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Graphical Abstract

Subject-Oriented Spatial Logic[1]

Przemysław Andrzej Wałęga, Michał Zawidzki

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Highlights

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- Research highlight 1
- Research highlight 2

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Subject-Oriented Spatial Logic[1]

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Abstract

We present a modal logic for subject-oriented representation and reasoning about a two-dimensional space. The space is represented with the polar coordinate system where the central point is occupied by the subject and modal operators are interpreted by relations defined relatively to the position and orientation of the subject, namely 'outwards', 'inwards', 'clockwise', 'counter-clockwise', and transitive closures of the first two of them. As we show, such a setting allows us to capture subject-oriented aspects of human-like reasoning while maintaining a relatively good compromise between expressivity and complexity. In particular, the logic allows us to express operators for the intrinsic relations: 'in front', 'behind', 'to the left', and 'to the right' of the subject, for the relative relations: 'behind an object', 'between the subject and an object', 'to the left of an object', and 'to the right of an object' from the point of view of the subject, as well as hybrid and distance operators. We show that the satisfiability problem in the logic is PSpace-complete, the same complexity holds over the classes of finite as well as over infinite models, however, if the size of models is bounded by a fixed constant, the problem becomes NP-complete.

Keywords: Spatial logics, Modal logics, Knowledge Representation

[1] This is an extended version of a conference paper [1]. The logical formalism in the current version is simpler yet more general. The new results include showing that a wide range of spatial operators are expressible in the logic, the PSpace-completeness result for satisfiability checking transfers to the new setting and holds over the classes of finite as well as infinite models, and the satisfiability problem for models with the size bounded by a fixed constant is NP-complete.

1. Introduction

Spatial reasoning is one of the most interesting abilities that humans possess and whose modelling is still a great challenge for AI. Research in this area is interesting from a theoretical point of view, for it may help us understand how people reason and use language, and from a practical perspective, as it may result in methods with a broad range of applications, for example, in robotics [2] or geographical information systems [3].

There is a great variety of logical approaches to spatial reasoning, which exploit the machinery of modal logics [4, 5, 6, 7, 8, 9, 10], relational algebras [11, 12, 13, 14, 15, 16, 17], and first-order theories [18, 13, 19], among others. Various aspects of space are modeled by these systems, such as topology, directions, distance, orientation, size, shape, etc.; some use quantitative and other qualitative methods. What is, however, common for many of these approaches is that a *frame of reference* used to determine the location of objects is *absolute*, namely the location of an object is described by using absolute coordinates or directions such as north, south, east, and west [4, 6, 9], for example 'the tree is to the south of the car.' Hence, the description does not depend on the subject's position or orientation [20]. On the other hand, various studies show that users of many languages (e.g., Indo-European languages like English or Dutch; or Japanese) mostly adopt the *intrinsic* frame of reference, or the *relative* frame of reference when performing both linguistic and non-linguistic tasks [21, 20, 22]. In the intrinsic frame of reference, the location of an object is determined by means of a binary relation between this object and a landmark object. The landmark object is 'parsed' into its major parts such as its front, back, left side, and right side, and then a named facet of the landmark object is used to describe the location of the first object [21], for example, 'the tree is in front of the car.' The subject itself can be treated as the landmark object, and so 'the tree is in front of me' is a description in the intrinsic frame of reference, too [20]. In the relative frame of reference the position of an object is described by a ternary relation involving this object, a landmark object, and the subject (say myself) [20], for instance 'the tree is between the car and myself.' When using the relative frame of reference one also needs to perform 'parsing' of objects, and so it seems that in natural language the relative frame of reference cannot occur without the intrinsic one [23, 20].

In the paper, we construct a subject-oriented modal spatial logic, denoted by SOSL, which allows us to reason about both intrinsic and relative relations. We represent the two-dimensional space using the polar coordinate system with the subject located at its center. Hence, every point in the space is denoted by a pair $\langle r, \theta \rangle$ of coordinates, where r is the distance from the subject (the radius-coordinate) and θ is the angle from the direction in which the subject is looking (the angle-coordinate). Then, we divide the space into two-dimensional cells of constant radius-size and angle-width. The modal operators occurring in the language of SOSL are interpreted as binary relations between cells. We provide a formal definition of the syntax and semantics of SOSL in Section 2. Next, in Section 3, we analyse the expressive power of the logic. In particular, we show that SOSL allows us to express hybrid operators (nominals and satisfaction operators), operators expressing the distance, and a range of operators corresponding to intrinsic and relative relations. It is worth noting that relative relations hold between the object being described, a landmark object, and the subject, and so they are ternary by their nature. However, as the subject is always implicitly involved in our representation of the space, there is no need to explicitly involve them as the third argument of a relation. As a result, we can interpret relative relations using binary relations between cells, and consequently, capture them with modal operators. Then, in Section 4, we investigate the computational properties of the logic. We show that the satisfiability problem for SOSL is PSPACE-complete over the classes of arbitrary, finite, and infinite models; whereas if the radius-size of a model is bounded by a fixed constant, then the satisfiability problem becomes NP-complete. In Section 5, we compare SOSL with other modal logics for spatial reasoning. Afterwards, in Section 6, we describe possible applications of the logic and in Section 7 we conclude the paper.

2. Syntax and Semantics

The language of SOSL consists of a countably infinite set PROP of propositional variables, Boolean connectives $\neg$ and $\vee$, SOSL-modal operators $\textcircled{\uparrow}$, $\textcircled{\downarrow}$, $\textcircled{\rightarrow}$, $\textcircled{\leftarrow}$, $\Diamond_{\uparrow}$, $\Diamond_{\downarrow}$, and parentheses (,). *Formulas* of SOSL are generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \otimes\varphi,$$

where $p \in \text{PROP}$ and $\otimes \in \{\textcircled{\uparrow}, \textcircled{\downarrow}, \textcircled{\rightarrow}, \textcircled{\leftarrow}, \Diamond_{\uparrow}, \Diamond_{\downarrow}\}$.

The other Boolean connectives are defined in a standard way, namely: $\top = p \vee \neg p$, $\bot = \neg\top$, $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$, $\varphi \to \psi = \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, where $p \in \text{PROP}$ and φ, ψ are SOSL-formulas.

Moreover, we define $\boxed{\uparrow}$ and $\boxed{\downarrow}$ as the operators dual to $\langle\uparrow\rangle$ and $\langle\downarrow\rangle$, respectively, so for every SOSL-formula φ, we set $\boxed{\uparrow}\varphi = \neg\langle\uparrow\rangle\neg\varphi$ and $\boxed{\downarrow}\varphi = \neg\langle\downarrow\rangle\neg\varphi$. We use $\textcircled{\uparrow}^n\varphi$ as an abbreviation for φ preceded by an n-times iteration of $\textcircled{\uparrow}$, and we adopt the analogous notation for iterations of $\textcircled{\downarrow}$, $\textcircled{\rightarrow}$, and $\textcircled{\leftarrow}$. We identify $\textcircled{\uparrow}^{-n}$ with $\textcircled{\downarrow}^n$ for any $n \in \mathbb{N}$ (we use $\mathbb{N}$ for non-negative integers, and $\mathbb{N}_+$ for positive integers), and we do similarly in the cases of $\textcircled{\downarrow}^{-n}$, $\textcircled{\rightarrow}^{-n}$, and $\textcircled{\leftarrow}^{-n}$.

Next, we define the semantics of SOSL. We consider an infinite plane with polar coordinates. The universe of a model consists of two-dimensional cells partitioning the space—where each cell has the radius length 1 and the angular width of 1 degree—and an additional circular cell located at the center of the space, which represents the subject. We depict such a universe in Figure 1(a), but to make the picture more readable we present cells of angular width of 5 instead of 1 degree.

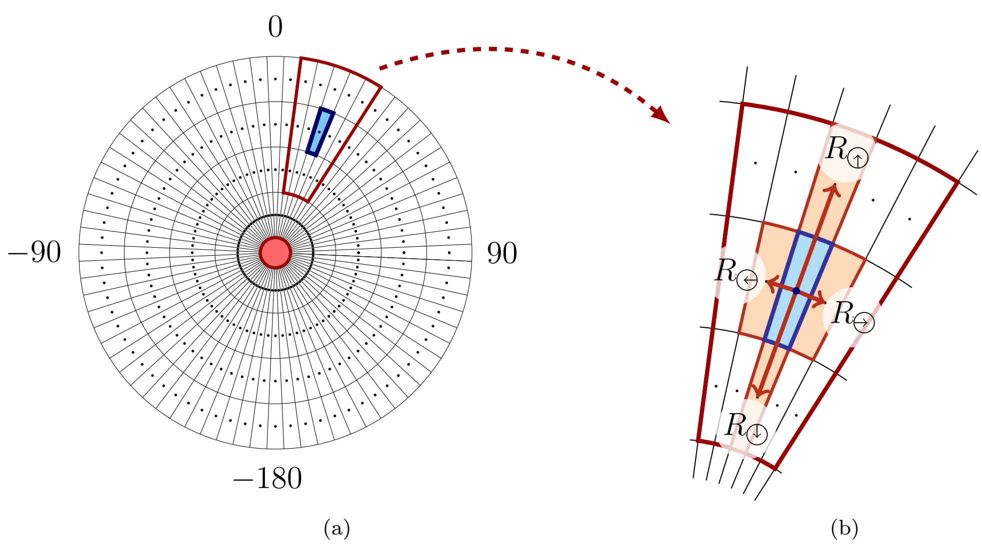

Figure 1: Universe in SOSL (a) and the relations $R_{\textcircled{\uparrow}}$, $R_{\textcircled{\downarrow}}$, $R_{\textcircled{\rightarrow}}$, and $R_{\textcircled{\leftarrow}}$ (b)

Each cell is identified with the pair of polar coordinates of its central point. The subject is denoted by $\langle 0, 0\rangle$ and is assumed to be oriented towards the angle of 0 degrees. Every non-subject cell is denoted by a pair $\langle r, \theta\rangle$ of polar coordinates, for r being a positive integer denoting the cell's distance from the subject and θ being an integer from the set $\{-180, \ldots, 179\}$ of 360 angle-values denoting the angular position with respect to the subject's orientation.

Then, a *universe* of a model is a set $\mathcal{C}$ of cells defined formally as

$$\mathcal{C} = \big\{\langle 0,0\rangle\big\} \cup \big\{\langle r,\theta\rangle \mid 1 \le r \le s \text{ and } -180 \le \theta \le 179\big\},$$

where $s \in \mathbb{N}_+ \cup \{\infty\}$ is referred to as the *size* of the model and $r, \theta \in \mathbb{Z}$. Note that s determines the number of cells in $\mathcal{C}$, which equals $1 + s \cdot 360$. Clearly, if $s \in \mathbb{N}_+$, then the number of cells in $\mathcal{C}$ is finite and if $s = \infty$, then the universe consists of infinitely many cells.

Each SOSL-modal operator $\otimes \in \{\textcircled{\uparrow}, \textcircled{\downarrow}, \textcircled{\rightarrow}, \textcircled{\leftarrow}, \diamond\!\!\uparrow, \diamond\!\!\downarrow\}$ is interpreted by a binary relation on $\mathcal{C}$, denoted by $R_\otimes$. The relations $R_{\textcircled{\uparrow}}$, $R_{\textcircled{\downarrow}}$, $R_{\textcircled{\rightarrow}}$, and $R_{\textcircled{\leftarrow}}$ hold between adjacent cells and have the intuitive meaning of, respectively, 'one cell outwards', 'one cell inwards', 'one cell clockwise', and 'one cell counterclockwise', as depicted in Figure 1(b). On the other hand, $R_{\diamond\!\uparrow}$ and $R_{\diamond\!\downarrow}$ mean 'any number of cells outwards' and 'any number of cells inwards.' Formally, we define these relations as follows.

