Towards a Logic of Essence and Ground

Benjamin Lev Pueo Brast-McKie

Supervisor: Timothy Williamson

Secondary: James Studd

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In dedication to my parents.

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Chapter 1

Introduction

This thesis is about the logic of essence and ground, where essence and ground are intended to regiment constitutive readings of the locutions 'necessary for' and 'sufficient for', respectively. For instance, consider the following:

`Gold:` The atom α having 79 protons is *necessary for* the atom α to be gold.

`Taller:` Sue being $5'4''$ and Marry being $5'9''$ is *sufficient for* Marry to be taller than Sue.

`Square:` The polygon being equilateral is *necessary for* the polygon to be a square.

`Pain:` John's C-fibres firing is *sufficient for* John to be in pain.

`Knowledge:` James believing that it is raining is *necessary for* James to know that it is raining.

`Red:` The ball b being crimson is *sufficient for* the ball b to be red.

`Action:` Cary intending to raise her arm is *necessary for* Cary to raise her arm.

`Wrong:` Sam intending to harm is *sufficient for* Sam's action to be wrong.

`Constitution:` The parts of the whole existing is *necessary for* the whole to exist.

`Existence:` Edward existing is *sufficient for* something to exist.

However contentious some of the claims above may be, these examples nevertheless witness the broad patterns of use of the locutions 'necessary for' and 'sufficient for' throughout a range of fields of inquiry. Whereas `Pain` belongs

philosophy of mind, `Knowledge` belongs to epistemology, `Action` belongs to the philosophy of action, `Wrong` belongs to normative ethics, and `Constitution` and `Existence` belong to metaphysics, we may observe that `Gold` belongs to Chemistry, `Square` belongs to geometry, and `Taller` and `Red` are common sense. It is the ambition of this thesis to provide a systematic means of reasoning with constitutive readings of 'necessary for' and 'sufficient for'.

For the sake of this introduction, I will rely on an intuitive grasp of the constitutive readings of 'necessary for' and 'sufficient for' as intended in the examples above, setting the stage for the chapters to come by contrasting a range of related notions. Nevertheless, it will help to clarify that by restricting consideration to constitutive readings of 'necessary for' and 'sufficient for', my intention is to exclude modal readings of these locutions. For instance:

> `Writing:` It being the case that I am writing is sufficient for 2+2=4.
>
> `Raining:` Grass being green or not green is necessary for it to be raining.

On a modal regimentation, 'sufficient for' and 'necessary for' express the strict conditional and its converse, respectively. Since every possibility in which I am writing is also a possibility in which 2+2=4, and every possibility in which it is raining is a possibility in which grass is green or not green, modal regimentations of `Writing` and `Raining` come out true. Although there are contexts in which modal readings of 'necessary for' and 'sufficient for' are most appropriate, modal readings fail to capture what is most naturally expressed by the ten examples presented above. At the very least, constitutive readings of 'necessary for' and 'sufficient for' require the antecedent to be wholly relevant to the consequent. Nevertheless, I will take the constitutive readings to entail the modal readings such that any counterexample to a modal reading of a 'necessary for'-claim or 'sufficient for'-claim will also be a counterexample to the corresponding constitutive reading of that claim. In this right, the modal readings may be said to approximate the constitutive readings.

Instead of speaking of some particulars in making 'necessary for'-claims or 'sufficient for'-claims as in the cases above, it is often more natural to make general assertions as indicated by the following examples:

`Gold`*: Having 79 protons is *necessary for* being gold.

`Red`*: Being crimson is *sufficient for* being red.

The claims above assert relationships between properties, where the occurrences of 'necessary for' and 'sufficient for' operate on predicates instead of sentences.[1] Although perfectly natural to assert, formalising such claims requires including a stock of predicates, and perhaps also variables and variable-binding operators in the language. By contrast, I will restrict attention to propositional languages with sentential operators '$\sqsubseteq$' and '$\leqslant$' for essence and ground respectively, remaining neutral on whether analogous notions which operate on predicates in place of sentences may be defined partly in these terms.

Given a propositional language which includes operators for essence and ground, we may ask which sentences of the language hold in full generality. In contrast to modal logic, the logic of essence and ground remains at an early stage of development. Principle among the pioneers of the logic of essence and ground is Kit Fine (1994a,b, 1995b,a, 2000, 2001, 2010a, 2012b,c, 2015a,b, 2017c). Additionally, Correia (2010, 2014, 2016a), Correia and Skiles (2019), and Lovett (2020) have developed logics closely related to Fine's later work. Despite their achievements, Fine, Correia, and Lovett work in systems in which the well-formed sentences of the language are restricted to atomic sentences of the form $\ulcorner A \star B \urcorner$, where '$\star$' stands in for either a grounding operator or propositional identity operator '$\equiv$'. Whereas in a *pure logic*, the arguments $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ are restricted to the sentence letters, in an *impure logic*, $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ are permitted to include at most the extensional operators '$\neg$', '$\wedge$', and '$\vee$' for negation, conjunction, and disjunction, respectively.

[1] Alternatively one might assert `Gold`†: To have 79 protons is *necessary to* be gold.

Even in working within an impure logic of ground, one cannot express rules of inference such as $A, A \leqslant B \vdash B$ or $\neg B, A \leqslant B \vdash \neg A$, not to mention modal interaction rules such as $A \leqslant B \vdash \Box(A \to B)$. Whereas the syntactic constraints that Fine and Correia impose are motivated by corresponding limitations of the existing semantics resources for interpreting such languages, the same cannot be said for Lovett's logic of ground on account of failing to provide any semantic theory at all.[2] By contrast, I will lift the artificial restrictions imposed on the well-formed sentences of the language, providing both a logic and semantics by which to accommodate consideration of the interactions principles which may be articulated in a propositional language with operators for ground and metaphysical modality in addition to the extensional operators, where essence may then be defined in terms of ground.

In contrast to the present attempt to regiment the locutions 'necessary for' and 'sufficient for', there is a growing body of literature which employs the label 'grounds' in connection with a binary relation between objects. For instance, Rosen (2010) considers a grounding relation between facts, where ⌜$[\![A]\!]$⌝ reads ⌜The fact that A⌝, and grounding claims take the form ⌜$[\![A]\!]$ *grounds* $[\![B]\!]$⌝. In addition to the stark difference in logical form between Rosen's grounding predicate and the sentential operators indicated above, it is not clear that Rosen has the same theoretical target in mind for his grounding predicate as is intended for '$\leqslant$'. We may begin to bring out the contrast by translating from Rosen's formalism into our own, where ⌜$[\![A]\!]$ *grounds* $[\![B]\!]$⌝ may be expressed as ⌜$A \leqslant B$⌝. In addition to committing to an ontology of facts, Rosen's formulation makes grounding *factive* insofar $[\![A]\!]$ *grounds* $[\![B]\!]$ entails A and B.[3] By contrast, 'sufficient for' and 'necessary for' admit of *non-factive* readings. For instance, one may assert `Red` while denying that b is red, and so conclude that b is not

[2] Although proof systems need not be tethered to a corresponding semantics, semantic considerations cannot figure in justifying the syntactic restrictions that Lovett imposes.

[3] See Correia's (2010, p. 254) discussion of the ontological neutrality of operational ground.

crimson. However, it makes little sense to claim that the fact that b is crimson grounds the fact that b is red, while at the same time denying that b is red. Accordingly, not every true claim of the form $\ulcorner A \leqslant B \urcorner$ will correspond to a true claim of the form $\ulcorner [\![A]\!] \textit{ grounds } [\![B]\!] \urcorner$.

Suppose that one were to attempt to provide a better means of translating from the operator formalism into Rosen's notation by defining non-factive grounding in terms of factive grounding. Although factive ground may be defined by non-factive ground with $A \leqslant_f B \coloneqq (A \leqslant B) \wedge A$, Fine (2012b, p. 49) points out that the same cannot be said in reverse. For instance, assuming that $A \leqslant B \coloneqq \Diamond(A \leqslant_f B)$ without requiring that $\Diamond A$ yields counterinstances to principles like $A \leqslant A$ and $A \leqslant A \vee B$ whenever $\neg\Diamond A$. However, restricting such principles to $\Diamond A \to (A \leqslant A)$ and $\Diamond A \to (A \leqslant A)$ significantly weakens the logic. Insofar as one wishes to admit of non-factive grounding in a manner analogous to Rosen's factive notion of grounding, facts may be replaced with states-of-affairs, where facts are identified with states-of-affairs that obtain. By letting $\ulcorner [\![A]\!]_s \urcorner$ abbreviate $\ulcorner$The state-of-affairs that $A\urcorner$, one may claim to freely translate between claims of the form $\ulcorner [\![A]\!]_s \textit{ grounds } [\![B]\!]_s \urcorner$ and $\ulcorner A \leqslant B \urcorner$.

Translating from the operator formalism into Rosen's predicate formalism comes at the cost of convoluting the logic for ground. Whereas it is natural to attempt to show that $A, A \leqslant B \vdash B$ is a valid rule of inference by proving that $A, A \leqslant B \vDash B$ for an appropriate definition of logical consequence, direct analogues of this attempt do not carry over into Rosen's notation. For instance, one cannot intelligibly write $[\![A]\!]_s, [\![A]\!]_s \textit{ grounds } [\![B]\!]_s \vdash [\![B]\!]_s$, for the syntactic and semantic entailment relations hold between sentences and not states-of-affairs. Instead, one may take $\ulcorner \mathcal{O}([\![A]\!]_s) \urcorner$ to read $\ulcorner [\![A]\!]_s$ obtains$\urcorner$, claiming that $\mathcal{O}([\![A]\!]_s), [\![A]\!]_s \textit{ grounds } [\![B]\!]_s \vdash \mathcal{O}([\![B]\!]_s)$ holds for any sentences $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ of the language, or perhaps that $A, [\![A]\!]_s \textit{ grounds } [\![B]\!]_s \vdash B$. However, given that what is at issue in asserting such a rule is a general pattern among the

sentences of the language, little is achieved by first mapping sentences to singular terms which are then taken to be the arguments of the predicates '*grounds*' and '$\mathcal{O}$'. Similar complications arise in attempting to provide a semantics for '*grounds*' which validates a compelling range of principles. For instance, assuming that singular terms are interpreted by objects in the domain, both $\ulcorner[\![A]\!]_s\urcorner$ and $\ulcorner[\![A \vee B]\!]_s\urcorner$ will be interpreted by objects. It remains to specify how the interpretation of $\ulcorner[\![A \vee B]\!]_s\urcorner$ is to be determined by the interpretation of $\ulcorner[\![A]\!]_s\urcorner$ and $\ulcorner[\![B]\!]_s\urcorner$, and what semantics for the grounding predicate could validate principles such as $[\![A]\!]_s$ *grounds* $[\![A \vee B]\!]_s$.[4] Given Rosen's concern to identify general laws for ground where the schematic variables take sentence position, it is natural to focus directly on the relationships between the sentences involved by taking grounding to be expressed by a sentential operator. Mapping sentences to singular terms which are taken to be the arguments of a grounding predicate is at best an indirect means of operating on sentences.

We also find reason to diverge from a predicate treatment of ground by Rosen's own lights, writing, "The thought is that when we cite grounds for $[\![p]\!]$, we cite facts that are strictly prior to $[\![p]\!]$ in a certain explanatory order" (p. 116). Insofar as the grounds are meant to explain the grounded, it is natural to deny that grounding claims ought to be expressed by a predicate since objects do not explain each other. For instance, the oxygen atom O does not explain the water molecule H_2O. This is not to deny that objects cannot participate in explanations. For instance, suppose that O, H, and H$'$ being arranged in manner W is sufficient for the H_2O molecule to be polar. Although the explanans and explanandum contain references to objects, these sentences do not themselves refer to objects. Rather that admitting an exception for objects which happen to be facts, I will take interpreted declarative sentences to express propositions, where propositions are not objects which fall within

[4] Rosen (2010, p. 117) defends the factive analogue $A \rightarrow ([\![A]\!]$ *grounds* $[\![A \vee B]\!])$.

the range of a first-order quantifier despite the fact that 'proposition' is a noun.[5] It is for this reason that explanatory locutions are best regimented by sentential operators rather than a predicate formalism.

Whereas Rosen (2010) takes grounding to be a binary relation between facts, Schaffer (2009, 2010, 2012, 2015, 2019) takes grounding to be a binary relation between entities of any sort whatsoever, writing:

> [M]etaphysics is the discipline that studies substances and their modes and kinds, by studying the fundamental entities and what depends on them. [...] The task of metaphysics is to say what grounds what. (2009, p. 351)

In place of Rosen's characterisation of ground as an explanatory order among facts, Schaffer takes 'ground' to be another name for 'ontological dependence', where I will maintain the latter title to avoid confusion. Whereas the locutions 'necessary for' and 'sufficient for' are employed across a wide range of scientific fields and philosophical disciplines, talk of ontological dependence is peculiar to metaphysics. For instance, in learning about the nature of gold, Chemistry students come to understand that having 79 protons is necessary but not sufficient for being gold, since a ball of 79 protons is not gold. By contrast, most educated adults have never heard of ontological dependence. Although this does not show that no sense can be made of the notion of ontological dependence, it goes some way towards showing that ontological dependence is considerably more obscure, and does not correspond to the same patterns of common use which both 'necessary for' and 'sufficient for' enjoy.

Despite the degree of divergence between ontological dependence and what 'necessary for' and 'sufficient for' each express, we may nevertheless attempt to define the former in terms of the latter. In particular, consider the following:

[5] Although there will be no need for such resources in what follows, I will assume that "quantification over propositions" is only rightly understood in a higher-order language with *sui generis* quantifiers which bind variables in sentence position. See Prior (1971), Williamson (2003), and Dorr (2016) for views of this kind.

Dependence: β *ontologically depends* on α *iff* α existing is necessary for β to exist.[6]

Insofar as Socrates' existence is necessary for the existence of {Socrates}, we may say by the lights of the definition above that {Socrates} ontologically depends upon Socrates. Similarly, insofar as the existence of the oxygen atom o is necessary for the existence of the water molecule w of which it is part, we may say that w ontologically depends upon o. In opposition to Schaffer's notion of ground, Wilson (2014) has claimed that ontological dependence lacks any theoretical unity, consolidating a miscellany of different dependence notions into one catch-all notion. Moreover, she claims that there is no theoretical work for the notion of ontological dependence to do which is not already served by one of the particular dependence relations of which ontological dependence is the sum. Rather than taking a stand on whether or not Wilson is right or not, it will suffice for my purposes to observe that the notion of ontological dependence which she aims to criticise cannot be identified with what either 'necessary for' or 'sufficient for' may be taken to express.

Setting Rosen and Schaffer's grounding predicates aside, we may turn to compare the various grounding operators which Fine (2012b,c) considers to the binary sentential operator '$\leqslant$' introduced above. In place of a single notion of ground, Fine considers a distinction between weak and strict grounds, as well as a distinction between full and partial grounds, where these two distinctions cut across each other, providing the following table:[7]

	Full	*Partial*
Weak	$\Delta \leqslant C$	$A \preceq C$
Strict	$\Delta < C$	$A \prec C$

Matters are complicated further by permitting full grounds to include many antecedents. Here $\ulcorner\Delta\urcorner$ indicates a set of sentences which may be either infinite

[6] See Fine (1995b) and Fine (2010b, p. 582) for related definitions and discussion.

[7] Fine (2012b, p. 51) also contrasts *mediate* with *immediate* ground, where only the former in transitive. As I will not presume that there are always immediate grounds— for instance, grounding chains could be dense— I will work exclusively with a mediate notion of ground.

or empty, and $\ulcorner A \urcorner$ and $\ulcorner C \urcorner$ are individual sentences. Whereas the strict notions of ground exclude the consequent from playing a role in grounding any of the antecedents, this is not required by the weak notions of ground. Additionally, the full notions of ground require that the truth of the antecedents be sufficient for the truth of the consequent, and the partial notions of ground only require that the truth of the antecedent together with the truth of some further sentences is sufficient for the truth of the consequent. In this taxonomy, I will be concerned with a weak-full binary grounding operator.

Without taking the grounding operator to be many-one, a similar result can be achieved with a binary grounding operator by taking the conjunction of the antecedents to ground the consequent. Insofar as we are to accommodate of grounding sentences with an infinite number of antecedents, an infinite conjunction operator '$\bigwedge$' must be included in the language. Additionally, grounding sentences with no antecedents may be captured by taking the consequent to be grounded in the *extremal proposition* $\bot$ which grounds every proposition.[8] Whereas a many-one grounding operator builds in these expressive resources, restricting consideration to a binary sentential operator has the effect of distinguishing between the expressive power included in the language by means of a grounding operator such as '$\leqslant$', and the expressive power which is added by including '$\bigwedge$' and '$\bot$'. It is for this reason that I will develop a logic for a binary grounding operator, where analogous considerations apply to the essence operator could also be added to the language.

Given the intention to regiment the locution 'sufficient for', I will adopt a reading of '$\leqslant$' which most closely corresponds to weak-full ground in Fine's terminology. In particular, I will take all instances of $\ulcorner A \leqslant A \urcorner$ to hold on account of the "weakness" of the operator, assuming that such claims as $\ulcorner A \leqslant A \wedge B \urcorner$ and $\ulcorner B \leqslant A \wedge B \urcorner$ will admit of counterinstances on account of

[8] As brought out in Chapter 4 and Chapter 5, there are two bottom elements and two top elements, where the space of propositions forms a *bilattice* rather than Boolean lattice.

the "fullness" of the operator. By including '$\leqslant$' in a language with extensional operators, we may also provide the following abbreviations:

Weak-Partial: $\ulcorner A \preceq B \urcorner$ abbreviates $\ulcorner A \wedge B \leqslant B \urcorner$.[9]

Strict-Full: $\ulcorner A < B \urcorner$ abbreviates $\ulcorner (A \leqslant B) \wedge (B \not\leq A) \urcorner$.

Strict-Partial: $\ulcorner A \prec B \urcorner$ abbreviates $\ulcorner (A \prec B) \wedge (B \not\prec A) \urcorner$.

Although these are by no means the only definitions which one might provide with a primitive grounding operator in the presence of the extensional operators, we may observe that the three notions of ground defined above do not regiment 'sufficient for'. Additionally, without drawing on further expressive resources, it is difficult to see how a weak notion of ground could be defined in terms of a strict notion, or how a full notion of ground could be defined in terms of a partial notion.[10] Together, these considerations provide reason to take '$\leqslant$' to be primitive, and the other notions of ground to be defined.

Having compared a range of alternative notions which go by the name 'ground', we may now compare alternative conceptions of essence. Just as Rosen and Schaffer's grounding predicates diverge in logical form from '$\leqslant$', there is a considerable body of literature on notions of essence which diverge in their logical form from '$\sqsubseteq$'. We may begin by considering Fine's (1994a,b, 1995a,b, 2000) essence operator '$\Box_$' which takes a singular term and sentence as arguments, where $\ulcorner \Box_\alpha A \urcorner$ reads $\ulcorner$ It lies in the nature of α that $A \urcorner$.[11] For instance, Fine claims that although it lies in the nature of {Socrates} that Socrates is a member of {Socrates}, it does not lie in the nature of Socrates that Socrates is a member of {Socrates}.

Instead of considering what is essential for a given proposition to obtain, Fine's essence operator provides a means of making claims about the essence

[9] Fine (2012b, p. 53) defines $A \preceq B \coloneqq \exists X(A \wedge X \leqslant B)$, however on plausible assumptions, these are equivalent. See Fine (2017b, p. 657) for the semantic analogue of this equivalence.
[10] See Fine (2012b, §1.5) for discussion.
[11] Fine (1995a) also considers a notation with an operator $\ulcorner \Box_F \urcorner$ for every predicate $\ulcorner F \urcorner$.

of an object. Despite these differences, one might attempt to relate the two notions by defining the latter in terms of the former. For instance, consider:

Objectual Essence: $\Box_x A \coloneqq A \sqsubseteq_y (x = y)$.

Generic Essence: $A \sqsubseteq_{\vec{v}} B \coloneqq \forall \vec{v}(A \sqsubseteq B)$.[12]

In the latter definition, $\ulcorner \sqsubseteq_{\vec{v}} \urcorner$ may operate on any open sentences whose unbound variables are included among $\vec{v}$. As Fine (2015b, p. 304) brings out, *Generic Essence* faces a number of problems. For instance, '$\Box_{\{s\}}(s \in y)$' is now an abbreviation of '$\forall y[(s \in y) \sqsubseteq (\{s\} = y)]$'. However, not only is unclear whether $(s \in y) \sqsubseteq (\{s\} = y)$ when $y \neq \{s\}$, it is also difficult to evaluate such claims as $(s \in \{s\}) \sqsubseteq (\{s\} = \{s\})$. Put informally, we may ask: Is s being a member of $\{s\}$ necessary for $\{s\}$ to be identical to $\{s\}$? Here Fine complains that nothing at all is necessary for something to be identical to itself.

