O-minimality, Nonclassical Modular Functions and Diophantine Problems

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For Dad, who taught me to love mathematics.

For Mum, Chloe, and my whole family,
who taught me all the rest.

Contents

Statement of Originality

This is to certify that the content of this thesis is my own work and has not, to the best of my knowledge, been submitted for any other degree.

I certify that the intellectual content of this thesis is my own, that all sources have been cited appropriately, and that any assistance I received throughout my work towards this document has been acknowledged.

Haden Charles Spence.

Abstract

There now exists an abundant collection of conjectures and results, of various complexities, regarding the diophantine properties of Shimura varieties. Two central such statements are the André-Oort and Zilber-Pink Conjectures, the first of which is known in many cases, while the second is known in very few cases indeed.

The motivating result for much of this document is the modular case of the André-Oort Conjecture, which is a theorem of Pila. It is most commonly viewed as a statement about the simplest kind of Shimura varieties, namely modular curves. Here, we tend instead to view it as a statement about the properties of the classical modular j-function. It states, given a variety $V \subseteq \mathbb{C}^n$, that V contains only finitely many maximal special subvarieties, where a special variety is one which arises from the arithmetic behaviour of the j-function in a certain natural way.

The central question of this thesis is the following: what happens if in such statements we replace the j-function with some other kind of modular function; one which is less well-behaved in one way or another? Such modular functions are naturally called nonclassical modular functions. This question, as we shall see, can be studied using techniques of o-minimality and point-counting, but some interesting new features arise and must be dealt with.

After laying out some of the classical theory, we go on to describe two particular types of nonclassical modular function: *almost holomorphic modular functions* and *quasimodular functions* (which arise naturally from the derivatives of the j-function). We go on to prove some results about the diophantine properties of these functions, including several natural André-Oort-type theorems, then conclude by discussing some bigger-picture questions (such as the potential for nonclassical variants of, say, Zilber-Pink) and some directions for future research in this area.

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Chapter 1

Introduction

1.1 Diophantine Problems on Shimura Varieties

Shimura varieties are central objects in arithmetic geometry and the theory of automorphic forms. One can view them as generalised versions of modular curves, but in general they are rather more complex objects; we begin with a rough description of Shimura varieties, in order to formulate some central conjectures and results about them, which motivate the work in this thesis. We will certainly brush over some details in this discussion, since the precise nature of a general Shimura variety will not be relevant to the thesis as a whole; the interested reader can see [52] for a nice survey of Shimura varieties.

Start with a hermitian symmetric domain X and a reductive linear algebraic group G, with $G(\mathbb{R})$ acting on X. For any discrete arithmetic subgroup Γ of $G(\mathbb{Q})$, the quotient $\Gamma\backslash X$ turns out to have the structure of a complex quasiprojective variety S. This variety S is the *Shimura variety*[1] associated to the data G, X, Γ. It implicitly comes with a canonical map

$$p : X \to S,$$

which is holomorphic and Γ-invariant.

The simplest case is captured by the famous j-function. Here the relevant hermitian symmetric domain is the complex upper half plane

$$\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}.$$

The group $\mathrm{GL}_2^+(\mathbb{R})$ of 2 by 2 matrices with real entries and positive determinant acts

[1]Strictly, what we are defining here is a *connected* Shimura variety; the full definition of a general Shimura variety will not concern us.

on $\mathbb{H}$ by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

This action factors through $\mathrm{SL}_2(\mathbb{R})$ in the obvious way, and we consider the quotient of the hermitian symmetric domain $\mathbb{H}$ by the discrete subgroup $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{R})$.

The modular j-function

$$j : \mathbb{H} \to \mathbb{C}$$

is a holomorphic map, invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, that is

$$j(\gamma\tau) = j(\tau), \text{ for all } \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Hence, the identification

$$j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$$

serves to exhibit $\mathbb{C}$ as a Shimura variety. Much of this document will focus on variants of this simple case; we will come back to it shortly.

A Shimura variety S also carries with it a collection $\sigma(S)$ of *special subvarieties.* These are the "sub-Shimura varieties" of S; the subvarieties of S which are themselves Shimura varieties, compatible with all the given data. That is, they are those subvarieties of S which arise as the image, under p, of suitable subdomains of X.

Diophantine statements about Shimura varieties come in many flavours. Common to a number of them is the goal of determining, given an arbitrary subvariety $V \subseteq S$, how much of the special structure described above can be seen by V. The André-Oort conjecture is a famous statement of this type.

Conjecture 1.1 (The André-Oort Conjecture)**.** *Let S be a Shimura variety and $V \subseteq S$ an algebraic subvariety. Define*

$$\mathrm{Sp}(V) = \bigcup\{A \subseteq V : A \in \sigma(S)\}.$$

Then $\mathrm{Sp}(V)$ is a variety. In other words, V contains only finitely many maximal special subvarieties.

Note. For the remainder of this document, except where otherwise noted, we identify varieties with their set of complex points, as above. We do not implicitly require a variety to be irreducible.

One might view this as an analogue of the Manin-Mumford conjecture in the Shimura setting. It arose as a common generalisation of conjectures of André [2], who made a conjecture corresponding to 1.1 for curves, and Oort [54], who conjectured a version of 1.1 applying to subvarieties of the Shimura variety $\mathcal{A}_g$ (which is the moduli space of principally polarised abelian varieties of genus g; we will discuss it a little in Chapter 7).

The first step towards the André-Oort Conjecture was made by André, who in [3] showed that it holds when the Shimura variety S is a product of two modular curves. Under GRH (for imaginary quadratic fields), the full André-Oort Conjecture is known thanks to work of Edixhoven, Klingler, Ullmo and Yafaev [28, 44, 82]. Further, largely thanks to the o-minimal techniques to be described below, the André-Oort conjecture is known unconditionally in many cases, with the remaining cases typically requiring only a little more Galois-theoretic information. Most of this is *ineffective*, though some cases are known effectively; see for instance [12] and [46].

A natural step beyond the André-Oort Conjecture is the more far-reaching Zilber-Pink Conjecture. This arose as the combination of various conjectures, perhaps most notably conjectures of Pink [69] (generalising André-Oort and Modell-Lang) and Zilber [91], though a number of other authors also contributed to its development, for instance Bombieri-Masser-Zannier [14].

The first of these motivating conjectures was that of Zilber. His conjecture from [91] was made in the exponential setting and was called the "Conjecture on Intersections with Tori", or CIT. In this context, the appropriate analogues of the special subvarieties of a Shimura variety are the *algebraic tori*; translates of Zariski closed subgroups of $(\mathbb{C}^\times)^n$. CIT says, given a variety $V \subseteq (\mathbb{C}^\times)^n$, that there should only be finitely many algebraic tori whose intersection with V is too large.

This idea translates directly across to the Shimura setting. For a subvariety V of a Shimura variety S, Zilber-Pink says that not only should V *contain* only a few of the special subvarieties, but moreover its intersection with most special subvarieties should not be too large.

We need to define, then, what is meant by "too large".

Definition 1.2. A subvariety W of $V \subseteq S$ is called *atypical* (for V in S) if it is a component of $A \cap V$, for some $A \in \sigma(S)$, with

$$\dim W > \dim A + \dim V - \dim S.$$

Conjecture 1.3 (The Zilber-Pink Conjecture). *Let S be a* mixed[2] *Shimura variety and $V \subseteq S$ an algebraic subvariety. Define*

$$\mathrm{Atyp}(V) = \bigcup \{W \subseteq V : W \textit{ atypical for } V \textit{ in } S\}.$$

Then $\mathrm{Atyp}(V)$ *is a variety. In other words, V contains only finitely many maximal atypical subvarieties.*

A short inductive argument shows that Zilber-Pink implies André-Oort, and indeed it is much more difficult; unlike André-Oort it is known only in some very limited cases.

These conjectures, together with a number of other diophantine questions, related functional transcendence problems and model-theoretic consequences, form a large and intricate picture. Using various techniques, some o-minimal in nature and some not so, many authors have investigated different aspects of this picture - see for instance, on Shimura variety aspects: André, Barroero, Bilu, Bombieri, Capuano Daw, Freitag, Habegger, Jones, Kühne, Masser, Orr, Pila, Ren, Scanlon, Schmidt, Tsimerman, Ullmo, Yafaev and Zannier [2, 3, 8, 14, 23, 24, 29, 31, 32, 36, 46, 55, 65, 83, 88, 90]. We will tend to be less concerned with the group variety aspects, but for these one can see, among many others, Bombieri-Masser-Zannier [13, 14], or Rémond and Viada [70, 71, 86]; for further references, see Zannier [90].

The central question motivating the work in this thesis is the following.

> **Question.** How much of the picture described above can be formulated when we replace the Shimura varieties S and their uniformising maps p by nonclassical variants thereof?

1.2 The Modular Case

Let us return to the simple case described above, where the ambient Shimura variety is a product $\mathbb{C}^n$ of level 1 modular curves $\mathbb{C}$ and the uniformising map is the (nth cartesian power of the) j-function. The first step is to describe the special subvarieties of $\mathbb{C}^n$.

[2] We will not discuss mixed Shimura varieties; suffice it to say that they generalise "pure" Shimura varieties.

1.2.1 Special Subvarieties of $\mathbb{C}^n$

As well as being invariant with respect to $\mathrm{SL}_2(\mathbb{Z})$, the j function exhibits nice behaviour with respect to a larger subgroup of $\mathrm{GL}_2^+(\mathbb{R})$, namely $\mathrm{GL}_2^+(\mathbb{Q})$, the group of 2 by 2 matrices with rational entries and positive determinant. Such matrices induce algebraic relations on j, as described in the following proposition.

Proposition 1.4. *Let $N \geq 1$. There is a polynomial $\Phi_N \in \mathbb{Z}[X, Y]$, called the Nth* modular polynomial*, such that*

$$\Phi_N(j(g\tau), j(\tau)) = 0$$

identically, whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ is a primitive integer matrix of determinant N.

Proof. Classical. See for instance [89, Proposition 23, page 68]. □

The polynomials Φ_N are known as the *modular polynomials*. Of course, Φ_1 is simply $X - Y$. For $N > 1$, the polynomials Φ_N are more complicated, but it is known that $\Phi_N(X, X)$ is monic up to sign (see for instance [89, Proposition 24, page 70]). Since every quadratic $\tau \in \mathbb{H}$ is fixed by an element of $\mathrm{GL}_2^+(\mathbb{Q})$ with positive determinant, we can deduce from this the classical fact that $j(\tau)$ is an algebraic integer for all quadratic τ. Such $j(\tau) \in \mathbb{C}$ are traditionally called *special points* or *singular moduli*; we will tend to call them j-special points, to distinguish them from the nonclassical special points discussed later. As the name suggests, the special points are precisely the zero-dimensional special subvarieties of the Shimura variety $\mathbb{C}$. Special subvarieties of the more general Shimura variety $\mathbb{C}^n$ (which is of course the image of $\mathbb{H}^n$ under the nth cartesian power of j) are loosely those subvarieties cut out by the modular polynomials Φ_N. The following definition makes this precise.

Definition 1.5. Let $n \in \mathbb{N}$ and let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a partition of $\{1, \ldots, n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose a point $j_s \in \mathbb{C}$. For each $i > 0$, let s_i be the least element of S_i and for each $s \in S_i \setminus \{s_i\}$ choose a positive integer $N_{i,s}$. A *weakly j-special* subvariety of $\mathbb{C}^n$ is an irreducible component of a subvariety of the form

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_s = j_s \text{ for } s \in S_0, \\ \Phi_{N_{i,s}}(z_{s_i}, z_s) = 0 \text{ for } s \in S_i \setminus \{s_i\}, i = 1, \ldots, k\}$$

for some given data S_i, j_s, $N_{i,s}$.

A weakly j-special variety is j-special if all of the constant factors j_s are singular moduli, ie. $j_s = j(\tau_s)$ for some quadratic $\tau_s \in \mathbb{H}$.

The special subvarieties of $\mathbb{C}^n$ afforded by the Shimura theory are precisely these j-special varieties. A zero-dimensional j-special variety is of course just an n-tuple of singular moduli.

Definition 1.6. Let $n \in \mathbb{N}$ and let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a partition of $\{1, \ldots, n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose any point $q_s \in \mathbb{H}$. For each $i > 0$, let s_i be the least element of S_i and for each $s \in S_i \setminus \{s_i\}$ choose a matrix $g_{i,s} \in \mathrm{GL}_2^+(\mathbb{Q})$. A *weakly* $\mathbb{H}$*-special* subvariety of $\mathbb{H}^n$ is a set of the form

$$\big\{(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n : \tau_s = q_s \text{ for } s \in S_0, \tau_s = g_{i,s}\tau_{s_i} \text{ for } s \in S_i \setminus \{s_i\}, i = 1, \ldots, k\big\},$$

for some given data S_i, q_s, $g_{i,s}$.

A weakly $\mathbb{H}$-special subvariety is $\mathbb{H}$-special if, for all $s \in S_0$, the constant factor $q_s \in \mathbb{H}$ is imaginary quadratic.

By the properties of the Φ_N, any (weakly) j-special variety is precisely the image of a (weakly) $\mathbb{H}$-special variety, and vice versa.

In this setting, the André-Oort conjecture becomes the following.

Theorem 1.7 (Pila, Modular André-Oort)**.** *Let $V \subseteq \mathbb{C}^n$ be an algebraic variety. Then V contains only finitely many maximal j-special subvarieties.*

Equivalently, in its more modern formulation:

Theorem 1.8. *Let $\Sigma \subseteq \mathbb{C}^n$ be a set of j-special points. Then the irreducible components of the Zariski closure of Σ are j-special.*

André was the first to prove a result of this type. In [3] he proved a version of 1.7 holding in the case where the ambient variety V is a curve. The theorem as stated, however, is due to Pila, who proved it in [62]. (It was proven first by Edixhoven [28], assuming GRH.) Pila's result was the first unconditional André-Oort result to go beyond that of André. The techniques used were o-minimal in nature; we will describe them shortly, but let us first lay out the central results of this thesis.

1.3 Statement of Results

Much of this document will be concerned with the following question, which is the analogue, in the modular setting, of the question we posed earlier about general Shimura varieties.

Question. If, in Theorem 1.7, j is replaced by a nonclassical modular function f, does the special structure described by Definitions 1.5 and 1.6 still have any meaning, and can one formulate and/or prove an analogue of 1.7 for f?

This question, of course, raises another: what do we mean by nonclassical modular function? To answer this, we must first recall the defining properties of classical modular functions. A map $f : \mathbb{H} \to \mathbb{C}$ is called a modular function if:

- It is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$; ie. $f(\gamma\tau) = f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
- It is meromorphic on $\mathbb{H}$.
- It is meromorphic "at ∞"; this is a technical condition requiring that $f(\tau)$ is not too badly behaved as $\operatorname{Im}\tau$ approaches infinity.

It is a classical fact that any function satisfying these three properties automatically lies in $\mathbb{C}(j)$. A nonclassical modular function, therefore, is a map which remains at least somewhat modular, but for which one or more of the three conditions above is relaxed. Of course there are a number of classes of such function; we will be focussing on two particular classes: the *almost holomorphic modular functions* and the *quasimodular functions*. These are described in detail in Chapter 2. Almost holomorphic modular functions are, unsurprisingly, modular functions which fail to be meromorphic in some controlled way, while quasimodular functions are meromorphic but are not fully invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

It turns out to be enough to consider one particular almost holomorphic modular function called χ^*, which we define in (2.2). Together with the j-function, χ^* generates the field of almost holomorphic modular functions, so it is natural to work with the pair of maps (j, χ^*), which we call π. This map comes with modular polynomials much like those for j, allowing us to define "π-special subvarieties" in 2.16. These varieties naturally live in $\mathbb{C}^{2n}$, so the natural André-Oort statement for π is the following, which is one of our central theorems regarding almost holomorphic modular functions; it is proven in Chapter 6.

Theorem 6.5 (André-Oort for π)**.** *Let $V \subseteq \mathbb{C}^{2n}$ be a variety. Then V contains only finitely many maximal π-special subvarieties.*

En route to proving this, we prove the following "Ax-Lindemann" result, which is of significant interest in its own right as well as being a crucial tool towards proving 6.5, as described in 1.4.2.

Corollary 4.12 (Ax-Lindemann for π)**.** *Let S be an arc of a real algebraic curve in $\mathbb{H}^n$ and suppose that $S \subseteq \pi^{-1}(V)$, where V is some algebraic variety in $\mathbb{C}^{2n}$. Then S is contained in a weakly $\mathbb{H}$-special variety G with $G \subseteq \pi^{-1}(V)$.*

These are our two main theorems about χ^*, but we will also prove a further result of a rather different style. To state it, we recall that $j(\tau)$ is algebraic when τ is quadratic. This is also true of χ^*, and in fact for quadratic τ we have

$$\mathbb{Q}(\chi^*(\tau)) \subseteq \mathbb{Q}(j(\tau)).$$

Conjecturally, these two fields are equal, and while we cannot prove this, it can be approached using o-minimal techniques, which yield the following result, proven in Chapter 5.

Theorem 5.4. *For every nonconstant $f \in \mathbb{Q}(j, \chi^*)$, there is a constant M such that*

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(f(\tau))] \leq M$$

for all quadratic $\tau \in \mathbb{H}$.

Using the full power of the André-Oort statement 6.5, we also prove a uniform version of this result, which we will not state here; it is Theorem 5.11.

The final results of this thesis are once again of André-Oort type, but this time they concern quasimodular functions. As we will see in Chapter 2, the classes of almost holomorphic modular and quasimodular functions are dual to one another in a natural way. In particular, the function χ^* has a quasimodular dual χ.

It is perhaps more natural, however, to consider a more well-known case. It turns out that the derivatives of the j-function have quasimodular properties. The second derivative j'' of j, to be specific, is a (weakly holomorphic) quasimodular *form*, while j' is a (weakly holomorphic) modular form. Our final results concern the diophantine properties of the triple of maps

$$J = (j, j', j'').$$

Our focus is on a conjecture of Pila, which is the natural André-Oort statement for J.

Conjecture 3.6 (Pila, "Modular André-Oort with Derivatives")**.** *Let $V \subsetneq \mathbb{C}^{3n}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of $\mathbb{H}$-special subvarieties $H \subsetneq \mathbb{H}^n$, such that every $\mathbb{H}$-special subvariety of $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

The precise definition of "$\mathrm{SL}_2(\mathbb{Z})$-finite" is given in 3.5; a collection of subsets is $\mathrm{SL}_2(\mathbb{Z})$-finite if it is finite up to the action of $\mathrm{SL}_2(\mathbb{Z})$.

Thanks to the duality between almost holomorphic modular and quasimodular forms, we can use known results about χ^* to approach such questions; in Chapter 6, we prove the following.

Theorem 6.11. *Let $V \subsetneq \mathbb{C}^{3n}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$, consisting of $\mathbb{H}$-special subvarieties $H \subsetneq \mathbb{H}^n$, with the property that every j'-generic $\mathbb{H}$-special point in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

The definition of j'-genericity is given in 3.8; according to Conjecture 3.9, which is a very special case of standard conjectures in number theory, almost all quadratic points should be j'-generic, so Theorem 6.11 tells us that Conjecture 3.9 implies Conjecture 3.6.

In Chapter 6 we also prove a number of more precise versions of André-Oort for J (usually assuming Conjecture 3.9), which are too involved to state here.

These results are also available in my papers [77], [78] and [79]; they all make heavy use of the theory of o-minimal structures. We give an introduction to this theory in the next section, which readers familiar with o-minimality can safely skip!

1.4 O-minimality

The theory of o-minimal structures is part of model theory, in turn a branch of mathematical logic. We will describe some of the basics here; for more details the reader can see any of a number of excellent surveys, for instance [84].

A starting notion for this topic is the idea of a structure in mathematical logic.

Definition 1.9. A *structure* is a tuple

$$\mathcal{M} = (M, \{f_i\}, \{R_i\}, \{c_i\}),$$

where M is a set, $\{f_i\}$ is a collection (not necessarily countable) of distinguished functions from M^{n_i} to M^{m_i}, for some natural numbers n_i, m_i, $\{R_i\}$ is a collection of relations living in cartesian powers M^{k_i} for natural numbers k_i, and $\{c_i\}$ is a collection of constants $c_i \in M$.

These structures are a fundamental object of study in model theory. Of course, a structure comes with a vast array of features and properties, but the notion of most interest to us here is that of a *definable set*. A definable set in $\mathcal{M}$ is a subset of some M^n which takes the form

$$\{(x_1, \ldots, x_m) \in M^n : \phi(x_1, \ldots, x_n, a_1, \ldots, a_m) \text{ holds in } M\},$$

where $a_1, \ldots, a_m \in M$ and ϕ is formula in the language $\{f_i\} \cup \{R_i\} \cup \{c_i\}$, in the sense of first order logic.

In other words, definable sets are those which can be described precisely by a finite number of conditions, mentioning only the functions, predicates and constants from $\{f_i\}$, $\{R_i\}$ and $\{c_i\}$, together with finitely many distinguished points a_i. (Technically, by allowing the points a_i, we are discussing sets definable *with parameters*. This allows for a more general notion of definable set, which will be crucial for us. There are model-theoretic drawbacks to this, but they have no impact here; the main thing to note is that the constants c_i do not affect the picture since they could always be replaced by the parameters a_i anyway.)

Given a structure $\mathcal{M}$, the collection of definable sets in $\mathcal{M}$ is often of great interest. The collection has many nice properties: for instance it is closed under projections, fibres and finite boolean combinations. Indeed, if it is more comfortable, it is fine for the purposes of this document to forget about Definition 1.9 and instead simply to identify a structure with its collection of definable sets. One can then say that a structure is simply a set M together with a collection of distinguished subsets of M^n (called the definable sets) satisfying a certain list of properties:

- For every n, the collection of definable subsets of M^n is a Boolean algebra.
- If $A \subseteq M^m$ and $B \subseteq M^n$ are definable, then so is $A \times B$.
- If $p : M^{n+1} \to M^n$ is a coordinate projection and $A \subseteq M^{n+1}$ is definable, then so is $p(A)$.
- For any $1 \leq i < j \leq n$, the set

  $$\{(x_1, \ldots, x_n) \in M^n : x_i = x_j\}$$

  is definable.
- For any $x \in M$, the set $\{x\}$ is a definable subset of M.

Of course, whichever definition one prefers to use, the notion of definability does not apply to functions, only to sets, but we will extend the notion in the obvious way, saying that a function $f : M^m \to M^n$ is definable if its graph in M^{m+n} is a definable set.

The 'o' in 'o-minimality' stands for 'order'. Unsurprisingly, then, we will henceforth assume that one of the distinguished relations R_i is an order relation. In fact, we will restrict further, assuming throughout that $\mathcal{M}$ is a structure whose underlying set M is equal to $\mathbb{R}$, and that one of the R_i is the usual order relation $<$ on $\mathbb{R}$.

Definition 1.10. We say that a structure $\mathcal{M}$ on $\mathbb{R}$ is "o-minimal" if every definable subset of $\mathbb{R}$ is a finite union of points and intervals.

Example 1.11. Let us see some examples of ordered structures.

- We denote by $\overline{\mathbb{R}}$ the structure

  $$(\mathbb{R}, \{+, \cdot\}, \{<\}),$$

  where $+$ and $\cdot$ are the usual addition and multiplication, and $<$ is the usual order relation. In this structure, a definable set is just a semialgebraic set (a consequence of the Tarski-Seidenberg theorem; see for instance [20] or [84]). The semialgebraic subsets of $\mathbb{R}$ are obviously just finite unions of points and intervals, so $\overline{\mathbb{R}}$ is o-minimal.

- We denote by $\mathbb{R}_{\text{exp}}$ the structure

  $$(\mathbb{R}, \{+, \cdot, \exp\}, \{<\}),$$

  where exp is the usual real exponential function from $\mathbb{R}$ to $\mathbb{R}$. The fact that this is an o-minimal structure is a highly celebrated result; it follows from work of Wilkie [87] and Khovanskii [38, 39].

- The structure $\mathbb{R}_{\text{sin}}$, defined by

  $$\mathbb{R}_{\text{sin}} = (\mathbb{R}, \{+, \cdot, \sin\}, \{<\}),$$

  is a perfectly good structure, but is not o-minimal. The set $\{x \in \mathbb{R} : \sin x = 0\}$ is definable in this structure, but clearly is not a finite union of points and intervals. However, the structure

  $$\mathbb{R}_{\text{an,exp}} = (\mathbb{R}, \{+, \cdot, \exp\} \cup F_{\text{an}}, \{<\}),$$

where

$$F_{\text{an}} = \{f|_{[0,1]^n} : f \text{ is real analytic on an open neighbourhood of } [0,1]^n\},$$

is o-minimal, an important result of van den Dries and Miller [85]. This last structure is the one we will most often use.

Note. We will often wish to discuss definable subsets not of $\mathbb{R}^n$ but of $\mathbb{C}^n$. To do so we simply identify $\mathbb{C}$ with $\mathbb{R}^2$ in the obvious way. We will do this without comment, regularly throughout the document.

It turns out that the function j, restricted a suitable fundamental domain, is definable in $\mathbb{R}_{\text{an,exp}}$ (the whole of j can never be definable in an o-minimal structure; its periodicity precludes this). This can be seen either directly from the existence of the q-expansion

$$j(\tau) = \sum_{n=-1}^{\infty} c_n e^{2\pi i n \tau}$$

or by applying a result of Peterzil and Starchenko [58] asserting the definability in $\mathbb{R}_{\text{an,exp}}$ of the Weierstrass $\wp$-function and going via the theory of elliptic curves.

In the forthcoming, we will at times need a number of basic results from the theory of o-minimal structures, which we state here in increasing order of complexity. The proofs of 1.12 and 1.13 can be found in [84, Pages 52-57].

Theorem 1.12 (Uniform Finiteness Theorem)**.** *Let $X \subseteq \mathbb{R}^{n+r}$ be definable in some o-minimal structure. Consider it as a definable family of fibres X_y, $y \in \mathbb{R}^r$. Suppose that X_y is finite for every y. Then there is a constant $M = M(X)$ such that*

$$\#X_y \leq M$$

for all y.

Theorem 1.13 (Cell Decomposition Theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be a definable set. Then X is a finite union of* cells*.*

A cell is a particularly nice definable subset. Cells are defined inductively as follows:

- A (0)-cell is a point in $\mathbb{R}$.
- A (1)-cell is an open interval in $\mathbb{R}$.
- Given $(i_1, \ldots, i_n) \in \{0,1\}^n$:

– An $(i_1, \ldots, i_n, 0)$-cell is the graph $G \subseteq \mathbb{R}^{n+1}$ of a definable function $f : X \to \mathbb{R}$.

– An $(i_1, \ldots, i_n, 1)$-cell is a region

$$\left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \begin{array}{c} (x_1, \ldots, x_n) \in X, \\ f(x_1, \ldots, x_n) < x_{n+1} < g(x_1, \ldots, x_n) \end{array} \right\},$$

where $f, g : X \to \mathbb{R}$ are definable[3],

where in each case X is some $(i_1, \ldots, i_n)$-cell.

- A *cell* X is a $(i_1, \ldots, i_n)$-cell for any n and any $i_j \in \{0, 1\}$. The *dimension* of X is $\sum i_j$.

Since cells have a good notion of dimension, the cell decomposition theorem therefore induces a good notion of dimension on definable sets; the dimension of a definable set X is simply the maximal dimension of a cell in its cell decomposition (this is independent of the choice of cell decomposition; see [84] for more details).

Thanks to 1.13, a 0-dimensional definable set must be finite. Since we are using $\mathbb{R}$ as our underlying set, it also follows that a positive-dimensional definable set is uncountable, yielding:

Proposition 1.14. *A definable subset of* $\mathbb{R}^n$ *consisting entirely of rational points is finite.*

One result we will use more often than any of the above, however, is the *Pila-Wilkie counting theorem.*

1.4.1 The Pila-Wilkie Theorems

The definition of an o-minimal structure is deceptively simple, yet the o-minimality of a structure has surprisingly far-reaching consequences for the structure as a whole. One such consequence of great significance is the Pila-Wilkie counting theorem, the original version of which was proven in [67], then strengthened in [60] to include the versions stated here.

Definition 1.15. Given a set $X \subseteq \mathbb{R}^n$, we define the *algebraic part* X^{alg} of X to be the union of all the positive-dimensional, connected, real semialgebraic subsets of X.

The transcendental part X^{tr} of X is $X \setminus X^{\mathrm{alg}}$.

[3]Here we also allow unbounded intervals, so that f may be the constant function $-\infty$ and g is allowed to be the constant function $+\infty$.

Definition 1.16. Given $x \in \overline{\mathbb{Q}}^n$, we write $\mathrm{Ht}(x)$ for the *absolute multiplicative height* of x, as defined for instance in [15, Definition 1.5.4].

All versions of the Pila-Wilkie theorem have approximately the same form; they state that the transcendental part of a definable set cannot contain too many algebraic points of a given (absolute multiplicative) height and degree. We will state two versions of the Pila-Wilkie theorem, both of which concern an o-minimal structure $\mathcal{M}$ on $\mathbb{R}$.

Theorem 1.17 (The Pila-Wilkie Theorem for Algebraic Points)**.** *Let $X \subseteq \mathbb{R}^n$ be a set definable in $\mathcal{M}$. Let $\epsilon > 0$ and let $k \in \mathbb{N}$. Then there is a constant $c = c(X, k, \epsilon)$ such that*

$$\#\{x = (x_1, \ldots, x_n) \in X^{\mathrm{tr}} \cap \overline{\mathbb{Q}}^n : \mathrm{Ht}(x) \leq T, \max[\mathbb{Q}(x_i) : \mathbb{Q}] \leq k\} \leq cT^{\epsilon}.$$

This result, which is Theorem 1.6 from [60], gives us on its own a lot of control over the diophantine properties of definable sets in o-minimal structures. It can be made even stronger, however; the following uniform version is (a slight weakening of) Theorem 3.1 from [60].

Theorem 1.18 (The Uniform Pila-Wilkie Theorem for Algebraic Points)**.** *Let $X \subseteq \mathbb{R}^{n+r}$ be a set definable in $\mathcal{M}$, considered as a family of fibres X_y, with $y \in \mathbb{R}^r$. Let $\epsilon > 0$ and $k \in \mathbb{N}$. Then there is a constant $c = c(X, k, \epsilon)$ such that, for all $y \in \mathbb{R}^r$,*

$$\#\{x = (x_1, \ldots, x_n) \in (X_y)^{\mathrm{tr}} \cap \overline{\mathbb{Q}}^n : \mathrm{Ht}(x) \leq T, \max[\mathbb{Q}(x_i) : \mathbb{Q}] \leq k\} \leq cT^{\epsilon}.$$

These theorems together serve as excellent tools for studying diophantine problems. We will freely use the Pila-Wilkie theorem in both of these forms throughout the document.

1.4.2 The Pila-Zannier Strategy

The overall strategy used in many approaches to the diophantine problems we are discussing is known as the Pila-Zannier strategy, having been pioneered by Pila and Zannier and used in Pila's proof of the modular André-Oort Conjecture (Theorem 1.7). Variants of this strategy have since been successfully used by a number of authors to approach a variety of other diophantine problems, some of which have been mentioned above.

With various amounts of adaptation, we will be using this strategy as a central tool in many of the results in this thesis. We therefore take the opportunity now to illustrate the main ideas, using the j-function and the modular André-Oort theorem as an example.

The central idea of the strategy is to play off lower bounds for the size of Galois orbits of special points against upper bounds provided by the Pila-Wilkie theorem. There are four main ingredients. In what follows, V will always be an algebraic variety living in $\mathbb{C}^n$, and we abuse notation as usual, writing $j : \mathbb{H}^n \to \mathbb{C}^n$ to represent the nth cartesian power of j. Since j-special points are algebraic, we may always assume that V is defined over some number field K.

Throughout, we will write $\mathbb{F}$ for a suitable fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$; see Chapter 2, Section 2.1.

1. **Ax-Lindemann.** An *Ax-Lindemann result* is a statement giving suitable control over the algebraic part of a set of the form $j^{-1}(V)$. Pila gave such a result as a significant step in his paper [62], proving:

   > Any real semialgebraic subset of $j^{-1}(V)$ is contained in some weakly $\mathbb{H}$-special subvariety of $j^{-1}(V)$.

   Such a functional transcendence result, as well as being the first step of the Pila-Zannier strategy, is of significant intrinsic interest.

   Interestingly, Pila's proof of the appropriate Ax-Lindemann theorem for j in [62] itself relied on o-minimality and point-counting, albeit using a rather different internal strategy. In the work to follow, our Ax-Lindemann theorems, rather than being proven using o-minimality, will use technical trickery to modify existing Ax-Lindemann theorems.

2. **Counting positive-dimensional pieces.** Here lies the first (and most elementary) application of o-minimality. Consider the collection of subsets of $\mathbb{H}^n$ cut out by matrices from $\mathrm{SL}_2(\mathbb{R})$; call such subvarieties *Möbius subvarieties* (see Definition 3.11 for a more precise definition). The collection of Möbius subvarieties M such that, for some $\mathbf{x}$, the variety $M \times \{\mathbf{x}\}$ lies in $j^{-1}(V)$ (and meets $\mathbb{F}^n$ in full dimension), corresponds to a definable subset S of some cartesian power $\mathrm{SL}_2(\mathbb{R})^k$. Applying the Ax-Lindemann result, it follows that any maximal such Möbius subvariety is in fact a weakly $\mathbb{H}$-special subvariety. Hence the definable set S consists only of points from $\mathrm{GL}_2^+(\mathbb{Q})$, whence it is finite by 1.14.

   In particular, there are, up the action of $\mathrm{SL}_2(\mathbb{Z})$, only finitely many $\mathbb{H}$-special subvarieties G whose *translates* $G \times \{\mathbf{x}\}$ account for all the weakly $\mathbb{H}$-special subvarieties of $j^{-1}(V)$. It then remains to count, for each G, how many of its special translates lie within $j^{-1}(V)$. This is then a problem of counting zero-dimensional objects, which is where the final two ingredients come in.

3. **Galois bounds.** If V contains the image $j(\tau)$ of some quadratic point τ, then it also contains all of its Galois conjugates over K. The idea here is to show that the number of such Galois conjugates grows as a positive power of the complexity of τ, measured by its height or (essentially equivalently) its discriminant. It is a consequence of Siegel's lower bound [75] for class numbers of quadratic orders (see 5.8) that, for $\eta > 0$,

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \gg_\eta D(\tau)^{\frac{1}{2}-\eta},$$

where $D(\tau)$ is the absolute value of the discriminant of τ. Hence, any sufficiently complex quadratic point in $j^{-1}(V)$ will lead to "many" such points. Crucially, a Galois conjugate of a special point $j(\tau)$ is another special point $j(\tau')$, with τ' having the same discriminant as τ. This allows us to play these bounds off against the upper bound provided by the Pila-Wilkie theorem.

4. **Counting points.** The final ingredient is centred around the Pila-Wilkie theorem, which is typically applied to the definable set $\mathcal{Z} = j^{-1}(V) \cap \mathbb{F}^n$. Should $\mathcal{Z}$ contain quadratic points of arbitrarily large discriminant, then by the Galois considerations, it contains a number of such points which grows like a positive power of the height. Then the Pila-Wilkie theorem tells us that $\mathcal{Z}$ must have non-empty algebraic part. That is, it should contain a real semialgebraic arc. Then an application of the Ax-Lindemann result tells us that $j^{-1}(V)$ must contain an $\mathbb{H}$-special subvariety; but these have been accounted for in Step 2. An easy induction allows us to conclude.

1.5 Document Plan

The plan for this document is as follows. In Chapter 2, we describe almost holomorphic modular and quasimodular forms and functions, and investigate some of their basic properties, concluding by stating our main almost holomorphic André-Oort and Ax-Lindemann theorems. In Chapter 3, we discuss the derivatives of the j-function, introducing a number of new technical notions and stating various conjectures and theorems of André-Oort type. In Chapter 4 we state and prove the various Ax-Lindemann problems that arise in this setting, and in Chapter 5 we collect some technical results, largely regarding the properties of our distinguished almost holomorphic modular function χ^*. In Chapter 6, we combine these technical results with the Ax-Lindemann results of Chapter 4 to prove our various André-Oort theorems. Finally, in Chapter 7, we will discuss some bigger-picture questions on these topics and a number of potential future directions for this work.

Chapter 2

Quasimodular and Almost Holomorphic Modular Forms

2.1 Constructions and Definitions

Let us begin this section by recalling the classical construction of the modular j-function. A typical method of construction is to go via Eisenstein series.

For even numbers $k \geq 4$ and $\tau \in \mathbb{H}$ let

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{1}{(m\tau + n)^k}.$$

This series converges absolutely, and defines a holomorphic function from $\mathbb{H}$ to $\mathbb{C}$. Using the absolute convergence of the series, it is easy to see that E_k is a modular form, of weight k, for all even $k \geq 4$. That is,

$$E_k(\gamma\tau) = (c\tau + d)^k E_k(\tau),$$

whenever $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Notation 2.1. Throughout this document, we will often need to know, given a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

the values of c and d. Hence, given any matrix $g \in \mathrm{GL}_2^+(\mathbb{R})$, we will often simply write c and d to represent the values in the bottom row of g, without explicitly writing out all the entries of g.

Since E_4 and E_6 have weights 4 and 6 respectively, it easily follows that the function

$$j = 1728 \cdot \frac{E_4^3}{E_4^3 - E_6^2}$$

is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. As the notation suggests, this is indeed the definition of the usual modular j-function. The denominator

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2),$$

which is called the discriminant function, is well-known to be nonvanishing on $\mathbb{H}$, so j is everywhere holomorphic on $\mathbb{H}$.

Given $\tau \in \mathbb{H}$, there is an elliptic curve E_τ corresponding to the torus formed by the quotient of $\mathbb{C}$ by the lattice $\langle 1, \tau \rangle \subseteq \mathbb{C}$ generated over $\mathbb{Z}$ by 1 and τ. This is a classical picture; the Weierstrass $\wp$-function

$$\wp(z, \tau) = \frac{1}{z^2} + \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left(\frac{1}{(z + m\tau + n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

and its derivative $\frac{\partial \wp}{\partial z}$ together parametrise the points of an elliptic curve E_τ. Every elliptic curve arises this way and the j-invariant of E_τ, unsurprisingly, is $j(\tau)$.

Notation 2.2. We denote by $\mathbb{F}^-$ the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, that is

$$\mathbb{F}^- = \mathbb{F}^l \cup \mathbb{F}^r,$$

where

$$\mathbb{F}^l = \left\{ \tau \in \mathbb{H} : -\frac{1}{2} \leq \operatorname{Re} \tau \leq 0 \text{ and } |\tau| \geq 1 \right\}$$

and

$$\mathbb{F}^r = \left\{ \tau \in \mathbb{H} : 0 < \operatorname{Re} \tau < \frac{1}{2} \text{ and } |\tau| > 1 \right\}.$$

The closure of this fundamental domain will be denoted $\mathbb{F}$, so that

$$\mathbb{F} = \left\{ \tau \in \mathbb{H} : -\frac{1}{2} \leq \operatorname{Re} \tau \leq \frac{1}{2} \text{ and } |\tau| \geq 1 \right\}.$$

Two elliptic curves E_τ and E_σ are isomorphic if and only if τ and σ are in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit. Hence, every elliptic curve is isomorphic to some E_τ with $\tau \in \mathbb{F}^-$. Moreover, the j-function is well-known to be a bijection between $\mathbb{F}^-$ and $\mathbb{C}$, from which one can see the classical fact that two elliptic curves are isomorphic if and only if they have the same j-invariant.

The action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{H}$ also has meaning in this elliptic curves picture. Elliptic curves E_τ and E_σ are isogenous if and only if $\tau = g\sigma$, with $g \in \mathrm{GL}_2^+(\mathbb{Q})$. The degree of the isogeny is the same as the determinant of g, when it is scaled to be a primitive integer matrix. Further still, an elliptic curve E_τ has complex multiplication if and only if τ is quadratic over $\mathbb{Q}$, whence the j-special points are precisely the j-invariants of CM elliptic curves.

For $k \leq 2$, the series E_k is not absolutely convergent. Nonetheless, the case $k = 2$ has an interesting theory. The series E_2 is conditionally convergent, and by choosing a suitable order of summation, we can use it to define a holomorphic function with interesting properties.

Let

$$E_2(\tau) = \frac{1}{2} \lim_{M \to \infty} \sum_{m=-M}^{M} \lim_{n \to \infty} \sum_{\substack{n=-N \\ (m,n)=1}}^{N} \frac{1}{(m\tau + n)^2},$$

which defines a holomorphic function on $\mathbb{H}$. This function E_2 is not a weight 2 modular form (it is well known that there are no modular forms of weight 2 and level 1), but instead satisfies

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d) \tag{2.1}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. For a proof, see for instance Zagier [89, Proposition 6]; many of the claims made in this section will come from that paper. The transformation rule (2.1) says that E_2 is an example of a *quasimodular form*.

Definition 2.3. A function $f : \mathbb{H} \to \mathbb{C}$ is a quasimodular (qm) form (of weight k and depth p) if there is a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that:

- $f(\tau)$ is holomorphic and bounded as $\operatorname{Im} \tau \to \infty$ for each fixed $\operatorname{Re} \tau$.
- For every $\gamma \in \Gamma$, we have:

$$\frac{f(\gamma\tau)}{(c\tau + d)^k} = f(\tau) + \sum_{r=1}^{p} f_r(\tau) \left(\frac{c}{c\tau + d} \right)^r,$$

  for some holomorphic functions $f_r : \mathbb{H} \to \mathbb{C}$, bounded as $\operatorname{Im} \tau \to \infty$ for each fixed $\operatorname{Re} \tau$.