Definition 1. The relations $R_{\textcircled{\uparrow}}$ and $R_{\textcircled{\rightarrow}}$ are defined as:

$$\begin{aligned} R_{\textcircled{\uparrow}} = {} & \Big\{\big(\langle r_1,\theta_1\rangle, \langle r_2,\theta_2\rangle\big) \mid r_2 = r_1 + 1 \text{ and } \theta_2 = \theta_1\Big\} \cup \\ & \Big\{\big(\langle 0,0\rangle, \langle 1,\theta\rangle\big) \mid -180 \le \theta \le 179\Big\}; \\ R_{\textcircled{\rightarrow}} = {} & \Big\{\big(\langle r_1,\theta_1\rangle, \langle r_2,\theta_2\rangle\big) \mid r_2 = r_1 \text{ and } \theta_2 = \theta_1 + 1\Big\}, \end{aligned}$$

where $\langle r_1,\theta_1\rangle$, $\langle r_2,\theta_2\rangle$, and $\langle 1,\theta\rangle$ are cells in $\mathcal{C}$. We define $R_{\textcircled{\downarrow}}$ and $R_{\textcircled{\leftarrow}}$ as the converse relations of $R_{\textcircled{\uparrow}}$ and $R_{\textcircled{\rightarrow}}$, respectively; whereas $R_{\diamond\!\uparrow}$ and $R_{\diamond\!\downarrow}$ as the transitive closures of $R_{\textcircled{\uparrow}}$ and $R_{\textcircled{\downarrow}}$, respectively.[2]

Observe that if the universe is infinite, then every non-subject cell has exactly one $R_{\textcircled{\uparrow}}$-successor and exactly one $R_{\textcircled{\downarrow}}$-successor; whereas if the universe is finite, the cells with the maximal radius-coordinate do not have $R_{\textcircled{\uparrow}}$-successors. Furthermore, in each model cells with the angular coordinates 179 and -180 do not have $R_{\textcircled{\rightarrow}}$- and $R_{\textcircled{\leftarrow}}$-successors, respectively.

A model and the satisfiability relation are defined in a standard way for modal logics:

[2] Let us recall that the converse of a binary relation R is the following relation $R^{-1} = \{(y,x) \mid (x,y) \in R\}$, whereas the transitive closure of the relation R is the relation $R^+ = \{(x_1,x_n) \mid \text{there are } x_1,\ldots,x_n, \text{ with } (x_m,x_{m+1}) \in R \text{ for all } m \in \{1,\ldots,n-1\}\}$.

Definition 2. An SOSL-*model* $\mathfrak{M}$ is a tuple $(\mathcal{C}, \mathcal{R}, V)$, where $\mathcal{C}$ is a universe, $\mathcal{R} = \{R_\otimes \mid \otimes \text{ is an } \mathsf{SOSL}\text{-operator}\}$, and $V : \text{PROP} \longrightarrow \mathcal{P}(\mathcal{C})$. The relation $\Vdash$ is defined as follows:

$$\begin{array}{lll}
\mathfrak{M}, \langle r, \theta \rangle \Vdash p & \text{iff} & \langle r, \theta \rangle \in V(p), \text{ for } p \in \text{PROP} \\
\mathfrak{M}, \langle r, \theta \rangle \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, \langle r, \theta \rangle \nVdash \varphi \\
\mathfrak{M}, \langle r, \theta \rangle \Vdash \varphi \vee \psi & \text{iff} & \mathfrak{M}, \langle r, \theta \rangle \Vdash \varphi \text{ or } \mathfrak{M}, \langle r, \theta \rangle \Vdash \psi \\
\mathfrak{M}, \langle r, \theta \rangle \Vdash \otimes\varphi & \text{iff} & \text{there exists } \langle r', \theta' \rangle \text{ such that } \langle r, \theta \rangle R_\otimes \langle r', \theta' \rangle \\
 & & \text{and } \mathfrak{M}, \langle r', \theta' \rangle \Vdash \varphi,
\end{array}$$

where $\langle r, \theta \rangle, \langle r', \theta' \rangle \in \mathcal{C}$, φ and ψ are SOSL-formulas, and $\otimes$ is an SOSL-modal operator. We say that a formula φ is *satisfiable* if there exist a model $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$ and a cell $\langle r, \theta \rangle \in \mathcal{C}$ such that $\mathfrak{M}, \langle r, \theta \rangle \Vdash \varphi$.

3. Expressive Power

In this section, we study the expressive power of SOSL. We prove that the logic allows us to express the hybrid machinery, namely the difference, somewhere, and everywhere operators, nominals, and satisfaction operators. Then, we use them to encode modal operators capturing a wide range of intrinsic and relative spatial relations. Finally, we present how to express distance operators.

We start by observing that for every cell of the universe we can write a formula which is satisfied exactly in this cell. Such formulas will be useful in several parts of the paper. We start by defining a formula $\mathsf{cell}_{\langle 0,0 \rangle}$ which holds exactly in the subject cell $\langle 0, 0 \rangle$:

$$\mathsf{cell}_{\langle 0,0 \rangle} = \neg \textcircled{\downarrow} \top.$$

Indeed, $\langle 0, 0 \rangle$ is the only cell which has no $R_{\textcircled{\downarrow}}$-successors, so $\neg \textcircled{\downarrow} \top$ holds only there. For every non-subject cell $\langle r, \theta \rangle$ we define a formula $\mathsf{cell}_{\langle r,\theta \rangle}$ as

$$\mathsf{cell}_{\langle r,\theta \rangle} = \textcircled{\downarrow}^r \, \mathsf{cell}_{\langle 0,0 \rangle} \wedge \textcircled{\leftarrow}^{\theta+180} \neg \textcircled{\leftarrow} \top.$$

Observe that the formula $\mathsf{cell}_{\langle r,\theta \rangle}$ holds exactly in the cell $\langle r, \theta \rangle$; the conjunct $\textcircled{\downarrow}^r \, \mathsf{cell}_{\langle 0,0 \rangle}$ assures that the radius-coordinate of the current cell is r, whereas $\textcircled{\leftarrow}^{\theta+180} \neg \textcircled{\leftarrow} \top$ guarantees that the angle-coordinate is θ.

3.1. Hybrid Operators

Next, we show how to express the hybrid machinery. First, we demonstrate the way of capturing the *difference* operator D, where $\mathsf{D}\varphi$ is satisfied in a cell $\langle r, \theta\rangle$ if and only if φ is satisfied in some cell distinct from $\langle r, \theta\rangle$.

Proposition 3. *The difference operator is expressible in* SOSL.

Proof. For any SOSL-formula φ we define

$$\mathsf{D}\varphi = \Diamond_{\uparrow}\varphi \vee \Diamond_{\downarrow}\varphi \vee \bigvee_{1 \leq |n| \leq 359} \odot_{\rightarrow}^{n}(\varphi \vee \Diamond_{\uparrow}\varphi \vee \Diamond_{\downarrow}\varphi).$$

We need to show that the formula above is satisfied in a cell $\langle r, \theta\rangle$ if and only if φ is satisfied in some cell distinct from $\langle r, \theta\rangle$. First, assume that $\langle r, \theta\rangle$ is not the subject cell $\langle 0, 0\rangle$. Then $\Diamond_{\uparrow}\varphi \vee \Diamond_{\downarrow}\varphi$ states that φ holds in a cell $\langle r', \theta'\rangle$, for some $r' \neq r$ and $\theta' = \theta$, or in $\langle 0, 0\rangle$. Furthermore $\bigvee_{1 \leq |n| \leq 359} \odot_{\rightarrow}^{n}(\varphi \vee \Diamond_{\uparrow}\varphi \vee \Diamond_{\downarrow}\varphi)$ states that φ holds in a cell $\langle r', \theta'\rangle$, for any r' and $\theta' \neq \theta$, or in $\langle 0, 0\rangle$.

It remains to consider the case when $\langle r, \theta\rangle$ is $\langle 0, 0\rangle$. Then, $\Diamond_{\uparrow}\varphi$ states that φ holds in some cell $\langle r', \theta'\rangle$ with $r' \neq r$ and any θ', that is, in some cell different than $\langle 0, 0\rangle$. Observe that $\Diamond_{\downarrow}\varphi \vee \bigvee_{1 \leq |n| \leq 359} \odot_{\rightarrow}^{n}(\varphi \vee \Diamond_{\uparrow}\varphi \vee \Diamond_{\downarrow}\varphi)$ is false in $\langle 0, 0\rangle$ since $\langle 0, 0\rangle$ has neither $R_{\Diamond_{\downarrow}}$- nor $R_{\odot_{\rightarrow}}$-successors. □

It is well known that D allows us to express the *somewhere* operator E stating that a formula holds in some cell, and its dual—the *everywhere* operator A—indicating that a formula holds in all cells [24]. Indeed, we express them as $\mathsf{E}\varphi = \varphi \vee \mathsf{D}\varphi$ and $\mathsf{A}\varphi = \neg\mathsf{E}\neg\varphi$. These operators can be used to simulate *nominals*, that is atoms each of which is satisfied in exactly one (but arbitrary) cell. We can simulate a nominal i with a propositional variable p_i by stipulating that p_i holds in exactly one cell: $\mathsf{E}p_i \wedge \mathsf{A}(p_i \to \neg\mathsf{D}p_i)$. Then, we can express a *satisfaction operator* $@_i$, where $@_i\varphi$ states that φ holds in the cell in which i holds: $@_i\varphi = \mathsf{E}(p_i \wedge \varphi)$.

Corollary 4. *The somewhere operator, everywhere operator, nominals, and satisfaction operators are expressible in* SOSL.

