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RELEVANCE AND SYMMETRY

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Relevance and Symmetry*

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Abstract

We define a behavioral concept of relevance in the context of decision making under uncertainty. We argue that this concept provides a sensible answer to the question "What probabilistic environments do an individual's preferences reveal as mattering to her decisions?" under a symmetry assumption. This question has important implications for economic modeling. It is often the case that a modeler desires to restrict the probabilistic environments a decision maker considers. Without a concept of relevant beliefs, it is impossible to check from preferences whether a model is reflecting what the modeler intended. This checking is essential to isolating the effect of changing information while holding tastes fixed. We show that a single concept of relevance delivers this for a wide range of models, including models that allow for ambiguity attitude. We also use symmetry and relevance to provide insight into the foundations of the α-MEU and smooth ambiguity models of decision-making under uncertainty.

Keywords: Symmetry, beliefs, ambiguity, comparative statics of information

JEL codes: D01, D80, D81, D83

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1. Introduction

We aim to define a preference based concept of "relevance" in the context of decision making – specifically we want the concept to provide a sensible answer to the question "What probabilistic environments do an individual's preferences reveal as mattering to her decisions?" Why is this question of interest? Answering this question has important implications for economic modeling and applications. It is often the case that a modeler desires to incorporate restrictions on the probabilistic environments the decision maker considers. For example, when considering climate change policy, a forecast corresponds to a probability distribution over variables of interest, such as warming rates. When modeling the preferences of a policy maker in this situation, one may want to require that preferences take into account a specific set of forecasts (e.g., those with enough scientific credibility) and only those forecasts. Without an answer to our question, it is impossible to check from the preferences whether this requirement has been successfully imposed, and thus whether the model is reflecting what the modeler intended. This checking is essential to investigating the way in which model outcomes change as information varies (either across agents or for a single agent across settings), while other aspects, such as agents' tastes, are held fixed – a fundamental comparative static. This is so, because to carry out this comparative static the modeler needs to know precisely which objects in the representation of the decision maker's preferences correspond to the probabilistic environments that matter. We show that a single concept of relevance delivers this for a wide range of models. Moreover, our theory makes possible an additional comparative static – the effect of varying the model of preferences while holding the information environment fixed in terms of relevant probabilities. This is helpful in guiding model choice.

We characterize the probabilities satisfying our definition of relevant measures for a general class of preferences, thus enabling the checking and comparative statics mentioned above. Recently, there has been a great deal of interest in models where the decision maker perceives ambiguity and has preferences sensitive to the perceived ambiguity.[1] We apply our general characterization to identify the relevant measures for several models appearing in that literature. Models we examine include the α-MEU model[2] (see e.g., Ghirardato, Maccheroni and

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[1]Ambiguity, as used in the previous sentence, refers to subjective uncertainty about probabilities over states of the world. This is the sense in which much of the literature following Ellsberg [14] uses the word.

[2]The α-MEU terminology comes from the fact that the representing functional is a convex

Marinacci [18]), the smooth ambiguity model (see e.g., Klibanoff, Marinacci and Mukerji [26], Nau [30], Seo [34]), the extended MEU with contraction model[3] (see e.g., Gajdos et. al. [17], Gajdos, Tallon and Vergnaud [16], Kopylov [28], Tapking [38]) and the vector expected utility model (see e.g., Siniscalchi [36]).

To describe why identifying relevant measures from preferences might be difficult and to explain our strategy for overcoming this difficulty, consider the following simple example. Fix an event E. How can monetary bets reveal the relevant probabilities of E? For a risk neutral expected utility individual (i.e., an expected value maximizer) the unique probability assigned to E is revealed by the constant marginal rate of substitution of money across the event and its complement. If increasing the payoff on E by \$1 is just compensated by reducing the payoff on E^c by \$4, then the revealed probability of E is 0.8. What if, instead, the individual has risk averse expected utility preferences? In this case, marginal rates of substitution of money across events are a function of both marginal utilities (and thus risk attitudes) and the probabilities (reflecting this, a normalized representation of these marginal rates of substitution is often referred to in the finance literature as a risk neutral measure). This shows that marginal rates of substitution across events may confound tastes and beliefs. One way around this is to instead examine marginal rates of substitution in terms of utility rather than money, thereby filtering out the contribution of risk attitudes to these rates of substitution. However, for preferences beyond expected utility, other aspects of taste, not captured through a von Neumann-Morgenstern utility u, may be present. These aspects, a leading example of which are attitudes toward ambiguity or imprecision, will generally affect marginal rates of substitution even in utility space, as our next example illustrates.

Consider an individual with preferences as in the extended MEU with contraction model. Such an individual would evaluate a bet paying x on E and y otherwise according to the formula

$$\min_{p\in[\underline{p},\overline{p}]} \beta\left(pu(x)+(1-p)u(y)\right)+(1-\beta)(qu(x)+(1-q)u(y)),$$

where $\beta, \underline{p}, \overline{p}, q \in [0,1]$. From Gajdos et. al. [17], β is a taste parameter representing aversion toward imprecision of information, while $\underline{p}$ and $\overline{p}$ are exogenously

combination of the maxmin expected utility (MEU) of Gilboa and Schmeidler [21] and the corresponding maxmax expected utility, with weights $\alpha \in [0,1]$ and $1-\alpha$ respectively.

[3] This model has a functional form that is a convex combination of MEU and expected utility with coefficients β and $1-\beta$ respectively.

imposed "objective" restrictions on the probability of E. These restrictions are communicated to the individual. Fix $\frac{1}{2} \leq \beta < 1$, $\underline{p} < \overline{p}$ and $q \in (\underline{p}, \overline{p})$. Notice that, because of the min operator, there is a range of marginal rates of substitution of utility across E and its complement reflected in these preferences. By itself, this poses little difficulty – there would simply be a corresponding range of probabilities of E the individual considers relevant. However, if we try to require that the individual take into account probabilities of E, say, between $\frac{1}{4}$ and $\frac{3}{4}$, by setting $[\underline{p}, \overline{p}] = [\frac{1}{4}, \frac{3}{4}]$, equating probabilities with marginal rates of substitution of utility as described will not deliver the interval $[\frac{1}{4}, \frac{3}{4}]$, but instead $\left[\beta\frac{1}{4} + (1-\beta)q, \beta\frac{3}{4} + (1-\beta)q\right] \subset [\frac{1}{4}, \frac{3}{4}]$. Because the taste parameter β affects the substitution rates even in utility space, this method fails to isolate a belief component of preferences. We emphasize that this is simply one example of the general difficulty of equating relevant probabilities with marginal rates of substitution or trade-offs in utility space.

The notion of relevant measures that we propose will, in contrast, *not* be affected by tastes. How do we achieve this? Under subjective expected utility, one concept of relevance not affected by taste is the following (motivated by de Finetti [10]): Suppose an individual faces uncertainty about the result of an infinite sequence of experiments and views these experiments as symmetric in that the valuation of any mapping from realizations of the experiments to payoffs (i.e., an act) is unchanged when the sequence of experiments is reordered (i.e., the individual is indifferent among all finite permutations of an act).[4] Following de Finetti's and Hewitt and Savage's [23] celebrated result, under these circumstances the probability measure in the representation of the individual's preferences must be able to be written as a probability measure (call it m) over i.i.d. processes. Thus, this individual acts as if the only probabilistic environments they consider are i.i.d. environments. Given this, the motivating question we began the paper with can be refined to "Which i.i.d processes do the individual's preferences reveal as mattering to her decisions?" Specifically, we propose that an i.i.d. measure is relevant if the individual is not indifferent between two acts that differ in the distributions over outcomes they generate under *this* i.i.d. measure but *not* under

[4] To make things concrete, suppose each experiment is an investment and each investment can either succeed or fail. An example of an act would be a bet paying $10 if the investment labeled number 1 succeeds and the investment labeled number 2 fails and $0 otherwise. One permutation of this act is the bet paying $10 if the investment labeled 100 succeeds and the investment labeled 1 fails and $0 otherwise. If a decision maker views the various investments symmetrically, they will value the original act and the permuted act the same.

other i.i.d. measures.[5] Applying this definition, the relevant i.i.d. environments are those in the support of m. In this sense, the measures in the support of m are the only i.i.d. processes that "make a difference" in the decision maker's valuation of acts, and so are "relevant" in a natural sense of the word. Notice that which measures are relevant does not vary with the utility function, and thus, in the context of expected utility, is independent of tastes.

Indeed, the property that our classification of relevant measures is not affected by tastes, including, for example, ambiguity attitudes, extends well beyond expected utility preferences, encompassing a wide range of models of decision makers under an appropriately strengthened notion of symmetry. As we will show, an i.i.d. measure is relevant under the definition above exactly when an individual displays (when evaluating at least some acts) a positive marginal rate of substitution toward the event consisting of sequences having limiting frequency corresponding to that i.i.d. measure. Thus, the identification of relevant measures, though in this sense based on marginal rates of substitution, depends only on whether this rate is ever positive for such an event and not on the magnitude of that rate. As was pointed out above, magnitudes of marginal rates of substitution will, in general, be affected by taste components, in addition to beliefs. It is important to our argument to define what we mean by taste components. We assume, for any state independent preference,[6] a defining property of tastes (as opposed to beliefs) is: if an event has a revealed zero marginal rate of substitution everywhere (i.e., for all acts), then changing tastes alone cannot alter this, and, conversely, if an event has a revealed positive marginal rate of substitution for some acts, changing tastes alone cannot produce a zero marginal rate of substitution everywhere (i.e., for all acts). This suggests that determining relevance by looking only at whether marginal rates of substitution are positive or zero is immune to the influence of tastes. In general, this strategy wouldn't go very far – it would rule out only measures putting weight on null events. But, as the paper shows, with the structure provided by symmetry, this strategy turns out to be powerful and takes one very far indeed.

We want to emphasize that our notion of relevant measures is not designed to necessarily capture *all* aspects of belief, but specifically which processes the individual considers relevant (for example, in the subjective expected utility context

[5]More technically, they do not differ in the distributions they generate under the i.i.d. measures outside an arbitrarily small open neighborhood around the relevant i.i.d. measure.

[6]Preferences are state independent if outcomes are ranked the same way on each event in the state space. All the preferences we consider are state independent.

there may be good reasons for viewing the whole probability measure as representing beliefs, whereas the relevant measures are the i.i.d. processes that are given weight in these beliefs). The key is that relevant measures (1) reflect *only* aspects of beliefs (and so are not influenced by tastes) and (2) those aspects of beliefs are easily connected with a natural and simple question about how the individual being modeled perceives their environment – "Which i.i.d. measures matter?".

Our notion of relevant additionally has the following desirable feature: Suppose we start with an exogenous (to the individual) specification of what i.i.d. processes should be relevant in a given setting (possibly based on statistical data or some other information), and impose that as a constraint on the i.i.d. processes appearing in the representation of the individual's preferences. In the case of expected utility, we would require the individual's subjective probability measure to be a measure over i.i.d. processes with this exogenously imposed support. Next, we could ask, using our definition of relevance in terms of "making a difference" that we introduced above, "What is this individual's subjectively relevant set of i.i.d processes?" We show that the answer is exactly the exogenously specified processes we started with. This type of "fixed point" property is important because it is exactly what is needed to be able to do comparative statics in information within preference models – with this fixed point property, the analyst can be confident that manipulating a given part of the preference representation (here, manipulating the support of m) corresponds exactly (and only) to manipulating an aspect of information/belief (here, what i.i.d. processes are allowed as possibilities).

Just as the property of relevant measures being invariant to tastes extends beyond expected utility, so does the "fixed point" property. To illustrate this, consider the following example. Let the space of outcomes of an infinite sequence of experiments be denoted by the product space S^∞. Consider preferences $\succsim$ over acts on S^∞ represented by an α-MEU model under a symmetry assumption, i.e., a non-constant vN-M utility function u, an $\alpha \in [0,1]$ and a finite set of probability measures D over S such that

$$V(f) \equiv \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp + (1-\alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp \qquad (1.1)$$

represents $\succsim$. Suppose we start with an exogenous specification of what finite set of i.i.d. processes should be relevant in a given setting, and try to impose that as a constraint on preferences by setting D equal to these processes. The "fixed point" property for this model is that, when we ask, "What is this individual's subjectively relevant set of i.i.d processes?" the answer, using our definition of relevance, is exactly the exogenously specified processes we started with. This justifies interpreting comparative statics in D as capturing only an effect of changing beliefs over S (note that we make no claims about comparative statics in beliefs over the larger space S^{∞}). Additionally, this "fixed point" property also motivates and allows us to provide preference foundations (in symmetric environments) for the α-MEU model using the concept of relevant measures to overcome criticism by Eichberger, Grant and Kelsey [12] and Eichberger et. al. [13] of earlier efforts by Ghirardato, Maccheroni and Marinacci [18] to provide such foundations. We also use our framework to provide an alternative to existing foundations for another model, the smooth ambiguity model.

Overall then, our strategy for identifying relevant measures combines two ideas. First, in order to filter out tastes, when using marginal rates of substitution across events it is important to restrict attention to whether such rates are ever positive or are always zero, and not to rely on the magnitudes of these rates. Second, for a very broad class of preferences, (a strengthening of) de Finetti's symmetry is sufficient to associate a unique limiting frequency event with each measure over S. Taking these together, we show that the measures associated with limiting frequency events sometimes (i.e., for some acts) given positive marginal rates of substitution are the relevant measures.

1.1. Related Literature

Before describing our formal model and results, we discuss the most closely related literature.

1.1.1. The Bewley Set Approach to Relevant Measures

Ghirardato and Siniscalchi [19], building on an approach pioneered by Ghirardato, Maccheroni and Marinacci [18] and Nehring ([31], [32]), also propose a definition of relevant measures and a method of identifying them. The set of measures picked out by all of these papers (as well as by Gilboa et. al. [20]), is, as the various papers show under different assumptions on preferences $\succsim$, the set that appears in a Bewley-style (Bewley [4]) representation result: it is the non-empty weak* compact convex set of probability measures C such that for all acts f and g,

$$f \succsim^{*} g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in C,$$

where $\succsim^*$ is defined by

$$f \succsim^* g \text{ if } \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h \text{ for all } \alpha \in [0,1] \text{ and acts } h$$

and u is a non-constant vN-M utility function. Call such a C a Bewley set.

Ghirardato and Siniscalchi [19] (again building on Ghirardato, Maccheroni and Marinacci [18]) characterize the Bewley set using the Clarke-Rockafellar differentials in utility space of a functional representation of $\succsim$, a generalization of the usual notion of differentiation and differential. Just as we interpret the usual derivative in utility space as reflecting marginal rates of substitution of utility across different events, a similar interpretation may be attached to their more general notion. When preferences have an expected utility representation, the prior, p, represents such marginal rates of substitution globally, as these marginal rates of substitution are constant (i.e., it does not matter at which point in the utility space the rates of substitution are evaluated). For more general preferences, marginal rates of substitution may not only vary according to the act being evaluated (i.e., location in utility space), but may also vary according to the direction along which the rates of substitution are evaluated, and thus a set of marginal rates of substitution may apply for a given preference. This is what is captured by the more general differential used by Ghirardato and Siniscalchi [19]. Fundamentally then, Ghirardato and Siniscalchi [19] are showing that the Bewley set C captures the set of marginal rates of substitution of utility across events associated with $\succsim$.

As Ghirardato and Siniscalchi [19] look at marginal rates of substitution in utility space, they filter out the contribution of (expected utility) risk attitudes to these rates of substitution. However, as we discussed earlier, for preferences beyond expected utility, other aspects of taste are typically present. In this sense, the relevant measures identified by Ghirardato and Siniscalchi [19] and other papers using the Bewley set approach will generally incorporate taste as well as belief components of preference.[7]

One may think of our notion of relevant measures as an approximation of beliefs "from below" in that it may leave out some aspect of beliefs, while a notion

[7] A similar remark applies to the notion of plausible prior proposed and investigated in Siniscalchi [35]. As Ghirardato and Siniscalchi [19, p. 11] explain, a prior is plausible essentially if it is the unique probability that provides an expected utility representation of the individual's preferences over a *subset* of acts. Siniscalchi [35, pp. 112-113] shows that, when preferences satisfy the axioms he identifies as needed for the existence of plausible priors, the Bewley set C can be obtained as the weak* closed, convex hull of the set of plausible priors. Therefore plausible priors also generally are affected by tastes as well as beliefs.

like Ghirardato and Siniscalchi's [19] is an approximation of beliefs "from above" in that it may go beyond beliefs to include taste aspects of preference. The importance of this distinction is that when an analyst varies part of preferences based on an approximation from below, he can be sure that only beliefs are changing, whereas this is not guaranteed when using an approximation from above.

To illustrate further, recall the α-MEU representation (1.1) from our earlier example. If we again ask "What is this individual's subjectively relevant set of measures?" and this time answer using Ghirardato and Siniscalchi [19]'s notion of relevance, we will get back, not the exogenously imposed set of processes D, but instead, the Bewley set which is[8] (the convex hull of)

$$\left\{ \begin{array}{c} q \in \Delta(S^\infty) : q = \alpha \ell_1^\infty + (1-\alpha)\ell_2^\infty \text{ such that there exists an act } g \text{ with} \\ \{\ell_1^\infty\} = \arg\min_{p \in \{\ell^\infty : \ell \in D\}} \int u\left(g\right) dp \text{ and } \{\ell_2^\infty\} = \arg\max_{p \in \{\ell^\infty : \ell \in D\}} \int u\left(g\right) dp. \end{array} \right\}$$

Thus, the "fixed point" property would not hold (unless $\alpha = 0$ or $\alpha = 1$ or the exogenous set of processes is a singleton) when plugging in for D in (1.1) due to the influence of α, usually thought of as a taste parameter, on the Bewley set.

An additional question raised by the above example is "Under what circumstances will these two approaches to identifying relevant measures agree?" In the example, this occurs only in very special cases – when α is 0 or 1 or the set D is a singleton. This turns out to be a general result for α-MEU preferences in our setting. More broadly, under symmetry assumptions on preferences, we answer this question, both by providing a preference condition characterizing when the two approaches become equivalent, and by deriving the functional form counterpart of this condition for several models of preferences. We call the preference condition "No Half Measures". This condition embodies the following insight: given a symmetric environment, our classification of relevant measures coincides with a classification according to what is included in the Bewley set if and only if it is the case that, for each i.i.d. measure, the measure is either given full weight (i.e., weight = 1) in the evaluation of some act or it is never given *any* weight (i.e., given a weight 0 for every act). In such a case, one might say that the approximations "from below" and "from above" agree, and so beliefs are identified by each concept.

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[8] See Siniscalchi ([35], [37]).

1.1.2. Symmetry

One ingredient in our strategy for identifying relevant measures is the use of a symmetry condition on preferences to associate a unique limiting frequency event with each measure over S. In recognizing this property of symmetry, we are following the work of de Finetti [10] and Hewitt and Savage [23] in the context of expected utility and recent extensions of this work to larger classes of preferences by Epstein and Seo [15] and de Castro and Al-Najjar ([5],[6]). Cerreia-Vioglio et. al. [7] recently generalized even further along these lines. None of these papers use symmetry, as we do, to explore the concept of which i.i.d. measures are relevant and the implications of this relevance. Our particular formalization of symmetry is a preference axiom we call Event Symmetry (see Section 3). The relationship between Event Symmetry and other preference based notions of symmetry in the literature is discussed in detail in Section 7.

In our theory, we use a state space, S^{∞}, that is rich enough to contain limiting frequency events. Under a literal interpretation, S^{∞} has the same status as any other state space, and assumptions on preferences, such as symmetry conditions, are testable restrictions. Within such an interpretation, our claim that identifying relevant measures allows for comparative statics in information should be interpreted as referring to comparative statics in information about probabilities over S, not probabilities over S^{∞}. In this interpretation, weaker versions of symmetry corresponding to processes beyond i.i.d. (e.g., Markov) would enlarge the universe of possible relevant measures (e.g., instead of the relevant measures lying in the set of all i.i.d. measures, they could lie in the set of all n-state Markov processes.) We do not pursue such extensions here, but note that weakenings of symmetry and the corresponding classes of measures have been investigated in de Castro and Al-Najjar [6] and Cerreia-Vioglio et. al. [7] and might be helpful in this regard.

There is also a useful cognitive interpretation of S^{∞} in which the "real" state space is S and the extension to replications of S is a hypothetical device. This extension provides, using structured thought experiments, a means to discriminate among different belief bases for a given preference over acts on S. For example, suppose $S = \{H, T\}$ and the individual is indifferent between betting on H and betting on T. This might be because the individual knows that the coin is unbiased. Or, the individual might know that the coin is biased either 60/40 or 40/60 and think both biases are equally likely. By thinking of the coin being flipped many times, these two possibilities can be distinguished. With the known unbiased coin, the individual would value a bet that the long-run frequency of H is close to $\frac{1}{2}$, while this bet would have essentially no value in the case of the coin biased in an unknown direction. If S is the real state space, why is it useful to distinguish between these scenarios? The answer is the ability to do comparative statics in information. For example, suppose the individual is informed that the coin is not 60/40 biased in favor of heads. For an individual who knows the coin is unbiased, this will not change preferences over acts on S (i.e., bets on a single flip). If, however, the coin was known to be either 60/40 or 40/60, the new information will change preferences substantially in the direction of favoring T over H. Under this interpretation, a strong symmetry assumption is a virtue as it provides the most natural and straightforward thought experiments discriminating among beliefs over S.

1.1.3. Objective sets of probabilities

Another approach to identifying probabilities that matter to an individual is to consider probabilities over the state space as an objective primitive. In effect, these probabilities are specified as the ones that matter by fiat. Such models include those in Gajdos et. al. [17], Gajdos, Tallon and Vergnaud [16], Kopylov [28] and Wang [40]. Here, our subjective approach provides a way to derive, independent of the objective information, what probabilities the individual's preferences reveal as relevant. We can then check if these relevant measures match the given objective information. When they do, this provides a useful linkage between the objective and subjective approaches to preference models. One illustration of this is the application of our Theorem 4.3 to show that when the objectively given set in the extended MEU with contraction model consists of i.i.d. measures, the relevant measures are exactly the given set. Additionally, our theory of relevant measures shows how these theories based on objective sets of probabilities may be applied in symmetric environments even if objective information were not available. In this sense, our approach extends the scope of application of these models.

Gilboa et. al. [20] use the Bewley set approach to interpret the subjective probabilities in the MEU model as objective probabilities. In a symmetric environment, our results show that such an interpretation is independent of tastes precisely because MEU preferences satisfy the No Half Measures condition. However, for more general symmetric preferences, as we have pointed out, the Bewley set approach will confound tastes and beliefs.