Instead of defining the variable-binding operator '$\sqsubseteq_{\vec{v}}$' in terms of the universal quantification, Fine (2015a,b) provides an alternative to *Generic Essence* which draws on his theory of arbitrary objects in place of universal quantification. By contrast, one may choose to follow Correia and Skiles (2019) in taking a generic analogue of propositional identity '$\equiv_{\vec{v}}$' to be primitive, defining generic forms of both essence and ground in its terms.[13] However, instead of entering further into the debate over exactly how generic essence is to be understood, I will take the definitions above to provide some indication of the possible relationships between objectual essence as expressed by $\ulcorner \Box_x \urcorner$, generic essence expressed by $\ulcorner \sqsubseteq_{\vec{v}} \urcorner$, and what in this context we may call *propositional essence* as expressed by the sentential operator '$\sqsubseteq$'. Since a full treatment of the relationship between objectual, generic, and propositional essence requires expressive resources which outstrip the propositional language with which I will

[12] Or alternatively: $A \sqsubseteq_{\vec{v}} B \coloneqq \forall \vec{v}[B \to (A \sqsubseteq B)]$.

[13] Although Correia and Skiles (2019) employ the resources of higher-order quantification to define $A \sqsubseteq_{\vec{v}} B \coloneqq \exists X(A \wedge X \equiv_{\vec{v}} B)$ and $A \leqslant_{\vec{v}} B \coloneqq \exists X(A \vee X \equiv_{\vec{v}} B)$, this is not necessary, for on plausible assumptions, their definitions are equivalent to $A \sqsubseteq_{\vec{v}} B \coloneqq A \wedge B \equiv_{\vec{v}} B$ and $A \leqslant_{\vec{v}} B \coloneqq A \vee B \equiv_{\vec{v}} B$.

be concerned, I will remain neutral on precisely what relationships objectual and generic essence may bear to propositional essence.

Having contrasted the propositional notions of essence and ground with a variety of alternatives which go by the same name, I will drop 'propositional' in what follows. Nevertheless, it is important to specify that propositions are intended to be "worldly", amounting to *things being certain ways* rather than representations of things being some way or other such as interpreted sentences or Fregean thoughts. Insofar as essence and ground operate on propositions and not anything representational, matters to do with semantic determinacy are orthogonal to the present investigation. Although it may certainly be admitted that much of our language is subject to the phenomenon of vagueness, whether one proposition is grounded in, or essential to another is independent of the linguistic forms that one might choose to employ in asserting essence and grounding claims, as well as the particular interpretations upon which speakers may attempt to coordinate. Rather, my aim is to study the abstract structure of essence and ground within the space of propositions. Thus I will assume precise interpretations of the sentences under consideration throughout what follows, factoring out any degree of vagueness which examples such as `Red` might otherwise invite. This assumption will come across even more clearly in providing a semantics for formal languages, since the models of such languages will not admit of any degree of vagueness whatsoever.

Before attempting to provide a logic of essence and ground, it is important to emphasise that specifying the pair of informal locutions 'necessary for' and 'sufficient for' which essence and ground are intended to regiment does not suffice to provide a sufficiently clear theoretical target. As stated above, the aim is to regiment *constitutive* readings of 'necessary for' and 'sufficient for', not any felicitous use within English.[14] It remains, however, to specify what

[14] Although it is well worth providing a natural language semantics for 'necessary for' and 'sufficient for', this will not be my aim in what follows. Rather, I will focus on formal languages

constitutive readings of 'necessary for' and 'sufficient for' amount to, calling for the development of a philosophical theory of essence and ground. Although such philosophical ambitions are greatly assisted by the methods of proof theory and model theoretic semantics, any attempt to provide either a logic or semantics for a yet unspecified notion cannot claim to amount to anything more than a merely formal exercise. Rather, developing, a logic and semantics provides a systematic means for clarifying notions which we already intuitively grasp and consider to be of philosophical significance.

With these considerations in mind, CHAPTER 2 will set about describing the theoretical role which essence and ground may be taken to play in giving an account of reduction (also called real definition). As I will show, essence, ground, reduction, and propositional identity are best studied together, and indeed interdefinable in a language with extensional operators. In setting up the logic of essence and ground, I will assume as a matter of convention that ground is primitive and the other hyperintensional operators are defined. I will then provide abductive support for the definitions that I provide by considering the symmetries which emerge among the theorems for essence and ground as well as in the resulting theory of propositional identity.

Rather than beginning with a primitive operator for ground, CHAPTER 3 develops a hyperintensional theory of propositional identity which distinguishes A from $A \vee (A \wedge B)$ on account of having different subject-matters. Assuming a minimal theory of subject-matter, I will provide motivation for rejecting standard Boolean theories of propositional identity, focusing on the invalidity of four suspect principle. I will then show that neither Kripke's possible worlds semantics nor Humberstone's possibility semantics possess enough resolution power to admit counterexamples to the suspect principles, thereby motivating the implementation of a Finean state semantics. The remainder of the chapter

which include the resources for regimenting such natural language claims, where regimentation does not amount to providing an account of the meaning of the English locutions.

will present arguments against adopting the distribution laws, as well as showing that Correia's (2016b) logic and semantics for propositional identity ought to be rejected on account of making negation an opaque operator. The chapter concludes by considering extensions of my preferred logic of identity which include an operator for subject-matter, providing three ways of filling out the minimal theory of subject-matter with which the chapter begins.

Whereas Chapter 3 restricts consideration to a non-modal language in which the well-formed sentences take the form $\ulcorner A \equiv B \urcorner$ for $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ which only include extensional operators, Chapter 4 provides the semantic resources needed to lift these restrictions. By considering a language with primitive operators for ground and metaphysical necessity in addition to the extensional operators which is free of artificial syntactic constraints on the set of well-formed sentences, I will define operators for essence, relevance, reduction, and propositional identity. After extending the state semantics from Chapter 3 in order to interpret such a language, I will establish the soundness of the logic given in Chapter 2 over the semantics. The chapter concludes by presenting a number of countermodels, including to a supplementation law for conjunction advocated by Lovett (2020), showing that the resulting space of propositions forms a non-interlaced bilattice where neither conjunction nor disjunction is monotonic over either essence or ground. Despite these successes, the semantics which I provide is not able to distinguish the propositions expressed by true grounding sentences, or true modal sentences. Accordingly, I have not included principles or rules in the logic which describe valid inferences in which either grounding or modal operators occur within each other's scope.

Having established soundness in Chapter 4, I provide a completeness result for a fragment of the logic which has been enriched with rules and axioms for a specificity operator \$. Whereas most propositions may obtain in different ways— for instance, I may sit in any number of different positions and still be

sitting— *specific propositions* can only obtain in exactly one way. In this right, the specificity operator is comparable to the *numerical possibility operator* '$\Box_1$' from modal logic where ⌜$\Box_1 A$⌝ reads ⌜There is exactly one possibility in which A⌝, as well as to the singularity predicate '$\mathtt{S}$' from plural logics where ⌜$\mathtt{S}(aa)$⌝ reads ⌜There is exactly one among the aas⌝. Although a specificity operator is of interest in its own right, the introduction of specificity operators finds a wholly technical motivation, easing the tremendous difficulty of establishing completeness. Despite both restricting to a non-modal language with specificity operators and imposing constraints on the well-formed sentences of the language, the completeness proof nevertheless helps to indicate the manner in which a Henkin style construction applies to a Finean state semantics.

It remains open for exploration in future work whether a proof system at least as strong as the logic considered in Chapter 2 and Chapter 4 can be shown to be complete over a natural class of models. Rather than attempting to provide the final word on the matter, this thesis aims to take the next step towards providing an adequate logic of essence and ground.

Chapter 2

A Theory of Reduction

Chapter Abstract

Reduction plays a critical role within many fields of inquiry. Questions such as 'What is it to be water?', 'What is it for sets to be identical?', and 'What is it for an action to be right?' are good questions with good answers. For instance, to be water is to be H_2O, for sets to be identical is for them to have the same members, and according to the utilitarian, for an action to be right is for that action to maximise utility. Given the importance of such questions and their answers, this chapter provides an account of reduction in terms of *essence* and *ground*. Rather than drawing on independently motivated theories of essence and ground, I will assume that essence, ground, and reduction are best studied together, where this assumption is to be supported abductively by reference to the merits of the resulting theory. Accordingly, §2.1 develops a broad philosophical target for the theory reduction, contrasting reduction with both property identity and conceptual analysis in §2.2. After characterising the desiderata for an adequate theory of reduction in §2.3, I derive corresponding constraints for a theory essence and ground in §2.4, presenting a logic which describes the relationships between reduction, essence, and ground in §2.5.

2.1 Introduction

Property reduction aims to provide an account of the particular property in question. More specifically, I will take property reduction to amount to answering questions of the form 'What is it to F?' where 'F' takes predicate position. For instance, consider the aim to reduce the normative property of being right to some non-normative property, or to reduce the mental property of being in pain to some non-mental property, or to reduce the intentional

property of being about a given subject matter to a non-intentional property. Generalising on the latter ambition, Fodor (1989) writes the following:

> I suppose that sooner or later the physicists will complete the catalogue they've been compiling of the ultimate and irreducible properties of things. When they do, the likes of spin, charm, and charge will perhaps appear on their list. But aboutness surely won't; intentionality simply doesn't go that deep... If the semantic and the intentional are real properties of things, it must be in virtue of their identity with (or maybe of their supervenience on?) properties that are themselves neither intentional nor semantic. If aboutness is real, it must be really something else. (p. 97)

Whether the program that Fodor describes above proves to be successful or not, it is worth clarifying what it is to reduce a given property, or to reduce all properties of a given kind. In particular, one may ask whether either property identity or necessary equivalence provides an adequate account of property reduction. In opposition to these proposals, this paper defines reduction in terms of essence and ground which regiment 'necessary for' and 'sufficient for', respectively. As we will see, the resulting account of reduction draws a stronger connection between the reductum and reducens than necessary equivalence, without going so far as to identify the reducens and reductum.

We may begin by observing that in attempting to provide an account of a given property of interest, it is natural try to identify properties whose instantiation is both necessary and sufficient for the property in question to be instantiated. Fine (2015b) sketches a view of this kind, writing:

> What I would like to suggest is that we interpret the claim that water is H_2O as the claim that something's being H_2O is constitutively necessary and sufficient for it to be water. (p. 308)

In this spirit, I will begin by assuming that an adequate account of reduction claims of the form 'To F is to G' ought to respect the equivalence:

> *Reduction:* To F is to G *iff* to G is both necessary and sufficient to F.

Such an account is by no means unprecedented. Rather, I take *Reduction* to accord with common practice insofar as it respects the long standing tradition

of attempting to provide full accounts of the various properties of interest by identifying properties whose instantiation is both necessary and sufficient for the instantiation of the properties in question.[1] Of course, much will turn on how the locutions 'necessary to' and 'sufficient to' are to be understood. For instance, were one to take the strict conditional and its converse to regiment 'sufficient to' and 'necessary to' respectively, *Reduction* would amount to taking necessary equivalence to provide an account of reduction.[2] Although necessary equivalence may be necessary for property reduction to hold, it is natural to deny that necessary equivalence is sufficient for reduction to hold. For instance, sitting or not sitting is necessarily equivalent to being green or not green, and yet it is natural to distinguish sitting or not sitting from being green or not green, while also denying that either property reduces to the other. Accordingly, necessary equivalence fails to do justice to the manner in which reduction claims of the form 'To F is to G' are often used.

Instead of employing the locutions 'necessary to' and 'sufficient to' in accounting for what is expressed by 'To F is to G' along the lines given in *Reduction*, Dorr (2016) takes 'To F is to G' to express higher-order identity which is taken to be primitive rather than defined.[3] Given that higher-order identity is symmetric, such accounts cannot capture the direction of explanation expressed by many reduction claims. For instance, as Fine (2015b), observes:

> Philosophers have often wanted to express statements of essence in what appears to be the form of identities. Thus, they have wanted to say in response to the question 'What is water?', that water is H_2O, although

[1] It is natural to contrast *scientific reduction* which is a technical notion belonging to the philosophy of science, and typically holds between theories.

[2] More specifically, one might claim that to F it is sufficient to G *iff* necessarily, everything is such that if it is G, then it is F. By then permuting 'G' and 'F' in the definiens, one may similarly provide a modal account of when to F it is necessary to G.

[3] Dorr (2016) focuses on locutions of the form 'To be F is to be G', though this does not easily accommodate such substitutions as 'run' for F. As further alternatives, Correia and Skiles (2019) focus on the variable binding operator 'For some things $x, y...$ to be such that p is for them to be such that q', calling such claims *generalised identities*, and Rayo (2013) focuses on 'just is'-claims of the form 'For it to be the case that p just is for it to be the case that q'. In §2.3, I will return attention to the locutions 'necessary for' and 'sufficient for'.

> they have not wanted to say in the same sense that H_2O is water since that is not an account of what H_2O is. But, of course, if these statements of essence are genuine identities then it should be as equally acceptable to say that H_2O is water as that water is H_2O. (p. 307)

Similar worries apply if reduction is taken to be necessary equivalence, since necessary equivalence is also symmetric. However, higher-order identity views face the further difficulty of accounting for the apparent connection between what is expressed by 'To F is to G' on the one hand, and both 'To F it is necessary to G' and 'To F it is sufficient to G' on the other. Insofar as higher-order identity is taken to be primitive, it is not clear how an identity view will account for the long-standing practice of seeking to evaluate the truth of 'To F is to G'-claims by evaluating the truth of the corresponding claims 'To F it is necessary to G' and 'To F it is sufficient to G'.

In what follows, I will provide a theory of essence and ground by which to provide an account of reduction along the lines given by *Reduction*. Rather than asserting a relationship of necessary equivalence, or else asserting something as strong as higher-order identity, I will take many 'To F is to G'-claims to assert a relationship of constitution between properties.[4] More specifically, the theory of reduction to be elaborated below will aim to capture the following desiderata while respecting the account given in *Reduction*:

Ordering: The reducens is prior in nature to the reductum.

Metaphysical: Reduction relates properties independently of the means by which those properties are represented.

In developing a theory which respects *Ordering*, the ambition is to account for the explanatory asymmetry between reducens and reductum, where the instantiation of the reducens is to provide an account of what it is for the

[4] In general, I will take 'To F is to G'-claims to be ambiguous between reduction and identity, where one may attempt to clarify which claim is intended by asserting instead either 'What it is to F is to G' for reduction, or 'To F just is to G' for identity. See Rosen (2010) for use of 'What it is'-claims, and Rayo (2013) for use of 'just is'-claims. In §2.5, I provide the formal resources needed for resolving such ambiguities.

reductum to be instantiated, and not *vice versa*. In addition to capturing such asymmetries in explanatory order, *Metaphysical* requires that it is the properties alone which are to be ordered by reduction, irrespective of the means by which those properties happen to be represented. For instance, in claiming that to be right is to maximise utility, the utilitarian aims to give an account of the property of being right, where such an account ought to be insensitive to the means of presenting the account. In what follows, I will sharpen these desiderata, providing a theoretical target which the theory of reduction given below will aim to capture. Rather than ruling out all other accounts of 'To F is to G'-claims, the account given below will be shown to enjoy a number of advantages by accommodating *Reduction*, *Ordering*, and *Metaphysical*.

Drawing inspiration from the connections which Fine (2015b) suggests above, this chapter aims to provide a philosophical account of reduction which may be leveraged in the service of developing a theory of essence and ground. In §2.2, I will provide a more detailed account of the theoretical role that reduction is meant to play by contrasting reduction with both property identity and conceptual analysis. After characterising the desiderata for an adequate theory of reduction in §2.3, I will provide a definition of reduction in terms of essence and ground in §2.4 along the lines given in *Reduction*, drawing out a range of implications for an adequate theory of essence and ground. I will conclude in §2.5 by presenting *The Modalised Logic of Essence and Ground* (EGM) which describes a natural means of reasoning with the notions of essence, ground, reduction, propositional identity, relevance, metaphysical necessity, and the extensional operators, showing that the resulting theory of reduction satisfies the desiderata presented in §2.3, maintaining the spirit of *Reduction*, *Ordering*, and *Metaphysical* given above. In Chapter 4, I will go on to provide a state semantics over which EGM may be shown to be sound.

2.2 Identity and Analysis

Suppose that one were to take reduction to be property identity.[5] For instance, a theorist of this stripe might claim that an intentional property like the property of representing that it is raining is identical to some microphysical property even if such a property is too complicated to articulate in a language of microphysics. In particular, the sentence 'It is raining' has the property of representing that it is raining, where this representational property might be said to be identical to some hard to articulate microphysical property of that sentence.[6] Assuming that the property of representing that it is raining is an intentional property, the microphysical property to which it reduces must also be intentional for there cannot be any difference between identicals. In general, one cannot identify an intentional property— like the property of representing that it is raining— with a non-intentional property.

Something similar may be said for identifying mental or normative properties with non-mental or non-normative properties. For instance, suppose that a utilitarian were to claim that the property of being right reduces to the property of maximising utility, while also assuming that the property of being right is a normative property— i.e., a property which assigns a moral valence to an action, or otherwise prescribes how things ought, or ought not be. On the identity theory of reduction, it follows that maximising utility must also be normative, for identicals cannot differ in their properties. More generally, identity theorists cannot reduce normative properties to non-normative properties, where similar considerations prevent identity theorists from reducing mental properties to

[5] In providing an account of 'To be F is to be G'-claims in terms of higher-order identity, Dorr (2016, FN 7) does not take 'To be F is to be G'-claims to express reduction, despite admitting that, "In other work I have used other labels for the same class of sentences: 'metaphysical analyses' Dorr (2004, 2005) and 'real definitions' Dorr (2007)" (p. 42). I will continue to refer to 'To F is to G'-claims as *reduction claims* throughout for consistency.

[6] Alternatively, one might consider the representational relation such a sentence bears to what it represents. I focus on the corresponding relational property for simplicity.

non-mental properties, and so on for other kinds of properties.

Rather than reducing properties of one kind to properties of another incompatible kind, identity theorists about reduction may claim that properties of one kind *compose* to form a property of another kind.[7] For instance, one might hold that, however complicated, some composition of non-intentional physical properties is identical to the intentional property of representing that it is raining. Or similarly, one might claim that an appropriate composition of non-biological chemical properties is identical to the biological property of expressing a certain gene. Of course, one cannot go on to claim that a given property reduces to any of the individual properties from which it is composed. For instance, the property of representing that it is raining cannot be said to reduce to any of the individual non-intentional properties from which it might be said to be composed. Rather, it is the composition of an appropriate range of properties, not any property in that range, which may be said to be identical to the property in question, thereby providing a reduction of that property.[8] Whether identity theorists about reduction choose to provide a compositional account of properties or not, it remains that the attempt to reduce properties of one kind to properties of another kind cannot be maintained. As we will see, this raises trouble for any identity theorist who attempts to maintain the spirit of both *Ordering* and *Metaphysical* considered above, thereby distinguishing the role that I will take reduction to play from higher-order identity.

In order to bring out the difficulty that identity theorists face of maintaining *Ordering* and *Metaphysical*, it will help to begin by considering a view which can easily accommodate both. More specifically, consider the following:

[7] Although I will remain neutral about how property composition is to be understood, here is one proposal: Given n-ary predicates 'F^n' and 'G^n', one may take the complex predicate '$\lambda\vec{x}.[F^n(\vec{x}) \wedge G^n(\vec{x})]$' to express the composition of the properties express by 'F^n' and 'G^n', where perhaps other logical operators may be included in the composition of properties.

[8] Alternatively, one might claim that reduction relates a plurality of properties of one kind to a single property of another kind, where that relation is something like composition. However, such a view is not an identity theory of reduction for the simple reason that identity takes arguments of the same type, and will not be explored further here.

P1 Reduction relates properties.[9]

P2 The properties that the reducens and reductum instantiate may be distinct.

P3 Identicals instantiate the same properties.

C1 Reduction is not property identity.

