 3 thus suggests that three distinct ideas underpin the STP as captured by D4, namely, to recap: a property that can be seen as a particularly basic dominance condition (A1); a kind of trade-off consistency requirement (A2); and an especially minimal ordinal state-independence axiom (A3).

Proposition 3 also helps identifying potential simplifications for Savage’s and, incidentally, the Anscombe-Aumann axiomatizations of SEU. To prepare for including the latter axiomatization to our analysis, we note that the Appendix (Lemma 5) provides a decomposition of D3.i comparable to the one which, given Proposition 1, Proposition 3 provides for D3. In particular, this decomposition establishes that the part of D3.i which P2 cannot deliver is WA3, the weakening of A3 introduced next. Unlike the stronger A3, WA3 excludes only strict preference reversals (as in $xEy > yEy$ but $x\bar{E}y < y\bar{E}y$) in the ordering of consequences across one essential event and its complement.

WA3. For all $x, y \in X$, essential $E \in \Sigma$, $xEy > y \Rightarrow yEx \geq y$.

We record the links between D3.i and WA3 as stated in Lemma 1.

Lemma 1. Assume P2. Then, D3.i holds if and only if WA3 holds.

Proof. See the Appendix. □

The following corollaries of Proposition 3 and Lemma 1 can now be stated.²⁶

Corollary 2.

- a. In the statement of Savage’s Theorem, keeping all the other axioms, P2 can be weakened to A2.
- b. In the statement of Savage’s Theorem, keeping all the other axioms, P3 can be weakened to A3.
- c. In the statement of the Anscombe-Aumann Theorem, keeping all the other axioms, D3.i can be weakened to WA3.

²³ Because we find it illuminating here, we thus use freely a phrase established for other purposes elsewhere in the literature; see in particular Köbberling and Wakker, 2003.

²⁴ The axiom ranges only over essential (rather than over all non-null) events. As the details of the proof of Proposition 3.b indicate, P3 could be immediately thus generalized as well.

²⁵ Proposition 3.b holds for general acts. Hence so does Corollary 2.b.

²⁶ As regards Corollary 2.b (and similarly for our later Corollary 3.a), recall that we here consider Savage’s Theorem for simple acts. In this result, unlike in Savage’s Theorem for general acts (see Hartmann, 2020), P3 is not redundant. Accordingly, weakening it is not otiose.

Proof. Immediate from Proposition 3, Lemma 1, and (for Corollary 2.c) the well known fact that in the Anscombe-Aumann framework, P2 is implied by the Weak Order and the von Neumann - Morgenstern Independence properties alone. □

In light of Corollary 2, and to echo the comments we made after Proposition 1, assuming P2 or A2—respectively, P3 or A3—in the statement of Savage’s Theorem is another “matter of taste” (Savage, 1954, p. 26), and a similar remark applies to D3.i, WA3, and the Anscombe-Aumann Theorem.²⁷

2.2.3. The STP, P3, and Obvious Dominance

In this subsection we zoom in on one particularly interesting implication of our analysis pertaining to P3, specifically. Given our previous results, our main result in this section (Lemma 2) indicates that starting from P2, there is a surprising way of bridging the full STP, alternative to assuming either the maximal axiom P3 or the minimal axiom A3.

Our starting point is the observation that in principle, variants of Corollary 2 can be obtained with P2 (respectively, P3; D3.i) being replaced by any of its weakening that would still be strong enough to imply A2 (respectively, A3; WA3). Indeed, an interesting illustration pertaining to P3 involves the dominance axiom stated next.²⁸

D5. For all $f, g \in F$, partition $\{E_1, \dots, E_n\}$ of S ,

- i. if $f(E_i) \geq g(E_j)$ for all E_i, E_j , then $f \geq g$;
- ii. if in addition $f(E_i) > g(E_i)$ for some non-null E_i , then $f > g$.

