

# Symmetric function theory and unitary invariant ensembles

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## ABSTRACT

Representation theory and the theory of symmetric functions have played a central role in random matrix theory in the computation of quantities such as joint moments of traces and joint moments of characteristic polynomials of matrices drawn from the circular unitary ensemble and other circular ensembles related to the classical compact groups. The reason is that they enable the derivation of exact formulas, which then provide a route to calculating the large-matrix asymptotics of these quantities. We develop a parallel theory for the Gaussian Unitary Ensemble (GUE) of random matrices and other related unitary invariant matrix ensembles. This allows us to write down exact formulas in these cases for the joint moments of the traces and the joint moments of the characteristic polynomials in terms of appropriately defined symmetric functions. As an example of an application, for the joint moments of the traces, we derive explicit asymptotic formulas for the rate of convergence of the moments of polynomial functions of GUE matrices to those of a standard normal distribution when the matrix size tends to infinity.

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## I. INTRODUCTION

Many important quantities in random matrix theory, such as joint moments of traces and joint moments of characteristic polynomials, can be calculated exactly for matrices drawn from the circular unitary ensemble and the other circular ensembles related to the classical compact groups using representation theory and the theory of symmetric polynomials. In the case of joint moments of the traces, this approach has proved highly successful, as in, for example, the work of Diaconis and Shahshahani.<sup>19</sup> Similarly, the joint moments of characteristic polynomials were calculated exactly in terms of Schur polynomials by Bump and Gamburd,<sup>14</sup> leading to expressions equivalent to those obtained using the Selberg integral and related techniques.<sup>2,16,40,41</sup> Lee and Oh<sup>46</sup> extended the work of Bump and Gamburd<sup>14</sup> and computed the correlation functions of characteristic polynomials in Sato–Tato groups as sums of characters of irreducible characters of the symplectic group  $Sp(N)$ . Our aim here is to develop a parallel theory for the classical unitary invariant Hermitian ensembles of random matrices, in particular, for the Gaussian (GUE), Laguerre (LUE), and Jacobi (JUE) unitary ensembles.

Characteristic polynomials and their asymptotics have been well studied for Hermitian matrices using orthogonal polynomials, supersymmetric techniques, and Selberg and Itzykson–Zuber integrals; see, for example, Refs. 1, 9, 12, and 29–31. Other properties, including universality,<sup>11,57</sup> and ensembles with external sources<sup>25,28</sup> have also been considered. Here, we give a symmetric-function-theoretic approach similar to that established by Bump and Gamburd<sup>14</sup> using generalized Schur polynomials<sup>53</sup> or multivariate orthogonal polynomials<sup>3,4</sup> to compute correlation functions of characteristic polynomials for  $\beta = 2$  ensembles.

Diaconis and Shahshahani<sup>19</sup> used group theoretic arguments and symmetric functions to calculate joint moments of traces of matrices for classical compact groups. Here, using multivariate orthogonal polynomials, we develop a similar approach to calculate joint moments of traces for Hermitian ensembles, leading to closed-form expressions using combinatorial and symmetric-function-theoretic methods.

Moments of Hermitian ensembles and their correlators have recently received considerable attention. Cunden *et al.*<sup>18</sup> showed that as a function of their order, the moments are hypergeometric orthogonal polynomials. Cunden, Dahlqvist, and O'Connell<sup>17</sup> showed that the cumulants of the Laguerre ensemble admit an asymptotic expansion in inverse powers of  $N$  of whose coefficients are the Hurwitz numbers. Dubrovin and Yang<sup>20</sup> computed the cumulant generating function for the GUE, while Gissonni, Grava, and Ruzza calculated the generating function of the cumulants of the LUE in Ref. 32 and the JUE in Ref. 33.

If  $M$  is drawn at random from the classical compact groups  $U(N)$ ,  $O(N)$ ,  $Sp(N)$  equipped with Haar measure, then  $\text{Tr } M^k$ ,  $k \in \mathbb{N}$ , converges to a complex normal random variable as  $N \rightarrow \infty$ . Johansson<sup>37</sup> was the first to prove a central limit theorem when  $M$  belongs to an ensemble of Hermitian matrices invariant under unitary conjugation. In this case, the analog of  $\text{Tr } M^k$  is played by  $\text{Tr } T_k(M)$ , where  $T_k$  is the Chebyshev polynomial of the first kind; see also Refs. 5, 6, 10, 26, 42, 44, 50, 51, 54, 55, and 62 and references therein. As an example of an application of the general approach we take here, we apply our results to establish explicit asymptotic formulas for the rate of convergence of the moments and cumulants of Chebyshev-polynomial functions of GUE matrices to those of a standard normal distribution when the matrix size tends to infinity. In a companion article,<sup>38</sup> we use the techniques developed in this paper to investigate the moments of the characteristic polynomials in the GUE, uncovering structure that had been overlooked in previous studies.

The theory of symmetric functions has been applied to orthogonal polynomials also outside the context of random matrices. In a spirit similar to that in this paper, generalizations of classical combinatorial identities, such as the Cauchy formula, have played a fundamental role. The Schur functions arise naturally in the representation theory of the Heisenberg algebra. Lam<sup>43</sup> showed that by choosing a particular representation, the Schur polynomials are replaced by a new class of symmetric functions that obey Pieri and Cauchy-like identities. In turn, these can be applied to the theory of Hall–Littlewood and Macdonald polynomials. More recently, Borodin<sup>8</sup> studied a one-parameter family of rational symmetric functions that generalize the Hall–Littlewood polynomials. He also showed that such symmetric functions satisfy Pieri and Cauchy-like identities.

This paper is structured as follows. In Sec. II, we introduce our main results. The preliminaries, multivariate orthogonal polynomials, and their properties are discussed in Sec. III. The correlation functions of characteristic polynomials are calculated in Sec. IV. We discuss the change in the basis among different symmetric functions in Sec. V and prove there the results for the moments of characteristic polynomials and the joint moments of traces for different ensembles. Finally, in Sec. VI, by way of an example, we apply the results to derive explicit asymptotic formulas for the rate of convergence of the moments and cumulants of Chebyshev-polynomial functions of GUE matrices to those of a standard normal distribution when the matrix size tends to infinity.

## II. STATEMENTS AND RESULTS

For the classical compact groups, Schur polynomials and their generalizations are the characters of  $U(N)$ ,  $O(N)$ , and  $Sp(N)$ . In this context, they have been used extensively to calculate correlation functions of characteristic polynomials and joint moments of the traces; see Refs. 14 and 19. Although group theoretic tools are not available for the set of Hermitian matrices, multivariate orthogonal polynomials play the role of Schur functions for the GUE, LUE, and JUE and can be used to study fundamental quantities, such as moments of traces and characteristic polynomials.

For a partition  $\mu$ , let  $\Phi_\mu$  be the multivariate symmetric polynomials with a leading coefficient equal to 1 that obey the orthogonality relation

$$\int \Phi_\mu(x_1, \dots, x_N) \Phi_\nu(x_1, \dots, x_N) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{j=1}^N w(x_j) dx_j = \delta_{\mu\nu} C_\mu \quad (2.1)$$

for a weight function  $w$ . Here, the lengths of the partitions  $\mu$  and  $\nu$  are less than or equal to the number of variables  $N$ , and  $C_\mu$  is a constant, which depends on  $N$ . We prove the following lemma, which is a generalization of the dual Cauchy identity.

**Lemma 2.1.** Let  $\Phi_\mu$  be multivariate polynomials given in (2.1). Let  $p, q \in \mathbb{N}$ , and for  $\lambda \subseteq (q^p) \equiv \underbrace{(q, \dots, q)}_p$ , let  $\tilde{\lambda} = (p - \lambda'_1, \dots, p - \lambda'_p)$ .

Then,

$$\prod_{i=1}^p \prod_{j=1}^q (t_i - x_j) = \sum_{\lambda \subseteq (q^p)} (-1)^{|\tilde{\lambda}|} \Phi_{\tilde{\lambda}}(t_1, \dots, t_p) \Phi_{\lambda}(x_1, \dots, x_q). \quad (2.2)$$

Here, partition  $\lambda = (\lambda_1, \dots, \lambda_l)$  such that  $\lambda_1 \geq \dots \geq \lambda_l$  is a sub-partition of the partition  $(q^p)$ , denoted by  $\lambda \subseteq (q^p)$  (see Sec. III A). This lemma appears in Ref. 24 (p. 625) for the Jacobi multivariate polynomials for arbitrary  $\beta$ . Here, we present a different proof for  $\beta = 2$ , which holds for the Hermite and Laguerre polynomials, too. A key difference in our approach is that we have closed-form expressions for multivariate polynomials as determinants of univariate classical orthogonal polynomials, while in the previous literature, their construction was based on recurrence relations. This means that in this paper, formula (2.2) becomes a powerful tool and plays a role analogous to that of the classical dual Cauchy identity for  $U(N)$ . It is worth noting that the identity in Lemma 2.1 is independent of the weight  $w$ . This is a consequence of the row and column operations on determinants [see Eqs. (4.4) and (4.5)] that are central to the proof. It is the analog of the well-known equality

for Vandermonde determinants,

$$\prod_{1 \leq i < j \leq N} (x_i - x_j) = \det [x_i^{N-j}]_{i,j=1,\dots,N} = \det [\varphi_{N-j}(x_i)]_{i,j=1,\dots,N}, \quad (2.3)$$

where  $\varphi_i$ ,  $i = 1, 2, \dots$ , is any sequence of monic orthogonal polynomials.

We focus, in particular, on when  $w(x)$  in (2.1) is a Gaussian, Laguerre, and Jacobi weight,

$$w(x) = \begin{cases} e^{-\frac{x^2}{2}}, & x \in \mathbb{R}, & \text{Gaussian,} \\ x^\gamma e^{-x}, & x \in \mathbb{R}_+, \quad \gamma > -1, & \text{Laguerre,} \\ x^{\gamma_1} (1-x)^{\gamma_2}, & x \in [0, 1], \quad \gamma_1, \gamma_2 > -1, & \text{Jacobi.} \end{cases} \quad (2.4)$$

The classical polynomials orthogonal with respect to these weights satisfy

$$\int_{\mathbb{R}} H_j(x) H_k(x) e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} j! \delta_{jk}, \quad (2.5a)$$

$$\int_{\mathbb{R}_+} L_m^{(\gamma)} L_n^{(\gamma)} x^\gamma e^{-x} dx = \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)} \delta_{nm}, \quad (2.5b)$$

$$\begin{aligned} & \int_0^1 J_n^{(\gamma_1, \gamma_2)}(x) J_m^{(\gamma_1, \gamma_2)}(x) x^{\gamma_1} (1-x)^{\gamma_2} dx \\ &= \frac{1}{(2n+\gamma_1+\gamma_2+1)} \frac{\Gamma(n+\gamma_1+1) \Gamma(n+\gamma_2+1)}{n! \Gamma(n+\gamma_1+\gamma_2+1)} \delta_{mn}. \end{aligned} \quad (2.5c)$$

The identity in (2.2) gives a compact way to calculate the correlation functions and moments of characteristic polynomials of unitary ensembles using symmetric functions. The results are as follows.

**Theorem 2.1.** Let  $M$  be an  $N \times N$  GUE, LUE, or JUE matrix, and  $t_1, \dots, t_p \in \mathbb{C}$ . Then,

$$\begin{aligned} \text{(a)} \quad \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^p \det(t_j - M) \right] &= \mathcal{H}_{(N^p)}(t_1, \dots, t_p), \\ \text{(b)} \quad \mathbb{E}_N^{(L)} \left[ \prod_{j=1}^p \det(t_j - M) \right] &= \left( \prod_{j=N}^{p+N-1} (-1)^j j! \right) \mathcal{L}_{(N^p)}^{(\gamma)}(t_1, \dots, t_p), \\ \text{(c)} \quad \mathbb{E}_N^{(J)} \left[ \prod_{j=1}^p \det(t_j - M) \right] &= \left( \prod_{j=N}^{p+N-1} (-1)^j j! \frac{\Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+1)} \right) \mathcal{J}_{(N^p)}^{(\gamma_1, \gamma_2)}(t_1, \dots, t_p). \end{aligned} \quad (2.6)$$

Here, subscripts  $(H)$ ,  $(L)$ ,  $(J)$  indicate Hermite, Laguerre, and Jacobi, respectively, and  $\mathcal{H}_\lambda$ ,  $\mathcal{L}_\lambda^\gamma$ , and  $\mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}$  are multivariate polynomials orthogonal with respect to the generalized weights in (2.1).

Similar to the case of the classical compact groups, correlations of traces of Hermitian ensembles can be calculated using the theory of symmetric functions. For a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ ,  $\sum_j \lambda_j \leq N$ , define

$$\begin{aligned} C_\lambda(N) &= \prod_{j=1}^N \frac{(\lambda_j + N - j)!}{(N - j)!}, \\ G_\lambda(N, \gamma) &= \prod_{j=1}^N \Gamma(\lambda_j + N - j + \gamma + 1). \end{aligned} \quad (2.7)$$

The constants  $C_\lambda(N)$  and  $G_\lambda(N, \gamma)$  have several interesting combinatorial interpretations, which are discussed in Sec. V A.

**Theorem 2.2.** Let  $M$  be an  $N \times N$  GUE, LUE, or JUE matrix, and let  $\mu = (\mu_1, \dots, \mu_l)$  be a partition such that  $|\mu| = \sum_{j=1}^l \mu_j \leq N$ . Then,

(a)

$$\mathbb{E}_N^{(H)} \left[ \prod_{j=1}^l \text{Tr } M^{\mu_j} \right] = \begin{cases} \frac{1}{2^{\frac{|\mu|}{2}} \frac{|\mu|!}{2}} \sum_{\lambda \vdash |\mu|} \chi_{(2^{|\mu|/2})}^\lambda \chi_\mu^\lambda C_\lambda(N), & |\mu| \text{ is even,} \\ 0, & \text{otherwise,} \end{cases} \quad (2.8)$$

which is a polynomial in  $N$ .

(b)

$$\mathbb{E}_N^{(L)} \left[ \prod_{j=1}^l \text{Tr } M^{\mu_j} \right] = \frac{1}{|\mu|!} \sum_{\lambda \vdash |\mu|} \frac{G_\lambda(N, \gamma)}{G_0(N, \gamma)} C_\lambda(N) \chi_{(1^{|\mu|})}^\lambda \chi_\mu^\lambda. \quad (2.9)$$

(c)

$$\mathbb{E}_N^{(J)} \left[ \prod_{j=1}^l \text{Tr } M^{\mu_j} \right] = \sum_{\lambda \vdash |\mu|} \frac{G_\lambda(N, \gamma_1)}{G_0(N, \gamma_1)} C_\lambda(N) \chi_\mu^\lambda D_{\lambda_0}^{(J)}, \quad (2.10)$$

where

$$D_{\lambda_0}^{(J)} = \det \left[ \mathbb{1}_{\lambda_i - i + j \geq 0} \frac{1}{(\lambda_i - i + j)!} \frac{\Gamma(2N - 2i + \gamma_1 + \gamma_2 + 2)}{\Gamma(2N + \lambda_i - i - j + \gamma_1 + \gamma_2 + 2)} \right]_{i,j=1,\dots,N}. \quad (2.11)$$

In the above equations,  $\chi_\mu^\lambda$  are the characters of the symmetric group  $\mathcal{S}_m$ ,  $m = |\lambda| = |\mu|$ , associated with the  $\lambda$ th irreducible representation on the  $\mu$ th conjugacy class.

Next, we now focus our attention on the GUE with rescaled matrices  $M_R = M/\sqrt{4N}$ . Define the random variables

$$X_k := \text{Tr } T_k(M_R) - \mathbb{E}_N^{(H)} [\text{Tr } T_k(M_R)]. \quad (2.12)$$

Here,  $T_k$  is the Chebyshev polynomial of degree  $k$ . Johansson proved the following multi-dimensional central limit theorem for  $X_k$ :<sup>37</sup>

$$(X_1, \dots, X_{2m}) \xrightarrow{d} \left( \frac{1}{2} r_1, \dots, \frac{\sqrt{2m}}{2} r_{2m} \right), \quad (2.13)$$

where  $r_j$  are independent standard normal random variables and  $\xrightarrow{d}$  means convergence in distribution.<sup>37</sup>

Define

$$\mathcal{E}_{n,k} := \mathbb{E}_N^{(H)} [X_k^n] - \left( \frac{\sqrt{k}}{2} \right)^n \mathbb{E}[r_k^n]. \quad (2.14)$$

The formalism that we developed to study moments of traces allow us to derive explicit estimates for the error  $\mathcal{E}_{n,k}$  as a function of matrix size  $N$ . For rescaled Gaussian matrices, the correlators of traces are Laurant polynomials in  $N$ . This fact can be seen from (2.8) when applied to rescaled matrices. Consequently, the moments of polynomial test functions are also Laurant polynomials in  $N$ .

Let  $f$  and  $g$  be real or complex valued functions defined on some subset of  $\mathbb{R}$ . In what follows, we will use the notations  $f(x) \lesssim g(x)$  and  $f(x) = O(g(x))$  as  $x \rightarrow \infty$  interchangeably. More precisely, we write  $f(x) \lesssim g(x)$  [ $f(x) = O(g(x))$ ] if and only if there exist  $x_0$  and a positive constant  $K$  such that

$$|f(x)| \leq K|g(x)| \quad \text{for all } x > x_0. \quad (2.15)$$

More in general, if  $t$  belongs to the extended real line, then  $f(x) \lesssim g(x)$  [ $f(x) = O(g(x))$ ] as  $x \rightarrow t$  if and only if

$$\limsup_{x \rightarrow t} \frac{|f(x)|}{|g(x)|} < \infty. \quad (2.16)$$

We have the following theorem for Chebyshev polynomials.

**Theorem 2.3.** Fix  $k \in \mathbb{N}$ , and let  $kn \leq N$ . With the notation introduced above, the following statements hold as  $N \rightarrow \infty$ .