Whereas **P1** is implicit in any theory of property reduction, and **P3** is beyond dispute, the identity theorist is forced to reject **P2**. Rather, an identity theorist may attempt to account for the ambition to reduce the intentional to the non-intentional, or the normative to the non-normative, or the mental to the non-mental, by claiming that we ought to replace **P2** with the following:

P2* The guises for the reducens and reductum may instantiate distinct properties.

Apart from assuming that guises determine reference and not *vice versa*, I will remain neutral about what guises are, where predicates, mental representations, or modes of presentation are among the possible accounts that such a theorist might employ. Instead of relating the property of being right to the distinct property of maximising utility, or the property of being in pain to the distinct property of having firing C-fibres, etc., an identity theorist may claim that in each case there is just one property represented by different guises.[10]

Suppose that an identity theorist about reduction were to adopt **P2***. Instead of reducing an intentional property to a non-intentional property, such a theorist may maintain that the property of representing that it is raining has both intentional and non-intentional guises. Whereas the intentional guise picks out the property by means of intentional predicates such as 'represents', or else the intentional concept thereby expressed, or etc., the non-intentional guise is to be restricted to the predicates, or corresponding concepts, or etc., included in a language of microphysics, however complicated. Such a theorist

[9] Given that properties are expressed by terms of type $<e>$, property reduction is a higher-order relation expressed by a term of type $<<e>, <e>>$, not terms of type $<e, e>$.

[10] I employ talk of *representation* instead of *reference* to maintain neutrality about guises.

might add that in some manner or other, this non-intentional guise ought to reflect the compositional structure of the intentional property in question. For instance, on a compositional identity theory of reduction, the non-intentional guise might be said to be composed of guises for the non-intentional properties from which the intentional property in question is composed.[11] Abstracting from the details to do with composition, the underlying strategy is to account for reductionist ambitions by appealing to the different guises for one and the same property. For instance, an identity theorist may claim that the normative reduces to the non-normative insofar as all properties with normative guises also have non-normative guises, e.g., whereas the is-right guise is normative, the maximising-utility guise is non-normative. Similarly, an identity theorist may take the being-in-pain guise to be mental, and the C-fibres-firing guise to be non-mental, though both guises represent the same property. In this way, an identity theorist may claim to accommodate the ambition to reduce the mental to the non-mental, and so on for the other reductionist aims.

By replacing **P2** with **P2***, an identity theorist may reject **C1**. However, merely gathering together the various guises which represent some common property fails to capture the explanatory asymmetry between the reductum and reducens. For instance, one might hope to affirm `(A)` while rejecting `(B)`:

`Water:` `(A)` To be water is to be H_2O.
`(B)` To be H_2O is to be water.

Since property identity is symmetric, identity theories of reduction must either accept both or neither of the claims above, where something similar may be said for any other reduction claim that one might wish to defend. Even in adopting **P2***, identity theories of reduction are unable to accommodate asymmetries between the guises for the reductum and reducens, for identity is insensitive

[11] Given the view sketched in footnote 7, reference might be taken to preserve the compositional structure of predicates whereby a complex predicate such as '$\lambda.Fx \wedge Gx$' refers to the complex property which is the composition of the properties represented by 'F' and 'G'.

to matters of guise. However, as brought out by *Ordering*, it is natural to think that the reducens ought to explain the reductum, and not *vice versa*. Accordingly, identity theories of reduction fail to satisfy *Ordering*.

Rather than giving up *Ordering* and attempting to maintain a symmetric theory of reduction, the present chapter aims to provide a theory of reduction whereby one may assert `(A)` while denying `(B)` in `Water`. More generally, I will assume that an adequate theory of reduction ought to satisfy the following:

Non-Symmetry: There are true reduction claims whose converses fail to hold.

If a theory of reduction is to accommodate the above, reduction cannot be property identity. In an attempt to maintain **Non-Symmetry**, one might depart from identity theories by taking reduction to order the guises for a given property rather than relating the properties expressed by different guises. More specifically, suppose that a theorist were to replace **P1** with the following:

P1* Reduction relates the guises by which a single property is represented.

Maintaining **P2** in addition to the above allows one to admit of asymmetries in reduction. For instance, one might locate the asymmetry inherent in certain reduction claims in the different explanatory powers possessed by the various guises for the property in question. For example, suppose that one were to affirm `(A)` while rejecting `(B)` in both `Water` as well as below:

`Alcohol:` `(A)` To be alcohol is to be C_2H_5OH.
`(B)` To be C_2H_5OH is to be alcohol.

Assume for the sake of the present example that the guises for a property are the predicates which refer to that property. Even if 'is alcohol' and 'is C_2H_5OH' both refer to the same property, one might deny that these predicates play the same explanatory roles.[12] For instance, 'H' and 'O' belong to both 'H_2O' and

[12] Of course, one may present analogues which employ concepts, or etc. in place of predicates.

'C_2H_5OH', bringing out systematic connections that the predicates 'is water' and 'is alcohol' otherwise obscure. Although far from offering a general account of a non-symmetric notion of reduction, such observations may be taken to provide evidence that the chemical guises considered above possess greater explanatory power than their folk analogues.[13]

I will refer to theories of reduction which adopt **P1*** in place of **P1** as *representational*.[14] Rather than relating properties, representational theories of reduction relate the guises by which a single property may be represented. For instance, given **P1***, the claim that what it is to be right is to maximise utility commits the utilitarian to the view that the is-right guise and the maximises-utility guise both represent the same property. However, in contrast to identity theories, representational accounts of reduction may also maintain **P2** so that the utilitarian may go on to deny that what it is to maximise utility is to be right. For instance, a utilitarian might explain that the maximises-utility guise makes salient systematic connections which are otherwise obscured by the is-right guise, and so claim that the former possess greater explanatory power than the latter. Even so, such a theorist cannot also claim that the is-right guise represents a normative property and that the maximises-utility guise represents a non-normative property, for given **P1***, both predicates represent the same property by means of different guises. Such a theorist could, however, claim that although both guises represent a normative property, the normative property is composed of non-normative properties, where the composition of the non-normative guise corresponds to the composition of the non-normative properties from which the normative property is composed.

Instead of ordering properties, representational theories of reduction order the guises by which a single property may be represented multiple times over.

[13] One might go on to claim that the chemical guises carve nature closer to the joints than their folk analogues even though both chemical and folk guises represent the same properties. See Lewis (1983) and Sider (2013) accounts of join-carving.

[14] Correia (2017) and Correia and Skiles (2019) develop such a view of real definition.

Accordingly, representational views of reduction do not provide an account of property reduction at all, but rather the reduction of one guise to another. However, in asking questions of the form "What is it to F?", it is natural to think that it is the property of F-ing which is the target of inquiry, not the guise by which that property happens to be represented. In opposition to representational views, I will refer to theories of reduction which affirm **P1** as *metaphysical* on account of the fact that it is the properties themselves which are to be reduced independently of the means by which those properties happen to be represented. More specifically, I will take metaphysical theories of reduction to be committed to the following:

> **Transparency:** Reduction is expressed by a transparent operator.

A sentential operator is *transparent* in a language just in case any sentence in which that operator takes wide-scope expresses the same proposition as the result of freely substituting co-referring expressions for that sentence's arguments, and *opaque* in that language otherwise. Taking reduction to be expressed by a transparent operator clarifies the sense in which reduction relates properties independently of the means by which those properties are represented as stated above in *Metaphysical*. More generally, it is natural to require the operators which express claims of central importance to metaphysics to be transparent.[15] I will take reduction to be one such instance.

Given **Transparency**, it follows from **Non-Symmetry** that **P1*** is false. For instance, let `Water` witness **Non-Symmetry** such that `(A)` is true and `(B)` is false. If one were to accept **P1***, it would follows from `(A)` that 'is water' and 'is H_2O' are co-referential. By **Transparency**, we may then conclude that what it is to be water is to be water, where `(B)` follows from a second application of **Transparency**, contradicting the above. Of course, a similar

[15] See Dorr (2016, p. 44) for further discussion of such claims.

argument could be provided for any witness of **Non-Symmetry**.[16]

In the service of developing a metaphysical rather than representational theory of reduction which can accommodate asymmetries between the reducens and reductum, I will maintain both **Non-Symmetry** and **Transparency** throughout what follows. Granting these assumptions not only requires one to reject **P1***, but also allows one to maintain both **P1** and **P2** so that it is the property being reduced— not merely some guise for that property— which is intentional, normative, or mental, while the property it reduces to is non-intentional, non-normative, or non-mental, respectively. Of course, such an account of reduction is compatible with properties having more than one guise. For instance, one may claim that 'is a vixen' and 'is a female fox' refer to the same property while denying similar claims for the predicates 'is right' and 'maximises utility', leaving room for a utilitarian to claim that the property of being right reduces in a sense distinct from property identity to the property of maximising utility, and not *vice versa*. Instead of displacing property identity, this paper maintains that there are some true reduction claims beyond those of property identity, remaining largely neutral with respect to particular reduction claims. Rather, the theory of reduction given below provides a range of general principles for reduction, constraining the use of the concept in considering which particular reduction claims one may wish to defend.

In developing a theory of reduction along the lines above, one need not deny that there is any interesting non-symmetric relation which holds between the different guises which represent a single property. For instance, even in assuming that 'is a vixen' and 'is a female fox' are co-referential, one might assert `(A)` while rejecting `(B)`, so long as both `(A)` and `(B)` are interpreted as

[16] Correia (2017, §3) provides a related argument given in terms of real definition, replacing my *modus tollens* with a *modus ponens*, and so concludes that real definition is opaque. Insofar as operators which play a central role in metaphysics are transparent, such an account yields the unfortunate result that real definition cannot play a central role in metaphysics.

answers to the question 'What does it mean to be a Vixen?':[17]

`Vixen:` `(A)` To be a vixen is to be a female fox.
`(B)` To be a female fox is to be a vixen.

Whereas reduction claims answer questions of the form 'What is it to F?', one may take the above to assert *conceptual analyses*.[18] For instance, one could take `(A)` to assert that the predicate 'female fox' provides an analysis of the predicate 'vixen', adding that although 'female fox' and 'vixen' are synonymous, 'female fox' more perspicuously represents the property to which it and 'vixen' both refer. Alternatively, one could take `(A)` to assert that the concept *female fox* provides an analysis of the distinct concept *vixen*, where the former provides an account of the meaning of the latter. Despite these differences, such theorists may agree that insofar as 'female fox' and 'vixen' co-refer, any asymmetry between the female-fox guise and the vixen-guise cannot show up at the level of the properties to which they refer.

Although it is not the aim of the present paper to develop a theory of conceptual analysis, it is nevertheless useful to contrast the broad outline of such an account with the ambition to provide a theory of reduction. In particular, suppose one were to adopt the following assumptions:

A1 Analysis relates the guises for a single property.

A2 There are some true analysis claims.

A3 The analysans is prior in meaning to the analysandum.

C2 Analysis is expressed by an opaque operator.

Instead of ordering properties, conceptual analysis orders the guises for each property, where the analysans is to provide an account of the meaning of

[17] One might parse this question as: 'What does 'Vixen' mean?'.

[18] It is important not to conflate talk of chemical analyses, physical analyses, biological analyses, structural analyses, or etc., with talk of conceptual analyses as intended here. Alternatively, one might adopt the scholastic labels 'real definition' and 'nominal definition' in place of 'reduction' and 'analysis', respectively.

the analysandum. We may then observe that if analysis were expressed by a transparent operator, `(A)` and `(B)` in `Vixen` would be inter-derivable given that 'is a vixen' and 'is a female fox' are co-referential. Insofar as a theory of analysis is to admit of asymmetries in meaning between co-referential expressions, analysis must be expressed by an opaque operator.[19]

Given the above characterisation of conceptual analysis, we may contrast the theoretical role set out for a metaphysical theory of non-symmetric reduction with that of conceptual analysis. For instance, compare the following:

`Right:` `(R)` What it is to be right is to maximise utility.
`(A)` What it means to be right is to maximise utility.

Whereas `(R)` asserts the metaphysical thesis about the nature of rightness in answer to the question 'What is it to be right?', `(A)` asserts a semantic thesis in answer to the question 'What does it mean to be right?'. Accordingly, a utilitarian is free to claim that to be right is to maximise utility independently of what the predicate 'right' happens to mean. Or to take another example, one might claim to reduce knowledge, explaining that to know is to possess a true justified belief. Even so, one might deny that the knowledge-guise admits of any analysis, denying that there is any better representation of knowledge.[20] More generally, one might complain that there are virtually no analyses which do not arise by explicit definition. For instance, although the meaning of 'vixen' may be analysed in terms of the meaning of 'female' and 'fox', this is only due to its definition. By contrast, it is much more difficult to provide analyses independent of such stipulated definitions. The same worry cannot, however, be raised for reduction since reduction is not a matter of meanings— stipulated or otherwise— but rather the nature of the property in question. As suggested

[19] One might replace **A1** with **A1′**: Analysis relates the guise for one property to the guise for another property to which it reduces. I will not consider such alternatives here.

[20] For instance, see Williamson (2002, pp. 4,31-2) for a rejection of the attempt to analyse knowledge in more basic terms, rather taking knowledge itself to be primitive.

by `Water`, Chemistry— among other fields— is rich in true reduction claims which need not be understood as conceptual analyses.

Whereas conceptual analysis orders the guises for a common property such that the analysans is prior in meaning— or *conceptually prior*— to the analysandum, reduction orders properties independently of any guise whereby the reducens is prior in nature— or *metaphysically prior*— to the reductum. Of course, this is not yet to provide an account of either conceptual or metaphysical priority. Rather, the aim has been to bring out the metaphysical character of reduction in contrast to the representational character of analysis. Despite these differences, the two notions are nevertheless easy to conflate. Indeed, one may take representational theories of reduction to make this very mistake, burdening a theory of analysis with ambitions best left to a theory of reduction. Since the present aim is not to develop a theory of analysis, I will remain neutral on whether analysis and reduction are to stand in some important relationship to each other. Rather, all that matters for my purposes here is that these two notions can be told apart. In the following section, I will further refine the theoretical target for a theory of reduction.

2.3 Towards a Theory of Reduction

Generalising on the above, the theoretical role which reduction aims to play is that of providing an account of the reductum. Although the examples above only concerned unary properties, I will assume that a full theory of reduction ought to accommodate n-ary properties for any finite n. For instance, consider:

> `Sets:` For the sets α and β to be identical is for α and β to have exactly the same members.

Similarly, one might defer to the legal definition of marriage within a given jurisdiction for the reduction of the tertiary relation of marriage holding

between two people and their jurisdiction.[21] In order to accommodate all such cases, I will attend to the task of providing an account of the reduction of one *proposition* to another. Whereas unary properties are the worldly analogue of unary predicates such as 'is standing', propositions are the worldly analogue of declarative sentences such as 'John is standing'. Put otherwise, propositions are *things being certain ways* rather than *ways for things to be*.[22] Accordingly, in place of claims such as 'To be water is to be H_2O', I will expand consideration to claims of the form 'For α to be water is for α to be H_2O', where α is some entity.[23] More generally, given some n entities and n-ary properties F^n and G^n, one might reduce the proposition that those n things are F^n to the proposition that those n things are G^n, preserving the order of the n entities involved. In order to capture all such cases, I will take reduction to be expressed by the binary sentential operator '$\Rightarrow$' with the same logical form as '$\vee$' and '$\wedge$' which takes two sentences as inputs and yields a single sentence as output.

Given that I will be wholly concerned with the reduction of propositions in what follows, I will henceforth drop the qualification 'of propositions', speaking simply of reduction. Accordingly, I will assume the following:

Conditions: Reduction relates propositions.[24]

The constraint above is a propositional analogue of **P1**. Like properties, I will take propositions to be worldly, rather than representations of worldly affairs. Given this generalisation in logical form, we may restate *Reduction* from §2.1:

[21] Alternatively, one might take marriage to be a binary relation holding between a plurality of exactly two individuals and the jurisdiction in question. I use this case only to illustrate.

[22] It will be convenient to reify properties and propositions, speaking of both as if they are first-order entities rather than higher-order entities as I will officially assume. See Prior (1971), Williamson (2003), and Dorr (2016) for elaboration of such higher-order views.

[23] In addition to constants one could include variables along with variable binding operators such as '$\forall x$' and 'λx', though I will avoid such complications below.

[24] Since propositions are expressed by terms of type $<>$, reduction is a propositional operator expressed by a term of type $<<>,<>>$, not $<<e>,<e>>$ as in footnote 9.

> **Reduction:** For it to be the case that B is for it to be the case that A *iff* it being the case that A is both necessary and sufficient for it to be the case that B.

Note that 'necessary to' and 'sufficient to' employed in **Reduction** have been replaced with 'necessary for' and 'sufficient for' in **Reduction**. For ease, I will often drop the 'it to be the case that' in asserting reduction claims. In order to contrast the intended readings of 'For... is for...' with the expressions of the form '... it is necessary for...' and '... it is sufficient for...', consider:

> `Right:` For Sam's action to be right is for Sam's action to maximise utility.
>
> `Vixen:` Sheila being a vixen is necessary for Sheila to be a fox.
>
> `Red:` The ball being red is sufficient for the ball to be crimson.

Instead of relating properties such as the property of being right to the property of maximising utility, `Right` draws a connection between the proposition that Sam's action is right and the proposition that his action maximises utility. It follows by **Reduction** that `Right` commits the utilitarian to the claim that maximising utility is both necessary and sufficient for Sam's action to be right. By contrast, neither `Vixen` nor `Red` can be strengthened to true reduction claims. Although being a fox is necessary for being a vixen, being a fox is insufficient for being a vixen since male foxes are not vixens. Additionally, the ball may be red without being crimson by being another shade of red, and so being crimson is sufficient but not necessary for being red.

As before, we may observe that **Reduction** fits with the practice of aiming to provide an account of a given proposition by identifying propositions which are both necessary and sufficient for the proposition in question. For instance, in order to provide an account of what it is for Sam's action to be right, it is necessary to specify a proposition which is both necessary and sufficient for Sam's action to be right. Moreover, identifying a proposition which is necessary and sufficient for Sam's action to be right is sufficient to provide an account of

what it is for Sam's action to be right. More generally, locating a proposition which is necessary and sufficient for another proposition is both necessary and sufficient for reducing the latter to the former.[25] Even so, it remains to provide a theory of what 'necessary for' and 'sufficient for' express.

As a first step towards developing a theory of the propositional operators expressed by 'necessary for' and 'sufficient for', it is natural to consider where the following modal regimentations of these locutions fall short:

Modal Necessity: A is necessary for B *iff* necessarily, if B, then A.

Modal Sufficiency: A is sufficient for B *iff* necessarily, if A, then B.

Given **Reduction**, it follows from the regimentation above that reduction is at least necessarily equivalent to necessary equivalence. I will refer to this account as the *modal theory of reduction*. Additionally, it follows from the modal account that 'necessary for' and 'sufficient for' express each other's converse, where reduction is symmetric and all necessary propositions reduce to each other. For instance, consider the following example:

`Arbitrary:` For 2+2=4 is for grass to be green or not green.

Although a modal theorist could bite the bullet— claiming that '2+2=4' and 'Grass is green or not green' express the same necessary proposition— this paper aims to provide a theory of reduction which avoids such unnatural consequences, claiming instead that `Arbitrary` is false.[26] More specifically, I will assume that an adequate theory of reduction ought to account for the manifest difference in explanatory significance between reduction claims such as `Right` or `Water` on the one hand, and `Arbitrary` on the other. Rather than appealing to guises

[25] I will remain neutral about whether **Reduction** provides a reduction of reduction, or an analysis of reduction, since it will suffice for my purposes to read the '*iff*' in **Reduction** as expressing a necessary equivalence.

[26] Similar troubles arise in taking reduction to be weak supervenience in the manner of Kim (1984, p. 158), where F weakly supervenes on G *iff* necessarily, for any x and y, if x and y are G-indiscernible, then x and y are F-indiscernible. It follows that necessary properties supervene on every property, and every property supervenes on impossible properties.

in order to account for such explanatory differences, the account developed below will appeal to a hyperintensional theory of propositions whereby distinct propositions may nevertheless share the same modal profile.

Given that the theoretical role of reduction is to provide an account of the reductum, the reducens must be wholly relevant to the reductum. This follows from the assumption that reduction is an especially strong form of explanation, where in general any explanans must be wholly relevant to its explanandum, at least in any true explanation. Thus I will adopt the following:

Relevance: The reducens is wholly relevant to the reductum.