1.2. Organization of the Paper

The remainder of the paper is organized as follows. Section 2 describes the formal setting and notation. Section 3 defines symmetric preferences and the notion of relevant measure and provides the fundamental results relating relevant measures to representations of $\succsim^*$ and $\succsim$. Section 4 applies these results to identify relevant measures in a variety of specific decision models and discusses the "fixed point" property in the context of exogenous restrictions on measures. The results on when the Bewley set approach is equivalent to ours are in Section 5. Section 6 contains the new foundations for the α-MEU and smooth ambiguity models under symmetry. The relation between our Event Symmetry axiom and other conditions in the literature is explored in Section 7. All proofs and related material are contained in an Appendix.

2. Setting and Notation

Let S be a compact metric space and $\Omega = S^\infty$ the state space with generic element $\omega = (\omega_1, \omega_2, ...)$. The state space Ω is also compact metric (Aliprantis and Border [1, Theorems 2.61 and 3.36]). Denote by Σ_i the Borel σ-algebra on the i-th copy of S, and by Σ the product σ-algebra on S^∞. An act is a simple Anscombe-Aumann act, a measurable $f : S^\infty \to X$ having finite range (i.e., $f(S^\infty)$ is finite) where X is the set of lotteries (i.e., finite support probability measures on an outcome space Z). The set of acts is denoted by $\mathcal{F}$, and $\succsim$ is a binary relation on $\mathcal{F} \times \mathcal{F}$. As usual, we identify a constant act (an act yielding the same element of X on all of S^∞) with the element of X it yields.

Denote by Π the set of all finite permutations on $\{1, 2, ...\}$ i.e., all one-to-one and onto functions $\pi : \{1, 2, ...\} \to \{1, 2, ...\}$ such that $\pi(i) = i$ for all but finitely many $i \in \{1, 2, ...\}$. For $\pi \in \Pi$, let $\pi\omega = \left(\omega_{\pi(1)}, \omega_{\pi(2)}, ...\right)$ and $(\pi f)(\omega) = f(\pi\omega)$.

For any topological space Y, $\Delta(Y)$ denotes the set of (countably additive) Borel probability measures on Y. Unless stated otherwise, a measure is understood as a countably additive Borel measure. For later use, $ba(Y)$ is the set of finitely additive bounded real-valued set functions on Y, and $ba^1_+(Y)$ the set of nonnegative probability charges in $ba(Y)$. A measure $p \in \Delta(S^\infty)$ is called symmetric if the order doesn't matter, i.e., $p(A) = p(\pi A)$ for all $\pi \in \Pi$, where $\pi A = \{\pi\omega : \omega \in A\}$. Denote by ℓ^∞ the i.i.d. measure with the marginal $\ell \in \Delta(S)$. Define $\int_{S^\infty} f dp \in X$ by $\left(\int_{S^\infty} f dp\right)(B) = \left(\int_{S^\infty} f(\omega)(B)\, dp(\omega)\right)$. (Since f is simple, this is well-defined.)

Fix $x_*, x^* \in X$ such that $x^* \succ x_*$. For any event $A \in \Sigma$, 1_A denotes the act giving x^* on A and x_* otherwise. Informally, this is a bet on A. A finite cylinder event $A \in \Sigma$ is any event of the form $\{\omega : \omega_i \in A_i \text{ for } i = 1, ..., n\}$ for $A_i \in \Sigma_i$ and some finite n.

Endow $\Delta(S)$, $\Delta(\Delta(S))$ and $\Delta(S^\infty)$ with the relative weak* topology. To see what this is, consider, for example, $\Delta(S)$. The relative weak* topology on $\Delta(S)$ is the collection of sets $V \cap \Delta(S)$ for weak* open $V \subseteq ba(S)$, where the weak* topology on $ba(S)$ is the weakest topology for which all functions $\ell \longmapsto \int \psi d\ell$ are continuous for all bounded measurable ψ on S. Also note that a net $\ell_\alpha \in ba(S)$ converges to $\ell \in ba(S)$ under the weak* topology if and only if $\int \psi d\ell_\alpha \to \int \psi d\ell$ for all bounded measurable ψ on S. For a set $D \subseteq \Delta(S)$, denote the closure of D in the relative weak* topology by $\overline{D}$.

The support of a probability measure $m \in \Delta(\Delta(S))$, denoted $\operatorname{supp} m$, is a relative weak* closed set such that $m((\operatorname{supp} m)^c) = 0$ and if $G \cap \operatorname{supp} m \neq \emptyset$ for relative weak* open G, $m(G \cap \operatorname{supp} m) > 0$. (See e.g., Aliprantis and Border [1, p.441].)

Definition 2.1. *Let $\Psi_n(\omega) \in \Delta(S)$ denote the empirical frequency operator $\Psi_n(\omega)(A) = \frac{1}{n}\sum_{t=1}^{n} I(\omega_t \in A)$ for each event A in S. Define the limiting frequency operator Ψ by $\Psi(\omega)(A) = \lim_n \Psi_n(\omega)(A)$ if the limit exists and 0 otherwise. Also, to map given limiting frequencies or sets of limiting frequencies to events in S^∞, we consider the natural inverses $\Psi^{-1}(\ell) = \{\omega : \Psi(\omega) = \ell\}$ and $\Psi^{-1}(L) = \{\omega : \Psi(\omega) \in L\}$ for $\ell \in \Delta(S)$ and $L \subseteq \Delta(S)$.*

3. Symmetry and Relevance

3.1. Symmetric Preferences

We start by stating the conditions on preferences over acts $\mathcal{F}$ that delineate the scope of our theory of relevance. The theory will apply to preferences satisfying the following axioms.

Axiom 1 (C-complete Preorder). *$\succsim$ is reflexive, transitive and the restriction of $\succsim$ to X is complete.*

Notice that we allow $\succsim$ to be incomplete. Some of our results will later invoke completeness.

Axiom 2 (Monotonicity). *If $f(\omega) \succsim g(\omega)$ for all $\omega \in S^\infty$, $f \succsim g$.*

Monotonicity rules out state-dependence of preferences over X. This allows us to focus on states purely as specifying the resolution of acts.

Axiom 3 (Risk Independence). *For all $x, x', x'' \in X$ and $\alpha \in (0,1)$, $x \succsim x'$ if and only if $\alpha x + (1-\alpha)\, x'' \succsim \alpha x' + (1-\alpha)\, x''$.*

This is the standard von Neumann-Morgenstern Independence axiom on lotteries. This rules out non-expected utility preferences over lotteries. It allows us to separate attitudes toward risk from other aspects of preferences in a simple way, using a familiar von Neumann-Morgenstern utility function.

Axiom 4 (Non-triviality). *There exist $x, y \in X$ such that $x \succ y$.*

To describe our remaining axioms, it is notationally convenient to introduce the binary relation $\succsim^*$ derived from $\succsim$:

$$f \succsim^* g \text{ if } \alpha f + (1-\alpha)\, h \succsim \alpha g + (1-\alpha)\, h \text{ for all } \alpha \in [0,1] \text{ and } h \in \mathcal{F}.$$

Ghirardato, Maccheroni and Marinacci [18, GMM hereafter] refer to $\succsim^*$ as an unambiguous preference. We will not use this terminology here for reasons that will become clear later. As they state, Klaus Nehring is the first one to suggest using this maximal independent restriction $\succsim^*$ of a given $\succsim$, in a 1996 talk. See also Nehring ([31], [32], [33]). Observe that, given Monotonicity and Risk Independence, $\succsim^*$ and $\succsim$ are identical when restricted to constant acts, while, for more general acts, $f \succsim^* g$ implies $f \succsim g$ but the converse may be false.

The key axiom delineating the domain of our theory is Event Symmetry which says that the ordinates of S^∞ are viewed as interchangeable.

Axiom 5 (Event Symmetry). *For all finite cylinder events $A \in \Sigma$ and finite permutations $\pi \in \Pi$, $1_A \sim^* 1_{\pi A}$.*

A natural notion of symmetry, as expressed through preferences, is that the decision maker is always indifferent between betting on an event and betting on its permutation. The use of the term "always" here means at least that this preference should hold no matter what other act the individual faces in combination with the bet. In an Anscombe-Aumann framework such as ours, this may be expressed by the statement that $\alpha 1_A + (1-\alpha)h \sim \alpha 1_{\pi A} + (1-\alpha)h$ for all $\alpha \in [0,1]$ and all acts h, which is exactly $1_A \sim^* 1_{\pi A}$. For preferences satisfying the usual independence axiom, $1_A \sim^* 1_{\pi A}$ is equivalent to $1_A \sim 1_{\pi A}$. As a main goal of our analysis is to accommodate preferences that may violate independence (e.g., because of ambiguity concerns), we cannot substitute the latter condition for the former. We note that this idea of symmetry is related to the concept of event exchangeability developed by Chew and Sagi [9], with the permuting of payoffs in a Savage framework replaced by substitution in mixtures in our Anscombe-Aumann framework. For further discussion of the relation to concepts in the literature, see section 7.

Remark 1. *As written, Event Symmetry seems to depend on the choice of x^*, x_* in defining 1_A. In fact, in the presence of our other axioms, Event Symmetry implies that the analogous property holds for any choice of $x_*, x^* \in X$.*

Combining all of these conditions defines the class of preferences we will work with:

Definition 3.1. $\succsim$ *satisfies* Symmetry *if it satisfies C-complete Preorder, Monotonicity, Risk Independence, Non-triviality, and Event Symmetry.*

When we say that $\succsim$ is Symmetric, we mean that it satisfies Symmetry.

In addition to Symmetry, we will often need some form of continuity of preference. Different forms of continuity will be more or less convenient for subsequent results. We now state three forms of continuity that are used in the paper. The first and second are standard mixture continuity requirements.

Axiom 6 (Mixture Continuity of $\succsim$). *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim h\}$ and $\{\lambda \in [0,1] : h \succsim \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$.*

Mixture continuity of $\succsim$ appears many places in the literature. A weakening of this requirement is the Mixture Continuity of $\succsim^*$:[9]

[9] To see that this is a weakening, observe that

$$\begin{aligned}\{\lambda &\in [0,1] : \lambda f + (1-\lambda) g \succsim^* h\} \\ &= \bigcap_{\alpha \in [0,1], f' \in \mathcal{F}} \{\lambda \in [0,1] : \lambda(\alpha f + (1-\alpha) f') + (1-\lambda)(\alpha g + (1-\alpha) f') \succsim \alpha h + (1-\alpha) f'\}.\end{aligned}$$

Mixture Continuity of $\succsim$ implies this set is closed since it is the intersection of closed sets. The same reasoning applies for the set $\{\lambda \in [0,1] : h \succsim^* \lambda f + (1-\lambda) g\}$.

Axiom 7 (Mixture Continuity of $\succsim^*$). *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0,1] : \lambda f + (1-\lambda) g \succsim^* h\}$ and $\{\lambda \in [0,1] : h \succsim^* \lambda f + (1-\lambda) g\}$ are closed in $[0,1]$.*

We will want additional continuity in order to restrict attention to countably additive measures. The standard approach to this in the literature is based on the application to $\succsim^*$ of the monotone continuity of Arrow [2], as in Ghirardato, Maccheroni and Marinacci [18, GMM hereafter].

Axiom 8 (Monotone Continuity of $\succsim^*$). *For all $x, x', x'' \in X$, if $A_n \searrow \emptyset$ and $x' \succ x''$, then $x' \succsim^* x A_n x''$ for some n.*

3.2. Relevance

We now formalize what it means for a measure $\ell \in \Delta(S)$ to be relevant according to preferences $\succsim$. For notational convenience, let $\mathcal{O}_\ell$ be the collection of open subsets of $\Delta(S)$ that contains ℓ. That is, for $\ell \in \Delta(S)$, $\mathcal{O}_\ell = \{L \subseteq \Delta(S) : L$ is open, $\ell \in L\}$.

Definition 3.2. *A measure $\ell \in \Delta(S)$ is* relevant *(according to preferences $\succsim$) if, for any $L \in \mathcal{O}_\ell$, there are $f, g \in \mathcal{F}$ such that $f \nsim g$ and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\widehat{\ell} \in \Delta(S) \backslash L$.*

In words, if two acts differ in the reduced lottery they generate only when ℓ governs the independent realization of each ordinate S, and yet the two acts are not indifferent, then ℓ is relevant. Since $\Delta(S)$ is uncountable, formally the definition uses open neighborhoods of ℓ rather than ℓ. Notice that equality of the lotteries generated by f and g is required only for i.i.d. measures, $\hat{\ell}^\infty$ (and by linearity of the integral, therefore, for any mixtures over these i.i.d. measures). However, when $\succsim$ satisfy Symmetry, Mixture Continuity of $\succsim^*$and Monotone Continuity of $\succsim^*$, we will show there is a natural sense in which mixtures over i.i.d. measures (i.e., exchangeable measures) will be the only ones that matter for preference. Furthermore, as Symmetry and Mixture Continuity imply expected utility on constant acts, one could replace $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ by the condition on expected utilities, $\int u(f) d\hat{\ell}^\infty = \int u(g) d\hat{\ell}^\infty$ without changing the definition.

Next we introduce an alternative notion of relevance (or, more precisely, irrelevance) based on bets on events generated by limiting frequencies. In reading the definition recall that, for $A \subseteq \Delta(S)$, $\Psi^{-1}(A)$ is the event that limiting frequencies over S lie in A.

Definition 3.3. *A measure* $\ell \in \Delta(S)$ *is* betting irrelevant *(according to preferences* $\succsim$*) if, for some* $L \in \mathcal{O}_\ell$, $1_{\Psi^{-1}(L)} \sim^* 1_\emptyset$.

Intuitively, in an i.i.d. environment, $\ell \in \Delta(S)$ is irrelevant when a bet on the limiting frequencies generated by ℓ is always treated as a bet that can never win. Why define betting irrelevant using $\sim^*$ rather than simply $\sim$? As in the discussion following the Event Symmetry axiom, to ensure a bet is *always* treated the same as a sure losing bet demands looking at indifference according to $\succsim^*$. The reason is that for many preferences (e.g., MEU), bets on events are insufficient to identify which events are Savage-null (i.e., never affect preference), although comparison of bets on events mixed with a common act are sufficient to do so. To see this for MEU, observe that for preferences represented by $\inf_{p\in D} \int u(f)\,dp$, an event E satisfies $1_E \sim 1_\emptyset$ if and only if $\inf_{p\in D} p(E) = 0$, while E is Savage-null if and only if $p(E) = 0$ for all $p \in D$, a more restrictive condition. However, letting $h = 1_{E^c}$ and $\alpha = \frac{1}{4}$, $\alpha 1_E + (1-\alpha)h \sim \alpha 1_\emptyset + (1-\alpha)h$ if and only if $p(E) = 0$ for all $p \in D$.

3.3. Relevance and Symmetric Preferences

Assuming preferences are Symmetric and appropriately continuous, we show that the two notions of relevance offered above agree, and we provide a representation of the set of relevant measures in $\Delta(S)$. We also show that any such preferences may be represented by an increasing functional on the expected utilities generated by the relevant measures. Furthermore, up to closure, all relevant measures are needed for such a representation.

We first provide a Bewley-style (Bewley [4]) representation result for the induced relation $\succsim^*$. Compared to similar results in the literature (e.g., Ghirardato, Maccheroni and Marinacci [18], Gilboa et. al. [20], Ghirardato and Siniscalchi [19], Nehring [31]) the key difference is that Symmetry (and in particular, Event Symmetry) allows a de Finetti-style decomposition of the representing set of measures, C, the Bewley set.

Lemma 3.4. *Suppose* $\succsim$ *is reflexive and transitive. Then* $\succsim$ *satisfies Symmetry, Mixture Continuity of* $\succsim^*$ *and Monotone Continuity of* $\succsim^*$ *if and only if there exist a non-empty weak* compact convex set* $M \subseteq \Delta(\Delta(S))$ *and a non-constant vN-M utility function* u *such that*

$$f \succsim^* g \text{ if and only if } \int u(f)\,dp \geq \int u(g)\,dp \text{ for all } p \in C, \tag{3.1}$$

where $C = \left\{ \int \ell^\infty dm(\ell) : m \in M \right\}$. *Furthermore* M *is unique.*

Given this representation, define the set $R \equiv \bigcup_{m \in M} \operatorname{supp} m \subseteq \Delta(S)$. The set R is our candidate for the set of relevant measures in $\Delta(S)$. De Finetti's theorem (see Hewitt and Savage [23]) says that if we (or an agent) have a subjective expected utility preference, and if we are indifferent among the orderings of experiments, then the agent's subjective probability measure can be decomposed into parameters, corresponding to i.i.d. measures, and a unique probability measure over them. Our result goes beyond expected utility, and even beyond probabilistic sophistication, and says that Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$, with Event Symmetry playing the role of indifference among the ordering of experiments, is equivalent to existence of a similar decomposition. Instead of a unique probability measure, when $\succsim$ is incomplete and/or violates the Independence axiom, our result delivers a compact convex set of probability measures, M, over parameters corresponding to i.i.d. measures. In this sense, R, the union of the supports of measures in M, is the set of parameters given weight under $\succsim$. Indeed, we now show that R is the set of relevant measures according to preferences $\succsim$.

Theorem 3.5. *Assume $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$, and take R accordingly. Then, R is the set of all relevant measures in $\Delta(S)$.*

Our next result shows that R is also the set of measures that are *not* betting irrelevant, and therefore our two notions of relevance agree.

Theorem 3.6. *Assume $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$, and take R accordingly. Then, R^c is the set of all betting irrelevant measures in $\Delta(S)$ and R is relative weak* closed.*

When R is finite, the same result holds without the use of neighborhoods in defining betting irrelevant:

Corollary 3.7. *If R is finite, then $1_{\Psi^{-1}(\ell)} \nsim^* 1_\emptyset$ if and only if $\ell \in R$.*

The above results justify thinking of R as the unique set of parameters viewed as subjectively possible since any other set of measures in $\Delta(S)$ will either leave out some relevant measures or include some irrelevant ones.

For complete preferences satisfying Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$, our next result shows that (up to closure) all relevant measures are needed to represent preferences and thus the i.i.d. measures generated from R, the set of all relevant measures, is the unique minimal closed set of i.i.d. measures to do so.

Theorem 3.8. *Suppose $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$ and admits a real-valued representation. Then, there is a non-constant vN-M utility function u on X and a weakly increasing functional G on*

$$\left\{\tilde{f} \in [u(X)]^R : \tilde{f}(\ell) = \int u(f)\, d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$$

such that

$$f \longmapsto G\left(\left(\int u(f)\, d\ell^\infty\right)_{\ell \in R}\right)$$

represents $\succsim$. Furthermore, the measures in the representation are essentially unique – if $D \subseteq \Delta(S)$ and every element in D is relevant, $\tilde{u}$ is a non-constant vN-M utility function, H is a functional on

$$\left\{\tilde{f} \in [\tilde{u}(X)]^D : \tilde{f}(\ell) = \int (\tilde{u}(f)) d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$$

and

$$f \longmapsto H\left(\left(\int (\tilde{u}(f)) d\ell^\infty\right)_{\ell \in D}\right)$$

represents $\succsim$, then $\overline{D} = R$ and $\tilde{u}$ is a positive affine transformation of u.

Under slightly different assumptions, the fact that the set of expected utilities with respect to all i.i.d. measures can be monotonically aggregated to represent preferences was shown in de Castro and Al-Najjar ([5],[6]). In this regard, the main contribution of Theorem 3.8 is that R generates the *unique closed subset of i.i.d. measures that are essential for such a representation.* It is worth remarking that Theorem 3.8 does *not* imply that the set of i.i.d. measures generated from the relevant measures is the minimal closed set of measures in $\Delta(S^\infty)$ needed to represent preferences. In particular, specific mixtures over these i.i.d. measures may suffice. Formally, this is reflected in the fact that the Bewley set C may be a strict subset of $\Delta\left(\bigcup_{m \in M} \operatorname{supp} m\right)$ in Lemma 3.4.

From Theorem 3.8, we see that all decision makers whose complete $\succsim$ satisfy Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$ will have their preferences fully described by specifying (1) the relevant measures (i.e., the set R), (2) risk attitudes (i.e., u) and (3) how the expectations of (utility) acts with respect to the i.i.d. products of elements of R should be aggregated (i.e., G). This third element may generally depend on some combination of (possibly imprecise) likelihood judgements and any aspect of tastes not captured by vNM risk attitude, for example, ambiguity attitude.

4. Relating Relevant measures with sets of measures in representations

In models whose representation includes a set of probability measures, it would be very useful for applications and as a guide to intuition to interpret these measures as reflecting the set of environments the individual considers possible. In fact, when such models are explained or applied such an interpretation is typically given. Despite this, with a few notable exceptions (e.g., Gajdos et. al. [17] and Hansen and Sargent [22]), the underlying theory has not typically investigated the link between information about the underlying environments and these sets of measures. In this section we show how, for a variety of models, imposing Event Symmetry and using our definition of relevant measures R, allows for the consistent use of such an interpretation. Moreover, this interpretation is not only consistent within a given preference model, but allows the analyst to hold the set of environments considered possible fixed across different models.

One advantage of having this interpretation available is the ability of the modeler to impose assumptions about what an economic agent considers possible. Without such an interpretation, it is unclear how such assumptions should be expressed in terms of the decision model. Our interpretation provides explicit direction on how to do so, through the set of relevant measures. An additional advantage is the ability to compare different models of preference in terms of the choices they imply given a fixed set of possible environments or a given change in the set of possible environments. We next work to make this operational by identifying the set of relevant measures in some popular decision models: α-MEU, the smooth ambiguity model, extended MEU with contraction and the Vector Expected Utility (VEU) model (Siniscalchi [36]).

4.1. Relevant measures in specific decision models

All of the results in this section are proved using the same basic strategy for each model. Given a closed set of measures $D \subseteq \Delta(S)$ taken from the functional form of the model, we first show that every element of D is a relevant measure. Then, we verify (sometimes aided by Lemma 3.4 and the necessity direction of representation theorems in Section 6) that the preferences satisfy Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$. Finally, we either (1) note that each representation is a weakly increasing function of $\left(\int u\left(f\right)d\ell^{\infty}\right)_{\ell\in D}$ and invoke Theorem 3.8, or (2) prove that all measures outside of D are betting irrelevant and invoke Theorems 3.5 and 3.6, to conclude that all relevant measures are in D, and thus D is the set of relevant measures.

4.1.1. The α-MEU model

Theorem 4.1. *If $\succsim$ is represented by*

$$V\left(f\right)\equiv\alpha\min_{p\in\{\ell^{\infty}:\ell\in D\}}\int u\left(f\right)dp+\left(1-\alpha\right)\max_{p\in\{\ell^{\infty}:\ell\in D\}}\int u\left(f\right)dp,$$

where $D\subseteq\Delta(S)$ is finite and u is a non-constant vN-M utility function and $\alpha\in[0,1]$, then $R=D$.