1. For  $k$  odd and  $k > 1$ ,

$$\mathcal{E}_{n,k} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ d_1(n,k) \frac{1}{N^2} + O\left(\frac{1}{N^4}\right) & \text{if } n \text{ is even,} \end{cases} \quad (2.17)$$

where

$$d_1(n,k) \lesssim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{7nk}{8} - \frac{13n}{8} + \frac{n}{6k}} k^{\frac{3n}{8}(k+2) + \frac{n}{8} + \frac{n}{4k}} n^{\frac{3n}{8}(k+1) - \frac{k}{4} + \frac{7}{8}} e^{-\frac{n}{8}(k+1) + \frac{9n}{4} + \frac{5n}{8k} + \pi\sqrt{\frac{n}{3}(k+1)}} \quad (2.18)$$

as  $n \rightarrow \infty$  and  $k$  fixed.

2. For  $k$  even,

$$\mathcal{E}_{n,k} = \begin{cases} d_2(n,k) \frac{1}{N} + O\left(\frac{1}{N^3}\right) & \text{if } n \text{ is odd,} \\ d_3(n,k) \frac{1}{N^2} + O\left(\frac{1}{N^4}\right) & \text{if } n \text{ is even,} \end{cases} \quad (2.19)$$

where

$$\begin{aligned} d_2(n,k) &\lesssim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{3nk}{8} - 3n + \frac{n}{6k}} k^{\frac{3nk}{8} + \frac{n}{2} + \frac{9n}{4k}} n^{\frac{3nk}{8} + \frac{2n}{k} - \frac{k}{2} - \frac{3}{8}} e^{-\frac{n}{8}(k-18) + \pi\sqrt{\frac{nk}{3} - \frac{19n}{8k}}}, \\ d_3(n,k) &\lesssim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{3nk}{8} - 3n + \frac{n}{6k}} k^{\frac{3nk}{8} + \frac{n}{2} + \frac{9n}{4k}} n^{\frac{3nk}{8} + \frac{2n}{k} - \frac{k}{2} + \frac{5}{8}} e^{-\frac{n}{8}(k-18) + \pi\sqrt{\frac{nk}{3} - \frac{19n}{8k}}} \end{aligned} \quad (2.20)$$

as  $n \rightarrow \infty$  and  $k$  fixed. Here,  $A = 1.2824 \dots$  is the Glaisher–Kinkelin constant.<sup>15</sup>

Along with the moments, we also give an estimate for the cumulants of random variables  $X_k$ . Computing cumulants from (2.8) is not straightforward. Instead, we employ the well-established connection between correlators of traces and the enumeration of ribbon graphs to estimate the cumulants. The results are elaborated in Sec. VI B.

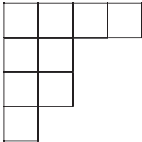
To summarize, for a fixed  $n$  and  $k$ , we show that the  $n$ th moment of  $X_k$  converges to the  $n$ th moment of independent scaled Gaussian variable as  $N^{-1}$  or  $N^{-2}$  depending on the parity of  $n$ , and the  $n$ th cumulant of  $X_k$  converges to 0 as  $N^{n-2}$  for  $n > 2$ . Theorem 2.3 provides explicit asymptotic estimates for the rate of convergence of the moments.

### III. BACKGROUND

Symmetric polynomials arise naturally in random matrix theory because the joint eigenvalue probability density function remains invariant under the action of the symmetric group. There has been a considerable focus on symmetric functions to study moments in various ensembles.<sup>14,19,21,49</sup> Here, we define some symmetric functions that will play a central role in our calculations and state some of their properties.

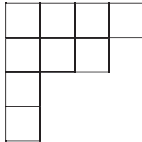
#### A. Review of symmetric functions

A partition  $\lambda$  is a sequence of non-negative integers such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ . We call the maximum  $l$  such that  $\lambda_l > 0$ , the length of the partition  $l(\lambda)$ , and  $|\lambda| = \sum_{i=1}^l \lambda_i$ , the weight. A partition can be represented with a Young diagram, which is a left adjusted table of  $|\lambda|$  boxes and  $l(\lambda)$  rows such that the first row contains  $\lambda_1$  boxes, the second row contains  $\lambda_2$  boxes, and so on. The conjugate partition  $\lambda'$  is defined by transposing the Young diagram of  $\lambda$ ,



Young diagram of  $\lambda$

→



Young diagram of  $\lambda'$

$$(3.1)$$

In the above example  $\lambda = (4, 2, 2, 1)$ ,  $|\lambda| = 9$  and  $l(\lambda) = 4$ . We denote a sub-partition  $\mu$  of  $\lambda$  by  $\mu \subseteq \lambda$  if the Young diagram of  $\mu$  is contained in the Young diagram of  $\lambda$ .

Another way to represent a partition is as follows: if  $\lambda$  has  $b_1$  1's,  $b_2$  2's, and so on, then  $\lambda = (1^{b_1} 2^{b_2} \dots k^{b_k})$ . In this representation, the weight  $|\lambda| = \sum_{j=1}^k j b_j$  and length  $l(\lambda) = \sum_{j=1}^k b_j$ . In the rest of this paper, we use both notations interchangeably and we do not distinguish partitions that differ only by a sequence of zeros; for example,  $(4, 2, 2, 1)$  and  $(4, 2, 2, 1, 0, 0)$  are the same partitions. We denote the empty partition by  $\lambda = 0$  or  $\lambda = ()$ .

The *elementary symmetric functions*  $e_r(x_1, \dots, x_N)$  are defined by

$$e_r(x_1, \dots, x_N) = \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}, \quad (3.2)$$

and the *complete symmetric functions*  $h_r(x_1, \dots, x_N)$  are defined by

$$h_r(x_1, \dots, x_N) = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \dots x_{i_r}. \quad (3.3)$$

Given a partition  $\lambda$ , we define

$$\begin{aligned} e_\lambda(x_1, \dots, x_N) &= \prod_j e_{\lambda_j}(x_1, \dots, x_N), \\ h_\lambda(x_1, \dots, x_N) &= \prod_j h_{\lambda_j}(x_1, \dots, x_N). \end{aligned} \quad (3.4)$$

The *Schur polynomials* are symmetric polynomials indexed by partitions. Given a partition  $\lambda$  such that  $l(\lambda) \leq N$ , we write

$$\begin{aligned} S_\lambda(x_1, \dots, x_N) &= \frac{\det [x_i^{\lambda_j + N - j}]_{i,j=1, \dots, N}}{\det [x_i^{N-j}]_{i,j=1, \dots, N}} \\ &= \frac{1}{\Delta(\mathbf{x})} \begin{vmatrix} x_1^{\lambda_1 + N - 1} & x_2^{\lambda_1 + N - 1} & \dots & x_N^{\lambda_1 + N - 1} \\ x_1^{\lambda_2 + N - 2} & x_2^{\lambda_2 + N - 2} & \dots & x_N^{\lambda_2 + N - 2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_N} & x_2^{\lambda_N} & \dots & x_N^{\lambda_N} \end{vmatrix}, \end{aligned} \quad (3.5)$$

where  $\Delta(\mathbf{x})$  is the Vandermonde determinant,

$$\Delta(\mathbf{x}) = \det [x_i^{N-j}]_{i,j=1, \dots, N} = \prod_{1 \leq i < j \leq N} (x_i - x_j). \quad (3.6)$$

If  $l(\lambda) > N$ , then  $S_\lambda = 0$ . The Jacobi–Trudi identities express Schur polynomials in terms of elementary and complete symmetric functions,

$$S_\lambda = \det [h_{\lambda_i + j - i}]_{i,j=1, \dots, l(\lambda)} = \det [e_{\lambda'_i + j - i}]_{i,j=1, \dots, l(\lambda')}. \quad (3.7)$$

Let  $\mu = 1^{b_1} 2^{b_2} \dots k^{b_k}$ , and write

$$p_j(\mathbf{x}) = \sum_{i=1}^N x_i^j, \quad j \in \mathbb{N}. \quad (3.8)$$

The *power sum* is defined by

$$P_\mu = \prod_{j=1}^k p_j^{b_j}. \quad (3.9)$$

Power sum and Schur functions are bases in the space of homogeneous symmetric polynomials, and they are related by

$$P_\mu = \sum_\lambda \chi_\mu^\lambda S_\lambda, \quad S_\lambda = \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} P_\mu, \quad (3.10)$$

$$z_\mu = \prod_j b_j^{b_j} b_j!,$$

where  $\chi_\mu^\lambda$  are the characters of the symmetric group  $\mathcal{S}_m$ ,  $m = |\lambda| = |\mu|$ , and  $z_\mu$  is the size of the centralizer of an element of conjugacy class  $\mu$ .

**Proposition 3.1** (Cauchy identity<sup>47</sup>). *Let  $t_1, t_2, \dots$  and  $x_1, x_2, \dots$  be two finite or infinite sequences of independent variables. Then,*

$$\prod_{i,j} (1 - t_i x_j)^{-1} = \sum_\lambda S_\lambda(\mathbf{t}) S_\lambda(\mathbf{x}). \quad (3.11)$$

When the sequences  $t_i$  and  $x_j$  are finite,

$$\prod_{i=1}^p \prod_{j=1}^q (1 - t_i x_j)^{-1} = \sum_\lambda S_\lambda(t_1, \dots, t_p) S_\lambda(x_1, \dots, x_q), \quad (3.12)$$

where  $\lambda$  runs over all partitions of length  $l(\lambda) \leq \min(p, q)$ . We also have the dual Cauchy identity,<sup>47</sup>

$$\prod_{i=1}^p \prod_{j=1}^q (1 + t_i x_j) = \sum_\lambda S_\lambda(t_1, \dots, t_p) S_{\lambda'}(x_1, \dots, x_q). \quad (3.13)$$

Since  $S_\lambda = 0$  or  $S_{\lambda'} = 0$  unless  $l(\lambda) \leq p$  or  $l(\lambda') \leq q$ ,  $\lambda$  runs over a finite number of partitions such that the Young diagram of  $\lambda$  fits inside a  $p \times q$  rectangle.

Changing  $x_j \rightarrow x_j^{-1}$  in the dual Cauchy identity and simplifying the fractions give

$$\prod_{i=1}^p \prod_{j=1}^q (t_i + x_j) = \sum_{\lambda \in (q^p)} S_\lambda(t_1, \dots, t_p) S_{\tilde{\lambda}}(x_1, \dots, x_q), \quad (3.14)$$

where  $(q^p) \equiv \underbrace{(q, \dots, q)}_p$  and  $\tilde{\lambda} = (p - \lambda'_q, \dots, p - \lambda'_1)$ . Since the Schur polynomials are homogeneous, we have

$$S_\mu(-x_1, \dots, -x_q) = (-1)^{|\mu|} S_\mu(x_1, \dots, x_q). \quad (3.15)$$

Thus, (3.14) becomes

$$\prod_{i=1}^p \prod_{j=1}^q (t_i - x_j) = \sum_{\lambda \in (q^p)} (-1)^{|\tilde{\lambda}|} S_\lambda(t_1, \dots, t_p) S_{\tilde{\lambda}}(x_1, \dots, x_q). \quad (3.16)$$

The Cauchy and dual Cauchy identities, combined with the fact that the Schur polynomials are the characters of  $U(N)$ , were essential tools in the proofs of the ratios of characteristic polynomials by Bump and Gamburd.<sup>14</sup> In order to prove Theorem 2.1, we use a similar approach, in which (3.16) is replaced by the generalized Cauchy dual identity (2.2).

## B. Multivariate orthogonal polynomials

Multivariate orthogonal polynomials can be defined by the determinant formula<sup>53</sup>

$$\Phi_\mu(\mathbf{x}) := \frac{1}{\Delta(\mathbf{x})} \begin{vmatrix} \varphi_{\mu_1+N-1}(x_1) & \varphi_{\mu_1+N-1}(x_2) & \dots & \varphi_{\mu_1+N-1}(x_N) \\ \varphi_{\mu_2+N-2}(x_1) & \varphi_{\mu_2+N-2}(x_2) & \dots & \varphi_{\mu_2+N-2}(x_N) \\ \vdots & \vdots & & \vdots \\ \varphi_{\mu_N}(x_1) & \varphi_{\mu_N}(x_2) & \dots & \varphi_{\mu_N}(x_N) \end{vmatrix}, \quad (3.17)$$



where  $l(\mu) \leq N$  and  $\varphi_i, i = 0, 1, \dots$  are a sequence of polynomials orthogonal with respect to the weight  $w(x)$ . One can check by a straightforward substitution that, up to a constant, multivariate polynomials (3.17) coincide with those in (2.1). When  $\varphi_j$  in (3.17) are the Hermite, Laguerre, and Jacobi polynomials, we have the multivariate generalizations  $\mathcal{H}_\mu$ ,  $\mathcal{L}_\mu^{(\gamma)}$ , and  $\mathcal{J}_\mu^{(\gamma_1, \gamma_2)}$ . These polynomials can be expressed as a linear combination of Schur polynomials, i.e.,

$$\Phi_\mu(\mathbf{x}) = \sum_{\nu \subseteq \mu} \kappa_{\mu\nu} S_\nu(\mathbf{x}). \quad (3.18)$$

For the Hermite, Laguerre, and Jacobi multivariate polynomials, we set the leading coefficient  $\kappa_{\mu\mu}$  in consistency with definitions (2.5) and (3.17),

$$\begin{aligned} \kappa_{\mu\mu}^{(H)} &= 1, \quad \kappa_{\mu\mu}^{(L)} = \frac{(-1)^{|\lambda| + \frac{1}{2}N(N-1)}}{G_\lambda(N, 0)}, \\ \kappa_{\mu\mu}^{(J)} &= \frac{(-1)^{|\lambda| + \frac{1}{2}N(N-1)}}{G_\lambda(N, \gamma_1 + \gamma_2) G_\lambda(N, 0)} \prod_{j=1}^N \Gamma(2N + 2\lambda_j - 2j + \gamma_1 + \gamma_2 + 1). \end{aligned} \quad (3.19)$$

The analogy between multivariate orthogonal polynomials and Schur functions becomes apparent by comparing definitions (3.5) with (3.17). In the literature, polynomials (3.17) are called generalized orthogonal polynomials or multivariate orthogonal polynomials<sup>3</sup> as well as generalized Schur polynomials.<sup>53</sup> The multivariate orthogonal polynomials also satisfy a generalization of the Jacobi–Trudi identities<sup>53</sup> similar to (3.7).

The classical Hermite, Laguerre, and Jacobi polynomials satisfy second-order Sturm Liouville problems. Similarly, their multivariate generalizations are eigenfunctions of second-order partial differential operators, known as Calogero–Sutherland Hamiltonians,

$$\begin{aligned} H^{(H)} &= \sum_{j=1}^N \left( \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^N \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \\ H^{(L)} &= \sum_{j=1}^N \left( x_j \frac{\partial^2}{\partial x_j^2} + (\gamma - x_j + 1) \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^N \frac{x_j}{x_j - x_k} \frac{\partial}{\partial x_j}, \\ H^{(J)} &= \sum_{j=1}^N \left( x_j(1 - x_j) \frac{\partial^2}{\partial x_j^2} + (\gamma_1 + 1 - x_j(\gamma_1 + \gamma_2 + 2)) \frac{\partial}{\partial x_j} \right) + 2 \sum_{\substack{j,k=1 \\ k \neq j}}^N \frac{x_j(1 - x_j)}{x_j - x_k} \frac{\partial}{\partial x_j}. \end{aligned} \quad (3.20)$$

These generalized orthogonal polynomials obey similar properties to their univariate counterparts.<sup>3</sup> The differential equations in (3.20) are also related to the Dyson Brownian motion.

#### IV. CORRELATION FUNCTIONS OF CHARACTERISTIC POLYNOMIALS

The main tool to compute correlations of characteristic polynomials and spectral moments is Lemma. 2.1, which we prove here.

**Proposition 4.1** (Laplace expansion). *Let  $\Xi_{p,q}$  consist of all permutations  $\sigma \in \mathcal{S}_{p+q}$  such that*

$$\sigma(1) < \dots < \sigma(p), \quad \sigma(p+1) < \dots < \sigma(p+q). \quad (4.1)$$

*Let  $A = a_{ij}$  be a  $(p+q) \times (p+q)$  matrix, and then Laplace expansion in the first  $p$  rows can be written as*

$$\det[a_{ij}] = \sum_{\sigma \in \Xi_{p,q}} \text{sgn}(\sigma) \begin{vmatrix} a_{1,\sigma(1)} & \dots & a_{1,\sigma(p)} \\ \vdots & & \vdots \\ a_{p,\sigma(1)} & \dots & a_{p,\sigma(p)} \end{vmatrix} \times \begin{vmatrix} a_{p+1,\sigma(p+1)} & \dots & a_{p+1,\sigma(p+q)} \\ \vdots & & \vdots \\ a_{p+q,\sigma(p+1)} & \dots & a_{p+q,\sigma(p+q)} \end{vmatrix}. \quad (4.2)$$

**Proposition 4.2.** *Let  $\lambda$  be a partition such that  $\lambda_1 \leq q$  and  $\lambda'_1 \leq p$ . Then, the  $p+q$  numbers*

$$\lambda_i + p - i \quad (1 \leq i \leq p), \quad p - 1 + j - \lambda'_j \quad (1 \leq j \leq q) \quad (4.3)$$

*are a permutation of  $\{0, \dots, p+q-1\}$ .*

Proposition 4.1 is a fact from Linear algebra, and the proof of Proposition 4.2 can be found in Ref. 47.

