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Sylow Subgroups of Index 2 in Their Normalizers

by

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Abstract

The following theorem is proved:

Theorem Let G be a finite group, P a Sylow p-subgroup of G, p odd.

Suppose (1) $|N(P)/P \cdot C(P)| = 2$;

(2) $cl(P) \leq 2$.

Then (i) if G is perfect, then P is necessarily cyclic;

(ii) if P is not cyclic, then either $O^p(G) < G$, or $O^2(G) < G$ with $G = O_{p'}(G) \cdot N(P)$.

A unified proof is given as far as possible, but the proof eventually splits into three cases, with hypothesis (2) strengthened by one of:

(2A) $|P'| = p$,

(2B) P is abelian but not cyclic ,

or (2C) $|P'| > p$.

Different methods are in fact required for each case.

Several corollaries are also discussed.

Introduction

The following is the main result of this thesis:

Theorem Let G be a finite group, P a Sylow p-subgroup of G, p odd.

Suppose (1) $|N(P)/P \cdot C(P)| = 2$,

(2) $cl(P) \leq 2$. (nilpotence class)

Then: (i) if G is perfect, then P is necessarily cyclic;

(ii) if P is not cyclic, then either (a) $O^p(G) < G$;

or (b) $O^2(G) < G$, with $G = O_{p'}(G) \cdot N(P)$.

The study of groups satisfying the normalizer assumption (1) is not new; in particular, many of the methods originate in the work of Brauer [3]—— but we obtain here a result not apparently attainable by those methods alone. Groups satisfying the hypotheses of Stewart's theorem [17] must in fact satisfy (1) and (2) for some abelian Sylow p-subgroup; so we see that the groups PSL(2,q) ($q \geq 4$) are examples of simple groups having (1) and (2) for a cyclic Sylow subgroup.

For the proof, we observe at the outset that if $O^p(G) < G$ or $O^2(G) < G$, then $G' < G$, so that (i) follows from (ii). In proving (ii), we assume that (a) does not hold - that is, we make the additional assumption

(3) $O^p(G) = G$

and show that (b) must hold. But then it suffices to show $G = O_{p'}(G) \cdot N(P)$, for then $O^2(G) < G$ follows. The proof will fall into three cases depending on the structure of P, and we correspondingly add to hypothesis (2) one of:

(2A) $|P'| = p$,

(2B) P is abelian but not cyclic,

or (2C) $|P'| > p$.

We also refer to the separate cases of the theorem correspondingly; that is, Theorem A is the theorem proved under hypothesis (2A), and so on.

The proof then proceeds as follows: we do preliminary work on the structure of G, valid for all three cases, in Sections 2 and 3. Then in Section 4 we show that Theorems A and C will follow from certain conditions on the values of the irreducible characters of the principal p-block of G. We therefore devote Section 5 to showing that this situation must hold under hypothesis (2A). We can readily deduce Theorem B from Theorem A in Section 6. We can also use Theorem A and induction to obtain under hypothesis (2C) the character-theoretic information required by Section 4; thus Section 7 proves Theorem C, the last case of the main theorem. The eighth and final section presents several corollaries of the main theorem, and some remarks on the failure of the theorem for p-groups of class larger than 2.

Many people have made important contributions tc this work, and I hope to acknowledge them here. It was Graham Higman who first proved Theorem A in the smallest case ($|P|=27$); and supervised with unfailing patience my efforts to obtain the general theorem. The original proof of Theorem A was done in partial collaboration with A. P. Tyrer; and all the later work through Theorem C certainly benefited from his reading and suggestions. I am grateful to George Glauberman (who visited Oxford during 1971-1972) for indicating the relevance

of p-solvability, which was to play a key role in the subsequent work. Theorem B itself was a product of the combined reflections of the author and all of the above.

Thanks are due to Walter Feit, who read the proof of Theorem A; and suggested the relevance of his work in [5], on which I have leaned heavily on the way to Theorem C. Finally, the support of the Rhodes Trust during my years of graduate study is gratefully acknowledged.

Early work on Theorem A appears in the Oxford University Diploma dissertations of Smith [13] and Tyrer [19], and in Smith-Tyrer [15]. Theorems B and C are to appear in Smith-Tyrer [16] and Smith [14].

Section 1. Notation and Definitions

All groups referred to in this work are assumed to be finite.

The basic notation we use is standard, as found in Gorenstein [7]; any notation not explicitly defined should be referred to that book. We consequently mention here a few deviations from Gorenstein's conventions:

We frequently write $N(P)$ instead of $N_G(P)$ when the group G referred to is clear; and similarly for $C(P)$. We use $\leqslant$ and $<$ for subgroup inclusion and proper inclusion, as in $H \leqslant G$; we use $\cong$ for group isomorphism. Set difference is indicated by - rather than $\setminus$, as in A-B. A group is said to be p-nilpotent if it has a normal p-complement.

The subgroup $O^p(G)$ is the characteristic subgroup generated by the p'-elements of G. Clearly $G/O^p(G)$ is the maximal p-factor of G. The subgroup P^G is the normal subgroup of G generated by the conjugates of P (that is, by all the p-elements of G).

Further notation and conventions are introduced as required. In particular, each section will begin with a listing of notation, definitions, and results from other work which are required for that section. We will produce a full statement of the less well-known results to be used; but standard theorems will be indicated by a reference only.

Section 2. Preliminary Group Theory

Notation: We establish the following conventions for the remainder of the work.

N the normalizer N(P).

t a fixed 2-element of $N\text{-}O_{p'}(N)$. Thus $N = \langle t, P \cdot C(P) \rangle$.

x an arbitrary element of P-P'.

z an arbitrary element of P'-1 (whenever P is not abelian).

We observe that t induces on P an automorphism of order 2. We use, repeatedly and without explicit reference, the standard theorems relating to this situation. These can be found in Gorenstein [7]: results 5.2.3, 5.3.2, 5.3.5, 5.3.15, and 10.4.1. Other results we require for p-local analysis are the Frattini argument 1.3.7, the usual theorems on the Frattini subgroup 5.1.1 and 5.1.3, and the Schur-Zassenhaus theorem 6.2.1. To handle fusion, we require Burnside's theorem 7.1.1 and the focal subgroup theorem 7.3.4.

P. Hall defined the notion of a regular p-group; in particular, p-groups of class ≤ 2 for p odd are regular (see Hall [8, p.183]). We will be interested in using the Hall-Wielandt transfer theorem, IV.8.1 of Huppert [10]:

(HW) Let Q be a Sylow p-subgroup of H, $M = N_H(Q)$, and suppose Q is regular. Then $H/O^p(H) \cong M/O^p(M)$.

We also note here that if Q is a Sylow p-subgroup of H, and $H = Q \cdot C_H(Q)$, then it follows (using the Schur-Zassenhaus theorem) that H is p-nilpotent.

We require a result on weak closure from the work of Glauberman in [6]. We define weak closure of subgroups in the standard way as in Gorenstein [7]. And we say an element y of P is <u>weakly closed</u> in P with respect to G if whenever $y^g \in P$ for some $g \in G$, we have in fact that $y = y^g$. We say a subgroup H <u>controls weak closure</u> of p-elements in G if $y \in P \cap Z(H)$ implies y is weakly closed in P with respect to G. We will be making use of Theorem 7.5 of [6]:

(WC) Suppose Q is a Sylow p-subgroup of H, p odd, and $cl(Q) < p$. Then $N_H(Q)$ controls weak closure of p-elements in H.

This is a simplification by special case of Glauberman's result; to observe that the theorem applies here, take the W of the theorem to be the identity functor $W(Q) = Q$.

We can now discuss certain p-local and global properties of G.

<u>Lemma 2.1</u> (i) $[P,t] = P$;

(ii) $C_P(t) = P'$;

and (iii) $P' \leq Z(N)$.

<u>Proof</u>: We are assuming by hypothesis (3) that $O^p(G) = G$. Now P is regular, so that in view of (HW), we must have $O^p(N)=N$; thus $P \leq N'$. It follows from the focal subgroup theorem that $P \cap N' = [P,N]$, so that $P = [P,N]$. Then $N = \langle t, P \cdot C(P) \rangle$ gives $[P,N] = [P,t]P'$. And certainly $P' \leq \Phi(P)$ the Frattini subgroup, so that $P = [P,t]$ as asserted.

Let $\bar{P} = P/P'$. Then (i) gives $[\bar{P},t] = \bar{P}$; and since $\bar{P}$ is an abelian group, we must have $\bar{P} = [\bar{P},t] \times C_{\bar{P}}(t)$ so that $C_{\bar{P}}(t) = \bar{1}$. Since $C_{\bar{P}}(t) = \overline{C_P(t)}$, we get $C_P(t) \leq P'$. Suppose then that $C_P(t) < P'$ (in case $P' \neq 1$). As $P' \leq Z(P)$, certainly $C_P(t) \trianglelefteq P$; and then the non-abelian group $P/C_P(t)$ has a fixed-

point-free automorphism t of order 2 ... a contradiction. Thus $C_P(t) = P'$. Since $N = \langle t, P \cdot C(P) \rangle$, we get $P' \leq Z(N)$, and all parts of the lemma are proved.

Lemma 2.2 Each element of P' is weakly closed in P with respect to G.

Proof: We have $cl(P) \leq 2 < p$, so that by (WC), N controls weak closure of p-elements in G. Now the assertion follows from 2.1(iii).

Lemma 2.3 N strongly controls fusion in G; that is,

if $A, B \subseteq P$ with $A^g = B$ for some $g \in G$,

then we can write $g = cn$ where $c \in C(A)$ and $n \in N$.

Proof: Let A and B be given as above. We observe in view of 2.2 that the subgroup P' itself is weakly closed in P with respect to G. As $P' \leq Z(P)$, certainly $P' \leq C(A)$; and $A \leq P^{g^{-1}}$ gives also $(P')^{g^{-1}} \leq C(A)$. So let $(P')^{g^{-1}}$ lie in Q, some Sylow p-subgroup of $C(A)$. We can find $c_1 \in C(A)$ with $(P')^{c_1^{-1}} \leq Q$. But there must be $h \in G$ with $Q^h \leq P$, so we conclude by weak closure that $(P')^{g^{-1}h} = P' = (P')^{c_1^{-1}h}$, and hence $(P')^{g^{-1}} = (P')^{c_1^{-1}}$. Then $g = c_1 n_1$ with $n_1 \in N(P')$.

Now we have $A^g = A^{n_1} = B$. From 2.2 we can see that $N(P') = C(P')$; so let $C = C(P')$ and $\overline{C} = C(P')/P'$. We have $\overline{A}^{\overline{n}_1} = \overline{B}$, and the Sylow p-subgroup $\overline{P}$ of $\overline{C}$ is abelian. Then by Burnside's theorem (or just an application of the argument we have just made) we get $\overline{n}_1 = \overline{c}_2 \overline{n}_2$ with $\overline{c}_2 \in C_{\overline{C}}(\overline{A})$ and $\overline{n}_2 \in N_{\overline{C}}(\overline{P})$. If we take pre-images in C, we have $n_1 = c_2 n_2 n_3$ where $c_2 \in C_C(\langle A \rangle P'/P')$, $n_2 \in N(P)$, and $n_3 \in P' \leq N(P) = N$. We now look more closely at the element c_2. Let $D = C_C(\langle A \rangle P'/P')$. We

observe that any p'-element of D stabilizes the normal series $\langle A\rangle P' \trianglerighteq P' \trianglerighteq 1$, and hence must lie in $C_C(\langle A\rangle P') = C_C(A)$. Thus $O^P(D) \trianglelefteq C_C(A) \trianglelefteq D$; and since we see that P is a Sylow p-subgroup of D, we have $D = C_C(A)\cdot P$. Then we can write $c_2 = c_3 n_4$ where $c_3 \in C_C(A)$ and $n_4 \in P \trianglelefteq N$. This gives $g = c_1(c_3 n_4)n_2 n_3$ with $c_1 c_3 \in C(A)$ and $n_4 n_2 n_3 \in N$; so 2.3 is proved.

Remarks: In case $P' = 1$, Lemma 2.2 is trivial, and Lemma 2.3 reduces to Burnside's theorem. We observe that 2.3 has the following consequence for p-local subgroups: If $Q \trianglelefteq P$, then $N(Q) = N_N(Q)\cdot C(Q)$.

We use this fact to prove:

Lemma 2.4 $C(x) = O_{p'}(C(x))\cdot C_P(x)$ (thus C(x) is p-nilpotent).

Proof: Recall that x is an element of P-P'. It follows from 2.1(ii) that $C_N(x) = C_P(x)\cdot C(P)$. We will want to show that $C_P(x)$ is a Sylow p-subgroup of C(x). To do this, it suffices to show that if $x \in Q = P^g$ for some g of G, then $|C_Q(x)| \trianglelefteq |C_P(x)|$. This is equivalent to showing that $|[x,Q]| = |[x^{g^{-1}},P]| \geq |[x,P]|$. But since $x^{g^{-1}} \in P$, we can use 2.3 to take $g^{-1} = cn$, $c \in C(x)$ and $n \in N$. Thus $x^{g^{-1}} = x^n$ and $|[x^n,P]| = |[x,P^{n^{-1}}]| = |[x,P]|$, which establishes what we wanted.

We now write Q for $C_P(x)$, and C for C(x). Now certainly $P' \trianglelefteq Q$, and then the structure of N forces $N \trianglelefteq N(Q)$. By the remark after 2.3, $N_C(Q) = C_N(x)\cdot C(Q)$. Since $C_N(x) = Q\cdot C(P)$ by our earlier comment, and $C(P) \trianglelefteq C(Q)$, we have in fact that $N_C(Q) = Q\cdot C(Q)$. But then $N_C(Q)$ is p-nilpotent; and we conclude by (HW) that C itself is p-nilpotent. This establishes 2.4.

We conclude this section with a global result:

Lemma 2.5 It suffices to prove either $O_{p'}(G) \neq 1$ or $P^G < G$.

Proof: We work in either case by induction on $|G|$. Suppose first $O_{p'}(G) \neq 1$. Let $\overline{G} = G/O_{p'}(G)$. We have $\overline{N} = \overline{N(P)} = N(\overline{P})$; and $[\overline{P},t] = \overline{P}$ by 2.1(i) so that $O^p(\overline{G}) = \overline{G}$. Then $\overline{G}$ satisfies (1), (2) and (3), and we conclude by induction that $\overline{G} = O_{p'}(\overline{G})\cdot\overline{N}$; but $O_{p'}(\overline{G}) = \overline{1}$ by definition, so that $G = O_{p'}(G)\cdot N$.

Suppose now $M = P^G < G$. If $t \in M$, then $N_M(P)$ satisfies (1), $P \trianglelefteq M$ gives (2), and $[P,t] = P$ gives (3) as before. Thus $M = O_{p'}(M)\cdot N_M(P)$ by induction, so that $O_{p',p}(M) = O_{p'}(M)\cdot P$. But $O_{p'}(M) \trianglelefteq O_{p'}(G)$, and so $O_{p',p}(G) = O_{p'}(G)\cdot P$. We then get $G = O_{p'}(G)\cdot N(P)$ by the Frattini argument. If on the other hand $t \notin M$, we must have $N_M(P) = P\cdot C_M(P)$. In this case, $N_M(P)$ is p-nilpotent, and by (HW) so is M. This gives $O_{p',p}(M) = O_{p'}(M)\cdot P$; and exactly as above, we have $G = O_{p'}(G).N(P)$. This completes the proof of Lemma 2.5.

Section 3. Preliminary Character Theory

We established in 2.2 that the elements of P' are weakly closed in P with respect to G. The character-theoretic part of our proof will deal with expressions involving the values of principal-block characters on the elements z of P'-1 in the case $P' \neq 1$. These values will turn out to be directly related to the values taken by the principal-block characters of N. The aim of this section is to describe the relevant characters of N in terms of characters of the subgroup P'; the possibility that $P' = 1$ is not excluded, although the results are well-known in this case.