Returning to the example above, a good account of why the proposition that 2+2=4 does not reduce to the proposition that grass is green or not green is that the latter does not provide an account of the former, where this follows from the fact that the latter fails to be wholly relevant to the former. Since the modal theory of reduction does not require the reducens to be wholly relevant to the reductum, the modal account does not satisfy **Relevance**, and so fails to provide an adequate theory of reduction.[27] Of course, it remains to clarify the intended sense of relevance. In sections §§2.4 – 2.5, I will provide a more detailed account of the intended notion of relevance, relying for the time being on an intuitive grasp of the **Relevance** constraint articulated above.

Although insufficient when taken on its own, we may nevertheless observe that sameness in modal profile is necessary for reduction to hold. Put otherwise, any counterexample to the necessary equivalence of two propositions may also serve as a counterexample to reduction holding between them. For instance, if one could show that there is any possibility in which Sam's action maximises utility without being right, or right without maximising utility, we may safely

[27] This is not to deny the adequacy of any theory with necessary equivalence at its core, but rather the adequacy of necessary equivalence on its own. For instance, one might combine necessary equivalence with a theory of analysis which requires the guise corresponding to the reducens to be wholly relevant to the guise corresponding to the reductum.

conclude that neither proposition reduces to the other. Thus I will assume:

Equivalence: The reducens and reductum are necessarily equivalent.[28]

In requiring the above to hold, it is important to note that reduction need not be restricted to either actual or possible propositions. For instance, we may coherently ask what it is for me to stand even if I am not standing, or what it is for the sets α and β to be identical even if α and β are not, nor could be identical. Accordingly, I will assume the following:

Non-Factive: Reduction may hold between propositions which do not obtain.

In addition to **Conditions**, **Reduction**, **Relevance**, **Equivalence**, and **Non-Factive** discussed above, I will assume that an adequate account of reduction must satisfy **Non-Symmetry** and **Transparency** given in §2.2. Together, these principles describe a target which a theory of reduction may aim to capture. It follows from **Reduction** that corresponding requirements carry over to whatever theory is to be provided for the propositional operators expressed by 'necessary for' and 'sufficient for'. Accordingly, the following section will be devoted to developing a theory of essence and ground which fits this job description, providing a means by which to regiment 'necessary for' and 'sufficient for' as they occur in **Reduction**, respectively. The adequacy of the resulting account of essence and ground will then be demonstrated in §2.5 by providing a modalised logic for essence and ground whose theorems satisfy the constraints set out above on an adequate theory of reduction.

2.4 Essence and Ground

Although a considerable amount of work has been done on notions bearing the titles 'essence' and 'ground', there is little consensus on even the grammatical

[28] Were one to take reduction to be weak supervenience as in footnote 26, reduction may hold without necessary equivalence holding. For instance, being green or not green supervenes on being red, and yet being green or not green fails to be necessarily equivalent with being red.

category for such notions, not to mention a unified logic with a reasonable degree of expressive power.[29] Accordingly, I will use the terms 'essence' and 'ground' not to import established notions in good standing, but rather as labels for specific theoretical duties each notion may aim to serve. In particular, I will take the regimentation of 'sufficient for' and 'necessary for' as they occur in **Reduction** to provide a target for the theory of essence and ground, constrained by the adequacy of the resulting theory of reduction.

Whereas reduction is to be defined in terms of essence and ground along the lines given in **Reduction**, essence and ground are at most interdefinable in a language with negation. Nevertheless, I will treat both essence and ground as primitive notions for the time being, contemplating which principles a logic of essence and ground ought to include as theorems. It will help to begin by fixing a notation, employing the sentential operators '$\sqsubseteq$' and '$\leqslant$' as follows:

Regimentation:
- (E) $\ulcorner A \sqsubseteq B \urcorner$ regiments ⌜For it to be the case that B it is necessary for it to be the case that A⌝.
- (G) $\ulcorner A \leqslant B \urcorner$ regiments ⌜For it to be the case that B it is sufficient for it to be the case that A⌝.

Even though 'A' and 'B' are schematic variables taking sentence position, it will often be convenient to read $\ulcorner A \sqsubseteq B \urcorner$ as ⌜A is essential to B⌝, and to read $\ulcorner A \leqslant B \urcorner$ as ⌜A grounds B⌝. Given these expressive resources, we may ask which essence and grounding claims hold in full generality. As a step towards providing an answer, the present section will review a number of characteristic principles for essence and ground which accord with the aim to regiment 'necessary for' and 'sufficient for' as they occur in **Reduction**.

In contrast to reduction claims such as `Right`, we may observe that both 'necessary for' and 'sufficient for' appear to be reflexive, such that every proposition is (trivially) necessary and sufficient for itself. Insofar as essence

[29] See however Fine (2012b, 2015b), Correia (2010), and Lovett (2020) for steps in this direction.

and ground are to regiment 'necessary for' and 'sufficient for', I will assume:

Reflexivity: (e) $A \sqsubseteq A$.
(g) $A \leqslant A$.

As with all of the principles for essence and ground to be considered below, the principles above are to hold of metaphysical necessity. Given Reflexivity, the operators '$\sqsubseteq$' and '$\leqslant$' may be said to express *weak* as opposed to *strict* (irreflexive) notions of essence and ground.[30] Given the reflexive notions of essence and ground, we may disambiguate **Reduction** as follows:

Weak Reduction: $A \Rightarrow B$ *iff* $(A \sqsubseteq B) \wedge (A \leqslant B)$.[31]

Strict Reduction: $A \Mapsto B$ *iff* $(A \Rightarrow B) \wedge (B \not\Rightarrow A)$.

Whereas weak reduction amounts to requiring A to be both weakly essential to, and to weakly ground B, strict reduction also requires that the converse claims do not both hold.[32] Given these definitions, **Non-Symmetry** may now be taken to require that there are some true strict reduction claims. It follows immediately that essence and ground cannot be each other's converse.[33] Although claims of strict reduction are typically of greater interest in considering particular cases, it is often easier to work with weak reduction in evaluating general principles for reduction.[34] Accordingly, I will continue to focus on weak reduction in what follows, often dropping 'weak' for convenience.

[30] It is easy to define strict essence as follows: $A \sqsubset B$ *iff* $(A \sqsubseteq B) \wedge (B \not\sqsubseteq A)$. Strict ground is similar: $A < B$ *iff* $(A \leqslant B) \wedge (B \not\leqslant A)$. See Fine (2012b) for a range of related notions.

[31] Given first-order quantifiers, one might define property reduction by means of the following: (1) $G \sqsubseteq^{\text{P}} F$ *iff* $\Box\forall x(Gx \sqsubseteq Fx)$; (2) $G \leqslant^{\text{P}} F$ *iff* $\Box\forall x(Gx \leqslant Fx)$; and (3) $G \Rightarrow^{\text{P}} F$ *iff* $(G \sqsubseteq^{\text{P}} F) \wedge (G \leqslant^{\text{P}} F)$. Factive analogues may also be defined as in Factive given below.

[32] Letting '$\sqsubset$' and '$<$' be defined as in footnote 30, we may alternatively define strict reduction as follows: $A \Mapsto B$ *iff* $[(A \sqsubset B) \wedge (A \leqslant B)] \vee [(A \sqsubseteq B) \wedge (A < B)]$.

[33] *Proof:* Assume for contradiction that $\sqsubseteq$ and $\leqslant$ are each other's converse, letting $A \Rightarrow B$ for arbitrary A and B. It follows that $A \sqsubseteq B$ and $A \leqslant B$, and so $B \leqslant A$ and $B \sqsubseteq A$ by assumption. Thus $B \Rightarrow A$, and so $A \Mapsto B$ does not hold. Generalising on A and B, we may conclude that there are no true strict reduction claims, contradicting the above. □

[34] As brought out below, $A \Rightarrow A \wedge (A \vee B)$ is a theorem, but $A \Mapsto A \wedge (A \vee B)$ is not, demonstrating the theoretical versatility of weak as opposed to strict reduction.

Having regimented **Reduction** in the notation above, we may observe that reduction is expressed by a sentential operator. Given that declarative sentences express propositions, we may conclude that **Conditions** is satisfied. Accordingly, such propositional notions of essence and ground must not be confused with objectual notions of essence and ground which relate objects to properties, or else relate objects to objects. For instance, instead of asking which propositions are essential for a given proposition to obtain, one may ask which properties are essential to a given individual.[35] In particular, one might ask what it is to be Socrates, where perhaps being human, or having the parents he in fact had, are to be included among his essential properties. Although I will not be concerned with objectual essence in what follows, this is not to claim that there is no relationship between objectual and propositional notions of essence. Indeed, one might attempt to define objectual essence in terms of propositional essence.[36] Something similar may be said for ground, where the propositional notion of ground developed below ought to be contrasted sharply with objectual notions such as ontological dependence which relate first-order entities rather than propositions.[37] Accordingly, objectual ground is to be expressed by a two-place predicate rather than a sentential operator. Although it is not the ambition of this paper to develop such connections, one might attempt to define ontological dependence in terms of propositional essence.[38] I mention these objectual notions of essence and ground in order to set them aside as neither accords with **Conditions** given above.

In addition to relating propositions, reduction must only relate necessarily equivalent propositions insofar as **Equivalence** is to hold. However, rather than requiring both essence and ground to only relate necessarily equivalent

[35] See Fine (1994a,b, 1995b,a) for such notions of essence.
[36] For instance, given the definitions in footnote 31, we may let: $F \sqsubseteq^{\mathrm{o}} \alpha$ *iff* $F \sqsubseteq^{p} \lambda x.(\alpha = x)$.
[37] See Schaffer (2009) and Rosen (2010) for objectual notions of ground.
[38] For instance, given the definitions in footnote 31, let: $\beta \leqslant^{\mathrm{o}} \alpha$ *iff* $\varphi(\beta, x) \sqsubseteq^{p} \lambda x.(\alpha = x)$. See Fine (2010b, p. 582) for a related definition of ontological priority.

propositions, we need only require as much of their conjunction. In particular, I will assume the following, where '$\Box$' expresses metaphysical necessity:

Strict Implication: (E) $(A \sqsubseteq B) \to \Box(B \to A)$.
(G) $(A \leqslant B) \to \Box(A \to B)$.

Given the principles above, strict and weak reduction satisfy **Equivalence**. Indeed, Strict Implication amounts to requiring that the modal readings of 'necessary for' and 'sufficient for' hold as a consequence of their readings in terms of essence and ground. Moreover, I will assume that such consequences are not arbitrary, but rather part of what it is for essence and ground to hold, respectively. Accordingly, I will also assume the following:

Essential Implication: (E) $\Box(B \to A) \sqsubseteq (A \sqsubseteq B)$.
(G) $\Box(A \to B) \sqsubseteq (A \leqslant B)$.

Whereas Strict Implication draws the necessary connections which result from true essence and grounding claims, Essential Implication explains why those necessary connections hold by deferring to the corresponding grounding or essence claim.[39] For instance, Sheila is a fox in every possibility in which she is a vixen because being a fox is part of what it is for Sheila to be a vixen.[40] Similarly, the ball is red in every possibility in which it is crimson because being crimson makes the ball red. Instead of ruling out objective possibilities, essence and ground describe the limits of objective possibility, even under the broadest conception of what is objectively possible.[41]

Despite satisfying Strict Implication, we may distinguish ground and essence from strict implication and its converse by examining why essence and ground are not converses of each other. In order to bring this out, consider:

[39] Such connections do not amount to reducing modality to essence. Somewhat more promising is $\Box A := A \equiv (A \vee \neg A)$, or more simply, $\Box A := \neg A \leqslant A$. However, insofar as the absorption laws are to be rejected, it is not clear that $\Box$ will have a normal modal logic on this definition.
[40] These examples demonstrate the manner in which 'because' can be used to express essence claims, whereas in other cases 'because' is more naturally interpreted in terms of ground.
[41] See Williamson (2016) for an account of metaphysical necessity as the strongest objective modal.

Whereas each conjunct of a conjunction is essential to that conjunction as stated in $\sqsubseteq$-Conjunction, each disjunct in a disjunction grounds the disjunction to which it belongs as stated in $\leqslant$-Disjunction.[42] For instance, the proposition that the ball is red (similarly round) is essential to the proposition that the ball is red and round. Additionally, the proposition that I am sitting (similarly writing) grounds the proposition that I am sitting or writing. Note that the readings given in Regimentation preserve these intuitions.

If essence and ground were converses of each other, the principles below would follow from $\sqsubseteq$-Conjunction and $\leqslant$-Disjunction, respectively:

On plausible assumptions, we may derive $A \vee (A \wedge B) \Rightarrow A$ and $A \wedge (A \vee B) \Rightarrow A$ from the above even though $A \vee (A \wedge B)$ and $A \wedge (A \vee B)$ need not be wholly relevant to A on account of B, thereby violating **Relevance**.[43] In order to maintain **Relevance** for reduction, we may impose corresponding relevance constraints on both essence and ground. In particular, I will assume that insofar as essence and ground are explanatory, their antecedents must be wholly relevant to their consequents.[44] Even though a conjunction strictly entails each of its conjuncts, and a disjunction is strictly entailed by each of its disjuncts, the former need not be wholly relevant to the latter. It is the failure to preserve relevance which explains why the $\leqslant$-Conjunction and $\sqsubseteq$-Disjunction principles given above do not hold.

[42] I employ talk of disjunctions and conjunctions as a shorthand for the propositions expressed by disjunctive or conjunctive sentences, where the propositions themselves need not be said to be conjunctive or disjunctive. Something similar may also be said for negation.

[43] *Proof:* Given that $A \leqslant A$ by Reflexivity and $A \wedge B \leqslant A$ by $\leqslant$-Conjunction, it follows by **R1** given in §2.5 that $A \vee (A \wedge B) \leqslant A$, and since $A \vee (A \wedge B) \sqsubseteq A$ by $\sqsubseteq$-Disjunction, it follows that $A \vee (A \wedge B) \Rightarrow A$. The derivation of $A \wedge (A \vee B) \Rightarrow A$ is similar. □

[44] Suppressing relativisation to an interpretation, I will assume throughout that the relevance of one sentence to another is parasitic on the propositions that those sentences express.

In order to articulate the relevance constraint on essence and ground, let ⌜$A \preceq B$⌝ read ⌜It being the case that A is *wholly relevant to* it being the case that B⌝, or for ease, ⌜A is *wholly relevant to* B⌝. We may then assert:

Essential Relevance: (E) $(A \preceq B) \sqsubseteq (A \sqsubseteq B)$.
(G) $(A \preceq B) \sqsubseteq (A \leqslant B)$.

The principles above state that tracking relevance is not merely an arbitrary consequence of essence and ground, but rather part of what it is for essence and ground to hold. Given Strict Implication, we may derive the following:

Strict Relevance: (E) $(A \sqsubseteq B) \rightarrow (A \preceq B)$.
(G) $(A \leqslant B) \rightarrow (A \preceq B)$.

If essence and ground were each other's converses, it would follow from any essence or grounding claim that both the antecedent and consequent would be wholly relevant to each other. However, this fails to hold in general. For instance, the ball being red is essential to the ball being red and round, but the ball being red and round does not ground the ball being red on account of including an irrelevant conjunct, i.e., the ball being round. Similarly, my sitting grounds my sitting or writing, but my sitting or writing is not essential to my sitting for the former fails to be wholly relevant to the latter on account of including an irrelevant disjunct, i.e., my writing. In this way, the relevance principles above explain why $\leqslant$-Conjunction and $\sqsubseteq$-Disjunction fail to hold, and so essence and ground fail to be each other's converse.

Given the relevance principles together with the definitions of weak and strict reduction, it is natural to expect that part of what it is for reduction to hold is for the reducens to be wholly relevant to the reductum. In order to derive this conclusion, we may admit either of the following principles:

Transitivity: (E) $(A \sqsubseteq B) \rightarrow [(B \sqsubseteq C)] \rightarrow (A \sqsubseteq C)]$.
(G) $(A \leqslant B) \rightarrow [(B \leqslant C) \rightarrow (A \leqslant C)]$.

The principles above accord with the intention to regiment 'necessary for' and 'sufficient for', respectively. Given $\sqsubseteq$-Conjunction, we may derive the following principles from either Essential Relevance (e) and Transitivity (e), or else from Essential Relevance (g) and Transitivity (g):

Reduction Relevance: (w) $(A \preceq B) \sqsubseteq (A \Rightarrow B)$.
(s) $(A \preceq B) \sqsubseteq (A \Mapsto B)$.

Given that the proposition that 2+2=4 fails to be wholly relevant to the proposition that grass is green or not green, Reduction Relevance accounts for why `Arbitrary` does not hold. These results accord with the **Relevance** constraint on the theoretical role of reduction given in §2.3, providing good reason to adopt the Essential Relevance principles.

Insofar as essence and ground are to succeed in tracking relevance, the present framework cannot individuate propositions by appealing to their modal profiles alone. After all, grass is green or not green in just the same possibilities in which 2+2=4 despite the fact that grass being green or not green has nothing to do with it being the case that 2+2=4. Were we to identify propositions which obtain in the same possibilities, 'Grass is green or not green' and '2+2=4' would express the same proposition, and surely every proposition is wholly relevant to itself. However, it was by appealing to the apparent failure of relevance between the propositions expressed by 'Grass is green or not green' and '2+2=4' that explained why `Arbitrary` was false in contrast to `Water` or `Right`. In this way, the present theory of essence and ground requires a hyperintensional theory of propositions which distinguishes necessarily equivalent propositions. However, if **Transparency** is to be preserved, one cannot appeal to guises in order to draw hyperintensional distinctions. Rather, essence and ground require a theory of propositions which draws hyperintensional distinctions in reality.[45] Nevertheless, I will assume in accordance with a conservative methodology

[45] See Chapter 3 for such a theory.

that departures from standard Boolean theories of propositional identity ought to be motivated by the present aim to provide an adequate theory of reduction *via* essence and ground as characterised in §2.3 above.

In order to provide a theory of reduction in terms of essence and ground which maintains **Transparency**, I will restrict consideration to *transparent languages* which only include transparent operators. Letting ⌜$A \equiv B$⌝ read ⌜For it to be the case that A *just is* for it to be the case that B⌝, or just ⌜A *is identical to* B⌝ for short, we may show on plausible assumptions that a language is transparent just in case all instances of the following schema hold:

Leibniz's Law: $(A \equiv B) \to (C \to C_{[A/B]})$.[46]

Put informally, the truth-value of all sentences in the language are required to be insensitive to their particular means of expression, and so reduction claims in particular will be insensitive to guise. For instance, consider the following:

`Frege Case:`
(h) For Hesperus to be rising is for the Earth to be rotating towards the planet Venus.
(p) For Phosphorus to be rising is for the Earth to be rotating towards the planet Venus.

Assuming that Hesperus rising is the same as Phosphorus rising, the following are inter-derivable by an instance of Leibniz's Law. Given that reduction is to be defined in terms of essence and ground, we may conclude that essence and ground are equally insensitive to matters of guise.

Having articulated a range of principles for essence and ground in the service of providing an adequate theory of reduction, I will conclude the present section by arguing that the notions of essence and ground of greatest theoretical interest are *non-factive*, where the *factive* notions of essence and ground satisfy:

Factivity:

[46] I provide a proof in §9 of Chapter 3.

$$(A \sqsubseteq B) \to A. \qquad (\text{G}_\text{L})\ \ (A \leqslant B) \to A.$$
$$(\mathbf{E}_\mathbb{R})\ \ (A \sqsubseteq B) \to B. \qquad (\text{G}_\text{R})\ \ (A \leqslant B) \to B.$$

Assuming Factivity, both the antecedent and consequent in any true essence or grounding claim must hold. However, given Reflexivity, it follows from Factivity that every proposition is necessary.[47] Of course there are contingent propositions, and so we must either reject Factivity, or restrict Reflexivity to just true propositions. Since the very same result may be derived from any other principle of the forms $A \sqsubseteq B$ or $A \leqslant B$, we must likewise restrict all such principles for essence and ground to true propositions, significantly weakening the logic. That we may avoid imposing such restrictions by rejecting Factivity provides a powerful reason to develop a logic of non-factive essence and ground rather than a logic for their factive analogues.