*Proof of Lemma 2.1.* Let the  $\varphi_j$ s be monic. Choosing a different normalization would affect Eq. (2.2) by an overall constant. Using the definition of generalized polynomials, Proposition 4.1, and Proposition 4.2, the right-hand side of (2.2) can be written as

$$\frac{1}{\Delta_p(\mathbf{t})} \frac{1}{\Delta_q(\mathbf{x})} \begin{vmatrix} \varphi_{p+q-1}(t_1) & \varphi_{p+q-2}(t_1) & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \varphi_{p+q-1}(t_p) & \varphi_{p+q-2}(t_p) & \dots & 1 \\ \varphi_{p+q-1}(x_1) & \varphi_{p+q-2}(x_1) & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \varphi_{p+q-1}(x_q) & \varphi_{p+q-2}(x_q) & \dots & 1 \end{vmatrix}. \quad (4.4)$$

Now, using column operations, we arrive at

$$\frac{1}{\Delta_p(\mathbf{t})} \frac{1}{\Delta_q(\mathbf{x})} \begin{vmatrix} t_1^{p+q-1} & t_1^{p+q-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ t_p^{p+q-1} & t_p^{p+q-2} & \dots & 1 \\ x_1^{p+q-1} & x_1^{p+q-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ x_q^{p+q-1} & x_q^{p+q-2} & \dots & 1 \end{vmatrix}. \quad (4.5)$$

The determinant in (4.5) can be evaluated using the formula for the Vandermonde determinant. We have

$$\prod_{1 \leq i < j \leq p} (t_i - t_j) \prod_{1 \leq i < j \leq q} (x_i - x_j) \prod_{i=1}^p \prod_{j=1}^q (t_i - x_j). \quad (4.6)$$

Combining Eqs. (4.4)–(4.6) proves the lemma. ■

If  $\varphi_j(-x) = (-1)^j \varphi_j(x)$ , as for Hermite polynomials, then

$$\Phi_\mu(-x_1, \dots, -x_N) = (-1)^{|\mu|} \Phi_\mu(x_1, \dots, x_N). \quad (4.7)$$

It follows that (2.2) becomes

$$\prod_{i=1}^p \prod_{j=1}^q (t_i + x_j) = \sum_{\lambda \in (q^p)} \Phi_\lambda(t_1, \dots, t_p) \Phi_\lambda(x_1, \dots, x_q). \quad (4.8)$$

*Proof of Theorem 2.1.* Unlike Hermite polynomials, the univariate Laguerre and Jacobi polynomials that obey (2.5) are not monic. This fact is reflected in the normalization in (3.19) and also in the following formulas:

$$\begin{aligned} \prod_{i=1}^p \prod_{j=1}^N (t_i - x_j) &= \sum_{\lambda \in (N^p)} (-1)^{|\lambda|} \mathcal{H}_\lambda(t_1, \dots, t_p) \mathcal{H}_\lambda(x_1, \dots, x_N), \\ \prod_{i=1}^p \prod_{j=1}^N (t_i - x_j) &= \left( \prod_{j=0}^{p+N-1} (-1)^j j! \right) \sum_{\lambda \in (N^p)} (-1)^{|\lambda|} \mathcal{L}_\lambda^{(y)}(t_1, \dots, t_p) \mathcal{L}_\lambda^{(y)}(x_1, \dots, x_N), \\ \prod_{i=1}^p \prod_{j=1}^N (t_i - x_j) &= \left( \prod_{j=0}^{p+N-1} (-1)^j j! \frac{\Gamma(j + \gamma_1 + \gamma_2 + 1)}{\Gamma(2j + \gamma_1 + \gamma_2 + 1)} \right) \\ &\quad \times \sum_{\lambda \in (N^p)} (-1)^{|\lambda|} \mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(t_1, \dots, t_p) \mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(x_1, \dots, x_N). \end{aligned} \quad (4.9)$$

After taking the expectation value, the non-zero contribution comes from  $\tilde{\lambda} = 0$  because of (2.1). Thus,  $\lambda' = (p^N)$ , which implies  $\lambda = (N^p)$ . Now, using

$$\mathcal{H}_0 = 1, \quad \mathcal{L}_0^{(\gamma)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!}, \quad \mathcal{J}_0^{(\gamma_1, \gamma_2)} = \prod_{j=0}^{N-1} \frac{(-1)^j}{j!} \frac{\Gamma(2j + \gamma_1 + \gamma_2 + 1)}{\Gamma(j + \gamma_1 + \gamma_2 + 1)} \quad (4.10)$$

proves the result. ■

## V. CORRELATIONS

In this section, we calculate moments of traces and characteristic polynomials of  $N \times N$  GUE, LUE, and JUE matrices. We can write

$$P_\mu(\mathbf{x}) = \prod_j (\text{Tr}(M^j))^{b_j}, \quad (5.1)$$

where  $\mathbf{x} = (x_1, \dots, x_N)$  are the eigenvalues of  $M$ . The power symmetric functions form a basis in the space of symmetric polynomials of degree  $|\mu|$ . The main idea is to express them in the basis of the multivariate orthogonal polynomials.

### A. Change of basis between symmetric functions

In this section, we give expressions for a change in the basis between multivariate orthogonal polynomials and other symmetric functions. We mainly focus on the GUE, but the same approach can be used for the LUE and the JUE.

#### 1. Gaussian ensemble

Let  $M$  be an  $N \times N$  GUE matrix. The *j.p.d.f.* of the eigenvalues is

$$\rho^{(H)}(x_1, \dots, x_N) = \frac{1}{Z_N^{(H)}} \Delta^2(\mathbf{x}) \prod_{i=1}^N e^{-\frac{x_i^2}{2}}, \quad (5.2)$$

$$Z_N^{(H)} = (2\pi)^{\frac{N}{2}} \prod_{j=1}^N j!.$$

Denote by  $H_n(x)$  the Hermite polynomials normalized according to (2.5a). Given a partition  $\lambda$  with  $l(\lambda) \leq N$ , the multivariate Hermite polynomials are given by

$$\mathcal{H}_\lambda(\mathbf{x}) = \frac{1}{\Delta(\mathbf{x})} \begin{vmatrix} H_{\lambda_1+N-1}(x_1) & H_{\lambda_1+N-1}(x_2) & \dots & H_{\lambda_1+N-1}(x_N) \\ H_{\lambda_2+N-2}(x_1) & H_{\lambda_2+N-2}(x_2) & \dots & H_{\lambda_2+N-2}(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ H_{\lambda_N}(x_1) & H_{\lambda_N}(x_2) & \dots & H_{\lambda_N}(x_N) \end{vmatrix} \quad (5.3)$$

and satisfy the orthogonality relation

$$\langle \mathcal{H}_\lambda, \mathcal{H}_\mu \rangle := \frac{1}{Z_N^{(H)}} \int_{\mathbb{R}^N} \mathcal{H}_\lambda(\mathbf{x}) \mathcal{H}_\mu(\mathbf{x}) \Delta^2(\mathbf{x}) \prod_i e^{-\frac{x_i^2}{2}} dx_i = C_\lambda(N) \delta_{\mu\lambda}, \quad (5.4)$$

$$C_\lambda(N) = \prod_{i=1}^N \frac{(\lambda_i + N - i)!}{(N - i)!}.$$

Since  $\lambda_i \geq 0$ , the constant  $C_\lambda(N)$  is a polynomial in  $N$  of degree  $|\lambda|$ . It turns out that it has a nice interpretation in terms of characters of the symmetric group. Let  $(i, j) \in \lambda$ ,  $1 \leq j \leq \lambda_i$ , denote a node in the Young diagram of  $\lambda$ . The roots of  $C_\lambda(N)$  are  $i - j$ , where  $i$  runs across the rows from top to bottom and  $j$  across the columns from left to right of the Young diagram. For example, if  $\lambda = (4, 3, 3, 1)$ , the roots of  $C_\lambda(N)$  are

0	-1	-2	-3
1	0	-1	
2	1	0	
3			

(5.5)

It is shown in Ref. 39 that

$$\begin{aligned} C_\lambda(N) &= \prod_{j=1}^{l(\lambda)} \frac{(\lambda_j + N - j)!}{(N - j)!} = \prod_{(ij) \in \lambda} (N - i + j) \\ &= \frac{|\lambda|!}{\dim V_\lambda} \sum_{\mu \vdash |\lambda|} \frac{\chi_\mu^\lambda}{z_\mu} N^{l(\mu)} = |\lambda|! \frac{S_\lambda(1^N)}{\dim V_\lambda}. \end{aligned} \quad (5.6)$$

The constant  $z_\lambda$  is defined in (3.10), and  $\dim V_\lambda$  is the dimension of the irreducible representation labeled by  $\lambda$  of the symmetric group  $\mathcal{S}_{|\lambda|}$ ,

$$\dim V_\lambda = |\lambda|! \frac{\prod_{1 \leq j < k \leq l(\lambda)} (\lambda_j - \lambda_k - j + k)}{\prod_{j=1}^{l(\lambda)} (\lambda_j + l(\lambda) - j)!} \quad (5.7)$$

and

$$S_\lambda(1^N) = \prod_{1 \leq j < k \leq N} \frac{\lambda_j - \lambda_k - j + k}{k - j}. \quad (5.8)$$

Schur polynomials can be expressed in terms of multivariate Hermite polynomials,

$$S_\lambda = \sum_{\nu \subseteq \lambda} \psi_{\lambda\nu}^{(H)} \mathcal{H}_\nu = \sum_{j=0}^{\lfloor \frac{|\lambda|}{2} \rfloor} \sum_{\nu \vdash g(j)} \psi_{\lambda\nu}^{(H)} \mathcal{H}_\nu, \quad g(j) = \begin{cases} 2j, & |\lambda| \text{ is even,} \\ 2j+1, & |\lambda| \text{ is odd.} \end{cases} \quad (5.9)$$

The function  $g(j)$  takes care of the fact that polynomials of odd and even degree do not mix similar to the one variable case. The first summation in (5.9) running over all lower order partitions takes care of the fact that  $\mathcal{H}_\lambda$  are, unlike  $S_\lambda$ , not homogeneous polynomials. For example, when  $|\lambda|$  is even, the only partitions that appear in (5.9) are those with weight  $|\nu| = |\lambda| - 2k$ ,  $k = 0, \dots, \frac{|\lambda|}{2}$ , and  $\nu \subseteq \lambda$ . The following proposition gives an explicit expression for the coefficients  $\psi_{\lambda\nu}^{(H)}$ .

**Proposition 5.1.** *If  $\lambda$  is a partition of length  $L$  and  $\nu$  is a sub-partition of  $\lambda$  such that  $|\lambda| - |\nu| = 0 \bmod 2$  and  $N \geq L$ , then  $\psi_{\lambda\nu}^{(H)}$  is the following polynomial in  $N$ :*

$$\psi_{\lambda\nu}^{(H)} = \frac{1}{2^{\frac{|\lambda| - |\nu|}{2}}} D_{\lambda\nu}^{(H)} \prod_{j=1}^L \frac{(\lambda_j + N - j)!}{(\nu_j + N - j)!}, \quad (5.10)$$

where

$$D_{\lambda\nu}^{(H)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k = 0 \bmod 2} \left( \left( \frac{\lambda_j - \nu_k - j + k}{2} \right)! \right)^{-1} \right]_{j,k=1,\dots,L}. \quad (5.11)$$

**Proof.** Let  $\lambda = (\lambda_1, \dots, \lambda_L, 0, \dots, 0)$  and  $\nu = (\nu_1, \dots, \nu_l, 0, \dots, 0)$ . Here,  $l$  is the length of  $\nu$  and  $N - l$  is the length of the sequence of zeros added to  $\nu$ . From (5.4) and the fact that  $\nu \subseteq \lambda$ ,  $l \leq L$ , it follows that

$$\begin{aligned} \psi_{\lambda\nu}^{(H)} &= \frac{\langle S_\lambda, \mathcal{H}_\nu \rangle}{\langle \mathcal{H}_\nu, \mathcal{H}_\nu \rangle} = \frac{1}{Z_N^{(H)} \langle \mathcal{H}_\nu, \mathcal{H}_\nu \rangle} \int_{\mathbb{R}^N} S_\lambda(\mathbf{x}) \mathcal{H}_\nu(\mathbf{x}) \Delta_N^2(\mathbf{x}) \prod_{i=1}^N e^{-\frac{x_i^2}{2}} dx_i, \\ &= \frac{1}{Z_N^{(H)} \langle \mathcal{H}_\nu, \mathcal{H}_\nu \rangle} \int_{\mathbb{R}^N} \prod_{i=1}^N e^{-\frac{x_i^2}{2}} dx_i \\ &\quad \times \begin{vmatrix} x_1^{\lambda_1+N-1} & \dots & x_1^{\lambda_L+N-L} & H_{N-L-1}(x_1) & \dots & 1 \\ x_2^{\lambda_1+N-1} & \dots & x_2^{\lambda_L+N-L} & H_{N-L-1}(x_2) & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_N^{\lambda_1+N-1} & \dots & x_N^{\lambda_L+N-L} & H_{N-L-1}(x_N) & \dots & 1 \end{vmatrix} \\ &\quad \times \begin{vmatrix} H_{\nu_1+N-1}(x_1) & \dots & H_{\nu_l+N-l}(x_1) & H_{N-l-1}(x_1) & \dots & 1 \\ H_{\nu_1+N-1}(x_2) & \dots & H_{\nu_l+N-l}(x_2) & H_{N-l-1}(x_2) & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ H_{\nu_1+N-1}(x_N) & \dots & H_{\nu_l+N-l}(x_N) & H_{N-l-1}(x_N) & \dots & 1 \end{vmatrix}. \end{aligned} \quad (5.12)$$

The last  $N - L$  and  $N - l$  columns in  $S_\lambda$  and in  $\mathcal{H}_v$ , respectively, are written in terms of the Hermite polynomials using column operations. In addition,  $\psi_{\lambda v}^{(H)}$  can be expanded as a sum over the permutations of  $N$ ,

$$\begin{aligned} \psi_{\lambda v}^{(H)} &= \frac{1}{Z_N^{(H)} \langle \mathcal{H}_v, \mathcal{H}_v \rangle} \sum_{\sigma \in S_N} \text{sgn}(\sigma) \int_{\mathbb{R}^N} \prod_{i=1}^N e^{-\frac{x_i^2}{2}} dx_i \left( x_{\sigma(1)}^{\lambda_1+N-1} \dots x_{\sigma(L)}^{\lambda_L+N-L} H_{N-L-1}(x_{\sigma(N-L-1)}) \dots H_0(x_{\sigma(0)}) \right) \\ &\quad \times \begin{vmatrix} H_{v_1+N-1}(x_1) & \dots & H_{v_1+N-l}(x_1) & H_{N-l-1}(x_1) & \dots & 1 \\ H_{v_1+N-1}(x_2) & \dots & H_{v_1+N-l}(x_2) & H_{N-l-1}(x_2) & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ H_{v_1+N-1}(x_N) & \dots & H_{v_1+N-l}(x_N) & H_{N-l-1}(x_N) & \dots & 1 \end{vmatrix}. \end{aligned} \quad (5.13)$$

Since the integrand is symmetric in  $x_i$ , every term in the above sum gives the same contribution and it is sufficient to consider only the identity permutation. All the factors can be absorbed into the determinant by multiplying the  $j$ th row with  $x_j^{\lambda_j+N-j}$  if  $j \leq L$  and with  $H_{N-j}(x_{N-j})$  if  $N \geq j > L$ . Then, using orthogonality of Hermite polynomials (2.5a) for the last  $N - L$  rows gives

$$\psi_{\lambda v}^{(H)} = \frac{N!}{Z_N^{(H)} \langle \mathcal{H}_v, \mathcal{H}_v \rangle} (2\pi)^{\frac{N-L}{2}} \prod_{i=L+1}^N (N-i)! \det \left[ \int_{\mathbb{R}} x_j^{\lambda_j+N-j} H_{v_k+N-k}(x_j) e^{-\frac{x_j^2}{2}} dx_j \right]_{j,k=1,\dots,L}. \quad (5.14)$$

Expanding monomials in terms of Hermite polynomials with the formula

$$x^n = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^m m! (n-2m)!} H_{n-2m}(x) \quad (5.15)$$

and using orthogonality lead to (5.10). The determinant  $D_{\lambda v}^{(H)}$  is independent of  $N$  and  $\psi_{\lambda v}^{(H)}$  is a polynomial in  $N$  since  $v \subseteq \lambda$ . ■

**Corollary 5.1.** *The roots of coefficients  $\psi_{\lambda v}^{(H)}$  are integers given by the content of the skew diagram  $\lambda/v$ .*

*Proof.* The skew diagram  $\lambda/v$  is the set-theoretic difference of the Young diagrams of  $\lambda$  and  $v$ : the set of squares that belong to the diagram of  $\lambda$  but not to that of  $v$ . Using (5.6),

$$\psi_{\lambda v}^{(H)} = \frac{1}{2^{\frac{|\lambda|-|v|}{2}}} \frac{C_\lambda(N)}{C_v(N)} D_{\lambda v}^{(H)}. \quad (5.16)$$

Since  $v \subseteq \lambda$ , the roots of  $\psi_{\lambda v}^{(H)}$  are integers and can be read from the skew diagram  $\lambda/v$  whenever  $D_{\lambda v}^{(H)} \neq 0$ . For example, if  $\lambda = (4, 1, 1)$  and  $v = (2)$ , then the roots of  $\psi_{\lambda v}^{(H)}$  are  $\{-3, -2, 1, 2\}$ ,

$$\begin{array}{|c|c|c|c|} \hline 0 & -1 & -2 & -3 \\ \hline 1 & & & \\ \hline 2 & & & \\ \hline \end{array} \quad (5.17)$$

■

**Corollary 5.2.** *The coefficient  $\psi_{\lambda \lambda}^{(H)} = 1$ .*

*Proof.* If  $v = \lambda$ ,

$$\psi_{\lambda \lambda}^{(H)} = \frac{N!}{Z_N^{(H)} \langle \mathcal{H}_\lambda, \mathcal{H}_\lambda \rangle} \det \left[ \int_{\mathbb{R}} x_j^{\lambda_j+N-j} H_{\lambda_k+N-k}(x_j) e^{-\frac{x_j^2}{2}} dx_j \right]_{j=1,\dots,N}. \quad (5.18)$$

By expanding monomials in terms of Hermite polynomials, only the diagonal terms survive. ■

*Proposition 5.2. The coefficient*

$$\psi_{\lambda 0}^{(H)} = \begin{cases} \frac{C_{\lambda}(N)}{2^{\frac{|\lambda|}{2}} \frac{|\lambda|!}{2}!} \chi_{(2^{|\lambda|/2})}^{\lambda}, & |\lambda| \text{ is even,} \\ 0, & |\lambda| \text{ is odd,} \end{cases} \quad (5.19)$$

where  $\chi_{(2^{|\lambda|/2})}^{\lambda}$  is the character of the  $\lambda$ th irreducible representation evaluated on the elements of cycle-type  $(2^{|\lambda|/2})$ .