We establish the following conventions for the remainder of the work: "character" will mean "irreducible complex character" unless otherwise specified;

Notation: B the principal block of characters of G,

b the principal block of N,

χ an arbitrary character of B,

η an arbitrary character of b,

ζ an arbitrary character of P,

ϕ an arbitrary character of P',

g_p the p-part of an element g of G,

and $g_{p'}$ the p'-part of g.

The basic reference for ordinary character theory will be Gorenstein [7]. In particular, we require the usual orthogonality relations (4.2.4 and 4.2.8) and Clifford's theorem 3.4.1. If Q is an abelian group, then the characters of Q are linear (3.2.4); and if Q has odd order, non-principal characters are non-real (4.3.6). We denote the usual inner product of characters α and β of a subgroup H by $(\alpha,\beta)_H$; omitting the subscript H

when the group is clear. We generally denote the restriction of a character α of H to a subgroup K by α_K, but we may write $\alpha|_K$ if the expression for α already involves subscripts. We also use the standard fact that if ψ is a character of H and $g \in H$, then $\psi(g)$ is congruent to $\psi(g_{p'})$ modulo a prime ideal containing p in $\mathbb{Z}[\omega]$, ω a primitive $|H|$-th root of unity.

The basic reference for modular character theory will be Feit [5], although a good deal of reference is made to the papers of Brauer [1,2,3]. Many of the required results are not in the relevant chapter of Curtis-Reiner [4]. In particular, we require the standard facts that $O_{p'}(H)$ is the intersection of the kernels of the principal-block characters of H, and $O_{p',p}(H)$ is the intersection of the kernels of the modular irreducible characters of the principal block of H (see IV(4.12) of [5]). Also, a core-free group of p-length 1 has only the principal block of characters (or, in Brauer's terminology, has deficiency class 0). Details of the proof of this statement can be found in Stewart [17, Lemma 3.11].

Remark: Thus the characters of b are just the characters of $N/O_{p'}(N) \tilde{=} \langle t \rangle P/\langle t^2 \rangle$.

Following Brauer in [1], we define a column inner product: let c_1 and c_2 be columns of complex numbers indexed by the characters ψ of the principal p-block of H, so that c_1 has ψ-th entry $c_{1\psi}$, and so on. Then let $(c_1,c_2)^H$ be defined by the sum $\sum_\psi c_{1\psi}\,\overline{c_{2\psi}}$. If H = G, we usually omit the superscript. We may wish to denote by $(c_1,c_2)^H_S$ the sum for $(c_1,c_2)^H$ evaluated using only the characters of a subset S of the principal - block characters of H. Whenever an element h of H appears in such an

inner product, it is meant to denote the column of values $\psi(h)$, where ψ runs over the principal-block characters of H. We make an exception, however, to avoid confusion with character inner products involving the trivial character: the column at 1 (the character degrees) will be denoted by deg. A particular result we will require is Brauer's formula $(g,g) = (g_{p'},g_{p'})^{C(g_p)}$. This appears as Proposition 2 of [2] (our notation differs from Brauer's).

We now proceed to investigate the characters of b. So choose some character η. Since $P' \leq Z(N)$, we see by Clifford's theorem that $\eta_{P'} = \eta(1)\cdot\phi$ for some character ϕ of P'. It will turn out to be convenient to tabulate character values on the sets defined by:

$$A(N,\phi) = \{\eta : \eta_{P'} = \eta(1)\cdot\phi\} .$$

If we also apply Clifford's theorem with respect to P, we see that η_P will be an integer multiple either of a single character ζ with $\zeta^t = \zeta$, or of a sum $\zeta + \zeta^t$ where $\zeta \neq \zeta^t$. In any case, we can consider the restriction $\zeta_{P'}$ in the same way as $\eta_{P'}$, so that our first task will be to investigate the sets:

$$A(P,\phi) = \{\zeta : \zeta_{P'} = \zeta(1)\cdot\phi\} .$$

For simplicity, we always write 1 for the trivial character of P'. We note then that A(P,1) is precisely the set of linear characters of P; and so gives all the characters of P when P is abelian.

The following lemma describes the required properties of the sets $A(P,\phi)$:

Lemma 3.1 Fix ϕ, and some π of $A(P,\phi)$. Let $K = \ker \phi$, $\overline{P} = P/K$, and $\overline{X} = Z(\overline{P})$. Then:

(i) $\overline{X} = \overline{Y} \times \overline{P}'$, where t inverts $\overline{Y}$.

(ii) π vanishes off $\overline{X}$.

(iii) $\pi_{\overline{Y}} = \pi(1).\theta$, where θ is some character of $\overline{Y}$.

(iv) $\pi(1)^2 = |\overline{P} : \overline{X}|$, and hence $|\overline{Y}| \cdot \pi(1)^2 = |P:P'|$.

and

(v) each character θ of $\overline{Y}$ appears exactly once in some character of $A(P,\phi)$ as described in (iii).

Warning: The subgroups K, $\overline{X}$, and $\overline{Y}$, and the characters θ of $\overline{Y}$, all depend on ϕ, though we make no notational point of this. We continue to use the characters θ in our later notation, but will always write them next to the particular ϕ to which they refer.

Proof: We get (i) from the action of t on abelian groups, and (iii) comes from Clifford's theorem as before.

For (ii), take $g \in P$ so that $\overline{g} \notin \overline{X}$. Then $[g,P] = L \not\leq K$. Let ψ represent an arbitrary character of P with L in its kernel. We have

$$\sum_{\psi} \psi(g)\overline{\psi}(g) = |C_{P/L}(g)| = |P:L| = |P:[g,P]| = |C_P(g)|.$$

To get the last equation, we have used the fact that P has class ≤ 2, so that $[g,P]$ has the same order as the conjugacy class of g in P (which is just the coset $g[g,P]$).

Now $|C_P(g)|$ is the full value of $\sum_{\zeta} \zeta(g)\overline{\zeta}(g)$. Since $(\ker \pi) \cap P' = K$ and $L \not\leq K$, π does not appear among the ψ, and hence we must have $\pi(g) = 0$. This gives (ii).

Now let ψ be the character of $\overline{X}$ determined by $\pi_{\overline{X}} = \pi(1) \cdot \psi$ in the standard way. In view of (ii), we can write:

$$1 = (\pi,\pi)_{\overline{P}} = \pi(1)^2 \frac{|\overline{X}|}{|\overline{P}|} (\psi,\psi)_{\overline{X}} = \pi(1)^2 \frac{|\overline{X}|}{|\overline{P}|} .$$

Then (iv) is immediate.

Now (iv) implies that all the characters of $A(P,\phi)$ have a common degree. In view of (ii) this means that no character θ may appear more than once in a character of $A(P,\phi)$ as in (iii). We also use (iv) to observe that:

$$|P| = \sum_{\zeta} \zeta(1)^2 = \sum_{\phi} |A(P,\phi)| \cdot |\overline{P}:\overline{X}| .$$

But equality can hold only if $|A(P,\phi)| = |\overline{Y}|$ for each ϕ; that is, if each character θ of $\overline{Y}$ appears exactly once in a character of $A(P,\phi)$ as in (iii). This completes the proof of Lemma 3.1.

This lemma enables us to specify characters via the notation:

Definition

ζ_{ϕ}^{θ} the character π of $A(P,\phi)$ with $\pi_{\overline{Y}} = \pi(1)\cdot\theta$;

$s(\phi)$ defined by $p^{s(\phi)}$ = the common degree of the characters of $A(P,\phi)$;

$r(\phi)$ defined by $2r(\phi) + 1 = |\overline{Y}| = |A(P,\phi)|$;

and $a(\phi)$ defined by $2a(\phi) + 1 = p^{s(\phi)}$.

Remarks: $r(\phi)$ is the number of non-conjugate non-principal characters of $\overline{Y}$. We may have $r(\phi) = 0$, when $Z(P/K) = P'/K$, and in this case $A(P,\phi)$ is the single character ζ_{ϕ}^{1} (again we have written 1 instead of $1_{\overline{Y}}$ for simplicity). We always have $r(1) \neq 0$; and when $P' \neq 1$, we have $|P:P'| \geq p^2$ so that $r(1) \geq 4$. Also $s(1) = a(1) = 0$; but if $\phi \neq 1$, then $s(\phi), a(\phi) \geq 1$.

We now observe that $(\zeta_\phi^\theta)^t = \zeta_\phi^{\theta^t} = \zeta_\phi^{\bar\theta}$ so that only ζ_ϕ^1 of $A(P,\phi)$ is fixed by t. Then from Clifford's theorem and the orthogonality relations we can deduce the degrees and restrictions to P of the characters of b. In fact, the values of the characters of $A(N,\phi)$ on the classes of $N/O_{p'}(N)$ are given by:

(3.2)

	$\bar P - \bar X$	$\bar Y$	$\bar P'$	t	zt
η_ϕ^+	0	$p^{s(\phi)}.1$	$p^{s(\phi)}.\phi$	1	$\phi(z)$
η_ϕ^-	0	$p^{s(\phi)}.1$	$p^{s(\phi)}.\phi$	-1	$-\phi(z)$
$(\theta\neq 1)$ η_ϕ^θ	0	$p^{s(\phi)}_{(\theta+\bar\theta)}$	$2p^{s(\phi)}.\phi$	0	0

Here, we have chosen notation + and - to distinguish the two characters of degree $p^{s(\phi)}$ by their values on the coset tP'. The other characters are identified by θ(or $\bar\theta$) representing the restriction term $(\theta+\bar\theta)$ on $\bar Y$.

A detailed verification of the correctness of (3.2) proceeds as follows. We will identify a character α of P with the unique character β of $P\cdot C(P)/O_{p'}(N)$ such that $\beta_P = \alpha$. Then we observe that $(\zeta_\phi^\theta)^N = \eta_\phi^\theta$ for $\theta \neq 1$, and the latter is in fact an irreducible character of $N/O_{p'}(N)$. For the class functions of $N/O_{p'}(N)$ defined by η_ϕ^+ and η_ϕ^-, we can apply Brauer's characterization of characters [7,4.7.1], and note that it suffices to consider the maximal elementary subgroups P and $\langle t\rangle \times P'$ of $N/O_{p'}(N)$. Certainly the restrictions of the two class functions to P are just ζ_ϕ^1, which is a character of P. So consider $Q = \langle t\rangle \times P'$. Let $\hat\phi$ be the character of $Q/\langle t\rangle$ with $\hat\phi_{P'} = \phi$, and π the character of $Q/\langle t^2\rangle\cdot P'$ with $\pi(t) = -1$. Then $\pi\hat\phi$ is also a character of Q. We see in fact that

$$\eta_\phi^+|_Q = (a(\phi) + 1)\cdot\hat{\phi} + a(\phi)\cdot\pi\hat{\phi}$$

and $$\eta_\phi^-|_Q = a(\phi)\cdot\hat{\phi} + (a(\phi) + 1)\cdot\pi\hat{\phi} \quad .$$

So η_ϕ^+ and η_ϕ^- are generalized characters of $N/O_{p'}(N)$. We conclude that they are irreducible characters by observing that they have positive degree, and evaluating $(\eta_\phi^*, \eta_\phi^*)_N = 1$ for $* = +$ or $-$. Finally, evaluating $\sum_{*,\phi} \eta_\phi^*(1)^2$ over all ϕ and all superscripts $*$ gives $|N/O_{p'}(N)|$, which allows us to conclude that all the characters of $N/O_{p'}(N)$ appear as in (3.2).

The notation we have chosen identifies the characters of b uniquely (we will use a similar notation for the characters of B, when we come to define our principal-block isometry from b). We frequently write * for a superscript, as in the above, to denote one of {+, -, $\theta \neq 1$} which need not be specified. We deviate from our convention slightly to refer to the trivial character of N as 1_N rather than η_1^+.

As we observed earlier, the proof of the theorem depends on a calculation involving values of the characters of B on p-singular elements with p-part of the form z. Under hypothesis (2A), the best we can do is to specify these values in terms of generalized decomposition columns (hence the format of the next section). The proof proceeds by showing that these columns have essentially the same entries as the columns of the ordinary decomposition matrix of N - - for a treatment of the relevant theory, see Feit [5, Chapter I, Sec. 17]. The decomposition values for N are given by the following:

Lemma 3.3 (i) The restrictions to p'-elements of 1_N and η_1^- give the irreducible Brauer characters of b.

(ii) The decomposition matrix of N, for the characters of $A(N,\phi)$ is:

(3.3)

	D_N^+	D_N^-
η_ϕ^+	$a(\phi) + 1$	$a(\phi)$
η_ϕ^-	$a(\phi)$	$a(\phi) + 1$
$(\theta=1)\ \eta_\phi^\theta$	$p^{s(\phi)}$	$p^{s(\phi)}$

where $\eta(g) = (D_N^+)_\eta \cdot 1 + (D_N^-)_\eta \cdot \eta_1^-(g)$ $\qquad$ $(g = t \text{ or } 1)$.

Proof: Both 1_N and η_1^- are linear and so irreducible; and are in fact all the characters of $N/P \cdot O_{p'}(N)$. But $O_{p',p}(N) = O_{p'}(N) \cdot P$, which must lie in the kernels of the modular irreducible characters of b; so 1_N and η_1^- (restricted to p'-elements) are the only Brauer characters that could be obtained from modular irreducibles This gives (i). And (ii) follows by observing that the columns in fact give rise to the character values we determined in (3.2). So Lemma 3.3 holds.

Remark: We note for convenience that we can in fact use the columns of (3.3) to express character values for all p-singular elements with p-part of form z. That is, for $\eta \in A(N,\phi)$ we have

(3.4)
$$\eta(z) = (D_N^+ + D_N^-)_\eta\ \phi(z)$$
$$\eta(zt) = (D_N^+ - D_N^-)_\eta\ \phi(z) \quad .$$

Section 4. Character Theory Leading to the Final Calculation

To describe the values of the characters of B on p-singular elements of G, we use the theory of generalized decomposition columns —— cf. Brauer [1, II.5] and Feit [5, Chapter IV, Section 6]. The fundamental result is that we can take a basic set $\psi^i_{g_p}$ (i = 1, ... , n) for the complex space generated by the characters of the principal block of $C(g_p)$ restricted to p'-elements; and we have decomposition columns $D^i_{g_p}$ correspondingly, so that χ(in the principal block of G now) has:

$$\chi(g) = \sum_{i=1}^{n} (D^i_{g_p})_\chi \cdot \psi^i_{g_p}(g_p,) \ .$$

This is of course the body of theory centering around Brauer's Second Main Theorem. We require several properties of these columns: if we form a "Cartan matrix" by multiplying the generalized decomposition matrix by its conjugate transpose, the result is the same as the ordinary Cartan matrix of $C(g_p)$; see [5, IV (6.2)]. It is one of the properties of the column inner product that if h_p is not conjugate to g_p in G, then (h,g) = 0; it follows from the result just quoted that in fact $(h,D^i_{g_p}) = 0$ for all i.

The notion of defect group is found in any of the references on modular character theory. In particular, we require the fact that a Sylow p-subgroup of a group is a defect group for the principal block of that group; and a character vanishing on all p-singular elements lies in a block of defect 0 (see IV(4.19) and (4.20) of [5]). Thus the principal block can have defect 0 only in a p'-group.

This section is devoted to showing that in order to prove Theorem A or Theorem C, it suffices to establish the following character-theoretic situation in B:

Hypothesis 4.1

(i) There exists a 1:1 correspondence between the characters of b and the characters of B, which we express by the notation:

$$\chi_\phi^* \text{ corresponds to } \eta_\phi^* .$$

(ii) For each z, we have a basic set $\{1_{C(z)}, \psi_z^-\}$ of two characters of the principal block of C(z), with generalized decomposition columns denoted by D_z^+ and D_z^-.

and (iii) There are fixed signs $\delta_\phi^* = \pm 1$ with the properties:

(a) $(D_z^+)_{\chi_\phi^*} = \delta_\phi^* \cdot (D_N^+)_{\eta_\phi^*} \cdot \phi(z) \quad (* = +, -, \theta)$,

$(D_z^-)_{\chi_\phi^*} = \delta_\phi^* \cdot (D_N^-)_{\eta_\phi^*} \cdot \phi(z)$;

(b) whenever $r(\phi) \neq 0$, the generalized character

$\delta_\phi^+ \cdot \chi_\phi^+ + \delta_\phi^- \cdot \chi_\phi^- - \delta_\phi^\theta \cdot \chi_\phi^\theta$ (for each $\theta \neq 1$)

vanishes on the p'-elements of G.