D5 is essentially Obvious Dominance (Li, 2017) suitably enriched with a strict clause (D5.ii). Its weak clause (D5.i) simply states that if the worst case of f is better than the best case of g , then f must be better than g overall.²⁹ Considered normatively, this makes for an especially compelling “sure-thing principle”, if any. In the context of decision theory, the axiom is of interest also because, under appropriate background conditions (see, e.g., Chambers and Echenique, 2016, Sec. 8.4; Zhang and Levin, 2017, Thm. 1), it proves to underpin the existence of an Act-Dependent SEU representation. Virtually all non-expected utility models fall within that category (on which see especially Cerreia-Vioglio et al. 2011, Cor. 3).

In the Appendix (Lemma 6), we provide decompositions of D5 and D5.i comparable to the ones previously provided for D3 and D3.i. Based on these decompositions, one may claim what follows.

Lemma 2. Assume P2. Then,

- a. P3 holds if and only if D5 holds;
- b. D3.i holds if and only if D5.i holds.

Proof. See the Appendix. □

²⁷ Admittedly, it might not always be wise to assume only the leanest axioms possible. For instance, P2 and P3 are arguably more conceptually transparent than A2 and A3. Such is especially true of P2, compared to the less demanding but more cumbersome—our proposed interpretation in terms of trade-off consistency notwithstanding—A2.

²⁸ By contrast, for a weakening of one of the two classical axioms P2, P3 that is not strong enough to imply the relevant minimal property referred to in Corollary 2, one may mention the co-monotonic restriction of P2, that underlies Choquet Expected Utility. It is too weak to imply WA1 (featured in Lemma 5). Consequently, replacing D3.i by WA3 in an axiomatization of Choquet Expected Utility would lead to a more general model.

²⁹ D5.i is equivalent to the following unconditional variant of Savage’s P7. (In appreciating the axiom, recall once again that we write $f(E)$ if and only if f is constant over E .) For any $f, g \in F$, if $f \geq g(E)$ for all $E \in \Sigma$, then $f \geq g$, and if $f(E) \geq g$ for all $E \in \Sigma$, then $f \geq g$.

The main interest of Lemma 2 may reside in Corollary 3 below, which is the previously announced illustration. Bearing in mind what we previously wrote about Act-Dependent SEU, note that the results to follow pertain, by contrast, to the existence of the classical Act-Independent SEU representation.

Corollary 3.

- a. In the statement of Savage’s Theorem, keeping all the other axioms, P3 can be weakened to D5.
- b. In the statement of the Anscombe-Aumann Theorem, keeping all the other axioms, D3.i can be weakened to D5.i.

Proof. Immediate from Lemma 2 (and the remark about the Anscombe-Aumann framework already made in the proof of Corollary 2). □

Lemma 2 thus also implies that starting from P2, to reach the full force of the STP as captured by D4, it suffices to assume the weak and therefore especially compelling dominance property D5.

3. Conclusion

In this note, we have investigated the diversity and vindicated the unity of the Sure-Thing Principle as originally envisaged by Savage—viz., over simple (finitely-valued) acts, as the combination of his separability postulate P2 and his state-independence postulate P3. We have done so without enriching his framework in any way, be it with primitive conditional preferences, with knowledge operators, or with anything additional to the structural assumptions made in the traditional Savage literature. Among other results, we have shown that each of P2 and P3 is equivalent to a dominance condition (Proposition 1); that they can be explicitly combined in a unified dominance condition that is a candidate formal statement for the full STP (Proposition 2); and that they strengthen in different directions a common, basic dominance axiom (Proposition 3). These results, that hold under Savage’s P1 weak order assumption, are visually summarized in Diagram 1, displayed next.