*Proof.* Since Hermite polynomials of odd and even degree do not mix,  $\psi_{\lambda 0}^{(H)} = 0$  when  $|\lambda|$  is odd. When  $|\lambda|$  is even,

$$D_{\lambda 0}^{(H)} = \det \left[ \mathbb{1}_{\lambda_j - j + k \equiv 0 \pmod{2}} \frac{1}{\left(\frac{\lambda_j - j + k}{2}\right)!} \right]. \quad (5.20)$$

Let  $n = |\lambda|/2$  and  $L = l(\lambda)$ . Let  $g(x_1, \dots, x_L)$  be a formal power series in variables  $x_i$  and  $(k_1, \dots, k_L)$  be a partition constructed from  $\lambda$  such that  $k_j = \lambda_j + L - j$ ,  $j = 1, \dots, L$ . Let

$$[g(x_1, \dots, x_L)]_{(k_1, \dots, k_L)} = \text{coefficient of } x_1^{k_1} \dots x_L^{k_L}. \quad (5.21)$$

Using the Frobenius formula for characters of the symmetric group,

$$\begin{aligned} \chi_{(2^n)}^{\lambda} &= [\Delta(x_1, \dots, x_L)(x_1^2 + \dots + x_L^2)^n]_{(k_1, \dots, k_L)} \\ &= \sum_{n_1 + \dots + n_L = n} \frac{n!}{n_1! \dots n_L!} [\det [x_i^{L-j}] x_1^{2n_1} x_2^{2n_2} \dots x_L^{2n_L}]_{(\lambda_1 + L - 1, \lambda_2 + L - 2, \dots, \lambda_L)}. \end{aligned} \quad (5.22)$$

After absorbing  $x_i^{2n_i}$  into the  $i$ th row of the determinant, for each  $n_i$ , at most one term in the  $i$ th row has the exponent  $\lambda_i + L - i$ , say, the  $(i, j)$ th element  $x_i^{2n_i + L - j}$ , which implies  $2n_i = \lambda_i - i + j$ . For  $L$ -tuples  $\{n_1, \dots, n_L\}$  such that there is exactly one term in each row that has the required exponent, the non-zero summands are given by  $n! \operatorname{sgn}(\sigma) \prod_i ((\lambda_i - i + \sigma(i)/2)!)^{-1}$ , where  $\sigma \in S_L$ . Considering all such  $L$ -tuples and using Laplace expansion for the determinant prove the proposition. ■

Therefore, the expansion of Schur polynomials in terms of multivariate Hermite polynomials can be written as

$$S_{\lambda}(x_1, \dots, x_N) = C_{\lambda}(N) \sum_{\nu \subseteq \lambda} \frac{1}{2^{\frac{|\lambda| - |\nu|}{2}}} \frac{1}{C_{\nu}(\lambda)} D_{\lambda \nu}^{(H)} \mathcal{H}_{\nu}(x_1, \dots, x_N). \quad (5.23)$$

In a similar way, by expanding Hermite polynomials in terms of monomials in the definition of  $\mathcal{H}_{\lambda}$ , multivariate Hermite polynomials can be written in the Schur basis as follows:

$$\mathcal{H}_{\lambda} = \sum_{\nu \subseteq \lambda} \kappa_{\lambda \nu}^{(H)} S_{\nu} = \sum_{j=0}^{\lfloor \frac{|\lambda|}{2} \rfloor} \sum_{\nu \vdash g(j)} \kappa_{\lambda \nu}^{(H)} S_{\nu}, \quad g(j) = \begin{cases} 2j, & |\lambda| \text{ is even,} \\ 2j + 1, & |\lambda| \text{ is odd,} \end{cases} \quad (5.24)$$

where

$$\kappa_{\lambda \nu}^{(H)} = \left(\frac{-1}{2}\right)^{\frac{|\lambda| - |\nu|}{2}} D_{\lambda \nu}^{(H)} \prod_{j=1}^L \frac{(\lambda_j + N - j)!}{(\nu_j + N - j)!}. \quad (5.25)$$

Alternatively,

$$\mathcal{H}_{\lambda}(x_1, \dots, x_N) = C_{\lambda}(N) \sum_{\nu \subseteq \lambda} \left(\frac{-1}{2}\right)^{\frac{|\lambda| - |\nu|}{2}} \frac{1}{C_{\nu}(N)} D_{\lambda \nu}^{(H)} S_{\nu}(x_1, \dots, x_N), \quad (5.26)$$

where  $|\lambda| - |\nu| \equiv 0 \pmod{2}$ .

## 2. Laguerre ensemble

Let  $M$  be an  $N \times N$  LUE matrix with eigenvalues  $x_1, \dots, x_N$ . For  $\gamma > -1$ , the *j.p.d.f.* of eigenvalues is

$$\rho^{(L)}(x_1, \dots, x_N) = \frac{1}{Z_N^{(L)}} \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^\gamma e^{-x_i}, \quad (5.27)$$

$$Z_N^{(L)} = N! G_0(N, \gamma) G_0(N, 0),$$

where  $G_\lambda(N, \gamma)$  is given in (2.7).

The multivariate Laguerre polynomials defined by

$$\mathcal{L}_\lambda^{(\gamma)}(\mathbf{x}) = \frac{1}{\Delta_N} \begin{vmatrix} L_{\lambda_1+N-1}^{(\gamma)}(x_1) & L_{\lambda_1+N-1}^{(\gamma)}(x_2) & \dots & L_{\lambda_1+N-1}^{(\gamma)}(x_N) \\ L_{\lambda_2+N-2}^{(\gamma)}(x_1) & L_{\lambda_2+N-2}^{(\gamma)}(x_2) & \dots & L_{\lambda_2+N-2}^{(\gamma)}(x_N) \\ \vdots & \vdots & \dots & \vdots \\ L_{\lambda_N}^{(\gamma)}(x_1) & L_{\lambda_N}^{(\gamma)}(x_2) & \dots & L_{\lambda_N}^{(\gamma)}(x_N) \end{vmatrix}, \quad (5.28)$$

$l(\lambda) \leq N$ , satisfy the orthogonality relation

$$\begin{aligned} \langle \mathcal{L}_\lambda^{(\gamma)}, \mathcal{L}_\mu^{(\gamma)} \rangle &:= \frac{1}{Z_N^{(L)}} \int_{\mathbb{R}_+^N} \mathcal{L}_\lambda^{(\gamma)}(\mathbf{x}) \mathcal{L}_\mu^{(\gamma)}(\mathbf{x}) \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^\gamma e^{-x_i} dx_i \\ &= \frac{G_\lambda(N, \gamma)}{G_0(N, \gamma)} \frac{1}{G_\lambda(N, 0)} \frac{1}{G_0(N, 0)} \delta_{\lambda\mu}. \end{aligned} \quad (5.29)$$

The polynomials in the determinant (5.28) are normalized according to (2.5b). The Schur polynomials can be expanded in terms of multivariate Laguerre polynomials as

$$S_\lambda = \sum_{\nu \subseteq \lambda} \psi_{\lambda\nu}^{(L)} \mathcal{L}_\nu^{(\gamma)}, \quad (5.30)$$

where

$$\begin{aligned} \psi_{\lambda\nu}^{(L)} &= (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N, \gamma)}{G_\nu(N, \gamma)} G_\lambda(N, 0) D_{\lambda\nu}^{(L)}, \\ D_{\lambda\nu}^{(L)} &= \det \left[ \mathbb{1}_{\lambda_i - \nu_j - i + j \geq 0} \frac{1}{(\lambda_i - \nu_j - i + j)!} \right]_{i,j=1, \dots, l(\lambda)}. \end{aligned} \quad (5.31)$$

The coefficients  $\psi_{\lambda\nu}^{(L)}$  in (5.31) can be computed in the same way as in Proposition 5.1. It is interesting to note that the quantity  $|\lambda|/\nu! D_{\lambda\nu}^{(L)}$  gives the number of standard Young tableaux (SYT) of shape  $\lambda/\nu$  (Ref. 56, p. 344).

Multivariate Laguerre polynomials can also be expanded in the Schur basis,

$$\begin{aligned} \mathcal{L}_\lambda^{(\gamma)} &= \sum_{\nu \subseteq \lambda} \kappa_{\lambda\nu}^{(L)} S_\nu, \\ \kappa_{\lambda\nu}^{(L)} &= (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N, \gamma)}{G_\nu(N, \gamma)} \frac{1}{G_\nu(N, 0)} D_{\lambda\nu}^{(L)}. \end{aligned} \quad (5.32)$$

Similar to the Hermite case,  $D_{\lambda 0}^{(L)}$  turns out to be a character of the symmetric group.

**Proposition 5.3.** We have

$$D_{\lambda 0}^{(L)} = \frac{\chi_{(1^{|\lambda|})}^\lambda}{|\lambda|!} = \frac{\dim V_\lambda}{|\lambda|!}. \quad (5.33)$$

*Proof.* Same as Proposition 5.2. Note that  $|\lambda|! D_{\lambda 0}^{(L)}$  gives the number of standard Young tableaux of shape  $\lambda$ . ■

### 3. Jacobi ensemble

Let  $M$  be an  $N \times N$  JUE matrix with eigenvalues  $x_1, \dots, x_N$ . For  $\gamma_1, \gamma_2 > -1$ , the  $j.p.d.f.$  of eigenvalues is

$$\begin{aligned} \rho^{(j)}(x_1, \dots, x_N) &= \frac{1}{Z_N^{(j)}} \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^{\gamma_1} (1-x_i)^{\gamma_2}, \\ Z_N^{(j)} &= N! \prod_{j=0}^{N-1} \frac{j! \Gamma(j+\gamma_1+1) \Gamma(j+\gamma_2+1) \Gamma(j+\gamma_1+\gamma_2+1)}{\Gamma(2j+\gamma_1+\gamma_2+2) \Gamma(2j+\gamma_1+\gamma_2+1)}. \end{aligned} \quad (5.34)$$

Classical Jacobi polynomials are given by

$$J_n^{(\gamma_1, \gamma_2)}(x) = \frac{\Gamma(n+\gamma_1+1)}{\Gamma(n+\gamma_1+\gamma_2+1)} \sum_{j=0}^n \frac{(-1)^j}{j!(n-j)!} \frac{\Gamma(n+j+\gamma_1+\gamma_2+1)}{\Gamma(j+\gamma_1+1)} x^j \quad (5.35)$$

and satisfy the orthogonality relation (2.5c). The multivariate Jacobi polynomials are

$$\mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(\mathbf{x}) = \frac{1}{\Delta_N} \begin{vmatrix} J_{\lambda_1+N-1}^{(\gamma_1, \gamma_2)}(x_1) & J_{\lambda_1+N-1}^{(\gamma_1, \gamma_2)}(x_2) & \dots & J_{\lambda_1+N-1}^{(\gamma_1, \gamma_2)}(x_N) \\ J_{\lambda_2+N-2}^{(\gamma_1, \gamma_2)}(x_1) & J_{\lambda_2+N-2}^{(\gamma_1, \gamma_2)}(x_2) & \dots & J_{\lambda_2+N-2}^{(\gamma_1, \gamma_2)}(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ J_{\lambda_N}^{(\gamma_1, \gamma_2)}(x_1) & J_{\lambda_N}^{(\gamma_1, \gamma_2)}(x_2) & \dots & J_{\lambda_N}^{(\gamma_1, \gamma_2)}(x_N) \end{vmatrix}, \quad (5.36)$$

$l(\lambda) \leq N$ , and obey the orthogonality relation

$$\begin{aligned} \left\langle \mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}, \mathcal{J}_\mu^{(\gamma_1, \gamma_2)} \right\rangle &:= \frac{1}{Z_N^{(j)}} \int_{[0,1]^N} \mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(\mathbf{x}) \mathcal{J}_\mu^{(\gamma_1, \gamma_2)}(\mathbf{x}) \Delta^2(\mathbf{x}) \prod_{i=1}^N x_i^{\gamma_1} (1-x_i)^{\gamma_2} dx_i \\ &= \frac{N!}{Z_N^{(j)}} \frac{G_\lambda(N, \gamma_1) G_\lambda(N, \gamma_2)}{G_\lambda(N, \gamma_1+\gamma_2) G_\lambda(N, 0)} \prod_{j=1}^N (2\lambda_j + 2N - 2j + \gamma_1 + \gamma_2 + 1)^{-1} \delta_{\lambda\mu}. \end{aligned} \quad (5.37)$$

The expansion of the Schur polynomials in terms of multivariate Jacobi polynomials is

$$S_\lambda = \sum_{\nu \subseteq \lambda} \psi_{\lambda\nu}^{(j)} \mathcal{J}_\nu^{(\gamma_1, \gamma_2)}, \quad (5.38)$$

where

$$\begin{aligned} \psi_{\lambda\nu}^{(j)} &= (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N, \gamma_1)}{G_\nu(N, \gamma_1)} G_\nu(N, \gamma_1 + \gamma_2) G_\lambda(N, 0) \\ &\quad \times \mathcal{D}_{\lambda\nu}^{(j)} \prod_{j=1}^N (2\nu_j + 2N - 2j + \gamma_1 + \gamma_2 + 1), \\ \mathcal{D}_{\lambda\nu}^{(j)} &= \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} ((\lambda_j - \nu_k - j + k)! \Gamma(2N + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 2))^{-1} \right]_{j,k=1}^N. \end{aligned} \quad (5.39)$$

When  $N = 1$ , (5.38) coincides with the one variable analog

$$x^n = n! \Gamma(n + \gamma_1 + 1) \sum_{j=0}^n \frac{(-1)^j}{(n-j)!} \frac{(2j + \gamma_1 + \gamma_2 + 1) \Gamma(j + \gamma_1 + \gamma_2 + 1)}{\Gamma(j + \gamma_1 + 1) \Gamma(n + j + \gamma_1 + \gamma_2 + 2)} J_j^{(\gamma_1, \gamma_2)}(x). \quad (5.40)$$

Multivariate Jacobi polynomials can be expanded in Schur polynomials via

$$\mathcal{J}_\lambda^{(\gamma_1, \gamma_2)} = \sum_{\nu \subseteq \lambda} \kappa_{\lambda\nu}^{(j)} S_\nu, \quad (5.41)$$



where

$$\kappa_{\lambda\nu}^{(J)} = (-1)^{|\nu| + \frac{1}{2}N(N-1)} \frac{G_\lambda(N, \gamma_1)}{G_\nu(N, \gamma_1)} \frac{1}{G_\lambda(N, \gamma_1 + \gamma_2) G_\nu(N, 0)} \tilde{\mathcal{D}}_{\lambda\nu}^{(J)},$$

$$\tilde{\mathcal{D}}_{\lambda\nu}^{(J)} = \det \left[ \mathbb{1}_{\lambda_j - \nu_k - j + k \geq 0} \frac{\Gamma(2N + \lambda_j + \nu_k - j - k + \gamma_1 + \gamma_2 + 1)}{(\lambda_j - \nu_k - j + k)!} \right]_{j,k=1}^N.$$
(5.42)

## B. Moments of Schur polynomials

### 1. Gaussian case

Similar to the moments of monomials with respect to the Gaussian weight,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{2n} e^{-\frac{x^2}{2}} dx = (-1)^n H_{2n}(0) = \frac{2n!}{2^n n!},$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{2n+1} e^{-\frac{x^2}{2}} dx = 0,$$
(5.43)

the moments of Schur polynomials associated with a partition  $\lambda$  are given by

$$\mathbb{E}_N^{(H)}[S_\lambda] = \begin{cases} (-1)^{\frac{|\lambda|}{2}} \mathcal{H}_\lambda(0^N), & |\lambda| \text{ is even,} \\ 0, & |\lambda| \text{ is odd,} \end{cases}$$
(5.44)

where

$$H_\lambda(0^N) = \frac{(-1)^{\frac{|\lambda|}{2}}}{2^{\frac{|\lambda|}{2}} \frac{|\lambda|!}{2}!} C_\lambda(N) \chi_{(2^{|\lambda|/2})}^\lambda.$$
(5.45)

This can be easily seen from (5.9), (5.19), and (5.26) and the fact that  $S_\lambda = 1$  for  $\lambda = ()$  and  $S_\lambda(0^N) = 0$  for any non-empty partition  $\lambda$ . Using (5.4),  $\mathbb{E}_N^{(H)}[S_\lambda]$  is a polynomial in  $N$  with integer roots given by the content of  $\lambda$  whenever  $\chi_{(2^{|\lambda|/2})}^\lambda$  is non-zero.

A few examples of moments of Schur polynomials corresponding to partitions of 4 are as follows:

$$\begin{aligned} \mathbb{E}_N^{(H)}[S_4] &= \frac{1}{8} N(N+1)(N+2)(N+3), & \mathbb{E}_N^{(H)}[S_{3,1}] &= -\frac{1}{8} (N-1)N(N+1)(N+2), \\ \mathbb{E}_N^{(H)}[S_{2,2}] &= \frac{1}{4} (N-1)N^2(N+1), & \mathbb{E}_N^{(H)}[S_{2,1,1}] &= -\frac{1}{8} (N-2)(N-1)N(N+1), \\ \mathbb{E}_N^{(H)}[S_{1^4}] &= \frac{1}{8} (N-3)(N-2)(N-1)N. \end{aligned}$$
(5.46)

### 2. Laguerre case

The univariate moments are

$$\frac{1}{\Gamma(\gamma+1)} \int_0^\infty x^{n+\gamma} e^{-x} dx = \frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)} = n! L_n^{(\gamma)}(0).$$
(5.47)

The moments of the Schur polynomials with respect to the Laguerre weight can be computed using (5.30),

$$\begin{aligned} \mathbb{E}_N^{(L)}[S_\lambda] &= \frac{C_\lambda(N)}{|\lambda|!} \frac{G_\lambda(N, \gamma)}{G_0(N, \gamma)} \chi_{(1^{|\lambda|})}^\lambda \\ &= (-1)^{\frac{N(N-1)}{2}} G_\lambda(N, 0) \mathcal{L}_\lambda^{(\gamma)}(0^N). \end{aligned}$$
(5.48)

Like in the Hermite case,  $\mathbb{E}_N^{(L)}(S_\lambda)$  are polynomials in  $N$  with roots  $i-j$  and  $i-j-\gamma$ , where  $(i, j) \in \lambda$  as discussed in Sec. V A.