The *level* of f is the level of Γ.

We will see an alternative definition shortly. It is easy to see from (2.1) that E_2 is a qm form of weight 2 and level 1. Indeed, it turns out that it is, in a sense, the only qm form which is not a modular form. The following is a result of Zagier, proven in [89, Proposition 20 ii)].

Proposition 2.4 (Structure Theorem for Quasimodular Forms)**.** *Let* Γ *be a congruence subgroup of* $\mathrm{SL}_2(\mathbb{Z})$*. Then every qm form for* Γ *is a polynomial in* E_2 *with coefficients which are (classical) modular forms for* Γ*.*

Remark. In the other parts of Proposition 20 of [89], Zagier also shows that the space of qm forms is closed under differentiation. In particular, the derivative of a classical modular form is a qm form; this will be important in other chapters.

In particular, we see that the graded algebra of level 1 qm forms is generated by E_2 together with the classical Eisenstein series E_4 and E_6. The level structure of qm forms will be of little interest to us in what follows, so we restrict our attention to this level 1 case.

Looking once more at (2.1), an easy calculation shows that the real analytic function $E_2^* : \mathbb{H} \to \mathbb{C}$ defined by

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau}$$

satisfies the usual weight 2 modular transformation law, that is:

$$E_2^*(\gamma\tau) = (c\tau + d)^2 E_2^*(\tau)$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. This, together with the fact that it is a polynomial in $(\operatorname{Im} \tau)^{-1}$ with holomorphic coefficients, says that E_2^* is an *almost holomorphic modular form.*

Definition 2.5. A function $f : \mathbb{H} \to \mathbb{C}$ is an almost holomorphic modular (ahm) form (of weight k and depth p) if there is a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ such that:

1. There is a polynomial expansion

$$f(\tau) = \sum_{r=0}^{p} \frac{f_r(\tau)}{(\operatorname{Im} \tau)^r}$$

for some holomorphic functions $f_r : \mathbb{H} \to \mathbb{C}$.

2. For each fixed $\operatorname{Re} \tau$, $f_r(\tau)$ is bounded as $\operatorname{Im} \tau \to \infty$.

3. For any $\gamma \in \Gamma$,

$$f(\gamma\tau) = (c\tau + d)^k f(\tau).$$

The *level* of f is the level of Γ.

So E_2^* is an ahm form of weight 2, depth 1 and level 1. The constant term of E_2^* with respect to $(\operatorname{Im}\tau)^{-1}$ is the function E_2, which as we noted earlier is a qm form of weight 2, depth 1 and level 1. This is no coincidence; given any ahm form, one can always take its constant term with respect to $(\operatorname{Im}\tau)^{-1}$ to get a qm form of the same weight, depth and level. Conversely, any qm form can be completed in a unique way, correcting for the modified transformation law to get an ahm form. These processes are clearly inverse to one another, and induce an isomorphism between the spaces of ahm and qm forms. In particular, thanks to 2.4, the graded algebra of level 1 ahm forms is generated by E_2^*, E_4 and E_6. As we did for qm forms, we restrict our attention to this space of level 1 forms.

Since the qm (resp. ahm) forms of level 1 are generated by E_2 (resp. E_2^*), E_4 and E_6, clearly there are no level 1 forms of weight 0. This is problematic for our purposes, but easily solved; one simply allows oneself to take the quotient of two forms of equal weight.

Definition 2.6. A function $f : \mathbb{H} \to \mathbb{C}$ is a level 1 *quasimodular* (qm) function if it may be written as a quotient of two level 1 qm forms of equal weight.

A function $f : \mathbb{H} \to \mathbb{C}$ is a level 1 *almost holomorphic modular* (ahm) function if it may be written as a quotient of two level 1 ahm forms of equal weight.

The field of level 1 qm (resp. ahm) functions shall be denoted $\widetilde{F}$ (resp. F^*).

Clearly one could analogously define qm/ahm functions of higher level, but these will largely not concern us. Henceforth, unless otherwise stated, by "qm/ahm function" we shall implicitly mean "qm/ahm function *of level 1*".

The fields $\widetilde{F}$ and F^* are clearly isomorphic via the obvious extension of the aforementioned "constant term" isomorphism. So to get a structure theorem analogous to 2.4 it suffices to consider only one side of the picture; we will deal with F^*.

Perhaps the most immediately obvious ahm function (other than j) is

$$\frac{E_2^* E_4}{E_6}.$$

This function has been studied in a few places before, perhaps most notably by Masser in [49, Appendix 1], to whose work we will refer several times. Following Masser, we will call this function ψ.

Lemma 2.7. $F^* = \mathbb{C}(j, \psi)$.

Proof. Beginning with the fact that ahm forms are generated by E_2^*, E_4 and E_6, one sees by inspection that F^* is generated over $\mathbb{C}$ by

$$\frac{E_4^3}{E_6^2}, \quad \frac{E_2^* E_6}{E_4^2}, \quad \frac{E_2^{*3}}{E_6}, \quad \frac{E_2^* E_4}{E_6}, \text{ and } \frac{E_2^{*2}}{E_4}.$$

Looking slightly more closely at these, they clearly all lie in $\mathbb{C}(j, \psi)$. □

The function ψ has the mildly unfortunate property of being singular at all points in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i. For convenience, we would like to remove this singularity, which can be done if we multiply ψ by $(j - 1728)$, yielding a function

$$\chi^* = \psi(j - 1728) = \frac{E_2^* E_4 E_6}{\Delta} = 1728 \cdot \frac{E_2^* E_4 E_6}{E_4^3 - E_6^2}, \tag{2.2}$$

where Δ is again the modular discriminant function. Since Δ is nonvanishing, we see that χ^* has no singularities in $\mathbb{H}$, only one "at infinity".

We have

$$\chi^*(\tau) = \left.\frac{E_2 E_4 E_6}{\Delta}\right|_\tau - \frac{3}{\pi \operatorname{Im} \tau} \cdot \left.\frac{E_4 E_6}{\Delta}\right|_\tau.$$

The constant term $\frac{E_2 E_4 E_6}{\Delta}$ is a qm function which we shall call χ. It will be convenient also to give the coefficient of $\frac{3}{\pi \operatorname{Im} \tau}$ a name; we will write $\xi = \dfrac{E_4 E_6}{\Delta}$ so that

$$\chi^*(\tau) = \chi(\tau) - \frac{3\xi(\tau)}{\pi \operatorname{Im} \tau}.$$

It is clear that $F^* = \mathbb{C}(j, \chi^*)$ and $\widetilde{F} = \mathbb{C}(j, \chi)$, and the "constant term" isomorphism between the two fixes j and sends χ^* to χ.

It is clear that χ^* is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. The same is not true of the qm function χ, which instead satisfies, for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\chi(\gamma\tau) = \chi(\tau) - \frac{6i}{\pi} \frac{c}{(c\tau + d)} \xi(\tau). \tag{2.3}$$

Although many of the problems explored in this document are aimed at ahm functions in general and χ^* in particular, nonetheless we will often make use of the duality between ahm and qm functions.

We conclude the section with a proposition.

Proposition 2.8. *The functions j and χ^* are algebraically independent over $\mathbb{C}$. Similarly for j and χ.*

Proof. The first claim follows trivially from the fact that j is meromorphic and χ^* is only real analytic. The rest follows from the isomorphism between F^* and $\widetilde{F}$, but may also be deduced using the modified transformation law (2.3) and the modularity of j. □

2.2 q-expansions

It is well known that modular forms have q-expansions, that is, Fourier series expansions of the form

$$f(\tau) = \sum c_n q^n,$$

where $q = e^{2\pi i\tau}$ and $c_n \in \mathbb{C}$ are constants. Quasimodular forms have the same property; although not $\mathrm{SL}_2(\mathbb{Z})$-invariant, they are 1-periodic (this is easily seen from the transformation law) and hence have q-expansions.

The following q-expansions are fairly well-known (see for instance Zagier [89, p. 17,19]).

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

where $\sigma_k(n)$ is the sum of the kth powers of the positive divisors of n. In general, for even k, we have

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where B_k is the kth *Bernoulli number*. This fact can also be found in [89, p. 16-19].

The well-known q-expansion for j,

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots,$$

can be easily (if tediously) deduced from the q-expansions for the Eisenstein series. Similarly, using division of power series, it is clear that χ and ξ have q-expansions starting at q^{-1}. The first few terms of these expansions are:

$$\chi(\tau) = q^{-1} - 264 - 135756q - 5117440q^2 - 23848830q^3 + 2054439936q^4 + \dots$$

and

$$\xi(\tau) = q^{-1} - 240 - 141444q - 8529280q^2 - 238758390q^3 - 4303488384q^4 - \dots.$$

Of course, ahm forms do not have q-expansions in the usual sense. Instead, they may be expressed as polynomials in $(\operatorname{Im}\tau)^{-1}$, whose coefficients are q-expansions.

Analogously for ahm *functions*; we know for instance that $\chi^* = \chi - \frac{3\xi}{\pi y}$, whence the q-expansions for χ and ξ yield an expansion

$$\chi^* = \sum a_n q^n - \frac{3}{\pi \operatorname{Im} \tau} \sum b_n q^n.$$

We will occasionally abuse terminology and refer to an expansion like this as the "q-expansion" of the relevant ahm function, when strictly we are referring to some element of $\mathbb{C}((q))\left[\frac{1}{\operatorname{Im}\tau}\right]$.

Theorem 2.9. *When restricted to the closed fundamental domain $\mathbb{F}$, the functions χ, ξ and χ^* are all definable in $\mathbb{R}_{\text{an,exp}}$.*

Proof. This follows immediately from the fact that each of the functions has a convergent q-expansion. Alternatively, one can use the fact that j is similarly definable and $\chi, \xi \in \mathbb{C}(j, j', j'')$, as we shall see in Chapter 3. □

An immediate consequence of this is that any qm or ahm function, suitably restricted, is definable in $\mathbb{R}_{\text{an,exp}}$. Such definability results will be crucial to much of what follows.

2.3 Modular Relations

Given any function which is well-behaved with respect to $\mathrm{SL}_2(\mathbb{Z})$, it is reasonable to hope that it also has good behaviour with respect to $\mathrm{GL}_2^+(\mathbb{Q})$, just as j does. In the case of qm and ahm functions, this hope turns out to be justified; the following proposition shows that ahm functions satisfy modular relations, a fact which does not seem to have been noted elsewhere.

Proposition 2.10. *For a positive integer N, let M_N be the set of primitive integer matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with determinant N. For each such N, there is a nonzero polynomial $\Psi_N \in \mathbb{Q}[X, Y, Z]$, irreducible over $\mathbb{C}$, such that*

$$\Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = 0$$

for each $g \in M_N$ and all $\tau \in \mathbb{H}$.

Proof. The set

$$D_N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{N}, ad = N, 0 \leq b < d, \gcd(a, b, d) = 1 \right\}$$

is a full set of representatives for M_N under the action of $\mathrm{SL}_2(\mathbb{Z})$. That is, for all $g \in M_N$ there is some $g' \in D_N$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma g' = g$. (This is a standard fact; see for instance Lang [48] or Diamond/Shurman [25, Exercise 1.2.11].)

We will consider a polynomial in X, defined by

$$\prod_{g \in D_N} (X - \chi^*(g\tau)). \tag{2.4}$$

Clearly (for each τ) this is 0 if and only if X is $\chi^*(h\tau)$, for some $h \in D_N$. Thanks to the invariance of χ^* under $\mathrm{SL}_2(\mathbb{Z})$, this holds if and only if X is $\chi^*(h\tau)$ for some $h \in M_N$.

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. For each $g \in D_N$, we have $g \cdot \gamma = \gamma' \cdot h$, for some other $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and some $h \in D_N$. So by the invariance of χ^*, we have

$$\chi^*(g \cdot \gamma\tau) = \chi^*(\gamma' \cdot h\tau) = \chi^*(h\tau).$$

Thus the map $\tau \mapsto \gamma\tau$ induces a permutation of the set

$$S_N = \{\chi^*(g\tau) : g \in D_N\}.$$

In fact, the described action of $\mathrm{SL}_2(\mathbb{Z})$ on S_N is transitive. Indeed, any $g \in D_N$ can be written as

$$g = \gamma h \gamma', \qquad \gamma, \gamma' \in \mathrm{GL}_2(\mathbb{Z}),$$

with h in Smith Normal Form, meaning it is a diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, with $A|D$ (see for instance [27, Exercise 19, page 470]). Further, h must be primitive since g is, whence $A = 1$ and $D = N$. Replacing γ, γ' by $\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma'$ if necessary, we can ensure that they are in fact elements of $\mathrm{SL}_2(\mathbb{Z})$. The claimed transitivity follows immediately[1].

Each coefficient of X in the polynomial (2.4) is a symmetric polynomial in the functions $\chi^*(g\tau)$, $g \in D_N$, so each coefficient must be invariant under $\tau \mapsto \gamma\tau$. Moreover, if $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in D_N$, then

$$\mathrm{Im}(g\tau) = \frac{a \,\mathrm{Im}\,\tau}{d}.$$

Hence each coefficient is a polynomial in $(\mathrm{Im}\,\tau)^{-1}$ with coefficients which are meromorphic functions on $\mathbb{H}$. Since they are also $\mathrm{SL}_2(\mathbb{Z})$-invariant, each coefficient is

[1] I thank David Speyer for showing me the proof of this fact, which is taken as read in many texts.

therefore an ahm function, so can be written as a quotient of complex polynomials in j and χ^*. In each such rational function, we can replace instances of j and χ^* with variables Y and Z. If we do this for each coefficient, we get a polynomial

$$\Psi_N^0(X,Y,Z) \in \mathbb{C}(Y,Z)[X]$$

with $\Psi_N^0(X, j(\tau), \chi^*(\tau)) = 0$ if and only if $X = \chi^*(g\tau)$ for some $g \in M_N$.

Using the q-expansion of χ^*, we have (writing $y = \operatorname{Im} \tau$):

$$\begin{aligned}\Psi_N^0(X, j(\tau), \chi^*(\tau)) &= \prod_{\substack{ad=N \\ d>0}} \prod_{\substack{0 \le b < d \\ (a,b,d)=1}} \left(X - \chi^* \left(\frac{a\tau + b}{d} \right) \right) \\ &= \prod_{\substack{ad=N \\ d>0}} \prod_{\substack{0 \le b < d \\ (a,b,d)=1}} \left(X - \sum_{n=-1}^{\infty} c_n \zeta_d^{nb} q^{na/d} + \frac{d}{a} \frac{3}{\pi y} \sum_{n=-1}^{\infty} c'_n \zeta_d^{nb} q^{na/d} \right),\end{aligned}$$

where $\zeta_d = e^{2\pi i/d}$ and $c_n, c'_n \in \mathbb{Z}$. The innermost product is a polynomial in $3/\pi y$, with coefficients which are Laurent series in $q^{1/d}$ with coefficients from $\mathbb{Z}\left[\frac{d}{a}, \zeta_d\right]$, and leading term no smaller than $-a/d$. But it is 1-periodic, so the fractional powers of q must cancel out. Further, the coefficients in the resulting q-expansions must be in $\mathbb{Z}[\frac{d}{a}]$, since every Galois conjugation $\zeta_d \mapsto \zeta_d^r$, where $r \in (\mathbb{Z}/d\mathbb{Z})^*$, fixes the inner product; the numbers b and rb range over the same set.

So each coefficient f_k of X^k in Ψ_N^0 is a polynomial in $3/\pi y$ with coefficients which are rational Laurent series in q. Each coefficient is also equal to a quotient of polynomials p_k and q_k in j and χ^*, thus

$$p_k(j, \chi^*) = f_k \cdot q_k(j, \chi^*).$$

Comparing the coefficients of $(3/\pi y)^k$ on each side, we get various equalities between q-expansions. The coefficients of those q-expansions are $\mathbb{Q}$-linear in the coefficients of p_k and q_k. So we get a homogeneous system of $\mathbb{Q}$-linear equations holding for the coefficients of p_k and q_k. This system certainly has a solution since p_k and q_k exist. By basic linear algebra, the solution can be chosen to be rational up to scaling, ie. p_k and q_k are in $\lambda\mathbb{Q}[Y, Z]$, for some λ. In particular, p_k/q_k can be rewritten as a quotient of rational polynomials.

Thus $\Psi_N^0 \in \mathbb{Q}(Y,Z)[X]$. Finally, since, as noted earlier, $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on S_N, no subproduct of

$$\prod_{g \in D_N} (X - \chi^* \circ g) = \Psi_N^0(X, j, \chi^*) \in F^*[X]$$

can have coefficients that are $\mathrm{SL}_2(\mathbb{Z})$-invariant. Hence $\Psi_N^0(X, j, \chi^*)$ is irreducible over F^*. In particular, $\Psi_N^0(X, Y, Z)$ is irreducible over $\mathbb{C}(Y, Z)$ as a polynomial in X. It is also monic in X, so if we clear the denominators in Y and Z exactly, we get an irreducible polynomial $\Psi_N \in \mathbb{Q}[X, Y, Z]$ having the required properties. □

Example 2.11. By a fairly tedious calculation involving q-expansions, we can calculate the first few polynomials Ψ_N. They are rather more complex than their classical counterparts Φ_N. $\Psi_1(X, Y, Z)$ is just $X - Z$, of course, but we have for instance

$$\begin{aligned}
\Psi_2(X,Y,Z) = X^3 \cdot Y\big(Y - 1728\big) \\
+ X^2 \cdot \frac{1}{2} Y\big(Y - 1728\big)\big(-YZ - Y^2 + 1464Y + 504Z - 142560\big) \\
- X \cdot Y \Big[17288964Z^2 - \big(Y - 1728\big)\big(216Y^2 - 480Z^2 \\
- 264YZ + 11613375Y - 5007420Z + 1693612800\big)\Big] \\
+ 31049568000Z^3 - 679536Z^3Y \\
+ Y\big(Y - 1728\big)\Big[2Z^3 - 3Y^2Z/2 + Y^3/2 - 171612Y^2 + 512244YZ \\
- 2328717600Z + 8057444544Y - 13413413376000\Big].
\end{aligned}$$

I have calculated the next two Ψ_N as well (after which point the calculations become impractical), but there does not seem to be much value in including them here.

Note. The proof of Proposition 2.10 follows the standard construction of the classical modular polynomials Φ_N very closely. As is typical in our treatment of the basics of ahm functions we have followed Zagier [89, p. 68-70].

Let $\tau \in \mathbb{H}$ be quadratic. If $\tau \in \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, e^{2\pi i/3}\}$, then $\chi^*(\tau) = 0$. Otherwise, there is some primitive integer $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with determinant $N > 1$, such that $g\tau = \tau$. Thus, if one could show (perhaps by careful inspection of the q-expansions) that $\Psi_N(X, j(\tau), X)$ is nontrivial, we would see that

$$0 = \Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = \Psi_N(\chi^*(\tau), j(\tau), \chi^*(\tau)),$$

whence $\chi^*(\tau) \in \overline{\mathbb{Q}}$ (since $j(\tau) \in \overline{\mathbb{Q}}$). We have not yet been able to prove that this polynomial is indeed nonzero, but fortunately the fact that $\chi^*(\tau) \in \overline{\mathbb{Q}}$ is known already, thanks to Masser [49, Appendix 1], who actually showed more, namely that

$$\chi^*(\tau) \in \mathbb{Q}(j(\tau)).$$

His work will arise again in Chapter 5.

So we've seen that much of the behaviour of χ^* mimics that of j very closely, and it has very nice arithmetic properties. By contrast, the imperfect transformation law of χ ensures that it very rarely takes algebraic values at quadratic points. Diaz collects a variety of results and conjectures about the transcendence properties of these points (among others) in his paper [26]; the results there imply that $\chi(\tau)$ is transcendental for all quadratic $\tau \in \mathbb{H}$ except those in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or $\rho = e^{2\pi i/3}$.

On reflection, this is not very surprising. The algebraicity of j at quadratic τ comes from the fact that τ is fixed by some $g \in \mathrm{GL}_2^+(\mathbb{Q})$, but this g, of course, will never be upper triangular. It is clear from (2.3) that χ behaves most nicely with respect to upper triangular matrices g, since these are the ones for which $c = 0$, yielding a much tidier version of (2.3).

Consider, for instance, the proof of Proposition 2.10. It makes essential use of the fact that M_N is represented (up to the action of $\mathrm{SL}_2(\mathbb{Z})$) by the finitely many upper triangular matrices in D_N. Since χ^* is $\mathrm{SL}_2(\mathbb{Z})$-invariant, it is enough that the relation

$$\Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = 0$$

holds for $g \in D_N$; this implies the relation for all $g \in M_N$. Trying to do the same for χ will fail; we cannot go from the upper triangular $g \in D_N$ to general g. The best we can do is the following.

Lemma 2.12. *Let $g_1 \ldots, g_k$ be* upper triangular *primitive integer matrices with positive determinant, and consider the two fields:*

$$A = F^*(j \circ g_1, \ldots, j \circ g_k, \chi^* \circ g_1, \ldots, \chi^* \circ g_k)$$

and

$$B = \widetilde{F}(j \circ g_1, \ldots, j \circ g_k, \chi \circ g_1, \ldots, \chi \circ g_k),$$

each considered as fields of real analytic functions, defined locally. Then A and B are isomorphic via the map

$$\chi^* \mapsto \chi, \qquad \chi^* \circ g_i \mapsto \chi \circ g_i,$$

fixing j and all of the $j \circ g_i$.

Proof. The map is clearly a well-defined bijection. If some $\chi^* \circ g_i$ and $j \circ g_i$ satisfy a polynomial equation $p(\chi^* \circ g_1, j \circ g_1, \ldots, \chi^* \circ g_k, j \circ g_k) = 0$, then (by comparing growth rates) every coefficient of $(\operatorname{Im} \tau)^{-1}$ on the left hand side must vanish. In particular, the constant term $p(\chi \circ g_1, j \circ g_1, \ldots, \chi \circ g_k, j \circ g_k)$ must vanish. That is, the same polynomial equation holds for the $\chi \circ g_i$ and $j \circ g_i$, so the map is indeed an isomorphism. $\square$

It follows that

$$\Psi_N(\chi(g\tau), j(\tau), \chi(\tau)) = 0,$$

for all upper triangular primitive integer matrices of determinant N. For *any* matrix which is not upper triangular the relation will fail; simply look at the transformation law (2.3) satisfied by χ. In fact one can easily see that no polynomial will work for general g.

Proposition 2.13. *Let $N \in \mathbb{N}$. Let M_N be the set of primitive integer matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with determinant N. There* does not *exist a nonzero polynomial $\Psi_N \in \mathbb{C}[X, Y, Z]$ such that whenever $g \in M_N$,*

$$\Psi_N(\chi(g\tau), j(\tau), \chi(\tau)) = 0$$

for all $\tau \in \mathbb{H}$.

Proof. Suppose, for some N, that there were such a nontrivial polynomial Ψ_N. For $n \in \mathbb{Z}$, set

$$\gamma_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

By the supposed property of Ψ_N, we have

$$\Psi_N(\chi(g \circ \gamma_n\tau), j(\gamma_n\tau), \chi(\gamma_n\tau)) = 0$$

for all n, all $\tau \in \mathcal{H}$ and all $g \in M_N$. By the invariance property of j, and the 1-periodicity of $\tilde{\chi}$, we thus have

$$\Psi_N(\chi(g \circ \gamma_n\tau), j(\tau), \chi(\tau)) = 0.$$

We make the following claim.

Claim. There is a fixed $g \in M_N$ such that $\chi(g \circ \gamma_n\tau)$ takes infinitely many different values as n varies.

This forces Ψ_N to be constant in its first variable, inducing an algebraic relation between j and χ, contradicting Proposition 2.8.

Proof of Claim. The g we choose is the matrix $\begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix}$. Note that

$$g \circ \gamma_n = \begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -N \\ 1 & n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & n \\ 0 & N \end{pmatrix}.$$

So by the transformation law (2.3) of χ, we have

$$\chi(g \circ \gamma_n \tau) = \chi\left(\begin{pmatrix}1 & n\\ 0 & N\end{pmatrix}\tau\right) - \frac{6i\xi\left(\begin{pmatrix}1 & n\\ 0 & N\end{pmatrix}\tau\right)}{\pi \cdot \begin{pmatrix}1 & n\\ 0 & N\end{pmatrix}\tau}.$$

Letting $n = kN$ for various $k \in \mathbb{Z}$, we get

$$\chi(g \circ \gamma_{kN}\tau) = \chi\left(\frac{\tau}{N} + k\right) - \frac{6i\xi\left(\frac{\tau}{N} + k\right)}{\pi \cdot \left(\frac{\tau}{N} + k\right)}.$$

The functions χ and ξ are 1-periodic, so finally we get

$$\chi(g \circ \gamma_{kN}\tau) = \chi\left(\frac{\tau}{N}\right) - \frac{6i\xi\left(\frac{\tau}{N}\right)}{\pi \cdot \left(\frac{\tau}{N} + k\right)},$$

which clearly takes infinitely many values as k varies. □

The idea of using the somewhat strange transformation law of χ to prove things about modular relations (or their nonexistence) is simple enough, but surprisingly useful in a few situations. To conclude the section, we use this idea to answer an obvious question about the polynomials Ψ_N: does $\Psi_N(X, Y, Z)$ actually depend on Y (the j-coordinate)?

Proposition 2.14. *Let $N > 1$ and let $S = \chi^{-1}\{0\}$. Then there is an upper triangular $g \in M_N$ such that the set*

$$\{\chi(gs) : s \in S\}$$

is infinite.

Proof. For any $\tau \in \mathbb{H}$ which is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i, the Eisenstein series E_6 is equal to 0. In particular, $\mathrm{SL}_2(\mathbb{Z}) \cdot i \subseteq S$. So we only need to show that (for some g) $\chi(g(\gamma \cdot i))$ takes infinitely many values as γ varies. This is easy to see by considering matrices of the form

$$g = \begin{pmatrix}N & 0\\ 0 & 1\end{pmatrix} \in D_N, \qquad \gamma_n = \begin{pmatrix}1 & -1\\ 1-nN & nN\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then we get

$$g \cdot \gamma_n = \begin{pmatrix}N & -1\\ 1-nN & n\end{pmatrix} \cdot \begin{pmatrix}1 & 0\\ 0 & N\end{pmatrix},$$

so using the transformation law for χ, we have

$$\chi(g(\gamma_n \cdot \tau)) = \chi(\tau/N) - \frac{6i\xi(\tau/N)}{\pi}\frac{1-nN}{(1-nN)(\tau/N)+n}.$$

Setting $\tau = i$, the above expression clearly takes infinitely many values as n varies, provided that $\xi(i/N) \neq 0$, which is clear since the only zeros of E_4 and E_6 are $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i or ρ. □

Corollary 2.15. *The modular polynomial $\Psi_N(X,Y,Z)$ is nonconstant in Y for all $N > 1$.*

Proof. Otherwise $\Psi_N = \Psi_N(X,Z)$, and then by 2.14, $\Psi_N(0,Z)$ has infinitely many solutions, and so is identically zero. Since Ψ_N is irreducible, this would mean that $\Psi_N(X,Z)$ is a constant multiple of X, which is clearly false. □

2.4 Special Varieties and Diophantine Problems

Just as for j, it is reasonable to call the images of quadratic points under χ^* *special.* To distinguish the various notions, we shall, for quadratic τ, say that points of the form $j(\tau)$ are j-special, and analogously that points of the form $\chi^*(\tau)$ are χ^*-special. Since $\chi(\tau)$ is typically not algebraic, we will avoid calling any points "χ-special"; though in Chapter 3 we will see that such points do have some nice properties and raise some interesting questions. The rest of this section focuses on diophantine questions about χ^*, leaving the questions about χ to Chapter 3.

Thanks to the modular polynomials Ψ_N, we can produce analogues of the classical j-special subvarieties in the ahm setting. Since the modular polynomials Ψ_N depend on a j-coordinate, it is most natural to work with j and χ^* simultaneously. We therefore, as in the introduction, introduce a map $\pi : \mathbb{H} \to \mathbb{C}^2$, defined as the cartesian product of j and χ^*. That is:

$$\pi(\tau) = (j(\tau), \chi^*(\tau)).$$

We will regularly abuse notation and use π also to refer to the cartesian powers $\pi : \mathbb{H}^n \to \mathbb{C}^{2n}$.

From the discussions in the previous section, we know that whenever $G \subseteq \mathbb{H}^n$ is a weakly $\mathbb{H}$-special variety (recall Definition 1.6), the image $\pi(G)$ fits inside a variety V cut out by the Ψ_N and Φ_N. If G is $\mathbb{H}$-special, then V will be defined over $\overline{\mathbb{Q}}$. Loosely, a π-special subvariety will be a subvariety of $\mathbb{C}^{2n}$ arising in this way. The precise definition will be given shortly; there is a technicality to be dealt with first.

Consider the variety

$$V'_N = \big\{(W,X,Y,Z) \in \mathbb{C}^4 : \Phi_N(W,Y) = 0, \\ \Psi_N(X,Y,Z) = 0, \Psi_N(Z,W,X) = 0\big\} \subseteq \mathbb{C}^4.$$

By counting conditions, $\dim_{\mathbb{C}} V'_N$ is at most 2. In fact, $\dim_{\mathbb{C}} V'_N = 2$. To see this, note that V'_N contains the set

$$S_g = \{(j(\tau), \chi^*(\tau), j(g\tau), \chi^*(g\tau)) : \tau \in \mathbb{H}\}$$

for any $g \in M_N$. Since j and χ^* are algebraically independent, S_g cannot be contained in any algebraic curve; hence $\dim_{\mathbb{C}} V'_N > 1$.

I believe that the variety V'_N is absolutely irreducible. This can be confirmed by direct calculation for small N, but a general proof remains elusive. I leave this as an open problem, which fortunately has no impact whatsoever on the wider picture: by real analytic continuation, V'_N has an irreducible component containing S_g. Call this component V_N; it is still 2-dimensional. Since it contains S_g, the variety V_N in fact contains all the S_h, $h \in M_N$, by the modularity of π.

The V_N will form the building blocks of π-special varieties, the precise definition of which we now state; note the similarity to Definition 1.5, where we defined j-special varieties.

Definition 2.16. Let $n \in \mathbb{N}$ and let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a partition of $\{1, \ldots, n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose $\tau_s \in \mathbb{H}$ and let $(j_s, c_s) = \pi(\tau_s) \in \mathbb{C}^2$. For each $i > 0$, let s_i be the least element of S_i and for each $s \in S_i \setminus \{s_i\}$ choose a positive integer $N_{i,s}$. A weakly π-special subvariety of $\mathbb{C}^{2n}$ is an irreducible component of a subvariety of the form

$$\begin{aligned}\{(w_1, z_1, \ldots, w_n, z_n) \in \mathbb{C}^{2n} :&(w_s, z_s) = (j_s, c_s) \text{ for } s \in S_0,\\ &(w_s, z_s, w_{s_i}, z_{s_i}) \in V_{N_{i,s}} \text{ for } s \in S_i \setminus \{s_i\}, i = 1, \ldots, k\},\end{aligned}$$

for some given data S_i, (j_s, c_s), $N_{i,s}$.

A weakly π-special variety is π-special if every constant factor (j_s, c_s) is of the form $\pi(\tau_s)$ for some quadratic point $\tau_s \in \mathbb{H}$.

A 0-dimensional π-special subvariety will be called a π-special point. These are simply tuples of the form $\pi(\tau_1, \ldots, \tau_n)$, with each $\tau_i \in \mathbb{H}$ quadratic.

There are countably many π-special varieties. So it is expected that they should be quite sparse: in particular, given a random variety in $\mathbb{C}^{2n}$, it should not contain too many π-special subvarieties. This is the content of the main theorem of Chapter 6, which we will state now.

Theorem 6.5 (André-Oort for π)**.** *Let $V \subseteq \mathbb{C}^{2n}$ be a variety. Then V contains only finitely many maximal π-special subvarieties.*

Note. The "maximal" is certainly necessary; any positive-dimensional π-special subvariety contains infinitely many π-special subvarieties.

This theorem is the most natural ahm analogue of the classical Modular André-Oort Theorem, 1.7. The proof is also fairly similar; by far the most difficult step along the way is the following functional transcendence problem, which is proven in Chapter 4 as a corollary to a similar, but more intricate, problem.

Corollary 4.12 (Ax-Lindemann for π)**.** *Let S be an arc of a real algebraic curve in $\mathbb{H}^n$ and let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety. Suppose that $S \subseteq \pi^{-1}(V)$. Then S is contained in a weakly $\mathbb{H}$-special variety G with $G \subseteq \pi^{-1}(V)$.*

In essence, this is saying that the only algebraic relations between coordinates τ_i, $\tau_j \in \mathbb{H}$ which can induce algebraic relations between $\pi(\tau_i)$ and $\pi(\tau_j)$ are those coming from elements of $\mathrm{GL}_2^+(\mathbb{Q})$. It is a direct analogue of the classical Modular Ax-Lindemann Theorem for j, proven by Pila in [62] as a major step en route to Theorem 1.7. We will discuss a variety of Ax-Lindemann-type results in Chapter 4.

In the classical case, André-Oort and Ax-Lindemann questions are only a small part of the much wider theory of Shimura varieties. In the bigger picture, there are a variety of conjectures extending and generalising these results. The obvious examples are the Zilber-Pink conjecture, a significant generalisation of André-Oort, and Ax-Schanuel questions, which are functional transcendence problems generalising Ax-Lindemann. N. Mok, Pila and Tsimerman [53] have recently announced the proof of Ax-Schanuel (in fact, "Ax-Schanuel with derivatives" - more later) for a general Shimura variety. It is reasonable to ask, in the presence of the π-special varieties, whether one can formulate analogues to these problems for π and perhaps, at least in some very simple cases, approach them using suitable variants of existing techniques. We will discuss such problems, along with some other big-picture questions, in Chapter 7.

Chapter 3

The Derivatives of j

3.1 Setup

Define a map $J : \mathbb{H} \to \mathbb{C}^3$ by

$$J = (j, j', j'') : \tau \mapsto (j(\tau), j'(\tau), j''(\tau)).$$

As for π earlier, we will frequently abuse notation, writing $J : \mathbb{H}^n \to \mathbb{C}^{3n}$ also for the nth cartesian power of J.

The function J is *not* a modular function. By differentiating the equation $j(\gamma\tau) = j(\tau)$ with respect to τ, one sees that

$$j'(\gamma\tau) = (c\tau + d)^2 j'(\tau). \tag{3.1}$$

So j' transforms like a modular form of weight 2. However, being unbounded at ∞, strictly it is a *meromorphic* modular form; that is, a function satisfying all the requirements of being a modular form except possibly having (nicely behaved) poles within $\mathbb{H}$ or "at ∞".

Differentiating again, we get

$$j''(\gamma\tau) = (c\tau + d)^4 j''(\tau) + 2c(c\tau + d)^3 j'(\tau). \tag{3.2}$$

So j'' transforms like a quasimodular form of weight 4 - recall Definition 2.3 - but again is strictly a *meromorphic* quasimodular form. The fact that the derivatives of j are quasimodular is no coincidence; by differentiating transformation laws like this, one can see that the space of (meromorphic) quasimodular forms is in fact closed under differentiation.

We also know that level 1 qm forms are generated by the Eisenstein series E_2, E_4 and E_6. Moreover, E_4 and E_6 are algebraically independent [89, Proposition 4] and therefore certainly E_2^*, E_4 and E_6 are algebraically independent, so by the isomorphism between qm and ahm forms, E_2, E_4 and E_6 are algebraically independent.

We deduce from all this that the space of level 1 (meromorphic) qm forms has transcendence degree 3 over $\mathbb{C}$. In particular, the functions j, j', j'', j''' (all of which are meromorphic qm forms) are algebraically dependent. In fact this dependence occurs over $\mathbb{Q}$; it is a classical fact (see for instance [50, Page 20]) that

$$\frac{j'j'''}{j'^2} - \frac{3}{2}\frac{j''^2}{j'^2} + \frac{j'^2(j^2 - 1968j + 2654208)}{2j^2(j-1728)^2} = 0.$$

This is why we have not included j''' in our map J; it does not add anything to the overall picture. On the other hand, since j, j' and j'' are algebraically independent, we can see that the map J encapsulates precisely the information in which we are interested, and no more.

Since J is only quasimodular, the usual arguments showing that $j(\tau)$ is algebraic at quadratic points cannot apply; indeed it is known that $j'(\tau)$ is transcendental for all quadratic τ except those in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or ρ, where j' vanishes. (This is variously attributed to Mahler or Chudnovskii [21], but can also be found in many surveys, for instance Diaz's paper [26].) However, J does have *some* nice behaviour at quadratic points, namely that $j''(\tau)$ is algebraic over $j'(\tau)$. The relation between such $j''(\tau)$ and $j'(\tau)$ is known as the *Masser relation*; it goes as follows.

First we write out the so-called *Ramanujan equations* (see [89, Proposition 15]):

$$E_2' = 2\pi i\frac{E_2^2 - E_4}{12}, \qquad E_4' = 2\pi i\frac{E_2E_4 - E_6}{3}, \qquad E_6' = 2\pi i\frac{E_2E_6 - E_4^2}{2}.$$

By repeatedly applying these to the Eisenstein series decompositions of j and χ^*, we see that, for all $\tau \in \mathbb{H}$,

$$j''(\tau) = \frac{1}{6}\frac{j'(\tau)^2(\chi^*(\tau) - 7j(\tau) + 6912)}{j(\tau)(j(\tau) - 1728)} + \frac{ij'(\tau)}{\operatorname{Im}\tau}. \tag{3.3}$$

We therefore define a rational function $p_c \in \mathbb{Q}\left[\frac{i}{c}\right](W, X, Z)$ by

$$p_c(W,X,Z) = \frac{1}{6}\frac{Z^2(X - 7W + 6912)}{W(W-1728)} + \frac{iZ}{c},$$

so that (3.3) becomes

$$j''(\tau) = p_{\operatorname{Im}\tau}(j(\tau), \chi^*(\tau), j'(\tau)). \tag{3.4}$$

Since, as earlier noted, $\chi^*(\tau)$ is algebraic when τ is quadratic, we see as desired that $\mathrm{tr.deg.}_{\mathbb{Q}}(J(\tau)) \leq 1$ for quadratic τ. (This fact is named for Masser because, as mentioned in Chapter 2, he seems to have been the first to show that ahm functions like χ^* are algebraic at quadratic points.) Of course the transcendence of $j'(\tau)$ implies that $\text{tr.deg.}_{\mathbb{Q}}(J(\tau)) = 1$ unless $\tau \in \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$.

So J has *some* nice behaviour at quadratic points, though it is less well-behaved than j. This theme continues when we consider the algebraic behaviour of J with respect to the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{H}$. Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$ be a matrix of determinant N. Since $\Phi_N(j(\tau), j(g\tau)) = 0$, it follows that

$$\left.\frac{\partial \Phi_N(X,Y)}{\partial X}\right|_{(j(\tau),j(g\tau))} \cdot j'(\tau) + \left.\frac{\partial \Phi_N(X,Y)}{\partial Y}\right|_{(j(\tau),j(g\tau))} \cdot j'(g\tau)\frac{d(g\tau)}{d\tau} = 0.$$

Otherwise written:

$$j'(g\tau) = -j'(\tau) m_g(\tau) \lambda_N(j(\tau), j(g\tau)), \tag{3.5}$$

where

$$\lambda_N(X,Y) = \frac{\frac{\partial}{\partial X}\Phi_N(X,Y)}{\frac{\partial}{\partial Y}\Phi_N(X,Y)}$$

and

$$m_g(\tau) = \left(\frac{d(g\tau)}{d\tau}\right)^{-1} = \frac{(c\tau+d)^2}{\det g}.$$

(Note that m_g is unaffected by scaling g, so is well-defined even as a function of $g \in \mathrm{PSL}_2(\mathbb{R})$.)

Similarly, by differentiating again one can find a rational function μ_N, in 7 variables and defined over $\mathbb{Q}$, such that

$$j''(g\tau) = \mu_N(j(\tau), j(g\tau), j'(\tau), j'(g\tau), j''(\tau), c, (c\tau+d)). \tag{3.6}$$

Notation 3.1. The notations λ_N, m_g and μ_N above will be fixed for the remainder of the document.

From all the above we deduce that

$$\text{tr.deg.}_{\mathbb{C}(\tau)}(J(\tau), J(g\tau)) = 3$$

if we consider τ as a variable in $\mathbb{H}$. If g is upper-triangular, (ie. $c = 0$), then the dependence on the variable τ in equations (3.5) and (3.6) vanishes, so we get

$$\text{tr.deg.}_{\mathbb{C}}(J(\tau), J(g\tau)) = 3.$$

Otherwise,

$$\text{tr.deg.}_{\mathbb{C}}(J(\tau), J(g\tau)) = 4.$$

In fact this holds even when the transcendence degree is taken over $\mathbb{Q}$ instead of $\mathbb{C}$.

Since the nature of these modular relations varies depending on whether g is upper-triangular, we are motivated to make the following definition.

Definition 3.2. A weakly $\mathbb{H}$-special variety G is called a geodesic upper-triangular (GUT) variety if all of the matrices $g_{i,s} \in \mathrm{GL}_2^+(\mathbb{Q})$ arising in its definition (see 1.6) are upper triangular.

GUT varieties will come up repeatedly later on, particularly in Chapter 4.