This demonstrates that when the set of measures in an α-MEU representation is a finite set of i.i.d. product measures, the marginals generating this set are the relevant measures, R. Note that the finiteness restriction is necessary for these α-MEU preferences to satisfy Monotone Continuity of $\succsim^*$, as will be shown in our representation theorem for α-MEU preferences (Theorem 6.1).

4.1.2. The Smooth Ambiguity model

Theorem 4.2. *Assume $\succsim$ is represented by*

$$U\left(f\right)\equiv\int_{\Delta(S)}\phi\left(\int u(f)d\ell^{\infty}\right)d\mu\left(\ell\right)$$

where u is a non-constant vN-M utility function, $\phi:u(X)\rightarrow\mathbb{R}$ is a strictly increasing continuous function and $\mu\in\Delta(\Delta(S))$ such that either (i) there are $m,M>0$ such that $m\left|a-b\right|\leq\left|\phi\left(a\right)-\phi\left(b\right)\right|\leq M\left|a-b\right|$ for all $a,b\in u\left(X\right)$ or, (ii) $\operatorname{supp}\mu$ is finite. Then, $R=\operatorname{supp}\mu$.

Thus, for such smooth ambiguity preferences satisfying either (i) or (ii), the relevant measures are exactly the support of the second-order measure μ. Note that the requirement that either (i) or (ii) is satisfied is necessary for these preferences to satisfy Monotone Continuity of $\succsim^*$, as will be shown in our representation theorem for smooth ambiguity preferences (Theorem 6.2).

4.1.3. The Extended MEU with contraction model

This model has a functional form that is a convex combination of MEU and expected utility.

Theorem 4.3. *If* $\succsim$ *is represented by*

$$W(f) \equiv \beta \min_{p \in \{\ell^{\infty}:\ell \in D\}} \int u(f)\, dp + (1-\beta) \int u(f)\, dq,$$

where $D \subseteq \Delta(S)$ *is finite,* $q \in co\{\ell^{\infty} : \ell \in D\}$, $0 < \beta \leq 1$ *and* u *is a non-constant vN-M utility function, then* $R = D$.

This demonstrates that for an Extended MEU with contraction representation using a finite set of i.i.d. product measures, the marginals generating this set are the relevant measures, R. Note that the finiteness restriction is sufficient for these preferences to satisfy Monotone Continuity of $\succsim^*$.

4.1.4. The Vector Expected Utility (VEU) model

Theorem 4.4. *Suppose* $\succsim$ *is represented by a VEU functional, that is,*

$$T(f) \equiv \int u(f)\, dp + A\left(\left(\int \zeta_i u(f)\, dp\right)_{1 \leq i \leq n}\right),$$

where p *is a probability measure on* S^{∞}, u *is a non-constant vN-M utility function,* ζ_i, *for each* i, *is a bounded, measurable real-valued function on* S^{∞} *such that* $\int \zeta_i dp = 0$, $A(0) = 0$, $A(a) = A(-a)$ *for all* $a \in \mathbb{R}^n$, *and* T *is weakly monotonic. If* n *is finite,* p *and the* ζ_i*'s are symmetric (i.e.,* $p = \int \ell^{\infty} dm(\ell)$ *for some* $m \in \Delta(\Delta(S))$ *and, for all* $\pi \in \Pi$, $\zeta_i(\omega) = \zeta_i(\pi\omega)$ p *almost-everywhere) and* A *is Lipschitz continuous (i.e., there is an* $M > 0$ *such that* $|A(a) - A(b)| \leq M \sup_{1 \leq i \leq n} |a_i - b_i|$ *for all* $a, b \in \mathbb{R}^n$*), then* $R = \operatorname{supp} m$.

Thus, for VEU preferences with Lipschitz continuous adjustment function A, symmetric baseline probability, p, and a finite number of symmetric adjustment factors, ζ_i, the relevant measures, R, are those $\ell \in \Delta(S)$ given weight by p. The symmetry conditions are imposed to ensure Event Symmetry, while n finite and the Lipschitz condition are imposed to ensure Monotone Continuity of $\succsim^*$.

4.2. Consistency with exogenous sets of measures

Given the above results, we now describe how exogenous assumptions about possible i.i.d. environments can be related consistently to the relevant i.i.d. environments identified through preferences.

Step 1: Start with an exogenously given (finite) set of possible i.i.d. environments (i.e., a finite subset of $\Delta(S)$). Call this set D.

Step 2: Write a preference representation "using" D satisfying Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$. Specifically, we would like to plug D into what, possibly naively, seems to be the component of the representation standing for the set of subjectively possible environments. To make this concrete we explain what such a component would be for several prominent models.

a. Expected Utility: $\succsim_1$ represented by $\int u(f)\,dp$ where p is in the interior of $co\{\ell^\infty : \ell \in D\}$.

b. α-MEU: $\succsim_2$ represented by[10]

$$\alpha \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp + (1-\alpha) \max_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp.$$

c. Smooth ambiguity model: $\succsim_3$ represented by

$$\int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)\,d\ell^\infty\right) d\mu(\ell)$$

with continuous ϕ where $\operatorname{supp}\mu = D$.

d. Extended MEU with contraction: $\succsim_4$ represented by

$$\beta \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp + (1-\beta) \int u(f)\,dq$$

where $q \in co\{\ell^\infty : \ell \in D\}$ and $0 < \beta \leq 1$.

e. VEU: $\succsim_5$ represented by

$$\int u(f)\,dp + A\left(\left(\int \zeta_i u(f)\,dp\right)_{1\leq i\leq n}\right)$$

[10]The special case $\alpha = 1$ is the MEU model.

where p is in the interior of $co\{\ell^\infty : \ell \in D\}$, the ζ_i are symmetric and A is Lipschitz continuous.

Step 3: For the preferences included in Step 2, we may ask what the set of relevant measures is in the sense of Definition 3.2. Applying our earlier results for these models, we find $R_{\succsim_1} = R_{\succsim_2} = R_{\succsim_3} = R_{\succsim_4} = R_{\succsim_5} = D$, where $R_{\succsim}$ is the set of relevant measures according to $\succsim$. (Note that if $\beta = 0$ in part d, then $R_{\succsim_4} = \{\ell \in D : \ell^\infty$ is given positive weight by $q\}$.) In this way, we see that our notion of relevant measures recovers the exogenously specified set of possible probabilities D. This is the "fixed point" property we referred to in the introduction. Furthermore, it makes formal the sense in which all of these preferences share the same set of subjectively possible environments.

4.3. Comparison with the Bewley set approach

For the sake of comparison, suppose we try to do a similar exercise to the steps above using the Bewley set, C, instead of the set of relevant measures, R. We will see that in order to ensure recovery of the set of measures specified in the first step, the nature of the "plugging in" in the second step becomes quite complex and involves more pieces of the representation.

Step $1'$: Start with an exogenously given closed, convex set of possible "beliefs" $F \subseteq co\{\ell^\infty : \ell \in D\}$ where D is a (finite) set of i.i.d. environments (i.e., a subset of $\Delta(S)$).

Step $2'$:

a. Expected Utility: When F is a singleton, $\succsim_{1'}$ represented by $\int u(f)\,dp$ where $p = F$.

b. α-MEU: $\succsim_{2'}$ represented by

$$\alpha \min_{p\in G} \int u(f)\,dp + (1-\alpha) \max_{p\in G} \int u(f)\,dp$$

where $G \subseteq \Delta(S^\infty)$ is such that

$$co \left\{ \begin{array}{l} q \in \Delta(S^\infty) : q = \alpha \arg\min_{p\in G} \int u(g)\,dp + (1-\alpha) \arg\max_{p\in G} \int u(g)\,dp \\ \text{for some act } g \text{ such that } \arg\min_{p\in G} \int u(g)\,dp \text{ and } \arg\max_{p\in G} \int u(g)\,dp \\ \text{are singletons.} \end{array} \right\} = F.$$

c. Smooth ambiguity model: $\succsim_{3'}$ represented by

$$\int_{\Delta(S)} \phi \left(\int_{S^\infty} u(f)\, d\ell^\infty \right) d\mu(\ell)$$

with ϕ continuously differentiable where

$$\overline{co} \left\{ \begin{array}{c} q \in \Delta(S^\infty) : q \propto \int_{\Delta(S)} \phi' \left(\int_{S^\infty} u(g)\, d\ell^\infty \right) \ell^\infty d\mu(\ell) \text{ for some act } g \\ \text{such that } u(g) \text{ is an interior, bounded utility-act.} \end{array} \right\} = F.$$

d. Extended MEU with contraction: $\succsim_{4'}$ represented by

$$\begin{aligned} & \beta \min_{p \in G} \int u(f)\, dp + (1-\beta) \int u(f)\, dq \\ = & \min_{p \in \{\beta r + (1-\beta) q : r \in G\}} \int u(f)\, dp, \end{aligned}$$

where $\{\beta r + (1-\beta) q : r \in G\} = F$.

e. VEU: $\succsim_{5'}$ represented by

$$\int u(f)\, dp + A \left(\left(\int \zeta_i u(f)\, dp \right)_{1 \le i \le n} \right)$$

with p and the ζ_i symmetric and A continuously differentiable where

$$\overline{co} \left\{ \begin{array}{c} q \in \Delta(S^\infty) : \text{For some } \lambda > 0 \text{ and act } g \text{ such that} \\ u(g) \text{ is an interior, bounded utility-act,} \\ q(E) = \lambda \left[p(E) + \sum_{i=1}^n \partial A_i \left(\left(\int \zeta_i u(g)\, dp \right)_{1 \le i \le n} \right) \int_E \zeta_i dp \right] \text{ for all } E \in \Sigma \ . \end{array} \right\} = F.$$

Step $3'$: For the preferences included in Step 2, we may calculate the Bewley set. Applying results from Ghirardato and Siniscalchi [19], Ghirardato, Maccheroni and Marinacci [18] and Siniscalchi ([35], [37]), we find $C_{\succsim_{1'}} = C_{\succsim_{2'}} = C_{\succsim_{3'}} = C_{\succsim_{4'}} = C_{\succsim_{5'}} = F$, where $C_{\succsim}$ is the Bewley set according to $\succsim$.

In comparing these "primed" steps to the ones using the set of relevant measures, we see that to get the same fixed point property, we need to allow the way in which the exogenous beliefs are plugged in to the representations in step $2'$ to vary depending on parameters or functionals (e.g., α, ϕ, β or A) usually associated with tastes (and satisfying the defining property of tastes we stated in the Introduction) within the representations. Hence, generally, it is not possible to remain consistent using the Bewley set approach while respecting the separation of beliefs and tastes as customarily attributed in these models. As our relevant measures approach does respect such separation, it would be useful to know when the two approaches coincide in order to better understand the source of this dependence or lack thereof on taste parameters. This is the question we address next.

5. When do Relevant measures and the Bewley set agree?

Recall the limiting frequency operator defined in Definition 2.1. Let Σ^{Ψ} be the σ-algebra generated by the sets

$$\Psi^{-1}(\ell) \equiv \{\omega : \Psi(\omega) = \ell\} \text{ for } \ell \in \Delta(S).$$

This algebra contains events based on limiting frequencies.

We will show that the following condition is equivalent to the set of Relevant measures fully determining the Bewley set.

Condition 1 (No Half Measures). *For $A \in \Sigma^{\Psi}$, at least one of the following holds:*

(i) *If $1_{S^\infty} \succ x \in X$, then $\alpha x + (1-\alpha) f \not\succsim \alpha 1_A + (1-\alpha) f$ for some $f \in \mathcal{F}$ and $\alpha \in (0,1]$;*

(ii) *For all $f \in \mathcal{F}$ and $\alpha \in (0,1]$, $\alpha 1_{\emptyset} + (1-\alpha) f \sim \alpha 1_A + (1-\alpha) f$.*

To understand what this condition says, fix any bet on an event $A \in \Sigma^{\Psi}$ and let $x^* \in X$ be the lottery you get if you win and x_* the lottery if you lose (with $x^* \succ x_*$). Imagine agreeing to lose for sure (i.e., having the bet shrunk from A to the empty set) in return for improvement in the losing lottery (call the new losing lottery w, with $x^* \succ w \succsim x_*$). Notice that this is a trade-off between utility on A and A^c – on A, x^* is lowered to w, while on A^c, x_* is raised to w. No Half Measures says that either there is always some mixture within which you do not prefer to make this trade-off (i.e., sometimes the increase on A^c is worth less than *any* reduction on A), or, for every mixture, you are willing to agree to substitute a sure loss without any improvement in the losing stakes (i.e., you require *no* compensation for the reduced payoff on A). When this is true, then identifying the measures ℓ that are relevant is the same as identifying the measures that sometimes have an infinite marginal rate of substitution (i.e., that get assigned full belief). Thus, under this condition, classifying measures into those that sometimes get positive weight and those that do not is the same as classifying measures according to the range of marginal rates of substitution they sometime receive.

When this condition fails, the latter classification will make use of variations in ranges of marginal rates of substitution, not just zero vs. positive. As we explained in the introduction, comparing magnitudes of positive marginal rates of substitution generally involves not only beliefs but also any tastes not captured by u, such as, for example, ambiguity attitudes.

Theorem 5.1. *Suppose $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$. Then $\succsim$ satisfies No Half Measures if and only if $C = \left\{ \int \ell^\infty dm\left(\ell\right) : m \in \Delta(R) \right\}$ in (3.1).*

Recall that, without No Half Measures, $C = \left\{ \int \ell^\infty dm\left(\ell\right) : m \in M \right\}$ where $M \subseteq \Delta(R)$, so that knowing R need not pin down C. Our theorem shows that the No Half Measures condition is identifying those cases in which C reflects *only* the set R of processes considered relevant and not any other taste/belief component. In this sense, when No Half Measures holds, the approach in Ghirardato, Maccheroni and Marinacci [18], Nehring [31] and Ghirardato and Siniscalchi [19] based on C agrees with our approach.

What restrictions are implied by No Half Measures? In light of the theorem above, answering this question allows us to see where an approach based on relevant measures reinforces a Bewley-style approach and where they differ.

The first restriction is that, under our other conditions on preferences, No Half Measures implies there is only a finite set of relevant measures.

Theorem 5.2. *If $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$, Monotone Continuity of $\succsim^*$ and No Half Measures then R is finite.*

What are the further implications of No Half Measures in the context of specific decision models? We begin with the MEU model. There, No Half Measures implies the set of measures in the representation are the i.i.d. products generated by the measures in R. We then examine the α-MEU model and show there, when R is non-singleton No Half Measures is only compatible with extreme values of α (α equals 0 or 1). Similarly, we conclude this subsection with an examination of the smooth ambiguity model – when R is non-singleton we show No Half Measures places non-trivial restrictions on the function ϕ.[11]

5.1. MEU Model

Theorem 5.3. *Suppose there is a non-constant vN-M utility function u, and a non-empty weak* compact convex set $M \subseteq \Delta(\Delta(S))$ such that, for $F \equiv \left\{\int \ell^{\infty} dm(\ell) : m \in M\right\}$,*

$$V(f) \equiv \min_{p \in F} \int u(f)\, dp$$

represents $\succsim$. Then $\succsim$ satisfies No Half Measures if and only if

$$V(f) = \min_{\ell \in R} \int_{S^{\infty}} u(f) d\ell^{\infty}$$

where R is finite.

The contribution of Theorem 5.3 is that in this symmetric MEU context, No Half Measures is equivalent to limiting the set of measures to the i.i.d. products generated by the relevant measures, R. The proof uses Ghirardato, Maccheroni and Marinacci [18, Theorem 14] to show that F is the Bewley set, and then applies Lemma 3.4 and Theorems 5.1 and 5.2 to obtain the result. Note that the supposition in the first sentence of the theorem could equivalently be replaced by assuming $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim$ and Monotone Continuity of $\succsim^*$ and the axioms in Gilboa and Schmeidler [21].

We can apply Theorem 5.3 to the Extended MEU with contraction model to get the following:

Corollary 5.4. *Suppose there is a non-constant vN-M utility function u, a $\beta \in [0,1]$, a finite set $D \subseteq \Delta(S)$ and a probability measure $q = \int \ell^{\infty} dm(\ell)$ for an $m \in \Delta(D)$, such that,*

$$\begin{aligned} V(f) &\equiv \beta \min_{p \in \{\ell^{\infty} : \ell \in D\}} \int u(f)\, dp + (1-\beta) \int u(f)\, dq \\ &= \min_{p \in \{\beta \ell^{\infty} + (1-\beta) q : \ell \in D\}} \int u(f)\, dp \end{aligned}$$

represents $\succsim$. Then $\succsim$ satisfies No Half Measures if and only if ($\beta = 1$ or ($q \in \{\ell^{\infty} : \ell \in D\}$ and ($D = \{q\}$ or $\beta = 0$))).

[11] Similarly, though we do not fully investigate it, No Half Measures imposed on the VEU model will imply non-trivial restrictions on the adjustment function A and the adjustment factors ζ_i.

To see that this is a strong condition, observe that the restriction ($\beta = 1$ or ($q \in \{\ell^\infty : \ell \in D\}$ and ($D = \{q\}$ or $\beta = 0$))) is equivalent to MEU without contraction ($\beta = 1$) or expected utility with an i.i.d. prior, ruling out all intermediate cases.

5.2. α-MEU Model

We have shown earlier that when the set of measures in the α-MEU representation consists of i.i.d. product measures that the set of relevant measures, R, may be recovered directly from the set appearing in the representation. Section 4.3 showed that the Bewley set, C, generally will depend on α as well as the set appearing in the representation. Since No Half Measures ensures C is fully determined by R, it must eliminate the dependence of C on α. When does this dependence occur? Our next example (see also Eichberger, Grant and Kelsey [12, Section 4] and Eichberger et. al. [13, section 3.2.2]), suggests that $\alpha \in (0,1)$ creates dependence.

Example 5.5. *Let $S = \{H, T\}$ and denote $\ell \in \Delta(S)$ by the corresponding probability of H. Consider the $\succsim$ represented by*

$$\frac{3}{4} \min_{p \in co\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}} \int u(f)\, dp + \frac{1}{4} \max_{p \in co\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}} \int u(f)\, dp.$$

Theorem 4.1 implies that $R = \{\frac{2}{3}, \frac{1}{4}\}$. Using the results of Siniscalchi ([35], [37]), the Bewley set C is given by the formula in Step $2'$, part b of Section 4.3. In this example, $C = co\{\frac{3}{4}\frac{2}{3}^\infty + \frac{1}{4}\frac{1}{4}^\infty, \frac{1}{4}\frac{2}{3}^\infty + \frac{3}{4}\frac{1}{4}^\infty\}$, the convex hull of some strict *convex combinations of $\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}$ (i.e., convex combinations that are not themselves i.i.d.) and is a proper subset of co $\{\ell^\infty : \ell \in R\}$. The reason for this is α being different from 0 and 1. Thus, in this example, the Bewley set C is influenced by α when $\alpha \in (0,1)$.*

Our next result shows that the troublesome case in the example (i.e., $\alpha \in (0,1)$ and R non-singleton) is exactly what is ruled out by No Half Measures. In this sense, No Half Measures turns out to be extremely restrictive in the context of α-MEU:

Theorem 5.6. *Suppose there is a non-constant vN-M utility function u, an $\alpha \in [0,1]$ and a finite set $D \subseteq \Delta(S)$ such that*

$$V(f) \equiv \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp + (1-\alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp$$

represents $\succsim$. Then No Half Measures is equivalent to ($\alpha = 0$ or $\alpha = 1$ or D singleton).

Eichberger et. al. [13, Theorem 2] prove a related result – for any finite state space, when C is non-singleton, any $\succsim$ having an α-MEU representation with Bewley set C as the set of measures has α equal to 0 or 1. Since Theorem 4.1 implies $D = R$ and, under No Half Measures, Theorem 5.1 tells us that $C = \{\int \ell^\infty dm\,(\ell) : m \in \Delta(R)\}$, if S^∞ were a finite state space, Theorem 5.6 would follow from Eichberger et. al. [13]'s result combined with our Theorems 4.1 and 5.1. Since S itself, let alone S^∞, need not be finite our result requires different arguments. In fact, Eichberger et. al. [13] provide an example with an infinite compact metric state space showing that their result may fail in such settings. In light of this, our theorem shows that Event Symmetry and our continuity together with the product structure of the state space is sufficient to extend their finite state space conclusion.

5.3. Smooth Ambiguity Model

Theorem 5.7. *Suppose there is a non-constant vN-M utility function u, a strictly increasing continuously differentiable function $\phi : u(X) \to \mathbb{R}$ and a Borel probability measure $\mu \in \Delta(\Delta(S))$ with $\operatorname{supp}\mu$ finite such that*

$$U(f) \equiv \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)d\ell^\infty\right) d\mu\,(\ell)$$

represents $\succsim$. Then No Half Measures is equivalent to ($\sup \bigcup_{r,t\in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$ or $\operatorname{supp}\mu$ singleton).

Some examples violating the condition $\sup \bigcup_{r,t\in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$ include $\phi(x) = x$ or $\phi(x) = 2x - \cos(x)$ with $u\,(X) = \mathbb{R}_{++}$ or $\mathbb{R}_+$, and $\phi(x) = x + \ln(1+x)$ with $u\,(X) = \mathbb{R}_{++}$ or $\mathbb{R}_+$. When utility is bounded both above and below (e.g., there is a best and a worst outcome), it is easy to find examples violating the condition. Examples satisfying the condition in the theorem include, $\phi\,(u) = -\exp\,(-u/\theta)$ or $\phi(x) = x - \exp\,(-x/\theta)$ with $u\,(X) = \mathbb{R}$ and $\phi\,(x) = \frac{x^{1-\gamma}}{1-\gamma}$ or $\phi(x) = x + \frac{x^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ with $u\,(X) = \mathbb{R}_{++}$ or $\mathbb{R}_+$.

Following the statement of the No Half Measures condition, we explained that this condition fails when those measures that sometimes get positive weight differ from those that sometimes get arbitrarily close to full weight (where "weight" is measured, as explained earlier, by utility trade-offs between the limiting frequency event corresponding to the measure and its complementary event). By characterizing No Half Measures in the context of the Extended MEU with contraction, α-MEU and the smooth ambiguity models, we have seen three concrete illustrations of when the Bewley approach, based on (magnitudes of) these positive weights or marginal rates of substitution, will involve parameters usually thought of as tastes (e.g., β, α and ϕ) and when it will not. The cases where it will involve these parameters are extensive: For Extended MEU with contraction, No Half Measures fails except for MEU with *no* contraction or expected utility with an i.i.d. prior. For α-MEU, No Half Measures fails in essentially all the cases beyond MEU and max-max EU. For the smooth ambiguity model, No Half Measures fails whenever the variation in the slope of ϕ is bounded and μ is non-degenerate.