### 3. Jacobi case

We have

$$\begin{aligned} \int_0^1 x^{n+\gamma_1} (1-x)^{\gamma_2} dx &= n! \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}{\Gamma(n+\gamma_1+\gamma_2+2)} J_n^{(\gamma_1, \gamma_2)}(0) \\ &= \frac{\Gamma(n+\gamma_1+1)\Gamma(\gamma_2+1)}{\Gamma(n+\gamma_1+\gamma_2+2)}. \end{aligned} \quad (5.49)$$

Similarly,

$$\begin{aligned} \mathbb{E}_N^{(J)}[S_\lambda] &= \frac{G_\lambda(N, \gamma_1)}{G_0(N, \gamma_1)} C_\lambda(N) D_{\lambda_0}^{(J)} \\ &= (-1)^{\frac{N(N-1)}{2}} \frac{D_{\lambda_0}^{(J)}}{\tilde{D}_{\lambda_0}^{(J)}} G_\lambda(N, \gamma_1 + \gamma_2) G_\lambda(N, 0) \mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(0^N), \end{aligned} \quad (5.50)$$

where  $D_{\lambda_0}^{(J)}$  and  $\tilde{D}_{\lambda_0}^{(J)}$  are given in (2.11) and (5.42), respectively, and

$$\mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(0^N) = (-1)^{\frac{N(N-1)}{2}} \frac{G_\lambda(N, \gamma_1)}{G_\lambda(N, \gamma_1 + \gamma_2) G_0(N, \gamma_1) G_0(N, 0)} \tilde{D}_{\lambda_0}^{(J)}. \quad (5.51)$$

### C. Moments of characteristic polynomials

The Cauchy identity can be written as

$$\prod_{i=1}^q \prod_{j=1}^N \frac{1}{(T_i - x_j)} = \frac{1}{\prod_{j=1}^q T_j^N} \sum_{\lambda} \sum_{\mu \subseteq \lambda} \psi_{\lambda\mu} S_\lambda(T_1^{-1}, \dots, T_q^{-1}) \Phi_\mu(x_1, \dots, x_N), \quad (5.52)$$

where  $\Phi_\mu$  is one of the generalized polynomials  $\mathcal{H}_\mu$ ,  $\mathcal{L}_\mu^{(\gamma)}$ , or  $\mathcal{J}_\mu^{(\gamma_1, \gamma_2)}$ . By using orthogonality of multivariate polynomials (5.4), (5.29), and (5.37), we have the following proposition.

**Proposition 5.4.** *Let  $t_1, \dots, t_p$  and  $T_1, \dots, T_q$  be two sets of variables. Then,*

$$\begin{aligned} \prod_{j=1}^p \prod_{k=1}^q \mathbb{E}_N^{(H)} \left[ \frac{\det(t_j - M)}{\det(T_k - M)} \right] &= \prod_{j=1}^q \frac{1}{T_j^N} \sum_{\substack{\lambda \subseteq (N^p) \\ \text{s.t. } \tilde{\lambda} = \nu}} \sum_{\mu} \sum_{\nu \subseteq \mu} \frac{(-1)^{|\nu|}}{2^{\frac{|\mu| - |\nu|}{2}}} C_\mu(N) D_{\mu\nu}^{(H)} \mathcal{H}_\lambda(\mathbf{t}) S_\mu(\mathbf{T}^{-1}), \\ \prod_{j=1}^p \prod_{k=1}^q \mathbb{E}_N^{(L)} \left[ \frac{\det(t_j - M)}{\det(T_k - M)} \right] &= \prod_{j=N}^{p+N-1} (-1)^j j! \prod_{k=1}^q \frac{1}{T_k^N} \\ &\quad \times \sum_{\substack{\lambda \subseteq (N^p) \\ \text{s.t. } \tilde{\lambda} = \nu}} \sum_{\mu} \sum_{\nu \subseteq \mu} \frac{G_\mu(N, \gamma)}{G_0(N, \gamma)} \frac{C_\mu(N)}{C_\nu(N)} D_{\mu\nu}^{(L)} \mathcal{L}_\lambda^{(\gamma)}(\mathbf{t}) S_\mu(\mathbf{T}^{-1}), \\ \prod_{j=1}^p \prod_{k=1}^q \mathbb{E}_N^{(J)} \left[ \frac{\det(t_j - M)}{\det(T_k - M)} \right] &= \prod_{j=N}^{p+N-1} (-1)^j j! \frac{\Gamma(j + \gamma_1 + \gamma_2 + 1)}{\Gamma(2j + \gamma_1 + \gamma_2 + 1)} \prod_{k=0}^{N-1} \Gamma(2k + \gamma_1 + \gamma_2 + 2) \prod_{l=1}^q \frac{1}{T_l^N} \\ &\quad \times \sum_{\substack{\lambda \subseteq (N^p) \\ \text{s.t. } \tilde{\lambda} = \nu}} \sum_{\mu} \sum_{\nu \subseteq \mu} \frac{G_\mu(N, \gamma_1)}{G_0(N, \gamma_1)} \frac{G_\nu(N, \gamma_2)}{G_0(N, \gamma_2)} \frac{C_\mu(N)}{C_\nu(N)} D_{\mu\nu}^{(J)} \mathcal{J}_\lambda^{(\gamma_1, \gamma_2)}(\mathbf{t}) S_\mu(\mathbf{T}^{-1}). \end{aligned} \quad (5.53)$$

Note that the RHS is a formal power series in the variables  $\mathbf{T}$ .

**Corollary 5.3.** Let  $\lambda = (N^p)$ . If  $t_i = t$  in Theorem 2.1, then

$$\begin{aligned}\mathbb{E}_N^{(H)}[(\det(t - M))^p] &= C_\lambda(p) \sum_{v \subseteq \lambda} \left(\frac{-1}{2}\right)^{\frac{|v|-|v|}{2}} \frac{\dim V_v}{|v|!} D_{\lambda v}^{(H)} t^{|v|}, \\ \mathbb{E}_N^{(L)}[(\det(t - M))^p] &= (-1)^{p(p+N-1)} G_\lambda(p, \gamma) \frac{G_\lambda(p, 0)}{G_0(p, 0)} \sum_{v \subseteq \lambda} \frac{(-1)^{|v|}}{|v|!} \dim V_v D_{\lambda v}^{(L)} t^{|v|}, \\ \mathbb{E}_N^{(J)}[(\det(t - M))^p] &= \left( \prod_{j=N}^{p+N-1} \frac{1}{\Gamma(2j + \gamma_1 + \gamma_2 + 1)} \right) (-1)^{p(p+N-1)} \frac{G_\lambda(p, \gamma_1) G_\lambda(p, 0)}{G_0(p, 0)} \\ &\quad \times \sum_{v \subseteq \lambda} \frac{(-1)^{|v|}}{|v|!} \dim V_v \tilde{D}_{\lambda v}^{(J)} t^{|v|},\end{aligned}\tag{5.54}$$

where  $\dim V_v$  is given in (5.7).

*Proof.* Let us consider the GUE case. We have

$$\mathbb{E}_N^{(H)}[(\det(t - M))^p] = \mathcal{H}_{(N^p)}(t^p).\tag{5.55}$$

Using (5.26) and calculating  $C_\lambda$  in (5.4) for  $\lambda = (N^p)$ ,

$$C_{(N^p)}(p) = \prod_{j=1}^p \frac{(N + p - j)!}{(p - j)!}\tag{5.56}$$

proves the statement. Similarly, the Laguerre and Jacobi cases can be computed in an identical way. ■

## D. Joint moments of traces

Recently, the study of moments and joint moments of Hermitian ensembles has attracted considerable interest.<sup>17,18,20,32,33</sup> Here, we give new and self-contained formulas for the joint moments of unitary ensembles in terms of characters of the symmetric group. We focus on the GUE, but exactly the same method applies to the LUE and JUE.

Using (3.10) and (5.9), power sum symmetric polynomials can be written in terms of multivariate Hermite polynomials

$$P_\mu = \sum_{\lambda} \sum_{v \subseteq \lambda} \chi_\mu^\lambda \psi_{\lambda v}^{(H)} \mathcal{H}_v.\tag{5.57}$$

*Proof of Theorem 2.2.* When  $|\mu|$  is odd,  $P_\mu$  is a sum of the product of monomials in  $x_i$  with the degree of at least one  $x_i$  being odd. Since the generalized weight  $\Delta_N^2(\mathbf{x}) \prod_{i=1}^N e^{-\frac{x_i^2}{2}}$  is an even function and  $P_\mu(\mathbf{x})$  is odd,  $\mathbb{E}_N^{(H)}[P_\mu]$  vanishes.

When  $|\mu|$  is even, writing  $P_\mu$  in terms of multivariate Hermite polynomials (5.57) and using orthogonality of the  $\mathcal{H}_v$  along with (5.19) prove the first line of (2.8). ■

*Remark.* When  $|\mu|$  is even, the orthogonality of characters indicates that  $\mathbb{E}_N^{(H)}[(\text{Tr } M^2)^{\frac{|\mu|}{2}}]$  is a polynomial in  $N$  of degree  $|\mu|$ . The polynomial degree of all other joint moments corresponding to the partitions of  $|\mu|$  is strictly less than  $|\mu|$ .

**Corollary 5.4.** Correlators of traces in the L.H.S. of (2.8) are either even or odd polynomials in  $N$ . More precisely, we have

$\mathbb{E}_N^{(H)}[P_\mu]$	$l(\mu)$	$ \mu /2$
Even polynomial	even	even
	odd	odd
Odd polynomial	even	odd
	odd	even

*Proof.* Let  $|\mu|$  be even. Since  $\mathbb{E}_N^{(H)}[S_\mu]$  is a polynomial in  $N$  of degree  $|\mu|$  and the characters  $\chi_\lambda^\mu$  are integers,  $\mathbb{E}_N^{(H)}[P_\lambda]$  is also a polynomial in  $N$ . Now, for any partitions  $\lambda$  and  $\mu$ ,

$$\begin{aligned}\chi_\mu^{\lambda'} &= (-1)^{|\mu|-l(\mu)} \chi_\mu^\lambda, \\ C_{\mu'}(N) &= C_\mu(-N).\end{aligned}\quad (5.58)$$

Thus,

$$\begin{aligned}\mathbb{E}_N^{(H)}[P_\mu] &= \frac{1}{2} \sum_\lambda \left( \chi_\mu^\lambda \mathbb{E}_N^{(H)}[S_\lambda] + \chi_\mu^{\lambda'} \mathbb{E}_N^{(H)}[S_{\lambda'}] \right) \\ &= \frac{1}{2^{\frac{|\mu|+2}{2}} \frac{|\mu|}{2}!} \sum_\lambda \chi_\mu^\lambda \chi_\mu^\lambda \left( C_\lambda(N) + (-1)^{\frac{|\mu|}{2}-l(\mu)} C_\lambda(-N) \right).\end{aligned}\quad (5.59)$$

The corollary is proved by noting that the symmetric and anti-symmetric combination of  $C_\lambda(N)$  and  $C_\lambda(-N)$  is an even and odd polynomial in  $N$ , respectively. ■

Since  $\mathbb{E}_N^{(H)}[P_\mu]$  are polynomials in  $N$ , the domain of  $N$  can be analytically continued from integers to the whole complex plane. In Ref. 18, it is shown that  $\mathbb{E}_N^{(H)}[\text{Tr } M^{2j}]$ ,  $j \in \mathbb{N}$ , are Meixner–Pollaczek polynomials, which are a family of orthogonal polynomials,

$$\begin{aligned}\mathbb{E}_N^{(H)}[\text{Tr } M^{2j}] &= N(2j-1)!! i^{-j} \frac{1}{j+1} P_j^{(1)}\left(iN, \frac{\pi}{2}\right) \\ &= N(2j-1)!! {}_2F_1\left(\begin{matrix} -j, 1-N \\ 2 \end{matrix}; 2\right),\end{aligned}\quad (5.60)$$

where  $P_k^{(1)}(iN, \pi/2)$  is a Meixner–Pollaczek polynomial and  ${}_2F_1(\dots)$  is a terminating hypergeometric series. These polynomials  $P_n^{(\lambda)}(x, \phi)$  satisfy

$$\int_{-\infty}^{\infty} P_m^{(\lambda)}(x, \phi) P_n^{(\lambda)}(x, \phi) |\Gamma(\lambda + ix)|^2 e^{(2\phi - \pi)x} dx = \frac{2\pi \Gamma(n + 2\lambda)}{(2 \sin \phi)^{2\lambda} n!} \delta_{nm}.\quad (5.61)$$

Clearly, the zeros of  $\mathbb{E}_N^{(H)}[\text{Tr } M^{2j}]$  lie on the line  $\text{Re}(N) = 0$ .

Correlators of traces are combinatorial objects as they are connected to enumeration of ribbon graphs.<sup>7,58,59</sup> This connection is briefly discussed in Appendix B. By counting ribbon graphs, it can be easily shown that

$$\begin{aligned}\mathbb{E}_N^{(H)}[\text{Tr } M^{2k-1} \text{Tr } M] &= (2k-1) \mathbb{E}_N^{(H)}[\text{Tr } M^{2k-2}] \\ &= N(2k-1)!! i^{-k+1} \frac{1}{k} P_{k-1}^{(1)}\left(iN, \frac{\pi}{2}\right).\end{aligned}\quad (5.62)$$

Thus,  $\mathbb{E}_N^{(H)}[P_\mu]$ ,  $\mu = (2k-1, 1)$ , is also a polynomial in  $N$ . A few examples of joint moments of traces corresponding to partitions of 6 are given below. Here,  $p_j = \text{Tr } M^j$ ,

$$\begin{aligned}\mathbb{E}_N^{(H)}[p_6] &= 5N^2(N^2 + 2), & \mathbb{E}_N^{(H)}[p_5 p_1] &= 5N(2N^2 + 1), \\ \mathbb{E}_N^{(H)}[p_4 p_2] &= N(2N^2 + 1)(N^2 + 4), & \mathbb{E}_N^{(H)}[p_4 p_1^2] &= N^2(2N^2 + 13), \\ \mathbb{E}_N^{(H)}[p_3^2] &= 3N(4N^2 + 1), & \mathbb{E}_N^{(H)}[p_3 p_2 p_1] &= 3N^2(N^2 + 4), \\ \mathbb{E}_N^{(H)}[p_3 p_1^3] &= 3N(3N^2 + 2), & \mathbb{E}_N^{(H)}[p_2^3] &= N^2(N^2 + 2)(N^2 + 4), \\ \mathbb{E}_N^{(H)}[p_2^2 p_1^2] &= N(N^2 + 2)(N^2 + 4), & \mathbb{E}_N^{(H)}[p_2 p_1^4] &= 3N^2(N^2 + 4), \\ \mathbb{E}_N^{(H)}[p_1^6] &= 15N^3.\end{aligned}\quad (5.63)$$

## VI. EIGENVALUE FLUCTUATIONS

### A. Moments

Here, we focus on the GUE, but the Laguerre and Jacobi ensembles can be studied in a similar way. Consider the rescaled GUE matrices  $M_R = M/\sqrt{4N}$  of size  $N$  with *j.p.d.f.*,

$$\frac{(4N)^{\frac{N^2}{2}}}{(2\pi)^{\frac{N}{2}} \prod_{j=1}^N j!} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{j=1}^N e^{-2Nx_j^2}. \quad (6.1)$$

The limiting eigenvalue density is

$$\rho_{sc}(x) = \frac{2}{\pi} \sqrt{1 - x^2}. \quad (6.2)$$

*Proposition 6.1.* We have

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R)^{2n}] = \frac{2n!}{2^{3n} n!}. \quad (6.3)$$

*Proof.* When  $\mu = (1^{2n})$  in (2.8), using (5.6) and the fact that  $\chi_{(1^{2n})}^\lambda = \dim V_\lambda$ ,

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R)^{2n}] = \frac{2n!}{2^{3n} n!} \frac{1}{N^n} \sum_{\lambda \vdash 2n} \chi_{(2^n)}^\lambda S_\lambda(1^N). \quad (6.4)$$

Using (3.10) and  $P_\nu(1^N) = N^{l(\nu)}$ ,

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R)^{2n}] = \frac{2n!}{2^{3n} n!} \frac{1}{N^n} P_{(2^n)}(1^N) = \frac{2n!}{2^{3n} n!}. \quad (6.5)$$

■

The R.H.S. is the  $2n$ th moment of  $r_1/2$  where  $r_1 \sim \mathcal{N}(0, 1)$ . This exact equality of moments with the moments of Gaussian normals is special to  $\mathbb{E}_N^{(H)}[(\text{Tr } M_R)^{2n}]$ . In general, one can consider moments of the form  $\mathbb{E}_N^{(H)}[(\text{Tr } g(M))^n]$  for a well-defined function  $g$ .

Johansson<sup>37</sup> showed that when  $g$  is the Chebyshev polynomial of the first kind of degree  $k$ , the random variable

$$X_k = \text{Tr } T_k(M_R) - \mathbb{E}_N^{(H)}[\text{Tr } T_k(M_R)], \quad k = 0, 1, \dots, \quad (6.6)$$

converges in distribution to the Gaussian variable  $\mathcal{N}(0, k/4)$ . In this section, we prove Theorem 2.3, which implies that

$$\mathbb{E}_N^{(H)}[X_k^n] = \left(\frac{\sqrt{k}}{2}\right)^n \frac{n!}{2^{\frac{n}{2}} \left(\frac{n}{2}\right)!} \eta_n + d(n, k) \frac{1}{N^{1+m_{k,n}}} + O(N^{-2}), \quad (6.7)$$

where  $\eta_n = 1$  if  $n$  is even and 0 otherwise and where  $m_{k,n}$  is either 0 or 1, with asymptotic estimates for  $d(n, k)$ . The results for  $k = 1$  are already discussed in Proposition 6.1. We first consider  $X_2$  and discuss results for general values of  $k$  in Sec. VI A 2.