3.2 Special Sets

We have seen that J has some good algebraic behaviour with respect to quadratic points and with respect to the action of $g \in \mathrm{GL}_2^+(\mathbb{Q})$. So it makes sense to work with the images, under J, of $\mathbb{H}$-special varieties, and, loosely, to call such objects J-special. But there is a complication. Given a typical $\mathbb{H}$-special variety G, the variety

$$J(G)^{\mathrm{Zar}}$$

will not, in general, be defined over $\overline{\mathbb{Q}}$, since $j'(\tau)$ is usually transcendental at quadratic points. This has two consequences: first, varieties not defined over $\overline{\mathbb{Q}}$ are very difficult to work with, and second, it seems questionable to call such a variety special. The solution to both of these problems is simple; we take Zariski closures over $\overline{\mathbb{Q}}$ instead. This idea, as well as the following associated notation, are both due to Pila.

Notation 3.3. For any subset $S \subseteq \mathbb{H}^n$, define $\langle\langle S\rangle\rangle$ to be the $\overline{\mathbb{Q}}$-Zariski closure of $J(S)$. That is, $\langle\langle S\rangle\rangle$ is the smallest algebraic subvariety of $\mathbb{C}^{3n}$, defined over $\overline{\mathbb{Q}}$, which contains $J(S)$.

We also use this notation to apply to individual points in $\mathbb{H}$, equating points $\tau \in \mathbb{H}$ with singleton sets $\{\tau\}$ in the obvious way.

This raises a question: when trying to define the notion of "J-special variety", should one use $J(G)^{\mathrm{Zar}}$ or $\langle\langle G\rangle\rangle$? Varieties of the form $\langle\langle G\rangle\rangle$ may be easier to work with, but that does not seem enough to justify choosing them over $J(G)^{\mathrm{Zar}}$. Fortunately, this question turns out not to cause problems. In most cases, we will be dealing with varieties V defined over $\overline{\mathbb{Q}}$, so in any case if $J(G) \subseteq V$ then $\langle\langle G\rangle\rangle \subseteq V$. Moreover, we will shortly see that it makes more sense to count $\mathbb{H}$-special subvarieties of $J^{-1}(V)$, rather than J-special subvarieties of V, whence, again, the precise definition of a J-special variety will not matter.

So let us choose one of these options now, for the purposes of discussion, on the understanding that the precise definition of a J-special variety will not be of much importance later on.

Definition 3.4. A J-special subvariety of $\mathbb{C}^{3n}$ is an irreducible component of any set of the form $\langle\langle G\rangle\rangle$, where G is an $\mathbb{H}$-special set.

We might naively conjecture a direct analogue of 1.7, with j-special varieties replaced by J-special ones. Since J is not truly modular, such a conjecture is doomed to fail. Consider the variety $V \subseteq \mathbb{C}^3$ defined by

$$X_1 = j(\tau),$$

for some fixed quadratic $\tau \in \mathbb{H} \setminus (\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\})$. By the modularity of j, V contains all the points $J(\gamma\tau)$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. In particular, taking Zariski closures, we have

$$\langle\langle\gamma\tau\rangle\rangle \subseteq V$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Since

$$\langle\langle\gamma\tau\rangle\rangle = \{(j(\tau), w, p_{\mathrm{Im}\,\gamma\tau}(j(\tau), \chi^*(\tau), w)) : w \in \mathbb{C}\},$$

it is clear that there are infinitely many distinct such $\langle\langle\gamma\tau\rangle\rangle$. So V contains infinitely many J-special varieties. They are maximal, since the only $\mathbb{H}$-special set which properly contains any of the $\gamma\tau$ is $\mathbb{H}$, and $\langle\langle\mathbb{H}\rangle\rangle = \mathbb{C}^3 \not\subseteq V$. (Note that this problem would still arise if we worked with $J(G)^{\mathrm{Zar}}$ instead of $\langle\langle G\rangle\rangle$; it is not our definition of J-special variety which is at fault.)

So we need a version of André-Oort which takes the action of $\mathrm{SL}_2(\mathbb{Z})$ into account. As alluded to earlier, we will look at André-Oort from the other direction, counting $\mathbb{H}$-special subvarieties of the preimage $J^{-1}(V)$ instead of J-special subvarieties of V.

Definition 3.5. Let $\mathcal{S}$ be a collection of subsets of $\mathbb{H}^n$. We say $\mathcal{S}$ is $\mathrm{SL}_2(\mathbb{Z})$-finite if there is some finite subcollection $\mathcal{T} \subseteq \mathcal{S}$ such that every $S \in \mathcal{S}$ takes the form

$$S = \gamma \cdot T,$$

for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$, $T \in \mathcal{T}$. Otherwise, $\mathcal{S}$ is $\mathrm{SL}_2(\mathbb{Z})$-infinite.

As in 3.3, we also apply this terminology to collections of points, equating points τ with singleton sets $\{\tau\}$.

With this definition, we can state a conjecture of Pila which is the prototype version of André-Oort for J.

Conjecture 3.6 (Pila, "Modular André-Oort with Derivatives")**.** *Let* $V \subseteq \mathbb{C}^{3n}$ *be a* proper[1] *algebraic variety defined over* $\overline{\mathbb{Q}}$*. There exists an* $\mathrm{SL}_2(\mathbb{Z})$*-finite collection* $\sigma(V)$*, consisting of* proper $\mathbb{H}$*-special varieties of* $\mathbb{H}^n$*, with the property that every* $\mathbb{H}$*-special subvariety of* $J^{-1}(V)$ *is contained in some* $G \in \sigma(V)$*.*

Remark 3.7. Recently, Aslanyan wrote a paper [5], in which he approaches this problem as part of a more general theory of "Zilber-Pink with Derivatives". He succeeds in proving a modified version of modular André-Oort with derivatives, in which J-special varieties are replaced by D-special varieties, a more general class of object coming from the model theory of differential fields. His theorem neither implies, nor is implied by, Conjecture 3.6. This is because, although D-special varieties are more general than J-special ones, the statement of his theorem restricts to *strongly* D*-special* varieties. These are the analogue, in Aslanyan's setting, of what we would call *basic* special varieties; varieties with no constant coordinate.

Aslanyan's result is also orthogonal to the André-Oort theorems for J that we will prove later: our results do not imply Aslanyan's, nor vice versa. Indeed the two results are of quite different flavours: his result is functional/differential-algebraic in nature, while our results are more number-theoretic. Similarly, the methods used are quite different; his approach is purely model-theoretic and does not appeal to o-minimality.

In Chapter 6 we will present the results we have proven towards Conjecture 3.6. For the rest of this chapter, we discuss some new notions and technicalities which will be crucial.

3.3 Technicalities

3.3.1 Sufficiently Generic Points

The methods we will use to approach Conjecture 3.6 rely heavily on the fact that j' takes transcendental values at (almost all) quadratic points. Loosely, the idea is the following.

Throughout, let $\tau \in \mathbb{H}^n$ be a quadratic point, V a variety defined over $\overline{\mathbb{Q}}$ and $J(\tau) \in V$. It follows that $\langle\langle\tau\rangle\rangle \subseteq V$. In the simplest case, when $n = 1$, the transcendence of $j'(\tau)$ and equation (3.4) together yield

$$\langle\langle\tau\rangle\rangle = \{(j(\tau), w, p_{\mathrm{Im}\,\tau}(j(\tau), \chi^*(\tau), w)) : w \in \mathbb{C}\}.$$

[1] I use the term "proper" as in "proper subset"; V is a proper subvariety of $\mathbb{C}^n$ if it is a subvariety and is not *equal* to $\mathbb{C}^n$.

So to deal with this situation we only need to control the behaviour of j and χ^*.

The next step up is to deal with a point of the form $\sigma = (\tau, g\tau)$, with $g \in \mathrm{GL}_2^+(\mathbb{Q})$. In this case, we can see using (3.5) that (provided $\tau, g\tau \notin \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$):

$$\langle\langle\sigma\rangle\rangle = \Bigg\{ \Big(j(\tau), w, p_{\operatorname{Im}\tau}(j(\tau), \chi^*(\tau), w), \\ j(g\tau), -w\lambda_N(j(\tau), j(g\tau))m_g(\tau), \\ p_{\operatorname{Im} g\tau}\Big(j(g\tau), \chi^*(g\tau), -w\lambda_N(j(\tau), j(g\tau))m_g(\tau)\Big)\Big) : w \in \mathbb{C} \Bigg\}. \quad (3.7)$$

Thus the problem is once again reduced to one involving only j and χ^*. This time there are copies of τ arising from the $m_g(\tau)$, but we shall see that these can be dealt with. Similarly, for any quadratic point of the form $\sigma = (\tau, g_1\tau, \dots, g_n\tau)$, $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$, we completely understand what $\langle\langle\sigma\rangle\rangle$ looks like.

On the other hand, points of the form (τ, σ), with $\sigma \notin \mathrm{GL}_2^+(\mathbb{Q}) \cdot \tau$, can be more problematic. In this situation, with no modular relations occurring, one does not expect that $j'(\sigma)$ and $j'(\tau)$ are algebraically related, but it is conceivable that some such relation exists. Not knowing the form of this hypothetical relation, it is impossible to determine the shape of $\langle\langle(\tau, \sigma)\rangle\rangle$.

Our techniques cannot get around this problem. So the only such points (τ, σ) that can be dealt with are those for which $j'(\tau)$ and $j'(\sigma)$ are algebraically independent, as we expect them to be. We call such points *j'-generic*; we can of course extend this definition to include points with more coordinates.

Definition 3.8. Let $\sigma = (\tau_1, \dots, \tau_n) \in \mathbb{H}^n$ be a quadratic point. Suppose that the τ_i lie in precisely k distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits. Then σ is called j'-generic if

$$\operatorname{tr.deg.}_{\mathbb{Q}}(j'(\tau_1), \dots, j'(\tau_n)) = k.$$

For instance, any quadratic point $(\tau, g_1\tau, \dots, g_n\tau)$, with $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$, is j'-generic provided $\tau \notin \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$, and a point $(\tau, g\tau, \sigma)$, with $\sigma \notin \mathrm{GL}_2^+(\mathbb{Q}) \cdot \tau$, is j'-generic if and only if $j'(\tau)$ and $j'(\sigma)$ are algebraically independent.

For any j'-generic special point $\sigma \in \mathbb{H}^n$, we completely understand what $\langle\langle\sigma\rangle\rangle$ looks like; it can be written out explicitly, just as in equation (3.7). But how many j'-generic points are there?

Conjecture 3.9. *Let $\tau_1, \dots, \tau_n \in \mathbb{H}$ be quadratic points, lying in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits, none of which lies in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of i or ρ. Then $j'(\tau_1), \dots, j'(\tau_n)$ are algebraically independent over $\overline{\mathbb{Q}}$.*

Otherwise put, we expect that all quadratic points in $\mathbb{H}^n$ are j'-generic, except those with a coordinate in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$.

Remark 3.10. Should we believe Conjecture 3.9? It is rather strong, but it does fit into the existing body of conjectures. See specifically the "Elliptic Conjecture" (CE) of Bertolin, [10, Page 207]. CE is a special case of the Grothendieck-André generalised period conjecture; put into the context of our situation, it states that

$$\operatorname{tr.deg.}_{\mathbb{Q}}\big(j(\tau_i), \omega_{i1}, \omega_{i2}, \eta_{i1}, \eta_{i2}\big)_{i=1,\dots,n} \geq 4\sum_{i=1}^{n}(\dim_{\mathbb{Q}} k_i)^{-1} - n + 1, \tag{3.8}$$

where:

- The ω_{ij} are the periods of an elliptic curve E_i, and $\tau_i = \omega_{i1}/\omega_{i2} \in \mathbb{H}$.
- The τ_i lie in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits.
- Each k_i is the field of endomorphisms of E_i.
- The η_{ij} are the *quasiperiods* of E_i.

The quasiperiods η_1, η_2 of an elliptic curve E are numbers arising from the Weierstrass $\wp$-function corresponding to the periods ω_1, ω_2 of E. Specifically, if we let ζ be the antiderivative of $-\wp$, it happens that

$$\zeta(z + \omega_i) = \zeta(z) + \eta_i \text{ for all } z \text{ and some } \eta_i,$$

and we let this property define the η_i. We have the *Legendre relation*:

$$\omega_2\eta_1 - \omega_1\eta_2 = 2i\pi.$$

For these and more details about quasiperiods, one can see for instance Diaz [26, Page 162] or Masser [49, Page VII].

In the case where all the τ_i are quadratic, the CE conjecture (3.8) becomes

$$\operatorname{tr.deg.}_{\mathbb{Q}}(\omega_{i1}, \eta_{i1}, \eta_{i2})_{i=1,\dots,n} \geq n + 1.$$

Using the Legendre relation this becomes

$$\operatorname{tr.deg.}_{\mathbb{Q}}(\omega_{i1}, \eta_{i1}, \pi)_{i=1,\dots,n} \geq n + 1.$$

The periods and quasiperiods of elliptic curves are intricately tied up with j' and j''. We find in [26, Page 165] the two equations

$$j'(\tau) = 18\frac{\omega_1^2}{2i\pi}\frac{g_3}{g_2}j(\tau)$$

and

$$j''(\tau) = j'(\tau)\left[-2\frac{\omega_1\eta_1}{2i\pi} + \left(12 + \frac{9j(\tau)}{j(\tau)-1728}\right)\frac{\omega_1^2}{2i\pi}\frac{g_3}{g_2}\right].$$

Here g_2 and g_3 are coefficients of the elliptic curve E corresponding to the periods ω_i; they are also linearly related to the Eisenstein series E_4 and E_6 respectively. When τ is quadratic, it follows from the algebraicity of j that $\frac{g_2^3}{g_3^2}$ is algebraic; by rescaling the periods ω_i one can therefore ensure that g_2 and g_3 are algebraic. Putting all this together, (3.8) finally becomes

$$\operatorname{tr.deg.}_{\mathbb{Q}}(j'(\tau_i), j''(\tau_i), \pi)_{i=1,\ldots,n} \geq n+1,$$

which in turn, by the Masser relation, says (unless $\tau_i \in \mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$)

$$\operatorname{tr.deg.}_{\mathbb{Q}}(j'(\tau_1), \ldots j'(\tau_n)) \geq n.$$

Thus, CE implies Conjecture 3.9. Of course, from all the reductions we've made to reach this point, it is clear that CE is significantly stronger, but we are satisfied in knowing that Conjecture 3.9 does fall within the existing body of standard conjectures.

In Chapter 6 we will prove, among other things, the following weakened version of Conjecture 3.6.

Theorem 6.11 (Modular André-Oort with Derivatives for Generic Points)**.** *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an* $\mathrm{SL}_2(\mathbb{Z})$*-finite collection $\sigma(V)$, consisting of* proper $\mathbb{H}$*-special varieties of $\mathbb{H}^n$, with the following property. Every j'-generic $\mathbb{H}$-special point in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

In particular, if we assume Conjecture 3.9, then Conjecture 3.6 will hold automatically.

In the final section of this chapter, we will describe the notion of "adjacency", which is the primary novelty in our proof of 6.11.

3.3.2 Adjacency

We begin with some motivation. Suppose that $\sigma \in \mathbb{H}^n$ is a j'-generic quadratic point. For simplicity we shall assume that

$$\sigma = (\tau, g_2\tau, \ldots, g_n\tau),\ g_i \in \mathrm{GL}_2^+(\mathbb{Q}).$$

Any quadratic point, up to permutation of coordinates, is a cartesian product of points of this form, and provided that the point is j'-generic, such extra structure has no effect except to make the notation more involved.

Suppose also that V is an algebraic variety defined over a number field K, with $J(\sigma) \in V$. Then $\langle\langle\sigma\rangle\rangle \subseteq V$, so writing $N_i = \det g_i$:

$$V \supseteq \langle\langle\sigma\rangle\rangle = \Bigg\{ \Bigg(j(\tau), w, p_{\operatorname{Im}\tau}(j(\tau), \chi^*(\tau), w), \ldots \\ \ldots, j(g_i\tau), -w\lambda_{N_i}(j(\tau), j(g_i\tau))m_{g_i}(\tau), \\ p_{\operatorname{Im} g_i\tau}\Big(j(g_i\tau), \chi^*(g_i\tau), -w\lambda_{N_i}(j(\tau), j(g_i\tau))m_{g_i}(\tau)\Big), \ldots \Bigg) : w \in \mathbb{C} \Bigg\}.$$

Consider a field automorphism θ, acting on the $j(\tau)$, $j(g_i\tau)$ and fixing the field $K(\tau, i\operatorname{Im}\tau, i\operatorname{Im} g_i\tau)$. Then $\theta(j(\tau)) = j(\tau')$ for some quadratic τ' and $\theta(j(g_i\tau)) = j(h_i\tau')$ for some $h_i = g_i\gamma_i$, $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$.

As we shall see in Proposition 5.7 later, $\chi^*(\tau) \in \mathbb{Q}(j(\tau))$ and moreover the Galois behaviour of the χ^*-special points is the same as that of j-special points. That is, $\theta(\chi^*(\tau)) = \chi^*(\tau')$ and $\theta(\chi^*(g_i\tau)) = \chi^*(h_i\tau')$. It follows that

$$V \supseteq \Bigg\{ \Bigg(j(\tau'), w, p_{\operatorname{Im}\tau}(j(\tau'), \chi^*(\tau'), w), \ldots \\ \ldots, j(h_i\tau'), -w\lambda_{N_i}(j(\tau'), j(h_i\tau'))m_{g_i}(\tau), \\ p_{\operatorname{Im} g_i\tau}\left(j(h_i\tau'), \chi^*(h_i\tau'), -w\lambda_{N_i}(j(\tau'), j(h_i\tau'))m_{g_i}(\tau)\right), \ldots \Bigg) : w \in \mathbb{C} \Bigg\}.$$

Now note that

$$-j'(\tau')\lambda_{N_i}(j(\tau'), j(h_i\tau'))m_{g_i}(\tau) = j'(h_i\tau')\frac{m_{g_i}(\tau)}{m_{h_i}(\tau')},$$

whence we see

$$\Bigg(j(\tau'), j'(\tau'), p_{\operatorname{Im}\tau}(j(\tau'), \chi^*(\tau'), j'(\tau')), \ldots \\ \ldots, j(h_i\tau'), j'(h_i\tau')\frac{m_{g_i}(\tau)}{m_{h_i}(\tau')}, p_{\operatorname{Im} g_i\tau}\left(j(h_i\tau'), \chi^*(h_i\tau'), j'(h_i\tau')\frac{m_{g_i}(\tau)}{m_{h_i}(\tau')}\right), \ldots \Bigg) \in V. \tag{3.9}$$

This last equation is at the heart of the concept of adjacency.

Note: if we were to replace each g_i in (3.9) by h_i and each τ by τ', we would get $J(\sigma') \in V$, where

$$\sigma' = (\tau', h_2\tau', \ldots, h_n\tau').$$

As it is, since the (g_i, τ) and (h_i, τ') do not match, we cannot deduce $\sigma' \in J^{-1}(V)$. We might say, though, that σ' *nearly* lies in $J^{-1}(V)$. This is the idea behind the notion of adjacency, but it turns out to be better to apply it to *subvarieties* of $\mathbb{H}^n$, rather than just points.

Definition 3.11. A subset of $\mathbb{H}^n$ is a *Möbius variety* if, up to permutation of coordinates, it takes the form

$$G = \left\{ \left(g_{i,j}\tau_i,\ t_m \right)_{\substack{i=1,\ldots,k \\ j=1,\ldots,r_i \\ m=1,\ldots,r_0}} : \tau_1, \ldots, \tau_k \in \mathbb{H} \right\},$$

where $t_m \in \mathbb{H}$ and $g_{i,j} \in \mathrm{SL}_2(\mathbb{R})$.

A Möbius variety is called *basic* if $r_0 = 0$. So a basic Möbius variety is defined by finitely many matrices $g_{i,j} \in \mathrm{SL}_2(\mathbb{R})$, together with a specified order of the coordinates. We will often suppress the permutation of coordinates, assuming for simplicity that coordinates are in numerical order:

$$g_{1,1}\tau_1, \ldots, g_{1,r_1}\tau_1, \quad \ldots \quad, g_{k,1}\tau_k, \ldots, g_{k,r_k}\tau_k, \quad t_1, \ldots, t_{r_0}.$$

Any Möbius variety $G \subseteq \mathbb{H}^n$ has an underlying basic variety associated with it, namely the $B \subseteq \mathbb{H}^{n-r_0}$ attained by ignoring the constant coordinates t_m. We say that G is a *translate* of B (by the t_m), or that B is the basic variety underlying G.

Any weakly $\mathbb{H}$-special variety is in particular a Möbius variety, specifically one in which all of the $g_{i,j}$ lie in $\mathrm{GL}_2^+(\mathbb{Q})$. A basic weakly $\mathbb{H}$-special variety is automatically $\mathbb{H}$-special.

We can now finally state the definition of adjacency.

Definition 3.12. Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety and $B \subseteq \mathbb{H}^d$ a basic Möbius variety, given by data $g_{i,j}$ as in Definition 3.11. For each i, j, take a complex number $z_{i,j}$ and a real $c_{i,j} > 0$. Also take $n - d$ triples of complex numbers (w_l, x_l, y_l).

We say B is *adjacent to V via the data $z_{i,j}, c_{i,j}, w_l, x_l, y_l$* if for all τ_i, we have, up to permutation of coordinates,

$$\left[j(g_{i,j}\tau_i), \frac{j'(g_{i,j}\tau_i)z_{i,j}}{m_{g_{i,j}}(\tau_i)}, p_{c_{i,j}}\left(j(g_{i,j}\tau_i), \chi^*(g_{i,j}\tau_i), \frac{j'(g_{i,j}\tau_i)z_{i,j}}{m_{g_{i,j}}(\tau_i)}\right), \ldots \right.$$
$$\left. \ldots, w_l, x_l, y_l\right]_{\substack{i=1,\ldots,k\\ j=1,\ldots,r_i\\ l=1,\ldots,n-d}} \in V.$$

If this holds for some choice of the data $z_{i,j}$, $c_{i,j}$, w_l, x_l, y_l, we simply say B is adjacent to V, and write

$$B \hookrightarrow V.$$

If this is the case and $G \subseteq \mathbb{H}^n$ is a translate of B, we will also say that G is adjacent to V (via the same data), and denote this by

$$G \hookrightarrow V.$$

The author hopes that the connection between the definition of adjacency and equation (3.9) is clear. There are some properties of adjacency which turn out to be crucial:

1. It arises naturally from Galois considerations, as described earlier.

2. It is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ on the $g_{i,j}$; if we replace each $g_{i,j}$ everywhere by $\gamma g_{i,j}$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, it is easy to see using the transformation laws of j, χ^* and j' that the resulting Möbius variety B' is still adjacent to V, via the same data.

3. It is a definable condition, both on the $g \in \mathrm{SL}_2(\mathbb{R})^n$ parametrising Möbius varieties and on the $\tau \in \mathbb{H}^n$, suitably restricted.

Our approach to Conjecture 3.6 has adjacency at its heart. Should a variety V contain too many points $J(\sigma)$, with σ a j'-generic quadratic point, then all of the Galois conjugates of these points satisfy an equation like (3.9). From this we will produce a positive-dimensional algebraic collection of points satisfying such an equation. The final step is to show that whenever such an algebraic collection exists, it must in fact be contained within a weakly $\mathbb{H}$-special set lying adjacent to V. This is a familiar idea; it is a question of Ax-Lindemann type, and will be dealt with along with various other Ax-Lindemann questions in the next chapter.

Chapter 4

Ax-Lindemann

4.1 Introduction and Setup

We have made passing references, in earlier chapters, to a number of statements of Ax-Lindemann type. In this chapter we will discuss one or two classical Ax-Lindemann statements, then prove some of our own.

An Ax-Lindemann statement for a map $f : X \to Y$, put in a rather general context, is an assertion of the form:

> Let $V \subseteq Y$ be an algebraic variety and A an algebraic subset of $f^{-1}(V)$. Then A is contained in some $G \subseteq f^{-1}(V)$ of a specific form.

When f is a modular function from $\mathbb{H}$ to some cartesian power $\mathbb{C}^d$, then, in the above, X should be $\mathbb{H}^n$, Y should be $\mathbb{C}^{dn}$ and G should be a weakly $\mathbb{H}$-special subvariety of $\mathbb{H}^n$.

For instance, the function f might be the classical modular j-function. In this case, the Ax-Lindemann statement is a theorem due to Pila, which we state here in a slightly different, but equivalent, form to that in the original paper [62].

Theorem 4.1. *Let $V \subseteq \mathbb{C}^n$ be an algebraic variety and $S \subseteq j^{-1}(V)$ an arc of a real algebraic curve. Then there is a weakly $\mathbb{H}$-special variety $G \subseteq j^{-1}(V)$ containing S.*

An Ax-Lindemann theorem is very often a crucial step towards proving an André-Oort statement for a given modular function, but it also carries significant intrinsic value. It is the functional-theoretic modular analogue of the Lindemann-Weierstrass theorem on the algebraic independence of exponentials of algebraic numbers. The Lindemann-Weierstrass theorem, of course, can be viewed as part of Schanuel's Conjecture on exponentials, and in turn, Ax-Lindemann can be viewed as weak version of

“Ax-Schanuel”, the functional-transcendence analogue of Schanuel’s conjecture. We will discuss this further in Chapter 7.

Pila proved Theorem 4.1 using o-minimal ideas. The approach relied heavily, however, on the holomorphicity of j. In the context of ahm functions (or the context of adjacency), we can no longer rely on holomorphicity, so the standard o-minimal approach does not immediately apply. Our approach is a rather more elementary one, if still quite technical. We will take some existing Ax-Lindemann statements and us various algebraic manipulations to transfer them across to our setting. Crucial to this approach is to recall that our central ahm function χ^* lies in $\mathbb{C}(j(\tau), j'(\tau), j''(\tau), \operatorname{Im}\tau)$. With enough trickery, we can use this to convert existing Ax-Lindemann statements for j, j' and j'' into ones involving χ^*.

The starting point is another theorem of Pila, from [63]. To state it, we will need a definition.

Definition 4.2. Let $\tau_1, \dots, \tau_n$ be elements of some algebraic function field $\mathbb{C}(W)$. Then $\tau_1, \dots, \tau_n$ are called *geodesically dependent* if either:

- For some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and some i, j, we have $\tau_i = g\tau_j$ whenever τ_i, τ_j take values in $\mathbb{H}$, *or*
- At least one of the τ_i is constant.

Otherwise, the τ_i are called *geodesically independent.*

Theorem 4.3 (Pila, Modular Ax-Lindemann with Derivatives)**.** *Suppose that* $\mathbb{C}(W)$ *is an algebraic function field and that*

$$\tau_1, \dots, \tau_n \in \mathbb{C}(W)$$

take values in $\mathbb{H}$ *at some* $P \in W$*, and are geodesically independent. Then the* $3n$ *functions*

$$j(\tau_1), \dots, j(\tau_n), \qquad j'(\tau_1), \dots, j'(\tau_n), \qquad j''(\tau_1), \dots, j''(\tau_n)$$

(considered as functions on W *locally near* P*) are algebraically independent over* $\mathbb{C}(W)$*.*

It is perhaps not entirely obvious how this fits the usual format of Ax-Lindemann statements. In the next section, we will formulate several strengthened versions of the theorem which more clearly qualify as Ax-Lindemann results. The remainder of this section deals with one more crucial technicality.

Definition 4.4. Let $Z \subseteq \mathbb{H}^n \times \mathbb{C}^m$ be an arbitrary subset. If A is a connected component of a set of the form

$$W \cap (\mathbb{H}^n \times \mathbb{C}^m),$$

for W an irreducible subvariety of $\mathbb{C}^{n+m}$, and it happens that $A \subseteq Z$, then we say that A is a *complex algebraic component* of Z.

A crucial component of much of the upcoming work is the ability, using analytic continuation, to take a real algebraic arc within a suitable set Z and produce a complex algebraic component of Z. This is encapsulated in the following Lemma, which uses a method of 'complexifying the parameter' due to Pila - see [61, Lemma 2.1].

Lemma 4.5. *Let $f_1, \ldots, f_k : \mathbb{H} \to \mathbb{C}$ be holomorphic functions and let p be a polynomial in $kn + m$ variables, for some n, m. Suppose that*

$$p\big(f_i(\tau_j), z_l\big)_{\substack{i=1,\ldots,k \\ j=1,\ldots,n \\ l=1,\ldots,m}} = 0 \tag{4.1}$$

for all $(\tau_1, \ldots, \tau_n, z_1, \ldots, z_m) \in S$, where S is an arc of a real algebraic curve in $\mathbb{H}^n \times \mathbb{C}^m$. Then (4.1) holds for all $(\tau_1, \ldots, \tau_n, z_1, \ldots, z_m) \in A$, where A is the smallest complex algebraic component of $\mathbb{H}^n \times \mathbb{C}^m$ containing S.

Proof. Let us parametrise S in terms of some real parameter t, as the image of a map $\phi : t \mapsto (\tau_1(t), \ldots, \tau_n(t), z_1(t), \ldots, z_m(t))$ around $t = 0$. Without loss of generality, suppose that τ_1 is nonconstant, so that all of the other functions τ_i, z_i are algebraic over τ_1. The functions τ_i, z_i may then be extended to complex t in some complex neighbourhood of 0.

The image of this complex neighbourhood under ϕ then necessarily lives in some irreducible complex algebraic curve C. Since (4.1) holds on $S \subseteq C$ and all of the f_i arising in (4.1) are complex analytic, it follows that (4.1) holds on the whole of $C \cap (\mathbb{H}^n \times \mathbb{C}^m)$.

The set $C \cap (\mathbb{H}^n \times \mathbb{C}^m)$ has a connected component A' which contains S. Since A is the smallest complex algebraic component containing S, we must have $A \subseteq A'$, whence (4.1) holds on A. □

The ability to interchange freely between real algebraic arcs and complex algebraic components, in the context of holomorphic functions, will be very helpful for much of the work to come.

4.2 More Holomorphic Ax-Lindemann Statements

To carry out the conversion from Theorem 4.3 to an Ax-Lindemann statement involving ahm functions, we will first need to modify 4.3 to get a slightly stronger Ax-Lindemann statement involving all the functions j, j', χ and ξ. Such a result will be the goal of this section.

Recall first, from 3.2, that a weakly $\mathbb{H}$-special variety is called a GUT variety if every matrix arising in its definition is upper-triangular.

Theorem 4.6 (Pila's Modular Ax-Lindemann with Derivatives, Stronger Form)**.** *Let F be an irreducible polynomial in $3n+1$ variables over $\mathbb{C}$. Let $A \subseteq \mathbb{H}^n$ be a complex algebraic component and let G be the smallest weakly $\mathbb{H}$-special variety containing A. Suppose that G is a GUT variety and that*

$$F(\tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \ldots, j(\tau_n), j'(\tau_n), j''(\tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in A$. Then in fact this holds for all $(\tau_1, \ldots, \tau_n) \in G$.

Proof. We will work by induction on n. The case $n = 1$ is immediate.

By definition, the algebraic component A is a connected component of $W \cap \mathbb{H}^n$ for some variety $W \subseteq \mathbb{C}^n$. Treating $\tau_1, \ldots, \tau_n$ as the coordinate functions on W, the hypotheses of the theorem imply that $j(\tau_1), j'(\tau_1), j''(\tau_1), \ldots, j(\tau_n), j'(\tau_n), j''(\tau_n)$, treated as functions locally near some $P \in A$, are algebraically dependent over $\mathbb{C}(W)$, whence Theorem 4.3 tells us that the τ_i are geodesically dependent.

By induction, we may assume that no τ_i is constant on A. Hence there are $1 \leq i, j \leq n$ and $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $\tau_i = g\tau_j$ on A. Since this is a symmetric condition, we may assume that $i \neq 1$. Then without loss of generality, $i = n$.

Since G is a GUT variety, g is upper triangular. Hence there are algebraic functions ϕ_1, ϕ_2, ϕ_3 (induced by the modular polynomials and their derivatives) such that:

$$j(\tau_n) = \phi_1(j(\tau_j)), \tag{4.2}$$

$$j'(\tau_n) = \phi_2(j(\tau_j), j'(\tau_j)), \tag{4.3}$$

and

$$j''(\tau_n) = \phi_3(j(\tau_j), j'(\tau_j), j''(\tau_j)). \tag{4.4}$$

Substituting this into F yields

$$F(\tau_1, j(\tau_1), \ldots, j''(\tau_{n-1}), \phi_1(j(\tau_j)), \phi_2(j(\tau_j), j'(\tau_j)), \phi_3(j(\tau_j), j'(\tau_j), j''(\tau_j))) = 0$$

whenever $(\tau_1, \ldots, \tau_{n-1}, g\tau_j) \in A$. We can then rewrite this as

$$\sigma(\tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \ldots, j(\tau_{n-1}), j'(\tau_{n-1}), j''(\tau_{n-1})) = 0,$$

for some algebraic function σ. This will hold for all $(\tau_1, \ldots, \tau_{n-1}) \in A'$, where A' is the projection of A onto the first $n-1$ coordinates.

It is possible that σ is the zero function. If so, then working backwards we see that F vanishes whenever (4.2), (4.3) and (4.4) hold. In particular, F vanishes whenever $\tau_i = g\tau_j$. Hence it must vanish on G, as required.

If $\sigma \neq 0$, we have more work to do. There is an irreducible polynomial p_σ such that

$$p_\sigma(\sigma(\mathbf{X}), \mathbf{X}) = 0$$

for all $\mathbf{X}$. In particular,

$$p_\sigma(0, \tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \ldots, j(\tau_{n-1}), j'(\tau_{n-1}), j''(\tau_{n-1})) = 0 \tag{4.5}$$

for all $(\tau_1, \ldots, \tau_{n-1}) \in A'$. Note that $p_\sigma(0, \mathbf{X})$ is not the zero polynomial.

We can now appeal to induction to see that (4.5) holds for all $(\tau_1, \ldots, \tau_{n-1}) \in G'$, where G' is the projection of G onto its first $n-1$ coordinates. Putting it in different terms: $X = 0$ is a solution to

$$p_\sigma(X, \tau_1, j(\tau_1), \ldots, j''(\tau_{n-1})) = 0 \tag{4.6}$$

whenever $(\tau_1, \ldots, \tau_{n-1}) \in G'$. We can choose a point $\mathbf{p} \in A'$, a G'-open neighbourhood V of $\mathbf{p}$ and a complex-open neighbourhood U of 0 such that: for all $\mathbf{q} \in V$, the only solution to (4.6) within U is the solution $X = 0$.

However, $X = \sigma(\tau_1, j(\tau_1), \ldots, j''(\tau_{n-1}))$ is always a solution to (4.6) and it vanishes on A'. Thus, whenever $(\tau_1, \ldots, \tau_{n-1}) \in V$ is sufficiently close to $\mathbf{p}$, we have $\sigma(\tau_1, \ldots, \tau_{n-1}) \in U$. By uniqueness (within U) of the solution $X = 0$ we then have

$$\sigma(\tau_1, j(\tau_1), \ldots, j''(\tau_{n-1})) = 0,$$

holding for all $(\tau_1, \ldots, \tau_{n-1})$ in some G'-open set containing $\mathbf{p}$. By analytic continuation, this holds for all $(\tau_1, \ldots, \tau_{n-1}) \in G'$. Recalling the definition of σ, we get

$$F(\tau_1, j(\tau_1), \ldots, j''(\tau_{n-1}), \phi_1(j(\tau_j)), \phi_2(j(\tau_j), j'(\tau_j)), \phi_3(j(\tau_j), j'(\tau_j), j''(\tau_j))) = 0$$

whenever $(\tau_1, \ldots, \tau_{n-1}) \in G'$. Hence

$$F(\tau_1, j(\tau_1), \ldots, j''(\tau_{n-1}), j(g\tau_j), j'(g\tau_j), j''(g\tau_j)) = 0$$

for all $(\tau_1, \ldots, \tau_{n-1}) \in G'$. In other words, F vanishes on G, as required. □

Theorem 4.6 is more obviously a statement of Ax-Lindemann type than 4.3. To apply it to χ^*, we need a version which discusses the component functions of χ^*, that is χ and ξ. For our statement about adjacency, we will need an even stronger version which also includes j and j'. To collect these functions together, we will for the rest of the chapter use the notation

$$\widetilde{\pi}(\tau_1,\ldots,\tau_n) = (j(\tau_1), j'(\tau_1), \chi(\tau_1), \xi(\tau_1), \ldots, j(\tau_n), j'(\tau_n), \chi(\tau_n), \xi(\tau_n)).$$

Theorem 4.7. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let G be the smallest weakly $\mathbb{H}$-special variety containing S. Suppose that G is a GUT variety.*

Let $V \subseteq \mathbb{C}^{4n+1}$ be an algebraic variety such that

$$(\tau_1, \widetilde{\pi}(\tau_1,\ldots,\tau_n)) \in V$$

for all $(\tau_1,\ldots,\tau_n) \in S$. Then in fact this holds for all $(\tau_1,\ldots,\tau_n) \in G$.

Proof. Let F be a defining polynomial of V. We will represent the various coordinates as follows.

- The τ_1 coordinate will be represented by a variable T.
- The j-coordinates (ie. the 2nd, 6th, coordinates, etc.) will be represented by variables $J_1,\ldots,J_n$.
- The j'-coordinates will be represented by variables $K_1,\ldots,K_n$.
- The χ-coordinates will be represented by variables $X_1,\ldots,X_n$.
- The ξ-coordinates will be represented by variables $F_1,\ldots,F_n$.

Since j, j', χ and ξ are algebraically *dependent*, there is an irreducible polynomial p with the property that

$$p(j(\tau), j'(\tau), \chi(\tau), \xi(\tau)) = 0$$

for all τ. Consider the variety $W \subseteq \mathbb{C}^{4n+1}$ defined by

$$p(J_i, K_i, X_i, F_i) = 0$$

for each $1 \le i \le n$. Clearly $\dim W = 3n+1$.

If we further impose the condition

$$F(T, J_1, K_1, X_1, F_1, \ldots, J_n, K_n, X_n, F_n) = 0,$$

there are two possibilities. Either the resulting variety W_F still has dimension $3n+1$ or it has dimension $3n$.

If $\dim W_F = 3n+1$, it is automatically the case that

$$F(\tau_1, \widetilde{\pi}(\tau_1, \dots, \tau_n)) = 0$$

for all $(\tau_1, \dots, \tau_n) \in \mathbb{H}^n$. On the other hand, if $\dim W_F = 3n$, then W_F amounts to the imposition of a relation between 3 of the 4 functions. That is, there are distinct $A, B, C \in \{J, K, X, F\}$ and a polynomial H, in $3n+1$ variables, such that W_F is defined by requiring precisely

$$(T, J_1, \dots F_n) \in W \quad \text{and} \quad H(T, \dots, A_i, B_i, C_i, \dots) = 0.$$

We then have, for the corresponding $f_A, f_B, f_C \in \{j, j', \chi, \xi\}$ that

$$H(\tau_1, \dots, f_A(\tau_i), f_B(\tau_i), f_C(\tau_i), \dots) = 0 \tag{4.7}$$

for all $(\tau_1, \dots, \tau_n) \in S$. By Lemma 4.5, equation (4.7) holds for all $(\tau_1, \dots, \tau_n) \in \mathcal{A}$, where $\mathcal{A}$ is the smallest complex algebraic component containing S.

Since f_A, f_B and f_C are algebraically independent and lie in $\mathbb{C}(j, j', j'')$, we can apply Theorem 4.6 to see that (4.7) holds for all $(\tau_1, \dots, \tau_n) \in G$. In particular,

$$F(\tau_1, \widetilde{\pi}(\tau_1, \dots, \tau_n)) = 0$$

for all $(\tau_1, \dots, \tau_n) \in G$.

This holds for each defining polynomial F of V, so we're done. □

Corollary 4.8. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let ϕ be an algebraic function in $4n+1$ variables. Let G be the smallest weakly $\mathbb{H}$-special variety containing S, and suppose that G is a GUT variety. Suppose also that, on some branch of ϕ,*

$$\phi(\tau_1, \widetilde{\pi}(\tau)) = 0$$

for all $\tau = (\tau_1, \dots, \tau_n) \in S$. Then this holds for all $\tau \in G$, excluding perhaps some exceptional set corresponding to branch points of ϕ.

Proof. There exists an irreducible polynomial p such that

$$p(\phi(\mathbf{X}), \mathbf{X}) = 0$$

for all $\mathbf{X}$. Then in particular we have

$$p(0, \tau_1, \widetilde{\pi}(\tau)) = 0$$

for all $\tau = (\tau_1, \ldots, \tau_n) \in S$. By Theorem 4.7, we have this relation for all $\tau \in G$.

We can pick a point $\mathbf{q} = (\tau_1, \ldots, \tau_n) \in S$, a G-open neighbourhood U of $\mathbf{q}$ and a $\mathbb{C}$-open neighbourhood V of 0 with the following property. Whenever $\tau \in U$, the only solution to

$$p(X, \tau_1, \widetilde{\pi}(\tau)) = 0 \tag{4.8}$$

lying in V is the solution $X = 0$. Now, for all $\tau \in \mathbb{H}^n$, we have

$$p(\phi(\tau_1, \widetilde{\pi}(\tau)), \tau_1, \widetilde{\pi}(\tau)) = 0$$

by definition of p. In other words, $X = \phi(\tau_1, \widetilde{\pi}(\tau))$ is a solution to (4.8). However, as $\tau \in U$ gets arbitrarily close to $\mathbf{q}$, the value of $\phi(\tau_1, \widetilde{\pi}(\tau))$ gets arbitrarily close to 0. Hence it eventually lies in V. The only solution to (4.8) within V is $X = 0$, whence for τ in some G-open neighbourhood of $\mathbf{q}$, we have

$$\phi(\tau_1, \widetilde{\pi}(\tau)) = 0.$$

By analytic continuation, this holds for all $\tau \in G$, perhaps excluding some exceptional set corresponding to the branch points and branch cuts of ϕ. □

4.3 Ax-Lindemann for Adjacency and ahm Functions

In this final section of the chapter we prove the Ax-Lindemann results which have been our goal. The main theorem, Theorem 4.13, does not quite fit the typical form of an Ax-Lindemann statement. Rather than giving us control over which algebraic sets can exist in the preimage of a variety V, it gives us control over which algebraic sets can be *adjacent* to V (as defined in Chapter 3). This turns out to be quite a strong statement; we will use it to deduce a more traditional Ax-Lindemann statement for χ^*.