In addition to weakening the logic, Factivity undermines an otherwise natural account of the necessary connections entailed by true essence and grounding claims, at least insofar as the 4 axiom is to hold for metaphysical modality. To see this, suppose that $A \sqsubseteq B$ is true in a world u, and consider a world u' accessible from u where A does not hold. By Factivity, $A \sqsubseteq B$ does not hold in u'. Since $\Box(B \to A)$ is true in u by Strict Implication, it follows by the 4 axiom that $\Box\Box(B \to A)$ is true in u, and so $\Box(B \to A)$ is true in u' even though we cannot account for the necessary connection between B and A in u' by relying on the truth of $A \sqsubseteq B$ in u'. Similarly, by letting $A \leqslant B$ be true in a world v, we may consider a world v' accessible from v where B does not hold. Given Factivity, $A \leqslant B$ does not hold in v'. Since $\Box(A \to B)$ is true in v by Strict Implication, it follows by the 4 axiom that $\Box\Box(A \to B)$ is true in v, and so $\Box(A \to B)$ is true in v'. However, we cannot rely on the truth of $A \leqslant B$ in v' in accounting for the necessary connection

[47] *Proof:* By Factivity, $(A \leqslant A) \to A$, and so $\Box[(A \leqslant A) \to A]$ by necessitation. Thus $\Box(A \leqslant A) \to \Box A$ by K, and so $\Box A$ since $\Box(A \leqslant A)$ by Reflexivity and necessitation. $\Box$

between A and B in v'. Accordingly, $\Box(A \to B)$ is an unexplained necessary connection in v' but not in v. Thus Factivity forces one to either accept some unexplained necessary connections or else give up the 4 axiom, weakening the logic for metaphysical modality in order to accommodate a factive logic for essence and ground.[48] Rather than making such compromises, I will reject Factivity, developing a logic for non-factive essence and ground.

In addition to these theoretical considerations, we may also observe that Factivity carries as a consequence that both weak and strict reduction are factive, contradicting **Non-Factive** given in §2.3. Accordingly, **Non-Factive** provides a decisive reason to reject Factivity given our present purposes. Nevertheless, it is important to emphasise that the arguments given above are not intended to impugn the coherence of factive notions of essence and ground. Indeed, by assuming the coherence of the non-factive notions of essence and ground discussed above, we may easily define factive analogues as follows:

Factive: (e) $A \sqsubseteq_{\mathrm{F}} B$ *iff* $(A \sqsubseteq B) \wedge B$.
(g) $A \leqslant_{\mathrm{F}} B$ *iff* $(A \leqslant B) \wedge A$.

We may derive analogues of Factivity for factive essence and ground in any normal modal logic with the T axiom.[49] Instead of showing the factive notions of essence and ground to be nonsense, the arguments above speak in favour of beginning with non-factive notions in developing a logic for essence and ground, and defining the factive notions in terms of the non-factive notions, rather than *vice versa*. Accordingly, one need not deny that some claims are more naturally regimented in terms of factive essence and ground.

[48] Given an objective modal $\Box_4$ for which the 4 axiom fails, we may define an ancestral modal $\Box_4^\infty A = \bigvee\{\Box_4^n A : n \in \mathbb{N}^+\}$ where $\Box_4^0 A = A$ and $\Box_4^{n+1} A = \Box_4 \Box_4^n A$ which obeys the 4 axiom. Accordingly, $\Box_4^\infty$ is a stronger modal than $\Box_4$. If metaphysical modality is the strongest objective modality, then $\Box_4^\infty$ holds better claim to being a metaphysical modal than $\Box_4$. Thus the 4 axiom cannot be rejected for metaphysical modality. See also Williamson (2016).

[49] *Proof:* Assuming $A \sqsubseteq_{\mathrm{F}} B$, both $A \sqsubseteq B$ and B, where $\Box(B \to A)$ follows from the former by Strict Implication. Given the T axiom, $B \to A$, and so A. Thus $(A \sqsubseteq_{\mathrm{F}} B) \to A$, where something similar may be shown for ground.

The considerations given above are by no means intended to be exhaustive. Rather than providing a complete theory of essence and ground, my aim has been to identify a number of central principles by which to characterise the distinct theoretical roles which correspond to the locutions 'necessary for' and 'sufficient for' as they occur in **Reduction**. Accordingly, the constraints which an adequate theory of reduction must satisfy impose corresponding constraints for a theory of essence and ground. In particular, **Conditions**, **Relevance**, **Equivalence**, **Transparency**, and **Non-Factive** provide reason to adopt the principles for essence and ground defended above. It remains to provide an axiom system which includes these principles among its theorems. Accordingly, the following section will present a modalised logic for essence and ground which provides a natural means of reasoning with the notions of essence and ground in a language with operators for propositional identity, metaphysical necessity, relevance, and the extensional operators. As I will show, given any distinct propositions A and B, we may locate a range of strict reduction claims which hold by virtue of their logical form alone, thereby witnessing **Non-Symmetry**. In this way, each of the constraints on an adequate theory of reduction will be satisfied, thereby providing an adequate theory of reduction.

2.5 Reduction Reconsidered

Although the principles considered above help to clarify the theoretical roles which essence and ground are intended to play, such principles taken on their own do not provide a systematic means of reasoning with the notions of essence and ground. Accordingly, this section will present a logic whose axioms and rules of inference provide a natural deduction system for essence and ground. In this way, we may derive all of the principles defended above, along with a range of some but not all of the Boolean identities, thereby shedding light on the fineness of grain of the present theory of propositions.

Instead of taking all of the operators considered above to be primitive, it will only be necessary to take ground and metaphysical necessity to be primitive in addition to the extensional operators. Letting $\mathcal{L} = \langle \mathbb{L}, \neg, \wedge, \vee, \Box, \leqslant \rangle$ be a propositional language with sentence letters $\mathbb{L} = \{p_i : i \in \mathbb{N}\}$, we may define the *well formed sentences* of $\mathcal{L}$ as follows, where $p \in \mathbb{L}$ is arbitrary:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A \mid \Box A \mid A \leqslant A.$$

We may then adopt the following abbreviations, relying on the theoretical virtues of the resulting theorems in support of these conventions:

Essence: $\ulcorner A \sqsubseteq B \urcorner$ abbreviates $\ulcorner \neg A \leqslant \neg B \urcorner$.

Weak Reduction: $\ulcorner A \Rightarrow B \urcorner$ abbreviates $\ulcorner (A \sqsubseteq B) \wedge (B \leqslant A) \urcorner$.

Strict Reduction: $\ulcorner A \Mapsto B \urcorner$ abbreviates $\ulcorner (A \Rightarrow B) \wedge (B \not\Rightarrow A) \urcorner$.

Identity: $\ulcorner A \equiv B \urcorner$ abbreviates $\ulcorner (A \Rightarrow B) \wedge (B \Rightarrow A) \urcorner$.

Relevance: $\ulcorner A \preceq B \urcorner$ abbreviates $\ulcorner A \wedge B \leqslant B \urcorner$.

Interdependence: $\ulcorner A \asymp B \urcorner$ abbreviates $\ulcorner (A \sqsubseteq B) \wedge (B \sqsubseteq A) \urcorner$.

Interdetermination: $\ulcorner A \cong B \urcorner$ abbreviates $\ulcorner (A \leqslant B) \wedge (B \leqslant A) \urcorner$.

Although I have adopted *Interdependence* and *Interdetermination* merely as useful expedients, there may well be applications for these notions. Setting these cases aside, it remains to provide abductive support for the remaining abbreviations given above. Accordingly, I will begin by presenting a logic for $\mathcal{L}$ which encodes all of the principles defended above.

In order to specify the manner in which the primitive operators in $\mathcal{L}$ interact, I will adopt the following axioms and rules, referring to the resulting axiom system as *The Modalised Logic of Essence and Ground* (EGM), where $A, B, C \in \mathtt{wfs}(\mathbb{L})$ and $\ulcorner A \to B \urcorner$ abbreviates $\ulcorner \neg A \vee B \urcorner$ as usual:

A1 $A \cong \neg\neg A$.
A2 $A \wedge (B \wedge C) \cong (A \wedge B) \wedge C$.
A3 $\neg(A \wedge B) \cong \neg A \vee \neg B$.
A4 $\neg(A \vee B) \cong \neg A \wedge \neg B$.
A5 $A \leqslant A \vee B$.
A6 $B \leqslant A \vee B$.
A7 $A \wedge B \leqslant B \wedge A$.
A8 $A \wedge B \leqslant A \vee B$.
A9 $A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)$.
A10 $A \vee (B \wedge C) \leqslant (A \vee B) \wedge (A \vee C)$.
A11 $\Box A \to A$.
A12 $\Diamond A \to \Box\Diamond A$.

In addition to the above, we may capture a natural means of reasoning by including the following rules of inference and meta-rules, where Γ is any set of well formed sentences of $\mathcal{L}$, and $\ulcorner\Box\Gamma\urcorner$ abbreviates $\ulcorner\{\Box A : A \in \Gamma\}\urcorner$:

R1 $A \to B,\ A \vdash B$.
R2 $A \leqslant B,\ B \leqslant C \vdash A \leqslant C$.
R3 $A \leqslant C,\ B \leqslant C \vdash A \vee B \leqslant C$.
R4 $A \leqslant B,\ A \leqslant C \vdash A \leqslant B \wedge C$.
R5 $A \leqslant B \vdash A \vee B \sqsubseteq B$.
R6 $A \leqslant B \vdash B \sqsubseteq A \vee B$.
R7 $A \leqslant B \vdash \Box(A \to B)$.
R8 $A \leqslant B \vdash \Box(A \leqslant B)$.

M1 If $\Gamma,\ A \vdash B$, then $\Gamma \vdash A \to B$.
M2 If $\Gamma,\ \neg A \vdash B$ and $\Gamma,\ \neg A \vdash \neg B$, then $\Gamma \vdash A$.
M3 If $\Gamma \vdash A$ and $\Gamma \vdash B$, then $\Gamma \vdash A \wedge B$.
M4 If $\Gamma \vdash A$, then $\Box\Gamma \vdash \Box A$.

I will take $\vdash_{\text{MEG}}$ to be the smallest relation to satisfy the axioms and rules for EGM which is closed under the standard structural rules, dropping '$_{\text{MEG}}$' from '$\vdash_{\text{MEG}}$' since there is no threat of ambiguity throughout the present chapter. Letting $\ulcorner\Gamma \vdash A\urcorner$ read '$\ulcorner A$ is *provable* from $\Gamma\urcorner$', the *theorems* of EGM are those sentences provable from the empty set. In Chapter 4, I provide a Finean state semantics over which EGM may be shown to be sound. Although Chapter 5 makes some headway, a completeness result remains elusive.

In order to provide a better sense of the deductive power of EGM, we may derive the following theorems and admissible rules, taking $\ulcorner C_{[A/B]}\urcorner$ to be the result of uniformly replacing all occurrences of B in C with A, and letting $\ulcorner C_{(A/B)}\urcorner$ be the result of replacing one or more occurrences of B in C with A:

Admissible Rules

AR1 $A,\ A \leqslant B\ \vdash B.$

AR2 $\neg B,\ A \leqslant B\ \vdash \neg A.$

AR3 $B,\ A \sqsubseteq B\ \vdash A.$

AR4 $\neg A,\ A \sqsubseteq B\ \vdash \neg B.$

AR5 $A \sqsubseteq B,\ A \sqsubseteq C\ \vdash A \sqsubseteq B \vee C.$

AR6 $A \sqsubseteq B,\ C \sqsubseteq D\ \vdash A \wedge C \sqsubseteq B \wedge D.$

AR7 $A \sqsubseteq C,\ B \sqsubseteq C\ \vdash A \wedge B \sqsubseteq C.$

AR8 $A \leqslant B,\ C \leqslant D\ \vdash A \vee C \leqslant B \vee D.$

AR9 $A \sqsubseteq B\ \vdash\ A \wedge B \leqslant B.$

AR10 $A \sqsubseteq B\ \vdash\ B \leqslant A \wedge B.$

AR11 $A \sqsubseteq B\ \vdash\ \Box(B \to A).$

AR12 $A \sqsubseteq B\ \vdash\ \Box(A \sqsubseteq B).$

AR13 $A \leqslant B\ \vdash A \preceq B.$

AR14 $A \sqsubseteq B\ \vdash A \preceq B.$

AR15 $A \sqsubseteq B,\ B \sqsubseteq C\ \vdash A \sqsubseteq C.$

LL $A \equiv B,\ C\ \vdash C_{(A/B)}.$

Essence Theorems

E1 $A \sqsubseteq A \wedge B.$

E2 $B \sqsubseteq A \wedge B.$

E3 $A \vee B \sqsubseteq A \wedge B.$

Identity Theorems

ID1 $A \equiv A.$

ID2 $A \equiv \neg\neg A.$

ID3 $A \wedge A \equiv A.$

ID4 $A \vee A \equiv A.$

ID5 $A \wedge B \equiv B \wedge A.$

ID6 $A \vee B \equiv B \vee A.$

ID7 $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C.$

ID8 $A \vee (B \vee C) \equiv (A \vee B) \vee C.$

ID9 $\neg(A \wedge B) \equiv \neg A \vee \neg B.$

ID10 $\neg(A \vee B) \equiv \neg A \wedge \neg B.$

Reduction Theorems

RT1 $A \Rightarrow A \wedge (A \vee B).$

RT2 $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C).$

RT3 $A \Rightarrow A \vee (A \wedge B).$

RT4 $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C).$

RT5 $A \Rightarrow A.$

RT6 $(A \Rightarrow B) \wedge (B \Rightarrow A)] \to (A \equiv B)].$

RT7 $\Box(A \leftrightarrow B) \sqsubseteq (A \Rightarrow B).$

RT8 $(A \preceq B) \sqsubseteq (A \Rightarrow B).$

RT9 $(A \Mapsto B) \to (B \not\Rightarrow A).$

RT10 $(A \Mapsto B) \to [(B \Mapsto C) \to (A \Mapsto C)].$

RT10 $\Box(A \leftrightarrow B) \sqsubseteq (A \Mapsto B).$

RT12 $(A \preceq B) \sqsubseteq (A \Mapsto B).$

In addition to the theorems above, we may also note that all tautologies of propositional logic are theorems of EGM.[50] We may also observe the manner in which the reduction theorems stand in place of the Boolean absorption and distribution laws for propositional identities.

I provide counterexamples to the absorption and distribution laws for propositional identity Chapter 3 and Chapter 4. However, even without drawing on such semantic resources, we may nevertheless employ EGM in order argue that the following identities do not hold in full generality:

#Abs1 $A \equiv A \wedge (A \vee B)$. **#Abs2** $A \equiv A \vee (A \wedge B)$.

In order to provide a counterinstance to **#Abs1**, choose some A and B for which $B \not\leq A$. For instance, let A be 'Grass is green or not green' and B be '2+2=4', assuming the latter fails to be wholly relevant to the former. We may nevertheless show that $B \preceq A \wedge (A \vee B)$.[51] However, if **#Abs1** were to hold, then $(B \preceq [A \wedge (A \vee B)]) \rightarrow (B \preceq A)$ follows by **LL**, and so $B \preceq A$, contradicting our assumption. Thus **#Abs1** does hold in general, where a similar argument shows that **#Abs2** also admits of exceptions. Given any A and B for which $A \equiv A \wedge (A \vee B)$ and $A \equiv A \vee (A \wedge B)$ do not hold, it follows from *Identity* that neither $A \wedge (A \vee B) \Rightarrow A$ nor $A \vee (A \wedge B) \Rightarrow A$. Given **RT1** and **RT2**, it then follows by *Strict Reduction* that both $A \Mapsto A \wedge (A \vee B)$ and $A \Mapsto A \vee (A \wedge B)$, thereby witnessing **Non-Symmetry**. In this way, the theorems above provide a general recipe for producing strict reduction claims given any propositions A and B for which $B \not\leq A$.

[50] *Proof:* By **E1**, $\vdash A \sqsubseteq A \wedge B$, where $A \sqsubseteq A \wedge B \vdash \Box(A \wedge B \rightarrow A)$ by **AR8**, and so $\vdash \Box(A \wedge B \rightarrow A)$. Given **A11**, $\vdash \Box(A \wedge B \rightarrow A) \rightarrow (A \wedge B \rightarrow A)$, and so $\vdash A \wedge B \rightarrow A$ by **R1**. Thus $A \wedge B \vdash A$ again by **R1**, thereby deriving a conjunction elimination rule. The disjunction introduction rules, and negation introduction rules are similar.

[51] *Proof:* By **A2**, $B \wedge [A \wedge (A \vee B)] \leqslant (B \wedge A) \wedge (A \vee B)$, where it follows by **A9** that $(B \wedge A) \wedge (A \vee B) \leqslant [(B \wedge A) \wedge A] \vee [(B \wedge A) \wedge B]$. It follows by **ID3**, **ID4**, **ID5**, **ID7**, and **LL** that $(B \wedge A) \wedge (A \vee B) \leqslant A \wedge B$, and so $B \wedge [A \wedge (A \vee B)] \leqslant A \wedge B$ by **R2**. However, $A \wedge B \leqslant A \vee (A \wedge B)$ by **A6**, and so $A \wedge B \leqslant A \wedge (A \vee B)$ by **ID1** and **LL**. Thus $B \wedge [A \wedge (A \vee B)] \leqslant A \wedge (A \vee B)$ by **R2**, and so $B \preceq A \wedge (A \vee B)$ holds by *Relevance*. □

Without implicating any particular subject matter, the reduction theorems specify reductions which hold in virtue of logical form alone. We may then contrast the ambition to articulate general principles for reduction with our explanatory aims in asserting particular reduction claims. For instance, given that `(A)` in `Water` is true and that `(B)` is false, it is more informative to assert a reading of `(A)` articulated in terms of strict rather than weak reduction. By contrast, in cases of weak reduction in which the converse claim also holds, it is more informative to assert the corresponding identity claim which follows by *Identity*. For instance, supposing that `(A)` and `(B)` in `Vixen` are both true on a weak reading of reduction, the proposition that Sheila is a vixen is identical to the proposition that Sheila is a female fox, and so we do better to assert as much rather than either `(A)` or `(B)`. More generally, for any true weak reduction claim, either its converse holds or it does not, and so it follows that for every true weak reduction claim, it is more informative to either assert a propositional identity claim or else a strict reduction claim. Whereas strict reduction and propositional identity are better suited to making claims about particular cases such as `(A)` in `Water` or `Vixen`, weak reduction is often better suited to articulating general principles since the strict analogues of such weak reduction principles do not always hold in general, nor do analogues where weak reduction has been replaced with propositional identity.

The considerations above clarify the distinct theoretical roles afforded to weak and strict reduction by their respective definitions. We may further justify adopting these definitions by recalling that in attempting to explain what it is for a given proposition to obtain, it is natural to seek to provide a proposition which is necessary and sufficient for the proposition in question. By taking essence and ground to regiment 'necessary for' and 'sufficient for' respectively, where both 'necessary for' and 'sufficient for' appear to be reflexive, we arrive at the definition given in *Weak Reduction*. However, as brought out above,

many reduction claims are asymmetric, making the reflexive notion of weak reduction ineligible as means of regimenting such reduction claims. Rather, we may take asymmetric reduction claims to both assert that the reductum in question weakly reduces to the given reducens as well as deny that the converse holds. This yields the definition given in *Strict Reduction*, where the aim is to do justice to our pre-theoretic conception of reduction.

Having defended the definition of both '$\Rightarrow$' and '$\Mapsto$' given above, we may move to consider the definitions stipulated for '$\sqsubseteq$', '$\preceq$', and '$\equiv$'. Beginning with essence, we may observe that if being crimson is sufficient for the ball to be red, then part of what it is for the ball to fail to be red is for it to fail to be crimson, and so not being crimson is necessary for the ball to not be red. Similarly, if having 79 protons is necessary for the atom to be gold, then failing to have 79 protons makes it the case that the atom is not gold, and so the atom not having 79 protons is sufficient for the atom to not be gold. Generalising on these examples, consider the following principles:

Interaction: (E) $(A \sqsubseteq B) \rightarrow (\neg A \leqslant \neg B)$.
(G) $(A \leqslant B) \rightarrow (\neg A \sqsubseteq \neg B)$.

Given **ID2**, **R2**, and **AR15**, we may derive the following duality principles:

Duality: (E) $(A \sqsubseteq B) \leftrightarrow (\neg A \leqslant \neg B)$.
(G) $(A \leqslant B) \leftrightarrow (\neg A \sqsubseteq \neg B)$.

It follows from the above that essence and ground are interdefinable. Although I have chosen to define essence in terms of ground as in *Essence*, this is no more than a matter of convention. In particular, one need not go on to claim that ground is somehow conceptually prior to essence, admitting instead that both notions hold claim to an equal degree of conceptual priority.[52]

[52] As Correia and Skiles (2019) suggest, one might also define both essence and ground in terms of propositional identity. See Chapter 4 for further discussion.