### B. Second degree

We have that

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R^2)^n] = \frac{1}{(4N)^n} \prod_{j=0}^{n-1} (N^2 + 2j). \quad (6.8)$$

For a fixed  $n$ , this can be obtained by substituting in the character values of  $S_{2n}$  in (2.8). Alternatively, a proof by counting topologically invariant ribbon graphs is sketched in Appendix B. Clearly,

$$\begin{aligned}
\mathbb{E}_N^{(H)}[X_2^n] &= \mathbb{E}_N^{(H)}\left[\left(2 \operatorname{Tr} M_R^2 - \frac{N}{2}\right)^n\right] \\
&= \sum_{j=0}^n \binom{n}{j} \left(-\frac{N}{2}\right)^{n-j} \mathbb{E}_N^{(H)}[(2 \operatorname{Tr} M_R^2)^j] \\
&= \frac{N^n}{2^{n+1}} \sum_{j=0}^n (-1)^{n-j} 2^j N^{2-2j} \binom{n}{j} \frac{\Gamma\left(\frac{N^2}{2} + j\right)}{\Gamma\left(\frac{N^2}{2} + 1\right)}.
\end{aligned} \tag{6.9}$$

The asymptotic expansion for the ratios of Gamma functions is<sup>27</sup>

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{l=0}^{\infty} \frac{1}{z^l} \binom{a-b}{l} B_l^{(a-b+1)}(a), \quad a, b \in \mathbb{C}, \quad z \rightarrow \infty, \tag{6.10}$$

where  $B_j^{(l)}$  are generalized Bernoulli polynomials. Hence,

$$\mathbb{E}_N^{(H)}[X_2^n] = \frac{N^n}{2^n} \sum_{j=1}^n \sum_{l=0}^{j-1} (-1)^{n-j+l} \frac{2^l}{N^{2l}} \binom{n}{j} \binom{j-1}{l} B_l^{(j)}(0). \tag{6.11}$$

In arriving at (6.11), we used

$$B_l^{(j)}(j) = (-1)^l B_l^{(j)}(0). \tag{6.12}$$

Here,  $B_l^{(j)}(0)$  are generalized Bernoulli numbers, and the first few numbers are given as follows:

$$\begin{aligned}
B_0^{(j)}(0) &= 1, \\
B_1^{(j)}(0) &= -\frac{j}{2}, \\
B_2^{(j)}(0) &= \frac{j^2}{4} - \frac{j}{12}, \\
B_3^{(j)}(0) &= -\frac{j^3}{8} + \frac{j^2}{8}.
\end{aligned} \tag{6.13}$$

By inserting (6.13) into (6.11),

$$\begin{aligned}
\text{Coef. of } N^n &: \frac{1}{2^n} \sum_{j=1}^n (-1)^{n-j} \binom{n}{j} = 0, \\
\text{Coef. of } N^{n-2} &: \frac{1}{2^n} \sum_{j=2}^n (-1)^{n-j} \binom{n}{j} j(j-1) \\
&= \frac{n}{2^n} \sum_{j=2}^n (-1)^{n-j} (j-1) \binom{n-1}{j-1} = 0.
\end{aligned} \tag{6.14}$$

Calculating the coefficient of  $N^{n-2l}$  for arbitrary  $n$  and  $l$  is not straightforward because there are no simple expressions for generalized Bernoulli numbers. Although these numbers can be written in terms of Stirling's numbers of first kind, the coefficients can be explicitly computed only for small values of  $l$ . It can be shown for a given  $n$  that

$$\begin{aligned}
\text{Coef. of } N^{n-2k} &= 0 \quad \text{for } 0 \leq k < \lfloor n/2 \rfloor, \\
\text{Coef. of } N^0 &= \frac{n!}{2^n \left(\frac{n}{2}\right)!} \eta_n,
\end{aligned} \tag{6.15}$$

where  $\eta_n = 1$  if  $n$  is even and 0 otherwise. Our goal is not to compute these coefficients more generally, but rather to give an estimate for the sub-leading term in (6.7). Since the Chebyshev polynomials of even and odd degree do not mix, the moments of  $X_k$  show a similar behavior as in Corollary 5.4,

$$\mathbb{E}_N^{(H)}[X_2^n] = \begin{cases} d_2(n, 2) \frac{1}{N} + O(N^{-3}) & \text{if } n \text{ is odd,} \\ \frac{n!}{2^n (\frac{n}{2})!} + d_3(n, 2) \frac{1}{N^2} + O(N^{-4}) & \text{if } n \text{ is even.} \end{cases} \quad (6.16)$$

Coefficients  $d_2(n, 2)$  and  $d_3(n, 2)$  can be estimated using (6.11). In Sec. VI C, we give an estimate of these coefficients for arbitrary values of  $n$  and  $k$ .

### C. General degree

The explicit expression for the joint moments of eigenvalues in Theorem 2.2 allows us to obtain Theorem 2.3. Consequently,  $X_k$  converges to a normal random variable. For a fixed  $k$  and  $n$ ,

$$X_k \rightarrow \frac{\sqrt{k}}{2} \mathcal{N}(0, 1) \quad \text{as } N \rightarrow \infty. \quad (6.17)$$

In reality, the correct bounds in Theorem 2.3 are much more smaller than given. This is due to sequential cancellations in the sum

$$\sum_{\lambda} \chi_{\mu}^{\lambda} \chi_{2^{|\mu|/2}}^{\lambda} C_{\lambda}(N) \quad (6.18)$$

and in the Chebyshev expansion

$$\text{Tr } T_k(M_R) = \frac{k^{\lfloor \frac{k}{2} \rfloor}}{2} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \frac{(k-j-1)!}{j!(k-2j)!} 2^{k-2j} M_R^{k-2j}. \quad (6.19)$$

The bounds in Theorem 2.3 are better for smaller moments.

To prove Theorem 2.3, we first need to estimate the coefficient of the  $1/N$  term in the Laurent series of  $\mathbb{E}_N^{(H)}[P_{\mu}]$  of rescaled matrices, which leads to estimating the characters of the symmetric group.

All the characters of the symmetric group are integers and satisfy

$$\frac{|\chi_{\mu}^{\lambda}|}{\chi_{(1^{|\mu|})}^{\lambda}} < 1. \quad (6.20)$$

It turns out that under suitable assumptions, the ratios  $|\chi_{\mu}^{\lambda}|/\chi_{(1^{|\mu|})}^{\lambda}$  are very small, sometimes exponentially and super-exponentially small.<sup>23,52</sup> Particularly useful bounds are

$$|\chi_{\mu}^{\lambda}| \leq (\chi_{(1^{|\mu|})}^{\lambda})^{a_{\mu}}, \quad (6.21)$$

where  $a_{\mu}$  depends on  $\mu$ .

The frequency representation of a partition  $\mu = (1^{b_1}, 2^{b_2}, \dots, k^{b_k})$  also represents a permutation cycle of an element in  $\mathcal{S}_{|\mu|}$ . The number  $b_j$  gives the number of  $j$ -cycles in  $\mu$ ,  $1 \leq j \leq k$ . For example, if  $b_1 = 0$ , then there are no one-cycles. In other words, there are no fixed points when  $b_1 = 0$ .

The only obstruction to the small character values of  $|\chi_{\mu}^{\lambda}|$  is when  $\mu$  has many short cycles.<sup>45</sup> With this information, the following holds:

*Proposition 6.2.*

(a) Given any  $\lambda \in \text{Irr}(\mathcal{S}_n)$ , let  $\mu = (m^{n/m})$ ; then,<sup>23</sup>

$$|\chi_{\mu}^{\lambda}| \leq c n^{\frac{1}{2}(1-\frac{1}{m})} \left( \chi_{(1^{|\mu|})}^{\lambda} \right)^{\frac{1}{m}}, \quad (6.22)$$

where  $c$  is an absolute constant.

(b) If  $\mu \in \mathcal{S}_n$  is fixed-point-free or has  $n^{o(1)}$  fixed points, then for any  $\lambda \in \text{Irr}(\mathcal{S}_n)$ ,<sup>45</sup>

$$|\chi_{\mu}^{\lambda}| \leq \left( \chi_{(1^{|\mu|})}^{\lambda} \right)^{\frac{1}{2}+o(1)}. \quad (6.23)$$

(c) Fix  $a \leq 1$ , and let  $\mu \in \mathcal{S}_n$  with at most  $n^a$  cycles. Then, for any  $\lambda \in \text{Irr}(\mathcal{S}_n)$ ,<sup>45</sup>

$$|\chi_\mu^\lambda| \leq \left( \chi_{(1^{|\mu|})}^\lambda \right)^{a+o(1)}. \quad (6.24)$$

**Proposition 6.3.** For a given  $\mu$ ,  $\mathbb{E}_N^{(H)}[P_\mu]$  is a Laurent polynomial in  $N$  with

$$\text{Coefficient of } 1/N^q \text{ in } \mathbb{E}_N^{(H)}[P_\mu] \lesssim 2^{-\frac{|\mu|}{2}-q-\frac{3}{2}} |\mu|^{\frac{3|\mu|}{4}-\frac{11}{8}+q} e^{-\frac{|\mu|}{4}+\pi\sqrt{\frac{2}{3}|\mu|}}, \quad q \in \mathbb{N}, \quad (6.25)$$

as  $|\mu| \rightarrow \infty$ .

*Proof.* For rescaled matrices, the expected value of  $P_\mu$  is

$$\mathbb{E}_N^{(H)} \left[ \prod_{j=1}^l \text{Tr } M_R^{\mu_j} \right] = \begin{cases} \frac{1}{(8N)^{\frac{|\mu|}{2}} \frac{|\mu|}{2}! \lambda_{\vdash|\mu|}} \sum \chi_{(2^{|\mu|/2})}^\lambda \chi_\mu^\lambda C_\lambda(N), & |\mu| \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \quad (6.26)$$

Using (5.6), we obtain

$$\frac{\Gamma(N+1)}{\Gamma(N-|\lambda|+1)} \leq C_\lambda(N) \leq \frac{\Gamma(N+|\lambda|)}{\Gamma(N)}. \quad (6.27)$$

Using the asymptotics of Gamma functions, as  $N \rightarrow \infty$ ,

$$\frac{\Gamma(N+|\lambda|)}{\Gamma(N)} \sim N^{|\lambda|} \sum_{l=0}^{\infty} \frac{1}{N^l} \binom{|\lambda|}{l} B_l^{(|\lambda|+1)}(|\lambda|), \quad (6.28)$$

where  $B_l^{(j)}(x)$  are generalized Bernoulli polynomials of degree  $l$ . Thus, the coefficient of  $1/N^q$  in (2.8) is bounded by

$$\text{Coefficient of } 1/N^q \text{ in } \mathbb{E}_N^{(H)}[P_\mu] \leq \frac{1}{8^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \binom{|\mu|}{\frac{|\mu|}{2}+q} B_{\frac{|\mu|}{2}+q}^{(|\mu|+1)}(|\mu|) \sum_{\lambda} |\chi_\mu^\lambda| |\chi_{2^{|\mu|/2}}^\lambda|. \quad (6.29)$$

Using (6.21) and (6.22), the R.H.S. of (6.29) is bounded from above by

$$\frac{c}{8^{\frac{|\mu|}{2}} \frac{|\mu|}{2}!} \binom{|\mu|}{\frac{|\mu|}{2}+q} |\mu|^{\frac{1}{4}} (\chi_{1^{|\mu|}}^\lambda)_{\max}^{a_\mu+\frac{1}{2}} \# \text{par}(|\mu|) B_{\frac{|\mu|}{2}+q}^{(|\mu|+1)}(|\mu|). \quad (6.30)$$

The maximum of the dimension of the irreducible representation<sup>48</sup>

$$(\chi_{1^{|\mu|}}^\lambda)_{\max} \leq (2\pi)^{\frac{1}{4}} |\mu|^{\frac{|\mu|}{2}+\frac{1}{4}} e^{-\frac{|\mu|}{2}} \quad (6.31)$$

and the number of partitions grow as<sup>34,60</sup>

$$\# \text{par}(|\mu|) \sim \frac{1}{4\sqrt{3}|\mu|} \exp\left(\pi\sqrt{\frac{2|\mu|}{3}}\right) \quad \text{as } |\mu| \rightarrow \infty. \quad (6.32)$$

Polynomials  $B_l^{(j)}(x)$  satisfy

$$B_l^{(j+1)}(x) = \left(1 - \frac{l}{j}\right) B_l^{(j)}(x) + l \left(\frac{x}{j} - 1\right) B_{l-1}^{(j)}(x). \quad (6.33)$$



Hence,

$$B_{\frac{|\mu|}{2}+q}^{(|\mu|+1)}(|\mu|) = \left(\frac{1}{2} - \frac{q}{|\mu|}\right) B_{\frac{|\mu|}{2}+q}^{(|\mu|)}(|\mu|) \lesssim \frac{1}{2^{\frac{|\mu|}{2}+q+1}} |\mu|^{\frac{|\mu|}{2}+q} \quad \text{as } |\mu| \rightarrow \infty. \quad (6.34)$$

Inserting  $a_\mu = 1$ , (6.31), and (6.34) in (6.30), we have that

$$\text{coefficient of } 1/N^q \text{ in } \mathbb{E}_N^{(H)}[P_\mu] \lesssim \frac{1}{2^{2|\mu|+q+3}} \frac{1}{\frac{|\mu|}{2}} \left(\frac{|\mu|}{2} + q\right) |\mu|^{\frac{5|\mu|}{4} - \frac{3}{8} + q} e^{-3\frac{|\mu|}{4} + \pi\sqrt{\frac{2}{3}}|\mu|} \quad (6.35)$$

as  $|\mu| \rightarrow \infty$ . Now, using Stirling's approximation proves (6.25). ■

We are now ready to prove Theorem 2.3.

*Proof of Theorem 2.3.* Using (6.26), it can be seen that the joint moments of traces of rescaled matrices are Laurent polynomial in  $N$  with rational coefficients. Thus, the central moments of traces of Chebyshev polynomials are also Laurent polynomials. Since  $X_k(M_R)$  converges in distribution to a normal random variable as  $N \rightarrow \infty$ ,  $\mathbb{E}_N^{(H)}[X_k^n]$  is a polynomial in  $1/N$  with a constant term given in (2.17) and (2.19) depending on whether  $k$  is odd and even, respectively.

To estimate the sub-leading term in  $\mathbb{E}_N^{(H)}[X_k^n]$ , we consider  $k$  even and odd cases separately.

(1) For  $k$  odd,  $\mathbb{E}_N^{(H)}[\text{Tr } T_k(M_R)] = 0$ . Using the expansion of Chebyshev polynomials of the first kind,

$$\begin{aligned} \mathbb{E}_N^{(H)}[X_k^n] &= \mathbb{E}_N^{(H)}[(\text{Tr } T_k(M_R))^n] \\ &= \mathbb{E}_N^{(H)}\left[\left(k \sum_{j=0}^{\frac{k-1}{2}} (-1)^{\frac{k-1}{2}-j} \frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!} 2^{2j} \text{Tr } M^{2j+1}\right)^n\right] \\ &= k^n \sum_{n_0+\dots+n_{\frac{k-1}{2}}=n} \binom{n}{n_0, \dots, n_{\frac{k-1}{2}}} \prod_{j=0}^{\frac{k-1}{2}} (-1)^{\frac{k-1}{2}n_j-jn_j} \left(\frac{(\frac{k-1}{2}+j)!}{(\frac{k-1}{2}-j)!(2j+1)!}\right)^{n_j} 2^{2jn_j} \\ &\quad \times \mathbb{E}_N^{(H)}[P_\mu], \end{aligned} \quad (6.36)$$

where

$$\mathbb{E}_N^{(H)}[P_\mu] = \mathbb{E}_N^{(H)}\left[\prod_{l=0}^{\frac{k-1}{2}} (\text{Tr } M_R^{2l+1})^{n_l}\right], \quad \mu = (1^{n_0}, 3^{n_1}, \dots, k^{\frac{n_{\frac{k-1}{2}}}{2}}). \quad (6.37)$$

The odd moments of  $\text{Tr } T_k(M_R)$  are identically zero because  $\mathbb{E}_N^{(H)}[P_\mu] = 0$  when  $|\mu|$  is odd; see (6.26). When  $n$  is even, the leading term is given by the  $n$ th moment of  $\sqrt{k}r_k/2$ ,  $r_k \sim \mathcal{N}(0, 1)$ , according to Szegő's theorem. For  $n$  even,  $l(\mu)$  is always even. Hence, the sub-leading term in (6.36) is  $O(N^{-2})$ . (See Corollary 5.4. Note that the matrix is now rescaled.)

The maximum possible degree of  $\mu$  is  $|\mu| = nk$  when  $n_{\frac{k-1}{2}} = n$ ,  $n_j = 0$  for  $j = 0, \dots, \frac{k-3}{2}$ , and the minimum degree is  $|\mu| = n$  when  $n_0 = n$ ,  $n_j = 0$  for  $j = 1, \dots, \frac{k-1}{2}$ . The coefficient of  $1/N^2$  in (6.36) is estimated using (6.25) by choosing an appropriate  $\mu$ . Note that the multinomial coefficient is maximum when all  $n_j$ 's are approximately equal. In this case,  $\mu = (1^{\frac{2n}{k+1}}, 3^{\frac{2n}{k+1}}, \dots, k^{\frac{2n}{k+1}})$  and  $|\mu| = n(k+1)/2$ . For a fixed  $k$  as  $n$  increases, the number of short cycles in  $\mu$  increases. Using (6.24),

$$|\chi_\mu^\lambda| \leq \chi_{1^{|\mu|}}^\lambda, \quad (6.38)$$

which implies  $a_\mu = 1$  in (6.25).

Let

$$d_1(n, k) = \left[\mathbb{E}_N^{(H)}[(\text{Tr } T_k(M_R))^n]\right]_{1/N^2} \quad (6.39)$$

denote the coefficient of  $1/N^2$  in  $\mathbb{E}_N^{(H)}[(\text{Tr } T_k(M_R))^n]$ . Putting  $q = 2$  in (6.25),

$$d_1(n, k) \sim k^n \frac{n!}{\left(\frac{2n}{k+1}\right)!^{\frac{k+1}{2}}} \left( \prod_{j=0}^{\frac{k-1}{2}} \frac{\left(\frac{k-1}{2} + j\right)!}{\left(\frac{k-1}{2} - j\right)!(2j+1)!} \right)^{\frac{2n}{k+1}} 2^{2|\mu|} [\mathbb{E}_N^{(H)}[P_\mu]]_{1/N^2} \quad \text{as } n \rightarrow \infty. \quad (6.40)$$

Now,

$$\prod_{j=0}^{\frac{k-1}{2}} \frac{\left(\frac{k-1}{2} + j\right)!}{\left(\frac{k-1}{2} - j\right)!(2j+1)!} = 2^{-\frac{5}{24} - \frac{1}{4}k(k+2)} e^{\frac{1}{8}\pi^{\frac{1}{4}}(k+2)} \frac{1}{A^{\frac{3}{2}}} \frac{G(k+1)}{G\left(\frac{k}{2} + 2\right)G\left(\frac{k+1}{2}\right)\left(G\left(\frac{k+3}{2}\right)\right)^2}, \quad (6.41)$$

where  $G(x)$  is the Barnes-G function and  $A = 1.2824 \dots$  is the Glaisher-Kinkelin constant.