We begin with a lemma, the motivation for which will become clear shortly.

Lemma 4.9. *Let $S \subseteq \mathbb{H}^n$ be an arc of a real algebraic curve and let ϕ be an algebraic function in $4n$ variables. Suppose that*

$$\operatorname{Im} \tau_1 = \phi(\widetilde{\pi}(\tau_1, \ldots, \tau_n))$$

for all $(\tau_1, \ldots, \tau_n) \in S$. Let G be the smallest weakly $\mathbb{H}$-special variety containing S, and suppose that G is a GUT variety. Then $\operatorname{Im} \tau_1$ is constant on S.

Proof. We will write $y = \operatorname{Im}\tau_1$ throughout. Suppose for a contradiction that y is nonconstant. Then S can be parametrised as

$$S = \{(x_1(y) + iy, x_2(y) + iy_2(y), \ldots, x_n(y) + iy_n(y)) : y \in U\}$$

for some interval $U \subseteq \mathbb{R}$ and algebraic functions x_i, y_i.

Now, take one of the polynomials $p(x_1, y_1, \ldots, x_n, y_n)$ defining S. We can write

$$p\big[\tau_1 - i\phi(\widetilde{\pi}(\tau)), \phi(\widetilde{\pi}(\tau)), x_2(\phi(\widetilde{\pi}(\tau))), y_2(\phi(\widetilde{\pi}(\tau))), \ldots, x_n(\phi(\widetilde{\pi}(\tau))), y_n(\phi(\widetilde{\pi}(\tau)))\big] = 0 \quad (4.9)$$

for all $\tau = (\tau_1, \ldots, \tau_n) \in S$. Otherwise put, we have an algebraic function ψ such that

$$\psi(\tau_1, \widetilde{\pi}(\tau)) = 0$$

for all $\tau \in S$. By Corollary 4.8, this holds for all $\tau \in G$, whence (4.9) holds for all $\tau \in G$.

Since y is assumed to be nonconstant on S, the variable τ_1 cannot be constant on G. So up to permutation of coordinates G looks like

$$\{(\tau_1, g_2\tau_1, \ldots, g_k\tau_1) : \tau_1 \in \mathbb{H}\} \times H$$

for some upper triangular matrices $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$ and some GUT variety H.

For any $\tau_1 \in \mathbb{H}$, $\tau' \in H$ and any $t \in \mathbb{Z}$, we then have

$$\tau_t := (\tau_1 + t, g_2(\tau_1 + t), \ldots, g_k(\tau_1 + t), \tau') \in G.$$

Since the g_i are upper triangular, we can find an integer N with the following property. For every $t \in \mathbb{Z}$ and for all $i \leq k$, there exists $m \in \mathbb{Z}$ such that

$$g_i(\tau_1 + tN) = g_i(\tau_1) + m.$$

By the periodicity of j, j', χ and ξ, it follows that

$$\widetilde{\pi}(\tau_{tN}) = \widetilde{\pi}(\tau_0)$$

for all $t \in \mathbb{Z}$. So for all $\tau = (\tau_1, \ldots, \tau_n) \in G$ and all $t \in \mathbb{Z}$ we have

$$p\big[\tau_1 + tN - i\phi(\widetilde{\pi}(\tau)), \phi(\widetilde{\pi}(\tau)), x_2(\phi(\widetilde{\pi}(\tau))), y_2(\phi(\widetilde{\pi}(\tau))), \ldots, x_n(\phi(\widetilde{\pi}(\tau))), y_n(\phi(\widetilde{\pi}(\tau)))\big] = 0.$$

In particular, whenever $\tau = (x_1(y) + iy, x_2(y) + iy_2(y), \ldots, x_n(y) + iy_n(y)) \in S$, we have

$$p(x_1(y) + tN, y, x_2(y), y_2(y), \ldots, x_n(y), y_n(y)) = 0.$$

This holds for *every* polynomial p defining S. Since S has only one real dimension, it must therefore be a horizontal line in the τ_1 coordinate. That is, y is constant, a contradiction. □

Now we will prove the central theorem underlying all of our Ax-Lindemann results for this chapter.

Theorem 4.10. *Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety. Let S be an arc of a real algebraic curve lying in $(\mathbb{H} \times \mathrm{SL}_2(\mathbb{R}))^n$. Define*

$$\widehat{S} = \{(g_1\tau_1, \ldots, g_n\tau_n) : (\tau_1, g_1, \ldots, \tau_n, g_n) \in S\}.$$

Suppose that

$$\left(\ldots, j(g_i\tau_i), \chi^*(g_i\tau_i), \frac{j'(g_i\tau_i)}{m_{g_i}(\tau_i)}, \ldots\right) \in V$$

for all $(\tau_1, g_1, \ldots, \tau_n, g_n) \in S$, and fix any $(\tau_1^0, g_1^0, \ldots, \tau_n^0, g_n^0) \in S$.

Then there is a weakly $\mathbb{H}$-special variety G with $\widehat{S} \subseteq G$ and

$$\left(\ldots, j(\sigma_i), \chi^*(\sigma_i), \frac{j'(\sigma_i)}{m_{g_i^0}(\tau_i^0)}, \ldots\right) \in V$$

for all $(\sigma_1, \ldots, \sigma_n) \in G$.

Idea of Proof. First, we attempt to parametrise the relevant algebraic arcs in terms of the imaginary part y_j of one of the variables. With suitable manipulations, we reach one of two outcomes: either a particular complex analytic relation involving just $\widetilde{\pi}$ holds, or y_j is equal to an algebraic function in $\widetilde{\pi}$. In the first case, the result comes fairly easily. In the second case, we apply Lemma 4.9 to see that y_j is constant, which situation is also easy to deal with.

Note. We will throughout write G for the weakly $\mathbb{H}$-special closure of $\widehat{S}$: the smallest weakly $\mathbb{H}$-special variety containing $\widehat{S}$. The functions $j(g_j\tau_j)$ and $\chi^*(g_j\tau_j)$ are unaffected if we replace g_j by γg_j for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and it is easy to see, using the transformation law for j', that the same is true of $j'(g_j\tau_j)/m_{g_j}(\tau_j)$. In particular, by applying suitable elements of $\mathrm{SL}_2(\mathbb{Z})$ to S, we may assume that G is a GUT variety. This will be useful several times throughout.

Proof of 2.8. Given $(\ldots, \tau_j, g_j, \ldots) \in S$, let us write

$$g_j\tau_j = \sigma_j = x_j + iy_j$$

and

$$m_{g_j}(\tau_j) = \rho_j = u_j + iv_j.$$

We wish to parametrise the real algebraic arc

$$\widetilde{S} = \{(\ldots, g_j\tau_j, m_{g_j}(\tau_j), \ldots) : (\ldots, \tau_j, g_j, \ldots) \in S\} \subseteq (\mathbb{H} \times \mathbb{C})^n$$

in terms of one of the y_j. Thus the content of the argument depends on whether or not the y_j are constant on $\widetilde{S}$. In fact there will be a number of possible cases, depending on whether the y_j and/or the ρ_j are constant on $\widetilde{S}$.

Case 1: The $y_j = \operatorname{Im}\sigma_j$ and ρ_j are all constant on $\widetilde{S}$.

Let us write $a_j = y_j$ to make it a little clearer that these are constant. We have

$$\left(\ldots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{\rho_j}, \ldots\right) \in V \tag{4.10}$$

for some constants ρ_j and all $\sigma = (\sigma_1, \ldots, \sigma_n) \in \widehat{S}$. So we can apply Theorem 4.7 to see that (4.10) holds for all $\sigma \in G$. Our goal is to show that it still holds if we replace each of the $\chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi a_j}$ by $\chi^*(\sigma_j)$. For this purpose, we may assume without loss of generality that G takes the form

$$\{(\tau, g_2\tau, \ldots, g_n\tau) : \tau \in \mathbb{H}\},$$

with each $g_i \in \mathrm{GL}_2^+(\mathbb{Q})$ an upper triangular matrix. (Any higher-dimensional GUT variety is simply a cartesian product of such varieties, along with some constant factors. We have $\operatorname{Im}\sigma_j = a_j$ for any constant factors σ_j, whence $\chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi a_j} = \chi^*(\sigma_j)$ already, so by induction we can ignore the constant factors also.)

Taking Zariski closures (over $\mathbb{C}$) in (4.10), we get

$$\left(\ldots, j(g_j\tau), \chi(g_j\tau) - \frac{3\xi(g_j\tau)}{\pi a_j}, w_j, \ldots\right) \in V$$

for all $\sigma = (\tau, g_2\tau, \ldots, g_n\tau) \in G$ and $(w_1, \ldots, w_n) \in W_{j(\sigma)}$. Here $W_{j(\sigma)}$ is a variety which depends only on the variables $j(\tau)$, $j(g_j\tau)$, and which contains the point

$$\left(\frac{j'(\tau)}{\rho_1}, \ldots, \frac{j'(g_n\tau)}{\rho_n}\right).$$

By Lemma 4.11, which we prove below, it follows that

$$\big(\dots, j(g_j\tau), \chi^*(g_j\tau), w_j, \dots\big) \in V$$

for all $\sigma = (\tau, g_2\tau, \dots, g_n\tau)$ and all $(w_1, \dots, w_n) \in W_{j(\sigma)}$. In particular we have

$$\left(\dots, j(\sigma_j), \chi^*(\sigma_j), \frac{j'(\sigma_j)}{\rho_j}, \dots\right) \in V,$$

for all $\sigma \in G$.

Finally, since the ρ_j are constant on $\widetilde{S}$, they are necessarily equal to the $m_{g_j^0}(\tau_j^0)$ chosen in the hypotheses of the theorem, so we're done.

Case 2: At least one of the y_j is nonconstant on $\widetilde{S}$.

Without loss of generality, we can suppose that y_1 is nonconstant; let us write $y = y_1$. We can parametrise

$$\widetilde{S} = \{(\dots, x_j(y) + iy_j(y), u_j(y) + iv_j(y), \dots,) : y \in I\},$$

for some interval $I \subseteq \mathbb{R}$ and algebraic functions x_j, y_j, u_j, v_j (setting $y_1(y) = y$). Letting F be a defining polynomial of V, we have

$$F\left[\dots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi y_j(y)}, \frac{j'(\sigma_j)}{u_j(y) + iv_j(y)}, \dots\right] = 0.$$

We can rewrite this; there is an algebraic function s such that the above holds if and only if

$$s(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma = (\sigma_1, \dots, \sigma_n) \in \widehat{S}$.

Since s is an algebraic function, there is a nontrivial irreducible polynomial p_s such that

$$p_s(s(\mathbf{X}), \mathbf{X}) = 0$$

for all $\mathbf{X}$. In particular,

$$p_s(0, y, \widetilde{\pi}(\sigma)) = 0$$

whenever $\sigma \in \widehat{S}$ and $y = \operatorname{Im} \sigma_1$. Since p_s is irreducible and nontrivial, we get a nontrivial $q_s(\mathbf{X}) = p_s(0, \mathbf{X})$. (It is clear that $p_s(t, \mathbf{X}) \neq t$.)

Now we apply the following iterative procedure to q_s, starting with $q = q_s$.

1. Inspect separately each coefficient r_k of T^k in $q(T, \dots)$. If

$$r_k(\widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and all k, then terminate. Otherwise, let q' be the polynomial produced by removing from q all terms $r_k T^k$ for which $r_k(\widetilde{\pi}(\sigma))$ vanishes on $\widehat{S}$.

2. If q' is irreducible, terminate. Otherwise, there is a factor q'' of q' with

$$q''(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and $y = \operatorname{Im} \sigma_1$.

3. We now have a polynomial q'', which retains the property that

$$q''(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$ and $y = \operatorname{Im} \sigma_1$. Repeat from step 1, with q'' instead of q.

This must eventually terminate, since step 2 will always reduce the degree of the polynomial in question. So we have two possibilities.

Subcase 2a: We terminated at step 1.

In this case, working backwards we see that every coefficient r_k of T^k in q_s has the property that

$$r_k(\widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$. Using the fact that G is a GUT variety, we can apply Theorem 4.7 to see that this holds for all $\sigma \in G$. In particular,

$$q_s(y, \widetilde{\pi}(\sigma)) = 0$$

for all $y \in \mathbb{C}$ and all $\sigma \in G$. Otherwise put, $X = 0$ is a solution to

$$p_s(X, y, \widetilde{\pi}(\sigma)) = 0 \tag{4.11}$$

for all $y \in \mathbb{C}$ and all $\sigma \in G$.

Now we proceed much as we did in the proof of Corollary 4.8. Let

$$\mathbf{a} = (g_1^0 \tau_1^0, \dots, g_n^0 \tau_n^0) \in \widehat{S}.$$

We can choose a G-open neighbourhood U of $\mathbf{a}$ and a $\mathbb{C}$-open neighbourhood W of 0 such that: whenever $\sigma = (\sigma_1, \dots, \sigma_n) \in U$, the only solution to (4.11) which lies in W is $X = 0$.

Now recall that

$$X = s(y, \widetilde{\pi}(\sigma))$$

is a solution to (4.11) for all y and all σ. (This is just the definition of p_s.) Since s vanishes at $\mathbf{a} \in \widehat{S}$, there is a G-open neighbourhood U' with

$$\mathbf{a} \in U' \subseteq U$$

such that

$$s(\operatorname{Im} g_1^0 \tau_1^0, \widetilde{\pi}(\sigma)) \in W$$

for all $\sigma \in U'$. But the only root of (4.11) lying in W is $X = 0$. So it must be the case that

$$s(\operatorname{Im} g_1^0 \tau_1^0, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in U'$ and hence for all $\sigma \in G$.

Recalling the definition of s, we see that

$$F\left[\ldots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi y_j}, \frac{j'(\sigma_j)}{\rho_j}, \ldots\right] = 0,$$

for all $\sigma = (\sigma_1, \ldots, \sigma_n) \in G$ and for *constants*

$$y_j = y_j(\operatorname{Im} g_1^0 \tau_1^0) \text{ and } \rho_j = u_j(\operatorname{Im} g_1^0 \tau_1^0) + i v_j(\operatorname{Im} g_1^0 \tau_1^0).$$

By construction, we have $\rho_j = m_{g_j^0}(\tau_j^0)$.

If we repeat this whole procedure for each defining polynomial of V, we get

$$\left[\ldots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi y_j}, \frac{j'(\sigma_j)}{m_{g_j^0}(\tau_j^0)}, \ldots\right] \in V$$

for all $\sigma \in G$ and *constants* $y_j = y_j(\operatorname{Im} g_1^0 \tau_1^0)$. Now we are in exactly the same position as we were for (4.10), so we conclude as we did earlier.

Subcase 2b: We terminated at step 2.

Then we have an irreducible polynomial q such that

$$q(y, \widetilde{\pi}(\sigma)) = 0$$

for all $\sigma \in \widehat{S}$. Moreover, for every k, the coefficient r_k of T^k in $q(T, \ldots)$ has the property that

$$r_k(\widetilde{\pi}(\sigma))$$

does *not* vanish identically for $\sigma \in \widehat{S}$. Hence we can extract an algebraic function ϕ such that

$$\operatorname{Im} \sigma_1 = \phi(\widetilde{\pi}(\sigma))$$

for all $\sigma = (\sigma_1, \ldots, \sigma_n) \in \widehat{S}$. Now we may apply Lemma 4.9 (once again using the fact that the weakly $\mathbb{H}$-special closure of $\widehat{S}$ is a GUT variety) and see that y is constant on $\widehat{S}$, a contradiction.

Case 3: All the y_j are constant on $\widetilde{S}$, but the ρ_j may vary.

Write $y_j = a_j$. We may assume that at least one of the σ_j is nonconstant on $\widetilde{S}$, since otherwise the conclusion is automatic. Without loss of generality, let us say it is σ_1.

We have the relation

$$\left[\ldots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{\rho_j}, \ldots\right] \in V, \tag{4.12}$$

holding for all $(\ldots, \sigma_j, \rho_j, \ldots) \in \widetilde{S}$. Letting A be the smallest complex algebraic component containing $\widetilde{S}$, Lemma 4.5 then tells us that (4.12) holds for all

$$(\ldots, \sigma_j, \rho_j, \ldots) \in A.$$

Since the weakly $\mathbb{H}$-special closure of $\widehat{S}$ is a GUT variety G, the projection of A onto the σ_j coordinates also has G as its weakly $\mathbb{H}$-special closure. Hence we can find a real algebraic arc

$$T \subseteq A \subseteq (\mathbb{H} \times \mathbb{C})^n$$

with the following properties:

- On T, the imaginary part of σ_1 is not constant.
- Whenever $(\ldots, \sigma_j, \rho_j, \ldots) \in T$, we have

$$\left[\ldots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{\rho_j}, \ldots\right] \in V. \tag{4.13}$$

- The projection of T onto the σ_j coordinates has G as its weakly $\mathbb{H}$-special closure.

We can then parametrise T in terms of $y = \operatorname{Im} \sigma_1$ exactly as in Case 2 earlier. We rewrite each polynomial relation (4.13) as an algebraic function s, yielding

$$s(y, \widetilde{\pi}(\sigma_1, \ldots, \sigma_n)) = 0$$

whenever $(\ldots, \sigma_j, \rho_j, \ldots) \in T$ and $y = \operatorname{Im} \sigma_1$. By the same analysis as in Case 2, we end up with

$$s(\operatorname{Im} g_1^0 \tau_1^0, \widetilde{\pi}(\sigma_1, \ldots, \sigma_n)) = 0$$

for all $(\sigma_1, \ldots, \sigma_n) \in G$. Substituting this back into (4.13) yields

$$\left[\ldots, j(\sigma_j), \chi(\sigma_j) - \frac{3\xi(\sigma_j)}{\pi a_j}, \frac{j'(\sigma_j)}{m_{g_1^0}(\tau_1^0)}, \ldots\right] \in V$$

for all $(\sigma_1, \ldots, \sigma_n) \in G$. Since a_j is constant, we can then conclude exactly as we did for (4.10). □

The proof is incomplete until we prove the following lemma to which we appealed above.

Lemma 4.11. *Let $V \subseteq \mathbb{C}^{3n}$ be a variety. Let*

$$G = \{(\tau, g_2\tau, \ldots, g_n\tau) : \tau \in \mathbb{H}\}$$

be a GUT variety and let $W_{j(\sigma)} \subseteq \mathbb{C}^n$ be a family of algebraic varieties fibred over tuples of the form $j(\sigma)$, where $\sigma = (\tau, g_2\tau, \ldots, g_n\tau) \in G$.

Suppose that

$$\left(\ldots, j(g_j\tau), \chi(g_j\tau) - \frac{3\xi(g_j\tau)}{\pi a_j}, w_j, \ldots\right) \in V \tag{4.14}$$

whenever $\sigma = (\tau, g_1\tau, \ldots, g_n\tau) \in G$ and $(w_1, \ldots, w_n) \in W_{j(\sigma)}$. Then in fact

$$(\ldots, j(g_j\tau), \chi^*(g_j\tau), w_j, \ldots) \in V$$

whenever $\sigma = (\tau, g_1\tau, \ldots, g_n\tau) \in G$ and $(w_1, \ldots, w_n) \in W_{j(\sigma)}$.

Proof. Consider a general upper triangular matrix $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$. Let $A = \gcd(b, d)$, let $D = ad/A$ and let k, m be integers such that $mb + kd = A$. For all integers t, we have

$$\begin{pmatrix} b/A & -k + tb \\ d/A & m + td \end{pmatrix} \cdot \begin{pmatrix} A & -ma \\ 0 & D \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}.$$

The leftmost matrix is in $\mathrm{SL}_2(\mathbb{Z})$, while the matrix $H(g) = \begin{pmatrix} A & -ma \\ 0 & D \end{pmatrix}$ has the same determinant as g. Note (taking $t = 0$ above) that

$$\begin{pmatrix} b/A & -k \\ d/A & m \end{pmatrix} H(g) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = g. \tag{4.15}$$

From these matrix equations and the transformation laws of j, χ and ξ, we can easily see that

$$j\left(g\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) = j(H(g)\tau), \qquad \text{for all } t.$$
$$\chi\left(g\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to \chi(H(g)\tau) \qquad \text{as } t\to\infty.$$
$$\xi\left(g\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to 0 \qquad \text{as } t\to\infty.$$

Also

$$j\left(\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) = j(\tau).$$
$$\chi\left(\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to \chi(\tau) \qquad \text{as } t\to\infty.$$
$$\xi\left(\begin{pmatrix}0 & -1\\ 1 & tD\end{pmatrix}\tau\right) \to 0 \qquad \text{as } t\to\infty.$$

Since equation (4.14) holds for all $\sigma \in G$ and $(w_1,\dots,w_n) \in W_{j(\sigma)}$, in particular it holds for points of the form

$$\sigma_t = \left(\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}\tau, g_1\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}\tau, \dots, g_n\begin{pmatrix}0 & -1\\ 1 & t\end{pmatrix}\tau\right) \in G,$$

whenever $(w_1,\dots,w_n) \in V_{j(\sigma_t)}$.

By the above considerations, if we choose D suitably and let $t = kD$ for increasingly large k, we can ensure:

$$j(\sigma_t) = j(\sigma_{kD}) \to j(\sigma'), \quad \chi(\sigma_t) = \chi(\sigma_{kD}) \to \chi(\sigma'), \quad \xi(\sigma_t) = \xi(\sigma_{kD}) \to 0$$

as $k \to \infty$, where

$$\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau).$$

Thus, by replacing σ by σ_{kD} in (4.14) and letting $k \to \infty$, we see by continuity that

$$\big(\dots, j(H(g_j)\tau), \chi(H(g_j)\tau), w_j, \dots\big) \in V$$

for all $\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau)$ and all $(w_1,\dots,w_n) \in W_{j(\sigma')}$. This is an algebraic condition on the points $j(\sigma'), \chi(\sigma')$, holding for all $\sigma' = (\tau, H(g_2)\tau, \dots, H(g_n)\tau)$, so by Lemma 2.12 we can replace the copies of χ by χ^*. That is

$$\big(\dots, j(H(g_j)\tau), \chi^*(H(g_j)\tau), w_j, \dots\big) \in V,$$

for all $\sigma' = (\tau, H(g_2)\tau, \ldots, H(g_n)\tau)$ and all $(w_1, \ldots, w_n) \in W_{j(\sigma')}$. Replacing τ by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau$ and using the modularity of χ^* and j, we see from (4.15) that

$$\big(\ldots, j(g_j\tau), \chi^*(g_j\tau), w_j, \ldots\big) \in V$$

for all $\sigma = (\tau, g_2\tau, \ldots, g_n\tau)$ and all $(w_1, \ldots, w_n) \in W_{j(\sigma)}$. □

To conclude the chapter, we present a number of corollaries to Theorem 4.10. The first is the desired Ax-Lindemann result for the map $\pi = (j, \chi^*)$ discussed in Chapter 2 (and to be discussed in Chapter 6). This was originally proven *before* 4.10, in my paper [79], using almost exactly the same techniques.

Corollary 4.12. *Let S be an arc of a real algebraic curve in $\mathbb{H}^n$ and let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety. Suppose that $S \subseteq \pi^{-1}(V)$. Then S is contained in a weakly $\mathbb{H}$-special variety G with $G \subseteq \pi^{-1}(V)$.*

Proof. Immediate from 4.10, applied to the arc $S' \subseteq (\mathbb{H} \times \mathrm{SL}_2(\mathbb{R}))^n$ defined by

$$S' = \left\{ \left(\tau_1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \tau_n, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) : (\tau_1, \ldots, \tau_n) \in S \right\},$$

and the variety $V' \subseteq \mathbb{C}^{3n}$ defined as V on the j and χ^* coordinates, with no restriction on the j' coordinates. □

Our next corollary is the main Ax-Lindemann result applying to the notion of "adjacency" discussed in Chapter 3. It will be crucial in our approach to André-Oort problems for the derivatives of j. It is for the purposes of proving this result that we have gone to the trouble of proving the more intricate 4.10 rather than approaching Corollary 4.12 directly.

Corollary 4.13 (Ax-Lindemann for Adjacency)**.** *Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety. Let S be an arc of a real algebraic curve lying in $(\mathbb{H} \times \mathrm{SL}_2(\mathbb{R}))^n$. Define*

$$\widehat{S} = \{(g_1\tau_1, \ldots, g_n\tau_n) : (\tau_1, g_1, \ldots, \tau_n, g_n) \in S\}.$$

Suppose that, for some $c_k \in \mathbb{R}$, we have

$$\left[\ldots, j(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k}(\tau_k)}, p_{c_k}\left(j(g_k\tau_k), \chi^*(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k}(\tau_k)} \right), \ldots \right] \in V$$

for all $(\tau_1, g_1, \ldots, \tau_n, g_n) \in S$, and fix any $(\tau_1^0, g_1^0, \ldots, \tau_n^0, g_n^0) \in S$.

Then there exists a weakly $\mathbb{H}$-special variety G such that

$$\widehat{S} \subseteq G,$$

and such that $G \hookrightarrow V$ via the following data:

- *Some*[1] $z_{i,j} \in \mathbb{C}$.
- *Real numbers* $c_{i,j}$ *which are, up to the obvious coordinate-wise relabelling, equal to the given* c_k.
- *Triples*

$$(w_l, x_l, y_l) = \left(j(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k^0}(\tau_k^0)}, p_{c_k}\left(j(g_k\tau_k), \chi^*(g_k\tau_k), \frac{j'(g_k\tau_k)}{m_{g_k^0}(\tau_k^0)} \right) \right),$$

corresponding to those coordinates k *on which* $\widehat{S}$ *(and hence* G*) is constant.*

Proof. Immediate from 4.10. □

The final two corollaries of the chapter will be crucial for some of the counting techniques in our discussions of André-Oort in Chapter 6. They are obvious consequences of Corollaries 4.12 and 4.13: rather than classifying the real algebraic sets which can lie in $\pi^{-1}(V)$ or lie adjacent to some V, they instead classify Möbius varieties with this property.

Corollary 4.14. *Let* $V \subseteq \mathbb{C}^{2k}$ *be a variety and let* B *be a Möbius variety with* $B \subseteq \pi^{-1}(V)$*. Suppose* B *is maximal with this property. Then* B *is weakly* $\mathbb{H}$*-special, ie. all of the* $g_{i,j}$ *defining* B *lie in* $\mathrm{GL}_2^+(\mathbb{Q})$*.*

Proof. Immediate from 4.12. □

Corollary 4.15. *Let* $V \subseteq \mathbb{C}^{3n}$ *be a variety and let* $B \subseteq \mathbb{H}^k$ *be a basic Möbius variety, adjacent to* V *via some data including real numbers* $c_{i,j}$ *and triples* $(w_l, x_l, y_l) \in \mathbb{C}^3$*. Suppose that* B *is maximal among the Möbius subvarieties of* $\mathbb{H}^k$ *with this property (adjacent via the same* $c_{i,j}$*,* (w_l, x_l, y_l)*, though the rest of the data may vary). Then* B *is* $\mathbb{H}$*-special, ie. all of the* $g_{i,j}$ *defining* B *lie in* $\mathrm{GL}_2^+(\mathbb{Q})$*.*

Proof. Immediate from 4.13. □

[1] These can be calculated in terms of G and the $m_{g_k^0}(\tau_k^0)$, but we will not need to know what they are.

Chapter 5

Assorted Technicalities and Number-Theoretic Results

5.1 Jacobians

In this chapter we compile a selection of useful technicalities and miscellaneous facts, focusing on the properties of the function χ^*. We will begin with a brief discussion of Jacobian maps, which will arise in several of the results to come.

Definition 5.1. By the *Jacobian* of a real analytic map $f : U \to \mathbb{C}$ (where U is an open subset of $\mathbb{C}$), we mean the real analytic map $J_f : U \to \mathbb{R}$ defined by

$$J_f(x+iy) = \det \begin{pmatrix} \dfrac{\partial}{\partial x} \operatorname{Re} f(x+iy) & \dfrac{\partial}{\partial y} \operatorname{Re} f(x+iy) \\ \\ \dfrac{\partial}{\partial x} \operatorname{Im} f(x+iy) & \dfrac{\partial}{\partial y} \operatorname{Im} f(x+iy) \end{pmatrix}.$$

Note. If f is a real analytic function and $D \subseteq U$ is such that $f{\restriction}_D$ is definable in some o-minimal expansion of $\mathbb{R}$, then $J_f{\restriction}_D$ is similarly definable.

It is a well-known fact that f is locally invertible near τ whenever $J_f(\tau)$ is nonzero. In particular, if f is constant on some real analytic arc S, then J_f *must* vanish upon that arc. This will be crucial to several of the results of this section; to make use of it we will need to know that the Jacobians in which we are interested do not vanish identically.

Lemma 5.2. *Let $h \in \mathbb{R}(j, \chi^*)$ be nonconstant. Then the Jacobian J_h of h is not identically zero.*

Proof. Such an h may be written as

$$h(\tau) = \frac{f}{g} = \frac{\sum_k f_k(\operatorname{Im}\tau)^{-k}}{\sum_k g_k(\operatorname{Im}\tau)^{-k}},$$

for some holomorphic functions f_k, g_k having q-expansions with real coefficients. A simple manipulation then shows, for $\tau = iy$, that J_h vanishes only if

$$f\sum_k g_k' y^{-k} = g\sum_k f_k' y^{-k} \tag{5.1}$$

or

$$f\left(\sum_k g_k' y^{-k} - \sum_k k g_k y^{-k-1}\right) = g\left(\sum_k f_k' y^{-k} - \sum_k k f_k y^{-k-1}\right). \tag{5.2}$$

By growth considerations, the coefficient of each y^k must vanish. In case (5.1) we compare the $k = 0$ terms to get

$$f_0 g_0' - g_0 f_0' = 0,$$

whence $(f_0/g_0)' = 0$ on $\tau = iy$ (and therefore everywhere), so $f_0 = \lambda g_0$ for some constant λ. By the isomorphism between qm and ahm forms, this implies $f = \lambda g$, which we are assuming is not the case.

The case (5.2) is exactly the same by comparing the $k = 0$ terms; the sums of the form $\sum k g_k y^{-k-1}$ contribute nothing to the $k = 0$ term. □

5.2 Surjectivity of χ^*

Our first question about χ^* for this chapter is an obvious one: does χ^* map surjectively onto $\mathbb{C}$? This is known to be the case for j, and with some careful analysis we can answer the question affirmatively.

Proposition 5.3. *χ^* maps surjectively onto $\mathbb{C}$.*

Proof. We consider the half-fundamental domain

$$\mathbb{F}^l = \{\tau \in \mathbb{H} : |\tau| \geq 1, -1/2 \leq \operatorname{Re}\tau \leq 0\}.$$

Since χ^* is conjugate-symmetric about the line $\operatorname{Re}\tau = 0$, it will be enough to show that $\chi^*(\mathbb{F}^l) = \mathbb{H} \cup \mathbb{R}$.

From the q-expansion and the modularity of χ^*, it follows that χ^* is real-valued on the boundary of $\mathbb{F}^l$. Moreover, along the line $\{-1/2 + iy\}$, χ^* goes to $-\infty$ as

$y \to \infty$. Along the imaginary axis, χ^* goes to ∞ as $y \to \infty$. It follows by continuity that χ^* takes all real values somewhere on the boundary of $\mathbb{F}^l$.

It suffices to show that for each individual $c > 0$, the line $L = \{x + ic : x \in \mathbb{R}\}$ lies in $\chi^*(\mathbb{F}^l)$. To do this we consider horizontal lines in $\mathbb{F}^l$ of the form

$$H_y = \{x + iy : -1/2 \leq x \leq 0\}.$$

For large y, using the q-expansion, $\chi^*(H_y)$ approximates a semicircle with radius tending to infinity as y grows. So for sufficiently large y, the point $\chi^*(-1/4 + iy)$ lies above L, whereas $\chi^*(-1/2 + iy)$ and $\chi^*(iy)$ are real, so $\chi^*(H_y)$ meets L at least once to the left of $\operatorname{Re}\tau = -1/4$ (first at a point $\chi^*(x_l + iy) \in L$) and at least to the right, first at a point $\chi^*(x_r + iy) \in L$.

For large enough y, by inspecting derivatives of the q-expansion, one can see that to the left of some $x_l + \epsilon$, $\operatorname{Im}\chi^*(x + iy)$ is increasing. Similarly, to the right of some $x_r - \epsilon$, $\operatorname{Im}\chi^*(x + iy)$ is decreasing. Hence $\chi^*(H_y)$ meets L *only* at $\chi^*(x_l + iy)$ and $\chi^*(x_r + iy)$.

We now fix a large enough y and divide the line L into three segments: the half-line P to the left of $\chi^*(x_l + iy)$, the half-line Q to the right of $\chi^*(x_r + iy)$ and the line segment S in between.

It is clear that the preimage of P contains (as well as potentially some other isolated pieces) at least one real analytic arc A_l meeting $x_l + iy$ and tending up towards ∞. So let A be the real analytic path formed by real analytic continuation of A_l. The image $\chi^*(A)$ is contained in L, so A cannot ever meet the boundary of $\mathbb{F}^l$ (where χ^* is real-valued). Starting at ∞, we follow A along A_l, past where it meets $x_l + iy$ and passes below the horizontal line H_y. Being a definable real analytic path, A cannot remain contained within the region in $\mathbb{F}^l$ below H_y, but must continue and eventually pass back above H_y, and escape to ∞.

Since $\chi^*(H_y)$ only meets L twice for all $Y > y$, the point at which A meets H_y can only be $x_r + iy$. Hence, on the path A, χ^* takes all values in P (while traversing A_l) and also takes the value $\chi^*(x_r + iy)$. By continuity it takes all the values on $P \cup S$. By a symmetric argument, we know that Q is also in the image of χ^*. Thus L lies in the image of χ^*, as required. □

5.3 Injectivity and Field Extensions

In a similar vein to the above, it is reasonable to ask whether χ^* is injective, at least when restricted to the fundamental domain $\mathbb{F}^-$. Of course it is not; we know that

χ^* vanishes at both i and ρ. More generally, then, is χ^* at least finite-to-one when restricted to $\mathbb{F}^-$? The nonholomorphicity of χ^* makes this question rather challenging, and in attempting to answer it we naturally encounter an approach towards a problem of an entirely different nature.

Let δ be a negative integer. Let P_δ^N be the (finite) set of primitive, reduced, integer quadratic forms

$$Q(x,y) = Ax^2 + Bxy + Cy^2,$$

with $N|A$, of discriminant δ. Let τ_Q be the unique root of $Q(\tau, 1)$ lying in the upper half plane. In their 2015 paper [51], Mertens and Rolen proved that, for a certain ahm function P of level 6, the so-called *class polynomial*

$$H_\delta^P = \prod_{Q \in P_\delta^6} (X - P(\tau_Q)) \in \mathbb{Q}[X]$$

is irreducible over $\mathbb{Q}$ whenever $\delta \equiv 1 \mod 24$, thus settling a question posed by Bruinier and Ono in [19]. In particular, we have

$$[\mathbb{Q}(P(\tau)) : \mathbb{Q}] = \# P_\delta^6$$

whenever τ is a root of a quadratic polynomial $Ax^2 + Bx + C$, $6|A$, with discriminant congruent to 1 mod 24.

This result is a direct analogue of the classical fact that the class polynomial for j, namely

$$H_\delta^j = \prod_{Q \in P_\delta^1} (X - j(\tau_Q)) \in \mathbb{Q}[X],$$

is irreducible over $\mathbb{Q}$ for all δ. In general, given a (perhaps nonclassical) modular function f of level N, one can define the class polynomial

$$H_\delta^f = \prod_{Q \in P_\delta^N} (X - f(\tau_Q)).$$

It is natural to ask, given some f, whether H_δ^f is irreducible, over various fields. In the case of level 1 ahm functions, since $\chi^*(\tau) \in \mathbb{Q}(j(\tau))$, this question over $\mathbb{Q}$ amounts to asking whether or not $\mathbb{Q}(f(\tau)) = \mathbb{Q}(j(\tau))$.

In a paper [17] extending the techniques of Mertens and Rolen, their students Braun, Buck and Girsch made significant progress towards a fairly general irreducibility result for ahm functions. They show that for each ahm function f in the image of a certain differential operator, there is an effectively computable D such that H_δ^f is irreducible over $\mathbb{Q}$ for all $|\delta| > D$. In theory, one can then calculate the H_δ^f for the finitely many "small" δ to achieve the result in general.

Of course, the collection of functions covered by the work of Braun, Buck and Girsch, while extensive, is far from complete; in particular it does not contain χ^*. (Though it does contain the function $\chi^* + 5j$!)

In the course of attempting to prove that χ^* is, say, finite-to-one, using o-minimal methods, one naturally approaches the following result. It is in some ways less satisfactory than the results of Mertens, Rolen, Braun, Buck and Girsch - most notably since it is ineffective - but it holds for a very wide class of level 1 ahm functions.

Theorem 5.4. *For every nonconstant $f \in \mathbb{Q}(j, \chi^*)$, there is a constant M such that*

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(f(\tau))] \leq M$$

for all quadratic $\tau \in \mathbb{H}$.

We will prove this shortly; first we discuss a few of the components of the proof, and describe how this arises when considering the question of the injectivity or otherwise of χ^*.

A crucial component is the following proposition, which arises from a careful inspection of Masser's proof in [49] that $\chi^*(\tau) \in \mathbb{Q}(j(\tau))$. This result will also be central to our André-Oort theorems in Chapter 6.

Notation 5.5. Throughout the remainder of the document, we will, given a quadratic point $\tau \in \mathbb{H}$, denote its discriminant by $\delta(\tau)$. Since $\delta(\tau)$ will always be negative, we will also write $D(\tau) = |\delta(\tau)|$, which will often be more convenient. Furthermore, if $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{H}^n$ has quadratic coordinates, we will write $\delta(\tau) = \min_i \delta(\tau_i)$, and $D(\tau) = \max_i D(\tau_i)$.

Lemma 5.6. *For each natural number n, there is a constant $c_{\mathrm{Ht}}(n)$ such that, whenever $\tau \in \mathbb{F}^n$ is a quadratic point, we have*

$$\mathrm{Ht}(\tau) \leq c_{\mathrm{Ht}}(n) D(\tau).$$

Proof. See [62, Proposition 5.7]. □

Proposition 5.7. *Let $\tau \in \mathbb{H}$ be a quadratic point and consider the algebraic numbers $j(\tau)$ and $\chi^*(\tau)$. Let θ be a Galois automorphism acting on $j(\tau)$, so that $\theta(j(\tau)) = j(\tau')$ for some quadratic τ'. Then $\chi^*(\tau') = \theta(\chi^*(\tau))$.*

Proof. Fix $D = D(\tau)$, and suppose that $D \neq 3k^2$ for any odd k. Define univariate rational functions $\beta^{\tau}_{i,k}$ such that the $\beta^{\tau}_{i,k}(j(\tau))$ are the coefficients of the Taylor expansion of Φ_D about the point $(j(\tau), j(\tau))$. This we can certainly do, and we get

$$\Phi_D(X,Y) = \sum_{(i,k)\neq(0,0)} \beta^{\tau}_{i,k}(j(\tau))(X - j(\tau))^i(Y - j(\tau))^k. \tag{5.3}$$

The rational functions $\beta^{\tau}_{i,k}$ will differ with τ, of course. However, we will show that, for the τ and τ' chosen in the hypotheses of the theorem, we do have $\beta^{\tau}_{i,k} = \beta^{\tau'}_{i,k}$.

Since Φ_D has rational coefficients, any field automorphism preserves the left hand side of (5.3). So we get

$$\begin{aligned}\Phi_D(X,Y) &= \sum \theta(\beta^{\tau}_{i,k}(j(\tau)))(X - \theta(j(\tau)))^i(Y - \theta(j(\tau)))^k \\ &= \sum \beta^{\tau}_{i,k}(j(\tau'))(X - j(\tau'))^i(Y - j(\tau'))^k.\end{aligned}$$

We also have

$$\Phi_D(X,Y) = \sum \beta^{\tau'}_{i,k}(j(\tau'))(X - j(\tau'))^i(Y - j(\tau'))^j,$$

so by uniqueness of Taylor coefficients, the rational functions $\beta^{\tau}_{i,k}$ and $\beta^{\tau'}_{i,k}$ are equal, as required.

On pages 118 and 119 of [49], the ahm function $\psi = \frac{\chi^*}{j-1728}$ is expressed as a fixed $\mathbb{Q}$-rational function p in the $\beta^{\tau}_{i,k}(j(\tau))$ and $j(\tau)$. The equality

$$\psi(\tau) = p(j(\tau), \beta^{\tau}_{i,k}(j(\tau))) \tag{5.4}$$

holds whenever $D(\tau) = D$. Since τ' and τ have the same discriminant (both satisfy $\Phi_D(j(\rho), j(\rho)) = 0$ and no such equation for Φ_N, $N < D$), this equation holds for both τ and τ'. Since $\beta^{\tau}_{i,k} = \beta^{\tau'}_{i,k}$ we get

$$\begin{aligned}\theta(\psi(\tau)) &= p(\theta(j(\tau)), \beta^{\tau}_{i,k}(\theta(j(\tau)))) \\ &= p(j(\tau'), \beta^{\tau}_{i,k}(j(\tau'))) \text{ by (5.4)} \\ &= p(j(\tau'), \beta^{\tau'}_{i,k}(j(\tau'))) \text{ since } \beta^{\tau}_{i,k} = \beta^{\tau'}_{i,k} \\ &= \psi(\tau') \text{ by (5.4)}.\end{aligned}$$

When $D(\tau)$ is $3k^2$ for some odd k, the exact same argument still goes through, except the rational function p is replaced by q, which is some other (still fixed and explicit) rational function. In either case we get $\theta(\psi(\tau)) = \psi(\tau')$. Since $\chi^* = (j - 1728)\psi$, we get $\theta(\chi^*(\tau)) = \chi^*(\tau')$ as required.