3.2. Intrinsic Relations

Now, we consider the following intrinsic relations: 'in front of the subject', 'behind the subject', 'to the right of the subject', and 'to the left of the subject.' Various meanings can be assigned to such relations as presented in

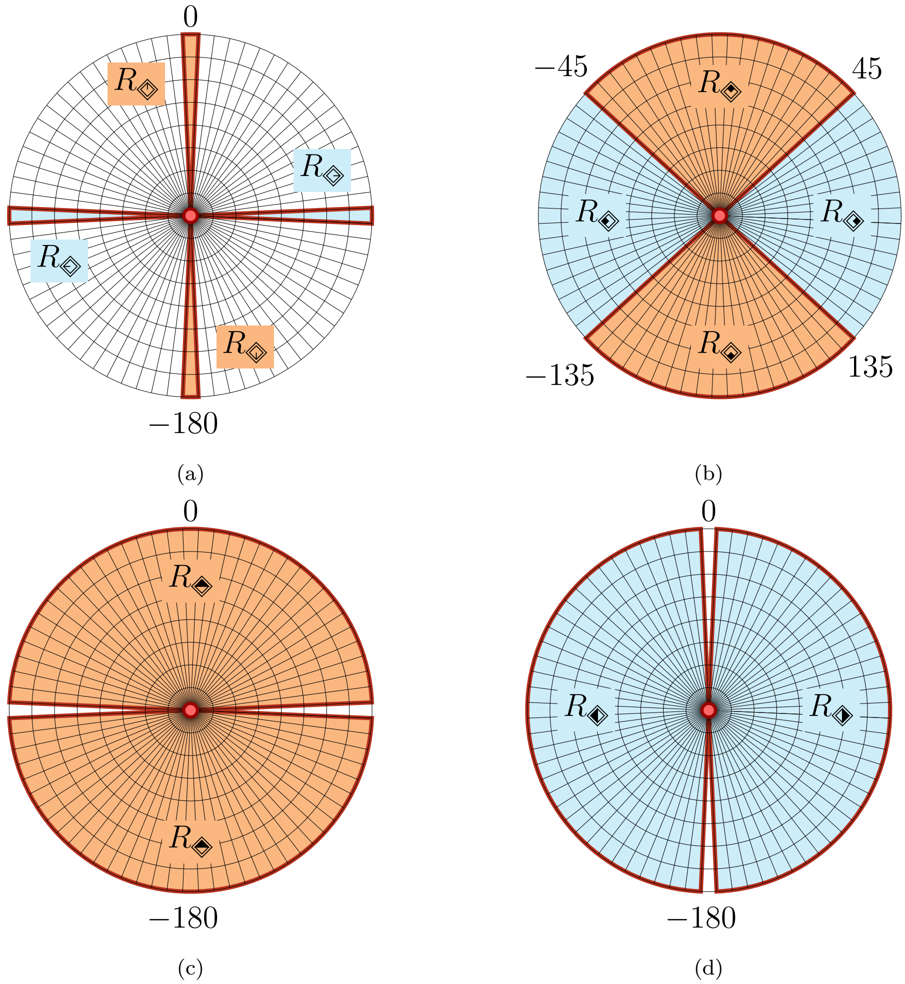

Figure 2: Different interpretations of the intrinsic relations ‘in front of the subject’, ‘behind the subject’, ‘to the right of the subject’, and ‘to the left of the subject’

Figure 2. As we show below, SOSL allows us to express modal operators corresponding to these exemplary meanings, but the class of intrinsic relations that can be captured in the logic is much broader.

One possible semantics for the intrinsic relations—as depicted in Figure 2(a)—results from associating a relation with a single angular coordinate, so that ‘in front of the subject’, ‘behind the subject’, ‘to the right of the subject’, and ‘to the left of the subject’ indicate that the angular coordinate of an object is 0, -180, 90, and -90 degrees, respectively. We denote such relations by $R_{\Diamond^{\uparrow}}, R_{\Diamond^{\downarrow}}, R_{\Diamond^{\rightarrow}}$, and $R_{\Diamond^{\leftarrow}}$, and define them formally as follows, for $\langle r, \theta \rangle$ ranging over non-subject cells:

$$R_{\Diamond^{\uparrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid \theta = 0 \Big\}; \qquad R_{\Diamond^{\downarrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid \theta = -180 \Big\};$$
$$R_{\Diamond^{\rightarrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid \theta = 90 \Big\}; \quad R_{\Diamond^{\leftarrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid \theta = -90 \Big\}.$$

Another semantics for the intrinsic relations is obtained by dividing the space surrounding the subject into four cones whose angular width is 90 degrees, as in Figure 2(b). Then, every cone represents one of the intrinsic relations, for example, the cone extending in the direction of the subject's orientation contains cells which are 'in front of the subject.' We denote such relations by $R_{\Diamond^{\blacktriangle}}, R_{\Diamond^{\blacktriangledown}}, R_{\Diamond^{\blacktriangleright}}$, and $R_{\Diamond^{\blacktriangleleft}}$, and define them below, for $\langle r, \theta \rangle$ being a non-subject cell:

$$R_{\Diamond^{\blacktriangle}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid |\theta| \leq 45 \Big\}; \quad R_{\Diamond^{\blacktriangledown}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid 135 \leq |\theta| \Big\};$$
$$R_{\Diamond^{\blacktriangleright}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid 45 < \theta < 135 \Big\};$$
$$R_{\Diamond^{\blacktriangleleft}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid -135 < \theta < -45 \Big\}.$$

One can also understand the 'in front of the subject' and 'behind the subject' relations as dividing the space into two half-planes, where the division line coincides with the 'shoulder line of the subject' and does not belong to any half, as depicted in Figure 2(c). Similarly, the 'to the right of the subject' and 'to the left of the subject' relations can be obtained by dividing the space with a line coinciding with the direction the subject is looking in, as in Figure 2(d). Note that in this case, the intrinsic relations are not disjoint, as, for instance, an object can be located simultaneously 'in front of the subject' and 'to the right of the subject.' Let $R_{\Diamond^{\Uparrow}}, R_{\Diamond^{\Downarrow}}, R_{\Diamond^{\Rightarrow}}$, and $R_{\Diamond^{\Leftarrow}}$ denote these relations. Hence, for $\langle r, \theta \rangle$ ranging over non-subject cells we define:

$$R_{\Diamond^{\Uparrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid |\theta| < 90 \Big\}; \quad R_{\Diamond^{\Downarrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid 90 < |\theta| \Big\};$$
$$R_{\Diamond^{\Rightarrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid 0 < \theta \Big\}; \qquad R_{\Diamond^{\Leftarrow}} = \Big\{ \big(\langle 0,0 \rangle, \langle r, \theta \rangle \big) \mid -180 < \theta < 0 \Big\}.$$

We introduce the modal operators $\Diamond^{\uparrow}, \Diamond^{\downarrow}, \Diamond^{\rightarrow}, \Diamond^{\leftarrow}, \Diamond^{\blacktriangle}, \Diamond^{\blacktriangledown}, \Diamond^{\blacktriangleright}, \Diamond^{\blacktriangleleft}, \Diamond^{\Uparrow}, \Diamond^{\Downarrow}, \Diamond^{\Rightarrow}$, and $\Diamond^{\Leftarrow}$ corresponding to the intrinsic relations presented above. The semantics of these operators is straightforward: for every SOSL-model $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$, every $\langle r, \theta \rangle \in \mathcal{C}$, every $\otimes \in \{\Diamond_{\uparrow}, \Diamond_{\downarrow}, \Diamond_{\rightarrow}, \Diamond_{\leftarrow}, ⬘, ⬙, ⬗, ⬖, ⬘, ⬙, ⬗, ⬖\}$, and every SOSL-formula φ it holds that:

$$\mathfrak{M}, \langle r, \theta \rangle \Vdash \otimes\varphi \quad \text{iff} \quad \text{there exists } \langle r', \theta' \rangle \in \mathcal{C} \text{ such that } \langle r, \theta \rangle R_{\otimes} \langle r', \theta' \rangle \text{ and } \mathfrak{M}, \langle r', \theta' \rangle \Vdash \varphi.$$

Proposition 5. *All the operators* $\Diamond_{\uparrow}, \Diamond_{\downarrow}, \Diamond_{\rightarrow}, \Diamond_{\leftarrow}, ⬘, ⬙, ⬗, ⬖, ⬘, ⬙, ⬗$*, and* $⬖$ *are expressible in* SOSL.

Proof. We can notice that for each operator $\otimes$ mentioned in the proposition, the relation $R_{\otimes}$ holds between the subject cell $\langle 0, 0 \rangle$ and cells that form a segment of a circle (see Figure 2). Such a segment can be divided into two sets of cells. The first set, denoted by $\mathcal{A}_{\otimes}$, consists of cells lying on the first ring around the subject and the second set consists of all cells that can be reached from cells in $\mathcal{A}_{\otimes}$ by moving outwards. We use this intuition to define the modal operators as follows, where φ is an arbitrary SOSL-formula:

$$\otimes\varphi = \mathsf{cell}_{\langle 0,0 \rangle} \wedge \mathsf{E} \bigvee_{\langle r, \theta \rangle \in \mathcal{A}_{\otimes}} \big(\mathsf{cell}_{\langle r, \theta \rangle} \wedge (\varphi \vee \Diamond_{\uparrow} \varphi) \big).$$

The first conjunct of the formula states that the current cell is $\langle 0, 0 \rangle$. The second conjunct ensures that φ holds in some cell belonging to $\mathcal{A}_{\otimes}$ or in a cell located outwards some cell in $\mathcal{A}_{\otimes}$. Note that it is essential that $\mathcal{A}_{\otimes}$ is finite, so the disjunction over $\langle r, \theta \rangle \in \mathcal{A}_{\otimes}$ is also finite. □

The proof of Proposition 5 suggests a method of defining a wide range of other intrinsic relations in SOSL. Indeed, the logic allows us to capture each intrinsic relation whose range can be described using two finite sets of cells, call them A and B, namely it consists of all cells in A and cells which are reachable by going outwards from cells in B. Observe that in the proof of Proposition 5 the sets A and B coincide (they are denoted by $\mathcal{A}_{\otimes}$), but in general we do not need to impose such a restriction.

3.3. Relative relations

Recall that spatial relations in the relative frame of reference allow us to determine mutual spatial arrangement of objects from the subject's perspective, for instance 'the house is to the left of the tree from my point of view.' We consider four relative relations: 'behind an object', 'between the subject and an object', 'to the right of an object', and 'to the left of an object.' As we show in this section, SOSL allows us to define modal operators for several exemplary meanings of these relative relations. Then, we observe that there is a much wider class of relations we can capture in the logic.

To describe relative relations it is useful to define additional notions pertaining to angular coordinates of cells. We will write $\theta_1 \lessdot \theta_2$, if the clockwise angle from θ_1 to θ_2 is smaller than the clockwise angle from θ_2 to θ_1, namely

$$\theta_1 \lessdot \theta_2 \quad \text{iff} \quad (\theta_2 - \theta_1) \bmod 360 \ < \ (\theta_1 - \theta_2) \bmod 360,$$

where $(\theta_2 - \theta_1) \bmod 360$ is the clockwise angle from the angle θ_1 to θ_2, and $(\theta_1 - \theta_2) \bmod 360$ is the clockwise angle from θ_2 to θ_1. If the opposite inequality holds, we write $\theta_1 \gtrdot \theta_2$. For example, in Figure 3(d), with the reference cell $\langle r_1, \theta_1 \rangle$, the area corresponding to $R_{\lozenge\!\blacktriangleright}$ contains all cells $\langle r_2, \theta_2 \rangle$ such that $\theta_1 \lessdot \theta_2$ and the area marked as $R_{\blacktriangleleft\!\lozenge}$ contains all cells $\langle r_2, \theta_2 \rangle$ such that $\theta_1 \gtrdot \theta_2$.

Next we define the *angular distance* between θ_1 and θ_2, denoted as $\Delta(\theta_1, \theta_2)$. Intuitively, the angular distance is the smaller of the two clockwise angles: form θ_1 to θ_2 and form θ_2 to θ_1. Formally

$$\Delta(\theta_1, \theta_2) = \min\big\{(\theta_2 - \theta_1) \bmod 360, (\theta_1 - \theta_2) \bmod 360\big\}.$$

Clearly, $\Delta(\theta_1, \theta_2)$ takes integer values from the set $\{0, \ldots, 180\}$.