In the remainder of this section, we assume that $P' \neq 1$ and that Hypothesis 4.1 holds. Thus we are given the specific relation of the characters of B to those of b. Now in N, we were able to describe values of the characters of b at p'-elements by means of the columns $D_N^\pm$ and the characters 1_N and η_1^- ... this was the point of Lemma 3.3. Hypothesis 4.1 provides us with columns $D_z^\pm$, related to D_N by (iii) (a), and we first show we can get analogous expressions in terms of the values of characters 1_G and χ_1^- on the p'-elements of G. So suppose g is a p'-element of G.

The condition (iii)(b) allows us to relate values of the characters corresponding to a given $A(N,\phi)$ whenever $r(\phi) \neq 0$. In particular, we always have $r(1) \neq 0$. We write E for $\delta_1^- \cdot \chi_1^-(g)$ for convenience; and then (iii)(b) gives:

$$\delta_1^\theta \cdot \chi_1^\theta(g) = 1 + E \quad .$$

We similarly write $E_\phi^\pm$ for $\delta_\phi^\pm \cdot \chi_\phi^\pm(g)$ for $\phi \neq 1$. Then (iii)(b) gives:

$$\delta_\phi^\theta \cdot \chi_\phi^\theta(g) = E_\phi^+ + E_\phi^- \quad .$$

We now show how to relate the values of all the $E_\phi^\pm$ to E. Since g_p and z are certainly not conjugate, we have $(g, D_z^\pm) = 0$. We shall therefore use (iii)(a) to write out the expression for $(g, D_z^+ + D_z^-) = 0$. We sum over the sets corresponding to the $A(N,\phi)$ separately, and use the previous relations when $r(\phi) \neq 0$, to get:

$$0 = \sum_\phi E_\phi^+ \cdot p^{s(\phi)} \cdot \bar{\phi}(z) + E_\phi^- \cdot p^{s(\phi)} \cdot \bar{\phi}(z) + 2r(\phi) \cdot p^{s(\phi)} \cdot (E_\phi^+ + E_\phi^-) \cdot \bar{\phi}(z)$$

$$= \sum_\phi (2r(\phi) + 1) \cdot p^{s(\phi)} \cdot (E_\phi^+ + E_\phi^-) \cdot \bar{\phi}(z) \quad .$$

We observe that $(2r(\phi) + 1) \cdot p^{s(\phi)} = (1/p^{s(\phi)}) \cdot |P:P'|$. We write out also the expression for $(g, D_z^+ - D_z^-) = 0$ to yield:

$$0 = \sum_\phi (E_\phi^+ - E_\phi^-) \cdot \bar{\phi}(z) \quad .$$

Now equations of these forms hold for all z of P'-1. But whenever we have an expression of the form $\sum_\phi c_\phi \cdot \phi$ vanishing on P'-1, we must in fact have a multiple of the regular character of P', and hence may set all coefficients $c_\phi = c_1$. Doing this in the above yields:

$$|P:P'| \ (E+1) = (1/p^{s(\phi)}) \ |P:P'| \ (E_\phi^+ + E_\phi^-)$$

$$1 - E = (E_\phi^+ - E_\phi^-) \quad .$$

The solution of these equations in terms of E is given by:

$$E_\phi^+ = a(\phi)(E+1) + 1$$

(4.2)

$$E_\phi^- = (a(\phi)+1)(E+1) - 1 \quad .$$

With these relations in hand, we proceed to the final calculation.

Let h be a p'-element of C(z), and g any involution of G. Then we can never have conjugates g_1 and g_2 of g so that $g_1 g_2 = zh$; for then g_1 and g_2 would invert zh and hence invert $(zh)_p = z$, contrary to the fact (Lemma 2.2) that z is not conjugate to its inverse in G. Then by the usual class-sum formula [7, 4.2.12] we have:

$$\sum_\psi \frac{\psi(g)^2\bar{\psi}(zh)}{\psi(1)} = 0$$

where the sum runs over all the characters ψ of G. But since the equation holds for all p'-elements h of C(z), we get the same equation by restricting the sum to the characters of B [5, IV(6.3)]:

(4.3)

$$\sum_\chi \frac{\chi(g)^2\bar{\chi}(zh)}{\chi(1)} = 0$$

Of course Hypothesis 4.1 allows us to write out this expression using the generalized decomposition columns $D_z^\pm$ to get an equation of the form

$$c_1 + c_2 \cdot \psi_z^-(h) = 0$$

where c_1 is an expression in the entries of the column D_z^+, and c_2 in D_z^-. Now ψ_z^- may not be trivial on all p'-elements of C(z), since in particular C(z) contains N, which is not p-nilpotent. This means that the separate expressions c_1 and c_2 must in fact vanish. So equality still holds if in (4.3) we replace the

column zh by the column $D_z^+ - D_z^-$. In view of the relations (4.2), we can re-write the equation in terms of $E = \delta_1^- \cdot \chi_1^-(1)$ and $F = \delta_1^- \cdot \chi_1^-(g)$ to get:

$$0 = \sum_{\phi} \left(\frac{(a(\phi)(F+1) + 1)^2}{a(\phi)(E+1) + 1} - \frac{((a(\phi)+1)(F+1) - 1)^2}{(a(\phi)+1)(E+1) - 1} \right) \bar{\phi}(z)$$

This holds for each z of P'-1, so we can equate coefficients as before:

$$(4.4) \qquad 1 - \frac{F^2}{E} = \frac{(a(\phi)(F+1) + 1)^2}{a(\phi)(E+1) + 1} - \frac{((a(\phi)+1)(F+1) - 1)^2}{(a(\phi)+1)(E+1) - 1}$$

Now equation (4.4) can be factored. Indeed, setting E = F visibly solves the equation. We differentiate with respect to F to get:

$$\frac{-2F}{E} = 2a(\phi)\frac{(a(\phi)(F+1) + 1)}{a(\phi)(E+1) + 1} - 2(a(\phi)+1)\frac{((a(\phi)+1)(F+1) - 1)}{(a(\phi)+1)(E+1) - 1}$$

Again setting E = F solves the equation. Thus (E - F) divides the corresponding polynomial in E and F at least twice. Setting E = 1 yields

$$1 - F^2 = \frac{1}{p^{s(\phi)}} \left(-p^{s(\phi)}(F+1)^2 + 2p^{s(\phi)}(F+1) \right)$$

But this is quickly seen to be an identity in F, so that (E - 1) also divides the relevant polynomial. Now if we clear denominators in (4.4), the terms of total degree 4 (that is, those in E^2F^2) vanish, giving total degree 3. As the leading coefficient (say, the one for E^3) is $a(\phi)(a(\phi)+1)$, we conclude that (4.4) must in fact reduce to:

$$0 = a(\phi)\ (a(\phi)+1)\ (E - F)^2\ (E-1)$$

Now $\phi \neq 1$ gives $a(\phi) \geq 1$, so that $a(\phi)$ and $(a(\phi)+1)$ do not vanish. Suppose $E = F$. Then by our relations (4.2), the columns for 1 and g are identical; that is, g is in the kernel of χ for all χ of B. Then $g \in O_{p'}(G)$ so that $O_{p'}(G) \neq 1$. Suppose on the other hand that $E = 1$. Character degrees are positive, so we must have $\delta_1^- = \chi_1^-(1) = 1$; that is, χ_1^- is a linear character of G. But then we can see from the relations (4.2) that χ_1^- is the only linear character of B other than 1_G; which implies that χ_1^- must be rational. But then $|G:\ker \chi_1^-| = 2$, so that $P \trianglelefteq \ker \chi_1^- < G$ and hence $P^G < G$. So in either case $E = F$ or $E = 1$, the theorem follows by induction as we observed in Lemma 2.5. The work of this section is complete.

Section 5. Proof of Theorem A

In this section we operate under the assumption (2A) $|P'| = p$, and prove Theorem A be reducing to the situation we have just considered. We introduce some shorthand notation for use only in this section.

Notation: $P' = \langle z \rangle$,

ω a fixed primitive p-th root of unity,

and $\bar{k}$ the Galois automorphism of $Q(\omega)$ over Q given by $(\omega)\bar{k} = \omega^k$ $(k = 1, \ldots, p-1)$.

We write the characters of $P' = \{\phi_o = 1, \phi_1, \ldots, \phi_{p-1}\}$ where we have $\phi_i(z^j) = \omega^{ij}$ for $0 \leq i,j \leq p-1$.

r_1 is $r(1)$. Note then $r_1 \geq 4$;

r $r(\phi_i)$

s $s(\phi_i)$

a $a(\phi_i)$

$i \neq 0$ (independent of i)

We define $\sigma(\phi)$ to be the character $\sum_{\zeta \in A(P,\phi)} \zeta$.

Remarks: (i) We have $a(1) = s(1) = 0$ as the characters of $A(P,1)$ are linear. In the case $r = r(\phi_i) = 0$, the character $\sigma(\phi_i)$ is the single character $\zeta^1_{\phi_i}$.

(ii) A generalized character ψ of P vanishes off P' if and only if ψ is of the form $\sum_\phi c_\phi \cdot \sigma(\phi)$ with $c_\phi \in \mathbb{Z}$. For we can observe from Lemma 3.1 that each $\sigma(\phi)$ vanishes off P'. Conversely $\psi_{P-P'} = 0$ implies that the multiplicity (ψ,ζ) is constant for all the characters ζ from a given $A(P,\phi)$.

We saw in Lemma 2.1 that $O^P(G) = G$ forced $[P,t] = P$; so any subgroup H containing P and t must also have $O^P(H) = H$. It follows that such an H must satisfy (1), (2), and (3), and

hence that the work of Section 2 (Lemma 2.5 is not particularly relevant) must be valid in the corresponding notation in H, and similarly the work of Section 3 for the characters of $N_H(P)$. We will want to apply parts of the work of this section to the subgroup C(z) (which contains N) as well as to G, so we collect here the hypotheses under which we establish the fundamental work of the section.

Hypothesis 5.1

(i) Hypotheses (1), (2) and (3) hold.

(ii) Lemmas 2.3 and 2.4 hold.

(iii) The characters of a Sylow p-subgroup are described by Lemma 3.1.

We now fix some ϕ such that $r(\phi) \neq 0$. Thus the next two lemmas will apply in any case to $A(P,1)$, and to $A(P,\phi_i)$ for $i \neq 0$ when $r \neq 0$. As Hypothesis 4.1 indicated, we will be working with complex columns indexed by the characters of B; we can use this approach to describe the constituents of the characters obtained by restricting to P the characters of B.

For any $\theta \neq 1$ of $\overline{Y}$, we define a column a^θ (or $a^{(\theta,\phi)}$ if we wish to specify the ϕ to which θ refers) by:

$$a^\theta_\chi = (\chi_P, \zeta^1_\phi - \zeta^\theta_\phi) .$$

We observe then that a^θ is a column of rational integers. Further, the generalized character $\zeta^1_\phi - \zeta^\theta_\phi$ vanishes on P', so that a^θ is a complex linear combination of the columns x for elements of P-P'. We show that the columns have the following properties:

Lemma 5.2 Let Hypothesis 5.1 be satisfied. Suppose ϕ is given with $r(\phi) \neq 0$. Then:

$$\text{(i)} \quad a^{\theta} = a^{\theta^t} = a^{\overline{\theta}} ;$$

$$\text{(ii)} \quad (g, a^{\theta}) = \begin{cases} 0 & \text{if } g_p \text{ is conjugate to no } x \text{ of } P\text{-}P' p, \\ (2.\zeta_{\phi}^{1} - \zeta_{\phi}^{\theta} - \zeta_{\phi}^{\overline{\theta}})(x) & \text{if } g \text{ is conjugate to some } x \text{ of } P\text{-}P' ; \end{cases}$$

$$\text{and (iii)} \quad (a^{\theta}, a^{\theta'}) = \begin{cases} 3 & \text{if } \theta' = \theta \text{ or } \overline{\theta}, \\ 2 & \text{otherwise} . \end{cases}$$

Proof: We observe that $\theta^t = \overline{\theta}$ and hence $a^{\theta^t} = a^{\overline{\theta}}$. Now

$$a_{\chi}^{\theta^t} = \frac{1}{|P|} \sum_{x} \chi(x) \cdot \overline{(\zeta_{\phi}^{1} - \zeta_{\phi}^{\theta^t})}(x) .$$

But $\chi = \chi^t$ and $(\zeta_{\phi}^{1})^t = \zeta_{\phi}^{1}$ so that we can rewrite the above as

$$a_{\chi}^{\theta^t} = \frac{1}{|P|} \sum_{x} \chi(x^t) \cdot \overline{(\zeta_{\phi}^{1} - \zeta_{\phi}^{\theta})}(x^t) = a_{\chi}^{\theta} .$$

(we have used the fact that x^t runs over P-P' as x does). So (i) holds.

Now take g as in (ii). If g_p is conjugate to no x of P-P', then we have $(g,x) = 0$ for all x, whence $(g,a^{\theta}) = 0$ by our remark just before 5.2. We suppose then that $g = x_1$ some element of P-P'. We still have $(x,x_1) = 0$ for x outside the conjugacy class of x_1. We see (from the class 2 hypothesis and the fact that 2.3 holds) that the conjugates of x_1 in P are precisely the cosets $x_1[x_1,P]$ and $x_1^t[x_1,P]$. For any x in this set, we have $(x,x_1) = (x_1,x_1)$. Since 2.4 holds, we have

$C(x_1) = O_{p'}(C(x_1))\cdot C_P(x_1)$, so that the principal-block characters of $C(x_1)$ are just the characters of $C_P(x_1)$. Then we use Brauer's formula to get $(x_1,x_1)=(\deg,\deg)^{C(x_1)}=|C_P(x_1)|$; and this value will occur in the expression for (x_1,a^θ) a total of $2|[x_1,P]|$ times. This allows us to write:

$$(x_1,a^\theta)=\frac{1}{|P|}\sum_x(\zeta_\phi^1-\zeta_\phi^\theta)(x)\cdot(x_1,x)=\frac{1}{|P|}(|P|\cdot(\zeta_\phi^1-\zeta_\phi^\theta)(x_i)+|P|\cdot(\zeta_\phi^1-\zeta_\phi^\theta)(x_1^t)$$

And (ii) follows directly from this.

We can then use (ii) in evaluating:

$$(a^\theta,a^{\theta'}) = \frac{1}{|P|}\sum_x(\overline{\zeta_\phi^1-\zeta_\phi^\theta})(x)\cdot(x,a^{\theta'}) = \frac{1}{|P|}\sum_x(2.\zeta_\phi^1-\zeta_\phi^{\theta'}-\zeta_\phi^{\overline{\theta}'})(x)\cdot(\overline{\zeta_\phi^1-\zeta_\phi^\theta})($$

Now since $(\zeta_\phi^1-\zeta_\phi^\theta)$ vanishes on P', we can consider the sum as being taken over all elements of P, so that we have the character inner product:

$$= (2.\zeta_\phi^1 - \zeta_\phi^{\theta'} - \zeta_\phi^{\overline{\theta}'}, \zeta_\phi^1 - \zeta_\phi^\theta)_P \quad .$$

Now (iii) follows immediately; so Lemma 5.2 is proved.

We consider therefore what can be the structure of $r(\phi)$ columns satisfying the requirements of 5.2(iii). We can make several quite elementary remarks. We will say that a column <u>involves</u> χ if its entry at χ is not zero. Since the a^θ are integral columns with $(a^\theta,a^\theta) = 3$, we see that each a^θ must involve precisely three characters of B, each with entry ±1. The requirement $(a^\theta,a^{\theta'}) = 2$ for $\theta' \neq \theta$ or $\overline{\theta}$ means that any two distinct columns must involve exactly two characters in common; and if χ is either of these characters, the entries

at χ in both columns must agree. We note in fact that both signs +1 and -1 must appear in each column; for we must have $(\deg, a^\theta) = 0$ by 5.2(ii), and all degree terms are strictly positive. When $r(\phi) \geq 2$, we have difference columns of the form $a^\theta - a^{\theta'}$ (θ' not θ or $\overline{\theta}$) of length 2, which therefore involve exactly two characters. Suppose then $a^\theta - a^{\theta'}$ involves χ_1 and χ_2, with χ_1 the character involved by a^θ but not by $a^{\theta'}$. Then we must have $(\deg, a^\theta - a^{\theta'}) = 0$; and since degree terms are positive, this can only happen if $a^\theta_{\chi_1} = a^{\theta'}_{\chi_2}$ and also $\chi_1(1) = \chi_2(1)$.