The rational functions p and q are written out on pages 118 and 119 of [49], but we will duplicate them here for completeness[1].

$$p(j, \beta_{i,k}) = \frac{9j(\beta_{2,0} - \beta_{1,1} + \beta_{0,2})}{\beta_{0,1}} + \frac{3(7j - 6912)}{2(j - 1728)}.$$

$$q(j, \beta_{i,k}) = \frac{9j(\beta_{4,0} - \beta_{3,1} + \beta_{2,2} - \beta_{1,3} + \beta_{0,4})}{\beta_{0,1}} + \frac{3(7j - 6912)}{2(j - 1728)}.$$

□

It is an easy consequence of this result that $H_\delta^{\chi^*}$ is always a power of an irreducible polynomial, that is

$$H_\delta^{\chi^*} = p_\delta^{k_\delta},$$

for some irreducible polynomial $p_\delta \in \mathbb{Q}[X]$ and some natural number k_δ. Further, it is easy to see that

$$k_\delta = \#\{\tau \in \mathbb{F}^- : \tau \text{ is quadratic of discriminant } \delta, \chi^*(\tau) = \chi^*(\tau_0), \},$$

for any fixed τ_0 of discriminant δ. Thus, if we could show that χ^* is injective on the set of quadratic $\tau \in \mathbb{F}^-$ of discriminant δ, we would get $k_\delta = 1$, whence $H_\delta^{\chi^*}$ would be irreducible. More generally, we can get a bound on k_δ, and therefore a bound on the degree

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(\chi^*(\tau))],$$

by bounding the size of preimages $(\chi^*)^{-1}(z) \cap \mathbb{F}$. This final idea lies behind the proof of Theorem 5.4.

Before we finally start working on 5.4, we need one more piece of Galois-theoretic information, which will also be crucial in our André-Oort results in later chapters. Thanks to Proposition 5.7, we know that the Galois behaviour of the special points of χ^* is very similar to that of the special points of j. So standard facts about the Galois behaviour of j are of interest; in particular the following fact gives us a lower bound on the size of $[\mathbb{Q}(j(\tau)) : \mathbb{Q}]$.

Theorem 5.8 (Siegel Bound)**.** *For every $\eta > 0$, there is a constant $c_\eta > 0$ such that*

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \geq c_\eta D(\tau)^{\frac{1}{2} - \eta}$$

whenever $\tau \in \mathbb{H}$ is quadratic.

[1] The reader may note a strange-looking asymmetry in p and q, namely the $\beta_{0,1}$ in the denominator. Why not $\beta_{1,0}$? Masser proves that $\beta_{0,1} = \beta_{1,0}$, so really there is no asymmetry.

Proof. Let $\delta = \delta(\tau)$ and $D = D(\tau) = |\delta|$. Since H^j_δ is irreducible of degree $\#P^1_\delta$, it follows that $\mathbb{Q}(j(\tau))$ has degree $\#P^1_\delta$. The size of this set is known to be equal to the size h_D of the class group corresponding to τ (a classical fact; see for instance [16, Page I-2]). It is a classical result of Siegel [75] that $h_D \geq c_\eta D^{\frac{1}{2}-\eta}$, whence

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = \#P^1_\delta = h_d \geq c_\eta D^{\frac{1}{2}-\eta},$$

as required. □

The bulk of the work in proving 5.4 lies in the following lemma, which uses all the Galois information obtained from the last two results.

Lemma 5.9. *For every nonconstant $f \in \mathbb{Q}(j, \chi^*)$, there is a positive integer M such that*

$$\#\{\tau \in \mathbb{F} \cap f^{-1}\{z\} : \tau \text{ is quadratic}\} \leq M$$

for all $z \in \mathbb{C}$.

Proof. It is well-known (see for instance [80, 3.10]) that dimension is definable in o-minimal structures, whence the set

$$Z = \left\{\tau \in \mathbb{F} : \dim_{\mathbb{R}} \left(\mathbb{F} \cap f^{-1}\{f(\tau)\}\right) = 1\right\}$$

is definable in $\mathbb{R}_{\text{an,exp}}$.

If f is constant on a connected set of positive real dimension, then J_f also vanishes on that set. So with the possible exception of some isolated points, Z is contained in the zero set of J_f.

By 5.2, J_f is not identically zero, so by analytic continuation the zero set of J_f can have no interior. Hence, by the Cell Decomposition Theorem, 1.13, the definable set Z must consist of just finitely many points and real analytic arcs. We note for later that, in particular, the image $f(Z)$ is always a finite set.

By the Cell Decomposition Theorem and the definability of f, it follows that f is finite-to-one on $\mathbb{F} \setminus Z$. By the Uniform Finiteness Theorem 1.12, there is $m \in \mathbb{N}$ such that

$$\#\left(\mathbb{F} \cap f^{-1}\{f(\tau)\}\right) \leq m \text{ unless } \tau \in Z.$$

So it will be sufficient to prove the following.

Claim. There are only finitely many quadratic points in Z.

Proof of Claim. By Corollary 4.12, the set Z can never contain an arc of a real algebraic curve. So let us write

$$Q(T) = \{\tau \in Z : \tau \text{ is quadratic, } \mathrm{Ht}(\tau) \leq T\}.$$

Then by the Pila-Wilkie Theorem, Theorem 1.17, we have (for all $\epsilon > 0$) a constant c_ϵ such that

$$\#Q(T) = \#\{\tau \in Z : \tau \text{ is quadratic, } \mathrm{Ht}(\tau) \leq T\} \leq c_\epsilon T^\epsilon. \tag{5.5}$$

We will be playing this bound off against the lower bound provided by 5.8.

Let $\tau \in Z$ be a quadratic point with $D(\tau) = D$, $\delta(\tau) = \delta$ and $f(\tau) = z$. Recall that $H_\delta^f = p_\delta^{k_\delta}$, where p_δ is an irreducible polynomial which must then be the minimal polynomial of z. Hence

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = k_\delta[\mathbb{Q}(f(\tau)) : \mathbb{Q}] = k_\delta[\mathbb{Q}(z) : \mathbb{Q}].$$

Applying Theorem 5.8 with $\eta = 1/4$ yields a constant $c = c_{1/4}$ such that

$$k_\delta \geq \frac{cD^{\frac{1}{4}}}{[\mathbb{Q}(z) : \mathbb{Q}]}.$$

Note: since $f(Z)$ is finite, $[\mathbb{Q}(z) : \mathbb{Q}]$ is uniformly bounded as $z = f(\tau)$ varies among quadratic $\tau \in Z$. So there is a constant α such that

$$k_\delta \geq \frac{cD^{\frac{1}{4}}}{[\mathbb{Q}(z) : \mathbb{Q}]} \geq \frac{cD^{\frac{1}{4}}}{\alpha},$$

for all discriminants δ.

Now, since z is a root of p_δ, we get, for each power of p_δ arising in H_δ^f, another quadratic point τ' of discriminant δ with $f(\tau') = z$. Hence there are at least k_δ distinct quadratic points τ, of discriminant δ, such that $f(\tau) = z$.

Each such point lies in $Q(T)$ for a suitable height T. Moreover, for quadratic $\tau \in \mathbb{F}$, by Lemma 5.6, we know $\mathrm{Ht}(\tau) \leq c_{\mathrm{Ht}} D(\tau)$ for some constant c_{Ht}. So if we take, say, $\epsilon = \frac{1}{8}$ in the Pila-Wilkie bound (5.5), we get

$$\frac{cD^{\frac{1}{4}}}{\alpha} \leq k_\delta \leq Q_f(c_{\mathrm{Ht}} D) \leq c_{\frac{1}{8}} c_{\mathrm{Ht}}^{\frac{1}{8}} D^{\frac{1}{8}}.$$

This clearly yields a contradiction for any D larger than $\kappa = c_{\mathrm{Ht}} \left(\frac{c_{\frac{1}{8}}\alpha}{c}\right)^8$.

So for quadratic $\tau \in Z$, the size of the discriminant $D(\tau)$ is bounded above by κ. In particular, there are only finitely many quadratic $\tau \in Z$. □

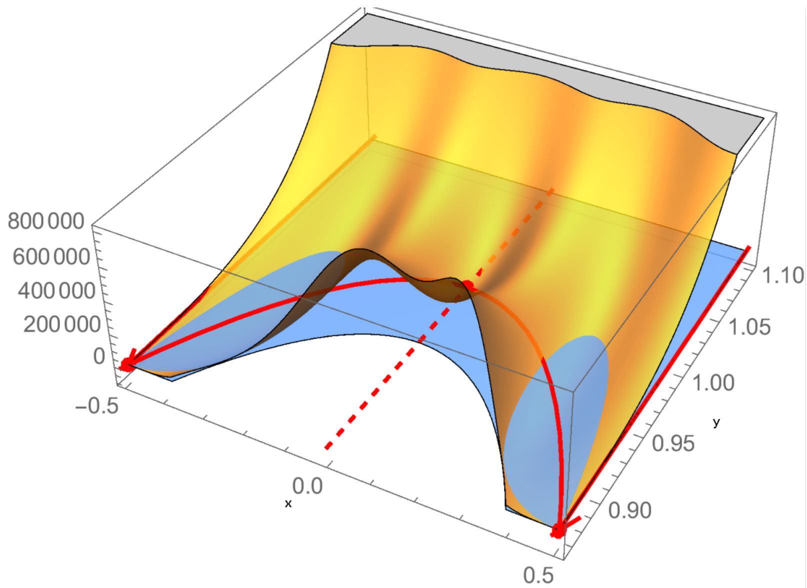


Figure 5.1: A graph of $J_{\chi^*}(x + iy)$ against x and y.

Aside. The author hopes that the statement of this lemma makes clear the connection between the Galois theoretic question answered by Theorem 5.4 and the question of how far χ^* is from being injective, and how Theorem 5.4 may have arisen from attempts to prove that χ^* is finite-to-one. Lemma 5.9 is as close as we have come to proving that χ^* is finite-to-one on $\mathbb{F}$. It is not clear how one would prove that there are no real analytic arcs upon which χ^* is constant, though numerical computation of the values of the Jacobian J_{χ^*} seems to indicate that there are none. See Figure 5.1, which depicts the graph of J_{χ^*} on a small part of $\mathbb{F}$, outside of which J_{χ^*} quickly blows up. The horizontal blue plane shown marks zero; it seems likely that the zero set of J_{χ^*} consists just of the zero-arcs we can see above. It is not possible that χ^* is constant on these arcs since they meet ρ or $\rho+1$, where χ^* vanishes, but χ^* vanishes nowhere else inside $\mathbb{F}$ except at i. This is thanks to the well-known fact that E_4 and E_6 only vanish on $\mathrm{SL}_2(\mathbb{Z}) \cdot \rho$ and $\mathrm{SL}_2(\mathbb{Z}) \cdot i$ respectively, combined with work of Masser [49, Proof of Lemma 3.2] who shows that E_2^* is also nonvanishing outside of $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$.

The proof of Theorem 5.4 is now easy.

Proof of Theorem 5.4. Let $f \in \mathbb{Q}(j, \chi^*)$ and consider the class polynomial of f:

$$H_\delta^f(x) = \prod_{Q \in P_\delta^1} (x - f(\tau_Q)),$$

which we recall is a power of an irreducible polynomial. Say $H^f_\delta = p_\delta^{k_\delta}$. Then p_δ is necessarily the minimal polynomial of $f(\tau)$ for some (any) quadratic τ with $\delta(\tau) = \delta$.

Since $H^f_\delta = p_\delta^{k_\delta}$, there must be a root of H^f_δ of order k_δ. So by definition of H^f_δ, there are at least k_δ distinct quadratic points $\tau_1, \ldots, \tau_{k_\delta} \in \mathbb{F}$ with $f(\tau_i) = f(\tau_1)$ for all i. By 5.9, there can be at most M such quadratic points, whence $k_\delta \leq M$.

For a quadratic point τ of discriminant δ, the degree of $j(\tau)$ over $\mathbb{Q}$ is equal to the degree of H^f_δ, while the degree of $f(\tau)$ over $\mathbb{Q}$ is equal to the degree of p_δ. Hence we must have

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(f(\tau))] = \frac{\deg H^f_\delta}{\deg p_\delta} = k_\delta \leq M,$$

as required. □

Remark 5.10. It is not clear that Theorem 5.4 is a new result; the techniques of Mertens, Rolen and others seem to be capable of producing a version of 5.4 for any given $f \in \mathbb{Q}(j, \chi^*)$. The primary benefit of our approach is that it can be made uniform.

For a field F and a natural number d, let $F^{\leq d}(X_1, \ldots, X_n)$ be the set of rational functions p/q in the variables X_i, with coefficients in F and with p and q having degree at most d.

With this notation we can state a uniform version of 5.4.

Theorem 5.11. *For each natural number d, there is a constant M_d such that, whenever $f \in \mathbb{Q}^{\leq d}(j, \chi^*)$ is nonconstant and $\tau \in \mathbb{H}$ is quadratic, we have*

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(f(\tau))] \leq M_d.$$

We will prove this in Chapter 6; although the concepts are the same as for 5.4, it turns out that we need a suitable André-Oort result for π, as well as some uniformity results of Scanlon. See Section 6.1.1.

The methods demonstrated here should be quite applicable to modular functions of level greater than 1, but there are some ingredients missing. The simplest case for higher-level ahm functions does not require much. Let $\Gamma_N^{\mathbb{Q}}$ be the field of meromorphic modular functions of level N, with rational q-expansions (including at infinity). Thanks to work of Mertens, Rolen [51, Proposition 3.2] and Schertz [74], the appropriate analogue of Proposition 5.7 holds for functions in $\Gamma_N^{\mathbb{Q}}(\chi^*)$, giving us the necessary Galois information. Definability follows from the q-expansions, so the only remaining detail here is to get some control on the Jacobians of the relevant functions.

A suitable analogue of Lemma 5.2 should not be too difficult to attain, at which point the methods would yield

$$[\mathbb{Q}(f(\tau)) : \mathbb{Q}] \geq \frac{\# P_\delta^N}{M}$$

where $f = p(\chi^*, g_1, \ldots, g_k)$. Here the g_i are the generators of $\Gamma_N^{\mathbb{Q}}$ and p is some rational function. Moreover, we should be able to ensure that the constant M would depend only upon N and the degree of p. The function P considered by Mertens and Rolen in [51] lies in $\Gamma_N^{\mathbb{Q}}(\chi^*)$, so the above would potentially yield a weak generalisation of their result.

Chapter 6

André-Oort Results

In this chapter we will prove several André-Oort results. The method to be used in each case is based on the o-minimal Pila-Zannier strategy described in Chapter 1, combining point-counting ideas with Ax-Lindemann results from Chapter 4. For the point counting, we will make use of results from Chapter 5, specifically Proposition 5.7 and Theorem 5.8.

We begin with the natural AO result for ahm functions, Theorem 6.5. For this problem, the usual Pila-Zannier approach works without significant modification.

6.1 AO for π

The first result of the chapter is a Lemma drawing together Proposition 5.7 and Theorem 5.8.

Lemma 6.1. *Let K be a number field and $\eta > 0$. There exists an absolute constant $c > 0$ and a constant $b = b(K, \eta) > 0$ with the following property.*

Let $\tau \in \mathbb{F}$ be a quadratic point, with $D(\tau) = D$. Then there are at least $bD^{\frac{1}{2}-\eta}$ distinct quadratic points $\tau' \in \mathbb{F}$, of height at most cD, such that $\pi(\tau')$ is a Galois conjugate, over K, of $\pi(\tau)$.

Proof. By Theorem 5.8, there is a constant $c_\eta > 0$ such that the number of distinct Galois conjugates of $j(\tau)$ over $\mathbb{Q}$ is at least

$$c_\eta D^{\frac{1}{2}-\eta}.$$

The number of Galois conjugates of $j(\tau)$ over K is therefore at least

$$bD^{\frac{1}{2}-\eta},$$

where $b = \frac{c_\eta}{[K:\mathbb{Q}]}$.

So we have some field automorphisms θ_i acting on $j(\tau)$ over K and some distinct $\tau_i \in \mathbb{F}$, such that $\theta_i(j(\tau)) = j(\tau_i)$. Moreover, $D(\tau_i) = D(\tau) = D$, for all i.

By Lemma 5.6, there is a constant $c = c_{\text{Ht}}$ such that, for any quadratic $\sigma \in \mathbb{F}$,

$$\text{Ht}(\sigma) \leq cD(\sigma).$$

Hence each τ_i has $\text{Ht}(\tau_i) \leq cD$.

By 5.7, we have $\theta_i(\pi(\tau)) = \pi(\tau_i)$. There are at least $bD^{\frac{1}{2}-\eta}$ of these points τ_i and they have height at most cD, as required. □

Notation 6.2. For the remainder of the section, V will be a subvariety of $\mathbb{C}^{2n}$ defined over a number field K. We will write V^{sp} for the union of all positive-dimensional π-special subvarieties of V. $\mathcal{Z}$ will be the preimage $\pi^{-1}(V)$ of V, and $Z = \mathcal{Z} \cap \mathbb{F}^n$. Thanks to the definability of j and of χ^*, note that Z is a definable set. We will also write $\mathcal{Z}^{\text{sp}} = \pi^{-1}(V^{\text{sp}})$ and $Z^{\text{sp}} = \mathcal{Z}^{\text{sp}} \cap \mathbb{F}^n$.

We need two more components before we come to the proof of André-Oort for π. The first is a proposition which will act as the engine driving an inductive argument later on.

Proposition 6.3. *Suppose that V^{sp} is a variety. Then $V \setminus V^{sp}$ contains only finitely many π-special points.*

Proof. Thanks to the Ax-Lindemann theorem, 4.12, the set $\mathcal{Z}^{\text{alg}}$ is a union of weakly $\mathbb{H}$-special varieties. It therefore consists of the $\mathbb{H}$-special varieties in $\mathcal{Z}^{\text{sp}}$, together with perhaps some weakly $\mathbb{H}$-special varieties which are not $\mathbb{H}$-special; but these last can contain no quadratic points.

So, writing $N(X,T)$ for the number of quadratic points in X up to height T, we have

$$N(Z \setminus Z^{\text{sp}}, T) \leq N(Z \setminus Z^{\text{alg}}, T) \ll_{\epsilon} T^{\epsilon}$$

for any $\epsilon > 0$; the last bound coming from the Pila-Wilkie Counting Theorem, 1.17.

Suppose for a contradiction that $V \setminus V^{\text{sp}}$ contains infinitely many π-special points. Then we can find quadratic points

$$\tau = (\tau_1, \ldots, \tau_n) \in Z \setminus Z^{\text{sp}}$$

with $D = D(\tau)$ arbitrarily large.

In 6.1, choose $\eta = \frac{1}{4}$, say. Then for some $b, c > 0$, there are at least $bD^{\frac{1}{4}}$ quadratic points $\tau' \in \mathbb{F}^n$, with height at most cD, such that $\pi(\tau')$ is a Galois conjugate of $\pi(\tau)$

over K. Each such point satisfies $\pi(\tau') \in V$, so we get at least $bD^{\frac{1}{4}}$ quadratic points (of height at most cD) in $Z \setminus Z^{\text{sp}}$. Thus we have

$$bD^{\frac{1}{4}} \leq N(Z \setminus Z^{\text{alg}}, cD) \ll_\epsilon (cD)^\epsilon,$$

which, choosing any $\epsilon < \frac{1}{4}$, yields a contradiction for sufficiently large D. □

Having dealt with the special points in V - the zero-dimensional π-special subvarieties - it remains to get control over the positive-dimensional π-special subvarieties. This is done by the following lemma; compare with Proposition 10.2 of [62], from which this lemma is adapted.

Lemma 6.4. *There is a finite collection $\mathcal{B}$ of basic $\mathbb{H}$-special varieties (each lying in some ambient $\mathbb{H}^k$) with the property that every maximal, positive-dimensional, weakly $\mathbb{H}$-special subvariety of $\mathcal{Z}$ is a translate of γB, for some $B \in \mathcal{B}$ (with $B \subseteq \mathbb{H}^k$, say) and $\gamma \in \mathrm{SL}_2(\mathbb{Z})^k$.*

Proof. The set of basic Möbius varieties in $\mathbb{H}^k$ corresponds to a semialgebraic subset of some $\mathbb{R}^m$; it is parametrised by the cartesian product of some elements of $\mathrm{SL}_2(\mathbb{R})$ with some Grassmannians. Since basic $\mathbb{H}$-special varieties correspond to elements of $\mathrm{SL}_2(\mathbb{R})$ which, up to scaling, lying in $\mathrm{GL}_2^+(\mathbb{Q})$, we can (by choosing coordinates appropriately) ensure that the points in $\mathbb{R}^m$ corresponding to basic $\mathbb{H}$-special varieties have rational coordinates.

Consider the collection $\mathcal{C}$ of basic Möbius varieties B (in any $\mathbb{H}^k$) such that, for some complex numbers $(w_1, x_1, \ldots, w_{n-k}, x_{n-k})$ we have

$$\pi(B) \times \{(w_1, \ldots, x_{n-k})\} \subseteq V \tag{6.1}$$

and such that B is maximal for these w_i, x_i among Möbius subvarieties of $\mathbb{H}^k$. Note that, in particular, if B is a Möbius subvariety having a translate which is a maximal Möbius subvariety of $\mathcal{Z}$, then, up to a permutation of coordinates, $B \in \mathcal{C}$. (Since there are only finitely many ways to permute the coordinates, we can safely ignore the permutation from now on.)

By Corollary 4.14, every $B \in \mathcal{C}$ must be $\mathbb{H}$-special. The collection $\mathcal{C}$ therefore corresponds to a subset $C \subseteq \mathbb{R}^m$ consisting entirely of algebraic points.

Now consider the restricted set

$$\widehat{\mathcal{C}} = \{B \in \mathcal{C} : B \text{ meets } \mathbb{F}^k \text{ in a set of dimension} \dim B.\}.$$

Then $\widehat{\mathcal{C}}$ corresponds to a definable subset $\widehat{C} \subseteq \mathbb{R}^m$. Definability of the dimension condition is a classical fact (see for instance [80, 3.10]) and the condition (6.1) can be checked definably on $\mathbb{F}^k$, and will then hold everywhere by real analytic continuation. Patently $\widehat{C} \subseteq C$, so it still consists entirely of rational points. A definable subset consisting entirely of rational points is finite (Proposition 1.14), whence $\widehat{C}$ (and therefore $\widehat{\mathcal{C}}$) is finite.

Now, any maximal positive-dimensional weakly $\mathbb{H}$-special subvariety of $\mathcal{Z}$ arises as a translate $B \times \{\tau\}$ of some basic $\mathbb{H}$-special $B \subseteq \mathbb{H}^k$. By 4.14 again, any such B is also maximal among Möbius subvarieties $M \subseteq \mathbb{H}^k$ with $\pi(M) \times \pi(\tau) \subseteq V$.

Any such B is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to one which meets $\mathbb{F}^n$ in its full dimension. By the invariance of π, we have $\gamma B \times \{\tau\} \subseteq \mathcal{Z}$. Putting all this together we get $\gamma B \in \widehat{\mathcal{C}}$, whence B is indeed $\mathrm{SL}_2(\mathbb{Z})$-equivalent to an element of the finite set $\widehat{\mathcal{C}}$. □

Finally, we combine 6.3 and 6.4 in an inductive argument to prove our main theorem. (Again, the reader can compare with the proof of Theorem 1.1 in [62, 11.3, Page 33].)

Theorem 6.5 (André-Oort for π)**.** *Let $V \subseteq \mathbb{C}^{2n}$ be a variety (no longer necessarily defined over $\overline{\mathbb{Q}}$). Then V contains only finitely many maximal π-special subvarieties.*

Proof. There is a subvariety $\widetilde{V} \subseteq V$, defined over $\overline{\mathbb{Q}}$, containing all the algebraic points of V. So we may assume that V is defined over $\overline{\mathbb{Q}}$, and thus over a number field K.

By 6.3, it suffices to show that V^{sp} is a variety. We will prove this by induction on n. The base case $n = 1$ is clear, since for $V \subseteq \mathbb{C}^2$, there are no positive-dimensional π-special subvarieties of V unless $V = \mathbb{C}^2$ (in which case $V^{\mathrm{sp}} = V = \mathbb{C}^2$).

So we need to show for $n > 1$ that V^{sp} is a variety, under the assumption that 6.5 holds for $V \subseteq \mathbb{C}^{2m}$, $m < n$.

By 6.4, there are finitely many basic $\mathbb{H}$-special varieties, $B \in \mathcal{B}$, such that every maximal, positive-dimensional, $\mathbb{H}$-special subvariety of $\pi^{-1}(V)$ is a translate of some γB. A maximal π-special subvariety of V is the Zariski closure of $\pi(S)$, for some maximal $\mathbb{H}$-special subvariety $S \subseteq \pi^{-1}(V)$. Therefore any maximal π-special subvariety of V is the translate (by some special points $\pi(\tau_i)$) of one of a finite collection $\mathcal{C}$ of basic π-special varieties. (The action of $\mathrm{SL}_2(\mathbb{Z})$ has no effect since π is $\mathrm{SL}_2(\mathbb{Z})$-invariant.)

So it is enough to show, given some basic π-special $C \in \mathcal{C}$, that there are only finitely many translates of C which are maximal π-special subvarieties of V.

For each such $C \subseteq \mathbb{C}^{2k}$, we will write V_C for the set of translates of C which are subvarieties of V, up to permutation of coordinates. That is:

$$V_C = \bigcup_{\omega \in \mathrm{Sym}(n)} \Big\{ (j_1, \chi_1, \ldots, j_{n-k}, \chi_{n-k}) : C \times \{(j_1, \chi_1, \ldots, j_{n-k}, \chi_{n-k})\} \subseteq \omega(V) \Big\},$$

where elements of $\mathrm{Sym}(n)$ act on subvarieties of $\mathbb{C}^{2n} = (\mathbb{C}^2)^n$ by permuting coordinates in pairs, in the obvious way.

The set V_C is an algebraic subvariety of $\mathbb{C}^{2(n-k)}$; each set in the union is clearly a variety and the union is a finite one. The translates of C which yield π-special subvarieties of V are the π-special points of V_C. The translates which yield *maximal* π-special subvarieties are the π-special points of $V_C \setminus (V_C)^{\mathrm{sp}}$.

By our inductive assumption, V_C contains only finitely many maximal π-special subvarieties. In particular $(V_C)^{\mathrm{sp}}$ is a variety, so by 6.3 there are only finitely many π-special points in $V_C \setminus (V_C)^{\mathrm{sp}}$. □

6.1.1 An Application: Proof of Theorem 5.11

Our goal for this section is to attain a uniform version of Theorem 5.4. Specifically, we are aiming to prove Theorem 5.11, which we stated towards the end of Chapter 5. Let us first recall the statement.

Theorem 5.11. *For each natural number d, there is a constant M_d such that, whenever $f \in \mathbb{Q}^{\leq d}(j, \chi^*)$ is nonconstant and $\tau \in \mathbb{H}$ is quadratic, we have*

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(f(\tau))] \leq M_d.$$

The first version of the result, Theorem 5.4, came very easily from Lemma 5.9, which told us that there are only finitely many quadratic $\tau \in \mathbb{F}$ in the preimage of any $z \in \mathbb{C}$ under a given $f \in \mathbb{Q}(j, \chi^*)$. This lemma can clearly be viewed as a result of André-Oort type; we will now use our full André-Oort result 6.5 to prove a uniform version of Lemma 5.9, which in turn will yield Theorem 5.11.

We will be using work of Scanlon as a central tool for uniformising our results. In [73] he proves that, in very general circumstances, André-Oort results can automatically be made uniform. We will not need the full strength of Scanlon's result; it will be enough to use two particular lemmas.

Lemma 6.6. *Let k be a field and K an algebraically closed field extension of k. Let X be a variety over k and X_K its base change to K. Let $A \subseteq X(K)$. Suppose that $Y \subseteq X_K$ is constructible. Then there is a natural number n, some constructible set $Z \subseteq X \times X^n$, defined over k, and some $a \in A^n$ such that $Z_a(K) \cap A = Y(K) \cap A$.*

Proof. See Lemma 3.1 from [73]. □

Lemma 6.7. *Let K be an algebraically closed field, X and B algebraic varieties over K, $Y \subseteq X \times B$ a constructible subset, and $A \subseteq X(K)$. There is a natural number n and a constructible set $Z \subseteq X \times X^n$ such that, for any parameter $b \in B(K)$, there is some $a \in A^n$ for which $Y_b(K) \cap A = Z_a(K) \cap A$.*

This last is Lemma 3.2 from [73]. It will be sufficient for our current purposes, but we will later need a slightly modified version, which we will prove now. The proof of this modified lemma is essentially identical to the original proof of 6.7; credit here certainly lies with Scanlon!

Definition 6.8 (Zariski Separation)**.** Let $k \subseteq K$ be fields, $s \in \mathbb{N}$ and let $\alpha, \beta \in K^s$. For $m \in \mathbb{N}$ and $i \leq m$, let $p_i^{(m)}$ be the ms-tuple

$$(\alpha, \ldots, \alpha, \beta, \alpha, \ldots, \alpha),$$

with the β arising in the ith place. Let $P_i^{(m)}$ be the k-Zariski closure, in K^{ms}, of the point $p_i^{(m)}$. Then α and β are k-Zariski separated if for every m and every $i \neq j$ we have

$$p_i^{(m)} \notin P_j^{(m)}.$$

Lemma 6.9. *Let $k \subseteq K$ be algebraically closed fields, $X = \mathbb{A}^s$, $B = \mathbb{A}^t$ and $Y \subseteq X \times B$ a constructible subset. Let $A \subseteq X(K)$. Suppose that there exist $\alpha, \beta \in A$ which are k-Zariski separated.*

Then there is a natural number n and a k-constructible set $Z \subseteq X \times X^n$ such that for any $b \in B(K)$, there is $a \in A^n$ for which $Y_b(K) \cap A = Z_a(K) \cap A$.

Proof. Throughout this proof we will suppress notation, writing $W = W(K)$ whenever W is a constructible set over K.

Consider K as a structure in the language $\mathcal{L} = (+, \cdot, P_A, \{a\}_{a \in K})$, where each constant symbol a is to be interpreted as the corresponding element $a \in K$ and P_A is an s-ary predicate to be interpreted as the set A. In this language, we can express the condition "$x \in A \cap W$" for any affine variety W over K.

Let T be the $\mathcal{L}$-theory of K, and then let $\mathcal{L}'$ be $\mathcal{L}$ together with some new constant symbols $b_1, \ldots, b_t$. Write $b = (b_1, \ldots, b_t)$.

Let $\mathcal{C}(k)$ be the set of k-constructible subsets of $X \times X^N$ (for some N) and consider the set

$$\Gamma = T \cup \{\forall c \in A^N \quad \exists x \in A \quad (x \in Y_b \setminus Z_c \vee x \in Z_c \setminus Y_b) : Z \in \mathcal{C}(k)\}.$$

Suppose that Γ is *not* finitely satisfiable. Let Γ_0 be a finite subset witnessing this. Since T is the theory of K, Γ_0 cannot be contained in T, so it mentions some finitely many k-constructible sets $Z_1, \dots, Z_l$, with $Z_i \subseteq X \times X^{N_i}$. Since Γ_0 is not satisfiable, we have:

$$\forall b \in B \quad \exists i \leq l \quad \exists c \in A^{N_i} \quad \forall x \in A \quad (x \notin Y_b \setminus Z_c \wedge x \notin (Z_i)_c \setminus Y_b).$$

In other words, for every $b \in B$, there is some Z_i and some $c \in A^{N_i}$ such that

$$A \cap (Z_i)_c = A \cap Y_b.$$

Now let $N = \max\{N_i : i \leq l\}$ and let $n = N + l$. Define

$$Z = \bigcup_{i=1}^{l} Z_i \times X^{N-N_i} \times P_i^{(l)},$$

with $P_i^{(l)}$ as in the definition of α, β being k-Zariski separated. This Z is a constructible set defined over k, and satisfies the conclusion of the lemma: if $b \in B$, then for some i, c, we have $A \cap (Z_i)_c = A \cap Y_b$. Letting $c' = (c, \alpha, \dots, \alpha, p_i^{(l)}) \in A^{N+l}$, we get

$$A \cap Z_{c'} = A \cap (Z_i)_c = A \cap Y_b,$$

as required.

So suppose on the other hand that Γ is finitely satisfiable. By the Compactness Theorem, it is satisfiable. So we have an algebraically closed extension $L \supseteq K$, a point $b \in B(L)$ and a set $A^* \subseteq X(L)$ such that, for every k-constructible $Z \subseteq X \times X^N$ and $c \in (A^*)^N$, we have

$$Y_b(L) \cap A^* \neq Z_c(L) \cap A^*.$$

This contradicts Lemma 6.6 (applied to k, L, A^*, X and Y_b). $\square$

We now use these results, together with 6.5, to prove a uniform version of Lemma 5.9.

Lemma 6.10. *For each natural number d, there is a constant M_d such that, whenever $f \in \mathbb{Q}^{\leq d}(j, \chi^*)$ is nonconstant and $z \in \mathbb{C}$, we have*

$$\#\{\tau \in \mathbb{F} : f(\tau) = z, \tau \textit{ is quadratic.}\} \leq M_d.$$

Proof. For each d, there is a natural number $n(d)$ such that $\mathbb{C}^{n(d)}$ parametrises elements of $\mathbb{C}^{\leq d}(X, Y)$ in the obvious way. For $b \in \mathbb{C}^{n(d)}$, we will write $f_b(X, Y)$ for the corresponding element of $\mathbb{C}^{\leq d}(X, Y)$.

Define a variety $V \subseteq \mathbb{C}^{2+n(d)+1}$ by

$$(X, Y, b, z) \in V \iff f_b(X, Y) = z.$$

Lemma 6.7 (applied with $X = \mathbb{C}^2$, $B = \mathbb{C}^{n(d)+1}$, $Y = V$, and A equal to the set of π-special points) yields a natural number N and a constructible set $Z \subseteq \mathbb{C}^{2+2N}$, such that for any parameter $(b, z) \in \overline{\mathbb{Q}}^{n(d)+1}$, there is a π-special point $a \in \mathbb{C}^{2N}$ such that $V_{b,z}$ and Z_a (both subvarieties of $\mathbb{C}^2$) have the same π-special points.

We can write

$$Z = \bigcup_{i=1}^{k} Z_i^{\text{in}} \setminus Z_i^{\text{out}}$$

for some varieties Z_i^{in}, Z_i^{out}.

Apply Theorem 6.5 to each of the Z_i^{in}. Each Z_i^{in} contains only finitely many maximal π-special subvarieties; call the collection of such subvarieties σ_i.

For some i, it might be the case that some $S \in \sigma_i$ contains $\mathbb{C}^2 \times a$, for some π-special point $a \in \mathbb{C}^{2N}$. However, by Theorem 5.4, whenever $b \in \mathbb{Q}^{n(d)}$ and $z \in \overline{\mathbb{Q}}$, the set $V_{b,z}$ contains only finitely many π-special points. Hence, if such an $S \in \sigma_i$ exists, then $(Z_i^{\text{out}})_a$ must also contain $\mathbb{C}^2 \times a$, removing S from consideration. We may therefore assume that no $S \in \sigma_i$ contains any $\mathbb{C}^2 \times a$.

The number of π-special points in Z_a is then bounded above by

$$M_d = \sum_{i \leq k} \sum_{S \in \sigma_i} \max\{\deg \Phi_N \in \mathbb{N} : \Phi_N \text{ occurs in the definition of } S\},$$

which, since each S involves only finitely many Φ_N, is a constant depending only on Z, and therefore only on d.

So take any $b \in \mathbb{Q}^{n(d)}$ and any $z \in \overline{\mathbb{Q}}$. Let $f = f_b(j, \chi^*) \in \mathbb{Q}^{\leq d}(j, \chi^*)$. Let $a \in \mathbb{C}^{2N}$ be the π-special point afforded by Lemma 6.7. The quadratic points $\tau \in \mathbb{F}$ such that $f(\tau) = z$ are in one-to-one correspondence with the π-special points of $V_{b,z}$. So the number of such τ is equal to the number of π-special points in Z_a, which is bounded above by M_d. □

Now the proof of 5.11 goes exactly as for Theorem 5.4.

Proof of Theorem 5.11. Let $f \in \mathbb{Q}^{\leq d}(j, \chi^*)$. Recall that, for each discriminant δ, we have $H_\delta^f = p_\delta^{k_\delta}$, where p_δ is irreducible and is therefore necessarily the minimal polynomial of $f(\tau)$ for some (any) quadratic τ with $\delta(\tau) = \delta$. It follows that $\{\tau_Q : Q \in P^1_\delta\}$ contains k_δ distinct quadratic points $\tau_1, \ldots, \tau_{k_\delta} \in \mathbb{F}$ with $f(\tau_i) = f(\tau_1)$ for all i. By Lemma 6.10, there are at most M_d such points, whence $k_\delta \leq M_d$.

Finally,

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}(f(\tau))] = \frac{\deg H^f_{\delta(\tau)}}{\deg p_{\delta(\tau)}} = k_\delta \leq M_d.$$

□

6.2 AO for J

We begin by recalling the André-Oort statements for J that we are working towards, as previously stated in Chapter 3. The motivating conjecture is the following.

Conjecture 3.6 (Modular André-Oort with Derivatives)**.** *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an* $\mathrm{SL}_2(\mathbb{Z})$*-finite collection $\sigma(V)$, consisting of* proper $\mathbb{H}$*-special varieties of $\mathbb{H}^n$, with the property that every* $\mathbb{H}$*-special subvariety of $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

Since quadratic points are topologically dense in $\mathbb{H}$-special subvarieties, this conjecture is unchanged if we instead assert that every *quadratic point* in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$. The André-Oort theorem we will prove in this section is formulated in such a way.

Theorem 6.11 (Modular André-Oort with Derivatives for Generic Points)**.** *Let $V \subseteq \mathbb{C}^{3n}$ be a proper algebraic variety defined over $\overline{\mathbb{Q}}$. There exists an* $\mathrm{SL}_2(\mathbb{Z})$*-finite collection $\sigma(V)$, consisting of* proper $\mathbb{H}$*-special varieties of* $\mathbb{H}^n$*, with the property that every j'-generic* $\mathbb{H}$*-special point in $J^{-1}(V)$ is contained in some $G \in \sigma(V)$.*

The method is much the same as that for the André-Oort result for π, from the previous section. If there are infinitely many quadratic points in $J^{-1}(V)$, then we can find such a τ of arbitrarily large discriminant. By taking Zariski closures over $\overline{\mathbb{Q}}$ and considering the Galois orbit of $\pi(\tau)$, this will force a certain definable set to contain too many quadratic points (provided that τ is j'-generic). Via the Pila-Wilkie theorem and Corollary 4.13 - "Ax-Lindemann for Adjacency" - this will yield some $\mathbb{H}$-special variety, containing τ and lying adjacent to V. An inductive argument using a variant of 6.4 then allows us to conclude.

All of this will require a number of number-theoretic estimates, some of which we have already discussed; one final piece we will need is the following, which gives us control over the heights of the matrices in $\mathrm{GL}_2^+(\mathbb{Q})$ occurring as relations between coordinates in a given special point.

Lemma 6.12. *Let $\tau, \sigma \in \mathbb{F}$ be quadratic points, with*

$$D(\tau), D(\sigma) \leq D.$$

Then there are positive constants c, δ such that: if $\tau \in \mathrm{GL}_2^+(\mathbb{Q}) \cdot \sigma$, then there is some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $\tau = g\sigma$ and

$$\mathrm{Ht}(g) \leq cD^{\delta}.$$

Proof. There are points $\tau' \in \mathrm{SL}_2(\mathbb{Z}) \cdot \tau$ and $\sigma' \in \mathrm{SL}_2(\mathbb{Z}) \cdot \sigma$, corresponding to elliptic curves E_τ and E_σ with coefficients in $\mathbb{Q}(j(\tau))$ and $\mathbb{Q}(j(\sigma))$ respectively. We have $D(\tau') = D(\tau)$ and $D(\sigma') = D(\sigma)$.

An isogeny estimate due to Pellarin [57, Theorem 2] tells us that in this situation there is an isogeny between E_τ and E_σ of degree at most

$$10^{78} d^4 \max(1, \log d)^2 \max(1, h_F(E_\tau))^2, \tag{6.2}$$

where d is the product of the degrees of $j(\tau)$ and $j(\sigma)$ over $\mathbb{Q}$ and $h_F(E_\tau)$ is the semi-stable Faltings height of E_τ. From [76, Proposition 2.1] (see also [30, Section 3]), we also know that

$$h_F(E_\tau) \leq 12\big(h(j(\tau)) + c \log \max\{2, h(j(\tau))\}\big),$$

where h is the logarithmic Weil height. (The heights discussed in this lemma are not relevant anywhere else so we will not define them.) Further, Habegger and Pila show in [32, Lemma 4.3] that

$$h(j(\tau)) \leq c_\epsilon D(\tau)^\epsilon$$

for any $\epsilon > 0$.