Now we can proceed to the exposition of concrete examples of relative relations, as depicted in Figure 3. One possible meaning of '$\langle r_2, \theta_2 \rangle$ is behind $\langle r_1, \theta_1 \rangle$ from the subject's point of view' is that $\langle r_2, \theta_2 \rangle$ and $\langle r_1, \theta_1 \rangle$ share the same angle-coordinate but $\langle r_2, \theta_2 \rangle$ is located further away from the subject than $\langle r_1, \theta_1 \rangle$. Then, '$\langle r_2, \theta_2 \rangle$ is between $\langle r_1, \theta_1 \rangle$ and the subject' if $\langle r_2, \theta_2 \rangle$ and $\langle r_1, \theta_1 \rangle$ share the same angle-coordinate and $r_2 < r_1$. Analogously, '$\langle r_2, \theta_2 \rangle$ is to the right of $\langle r_1, \theta_1 \rangle$ from the subject's point of view' if the cells are in the same distance from the subject and $\theta_1 \lessdot \theta_2$; and '$\langle r_2, \theta_2 \rangle$ is to the left of $\langle r_1, \theta_1 \rangle$ from the subject's point of view' is the inverse of the 'to the right' relation. We will denote these relations by $R_{\Diamond_{\uparrow}}$, $R_{\Diamond_{\downarrow}}$, $R_{\Diamond_{\rightarrow}}$, and $R_{\Diamond_{\leftarrow}}$, respectively. We present them in Figure 3(a) and define formally as follows, for $\langle r_1, \theta_1 \rangle$ and $\langle r_2, \theta_2 \rangle$ ranging over non-subject cells:

$$R_{\Diamond_{\uparrow}} = \Big\{\big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle\big) \mid r_1 < r_2 \text{ and } \theta_1 = \theta_2\Big\};$$

$$R_{\Diamond_{\downarrow}} = \Big\{\big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle\big) \mid r_1 > r_2 \text{ and } \theta_1 = \theta_2\Big\};$$

$$R_{\Diamond_{\rightarrow}} = \Big\{\big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle\big) \mid \theta_1 \lessdot \theta_2 \text{ and } r_1 = r_2\Big\};$$

$$R_{\Diamond_{\leftarrow}} = \Big\{\big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle\big) \mid \theta_1 \gtrdot \theta_2 \text{ and } r_1 = r_2\Big\}.$$

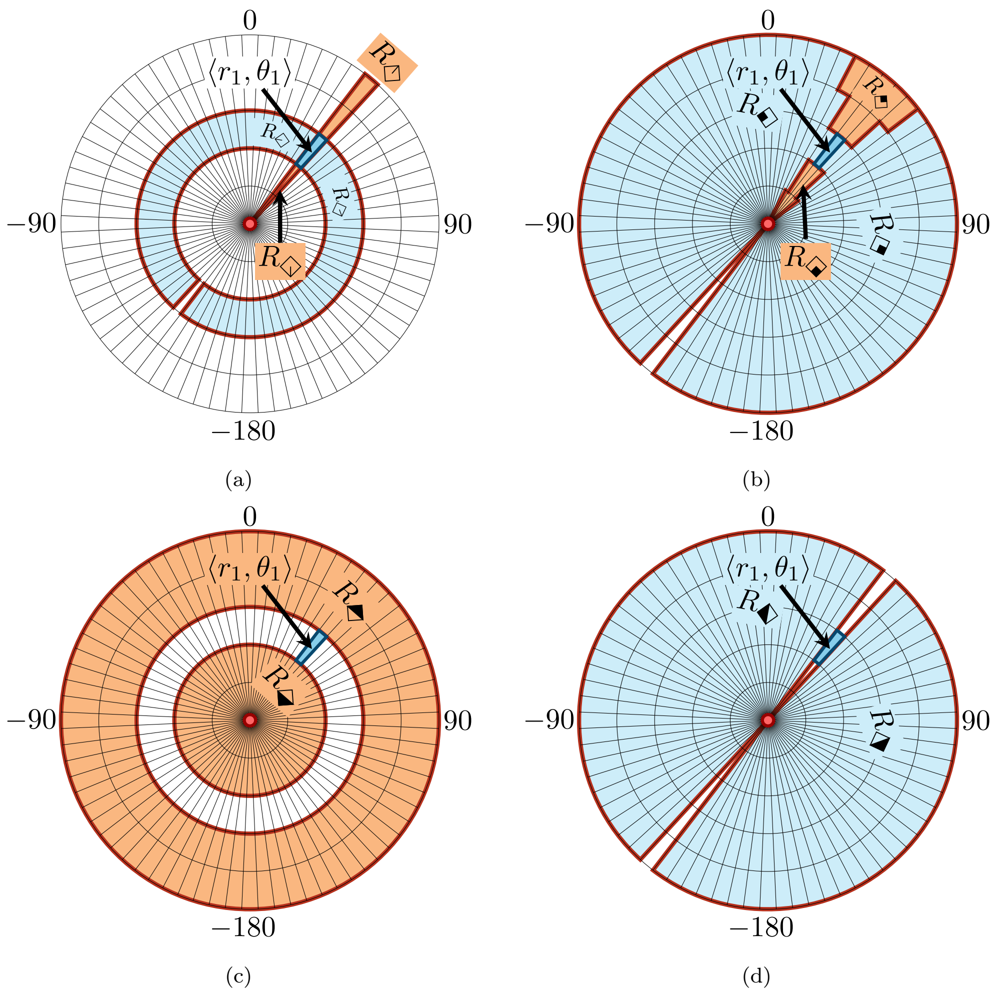

Figure 3: Different interpretations of the relative relations 'behind an object', 'between the subject and an object', 'to the right of an object', and 'to the left of an object'

Alternatively, the relative relations can be understood as resulting from a partitioning of a plane into cone-shape areas, as presented in Figure 3(b). We denote such relations by $R_{⬘}$, $R_{⬙}$, $R_{⬗}$, and $R_{⬖}$, and define them below, for $\langle r_1, \theta_1 \rangle$ and $\langle r_2, \theta_2 \rangle$ ranging over non-subject cells:

$$\begin{aligned}
R_{\Diamond_{\blacktriangle}} &= \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid r_1 < r_2 \text{ and } \Delta(\theta_1, \theta_2) \leq r_2 - r_1 \Big\}; \\
R_{\Diamond_{\blacktriangledown}} &= \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid r_1 > r_2 \text{ and } \Delta(\theta_1, \theta_2) \leq r_1 - r_2 \Big\}; \\
R_{\Diamond_{\blacktriangleright}} &= \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid \theta_1 \lessdot \theta_2 \text{ and } |r_1 - r_2| < \Delta(\theta_1, \theta_2) \Big\}; \\
R_{\Diamond_{\blacktriangleleft}} &= \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid \theta_1 \gtrdot \theta_2 \text{ and } |r_1 - r_2| < \Delta(\theta_1, \theta_2) \Big\}.
\end{aligned}$$

Another way of defining the relative relations is depicted in Figures 3(c) and 3(d), where 'behind a cell $\langle r_1, \theta_1 \rangle$' is understood as 'in a cell that is further away from the subject than $\langle r_1, \theta_1 \rangle$' and 'between a cell and the subject' as 'in a cell that is closer to the subject than $\langle r_1, \theta_1 \rangle$.' Similarly, we can define 'to the right of a cell' as 'in a cell with a $\lessdot$-greater angle-coordinate', and to the left of a cell' as 'in a cell with a $\lessdot$-less angle-coordinate.' We denote these relations by $R_{\text{⬘}}$, $R_{\text{⬙}}$, $R_{\text{⬗}}$, and $R_{\text{⬖}}$; and define them as follows, where $\langle r_1, \theta_1 \rangle$ and $\langle r_2, \theta_2 \rangle$ are non-subject cells:

$$\begin{aligned}
R_{\text{⬘}} &= \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid r_1 < r_2 \Big\}; \; R_{\text{⬙}} = \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid r_1 > r_2 \Big\}; \\
R_{\text{⬗}} &= \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid \theta_1 \lessdot \theta_2 \Big\}; \; R_{\text{⬖}} = \Big\{ \big(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle \big) \mid \theta_1 \gtrdot \theta_2 \Big\}.
\end{aligned}$$

We denote the modal operators corresponding to the relative relations by $\Diamond^{\uparrow}$, $\Diamond^{\downarrow}$, $\Diamond^{\rightarrow}$, $\Diamond^{\leftarrow}$, $\Diamond_{\blacktriangle}$, $\Diamond_{\blacktriangledown}$, $\Diamond_{\blacktriangleright}$, $\Diamond_{\blacktriangleleft}$, ⬘, ⬙, ⬗, and ⬖. Thus, for every model $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$, every $\langle r, \theta \rangle \in \mathcal{C}$, every $\otimes \in \{\Diamond^{\uparrow}, \Diamond^{\downarrow}, \Diamond^{\rightarrow}, \Diamond^{\leftarrow}, \Diamond_{\blacktriangle}, \Diamond_{\blacktriangledown}, \Diamond_{\blacktriangleright}, \Diamond_{\blacktriangleleft}, \text{⬘}, \text{⬙}, \text{⬗}, \text{⬖}\}$, and every formula φ it holds that:

$$\mathfrak{M}, \langle r, \theta \rangle \Vdash \otimes\varphi \quad \text{iff} \quad \begin{array}{l} \text{there exists } \langle r', \theta' \rangle \in \mathcal{C} \text{ such that } \langle r, \theta \rangle R_{\otimes} \langle r', \theta' \rangle \\ \text{and } \mathfrak{M}, \langle r', \theta' \rangle \models \varphi. \end{array}$$

Proposition 6. *All the operators* $\Diamond^{\uparrow}$, $\Diamond^{\downarrow}$, $\Diamond^{\rightarrow}$, $\Diamond^{\leftarrow}$, $\Diamond_{\blacktriangle}$, $\Diamond_{\blacktriangledown}$, $\Diamond_{\blacktriangleright}$, $\Diamond_{\blacktriangleleft}$, ⬘, ⬙, ⬗, *and* ⬖ *are expressible in* SOSL.

Proof. For each operator $\otimes$ mentioned in the proposition the relation $R_{\otimes}$ can be characterised using three finite subsets of $\mathbb{Z} \times \mathbb{Z}$, which we denote by $\Delta^1_{\otimes}$, $\Delta^2_{\otimes}$, and $\Delta^3_{\otimes}$. These sets determine $R_{\otimes}$ in the sense that for all cells $\langle r_1, \theta_1 \rangle$ and $\langle r_2, \theta_2 \rangle$, we have $\langle r_1, \theta_1 \rangle R_{\otimes} \langle r_2, \theta_2 \rangle$ if and only if one of the following statements hold:

1. There is $(n, m) \in \Delta^1_{\otimes}$ such that $r_2 = r_1 + n$ and $\theta_2 = \theta_1 + m$;
2. There is $(n, m) \in \Delta^2_{\otimes}$ such that $\langle r_1 + n, \theta_1 + m\rangle \in \mathcal{C}$, $r_2 > r_1 + n$, and $\theta_2 = \theta_1 + m$;
3. There is $(n, m) \in \Delta^3_{\otimes}$ such that $\langle r_1 + n, \theta_1 + m\rangle \in \mathcal{C}$, $r_2 < r_1 + n$, and $\theta_2 = \theta_1 + m$.

Intuitively, given a cell $\langle r_1, \theta_1\rangle$, the sets $\Delta^1_{\otimes}$, $\Delta^2_{\otimes}$, and $\Delta^3_{\otimes}$ divide the image of $\langle r_1, \theta_1\rangle$ under $R_{\otimes}$ into three subsets, corresponding to Statements 1, 2, and 3, respectively, where the first subset needs to be finite, whereas the other two can be infinite.