In order to provide a justification for adopting *Relevance*, we may first recall that $\leqslant$-Conjunction and $\sqsubseteq$-Disjunction were rejected on account of the antecedent failing to be wholly relevant to the consequent in each case. In particular, $A \wedge B \leqslant B$ and $A \vee B \sqsubseteq B$ were said not to hold in full generality, for A may fail to be relevant to B, i.e., $A \not\preceq B$. Accordingly, if $A \preceq B$, then the same worry cannot be raised, and so we ought to conclude that both $A \wedge B \leqslant B$ and $A \vee B \sqsubseteq B$. Moreover, if $A \wedge B \leqslant B$ and $A \vee B \sqsubseteq B$, then A must be wholly relevant to B, and so $A \preceq B$. Putting these two direction together, we find reason to take $\ulcorner (A \wedge B \leqslant B) \wedge (A \vee B \sqsubseteq B) \urcorner$ to abbreviate $\ulcorner A \preceq B \urcorner$. However, it turns out that $(A \wedge B \leqslant B) \leftrightarrow (A \vee B \sqsubseteq B)$, and so I have adopted the abbreviation given in *Relevance* for simplicity.

Whereas *Strict Reduction* and *Essence* may claim to have some pre-theoretic backing— sharpening concepts with which we are already familiar— the same cannot be said of *Relevance*, amounting to no more than a harmless abbreviation. Similarly, it is natural to deny that *Identity* provides an analysis of the concept of propositional identity. Rather, I will assume that propositional identity is conceptually basic. Instead of shedding light on propositional identity, *Identity* may be justified abductively by appealing to the range of resulting identity theorems in EGM. It is on this basis that *Identity* may be adopted as extensionally adequate, while simplifying the presentation of EGM.

Having defended the abbreviations given above, I will close by noting that each of the principles for essence and ground defended in §2.4 has either been included as an axiom or else may be derived as a theorem of EGM. Moreover, these principles were motivated by the ambition to provide an account of reduction which satisfies **Conditions**, **Relevance**, **Equivalence**, **Non-Factive**, **Non-Symmetry**, and **Transparency**. Whereas **LL** witnesses *Metaphysical* given in §2.1, **RT9** – **RT12** give expression to *Ordering* by showing that strict reduction is transitive and asymmetric where part of what

it is for strict reduction to hold is for the reducens to be necessarily equivalent as well as wholly relevant to the reductum. Although these principles lend credence to the idea that reduction is an explanatory order, merely satisfying the principles above does not make reduction explanatory. Instead of looking to the principles which reduction satisfies, the only fully adequate account of the explanatory nature of reduction appeals to its definition in terms of essence and ground. Although interdefinable, essence and ground are not to be defined in terms of anything else, but rather axiomatised in the manner given by EGM. Thus it is by looking to EGM together with the definitions of weak and strict reduction that we may claim to have provided an adequate theory of reduction.

Chapter 3

Identity and Aboutness

Chapter Abstract

A number of authors have recently presented hyperintensional theories of propositions which depart from standard intensional theories. Such accounts do not always make clear what is to constrain the fineness of grain of the resulting space of propositions, nor what semantic framework is to replace a standard possible worlds semantics. By contrast, I will defend a number of principles which an adequate theory of subject-matter ought to include, where I will take differences in subject-matter to indicate differences between propositions. The remainder of the chapter will draw on these considerations to motivate a Finean state semantics for a hyperintensional theory of propositions guided by the aim to individuate propositions by their subject-matter in addition to their modal profile. After contrasting the resulting logic of propositional identity (LPI^-) with Correia (2016b) and Correia and Skiles' (2019) logic of generalised identity (GI), I will conclude by extending LPI^- to include axioms and rules for a subject-matter operator which provide a broader theory of subject-matter.

3.1 Intensionalism

When do two sentences express the same proposition in virtue of their logical form? It is important to stress that the present understanding of propositions is intended to be worldly: propositions are, so to speak, *things being a certain way*, rather than mere representations of things being some way or other. *Intensionalism* is the view that what it is for propositions to be identical is for them to be necessarily equivalent. For instance, reading ⌜$\varphi \equiv \psi$⌝ as ⌜For it to be the case that φ *just is* for it to be the case that ψ⌝, Rayo (2013, p. 66)

writes, "'$\phi \equiv \psi$' should be thought of as equivalent to '$\Box(\phi \leftrightarrow \psi)$'." Part of what makes intensionalism appealing is that a simple and strong range of necessary equivalences may be derived in any normal modal logic, shedding light on the nature of propositional identity and the abstract structure of the space of propositions. Intensionalism is to be contrasted with the claim that necessary equivalence may serve to approximate propositional identity despite failing to be co-extensive with propositional identity, or else given some specific application, that necessary equivalence may prove to be of some utility. Certainly it should be admitted that an intensional theory of propositions has been of great utility in philosophy, logic, linguistics, and computer science. Nevertheless, the present inquiry concerns the nature of propositional identity itself, and not some approximation or useful application.

Instead of defining propositional identity as the intensionalist does, or in yet some further way, *primitivism* claims that propositional identity is conceptually basic, and so an informative definition in other terms cannot be supplied. Accordingly, primitivists cannot follow intensionalists in employing a modal logic to derive a range of theorems for propositional identity from its definition. In attempt to characterise propositional identity, a primitivist is forced to make a fresh start, axiomatising propositional identity anew. Of course, free from all constraints, little progress can be made, leaving one with no more than suspicions about which identities hold in full generality. For instance, given any sentence $\ulcorner A \urcorner$, does $\ulcorner A \wedge A \urcorner$ express the same or different proposition as $\ulcorner A \urcorner$? What about the sentences $\ulcorner A \urcorner$ and $\ulcorner A \vee (A \wedge B) \urcorner$?

Even if propositional identity cannot be defined in other terms, a primitivist must nevertheless look to some prior conception of the theoretical role that propositions are meant to play in order to guide the ambition to axiomatise propositional identity. For instance, suppose that: (1) the proposition expressed by a sentence on a given interpretation is conceived of as the set of that

sentence's truth-conditions; where (2) truth-conditions are in turn conceived of as possible worlds. It of course follows that propositional identity will be co-extensive with necessary equivalence. In particular, $\ulcorner A \urcorner$ is true in exactly the same worlds as $\ulcorner A \vee (A \wedge B) \urcorner$, and so given (1) and (2), the propositions expressed by these two sentences will be identical. Insofar as the theoretical role of a proposition does not require more of a proposition than the modal profile of an interpreted sentence, then such conclusions are easy to accept. However, it is natural to object that although $\ulcorner A \urcorner$ and $\ulcorner A \vee (A \wedge B) \urcorner$ are true in the same possible worlds, they need not share the same subject-matter, where the subject-matter of a sentence is what that sentence is about. For instance, the sentence 'It is raining or both raining and snowing' is partly about it snowing, where the same cannot be said of the sentence 'It is raining'. Without giving up on a truth-conditional account of interpreted sentences as in (1), we may nevertheless refine our conception of a sentence's truth-conditions, replacing (2) with some alternative, guided by the aim to accommodate sameness in subject-matter of the sentences flanking a propositional identity sign.

Setting aside whether primitivism is the right view of propositional identity or not, I will take the propositional identity operator '$\equiv$' to be a primitive term for the purposes of this chapter. Instead of offering a definition, the present aim will be to motivate and defend a logic of propositional identity guided by the dual aim to respect sameness of subject-matter in addition to necessary equivalence of the sentences flanking a propositional identity sign. Accordingly, the resulting space of propositions will draw distinctions which an intensional theory of propositions does not, providing novel theoretical resources. Rather than exploring any particular application, the present chapter will be concerned to map out the contours of a hyperintensional theory of propositional identity which is both motivated and constrained by the ambition to distinguish propositions which differ in subject-matter.

It is worth noting that although an intensionalist could try to appeal to *guises* in order to capture the differences in subject-matter between necessarily equivalent interpreted sentences, doing so makes subject-matter concern the means of expressing propositions rather than the propositions expressed. By contrast, I will assume that the subject-matter of an interpreted sentence is what that interpreted sentence is about, admitting that at least many sentences are not about guises or anything representational. Alternatively, an intensionalist could choose to accept a coarse-grained theory of subject-matter, where necessarily equivalent interpreted sentences share the same subject-matter, perhaps claiming that differences in guise generate the appearance of a difference in subject-matter, though such appearances systematically mislead. For instance, although 'It is raining or not raining' and 'Grass is green or not green' have the same subject-matter, these sentences cast the illusion that what they are about is distinct. Instead of taking on error theories of this form, the present chapter will seek to provide a live alternative to intensionalism whereby differences in subject-matter may be taken to draw hyperintensional distinctions between necessarily equivalent propositions.

I will structure the chapter as follows. In §3.2, I will present a rough outline of a theory of subject-matter by which to guide our departure from Boolean theories of propositional identity. In §3.3, I will review the theory which Dorr (2016) provides in setting out an account of higher-order identity, as well as the considerations which animate his tentative departure from Booleanism. Contrasting Dorr's ambitions with my own will help to focus the aim of this chapter while also motivating a semantic approach to the topic. Accordingly, I will consider a range of alternatives to the possible worlds frameworks by which to develop an adequate theory of propositions.

By reviewing Humberstone's (1981) possibility semantics in §3.4, I will provide motivation for adopting the nonmonotonic notions of exact verification

and falsification developed by Fine (2016, 2017b,c,d) in §5.3. Whereas Fine (2017d, 2020) has provided extensive analysis of the pitfalls that other semantic frameworks face as viewed from within the perspective of his own semantics, my aim in §§3.4 – 5.3 is to show how our present ambition to develop a theory of propositional identity which respects subject-matter provides natural motivation for the resources which Fine has developed.

After having motivated my preferred version of Fine's state semantics, §3.6 will present counterexamples to the distribution laws for conjunction and disjunction, arguing that it is only by considerable contrivance that such counterexamples may be excluded, discouraging Fine's convexity constraint on propositions. In §3.7, I present a logic for propositional identity (LPI$^-$), contrasting a closely related logic (GI) given by Correia (2016b) and Correia and Skiles (2019) which has the unwanted consequence of making negation an opaque operator. I conclude in §3.8 by extending LPI$^-$ to include axioms and rules for a subject-matter operator, thereby providing a broader theory of subject-matter. In §3.9, I establish a few formal results.

3.2 Subject-Matter

I claimed above that the subject-matter of an interpreted sentence is what that sentence is about on its given interpretation. Insofar as sentences are to be interpreted by propositions, we may speak more directly of the subject-matter of the proposition expressed by a sentence on an interpretation. Accordingly, I will begin by assuming the following principle, letting $\dot{\sigma}A$ be the subject-matter of A, and leaving the interpretation tacit throughout much of what follows:

Objectivity: $(A \equiv B) \rightarrow (\dot{\sigma}A = \dot{\sigma}B)$.

Plausibly, the converse fails, for as Fine (2020) points out, it is natural to assume that $\dot{\sigma}(A \vee B) = \dot{\sigma}(A \wedge B)$ for any A and B, and yet deny that $A \vee B \equiv A \wedge B$

holds in full generality. Rather, *Objectivity* takes sameness of subject-matter to provide a necessary but not sufficient condition for propositional identity. Given *Objectivity*, what an interpreted sentence is about is a function solely of the proposition expressed by that sentence on its interpretation, and not at all a function of the sentence by which that proposition is expressed. For instance, given that for Hesperus to be rising just is for Phosphorus to be rising, it follows from *Objectivity* that the subject matter of Hesperus rising is the same as the subject-matter of Phosphorus rising. The present view is to be contrasted with accounts in which the subject-matter of a proposition is a function of the means by which that proposition is expressed.[1]

Given that propositional identity figures in the antecedent of *Objectivity*, we cannot look to *Objectivity* for positive determinations of which propositional identities hold. Nevertheless, given *Objectivity*, discrepancies in the subject-matter of A and B will entail the distinctness of A and B. Thus we may look to a theory of subject-matter in order to inform our present aim to provide a theory of propositional identity. Additionally, I will draw a connection between the subject-matter of A and the range of propositions which are wholly relevant to A. Letting $\ulcorner A \preceq B \urcorner$ read $\ulcorner$It being the case that A is wholly relevant to it being the case that $B\urcorner$, I will assume all instances of the following schema:

Relevance: $(\dot\sigma A = \dot\sigma B) \to [(X \preceq A) \leftrightarrow (X \preceq B)]$.[2]

The schema above gives voice to the idea that what A is about is every proposition wholly relevant to A. Insofar as the subject-matter of a proposition is what that proposition is about, a difference in what is relevant to A and B entails a difference in their subject-matters. Together with *Objectivity*, the *Relevance* principle provides a means by which to evaluate sameness of subject-

[1] See also Yablo (2014), for an equally objective account of subject-matter, and Fine (2020) for extensive comparison.

[2] As brought out below, I will restrict consideration to transparent languages. Were one to include a *sui generis* higher-order quantifier binding variables in sentence position, then arguably $(\dot\sigma A = \dot\sigma B) \leftrightarrow \forall X[(X \preceq A) \leftrightarrow (X \preceq B)]$ will also hold.

matter, for if $\ulcorner X \preceq A \urcorner$ and $\ulcorner X \preceq B \urcorner$ differ in truth-value for any sentence $\ulcorner X \urcorner$, then we may conclude that A and B differ in subject-matter, and so given *Objectivity*, it follows that $A \not\equiv B$. Of course, such evaluations will depend on judgements about relevance. Nevertheless, *Relevance* provides an additional basis by which to evaluate propositional identity claims.

Recall from before the claim that although $A \vee \neg A$ and $B \vee \neg B$ are true in the same possible worlds, they need not share the same subject-matter. We may now observe that even without giving a full theory of relevance, it is natural to expect that the following relevance principles hold in full generality:

L$\vee$Rel: $A \preceq A \vee B$ **L$\wedge$Rel:** $A \preceq A \wedge B$

R$\vee$Rel: $B \preceq A \vee B$ **R$\wedge$Rel:** $B \preceq A \wedge B$

Given some A and B for which $A \not\preceq B \vee \neg B$, we know that $A \preceq A \vee \neg A$ by **L$\vee$Rel**, and so $(A \vee \neg A) \not\equiv (B \vee \neg B)$ follows by *Relevance* and *Objectivity*. For instance, my sitting fails to be wholly relevant to grass being green or not green, though of course my sitting is relevant to me sitting or not sitting. Thus it follows by *Relevance* and *Objectivity* that it is not the case that for me to be sitting or not sitting just is for grass to be green or not green. Similarly, given any A and B for which $B \not\preceq A \wedge \neg A$, we know by **L$\wedge$Rel** that $A \not\preceq A \wedge \neg A$, and so $(A \wedge \neg A) \not\equiv (B \wedge \neg B)$ by *Relevance* and *Objectivity*. In particular, it is not the case that for me to be sitting and not sitting just is for grass to be green and not green. I take such claims to accord with common sense.

Analogous arguments may be given against adopting the absorption laws. Assuming there is some A and B where $B \preceq [A \vee (A \wedge B)]$ but $B \not\preceq A$, it follows by *Relevance* and *Objectivity* that $A \not\equiv A \vee (A \wedge B)$. For instance, although it snowing is wholly relevant to it raining or both raining and snowing, it snowing fails to be wholly relevant to it raining. Thus it follows by *Relevance* and *Objectivity* that it is not the case that for it to be raining just is for it to be

raining or both raining and snowing. Given that $B \not\trianglelefteq A$ for some A and B, a similar argument concludes that $A \not\equiv A \wedge (A \vee B)$ given that $B \trianglelefteq [A \wedge (A \vee B)]$ holds in general. In particular, it is not the case that for it to be raining just is for it to be raining and either raining or snowing.

At most, the considerations above may be taken to provide a partial theory of relevance and subject-matter. Nevertheless, the *Relevance* and *Objectivity* principles provide a natural place to begin developing a theory of subject-matter. In §3.8, I will present three alternatives for the semantics of subject-matter, along with three corresponding extensions of the logic for propositional identity (LPI^-) with a subject-matter operator. As I will show, the strongest of the three logics will be co-extensive with Fine's (2020) account of subject-matter. For now, it will suffice to take exception to the following principles in attempting to provide a hyperintensional theory of propositional identity:

#Necs: $(A \vee \neg A) \equiv (B \vee \neg B)$. **#Abs1:** $A \equiv A \vee (A \wedge B)$.

#Imps: $(A \wedge \neg A) \equiv (B \wedge \neg B)$. **#Abs2:** $A \equiv A \wedge (A \vee B)$.

The identities above belong to Boolean theories of propositional identity, where intensionalism provides one such example. However, even in rejecting the principles above, we may readily admit that the sentences flanking the propositional identity operators in the principles above succeed in expressing necessarily equivalent propositions, and so satisfy the following:

Necessary Equivalence: $(A \equiv B) \rightarrow \Box(A \leftrightarrow B)$.

Although necessary equivalence is necessary for propositional identity to hold, I take the arguments from *Relevance* and *Objectivity* to show that necessary equivalence is not sufficient for propositional identity to hold. Put otherwise, one may expect that any theory of propositional identity which takes sameness of subject-matter into serious consideration will admit counterexamples to

#Necs, **#Imps**, **#Abs1**, and **#Abs2**, thereby departing from traditional Boolean theories of propositional identity. In giving up Booleanism, it remains to both locate and defend an alternative theory which at least describes the interaction between propositional identity and the extensional operators.

3.3 Beyond Booleanism

Instead of taking '$\equiv$' to regiment 'just is'-claims, Dorr (2016) considers claims of the form 'To be F is to be G', where 'F' and 'G' take predicate position. Despite referring to such 'To be F is to be G'-claims as *identifications*, Dorr (p. 42) admits that, "In other work I have used other labels for the same class of sentences: 'metaphysical analyses' Dorr (2004, 2005) and 'real definitions' Dorr (2007)," suggesting a degree of asymmetry rather than the symmetry characteristic of identity. Nevertheless, I will begin by taking Dorr (2016) at his word to be developing a theory of higher-order identity, and so in this respect comparable to related accounts provided by Rayo (2013), Correia (2016b), and Correia and Skiles (2019), all of which employ the notation '$A \equiv_{\vec{x}} B$' where '$\equiv_{\vec{x}}$' may bind any free variables $\vec{x}$ which occur in A and B. Since I will only be concerned with propositional languages, it will be convenient to drop the subscripted variables, writing '$\equiv$' in place of '$\equiv_{\vec{x}}$' as before.

Restriction consideration to a propositional language, the present section will consider Dorr's (2016) sententialist account of higher-order identity, postponing consideration of Correia (2016b) and Correia and Skiles' (2019) account to §3.7.[3] Despite raising 'metaphysical analysis' and 'real definition' as labels for '$\equiv$', Dorr encourages an identity reading of '$\equiv$' by adopting the principles below, where $\ulcorner C_{(B/A)} \urcorner$ is the result of freely replacing $\ulcorner A \urcorner$ in $\ulcorner C \urcorner$ with $\ulcorner B \urcorner$:

[3] Dorr (2016, p. 46) mostly focuses on a *predicationalist* account, where '$\equiv$' may be flanked by any two predicates in the type hierarchy of the same type.

It is natural to assume that anything deserving of the title 'identity' ought to be an equivalence relation, and so **Ref**, **Sym**, and **Trans** must hold for propositional identity.[4] Of course, as Dorr notes, **Sym** and **Trans** may be derived from **Ref** given **LL**. Additionally, **LL** gives expression to Leibniz's Law of the indiscernibility of identicals, whereby identicals satisfy the same conditions. As Dorr (p. 49) points out, **LL** will admit of exceptions as soon as opaque contexts such as 'John knows that Phosphorus is rising' are included in the language. For instance, John may know that Phosphorus is rising without knowing that Hesperus is rising, but not due to an difference between Hesperus rising and Phosphorus rising. Dorr avoids complicating **LL** with restrictions by instead restricting consideration to formal languages which do not include opaque operators which generate opaque contexts.

Rather than taking the concept of opacity for granted, it is worth introducing a number of definitions from which **LL** may be derived, given an appropriate language. To begin with, we may say that an operator $\mathcal{Q}$ is *transparent* in a language $\mathcal{L}$ just in case there are no counterinstances to the following, where $\ulcorner\vec{A}_{(B_i/A_i)}\urcorner$ is the result of freely substituting $\ulcorner B_i\urcorner$ for $\ulcorner A_i\urcorner$ in the sequence $\ulcorner\vec{A}\urcorner$:

Func: $(A \equiv B) \to [\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A}_{(B/A)})]$.