The above remarks show in particular that if any two columns involve a common character, then the column entries in that row agree; thus all the non-zero entries in a given row agree in sign for all the a^θ (and fixed ϕ). Since this is the case, we establish the convention of attaching a sign to the character corresponding to a row, so that we need only consider arrays in +1 and 0 satisfying 5.2(iii). This approach yields:

Lemma 5.3 The only solutions for 5.2 are given by:

(i) There are signs δ^+_ϕ, δ^-_ϕ, δ^θ_ϕ (not all the same) and characters of B so that:

	$a^\theta 1$	$a^\theta 2$	...	$a^\theta r(\phi)$
$\delta^+_\phi \cdot \chi^+_\phi$	1	1	...	1
$\delta^-_\phi \cdot \chi^-_\phi$	1	1	...	1
$\delta^\theta_\phi \cdot \chi^\theta_\phi 1$	1	0	...	0
	0	1	...	0
	⋮	⋮	⋱	
	0	0	...	1

(we have chosen some ordering $\theta_1,\ldots,\theta_{r(\phi)}$ of non-conjugate non-principal characters of $\overline{Y}$) ; and

(ii) $r(\phi) = 3$, with a sign δ and characters $\chi_1,\ldots,\chi_4$ so that

	$a^{\theta}1$	$a^{\theta}2$	$a^{\theta}3$
$\delta\chi_1$	1	1	1
$-\delta\chi_2$	1	1	0
$-\delta\chi_3$	1	0	1
$-\delta\chi_4$	0	1	1

Remark: The solution (ii) is an arithmetical accident - we will be able at a later stage to show that it cannot arise.

Proof: When $r(\phi) = 1$, the single column involves three characters, so we may as well say that solution (i) occurs - in this case the ordering $+,-,\theta$ of the three characters is not determined. When $r(\phi) = 2$, the previous remarks suffice to establish that (i) holds - that is, the characters $\chi_\phi^{\theta}1$ and $\chi_\phi^{\theta}2$ are determined by their multiplicities (although the order of χ_ϕ^{+} and χ_ϕ^{-} is not determined - but in fact, the order is not determined in any case). Suppose then $r(\phi) \geq 3$. Since each pair of distinct columns involves two common characters, each set of 3 columns must involve at least one common character. If there are three columns involving only one common character, we claim that (ii) must occur. Certainly this is the only way multiplicities can occur for the specified three columns to satisfy 5.2(iii). Our remarks after 5.2 suffice to determine the signs as given. Suppose then we try to fit a fourth column in with solution (ii). Then by

$(\deg, a^\theta i - a^\theta 4) = 0$ for $i = 1,2,3$, we see that $a^\theta 4$ must involve each of χ_2, χ_3, and χ_4. But this in view of our previous remarks would yield that $a^\theta 4$ involves exactly these three characters; and then $(a^\theta 1 - a^\theta 4, \deg) = 0$ forces the entries in $a^\theta 1$ at χ_1 and in $a^\theta 4$ at χ_4 to be equal - but $\delta \neq -\delta$. Thus (ii) can occur only when $r(\phi) = 3$. We suppose now that no three columns occur as in (ii). Then every set of three columns must involve two common characters; and the non-occurrence of (ii) forces the two characters to be common to all the columns for arbitrary $r(\phi) \geq 3$. This establishes the characters as in (i), and the distribution of signs follows from our earlier remarks. So Lemma 5.3 is established.

Our choice of notation for the characters in solution 5.3(i) indicates the manner in which we wish to associate them to the characters of $A(N,\phi)$. The order of χ_ϕ^+ and χ_ϕ^- is arbitrary, and we may wish to interchange the names so that the notation is suitable. We will refer to the set of characters described in 5.3 as $A(G,\phi)$ (in the case $r(\phi_i) = 0$, we find a different way to define a set $A(G,\phi_i)$ to correspond to $A(N,\phi_i)$ for the purposes of Hypothesis 4.1). For present needs, it is enough to remark here that the set $A(G,\phi)$ gives us precisely the characters χ of B with $(\chi_P, \zeta_\phi^1) \neq (\chi_P, \zeta_\phi^\theta)$ for some θ. Thus if $\chi \notin A(G,\phi)$, the contribution to χ_P from $A(P,\phi)$ is a multiple of $\sigma(\phi)$. If χ is in no $A(G,\phi)$, then necessarily χ vanishes on P-P' by our earlier remark that all the $\sigma(\phi)$ vanish on P-P'. The case $r = 0$ is no exception, since then we have $\sigma(\phi_i) = \zeta_{\phi_i}^1$ $(i \neq 0)$ vanishing on P-P', so that any χ of B-A(G,1) in fact vanishes on P-P'.

The calculations of Lemmas 5.2 and 5.3 are fundamental to the work of this section, as they give us a means exhibiting characters of B. We show first how we can describe the generalized decomposition columns $D_z^{\pm}$ required by Hypothesis 4.1. Thus we are led to consider the subgroup C(z): it contains N, so that it satisfies (1), (2), and (3) and hence all of Hypothesis 5.1. Now we have no a priori reason to think that C(z) < G, so that our calculations must all be valid in the case C(z) = G.

We find it easier to work first with the quotient $\bar{C} = C(z)/\langle z \rangle$. It is easily seen that $\bar{C}$ satisfies Hypothesis 5.1 just as C(z) itself does. Our main result for $\bar{C}$ is the following:

Lemma 5.4 The principal block of $\bar{C}$ consists of characters $1_{\bar{C}}$, ψ_z^-, $\psi_z^{\theta}1$, ... , $\psi_z^{\theta^{r-1}}$ with a sign δ so that:

(i) ψ_z^- is rational,

and (ii) if g is a p'-element of C(z), then $\psi_z^{\theta}(\bar{g}) = \psi_z^-(\bar{g}) + \delta$

Proof: The characters of the Sylow p-subgroup $\bar{P}$ of $\bar{C}$ are just the linear characters A(P,1) of P. So we apply Lemma 5.3 to $\bar{C}$ with $\phi = 1$ to yield a set of characters $A(\bar{C},1)$ of the principal p-block of $\bar{C}$. By the above arguments, a principal-block character outside $A(\bar{C},1)$ would vanish on $\bar{P}-\bar{1}$ and hence lie in a block of defect 0 rather than the principal block; so

$A(\bar{C},\phi)$ gives all of the principal block. Now since $r_1 \geq 4$, only solution 5.3(i) may occur, and this gives rise to the correct number of characters for 5.4. Since we know the restriction $1_{\bar{C}}|_{\bar{P}} = 1_{\bar{P}}$, we see we can take $1_{\bar{C}}$ for the character χ_1^+ given by 5.3. We then assign the names ψ_z^* to the characters χ_1^* (for * = - or θ) provided by 5.3, and δ to δ_1^-. Clearly the sign δ_1^+ corresponding to $1_{\bar{C}}$ is +1. If we write out $(\deg,a^\theta)^{\bar{C}} = 0$, we see we must in fact have the δ_1^θ all equal to $-\delta$, because $1_{\bar{C}}$ is linear. Thus we must also get (ii) if we write out $(\bar{g},a^\theta)^{\bar{C}} = 0$. From $(\deg,a^\theta)^{\bar{C}} = 0$, we conclude in particular that ψ_z^- differs in degree from all the other non-principal characters of $A(\bar{C},1)$. And since the principal block is an algebraically conjugate set, we must have ψ_z^- rational. So all of 5.4 holds.

We consider the restrictions to p'-elements of the characters of $A(\bar{C},1)$ in the light of 5.4(ii). It is evident that we may take $\{1_{\bar{C}}, \delta.\psi_z^-\}$ as a basic set for the principal block, with decomposition columns given by:

	$D_{\bar{C}}^+$	$D_{\bar{C}}^-$
$1_{\bar{C}}$	1	0
$\delta.\psi_z^-$	0	1
$\delta.\psi_z^\theta$	1	1

for all $\theta \neq 1$.

These columns give rise to a Cartan matrix:

$$\begin{pmatrix} r_1+1 & r_1 \\ r_1 & r_1+1 \end{pmatrix}$$

Now z is an element of the center Z(C(z)) so that by Brauer's work [1, Part II, Sec. 5] the set $\{1_{\overline{C}} = 1_{C(z)}, \delta\psi_z^-\}$, considered as characters of C(z),gives a basic set for the principal block of C(z); yielding a Cartan matrix:

$$(5.5) \qquad \begin{pmatrix} p(r_1+1) & pr_1 \\ pr_1 & p(r_1+1) \end{pmatrix} .$$

The point of obtaining this is that if we take generalized decomposition columns D_z^+ and D_z^- for G with respect to this basic set, the result of pre-multiplying the generalized decomposition matrix by its conjugate transpose is just the Cartan matrix (5.5). So it is these values which will be the starting point for our efforts to work back to the actual entries in the generalized decomposition columns. We have established at least the basic sets required by Hypothesis 4.1(ii).

Now the entries in the columns $D_z^{\pm}$ are defined by the equations:

$$(5.6) \qquad \chi(zg) = (D_z^+)_\chi + (D_z^-)_\chi . \psi_z^-(g)$$

where g is a p'-element of C(z). Taking some k, $1 \leq k \leq p-1$, we can see by considering the representation* underlying χ that $\chi(z^k g) = (\chi(zg))\overline{k}$. So on observing in (5.6) that ψ_z^- is rational, we conclude that:

$$D_z^{\pm}k = (D_z^{\pm})\overline{k} \qquad k = 1, \ldots, p-1.$$

In view of these relations, we will write D^+ and D^- for the decomposition columns in the remainder of the section, assuming that they refer to z, and getting the columns for z^k via the relevant Galois automorphism. Now the Cartan matrix (5.5) gives us the values of $(D^{\pm}, D^{\pm})$; but since z is not

*Here we work in the field of the $|C(z)|$-th roots of unity; and identify $\overline{k}$ with an automorphism fixing the $|C(z)|_{p'}$-th roots and taking ω to ω^k.

conjugate in G to any of its non-trivial powers by 2.2, we must have $(D^{\pm},(D^{\pm})\bar{k})=0$ for $k\neq 1$. We get in fact that $((D^{\pm})\bar{j},(D^{\pm})\bar{k})=0$ unless $j=k$, when it takes the value $(D^{\pm},D^{\pm})\bar{k} = (D^{\pm},D^{\pm})$.

Now the entries in the columns $D^{\pm}$ are algebraic integers in $\mathbb{Q}(\omega)$, that is, lie in $\mathbb{Z}[\omega]$. It will be convenient to define columns by:

$$D^{+} = \sum_{i=1}^{p-1} D_i^{+}\cdot\omega^{i} \quad \text{and} \quad D^{-} = \sum_{i=1}^{p-1} D_i^{-}\cdot\omega^{i} \ ,$$

so that the columns $D_i^{\pm}$ will have entries which are rational integers. The advantage of this procedure will be evident when we have translated (5.5) into:

Lemma 5.7

(i) $(D_i^{+},D_j^{+}) = (D_i^{-},D_j^{-}) = (1+\delta_{ij})(r_1+1)$ (δ_{ij}=Kronecker delt

and (ii) $(D_i^{+},D_j^{-}) = (D_i^{-},D_j^{+}) = (1+\delta_{ij})r_1$.

Proof: It is straightforward to verify that these values give rise to the correct values of inner products $((D^{\pm})\bar{j}, (D^{\pm})\bar{k})$; the only issue is the uniqueness of the solution. We consider only the inner products (D_i^{+},D_j^{+}). The equations $((D^{+})\bar{k},D^{+}) = \sum_{j=1}^{p-1} (D_j^{+},D^{+})\cdot\omega^{jk}$ as k varies can be viewed as simultaneous equations in the (D_j^{+},D^{+}). The matrix of coefficients has (j,k)-th entry $\omega^{jk} = (\omega^{j})\bar{k}$, the determinant of which (just the discriminant of the basis $\{\omega^{j}\}$) is not zero. Thus the values (D_j^{+},D^{+}) are uniquely determined. We fix some j, and write $(D_j^{+},(D^{+})\bar{k}) = \sum_{i=1}^{p-1} (D_j^{+},D_i^{+})\cdot\bar{\omega}^{ik}$, and apply the same argument, to conclude that the (D_j^{+},D_i^{+}) are uniquely determined. The argument is the same for all other cases, so that 5.7 is proved.

Lemma 5.7 shows that the columns $D_i^{\pm}$ are "shorter" than the $D^{\pm}$, and since they are columns of integers, we are in a much stronger position to determine what can be the actual entries in the columns. We can get started by doing this for the columns of A(G,1), where the task is not so difficult. Note that if C(z) = G, this is already done by the work of Lemma 5.4. In the general case we have:

Lemma 5.8

(i) Solution 5.3(i) occurs for A(G,1). We may take 1_G for χ_1^+, $\delta_1^+ = +1$, and we have $\delta_1^\theta = \delta_1^-$ for all $\theta \neq 1$.

(ii) The generalized character $1_G + \delta_1^- . \chi_1^- - \delta_1^- . \chi_1^\theta$ vanishes at g if g_p is conjugate to no x of P-P'.

(iii) A(G,1) is an algebraically conjugate set, and χ_1^- is rational.

(iv) For x of P-P' we have $\delta_1^* . \chi_1^*(x) = \eta_1^*(x)$ $(* = +, -, \theta)$.

(v) The characters of A(G,1) satisfy the requirements of Hypothesis 4.1(iii).

Proof: The proof of (i) and (ii) is exactly the same as the first part of the proof of Lemma 5.4; (ii) is just a re-phrasing of the assertion analogous to 5.4(ii).

For (iii), we know at least that B is an algebraically conjugate set. If $\chi \in B-A(G,1)$, then the sum of the linear constituents of χ_P is a multiple of $\sigma(1)$. Certainly A(P,1) is a conjugate set, so that any conjugate of χ preserves this condition, and hence B-A(G,1) is an algebraically conjugate set. But then so is A(G,1); and the rationality of χ_1^- follows by a degree argument as in 5.4. So (iii) holds.

Now take $x \in P-P'$. We shall use the letter n for an error term in (iv), that is, we write $\delta_1^-.\chi_1^-(x) = 1+n = \eta_1^-(n)+n$. By applying 5.2(ii) we get $\delta_1^-.\chi_1^\theta(x) = (\theta+\bar{\theta})(x) + n = \eta_1^\theta(x) + n$. We see that (iv) follows if we show that n = 0. Now by (iii) n must be a rational integer. We calculate in terms of n that:

$$(x,x)_{A(G,1)} = (x,x)^N_{A(N,1)} + 2n\Big(\sum_{\theta \text{ of } P/P'} \theta(x)\Big) + (r_1+1)n^2 = (x,x)^N_{A(N,1)} + (r_1+1)n^2$$

There are certainly elements x of P-Z(P), and for any such x, we have $(x,x) = (1/p).|P| = (x,x)^N_{A(N,1)}$ by considering (3.2); so that n = 0 for this case. If there are elements x of Z(P)-P', the case just considered guarantees at least that $\delta_1^-.\chi_1^-(x) \equiv 1 \pmod p$ so that $n \equiv 0 \pmod p$. In this case, $(x,x) = |P|$ and $(x,x)^N_{A(N,1)} = (1/p)\,|P|$. Then if $n \neq 0$, we get

$$(x,x)_{A(G,1)} \geq (1/p)|P| + p^2(r_1+1) > |P| = (x,x) ,$$

a contradiction. Thus n = 0 in all cases, and (iv) is established.