Moreover, since τ, σ have $D(\tau), D(\sigma) \leq D$, there are integer matrices g, h, of determinant at most D with $g\tau = \tau$ and $h\sigma = \sigma$. Namely, $g = \begin{pmatrix} -B & -2C \\ 2A & B \end{pmatrix}$, where

$$Ax^2 + Bx + C$$

is the minimal integer polynomial of τ, with $(A, B, C) = 1$, whence

$$D(\tau) = |B^2 - 4AC| = 4AC - B^2 = \det g,$$

and similarly for h and σ.

Now $\Phi_{\det g}(j(\tau), j(\tau)) = 0$, and it is clear from the classical construction of Φ_N (see eg. Zagier [89, Proposition 23]) that $\deg \Phi_N \ll N^2$, so the degrees of $j(\tau)$, $j(\sigma)$ over $\mathbb{Q}$ are $O(D^2)$. Putting all this together with (6.2) tells us that there is an isogeny between E_τ and E_σ of degree M, with

$$M \ll_\epsilon D^{16+\epsilon},$$

and in particular $M = O(D^{17})$.

On the $\mathbb{H}$ side, this tells us that we have a primitive integer matrix α', of determinant M, with $\alpha'\tau' = \sigma'$. Consequently there is such a matrix α, still of determinant M, with $\alpha\tau = \sigma$. By work either of Orr [56, Proposition 1.4] or Habegger-Pila [32, Lemma 5.2], in this situation there is a positive δ and a $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$ with $\tau = \beta\sigma$, $\det \beta = M$ and

$$\mathrm{Ht}(\beta) \ll M^\delta,$$

whence

$$\mathrm{Ht}(\beta) \ll D^{17\delta},$$

as required. □

Now we can begin work on the body of the proof of 6.11. First we prove the following theorem encapsulating all of the Galois aspects of the problem. Thanks to the uniform Pila-Wilkie theorem, these aspects can easily be made uniform; we will be using this uniformity to get a uniform version of Theorem 6.11.

Definition 6.13. Two weakly $\mathbb{H}$-special varieties G and H are called *internally* $\mathrm{SL}_2(\mathbb{Z})$*-equivalent* if H can be obtained from G by replacing each of the matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ occurring in the definition of G by some $g\gamma$, with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ (which is allowed to vary with the coordinates).

Theorem 6.14. *Let $V \subseteq \mathbb{C}^{3n+r}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Consider it as a family of algebraic varieties V_a, with $a \in \mathbb{C}^r$. Let d be a positive integer. Then there is a number $D = D(V, d)$ with the following property.*

Whenever $a \in \mathbb{C}^r$ is algebraic of degree at most d and $\tau \in J^{-1}(V_a)$ is a j'-generic quadratic point with $D(\tau) > D$, none of whose coordinates lie in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$, there are two positive-dimensional $\mathbb{H}$-special varieties G and H, internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent, with

$$\tau \in G \text{ and } H \hookrightarrow V_a.$$

Moreover, if $G = \mathbb{H}^k \times \{(\tau_{k+1}, \ldots, \tau_n)\}$ for some k and some τ_i (so that $G = H$), then in fact $\tau \in G \subseteq J^{-1}(V_a)$.

Proof. Let K be a number field containing a field of definition for V.

Consider a partition of $\{1, \dots, n\}$, written as

$$S_1 \cup \cdots \cup S_k,$$

with each $S_i \neq \emptyset$. For each i, let $s_i = \min S_i$ and $r_i = \#S_i - 1$. Given

$$\sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{H}^k$$

and

$$g = (g_{1,1}, \dots, g_{1,r_1}, \dots, g_{k,1}, \dots, g_{k,r_k}) \in \mathrm{SL}_2(\mathbb{R})^{n-k},$$

define the following set:

$$\begin{aligned}\mathcal{Z}_{a,\sigma,g} = \Bigg\{ &(\tau, h) \in \mathbb{H}^k \times \mathrm{GL}_2^+(\mathbb{R})^{n-k} : \det h_{i,j} = \det g_{i,j}, \\ &\Big[\dots, j(\tau_i), j'(\tau_i), p_{\mathrm{Im}\,\sigma_i}(j(\tau_i), \chi^*(\tau_i), j'(\tau_i)), \dots \\ &\dots, j(h_{i,j}\tau_i), j'(h_{i,j}\tau_i)\frac{m_{g_{i,j}}(\sigma_i)}{m_{h_{i,j}}(\tau_i)}, \\ &p_{\mathrm{Im}\,g_{i,j}\sigma_i}\left(j(h_{i,j}\tau_i), \chi^*(h_{i,j}\tau_i), j'(h_{i,j}\tau_i)\frac{m_{g_{i,j}}(\sigma_i)}{m_{h_{i,j}}(\tau_i)}\right), \dots \Big] \in V_a \Bigg\}\end{aligned}$$

Consider this as a family of sets, fibred over $\mathbb{C}^r \times \mathbb{H}^k \times \mathrm{SL}_2(\mathbb{R})^{n-k}$. There is one such family for each of the finitely many partitions of $\{1, \dots, n\}$, and we consider them all together. They are certainly *not* definable families. However, for a given partition, the family

$$Z_{a,\sigma,g} = \{(\tau, h) \in \mathcal{Z}_{a,\sigma,g} : \tau_i, h_{i,j}\tau_i \in \mathbb{F}, i \leq k, j \leq r_k\}$$

is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

Given some algebraic $a \in \mathbb{C}^r$, let us consider a j'-generic special point $\tau \in J^{-1}(V_a)$. Assume no coordinate of τ lies in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$ and set $D = D(\tau)$.

Up to permutation of coordinates, τ looks like

$$\tau = (\sigma_1, g_{1,1}\sigma_1, \dots, g_{1,r_1}\sigma_1, \dots, \sigma_k, g_{k,1}\sigma_k, \dots, g_{k,r_k}\sigma_k), \tag{6.3}$$

with the σ_i lying in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits, and $g_{i,j} \in \mathrm{GL}_2^+(\mathbb{Q})$. So τ corresponds to a partition of $\{1, \dots, n\}$ in the obvious way. Writing the $g_{i,j}$ as primitive integer matrices, let $N_{i,j} = \det g_{i,j}$. The j'-genericity of τ means that

$$j'(\sigma_1), \dots, j'(\sigma_k)$$

are algebraically independent over $\overline{\mathbb{Q}}$. So the $\overline{\mathbb{Q}}$-Zariski closure $\langle\langle\tau\rangle\rangle$ of $J(\tau)$ is the set of all points of the form

$$\Big[\dots, j(\sigma_i), w_i, p_{\operatorname{Im}\sigma_i}(j(\sigma_i), \chi^*(\sigma_i), w_i), \dots$$
$$\dots, j(g_{i,j}\sigma_i), -w_i\lambda_{N_{i,j}}\big(j(\sigma_i), j(g_{i,j}\sigma_i)\big)m_{g_{i,j}}(\sigma_i),$$
$$p_{\operatorname{Im} g_{i,j}\sigma_i}\Big(j(g_{i,j}\sigma_i), \chi^*(g_{i,j}\sigma_i), -w_i\lambda_{N_{i,j}}\big(j(\sigma_i), j(g_{i,j}\sigma_i)\big)m_{g_{i,j}}(\sigma_i)\Big), \dots\Big],$$

for some $w_1, \dots, w_k \in \mathbb{C}$.

We will show that the existence of this τ implies that $Z_{a,\sigma,g}$ contains "many" quadratic points of bounded height. As in the previous section, these "many" points will arise from taking Galois conjugates of the point $\pi(\tau)$. We will define a variety which serves to translate the problem from the realm of J into the realm of π, making this approach possible.

To define said variety, we will write a general element of $\mathbb{C}^{2n}$ as

$$(\dots, X_i, Y_i, \dots, X_{i,j}, Y_{i,j}, \dots),$$

with $i \leq k$ and $j \leq r_i$, matching the structure of the underlying partition of $\{1, \dots, n\}$.

With this notation, define a variety $V_{a,\sigma,g} \subseteq \mathbb{C}^{2n}$ by

$$V_{a,\sigma,g} = \Big\{(\mathbf{X}, \mathbf{Y}) \in \mathbb{C}^{2n} : \forall w_1, \dots, w_k \in \mathbb{C},$$
$$\Big[\dots, X_i, w_i, p_{\operatorname{Im}\sigma_i}(X_i, Y_i, w_i), \dots, X_{i,j}, -w_i\lambda_{N_{i,j}}(X_i, X_{i,j})m_{g_{i,j}}(\sigma_i),$$
$$p_{\operatorname{Im} g_{i,j}\sigma_i}\big(X_{i,j}, Y_{i,j}, -w_i\lambda_{N_{i,j}}(X_i, X_{i,j})m_{g_{i,j}}(\sigma_i)\big), \dots\Big] \in V_a\Big\}.$$

Then $V_{a,\sigma,g}$ is a subvariety of $\mathbb{C}^{2n}$ defined over $K(a, \sigma, i\operatorname{Im}\sigma)$.

The definition of $V_{a,\sigma,g}$ is designed to mirror the structure of $\langle\langle\tau\rangle\rangle$, so that $\langle\langle\tau\rangle\rangle \subseteq V_a$ implies that $\pi(\tau) \in V_{a,\sigma,g}$. Hence $V_{a,\sigma,g}$ also contains every Galois conjugate (over $K(a, \sigma, i\operatorname{Im}\sigma)$) of $\pi(\tau)$. By Proposition 5.7, such a Galois conjugate must take the form $\pi(\tau')$, for some quadratic τ' with $D(\tau') = D(\tau)$. Moreover, thanks to the modular polynomials Φ_N, τ' must have the same $\mathrm{GL}_2^+(\mathbb{Q})$-structure as τ. That is:

$$\tau' = (\sigma_1', g_{1,1}'\sigma_1', \dots, g_{1,r_1}'\sigma_1', \dots, \sigma_k', g_{k,1}'\sigma_k', \dots, g_{k,r_k}'\sigma_k'), \tag{6.4}$$

where the σ_i' are quadratic points and $g_{i,j}' = g_{i,j}\gamma_{i,j}$, for some $\gamma_{i,j} \in \mathrm{SL}_2(\mathbb{Z})$. By modularity of π, we can also choose the σ_i' to lie in $\mathbb{F}$.

For each τ' arising this way, let us take $w_i = j'(\sigma_i')$ in the definition of $V_{a,\sigma,g}$. Noting that

$$-j'(\sigma_i') \cdot \lambda_{N_{i,j}}\big(j(\sigma_i'), j(g_{i,j}'\sigma_i')\big) \cdot m_{g_{i,j}}(\sigma_i) = j'(g_{i,j}'\sigma_i')\frac{m_{g_{i,j}}(\sigma_i)}{m_{g_{i,j}'}(\sigma_i')},$$

we see that $\big(\sigma_i', g_{i,j}'\big)_{i,j} \in \mathcal{Z}_{a,\sigma,g}$. Further, there is $\gamma_{i,j}' \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_{i,j}'g_{i,j}'\sigma_i' \in \mathbb{F}$. This yields $\big(\sigma_i', \gamma_{i,j}'g_{i,j}'\big)_{i,j} \in Z_{a,\sigma,g}$.

By 5.8, we know (setting $\eta = \frac{1}{4}$, say) that

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] \geq c_{\frac{1}{4}} D^{\frac{1}{4}},$$

for some absolute constant $c_{\frac{1}{4}}$. So if $a \in \overline{\mathbb{Q}}^r$ has degree at most d, then

$$\begin{aligned}
[\mathbb{Q}(j(\tau)) : K(a, \sigma, i\operatorname{Im}\sigma)] &= \frac{[\mathbb{Q}(j(\tau)) : \mathbb{Q}]}{[K(a, \sigma, i\operatorname{Im}\sigma) : \mathbb{Q}]}\\
&\geq \frac{[\mathbb{Q}(j(\tau)) : \mathbb{Q}]}{[K : \mathbb{Q}](rd + 2n + 2n)}\\
&\geq \frac{c_{\frac{1}{4}} D^{\frac{1}{4}}}{[K : \mathbb{Q}](rd + 4n)}\\
&= c(V, d)D^{\frac{1}{4}},
\end{aligned}$$

where $c(V, d) = \dfrac{c_{\frac{1}{4}}}{[K : \mathbb{Q}](rd + 4n)}$.

Each of the Galois conjugates of $\pi(\tau)$ over $K(a, \sigma, i\operatorname{Im}\sigma)$ corresponds to a different $\big(\sigma_i', \gamma_{i,j}'g_{i,j}'\big)_{i,j} \in Z_{a,\sigma,g}$, so there are at least $c(V, d)D^{\frac{1}{4}}$ points of degree at most 2 in $Z_{a,\sigma,g}$.

By Lemma 5.6, we know $\mathrm{Ht}(\sigma') \leq c_1 D$, for some $c_1 > 0$. Further, by Lemma 6.12, the corresponding matrices $\gamma_{i,j}'g_{i,j}\gamma_{i,j}$ can be chosen such that $\mathrm{Ht}\,\big(\gamma_{i,j}'g_{i,j}'\big)_{i,j} \leq c_2 D^{c_3}$, for positive c_2 and c_3. (In fact we know $c_3 > 1$, so clearly we can ensure $c_1 D \leq c_2 D^{c_3}$.)

Putting all this together tells us that

$$N(Z_{a,\sigma,g}, c_2 D^{c_3}) \geq \#\left\{(\tau, h) \in Z_{a,\sigma,g} : \begin{array}{c}\tau \text{ quadratic},\\ h \in \mathrm{GL}_2^+(\mathbb{Q}),\\ \mathrm{Ht}(\tau, h) \leq c_2 D^{c_3}\end{array}\right\} \geq c(V, d)D^{\frac{1}{4}}. \tag{6.5}$$

On the other hand, the uniform Pila-Wilkie Theorem 1.18 gives us, for each $\epsilon > 0$, a constant $b(V, \epsilon) > 0$ such that, for any a, σ and g:

$$N(Z_{a,\sigma,g} \setminus Z_{a,\sigma,g}^{\mathrm{alg}}, T) \leq b(V, \epsilon)T^{\epsilon}.$$

In this inequality, we can set $T = c_2 D^{c_3}$ and choose, say, $\epsilon = \frac{1}{8c_3}$, to get

$$N(Z_{a,\sigma,g} \setminus Z_{a,\sigma,g}^{\mathrm{alg}}, c_2 D^{c_3}) \leq b(V)D^{\frac{1}{8}}. \tag{6.6}$$

Now, if the chosen point $\tau \in J^{-1}(V)$ has

$$D(\tau) > \frac{b(V)^8}{c(V,d)^8},$$

then the two inequalities (6.5) and (6.6) are inconsistent unless the corresponding $Z^{\text{alg}}_{a,\sigma,g}$ is nonempty. In fact, we still get the same inconsistency in these inequalities unless $\left(\sigma'_i, \gamma'_{i,j} g'_{i,j}\right)_{i,j} \in Z^{\text{alg}}_{a,\sigma,g}$, where $\left(\sigma'_i, \gamma'_{i,j} g'_{i,j}\right)_{i,j}$ corresponds to one of the Galois conjugates $\pi(\tau')$ of $\pi(\tau)$. In particular, we have a real semialgebraic arc $T \subseteq (\mathbb{H} \times \mathrm{GL}_2^+(\mathbb{R}))^n$ with

$$\left(\sigma'_i, \gamma'_{i,j} g'_{i,j}\right)_{i,j} \in T \subseteq Z_{a,\sigma,g}.$$

We would like to apply Theorem 4.13 to some algebraic arc constructed from T. Let

$$S = \left\{ \left(\dots, \tau_i, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \dots, \tau_i, \overline{h_{i,j}}, \dots \right) : (\tau, h) \in T \right\},$$

where $\overline{h}$ is the element of $\mathrm{SL}_2(\mathbb{R})$ corresponding to the image of h as an element of $\mathrm{PGL}_2(\mathbb{R})$. Also let

$$\widehat{S} = \{(\dots, \tau_i, \dots, h_{i,j}\tau_i, \dots) : (\tau, h) \in T\}.$$

Since $\left(\sigma'_i, \gamma'_{i,j} g'_{i,j}\right)_{i,j} \in T$, we have $\gamma\tau' \in \widehat{S}$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$.

Note that $\widehat{S}$ is indeed an arc, rather than just a point. This is easy to see; if τ_i and $h_{i,j}\tau_i$ are all constant on $\widehat{S}$, then h must be constant, up to determinant, on T. Since the determinant of $h_{i,j}$ is fixed in the definition of $Z_{a,\sigma,g}$, it follows that τ_i and $h_{i,j}$ are both constant on T, whence T itself is just a point, which is a contradiction.

Corollary 4.13 yields an $\mathbb{H}$-special set H_0 with

$$\widehat{S} \subseteq H_0 \hookrightarrow V_a.$$

Since $\widehat{S}$ is not just a point, H_0 is certainly positive-dimensional. Hence, some $\mathrm{SL}_2(\mathbb{Z})$-translate H_1 of H_0 contains τ', and remains adjacent to V_a.

Let us write, without loss of generality,

$$H_1 = B \times \{\tau'_{k+1}, \dots, \tau'_n\} \hookrightarrow V_a,$$

for some basic $\mathbb{H}$-special variety B. Since the definition of adjacency does not depend on any constant coordinates, we also have

$$H = B \times \{\tau_{k+1}, \dots, \tau_n\} \hookrightarrow V_a,$$

where the τ_i now correspond to suitable coordinates of our original point τ. Since τ and τ' had the same $\mathrm{GL}_2^+(\mathbb{Q})$-structure (see (6.3) and (6.4)), it follows that there is an $\mathbb{H}$-special G, internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to H, which contains τ, as required.

It remains to show that if $H = G = \mathbb{H}^k \times \{\tau_{k+1}, \ldots, \tau_n\}$, then in fact $G \subseteq J^{-1}(V_a)$. For this we must refer to the finer detail of Corollary 4.13. In this situation it tells us that $\mathbb{H}^k$ is adjacent to V_a via:

- Some positive reals c_i,
- Some $z_i \in \mathbb{C}$,
- Triples of the form

$$(w, x, y) = \Bigg(j(g'_{i,j}\sigma'_j), j'(g'_{i,j}\sigma'_j)\frac{m_{g_{i,j}}(\sigma_j)}{m_{g'_{i,j}}(\sigma'_j)}, \\ p_{\mathrm{Im}\, g_{i,j}\sigma_j}\left(j(g'_{i,j}\sigma'_j), \chi^*(g'_{i,j}\sigma'_j), j'(g'_{i,j}\sigma'_j)\frac{m_{g_{i,j}}(\sigma_j)}{m_{g'_{i,j}}(\sigma'_j)} \right) \Bigg).$$

From the definition of adjacency and the construction of the polynomial p_{c_i}, it follows that

$$J(\sigma_1, \ldots, \sigma_k) \times \{(\mathbf{w}, \mathbf{x}, \mathbf{y})\} \in V_a$$

for any σ_i with $\mathrm{Im}\, \sigma_i = c_i$. From the algebraic independence of the $J(\sigma_i)$ we get

$$\mathbb{C}^{3k} \times \{(\mathbf{w}, \mathbf{x}, \mathbf{y})\} \subseteq V_a.$$

Finally, recall that we had a field automorphism which had the effect of sending

$$\pi(g_{i,j}\sigma_j) \text{ to } \pi(g'_{i,j}\sigma'_j)$$

and

$$j'(g_{i,j}\sigma_j) \text{ to } j'(g'_{i,j}\sigma'_j)\frac{m_{g_{i,j}}(\sigma_j)}{m_{g'_{i,j}}(\sigma'_j)}.$$

Applying the inverse automorphism to $(\mathbf{w}, \mathbf{x}, \mathbf{y})$ now yields

$$\mathbb{C}^{3k} \times \big\{ \big[\ldots, j(g_{i,j}\sigma_j), j'(g_{i,j}\sigma_j), p_{\mathrm{Im}\, g_{i,j}\sigma_j}\left(j(g_{i,j}\sigma_j), \chi^*(g_{i,j}\sigma_j), j'(g_{i,j}\sigma_j) \right), \ldots \big] \big\} \\ = \mathbb{C}^{3k} \times \{[\ldots, J(g_{i,j}\sigma_j), \ldots]\} \subseteq V_a,$$

which, unpacking the notation, yields $J(H) \subseteq V_a$, as required. $\square$

The above result is one of two crucial pieces of "counting" we need in order to prove 6.11. The other half is a proposition counting the "positive-dimensional pieces" lying adjacent to V, namely the basic $\mathbb{H}$-special varieties. The necessary result is a direct analogue of Lemma 6.4, which counted the basic $\mathbb{H}$-special varieties in some $\pi^{-1}(V)$.

Proposition 6.15. *Let $V \subseteq \mathbb{C}^{3n+r}$ be a variety, considered as a family of varieties V_a, $a \in \mathbb{C}^r$. Let $\mathcal{B}$ be the collection of basic $\mathbb{H}$-special varieties $B \subseteq \mathbb{H}^k$ (for any k) having the following properties:*

- *For some $a \in \mathbb{C}^r$, B is adjacent to V_a via:*
  - *Some real numbers $c_{i,j}$,*
  - *Some complex numbers $z_{i,j}$,*
  - *Some triples (w_l, x_l, y_l).*
- *For the given a, $c_{i,j}$ and (w_l, x_l, y_l), the variety B is maximal with this property among $\mathbb{H}$-special subvarieties of $\mathbb{H}^k$ (the $z_{i,j}$ may vary).*

Then $\mathcal{B}$ is $\mathrm{SL}_2(\mathbb{Z})$-finite.

Proof. This goes through exactly as for Lemma 6.4. The collection $\mathcal{B}'$ of basic Möbius varieties with the given properties, which meet $\mathbb{F}^k$ in their full dimension, is definable among the Möbius subvarieties. By Corollary 4.15, such Möbius varieties are automatically $\mathbb{H}$-special, so the definable set is parametrised by some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and therefore consists entirely of rational points, so is finite.

The collection $\mathcal{B}$ then consists simply of all the $\mathrm{SL}_2(\mathbb{Z})$-translates of elements of $\mathcal{B}'$. □

It will be useful to note the following fairly obvious corollary:

Corollary 6.16. *The collection $\mathcal{C}$ of basic $\mathbb{H}$-special varieties C which are internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $B \in \mathcal{B}$ (as described in Proposition 6.15) is $\mathrm{SL}_2(\mathbb{Z})$-finite.*

Proof. We saw earlier, in Chapter 2 (Proposition 2.10), that there are only finitely many primitive integer matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ of any given determinant, up to the action of $\mathrm{SL}_2(\mathbb{Z})$. Hence, given any $\mathbb{H}$-special G, defined by some matrices g_i, there are only $\mathrm{SL}_2(\mathbb{Z})$-finitely many possibilities for $g_i\gamma_i$, $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$, and therefore only finitely many $\mathbb{H}$-special varieties internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to G.

With this in mind, the assertion is immediate from 6.15. □

The proof of Theorem 6.11 now goes much as the proof of André-Oort for π, combining the point-counting result of 6.14 with the results 6.15 and 6.16 which count basic $\mathbb{H}$-special varieties adjacent to V.

We will prove 6.11 by way of the following stronger, uniform result. We will later use this uniform version of 6.11 to prove a stronger version of André-Oort for J; Theorem 6.11 clearly follows from 6.17.

Theorem 6.17. *Let $V \subseteq \mathbb{C}^{3n+r}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$, considered as an algebraic family of varieties,*

$$V_a \subseteq \mathbb{C}^{3n}, \quad a \in \mathbb{C}^r.$$

For each positive integer d, there is an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma_d(V)$, consisting of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$, with the following property.

Whenever $a \in \overline{\mathbb{Q}}^r$ satisfies $\max[\mathbb{Q}(a_i) : \mathbb{Q}] \leq d$ and V_a is a proper *subvariety of $\mathbb{C}^{3n}$, every j'-generic quadratic point in $J^{-1}(V_a)$ is contained in some $G \in \sigma_d(V)$.*

Proof. First, let us note: we can safely ignore any τ which have a coordinate lying in $\mathrm{SL}_2(\mathbb{Z}) \cdot \{i, \rho\}$, since these all clearly lie in $\mathrm{SL}_2(\mathbb{Z})$-finitely many proper $\mathbb{H}$-special subvarieties.

We work by induction on n. When $n = 1$, the result follows immediately from Theorem 6.14.

For larger n, the first stage is to construct a variety V^* which is designed to account for all possible positive-dimensional special subvarieties of any V_a.

Let $\mathcal{B}$ be the collection of basic $\mathbb{H}$-special subvarieties afforded by applying Proposition 6.15 to V. Let $\mathcal{C}$ be the collection of $\mathbb{H}$-special subvarieties internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent to some $B \in \mathcal{B}$. If any of these are equal to $\mathbb{H}^k$, for some k, remove them from $\mathcal{C}$.

Now let

$$\mathcal{G}_1^* = \left\{ \omega\big(B \times \mathbb{H}^{n-k}\big) : \begin{array}{c} B \in \mathcal{C}, 1 \leq k \leq n, B \subseteq \mathbb{H}^k, \\ \omega \text{ a permutation of the coordinates} \end{array} \right\}$$

Since, by 6.16, $\mathcal{C}$ is $\mathrm{SL}_2(\mathbb{Z})$-finite, $\mathcal{G}_1^*$ is too.

Next, consider the variety $V_k \subseteq \mathbb{C}^{3(n-k)+r}$, defined over $\overline{\mathbb{Q}}$ for $1 \leq k < n$ by

$$V_k = \{(\mathbf{X}, a) \in \mathbb{C}^{3(n-k)+r} : \text{Up to permutation of coordinates, } \mathbb{C}^{3k} \times \{\mathbf{X}\} \subseteq V_a\}.$$

Clearly V_k is a proper subvariety of $\mathbb{C}^{3(n-k)+r}$. By our inductive assumption, there is some $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\mathcal{F}_k$ of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^{n-k}$. Every

j'-generic $\mathbb{H}$-special point in $J^{-1}((V_k)_a)$ is contained in some $F \in \mathcal{F}_k$, provided that $(V_k)_a \neq \mathbb{C}^{3(n-k)}$.

Let

$$\mathcal{G}_2^* = \{\omega(\mathbb{H}^k \times F) : F \in \mathcal{F}_k, 1 \leq k < n, \omega \text{ a permutation of coordinates}\}.$$

Then $\mathcal{G}_2^*$ is $\mathrm{SL}_2(\mathbb{Z})$-finite.

Let

$$\mathcal{G}^* = \mathcal{G}_1^* \cup \mathcal{G}_2^*,$$

and let

$$V^* = \left\langle \left\langle \bigcup \mathcal{G}^* \right\rangle \right\rangle.$$

Since $\mathcal{G}^*$ consists of $\mathrm{SL}_2(\mathbb{Z})$-finitely many proper $\mathbb{H}$-special subvarieties, V^* is a proper subvariety of $\mathbb{C}^{3n}$.

Suppose that, for some a of degree at most d, the set $J^{-1}(V_a \setminus V^*)$ contains a j'-generic quadratic τ with $D(\tau) > D = D(V,d)$. By 6.14, there are positive-dimensional $\mathbb{H}$-special varieties G and H, internally $\mathrm{SL}_2(\mathbb{Z})$-equivalent, with

$$\tau \in G, \qquad H \hookrightarrow V_a,$$

and maximal with these properties.

Assume first that H does not take the form $\mathbb{H}^k \times \{(\tau_{k+1}, \dots, \tau_n)\}$. Then since $H \hookrightarrow V_a$, we have $H \in \mathcal{B}$, whence $G \in \mathcal{C}$. It follows that $\tau \in \bigcup \mathcal{G}_1^*$, whence $J(\tau) \in V^*$, which is a contradiction.

If $G = H = \mathbb{H}^k \times \{(\tau_{k+1}, \dots, \tau_n)\}$ for some k, then 6.14 tells us that $\tau \in G \subseteq J^{-1}(V_a)$. Taking Zariski closures, it follows that

$$\mathbb{C}^{3k} \times \{J(\tau_{k+1}, \dots, \tau_n)\} \subseteq V_a,$$

whence $(\tau_{k+1}, \dots, \tau_n)$ is a j'-generic quadratic point of $J^{-1}((V_k)_a)$. This implies $G \in \mathcal{G}_2^*$, whence $\tau \in V^*$, which is a contradiction.

Hence, for every a with $\deg a \leq d$, every $\tau \in J^{-1}(V_a \setminus V^*)$ has $D(\tau) \leq D$. In particular, the collection $\mathcal{H}^*$ of j'-generic τ which lie in *any* of the $V_a \setminus V^*$ is $\mathrm{SL}_2(\mathbb{Z})$-finite. So whenever $\deg a \leq d$ and $V_a \neq \mathbb{C}^{3n}$, every j'-generic quadratic point in $J^{-1}(V_a)$ is contained either in an element of $\mathcal{G}^*$ or $\mathcal{H}^*$. Both $\mathcal{G}^*$ and $\mathcal{H}^*$ are $\mathrm{SL}_2(\mathbb{Z})$-finite, so we're done. □

6.2.1 A More Precise Statement

Readers familiar with the normal shape of André-Oort statements may have noticed that Conjecture 3.6 is weaker than a typical André-Oort statement. Even taking into account the action of $\mathrm{SL}_2(\mathbb{Z})$, a more natural analogue of the classical case might look like the following.

Statement 6.18. *Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Then the collection of maximal J-special subvarieties of V is $\mathrm{SL}_2(\mathbb{Z})$-finite.*

This turns out to be false. The reason for its failure is fairly simple; the modular relations that relate $j'(g\tau)$ (for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$) with $j'(\tau)$ include not just $j(\tau)$, $j'(\tau)$, $j(g\tau)$ and $j'(g\tau)$, but also include instances of $m_g(\tau) = (c\tau + d)^2/N$. With the right polynomial, one can therefore enforce arbitrary relations between $m_g(\tau)$ and other $m_h(\sigma)$ arising in other coordinates.

Similarly, the modular relation for j'' introduces new variables to the equation. Recall that there is a rational function μ_N in 7 variables such that

$$j''(g\tau) = \mu_N(j(\tau), j(g\tau), j'(\tau), j'(g\tau), j''(\tau), c, (c\tau + d)),$$

whenever g is a primitive integer matrix of determinant N. Moreover, μ_N is linear in the c-coordinate. Hence we are able to enforce relations on the c that can arise, as well as the $m_g(\tau)$.

Example 6.19. Let us see some examples to illustrate this issue.

1. Let $W \subseteq \mathbb{C}^2$ be an algebraic variety defined over $\mathbb{Q}$. Suppose that W has at least one solution (x, y) where x and y are both squares of quadratic points in $\mathbb{H}$. Fix two positive integers M and N.

   Writing a general element of $\mathbb{C}^{12}$ as $(X_1, Y_1, Z_1, \ldots, X_4, Y_4, Z_4)$, consider the variety $V \subseteq \mathbb{C}^{12}$ defined (over $\overline{\mathbb{Q}}$) by

   $$\Phi_M(X_1, X_2) = 0, \quad \Phi_N(X_3, X_4) = 0,$$
   $$\left(\frac{-Y_2}{Y_1 \lambda_M(X_1, X_2)}, \frac{-Y_4}{Y_3 \lambda_N(X_3, X_4)}\right) \in W.$$

   Then the special points of $J^{-1}(V)$ are precisely the points $(\tau, g\tau, \sigma, h\sigma)$, where g and h are (arbitrary) primitive integer matrices with determinant M and N respectively, and τ, σ are quadratic points satisfying

   $$(m_g(\tau), m_h(\sigma)) \in W.$$

Since W has at least one solution which is a square of a quadratic point, we can certainly find τ and σ to solve this equation. Indeed, by modifying τ and σ we can solve this equation for any g and h of suitable determinant. The resulting collection of special points is certainly $\mathrm{SL}_2(\mathbb{Z})$-infinite, but no positive-dimensional $\mathbb{H}$-special variety is contained in $J^{-1}(V)$.

This example therefore serves as a counterexample to the hypothetical statement 6.18. Since the points all lie within the $\mathrm{SL}_2(\mathbb{Z})$-translates of the $\mathbb{H}$-special set

$$\{(\tau_1, g\tau_1, \tau_2, h\tau_2) : \tau_i \in \mathbb{H}\},$$

this example does *not* serve as a counterexample to Conjecture 3.6 (or Theorem 6.11!).

2. Fix a quadratic point $\sigma \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$(c\sigma + d)^2 = m_\gamma(\sigma) = \frac{j'(\gamma\sigma)}{j'(\sigma)}$$

and

$$c = \frac{j''(\gamma\sigma) - j''(\sigma)(c\sigma+d)^4}{2(c\sigma+d)^3 j'(\tau)},$$

whence

$$\begin{aligned} c^2 &= \frac{\left(j''(\gamma\sigma) - j''(\sigma)(c\sigma+d)^4\right)^2}{4(c\sigma+d)^6 j'(\sigma)^2} \\ &= \frac{\left(j''(\gamma\sigma) - j''(\sigma)\left(\frac{j'(\gamma\sigma)}{j'(\sigma)}\right)^2\right)^2}{4\left(\frac{j'(\gamma\sigma)}{j'(\sigma)}\right)^3 j'(\sigma)^2}. \end{aligned}$$

So for the appropriate rational function q we have

$$c^2 = q(j'(\sigma), j'(\gamma\sigma), j''(\sigma), j''(\gamma\sigma)).$$

Given a variety $W \subseteq \mathbb{C}^2$, defined over $\overline{\mathbb{Q}}$, we can then define $V \subseteq \mathbb{C}^9$ by

$$\Phi_N(X_1, X_2), \quad X_3 = j(\sigma),$$

$$\forall w \in \mathbb{C}, \left(\frac{-Y_2}{Y_1 \lambda_N(X_1, X_2)}, q\left[w, Y_3, p_{\operatorname{Im}\sigma}(j(\sigma), \chi^*(\sigma), w), Z_3\right]\right) \in W.$$

Then the $\mathbb{H}$-special points of $J^{-1}(V)$ are exactly those points

$$(\tau, g\tau, \gamma\sigma),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any primitive integer matrix of determinant N, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and

$$((c\tau + d)^2, C^2) \in W.$$

Once again, if W is suitable then this is an $\mathrm{SL}_2(\mathbb{Z})$-infinite collection, but no positive dimensional $\mathbb{H}$-special set lies in $J^{-1}(V)$.

With variants of these examples, one can produce varieties which impose near-arbitrary conditions on the variables c, $(c\tau + d)$, C and D, where:

- c and $(c\tau + d)$ correspond to matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ occurring in the definition of some $\mathbb{H}$-special variety.
- C and D correspond to some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, which is hypothetically acting on some constant coordinate of an $\mathbb{H}$-special variety. (So we are picking out particular $\mathrm{SL}_2(\mathbb{Z})$-translates of a fixed $\mathbb{H}$-special variety.)

In this section I will prove a stronger version of Conjecture 3.6 (assuming the conjecture on j'-generic points, 3.9). The idea of this stronger result is that relations of the type described above should be the only obstruction to getting a result like Statement 6.18. To state this precisely, we will first need to go through a number of technicalities.

Given a proper $\mathbb{H}$-special set $G \subseteq \mathbb{H}^n$, there is an underlying partition of $\{1, \dots, n\}$ which can be written as

$$S_0 \cup S_1 \cup \cdots \cup S_h \cup T_1 \cup \cdots \cup T_k,$$

with $\#S_i > 1$ for $i > 0$ and $\#T_j = 1$ for all j. (The condition that G is proper is equivalent to requiring that $k < n$.) For $i > 0$, let $r_i = \#S_i - 1$ and let s_i be the smallest element of S_i. Also associated to G are some matrices $g_{i,1}, \dots, g_{i,r_i} \in \mathrm{GL}_2^+(\mathbb{Q})$, so that the τ corresponding to each element of $S_i \setminus \{s_i\}$ is defined by $\tau = g_{i,j}\tau_{s_i}$.

Given such a G and given a tuple of matrices

$$\gamma' = (\gamma_1, \dots, \gamma_{\#S_0}, \gamma_{1,1}, \dots, \gamma_{1,r_1}, \dots, \gamma_{h,1}, \dots, \gamma_{h,r_h}) \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0 + \sum r_i}, \qquad (6.7)$$

let c_i, d_i be the bottom row of γ_i, and $c_{i,j}, d_{i,j}$ be the bottom row of the matrix $\gamma_{i,j} g_{i,j}$. Then we make the following definition.

Definition 6.20. A variety $W \subseteq \mathbb{C}^{2\#S_0+2\sum r_i}$, defined over $\overline{\mathbb{Q}}$, is called a G-variety. For a G-variety W and a given $\gamma' \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0+\sum r_i}$ (as above), we define $W^{\gamma'} \subseteq \mathbb{H}^h$ to be the set of $(\tau_{s_1}, \ldots, \tau_{s_h})$ such that

$$(\ldots, c_i, d_i, \ldots, c_{i,j}, c_{i,j}\tau_{s_i} + d_{i,j}, \ldots) \in W.$$

If γ is an element of the full group $\mathrm{SL}_2(\mathbb{Z})^n$, then it consists of $\gamma' \in \mathrm{SL}_2(\mathbb{Z})^{\#S_0+\sum r_i}$, as above, together with some more matrices

$$\alpha_{s_1}, \ldots, \alpha_{s_h} \in \mathrm{SL}_2(\mathbb{Z}),$$

corresponding to the s_i-coordinates, and

$$\beta_1, \ldots \beta_k \in \mathrm{SL}_2(\mathbb{Z})$$

corresponding to the singleton coordinates in the T_i. For such a $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$, we will write

$$W^{\gamma} = \{(\alpha_{s_1}\tau_{s_1}, \ldots, \alpha_{s_h}\tau_{s_h}) : (\tau_{s_1}, \ldots, \tau_{s_h}) \in W^{\gamma'}\}.$$

Notation 6.21. We will write

$$\mathrm{Sp}(\gamma, W) = \{(\tau_1, \ldots, \tau_n) \in \mathbb{H}^n : (\tau_{s_1}, \ldots, \tau_{s_h}) \in W^{\gamma} \text{ and every } \tau_i \text{ is quadratic}\}.$$

In the case where $h = 0$, so that we have no τ-coordinates to work with, then W only enforces conditions on the γ corresponding to the S_0-coordinates. In this case, we will use the convention that

$$\mathrm{Sp}(\gamma, W) = \{\tau \in \mathbb{H}^n : \tau \text{ is quadratic.}\}$$

if $(\ldots, c_i, d_i, \ldots) \in W$, and

$$\mathrm{Sp}(\gamma, W) = \emptyset$$

otherwise.

Before we can state our more precise version of 3.6, we need one more definition.

Definition 6.22. A pair (G, W), with G a proper $\mathbb{H}$-special set and W a G-variety is said to be *geodesically minimal* if

$$\bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^n} \gamma \cdot G \cap \mathrm{Sp}(\gamma, W)$$

is *not* contained in any $\mathrm{SL}_2(\mathbb{Z})$-finite collection of proper $\mathbb{H}$-special subvarieties of G.

Theorem 6.23 (Precise Modular André-Oort with Derivatives). *Assume Conjecture 3.9. Let $V \subseteq \mathbb{C}^{3n}$ be an algebraic variety defined over $\overline{\mathbb{Q}}$. Then there is a finite collection $\sigma(V)$ of $\mathbb{H}$-special subvarieties of $\mathbb{H}^n$, and for each $G \in \sigma(V)$ an associated G-variety W_G, with the following properties.*

- *For every $G \in \sigma(V)$, the pair (G, W_G) is geodesically minimal.*
- *The set of quadratic points in $J^{-1}(V)$ is precisely*

$$\bigcup_{\substack{G\in\sigma(V)\\ \gamma\in\mathrm{SL}_2(\mathbb{Z})^n}} \gamma \cdot G \cap \mathrm{Sp}(\gamma, W_G).$$

Proof. Under the assumption of Conjecture 3.9, Theorem 6.11 yields a finite collection $\sigma(V)$ of proper $\mathbb{H}$-special subvarieties, such that the special points of $J^{-1}(V)$ are contained in

$$\bigcup_{\substack{G\in\sigma(V)\\ \gamma\in\mathrm{SL}_2(\mathbb{Z})^n}} \gamma \cdot G.$$

Let us look first at a single $G \in \sigma(V)$, and associate some data to it, as in the definitions above. Associated to G is a partition of $\{1, \dots, n\}$,

$$S_0 \cup S_1 \cup \cdots \cup S_h \cup T_1 \cup \cdots \cup T_k,$$

with T_i singletons and $\#S_i > 1$. As above, we have some associated data

$$s_i = \min S_i, \qquad r_i = \#S_i - 1 \text{ for } i > 0,$$

$$\sigma_1, \dots, \sigma_{\#S_0} \in \mathbb{H} \text{ quadratic}$$

and

$$g_{i,j} \in \mathrm{GL}_2^+(\mathbb{Q}), \text{ a primitive integer matrix, for } 1 \le i \le h, 1 \le j \le r_i.$$

Write $N_{i,j} = \det g_{i,j}$. For ease of notation, we will assume that the coordinates are ordered nicely, with the first few coordinates in S_0, the next few in S_1, and so on, with $\tau_{n-k+1}, \dots \tau_n$ corresponding to the T_i coordinates.

Recall: for each N, there is a rational function μ_N with the property that

$$j''(g\tau) = \mu_N(j(\tau), j(g\tau), j'(\tau), j'(g\tau), j''(\tau), c, (c\tau + d))$$

whenever g is a primitive integer matrix of determinant N.