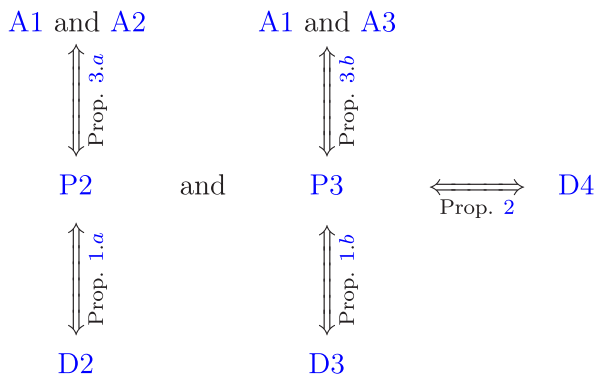


Diagram 1. Main results.

From these and our other results (see especially our surprising Corollary 3, that involves Obvious Dominance) emerges a clearer conceptual picture of the variety of dominance conditions imposed by Subjective Expected Utility, as well as a better understanding of its axiomatic underpinnings. Directions for future research include, of course, investigating the Sure-Thing Principle over general (possibly infinitely-valued) acts. This is natural since many but not all our proofs hold unchanged for general acts, and because Savage also uses the Sure-Thing Principle to motivate his axiom P7, on which his theorem relies when such acts are at stake. The main result in Hartmann, 2020 may be taken to suggest that in that case, one could focus on Savage’s P2 and P7, i.e. ignore P3, on the account that the latter axiom demonstrably follows from the former two. But this is true if one is ready to

assume also Savage’s P4, which is arguably (also in Savage’s own view) unrelated to the Sure-Thing Principle. Another more methodological direction for future research is to further investigate the intersections of the logically independent axioms underpinning Subjective Expected Utility (or its generalizations), as we did here focusing on P2 and P3. For instance, a close inspection of the proof of Hartmann, 2020 reveals that it hinges on the identification of an intersection between Savage’s P3 and P4. The rest of Savage’s axiomatic system could be systematically revisited, looking for such intersections and the further light they may shed on the foundations of Subjective Expected Utility (or its generalizations).

Declaration of competing interest

None

Data availability

No data was used for the research described in the article.

Appendix

Proof of Proposition 1

Proof.

- a. (⇒) Assume $f E_i h \geq g E_i h$ for all E_i . By P2, take $h_1 = f$ and, for all $i \in \{2, \dots, n\}$, $h_i = g E_1 \cup \dots \cup E_{i-1} f$ to conclude by P1 that $f \geq g$, with a strict consequent if any of the antecedents is strict. In more detail, the first preference reads as $f \geq g E_1 f$, the second as $g E_1 f \geq g E_1 \cup E_2 f$, and so on, with the final one reading as $g E_1 \cup \dots \cup E_{i-1} f \geq g$, so that $f \geq g$ (or, if any of the preceding preferences is strict, $f > g$) follows by P1.
(⇐) Assume $f E h \geq g E h$. By P1, $h E h' \geq h E h'$. Hence, by D2, $f E h' \geq g E h'$.
- b. (⇒) Assume $f(E_i) \geq g(E_i)$ for all E_i . By P3 or the definition of null events, $f(E_i) E_i h_i \geq g(E_i) E_i h_i$ for all E_i , with a strict preference if $f(E_i) > g(E_i)$ and E_i is non-null, and $\{h_1, \dots, h_n\}$ any collection of acts. So take $h_1 = f$ and, for all $i \in \{2, \dots, n\}$, $h_i = g E_1 \cup \dots \cup E_{i-1} f$ to conclude by P1 that $f \geq g$, with a strict preference $f > g$ if $f(E_j) E_j h_j > g(E_j) E_j h_j$ for some E_j .
(⇐)[⇒] Assume $x \geq y$. By D3, $x E f \geq y E f$ for any E , therefore for any non-null E in particular. [⇐] Assume $x E f \geq y E f$ for some non-null E . If $y > x$, by D3, $y E f > x E f$, a contradiction. Hence, it must be the case that $x \geq y$. □

Proof of Lemma 3

LRP3. For all $x, y \in X$, non-null $E \in \Sigma$, $h \in F$, $x \geq y \Rightarrow x E h \geq y E h$.