Intuitively, a sentential operator is transparent in a language $\mathcal{L}$ just in case it expresses a propositional function, where the output proposition is determined solely by the input propositions independent of their means of expression. An operator is *opaque* in $\mathcal{L}$ just in case it is not transparent in $\mathcal{L}$. A language $\mathcal{L}$ is *transparent* just in case all of its operators are transparent in $\mathcal{L}$, and *opaque* in $\mathcal{L}$ otherwise. We may then consider the following principle:

[4] Dorr (2016, p. 49) also include *Alphabetic Variation:* $(A \equiv_{\vec{v}} B) \to ([u_i/v_i]A \equiv_{\vec{u}} [u_i/v_i]B)$. However, consideration of this principle will not be needed for what follows.

Imp: $(A \equiv B) \rightarrow (A \rightarrow B)$.

Assuming **Ref** and **Imp** in a classical propositional logic, **P2** in §3.9 establishes that a language $\mathcal{L}$ is transparent just in case **LL** holds without exception in $\mathcal{L}$. Thus instead of adopting **LL** as an independent assumption, we may derive **LL** from the stipulation that the languages under consideration are transparent where both **Ref** and **Imp** may be taken to hold without exception. I will refer the principles above as the *core principles* of propositional identity.

Given the ambition to provide a theory of propositional identity in a transparent language, I will take the principles specified above to be beyond dispute. It remains to provide interaction principles for propositional identity and the extensional operators. Consider the following principles:

$\wedge$-Idem: $A \wedge A \equiv A$ **Invo:** $\neg\neg A \equiv A$

$\vee$-Idem: $A \vee A \equiv A$

Although it is perhaps equally difficult to conceive of counterexamples to any of the three principles articulated above, Dorr (2016) adopts **Invo** while asserting the negation of **$\wedge$-Idem**, commenting as follows:

> I admit that this feature is quite surprising: especially given that we are endorsing *Involution*, you might have expected other especially "trifling" equivalences in propositional logic to correspond to true identifications. [...] But I insist that despite the fact that the present theory rejects *Idempotence*, it is still intended as a theory of a kind of claim just as "worldly" as ordinary identity claims. The hypothesis is that in the very sense in which it is true that to be a water molecule is to be a H_2O molecule, it is just not true that to be red is to be red and red. I insist that these claims are not obviously false. They are "edge cases" of the sort one would only ever consider as part of the kind of systematic logical investigation we are currently engaged in, and as such they should be settled on the basis of broader systematic considerations. (p. 77)

Here we may observe the manner in which our pre-theoretic suspicions fail to provide an adequate basis upon which to theorise about the most general truths concerning propositional identity. Rather, as Dorr highlights, such

determinations about the abstract structure of the space of propositions ought to be settled on the basis of systematic theorising. Of course, systematicity alone will hardly suffice, requiring in addition that any attempt to provide a theory of propositional identity be organised around a guiding vision. By contrast, the core principles for propositional identity cannot be given up without changing the topic under consideration, or else admitting opacity into the language in question, contrary to the stipulation above.

As an alternative to adopting a Boolean theory of propositional identity, Dorr (p. 70) takes direction from the definitional role which claims of the form ⌜To be F is to be G⌝ appear to play in the following case:

> Grue: To be grue is to be either green and observed before t, or blue and not observed before t (i.e., $G'x \equiv_x [(Gx \wedge Ox) \vee (Bx \wedge \neg Ox)]$).
>
> Bleen: To be bleen is to be either blue and observed before t, or green and not observed before t (i.e., $B'x \equiv_x [(Bx \wedge Ox) \vee (Gx \wedge \neg Ox)]$).
>
> Green: To be green is to be either grue and observed before t, or bleen and not observed before t (i.e., $Gx \equiv_x [(G'x \wedge Ox) \vee (B'x \wedge \neg Ox)]$).

Assuming a Boolean theory of propositional identity, Dorr (p. 71) shows that Green follows from Grue and Bleen, making use of the following identity:

> Taut: $Gx \equiv_x [([(Gx \wedge Ox) \vee (Bx \wedge \neg Ox)] \wedge Ox) \vee ([(Bx \wedge Ox) \vee (Gx \wedge \neg Ox)] \wedge \neg Ox)]$.

It is worth noting that although the sentences flanking '$\equiv_x$' above are necessarily equivalent, these sentences differ in subject-matter. Instead of ruling out Taut by requiring propositional identity to preserve subject-matter, or else by some other means, Dorr hazards the following diagnosis:

> The idea that there is something 'vicious' (i.e. impossible) about a kind of circularity that would be exhibited by this set of identifications is encouraged if we think of identifications as 'real definitions'. [...] If I have stipulatively defined a simple expression using a certain complex expression, I am not allowed later to stipulatively define one of the simple constituents of that complex expression using a different complex expression that contains the originally defined simple expression. (p. 72)

The quote above brings out the manner in which Dorr attributes to identity the theoretical role of providing definitions, motivating a "no circularity" constraint by these means.[5] However, this attribution is worth questioning. In particular, it is natural to take the theoretical role of definition to be best filled by an asymmetric relation, whereby the definiens is in some sense prior to the definiendum. By contrast, I take identity to be a paragon of symmetry. Such considerations raise doubts that Dorr is solely concerned with identity, running together the theoretical roles for identity and real definition.[6]

Without following Dorr's reasoning all the way to the assertion of the negation of $\wedge$**-Idem**, the case above illustrates the manner in which taking identity to fill the theoretical role of definition motivates Dorr's departure from a Boolean theory of identity. By contrast, I will attribute no such definitional role to propositional identity. Rather, I will assume that asserting $\ulcorner A \equiv B \urcorner$ amounts to the claim that $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ both express the same proposition where neither $\ulcorner A \urcorner$ nor $\ulcorner B \urcorner$ should be designated as the definiens nor definiendum. Nevertheless, I will assume that propositional identity requires sameness of subject-matter and modal profile, motivating a departure from Boolean theories of propositional identity by these means. Accordingly, I will seek to weaken the logic of propositional identity in a principled manner, guided by the aim to preserve sameness of subject-matter without giving up anything unnecessary, maintaining as many of the Boolean identities as possible.

Having contrasted the present vision of propositional identity with the definitional role which Dorr takes identity to play, it remains to provide a systematic means of developing a theory of propositional identity that respects sameness in subject-matter in addition to sameness in modal profile. Whereas an intensionalist about propositional identity may justify a Boolean theory of

[5] Dorr (2016, p. 74) ends up defending a "Only Logical Circles" constraint.

[6] In Chapter 2, I develop a theory of strict reduction (or real definition) by which to regiment 'To F is to G'-claims, contrasting strict reduction with propositional identity.

propositional identity by way of modal logic together with the intensionalist's definition of propositional identity, a primitivist is saddled with the hard work of providing an independent basis from which to evaluate propositional identity claims. Instead of looking to a particular proof system, the following sections will explore the model-theoretic resources needed to interpret propositional identity claims in a manner which is sensitive to subject-matter. As I will argue in §5.3, Fine's state semantics provides an ideal framework in which to approach this task. By way of motivating the state semantics, I will begin by reviewing the shortcomings of both Kripke's (1963) possible worlds semantics and Humberstone's (1981) possibility semantics in the following section.

3.4 Possible Worlds and Their Parts

Rather than attempting to axiomatise propositional identity by means of introspection alone, or else by attributing a definitional role to identity, I will draw on the resources of Fine's truthmaker semantics in order to argue for a non-Boolean theory of propositional identity which attempts to encode the differences in subject-matter brought out above. However, instead of beginning with a Finean state semantics, I will motivate the adoption of a state semantics by way of examining the shortcomings of both Kripke's (1963) possible worlds semantics and Humberstone's (1981) possibility semantics.

For consistency, I will focus attention throughout what follows on the propositional language $\mathcal{L} = \langle \mathbb{L}, \neg, \vee, \wedge, \equiv \rangle$ where $\mathbb{L} = \{p_i : i \in \mathbb{N}\}$ where the *extensional sentences* of $\mathcal{L}$ are defined as usual with $p \in \mathbb{L}$:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A.$$

Letting $\texttt{ext}(\mathbb{L})$ be the set of extensional sentences in $\mathcal{L}$, $A \equiv B$ is an *identity sentence* in $\mathcal{L}$ for any $A, B \in \texttt{ext}(\mathbb{L})$, where $\texttt{id}(\mathbb{L})$ is the set of identity sentences

in $\mathcal{L}$.[7] The following sections aim to provide a model-theoretic semantics by which to evaluate the sentences of $\mathtt{id}(\mathbb{L})$ for validity.

Beginning with a possible worlds semantics, $\mathcal{M} = \langle W, I\rangle$ is a *world-model* just in case W is a nonempty set and I is a total function from sentence letters in $\mathbb{L}$ paired with worlds in W to the truth-values T or F. Given that $\mathcal{L}$ is non-modal, there is no need to include an accessibility relation in $\mathcal{M}$. We may then provide the following semantic clauses, for $A, B \in \mathtt{ext}(\mathbb{L})$:

(p) $\mathcal{M}, w \models p$ *iff* $V(p, w) = T$.

($\neg$) $\mathcal{M}, w \models \neg A$ *iff* $\mathcal{M}, w \not\models A$.

($\wedge$) $\mathcal{M}, w \models A \wedge B$ *iff* $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models B$.

($\vee$) $\mathcal{M}, w \models A \vee B$ *iff* $\mathcal{M}, w \models A$ or $\mathcal{M}, w \models B$.

($\equiv$) $\mathcal{M} \models A \equiv B$ *iff* $\mathcal{M}, w \models A$ just in case $\mathcal{M}, w \models B$ for all $w \in W$.

Given our present concern to evaluate the sentences in $\mathtt{id}(\mathbb{L})$ for validity, only the extensional sentences in $\mathtt{ext}(\mathbb{L})$ will be interpreted at model-world pairs, where propositional identities sentences in $\mathtt{id}(\mathbb{L})$ will be evaluated at models. Although one could interpret $\varphi \in \mathtt{id}(\mathbb{L})$ at model-world pairs, little would be gained for present purposes.[8] For any $\{\varphi\} \cup \Gamma \subseteq \mathtt{id}(\mathbb{L})$, we may let φ be a *logical consequence*$_{\text{w}}$ of Γ, i.e. $\Gamma \models \varphi$, just in case $\mathcal{M} \models \varphi$ whenever $\mathcal{M} \models \chi$ for all $\chi \in \Gamma$, and φ is *valid*$_{\text{w}}$ just in case $\models \varphi$. Letting $I(A) = \{w \in W : \mathcal{M}, w \models A\}$ be the proposition that $A \in \mathtt{ext}(\mathbb{L})$ expresses in $\mathcal{M} = \langle W, I\rangle$, it follows that $\mathcal{M} \models A \equiv B$ just in case $I(A) = I(B)$. Thus a possible worlds semantics requires identical propositions to have the same modal profile.

Given any world-model $\mathcal{M} = \langle W, I\rangle$, we may conclude from the totality and functionality of I together with the fact that T and F are the only truth-values that both of the following claims hold:

[7] I will often omit corner quotes, taking use-mention ambiguities to be resolved by context.
[8] Chapter 4 provides a semantics in which identities may be evaluated at model-state pairs.

Consistency: For any $p \in \mathbb{L}$ and $w \in W$, either $I(p, w) \neq T$ or $I(p, w) \neq F$.

Maximality: For any $p \in \mathbb{L}$ and $w \in W$, either $I(p, w) = T$ or $I(p, w) = F$.

It follows from the semantics that *Consistency* and *Maximality* may be extended to all $A \in \texttt{ext}(\mathbb{L})$ by way of the following definition, such that every $\mathcal{M} = \langle W, I \rangle$ and $w \in W$ determines a unique truth-value for every sentence $A \in \texttt{ext}(\mathbb{L})$:

Falsity: $\mathcal{M}, w \dashv A$ *iff* $\mathcal{M}, w \not\models A$.

Since falsity may be defined in terms of truth as above, there is no need to state separate clauses for the falsity-conditions for the sentences of $\mathcal{L}$. We may then observe that the semantics for negation satisfies the following principles:

Negation: (T) The truth-conditions for $\neg A$ are the same as the falsity-conditions for A.
(F) The falsity-conditions for $\neg A$ are the same as the truth-conditions for A.

I take it that any adequate semantics for negation ought to satisfy the principles above. Given that (T) and (F) follow from ($\neg$) together with *Falsity*, the possible worlds semantics for negation passes this elementary test.

Although *Maximality* and *Consistency* are essential to the possible worlds framework, these conditions also serve to explain why the possible worlds framework is unable to track subject-matter in addition to modal profile without adopting a very coarse-grained theory of subject matter. Given any sentence A, world-model $\mathcal{M} = \langle W, I \rangle$, and world $w \in W$, it follows that either $w \in I(A)$ or $w \in I(\neg A)$, but not both, and so $I(\neg A) = W / I(A)$. Additionally, by the semantics for conjunction and disjunction, it follows that $I(A \vee \neg A) = I(A) \cup I(\neg A) = W$ and $I(A \wedge \neg A) = I(A) \cap I(\neg A) = \varnothing$ for all $A \in \texttt{ext}(\mathbb{L})$. Thus $I(A \vee \neg A) = I(B \vee \neg B)$ and $I(A \wedge \neg A) = I(B \wedge \neg B)$ for any $A, B \in \texttt{ext}(\mathbb{L})$, making **#Necs** and **#Imps** valid_{W}, where something similar may be shown for **#Abs1** and **#Abs2**. Given the concern raised above to maintain sameness of subject-matter across propositional identity claims, we found reason to suspect that **#Imps**, **#Necs**, **#Abs1**, and **#Abs2** admit

of counterexamples. Such considerations motivate a departure from orthodoxy in search of a semantic framework with greater resources for distinguishing between necessarily equivalent propositions than possible worlds.

In order to increase the discriminating power of the semantics, one may relax *Maximality*, evaluating sentences at what may be considered "parts" of possible worlds. In particular, the situation semantics of Barwise and Perry (1981), possibility semantics of Humberstone (1981), and inexact semantics of Fine (2017b,d) each develop related visions of such "parts" of possible worlds. Despite their differences, these theories agree in imposing mereological structure over the space of verifiers at which sentences are to be evaluated, where such verifiers need not determine a truth-value for every sentence of the language, thereby jettisoning the *Maximality* constraint given above. Additionally, the verifiers for sentences must satisfy a monotonicity constraint such that if A is true at s, then A must also be true at every t of which s is a part.

Evaluating sentences at parts of worlds makes sense given our purposes since $A \vee \neg A$ and $B \vee \neg B$ may then diverge in the parts of worlds at which they are true. For instance, consider a part of a world u in which Davide is home that does not specify Valentina's whereabouts, and another part of a world v in which Valentina is home that does not specify Davide's whereabouts. We may then expect that both 'Davide is home' and 'Davide is home or not home' are true at u, where similarly both 'Valentina is home' and 'Valentina is home or not home' are true at v. However, the same cannot be said were u and v reversed, for neither 'Valentina is home' nor 'Valentina is home or not home' is true at a part of a world in which Davide is home such as u, where something similar may be said in evaluating 'Davide is home' and 'Davide is home or not home' at v. In this way, evaluating sentences at parts of worlds which determine the truth-values of some but not all sentences in $\mathcal{L}$ promises to offer greater resources for drawing hyperintensional distinctions.

Instead of attempting to maintain neutrality across these alternatives, I will follow Humberstone (1981) in letting $\mathcal{M} = \langle P, \sqsubseteq, V \rangle$ be a *possibility-model* of $\mathcal{L}$ just in case P is a nonempty set of *possibilities*, $\sqsubseteq$ is a weak partial order on P where '$s \sqsubseteq t$' reads 't is a *refinement* of s', and V is a partial function which takes sentence letters paired with possibilities to truth-values, and satisfies:[9]

Persistence: For any $p \in \mathbb{L}$ and $s, t \in P$, if $V(p, s)$ is defined and $s \sqsubseteq t$, then $V(p, s) = V(p, t)$.

Refinability: For any $p \in \mathbb{L}$ and $s \in P$, if $V(p, s)$ is undefined, then there exists some $t, d \in P$ where $s \sqsubseteq t$ with $V(p, t) = T$ and $s \sqsubseteq d$ with $V(p, d) = F$.

Given the functionality of V, an analogue of *Consistency* continues to hold: no sentence letter $p \in \mathbb{L}$ is both true and false at any possibility $s \in P$. However, by letting V be a partial function, an analogue of *Maximality* need not hold: given any sentence letter $p \in \mathbb{L}$ and possibility $s \in P$, the truth-value of p at s in $\mathcal{M}$ may fail to be defined. Accordingly, the falsity-conditions cannot be identified with the complement of the truth-conditions, and so an analogue of *Falsity* does not carry over. For instance, the possibility u of Davide being at home says nothing of whether Valentina is at home, and so 'Valentina is home' is not defined at u in $\mathcal{M}$. Nevertheless, it follows by *Refinability* that u may be refined so as to specify that Valentina is home, or alternatively, to specify that Valentina is not home. In mereological terms, *Refinability* requires that u is a part of some larger possibilities t and d, where t includes a part v in which Valentina is home, and d includes a part v' in which Valentina is not home. Despite this difference between t and d, it follows from *Persistence* that Davide is at home in both t and d. Generalising beyond this case, evaluation at possibilities is monotonic such that once a possibility s determines the truth-value of a given sentence letter p, any further refinement t of s to specify something more will leave the truth-value of p at t unchanged.

[9] Given the present aim to interpret the non-modal language $\mathcal{L}$, I have left out Humberstone's accessibility relation R, taking certain liberties in the notation that I use.

Rather than interpreting a language such as $\mathcal{L}$ with primitive terms for negation, conjunction, and disjunction, Humberstone defines disjunction in terms of negation and conjunction, letting $\ulcorner A \vee B \urcorner$ abbreviate $\ulcorner \neg(\neg A \wedge \neg B) \urcorner$. For consistency with what follows, I will continue to take disjunction to be a primitive of $\mathcal{L}$ rather than defined, while maintaining the semantics which Humberstone derives for disjunction. In particular, consider the following:

$(p)_{\text{P}}$ $\quad \mathcal{M}, s \rhd p$ *iff* $V(p, s) = T$.

$(\neg)_{\text{P}}$ $\quad \mathcal{M}, s \rhd \neg A$ *iff* there is no $t \in P$ where $s \sqsubseteq t$ and $\mathcal{M}, t \rhd A$.

$(\wedge)_{\text{P}}$ $\quad \mathcal{M}, s \rhd A \wedge B$ *iff* $\mathcal{M}, s \rhd A$ and $\mathcal{M}, s \rhd B$.

$(\vee)_{\text{P}}$ $\quad \mathcal{M}, s \rhd A \vee B$ *iff* for all $t \in P$ where $s \sqsubseteq t$, there is some $d \in P$ where $t \sqsubseteq d$ and either $\mathcal{M}, d \rhd A$ or $\mathcal{M}, d \rhd B$.

As brought out above, the possible worlds semantics makes the falsity-conditions for a sentence the complement in W of the truth-conditions for that sentence. The same does not hold for the possibility semantics since V is permitted to be partial: just because p is not true at s, it does not follow that p is false at s, for p could be neither true nor false at s. Nevertheless, the falsity-conditions for a sentence A can still be recovered from its truth-conditions given *Persistence* and *Refinability* in place of *Maximality*. In particular, consider the following:

`Falsity:` $\mathcal{M}, s \lhd A$ *iff* there is no $t \in P$ where $s \sqsubseteq t$ and $\mathcal{M}, t \rhd A$.

We may then extend *Persistence* and *Refinability* to all $A \in \texttt{ext}(\mathbb{L})$.[10] Whereas (T) in *Negation* follows as an immediate consequence of $(\neg)_{\text{P}}$, (F) follows from the generalisations of *Persistence* and *Refinability*.[11] Even though the falsity-

[10] Both proofs follow by a straightforward induction proof.

[11] *Proof:* Assuming $\mathcal{M}, s \lhd \neg A$, it follows by the generalisation of *Persistence* that (1) $t \lhd \neg A$ for every $t \in P$ where $s \sqsubseteq t$. Additionally, (2) $t \not\rhd \neg A$ for every $t \in P$ where $s \sqsubseteq t$ by `Falsity`, and so $s \not\rhd \neg A$ since $s \sqsubseteq s$. By $(\neg)_{\text{P}}$, $\mathcal{M}, t \rhd A$ for some $t \in P$ where $s \sqsubseteq t$. Assuming for contradiction that $\mathcal{M}, s \lhd A$, it follows by the generalisation of *Persistence* that $\mathcal{M}, t \lhd A$, and so $\mathcal{M}, d \not\rhd A$ for all $d \in P$ where $t \sqsubseteq d$ by `Falsity`. Thus $\mathcal{M}, t \not\rhd A$ in particular, contradicting the above, and so $\mathcal{M}, s \not\lhd A$. Assuming again for contradiction that $\mathcal{M}, s \not\rhd A$, it follows by the generalisation of *Refinability* that there are some $t, d \in P$ where $s \sqsubseteq t, d$ and $\mathcal{M}, t \rhd A$ and $\mathcal{M}, d \lhd A$. By (1), $\mathcal{M}, d \lhd \neg A$, and so both (3) $r \not\rhd \neg A$ for every

conditions have not been identified with the complement of the truth-conditions, the path between truth and falsity remains determined.