Using asymptotics of Barnes-G functions and Stirling's approximation,

$$\begin{aligned} \left( \prod_{j=0}^{\frac{k-1}{2}} \frac{\left(\frac{k-1}{2} + j\right)!}{\left(\frac{k-1}{2} - j\right)!(2j+1)!} \right)^{\frac{2n}{k+1}} &\sim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{nk}{2} - n + \frac{n}{6k}} k^{-\frac{3n}{2} + \frac{n}{4k}} e^{\frac{9n}{4} + \frac{5n}{8k}}, \\ \frac{n!}{\left(\frac{2n}{k+1}\right)!^{\frac{k+1}{2}}} &\sim \frac{1}{\pi^{\frac{k-1}{4}}} \frac{1}{2^{n+\frac{k}{2}}} n^{-\frac{k-1}{4}} (k+1)^{n+\frac{k+1}{4}} \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (6.42)$$

By combining the previous equations, we arrive at

$$d_1(n, k) \lesssim A^{\frac{3n}{k}} \pi^{-\frac{1}{4}(2n+k)} 2^{\frac{7nk}{8} - \frac{13n}{8} + \frac{n}{6k} - \frac{k}{2}} n^{\frac{3n}{8}(k+1) - \frac{k}{4} + \frac{7}{8}} k^{\frac{3n}{8}(k+2) + \frac{n}{8} + \frac{n}{4k} + \frac{k}{4} + \frac{7}{8}} e^{-\frac{n}{8}(k+1) + \frac{9n}{4} + \frac{5n}{8k} + \pi\sqrt{\frac{n}{3}(k+1)}} \quad (6.43)$$

as  $n \rightarrow \infty$ . We are interested to find the order of the coefficient of  $1/N$  as  $n$  increases for a fixed  $k$ . To capture the right behavior, it is sufficient to approximate (6.43) to

$$d_1(n, k) \lesssim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{7nk}{8} - \frac{13n}{8} + \frac{n}{6k}} k^{\frac{3n}{8}(k+2) + \frac{n}{8} + \frac{n}{4k}} n^{\frac{3n}{8}(k+1) - \frac{k}{4} + \frac{7}{8}} e^{-\frac{n}{8}(k+1) + \frac{9n}{4} + \frac{5n}{8k} + \pi\sqrt{\frac{n}{3}(k+1)}}. \quad (6.44)$$

(2) When  $k$  is even, let

$$c_k = \frac{1}{N} \mathbb{E}_N^{(H)}[\text{Tr } T_k(M_R)]. \quad (6.45)$$

We have

$$\begin{aligned} \mathbb{E}_N^{(H)}[X_k^n] &= \mathbb{E}_N^{(H)}[(\text{Tr } T_k(M_R) - Nc_k)^n] \\ &= \mathbb{E}_N^{(H)}\left[\left(N\left((-1)^{\frac{k}{2}} - c_k\right) + k \sum_{j=1}^{\frac{k}{2}} (-1)^{\frac{k}{2}-j} \frac{\left(\frac{k}{2} + j - 1\right)!}{\left(\frac{k}{2} - j\right)!(2j)!} 2^{2j-1} \text{Tr } M_R^{2j}\right)^n\right] \\ &= \sum_{n_0 + \dots + n_{\frac{k}{2}} = n} \binom{n}{n_0, \dots, n_{\frac{k}{2}}} N^{n_0} \left((-1)^{\frac{k}{2}} - c_k\right)^{n_0} \\ &\quad \times \prod_{j=1}^{\frac{k}{2}} (-1)^{\frac{k}{2}n_j - jn_j} k^{n_j} \left(\frac{\left(\frac{k}{2} + j - 1\right)!}{\left(\frac{k}{2} - j\right)!(2j)!}\right)^{n_j} 2^{(2j-1)n_j} \mathbb{E}_N^{(H)}[P_\mu], \end{aligned} \quad (6.46)$$

where

$$\mathbb{E}_N^{(H)}[P_\mu] = \mathbb{E}_N \left[ \prod_{l=0}^{\frac{k}{2}} (\text{Tr } M_R^{2l})^{n_l} \right], \quad \mu = (2^{n_1}, 4^{n_2}, \dots, k^{n_{\frac{k}{2}}}). \quad (6.47)$$

According to Szegő's theorem, when  $n$  is even, the leading order term in the R.H.S. of (6.46) is given by  $\mathbb{E}[(\sqrt{kr_k}/2)^n]$ ,  $r_k \sim \mathcal{N}(0, 1)$ . The sub-leading term is  $d_3(n, k)N^{-2}$ . When  $n$  is odd, the leading term in the R.H.S. is given by  $d_2(n, k)N^{-1}$ . Next, we compute the coefficients  $d_2(n, k)$  and  $d_3(n, k)$ .

Coefficient  $d_2(n, k)$ :  $c_k$  decays as  $1/N^2$  for  $k > 2$ , so we neglect it in (6.46). Note that  $\mu$  in (6.47) does not have any one-cycles. Hence,  $\mu$  is fixed-point-free and (6.23) can also be used to estimate characters  $\chi_\mu^\lambda$  in Proposition 6.3. Here, we just use (6.25) for  $q = 2$  and follow the exact same calculation as  $k$  odd. In the limit  $n \rightarrow \infty$ , this leads to

$$d_2(n, k) \lesssim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{3nk}{8} - 3n + \frac{n}{6k}} k^{\frac{3nk}{8} + \frac{n}{2} + \frac{9n}{4k}} n^{\frac{3nk}{8} + \frac{2n}{k} - \frac{k}{2} - \frac{3}{8}} e^{-\frac{n}{8}(k-18) + \pi\sqrt{\frac{nk}{3} - \frac{19n}{8k}}}. \quad (6.48)$$

Similarly,  $d_3(n, k)$  can be approximated as

$$d_3(n, k) \lesssim A^{\frac{3n}{k}} \pi^{-\frac{n}{2}} 2^{\frac{3nk}{8} - 3n + \frac{n}{6k}} k^{\frac{3nk}{8} + \frac{n}{2} + \frac{9n}{4k}} n^{\frac{3nk}{8} + \frac{2n}{k} - \frac{k}{2} + \frac{5}{8}} e^{-\frac{n}{8}(k-18) + \pi\sqrt{\frac{nk}{3} - \frac{19n}{8k}}}. \quad (6.49)$$

■

Since  $X_k$  converges to independent Gaussian normals, the correlators of  $X_k$  also converge to random Gaussian variables as  $N \rightarrow \infty$ . For instance,

$$\mathbb{E}_N^{(H)}[X_i X_j] = \frac{\sqrt{ij}}{4} \mathbb{E}[r_i r_j] + O(N^{-1}). \quad (6.50)$$

Given below are the moments corresponding to partitions of 6,

$$\begin{aligned} \mathbb{E}_N^{(H)}[X_6] &= 0, & \mathbb{E}_N^{(H)}[X_5 X_1] &= \frac{5}{4N^2}, \\ \mathbb{E}_N^{(H)}[X_4 X_2] &= \frac{1}{N^2}, & \mathbb{E}_N^{(H)}[X_4 X_1^2] &= \frac{1}{2N}, \\ \mathbb{E}_N^{(H)}[X_3^2] &= \frac{3}{4} + \frac{3}{4N^2}, & \mathbb{E}_N^{(H)}[X_3 X_2 X_1] &= \frac{3}{4N}, \\ \mathbb{E}_N^{(H)}[X_3 X_1^3] &= \frac{3}{8N^2}, & \mathbb{E}_N^{(H)}[X_2^3] &= \frac{1}{N}, \\ \mathbb{E}_N^{(H)}[X_2^2 X_1^2] &= \frac{1}{8} + \frac{1}{2N^2}, & \mathbb{E}_N^{(H)}[X_2 X_1^4] &= \frac{3}{8N}, \\ \mathbb{E}_N^{(H)}[X_1^6] &= \frac{15}{64}. \end{aligned} \quad (6.51)$$

In this paper, we do not pursue correlations of  $X_k$  any further.

## D. Cumulants

In general, the moments and the cumulants are related by the recurrence relation

$$\kappa_n = m_n - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_j m_{n-j}. \quad (6.52)$$

Cumulants and moments can also be expressed in terms of each other through a more elegant formula. Let  $\lambda = (\lambda_1, \dots, \lambda_l) \equiv (1^{b_1}, 2^{b_2}, \dots, r^{b_r})$  be a partition of  $n$ . Define

$$\kappa_\lambda := \prod_{j=1}^l \kappa_j = \prod_{j=1}^r \kappa_j^{b_j}, \quad m_\lambda := \prod_{j=1}^l m_j = \prod_{j=1}^r m_j^{b_j}. \quad (6.53)$$

We have

$$\begin{aligned} m_n &= \sum_{\lambda} d_{\lambda} \kappa_{\lambda}, \\ \kappa_n &= \sum_{\lambda} (-1)^{l(\lambda)-1} (l(\lambda)-1)! d_{\lambda} m_{\lambda}, \end{aligned} \quad (6.54)$$

where

$$d_{\lambda} = \frac{n!}{(1!)^{b_1} b_1! \dots (r!)^{b_r} b_r!} \quad (6.55)$$

is the number of decompositions of a set of  $n$  elements into disjoint subsets containing  $\lambda_1, \dots, \lambda_l$  elements.

In this section, we give an estimate on the cumulants of random variables  $X_k$ , and to do so, we rely on the well-studied connection between GUE correlators and enumerating ribbon graphs, which has briefly been discussed in [Appendix B](#).

Consider the formal matrix integral over the space of  $N \times N$  rescaled GUE matrices,

$$Z_N(\mathbf{s}, \xi) = e^{s_0 N \xi} \int e^{-2N \text{Tr } M^2} e^{\xi \text{Tr } V(M)} dM. \quad (6.56)$$

Here, the formal series  $V(M)$  depending on the parameters  $\mathbf{s} = \{s_0, s_1, \dots, s_k\}$  has the form

$$V(M) = \sum_{j=1}^k s_j M^j. \quad (6.57)$$

The integral in (6.56) can be considered as a formal expansion in the set of parameters  $s_j$  and  $\xi$ . Now,

$$\begin{aligned} \frac{Z(\mathbf{s}, \xi)}{Z(0, \xi)} &= \sum_{n_0, n_1, \dots, n_k} \xi^{\sum n_j} \frac{(s_0 N)^{n_0}}{n_0!} \frac{s_1^{n_1}}{n_1!} \dots \frac{s_k^{n_k}}{n_k!} \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^k (\text{Tr } M_R^j)^{n_j} \right] \\ &= \sum_{n \geq 0} \xi^n \sum_{n_0 + \dots + n_k = n} \frac{(s_0 N)^{n_0}}{n_0!} \frac{s_1^{n_1}}{n_1!} \dots \frac{s_k^{n_k}}{n_k!} \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^k (\text{Tr } M_R^j)^{n_j} \right]. \end{aligned} \quad (6.58)$$

By choosing  $s_j$  to be the coefficients of Chebyshev polynomials in (6.58), we recover the moments of  $X_k$ . Thus, (6.58) is the moment generating function of  $X_k$ . For a given  $k$ , by fixing  $s_j$  to be the Chebyshev coefficients in  $T_k$ ,

$$\mathbb{E}_N^{(H)} \left[ e^{\xi X_k} \right] = \sum_{n \geq 0} \frac{\xi^n}{n!} \mathbb{E}_N^{(H)} [X_k^n] = \frac{Z(\mathbf{s}, \xi)}{Z(0, \xi)}. \quad (6.59)$$

By matching the terms in the L.H.S. and R.H.S. of (6.59) by powers in  $\xi$ , we recover the moments of  $X_k$ . The correlators of  $\text{Tr } M_R^j$  are connected to the problem of enumerating ribbon graphs. For a brief introduction, see [Appendix B](#), and for more details, see Refs. [22](#) and [36](#) and references within. The trace correlators count ribbon graphs that are connected and also multiplicatively count ribbon graphs that are disconnected. When we have a generating function that counts disconnected objects multiplicatively, taking the logarithm counts only the connected objects.<sup>35</sup> Hence, the cumulant generating function

$$\begin{aligned} \log \mathbb{E}_N^{(H)} \left[ e^{\xi X_k} \right] &= \log \frac{Z(\mathbf{s}, \xi)}{Z(0, \xi)} = \sum_{n \geq 1} \frac{\xi^n}{n!} \kappa_n \\ &= s_0 N \xi + \sum_{n \geq 1} \xi^n \sum_{n_1 + \dots + n_k = n} \frac{s_1^{n_1}}{n_1!} \dots \frac{s_k^{n_k}}{n_k!} \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^k (\text{Tr } M_R^j)^{n_j} \right]_c \end{aligned} \quad (6.60)$$

keeps only connected ribbon graphs indicated by subscript  $c$ . For  $\mu = (1^{n_1}, \dots, k^{n_k}) \equiv (\mu_1, \dots, \mu_l)$ , the connected correlators are given by

$$\mathbb{E}_N^{(H)} \left[ \prod_{j=1}^l \text{Tr } M_R^{\mu_j} \right]_c \equiv \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^k (\text{Tr } M_R^j)^{n_j} \right]_c = \sum_{0 \leq g \leq \frac{|\mu|}{4} - \frac{l}{2} + \frac{1}{2}} \frac{1}{2^{|\mu|}} a_g(\mu_1, \dots, \mu_l) N^{2-2g-l}, \quad |\mu| \text{ is even.} \quad (6.61)$$

Here,

$$\begin{aligned} a_g(\mu_1, \dots, \mu_l) &= \#\{\text{connected oriented labeled ribbon graphs} \\ &\quad \text{of genus } g \text{ with } l \text{ vertices of valencies } \mu_1, \dots, \mu_l\} \\ &= l! \sum_{\Gamma} \frac{1}{\#\text{Sym}(\Gamma)}, \end{aligned} \quad (6.62)$$

where  $\Gamma$  is a connected (unlabeled) ribbon graph of genus  $g$  with  $l$  vertices of valencies  $\mu_1, \dots, \mu_l$ ,  $\#\text{Sym}(\Gamma)$  is the order of the symmetry group of  $\Gamma$ , and the last summation is taken over all such  $\Gamma$ . For the explicit expressions of connected correlators of the GUE, see Ref. [20](#).

We are now ready to estimate the cumulants of  $X_k$ . We treat  $k$  even and odd cases separately.

(1) *k odd*: In this case, the parameters  $s_{2j} = 0$  for  $0 \leq j \leq \frac{k-1}{2}$  and

$$s_{2j+1} = (-1)^{\frac{k-1}{2}-j} k \frac{\left(\frac{k-1}{2} + j\right)!}{\left(\frac{k-1}{2} - j\right)!(2j+1)!} 2^{2j}, \quad 0 \leq j \leq \frac{k-1}{2}. \quad (6.63)$$

When  $k$  is odd, all the odd moments are zero. Hence, all the odd cumulants are also zero. By inserting (6.61) into (6.60), the even cumulants are given by

$$\kappa_{2n} = \frac{2n!}{N^{2n-2}} \sum_g \sum_{n_1+n_3+\dots+n_k=2n} \frac{1}{2^{\sum_j j n_j}} \frac{a_g}{N^{2g}} \frac{s_1^{n_1}}{n_1!} \frac{s_3^{n_3}}{n_3!} \dots \frac{s_k^{n_k}}{n_k!}. \quad (6.64)$$

(2) *k even*: In this case, the parameters  $s_{2j+1} = 0$  for  $0 \leq j \leq \frac{k}{2} - 1$  and

$$\begin{aligned} s_0 &= (-1)^{\frac{k}{2}} - c_k, \\ s_{2j} &= (-1)^{\frac{k}{2}-j} k \frac{\left(\frac{k}{2} + j - 1\right)!}{\left(\frac{k}{2} - j\right)!(2j)!} 2^{2j-1}, \quad 1 \leq j \leq \frac{k}{2}. \end{aligned} \quad (6.65)$$

The first cumulant is zero by definition of  $X_k$ . Therefore, the first term in (6.60) is canceled by  $n = 1$  contribution from the second term. Hence,

$$\log \mathbb{E}_N^{(H)} \left[ e^{\xi X_k} \right] = \sum_{n \geq 2} \xi^n \sum_{n_2+\dots+n_k=n} \frac{s_2^{n_2}}{n_2!} \frac{s_4^{n_4}}{n_4!} \dots \frac{s_k^{n_k}}{n_k!} \mathbb{E}_N^{(H)} \left[ \prod_{j=1}^{k/2} (\text{Tr } M_R^{2j})^{n_{2j}} \right]. \quad (6.66)$$

By inserting (6.61) into (6.60), the cumulants are given by

$$\kappa_n = \frac{n!}{N^{n-2}} \sum_g \sum_{n_2+\dots+n_k=n} \frac{1}{2^{\sum_j j n_j}} \frac{a_g}{N^{2g}} \frac{s_2^{n_2}}{n_2!} \frac{s_4^{n_4}}{n_4!} \dots \frac{s_k^{n_k}}{n_k!}, \quad n \geq 2. \quad (6.67)$$

Third and higher order cumulants of Gaussian random variable are identically zero. Since  $X_k$  converges to  $\mathcal{N}(0, k/4)$  as  $N \rightarrow \infty$ , cumulants of  $X_k$ ,  $\kappa_n \rightarrow 0$  as  $N \rightarrow \infty$  for all  $n \geq 3$ . For a fixed  $n$ , we see from (6.64) and (6.67) that  $\kappa_n$  decay as  $N^{-n+2}$ .