Lemma 3. LRP3 holds if and only if D3.i holds.

Proof.

- (⇒) Similar to the proof of (⇒) in Proposition 1.b.
- (⇐) We prove the contrapositive. Assume $y E h > x E h$. If $x \geq y$, then by D3.i, $x E h \geq y E h$, a contradiction. Hence, $y > x$. □

Proof of Lemma 4

D1 For all $f, g, h \in F$, partition $\{E_1, \dots, E_n\}$ of S ,

- i. if $f(E_i)E_i h \geq g(E_i)E_i h$ for all E_i , then $f \geq g$;
- ii. if in addition $f(E_i)E_i h > g(E_i)E_i h$ for some E_i , then $f > g$.

Lemma 4. *A1 holds if and only if D1 holds.*

Proof.

(\Rightarrow) Similar to the proof of (\Rightarrow) in Proposition 1.a.
 (\Leftarrow) Assume $xEf \geq yEf$. From this assumption and the fact that $fEx \geq fEx$ by P1, it follows from D1 that $x \geq yEx$. Now assume $yEg > xEg$ for a contradiction. From this assumption and the fact that $gEx \geq gEx$ by P1, it follows from D1 that $yEx > x$, a contradiction. Thus, it must be the case that $xEg \geq yEg$. \square

Proof of Proposition 3

Proof.

- a. (\Rightarrow) Trivial.
 (\Leftarrow) The proof is by induction on the number of consequences that are possibly non-common between fEh and gEh .

Induction basis. The base case is $fEh \geq gEh$ with f and g having one possibly non-common consequence. It follows from A1 that $fEh' \geq gEh'$.

Induction step. Assume P2 holds for all acts having n possibly non-common consequences. Consider the case where $n + 1$ consequences are possibly non-common. So assume $fEh = aE_1xE_2h \geq bE_1yE_2h = gEh$, with $\{E_1, E_2\}$ a partition of E and a and b having n possibly non-common consequences on E_1 . Either $aE_1h \geq bE_1h$ or $xE_2h \geq yE_2h$ holds, for otherwise by P1 and the induction hypothesis $bE_1yE_2h > aE_1xE_2h$ would follow, a contradiction. If both $aE_1h \geq bE_1h$ and $xE_2h \geq yE_2h$ hold, then $aE_1xE_2h' \geq bE_1xE_2h'$ by P1 and the induction hypothesis. If $aE_1h < bE_1h$ and $xE_2h \sim yE_2h$ or if $aE_1h \sim bE_1h$ and $xE_2h < yE_2h$, then by P1 and the induction hypothesis $bE_1yE_2h > aE_1xE_2h$, a contradiction as above. So the only remaining cases to consider are $aE_1h < bE_1h$ and $xE_2h > yE_2h$, and $aE_1h > bE_1h$ and $xE_2h < yE_2h$. In both cases A2 implies from $aE_1xE_2h \geq bE_1yE_2h$ that $aE_1xE_2h' \geq bE_1yE_2h'$. Thus $fEh' \geq gEh'$ holds in all cases.

- b. (\Rightarrow) Trivial.
 (\Leftarrow)
 [\Rightarrow] Assume $yEf > xEf$. Consider first the case where E is essential. A1 then implies both $y > xEy$ and $yEx > x$. From the latter, A3 implies $xEy > x$. The former and P1 then imply $y > x$. Consider next the case where \bar{E} is null. Then by definition for all f, g, h , $fEh \sim gEh$. Thus it holds in particular that $y \sim yEf$ and $x \sim xEf$, so that given $yEf > xEf$, by P1, $y > x$ also follows.
 [\Leftarrow] Assume $xEf \geq yEf$ with E non-null. If E is essential, A1 implies both $x \geq yEx$ and $xEy \geq y$. From the latter, A3 implies $yEx \geq y$. The former and P1 then imply $x \geq y$. Consider next the case where \bar{E} is null. Then by definition for all f, g, h , $fEh \sim gEh$. Thus it holds in particular that $x \sim xEf$ and $y \sim yEf$, so that given $xEf \geq yEf$, by P1, $x \geq y$ also follows. \square