Given $\gamma \in \mathrm{SL}_2(\mathbb{R})^n$, and a point $\tau = (\tau_1, \ldots, \tau_h) \in \mathbb{H}^h$, define a variety

$$V_{\gamma,\tau} \subseteq \mathbb{C}^{3(h+k)}$$

as follows. First, write γ as in (6.7). That is, γ consists of

$$\gamma' = (\gamma_1, \ldots, \gamma_{\#S_0}, \gamma_{1,1}, \ldots, \gamma_{1,r_1}, \ldots, \gamma_{h,1}, \ldots, \gamma_{h,r_h}) \in \mathrm{SL}_2(\mathbb{R})^{\#S_0 + \sum r_i},$$

some

$$\alpha_{s_1}, \ldots, \alpha_{s_h} \in \mathrm{SL}_2(\mathbb{R})$$

corresponding to the s_i coordinates, and

$$\beta_1, \ldots, \beta_k \in \mathrm{SL}_2(\mathbb{R})$$

corresponding to the singleton coordinates in the T_i. Let c_i, d_i be the bottom row of γ_i and $c_{i,j}, d_{i,j}$ the bottom row of $h_{i,j} = \gamma_{i,j} g_{i,j}$.

Define $V'_{\gamma,\tau}$ by

$$(X_1, Y_1, Z_1, \ldots, X_{h+k}, Y_{h+k}, Z_{h+k}) \in V'_{\gamma,\tau}$$

$$\Longleftrightarrow$$

$$\begin{aligned}
&\forall w_{i,j} \in \mathbb{C} \text{ with } \Phi_{N_{i,j}}(X_i, w_{i,j}), i \leq h, j \leq r_i, \\
&\quad \Big[\ldots, j(\sigma_i), j'(\gamma_i\sigma_i), j''(\gamma_i\sigma_i), \ldots, X_i, Y_i, Z_i, \ldots, w_{i,j}, -Y_i m_{h_{i,j}}(\tau_{s_i})\lambda_{N_{i,j}}(X_i, w_{i,j}), \\
&\qquad \mu_{N_{i,j}}\big(X_i, w_{i,j}, Y_i, -Y_i m_{h_{i,j}}(\tau_{s_i})\lambda_{N_{i,j}}(X_i, w_{i,j}), Z_i, c_{i,j}, c_{i,j}\tau_{s_i} + d_{i,j}\big), \ldots \\
&\qquad\qquad \ldots, X_{h+i}, Y_{h+i}, Z_{h+i}, \ldots\Big] \in V.
\end{aligned}$$

Taking $\overline{\mathbb{Q}}$-Zariski closures replaces the $j'(\gamma_i\sigma_i)$ and $j''(\gamma_i\sigma_i)$ by suitable rational functions involving σ_i, $\operatorname{Im}\sigma_i$, $\chi^*(\sigma_i)$, c_i, d_i, and some complex numbers w which are allowed to be arbitrary. Making these replacements we get a variety $V_{\gamma,\tau}$, defined over $\overline{\mathbb{Q}}$, depending polynomially on $c_i, d_i, c_{i,j}, (c_{i,j}\tau_i + d_{i,j})$ (and nothing else). Thus each $V_{\gamma,\tau}$ is a fibre of an algebraic family of varieties $\widehat{V}$, defined over $\overline{\mathbb{Q}}$.

We now apply Theorem 6.17 to $\widehat{V}$. We get an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma_2\big(\widehat{V}\big)$, consisting of $\mathbb{H}$-special subvarieties of $\mathbb{H}^{(h+k)}$, such that for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ and all quadratic $\tau \in \mathbb{H}^h$, either

$$V_{\gamma,\tau} = \mathbb{C}^{3(h+k)}$$

or the $\mathbb{H}$-special subvarieties of $J^{-1}(V_{\gamma,\tau})$ are accounted for by $\sigma_2\big(\widehat{V}\big)$.

For each $H \in \sigma_2\big(\widehat{V}\big)$, we produce a proper $\mathbb{H}$-special subvariety $G_H \subsetneq G$ via the construction

$$G_H = \{\tau \in G : (\tau_{s_1}, \ldots, \tau_{s_h}, \tau_{n-k+1}, \ldots, \tau_n) \in H\}.$$

So $\sigma_2(\widehat{V})$ corresponds to an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\mathcal{G}$ of proper $\mathbb{H}$-special subvarieties $G_H \subsetneq G$. We add all these $G_H \in \mathcal{G}$ to the overarching collection $\sigma(V)$.

Now, a quadratic point lying in

$$\bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^n} \gamma \cdot G \cap J^{-1}(V)$$

corresponds to a quadratic point $\tau = (\tau_1, \dots, \tau_n) \in G$ together with $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ such that $\gamma\tau \in J^{-1}(V)$. Such a pair (τ, γ) necessarily satisfies

$$J(\tau') \in V_{\gamma,\tau},$$

where

$$\tau' = (\alpha_{s_1}\tau_{s_1}, \dots, \alpha_{s_h}\tau_{s_h}, \beta_1\tau_{n-k+1}, \dots, \beta_k\tau_n).$$

By the properties of $\sigma_2(\widehat{V})$, either $\tau' \in H$ for some $H \in \sigma_2(\widehat{V})$, or $V_{\gamma,\tau} = \mathbb{C}^{3(h+k)}$. In the first case, we have $\tau \in G_H$, for some $G_H \in \mathcal{G}$.

Now we define a 'remainder set' R, to be the set of remaining special points not accounted for by $\mathcal{G}$. That is,

$$R = \left\{ (\tau, \gamma) \in \mathbb{H}^n \times \mathrm{SL}_2(\mathbb{Z})^n : \begin{array}{c} \tau \text{ is quadratic}, \tau \in G, J(\gamma\tau) \in V, \\ \tau \notin \bigcup_{G_H \in \mathcal{G}} G_H \end{array} \right\}.$$

If R is empty, we can stop here, removing G from $\sigma(V)$ entirely; it contributes no special points other than those already accounted for by $\mathcal{G}$.

If $R \neq \emptyset$, we continue. By the properties of $\mathcal{G}$, every $(\tau, \gamma) \in R$ must satisfy

$$V_{\gamma,\tau} = \mathbb{C}^{3(h+k)}.$$

Hence, defining a G-variety W_G such that

$$W_G^\gamma = \{\tau \in \mathbb{H}^h : V_{\gamma,\tau} = \mathbb{C}^{3(h+k)}\},$$

we get

$$(\tau, \gamma) \in R \implies (\alpha_{s_1}\tau_{s_1}, \dots, \alpha_{s_h}\tau_{s_h}) \in W_G^\gamma.$$

Conversely, for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, any quadratic $z \in W_G^\gamma$ induces a quadratic element of $\gamma \cdot G \cap J^{-1}(V)$. Indeed, with γ decomposed into components α, β, γ'

as before, writing $z = (\alpha_{s_1} z_1, \dots, \alpha_{s_h} z_h) \in W_G^\gamma$ and $h_{i,j} = \gamma_{i,j} g_{i,j}$, we know since $V_{\gamma,z} = \mathbb{C}^{3(h+k)}$ that

$$\Bigg[\dots, j(\sigma_i), j'(\gamma_i\sigma_i), j''(\gamma_i\sigma_i), \dots, J(\alpha_{s_i} z_i), \dots$$
$$\dots, j(h_{i,j}\alpha_{s_i} z_i), -j'(\alpha_{s_i} z_i) m_{h_{i,j}}(\alpha_{s_i} z_i)\lambda_{N_{i,j}}(j(\alpha_{s_i} z_i), j(h_{i,j}\alpha_{s_i} z_i)),$$
$$\mu_{N_{i,j}}\Bigg(j(\alpha_{s_i} z_i), j(h_{i,j}\alpha_{s_i} z_i), j'(\alpha_{s_i} z_i), -j'(\alpha_{s_i} z_i) m_{h_{i,j}}(\alpha_{s_i} z_i)\lambda_{N_{i,j}}(j(\alpha_{s_i} z_i), j(h_{i,j}\alpha_{s_i} z_i)),$$
$$j''(\alpha_{s_i} z_i), c_{i,j}, c_{i,j} z_i + d_{i,j}\Bigg), \dots$$
$$\dots, J(z_{h+i}), \dots\Bigg] \in V$$

for any $z_{h+1}, \dots, z_{h+k} \in \mathbb{H}$.

By the properties of λ_N and μ_N, we get

$$\Big[\dots, J(\gamma_i\sigma_i), \dots, J(\alpha_{s_i} z_i), \dots, J(h_{i,j}\alpha_{s_i} z_i), \dots, J(z_{h+i}), \dots\Big] \in V,$$

for every $(z_{h+1}, \dots, z_{h+k}) \in \mathbb{H}^k$. So z induces a quadratic point

$$(\dots, \gamma_i\sigma_i, \dots, \alpha_{s_i} z_i, \dots, h_{i,j}\alpha_{s_i} z_i, \dots, z_{h+i}, \dots) \in \gamma \cdot G \cap J^{-1}(V),$$

for any choice of quadratic z_{h+i}.

To sum up, we have seen that the quadratic points in $\gamma \cdot G \cap J^{-1}(V)$ consist precisely of:

1. Those points that correspond to quadratic solutions of W_G^γ, that is

$$\gamma \cdot G \cap \mathrm{Sp}(\gamma, W_G),$$

   and:

2. Those quadratic points that lie in some $G_H \in \mathcal{G}$.

We do not claim that these possibilities are mutually exclusive!

Before we are done with G, we must check whether (G, W_G) is geodesically minimal. If it does happen to be geodesically minimal, we are done. Otherwise, by definition,

$$\bigcup_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^n} \mathrm{Sp}(\gamma, W)$$

is contained in some $\mathrm{SL}_2(\mathbb{Z})$-finite collection of proper $\mathbb{H}$-special subvarieties of $\mathbb{H}^{h+k}$. This is not a problem; we simply remove G from $\sigma(V)$ entirely and replace it by the corresponding finite collection of proper subvarieties of G.

This is as much as we can do with a given $G \in \sigma(V)$. For this G, our process has done two things.

1. Either:
   (a) removed G from $\sigma(V)$ entirely, or
   (b) associated to G a G-variety W_G such that (G, W_G) is geodesically minimal.
2. Added to $\sigma(V)$ some finite collection $\mathcal{G}$ of proper subvarieties of G.

The $\mathbb{H}$-special points of $\gamma \cdot G \cap J^{-1}(V)$ are precisely the $\mathbb{H}$-special points in

$$\bigcup_{G_H \in \mathcal{G}} \gamma \cdot G_H \cap J^{-1}(V),$$

together (if we are in case 1.(b) above) with

$$\gamma \cdot G \cap \mathrm{Sp}(\gamma, W_G).$$

This is enough to conclude the theorem. Perform the above process on each $G \in \sigma(V)$ in turn, taking them in *descending order of dimension*. Since each G can add to $\sigma(V)$ only finitely many varieties of strictly smaller dimension, the process will eventually terminate. □

Remark 6.24. As discussed earlier, the classical André-Oort Conjecture fits into the wider picture of the Zilber-Pink Conjecture. The statement of Zilber-Pink discusses not only a variety $V \subseteq \mathbb{C}^n$, but also depends on the j-special subvariety $U \subseteq \mathbb{C}^n$, containing V, inside which V is considered. This is typically encapsulated by saying that "Zilber-Pink depends on the *level structure*".

On the other hand, the classical André-Oort statement does *not* depend on the level structure; the special subvarieties of V do not vary depending on the j-special subvariety inside which V is considered. This is also true of our André-Oort theorem for π, Theorem 6.5. In the J-case discussed in this chapter, however, the level structure is crucial. Thanks to the examples in 6.19, if we are given $V \subseteq \langle\langle G \rangle\rangle$ for some $\mathbb{H}$-special subvariety $G \subsetneq \mathbb{H}^n$, it is not possible to ensure that the special points of V live inside $\mathrm{SL}_2(\mathbb{Z})$-finitely many proper $\mathbb{H}$-special subvarieties of G. Theorem 6.23 can be interpreted as giving a precise description of how the level structure influences the special subvarieties of such V.

6.2.2 Uniformity in Theorem 6.11

In this last section on the topic of André-Oort results, I will discuss uniform versions of the our two "Modular André-Oort with Derivatives" results, Theorems 6.11 and 6.23. For this we once again use the uniformity results of Scanlon; more specifically, recall Lemma 6.9, which is our slightly modified version of Lemma 3.2 from [73].

Lemma 6.9. *Let $k \subseteq K$ be algebraically closed fields, $X = \mathbb{A}^s$, $B = \mathbb{A}^t$ and $Y \subseteq X \times B$ a constructible subset. Let $A \subseteq X(K)$. Suppose that there exist $\alpha, \beta \in A$ which are k-Zariski separated.*

Then there is a natural number n and a k-constructible set $Z \subseteq X \times X^n$ such that for any $b \in B(K)$, there is $a \in A^n$ for which $Y_b(K) \cap A = Z_a(K) \cap A$.

Using this, we would like to get a uniform version of Theorem 6.23. This can indeed be done, though perhaps not in exactly the manner we might like.

Given an algebraic family $V \subseteq \mathbb{C}^{3n+k}$, we can apply 6.9, with

$$A = \{J(\tau) : \tau \in \mathbb{H}^n \text{ quadratic}\}.$$

For the resulting constructible set Z, we can write

$$Z = \bigcup_{i=1}^{r} X_i \setminus Y_i$$

for some varieties X_i, Y_i defined over $\overline{\mathbb{Q}}$. If we apply 6.23 (under the assumption of the j'-genericity conjecture, 3.9) to each of the X_i and Y_i, we get a finite collection $\sigma(V)$, consisting of *pairs* (G, H) of $\mathbb{H}$-special varieties, with corresponding G-varieties (W_G, W_H), such that the set of $\mathbb{H}$-special points of $J^{-1}(V_b)$, for any fibre b, is precisely the fibre of the set

$$\bigcup_{(G,H)\in\sigma(V)} \left[\bigcup_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} \gamma G \cap \mathrm{Sp}(\gamma, W_G) \Bigg\backslash \bigcup_{\gamma\in \mathrm{SL}_2(\mathbb{Z})} \gamma H \cap \mathrm{Sp}(\gamma, W_H) \right]$$

at some quadratic $\tau = \tau(b)$.

This is not quite as good as we might like; one would prefer not to have the $\mathbb{H}$-special sets H in the picture. These arise thanks to the fact that Z is only constructible, rather than Zariski closed. This is a problem that can be circumvented in some contexts, using the full strategy outlined in Scanlon's paper [73]. In our context, however, the problem seems unavoidable.

Let G and H be $\mathbb{H}$-special varieties and take a corresponding G-variety W_G and H-variety W_H. Scanlon's full method, applied in our context, would only work if it were the case that $(G \cap H, W_G \cap W_H)$ was geodesically minimal whenever (G, W_G) and (H, W_H) were. Since this does not hold in general, the rest of Scanlon's work does not apply, so in the absence of another approach, it does not seem possible to remove the $\mathbb{H}$-special sets H from the above picture. Hence the above seems likely to be the best possible uniform version of 6.23.

The weaker Theorem 6.11, however, can be uniformised more cleanly.

Theorem 6.25. *Assume Conjecture 3.9. Let $V \subseteq \mathbb{C}^{3n+r}$ be an algebraic variety, considered as an algebraic family of $V_b \subseteq \mathbb{C}^{3n}$, $b \in \mathbb{C}^r$. There is a natural number N, and an $\mathrm{SL}_2(\mathbb{Z})$-finite collection $\sigma(V)$ of $\mathbb{H}$-special subvarieties of $\mathbb{H}^{n+N}$, with the following property.*

For every $b \in \mathbb{C}^r$ with $V_b \neq \mathbb{C}^{3n}$, the $\mathbb{H}$-special points of $J^{-1}(V_b)$ are contained in

$$\bigcup_{G \in \sigma(V)} G_\tau,$$

where G_τ is the fibre of G at some quadratic $\tau = \tau(b) \in \mathbb{H}^N$. This τ can be chosen so that all of the G_τ are proper *subvarieties of $\mathbb{H}^n$.*

Proof. Let $V \subseteq \mathbb{C}^{3n+r}$ be a variety, considered as an algebraic family of fibres $V_b \subseteq \mathbb{C}^{3n}$. We will apply Lemma 6.9, with $X = \mathbb{C}^{3n}$, $Y = V$, $K = \mathbb{C}$, $k = \overline{\mathbb{Q}}$ and

$$A = \{J(\tau) : \tau \in \mathbb{H}^n \text{ quadratic}\}.$$

To apply the lemma, we need to find two $\overline{\mathbb{Q}}$-Zariski separated points $\alpha, \beta \in A$. This is easy; we only need the j-coordinates of α and β to be distinct. So Lemma 6.9 does apply; we get a $\overline{\mathbb{Q}}$-constructible set $Z \subseteq \mathbb{C}^{3(n+dn)}$ such that, for every $b \in \mathbb{C}^r$, there is some $a \in A^d$ such that

$$Z_a \cap A = V_b \cap A.$$

Write

$$Z = \bigcup_{i=1}^{m} X_i \setminus Y_i,$$

for some $\overline{\mathbb{Q}}$-varieties X_i and Y_i. We will apply Theorem 6.23 to each X_i and Y_i separately. We get some finite sets $\sigma(X_i)$ and $\sigma(Y_i)$ of $\mathbb{H}$-special varieties, with associated G-varieties, exactly describing the special subvarieties of the Z_i in the manner described in the statement of Theorem 6.23.

Now, given $b \in \mathbb{C}^r$, let $a \in A^d$ be such that $Z_a \cap A = V_b \cap A$. Let $\tau \in \mathbb{H}^d$ satisfy $J(\tau) = a$.

First suppose that no $\sigma(X_i)$ contains any G such that, for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^N$, $G_{\gamma\tau} = \mathbb{H}^n$. Then we are done; the special points of $J^{-1}(V_b)$ are contained in the $\mathrm{SL}_2(\mathbb{Z})$-finite collection of proper $\mathbb{H}$-special varieties

$$\left\{ \gamma' \cdot G_{\gamma\tau} : \begin{array}{c} G \in \sigma(X_i), i \leq r, \\ \gamma \in \mathrm{SL}_2(\mathbb{Z})^N, \gamma' \in \mathrm{SL}_2(\mathbb{Z})^n \end{array} \right\}.$$

On the other hand, suppose that some $\sigma(X_i)$ contains G such that, for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^N$, $G_{\gamma\tau} = \mathbb{H}^n$. Then the G-variety associated to G cannot impose any condition on the coordinates corresponding to $\mathbb{H}^n$. Hence by the properties laid out in 6.23, we must have $(X_i)_a = \mathbb{C}^{3n}$.

Now apply the same argument to Y_i. There are 2 possibilities. Either:

- The special points of $(Y_i)_a$ are contained in an $\mathrm{SL}_2(\mathbb{Z})$-finite collection

  $$\{\gamma' \cdot G_{\gamma\tau} : G \in \sigma(Y_i), i \leq r, \gamma \in \mathrm{SL}_2(\mathbb{Z})^N, \gamma' \in \mathrm{SL}_2(\mathbb{Z})^n\},$$

  with each $G_{\gamma\tau}$ being a *proper* $\mathbb{H}$-special subvariety of $\mathbb{H}^n$, or

- $(Y_i)_a = \mathbb{C}^{3n}$.

In the first case, since the special points of $(Y_i)_a$ are contained in a lower-dimensional set, it follows that the special points of Z_a are Zariski dense, whence $V_b = \mathbb{C}^{3n}$. In the second case, $(X_i)_a \setminus (Y_i)_a$ contributes no new special points, so we can ignore this i and move on.

Finally, note by the properties of $\mathbb{H}$-special varieties that if $G_\tau = \mathbb{H}^n$ for some τ, then $G_{\tau'} = \mathbb{H}^n$ or $G_{\tau'} = \emptyset$ for every τ'. Thus, if some $G \in \sigma(X_i)$ has the property that (for some τ), $G_\tau = \mathbb{H}^n$, we can safely remove it. By the previous arguments, all the special points will still be covered by the rest of the $G \in \bigcup \sigma(X_i)$, except in the case where $V_b = \mathbb{C}^{3n}$.

Hence the collection

$$\sigma(V) = \left\{ \gamma \cdot G : \begin{array}{c} \gamma \in \mathrm{SL}_2(\mathbb{Z})^n, G \in \sigma(X_i), i \leq r \\ \text{For every } \tau \in \mathbb{H}^N, G_\tau \neq \mathbb{H}^n \end{array} \right\}$$

satisfies the conclusion of the theorem. □

To conclude the chapter, we state one final corollary.

Corollary 6.26. *Assume Conjecture 3.9. Let* $V \subseteq \mathbb{C}^{3n}$ *be a proper algebraic variety (which need not be defined over* $\overline{\mathbb{Q}}$*). There exists an* $\mathrm{SL}_2(\mathbb{Z})$*-finite collection* $\sigma(V)$*, consisting of* proper $\mathbb{H}$*-special varieties of* $\mathbb{H}^n$*, such that every* $\mathbb{H}$*-special point in* $J^{-1}(V)$ *is contained in some* $G \in \sigma(V)$*.*

Proof. Immediate. □

Chapter 7

Further Directions and Outlook

In this final chapter of the document, we will draw things together by discussing some big-picture questions and some possible directions for further exploration. Much of this discussion will also highlight some open problems from the existing work, tying these in to the bigger picture.

We begin with a discussion of Zilber-Pink (and relatedly Ax-Schanuel) in the ahm setting. We will not discuss these issues for J; they have been thoroughly dealt with in other places, notably (for Ax-Schanuel) by Pila/Tsimerman [66], and (for Zilber-Pink) by Pila, in unpublished notes.

7.1 AHM Zilber-Pink, Schanuel's Conjecture and Ax-Schanuel

Having approached André-Oort results in a couple of nonclassical settings, it is only natural to consider Zilber-Pink problems in the same settings. We mentioned Zilber-Pink briefly in Chapter 1; as discussed there, it is a conjecture which originally arose as a combination of conjectures of Pink [69] and Zilber [91]. The Zilber-Pink conjecture is closely connected with a number of other conjectures in the area, including André-Oort, Manin-Mumford, Mordell-Lang and Schanuel's Conjecture; this intricate paradigm is due primarily to Zilber, starting with his work in [91].

In this section we will discuss the extent to which this wider classical picture can be carried over to the ahm setting; we begin by discussing nonclassical versions of the Zilber-Pink conjecture in the context of the map $\pi = (j, \chi^*)$.

The Zilber-Pink conjecture discusses varieties which have atypically large intersection with special varieties. As such, to phrase a Zilber-Pink problem, we will first need to define what it means for such an intersection to be atypical. Here lies the first issue in working with Zilber-Pink for π; thanks to the fact that π does not map *onto* $\mathbb{C}^2$, we end up with several potential definitions for atypicality.

Definition 7.1 (Notions of Atypicality)**.** Let $U \subseteq \mathbb{H}^n$ be $\mathbb{H}$-special, and let $V \subseteq \pi(U)^{\mathrm{Zar}}$ be a subvariety. Assume that $\pi^{-1}(V) \neq \emptyset$ and let $A \subseteq V$ be a subvariety.

- We say A is left-atypical (for V in U) if it is an irreducible component of $\pi(G)^{\mathrm{Zar}} \cap V$ for some $\mathbb{H}$-special $G \subseteq U$, and

$$\dim_{\mathbb{R}} \pi^{-1}(A) > \dim_{\mathbb{R}} \pi^{-1}(V) + \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} U.$$

- We say A is right-atypical (for V in U) if it is an irreducible component of $\pi(G)^{\mathrm{Zar}} \cap V$ for some $\mathbb{H}$-special $G \subseteq U$, and

$$\dim A > \dim V + \dim \pi(G)^{\mathrm{Zar}} - \dim \pi(U)^{\mathrm{Zar}}.$$

Now let $B \subseteq \pi^{-1}(V)$.

- We say B is left-$\mathbb{H}$-atypical (for V in U) if it is a real-analytically irreducible component of $\pi^{-1}(V) \cap G$, for some $\mathbb{H}$-special $G \subseteq U$, and

$$\dim_{\mathbb{R}} B > \dim_{\mathbb{R}} \pi^{-1}(V) + \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} U.$$

- We say B is right-$\mathbb{H}$-atypical (for V in U) if it is a real-analytically irreducible component of $\pi^{-1}(V) \cap G$, for some $\mathbb{H}$-special $G \subseteq U$, and

$$\dim \pi(B)^{\mathrm{Zar}} > \dim V + \dim \pi(G)^{\mathrm{Zar}} - \dim \pi(U)^{\mathrm{Zar}}.$$

It is not immediately clear which of these definitions is the “correct” notion of atypicality.

Clearly, for any left-atypical component A of $\pi(G)^{\mathrm{Zar}} \cap V$, there will be a component B of $G \cap \pi^{-1}(V)$ containing a full-dimensional component of $\pi^{-1}(A)$. By definition, this B will be left-$\mathbb{H}$-atypical. Conversely, any left-$\mathbb{H}$-atypical component B of $G \cap \pi^{-1}(V)$ will induce a left-atypical component A of $\pi(G)^{\mathrm{Zar}} \cap V$ with $\pi(B)^{\mathrm{Zar}} \subseteq A$.

On the other hand, while a right-$\mathbb{H}$-atypical component B yields a right-atypical component containing $\pi(B)^{\mathrm{Zar}}$, the converse does not necessarily hold, since, for a right-atypical component A, the preimage $\pi^{-1}(A)$ could have relatively small dimension (indeed, it might be empty), and in particular $\pi(\pi^{-1}(A))^{\mathrm{Zar}}$ need not equal A.

The relationship between left- and right- atypicality depends on the relative sizes of the quantities

$$\delta(V) = \dim_{\mathbb{R}} V - 2\dim_{\mathbb{R}} \pi^{-1}(V)$$

and

$$\delta(A) = \dim_{\mathbb{R}} A - 2\dim_{\mathbb{R}} \pi^{-1}(A).$$

Since, for $\mathbb{H}$-special G, we have

$$\dim \pi(G)^{\mathrm{Zar}} = 2\dim G,$$

it follows that if A is right-atypical and $\delta(V) \geq \delta(A)$, then A is left-atypical. Conversely, if A is left-atypical and $\delta(V) \leq \delta(A)$, then A is right-atypical. We have a similar situation for $\mathbb{H}$-atypicality, with $\delta(A)$ replaced by

$$\delta(B) = \dim_{\mathbb{R}} \pi(B)^{\mathrm{Zar}} - 2\dim_{\mathbb{R}} B.$$

For a variety $V \subseteq \pi(U)^{\mathrm{Zar}}$, we will suppress U and write $\mathrm{Atyp}_L(V)$ for the union of all left-atypical components of a variety V, and similarly $\mathrm{Atyp}_R(V)$, $\mathrm{Atyp}_{L\mathbb{H}}(V)$ and $\mathrm{Atyp}_{R\mathbb{H}}(V)$ for the right-, left-$\mathbb{H}$- and right-$\mathbb{H}$-atypical components respectively.

Conjecture 7.2. *Let $U \subseteq \mathbb{C}^{2n}$ be $\mathbb{H}$-special, and let $V \subseteq \pi(U)^{\mathrm{Zar}}$ be a subvariety. We make four distinct conjectures:*

- *ZPL is the conjecture that $\mathrm{Atyp}_L(V)$ is a finite union of left-atypical components.*
- *ZPR is the conjecture that $\mathrm{Atyp}_R(V)$ is a finite union of right-atypical components.*
- *ZPL$\mathbb{H}$ is the conjecture that $\mathrm{Atyp}_{L\mathbb{H}}(V)$ is an $\mathrm{SL}_2(\mathbb{Z})$-finite union of left-$\mathbb{H}$-atypical components.*
- *ZPR$\mathbb{H}$ is the conjecture that $\mathrm{Atyp}_{R\mathbb{H}}(V)$ is an $\mathrm{SL}_2(\mathbb{Z})$-finite union of right-$\mathbb{H}$-atypical components.*

Clearly ZPL implies ZPL$\mathbb{H}$. It is less clear that ZPR implies ZPR$\mathbb{H}$, but it does; to get there we go via yet another formulation of ZP.

Proposition 7.3. *Assume that $\mathrm{Atyp}_R(V) \cap \pi(U)$ is a finite union of sets of the form $A \cap \pi(U)$, where A is right-atypical. Then ZPR$\mathbb{H}$ holds (for V in U).*

Proof. Suppose we have a finite collection of right-atypical components A_i such that

$$\mathrm{Atyp}_L(V) \cap \pi(U) = \bigcup A_i \cap \pi(U).$$

Let B be a maximal right-$\mathbb{H}$-atypical component of V; a component of $G \cap \pi^{-1}(V)$ for some special G.

We have

$$\pi(B)^{\mathrm{Zar}} \subseteq \pi(G)^{\mathrm{Zar}} \cap V.$$

We can take a component A of $\pi(G)^{\mathrm{Zar}} \cap V$ containing $\pi(B)^{\mathrm{Zar}}$. Then $\dim A > \dim \pi(B)^{\mathrm{Zar}}$, so A is right-atypical by definition, whence $A \cap \pi(U) = A_i \cap \pi(U)$ for some i.

Now take a component C of $\pi^{-1}(A)$ which contains B. Clearly

$$\pi(B)^{\mathrm{Zar}} \subseteq \pi(C)^{\mathrm{Zar}} \subseteq A \subseteq \pi(G)^{\mathrm{Zar}} \cap V.$$

It follows that C is a component of $G \cap \pi^{-1}(V)$, so clearly it is right-$\mathbb{H}$-atypical, whence by maximality of B we have $B = C$.

Hence $\mathrm{Atyp}_{R\mathbb{H}}(V)$ consists of (some of) the components of the $\pi^{-1}(A_i)$. Since there are finitely many A_i, there are $\mathrm{SL}_2(\mathbb{Z})$-finitely many such components. □

Conjecture 7.4. *ZPR' is the conjecture that*

$$\mathrm{Atyp}_R(V) \cap \pi(U) = \bigcup_{i=1}^{k} A_i \cap \pi(U)$$

with each A_i right-atypical.

Clearly ZPR $\Longrightarrow$ ZPR', so by 7.3 we have

$$\text{ZPR} \Longrightarrow \text{ZPR'} \Longrightarrow \text{ZPR}\mathbb{H}.$$

For left-atypicality, on the other hand, there is no need for a separate conjecture called ZPL', thanks to the following result.

Proposition 7.5. *ZPL$\mathbb{H}$ is equivalent to the statement that $\mathrm{Atyp}_L(V) \cap \pi(U)$ is a finite union of sets of the form $A \cap \pi(U)$, where A is left-atypical.*

Proof. Suppose ZPL$\mathbb{H}$ holds. Then $\mathrm{Atyp}_{L\mathbb{H}}$ is an $\mathrm{SL}_2(\mathbb{Z})$-finite union $\bigcup B_i$. Let $A \cap \pi(U)$ be maximal with the property that A is left-atypical, so that A is a component of some $\pi(G)^{\mathrm{Zar}} \cap V$. Without loss of generality, we will assume that $A \cap \pi(U) \neq \emptyset$.

Take a component B of $G \cap \pi^{-1}(V)$ containing a full-dimensional component $C \subseteq \pi^{-1}(A)$. By definition, B is left-$\mathbb{H}$-atypical. Further, if $B \subseteq B'$ with B' left-$\mathbb{H}$-atypical for V, then $\pi(B')^{\mathrm{Zar}} \cap \pi(U) \supseteq A \cap \pi(U)$, which by maximality of A forces $A = \pi(B')^{\mathrm{Zar}}$, whence $B = B'$. That is, B is maximal left-$\mathbb{H}$-atypical, so is equal to one of the $\mathrm{SL}_2(\mathbb{Z})$-finitely many B_i.

By dimensional considerations, $\pi(C) = A \cap \pi(U)$, and $B \supseteq C$, so $\pi(B)^{\mathrm{Zar}} \cap \pi(U) \supseteq A \cap \pi(U)$. Moreover, $\pi^{-1}(\pi(B)^{\mathrm{Zar}}) \supseteq B$, so $\dim_{\mathbb{R}} \pi^{-1}(\pi(B)^{\mathrm{Zar}}) \geq \dim_{\mathbb{R}} \pi^{-1}(A)$, whence $\pi(B)^{\mathrm{Zar}}$ is left-atypical. By maximality of $A \cap \pi(U)$, we have $\pi(B)^{\mathrm{Zar}} \cap \pi(U) = A \cap \pi(U)$.

So each of the $A \cap \pi(U)$ is equal to some $\pi(B_i)^{\mathrm{Zar}} \cap \pi(U)$, and moreover the $\pi(B_i)^{\mathrm{Zar}}$ arising this way are left-atypical. Therefore, for a suitable subset $\mathcal{B}$ of the B_i, we have

$$\mathrm{Atyp}_L(V) \cap \pi(U) = \bigcup_{B \in \mathcal{B}} \pi(B)^{\mathrm{Zar}} \cap \pi(U),$$

as required.

The converse direction is very similar to the proof of Proposition 7.3. □

In the classical setting, Zilber-Pink implies André-Oort in a completely formal manner. We should confirm, then, that this is also the case for our various versions of Zilber-Pink. We will write AHMAO to represent the ahm André-Oort statement (Theorem 6.5). The arguments in the upcoming propositions follow standard such arguments quite closely; see for instance [91, Proposition 2].

Proposition 7.6. *ZPL$\mathbb{H}$ implies AHMAO.*

Proof. Let $V \subseteq \mathbb{C}^{2n}$ be a variety, which we can assume is absolutely irreducible. We will proceed by induction on the dimension V; the base case is trivial.

Let $\pi(U)^{\mathrm{Zar}}$ be the smallest π-special variety containing V. Note that

$$\pi^{-1}(\pi(U)^{\mathrm{Zar}}) = \mathrm{SL}_2(\mathbb{Z})^n \cdot U$$

(a consequence of the definition of the modular polynomials Φ_N, Ψ_N), so that

$$\pi^{-1}(V) \subseteq \mathrm{SL}_2(\mathbb{Z})^n \cdot U.$$

Hence $\dim_{\mathbb{R}} \pi^{-1}(V) \leq \dim_{\mathbb{R}} U$. We can assume without loss of generality that

$$\dim_{\mathbb{R}} \pi^{-1}(V) < \dim_{\mathbb{R}} U,$$

since otherwise $\pi^{-1}(V) = \mathrm{SL}_2(\mathbb{Z})^n \cdot U$, whence $V = \pi(U)^{\mathrm{Zar}}$ and we conclude immediately.

Now let $\pi(G)^{\mathrm{Zar}} \subseteq V$ be maximal π-special. Then $G \subseteq \pi^{-1}(V)$ and so G is a real-analytically irreducible component of $G \cap \pi^{-1}(V)$. We clearly have

$$\dim_{\mathbb{R}} G > \dim_{\mathbb{R}} \pi^{-1}(V) + \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} U,$$

whence G is left-$\mathbb{H}$-atypical (for V in U). Therefore $G \subseteq B$ for some maximal left-$\mathbb{H}$-atypical component B. By hypothesis, $B = \gamma B_i$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ and one of finitely many left-$\mathbb{H}$-atypical components B_i.

By definition, B_i is a component of $G_i \cap \pi^{-1}(V)$ for some $\mathbb{H}$-special G_i with

$$\dim_{\mathbb{R}} B_i > \dim_{\mathbb{R}} \pi^{-1}(V) + \dim_{\mathbb{R}} G_i - \dim_{\mathbb{R}} U. \tag{7.1}$$

Note: if $\pi(G_i)^{\mathrm{Zar}} \supseteq V$, then by the assumption on U we have $\gamma U \subseteq G_i$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$, whence $\dim_{\mathbb{R}} B_i = \dim_{\mathbb{R}} \pi^{-1}(V)$, contradicting (7.1). Therefore $\pi(G_i)^{\mathrm{Zar}} \cap V$ has lower dimension than V.

So every maximal π-special subvariety of V is a maximal π-special subvariety of $\pi(G_i)^{\mathrm{Zar}} \cap V$ for some i. By induction, there are only finitely many such subvarieties for each (of the finitely many) i, so we're done. □

The same argument will not go through for ZPR$\mathbb{H}$, since it is possible that *every* component of $\pi^{-1}(V)$ is right-$\mathbb{H}$-atypical for V in *every* possible U. So the above argument stalls, since every $\mathbb{H}$-special subvariety of $\pi^{-1}(V)$ is simply contained in a component of $\pi^{-1}(V)$ which is maximal right-$\mathbb{H}$-atypical and we cannot force the dimension of the ambient variety to drop.

We do have the following proposition, however, the proof of which is very similar to the proof of 7.6.

Proposition 7.7. *ZPR' implies AHMAO.*

Proof. Let $V \subseteq \mathbb{C}^{2n}$ be a variety, which we can assume is absolutely irreducible. We will proceed by induction on the dimension V; the base case is trivial.

Let $\pi(U)^{\mathrm{Zar}}$ be the smallest π-special variety containing V. We can clearly assume that $V \neq \pi(U)^{\mathrm{Zar}}$, whence $\dim V < \dim \pi(U)^{\mathrm{Zar}}$.

Now let $W \subseteq V$ be maximal π-special. We clearly have

$$\dim W > \dim V + \dim W - \dim \pi(U)^{\mathrm{Zar}},$$

whence W is right-atypical (for V in U). Therefore $W \subseteq A$ for some maximal right-atypical component A. By hypothesis, $A \cap \pi(U) = A_i \cap \pi(U)$ for one of finitely many right-atypical components A_i. It follows that W is a maximal π-special subvariety of A_i, since $W = \pi(G)^{\text{Zar}}$ for some $\mathbb{H}$-special $G \subseteq U$, so that $\pi(G) \subseteq A \cap \pi(U) = A_i \cap \pi(U)$, whence

$$W = \pi(G)^{\text{Zar}} \subseteq (A_i \cap \pi(U))^{\text{Zar}} \subseteq A_i \cap \pi(U)^{\text{Zar}}.$$

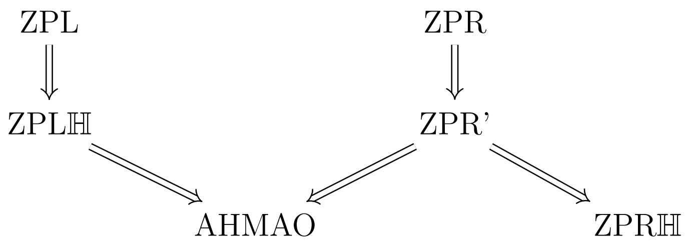
By definition, A_i is a component of $W_i \cap V$ for some π-special W_i with

$$\dim A_i > \dim V + \dim W_i - \dim \pi(U)^{\text{Zar}}. \tag{7.2}$$

If $V \subseteq W_i$, then $A_i = V$ and by the assumption on U we have $\pi(U)^{\text{Zar}} \subseteq W_i$, contradicting (7.2). Hence $\dim V \cap W_i < \dim V$.

So every maximal π-special subvariety of V is a maximal π-special subvariety of $V \cap W_i$ for some i. By induction, there are only finitely many such subvarieties for each i, so we're done. □

To summarise all of this, we have the following diagram of conjectures (and one theorem!):



7.1.1 Schanuel-type Conjectures

In the classical setting, an important feature of Zilber-Pink is that it serves to produce uniform versions of Schanuel-type statements. We will take the time here to briefly consider the appropriate analogue in our ahm setting.

The classical Schanuel conjecture [47, p. 30-31], of course, is a statement about the exponentials of complex numbers.

Conjecture 7.8 (Schanuel's Conjecture)**.** *Let $z_1, \dots, z_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots e^{z_n}) \geq n.$$

This seems quite out of reach at the moment, but has recently been shown to be intricately connected to several areas of model theory, prompting more research. (Macintyre and Wilkie, for instance, show that Schanuel's conjecture would imply the decidability of the real exponential field, while Zilber's theory of pseudo-exponentiation

[92] gives a strong connection between the model theory of real exponentiation and Schanuel's Conjecture; this theory has since been expanded by Kirby and Zilber [42].)

There is also a modular version of Schanuel's conjecture, which translates directly from the exponential conjecture; it states that for $z_1, \dots z_n \in \mathbb{H}$, non-quadratic and lying in distinct $\mathrm{GL}_2^+(\mathbb{Q})$-orbits, we have

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}(z_1, \dots, z_n, j(z_1), \dots, j(z_n)) \geq n.$$

One can make this more general, though; if some of the z_i are quadratic or lie in the same $\mathrm{GL}_2^+(\mathbb{Q})$-orbit, then $z = (z_1, \dots, z_n) \in \mathbb{H}^n$ lies in some proper $\mathbb{H}$-special variety $G \subsetneq \mathbb{H}^n$. By the properties of j, one then expects that

$$\operatorname{tr.deg.}_{\mathbb{Q}} \mathbb{Q}(z, j(z)) \geq \dim G,$$

provided that G is chosen minimal.

We'd like an ahm version of the modular Schanuel conjecture; in this case, since we are working with two algebraically independent maps, j and χ^*, the appropriate Schanuel statement is:

Conjecture 7.9 (AHM Schanuel Conjecture, "AHMSC")**.** *Let $z = (z_1, \dots, z_n) \in \mathbb{H}^n$. Let $G \subseteq \mathbb{H}^n$ be the smallest $\mathbb{H}$-special set containing z. Then*

$$\operatorname{tr.deg.}_{\mathbb{Q}}(z, j(z), \chi^*(z)) \geq 2 \dim G.$$

Just as for classical Schanuel statements, we might hope that this has some model-theoretic connections. Here, we would like to discuss the connection to Zilber-Pink. In the classical setting, the Zilber-Pink conjecture serves as a bridge between the modular Schanuel conjecture and a uniform version thereof. This is an idea of Zilber, who in [91] showed that his Conjecture on Intersections with Tori, "CIT" (which is the exponential version of Zilber-Pink), serves as a bridge between the exponential Schanuel Conjecture and a uniform version thereof. (Kirby and Zilber [41] also showed that the (exponential) Schanuel conjecture over the reals implies the uniform conjecture over the reals without appealing to CIT.)

In our ahm setting, the appropriate uniform statement is the following.