6. Using Event Symmetry and Relevance to provide foundations for decision models

In this section, we characterize the α-MEU and smooth ambiguity models under our symmetry and continuity assumptions. In this setting, our α-MEU characterization improves on the existing result of Ghirardato, Maccheroni and Marinacci [18, Proposition 19] who characterize α-MEU when the set of measures appearing in the representation is the Bewley set. Recalling the discussion in Section 5.2, their characterization implies $\alpha = 0$ or 1 or preferences are expected utility. Ours allows the full range of $\alpha \in [0, 1]$ by taking an approach based on relevant measures rather than the Bewley set.

Our smooth ambiguity model characterization is entirely in terms of preferences over acts. An advantage relative to that in Klibanoff, Marinacci and Mukerji [26] is that their second order acts are not required, while an advantage relative to Seo [34] is that failure to reduce objective compound lotteries is no longer implied by non-neutral attitudes to ambiguity. The objects playing the role of second order acts are the acts whose payoffs depend only on events based on limiting frequencies (i.e., events in Σ^{Ψ}). Independently, de Castro and Al-Najjar [6] and Cerreia-Vioglio et. al. [7] have recently developed results similar to our smooth ambiguity model characterization.

An advantage of characterizing these two models in the same framework, is that it becomes possible to compare them on their whole domain of preferences. As we will see below, the difference between the two models is that α-MEU satisfies the Certainty Independence axiom of Gilboa and Schmeidler [21] and an axiom we call Relevant Range, while the smooth ambiguity model need not, and when it is not expected utility, cannot. Conversely, the smooth ambiguity model must satisfy the axioms of expected utility when restricted to acts whose payoffs depend only on events based on limiting frequencies, while the α-MEU model need not. In particular, when restricted to those acts, the α-MEU model reduces to a representation of preferences under complete ignorance proposed by Hurwicz [24], [25] and axiomatized by Arrow and Hurwicz [3] with state space equal to the set of relevant measures:[12]

$$V(f) \equiv \alpha \min_{\ell \in R} u(f) + (1-\alpha) \max_{\ell \in R} u(f). \tag{6.1}$$

Thus an important aspect of the comparison of the two models is the comparison of expected utility with Hurwicz preferences on the domain of acts based on limiting frequencies.

6.1. Alpha-MEU Model

We will show that under Symmetry, Mixture Continuity of $\succsim$ and Monotone Continuity of $\succsim^*$, α-MEU is what results from strengthening C-completeness to Completeness, Risk Independence to Gilboa and Schmeidler [21]'s Certainty Independence (stated below) and adding an axiom making use of the following definition:

$$\begin{aligned} C^*(f) \quad \equiv \quad & \{x \in X : x \succsim \int f d\ell^\infty \text{ for at least one relevant } \ell \\ & \text{and } \int f d\ell^\infty \succsim x \text{ for at least one relevant } \ell\}. \end{aligned}$$

The set $C^*(f)$ consists of the lotteries that (in terms of preference) lie in the range of valuations of f by relevant measures (i.e., between the best and worst lotteries formed by combining f with ℓ^∞ for $\ell \in R$). It is straightforward to see that, when $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$,

$$u(C^*(f)) = [\min_{\ell \in R} \int u(f)\, d\ell^\infty, \max_{\ell \in R} \int u(f)\, d\ell^\infty].$$

[12] Since these acts are measurable with respect to limiting frequency events, $u(f)$ evaluated at $\ell \in \Delta(S)$ is well-defined and equal to $u(f(\Psi^{-1}(\ell)))$.

The new axiom needed to characterize α-MEU says that if two acts have the same sets C^* then the individual must be indifferent between them. In other words, equality of the above range is sufficient to determine indifference.

Axiom 9 (Relevant Range). *$C^*(f) = C^*(g)$ implies $f \sim g$*

We also need the following strengthening of Risk Independence, introduced by Gilboa and Schmeidler [21],

Axiom 10 (Certainty Independence). *For all $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0,1)$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)\, x \succsim \alpha g + (1-\alpha)\, x$.*

The next result shows that Symmetry, Mixture Continuity of $\succsim$ and Monotone Continuity of $\succsim^*$, when strengthened by adding Relevant Range, replacing C-completeness with Completeness and Risk Independence with Certainty Independence, characterizes the α-MEU model in symmetric environments.

Theorem 6.1. *The following are equivalent: (1) $\succsim$ satisfies Symmetry with C-complete preorder replaced by Complete Preorder and Risk Independence replaced by Certainty Independence, and satisfies Mixture Continuity of $\succsim$, Monotone Continuity of $\succsim^*$ and Relevant Range; (2) There is a non-constant vN-M utility function u, an $\alpha \in [0,1]$ and a finite set $D \subseteq \Delta(S)$ such that*

$$V(f) \equiv \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp + (1-\alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f)\, dp \tag{6.2}$$

represents $\succsim$.
Furthermore, D is unique and $D = R$, α is unique if D is non-singleton and u is unique up to positive affine transformations.

This characterizes the α-MEU model, albeit limited to symmetric environments and finitely generated sets of countably additive measures. We note that the only previous characterization of the α-MEU model is that in Ghirardato, Maccheroni and Marinacci [18, Proposition 19]. Their Axiom 7 is like Relevant Range except that it uses the range generated by measures in the Bewley set C rather than measures in R. Since, in our setting, $C \subseteq \{\int \ell^\infty dm(\ell) : m \in \Delta(R)\}$, constructing the set C^* using C rather than R will shrink C^*. Combining Ghirardato, Maccheroni and Marinacci [18, Proposition 19] with Theorem 6.1, we can conclude that the additional restriction imposed by their Axiom 7 in terms of the representation (6.2) is exactly that $C = \{\int \ell^{\infty} dm(\ell) : m \in \Delta(R)\}$. As follows from Theorems 5.1 and 5.6, whenever this restriction holds, either α is 0 or 1 or the set of measures in the representation is a singleton. That is, under their Axiom 7, $\succsim$ must be MEU or max-max EU, in which cases the earlier characterization results of Gilboa and Schmeidler [21] already apply. In this sense, our result shows that in a symmetric environment, the difference between the α-MEU model and the union of the MEU and max-max EU models is exactly the difference between Relevant Range and Ghirardato, Maccheroni and Marinacci [18]'s Axiom 7.

6.2. Smooth Ambiguity Model

The following two assumptions will be the key additions needed for our smooth ambiguity representation.

Recall that Σ^{Ψ} is the σ-algebra generated by the sets

$$\Psi^{-1}(\ell) \equiv \{\omega : \Psi(\omega) = \ell\} \text{ for } \ell \in \Delta(S).$$

Let $\mathcal{F}^{\Psi}$ be the set of simple Σ^{Ψ}-measurable acts in $\mathcal{F}$, i.e. bets on the empirical frequency limit.

Assumption 1 (Expected utility on $\mathcal{F}^{\Psi}$). $\succsim$ on $\mathcal{F}^{\Psi}$ satisfies Wakker [39, Theorem V.6.1]'s axioms when acts in $\mathcal{F}^{\Psi}$ are viewed as acts from $\Delta(S)$ to X.

To apply Wakker [39, Theorem V.6.1] we need to specify an algebra on $\Delta(S)$ and a topology on X. For this purpose, take the Borel σ-algebra of sets in $\Delta(S)$ generated by the relative weak* open sets, and endow X with the weak convergence (wc) topology. The wc topology on X is the weakest topology for which all functions $x \longmapsto \int \psi dx$ are continuous for all bounded continuous ψ on Z. Also note that a sequence $x_n \in X$ converges to $x \in X$ under the wc topology if and only if $\int \psi dx_n \to \int \psi dx$ for all bounded continuous ψ on Z.

Expected utility on $\mathcal{F}^{\Psi}$ plays a role analogous to Klibanoff, Marinacci and Mukerji [26]'s expected utility assumption on second order acts. It is important to note that Expected utility on $\mathcal{F}^{\Psi}$ does *not* imply that the individual views events of the form $\Psi^{-1}(\ell)$ as unambiguous. This can be made formal, for example, by applying the preference-based definition of unambiguous events given by Klibanoff, Marinacci and Mukerji [26, Definition 7].

Our second assumption is a continuity requirement. Let $\hat{\mathcal{F}}$ denote the set of all bounded and measurable functions from Ω to X.[13] Ghirardato and Siniscalchi [19] propose a notion of convergence that they show corresponds to sup-norm convergence in the space of utility acts. Following them, we say $f_k \in \hat{\mathcal{F}}$ *norm-converges* to $f \in \hat{\mathcal{F}}$ if for all $x, y \in X$ with $x \succ y$, there exists K such that $k \geq K$ implies for all $\omega \in \Omega$

$$\tfrac{1}{2}f(\omega) + \tfrac{1}{2}y \prec \tfrac{1}{2}f_k(\omega) + \tfrac{1}{2}x \text{ and } \tfrac{1}{2}f_k(\omega) + \tfrac{1}{2}y \prec \tfrac{1}{2}f(\omega) + \tfrac{1}{2}x.$$

Ghirardato and Siniscalchi [19] propose the following continuity condition using norm-convergence:

Assumption 2 (Cauchy continuity). Consider sequences $f_k \in \mathcal{F}$, $x_k \in X$ such that f_k norm-converges to $f \in \hat{\mathcal{F}}$. If $f_k \sim x_k$ for all k, then there exists $x \in X$ such that x_k norm-converges to x.

In proving the theorem below, it is useful to work with preference on the larger space $\hat{\mathcal{F}}$. In the presence of other standard conditions, Ghirardato and Siniscalchi [19] show that Cauchy continuity is necessary and sufficient for the existence of a continuous and monotonic extension of $\succsim$ from $\mathcal{F}$ to $\hat{\mathcal{F}}$.

Theorem 6.2. *$\succsim$ satisfies Symmetry with C-complete preorder replaced by Complete Preorder, Mixture Continuity of $\succsim$, Monotone Continuity of $\succsim^*$, Expected utility on $\mathcal{F}^\Psi$ and Cauchy continuity if and only if there is a non-constant wc continuous vN-M utility function u, a strictly increasing continuous function $\phi : u(X) \to \mathbb{R}$ and a Borel probability measure $\mu \in \Delta(\Delta(S))$ such that*

$$U(f) = \int_{\Delta(S)} \phi \left(\int_{S^\infty} u(f)\, d\ell^\infty \right) d\mu(\ell) \tag{6.3}$$

represents $\succsim$ and either (i) there are $m, M > 0$ such that $m|a-b| \leq |\phi(a) - \phi(b)| \leq M|a-b|$ for all $a, b \in u(X)$ or, (ii) $\operatorname{supp}\mu$ is finite. Moreover, μ is unique, u is unique up to a positive affine transformation, and, given a normalization of u, ϕ is unique up to a positive affine transformation.

[13] More precisely, $\hat{\mathcal{F}}$ is the collection of functions $f : \Omega \to X$ that satisfy the following two properties:
(i) for all $x \in X$, $\{\omega : f(\omega) \succ x\} \in \Sigma$; and
(ii) there exist $x, y \in X$ such that $x \succsim f(\omega) \succsim y$ for all $\omega \in \Omega$.

The above provides foundations for the smooth ambiguity model using the Event Symmetry requirement. Note that the restriction that (i) or (ii) holds is solely to ensure Monotone Continuity of $\succsim^*$. Theorem 6.2 is completely analogous to the smooth ambiguity representation theorems in Klibanoff, Marinacci and Mukerji [26] and Seo [34] with the additional assumption that the environment is known to be symmetric. For each, the key assumptions are (1) conditions equivalent to expected utility over lotteries, (2) conditions equivalent to expected utility over acts in $\mathcal{F}^\Psi$ (resp. second order acts and lotteries over acts) and (3) Event Symmetry, which we show later in Theorem 7.2 is equivalent to strengthening Klibanoff, Marinacci and Mukerji [26]'s Consistency or Seo [34]'s Dominance to reflect symmetry. Thus, our representation result shows how, in a symmetric setting, objects like second order acts or lotteries over acts can be replaced by particular standard acts related to frequencies. See the discussion in Klibanoff, Marinacci and Mukerji ([26, pp. 1854 and 1856], [27, p. 937]) for the idea that objects like second order acts could, with enough invariance, be replaced by acts based on long run outcomes of repeated trials. An advantage relative to Klibanoff, Marinacci and Mukerji [26] is that second order acts are made more concrete, while an advantage relative to Seo [34] is that failure to reduce objective compound lotteries is no longer implied by non-neutral attitudes to ambiguity.

We see that μ is uniquely determined by expected utility preferences over "frequency acts" (i.e., Σ^Ψ-measurable acts) and thus, it expresses beliefs over the events in Σ^Ψ in the same sense as the prior in an expected utility representation. Notice that ϕ is unique up to positive affine transformations only *given* a normalization of u. Should one worry that normalization of u is needed to pin down ϕ, and thus to pin down ambiguity attitude? The answer is no. Expected utility preferences over monetary lotteries have their risk aversion as measured by the Arrow-Pratt index depend on the currency used to denominate money. This in no way means that risk attitudes are non-unique. Similarly, the Arrow-Pratt index of ϕ, identified by Klibanoff, Marinacci and Mukerji [26] as measuring ambiguity attitude, depends on the units used to measure utility, and this does not affect the unique identification of ambiguity attitudes.

In Klibanoff, Marinacci and Mukerji [26], the joint uniqueness of μ and ϕ relied on representing preferences over second order acts as well as acts. In the present result, uniqueness is obtained solely through preferences over acts. To make the analogy clear, (μ, ϕ, u) is the unique (as in the statement of Theorem 6.2) triple such that (i) $\succsim$ restricted to acts depending only on S are represented by

$$\int_{\Delta(S)} \phi\left(\int_S u(f)\, d\ell\right) d\mu(\ell)$$

and (ii) $\succsim$ restricted to acts in $\mathcal{F}^{\Psi}$ (i.e., frequency acts) are represented by

$$\int_{\Delta(S)} \phi\left(u\left(f(\Psi^{-1}(\ell))\right)\right) d\mu(\ell).$$

Thus (i) and (ii) are exactly the same as in Klibanoff, Marinacci and Mukerji [26] with acts in $\mathcal{F}^{\Psi}$ substituting for their second order acts in (ii) and S substituting for their state space on which acts were defined in (i). The role that Event Symmetry plays in enabling this substitution is that it allows μ to be a measure over S rather than S^{∞}.

7. Relating Event Symmetry to the literature

A key to our results is Event Symmetry. We now show that this condition relates quite closely to a variety of other conditions from the literature, including strengthenings of de Finetti [10]'s Exchangeability, Hewitt and Savage [23]'s Symmetry, of Seo [34]'s Dominance and of Klibanoff, Marinacci and Mukerji [26]'s Consistency. One of those conditions (condition (viii) below) requires some additional definitions.

Definition 7.1. *For $f \in \mathcal{F}$, f^{Ψ} is the (not necessarily simple) act uniquely defined as follows:*

$$f^{\Psi}(\omega) = \begin{cases} \int_{S^{\infty}} f d\ell^{\infty} & \text{if } \ell = \Psi(\omega) \in \Delta(S) \\ \delta_{x^*} & \{\omega : \Psi(\omega) \text{ is not defined}\} \end{cases}.$$

Note this definition associates with each act f an act f^{Ψ} that, for each event $\{\omega : \Psi(\omega) = \ell\}$ corresponding to the limiting frequencies generated by ℓ, yields the lottery generated by f under the assumption that the i.i.d. process ℓ^{∞} governs the realization of the state.

Since f^{Ψ} need not be simple, but is an element of the space $\hat{\mathcal{F}}$ (defined in Section 6.2) of all bounded and measurable functions from Ω to X, it is necessary to consider extending $\succsim$ to $\hat{\mathcal{F}}$. In particular, we consider extensions continuous in the following sense: $\hat{\succsim}$ on $\hat{\mathcal{F}}$ satisfies *Norm Continuity* if $f \hat{\succsim} g$ whenever $f_k \hat{\succsim} g_k$ for all $k = 1, 2, ...$ and f_k and g_k norm-converge to f and g respectively.

Theorem 7.2. *The following conditions are equivalent under Preorder and Mixture Continuity of* $\succsim$*:*
(i) for every $f \in \mathcal{F}$ *and* $\pi \in \Pi$, $f \sim \frac{1}{2}f + \frac{1}{2}\pi f$,
(ii) for every $f \in \mathcal{F}$, $\pi \in \Pi$ *and* $\alpha \in [0,1]$, $f \sim \alpha\pi f + (1-\alpha) f$,
(iii) for every $f \in \mathcal{F}$ *and* $\pi_i \in \Pi$, $f \sim \frac{1}{n}\sum_{i=1}^{n} \pi_i f$,
(iv) for every $f \in \mathcal{F}$, $\pi_i \in \Pi$ *and* $\alpha_i \in [0,1]$ *with* $\sum_{i=1}^{n} \alpha_i = 1$, $f \sim \sum_{i=1}^{n} \alpha_i \pi_i f$, *and*
(v) for every $f \in \mathcal{F}$ *and* $\pi \in \Pi$, $f \sim^* \pi f$.
Moreover, the above are equivalent to each of the following under C-complete Preorder, Mixture Continuity of $\succsim$*, Monotonicity, Risk Independence, Non-triviality and Monotone Continuity of* $\succsim^*$*:*
(vi) Event Symmetry,
(vii) for every $f, g \in \mathcal{F}$*, if* $\int f dp \succsim \int g dp$ *for all symmetric* $p \in \Delta(S^\infty)$*, then* $f \succsim g$.
Finally, if, in addition, there exists an extension of $\succsim$ *to* $\hat{\mathcal{F}}$ *satisfying Preorder and Norm Continuity, then the following is equivalent to all of the above:*
(viii) for $f, g \in \mathcal{F}$, $f \succsim g$ *if and only if* $f^\Psi \hat{\succsim} g^\Psi$*, if* $\hat{\succsim}$ *is any such extension.*

All of these conditions are strengthenings of Hewitt and Savage [23]'s symmetry: given $p \in \Delta(S^\infty)$, $p(A) = p(\pi A)$ for all finite cylinder events $A \in \Sigma$. In terms of preference, this translates into $1_A \sim 1_{\pi A}$ for all such events and all permutations $\pi \in \Pi$. Event Symmetry strengthens this by requiring the indifference to be preserved under mixture with any common third act. Under Preorder and Mixture Continuity of $\succsim$, conditions (i)-(v) each imply Event Symmetry.

Condition (ii) is closely related to Epstein and Seo [15]'s Strong Exchangeability, the first behavioral axiom in the literature that captures the idea that the agent views all experiments as identical, i.e., i.i.d. (See Epstein and Seo [15] for a behavioral interpretation of condition (ii). A similar interpretation applies to (i), (iii) and (iv).) Their axiom states that condition (ii) holds when f depends only on a finite number of experiments. However, under their regularity axiom, their Strong Exchangeability extends to every act f and hence is equivalent to condition (ii).

Condition (i) is a special case of condition (ii) when $\alpha = \frac{1}{2}$. Since Epstein and Seo [15] consider MEU models, they could restrict to the case $\alpha = \frac{1}{2}$. The above theorem shows that $\alpha = \frac{1}{2}$ is sufficient to capture the same idea in general as long as Preorder and Mixture Continuity of $\succsim$ hold.

De Castro and Al-Najjar ([5], [6]) use condition (iii) and its generalization to collections of transformations Γ other than the finite permutations Π. They provide conditions on Γ under which complete, transitive, monotonic, continuous and risk independent preferences satisfying (iii) with respect to Γ are such that each act is indifferent to an associated act based on limiting frequencies (where the notion of limiting frequencies uses the given Γ) where the associated act is constructed much like f^{Ψ} above with Γ-ergodic measures replacing the i.i.d. measures as parameters. As was mentioned in Section 6.2, they also prove a representation result similar to Theorem 6.2 using an expected utility assumption on parameter-based acts added to the conditions on preferences and on Γ mentioned in the previous sentence.

Condition (iv) strengthens condition (iii).

Condition (v) is stronger than Event Symmetry in that f is not necessarily a binary act.

Condition (vii) is analogous to Seo [34]'s Dominance and Cerreia-Vioglio et. al.[7]'s Consistency, and condition (viii) to Klibanoff, Marinacci and Mukerji [26]'s Consistency. Seo's Dominance is stated with lotteries over acts, objects that are not available in the domain of this paper, and condition (vii) restricts $p \in \Delta(S^{\infty})$ to be symmetric while p is unrestricted in Seo's Dominance. Thus, the difference between the two is that Seo's Dominance requires f to induce a better lottery than g under all processes, not just symmetric ones. The reason for the additional restriction here is to reflect the fact that the experiments are symmetric. We want to include as reflecting dominance, for example, the following case:

$$f(\omega) = x^* H_1 x_* \text{ and } g(\omega) = \left(\tfrac{1}{2}x^* + \tfrac{1}{2}x_*\right) H_2 x_*$$

where $H_i = \{\omega \in S^{\infty} : \omega_i = H\}$ and $S = \{H, T\}$. The act f is a bet that the first coin comes up heads, and the act g is a bet that the second coin comes up heads but with a less valuable reward for winning. Under symmetry of the experiments, it is intuitively clear that f is better than g. Condition (vii) indeed implies that $f \succsim g$, while Seo's Dominance would not – for example, when $p = \delta_{THTTT....}$, $\int g dp \succ \int f dp$.

Cerreia-Vioglio et. al. [7]'s Consistency is similar to condition (vii), but instead of considering transformations (e.g. permutations) and/or all symmetric measures, they assume a family of objectively rational probability measures on a general state space and require $\int f dp \succsim \int g dp$ for all measures p in that family. Taking the family to be all symmetric measures makes Cerreia-Vioglio et. al.'s Consistency exactly condition (vii). With their Consistency they provide representation theorems for Bewley preference, Choquet expected utility, variational preferences, uncertainty averse preferences and, as was mentioned earlier, the smooth ambiguity model. Also, they prove that the generalization of condition (v) to other collections of transformations Γ together with some (mild) axioms imply their Consistency when the family of objectively rational measures is the Γ-invariant measures.