Proof of Lemma 1

Proof. Immediate from Lemma 5 below. \square

WA1. For all $x, y \in X$ s.t. $x \geq y$, $E \in \Sigma$, $h, h' \in F$, $xEh \geq yEh \Rightarrow xEh' \geq yEh'$.

Lemma 5. *D3.i holds if and only if WA1 and WA3 hold.*

Proof.

(\Rightarrow) Trivial.
 (\Leftarrow) Consider the basic case where f and g differ on at most one event of the partition. So consider $x, y \in X$, $f \in F$ and $E \in \Sigma$, and assume $x \geq y$. P1 implies that either $x \geq xEy$ or $xEy \geq y$ must hold, for otherwise $y > x$ would hold. Assume that $xEy \geq y$ holds. Then WA1 implies $xEf \geq yEf$. Assume that $x \geq xEy$ holds. If $x \sim xEy$, then by P1 $xEy \geq y$ follows, so that again $xEf \geq yEf$ follows from WA1. If $x > xEy$, then $x \geq yEx$ follows either from the definition of null events, if E is null, or from WA3 (contraposed, with substitution of variables), if E is non-null. WA1 then implies $xEf \geq yEf$. So $x \geq y$ implies $xEf \geq yEf$ in the above basic case. From that, the general case follows by iteratively using P1 finitely many times. Specifically, the baseline argument delivers $f \geq gE_1f$, then $gE_1f \geq gE_1 \cup E_2f$, and so on, finally $gE_1 \cup \dots \cup E_{i-1}f \geq g$, so that $f \geq g$ follows from P1. \square

Proof of Lemma 2

Proof. Immediate from Lemma 6 below. \square

A4. For any $E \in \Sigma$, for any $f, g \in F$ and $x, y \in X$ such that for all $A, B \in \Sigma$, $f(A) \geq x \geq y \geq g(B)$, $fEx \geq gEx \Leftrightarrow fEy \geq gEy$ and $gEx \geq fEx \Leftrightarrow gEy \geq fEy$.

WA4. For any $E \in \Sigma$, for any $f, g \in F$ and $x, y \in X$ such that for all $A, B \in \Sigma$, $f(A) \geq x \geq y \geq g(B)$, $fEx \geq gEx \Leftrightarrow fEy \geq gEy$.

Lemma 6.

- a. *D5 holds if and only if A4 and A3 hold;*
- b. *D5.i holds if and only if WA4 and WA3 hold.*

Proof.

- b. (\Rightarrow) Consider first WA3. Assume $xEy > y$ with E essential. D5.i (contraposed) and P1 imply $x > y$, so that D5.i implies $yEx \geq y$. Consider next WA4. On the assumptions of the axiom, D5.i implies the conjunction, hence a fortiori the equivalence, of $fEx \geq gEx$ and $fEy \geq gEy$.
 (\Leftarrow) For an act $h = (x_1, E_1; x_2, E_2; \dots; x_n, E_n)$ with (without loss of generality) $x_1 \geq x_2 \geq \dots \geq x_n$, we say that the *weak internality condition* holds if $x_1 \geq h \geq x_n$. To show D5.i, it suffices to show that every act satisfies the weak internality condition. We show that the weak internality condition holds via induction on the size of the partition of the state space with respect to which the act is defined.

Induction base: We need to show that for all $x, y \in X$, if $x \geq y$ then for all $E \in \Sigma$, we have $x \geq xEy \geq y$.