Conjecture 7.10 (Uniform AHM Schanuel Conjecture, "UAHMSC")**.** *Let $V \subseteq \mathbb{C}^n \times \mathbb{C}^{2n}$ be a variety defined over $\mathbb{Q}$, with $\dim V < 2n$. Then there is an $\mathrm{SL}_2(\mathbb{Z})$-finite set $\mu(V)$ of proper $\mathbb{H}$-special sets such that whenever $z \in \mathbb{H}^n$ and*

$$(z, \pi(z)) \in V,$$

we have $z \in G \in \mu(V)$ for some G.

As in the classical setting, some of our ahm Zilber-Pink statements can bridge the gap between AHMSC and UAHMSC. This is proven exactly as for the original theorem of Zilber [91]; we also recommend the notes of Pila [64], which give a thorough survey of these topics.

Proposition 7.11. *ZPR' and AHMSC together imply UAHMSC.*

Proof. Let V be as in the setup of UAHMSC and assume without loss of generality that V is irreducible. Let W be the projection of V onto $\mathbb{C}^{2n}$ (the 2nd and 3rd copies of $\mathbb{C}^n$). Let d be the dimension of the generic fibre of this projection and V' the proper subvariety of V where the fibre dimension exceeds d.

If $(z, \pi(z)) \in V \setminus V'$, then let G be the smallest $\mathbb{H}$-special set containing z. Thanks to AHMSC, we know

$$2 \dim G \leq \operatorname{tr.deg.}_{\mathbb{Q}}(z, \pi(z)).$$

Let $S = \pi(G)^{\mathrm{Zar}}$ and note that $\dim S = 2 \dim G$. Take a component A of $W \cap S$ containing $\pi(z)$, and let $P \subseteq W$ be its preimage under the projection from V onto W. Note that $\dim P = d + \dim A$ and $(z, \pi(z)) \in P$, so that

$$\begin{aligned} 2 \dim G \leq \operatorname{tr.deg.}_{\mathbb{Q}}(z, \pi(z)) &\leq d + \dim A \\ &= \dim V - \dim W + \dim A < 2n - \dim W + \dim A. \end{aligned}$$

Hence

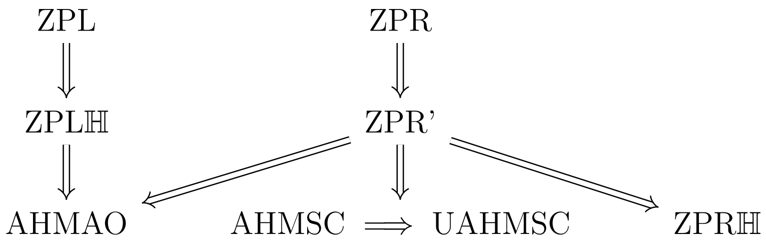
$$\dim A > \dim S + \dim W - \dim \pi(\mathbb{H}^n)^{\mathrm{Zar}},$$

so A is right-atypical for W in $\mathbb{H}^n$.

From ZPR', we know that $\mathrm{Atyp}_R(W) \cap \pi(\mathbb{H}^n)$ is a finite union, so that $\pi(z) \in A \cap \pi(\mathbb{H}^n)$ is contained in one of finitely many π-special varieties S_i. We can ensure the S_i are proper: if W is right-atypical for itself in $\mathbb{H}^n$, then W must by definition be contained in some proper π-special subvariety of $\mathbb{C}^{2n}$.

On the other hand, if $(z, \pi(z)) \in V'$, then we can conclude by induction on the dimension of V, since $\dim V' < \dim V$. □

The other versions of Zilber-Pink, however, do not appear to yield a uniformisation of AHMSC. So we can extend our earlier diagram:



As the only conjectures that imply both the André-Oort statement and the appropriate uniformisation of the Schanuel conjecture, it seems likely that either ZPR or ZPR' is the "correct" version of Zilber-Pink for π. This will likely come as no surprise to the knowledgeable reader; these are the versions of Zilber-Pink which most closely mimic classical Zilber-Pink statements.

7.1.2 Ax-Schanuel

It is conceivable that some very special cases of one or more of these Zilber-Pink conjectures might be approachable via o-minimal methods. Typically, in order to attempt this, one needs a suitable Ax-Schanuel-type result.

The terminology "Ax-Schanuel" stems from a theorem of Ax, who showed in [6] that for formal power series $f_1, \ldots, f_n \in \mathbb{C}[[t]]$,

$$\operatorname{tr.deg.}_{\mathbb{C}(t)} \mathbb{C}(t, f_1, \ldots, f_n, \exp(f_1), \ldots \exp(f_n)) \geq n$$

unless the f_i are linearly dependent. (There is also a version of this result phrased in the setting of differential fields, but this "power series version" more easily translates into our setting.) This is easily seen to be a power series analogue of Schanuel's conjecture on the transcendence of exponentials. The term Ax-Schanuel has come to refer to any such functional analogue of a Schanuel-type conjecture, in a variety of settings. (Occasionally in the literature the term "hyperbolic Ax-Schanuel" is used to distinguish statements about Shimura varieties from other Ax-Schanuel-type results.) The terminology "Ax-Lindemann" has a similar derivation, being a functional analogue of the part of Schanuel's conjecture corresponding to Lindemann's theorem; typically, Ax-Schanuel statements can be viewed as direct generalisations of suitable Ax-Lindemann results.

In the ahm setting, there are two distinct variants of Ax-Schanuel.

Conjecture 7.12 (AHM Ax-Schanuel, 1-sorted version)**.** *Let $U \subseteq \mathbb{H}^n$ be $\mathbb{H}$-special and let $\Gamma \subseteq U \times \pi(U)^{Zar}$ be the graph of π. Let V be a complex algebraic component of $U \times \pi(U)^{Zar}$ and let A be a real-analytically irreducible component of $V \cap \Gamma$. If the projection of A onto the U-factor is not contained in a proper weakly $\mathbb{H}$-special subvariety of U, then*

$$\dim_{\mathbb{R}} A \leq \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} \pi(U)^{Zar}.$$

This version of Ax-Schanuel comes from the following statement about transcendence degrees, which is more obviously analogous to AHMSC.

We suppose that $x_1, y_1, \ldots, x_n, y_n$ are real power series in variables $t_1, \ldots, t_m$, with $y_j > 0$ near some point P. If the functions $\tau_j = x_j + iy_j$ are nonconstant and not related to each other by elements of $\mathrm{GL}_2^+(\mathbb{Q})$, we expect

$$2\,\mathrm{tr.\,deg.}_{\mathbb{C}}\,\mathbb{C}(\tau_1, \ldots, \tau_n, \pi(\tau_1), \ldots, \pi(\tau_n)) \geq 4n + \mathrm{rank}\left(\frac{\partial x_i}{\partial t_j}\right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}. \tag{7.3}$$

Conjecture 7.12 is simply a translation of this transcendence-degree statement, taking into account the level structure.

The second version of Ax-Schanuel comes from a weaker version of (7.3), requiring instead that

$$2\,\mathrm{tr.\,deg.}_{\mathbb{C}}\,\mathbb{C}(\tau_1, \ldots, \tau_n) + 2\,\mathrm{tr.\,deg}_{\mathbb{C}}\,\mathbb{C}(\pi(\tau_1), \ldots, \pi(\tau_n)) \geq 4n + \mathrm{rank}\left(\frac{\partial x_i}{\partial t_j}\right)_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}.$$

In the language of Conjecture 7.12, this corresponds to the following.

Conjecture 7.13 (AHM Ax-Schanuel, 2-sorted version)**.** *Let $U \subseteq \mathbb{H}^n$ be $\mathbb{H}$-special, $W \subseteq U$ a complex algebraic component, and $V \subseteq \pi(U)^{Zar}$ an algebraic variety. Let A be a real-analytically irreducible component of $W \cap \pi^{-1}(V)$. If A is not contained in any proper weakly $\mathbb{H}$-special subvariety of U, then*

$$\dim_{\mathbb{R}} A \leq \dim_{\mathbb{R}} V + \dim_{\mathbb{R}} W - \dim_{\mathbb{R}} \pi(U)^{Zar}.$$

These versions of Ax-Schanuel are interesting in their own right, but also tend to be crucial in approaching Zilber-Pink statements. In the usual approach to Zilber-Pink, only "2-sorted Ax-Schanuel", Conjecture 7.13 is needed, but unfortunately even that weaker version is rather strong. We might hope to be able to approach it via the derivatives of j, using much the same sort of argument we used to get Ax-Lindemann for π, but the potentially bad behaviour of the real-analytic sets involved makes it difficult to get good control on the various dimensions.

Proposition 7.14. *Assume Conjecture 7.13. Then χ^* is finite-to-one on $\mathbb{F}$.*

Proof. In 7.13, let $W = U = \mathbb{H}$, and $V \subseteq \mathbb{C}^2$ defined by

$$V = \{(z, x_0) : z \in \mathbb{C}\}$$

for some complex x_0. According to 7.13, any component of $\pi^{-1}(V)$ has dimension at most

$$\dim_{\mathbb{R}} V + \dim_{\mathbb{R}} W - \dim_{\mathbb{R}} \mathbb{C}^2 = 2 + 2 - 4 = 0,$$

whence by the definability of χ^* on $\mathbb{F}$, the preimage $\pi^{-1}(V) \cap \mathbb{F}$ must be finite. In particular, there are only finitely many $\tau \in \mathbb{F}$ such that $\chi^*(\tau) = x_0$. □

Since we have been unable to prove the conclusion of 7.14, even 2-sorted Ax-Schanuel seems to be out of reach for now. To conclude this section we note one other implication.

Proposition 7.15. *Conjecture 7.13 implies the Ax-Lindemann statement for π, ie. it implies Theorem 4.12.*

Proof. Let $V \subseteq \mathbb{C}^{2n}$ be an algebraic variety, and $\pi(U)^{\text{Zar}}$ the smallest π-special variety containing V. We may assume $V \neq \pi(U)^{\text{Zar}}$ since otherwise any subset of $\pi^{-1}(V)$ is contained in an $\mathbb{H}$-special variety of the form $\gamma \cdot U$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$.

Let $S \subseteq \pi^{-1}(V)$ be a real semialgebraic arc. We may assume by induction that S is nonconstant on every coordinate. Being a real semialgebraic arc, S must be contained in some complex algebraic component $W \subseteq U$ with $\dim_{\mathbb{C}} W = 1$. Let A be a component of $W \cap \pi^{-1}(V)$ containing S, and assume for a contradiction that A is not contained in any proper weakly $\mathbb{H}$-special subvariety of U. We have by 7.13 that

$$\dim_{\mathbb{R}} S \leq \dim_{\mathbb{R}} A \leq \dim_{\mathbb{R}} V + \dim_{\mathbb{R}} W - \dim_{\mathbb{R}} \pi(U)^{\text{Zar}}.$$

Since $\dim_{\mathbb{R}} W = 2$ and the real codimension of V in $\pi(U)^{\text{Zar}}$ is at least 2, we get

$$\dim_{\mathbb{R}} S \leq 0,$$

a contradiction.

So A (hence S) is contained in a proper weakly $\mathbb{H}$-special subvariety $G \subsetneq U$. Since S has no constant coordinates, G is in fact $\mathbb{H}$-special, and we have

$$S \subseteq \pi^{-1}(V \cap \pi(G)^{\text{Zar}}).$$

By induction, we may assume that Ax-Lindemann holds for $V \cap \pi(G)^{\text{Zar}}$, that is, there is a weakly $\mathbb{H}$-special H with

$$S \subseteq H \subseteq \pi^{-1}(V \cap \pi(G)^{\text{Zar}}) \subseteq \pi^{-1}(V),$$

as required. □

7.2 Model-Theoretic Questions

In their paper [22], Daw and Harris worked on a fairly general question regarding the categoricity of Shimura varieties. They consider Shimura varieties as 2-sorted structures (in the sense of first-order logic):

$$\langle D, X, p \rangle,$$

where X is the Shimura variety in question, D is the underlying hermitian symmetric domain and $p : D \to X$ is the uniformising map.

The domain D is given the following structure:

$$\langle D, \{f_g : g \in G^{\mathrm{ad}}(\mathbb{Q})\} \rangle.$$

That is, we have a function symbol f_g for each element of $G^{\mathrm{ad}}(\mathbb{Q})^+$ (the analogue of $\mathrm{GL}_2^+(\mathbb{Q})$ in the general Shimura setting), representing the action of g on D.

The variety structure X is:

$$\langle X, \{P_V : V \in S_{E^{\mathrm{ab}}(\Sigma)}\} \rangle,$$

where each P_V is a predicate symbol representing a Zariski closed subset of X^n (for any n). Specifically, the set $S_{E^{\mathrm{ab}}(\Sigma)}$ is the set of all Zariski closed subsets of any X^n defined over $E^{\mathrm{ab}}(\Sigma)$. Here E is the so-called *reflex field* of X, E^{ab} is its maximal abelian extension, and Σ is the set of coordinates of special points of X.

The object of Daw and Harris's investigation is to investigate the categoricity (or otherwise) of this structure. More precisely, they give a certain $\mathcal{L}_{\omega_1,\omega}$-sentence

$$\mathrm{SF} = \forall x \forall y \in D \left(p(x) = p(y) \implies \bigvee_{\gamma \in \Gamma} x = \gamma y \right)$$

saying that p is one-to-one up to the action of $\Gamma \subseteq G^{\mathrm{ad}}(\mathbb{Q})^+$ (the analogue of $\mathrm{SL}_2(\mathbb{Z})$ in this setting). They then give necessary and sufficient conditions for $\mathrm{Th}\langle D, X, p\rangle \cup \{\mathrm{SF}\}$ to be *categorical*, ie. for it to have a unique model, up to isomorphism, in each infinite cardinality. (By $\mathrm{Th}\langle D, X, p\rangle$ we mean the first-order theory of the structure $\langle D, X, p\rangle$ described above.)

They also prove that, in the case where $X = \mathbb{C}$ and $p = j$, the structure satisfies their sufficient conditions, so that

$$\mathrm{Th}\langle \mathbb{H}, \mathbb{C}, j\rangle \cup \{\mathrm{SF}\}$$

has a unique model in each infinite cardinality. Restricting to this simple case, it is reasonable to ask whether similar arguments can be used to get meaningful model-theoretic information about the theory of the structure

$$\langle \mathbb{H}, \mathbb{C}, j, \chi^* \rangle,$$

in some similar language.

For this discussion we will choose to give $\langle \mathbb{H}, \mathbb{C}, j, \chi^* \rangle$ almost the same language as Daw and Harris use for $\langle \mathbb{H}, \mathbb{C}, j \rangle$. We will represent j and χ^* by function symbols p and q, and give $\mathbb{H}$ exactly the same language as before. We give $\mathbb{C}$ almost the same language as before, with one modification: to have any hope that this structure will have quantifier elimination, we add a binary predicate Ran to the language with the intent that it should represent the range of $\pi = (j, \chi^*)$, that is, $\mathrm{Ran}(x, y) \iff (x, y) \in (p, q)(\mathbb{H})$. The field $E^{\mathrm{ab}}(\Sigma)$, in this case, is

$$E = E^{\mathrm{ab}}(\Sigma) = \mathbb{Q}^{\mathrm{ab}}(j(\tau) : \tau \text{ quadratic}),$$

with no need to add the special $\chi^*(\tau)$ since $\mathbb{Q}(\chi^*(\tau)) \subseteq \mathbb{Q}(j(\tau))$.

7.2.1 Axiomatisation, Homogeneity and Completeness

In [22], an axiomatisation is given for $\mathrm{Th}\langle \mathbb{H}, \mathbb{C}, j \rangle$. For each tuple $\overline{g} = (g_1, \ldots, g_k) \in \mathrm{GL}_2^+(\mathbb{Q})^k$, let $Z_{\overline{g}}$ be the j-special subvariety corresponding to $\overline{g}$, that is

$$Z_{\overline{g}} = \{(j(g_1\tau), \ldots, j(g_k\tau)) : \tau \in \mathbb{H}\}.$$

Consider $\mathcal{L}$-formulae

$$\mathrm{MOD}^1_{\overline{g}} = \forall x \in \mathbb{H} \quad (p(g_1x), \ldots, p(g_kx)) \in Z_{\overline{g}},$$

$$\mathrm{MOD}^2_{\overline{g}} = \forall z \in Z_{\overline{g}} \quad \exists x \in \mathbb{H} \quad (p(g_1x), \ldots, p(g_kx)) = z.$$

For quadratic $x \in \mathbb{H}$, let $g_x \in \mathrm{GL}_2^+(\mathbb{Q})$ fix x (and only x) and for each such x consider the $\mathcal{L}$-formula

$$\mathrm{SP}_x = \forall y \in \mathbb{H} \quad (g_xy = y \implies p(y) = j(x)),$$

which is an $\mathcal{L}$-formula since $j(x) \in E$, whence $\{j(x)\}$ is a Zariski-closed subset of $\mathbb{C}$ over E.

Then $\mathrm{Th}\langle \mathbb{H}, \mathbb{C}, j \rangle$ is axiomatised by:

- $\mathrm{Th}\langle \mathbb{H}, \mathrm{GL}_2^+(\mathbb{Q}) \rangle$ (as a set with a group action).

- $\mathrm{Th}\langle\mathbb{C}, S_E\rangle$ (interpreted, if we prefer, in the language of rings with constants for elements of E).
- The set of formulae $\left\{\mathrm{MOD}^1_{\overline{g}}, \mathrm{MOD}^2_{\overline{g}} : \overline{g} \text{ is a tuple of elements of } \mathrm{GL}_2^+(\mathbb{Q})\right\}$.
- The set of formulae $\{\mathrm{SP}_x : x \in \mathbb{H} \text{ quadratic}\}$.

Now we want to add χ^* to this picture, and investigate $\mathrm{Th}\langle\mathbb{H}, \mathbb{C}, j, \chi^*\rangle$ (in the language described above). For each tuple $\overline{g} \in \mathrm{GL}_2^+(\mathbb{Q})^k$, let $X_{\overline{g}} \subseteq \mathbb{C}^{2k}$ be the Zariski closure, over E, of

$$\{(\pi(g_1\tau), \ldots, \pi(g_k\tau)) : \tau \in \mathbb{H}\}.$$

It is clear from the construction of the polynomials Φ_N and Ψ_N defining π-special subvarieties that

$$X_{\overline{g}} \cap \pi(\mathbb{H}^k) = \{(\pi(g_1\tau), \ldots, \pi(g_k\tau)) : \tau \in \mathbb{H}\}.$$

For much of the rest of this discussion, we will need something a little stronger. Let us assume the following:

Conjecture 7.16. *For any $\overline{g}$, we have*

$$X_{\overline{g}} \cap \left(\pi(\mathbb{H}) \times \mathbb{C}^{2(k-1)}\right) = \{(\pi(g_1\tau), \ldots, \pi(g_k\tau)) : \tau \in \mathbb{H}\}.$$

Then $\mathrm{Th}\langle\mathbb{H}, \mathbb{C}, j, \chi^*\rangle$ clearly contains the following:

- The sentences

  $$\mathrm{RAN} = \forall x \forall y \quad (\mathrm{Ran}(x, y) \iff \exists z \quad (x, y) = (p(z), q(z)))$$

  and

  $$\mathrm{SURJ} = \forall z \in \mathbb{C} \quad \exists x \in \mathbb{H} \quad q(x) = z.$$

- For each tuple $\overline{g}$, the sentences:

  $$\mathrm{MOD}^3_{\overline{g}} = \forall x \in \mathbb{H} \quad (p(g_1x), q(g_1x), \ldots, p(g_kx), q(g_kx)) \in X_{\overline{g}}.$$

  and

  $$\mathrm{MOD}^4_{\overline{g}} = \forall z \in X_{\overline{g}} \quad \left[\mathrm{Ran}(z_1, z_2) \implies \exists y \in \mathbb{H} \quad (p(g_1y), q(g_1y), \ldots, p(g_ky), q(g_ky)) = z\right].$$

  (This last is simply a restatement of Conjecture 7.16.)

- For each k-tuple $\overline{g}$ and each $V \subseteq \mathbb{C}^{2k}$ defined over E, the sentence

$$\mathrm{MOD}^5_{\overline{g},V} = (\forall x \in \mathbb{H} \quad (p(g_1x), q(g_1x), \ldots, p(g_kx), q(g_kx)) \in V) \implies X_{\overline{g}} \subseteq V.$$

- For each quadratic $x \in \mathbb{H}$, fixed by g_x as before, the sentence:

$$\mathrm{SP}^2_x = \forall y \in \mathbb{H} \quad (g_xy = y \implies q(y) = \chi^*(x)).$$

So let

$$\begin{aligned} T_\pi = \mathrm{Th}\langle \mathbb{H}, \mathrm{GL}_2^+(\mathbb{Q})\rangle \cup \mathrm{Th}\langle \mathbb{C}, S_E\rangle \\ &\cup \{\mathrm{MOD}^i_{\overline{g}} : 1 \leq i \leq 5, \overline{g} \text{ a tuple of elements in } \mathrm{GL}_2^+(\mathbb{Q})\} \\ &\cup \{\mathrm{SP}_x, \mathrm{SP}^2_x : x \in \mathbb{H} \text{ quadratic}\} \\ &\cup \{SURJ, RAN\}. \end{aligned}$$

We'd like to investigate the properties of this theory; in particular, we would like to know whether it is complete, and therefore whether it is a complete axiomatisation of $\mathrm{Th}\langle \mathbb{H}, \mathbb{C}, j, \chi^*\rangle$.

The approach used by Daw and Harris to approach these problems is a fairly standard application of the back-and-forth method. When χ^* is added to the picture, some significant difficulties arise.

We start with two $\mathcal{L}$-structures

$$S = \langle H, C, p, q\rangle, \quad S' = \langle H', C', p', q'\rangle,$$

models of T_π, and we have a partial isomorphism ρ between finitely generated substructures $U \subseteq S$ and $U' \subseteq S'$. So $U \cap H$ consists of the $\mathrm{GL}_2^+(\mathbb{Q})$-orbits of finitely many x_i, and $U \cap C$ is a field generated over E by the $p(x_i)$, $q(x_i)$ and finitely many other points. Then

$$\rho|_{U\cap H} : U \cap H \to U' \cap H'$$

is a $\mathrm{GL}_2^+(\mathbb{Q})$-equivariant injection, and

$$\rho|_{U\cap C} : U \cap C \to U' \cap C'$$

is a field embedding fixing E pointwise. Moreover, ρ preserves p, q and Ran.

Given $x \in H \setminus U$ (or in $C \setminus U$), we would like to extend ρ to include x in its domain. We need to do so consistently, meaning in particular that we need to find $x' \in H'$ (respectively C') which satisfies

$$\rho(\mathrm{qftp}(x/U)).$$

We'll start with the case when $x \in H$. There are various countably infinite collections $z_i \in H$, indexed by $\mathbb{Z}$, with $z_0 = x$, such that

$$p(z_i) = q(z_{i+1})$$

for all $i \in \mathbb{Z}$. For any such sequence z_i, if we let $I \subseteq \mathbb{Z}$ be the maximal interval in $\mathbb{Z}$ containing 0 such that $p(z_i) \in U$ for all $i \in I \setminus \{0\}$, then $\mathrm{qftp}(x/U)$ can see all of the z_i, $i \in I$. We therefore need to choose $x' \in H'$ consistently with this, that is, we need to find x' such that $p(x') = \rho(q(z_1))$ (if $1 \in I$) and $q(x') = \rho(p(z_{-1}))$ (if $-1 \in I$).

The case $x \in C$ is similar; we just need to start by finding z_{-1} and z_0 such that $p(z_{-1}) = x$ and $q(z_0) = x$, then choose a sequence z_i, exactly as before.

It is not at all clear that we can always choose x' to satisfy these requirements. In order to do so we would need the following conjecture, which applies to two models S, S' as above.

Conjecture 7.17 (One-Step Extension, "$\mathrm{OSE}_{S,S'}$"). *Let ρ a partial isomorphism between finitely generated substructures $U \subseteq S$ and $U' \subseteq S'$. Let $x \in C$. Then there is $x' \in C'$ satisfying $\rho(\mathrm{qftp}_{\mathbb{C}}(x/U))$ such that:*

- *For all $x_{-1} \in U \cap C$ with $\mathrm{Ran}(x_{-1}, x)$, we have $\mathrm{Ran}(\rho(x_{-1}), x')$.*
- *For all $x_1 \in U \cap C$ with $\mathrm{Ran}(x, x_1)$, we have $\mathrm{Ran}(x', \rho(x_1))$.*

This is quite a strong conjecture, telling us a lot about the Galois behaviour of point $(p(z), q(z))$ in arbitrary models. Nonetheless, it seems likely to be necessary in order for T_π to have nice model-theoretic properties.

Proposition 7.18. *Let S, S' be $\aleph_0$-saturated models of T_π. Then the set of finite partial isomorphisms from S to S' satisfies the back-and-forth property if and only if $\mathrm{OSE}_{S,S'}$ holds.*

Proof. If $\mathrm{OSE}_{S,S'}$ fails, then its failure immediately gives us a finite partial isomorphism between S and S' which cannot be extended.

So suppose $\mathrm{OSE}_{S,S'}$ holds. If we wish to extend ρ to include $x \in C \setminus U$, then it follows immediately from $\mathrm{OSE}_{S,S'}$ that we can find x' to satisfy

$$\rho(\mathrm{qftp}_{\mathcal{L}\setminus \mathrm{GL}_2^+(\mathbb{Q})}(x/U)).$$

If $x \in H \setminus U$, then $p(x)$, $q(x) \in C$. Applying $\mathrm{OSE}_{S,S'}$ twice, we can again find $x' \in H'$ such that $(p'(x'), q'(x'))$ satisfies

$$\rho(\mathrm{qftp}_{\mathcal{L}\setminus \mathrm{GL}_2^+(\mathbb{Q})}(p(x), q(x)/U)).$$

It remains to ensure that we can do this consistently with the $\mathrm{GL}_2^+(\mathbb{Q})$-action. This goes through almost exactly as in Propositions 3.2 to 3.5 of [22], using the axioms $\mathrm{MOD}_{\overline{g}}^i$, SP_x and SP_x^2 together with the fact that $X_{\overline{g}}$ is defined over E. □

Using standard model-theoretic facts, we can then deduce the following.

Proposition 7.19. *Suppose that there exist $\aleph_0$-saturated models S, S' of T_π for which $\mathrm{OSE}_{S,S'}$ fails. Then T_π does not admit quantifier elimination.*

Proof. See for instance [68, Proposition 2.29]. □

Proposition 7.20. *If $\mathrm{OSE}_{S,S'}$ holds for all $\aleph_0$-saturated models S, S' of T_π, then T_π admits quantifier elimination and is complete and model-complete; in particular, $T_\pi \models \mathrm{Th}\langle \mathbb{H}, \mathbb{C}, j, \chi^* \rangle$.*

Proof. See for instance [68, Proposition 2.29 and 2.30, Remark 2.38]. □

For the remainder of our discussion, we will assume that $\mathrm{OSE}_{S,S'}$ does indeed hold for $\aleph_0$-saturated models, so that T_π is complete, model-complete and admits quantifier elimination.

7.2.2 Categoricity and the $\mathcal{L}_{\omega_1,\omega}$-theory

We want to have some suitable analogue of the $\mathcal{L}_{\omega_1,\omega}$-sentence SF, which specified in the j setting that j was one-to-one up to the action of $\mathrm{SL}_2(\mathbb{Z})$. Here lies an issue: as discussed earlier, it is unknown whether χ^* is even finite-to-one up to $\mathrm{SL}_2(\mathbb{Z})$ (it is *not* one-to-one). There is no way around this; for the purposes of discussion let us assume:

Conjecture 7.21. *The function χ^* is finite-to-one up to the action of $\mathrm{SL}_2(\mathbb{Z})$.*

Then by definability it is uniformly finite-to-one; that is, there is some N such that

$$|(\chi^*)^{-1}(z) \cap \mathbb{F}| \leq N$$

for all $z \in \mathbb{C}$. It follows that $\langle \mathbb{H}, \mathbb{C}, j, \chi^* \rangle$ is a model of the following $\mathcal{L}_{\omega_1,\omega}$-sentence:

$$\mathrm{SF}^2 = \forall x_1 \ldots \forall x_{N+1} \in \mathbb{H} \left(q(x_1) = \cdots = q(x_{N+1}) \implies \bigvee_{i \neq j} \bigvee_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} x_i = \gamma x_j \right).$$

There are also $\mathrm{L}_{\omega_1,\omega}$-sentences $\mathrm{SP\text{-}SIZE}_x$ (one for each quadratic x), expressing the exact cardinality of $|q^{-1}(\chi^*(x)) \cap \mathbb{F}|$, and $\mathrm{L}_{\omega_1,\omega}$-sentences $SIZE_n$, $n \leq N$, specifying the (possibly infinite) number of distinct z such that $n = |q^{-1}(z) \cap \mathbb{F}|$.

Let

$$T_\pi^\infty = T_\pi \cup \{\mathrm{SF}, \mathrm{SF}^2\} \cup \{\text{SP-SIZE}_x : x \in \mathbb{H} \text{ quadratic}\} \cup \{\mathrm{SIZE}_n : n \leq N\}.$$

Before we make any statements about T_π^∞, we should first emphasise that all of this is very much dependent on Conjectures 7.16 and 7.21, without which T_π^∞ seems not to have any real meaning.

Theorem 7.22. *A model M of T_π^∞ is $\aleph_0$-homogeneous over the empty set and over countable submodels if and only if $\mathrm{OSE}_{M,M}$ holds.*

Proof. If $\mathrm{OSE}_{M,M}$ fails, then by definition $\mathrm{OSE}_{M,M}$ cannot be $\aleph_0$-homogeneous.

The converse goes exactly as for Lemmas 4.11 and 4.12 of [22]; the appropriate analogues of Conditions 4.7 and 4.8 from [22] hold as a result of the fact that they hold for j (proven in [22]), since MOD^4 holds for M. □

Conjecture 7.23. *T_π^∞ is κ-categorical for all infinite cardinalities κ. (In particular, it is the complete $\mathcal{L}_{\omega_1,\omega}$-theory of $\langle \mathbb{H}, \mathbb{C}, j, \chi^* \rangle$.)*

We can attempt to prove this (assuming all of the conjectures we've already mentioned) using the theory of quasiminimal classes developed by Kirby and others: [9], [40]. This is the approach used by Daw and Harris in [22]. Thanks to 7.22, the only missing piece of this is to have a closure operator on models of T_π^∞ which induces a suitable pregeometry. The only obvious such closure operator is the following:

For a subset $A \subseteq H \cup C$ in some $\mathcal{L}$-structure $\langle H, C, p, q \rangle$, let

$$A_0^l = A \cap H$$

and

$$A_0^r = \overline{E(A \cap C, p(A_0^l), q(A_0^l))}.$$

Then in general define

$$A_{i+1}^l = p^{-1}(A_i^r) \cup q^{-1}(A_i^r)$$

and

$$A_{i+1}^r = \overline{E(p(A_{i+1}^l), q(A_{i+1}^l))}.$$

Finally define

$$\mathrm{cl}(A) = \bigcup_{i=0}^{\infty} A_i^l \cup A_i^r.$$

This is clearly a pregeometry satisfying all the conditions of [40] (the countable closure property follows from Conjecture 7.21), except possibly condition I.2. Here lies the issue; it is not at all clear that, for arbitrary A, $\mathrm{cl}(A)$ is a model of T_π^∞. The problem is with the $\mathcal{L}_{\omega_1,\omega}$-sentences SIZE_n, which may well fail to hold in such a structure $\mathrm{cl}(A)$.

To proceed with this approach, then, we would need to either come up with some new closure operator which ensures that $\mathrm{SP\text{-}SIZE}_x$ and SIZE_n hold in $\mathrm{cl}(A)$, or prove that this is already true of our existing cl. We will leave this problem, as well as the various conjectures we have made to reach this point, as an open avenue for future work.

7.3 Almost Holomorphic Siegel Forms and Other Nonclassical Modular Functions

Almost holomorphic modular and quasimodular forms/functions are by no means the only nonclassical modular objects of note. There is an ever-growing collection of different kinds of nearly-modular or partially-modular objects which arise in all manner of contexts. Several times in this document we have referred to the excellent paper of Zagier, [89], which surveys a wide variety of nonclassical modular objects as well as the quasimodular and ahm forms which have been our focus here.

One class of objects on which we could focus is the class of Maass forms, which are real analytic forms which are eigenfunctions for the weight-k hyperbolic Laplacian operator

$$(L_k f)(x+iy) = -y^2\left(\frac{\partial^2 f(x+iy)}{\partial x^2} + \frac{\partial^2 f(x+iy)}{\partial y^2}\right) + iky\frac{\partial f(x+iy)}{\partial x},$$

where k is the weight of the Maass form in question. A Maass form must also satisfy a growth condition at the cusp, as usual.

These forms, as well as many other classes such as the Generalised Modular Forms of Knopp and Mason [45], the Modular Graph Functions of Brown [18], and various notions of vector-valued modular form, are all potential avenues of exploration. As a starting point, one would need to know whether any of these, suitably restricted are definable in an o-minimal structure.

Two classes of particular interest are harmonic weak Maass forms and mock modular forms. A harmonic weak Maass form is a weak Maass form ("weak" here meaning that the growth condition at the cusp is relaxed) whose eigenvalue under the Laplacian is $(1-\frac{k}{2})\frac{k}{2}$ (where k is the weight). A harmonic weak Maass form has a "holomorphic part", and the holomorphic part of a harmonic weak Maass form is called a mock modular form.

This duality between the space of harmonic weak Maass forms and the space of mock modular forms is much the same as the duality between ahm and qm forms. This gives us some hope that some of the work done for qm and ahm forms could work for harmonic weak Maass and mock modular forms as well.

Perhaps, rather than looking for entirely new classes of form, it is more natural to look for analogues of ahm and qm functions in more complicated settings, ie. in the context of more complex Shimura varieties. The most natural step up from the setting of modular curves is either to consider Shimura curves or to consider $\mathcal{A}_g$, the moduli space of principally polarised abelian varieties.

For Shimura curves, this problem has not been investigated much; we leave these as an opportunity for further investigation. For $\mathcal{A}_g$, however, we do have some basic knowledge about analogues of ahm functions.

Definition 7.24. For $g \in \mathbb{N}$, let $\mathbb{H}_g$ be the Siegel upper-half space

$$\mathbb{H}_g = \{\tau \in \mathrm{Mat}_g(\mathbb{C}) : \tau^t = \tau, \mathrm{Im}\, \tau \text{ positive definite}\},$$

where τ^t denotes the transpose of τ, here and throughout. $\mathbb{H}_g$ admits an action of the symplectic group

$$\mathrm{Sp}_{2g}(\mathbb{R}) = \left\{ \gamma \in \mathrm{Mat}_{2g}(\mathbb{R}) : \gamma^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \gamma = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\},$$

where I_g is the g by g identity matrix. The action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = (a\tau + b)(c\tau + d)^{-1},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R})$, with a, b, c, $d \in \mathrm{Mat}_g(\mathbb{R})$.

A holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ is called a (classical) Siegel modular form of weight k if

$$f(\gamma\tau) = \det(c\tau + d)^k f(\tau) \tag{7.4}$$

for all $\tau \in \mathbb{H}_g$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z})$.

More generally, if V is a finite-dimensional complex vector space and $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{Aut}(V)$ is a representation, a function $f : \mathbb{H}_g \to V$ is called a vector-valued Siegel modular form of weight ρ if

$$f(\gamma\tau) = \rho(c\tau + d) \cdot f(\tau).$$

A function $f : \mathbb{H}_g \to \mathbb{C}$ which satisfies (7.4) and can be written as a polynomial in the entries of $(\operatorname{Im}\tau)^{-1}$ with holomorphic coefficients, is called a Siegel almost holomorphic modular (ahm) form (of weight k). The *depth* of such an f is its degree as a polynomial in $(\operatorname{Im}\tau)^{-1}$.

Classical Siegel modular forms parametrise the moduli space $\mathcal{A}_g$ of principally polarised abelian varieties, and therefore we can ask André-Oort questions about them. The classical André-Oort statement is known for $\mathcal{A}_g$ thanks to work of Pila and Tsimerman [65], [81]; just as we did for ahm forms on $\mathbb{H}$, we would like to know whether it is possible to formulate André-Oort statements for Siegel ahm functions.

The first step we would need to begin work on this would be to get some understanding of the structure of the space of Siegel ahm forms. A start has already been made on this by Pitale, Saha, Schmidt [1] and Klemm, Poretschkin, Schimannek, Westerholt-Raum [43]. Their results only apply to $\mathbb{H}_2$, so we will restrict to this case for the remainder of the discussion.

Theorem 7.25 (Klemm, Poretschkin, Schimannek, Westerholt-Raum, building on work by Pitale, Saha, Schmidt)**.** *The space $M_2^*(k,d)$ of Siegel ahm forms of weight k and depth d on $\mathbb{H}_2$ is given by*

$$M_2^*(k,d) = \sum_{j=0}^{d} \operatorname{span} R_j \left[M_2 \left((\operatorname{sym}^{\vee 2})^j \det{}^k \right) \right],$$

where $M_2(\rho)$ is the space of vector-valued Siegel modular forms of weight ρ, and R_j is a particular differential operator which, applied to a vector-valued Siegel form of weight $(\operatorname{sym}^{\vee 2})^j \otimes \rho$, yields a finite set of ahm Siegel forms of weight ρ.

In order to get anywhere, then, we need to understand $M_2\left((\operatorname{sym}^{\vee 2})^j \det^k\right)$ fairly well. In some cases, this space is indeed quite well-understood. For small j, various authors such as Aoki [4], Ibukiyama [33] and Satoh [72] have given a description of $M_2\left((\operatorname{sym}^{\vee 2})^j \det^k\right)$ for $j = 2, 4, 6$. Not only do they give generators for this space, but the generators all arise from a certain differential operator acting on classical Siegel modular forms.

Since classical Siegel modular forms of genus 2 are well understood (thanks to work of Igusa [34, 35]), we know the generators of $M_2^*(k,d)$ exactly, for small d. Since the generators all come from derivatives of classical Siegel modular forms, we know that such ahm Siegel forms, suitably restricted, are definable in $\mathbb{R}_{\mathrm{an,exp}}$ (again, only for small d). Moreover, thanks to work of N. Mok, Pila and Tsimerman, who have recently announced a result on Ax-Schanuel with derivatives for general Shimura varieties (an extension of their paper [53]), it is conceivable that we could use a similar strategy to that laid out in Chapter 4 of this thesis to get an Ax-Lindemann statement for ahm Siegel forms of small depth. Work to carry this out is ongoing.

In general, we would like the space of all ahm Siegel forms, $M_2^* = \bigcup_k M_2^*(k)$, to be finitely generated as a graded algebra. This would follow if we knew that all the vector-valued Siegel forms in $M_2\left((\mathrm{sym}^{\vee 2})^j \det^k\right)$ came from derivatives of classical Siegel forms (since the space of derivatives of classical Siegel forms is finitely generated; see [11]). In the absence of such a result, we leave this discussion here.

7.4 Effectivity

The work done towards André-Oort problems in this thesis has relied heavily on the theory of o-minimality. As a consequence of this, all the André-Oort results are ineffective, as they currently stand. The cell decomposition theorem, for instance, is ineffective in general, as is the Pila-Wilkie theorem. Siegel's lower bound for Galois orbits, Theorem 5.8, is also ineffective.

It seems unlikely that one would be able to effectivise the full André-Oort statements for (say) $\pi = (j, \chi^*)$, since there do not exist effective André-Oort statements in full generality even for j alone. It is conceivable, however, that some weak effective André-Oort statements might be attainable.

Consider, for instance, the André-Oort problem for a curve $C \subseteq \mathbb{C}^{2n}$, defined over $\mathbb{Q}$, say. We would like to know, effectively, how many π-special points can live on C. In the j setting, the analogous problem (for a curve in $\mathbb{C}^n$) was originally solved by André (ineffectively, [3]), then effectively by Kühne [46] and independently by Bilu-Masser-Zannier [12]. Their results rely heavily on famous and powerful results giving effective bounds on linear forms in logarithms of algebraic numbers; specifically Baker's theorem [7].

Unfortunately, in our case, these results on linear forms in logarithms cannot be applied, thanks to the presence of π (the transcendental number, not the map!) in the definition of the function χ^*. It seems quite difficult to get around this.

Another potential approach to effectivity would be to use the theory of Pfaffian functions. There are a variety of results providing an effective method of counting solutions to systems of equations in Pfaffian functions, originating with work of Khovanskii [38] and continued by a number of authors including for instance Pila [59], Jones and Thomas [37].

It seems unlikely that either j or χ^*, unrestricted, is a Pfaffian function. It is conceivable, however, that some restrictions of j are Pfaffian.

Proposition 7.26. *Assume that the restrictions of j to the vertical lines $\operatorname{Re}\tau = 0$ and $\operatorname{Re}\tau = \frac{1}{2}$ are both Pfaffian functions.*

Let $C \subseteq \mathbb{C}^{2n}$ be a curve defined over $\mathbb{Q}$. Then there is an effectively computable constant $d(V) > 0$ such that whenever $\pi(\tau) \in V$ is a special point, the discriminant $D(\tau)$ is at most $d(V)$.

Proof. First, by projecting onto each copy of $\mathbb{C}^2$ separately, we can assume that $n = 1$.

If the restriction of j to the given lines is Pfaffian, then so is χ^*, by definition. Now, any special point $\pi(\tau)$ has a Galois conjugate $\pi(\tau')$ (over $\mathbb{Q}$) with $\operatorname{Re}\tau' = 0$ or $\frac{1}{2}$. Clearly $\pi(\tau') \in C$, so standard results on Pfaffian functions then give us an effective bound on $D(\tau')$, from which we can conclude since $D(\tau') = D(\tau)$. $\square$

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