The content of condition (viii) is that given the "knowledge" that everything is driven by some (as yet unknown) i.i.d. process, it seems reasonable that when evaluating an act, the individual would ultimately care only about the induced mapping from the space of i.i.d. processes to the lotteries generated under each process. Klibanoff, Marinacci and Mukerji [26]'s Consistency assumption says that when evaluating an act, an individual cares only about the induced mapping from probability measures on the state space to lotteries. Their assumption was stated in terms of acts and "second order acts" (maps from probability measures on the state space to outcomes). The latter objects do not appear as such in the present paper, but their role is played by the subset of acts measurable with respect to limiting frequency events, the acts $f^{\Psi} \in \hat{\mathcal{F}}$. The identification (in terms of preference) of f with f^{Ψ} stated in condition (viii) is analogous to the identification of f with an "associated second order act" in Klibanoff, Marinacci and Mukerji [26]'s Consistency with the qualification that f^{Ψ} only induces the same mapping from probability measures to lotteries as f for i.i.d. probability measures. Thus, condition (viii) strengthens Klibanoff, Marinacci and Mukerji [26]'s Consistency to incorporate the known symmetry of the ordinates in the same way that condition (vii) strengthened Seo [34]'s Dominance. Theorem 7.2 says that, under our other axioms, each of these strengthenings is equivalent to Event Symmetry. One can view the equivalence of condition (iii) and condition (viii) as following from the i.i.d. case of the sufficient statistic result in de Castro and Al-Najjar [6]. The equivalence of condition (viii) to the other conditions informed our discussion of Theorem 6.2 in Section 6.2.

8. Appendix: Proofs

Denote by $B(S)$ the set of bounded measurable functions on S. Similarly for $B(\Delta(S))$ and $B(S^{\infty})$. When B is replaced by B_0 this denotes the restriction to simple measurable functions. Similarly, when B_0 is replaced by B_0^l (resp. B_0^u) this denotes the restriction to lower (resp. upper) semicontinuous simple functions. For $b \in B(S^{\infty})$, we write $\|b\|$ for the sup-norm of b (i.e., $\sup_{\omega} |b(\omega)|$).

8.1. Proof of Lemma 3.4

We prove sufficiency of the stated axioms, first. We first show that $\succsim^*$ satisfies the properties assumed in Gilboa et. al. [20, Theorem 1]. Preorder, Monotonicity, Mixture Continuity, Non-triviality, C-Completeness and Independence of $\succsim^*$ follow directly from the axioms we assume and the definition of $\succsim^*$. Therefore, by Gilboa et. al. [20, Theorem 1], there exists a unique non-empty weak* closed and convex set $C \subseteq ba_1^+ (S^\infty)$ and a non-constant vN-M utility function, $u : X \to \mathbb{R}$, such that

$$f \succsim^* g \text{ if and only if } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in C.$$

By Alaoglu's Theorem, C is weak* compact. Monotone Continuity of $\succsim^*$ implies $C \subseteq \Delta(S^\infty)$ by Ghirardato, Maccheroni and Marinacci [18, Remark 1]. Moreover, Event Symmetry implies every $p \in C$ is symmetric on finite cylinder events and hence every $p \in C$ is of the form $\int \ell^\infty dm(\ell)$ for some $m \in \Delta(\Delta(S))$ by the de Finetti Theorem (see e.g., Hewitt and Savage [23]). Thus, $C = \left\{ \int \ell^\infty dm(\ell) : m \in M \right\}$ for some non-empty $M \subseteq \Delta(\Delta(S))$. It is clear that M is convex.

To see that M is weak* compact, take any net $m_\alpha \in M$. Since C is weak* compact, there is a converging subnet m'_λ of m_α such that

$$\int \left(\int \varphi d\ell^\infty \right) dm'_\lambda(\ell) \to \int \left(\int \varphi d\ell^\infty \right) dm'(\ell) \text{ for each } \varphi \in B(S^\infty).$$

Recall that since $\int \phi d\hat{m} = \int \left(\int \phi \circ \Psi d\ell^\infty \right) d\hat{m}$ and $\phi \circ \Psi \in B(S^\infty)$ for each $\hat{m} \in \Delta(\Delta(S))$ and $\phi \in B(\Delta(S))$, m'_λ converges to m'.

Uniqueness of M follows from uniqueness of C.

To show necessity, assume such a set M. It is clear that $\succsim^*$satisfies Monotonicity and Risk Independence and thus $\succsim$ inherits these properties as well. Event Symmetry follows since each element of C is of the form $\int \ell^\infty dm(\ell)$ for some $m \in M$. Non-triviality of $\succsim$ follows from non-constancy of u. Monotone Continuity of $\succsim^*$ follows from weak* compactness of C, which is implied by that of M. Mixture Continuity of $\succsim^*$ follows from Mixture Continuity of expected utility and the fact that intersections of closed sets are closed. □

8.2. Proof of Theorem 3.6

We begin by showing that R is relative weak* closed. The set R is relative weak* closed if it equals $\Delta(S) \cap K$ for some weak* closed $K \subseteq ba(S)$. Consider K equal to the weak* closure of R. That $R \subseteq \Delta(S) \cap K$ is direct. To show $\Delta(S) \cap K \subseteq R$, consider any limit point $\widehat{\ell} \in \Delta(S)$ of R. Lemma 3.4 implies that no ℓ outside of R is relevant – if $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in R$ then $\int f dp = \int g dp$ for all $p \in C$ and thus $f \sim^* g$ implying $f \sim g$. To show that $\widehat{\ell} \in R$, it therefore suffices to show that $\widehat{\ell}$ is relevant. Fix $L \in \mathcal{O}_{\widehat{\ell}}$. Then, $\left(L \backslash \left\{\widehat{\ell}\right\}\right) \cap R \neq \emptyset$. Choose any $\widetilde{\ell} \in \left(L \backslash \left\{\widehat{\ell}\right\}\right) \cap R$. Since $\widetilde{\ell}$ is relevant, for $\widetilde{L} \in \mathcal{O}_{\widetilde{\ell}}$, there are $f, g \in \mathcal{F}$ such that $f \nsim g$ and $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash \widetilde{L}$. Note that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash L \subseteq \Delta(S) \backslash \widetilde{L}$. Since L is an arbitrary set in $\mathcal{O}_{\widehat{\ell}}$, $\widehat{\ell}$ is relevant. Thus $R = \Delta(S) \cap K$ and R is relative weak* closed.

We next show that every $\ell \in R^c$ is betting irrelevant. Since, R^c is open, $\ell \in R^c$ implies there exists $L \in \mathcal{O}_\ell$ such that $L \subseteq R^c$. Note that $\int 1_{\Psi^{-1}(L)} d\hat{\ell}^\infty = 0 = \int 1_\emptyset d\hat{\ell}^\infty$ for all $\hat{\ell} \in R$. Thus, $\int 1_{\Psi^{-1}(L)} dp = 0 = \int 1_\emptyset dp$ for all $p \in C$. By Lemma 3.4, $1_{\Psi^{-1}(L)} \sim^* 1_\emptyset$, showing that ℓ is betting irrelevant.

Next we show no $\ell \in R$ is betting irrelevant. Take any $\ell \in R$ and $L \in \mathcal{O}_\ell$. By Lemma 3.4, $1_{\Psi^{-1}(L)} \succsim^* 1_\emptyset$ since $\int 1_{\Psi^{-1}(L)} dp \geq \int 1_\emptyset dp$ for all $p \in \Delta(S^\infty)$. It remains to show that $1_{\Psi^{-1}(L)} \not\precsim^* 1_\emptyset$. Note that by definition of R there is $m \in M$ such that $L \cap \operatorname{supp} m \neq \emptyset$. Let $p = \int \hat{\ell}^\infty dm\left(\hat{\ell}\right)$ and compute

$$\int 1_{\Psi^{-1}(L)} dp = m(L) > 0 = \int 1_\emptyset dp.$$

By Lemma 3.4, $1_{\Psi^{-1}(L)} \not\precsim^* 1_\emptyset$. □

8.3. Proof of Corollary 3.7

If R is finite, $\ell \in R$ implies there is $m \in M$ such that $m(\ell) > 0$, thus, by Lemma 3.4, $1_{\Psi^{-1}(\ell)} \nsim^* 1_\emptyset$. If $\ell \notin R$, $1_{\Psi^{-1}(\ell)} \sim^* 1_\emptyset$ by Theorem 3.6.

8.4. Proof of Theorem 3.5

Observe that Lemma 3.4 implies that no ℓ outside of R is relevant – if $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in R$ then $\int f dp = \int g dp$ for all $p \in C$ and thus $f \sim^* g$ implying $f \sim g$. We now show that every element of R is relevant. Take any $\widehat{\ell} \in R$. By Theorem 3.6, $1_{\Psi^{-1}(L)} \nsim^* 1_\emptyset$ for all $L \in \mathcal{O}_{\widehat{\ell}}$. This implies that, for all $L \in \mathcal{O}_{\widehat{\ell}}$, it can't be true that $1_{\Psi^{-1}(L)} \precsim^* 1_\emptyset$. Hence, for all $L \in \mathcal{O}_{\widehat{\ell}}$, there is $h_L \in \mathcal{F}$ such that it is not true that $\alpha 1_{\Psi^{-1}(L)} + (1-\alpha) h_L \precsim \alpha 1_{\emptyset} + (1-\alpha) h_L$, which implies

$$\alpha 1_{\Psi^{-1}\left(\widehat{\ell}\right)} + (1-\alpha) h_L \nsim \alpha 1_{\emptyset} + (1-\alpha) h_L.$$

But for any $L \in \mathcal{O}_{\widehat{\ell}}$ and all $\ell \in \Delta(S) \backslash L$, $\int 1_{\Psi^{-1}(L)} d\ell^\infty = 0 = \int 1_{\emptyset} d\ell^\infty$. Thus, $\widehat{\ell}$ is relevant. This proves that R is the set of all relevant measures in $\Delta(S)$.

8.5. Proof of Theorem 3.8

Let $U : \mathcal{F} \rightarrow \mathbb{R}$ represent $\succsim$. Recall Lemma 3.4 guarantees the existence of a non-constant affine utility $u : X \rightarrow \mathbb{R}$ and a set C derived there from $\succsim$. Define G on $\left\{\tilde{f} \in [u(X)]^R : \tilde{f}(\ell) = \int u(f) d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$ by $G\left(\left(\int u(f) d\ell^\infty\right)_{\ell \in R}\right) = U(f)$, which is well-defined because $\int u(f) d\ell^\infty = \int u(g) d\ell^\infty$ for all $\ell \in R$ implies $\int u(f) dp = \int u(g) dp$ for all $p \in C$, which, by Lemma 3.4, implies $f \sim g$. Thus $f \longmapsto G\left(\left(\int u(f) d\ell^\infty\right)_{\ell \in R}\right)$ represents $\succsim$. Suppose

$$\hat{f}, \hat{g} \in \left\{\tilde{f} \in [u(X)]^R : \tilde{f}(\ell) = \int u(f) d\ell^\infty \text{ for some } f \in \mathcal{F}\right\}$$

are such that $\hat{f}(\ell) \geq \hat{g}(\ell)$ for all $\ell \in R$ and fix some corresponding acts f, g that generate these expected utilities. Since $\int u(f) d\ell^\infty \geq \int u(g) d\ell^\infty$ for all $\ell \in R$, $\int u(f) dp \geq \int u(g) dp$ for all $p \in C$. By Lemma 3.4, this implies $f \succsim g$. Therefore $G\left(\tilde{f}\right) = U(f) \geq U(g) = G(\tilde{g})$ which shows G is weakly increasing.

Uniqueness is shown as follows. Since every element in D is relevant, $D \subseteq R$ by Theorem 3.5. Since R is closed, $\overline{D} \subseteq R$. We show that $R \subseteq \overline{D}$. Suppose that $\widehat{\ell} \notin \overline{D}$ for some $\widehat{\ell} \in \Delta(S)$. Since $\overline{D}$ is closed, there exists $L \in \mathcal{O}_{\widehat{\ell}}$ such that $L \subseteq \Delta(S) \backslash \overline{D}$. Since $f \longmapsto H\left(\left(\int (\tilde{u}(f)) d\ell^\infty\right)_{\ell \in D}\right)$ represents $\succsim$, if $\int (\tilde{u}(f)) d\ell^\infty = \int (\tilde{u}(g)) d\ell^\infty$ for all $\ell \in D \subseteq \overline{D} \subseteq \Delta(S) \backslash L$, $f \sim g$. If $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in D \subseteq \overline{D} \subseteq \Delta(S) \backslash L$, then, because $\tilde{u}$ is affine, $\int (\tilde{u}(f)) d\ell^\infty = \int (\tilde{u}(g)) d\ell^\infty$ for all $\ell \in D \subseteq \overline{D} \subseteq \Delta(S) \backslash L$. Therefore, $\widehat{\ell}$ can't be relevant, and thus $\widehat{\ell} \notin R$ by Theorem 3.5. Uniqueness of u up to positive affine transformation is standard, as $\succsim$ restricted to constant acts is expected utility. $\square$

8.6. Proof of Theorem 4.1

Suppose $\succsim$ is represented by such a $V(f)$. We first show that all measures in D are relevant. Suppose $\hat{\ell} \in D$ and fix any open $K \subseteq \Delta(S)$ such that $\hat{\ell} \in K$. Consider

$f=1_{\Psi^{-1}(K)}$ and $g=1_{\emptyset}$ and observe that $\int fd\ell^{\infty}=\int gd\ell^{\infty}$ for all $\ell\in\Delta(S)\backslash K$. Note that $\int u(f)\,d\ell^{\infty}>\int u(g)\,d\ell^{\infty}$ for all $\ell\in K$ while $\int u(f)\,d\ell^{\infty}\geq\int u(g)\,d\ell^{\infty}$ for all $\ell\in D$. Thus, if $\alpha\in[0,1)$, $f\succ g$ and $\hat{\ell}$ is relevant. If $\alpha=1$, consider instead $f=\frac{1}{2}1_{\Psi^{-1}(K)}+\frac{1}{2}1_{\Psi^{-1}(\Delta(S)\backslash K)}$ and $g=\frac{1}{2}1_{\emptyset}+\frac{1}{2}1_{\Psi^{-1}(\Delta(S)\backslash K)}$ and observe that $\int fd\ell^{\infty}=\int gd\ell^{\infty}$ for all $\ell\in\Delta(S)\backslash K$ while $\min_{\ell\in D}\int u(f)\,d\ell^{\infty}=\frac{1}{2}u(x^{*})+\frac{1}{2}u(x_{*})>u(x_{*})=\min_{\ell\in D}\int u(g)\,d\ell^{\infty}$ so that $f\succ g$ and again $\hat{\ell}$ is relevant.

By Theorem 6.1, $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^{*}$ and Monotone Continuity of $\succsim^{*}$. Observe that $V(f)$ can be re-written as

$$\begin{aligned}&H\left(\left(\int u(f)\,d\ell^{\infty}\right)_{\ell\in D}\right)\\ \equiv\ &\alpha\min_{\ell\in D}\int u(f)\,d\ell^{\infty}+(1-\alpha)\max_{\ell\in D}\int u(f)\,d\ell^{\infty},\end{aligned}$$

and H so defined is weakly increasing. Therefore we may apply the uniqueness result in Theorem 3.8 to conclude $\overline{D}=R$. Since D is finite, $\overline{D}=D$. □

8.7. Proof of Theorem 4.2

Suppose $\succsim$ is represented by such a $U(f)$. We first show that all measures in $\operatorname{supp}\mu$ are relevant. Suppose $\hat{\ell}\in\operatorname{supp}\mu$ and fix any open $L\subseteq\Delta(S)$ such that $\hat{\ell}\in L$. Consider $f=1_{\Psi^{-1}(L)}$ and $g=1_{\emptyset}$ and observe that $\int fd\ell^{\infty}=\int gd\ell^{\infty}$ for all $\ell\in\Delta(S)\backslash L$. Since ϕ is strictly increasing, $\phi\left(\int u(f)\,d\ell^{\infty}\right)>\phi\left(\int u(g)\,d\ell^{\infty}\right)$ for all $\ell\in L$ and $\phi\left(\int u(f)\,d\ell^{\infty}\right)\geq\phi\left(\int u(g)\,d\ell^{\infty}\right)$ for all $\ell\in\operatorname{supp}\mu$. By the definition of $\operatorname{supp}\mu$, $\mu(L)>0$. Thus, $f\succ g$ and $\hat{\ell}$ is relevant.

Note that U meets all the requirements of the representation in Theorem 6.2, except possibly the wc continuity of u. This lack of continuity does not affect whether $\succsim$ satisfies Symmetry. Inspection of the necessity direction of the proof of Theorem 6.2 verifies that this continuity also plays no role in $\succsim$ satisfying Monotone Continuity of $\succsim^{*}$. We show directly that $\succsim$ satisfies Mixture Continuity of $\succsim$ (and thus Mixture Continuity of $\succsim^{*}$): Fix acts $f,g,h\in\mathcal{F}$ and consider a sequence $\lambda_n\in[0,1]$ such that $\lambda_n\to\lambda$ and $\lambda_n f+(1-\lambda_n)g\succsim h$ for all n. Therefore, for all n,

$$\begin{aligned}&\int_{\Delta(S)}\phi\left(\lambda_n\int_{S^{\infty}}u(f)d\ell^{\infty}+(1-\lambda_n)\int_{S^{\infty}}u(g)d\ell^{\infty}\right)d\mu(\ell)\\ \geq\ &\int_{\Delta(S)}\phi\left(\int_{S^{\infty}}u(h)d\ell^{\infty}\right)d\mu(\ell)\,.\end{aligned}$$

Since ϕ is continuous, by the Dominated Convergence Theorem (e.g., Aliprantis and Border [1, Theorem 11.21])

$$\begin{aligned} &\int_{\Delta(S)} \phi\left(\lambda_n \int_{S^\infty} u(f)d\ell^\infty + (1-\lambda_n)\int_{S^\infty} u(g)d\ell^\infty\right) d\mu\left(\ell\right) \\ \to\ &\int_{\Delta(S)} \phi\left(\lambda \int_{S^\infty} u(f)d\ell^\infty + (1-\lambda)\int_{S^\infty} u(g)d\ell^\infty\right) d\mu\left(\ell\right) \end{aligned}$$

so that $\lambda f + (1-\lambda)g \succsim h$.

Next, observe that $U(f)$ can be re-written as

$$\begin{aligned} &H\left(\left(\int u\left(f\right)d\ell^\infty\right)_{\ell \in \operatorname{supp}\mu}\right) \\ \equiv\ &\int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)d\ell^\infty\right) d\mu\left(\ell\right), \end{aligned}$$

and H so defined is weakly increasing. Therefore we may apply the uniqueness result in Theorem 3.8 to conclude $R = \overline{\operatorname{supp}\mu}$. Since $\operatorname{supp}\mu$ is relative weak* closed by definition, $\overline{\operatorname{supp}\mu} = D$. □

8.8. Proof of Theorem 4.3

Suppose $\succsim$ is represented by such a $W(f)$. We first show that all measures in D are relevant. Suppose $\hat{\ell} \in D$ and fix any open $L \subseteq \Delta(S)$ such that $\hat{\ell} \in L$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_{\emptyset}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash L$. Observe that $\int u(f)\, d\ell^\infty > \int u(g)\, d\ell^\infty$ for all $\ell \in L$ while $\int u(f)\, d\ell^\infty \geq \int u(g)\, d\ell^\infty$ for all $\ell \in D$ and thus also $\int u(f)\, dq \geq \int u(g)\, dq$. Therefore, if $\beta < 1$ and $q(\Psi^{-1}(L)) > 0$, $f \succ g$ and $\hat{\ell}$ is relevant. If either $\beta = 1$ or $q(\Psi^{-1}(L)) = 0$, consider instead $f = \frac{1}{2}1_{\Psi^{-1}(L)} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\backslash L)}$ and $g = \frac{1}{2}1_{\emptyset} + \frac{1}{2}1_{\Psi^{-1}(\Delta(S)\backslash L)}$ and observe that $\int f d\ell^\infty = \int g d\ell^\infty$ for all $\ell \in \Delta(S) \backslash L$ while $\min_{\ell \in D} \int u(f)\, d\ell^\infty = \frac{1}{2}u(x^*) + \frac{1}{2}u(x_*) > u(x_*) = \min_{\ell \in D} \int u(g)\, d\ell^\infty$ so that $f \succ g$ and again $\hat{\ell}$ is relevant.

We now show that $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$ via Lemma 3.4. To invoke the lemma, we demonstrate that $\succsim^*$ may be represented as in (3.1). Suppose $\int u(f)\, dp \geq \int u(g)\, dp$ for all $p \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}$. Fix any $\lambda \in [0,1]$ and acts $f, g, h \in \mathcal{F}$, and let

$\hat{\ell}^\infty \in \arg\min_{p\in\{\ell^\infty:\ell\in D\}} \int u\left(\lambda f+(1-\lambda)h\right)dp$. Then

$$\begin{aligned} & W\left(\lambda f+(1-\lambda)h\right) \\ = & \int u\left(\lambda f+(1-\lambda)h\right)d\left(\beta\hat{\ell}^\infty+(1-\beta)q\right) \\ \geq & \int u\left(\lambda g+(1-\lambda)h\right)d\left(\beta\hat{\ell}^\infty+(1-\beta)q\right) \\ \geq & W\left(\lambda g+(1-\lambda)h\right) \end{aligned}$$

so that $f \succsim^* g$. Going the other direction, suppose $f \succsim^* g$ and that there exists a $\hat{p} \in co\{\beta\ell^\infty+(1-\beta)q : \ell \in D\}$ such that $\int u(f)\,d\hat{p} < \int u(g)\,d\hat{p}$. This implies that there exists an $\hat{\ell} \in D$ such that $\int u(f)\,d\left(\beta\hat{\ell}^\infty+(1-\beta)q\right) < \int u(g)\,d\left(\beta\hat{\ell}^\infty+(1-\beta)q\right)$. Let $\hat{h} = 1_{\Psi^{-1}(D\setminus\hat{\ell})}$. Choose $\hat{\lambda} \in (0,1)$ small enough to satisfy

$$\begin{aligned} & (1-\hat{\lambda})(u(x^*)-u(x_*)) \\ > & \hat{\lambda}\max[\int u(f)\,d\hat{\ell}^\infty - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(f)\,dp, \int u(g)\,d\hat{\ell}^\infty - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(g)\,dp]. \end{aligned}$$

Then

$$\begin{aligned} & \min_{p\in\{\ell^\infty:\ell\in D\}}\int u\left(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\right)dp \\ = & \int u\left(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\right)d\hat{\ell}^\infty \\ < & \int u\left(\hat{\lambda}g+(1-\hat{\lambda})\hat{h}\right)d\hat{\ell}^\infty \\ = & \min_{p\in\{\ell^\infty:\ell\in D\}}\int u\left(\hat{\lambda}g+(1-\hat{\lambda})\hat{h}\right)dp. \end{aligned}$$

Therefore, as $\beta > 0$,

$$\begin{aligned} & W\left(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\right) \\ = & \int u\left(\hat{\lambda}f+(1-\hat{\lambda})\hat{h}\right)d\left(\beta\hat{\ell}^\infty+(1-\beta)q\right) \\ < & \int u\left(\hat{\lambda}g+(1-\hat{\lambda})\hat{h}\right)d\left(\beta\hat{\ell}^\infty+(1-\beta)q\right) \\ = & W\left(\lambda g+(1-\lambda)h\right) \end{aligned}$$

contradicting $f \succsim^* g$. Summarizing, we have shown that

$$f \succsim^* g \text{ if and only if } \int u(f)\,dp \geq \int u(g)\,dp \text{ for all } p \in co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}.$$

Therefore, applying Lemma 3.4 and noting that $co\{\beta\ell^\infty + (1-\beta)q : \ell \in D\}$ is weak* compact because D is finite, $\succsim$ represented by $W(f)$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$.