It is not hard to observe that for each operator mentioned in the proposition we can indeed introduce such $\Delta^1_{\otimes}$, $\Delta^2_{\otimes}$, and $\Delta^3_{\otimes}$. For example consider the operator $\Diamond$, then we can set $\Delta^1_{\Diamond} = \emptyset$, $\Delta^2_{\Diamond} = \{(0, 0)\}$, and $\Delta^3_{\Diamond} = \emptyset$. For the other operators these sets can be more complex; for example in the case of $\blacklozenge$ we set $\Delta^1_{\blacklozenge} = \{(n, m) \mid 1 \le n \le 180 \text{ and } (0 \le m < n \text{ or } 1 \le 360 + m \le n)\}$, $\Delta^2_{\blacklozenge} = \{(180, m) \mid |m| \le 359\}$, and $\Delta^3_{\blacklozenge} = \emptyset$.

Given sets $\Delta^1_{\otimes}$, $\Delta^2_{\otimes}$, and $\Delta^3_{\otimes}$ satisfying Statements 1–3, we can define the operator $\otimes$, as follows, where φ is an arbitrary SOSL-formula:

$$\otimes\,\varphi = \neg\mathsf{cell}_{\langle 0,0\rangle} \wedge \Bigg(\bigvee_{(n,m)\in\Delta^1_{\otimes}} \textcircled{\scriptsize$\uparrow$}^n\, \textcircled{\scriptsize$\rightarrow$}^m (\neg\mathsf{cell}_{\langle 0,0\rangle} \wedge \varphi) \vee \bigvee_{(n,m)\in\Delta^2_{\otimes}} \textcircled{\scriptsize$\uparrow$}^n\, \textcircled{\scriptsize$\rightarrow$}^m\, \Diamond_{\uparrow}(\neg\mathsf{cell}_{\langle 0,0\rangle} \wedge \varphi) \vee \bigvee_{(n,m)\in\Delta^3_{\otimes}} \textcircled{\scriptsize$\uparrow$}^n\, \textcircled{\scriptsize$\rightarrow$}^m\, \Diamond_{\downarrow}\big(\neg\mathsf{cell}_{\langle 0,0\rangle} \wedge \varphi\big) \Bigg).$$

The first conjunct ensures that the current cell is not $\langle 0, 0\rangle$, as $\langle 0, 0\rangle$ is never the first argument of $R_{\otimes}$. The second conjunct is composed of three disjuncts corresponding to Statements 1–3. Note also that $\langle 0, 0\rangle$ cannot be the second argument of $R_{\otimes}$, which we ensure explicitly in each of the three disjuncts. Finally, observe that our requirement that each of the sets $\Delta^1_{\otimes}$, $\Delta^2_{\otimes}$, and $\Delta^3_{\otimes}$ is finite is necessary to make the disjunctions in the formula above finite. □

Like in the case of the intrinsic relations, SOSL enables to define a much wider family of relative relations, and the proof of Proposition 6 reveals what property such relations need to satisfy to be expressible. In particular, they need to be characterisable by finite sets $\Delta^1_{\otimes}$, $\Delta^2_{\otimes}$, and $\Delta^3_{\otimes}$ satisfying the requirements formulated in the proof above.

3.4. Distance Operators

Next, we show how to define operators expressing the distance between the central points of two cells. We can compute the distance between the points $\langle r_1, \theta_1 \rangle$ and $\langle r_2, \theta_2 \rangle$, denoted here by $dist(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle)$, as follows:

$$dist(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) = \sqrt{r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot cos(\theta_1 - \theta_2)}.$$

Then, for d being a rational number we introduce the operator $\Diamond_{=d}$, where $\Diamond_{=d}\, \varphi$ holds in $\langle r_1, \theta_1 \rangle$ if and only if φ holds in some $\langle r_2, \theta_2 \rangle$ such that $dist(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) = d$. Similarly, the operators $\Diamond_{<d}$ and $\Diamond_{>d}$ state that the aforementioned distance is, respectively, less or greater than d.

Proposition 7. *For every rational number d, the operators $\Diamond_{=d}$, $\Diamond_{<d}$, and $\Diamond_{>d}$ are expressible in* SOSL.

Proof. First, we show that for every formula φ we can define $\Diamond_{=d}\, \varphi$ as

$$\Diamond_{=d}\, \varphi = \textcircled{\uparrow}^d \varphi \vee \textcircled{\downarrow}^d \varphi \vee \tag{1}$$

$$\bigvee_{\langle r_1, \theta_1 \rangle \in \mathcal{C} \ \mid\ r_1 < last(d)} \Big(\mathsf{cell}_{\langle r_1, \theta_1 \rangle} \wedge \tag{2}$$

$$\bigvee_{\langle r_2, \theta_2 \rangle \in \mathcal{C} \ \mid\ dist(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) = d} \mathsf{E}(\mathsf{cell}_{\langle r_2, \theta_2 \rangle} \wedge \varphi) \Big) \tag{3}$$

where $last(d) = d + \min\{n \in \mathbb{N}_+ \mid dist(\langle n, 0 \rangle, \langle n, 1 \rangle) > d\}$ and we identify $\textcircled{\uparrow}^d \varphi$ and $\textcircled{\downarrow}^d \varphi$ with $\bot$, if d is not a natural number. It is relatively easy to see that if Formulas (1)–(3) hold in $\langle r_1, \theta_1 \rangle$, there exists a cell $\langle r_2, \theta_2 \rangle$ in which φ holds and such that the distance between the cells is d.

For the opposite implication assume that φ holds in $\langle r_2, \theta_2 \rangle$ which is in the distance d from $\langle r_1, \theta_1 \rangle$. If $\theta_1 = \theta_2$, then as the distance between $\langle r_1, \theta_1 \rangle$ and $\langle r_2, \theta_2 \rangle$ equals d, we have $|r_2 - r_1| = d$, which is expressed by Formula (1). Now, assume that $\theta_1 \neq \theta_2$. We show that r_1 needs to be small. Suppose towards a contradiction that $r_1 \geq last(d)$. Then, by the definition of $last(d)$, for every $n \in \mathbb{N}$ such that $n \geq r_1 - d$ we have $dist(\langle n, 0 \rangle, \langle n, 1 \rangle) > d$. Since $dist(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) = d$ and $\theta_1 \neq \theta_2$, we obtain that $r_2 > r_1 - d$, and so $dist(\langle r_2, 0 \rangle, \langle r_2, 1 \rangle) > d$. Now, let r be the minimum of r_1 and r_2, thus $dist(\langle r, 0 \rangle, \langle r, 1 \rangle) > d$. Since $\theta_1 \neq \theta_2$, we have $\Delta(\theta_1, \theta_2) \geq 1$, and so $dist(\langle r, \theta_1 \rangle, \langle r, \theta_2 \rangle) > d$. On the other hand, since $dist(\langle r_1, \theta_1 \rangle, \langle r_2, \theta_2 \rangle) = d$, we have $dist(\langle r, \theta_1 \rangle, \langle r, \theta_2 \rangle) \leq d$, which raises a contradiction, so $r_1 < last(d)$.

To determine r_1 and θ_1 it suffices to check which of the formulas of the form $\mathsf{cell}_{\langle r_1,\theta_1\rangle}$, for $r_1 < last(d)$ and $-180 \leq \theta_1 \leq 179$, is satisfied in the current cell, which is expressed by Formula (2). Note that the number of candidate values for r_1 and θ_1 is bounded, so the disjunction in Formula (2) is finite. Assuming that we know the values of r_1 and θ_1, it is sufficient to find out if there are values for r_2 and θ_2 such that $dist(\langle r_1,\theta_1\rangle,\langle r_2,\theta_2\rangle) = d$, and φ holds in $\langle r_2,\theta_2\rangle$. Observe that for any $\langle r_1,\theta_1\rangle$ and d there is a bounded number of $\langle r_2,\theta_2\rangle$ such that $dist(\langle r_1,\theta_1\rangle,\langle r_2,\theta_2\rangle) = d$. Hence, the size of the disjunction from Formula (3) is bounded. Now, for each $\langle r_2,\theta_2\rangle$ such that $dist(\langle r_1,\theta_1\rangle,\langle r_2,\theta_2\rangle) = d$ it is enough to verify whether φ is true in the cell in which $\mathsf{cell}_{\langle r_2,\theta_2\rangle}$ holds, which is expressed in Formula (3).

Analogously, we can define $\Diamond_{<d}$ as follows:

$$\Diamond_{<d}\,\varphi = \bigvee_{n\in\mathbb{N}\,\mid\,n<d} \circledcirc\!\!\uparrow^n\varphi \vee \bigvee_{n\in\mathbb{N}\,\mid\,n<d} \circledcirc\!\!\downarrow^n\varphi \vee$$
$$\bigvee_{\langle r_1,\theta_1\rangle\in\mathcal{C}\,\mid\,r_1<last(d)} \Big(\mathsf{cell}_{\langle r_1,\theta_1\rangle} \wedge \bigvee_{\langle r_2,\theta_2\rangle\in\mathcal{C}\,\mid\,dist(\langle r_1,\theta_1\rangle,\langle r_2,\theta_2\rangle)<d} \mathsf{E}(\mathsf{cell}_{\langle r_2,\theta_2\rangle} \wedge \varphi)\Big).$$

Using a similar idea we can also define the operator $\Diamond_{>d}$, but due to its combinatorial nature, the formula is long, so we omit its presentation. □

4. Computational Complexity

In this section, we study the computational complexity of the satisfiability problem in SOSL. As we show, the problem is PSPACE-complete. We obtain the lower bound by reducing the satisfiability problem in linear temporal logic (LTL in short) [25]; we simulate an LTL-model with one ray of an SOSL-model, that is, with a set of cells with the same angle-coordinate. Then, we show the matching upper bound by constructing a Büchi automaton, where every state describes formulas satisfied in cells with the same radius-coordinate. We prove that checking satisfiability over only finite as well as over only infinite models remains PSPACE-complete. On the other hand, if the size of a model is bounded by a fixed constant, the problem becomes NP-complete.

We start by showing the NP-completeness of the satisfiability problem over models whose size does not exceed a fixed constant c. Importantly, c is not treated as a part of the input to the problem.

Theorem 8. *Let c be fixed constant. Then, checking* SOSL*-satisfiability over models of size at most c is* NP*-complete.*

Proof. The lower bound is seamlessly inherited from the well-known NP-completeness of propositional calculus [26]. To show the matching upper bound we can guess a model $\mathfrak{M}$ of size at most c over the signature of the input formula φ and then check whether φ is satisfied in $\mathfrak{M}$. The number of cells in $\mathfrak{M}$ does not depend on the input (i.e., on φ), and so the valuation of propositional variables in $\mathfrak{M}$ is polynomial (in the size of φ). Thus, $\mathfrak{M}$ can be guessed in NP. To check whether φ is satisfied in $\mathfrak{M}$ we determine inductively, for every subformula ψ of φ, the cells of $\mathfrak{M}$ in which ψ holds. The valuation of propositional variables is already determined by $\mathfrak{M}$. To check in which cells a compound formula ψ holds it is sufficient to know where the subformulas of ψ hold, and so the time consumed by such a check does not depend on φ. The inductive procedure has as many steps as there are subformulas of φ, and so it is feasible in polynomial time in the size of φ. ☐

Next, we show the PSPACE-hardness for checking satisfiability in infinite, finite, and arbitrary SOSL-models. First, we prove that checking satisfiability over infinite models is PSPACE-hard by reducing the satisfiability problem in LTL over finite traces, denoted by LTL_f [27]. Since we are considering infinite SOSL-models, it would be more natural to reduce the satisfiability problem in LTL and not in LTL_f, however, the advantage of our approach is that we can essentially reuse it when considering finite SOSL-models. The LTL_f-formulas are generated by the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid X\varphi \mid G\varphi,$$

where $p \in \textsc{prop}$, X is the 'next point', and G the 'everywhere in the future' operator. An LTL_f-model is a pair $(\mathbb{N}_{\leq k}, V)$, where $\mathbb{N}_{\leq k}$ is the initial segment of natural numbers which are smaller or equal to some $k \in \mathbb{N}$—such k is the *size* of the model—and $V : \textsc{prop} \longrightarrow \mathcal{P}(\mathbb{N}_{\leq k})$. The forcing relation is defined as in standard LTL and the satisfiability of LTL_f-formulas is known to be PSPACE-complete [27].