*Example.* The simplest non-trivial example is to calculate the cumulants of  $X_2$ . By mapping the problem to counting ribbon graphs (see Appendix B),

$$\mathbb{E}_N^{(H)} \left[ (\text{Tr } M_R^2)^n \right]_c = \frac{1}{(4N)^n} 2^{n-1} (n-1)! N^2 = (n-1)! \frac{1}{2^{n+1}} \frac{1}{N^{n-2}}. \quad (6.68)$$

For  $X_2$ ,  $s_0 = -\frac{1}{2}$ ,  $s_2 = 2$ , and  $s_j = 0$  for  $j \neq 0, 2$ . Hence,

$$\kappa_n = s_2^n \mathbb{E}_N^{(H)} \left[ \text{Tr } (M_R^2)^n \right]_c = \frac{1}{2} \frac{(n-1)!}{N^{n-2}}. \quad (6.69)$$

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## APPENDIX A: SOME PROPERTIES OF MULTIVARIATE ORTHOGONAL POLYNOMIALS

## 1. Gaussian case

For  $N = 1$ , the multivariate Hermite polynomials coincide with the classical polynomials,

$$H_n(x) = n! \sum_{j=0}^n \mathbb{1}_{n-j \equiv 0 \pmod{2}} \frac{1}{\left(\frac{n-j}{2}\right)!} \frac{(-1)^{\frac{n-j}{2}}}{2^{\frac{n-j}{2}} j!} x^j, \quad (\text{A1})$$

which have the generating function

$$\sum_j \frac{H_j(x)}{j!} t^j = e^{xt - \frac{t^2}{2}}. \quad (\text{A2})$$

By comparing (5.26) and (A1), we can see the analogies between classical Hermite polynomials and their multivariate counterparts: the sum over  $j$  is replaced by the sum over partitions; the role of monomials is played by Schur polynomials; the factorials are replaced with  $C_\lambda(N)$ . With this analogy, the generating function of  $\mathcal{H}_\lambda$  is<sup>3</sup>

$$\sum_\lambda \frac{\mathcal{H}_\lambda(\mathbf{x})}{C_\lambda(N)} S_\lambda(\mathbf{t}) = \left( \sum_\mu \frac{S_\mu(\mathbf{x}) S_\mu(\mathbf{t})}{C_\mu(N)} \right) \left( \sum_{n=0}^{\infty} \sum_{\nu \vdash 2n} \frac{(-1)^{\frac{|\nu|}{2}}}{2^{\frac{|\nu|}{2}}} S_\nu(\mathbf{t}) D_{\nu 0}^{(H)} \right). \quad (\text{A3})$$

The validity of the above formula can be easily verified for lower order partitions (say  $|\lambda| = 2, 4$ ) using Pieri's formula, but the second factor in (A3) can be simplified further.

*Proposition A.1.* Let  $t_1, t_2, \dots$  be a set of variables; then,

$$\prod_j \exp\left(-\frac{t_j^2}{2}\right) = \sum_{n=0}^{\infty} \sum_{\nu \vdash 2n} \frac{(-1)^{\frac{|\nu|}{2}}}{2^{\frac{|\nu|}{2}}} S_\nu(\mathbf{t}) D_{\nu 0}^{(H)}. \quad (\text{A4})$$

*Proof.* For a fixed  $n$ ,  $|\nu| = 2n$ . Comparing (5.10) with (5.19),

$$D_{\nu 0}^{(H)} = \frac{1}{n!} \chi_{(2^n)}^\nu. \quad (\text{A5})$$

Now, using (3.10) proves the proposition. ■

*Proposition A.2.* Let  $x_1, \dots, x_N$  and  $t_1, \dots, t_N$  be two sets of variables. The multivariate Hermite polynomials defined in (5.3) have the following generating function:<sup>3</sup>

$$\sum_\lambda \frac{\mathcal{H}_\lambda(\mathbf{x})}{C_\lambda(N)} S_\lambda(\mathbf{t}) = \left( \sum_\mu \frac{S_\mu(\mathbf{x}) S_\mu(\mathbf{t})}{C_\mu(N)} \right) \prod_j \exp\left(-\frac{t_j^2}{2}\right). \quad (\text{A6})$$

Several other analogs of properties of the classical Hermite polynomials, including an integral representation, summation, integration, and differentiation formulas, are given for  $\beta$ -ensembles in Ref. 3. Note that in Ref. 3,  $C_\mu^\alpha$  ( $\alpha \in \mathbb{R}$ ) is used to denote Schur polynomials with a specific normalization, whereas in this work,  $C_\mu(N)$  is a constant in  $N$  given in (5.6).

## 2. Laguerre case

When  $N = 1$ ,  $\mathcal{L}_\lambda^{(\gamma)}$  coincides with the classical Laguerre polynomials

$$L_n^{(\gamma)}(x) = \sum_{j=0}^n (-1)^j \frac{\Gamma(n+\gamma+1)}{\Gamma(j+\gamma+1)(n-j)!} \frac{x^j}{j!} \quad (\text{A7})$$

whose generating function is

$$\sum_{j=0}^{\infty} \frac{1}{\Gamma(j+\gamma+1)} L_j^{(\gamma)}(x) t^j = e^{t J_\gamma(2\sqrt{tx})} = e^t \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\gamma+1)} (tx)^m, \quad (\text{A8})$$

where  $J_\gamma$  is the Bessel function. By comparing (5.32) and (A7), the generating function for multivariate Laguerre polynomials<sup>3</sup> is

$$\sum_{\nu} \frac{1}{G_{\nu}(N, \gamma)} \mathcal{L}_{\nu}^{(\gamma)}(\mathbf{x}) S_{\nu}(\mathbf{t}) = (-1)^{\frac{N(N-1)}{2}} \left( \sum_{\lambda} S_{\lambda}(\mathbf{t}) D_{\lambda 0}^{(L)} \right) \left( \sum_{\mu} \frac{(-1)^{|\mu|}}{G_{\mu}(N, \gamma)} \frac{S_{\mu}(\mathbf{x}) S_{\mu}(\mathbf{t})}{G_{\mu}(N, 0)} \right), \quad (\text{A9})$$

or equivalently, using (5.33),

$$\sum_{\nu} \frac{1}{G_{\nu}(N, \gamma)} \mathcal{L}_{\nu}^{(\gamma)}(\mathbf{x}) S_{\nu}(\mathbf{t}) = (-1)^{\frac{N(N-1)}{2}} \left( \sum_{\mu} \frac{(-1)^{|\mu|}}{G_{\mu}(N, \gamma)} \frac{S_{\mu}(\mathbf{x}) S_{\mu}(\mathbf{t})}{G_{\mu}(N, 0)} \right) \prod_{j=1}^N e^{t_j}. \quad (\text{A10})$$

## APPENDIX B: RIBBON GRAPHS AND MATRIX INTEGRALS

Let  $\mathbf{x} = (x_1, \dots, x_N)$  be an  $N$ -dimensional random variable. Consider the normalized Gaussian measure

$$d\mu(\mathbf{x}) = (2\pi)^{-\frac{N}{2}} \sqrt{\det A} e^{-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j} \prod_k dx_k, \quad (\text{B1})$$

where  $A$  is a positive definite symmetric matrix. The inverse

$$B_{ij} = (A^{-1})_{ij} \quad (\text{B2})$$

is called the propagator.

Correlations of Gaussian random variables can be computed in a combinatorial way using *Wick's theorem*,<sup>61</sup> also known as *Isserlis' theorem*, which is stated below.

**Theorem B.1** (Wick's theorem). *The expectation value of the product of Gaussian random variables is*

$$\mathbb{E}[x_{i_1} x_{i_2} \dots x_{i_n}] = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ B_{i_1 i_2} & \text{if } n = 2, \\ \sum_{\text{pairings of } (i_1, \dots, i_n)} \prod_{\text{pairs } (k, l)} B_{i_k i_l} & \text{if } n \geq 2 \text{ and even.} \end{cases} \quad (\text{B3})$$

For example,

$$\mathbb{E}[x_{i_1} x_{i_2} x_{i_3} x_{i_4}] = B_{i_1 i_2} B_{i_3 i_4} + B_{i_1 i_3} B_{i_2 i_4} + B_{i_1 i_4} B_{i_2 i_3}. \quad (\text{B4})$$

Wick's theorem becomes particularly useful when the indices  $i_j$  are repeated. The problem of computing the expectation values  $\mathbb{E}[x_{i_1}^{b_1} \dots x_{i_n}^{b_n}]$  can be mapped to counting the number of ways of gluing  $n$  vertices with valencies  $b_1, \dots, b_n$  whose weights are determined by the propagators that correspond to their edges,

$$\mathbb{E}[x_{i_1}^{b_1} \dots x_{i_n}^{b_n}] = \sum_{\substack{\text{Graphs } G \text{ with } n \text{ vertices} \\ \text{of valencies } b_j}} \prod_{(i_k, i_l) \text{ edge of } G} B_{i_k i_l}. \quad (\text{B5})$$

For example,

$$\mathbb{E}[x_{i_1}^2 x_{i_2}^2] = B_{i_1 i_1} B_{i_2 i_2} + 2B_{i_1 i_2}^2. \quad (\text{B6})$$

Clearly, many graphs in (B5) are topologically identical and have the same weight because of the symmetries among the edges and vertices. Let  $\mathbf{G}$  be the group of these symmetries,  $\#gluings$  be the number of gluings of obtaining a graph, and  $\text{Aut}(G)$  be the automorphism group of the graph. By the orbit-stabilizer theorem,

$$\#\text{Aut}(G) \times \#gluings = \#\mathbf{G}, \quad (\text{B7})$$

where  $\#G$  is the order of group relabeling. Wick's theorem can be written only in terms of non-equivalent graphs as follows:

$$\frac{1}{\#G} \mathbb{E} \left[ \prod_j x_{ij}^{b_j} \right] = \sum_{\text{Non-equivalent graphs } G} \frac{1}{\# \text{Aut}(G)} \prod_{(i,j) \text{ edge of } G} B_{ij}. \quad (\text{B8})$$

In the case of Gaussian matrix integrals, Wick's theorem can be applied to compute correlators of traces by studying *fat graphs* also called *ribbon graphs*.

Consider the Hermitian Gaussian matrix model with the probability measure

$$d\mu_0(M_R) = \frac{1}{Z_0} e^{-2N \text{Tr} M_R^2} \prod_{j=1}^N dM_{R_{jj}} \prod_{j < k} d\text{Re} M_{R_{jk}} d\text{Im} M_{R_{jk}}, \quad (\text{B9})$$

where

$$Z_0 = \frac{1}{2^{N(N-1)}} \left( \frac{\pi}{N} \right)^{\frac{N^2}{2}}. \quad (\text{B10})$$

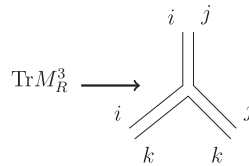
Wick's propagator is

$$\mathbb{E}_N^{(H)} [M_{R_{ij}} M_{R_{kl}}] \equiv \langle M_{R_{ij}} M_{R_{kl}} \rangle = \frac{1}{4N} \delta_{il} \delta_{jk}. \quad (\text{B11})$$

As an example, consider

$$\mathbb{E}_N^{(H)} [(\text{Tr} M_R^3)^2] = \sum_{\substack{i,j,k, \\ l,m,n}} \mathbb{E}_N^{(H)} [M_{R_{ij}} M_{R_{jk}} M_{R_{ki}} M_{R_{lm}} M_{R_{mn}} M_{R_{nl}}]. \quad (\text{B12})$$

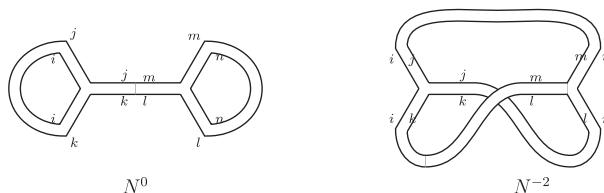
To map the problem to counting graphs, associate a vertex with each trace. The power of the matrix inside the trace gives the number of half-edges as double lines with index associated with each single line. The propagator in (B11) can be used to glue these half-edges together



to form a double line edge of the graph. Thus,

$$\begin{aligned} \mathbb{E}_N^{(H)} [(\text{Tr} M_R^3)^2] &= \sum_{\substack{i,j,k, \\ l,m,n}} \langle M_{R_{ij}} M_{R_{jk}} \rangle \langle M_{R_{ki}} M_{R_{lm}} \rangle \langle M_{R_{mn}} M_{R_{nl}} \rangle + \langle M_{R_{ij}} M_{R_{ki}} \rangle \langle M_{R_{jk}} M_{R_{lm}} \rangle \langle M_{R_{mn}} M_{R_{nl}} \rangle + \dots \\ &= \frac{1}{(4N)^3} \sum_{\substack{i,j,k, \\ l,m,n}} \delta_{ik} \delta_{km} \delta_{il} \delta_{ml} + \delta_{jk} \delta_{jm} \delta_{kl} \delta_{ml} + \dots \\ &= \frac{1}{4^3} \left( 12 + \frac{3}{N^2} \right). \end{aligned} \quad (\text{B13})$$

There are in total  $5!! = 15$  graphs in (B13) with only two topologically distinct graphs shown as follows:





If we attach to each vertex a factor of  $N$ , the  $N$  dependence of a graph is as follows: There is a factor  $N$  per vertex, a factor  $N^{-1}$  per edge, and a factor  $N$  for each single line when summed over indices. The number of single lines remaining at the end is the number of faces of the graph. Hence, the total  $N$  dependency of a graph is

$$N^{\#\text{vertices}-\#\text{edges}+\#\text{faces}} = N^{\chi(G)}, \quad (\text{B14})$$

where  $\chi(G)$  is the topological invariant of the graph called its Euler-characteristic.

This notion of counting ribbon graphs can be extended to compute correlators of the form  $\mathbb{E}_N^{(H)}[\prod_j (\text{Tr } M_R^j)^{b_j}]$ . When divided by  $\prod_j j^{b_j} b_j!$ , the order of group relabeling, matrix integrals takes a form similar to (B8). This formula is due to Brezin, Itzykson, Parisi, and Zuber in 1978,<sup>13</sup>

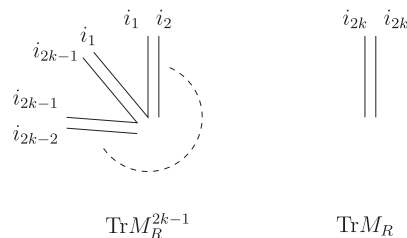
$$\mathbb{E}_N^{(H)}\left[\prod_{j=1}^n \frac{1}{b_j!} \left(\frac{N}{j} \text{Tr } M_R^j\right)^{b_j}\right] = \sum_{\text{Ribbon Graphs } G} \frac{1}{\#\text{Aut}(G)} 4^{-\#\text{edges}} N^{\chi(G)}, \quad (\text{B15})$$

where the sum is over non-topologically equivalent ribbon graphs and  $\#\text{Aut}(G)$  is the number of automorphisms of  $G$ . There are a total of  $(\sum_j j b_j - 1)!!$  graphs (counting equivalent and non-equivalent graphs). The total number of vertices is  $b = \sum_j b_j$  with  $j$  valencies for each vertex, and the total number of edges is  $(\sum_j j b_j)/2$ .

## 1. Special cases

Here, we consider two cases: (i)  $\mathbb{E}_N^{(H)}[\text{Tr } M_R^{2k-1} \text{Tr } M_R]$  and (ii)  $\mathbb{E}_N^{(H)}[(\text{Tr } M_R^2)^n]$ .

(i)  $\mathbb{E}_N^{(H)}[\text{Tr } M_R^{2k-1} \text{Tr } M_R]$ : We represent  $\text{Tr } M_R^{2k-1} \text{Tr } M_R$  as two vertices with  $2k-1$  and one valencies, respectively,

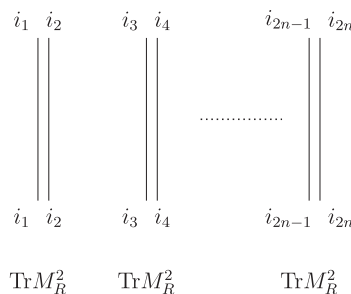


Since index  $i_{2k}$  has  $2k-1$  choices, by gluing the half-edges using (B11),

$$\begin{aligned} \mathbb{E}_N^{(H)}[\text{Tr } M_R^{2k-1} \text{Tr } M_R] &= \frac{(2k-1)}{4N} \mathbb{E}_N^{(H)}[\text{Tr } M_R^{2k-2}] \\ &= \frac{N}{k(4N)^k} (2k-1)!! i^{-k+1} P_{k-1}^{(1)}\left(iN, \frac{\pi}{2}\right), \end{aligned} \quad (\text{B16})$$

where  $P_{k-1}^{(1)}\left(iN, \frac{\pi}{2}\right)$  is a Meixner–Pollaczek polynomial.

(ii)  $\mathbb{E}_N^{(H)}[(\text{Tr } M_R^2)^n]$ : Here, we sketch the idea to calculate moments of  $\text{Tr } M_R^2$ . We represent  $(\text{Tr } M_R^2)^n$  as  $n$  vertices each with two valencies as shown below. There are several ways of gluing this set of vertices and half-edges. Trivially,  $i_j$  can be glued with itself for  $j = 1, \dots, 2n$ , which gives a total contribution of  $N^{2n}/(4N)^n$ .



The next non-trivial contribution comes from choosing any two vertices and gluing their valencies to form an edge between them. There are  $\binom{n}{2}$  ways of choosing two vertices. Let  $(i_p, i_{p+1})$  and  $(i_q, i_{q+1})$ ,  $1 \leq p, q \leq 2n$ , be the indices of the valencies of these two vertices.

There are two ways to pair  $(i_p, i_{p+1})$  and  $(i_q, i_{q+1})$ . This gives a contribution of  $n(n-1)N^2/(4N)^2$ . The remaining  $n-2$  disconnected graphs multiplicatively gives  $N^{2n-4}/(4N)^{n-2}$ . Hence, the first two leading terms are

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R^2)^n] = \frac{1}{(4N)^n} (N^{2n} + n(n-1)N^{2n-2} + \dots). \quad (\text{B17})$$

Remaining terms in the  $n$ th moment can be likewise computed,

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R^2)^n] = \frac{1}{(4N)^n} \prod_{j=0}^{n-1} (N^2 + 2j). \quad (\text{B18})$$

Similar arguments can be used to show that

$$\mathbb{E}_N^{(H)}[(\text{Tr } M_R^2)^k (\text{Tr } M_R)^{2n-2k}] = (2n-2k-1)!! \frac{1}{(4N)^n} N^{n-k} \prod_{l=n-k}^{n-1} (N^2 + 2l) \quad (\text{B19})$$

for  $k = 1, \dots, n-1$ .

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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