Remark: 5.8(iv) shows that the generalized character $\delta_1^*.\chi_1^*|_p - \eta_1^*|_p$ vanishes off P'. In view of our remark (ii) at the beginning of the section, this shows that no character χ of A(G,1) may appear in $A(G,\phi)$ for $\phi \neq 1$ — since all the characters of $A(P,\phi)$ must appear in χ_P with the same multiplicity.

To prove (v), we define columns $D_i = D_i^+ - D_i^-$ (i = 1,..., p-1). We obtain from Lemma 5.7 the relevant inner products:

$$(D_i, D_j) = 2 + 2.\delta_{ij} \quad .$$

So these new columns have length 4, and it is this shortness which will allow us to pin down the necessary column entries.

In particular, we have $(D^+)_{1_G} = 1$ and $(D^-)_{1_G} = 0$. Now as χ_1^- is rational, if we set $n^+ = (D^+)_{\chi_1^-}$ and $n^- = (D^-)_{\chi_1^-}$, then the numbers $n^\pm$ are rational integers. We apply 5.2(ii) the elements g with $g_p = z$ to get:

$$\delta_1^-.(D^+)_{\chi_1\theta} = 1 + \delta_1^-.n^+$$

and

$$\delta_1^-.(D^-)_{\chi_1\theta} = \delta_1^-.n^- \quad .$$

In particular, if we have $1 + \delta_1^-.n^+ \neq \delta_1^-.n^-$, then we get:

$$(D_i,D_i)_{A(G,1)} \geq r_1 + 1 > 4 = (D_i,D_i) \quad ,$$

a contradiction. So $1 + \delta_1^-.n^+ = \delta_1^-.n^-$, or $n^- = n^+ + \delta_1^-$. By (iv) we have that $\delta_1^-.\chi_1^-(x) \equiv 1 \pmod p$ so that also $\delta_1^-.\chi_1^-(z) \equiv 1 \pmod p$. But we can make the argument for (iv) in $\overline{C}$ (by just adding it to the proof of 5.4) to conclude similarly that $\delta.\psi_z^-(1) \equiv 1 \pmod p$. Since $\chi_1^-(z) = n^+ + n^-.(\delta.\psi_z^-(1))$, we can deduce that $\delta_1^-(n^+ + n^-) \equiv 1 \pmod p$. Substituting $n^- = n^+ + \delta_1^-$, we get $2\delta_1^-.n^+ + 1 \equiv 1 \pmod p$, or $n^+ \equiv 0 \pmod p$. Now we observe that:

$$(D^-,D^-)_{A(G,1)} = (D_i^-,D_i^-)_{A(G,1)} = (r_1+1)(n^-)^2 \quad .$$

From 5.7 we have that $(D_i^-,D_i^-) = 2(r_1+1)$, so that necessarily $|n^-| \leq 1$. This forces $|n^+| \leq 2$ in view of $n^- = n^+ + \delta_1^-$. Since also $n^+ \equiv 0 \pmod p$ and p is odd, we have $n^+ = 0$ and $n^- = \delta_1^-$. One then checks by setting $\phi = 1$ in the table of Lemma 3.3 that these values yield $\delta_1^*.(D^\pm)_{\chi_1^*} = (D_N^\pm)_{\eta_1^*}$. This gives part (a) of 4.1(iii), and part (b) comes from (ii). This proves (v) and completes the proof of Lemma 5.8.

So the remainder of the section will be devoted to bringing the other characters of B into line with Hypothesis 4.1. In case $r \neq 0$, the sets $A(G,\phi_i)$ $(i \neq 0)$ are defined, and it is fairly clear how we wish to define the character correspondence. When $r = 0$, we find another way of selecting two characters to play the role of $A(G,\phi_i)$.

The following congruences will be critical in the determination of column entries:

Lemma 5.9

(i) $\delta.\psi_1^-(1) \equiv \delta_1^-.\chi_1^-(1) \equiv 1 \pmod{\frac{|P:P'|}{p^s}}$;

and (ii) If $\chi \in B\text{-}A(G,1)$, then $(D_i^+ + D_i^-)_\chi \equiv 0 \pmod{p^s}$ for all i.

Proof: It follows from 5.8(iv) that $\delta_1^-.\chi_1^-|_p - 1_p$ vanishes on P-P', and hence has the form $\sum c_\phi.\sigma(\phi)$ with $c_\phi \in \mathbb{Z}$, the degree of which (cf. Lemma 3.1) is divisible by $(1/p^s)\ |P:P'|$. Similarly, work analogous to 5.8(iv) is valid for the characters of $A(C(z),1)$, so that $\delta.\psi_z^-|_p - 1_p$ has the same form. Thus (i) is proved.

Now let $\chi \in B\text{-}A(G,1)$. Then χ_P is of form $c_1\sigma(1) + \pi$ where c_1 is an integer and $\sigma(1)$ has degree $|P:P'|$, and π is a sum of non-linear characters, whose degrees are all equal to p^s. Now $\chi(z) = \sum_{i=1}^{p-1} (D_{i\chi}^+ + D_{i\chi}^-.\delta.\psi_z^-(1)).\omega^i$, so we must have each of the integer coefficients $D_{i\chi}^+ + D_{i\chi}^-.\delta.\psi_z^-(1) \equiv 0 \pmod{p^s}$. In view of (i) and $(1/p^s)\ |P:P'| \geq p^s$, we get $(D_i^+ + D_i^-)_\chi \equiv 0 \pmod{p^s}$. So Lemma 5.9 is proved.

Now from 5.8 we can observe that $(D_i,D_i)_{A(G,1)} = 2$; so that there must be some characters of B not contained in $A(G,1)$, namely, all the other characters χ of B with $(D_i^+)_\chi \neq (D_i^-)_\chi$ for

some i. We discuss the occurrence of such characters in the following lemma.

Lemma 5.10 (i) There are characters χ_i^p and χ_i^m of B(i=1,...,p-1) and signs δ^+ and δ^- so that:

$$(D_i)\chi_j^p = \delta^+.\delta_{ij}$$

$$\text{and } (D_i)\chi_j^m = \delta^-.\delta_{ij}.$$

(ii) We have $(\delta^+.\chi_i^p + \delta^-.\chi_i^m)(x) = 0$ for $x \in P-P'$ and any i.

Proof: Lemma 5.8 gives us that $(D_i,D_j)_{A(G,1)} = 2$ so that $(D_i,D_j)_{B-A(G,1)} = 2.\delta_{ij}$. Hence each D_i will involve exactly two more characters, so that there are at most 2(p-1) further characters χ with $D_\chi^+ \neq D_\chi^-$. To prove (i), we want to show that if $i \neq j$, then D_i and D_j may not involve any common character. If they did, then in order to satisfy $(D_i,D_j)_{B-A(G,1)} = 0$, there would have to be signs δ_1,δ_2 and characters χ_1,χ_2 so that

	D_i	D_j
$\delta_1\chi_1$	1	1
$\delta_2\chi_2$	1	-1

This would correspond to the situation:

	D_i^+	D_j^+	D_i^-	D_j^-
$\delta_1\chi_1$	m_1	m_2	m_1-1	m_2-1
$\delta_2\chi_2$	n_1	n_2	n_1-1	n_2+1

with $D_i^+ = D_i^-$ and $D_j^+ = D_j^-$ for all other characters of B except 1_G and χ_1^-. Then to investigate the equation $(D_i^+,D_i^+) = (D_i^-,D_i^-)$,

we may just set equal the contributions from the four characters on which the column entries may differ:

$$1 + 0 + (m_1)^2 + (n_1)^2 = 0 + 1 + (m_1-1)^2 + (n_1-1)^2$$

which reduces to $m_1 + n_1 = 1$. Comparing similarly in the equation $(D_i^+, D_j^+) = (D_i^+, D_j^-) + 1$ gives:

$$1 + 0 + m_1 m_2 + n_1 n_2 = 0 + 0 + m_1(m_2-1) + n_1(n_2+1) + 1$$

which reduces to $m_1 = n_1$. But then $m_1 = n_1 = \frac{1}{2}$, contradicting the fact that the $D_i^\pm$ are columns of integers. Thus the distinct columns involve distinct pairs of characters; and the action of the Galois automorphisms $\bar{k}$ ($k = 1, \ldots, p-1$) will establish signs δ^+ and δ^- and an ordering so that (i) holds (the order of + and - remains arbitrary).

For (ii), we have $(x,z) = 0$, which gives $(x,D^\pm) = 0$, whence $(x,D_i^\pm) = 0$ for each i and hence $(x,D_i) = 0$. And (ii) gives the equation for $(x,D_i) = 0$ in the notation of (i). So Lemma 5.10 is proved.

With this result in hand, we can accomplish most of the remaining work in:

Lemma 5.11

(i) Suppose $r=0$. We define $A(G,\phi_i) = \{\chi_i^p, \chi_i^m\}$, and the requirements of 4.1(iii)(a) are satisfied by this definition.

(ii) Suppose $r \neq 0$. Then χ_i^p, $\chi_i^m \in A(G,\phi_i)$, and $A(G,\phi_i)$ satisfies the requirements of 4.1(iii)(a) and (b).

(iii) In either case, $B = \bigcup_\phi A(G,\phi)$ disjointly.

Proof: We make two preliminary remarks:

(a) We shall use the notation $\delta^+.(D_i^+)_{\chi_i^p} = n+1$. Then we have:

	D_i^+	D_i^-
$\delta^+.\chi_i^p$	n+1	n
$-\delta^-.\chi_i^m$	n	n+1

For if we use 5.9 to write out $(D_i, D_i^+) = 2$, we obtain

$$1 + 0 + \delta^+.(D_i^+)_{\chi_i^p} + \delta^-.(D_i^+)_{\chi_i^m} = 2$$

from which $-\delta^-.(D_i^+)_{\chi_i^m} = n$ is immediate. We obtain the values in D_i^- from our knowledge of $D_i = D_i^+ - D_i^-$.

(b) We claim that $r = 0$ if and only if the characters χ_i^p, χ_i^m vanish on P-P'. For we have already remarked that if $r = 0$, then the characters of B-A(G,1) vanish on P-P'. On the other hand, if χ vanishes on P-P', then χ_P has the form $\sum_\phi c_\phi.\sigma(\phi)$ ($c_\phi \in \mathbb{Z}$); so that arguing as in 5.9(ii) yields that $(1/p^s)\,|P:P'|$ divides $(D_i^+ + D_i^-)_\chi$. If we have $\chi = \chi_i^p$, then

$$(D_i^+, D_i^+)_{A(G,1),\chi_i^p,\chi_i^m} = (r_1+1) + (n+1)^2 + n^2 \leq 2(r_1+1) = (D_i^+, D$$

or: $$(2n+1)^2 \leq |P:P'| .$$

Now as $|P:P'|(1/p^s)$ divides $(2n+1)$ by the above, we must in fact have $p^{2s} = |P:P'|$, which says exactly that $r = 0$.

To prove 5.11, we first take up the case $r \neq 0$, and consider χ_1^p. If χ_1^p were outside all $A(G,\phi_i)$, it would vanish on P-P', contrary to (b). So there is some i so that $\chi_1^p \in A(G,\phi_i)$.

Suppose that a^θ is a column for $A(G,\phi_i)$ that involves χ_1^p; it follows from 5.10(ii) that the entry in a^θ for χ_1^m is $-\delta^+\delta^-$ times the entry for χ_1^p, so that a^θ involves χ_1^m, and $\chi_1^m \in A(G,\phi_i)$. We can now say that only solution 5.3(i) may occur for $A(G,\phi_i)$ -- since the array 5.3(ii) has no two rows agreeing up to sign that could correspond to χ_1^m and χ_1^p. If $r \neq 1$, we can see also that χ_1^p and χ_1^m must be the two top rows of 5.3(i), as these are the only two that agree up to sign. In case $r = 1$, this remark is not necessary.

Now 5.10(ii) further relates the signs that may arise in $A(G,\phi_i)$. We conclude that there are signs δ_1, δ_2 arising in $A(G,\phi_i)$ via 5.3 so that:

	a^θ
$\delta^+\chi_1^p$	$\delta^+\delta_1$
$-\delta^-\chi_1^m$	$\delta^+\delta_1$
.	.
.	.
.	0
.	.
$-\delta^+\delta_1\delta_2\chi^\theta_{\phi_i}$	$-\delta^+\delta_1$
.	.
.	.
.	0
.	.

We can conclude from the relation $(D_1^+, a^\theta) = 0$ that

$$(5.12) \qquad -\delta^+\delta_1\delta_2 \cdot (D_1^+)_{\chi^\theta_{\phi_1}} = \delta^+(D_1^+)_{\chi_1^p} - \delta^-(D_1^+)_{\chi_1^m} .$$

We will now be able to determine the values of the entries in D_1^+ for the characters of $A(G,\phi_i)$ -- the calculation will also be valid for the case $r = 0$ and $i = 1$ by definition. We recall in the case $r \neq 0$ that $A(G,1)$ does not intersect

$A(G,\phi_i)$ for $i \neq 0$; and in case $r = 0$ that the χ_i^p and χ_i^m lie outside $A(G,1)$. We write $n+1$ for $\delta^+(D_1^+)_{\chi_1^p}$ as in (a); and use (a) and the relation (5.12) in:

$$(D_1^+,D_1^+)_{A(G,1),A(G,\phi_i)} = (r_1+1)+(n+1)^2+n^2+r(2n+1)^2 \leq 2(r_1+1) = (D_1^+,D_1^+)$$

or: $\quad (2r+1)(2n+1)^2 \leq |P:P'|$.

Now by 5.9(ii), p^s divides $(2n+1)$, and by definition $(2r+1)p^{2s} = |P:P'|$. We conclude we must have exactly $p^s = \delta^*(2n+1)$ ($\delta^* = \pm 1$); and all the non-zero entries in D_1^+ must lie in $A(G,1)$ and $A(G,\phi_i)$. Now if necessary we interchange the roles of $(+,-)$ and (p,m) in 5.10 (it is only necessary if $\delta^* = -1$) so that we have:

(5.13)

	D_1^+	D_1^-
$\delta^*\delta^+\chi_1^p$	a+1	a
$-\delta^*\delta^-\chi_1^m$	a	a+1
$-\delta^*\delta^+\delta_1\delta_2\chi_{\phi_i}^\theta$	p^s	p^s

For the purposes of 4.1(iii), we can now define $\delta_{\phi_i}^+ = \delta^*\delta^+$, $\delta_{\phi_i}^- = -\delta^*\delta^-$, $\chi_{\phi_i}^+ = \chi_1^p$, and $\chi_{\phi_i}^- = \chi_1^m$. In the case $r = 0$, we have $i = 1$, and we verify directly that the definition of $A(G,\phi_i)$ as $\{\chi_{\phi_i}^+, \chi_{\phi_i}^-\}$ satisfies the requirements of 4.1(iii)(a) - by comparing with (5.13).

So we again assume that $r \neq 0$; we first show that the i we have been calculating with must have been 1 in the first place. We have just established that $\delta^*\delta^+\chi_1^p(z) = ((a+1)+a.\delta.\psi_z^-(1))$. Using 5.9(ii) with $(1/p^s)|P:P'| > p^s$ since $r \neq 0$, we see that the coefficient of ω is congruent to $2a+1 = p^s \not\equiv 0 \pmod{(1/p^s)|P:P'|}$. On the other hand, the structure of $A(G,\phi_i)$ requires that

$\delta_1\chi_1^p|_p = \zeta_{\phi_i}^1 + \sum_\phi c_\phi \cdot\sigma(\phi)$ ($c_\phi \in \mathbb{Z}$), so that the coefficient of ω^i in $\delta_1\chi_1^p(z)$ is also congruent to $p^s \pmod{(1/p^s)|P:P'|}$. Since all other coefficients are divisible by $(1/p^s)|P:P'|$, we conclude that we must indeed have $i = 1$, with $\delta^* \delta^+ = \delta_1$. So in addition to the above definitions, we put $\delta^\theta_{\phi_1} = -\delta_2$ for the purposes of 4.1. We claim that similar work will hold for each of the $A(G,\phi_i)$ -- for each is conjugate to $A(G,\phi_1)$ under the action of the Galois group.