Theorem 9. SOSL-*satisfiability over infinite models is* PSPACE-*hard.*

Proof. We define a translation τ of LTL_f-formulas into SOSL-formulas as:

$$\begin{array}{rclcrcl}
\tau(p) & = & p & \quad & \tau(\varphi \wedge \psi) & = & \tau(\varphi) \wedge \tau(\psi) \\
\tau(\neg p) & = & \neg p & & \tau(X\psi) & = & \textcircled{\uparrow}(q \wedge \tau(\psi)) \\
\tau(\psi \vee \chi) & = & \tau(\psi) \vee \tau(\chi) & & \tau(G\psi) & = & \boxed{\uparrow}(q \rightarrow \tau(\psi))
\end{array}$$

where $p \in \textsc{prop}$, q is a fresh propositional variable, and ψ, χ are LTL_f-formulas. Then, given an LTL_f formula φ we define φ' as:

$$\tau(\varphi) \wedge \mathsf{cell}_{\langle 1,0\rangle} \wedge q \wedge \mathsf{A}(q \to \bigwedge_{1\le i\le 179} \text{\textcircled{$\to$}}^i \neg q \wedge \bigwedge_{1\le i\le 180} \text{\textcircled{$\leftarrow$}}^i \neg q) \wedge \\ \mathsf{A}(\neg q \to \boxed{\uparrow} \neg q) \wedge \mathsf{E}(q \wedge \neg \text{\textcircled{$\uparrow$}}\, q),$$

where $\mathsf{cell}_{\langle 1,0\rangle}$, A, and E are as defined in Section 3. Formula φ' states that there is $k \in \mathbb{N}$ such that q holds in all cells $\langle r, 0\rangle$, for $1 \le r \le k$; and $\neg q$ holds in all other cells. Clearly, φ' can be constructed in logarithmic space; in particular to compute $\tau(\varphi)$ it suffices to scan φ and replace the operators X and G with appropriate symbols. Hence, it remains to show that φ is LTL_f-satisfiable if and only if φ' is SOSL-satisfiable in an infinite model.

First, assume that $\mathcal{M} = (\mathbb{N}_{\le k}, V)$ is an LTL_f-model, satisfying φ, so $\mathcal{M}, 0 \Vdash \varphi$. Let $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V')$ be an infinite SOSL-model such that for all $\langle r, \theta\rangle \in \mathcal{C}$, $p \in \textsc{prop}$, and q a propositional variable not occurring in φ:

$$\begin{aligned} \langle r,\theta\rangle \in V'(p) \quad &\text{iff} \quad r-1 \in V(p) \text{ and } \theta = 0,\\ \langle r,\theta\rangle \in V'(q) \quad &\text{iff} \quad r \le k \text{ and } \theta = 0. \end{aligned}$$

By the definition of $\mathfrak{M}$ every conjunct of φ' which is different than $\tau(\varphi)$ holds in $\mathfrak{M}$ in the cell $\langle 1, 0\rangle$. To show $\mathfrak{M}, \langle 1, 0\rangle \Vdash \tau(\varphi)$, we prove inductively on the complexity of a formula that for every LTL_f formula ψ, and every $n \le k$, if $\mathcal{M}, n \Vdash \psi$, then $\mathfrak{M}, \langle n+1, 0\rangle \Vdash \tau(\psi)$. Thus, $\mathfrak{M}, \langle 1, 0\rangle \Vdash \varphi'$.

For the opposite direction assume that $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V')$ is an infinite SOSL-model, with $\mathfrak{M}, \langle r, \theta\rangle \Vdash \varphi'$. Thus, it needs to be the case that $r = 1$, $\theta = 0$, and there is k such that $V'(q) = \{\langle r, 0\rangle \mid r \le k\}$. We define an LTL_f-model $\mathcal{M} = (\mathbb{N}_{\le k}, V)$ such that for all $n \le k$ and $p \in \textsc{prop}$ distinct from q

$$n \in V(p) \quad \text{iff} \quad \langle n+1, 0\rangle \in V'(p).$$

We can show by induction on the complexity of a formula that for every SOSL-formula ψ and $r \in \mathbb{N}$ such that $1 \le r \le k+1$, if $\mathfrak{M}, \langle r, 0\rangle \Vdash \tau(\psi)$, then $\mathcal{M}, r-1 \Vdash \psi$. In particular, $\mathcal{M}, 0 \Vdash \varphi$. $\square$

We can show that the same reduction works also for finite SOSL-models:

Theorem 10. SOSL-*satisfiability over finite models is* PSPACE-*hard.*

Proof. For every LTL_f-formula φ, we construct an SOSL-formula φ' in the same way as in the proof of Theorem 9. Then, we need to prove that φ is LTL_f-satisfiable if and only if φ' is SOSL-satisfiable over finite models. This, however, can be done similarly to the method from the proof of Theorem 9, namely for every LTL_f-model of size k we construct a corresponding SOSL-model of size $k+1$, and vice versa. The only difference is that now, the SOSL-model is finite, which actually makes the proof even easier. $\square$

As the translation of an LTL_f-formula to an SOSL-formula is exactly the same in the reductions used to prove Theorems 9 and 10, we obtain:

Corollary 11. SOSL*-satisfiability (over arbitrary models) is* PSPACE*-hard.*

Next, we show the matching PSPACE upper bounds. The main difficulty is to prove that SOSL-satisfiability over arbitrary models is in PSPACE (whereas to show that the same upper bounds hold for finite and infinite models it is sufficient to introduce formulas ensuring that a model is finite or infinite, respectively). Clearly, to check whether φ is SOSL-satisfiable it suffices to verify whether $\mathsf{E}\varphi$ can be satisfied in $\langle 0, 0\rangle$. To that end we build a generalized Büchi automaton which to a certain extent resembles the standard construction for LTL [28], as $\Diamond\!\!\!\uparrow$ and $\Diamond\!\!\!\downarrow$ have a similar meaning to the LTL-modal operators 'somewhere in the future' and 'somewhere in the past', respectively.

Thus, our aim is to check, for an arbitrary SOSL-formula φ, if it can be satisfied in $\langle 0, 0\rangle$. First, we guess in NP a set K (called *kernel*) of formulas which are satisfied in $\langle 0, 0\rangle$ (thus $\varphi \in K$). Then, we construct a generalized Büchi automaton $G_{\varphi,K}$, depending on K, whose states determine formulas which are satisfied in cells sharing the same radius coordinate (a set of such cells forms a *ring* around $\langle 0, 0\rangle$); each symbol of the automaton's alphabet describes 360 sets of propositional variables (which encodes a valuation in cells located within one ring); the transition relation forces consecutive rings to match each other; and the accepting sets assure that every 'promise' (e.g., that p holds somewhere behind an object) will eventually be kept. As we show, an infinite word w is accepted by $G_{\varphi,K}$ (i.e., there is a run of $G_{\varphi,K}$ on w where at least one state from each accepting set occurs infinitely often) if and only if w describes an SOSL-model in which φ holds in $\langle 0, 0\rangle$.

In the construction below, up to Lemma 16, let φ be a fixed yet arbitrary SOSL-formula. We use the standard notion of *closure of* φ, denoted by $\mathsf{cl}(\varphi)$, which stands for the set of all subformulas of φ and their negations (as usual, we identify $\neg\neg\psi$ with ψ, for any formula ψ). For a set $Y \subseteq \mathsf{cl}(\varphi)$ of formulas we use the following notions: Y is (propositionally) *consistent* if for all ψ and $\varphi_1 \vee \varphi_2$ in $\mathsf{cl}(\varphi)$ we have (i) if $\psi \in Y$, then $\neg\psi \notin Y$ and (ii) $\varphi_1 \vee \varphi_2 \in Y$ iff $\varphi_1 \in Y$ or $\varphi_2 \in Y$; Y is *maximal* if for all $\psi \in \mathsf{cl}(\varphi)$, $\psi \notin Y$ implies $\neg\psi \in Y$. Then, we define a *kernel* for φ as follows

Definition 12. A φ-*kernel* is a consistent and maximal set $K \subseteq \mathsf{cl}(\varphi)$ such that $\varphi \in K$ and $\textcircled{\downarrow}\psi, \textcircled{\rightarrow}\psi, \textcircled{\leftarrow}\psi, \Diamond\!\!\!\downarrow\psi \notin K$, for all $\psi \in \mathsf{cl}(\varphi)$.

Intuitively, a φ-kernel is set of formulas which are to be satisfied in $\langle 0, 0\rangle$, and so it contains φ and it does not contain formulas which clearly cannot be satisfied in $\langle 0, 0\rangle$. Next, we define *rings* which intuitively contain formulas satisfied in cells located in consecutive rings around $\langle 0, 0\rangle$. For brevity, we denote the set $\{-180, \ldots, 179\}$ of angle-coordinates by ang.

Definition 13. A φ-*ring* is a function ρ which maps elements of ang to subsets of $\mathsf{cl}(\varphi)$, such that one of the following conditions holds:

- for each $\theta \in \mathsf{ang}$ we have $\rho(\theta) = \emptyset$ (such a ρ is called *empty ring* and denoted by $\rho_\emptyset$); or
- for each $\theta \in \mathsf{ang}$ the set $\rho(\theta)$ is maximal and consistent and for all $\textcircled{\rightarrow}\psi, \textcircled{\leftarrow}\chi \in \mathsf{cl}(\varphi)$ and all $m, n \in \mathbb{N}$:

$$\textcircled{\rightarrow}\psi \in \rho(\theta) \text{ iff } \psi \in \rho(\theta+1) \quad \text{and} \quad \textcircled{\leftarrow}\chi \in \rho(\theta) \text{ iff } \chi \in \rho(\theta-1).$$

The above definition guarantees that each non-empty ring ρ is locally consistent with regard to formulas with $\textcircled{\rightarrow}$ and $\textcircled{\leftarrow}$, namely formulas of the form $\textcircled{\rightarrow}\psi$ and $\textcircled{\leftarrow}\psi$ belong to $\rho(\theta)$ if and only if they have witnesses in the same ring. Next, we define the notions of ring being a *successor* of a kernel, and ring being a *successor* of another ring. The former notion will be crucial to define the initial state of the automaton and the latter to define the transition function of the automaton.