Case 1: $x \sim y$. Assume, for a contradiction, $xEy > y$. WA4 then implies $x > yEx$ hence $x \geq yEx$. Also from $xEy > y$, WA3 (or the definition of null events if \bar{E} is null) implies $yEx \geq y$. As $x \sim y$, $x \geq yEx$ together with $yEx \geq y$ implies by P1 that $xEy \sim y$, a contradiction to the assumption we started with. Thus, $y \geq xEy$ must hold. The claim $xEy \geq y$

is proved similarly. Thus, $xEy \sim y$, so that by **P1** and $x \sim y$, $x \geq xEy \geq y$.

Case 2: $x > y$. By **P1** either $x > yEx$ or $yEx > y$ must hold, for otherwise $y \geq x$ would follow. If $x > yEx$ holds, then $x \geq xEy$ follows either from the definition of null events, if \bar{E} is null, or from **WA3** (contraposed, with substitution of variables), if \bar{E} is non-null. Also from $x > yEx$, **WA4** implies $xEy \geq y$ (for if not, **WA4** contraposed would imply $yEx > x$, a contradiction). If $yEx > y$ holds, then **WA4** implies $x > xEy$ hence $x \geq xEy$ more generally. Also from $yEx > y$, **WA3** (or the definition of null events) implies $xEy \geq y$. Thus, the weak internality condition always holds.

Induction step: We need to show that if for all acts defined with respect to an i -ary partition of the state space, the weak internality condition holds, then it also holds for all acts defined with respect to an $i+1$ -ary partition of the state space. So consider any act $h = (x_1, E_1; x_2, E_2; \dots; x_i, E_i; x_{i+1}, E_{i+1})$ such that $x_1 \geq \dots \geq x_{i+1}$. We need to show $h \geq x_{i+1}$. ($x_1 \geq h$ is similar.) The induction step implies that $x_i E_{i+1} h \geq x_i$. As $x_i \geq x_{i+1}$, the induction base implies that $x_i \geq x_i E_{i+1} x_{i+1}$. Thus, **P1** implies that $x_i E_{i+1} h \geq x_i E_{i+1} x_{i+1}$. As by the induction step and **P1** $x_i E_{i+1} h \geq x_i \geq x_{i+1}$, by **WA4**, $h \geq x_{i+1}$ follows (for if not, **WA4** contraposed would imply $x_i E_{i+1} x_{i+1} > x_i E_{i+1} h$, a contradiction).

- a. (\Rightarrow) Consider first **A3**. Assume $xEy \geq y$ with E essential. If $y > x$, then by **D5.ii** $y > xEy$ would follow. So $x \geq y$ must hold, so that by **D5.i** $yEx \geq y$ follows. The converse is proved similarly.