Observe that, since $q \in co\{\ell^\infty : \ell \in D\}$, $W(f)$ can be re-written as

$$\begin{aligned} & H\left(\left(\int u(f)\,d\ell^\infty\right)_{\ell\in D}\right) \\ \equiv\ & \beta \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\,dp + (1-\beta)\int u(f)\,dq, \end{aligned}$$

and H so defined is weakly increasing. Therefore we may apply the uniqueness result in Theorem 3.8 to conclude $\overline{D} = R$. Since D is finite, $\overline{D} = D$. □

8.9. Proof of Theorem 4.4

First we show that each measure in $\operatorname{supp} m$ is relevant. Suppose $\hat{\ell} \in \operatorname{supp} m$ and fix any open $L \subseteq \Delta(S)$ such that $\hat{\ell} \in L$. Take $x_1, x_2, x_3 \in X$ such that $x_2 \sim \frac{1}{2}x_1 + \frac{1}{2}x_3$ and $x_1 \succ x_3$. Define two acts f and g by

$$f(\omega) = \begin{cases} x_1 & \text{if } \Psi(\omega) \in L \\ x_2 & \text{otherwise} \end{cases} \text{ and } g(\omega) = \begin{cases} x_3 & \text{if } \Psi(\omega) \in L \\ x_2 & \text{otherwise} \end{cases}.$$

Since $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\widehat{\ell} \in \Delta(S)\backslash L$, it suffices to show that $f \nsim g$. Assume $f \sim g$. Then, for each $i = 1, ..., n$,

$$\begin{aligned} \int \zeta_i u(f)\,dp &= \int_{\Psi^{-1}(L)} \zeta_i u(x_1)\,dp + \int_{\Omega\backslash\Psi^{-1}(L)} \zeta_i u(x_2)\,dp \\ &= \int_{\Psi^{-1}(L)} \zeta_i [u(x_1) - u(x_2)]\,dp + \int_\Omega \zeta_i u(x_2)\,dp \\ &= \int_{\Psi^{-1}(L)} \zeta_i [u(x_1) - u(x_2)]\,dp \\ &= \int_{\Psi^{-1}(L)} \zeta_i [u(x_2) - u(x_3)]\,dp = -\int \zeta_i u(g)\,dp. \end{aligned}$$

The third equality follows because $\int \zeta_i dp = 0$, and the fourth comes from $x_2 \sim \frac{1}{2}x_1 + \frac{1}{2}x_3$. Then, $f \sim g$ implies

$$\int u\left(f\right)dp + A\left(\left(\int \zeta_i u\left(f\right)dp\right)_{1\leq i\leq n}\right) = \int u\left(g\right)dp + A\left(\left(\int \zeta_i u\left(g\right)dp\right)_{1\leq i\leq n}\right).$$

As $A\left(a\right) = A\left(-a\right)$, this yields $\int u\left(f\right)dp = \int u\left(g\right)dp$ which contradicts $m\left(L\right) > 0$ since $x_1 \succ x_3$. Thus, $f \nsim g$ and each measure in $\operatorname{supp} m$ is relevant.

Next, we show that all measures in $\Delta(S)\setminus\operatorname{supp} m$ are betting irrelevant. Suppose $\hat{\ell} \in \Delta(S)\setminus\operatorname{supp} m$. There exists an open $L \subseteq \Delta\left(S\right)$ such that $\hat{\ell} \in L$ and $L \subseteq \Delta(S)\setminus\operatorname{supp} m$. Consider $f = 1_{\Psi^{-1}(L)}$ and $g = 1_{\emptyset}$ and, for convenience, normalize u such that $u(x^*) = 1$ and $u(x_*) = 0$. Observe that $f \sim^* g$ if and only if $T(\alpha f + (1-\alpha)h) = T(\alpha g + (1-\alpha)h)$ for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$. Since

$$\begin{aligned} T(\alpha f + (1-\alpha)h) &= (1-\alpha)\int u(h)dp + A\left(\left((1-\alpha)\int \zeta_i u(h)dp\right)_{1\leq i\leq n}\right) \\ &= T(\alpha g + (1-\alpha)h), \end{aligned}$$

$\hat{\ell}$ is betting irrelevant. Thus, all measures in $\Delta(S)\setminus\operatorname{supp} m$ are betting irrelevant.

Next, we show that $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$. The form assumed for p and the symmetry property assumed for each ζ_i ensure that Event Symmetry is satisfied. The other properties in Symmetry along with Mixture Continuity of $\succsim$ follow directly from the properties of VEU (see Siniscalchi [36]). Mixture Continuity of $\succsim$ implies Mixture Continuity of $\succsim^*$. To see Monotone Continuity of $\succsim^*$, observe that $x' \succsim^* xA_kx''$ if and only if, for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$,

$$\begin{aligned} & \alpha u(x') + A\left(\left((1-\alpha)\int u(h)\zeta_i dp\right)_{1\leq i\leq n}\right) \\ \geq\ & \alpha\left(p(A_k)u(x) + (1-p(A_k)u(x''))\right) \\ & + A\left(\left(\alpha\left[u(x)\int_{A_k}\zeta_i dp + u(x'')\int_{A_k^c}\zeta_i dp\right] + (1-\alpha)\int u(h)\zeta_i dp\right)_{1\leq i\leq n}\right). \end{aligned}$$

Since p is countably additive and ζ_i is bounded and measurable, $A_k \searrow \emptyset$ implies $p(A_k) \to 0$ and $\int_{A_k}\zeta_i dp \to 0$ and $\int_{A_k^c}\zeta_i dp \to \int_{S^\infty}\zeta_i dp = 0$. Therefore, since n is finite and A is Lipschitz continuous, there exists a k such that A_k is small enough so that $x' \succsim^* xA_kx''$. This proves Monotone Continuity of $\succsim^*$.

Finally, applying Theorems 3.5 and 3.6, the fact that all measures in $\Delta(S)\backslash \operatorname{supp} m$ are betting irrelevant implies no measures in $\Delta(S)\backslash \operatorname{supp} m$ are relevant. Therefore $R = \operatorname{supp} m$. □

8.10. Proofs for Section 5

It is sometimes convenient in the proofs to use an alternative formulation of No Half Measures. The following lemma gives the alternative formulation and shows that it is equivalent to No Half Measures.

Lemma 8.1. *The following are equivalent:*

(a) *No Half Measures;*

(b) *For $A \in \Sigma^{\Psi}$, if there exists an $x \in X$ such that $1_{S^\infty} \succ x \succsim^* 1_A$, then $1_A \sim^* 1_\emptyset$.*

Proof of Lemma 8.1

(a)⇒(b): Fix $A \in \Sigma^{\Psi}$. Suppose for each $x \in X$ with $1_{S^\infty} \succ x$, there exists $f \in \mathcal{F}$ and $\alpha \in (0,1]$ such that $\alpha x + (1-\alpha) f \not\succsim \alpha 1_A + (1-\alpha) f$. Then $x \not\succsim^* 1_A$ and (b) is vacuously satisfied. Suppose instead that for all $f \in \mathcal{F}$ and $\alpha \in (0,1]$, $\alpha 1_\emptyset + (1-\alpha) f \sim \alpha 1_A + (1-\alpha) f$. Then $1_A \sim^* 1_\emptyset$ and (b) is satisfied.

(b)⇒(a): Fix $A \in \Sigma^{\Psi}$. Suppose (b) holds. Can it be that neither of the possibilities in (a) hold? If neither holds, $1_A \nsim^* 1_\emptyset$ and for some $x \in X$ with $1_{S^\infty} \succ x$, for all $f \in \mathcal{F}$ and $\alpha \in (0,1]$, $\alpha x + (1-\alpha) f \succsim \alpha 1_A + (1-\alpha) f$. But then, $1_{S^\infty} \succ x \succsim^* 1_A$ and $1_A \nsim^* 1_\emptyset$ contradicting (b). □

8.10.1. Proof of Theorem 5.1

Normalize u so that $u(x^*) = 1$ and $u(x_*) = 0$.

(if): We show No Half Measures. Fix any $A \in \Sigma^{\Psi}$ and assume $1_{S^\infty} \succ x \succsim^* 1_A$. Then, $u(x) \geq \int u(1_A) \, d\ell^\infty = \ell^\infty(A)$ for all $\ell \in R$. Note that $\ell^\infty(A) \in \{0,1\}$ for every $A \in \Sigma^{\Psi}$. Since $1_{S^\infty} \succ x$, $1 > u(x)$ and thus $\ell^\infty(A) = 0$ for all $\ell \in R$. Then, $1_A \sim^* 0$.

(only if): Since $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$, Lemma 3.4 delivers sets C, R and M. By definition, $C \subseteq \left\{ \int \ell^\infty dm(\ell) : m \in \Delta(R) \right\}$. For the other inclusion, follow the steps.

Step 1. For $\widehat{\ell} \in \Delta(S)$, if $\delta_{\widehat{\ell}} \notin M$, then $\max_{m \in M} m(L) < 1$ for some $L \in \mathcal{O}_{\widehat{\ell}}$. We show this by contradiction. Suppose that, for every $L \in \mathcal{O}_{\widehat{\ell}}$, there is $m_L \in M$ such that $m_L(L) = 1$. We can view $\mathcal{O}_{\widehat{\ell}}$ as a directed set with the inverse inclusion.

Then, $\{m_L\}$ is a net and it has a limit point in M because M is relative weak* compact. The limit point has to be $\delta_{\widehat{\ell}}$, a contradiction.
Step 2. $\widehat{\ell}^{\infty} \in C$ for every $\widehat{\ell} \in R$. To see this, again use proof by contradiction. Suppose not. Then, $\delta_{\widehat{\ell}} \notin M$ for some $\widehat{\ell} \in R$. By Step 1, $a = \max_{m \in M} m(L) < 1$ for some $L \in \mathcal{O}_{\widehat{\ell}}$. Let $A = \Psi^{-1}(L)$ and $x = ax^* + (1-a)x_*$. Then, $u(x) = a < 1$ and, for any $m \in M$,

$$\begin{aligned} u(x) &= a \geq m(L) \\ &= \int_L \ell^{\infty}(A)\, dm + \int_{\Delta(S)\setminus L} \ell^{\infty}(A)\, dm \quad (\text{since } \ell^{\infty}(A) = 1 \text{ for } \ell \in L, \text{ and } 0 \text{ otherwise}) \\ &= \int_L \int_{S^{\infty}} u(1_A)\, d\ell^{\infty} dm + \int_{\Delta(S)\setminus L} \int_{S^{\infty}} u(1_A)\, d\ell^{\infty} dm = \int_{\Delta(S)} \int_{S^{\infty}} u(1_A)\, d\ell^{\infty} dm(\ell). \end{aligned}$$

Thus, $1 \succ x \succsim^* 1_A$. By No Half Measures, $m(L) = \int \int u(1_A)\, d\ell^{\infty} dm(\ell) = 0$ for every $m \in M$. But since $\widehat{\ell} \in L$ and L is open, this implies $\widehat{\ell} \notin \operatorname{supp} m$ for every $m \in M$ and thus $\hat{\ell} \notin R$. This is a contradiction.
Step 3. $C \supseteq \left\{\int \ell^{\infty} dm(\ell) : m \in \Delta(R)\right\}$. Denote by $\overline{co}^{w*}(\cdot)$ the weak* closed convex hull in $ba(S^{\infty})$. Observe that

$$\min_{p \in \overline{co}^{w*}(\{\ell^{\infty}:\ell \in R\})} \int \psi dp = \inf_{p \in \{\ell^{\infty}:\ell \in R\}} \int \psi dp = \inf_{p \in \left\{\int \ell^{\infty} dm(\ell): m \in \Delta(R)\right\}} \int \psi dp$$

for any bounded measurable ψ on S^{∞}. Since the set of measures in the MEU functional is unique up to the weak* closed convex hull (Gilboa and Schmeidler [21]),

$$\overline{co}^{w*}(\{\ell^{\infty} : \ell \in R\}) \supseteq \left\{\int \ell^{\infty} dm(\ell) : m \in \Delta(R)\right\}.$$

By this inclusion, Step 2 and weak* closedness and convexity of C, we have

$$C \supseteq \overline{co}^{w*}(\{\ell^{\infty} : \ell \in R\}) \supseteq \left\{\int \ell^{\infty} dm(\ell) : m \in \Delta(R)\right\}.$$

This completes the proof. □

8.10.2. Proof of Theorem 5.2

Suppose R is not finite. Then, we can take distinct $\ell_n \in R$ for each n. Let $A_n = \bigcup_{k \geq n} \Psi^{-1}(\ell_k)$. Then, $A_n \searrow \emptyset$. To try to verify Monotone Continuity of

$\succsim^*$, take three constant acts $x \succ x' \succ x''$. Let $p_n = \ell_n^\infty$. Then, $p_n(A_n) = 1$ and $u(x') < p_n(A_n)\,u(x) + (1 - p_n(A_n))\,u(x'')$. By Theorem 5.1, $p_n \in C$ and Lemma 3.4 implies that it is not possible that $x' \succsim^* xA_nx''$ for any n. This contradicts Monotone Continuity of $\succsim^*$. $\square$

8.10.3. Proof of Theorem 5.3

Fix the representation V.

(if) If $\succsim$ is represented by $\min_{\ell \in R} \int_{S^\infty} u(f) d\ell^\infty$ then the fact that $\succsim$ satisfies No Half Measures follows from one direction of Theorem 5.6.

(only if) By Ghirardato, Maccheroni and Marinacci [18, Theorem 14], the Bewley set, C, associated with $\succsim$ is F. By Lemma 3.4, $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$. Suppose No Half Measures. Then, by Theorems 5.1 and 5.2, $C = \left\{\int \ell^\infty dm(\ell) : m \in \Delta(R)\right\}$ and R finite. Thus $V(f) = \min_{p \in F} \int u(f)\,dp = \min_{p \in \left\{\int \ell^\infty dm(\ell) : m \in \Delta(R)\right\}} \int u(f)\,dp = \min_{\ell \in R} \int_{S^\infty} u(f) d\ell^\infty$. $\square$

8.10.4. Proof of Theorem 5.6

See the proof of Theorem 6.1.

8.10.5. Proof of Theorem 5.7

Normalize $u(x^*) = 1$ and $u(x_*) = 0$.

(if): Suppose $\sup \bigcup_{r,t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$. No Half Measures says that for $A \in \Sigma^\Psi$, if there exists an $x \in X$ such that $1_{S^\infty} \succ x \succsim^* 1_A$, then $1_A \sim^* 1_\emptyset$. We prove this by showing that, for $A \in \Sigma^\Psi$, $1_A \nsim^* 1_\emptyset$ implies there does not exist an $x \in X$ such that $1_{S^\infty} \succ x \succsim^* 1_A$. Suppose $1_A \nsim^* 1_\emptyset$. In terms of the representation, using the definition of $\succsim^*$, this nonindifference may be written as follows:

$$\int_{\Delta(S)} \phi\left(\lambda \ell^\infty(A) + (1-\lambda)\int_{S^\infty} u(h) d\ell^\infty\right) d\mu(\ell) \neq \int_{\Delta(S)} \phi\left((1-\lambda)\int_{S^\infty} u(h) d\ell^\infty\right) d\mu(\ell)$$

for some $\lambda \in [0,1]$ and act h. From the facts that ϕ is strictly increasing and $\ell^\infty(A) \in \{0,1\}$, this is equivalent to the requirement that $\mu(B) > 0$ for the set $B \equiv \{\ell \in \Delta(S) : \ell^\infty(A) = 1\}$. Now we show that $1_{S^\infty} \succ x \succsim^* 1_A$ is impossible.

Observe that $1_{S^\infty} \succ x$ means simply $u(x) < 1$. Similarly, $x \succsim^* 1_A$ if and only if

$$\int_{\Delta(S)} \phi\left(\lambda u(x) + (1-\lambda)\int_{S^\infty} u(h)d\ell^\infty\right) d\mu(\ell) \geq \int_{\Delta(S)} \phi\left(\lambda \ell^\infty(A) + (1-\lambda)\int_{S^\infty} u(h)d\ell^\infty\right) d\mu(\ell) \tag{8.1}$$

for all $\lambda \in [0,1]$ and all acts h. Notice that when $\lambda = 0$, (8.1) is automatically satisfied with equality. Therefore a necessary condition for $x \succsim^* 1_A$ to hold is that when both sides of (8.1) are differentiated with respect to λ and evaluated at $\lambda = 0$ the derivative on the left-hand side is weakly larger than that on the right-hand side. Doing this differentiation and evaluating at $\lambda = 0$ yields

$$\begin{aligned} & \int_{\Delta(S)} \left(u(x) - \int_{S^\infty} u(h)d\ell^\infty\right) \phi'\left(\int_{S^\infty} u(h)d\ell^\infty\right) d\mu(\ell) \\ \geq & \int_{\Delta(S)} \left(\ell^\infty(A) - \int_{S^\infty} u(h)d\ell^\infty\right) \phi'\left(\int_{S^\infty} u(h)d\ell^\infty\right) d\mu(\ell), \end{aligned}$$

which simplifies to

$$\begin{aligned} & u(x)\int_{\Delta(S)} \phi'\left(\int_{S^\infty} u(h)d\ell^\infty\right) d\mu(\ell) \\ \geq & \int_{\Delta(S)} \ell^\infty(A)\, \phi'\left(\int_{S^\infty} u(h)d\ell^\infty\right) d\mu(\ell). \end{aligned} \tag{8.2}$$

Consider acts h that are bets on A (or A^c) such that $u(h) = a_A b$ for $a, b \in u(X)$. Such acts exist for any $a, b \in u(X)$. For such an h, (8.2) simplifies to

$$u(x)\left[\phi'(a)\,\mu(B) + \phi'(b)\,(1-\mu(B))\right] \geq \phi'(a)\,\mu(B). \tag{8.3}$$

Since $\mu(B) > 0$ and $\sup \bigcup_{r,t\in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$, we can choose a and b to make $\frac{\phi'(a)}{\phi'(b)}$ as large as desired, leading the right-hand side of (8.3) to be as close as desired to 1 and violating the inequality (since $u(x) < 1$). This completes the "if" direction of the proof.

(only if): We now show the converse under the assumption that $\operatorname{supp}\mu$ is non-singleton (if $\operatorname{supp}\mu$ is a singleton, then properties of ϕ beyond the fact that it is strictly increasing cannot affect preferences). By Theorem 6.2 and the fact that $\operatorname{supp}\mu$ is finite, $\succsim$ satisfies Symmetry, Mixture Continuity of $\succsim^*$ and Monotone Continuity of $\succsim^*$. Suppose No Half Measures holds. By Theorem 5.1, $C = \left\{\int \ell^\infty dm(\ell) : m \in \Delta(R)\right\}$. By Theorem 4.2, $R = \operatorname{supp}\mu$. Therefore

$C = \left\{\int \ell^\infty dm\,(\ell) : m \in \Delta(\operatorname{supp}\mu)\right\}$. Fix any $\ell_1 \in \operatorname{supp}\mu$. Note that $\ell_1^\infty \in C$ and is an extreme point of C. By the characterization of C from Proposition 17 of Ghirardato and Siniscalchi [19], simplified using the i.i.d. structure and the continuous differentiability of ϕ,

$$C = \overline{co}\left(\left\{\frac{\int \phi'\left(\int ed\ell^\infty\right)\ell^\infty d\mu\,(\ell)}{\int \phi'\left(\int ed\ell^\infty\right)d\mu\,(\ell)} : \ell \in \Delta\,(S)\,, e \in \operatorname{int} B_b\,(\Sigma, u\,(X))\right\}\right),$$

where $\operatorname{int} B_b\,(\Sigma, u\,(X))$ is the interior of the set of all Σ-measurable functions $a : S^\infty \to \mathbb{R}$ for which there exist $\alpha, \beta \in u\,(X)$ satisfying $\alpha \geq a(\omega) \geq \beta$ for all $\omega \in S^\infty$ (i.e., informally, the interior of the set of bounded utility-acts). Since $\ell_1^\infty \in C$ and is an extreme point of C, it must be that there exists a sequence of measures in $\left\{\frac{\int \phi'\left(\int ed\ell^\infty\right)\ell^\infty d\mu(\ell)}{\int \phi'\left(\int ed\ell^\infty\right)d\mu(\ell)} : \ell \in \Delta\,(S)\,, e \in \operatorname{int} B_b\,(\Sigma, u\,(X))\right\}$ converging to ℓ_1^∞. If $L \subseteq \Delta(S)$ is an open set containing ℓ_1, $\mu(L) > 0$. Since $\operatorname{supp}\mu$ is non-singleton, there exist open sets containing ℓ_1 such that $\mu(L) < 1$. Let $\hat{L}$ be such a set and consider the event $A = \Psi^{-1}(\hat{L})$. Suppose $\sup \bigcup_{r,t\in u(X)} \frac{\phi'(t)}{\phi'(r)} = K < +\infty$. Then

$$\begin{aligned}
&\sup\left\{\frac{\int \phi'\left(\int ed\ell^\infty\right)\ell^\infty(A)d\mu\,(\ell)}{\int \phi'\left(\int ed\ell^\infty\right)d\mu\,(\ell)} : \ell \in \Delta\,(S)\,, e \in \operatorname{int} B_b\,(\Sigma, u\,(X))\right\} \\
=\;& \sup\left\{\frac{\int_{\hat{L}} \phi'\left(\int ed\ell^\infty\right)d\mu\,(\ell)}{\int_{\hat{L}} \phi'\left(\int ed\ell^\infty\right)d\mu\,(\ell) + \int_{\Delta(S)\setminus\hat{L}} \phi'\left(\int ed\ell^\infty\right)d\mu\,(\ell)} : \ell \in \Delta\,(S)\,, e \in \operatorname{int} B_b\,(\Sigma, u\,(X))\right\} \\
\leq\;& \frac{K}{K+1} < 1 = \ell_1^\infty(A)
\end{aligned}$$

so that no sequence in $\left\{\frac{\int \phi'\left(\int ed\ell^\infty\right)\ell^\infty(A)d\mu(\ell)}{\int \phi'\left(\int ed\ell^\infty\right)d\mu(\ell)} : \ell \in \Delta\,(S)\,, e \in \operatorname{int} B_b\,(\Sigma, u\,(X))\right\}$ can converge to ℓ_1^∞, a contradiction. Thus, $\sup \bigcup_{r,t\in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$, completing the proof. □

8.11. Proof of Theorem 6.1

We begin by showing (1) ⇒ (2). Observing that Mixture Continuity of $\succsim$ implies Ghirardato, Maccheroni and Marinacci [18]'s Archimedean axiom, we see that $\succsim$ are Invariant Biseparable preferences (i.e., satisfy axioms 1-5 in Ghirardato, Maccheroni and Marinacci [18]). By Proposition 7 in Ghirardato, Maccheroni and Marinacci [18], $\succsim$ has a representation, $I(u(f))$, where I is monotonic and constant linear and u is non-constant and affine, lying between $\min_{p\in C}\int u(f)\,dp$ and $\max_{p\in C}\int u(f)\,dp$ (i.e., for all simple acts f, $\min_{p\in C}\int u(f)\,dp \leq I(u(f)) \leq \max_{p\in C}\int u(f)\,dp$) where C is the Bewley set from Lemma 3.4. Since, by Lemma 3.4, $C = \left\{\int \ell^{\infty} dm(\ell) : m \in M\right\}$ and $R \equiv \bigcup_{m\in M} \operatorname{supp} m$, $\min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp \leq \min_{p\in C}\int u(f)\,dp$ and $\max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp \geq \max_{p\in C}\int u(f)\,dp$. Therefore $\min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp \leq I(u(f)) \leq \max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp)$. Now consider Relevant Range. Observe that $C^*(f)$ can be written $\{x \in X : \max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp \geq u(x) \geq \min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp\}$. Thus, $C^*(f) = C^*(g)$ if $\max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp = \max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(g)\,dp$ and $\min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp = \min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(g)\,dp$. Relevant Range therefore implies that $I(u(f))$ must be able to be expressed as a function of $\max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp$ and $\min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp$ only. Since $\min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp \leq I(u(f)) \leq \max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp$ and $I(u(f))$ depends only on $\max_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp$ and $\min_{p\in\{\ell^{\infty}:\ell\in R\}}\int u(f)\,dp$, we may apply Lemma B.5 in Ghirardato, Maccheroni and Marinacci [18] to conclude that

$$I(u(f)) = \alpha \min_{p\in\{\ell^{\infty}:\ell\in R\}} \int u(f)\,dp + (1-\alpha) \max_{p\in\{\ell^{\infty}:\ell\in R\}} \int u(f)\,dp$$

for an $\alpha \in [0,1]$.