Definition 14. A φ-ring ρ is a *successor* of a φ-kernel K if:

- for each $\textcircled{\uparrow}\psi \in \mathsf{cl}(\varphi)$: $\textcircled{\uparrow}\psi \in K$ iff there is $\theta \in \mathsf{ang}$ such that $\psi \in \rho(\theta)$;
- for each $\otimes\psi \in \mathsf{cl}(\varphi)$ with $\otimes \in \{\textcircled{\downarrow}, \Diamond\!\!\!\downarrow\}$: either $\otimes\psi \in \rho(\theta)$ for all $\theta \in \mathsf{ang}$ or $\otimes\psi \notin \rho(\theta)$ for all $\theta \in \mathsf{ang}$;
- for each $\otimes\psi \in \mathsf{cl}(\varphi)$ with $\otimes \in \{\textcircled{\downarrow}, \Diamond\!\!\!\downarrow\}$: $\psi \in K$ iff $\otimes\psi \in \rho(\theta)$ for all $\theta \in \mathsf{ang}$.

A φ-ring ρ' is a *successor* of a φ-ring ρ if either $\rho = \rho' = \rho_\emptyset$, or $\rho \neq \rho_\emptyset$ and for all $\theta \in \mathsf{ang}$:

- for each $\textcircled{\uparrow}\psi \in \mathsf{cl}(\varphi)$: $\textcircled{\uparrow}\psi \in \rho(\theta)$ iff $\psi \in \rho'(\theta)$;
- for each $\textcircled{\downarrow}\psi \in \mathsf{cl}(\varphi)$: if $\rho' \neq \rho_\emptyset$, then $\textcircled{\downarrow}\psi \in \rho'(\theta)$ iff $\psi \in \rho(\theta)$;
- for each $\Diamond\!\!\!\uparrow\psi \in \mathsf{cl}(\varphi)$: $\Diamond\!\!\!\uparrow\psi \in \rho(\theta)$ iff $\psi \in \rho'(\theta)$ or $\Diamond\!\!\!\uparrow\psi \in R'(\theta)$;
- for each $\Diamond\!\!\!\downarrow\psi \in \mathsf{cl}(\varphi)$: if $\rho' \neq \rho_\emptyset$, then $\Diamond\!\!\!\downarrow\psi \in \rho'(\theta)$ iff we have $\psi \in \rho(\theta)$ or $\Diamond\!\!\!\downarrow\psi \in \rho(\theta)$.

In the following part of the section we use $\textsc{prop}(\varphi)$ to denote the set of all propositional variables occurring in φ.

Definition 15. For a φ-kernel K we define a generalized nondeterministic Büchi automaton $G_{\varphi,K} = (\Sigma, Q, Q_0, \delta, \mathcal{F})$, where:

- Σ is the set of all functions $\sigma : \mathsf{ang} \longrightarrow \mathcal{P}(\textsc{prop}(\varphi))$;
- Q is the set of all φ-rings;
- Q_0 is the set of all φ-rings $\rho \in Q$ such that ρ is a successor of K;
- $\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$ is such that for each $\rho \in Q$ and $\sigma \in \Sigma$:

$$\delta(\rho, \sigma) = \{\rho' \in Q \mid \rho' \text{ is a successor of } \rho \text{ and } \rho'(\theta) \cap \textsc{prop}(\varphi) = \sigma(\theta), \text{ for all } \theta \in \mathsf{ang}\};$$

- $\mathcal{F}$ that consists of sets of the form:

$$\mathcal{F}^{\theta}_{\Diamond\!\!\!\uparrow\psi} = \{\rho \in Q \mid \Diamond\!\!\!\uparrow\psi \notin \rho(\theta) \text{ or } \psi \in \rho(\theta)\} \cup \{\rho_\emptyset\},$$

  for each $\theta \in \mathsf{ang}$ and $\Diamond\!\!\!\uparrow\psi \in \mathsf{cl}(\varphi)$.

Intuitively, the definition of a state of the automaton allows us to capture the semantics of formulas preceded by $\textcircled{\rightarrow}$ and $\textcircled{\leftarrow}$; the transition relation captures the semantics of formulas preceded by $\textcircled{\uparrow}$, $\textcircled{\downarrow}$, and $\Diamond\!\!\!\downarrow$; whereas the accepting conditions correspond to the semantics of formulas preceded by $\Diamond\!\!\!\uparrow$. As a result, we can show that the automaton accepts exactly those words that represent models in which φ holds in $\langle 0, 0\rangle$.

Lemma 16. *For every* SOSL*-formula* φ *the following are equivalent:*

1. *There exists a φ-kernel K such that the language of $G_{\varphi,K}$ is non-empty.*
2. *There exists an* SOSL*-model* $\mathfrak{M}$ *such that* $\mathfrak{M}, \langle 0, 0\rangle \Vdash \varphi$.

Proof. ($1 \Rightarrow 2$) Let $w = \sigma_2\sigma_3\ldots$ be an infinite word accepted by $G_{\varphi,K}$ and let $\rho_1, \rho_2, \ldots$ be an infinite sequence of states in an accepting run of $G_{\varphi,K}$ on w. Note that if $\rho_n = \rho_\emptyset$ for some $n \in \mathbb{N}_+$, then $\rho_m = \rho_\emptyset$ for all $m > n$. We define a model $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$ corresponding to the non-empty rings in the run, namely, the size of $\mathfrak{M}$ is the largest n such that $\rho_n \neq \rho_\emptyset$ or ∞ if no such n exists, and for each $p \in \textsc{prop}(\varphi)$ the following hold:

- $\langle 0, 0\rangle \in V(p)$ iff $p \in K$, and
- for each $\langle r, \theta\rangle \in \mathcal{C}$ distinct from $\langle 0, 0\rangle$ we have $\langle r, \theta\rangle \in V(p)$ iff $p \in \rho_r(\theta)$.

We can show by simultaneous induction on the complexity of a formula that for each $\psi \in \mathsf{cl}(\varphi)$ the following statements hold:

- If $\psi \in K$, then $\mathfrak{M}, \langle 0, 0\rangle \Vdash \psi$;
- For each $r \in \mathbb{N}_+$ and $\theta \in \mathsf{ang}$, if $\psi \in \rho_r(\theta)$, then $\mathfrak{M}, \langle r, \theta\rangle \Vdash \psi$.

By Definition 12, we have $\varphi \in K$, and so, by the first of the statements above, we obtain $\mathfrak{M}, \langle 0, 0\rangle \Vdash \varphi$.

($1 \Leftarrow 2$) Let $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, V)$ be an SOSL-model (which is finite or infinite) such that $\mathfrak{M}, \langle 0, 0\rangle \Vdash \varphi$. We define a set $K = \{\psi \in \mathsf{cl}(\varphi) \mid \mathfrak{M}, \langle 0, 0\rangle \Vdash \psi\}$. Since $\mathfrak{M}$ is an SOSL-model, the set K satisfies the conditions from Definition 12, so K is a φ-kernel. Next, we show that the language of $G_{\varphi,K} = (\Sigma, Q, Q_0, \delta, \mathcal{F})$ is non-empty. Let $w = \sigma_2\sigma_3\ldots$ be an infinite word such that for each $r \geq 2$ which is smaller or equal to the the size of $\mathfrak{M}$ and for each $\theta \in \mathsf{ang}$ we set $\sigma_r(\theta) = \{p \in \textsc{prop}(\varphi) \mid \mathfrak{M}, \langle r, \theta\rangle \Vdash p\}$. Next, let $\mathfrak{R} = \rho_1, \rho_2, \ldots$ be an infinite sequence of functions such that for each $r \geq 1$ which is smaller or equal to the size of $\mathfrak{M}$ and for each $\theta \in \mathsf{ang}$ we set $\rho_r(\theta) = \{\psi \in \mathsf{cl}(\varphi) \mid \mathfrak{M}, \langle r, \theta\rangle \Vdash \psi\}$. If r is greater than the size of $\mathfrak{M}$, we put $\rho_r = \rho_\emptyset$.

Each ρ_r satisfies the conditions from Definition 13, so $\mathfrak{R}$ is a sequence of φ-rings, that is, a sequence of states of $G_{\varphi,K}$. Then, we can show that $\mathfrak{R}$ is an accepting run of $G_{\varphi,K}$ on w, namely, (i) $\rho_1 \in Q_0$, (ii) for each $r \in \mathbb{N}_+$ we have $\rho_{r+1} \in \delta(\rho_r, \sigma_{r+1})$, and (iii) for every set in $\mathcal{F}$ there are infinitely many elements of $\mathfrak{R}$ belonging to this set. As a result, the language of $G_{\varphi,K}$ is non-empty. □

Finally, we use Lemma 16 to show the matching PSpace upper bound.

Theorem 17. SOSL-*satisfiability is* PSPACE-*complete.*

Proof. The lower bound is given in Corollary 11. To show the matching upper bound assume that we want to check whether a formula φ is SOSL-satisfiable. To this end it suffices to verify whether there is a model in which $\varphi' = \mathsf{E}\varphi$ is satisfied in $\langle 0, 0\rangle$. Hence, by Lemma 16 we need to check whether there is a φ'-kernel K such that the language of $G_{\varphi',K}$ is non-empty. Since $\mathsf{cl}(\varphi')$ is of polynomial size (in the size of φ) and $K \subseteq \mathsf{cl}(\varphi')$, the size of K is also polynomial, and so we can guess it in NP. On the other hand, the size of $G_{\varphi',K}$ is exponential, so we cannot construct it in PSPACE, but as the size of every state of the automaton is polynomial, we can use the standard 'on-the-fly' approach [28] to check in PSPACE whether the language of $G_{\varphi',K}$ is non-empty. Hence, the whole procedure is in PSPACE. □

We showed the upper bound for the class of all models. Now, observe that $\mathsf{E}\neg \circledS \top$ (respectively, $\mathsf{A} \circledS \top$) is satisfied in a model if and only if it is finite (respectively, infinite). Thus, the PSPACE upper bound for the satisfiability problem holds also for the classes of finite and infinite models. Since we have already established that these problems are PSPACE-hard (Theorems 9 and 10), we obtain the following tight complexity result:

Theorem 18. SOSL-*satisfiability over finite as well as over infinite models is* PSPACE-*complete.*

5. Related Work on Space Representation

In this section, we consider space representations used in modal logics which seem to be the most closely related to SOSL; in particular we restrict attention to compass logic [4], spatial propositional neighborhood logic (SpPNL) [6], and cone logic [9]. Although spatial objects (constituting the universe of a model) and relations between them (interpreting modal operators) differ in these logics, the location of an object is always described in the absolute frame of reference using the Cartesian system.

In compass logic [4, 5] spatial objects are represented as points in the two-dimensional space with Cartesian coordinates. Modal operators express the relations between points: 'on the same vertical line and above', 'on the same vertical line and below', 'on the same horizontal line and to the left', and 'on the same horizontal line and to the right'. It is shown that the halting problem of a Turing machine reduces to the satisfiability problem of compass logic formulas, and so the latter problem is undecidable [5, 29, 30]. Similarly to compass logic, SpPNL uses the two-dimensional space with Cartesian coordinates. However, formulas are evaluated over rectangles rather than points. Intuitively, each spatial object is approximated with its minimum bounding rectangle and modal operators express the relations between rectangles: 'adjacent from above', 'adjacent from below', 'adjacent to the right', and 'adjacent to the left', which correspond to relations from rectangle algebra [11]. Every formula of compass logic can be translated into an equisatisfiable SpPNL-formula, and so the satisfiability problem in SpPNL is also undecidable. In cone logic [9] a spatial object is an arbitrary subset of the two-dimensional space. Each point is denoted by a pair of Cartesian coordinates and modal operators express the cone-shaped relations between points: 'to the north', 'to the south', 'to the east', and 'to the west.' The satisfiability problem in cone logic over the rational plane $\mathbb{Q} \times \mathbb{Q}$ is shown to be PSPACE-complete by using the tree (pseudo) model property [9].