We must also show that the $A(G,\phi_i)$ are disjoint -- which follows if we show that for $i \neq j$, the columns D_i^+ and D_j^+ involve no common character of B outside A(G,1). We have established so far that the characters of A(G,1) and the $\chi^\pm_{\phi_k}$ are distinct. The work of 5.8 and 5.10 shows that the only type of intersection that might occur is that some of the $\chi^\theta_{\phi_i}$ might occur among the $\chi^\theta_{\phi_j}$. Suppose that this happens in m places. We see by algebraic conjugacy that the value of D_i^+ at $\chi^\theta_{\phi_i}$ is the same as the value of D_j^+ at $\chi^\theta_{\phi_j}$. So the column $D_i^+ - D_j^+$ can have non-zero entries only for the $\chi^\pm_{i,j}$ and the 2(r-m) characters not involved in common. Then we have:

$$(D_i^+ - D_j^+, D_i^+ - D_j^+) = 2(r_1+1) = (a+1)^2 + a^2 + (a+1)^2 + a^2 + 2(r-m)(2a+1)^2 .$$

But since $(a+1)^2 + a^2 + r(2a+1)^2 = r_1 + 1$, we must have $m = 0$. Thus the $A(G,\phi_i)$ are mutually disjoint.

This means that for any i, the column D_1^+ has non-zero entries only in A(G,1) and $A(G,\phi_i)$. Then if $\chi \in A(G,\phi_i)$, we have

$$(D^+)_\chi = (D_i^+)_\chi \cdot \omega^i = (D_i^+)_\chi \cdot \phi_i(z) .$$

Then in view of (5.13) our definitions of the $\delta^*_{\phi_i}$ and $\chi^*_{\phi_i}$ will satisfy the requirements of 4.1(iii)(a). The fact that

$(g,a^\theta) = 0$ for g a p'-element of G will yield that

$$\delta^+_{\phi_i}\cdot\chi^+_{\phi_i} + \delta^-_{\phi_i}\cdot\chi^-_{\phi_i} - \delta^\theta_{\phi_i}\cdot\chi^\theta_{\phi_i} = \delta_1\cdot\chi_i^p - \delta^+\delta^-\delta_1\cdot\chi_i^m + \delta_2\cdot\chi^\theta_{\phi_i}$$

vanishes at g. So 4.1(iii)(b) is satisfied. This completes the proof of part (ii).

Finally suppose we had $\chi \in B$ but not in any $A(G,\phi_i)$. Then $\chi_{P-P'} = 0$ by early remarks; and we have $(D^+)_\chi = (D^-)_\chi = 0$ so that also $\chi_{P'-1} = 0$. But then χ would lie in a block of defect 0 rather than B. So we must have $B = \bigcup_\phi A(G,\phi)$, disjointly by previous remarks. This completes the proof of Lemma 5.11.

To end the work of this section, we need only note that the correspondence associating η^*_ϕ and χ^*_ϕ is 1:1, and thus our construction satisfies all the requirements of Hypothesis 4.1. So by the work of Section 4, the proof of Theorem A is complete.

Section 6. Deduction of Theorem B

We now operate under Hypothesis (2B), that P is abelian but not cyclic. We require a theorem of Swan dealing with local control of group cohomology [18]:

> Let Q be a Sylow p-subgroup of the finite group H, and suppose H is p-normal (that is, Z(Q) is weakly closed in Q with respect to H). Then the p-part of the Schur multiplier of H is isomorphic to the p-part of the Schur multiplier of $N_H(Z(Q))$.

A treatment of the theory of Schur multipliers can be found in Huppert [10].

To prove Theorem B, we first claim that the p-part of the multiplier of N is not trivial. Indeed, as P is not cyclic, we can write $P = \langle a_1 \rangle \times \langle a_2 \rangle \times A$ (where A may contain other cyclic direct factors). We construct the group $Q = B \times A$ where $B = \langle b_1, b_2 : |b_i| = |a_i| (i=1,2)\ ,\ [b_1,b_2] = z,\ z^p = [b_1,z] = [b_2,z] = 1 \rangle$. Then Q is a central extension of P by $\langle z \rangle$. We define $N^* = \langle t \rangle . (Q \times O_{p'}(N))$ by having t invert $Q/\langle z \rangle$ and centralize z. Since t already has a natural action on A and on $O_{p'}(N)$, we can do this by simply letting t invert the generators b_1 and b_2 of B. Then N^* is a suitable extension of N, so that the p-part of the multiplier of N is not trivial.

Now as P is abelian, Z(P) = P, so that G is certainly p-normal. Then by Swan's theorem, we can find an extension H of G by a cyclic group $\langle z \rangle$ of order p, such that $z \in H' \cap Z(H)$. Since the extension is non-splitting, it does not split when restricted to a Sylow p-subgroup Q of H (by a theorem of Gaschütz [8, p.245]). Since $H/\langle z \rangle \cong G$, we must have $N(Q)/\langle z \rangle \cong N(P)$,

so that $|N_H(Q)/Q.C_H(Q)| = 2$. But Q does not split over z, so that $z \in Q'$; and $Q/\langle z\rangle$ abelian forces $Q' = \langle z\rangle$ of order p. We assumed that $O^p(G) = G$, so that also $O^p(H) = H$. Then H satisfies (1), (2) and (2A), and (3), so that by Theorem A we can conclude $H = O_{p'}(H).N_H(Q)$. And factoring out $\langle z\rangle$ gives $G = O_{p'}(G).N(P)$. So Theorem B holds.

Section 7. Proof of Theorem C

In this section we assume that P has class 2 with $|P'|>p$. We work as always by induction on $|G|$, but we can now make use of Theorem A if we need to deal with a section satisfying the relevant hypotheses.

The work of this section will make use of the results of Feit in Section 7 ("Isometries") of Chapter V of his Yale notes [5]. This work is essentially a generalization of the techniques of Suzuki to take account of the "Brauer correspondence" between the p-blocks of a group, and the p-blocks of certain subgroups. We note here that Reynolds [12] does similar work which could equally well be applied in our case. We begin this section by presenting without proof as much of the theory of Feit as is necessary for present purposes. Some of our previous notation has been arranged to agree with Feit's, and we continue to do this when possible. In particular, we will follow the numbering of his results from Chapter V, Section 7 of [5] .... these will be results (7.1) through (7.6) of this section. Numbering of intermediate lemmas then required for the proof of Theorem C will be consecutive with these. We will occasionally simplify Feit's statements when we do not require the full generality.

We first establish the following conventions as in [5]:

Notation: A the set of p-singular elements of N;

$M(N,A)$ the generalized characters of N vanishing on $N-A$;

$M(N,A:b)$ the elements of $M(N,A)$ whose constituents come from b;

and $M_p(N,A)$ the elements α of $M(N,A)$ satisfying:

Whenever $g \in A$ and $g_1 \in O_{p'}(C_N(g_p))$, we have $\alpha(gg_1) = \alpha(g)$.

We observe that under our hypotheses, $M_p(N,A) = M(N,A;b)$. For $P \lhd N$ and $g \in A$ forces $g_p \in P$. Let Q be a Sylow p-subgroup of $C_N(g_p)$. Then certainly $Q > P'$, so that if $t \in C_N(g_p)$ (that is, $g_p \in P'$) we have $[Q,t]$ is a p-group because of 2.1, so that $t \notin O_{p'}(C_N(g_p))$. Thus in any case, $O_{p'}(C_N(g_p)) \lhd O_{p'}(N)$ by the structure of N. But $O_{p'}(N) \lhd C(P) \lhd C_N(g_p)$, so that we must in fact have $O_{p'}(C_N(g_p)) = O_{p'}(N)$. Now the characters of b are precisely the characters of N trivial on $O_{p'}(N)$, and this will give us $M_p(N,A) = M(N,A;b)$ in view of the definitions.

We go on to Feit's results:

(7.1)(Hypothesis) (i) N controls the fusion in G of the p-elements of A,

and (ii) for g a p-element of A, we have $C(g) = O_{p'}(C(g)).C_N(g)$.

We see we have (i) already by Lemma 2.3, and part of (ii) by 2.4. We deal with the rest of (ii) in Lemma 7.8. We will not require the hypothesis (7.2) of [5], so we go directly on to:

(7.3) Suppose (7.1) is satisfied. If g of G has $g_p \in A$, then we can write

$$g_{p'} = g_1 g_2 \quad \text{where } g_1 \in C_N(g_p) \text{ and } g_2 \in O_{p'}(C(g_p)).$$

If also $g_p^h \in A$ for some h of G, and $g_{p'}^h = g_1' g_2'$ as above, then

$$\alpha(g_p g_1) = \alpha(g_p^h g_1') \qquad \text{for any } \alpha \text{ of } M_p(N,A).$$

This enables us to follow Feit in defining our character isometry τ:

Definition Let $\alpha \in M_p(N,A)$. We define α^τ on G by:

$$\alpha^\tau(g) = \begin{cases} 0 & \text{if no conjugate of } g_p \text{ lies in } A, \\ \alpha(g_p^h g_1) & \text{if } g_p^h \in A \text{ and we write } g_{p'}^h = g_1 g_2 \text{ as in (7.3).} \end{cases}$$

It follows from (7.3) that α^τ is a well-defined class function on G, with $(\alpha^\tau)_N = \alpha$. We get the standard sorts of properties:

(7.4) (i) if $\alpha \in M_p(N,A)$ and $\chi \in B$, then $(\alpha^\tau,\chi)_G = (\alpha,\chi_N)_N$,

and (ii) if $\alpha_1,\alpha_2 \in M_p(N,A)$ then $(\alpha_1^\tau,\alpha_2^\tau)_G = (\alpha_1,\alpha_2)_N$.

It follows that τ defines an isometry from the complex space spanned by $M_p(N,A)$ into the space of class functions on G.

We need some further definitions from [5]. Let Q be a defect group of a block C of some subgroup H. The kernel of C is the intersection of the characters of C. The germ of C with respect to Q is the unique block c of $N_H(Q)$ satisfying $c^H = C$ (the superscript indicates the Brauer mapping; cf. II, Sec. 7 of [5]).

Remarks: P is a defect group both for b and for B. Of course $O_{p'}(N)$ is the kernel of b. The germ of B with respect to P is just b; and b is its own germ with respect to P.

These facts give us a very special case of Feit's more general situation:

(7.5) Let $\hat{b}$ be the germ of b with respect to D, D a defect group of b. Then there is a unique block $\tilde{b}$ of $N_N(D).C(D)$ so that $\hat{b}$ and $\tilde{b}$ agree as blocks of the quotient $N_N(D)/O_{p'}(N_N(D))$.

In our case, we can take D = P, and then $N = N_N(D)$, so that $b = \hat{b} = \tilde{b}$; that is, the correspondence is identical.

(7.6) Let $\tilde{b}$ be defined for b as in (7.5), and $\{b_i\}$ be the blocks of N with $\tilde{b}_i^G = B$. Let $\alpha \in M_p(N,A) \cap M(N,A; \bigcup_i b_i)$.

Then $\alpha^\tau = (\alpha^G)^B$,

where the superscript B indicates the sum of the B-constituents of the generalized character α^G of G induced by α.

Now by Brauer's Third Main Theorem (cf. [5, V.5.4]) only b among the blocks of N satisfies $b^G = B$, and we just observed that $b = \tilde{b}$ in the notation of (7.5). We can take $b = \bigcup_i b_i$ in (7.6), which then tells us that for $\alpha \in M_p(N,A)$, we must have $\alpha^\tau = (\alpha^G)^B$, that is, α^τ is a generalized character of G with constituents from B. It is by working with characters α of sufficiently small weight and using (7.4) that we can begin to describe the characters of B.

Finally, we require the standard argument from the theory of "special classes" (see [9], for instance) to describe actual character values once we have applied (7.6).

(7.7) Assume we have an isometry τ satisfying (7.4) and (7.6); assume $7.9: (g,h) = (g,h)^W$ for $g,h \in A$, and that we are given elements $\alpha_1, \ldots, \alpha_n$ of $M_p(N,A)$ which form a basis for the $\mathbb{C}$-space spanned by $M_p(N,A)$, as in Lemma 7.10. Define coefficients by $\alpha_i = \sum_\eta a_{i\eta}.\eta$ and $\alpha_i^\tau = \sum_\chi b_{i\chi}.\chi$. Then for each g of A, there are complex numbers c_{gi} determined uniquely by $\eta(g) = \sum_i c_{gi}.a_{i\eta}$, and also satisfying $\chi(g) = \sum_i c_{gi}.b_{i\chi}$.

Proof: Since the α_i lie in $M_p(N,A)$, the $a_{i\eta}$ are rational integers; and then by (7.6) so are the $b_{i\chi}$. We reproduce, mutatis mutandis, the entire calculation from [9].

We observe in 7.10 that the complex space spanned by $M_p(N,A)$ is just the space of all class functions of $N/O_{p'}(N)$ vanishing on N-A. For $g \in A$, the class functions $\sum_\eta \eta(g).\eta$ vanishes in fact off the class of g. So there are uniquely determined c_{gi} with:

$$\sum_\eta \eta(g).\eta = \sum_i c_{gi}.\alpha_i = \sum_{\eta,i} c_{gi}a_{i\eta}.\eta \quad .$$

Thus $\eta(g) = \sum_i c_{gi}.a_{i\eta}$. Conversely, if we are given d_{gi} with $\eta(g) = \sum_i d_{gi}.a_{i\eta}$, it follows that $\sum_\eta \eta(g).\eta = \sum_i c_{gi}.\alpha_i$, whence $d_{gi} = c_{gi}$; so that the c_{gi} are uniquely determined. It remains to prove:

(*) $\chi(g) = \sum_i c_{gi} \cdot b_{i\chi}$.

We use the fact that $\alpha_i^\tau|_N = \alpha_i$, which we express in the form:

(A) $\sum_\chi b_{i\chi} \cdot \chi(g) = \sum_\eta a_{i\eta} \cdot \eta(g)$, for $g \in A$.

We also have 7.9, which we express in the more usual way:

(B) $\sum_\chi \chi(g)\overline{\chi}(h) = \sum_\eta \eta(g)\overline{\eta}(h)$, for $g,h \in A$.