We now show that R is finite. Consider $\alpha \in [0,1)$ first. Suppose R is not finite. Then, we can take distinct $\ell_n \in R$ for each n. Let $A_n = \bigcup_{k>n} \Psi^{-1}(\ell_k)$. Then, $A_n \searrow \emptyset$. Without loss of generality, assume $[0,1] \subseteq u(X)$. Let $u(x) = 1 > u(x') = \frac{1}{k} > u(x'') = 0$. By Monotone Continuity of $\succsim^*$, for each integer $k > 1$, there is $n(k) > 0$ such that $V\left(xA_{n(k)}x''\right) < \frac{1}{k}$. Since $V\left(xA_{n(k)}x''\right)$ is decreasing in $n(k)$, $V\left(xA_{n(k)}x''\right) \to 0$. However, computation shows $V\left(xA_{n(k)}x''\right) \geq (1-\alpha)\max_{p\in\{\ell^{\infty}:\ell\in R\}} p(A_n) = 1-\alpha > 0$, a contradiction.

Now let $\alpha = 1$. Take $\ell_n \in R$ and $A_n \subset S^{\infty}$ as above, and also let $u(x) = 1 > u(x') = \frac{1}{k} > u(x'') = 0$. By Monotone Continuity of $\succsim^*$, for each $k > 1$, there is $n(k) > 0$ such that $x' \succsim^* xA_{n(k)}x''$. By Lemma 3.4, $\frac{1}{k} \geq p\left(A_{n(k)}\right)$ for all $p \in C$. Equivalently, $1-\frac{1}{k} \leq p\left(S^{\infty}\backslash A_{n(k)}\right)$ for all $p \in C$. This implies $V\left(xS^{\infty}\backslash A_{n(k)}x''\right) \in \left[1-\frac{1}{k}, 1\right]$. Since $V\left(xS^{\infty}\backslash A_{n(k)}x''\right)$ is increasing in $n(k)$, $V\left(xS^{\infty}\backslash A_{n(k)}x''\right) \to 1$. However, $(\ell_{n+1})^{\infty}(S^{\infty}\backslash A_n) = 0$ for all n, and hence $V\left(xS^{\infty}\backslash A_{n(k)}x''\right) \to 0$, a contradiction.

This proves (2).

We now show uniqueness. That $D = R$ follows from Theorem 4.1. Uniqueness of D then follows from uniqueness of R. Uniqueness of α when D is non-singleton is a conclusion of Lemma B.5 in Ghirardato, Maccheroni and Marinacci [18]. Uniqueness of u up to positive affine transformations is standard.

Finally, we show (2) $\Rightarrow$ (1). That Relevant Range is satisfied given (2) can be seen by noting that $D = R$ by Theorem 4.1 and that $C^*(f) = \{x \in X : \max_{p\in\{\ell^\infty:\ell\in R\}} \int u(f)\, dp \geq u(x) \geq \min_{p\in\{\ell^\infty:\ell\in R\}} \int u(f)\, dp\}$. The remaining axioms except Monotone Continuity of $\succsim^*$ are straightforward to verify.

We now show Monotone Continuity of $\succsim^*$. Consider $V_1(f) \equiv \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\, dp$ first. The Bewley set of V_1 is $co(\{\ell^\infty : \ell \in D\})$ and it is weak* compact since D is finite. Thus, V_1 satisfies Monotone Continuity of $\succsim^*$. Similarly, $V_0(f) = \max_{p\in\{\ell^\infty:\ell\in D\}} \int u(f)\, dp$ also satisfies Monotone Continuity of $\succsim^*$. Take $A_n \searrow \emptyset$ and $x, x', x'' \in X$ such that $u(x') > u(x'')$. Then, there is $\bar{n}_1$ and $\bar{n}_0$ such that

$$V_1(\lambda x' + (1-\lambda) h) \geq V_1(\lambda x A_n x'' + (1-\lambda) h)$$

for all $\lambda \in [0,1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_1$, and

$$V_0(\lambda x' + (1-\lambda) h) \geq V_0(\lambda x A_n x'' + (1-\lambda) h)$$

for all $\lambda \in [0,1]$, $h \in \mathcal{F}$ and $n \geq \bar{n}_2$. Since $V = \alpha V_1 + (1-\alpha) V_0$,

$$V(\lambda x' + (1-\lambda) h) \geq V(\lambda x A_n x'' + (1-\lambda) h) \text{ for } n = \max(\bar{n}_1, \bar{n}_2).$$

Thus, Monotone Continuity of $\succsim^*$ is satisfied.

We now show that adding No Half Measures to the requirements is equivalent to the statement that ($\alpha = 0$ or $\alpha = 1$ or D singleton). We first show ($\alpha = 0$ or $\alpha = 1$ or D singleton) implies No Half Measures. Recall that No Half Measures concerns only events $A \in \Sigma^\Psi$. If D singleton, then $1_{S^\infty} \succ x \succsim 1_A$, implies $\ell^\infty(A) = 0$ for the $\ell \in D$, in which case $1_A \sim^* 1_\emptyset$ is obvious. To see the other cases, observe that $1_{S^\infty} \succ x$ if and only if $u(x) < 1$ and $x \succsim^* 1_A$ if and only if

$$\begin{aligned} & \lambda u(x) + (1-\lambda)\left[\alpha \min_{p\in\{\ell^\infty:\ell\in D\}} \int u(h)\, dp + (1-\alpha) \max_{p\in\{\ell^\infty:\ell\in D\}} \int u(h)\, dp\right] \\ \geq\ & \alpha \min_{p\in\{\ell^\infty:\ell\in D\}} (\lambda p(A) + (1-\lambda) \int u(h)\, dp) + (1-\alpha) \max_{p\in\{\ell^\infty:\ell\in D\}} (\lambda p(A) + (1-\lambda) \int u(h)\, dp) \end{aligned} \tag{8.4}$$

for all $\lambda \in [0,1]$ and all acts h. Furthermore, $1_A \sim^* 1_\emptyset$ if and only if

$$\begin{aligned} & \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} (\lambda p(A) + (1-\lambda) \int u(h)\, dp) \\ & +(1-\alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} (\lambda p(A) + (1-\lambda) \int u(h)\, dp) \\ = \; & \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} ((1-\lambda) \int u(h)\, dp) + (1-\alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} ((1-\lambda) \int u(h)\, dp) \end{aligned} \tag{8.5}$$

for all $\lambda \in [0,1]$ and all acts h. In fact, $1_A \sim^* 1_\emptyset$ if and only if $p(A) = 0$ for all $p \in \{\ell^\infty : \ell \in D\}$. To see this last statement, note that the "if" direction is immediate, the "only if" direction follows for $\alpha \neq 1$ by setting $\lambda = 1$ in (8.5), and if $\alpha = 1$, it follows by setting $\lambda = \frac{1}{3}$ and $h = 1_{A^c}$ in (8.5). Therefore, in the presence of the other requirements in the theorem, No Half Measures is equivalent to the statement: For all $A \in \Sigma^\Psi$, $p(A) = 1$ for some $p \in \{\ell^\infty : \ell \in D\}$ implies there does not exist an $x \in X$ such that $u(x) < 1$ and (8.4) satisfied. Now consider the case $\alpha = 0$ and let $h = 1_A$. Then the right-hand side of (8.4) becomes 1 and therefore greater than any $u(x) < 1$ violating (8.4). Finally consider $\alpha = 1$ and let $\lambda = \frac{1}{3}$ and $h = 1_{A^c}$ in (8.4). Simplifying yields

$$\begin{aligned} & \frac{1}{3}u(x) + \frac{2}{3}\left[\min_{p \in \{\ell^\infty : \ell \in D\}} (1 - p(A))\right] \\ \geq \; & \min_{p \in \{\ell^\infty : \ell \in D\}} (\frac{1}{3}p(A) + \frac{2}{3}(1 - p(A))) \end{aligned}$$

which simplifies to $u(x) \geq 1$, thus showing No Half Measures holds.
We next show the other direction – that No Half Measures implies ($\alpha = 0$ or $\alpha = 1$ or D singleton). To see this, we assume D non-singleton and $\alpha \in (0,1)$ and show that No Half Measures is violated.

No Half Measures violated is equivalent, by the arguments above, to, for some $A \in \Sigma^\Psi$, $p(A) = 1$ for some $p \in \{\ell^\infty : \ell \in D\}$ while (8.4) is satisfied for some $u(x) < 1$. To this end, consider $A = \Psi^{-1}(\hat{\ell})$ for some $\hat{p} = \left(\hat{\ell}\right)^\infty \in \{\ell^\infty : \ell \in D\}$. Observe that $\hat{p}(A) = 1$, $p(A) = 0$ for all $p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}$ and $\{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}$ is non-empty (since D non-singleton). For this event A, we will show that (8.4) is satisfied for $u(x) = \frac{1+\max[\alpha, 1-\alpha]}{2}$. Note that $\alpha \in (0,1)$ implies

$\max[\alpha,1-\alpha] < u(x) < 1$. Rewriting (8.4) using this A and $u(x)$, we want to show

$$\begin{aligned}
&\alpha \min\left[\lambda+(1-\lambda)\int u(h)\,d\hat{p},\ \min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\,dp\right] \\
&+(1-\alpha)\max\left[\lambda+(1-\lambda)\int u(h)\,d\hat{p},\ \max_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\,dp\right] \qquad (8.6)\\
\leq\ &\lambda\frac{1+\max[\alpha,1-\alpha]}{2}+(1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp+(1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right]
\end{aligned}$$

holds for all acts h and all $\lambda\in[0,1]$. If $\lambda=0$, (8.6) holds with equality. If $\lambda=1$, (8.6) simplifies to $1-\alpha\leq\frac{1+\max[\alpha,1-\alpha]}{2}$, which is true. For the remainder of the argument we assume $\lambda\in(0,1)$. We proceed by considering three mutually exclusive and exhaustive cases: (1) $\hat{p}\notin\arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$, (2) $\hat{p}\in\arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$ and $\frac{\lambda}{1-\lambda}+\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\leq\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$ and (3) $\hat{p}\in\arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$ and $\frac{\lambda}{1-\lambda}+\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp>\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$. For each case we show (8.6) holds.

Case (1): If $\hat{p}\notin\arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$, then $p(A)$ and $\int u(h)\,dp$ have a common minimizer located in $\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}$ and

$$\begin{aligned}
&\alpha \min\left[\lambda+(1-\lambda)\int u(h)\,d\hat{p},\ \min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\,dp\right] \\
&+(1-\alpha)\max\left[\lambda+(1-\lambda)\int u(h)\,d\hat{p},\ \max_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\,dp\right] \\
=\ &\alpha(1-\lambda)\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp+(1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\left(\lambda p(A)+(1-\lambda)\int u(h)\,dp\right)\\
\leq\ &\lambda(1-\alpha)+(1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp+(1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right]\\
<\ &\lambda\frac{1+\max[\alpha,1-\alpha]}{2}+(1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp+(1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right].
\end{aligned}$$

Case (2): If $\hat{p}\in\arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$ and $\frac{\lambda}{1-\lambda}+\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\leq\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp$, then, noting that $\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp=\max_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}\int u(h)\,dp$, we have

$$
\begin{aligned}
& \alpha \min\left[\lambda + (1-\lambda)\int u(h)\, d\hat{p}, \min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\, dp\right] \\
& +(1-\alpha)\max\left[\lambda + (1-\lambda)\int u(h)\, d\hat{p}, \max_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\, dp\right] \\
= \; & \alpha \min\left[\lambda + (1-\lambda)\int u(h)\, d\hat{p}, \min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\, dp\right] \\
& +(1-\alpha)(1-\lambda)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp \\
\leq \; & \lambda\alpha + (1-\lambda)\left[\alpha\int u(h)\, d\hat{p} + (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp\right] \\
< \; & \lambda\frac{1+\max[\alpha, 1-\alpha]}{2} + (1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp + (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp\right].
\end{aligned}
$$

Case (3): If $\hat{p} \in \arg\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp$ and $\frac{\lambda}{1-\lambda} + \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp >$ $\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\, dp$, then

$$
\begin{aligned}
& \alpha \min\left[\lambda + (1-\lambda)\int u(h)\, d\hat{p}, \min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\, dp\right] \\
& +(1-\alpha)\max\left[\lambda + (1-\lambda)\int u(h)\, d\hat{p}, \max_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}(1-\lambda)\int u(h)\, dp\right] \\
= \; & \alpha(1-\lambda)\min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}\int u(h)\, dp + (1-\alpha)\left(\lambda + (1-\lambda)\int u(h)\, d\hat{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda(1-\alpha) + (1-\lambda)\left[\alpha\left(\min_{p\in\{\ell^\infty:\ell\in D\}\setminus\{\hat{p}\}}\int u(h)\,dp\right.\right.\\
&\quad + \left.\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)\\
&\quad +(1-\alpha)\left(\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right.\\
&\quad \left.\left.+ \max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp - \max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)\right]\\
&= \lambda(1-\alpha)\\
&\quad +(1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp + (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right]\\
&\quad +(1-\lambda)\left[(\alpha-1)\left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)\right.\\
&\quad \left.+\alpha\left(\min_{p\in\{\ell^\infty:\ell\in D\setminus\{\hat{p}\}\}}\int u(h)\,dp - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda(1-\alpha) + (1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp + (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right]\\
&\quad +(1-\lambda)(2\alpha-1)\left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)\\
&= \lambda\frac{1+\max[\alpha,1-\alpha]}{2}\\
&\quad +(1-\lambda)\left[\alpha\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp + (1-\alpha)\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right]\\
&\quad +\lambda\left(1-\alpha-\frac{1+\max[\alpha,1-\alpha]}{2}\right)\\
&\quad +(1-\lambda)(2\alpha-1)\left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right).
\end{aligned}
$$

To complete Case (3), we must show that the last two terms in the final expression above have a non-positive sum. Observe that the hypothesis of Case (3) implies

$$
0 \leq \left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp - \min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right) < \frac{\lambda}{1-\lambda}.
$$

Since $\left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp-\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)$ enters linearly, the sum of the last two terms is bounded above by the maximum of the sum when substituting 0 for $\left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp-\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)$and the sum when substituting $\frac{\lambda}{1-\lambda}$. Carrying out these substitutions and simplifying yields

$$\begin{aligned}
&\lambda\left(1-\alpha-\frac{1+\max[\alpha,1-\alpha]}{2}\right)\\
&+(1-\lambda)(2\alpha-1)\left(\max_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp-\min_{p\in\{\ell^\infty:\ell\in D\}}\int u(h)\,dp\right)\\
\leq\;&\max\left[\lambda\left(1-\alpha-\frac{1+\max[\alpha,1-\alpha]}{2}\right),\lambda\left(\alpha-\frac{1+\max[\alpha,1-\alpha]}{2}\right)\right]\\
=\;&\lambda\left(\frac{\max[\alpha,1-\alpha]-1}{2}\right)<0.
\end{aligned}$$

□

8.12. Proof of Theorem 6.2

We start with necessity of Monotone Continuity of $\succsim^*$ and Expected utility on $\mathcal{F}^\Psi$. Ghirardato and Siniscalchi [19] show necessity of Cauchy continuity. Necessity of the remaining axioms is straightforward.

Monotone Continuity of $\succsim^*$: Suppose that case (i) holds, so there are $m, M>0$ such that $m\,|a-b|\leq|\phi(a)-\phi(b)|\leq M\,|a-b|$ for all $a,b\in u(X)$. Fix any $x,x',x''\in X$ with $x'\succ x''$. The only non-trivial case is $x\succ x'$. Without loss of generality, assume $u(x)=1>u(x')=t'>u(x'')=0$ and $[0,1]\subseteq u(X)$. Suppose $A_n\searrow\emptyset$. Take $\varepsilon',\varepsilon>0$ so that

$$\varepsilon'<t' \text{ and } m\,(t'-\varepsilon')\,(1-\varepsilon)\geq M\,(1-t')\,\varepsilon.$$

Define $\zeta_n:\Delta(S)\to\mathbb{R}$ by $\zeta_n(\ell)=\ell^\infty(A_n)$, and temporarily equip $\Delta(S)$ with the wc topology. Since wc open sets are weak* open, μ is well-defined on the Borel σ-algebra generated by wc open sets. Then, by Lusin's theorem (Aliprantis and Border [1, Theorem 12.8]), there is a wc compact set $L\subseteq\Delta(S)$ such that $\mu(L)>1-\varepsilon$ and all ζ_n are wc continuous. Note that ζ_n converges monotonically to 0 pointwise. Then by Dini's Theorem (Aliprantis and Border [1, Theorem 2.66]), ζ_n on L converges uniformly to 0. Hence there is $N>0$ such that $\zeta_N=\ell^\infty(A_N)<\varepsilon'$

for all $\ell \in L$. To see $x' \succsim^* xA_N x''$, and thus Monotone Continuity of $\succsim^*$, compute, for any $\alpha \in [0,1]$ and $h \in \mathcal{F}$,

$$\begin{aligned}
& U(\alpha x' + (1-\alpha) h) - U(\alpha x A_N x'' + (1-\alpha) h) \\
= & \int_L \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha \ell^\infty (A_N) + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
& + \int_{\Delta(S)\setminus L} \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha \ell^\infty (A_N) + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
> & \int_L \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha \varepsilon' + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
& + \int_{\Delta(S)\setminus L} \phi\left(\alpha t' + (1-\alpha)\int h d\ell^\infty\right) - \phi\left(\alpha + (1-\alpha)\int h d\ell^\infty\right) d\mu(\ell) \\
\geq & \int_L \alpha m (t' - \varepsilon') d\mu(\ell) + \int_{\Delta(S)\setminus L} \alpha M (t' - 1) d\mu(\ell) \\
= & \alpha [m (t' - \varepsilon') \mu(L) - M (1 - t') (1 - \mu(L))] \\
\geq & \alpha [m (t' - \varepsilon') (1 - \varepsilon) - M (1 - t') \varepsilon] \geq 0.
\end{aligned}$$

Turn to the case where (ii) holds, so that $\operatorname{supp}\mu$ is finite. Again suppose $A_n \searrow \emptyset$ and $x \succ x' \succ x''$. Since $\operatorname{supp}\mu$ is finite, $\sup_{\ell \in \operatorname{supp}\mu} \ell^\infty(A_n) \to 0$. Thus, for $\varepsilon > 0$ satisfying $u(x') > \varepsilon u(x) + (1-\varepsilon) u(x'')$, there is $n > 0$ such that $\ell^\infty(A_n) < \varepsilon$ for all $\ell \in \operatorname{supp}\mu$. This implies

$$\begin{aligned}
& U(\alpha x' + (1-\alpha) h) - U(\alpha x A_n x'' + (1-\alpha) h) \\
= & \int \phi\left(\alpha u(x') + (1-\alpha)\int u(h) d\ell^\infty\right) \\
& -\phi\left(\alpha (\ell^\infty (A_n) u(x) + (1 - \ell^\infty (A_n)) u(x'')) + (1-\alpha)\int u(h) d\ell^\infty\right) d\mu(\ell) \\
\geq & 0
\end{aligned}$$

for all $\alpha \in [0,1]$, $h \in \mathcal{F}$, and $\ell \in \operatorname{supp}\mu$. Therefore, $x' \succsim^* xA_n x''$ and Monotone Continuity of $\succsim^*$ holds.

Expected utility on $\mathcal{F}^\Psi$: For $f \in \mathcal{F}^\Psi$, f is constant on $\Psi^{-1}(\ell)$, so

$$\begin{aligned}
U(f) &= \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f) d\ell^\infty\right) d\mu(\ell) \\
&= \int_{\Delta(S)} \phi\left(u\left(f \circ \Psi^{-1}(\ell)\right)\right) d\mu(\ell)
\end{aligned}$$

represents $\succsim$ on $\mathcal{F}^{\Psi}$. Viewing $f \circ \Psi^{-1}(\ell)$ as an act from $\Delta(S)$ to X, this is an expected utility representation with countably additive μ and wc continuous utility function $v \equiv \phi \circ u$. Therefore, by Wakker [39, Theorem V.6.1], this implies Expected utility on $\mathcal{F}^{\Psi}$ is satisfied.