To see how spatial representations differ in the above-discussed logics let us consider the configuration from Figure 4(a), where a subject is oriented to the north, and on the right side they see a flower behind which there is a door. In compass logic 'the flower is below and to the left of the door' (Figure 4(b)); in SpPNL 'the minimal rectangle containing the flower is (completely) to the left and (completely) below the minimal rectangle containing the door' (Figure 4(c)); in cone logic 'the flower is to the south of the door' (Figure 4(d)). In SOSL the location of the subject, flower and door can be described using various relations expressible in the logic (see Section 3), for example the flower is in the relation $R_{\Diamond}$ with the subject, interpreted as 'the flower is in front of the subject' (Figure 4(e)) and the flower is in the relation $R_{\diamond}$ with the door, meaning that 'the flower is between the door and the subject' (Figure 4(f)).

Observe that in compass logic, SpPNL, and cone logic the subject is not distinguished, and so their position has no influence on the spatial representation of a scene. On the other hand, the position of the subject is crucial when determining the relation between two objects in SOSL. For instance, if in the scene from Figure 4 the subject was turned towards the west (instead of north), we would have 'the flower is to the right of the subject' and if the subject was standing on the stairs, we would have 'the flower is to the right of the door'.

There is a number of other modal logics for spatial reasoning which, however, are less related to SOSL. For instance, $\mathsf{PDL}^{\mathsf{F}}_{\mathsf{M}}$ [10] is an extension of

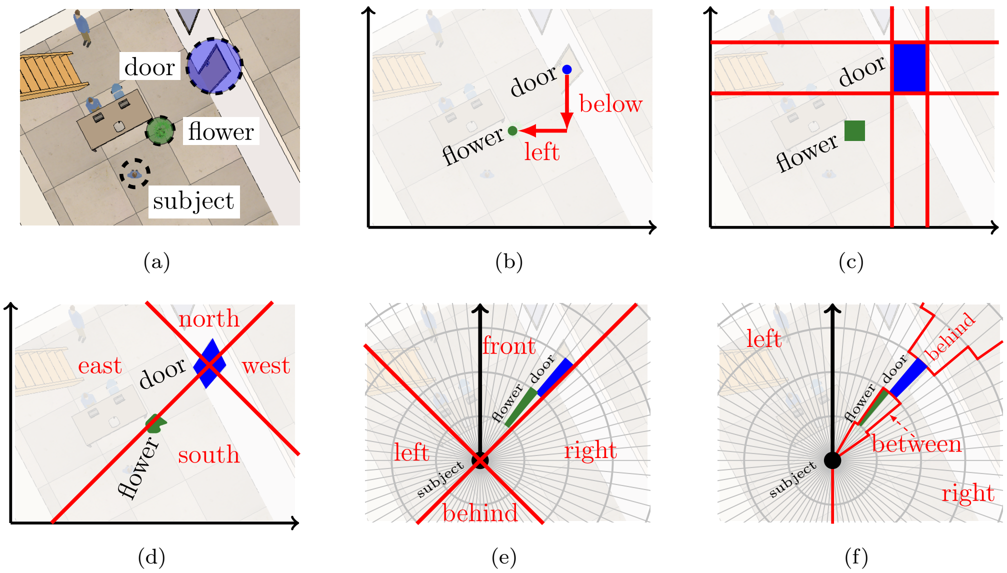

Figure 4: Representation of the same spatial configuration from (a) in compass logic (b), SpPNL (c), cone logic (d), SOSL (intrinsic frame of reference) (e), and SOSL (relative frame of reference) (f)

propositional dynamic logic [31] in which the movement of one object with respect to another is represented by means of a tuple of parameters. There are also logics based on topological spaces [8, 7, 32], where the closure and interior operators are used to express topological relations such as 'disconnected', 'overlaps', and 'inside' known from region connection calculus [13]. There are also modal logics over affine spaces (involving, e.g., the betweenness operator), logics of nearness, and logics of convexity, among others [33].

6. Applications

In this section, we present scenarios which exhibit the expressive power of SOSL and show some fields where SOSL could potentially be applied. In all presented examples we use $R_{\Diamond}$, $R_{\Diamond}$, $R_{\Diamond}$, and $R_{\Diamond}$ to represent the intrinsic relations and $R_{\diamond}$, $R_{\diamond}$, $R_{\diamond}$, and $R_{\diamond}$ for the relative relations.

First, we consider modeling human-centered cognition, which is required in various applications like computer-aided architecture design (CAAD) systems to address problems such as indoor spatial awareness, visibility analysis, or wayfinding [34, 35]. The subject-oriented representation of space makes SOSL an adequate tool for the above-mentioned applications. For the sake of example consider the following SOSL-formulas specifying how household goods should be located with respect to a person sitting on a sofa, where sofa, tv, and window are propositional variables and $d \in \mathbb{Q}$:

$$\mathsf{sofa} \wedge \text{◈}(\mathsf{tv} \wedge \text{⬗}\,\mathsf{window}) \wedge \Diamond_{\leq d}\, \mathsf{lamp};$$
$$\mathsf{A}((\mathsf{tv} \vee \mathsf{window}) \rightarrow \neg\, \text{⬙}\,\mathsf{lamp}) \wedge \mathsf{A}(\mathsf{tv} \rightarrow \neg\, \text{⬘}\,\mathsf{window}).$$

The first formula states that the configuration is described from the perspective of a person sitting on a sofa. There should be a tv standing in front of the sofa and a window located to the right of this tv with respect to the sofa. There should be also a lamp located in a distance at most d from the sofa. The second formula describes the following universal constraints: no lamp can stand between any tv and the sofa or between any window and the sofa (not to block the view); and no window can be located behind any tv (for a better visibility). Then, determining if the specification is spatially possible reduces to checking if the conjunction of the presented formulas is satisfiable.

Next, we consider preprocessing of spatial data. Such data, when gathered by artificial visual systems (e.g., in robotics), is often represented in the polar coordinate system [36], hence SOSL can be used to reason about it. Indeed, the following example shows how SOSL-formulas allow us to filter noisy data and detect borders of obstacles, where obst is a propositional variable satisfied in cells where an obstacle was detected:

$$\begin{aligned}
\mathsf{noise} &\leftrightarrow \mathsf{obst} \wedge \neg \textcircled{\uparrow}\, \mathsf{obst} \wedge \neg \textcircled{\downarrow}\, \mathsf{obst} \wedge \neg \textcircled{\rightarrow}\, \mathsf{obst} \wedge \neg \textcircled{\leftarrow}\, \mathsf{obst};\\
\mathsf{filt\text{-}obst} &\leftrightarrow \mathsf{obst} \wedge \neg \mathsf{noise};\\
\mathsf{bord\text{-}obst} &\leftrightarrow \mathsf{filt\text{-}obst} \wedge (\textcircled{\uparrow} \neg \mathsf{filt\text{-}obst} \vee \textcircled{\downarrow} \neg \mathsf{filt\text{-}obst} \vee \textcircled{\leftarrow} \neg \mathsf{filt\text{-}obst} \vee \textcircled{\rightarrow} \neg \mathsf{filt\text{-}obst}).
\end{aligned}$$

The first formula marks as noise each cell recognized as an obstacle, for which there are no adjacent cells which are also recognized as obstacles. In the second one, the input data is filtered by marking all non-noise cells which were recognized as obstacles with $\mathsf{filt\text{-}obst}$. Then, the third formula determines the borders of obstacles by marking with $\mathsf{bord\text{-}obst}$ the cells which are marked as $\mathsf{filt\text{-}obst}$ and have at least one adjacent cell which is not marked as $\mathsf{filt\text{-}obst}$.

The modal operators definable in SOSL allow us to express qualitative as well as quantitative relations, and combine them when performing reasoning. Indeed, for any $d, d1, d2 \in \mathbb{Q}$, any nominals objA, objB, and objC, the following formulas are tautologies in SOSL:

$$\begin{aligned}
\mathsf{cell}_{\langle 0,0\rangle} \wedge @_{\mathsf{objA}} \text{⬘}\, \mathsf{objB} \wedge \Diamond_{=d}\, \mathsf{objA} \;&\rightarrow\; \Diamond_{>d}\, \mathsf{objB};\\
(\mathsf{cell}_{\langle 0,0\rangle} \wedge \Diamond_{=d}\, \mathsf{objA} \wedge \Diamond_{=d}\, \mathsf{objB} \wedge &\\
\neg @_{\mathsf{objA}}(\mathsf{objB} \vee \text{⬖}\, \mathsf{objB} \vee \text{⬗}\, \mathsf{objB})) \;&\rightarrow\; @_{\mathsf{objA}} \Diamond_{=2d}\, \mathsf{objB};\\
@_{\mathsf{objA}} \Diamond_{=d1}\, \mathsf{objB} \wedge @_{\mathsf{objB}} \Diamond_{=d2}\, \mathsf{objC} \;&\rightarrow\; @_{\mathsf{objA}} \Diamond_{\leq d1+d2}\, \mathsf{objC}.
\end{aligned}$$

The first formula states that if objB is behind objA, then objB is further away from the subject than objA. The second one states that if objA and objB are in the same distance d from the subject, and moreover, objB neither occupies the same spot, nor is to the left nor to the right of objA, then the distance between objA and objB is $2d$ (objA and objB are on the opposite sides of the subject). The third formula expresses the triangle inequality theorem.

7. Conclusions

We have constructed a two-dimensional modal logic for subject-oriented spatial reasoning, denoted by SOSL. The approach employs the polar coordinate system to divide an infinite plane into cells of constant length and constant angle-width. The modal operators of the logic correspond to the binary relations between adjacent cells, namely 'outwards', 'inwards', 'clockwise', 'counterclockwise', and the transitive closures of the first two. We have shown that such a representation of space allows us to reflect the subject's perspective and express a wide class of spatial relations, in particular the intrinsic relations 'in front', 'behind', 'to the left', and 'to the right' of the subject; and the relative relations 'behind', 'to the left of', and 'to the right of' an object with respect to the subject, as well as 'between the subject and an object.' Such relations are commonly used in Indo-European languages to represent and reason about space. The framework of SOSL allows a wide range of different interpretations of these relations. We prove that the logic also enables to express hybrid and distance operators, which may be useful in various applications. We show that the satisfiability problem in the logic is PSpace-complete over only infinite, only finite, and over arbitrary models. On the other hand, if the size of models is bounded by a fixed constant, then the satisfiability problem becomes NP-complete.

Since SOSL is subject-centered, it makes it an adequate tool for applications such as computer-aided architecture design which addresses the problems of spatial awareness, visibility analysis, or wayfinding [34, 35], or processing spatial data gathered by artificial visual systems (e.g., in robotics), which is often represented in the polar coordinate system [36].

In future we plan to consider a modification of SOSL in which universe is dense, and its computational properties. Moreover, we would like to perform empirical experiments revealing the meaning ascribed by humans to the basic intrinsic and relative relations they use in everyday life.

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