Consider next **A4**. Given we already know that **D5.i** implies **WA4**, only the second clause of **A4** needs arguing for here. We show $fAx > gAx \Rightarrow fAy > gAy$. (The converse is similar.) To show this, the following claim is key. Under the assumptions of the axiom, if $fAx > gAx$, then with $\{E_1, \dots, E_n\}$ the coarsest partition of S common to f and g , there exists some E_j , with $E_j \cap A \neq \emptyset$ and $E_j \cap A$ non-null, such that $f(E_j) > g(E_j)$. Assume otherwise. Then, denoting $\{E_k, \dots, E_m\}$ the elements of the partition having non-empty intersection with A , for all $i = k, \dots, m$, either $g(E_i) \geq f(E_i)$, or $f(E_i) > g(E_i)$ but E_i is null. We start by showing that the case where $g(E_i) \geq f(E_i)$ for all $i = k, \dots, m$ is impossible. As we know that $f(E_i) \geq g(E_j)$ for all $i, j = 1, \dots, n$, this case would imply that $g(E_i) \geq f(E_i) \geq g(E_j) \geq f(E_j)$ hence $g(E_i) \geq f(E_j)$ for all $i, j = k, \dots, m$. Besides, as $f(E_i) \geq x$ for all $i = 1, \dots, n$ hence $g(E_i) \geq x$ for all $i = k, \dots, m$, it would thus hold that for all $E, E' \in \{E_k, \dots, E_m, A^c\}$, $(gAx)(E) \geq (fAx)(E')$. Thus by **D5.i** $gAx \geq fAx$ would follow, contradicting that $fAx > gAx$. So for our assumption to hold, it must be that for some $i = k, \dots, m$, $f(E_i) > g(E_i)$, but any such E_i is null. We show that this case also leads to a contradiction. Let A_1 stand for the union of all the indices satisfying the preceding condition, with A_2 its complement with respect to A . (If A_2 is empty, then by definition of null events the contradiction with $fAx > gAx$ is immediate.) Then by definition of null events, with z the worst consequence obtained under g over S , $zA_1 f A_2 x \sim fAx$. But following the same reasoning as in the first case, for all $E, E' \in \{E_k, \dots, E_m, A^c\}$, $(gAx)(E) \geq (zA_1 f A_2 x)(E')$. Thus by **D5.i** $gAx \geq zA_1 f A_2 x$ would follow, again contradicting the assumption that $fAx > gAx$ (since $zA_1 f A_2 x \sim fAx$ has already been established). So there must be some E_i having non-empty and non-null intersection with A such that $f(E_i) > g(E_i)$. Thus, together with the assumption that $f(E_i) \geq y \geq g(E_j)$ for all $i, j = 1, \dots, n$, by **D5.ii**, $fAy > gAy$ follows, as was to be shown.

(\Leftarrow) Given that **A3** \Rightarrow **WA3** and **A4** \Rightarrow **WA4**, in light of **Lemma 6.b**, only the strict clause of **D5** needs arguing for.

Consider two acts $f, g \in F$, with their coarsest common partition $\{E_1, \dots, E_n\}$ of S , such that (first half of the assumption) $f(E_i) \geq g(E_j) \forall i, j \in \{1, \dots, n\}$ and assume that (second half of the assumption) there exists a non-null $k \in \{1, \dots, n\}$ such that $f(E_k) > g(E_k)$. Assume first that $f(E_i) > g(E_j)$ for all $i, j \in \{1, \dots, n\}$. Then from **P1**, **A3** \Rightarrow **WA3** and **A4** \Rightarrow **WA4**, and **Lemma 6.b**, one can conclude $f > g$ without even considering the second half of the assumption. So assume $f(E_i) \sim g(E_j)$ for some $i, j \in \{1, \dots, n\}$ and consider now the second half of the assumption. From $f(E_i) \sim g(E_j)$, it must be that for all $l, m \in \{1, \dots, n\}$, $f(E_l) \geq f(E_j)$ and $g(E_l) \geq g(E_m)$, for otherwise $g(E_m) > f(E_l)$ would follow, in contradiction with the assumption that $f(E_i) \geq g(E_j) \forall i, j \in \{1, \dots, n\}$. Furthermore, from $f(E_k) > g(E_k)$, it must be that $f(E_k) > f(E_i)$ or $g(E_j) > g(E_k)$, for otherwise $g(E_k) \geq f(E_k)$ would follow. Consequently, to establish the strict conclusion of **D5**, it suffices to establish that the following *strict internality* condition holds: For any act $h = (x_1, E_1; x_2, E_2; \dots; x_n, E_n)$, with $\{E_1, \dots, E_n\}$ all non-null and $x_1 \geq x_2 \geq \dots \geq x_n$, if $x_1 > x_n$, then $x_1 > h > x_n$. In proving the condition, all the E_i can indeed be taken non-null without loss of generality. For instance, if E_1 is null, by definition of null events, the act $h = (x_1, E_1; x_2, E_2; x_3, E_3; \dots; x_n, E_n)$ is indifferent to the act $h' = (x_2, E_1 \cup E_2; x_3, E_3; \dots; x_n, E_n)$. Accordingly, instead of h , defined with respect to a partition of the state space not all the cells of which are non-null, one may always indifferently consider h' , defined with respect to a partition of the state space all the cells of which are non-null. Now, under the strict internality condition, if $f(E_k) > f(E_i)$, then $f > f(E_i)$ so that given that $f(E_i) \geq g(E_i)$ and $g(E_i) \geq g$ by the weak internality condition, $f > g$ follows. Similarly, if $g(E_j) > g(E_k)$, then $g(E_j) > g$ by the strict internality condition so that given that $f \geq f(E_i)$ by the weak internality condition and $f(E_i) \geq g(E_j)$, $f > g$ follows.