We observe first that (*) satisfies (A) and (B). Indeed, substituting it into (A):

$$\sum_\chi b_{i\chi} \cdot \chi(g) = \sum_{\chi,j} b_{i\chi} c_{gj} b_{j\chi} = \sum_j c_{gj} \sum_\chi b_{i\chi} b_{j\chi} .$$

Now (7.4) tells us that $(\alpha_i^\eta, \alpha_j^\tau) = (\alpha_i, \alpha_j)_N$; in the present notation, this is just $\sum_\chi b_{i\chi} b_{j\chi} = \sum_\eta a_{i\eta} a_{j\eta}$. So we can use this fact in the previous expression to get:

$$\sum_\chi b_{i\chi} \cdot \chi(g) = \sum_j c_{gj} \sum_\eta a_{i\eta} a_{j\eta} = \sum_{\eta,j} a_{i\eta} c_{gj} a_{j\eta} = \sum_\eta a_{i\eta} \cdot \eta(g) ,$$

completing (A). Now using (*) in (B):

$$\sum_\chi \chi(g)\overline{\chi}(h) = \sum_{\chi,j,k} (c_{gj} b_{j\chi})(\overline{c_{hk}} b_{k\chi})$$

$$= \sum_{j,k} c_{gj}\overline{c_{hk}} \sum_\chi b_{j\chi} b_{k\chi} = \sum_{j,k} c_{gj}\overline{c_{hk}} \sum_\chi a_{j\eta} a_{k\eta}$$

$$= \sum_{\eta,j,k} (c_{gj} a_{j\eta})(\overline{c_{hk}} a_{k\eta}) = \sum_\eta \eta(g)\overline{\eta}(h) ,$$

as desired. We wish then to show that (*) provides a unique solution to these relations. So we introduce an error term:

(**) $\chi(g) = \sum_k c_{gk} b_{k\chi} + \epsilon_{\chi g}$;

and require that (**) provide a solution of (A) and (B). If (A) holds, we must have:

$$\sum_\eta a_{i\eta} \cdot \eta(g) = \sum_\chi b_{i\chi} \left(\sum_k c_{gk} b_{k\chi} + \epsilon_{\chi g} \right) .$$

Now as (*) solves (A), the above reduces to:

$$\sum_\eta a_{i\eta} \cdot \eta(g) = \sum_\eta a_{i\eta} \cdot \eta(g) + \sum_\chi b_{i\chi} \epsilon_{\chi g} ,$$

from which we conclude that $\sum_\chi b_{i\chi} \epsilon_{\chi g} = 0$. We then substitute (**) into (B):

$$\sum_\eta \eta(g)\overline{\eta}(h) = \sum_\chi \left(\sum_i c_{gi} b_{i\chi} + \epsilon_{\chi g} \right) \left(\sum_j \overline{c_{hj}} b_{j\chi} + \overline{\epsilon_{\chi h}} \right) .$$

The expression on the right gives four terms:

$$\sum_{\eta} \eta(g)\overline{\eta}(h) = \sum_{\chi,i,j} (c_{gi}b_{i\chi})(\overline{c_{hj}}\,\overline{b_{j\chi}}) + \sum_{\chi,i} c_{gi}b_{i\chi}\overline{\epsilon_{\chi h}}$$

$$+ \sum_{\chi,j} \overline{c_{hj}}b_{j\chi}\epsilon_{\chi g} + \sum_{\chi} \epsilon_{\chi g}\overline{\epsilon_{\chi h}} \quad .$$

Since (*) solves (B), the first term is just $\sum_{\eta}\eta(g)\overline{\eta}(h)$. We have $\sum_{\chi} b_{i\chi}\overline{\epsilon_{\chi h}} = \overline{(\sum_{\chi} b_{i\chi}\epsilon_{\chi h})} = 0$ and $\sum_{\chi} b_{j\chi}\epsilon_{\chi g} = 0$ from the previous calculation, so that the second and third terms vanish. Consequently we must have $\sum_{\chi}\epsilon_{\chi g}\overline{\epsilon_{\chi h}} = 0$. If we set $g = h$, we conclude we must have each $\epsilon_{\chi g} = 0$; this reduces (**) to (*), and completes the proof of (7.7).

We can now start the proof of Theorem C.

Lemma 7.8 We may assume by induction that $C(z) = N.O_{p'}(C(z))$ for each z.

Proof: Suppose we have $C(z) = G$, so that $z \in Z(G)$. Then let $\overline{G} = G/\Omega_1(\langle z\rangle)$ with Sylow p-subgroup $\overline{P}$. Since $|P'| > p$, we have $|\overline{P}'| \neq 1$. Thus $\overline{G}$ will satisfy (1), (2), and (3); and in fact must have either (2A) or (2C). Thus by Theorem A or by induction in the present case, we have $\overline{G} = O_{p'}(\overline{G}).N(\overline{P})$. If K is the pre-image of $O_{p'}(\overline{G})$, we have $z \in Z(G)$ so that $K = \Omega_1(\langle z\rangle) \times O_{p'}(G)$. Then in fact $G = O_{p'}(G).N$. So we may henceforth assume that $C(z) < G$. Then C(z) contains N, and hence satisfies (1), (2), and (2C), and (3); so that by induction we have $C(z) = O_{p'}(C(z)).N$, as required.

We observe that we have now satisfied Hypothesis (7.1) for all the p-elements of A, that is, for all the p-elements of N.

In contrast to Section 5, we require in this section only basic information about column inner products:

Lemma 7.9 For $g,h \in A$, we have $(g,h) = (g,h)^N$.

Proof: We observe first that if g and h lie in different p-sections, both these inner products vanish; so we may as well assume $g_p = h_p$. We demonstrate an easy extension

of Brauer's formula in this case, namely that:

$$(g,h) = (g_{p'},h_{p'})^{C(g_p)} .$$

For let ϕ_i be the irreducible Brauer characters of the principal block of $C(g_p)$, with generalized decomposition columns d_i correspondingly. Then:

$$(g,h) = \sum_{\chi} \chi(g)\overline{\chi}(h) = \sum_{\chi,i,j} (d_{i\chi}\cdot\phi_i(g_{p'}))(\overline{d_{j\chi}}\cdot\overline{\phi_j}(h_{p'}))$$

$$= \sum_{i,j} \phi_i(g_{p'})\overline{\phi_j}(h_{p'}) \sum_{\chi} d_{i\chi}\overline{d_{j\chi}} .$$

Now the given summation over χ is equal to the Cartan invariant c_{ij}; which is again equal to $\sum_{\psi} D_{i\psi}D_{j\psi}$, where D_i is the ordinary decomposition column corresponding to ϕ_i, and ψ runs over the ordinary characters of the principal block of $C(g_p)$. So:

$$(g,h) = \sum_{i,j} \phi_i(g_{p'})\overline{\phi_j}(h_{p'}) \sum_{\psi} D_{i\psi}D_{j\psi}$$

$$= \sum_{\psi,i,j} (D_{i\psi}\cdot\phi_i(g_{p'}))(D_{j\psi}\cdot\overline{\phi_j}(h_{p'}) = \sum_{\psi} \psi(g_{p'})\overline{\psi}(h_{p'}) ,$$

which is $(g_{p'},h_{p'})^{C(g_p)}$, as we asserted.

To prove 7.9, first let g_p be of the form x. Then $g_{p'},h_{p'} \in O_{p'}(C(g_p))$ by 2.4, so that $(g,h) = (g_{p'},h_{p'})^{C(g_p)} = (deg,deg)^{C(g_p)}$. On the other hand, $g_{p'}$ and $h_{p'}$ lie in N; so they lie in $O_{p'}(C_N(g_p))$, which is equal to $O_{p'}(N)$ by the remark before (7.1). Thus:

$$(g,h)^N = (g_p,g_p)^N = |C_N/O_{p'}(N)(g_p)| = |C_P(g_p)| .$$

Now from 2.4, $C(g_p) = O_{p'}(C(g_p)).C_P(g_p)$, so that also $(deg,deg)^{C(g_p)} = |C_P(g_p)|$; and 7.9 is handled in this case.

Now let g_p be of the form z. We observe in view of 7.8 that the principal-block characters of $C(g_p)$, restricted to N, are just the characters of b (cf. V(6.4) of [5]). Then

$$(g,h) = (g_{p'},h_{p'})^{C(g_p)} = (g_{p'},h_{p'})^N = (g,h)^N .$$

This completes the proof of 7.9.

We exhibit now the set of generalized characters to which we will be applying (7.6) and (7.7).

Lemma 7.10 The generalized characters of the following forms give a basis for the complex space spanned by M(N,A;b):

$$(7.10)\quad \begin{aligned} &\text{(i)} && \eta_\phi^+ + \eta_\phi^- - \eta_\phi^\theta && (r(\phi) \neq 0)\ ; \\ &\text{(ii)} && (1_N - \eta_1^-) - (\eta_\phi^+ - \eta_\phi^-) && \\ &\text{(iii)} && p^{s(\phi)}.(1_N + \eta_1^-) - (\eta_\phi^+ + \eta_\phi^-) && \end{aligned} \Bigg\} \quad \phi \neq 1 \ .$$

Proof: That these characters vanish off A follows directly from the table of values (3.3). One verifies via the linear independence of the characters of b that the above generalized characters are independent. We observe that there are $\sum_\phi r(\phi) + \sum_{\phi \neq 1} 2$ of them. On the other hand, there are only 2 p'-classes (corresponding to 1 and t) in $N/O_{p'}(N)$, which itself has $\sum_\phi (2 + r(\phi))$ classes (since this is the number of irreducible characters). Thus the set in (7.10) must be a maximal linearly independent set in $M(N/O_{p'}(N),A) = M(N,A;b)$ and so must form a basis of the space spanned by M(N,A;b). So 7.10 is proved.

The technically-minded reader will notice that if we had defined A to be just the p-singular elements of N with p-part of the form x, we could apply the above theory using only the elements of 7.10(i) for the relevant basis to discuss the values of the characters of B on P-P'; and this taetic would work equally well in Section 5. The deficiency of the approach is that for $\phi \neq 1$, the arithmetic generates an exceptional solution exactly analogous to 5.3(ii). Now in Section 5, the shortness of certain columns allowed us to eliminate the possibility in a natural way; but in this general case (and working with either rows or columns), the exception does not seem to eliminate itself if we restrict attention to P-P' instead of considering all of P-1.

Even when we have defined A to be all the p-singular elements of N, the exceptional case is so inconvenient that we do not apply (7.6) directly to the generalized characters of 7.10 . Instead we do our preliminary inductions from the following generalized characters:

(7.11) (i) $1_N + \eta_1^- - \eta_1^\theta$;

(ii) $\eta^*_{\phi_1} - \eta^*_{\phi_2}$ $\quad * = +$ or $-$; $s(\phi_1) = s(\phi_2)$;

and (iii) $\eta^\theta_{\phi_1} - \eta^{\theta'}_{\phi_2}$ $\quad s(\phi_1) = s(\phi_2); r(\phi_1), r(\phi_2) \neq 0$.

We check as usual from (3.3) that these are in fact in $M(N,A;b)$. To deal with types (ii) and (iii), it is convenient to introduce notation:

Definition $S(\phi) = \{\phi' \text{ of } P' : s(\phi') = s(\phi)\}$,

Since $S(1) = \{1\}$, we are not interested in type (ii) for $\phi = 1$ (although we do get type (iii) with $\phi_1 = \phi_2 = 1$ and $\theta \neq \theta'$). For $\phi \neq 1$, the complex conjugate $\bar{\phi} \neq \phi$ lies in $S(\phi)$, so that $|S(\phi)| \geq 2$. Hence there will always be non-zero generalized characters of the forms (ii) and (iii) for $\phi \neq 1$.

There are several advantages of this approach. When $\phi \neq 1$, we do not need to worry (as we would in 7.10) about what to do when $r(\phi) = 0$. Secondly, we work in (7.11)(ii) and (iii) with generalized characters of weight 2. In this case, the "exceptional characters" calculation is standard (see 4.5.4 of [7]) and this allows us to produce the correct number of characters of B with no further uniqueness arguments. Once we have done the work for (7.11), we can get sufficient information on the restrictions of the characters of B to N to show how they must fit into the scheme of 7.10.

We first consider the characters of forms (i) and (iii) for $\phi = 1$. The usual calculation referred to above gives a sign δ_1^θ independent of θ and characters χ_1^θ so that $(\eta_1^\theta - \eta_1^{\theta'})^\tau = \delta_1^\theta(\chi_1^\theta - \chi_1^{\theta'})$. For type (i), we abbreviate $1_N + \eta_1^- - \eta_\phi^\theta$ by $\alpha(\theta)$. By the reciprocity law (7.4)(i), we get a contribution of 1_G exactly in each $\alpha(\theta)^\tau$. Now $\alpha(\theta) - \alpha(\theta') = \eta_1^{\theta'} - \eta_1^\theta$. This means that each $\alpha(\theta)^\tau$ must involve one or two of the $\chi_1^{\theta'}$ (no more, since $(\alpha(\theta)^\tau, \alpha(\theta)^\tau) = 3$). But the case where each involves two of the $\chi_1^{\theta'}$ can occur only when $r(1) = 3$ (for the same arithmetical reasons that cause solution 5.3(ii) to arise), which is never the case since $r(1) \geq 4$. Thus we conclude there must be another character χ_1^- and sign δ_1^- so that $\alpha(\theta)^\tau = 1_G + \delta_1^- \cdot \chi_1^- - \delta_1^\theta \cdot \chi_1^\theta$. Since $\alpha(\theta)$ has degree 0, it follows by the type of argument we used in Section 5 that $\delta_1^\theta = \delta_1^-$. As we did for N, we shall continue to write 1_G for the trivial character, although it would more properly be χ_1^+ in conformity with the notation. We write A(G,1) for the set of characters we have just exhibited. Then we have the conditions:

(7.12) (i) The characters of A(G,1) are exactly those χ of B with:

$$(\chi_N, 1_N + \eta_1^-) \neq (\chi_N, \eta_1^\theta) \quad \text{for some } \theta .$$

(ii) χ_1^θ is uniquely determined by satisfying (for some constant k_1^θ):

$$(\chi_1^\theta|_N , \eta_1^{\theta'}) = \begin{cases} k_1^\theta + \delta_1^\theta & \text{if } \theta' = \theta , \\ k_1^\theta & \text{otherwise .} \end{cases}$$

We now consider the forms (ii) and (iii) of (7.11) when $\phi \neq 1$. We have non-zero generalized characters (for $\phi \neq \phi'$ of $S(\phi)$) of the forms (ii) and (iii); and when $r(\phi) \geq 2$, there are also

some non-zero forms (iii) with $\phi = \phi_1 = \phi_2$. The usual calculation produces signs and characters so that:

$$(\eta^*_\phi - \eta^*_{\phi'})^\tau = \delta^*_\phi(\chi^*_\phi - \chi^*_{\phi'}) \qquad * = + \text{ or } - \text{ and } \phi \neq \phi' ,$$

and $$(\eta^\theta_\phi - \eta^{\theta'}_{\phi'})^\tau = \delta^\theta_\phi(\chi^\theta_\phi - \chi^{\theta'}_{\phi'}) \qquad \text{either } \phi\neq\phi' \text{ or } \theta\neq\theta' .$$

The signs are independent of ϕ' in $S(\phi)$; and if $r(\phi) \geq 2$, independent of superscript θ' as well. Now in the case that $|S(\phi)| = 2$, the signs $\delta^\pm_\phi$ are not uniquely determined; in fact, if we replace δ^*_ϕ by $-\delta^*_\phi$, and interchange the definitions of χ^*_ϕ and $\chi^*_{\overline{\phi}}$, then the conditions arising from (7.4) and (7.6) are still satisfied. Similar considerations apply to the case $|S(\phi)| = 2$ and $r(\phi) = 1$ with respect to the signs δ^θ_ϕ and characters χ^θ_ϕ. The ambiguity will not alter the validity of our calculation; and we may wish later to perform such a switch for notational convenience.

We obtain conditions analogous to (7.12) for these characters. First let * = + or -. Then:

(7.13) (i) If $\chi \in B$ is not one of the $\chi^*_\phi, (\phi' \in S(\phi))$ then $(\chi_N, \eta^*_{\phi'})$ is independent of $\phi' \in S(\phi)$.

(ii) χ^*_ϕ is uniquely determined by satisfying (for an integer constant k^*_ϕ):

$$(\chi^*_\phi|_N, \eta^*_{\phi'}) = \begin{cases} k^*_\phi + \delta^*_\phi & \text{if } \phi' = \phi , \\ k^*_\phi & \text{any other } \phi' \text{ of } S(\phi) . \end{cases}$$

Similarly:

(7.14) (i) If $\chi \in B$ is not one of the $\chi^{\theta'}_{\phi'}$ $(\phi' \in S(\phi))$ then $(\chi_N, \eta^{\theta'}_{\phi'})$ is independent of (θ', ϕ') for $\phi' \in S(\phi)$.

(ii) χ^θ_ϕ is uniquely determined by satisfying (for an integer constant k^θ_ϕ):

$$(\chi_\phi^\theta|_N \ , \ \eta_{\phi'}^{\theta'}) \ = \ \begin{cases} k_\phi^\theta + \delta_\phi^\theta \text{ if } (\theta',\phi') = (\theta,\phi) \ , \\ k_\phi^\theta \qquad \text{any other } (\theta',\phi') \text{ with } \phi' \text{ in } S(\phi) \ . \end{cases}$$

We observe that this procedure has associated to each character of b a character of B, in the way described by the notation in Hypothesis 4.1. We start to justify this by showing that the correspondence is 1:1 -- that is, that characters we have given distinct names in the above are in fact distinct. We show later that the correspondence covers all of B.

Thus suppose we have $\chi = \chi_{\phi_1}^* = \chi_{\phi_2}^{*'}$ with $(*,\phi_1) \neq (*',\phi_2)$. The method of construction above guarantees a certain amount of distinctness, namely that the following combinations could not occur:

(i) $\phi_1 = \phi_2 = 1$;

(ii) $S(\phi_1) = S(\phi_2)$ and $* = *' = +$ (or $-$) ;

or (iii) $S(\phi_1) = S(\phi_2)$, $*$ and $*'$ are of the form θ and θ'.