Now turn to sufficiency. By Complete Preorder and Mixture Continuity of $\succsim$, $\succsim$ has a real-valued representation. By Theorem 3.8, the event $\{\omega : \Psi(\omega) \text{ is not defined}\}$ never matters for $\succsim$ (i.e., it is Savage null) since $\ell^{\infty}(\{\omega : \Psi(\omega) \text{ is not defined}\}) = 0$ for all $\ell \in \Delta(S)$. Thus, by Expected utility on $\mathcal{F}^{\Psi}$ and Wakker [39, Theorem V.6.1], $\succsim$ on $\mathcal{F}^{\Psi}$ can be represented by

$$V(f) = \int_{\Delta(S)} v\left(f \circ \Psi^{-1}(\ell)\right) d\mu(\ell)$$

for a wc continuous v on X and a countably additive measure $\mu \in \Delta(\Delta(S))$. Since $\succsim$ on X is vN-M, there is a mixture linear function u on X, representing $\succsim$ on X. Thus, $v = \phi \circ u$ for some strictly increasing function ϕ on $u(X)$. By Mixture Continuity of $\succsim$, $\alpha \mapsto u(\alpha x + (1-\alpha) y)$ is continuous on $[0,1]$. Since v is wc continuous, ϕ is continuous. Moreover, $u = \phi^{-1} \circ v$ is wc-continuous. Non-triviality implies u is non-constant.

Note that, for $f \in \mathcal{F}^{\Psi}$,

$$u\left(f \circ \Psi^{-1}(\ell)\right) = \int_{\Psi^{-1}(\ell)} u(f)\, d\ell^{\infty} = \int_{S^{\infty}} u(f)\, d\ell^{\infty}.$$

Thus,

$$V(f) = \int_{\Delta(S)} \phi\left(\int_{S^{\infty}} u(f)\, d\ell^{\infty}\right) d\mu(\ell) \text{ for } f \in \mathcal{F}^{\Psi}.$$

It remains to extend to the entire $\mathcal{F}$. By Proposition 1 in Ghirardato and Siniscalchi [19], Cauchy continuity ensures the existence of a complete, monotonic and norm-continuous extension of $\succsim$ from $\mathcal{F}$ to $\hat{\mathcal{F}}$. Denote this extension by $\hat{\succsim}$, which is represented on the bounded, Σ^{Ψ}-measurable functions from Ω to X by

$$V(f) = \int_{\Delta(S)} \phi\left(\int_{S^{\infty}} u(f)\, d\ell^{\infty}\right) d\mu(\ell).$$

Given such an extension, we can invoke the equivalence of (vi) and (viii) in Theorem 7.2. By this equivalence, for any act $f \in \mathcal{F}$, $f \sim x \in X$ if and only if $f^{\Psi} \hat{\sim} x^{\Psi} = x$ (f^{Ψ} is defined just before the statement of Theorem 7.2). Therefore, for any act $f \in \mathcal{F}$, $f \sim f^{\Psi}$. Defining $U(f)$ by $U(f) = V\left(f^{\Psi}\right)$, we see that U represents $\succsim$ on $\mathcal{F}$. Since, by construction of f^{Ψ}, $\int_{S^\infty} u\left(f^{\Psi}\right) d\ell^\infty = \int_{S^\infty} u(f)\, d\ell^\infty$ for all $\ell \in \Delta(S)$,

$$\begin{aligned} U(f) &= V\left(f^{\Psi}\right) = \int_{\Delta(S)} \phi\left(\int_{S^\infty} u\left(f^{\Psi}\right) d\ell^\infty\right) d\mu(\ell) \\ &= \int_{\Delta(S)} \phi\left(\int_{S^\infty} u(f)\, d\ell^\infty\right) d\mu(\ell) \text{ for } f \in \mathcal{F}. \end{aligned}$$

To complete sufficiency, we suppose that (i) in the statement of the theorem does not hold and show that $\operatorname{supp}\mu$ is finite. To prove by contradiction, suppose $\operatorname{supp}\mu$ is infinite and thus we can take distinct elements $\ell_n \in \operatorname{supp}\mu$, for all integer $n \geq 1$. Let $A_n = \bigcup_{k \geq n} \Psi^{-1}(\ell_k)$. Then, $A_n \searrow \emptyset$ and for each n, $(\ell_n)^\infty(A_n) = 1 > \varepsilon \equiv \frac{1}{2}$. Fix $x \succ x' \succ x''$ and without loss of generality, assume $u(x) = 1 > u(x') = \frac{\varepsilon}{2} > u(x'') = 0$. Since $A_n \searrow \emptyset$, Monotone Continuity of $\succsim^*$ implies that there is n such that

$$\int \phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)\int u(h)\, d\ell^\infty\right) d\mu(\ell) \geq \int \phi\left(\alpha\ell^\infty(A_n) + (1-\alpha)\int u(h)\, d\ell^\infty\right) d\mu(\ell) \tag{8.7}$$

for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$. To pick a helpful h, note that $\ell \mapsto \ell^\infty(A_n)$ is relatively weak* continuous and hence there is a relatively weak* open $L \subseteq \Delta(S)$ containing ℓ_n such that $\ell^\infty(A_n) > \varepsilon$ for all $\ell \in L$. Since $\ell_n \in \operatorname{supp}\mu$, $\mu(L) > 0$. Take any $a, b \in u(X)$ and define h by

$$u(h(\omega)) = a \text{ if } \omega \in \Psi^{-1}(L), \text{ and } u(h(\omega)) = b \text{ otherwise.}$$

Then, the left-hand side of (8.7) reduces to

$$\mu(L)\,\phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)a\right) + (1-\mu(L))\,\phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)b\right)$$

and the right-hand side of (8.7) becomes

$$\begin{aligned} &\int_L \phi\left(\alpha\ell^\infty(A_n) + (1-\alpha)a\right) d\mu(\ell) + \int_{\Delta(S)\setminus L} \phi\left(\alpha\ell^\infty(A_n) + (1-\alpha)b\right) d\mu(\ell) \\ \geq\ & \mu(L)\,\phi\left(\alpha\varepsilon + (1-\alpha)a\right) + (1-\mu(L))\,\phi\left((1-\alpha)b\right). \end{aligned}$$

Therefore, (8.7) implies

$$\begin{aligned} & (1-\mu(L))\left[\phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)b\right) - \phi\left((1-\alpha)b\right)\right] \\ \geq\ & \mu(L)\left[\phi\left(\alpha\varepsilon + (1-\alpha)a\right) - \phi\left(\alpha\tfrac{\varepsilon}{2} + (1-\alpha)a\right)\right]. \end{aligned}$$

Then,

$$\mu(L) \le (1-\mu(L)) \frac{\phi\left(\alpha\frac{\varepsilon}{2}+(1-\alpha)b\right)-\phi((1-\alpha)b)}{\phi(\alpha\varepsilon+(1-\alpha)a)-\phi\left(\alpha\frac{\varepsilon}{2}+(1-\alpha)a\right)}.$$

Since α, a and b were arbitrary, we have $\mu(L) \le (1-\mu(L))K$ where

$$K = \inf\left\{\frac{\phi(a'+\delta)-\phi(a')}{\phi(b'+\delta)-\phi(b')} : a', b', a'+\delta, b'+\delta \in u(X), 0<\delta\le\tfrac{\varepsilon}{2}\right\}.$$

Recall that $\mu(L) > 0$ and hence $K > 0$. Thus, to show a contradiction, it suffices to show that $K = 0$. Let $\rho(t,t') = [\phi(t')-\phi(t)]/(t'-t)$. Assume the lower inequality in (i) fails – that is, for any $\gamma > 0$, $\rho(t,t') < \gamma$ for some $t < t' \in u(X)$. (The case where the upper inequality in (i) fails can be proved similarly.) Thus, for any $\delta \in (0, t'-t]$, there is $t'' \in u(X)$ such that $\rho(t'', t''+\delta) < \gamma$, because otherwise $\rho(t,t') < \gamma$ can't be true. Next take any $r < r' \in u(X)$ and let $\bar{\rho} = \rho(r,r') > 0$. By similar reasoning, for any $\delta \in (0, r'-r] > 0$, there is $r'' \in u(X)$ such that $\rho(r'', r''+\delta) \ge \bar{\rho}$. Thus,

$$\inf\left\{\frac{\rho(t'',t''+\delta)}{\rho(r'',r''+\delta)} : t'', r'', t''+\delta, r''+\delta \in u(X), 0<\delta\le\min[\tfrac{\varepsilon}{2}, r'-r, t'-t]\right\} = 0.$$

This infimum is at least K, thus $K = 0$, a contradiction.

Uniqueness of u up to positive affine transformation is standard. Uniqueness of μ and ϕ follows by the construction – expected utility preference on acts in $\mathcal{F}^{\Psi}$ uniquely pin down μ and make ϕ unique up to positive affine transformations given a normalization of u. □

8.13. Proof of Theorem 7.2

i)⇒v): Let $f, h \in \mathcal{F}$ and $\pi \in \Pi$. Since π is a finite permutation, π^N is the identity for some N. Thus, for any $g \in \mathcal{F}$,

$$g^*(\omega) \equiv \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \pi^{i-1} g(\omega) = \frac{1}{N}\sum_{i=1}^{N} \pi^{i-1} g(\omega)$$

is well defined everywhere.

Step 1. $g \sim g^*$.

By repeatedly applying i),

$$\begin{aligned} g &\sim \frac{1}{2}g + \frac{1}{2}\pi g \sim \frac{1}{2}\left(\frac{1}{2}g + \frac{1}{2}\pi g\right) + \frac{1}{2}\pi^2\left(\frac{1}{2}g + \frac{1}{2}\pi g\right) \\ &= \frac{1}{2^2}\sum_{i=1}^{2^2}\pi^{i-1}g \sim \frac{1}{2^k}\sum_{i=1}^{2^k}\pi^{i-1}g, \end{aligned}$$

for all k.

For any positive integer n, let $q(n)$ be the quotient when we divide n by N, and $r(n)$ the remainder. (That is, $n = q(n) \cdot N + r(n)$.) Then,

$$\begin{aligned} \frac{1}{2^k}\sum_{i=1}^{2^k}\pi^{i-1}g &= \frac{q\left(2^k\right)}{2^k}\sum_{i=1}^{N}\pi^{i-1}g + \frac{1}{2^k}\sum_{i=1}^{r\left(2^k\right)}\pi^{i-1}g \\ &= \frac{q\left(2^k\right)N}{2^k}g^* + \frac{1}{2^k}\sum_{i=1}^{r\left(2^k\right)}\pi^{i-1}g. \end{aligned}$$

(Here $\sum_{i=1}^{0} a_i$ is understood to be 0.) Since $r\left(2^k\right)$ can take at most finite number of integers, we can find a subsequence $2^{j(k)}$ of 2^k such that there is $\bar{K} = r\left(2^{j(k)}\right)$ for all k. Thus,

$$g \sim \alpha_k g^* + (1-\alpha_k)\left(\frac{1}{\bar{K}}\sum_{i=1}^{\bar{K}}\pi^{i-1}g\right) \text{ for all } k,$$

where $\alpha_k = \frac{q\left(2^{j(k)}\right)N}{2^{j(k)}}$. (Note that $1-\alpha_k = \frac{\bar{K}}{2^{j(k)}}$.) Since $\alpha_k \to 1$, Mixture Continuity implies $g \sim g^*$.

Step 2. $\alpha f + (1-\alpha)h \sim \alpha\pi f + (1-\alpha)h$.

By Step 1,

$$\alpha f + (1-\alpha)h \sim (\alpha f + (1-\alpha)h)^* = \alpha f^* + (1-\alpha)h^*.$$

Similarly,

$$\alpha\pi f + (1-\alpha)h \sim \alpha(\pi f)^* + (1-\alpha)h^* = \alpha f^* + (1-\alpha)h^*.$$

By Preorder, this step is proved.

v)⇒iv): By v),

$$\alpha f + (1-\alpha) h \sim \alpha \pi f + (1-\alpha) h \tag{8.8}$$

for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$. Note that $f \sim \pi f$ when $\alpha = 1$.

Setting $h = f$ gives,

$$f \sim \alpha \pi f + (1-\alpha) f \text{ for all } \alpha \in [0,1]. \tag{8.9}$$

Moreover, setting $h = \beta f + (1-\beta) \pi' f$ leads to

$$\begin{aligned} f &\sim (\alpha + (1-\alpha)\beta) f + (1-\alpha)(1-\beta) \pi' f \quad \text{(by (8.9))} \\ &= \alpha f + (1-\alpha)(\beta f + (1-\beta)\pi' f) \\ &\sim \alpha \pi f + (1-\alpha)(\beta f + (1-\beta)\pi' f) \quad \text{(by (8.8))} \\ &= (1-\alpha)\beta f + \alpha \pi f + (1-\alpha)(1-\beta)\pi' f. \end{aligned}$$

Since α and β can be taken arbitrarily, $f \sim \alpha f + \beta \pi f + (1-\alpha-\beta)\pi' f$ for all $\alpha, \beta \in [0,1]$. Since $f \sim \pi'' f$ for $\pi'' \in \Pi$,

$$f \sim \pi'' f \sim \alpha \pi'' f + \beta \pi \pi'' f + (1-\alpha-\beta) \pi' \pi'' f.$$

For any $\pi_1, \pi_2, \pi_3 \in \Pi$, take $\pi = \pi_2 (\pi_1)^{-1}$, $\pi' = \pi_3 (\pi_1)^{-1}$ and $\pi'' = \pi_1$. Then, we get iv) for $n = 3$. Repeat the argument to show iv) for any n.

Clearly, iv) implies ii) and iii). Moreover, ii) implies i) and so does iii). Thus, we have shown that i)-v) are equivalent.

Assume the additional axioms to show equivalence of i)-vii).

v)⇔vi): Clearly, v) implies vi). By Lemma 3.4, the converse holds.

vi)⇔vii): vi) implies vii) by Lemma 3.4. The converse holds since

$$\int (\lambda f + (1-\lambda) h)\, dp = \int (\lambda \pi f + (1-\lambda) h)\, dp \text{ for all symmetric } p \in \Delta(S^\infty).$$

Assume the existence of an extension $\hat{\succsim}$ satisfying Preorder and Norm Continuity to show the equivalence of i)-viii).

vi)⇔viii): The easy direction is viii) implies vi): Note that

$$(\lambda f + (1-\lambda) h)^\Psi = (\lambda \pi f + (1-\lambda) h)^\Psi$$

for all $f, h \in \mathcal{F}$ and $\lambda \in [0,1]$. Thus, viii) implies $\lambda f + (1-\lambda) h \sim \lambda \pi f + (1-\lambda) h$ for all $f, h \in \mathcal{F}$ and $\lambda \in [0,1]$, which implies Event Symmetry.

We now turn to the other direction. Let $\hat{\succsim}$ be any extension of $\succsim$ to $\hat{\mathcal{F}}$ satisfying Preorder and Norm Continuity. Define $\hat{\succsim}^*$ on $\hat{\mathcal{F}}$ by

$$f \hat{\succsim}^* g \text{ if } \alpha f + (1-\alpha) h \hat{\succsim} \alpha g + (1-\alpha) h \text{ for all } \alpha \in [0,1] \text{ and } h \in \hat{\mathcal{F}}. \quad (8.10)$$

Note that $\succsim$, $\hat{\succsim}$, $\succsim^*$ and $\hat{\succsim}^*$ on X all agree and can be represented by some non-constant mixture linear function u on X.

Our argument proceeds by steps along the following lines: In steps 1-3, given an arbitrary $h' \in \hat{\mathcal{F}}$, we construct sequences of simple acts that norm-converge to h' from above and from below. In steps 4-7, we show that $\hat{\succsim}^*$ is the unique extension of $\succsim^*$ satisfying C-complete Preorder, Norm Continuity, Monotonicity, Independence and Non-triviality. In step 8, we use the representation of $\succsim^*$ in Lemma 3.4 to construct a representation for $\hat{\succsim}^*$, and use that representation to show vi) implies viii).

Step 1. For any $h' \in \hat{\mathcal{F}}$, and $x, y \in X$ with $x \succ y$, there is $h \in \mathcal{F}$ such that $\frac{1}{2}h(\omega) + \frac{1}{2}y \prec \frac{1}{2}h'(\omega) + \frac{1}{2}x$ and $h(\omega) \succsim h'(\omega)$ for all $\omega \in \Omega$: Take $..., x_{-1}, x_0, x_1, ... \in X$ such that $x_0 = x$, $x_{-1} = y$ and $u(x_i) - u(x_{i-1}) = u(x) - u(y)$. For each $\omega \in \Omega$, set $h(\omega) = x_i$ if $x_{i-1} \prec h'(\omega) \precsim x_i$. Notice that $h \in \mathcal{F}$ since h' is bounded above and below and $u(x) - u(y) > 0$. Then h does the job since $h(\omega) \succsim h'(\omega)$ by construction and

$$\begin{aligned} \tfrac{1}{2}u(h(\omega)) + \tfrac{1}{2}u(y) &= \tfrac{1}{2}u(h(\omega)) - \tfrac{1}{2}u(x_i) + \tfrac{1}{2}u(x_{i-1}) + \tfrac{1}{2}u(x) \\ &< \tfrac{1}{2}u(h'(\omega)) + \tfrac{1}{2}u(x). \end{aligned}$$

Step 2. For any $h' \in \hat{\mathcal{F}}$, there is $h_k \in \mathcal{F}$ such that h_k norm-converges to h' and $h_k(\omega) \succsim h'(\omega)$ for all $\omega \in \Omega$: Take $z_k, z \in X$ such that $u(z_k) \searrow u(z)$. This is possible by Mixture Continuity. By Step 1, for each k, since $z_k \succ z$, there is $h_k \in \mathcal{F}$ such that $h_k(\omega) \succsim h'(\omega)$ and $\frac{1}{2}h_k(\omega) + \frac{1}{2}z \prec \frac{1}{2}h'(\omega) + \frac{1}{2}z_k$ for all $\omega \in \Omega$. Thus, $\sup_{\omega\in\Omega} |u(h_k(\omega)) - u(h'(\omega))| \le u(z_k) - u(z)$. For any $x \succ y$, there exists K such that for all $k > K$, $u(z_k) - u(z) \le u(x) - u(y)$. Thus, h_k norm-converges to h'.

Step 3. For any $h' \in \hat{\mathcal{F}}$, there is $h_k \in \mathcal{F}$ such that h_k norm-converges to h' and $h'(\omega) \succsim h_k(\omega)$ for all $\omega \in \Omega$: Slightly change Steps 1 and 2.

Step 4. $\hat{\succsim}$ satisfies Monotonicity: Take $f, g \in \hat{\mathcal{F}}$ such that $f(\omega) \hat{\succsim} g(\omega)$ for all $\omega \in \Omega$. By Steps 2 and 3, there are $f_k, g_k \in \mathcal{F}$ such that f_k, g_k norm-converge to f, g respectively, and $f_k(\omega) \hat{\succsim} f(\omega)$ and $g(\omega) \hat{\succsim} g_k(\omega)$ for all $\omega \in \Omega$. Then, $f_k(\omega) \succsim g_k(\omega)$ for all $\omega \in \Omega$, and Monotonicity of $\succsim$ implies $f_k \succsim g_k$, hence $f_k \hat{\succsim} g_k$. Norm Continuity guarantees $f \hat{\succsim} g$.

Step 5. $\hat{\succsim}^*$ satisfies C-complete Preorder, Norm Continuity, Monotonicity, Independence and Non-triviality: $\hat{\succsim}^*$ inherits C-complete Preorder, Norm Continuity, Monotonicity and Non-triviality from the corresponding properties of $\hat{\succsim}$ and satisfies Independence by (8.10).
Step 6. $\hat{\succsim}^*$ extends $\succsim^*$: Take $f, g \in \mathcal{F}$ such that $f \succsim^* g$, that is, $\alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h$ for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$ (and thus $\alpha f + (1-\alpha) h \hat{\succsim} \alpha g + (1-\alpha) h$ for all $\alpha \in [0,1]$ and $h \in \mathcal{F}$ since $\hat{\succsim}$ is an extension of $\succsim$). Now fix $h' \in \hat{\mathcal{F}}$. By Step 2, there is $h_k \in \mathcal{F}$ norm-converging to h'. Moreover, by the mixture linearity of u, $\alpha f + (1-\alpha) h_k$ and $\alpha g + (1-\alpha) h_k$ norm-converge to $\alpha f + (1-\alpha) h'$ and $\alpha g + (1-\alpha) h'$ respectively. Since $\alpha f + (1-\alpha) h_k \hat{\succsim} \alpha g + (1-\alpha) h_k$ for all $\alpha \in [0,1]$ and $k = 1, 2, ...$, Norm Continuity implies $\alpha f + (1-\alpha) h' \hat{\succsim} \alpha g + (1-\alpha) h'$ for all $\alpha \in [0,1]$. Since h' is arbitrary, $f \hat{\succsim}^* g$.
Step 7. All extensions of $\succsim^*$ satisfying the axioms in Step 5 are the same: Assume $\succsim_1^*, \succsim_2^*$ on $\hat{\mathcal{F}}$ are two such extensions. It is enough to show that $f \succsim_1^* g$ for $f, g \in \hat{\mathcal{F}}$ implies $f \succsim_2^* g$ since the labeling of the extensions is arbitrary. By Steps 2 and 3, there are $f_k, g_k \in \mathcal{F}$ such that f_k and g_k norm-converge to f and g respectively and $f_k(\omega) \succsim_1^* f(\omega)$, $g(\omega) \succsim_1^* g_k(\omega)$ for all $\omega \in \Omega$. Thus, $f_k \succsim_1^* f \succsim_1^* g \succsim_1^* g_k$. Since $\succsim_1^*, \succsim_2^*$ coincide on $\mathcal{F}$, $f_k \succsim_2^* g_k$. Norm Continuity of $\succsim_2^*$ implies $f \succsim_2^* g$.
Step 8. vi) implies viii): By Lemma 3.4, there is $M \subseteq \Delta(\Delta(S))$ such that for all $f, g \in \mathcal{F}$, $f \succsim^* g$ iff $\int u(f)\, dp \geq \int u(g)\, dp$ for all $p \in \left\{\int \ell^\infty dm(\ell) : m \in M\right\}$. Define an extension $\succsim_1^*$ of $\succsim^*$ to $\hat{\mathcal{F}}$ by

$$f \succsim_1^* g \text{ iff } \int u(f)\, dp \geq \int u(g)\, dp \text{ for all } p \in \left\{\int \ell^\infty dm(\ell) : m \in M\right\}.$$

Then, one can check that $\succsim_1^*$ satisfies all the axioms in Step 5. By Step 7, $\succsim_1^* = \hat{\succsim}^*$. Therefore, $f \hat{\sim}^* f^\Psi$ and $g \hat{\sim}^* g^\Psi$ for any $f, g \in \mathcal{F}$ and hence $f \hat{\sim} f^\Psi$ and $g \hat{\sim} g^\Psi$. Transitivity of $\hat{\succsim}$ implies viii). □

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