One may establish that the strict internality condition holds by induction on the size of the partition of the state space with respect to which the act is defined.

Induction base: Assume $x > y$. We need to show that for any essential $E \in \Sigma$, $x > xEy > y$. By **P1**, for any E , either $x > xEy$ or $xEy > y$ must hold, for otherwise $y \geq x$ would follow. Assume $x > xEy$. Then (the second clause of) **A4** implies $yEx > y$. **A3** then implies $xEy > y$. Assume $xEy > y$. Then $x > xEy$ similarly follows from **A4** and **A3**. So $x > xEy > y$ holds in all cases.

Induction step: We need to show that if for all acts defined with respect to an i -ary partition of the state space, the strict internality condition holds, then it also holds for all acts defined with respect to an $i+1$ -ary partition of the state space. So consider an act $h = (x_1, E_1; \dots; x_i, E_i; x_{i+1}, E_{i+1})$, with $\{E_1, \dots, E_i, E_{i+1}\}$ all non-null, $x_1 \geq x_2 \geq \dots \geq x_i \geq x_{i+1}$, and $x_1 > x_{i+1}$. We need to show that $x_1 > h > x_{i+1}$ holds.

Consider first the claim that $h > x_{i+1}$ must hold. By **P1**, one of $x_1 > x_i$ or $x_i > x_{i+1}$ must hold, for otherwise $x_{i+1} \geq x_1$ would follow. Assume first $x_1 > x_i$. By the induction hypothesis, $x_i E_{i+1} h > x_i$. By **Lemma 6.b**, $x_i \geq x_i E_{i+1} x_{i+1}$. Hence by **P1**, $x_i E_{i+1} h > x_i E_{i+1} x_{i+1}$. **A4** then implies $h > x_{i+1}$ (for otherwise, given that $x_i E_{i+1} h \geq x_i \geq x_{i+1}$, the second clause of **A4** would imply $x_i E_{i+1} x_{i+1} \geq x_i E_{i+1} h$,

a contradiction). Now assume $x_i > x_{i+1}$. By the induction hypothesis, $x_i > x_i E_{i+1} x_{i+1}$. By Lemma 6.b, $x_i E_{i+1} h \geq x_i$. Hence by P1, $x_i E_{i+1} h > x_i E_{i+1} x_{i+1}$. A4 again implies $h > x_{i+1}$.

Consider next the claim that $x_1 > h$ must hold. By P1, one of $x_1 > x_2$ or $x_2 > x_{i+1}$ must hold, for otherwise $x_{i+1} \geq x_1$ would follow. Assume first $x_1 > x_2$. By the induction hypothesis, $x_2 E_1 x_1 > x_2$. By Lemma 6.b, $x_2 \geq x_2 E_1 h$. Hence by P1, $x_2 E_1 x_1 > x_2 E_1 h$. A4 implies $x_1 > h$. Now assume $x_2 > x_{i+1}$. By the induction hypothesis, $x_2 > x_2 E_1 h$. By Lemma 6.b, $x_2 E_1 x_1 \geq x_2$. Hence by P1, $x_2 E_1 x_1 > x_2 E_1 h$. A4 again implies $x_1 > h$. \square

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