In particular we may assume that $\phi_1 \neq 1$. Take ϕ_1' a distinct algebraic conjugate of ϕ_1. Then we have $\ker \phi_1 = \ker \phi_1'$ so that $\eta_{\phi_1'}^*$ and hence $\chi_{\phi_1'}^*$ are defined; with $\chi \neq \chi_{\phi_1'}^*$. So we have

$$\delta_{\phi_1}^* (\chi_{\phi_1}^* - \chi_{\phi_1'}^*)_N \ = \ \eta_{\phi_1}^* - \eta_{\phi_1'}^* \ . \tag{7.15}$$

So suppose that also $\phi_2 \neq 1$ and $*' = +$ or $-$. Then (7.15) yields

$$(\chi_N, \eta_{\phi_2'}^{*'}) \ = \ (\chi_{\phi_1'}^*|_N, \eta_{\phi_2'}^{*'}) \quad \text{for any } \phi_2' \text{ of } S(\phi_2)$$

This contradicts the unique determination of $\chi = \chi_{\phi_2}^{*'}$ by multiplicities in (7.13). Suppose on the other hand that $r(\phi_2) \neq 0$ and $*' = \theta$. Then (7.15) yields:

$$(\chi_N \ , \eta_{\phi_2'}^{\theta'}) = (\chi_{\phi_1'}^*|_N \ , \ \eta_{\phi_2'}^{\theta'}) \text{ for any } (\theta',\phi_2') \text{ with } \phi_2' \text{ in } S(\phi_2).$$

This contradicts the unique determination of $\chi = \chi^{\theta}_{\phi_2}$ by multiplicities in either (7.12) (in case $\phi_2 = 1$) or (7.14). We certainly cannot have 1_G appearing as any character other than χ^{+}_1. So the only case we are left with is $\phi_2 = 1$ and $*' = -$. In this situation, we know we have δ^{-}_1 so that:

$$\delta^{-}_1(\chi^{-}_1 - \chi^{\theta}_1)_N = \eta^{-}_1 - \eta^{\theta}_1 .$$

We now consider the possibilities for $*$. If $* = +$ or $-$, then by the previous line,

$$(\chi_N, \eta^{*}_{\phi_1'}) = (\chi^{\theta}_1|_N, \eta^{*}_{\phi_1'}) \quad \text{for any } \phi_1' \text{ of } S(\phi_1) ,$$

If $* = \theta'$, then

$$(\chi_N, \eta^{\theta''}_{\phi_1'}) = (\chi^{\theta}_1|_N, \eta^{\theta''}_{\phi_1'}) \text{ for any } (\theta'', \phi_1') \text{ with } \phi_1' \text{ in } S(\phi_1) ,$$

Thus we get a contradiction to the unique determination of $\chi = \chi^{*}_{\phi_1}$ by multiplicities in either (7.13) or (7.14). This proves the 1:1 nature of our correspondence.

At this point, we have enough information to show that the characters we have produced appear in the α^{τ} for α of the forms in 7.10 in exactly the way we want them to . For the characters of 7.10(i) with $\phi = 1$, this is already done. So let α be of form 7.10(i) with $\phi \neq 1$. Then $(\alpha^{\tau}, \alpha^{\tau}) = 3$. Now we observe that $\alpha^{\tau}|_N = \alpha$ satisfies 7.13(i), and so this condition tells us that there must be ϕ' and ϕ'' in $S(\phi)$ so that $\chi^{+}_{\phi'}$ and $\chi^{-}_{\phi''}$ are involved in α^{τ}. In case $|S(\phi)| \geq 3$, we can conclude that $\phi' = \phi'' = \phi$; while in case $|S(\phi)| = 2$, it may be $\overline{\phi}$ that appears, so that we can re-define notation as described earlier in order that in the new system χ^{+}_{ϕ} and χ^{-}_{ϕ} are the characters that appear. Similarly, when $r(\phi) \neq 1$, we see from (7.14)(i) and (ii) that χ^{θ}_{ϕ} also appears in α^{τ}; in case $r(\phi) = 1$ and $|S(\phi)| = 2$, we may have to interchange χ^{θ}_{ϕ} and $\chi^{\theta}_{\overline{\phi}}$ in order for the notation to be correct. Once this is done, these are the three characters that appear in α^{τ}, and the

conditions (7.13)(ii) and (7.14)(ii) determine the signs so that $\alpha^\tau = \delta_\phi^+ \cdot \chi_\phi^+ + \delta_\phi^- \cdot \chi_\phi^- - \delta_\phi^\theta \cdot \chi_\phi^\theta$.

Next consider a character α of form (7.10)(iii). We get a constituent $p^{s(\phi)} \cdot 1_G$ in α^τ by applying (7.4)(i). Consideration of (7.13)(i) shows that two of the constituents must be of the form $\chi_{\phi'}^+$ and $\chi_{\phi''}^-$ for ϕ' and ϕ'' in $S(\phi)$. And we must still get a contribution of $p^{s(\phi)} \cdot \eta_1^-$ in the restriction $\alpha^\tau|_N = \alpha$. We observe that any character χ outside $A(G,1)$ satisfies $(\chi_N, 1_N + \eta_1^- - \eta_1^\theta) = 0$ whereas α itself does not. It follows then that α^τ must involve a generalized character β with constituents from $A(G,1)$ and of weight $\leq p^{2s(\phi)}$, such that β_N has the form $p^{s(\phi)} \cdot \eta_1^- + \gamma$, where $(\gamma, 1_N + \eta_1^- - \eta_1^\theta) = 0$. We can conclude from (7.12)(i) that the only possibility for β is $p^{s(\phi)}(\pm \chi_1^-)$. Now consideration of (7.12) and (7.13) shows that the only linear combination of characters with suitable weight and restriction to N is $p^{s(\phi)}(1_G + \delta_1^- \cdot \chi_1^-) - (\delta_\phi^+ \cdot \chi_\phi^+ + \delta_\phi^- \cdot \chi_\phi^-)$. The case $|S(\phi)| = 2$ will not cause any conflict with our choice of notation in the previous paragraph: for restricting to N shows that the only possible substitute for $-\delta_\phi^+ \cdot \chi_\phi^+$ in α^τ would be $\delta_\phi^+ \cdot \chi_{\bar{\phi}}^+$. However, it would not then be possible to satisfy the equation $(\alpha^\tau, \delta_\phi^+ \cdot \chi_\phi^+ + \delta_\phi^- \cdot \chi_\phi^- - \delta_\phi^\theta \cdot \chi_\phi^\theta) = -2$.

Finally, let α be a generalized character of the form (7.10)(ii). Then α^τ has weight 4, and by the reciprocity formula (7.4)(i) involves 1_G exactly once. Consideration of (7.13)(i) shows that χ_ϕ^+ and χ_ϕ^- are involved; except possibly when $|S(\phi)| = 2$, and we handle this case in the following paragraph. Once χ_ϕ^+ and χ_ϕ^- are involved, the conditions (7.12) through (7.14) in conjunction with weight 4 show that only the combination $(1_G - \delta_1^- \cdot \chi_1^-) - (\delta_\phi^+ \cdot \chi_\phi^+ - \delta_\phi^- \cdot \chi_\phi^-)$ has a suitable restriction to N.

In case $|S(\phi)| = 2$, there is one other possibility for α^τ, obtained by replacing $(\delta^+_\phi \cdot \chi^+_\phi - \delta^-_\phi \cdot \chi^-_\phi)$ by $(-\delta^+_\phi \cdot \chi^+_{\bar\phi} + \delta^-_\phi \cdot \chi^-_{\bar\phi})$; which would not contradict the relations (7.12) through (7.14). The elimination of this exceptional possibility -- which could only happen for some of the ϕ when $p = 3$ -- is not in the spirit of the main part of the proof; and hence the details of the computations involved will be omitted, and only the results given. The coefficients c_{gi} of (7.7) are uniquely determined; and one can verify from the definition that they are given by the following table:

	x	z	zt
type (i)	$-\eta^\theta_\phi(x)$	$-2p^{s(\phi)}\phi(z)$	0
type (ii)	0	0	$-\phi(z)$
type (iii)	0	$\frac{-\lvert P:P'\rvert}{p^{s(\phi)}}\phi(z)$	0

We use these in (7.7) to compute values, and hence decomposition columns. We find that $\delta^+_\phi \cdot \chi^+_\phi$ and $\delta^-_\phi \cdot \chi^-_\phi$ agree with η^+_ϕ and η^-_ϕ on P-1; the differences are given by:

	zt	$D^+_z + D^-_z$	$D^+_z - D^-_z$
$\delta^+_\phi \cdot \chi^+_\phi$	$-\bar\phi(z)$	$p^{s(\phi)}\phi(z)$	$-\bar\phi(z)$
$\delta^-_\phi \cdot \chi^-_\phi$	$\bar\phi(z)$	$p^{s(\phi)}\phi(z)$	$\bar\phi(z)$

The rest of the character theory is as described in 4.1; in particular, $\delta^+_\phi \cdot \chi^+_\phi + \delta^-_\phi \cdot \chi^-_\phi - \delta^\theta_\phi \cdot \chi^\theta_\phi$ vanishes at p-regular elements, because $\eta^+_\phi + \eta^-_\phi - \eta^\theta_\phi$ does, and τ preserves this property. The effect of the above in the calculation for (4.2) is to interchange the values of E^+_ϕ and E^-_ϕ. But this switch compensates for replacing $\phi(z)$ by $-\bar\phi(z)$ in the calculation for (4.4). So the same factorization takes place, and we conclude as there that the theorem must hold in G. But then the characters of B, restricted to N, are just the characters of b. And the values given above do not occur in b, a contradiction. So the exceptional case cannot occur when

$|S(\phi)| = 2$. Thus we have shown that the notation we chose causes characters of B to appear as constituents of the α^τ(α of form 7.10) in the natural way.

This process completes the determination of the coefficients $b_{i\chi}$ of (7.7). By the above, we have associated to η_ϕ^* a character χ_ϕ^* and a sign δ_ϕ^* so that

$$b_{i,\chi_\phi^*} = \delta_\phi^* \cdot a_{i,\eta_\phi^*} .$$

It follows then from (7.7) that for g a p-singular element of N, we have

(7.16) $$\chi_\phi^*(g) = \delta_\phi^* \cdot \eta_\phi^*(g) .$$

This formula puts us in a position to show that our correspondence satisfies the requirements of Hypothesis 4.1.

First we show that all the characters of B are covered by the correspondence. Let B' be the set of all χ_ϕ^* arising from the above procedure. For g a p-singular element of N, we see by (7.16) that:

$$(g,g)_{B'} = (g,g)^N .$$

But by 7.9, $(g,g) = (g,g)^N$. Thus a character of B-B' would vanish on all p-singular elements of N, and thus lie in a block of defect 0 rather than B. So B = B'.

We observed earlier that the principal-block characters of C(z), restricted to N, are just the characters of b. For the character ψ_z^- required by 4.1(ii), simply choose the unique character ψ of the principal block of C(z) with $\psi_N = \eta_1^-$. Then (7.16) guarantees that the decomposition columns D_z^+ and D_z^- will follow the pattern (up to sign) of the columns D_N^+ and D_N^- in 3.3 and 3.4, and hence will satisfy the requirements of 4.1(iii)(a). Finally, when $r(\phi) \neq 0$, the definition of τ guarantees that $\alpha^\tau = \delta_\phi^+ \cdot \chi_\phi^+ + \delta_\phi^- \cdot \chi_\phi^- - \delta_\phi^\theta \cdot \chi_\phi^\theta$ vanishes on p'-elements since $\alpha = \eta_\phi^+ + \eta_\phi^- - \eta_\phi^\theta$ does. Thus 4.1(iii)(b) is satisfied.

So by the work of Section 4, the proof of Theorem C is complete.

Section 8. Corollaries and Further Discussion

There are many examples of simple groups satisfying (1), (2) and (3) with respect to a cyclic Sylow subgroup. Among them are all the PSL(2,q) ($q \geq 4$) and hence groups of Ree type; and also any simple group with a cyclic Sylow 3-subgroup.

We cannot generalize the theorem by directly extending hypothesis (2). The groups A_9 and PSU(4,2) have Sylow 3-subgroups of order 81 and class 3, of index 2 in their normalizers. A similar structure will occur for any PSL(3,q) with $q \equiv 1 \pmod 9$, possibly for a larger Sylow 3-subgroup. These examples are all for $p = 3$, and I do not know of any similar examples for any other odd prime.

The first place that the proof of the theorem breaks down in the case $cl(P) > 2$ is that P need no longer be a regular p-group, so that the Hall-Wielandt theorem cannot be applied. Thus we cannot assume $[P,t] = P$, and consequently cannot begin to handle fusion as before. There are in fact other difficulties, which will not be gone into here.

We now proceed to discuss corollaries of the main theorem. The first is the problem from which the entire work originally arose.

Corollary 8.1 Let G be a finite perfect group with a self-normalizing cycle x of order 2p (p an odd prime). Then a Sylow p-subgroup of $N(\langle x^2 \rangle)$ is abelian.

Proof: Consider $H = N(\langle x^2 \rangle)$, and $\bar{H} = H/\langle x^2 \rangle$. Then $C_{\bar{H}}(\bar{x}^P) = \overline{N(\langle x \rangle)} = \langle \bar{x} \rangle = \langle \bar{x}^P \rangle$, so that in $\bar{H}$, $\bar{x}^P$ acts as a self-centralizing involution. It follows that $\bar{H} = \langle \bar{x}^P \rangle . \bar{K}$ where $\bar{K}$ is

an abelian group of odd order. If P is a Sylow p-subgroup of H, then $\overline{P}$ abelian requires $P' \leq \langle x^2 \rangle$. We assume by way of contradiction that P is not abelian, so that $P' = \langle x^2 \rangle$ of order p. Also $N(P) \leq N(P') = H$. Thus if Q is a Sylow p-subgroup of G containing P, we have $N_Q(P) = P$, which forces $Q = P$. Thus $N(P) = H$, and we have $|N(P)/P.C(P)| = 2$ with $|P'| = p$. It follows from Theorem A that G is not perfect. This proves 8.1.

It is conjectured that a finite group with a self-normalizing cycle of order 2p may not be perfect. Examples of minimal non-solvable groups satisfying the condition are the groups $SL(2,2^p)$ extended by the field automorphism of order p.

The other corollaries follow from the elementary lemma:

Lemma 8.2 If t is an automorphism of order 2 of a p-group P, p odd, then $P' \leq C_p(t)^P$.

Proof: Suppose $C_p(t) \leq N \trianglelefteq P$ but $P' \not\leq N$. Then t induces a fixed-point-free automorphism of order 2 on the non-abelian group P/N, a contradiction. So the lemma holds.

Corollary 8.3 Suppose G is a finite group satisfying (1) and either:

(i) P is a regular p-group with $C_p(t) \trianglelefteq P$,

or (ii) P' is cyclic.

Then G satisfies (2), and hence the main theorem is applicable.

Proof: In case P is a regular p-group, it follows as in 2.1 that $[P,t] = P$, and hence $C_p(t) \leq P'$. But then $C_p(t) \trianglelefteq P$ forces $C_p(t) = P'$ in view of Lemma 8.2. We can apply the three-subgroups lemma of P. Hall [7,2.2.3]:

$$[P,P',t] \leq [P',t] = 1 ,$$

$$[P',t,P] \leq [1,P] = 1 ,$$

so that $$[t,P,P'] = [P,P'] = 1.$$

Thus $P' \leq Z(P)$, so that P has class ≤ 2.

Now for (ii). Since P' is cyclic, it follows that P is regular ... see III.10.2 of Huppert [10]. Then $[P,t] = P$ and $C_P(t) \leq P'$ as in 2.1. But $C_P(t)$ char P' as P' is cyclic, and P' char P, so that $C_P(t) \trianglelefteq P$. Thus we have reduced to the situation of (i). This completes the proof of the corollary.

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