Even though the falsity-conditions for a sentence may be determined by its truth-conditions, it will facilitate comparison with what is to come to take the proposition that $A \in \texttt{ext}(\mathbb{L})$ expresses in $\mathcal{M}$ to be $V(A) = \langle V^+(A), V^-(A)\rangle$ where $V^+(A) = \{s \in P : \mathcal{M}, s \rhd A\}$ and $V^-(A) = \{s \in P : \mathcal{M}, s \lhd A\}$. We may then provide the following semantic clause for propositional identity:

$(\equiv)_{\text{P}}$ $\quad \mathcal{M} \rhd A \equiv B$ *iff* $V(A) = V(B)$.

Given any $\{\varphi\} \cup \Gamma \subseteq \texttt{id}(\mathbb{L})$, we may take φ to be a *logical consequence*$_{\text{P}}$ of Γ, i.e. $\Gamma \rhd_{\text{P}} \varphi$, just in case for any possibility-model $\mathcal{M}$, if $\mathcal{M} \rhd \chi$ for all $\chi \in \Gamma$, then $\mathcal{M} \rhd \varphi$. Additionally, φ is *valid*$_{\text{P}}$ just in case $\rhd_{\text{P}} \varphi$. We may then observe that although $V^+(\neg A) = P/V^+(A)$ need not hold, the semantics for disjunction and conjunction nevertheless make $V^+(A \vee \neg A) = V^-(A \wedge \neg A) = P$ and $V^+(A \wedge \neg A) = V^-(A \vee \neg A) = \varnothing$ for any $A \in \texttt{ext}(\mathbb{L})$ and possibility-model $\mathcal{M} = \langle P, \sqsubseteq, V\rangle$, and so both **#Necs** and **#Imps** come out valid$_{\text{P}}$.

Recall that our motivation for relaxing *Maximality* by considering "parts" of worlds instead of worlds in their entirety was in an attempt distinguish between necessary propositions as well as between impossible propositions. In particular, the hope was that the verifiers for $A \vee \neg A$ and $B \vee \neg B$ could in principle diverge. However, we may show that no possibility falsifies $A \vee \neg A$, and so by the generalisation of *Refinability*, every possibility verifies $A \vee \neg A$.[12]

$r \in P$ where $d \sqsubseteq r$, and (4) $r \not\rhd A$ for every $r \in P$ where $d \sqsubseteq r$. From (3), $\mathcal{M}, d \not\rhd \neg A$ since $d \sqsubseteq d$, and so by $(\neg)_{\text{P}}$, $\mathcal{M}, r \rhd A$ for some $r \in P$ where $d \sqsubseteq r$. However, from (4), $\mathcal{M}, r \not\rhd A$ contradicting the above, and so $\mathcal{M}, s \rhd A$ as desired.

Assuming for contraposition that $\mathcal{M}, s \not\lhd \neg A$, it follows by `Falsity` that (5) $\mathcal{M}, t \rhd \neg A$ for some $t \in P$ where $s \sqsubseteq t$. By $(\neg)_{\text{P}}$, $\mathcal{M}, d \not\rhd A$ for all $d \in P$ where $t \sqsubseteq d$, and so $\mathcal{M}, t \not\rhd A$ since $t \sqsubseteq t$. Assume for contradiction that $\mathcal{M}, s \rhd A$, and so $\mathcal{M}, t \rhd A$ by the generalisation of *Persistence*, contradicting the above. Thus $\mathcal{M}, s \not\rhd A$ as desired. □

[12] *Proof:* Assuming $\mathcal{M}, s \lhd A \vee \neg A$, it follows that there is no $t \in P$ where $s \sqsubseteq t$ and $\mathcal{M}, t \rhd A \vee \neg A$, and so $\mathcal{M}, s \not\rhd A \vee \neg A$ in particular. By $(\vee)_{\text{P}}$ there is some $t \in P$ where $s \sqsubseteq t$ and for all $d \in P$, if $t \sqsubseteq d$, then $\mathcal{M}, d \not\rhd A$ and $\mathcal{M}, d \not\rhd \neg A$. So $\mathcal{M}, t \not\rhd A$ and $\mathcal{M}, t \not\rhd \neg A$ since $t \sqsubseteq t$. By $(\neg)_{\text{P}}$, there is some $r \in P$ where $t \sqsubseteq r$ and $\mathcal{M}, r \rhd A$. Since $t \sqsubseteq r$, it follows that $\mathcal{M}, r \not\rhd A$ by the above. Thus $\mathcal{M}, s \not\lhd A \vee \neg A$ for any $s \in P$. □

These results cannot be avoided by replacing Humberstone's semantics for disjunction with the following alternative without also giving up *Refinability*:

$(\vee)^*_{\text{P}}$ $\quad \mathcal{M}, s \rhd A \vee B$ *iff* $\mathcal{M}, s \rhd A$ or $\mathcal{M}, s \rhd B$.[13]

Despite having given up *Maximality* and evaluating sentence letters at parts of worlds rather than worlds taken in their entirety, imposing *Refinability* requires the possibility semantics to make every possibility a verifier for $A \vee \neg A$ and a falsifier for $A \wedge \neg A$. Accordingly, there is no latitude for difference between the propositions expressed by $A \vee \neg A$ and $B \vee \neg B$, or by $A \wedge \neg A$ and $B \wedge \neg B$. It is for this reason that both **#Necs** and **#Imps** come out valid$_{\text{P}}$.

In order to increase the discriminating power of the semantics, suppose that one were to revise the possibility semantics by rejecting *Refinability* in attempt to maintain $(\vee)^*_{\text{P}}$ in place of $(\vee)_{\text{P}}$. Accordingly, $A \vee \neg A$ would be verified by just those possibilities which verify either disjunct. Since not every possibility verifies either A or $\neg A$, the verifiers for $A \vee \neg A$ and $B \vee \neg B$ may diverge, providing counterexamples to **#Necs** as desired.[14] Despite these advantages, we may nevertheless observe that both **#Abs1** and **#Abs2** come out valid$_{\text{P}}$. Focusing on the former, consider the following biconditionals:

$$\begin{aligned} \mathcal{M}, s \rhd A \vee (A \wedge B) \;\; &\textit{iff} \;\; \mathcal{M}, s \rhd A \text{ or } \mathcal{M}, s \rhd A \wedge B \\ (*) \;\; &\textit{iff} \;\; \mathcal{M}, s \rhd A \text{ or both: } \mathcal{M}, s \rhd A \text{ and } \mathcal{M}, s \rhd B \\ &\textit{iff} \;\; \mathcal{M}, s \rhd A. \end{aligned}$$

The transition from the truth of $A \wedge B$ at s to the truth of both A and B at s

[13] We may produce a counterexample to *Refinability* by letting $\mathcal{M}_C = \langle P_C, \subseteq, V_C \rangle$ be a possibility-model where $P_C = \{\{a\}, \{a,b\}, \{a,c\}\}$ and $V_C(p, \{a,b\}) = T$, $V_C(p, \{a,c\}) = F$, $V_C(q, \{a,b\}) = F$, $V_C(q, \{a,c\}) = T$, but both $V_C(p, \{a\})$ and $V_C(q, \{a\})$ are undefined. It follows from $(\vee)^*_{\text{P}}$ that $\mathcal{M}_C, \{a\} \not\rhd p \vee q$ and $\mathcal{M}_C, \{a\} \not\lhd p \vee q$ by `Falsity`, but there is no extension t of $\{a\}$ where $\mathcal{M}_C, t \lhd p \vee q$, thereby contradicting *Refinability*. ◇

[14] The same cannot be said for the falsifiers for $A \wedge \neg A$, indicating an awkward asymmetry which arises from the semantics for negation. Nevertheless, consideration of this alternative to Humberstone's semantics will help to motivate key features of Fine's state semantics.

in $(*)$ follows from the possibility semantics for conjunction $(\wedge)_{\text{P}}$. For instance, assuming that 'Valentina and Davide are home' is true at the possibility in which Valentina and Davide are both home, it follows by $(\wedge)_{\text{P}}$ that both 'Valentina is home' and 'Davide is home' are true at s. Although s certainly includes enough for both 'Valentina is home' and 'Davide is home' to be true, we may observe that in each case s also includes something extraneous: s not only specifies that Valentina is home, but also specifies that Davide is home, where the latter does not participate in verifying 'Valentina is home'. Far from an isolated instance, it follows from the generalisation of *Persistence* that A is true at t whenever A is true at s and s is a part of t even if t includes parts irrelevant to A. Thus possibilities need only include enough for the sentence in question to be true, where not every part of a possibility is required to participate in verifying the sentence in question. Something similar holds for falsification, whereby **#Abs2** may also be shown to be valid_{P}.

Given that *Persistence* applies to all $A \in \texttt{ext}(\mathbb{L})$, no simple tweak can be made to the semantics which avoids validating **#Abs1** and **#Abs2**, at least insofar as $(\vee)^*_{\text{P}}$ is to be maintained. After all, if both A and B are true at s, it must be admitted that $A \wedge B$ is true at s. Conversely, if $A \wedge B$ is true at some possibility s, then there must be some parts $t, d \sqsubseteq s$ where A is true t and B is true at d, and so both A and B will be true at s by the generalisation of *Persistence*. Thus if one accepts *Persistence*, the semantics for conjunction given above must be maintained, and so $(*)$ in the argument above follows as a consequence. By contrast, one may block $(*)$ by claiming that although $A \wedge B$ is true at s, neither A nor B need be true at s on account of s possibly including parts which are irrelevant to A and B considered individually. Additionally, imposing the requirement that verifiers and falsifiers be wholly relevant to the sentences that they verify or falsify provides a principled reason to reject both *Persistence* and *Refinability*. As observed above, it follows from

Persistence that possibilities will typically contain parts which are irrelevant to the sentences that they verify and falsify. Similarly, given any possibility s at which a sentence A is undefined, it follows by the generalisation of *Refinability* that there are extensions t and d of s at which A is true and false respectively, even though s may play no role in t and d verifying and falsifying A.

Given the aim to distinguish necessarily equivalent propositions which nevertheless differ in subject-matter from §3.2, where differences in relevance indicate differences in subject-matter, it is not surprising that considerations of relevance will figure in an appropriate semantics. After all, the classical semantic clauses for negation, conjunction, and disjunction all defer to corresponding notions in the meta-language, and so it is reasonable to expect that something similar will hold for relevance. However, in imposing a relevance constraint and giving up *Persistence* and *Refinability*, one can no longer derive (F) in *Negation* from `Falsity` as above, and so the path between the truth-conditions and falsity-conditions for each sentence cannot be maintained.

In light of these considerations, we may conclude that although the modified possibility semantics makes progress towards providing a semantic framework which can track subject-matter, the possibility semantics does not go far enough. In particular, it is reasonable to expect that an adequate semantics will require the verifiers and falsifiers to be wholly relevant to the sentences that they verify or falsify respectively, at least insofar as **#Abs1** and **#Abs2** are to admit of counterexamples. Accordingly, analogues of *Persistence* and *Refinability* should not be maintained, where it follows as a result that *Negation* cannot be derived from an analogue of `Falsity`. However, instead of defining falsification in terms of verification in a manner which satisfies *Negation*, one may seek instead to provide a *bilateral semantics* which includes recursive clauses for both verification and falsification which takes *Negation* itself to provide a blueprint of the semantics for negation. These considerations motive

the adoption of the nonmonotonic notions of exact verification and exact falsification developed by Fine (2017b,c,d) which are sensitive to considerations of relevance, providing recursive clauses for exact falsification in addition to exact verification which nevertheless satisfy *Negation*. In the following section, I will develop a particular version of Fine's state semantics, presenting counterexamples to **#Necs**, **#Imps**, **#Abs1**, and **#Abs2** as desired.

3.5 State Semantics

Instead of merely requiring the verifiers and falsifiers for a sentence to be possibilities which include enough for the sentence in question to be true or false respectively, Fine (2017b,c,d) has developed the stricter notions of *exact verification* and *exact falsification*, where the exact verifiers and falsifiers must be wholly relevant to the sentences which they exactly verify or falsify.[15] By contrast, Fine (2017d) refers to weaker notions of verification and falsification closely related to the possibility semantics above as *inexact*, writing:

> With inexact verification, the state should at least be partially relevant to the statement; and with exact verification, it should be wholly relevant. Thus the presence of rain will be an exact verifier for the statement 'it is rainy'; the presence of wind and rain will be an inexact verifier for the statement 'it is rainy,' though not an exact verifier. (p. 558)

Given the added relevance requirement, it follows that exact verification and falsification are nonmonotonic, and so analogues of *Persistence* and *Refinability* do not hold. Given any exact verifier s for a sentence A and extension t of s, we cannot expect that t will also be wholly relevant to A, and so t may fail to exactly verify A. Similarly, given any s at which A is undefined, we should not expect that there will be extensions t and d of s which exactly verify and falsify A respectively, for s may not be wholly relevant to A.

[15] See Fine (2012c,a, 2014, 2013, 2016, 2017a, 2018a,b, 2020) for further development and applications of the state semantics.

In order to appreciate the differences between Fine's (2017b,c,d) state semantics and the possibility semantics, I will begin with the core definitions, successively motivating a range of additional constraints. Letting a *state space* be any ordered pair $\mathcal{S} = \langle S, \sqsubseteq \rangle$, where S is the set of *states* and $\sqsubseteq$ is a complete partial order on S, we may identify the *fusion* of any set of states $X \subseteq S$ with the least upper bound of X, articulating a further constraint in these terms:

Upper Bound: s is an *upper bound* of $X \subseteq S$ *iff* $x \sqsubseteq s$ for every $x \in X$.

Least Upper Bound: s is a *least upper bound* of $X \subseteq S$ *iff* s is an upper bound of X, and $s \sqsubseteq y$ for every upper bound y of X.

When it exists, we may take $\bigsqcup X$ to be *the* fusion of states in X, as uniqueness is readily proven. Given a state space $\mathcal{S} = \langle S, \sqsubseteq \rangle$, an $\mathcal{S}$-model is any ordered triple $\mathcal{M} = \langle S, \sqsubseteq, |\cdot| \rangle$, where $|p| = \langle |p|^+, |p|^- \rangle$ with $|p|^\pm \subseteq S$ for every $p \in \mathbb{L}$. We may then consider the following additional conditions on $\mathcal{S}$-models:

Full: An $\mathcal{S}$-model is *full iff* $\bigsqcup X \in S$ for any $X \subseteq S$.

Closed: An $\mathcal{S}$-model is *closed iff* $\bigsqcup X \in |p|^\pm$ for all nonempty $X \subseteq |p|^\pm$.

Disjoint: An $\mathcal{S}$-model is *disjoint iff* $|p|^+ \cap |p|^- = \varnothing$ for all $p \in \mathbb{L}$.

Nonvacuous: An $\mathcal{S}$-model is *nonvacuous iff* both $|p|^\pm$ are nonempty for all $p \in \mathbb{L}$.

Given any possibility-model $\mathcal{M} = \langle P, \sqsubseteq, V \rangle$, we may define a corresponding model $\mathcal{M}^* = \langle P, \sqsubseteq, |\cdot|_V \rangle$ in the state semantics where for every $p \in \mathbb{L}$, both $|p|^+_V = \{s \in P : V(p, s) = T\}$ and $|p|^-_V = \{s \in P : V(p, s) = F\}$. Given that V is a function, it follows that $|p|^+_V \cap |p|^-_V = \varnothing$ for all $p \in \mathbb{L}$, and so $\mathcal{M}^*$ is disjoint. Conversely, for any disjoint model $\mathcal{M} = \langle S, \sqsubseteq, |\cdot| \rangle$ in the state semantics, we may define a corresponding possibility-model $\mathcal{M}^\dagger = \langle S, \sqsubseteq, V_{|\cdot|} \rangle$ where for every $p \in \mathbb{L}$: (1) $V(p, s) = T$ if $s \in |p|^+$; (2) $V(p, s) = F$ if $s \in |p|^-$; and (3) $V(p, s)$ is undefined otherwise. By contrast, non-disjoint state semantics models do not correspond to any possibility-model in the possibility semantics for insofar as V is a function, it cannot be that both $V(p, s) = T$ and $V(p, s) = F$. Accordingly,

the state semantics admits of a much wider range of models of $\mathcal{L}$ than the possibility semantics was designed to accommodate, where these extra degrees of freedom will play a crucial role in the state semantics below.

In order to motivate the requirement that state spaces be complete, it will be important to attend to the manner in which the framework above is intended to accommodate matters of relevance. Instead of considering which sentences are true or false in which possibilities, exact verification and falsification is a matter of which sentences are "made" true or false by which states, where the truth-makers and falsity-makers must be wholly relevant to the sentences they make true or false, respectively. For instance, the state t of Julieta thinking exactly verifies the sentence 'Julieta is thinking'. Similarly, the state d of Julieta writing exactly verifies the sentence 'Julieta is writing'. Given the states t and d, we may contemplate their fusion $\bigsqcup\{t, d\}$ which we may abbreviate as $t.d$. Although $t.d$ fails to be wholly relevant to either 'Julieta is thinking' or 'Julieta is writing', we may nevertheless admit that $t.d$ exactly verifies the conjunction 'Julieta is thinking and writing'. More generally, given an exact verifier u for a sentence A, and an exact verifier v a sentence B, the fusion $u.v$ exactly verifies $A \wedge B$, and so $u.v \in S$. Generalising on u and v justifies the requirement that state spaces be complete partial orders as claimed above.

It is worth emphasising that in contrast to the possibility semantics, state spaces will typically include a vast array of impossible states on account of completeness. For instance, given a state u which exactly verifies 'I am sitting', and a state v which exact verifies 'I am standing', there will be a unique fusion state $u.v$ which exactly verifies 'I am sitting and standing'. This is not to claim that $u.v$ could obtain, or that 'I am sitting and standing' could be true. Rather, one may explain why 'I am sitting and standing' on its intended interpretation is necessarily false by appeal to the fact that 'I am sitting and standing' only has impossible verifiers. By contrast, there are no possibilities in which I am

sitting and standing despite the fact that there are possibilities in which I am sitting, as well as possibilities in which I am standing.

Given that analogues of *Persistence* and *Refinability* have not been extended to the state semantics, the falsity-conditions cannot be determined from the truth-conditions in the manner given by `Falsity`. Rather, we should expect to provide separate semantic clauses for both truth and falsity throughout the semantics. With these expectations in mind, we may follow Fine (2017d, §5) in recursively defining exact verification $\Vdash$ and exact falsification $\dashv\Vdash$ as distinct primitives, referring to the result as the *exclusive semantics* for $\mathcal{L}$:

$(p)^+$ $\mathcal{M}, s \Vdash p$ *iff* $s \in |p|^+$. $(\neg)^+$ $\mathcal{M}, s \Vdash \neg A$ *iff* $\mathcal{M}, s \dashv\Vdash A$.

$(p)^-$ $\mathcal{M}, s \dashv\Vdash p$ *iff* $s \in |p|^-$. $(\neg)^-$ $\mathcal{M}, s \dashv\Vdash \neg A$ *iff* $\mathcal{M}, s \Vdash A$.

$(\wedge)^+$ $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = t.d$ where $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$.

$(\wedge)^-_{\text{E}}$ $\mathcal{M}, s \dashv\Vdash A \wedge B$ *iff* $\mathcal{M}, s \dashv\Vdash A$ or $\mathcal{M}, s \dashv\Vdash B$.

$(\vee)^+_{\text{E}}$ $\mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$.

$(\vee)^-$ $\mathcal{M}, s \dashv\Vdash A \vee B$ *iff* $s = t.d$ where $\mathcal{M}, t \dashv\Vdash A$ and $\mathcal{M}, d \dashv\Vdash B$.

Rather than deriving *Negation* from Humberstone's semantics for negation together with `Falsity`, *Persistence*, and *Refinability*, the clauses for negation directly encode the spirit of *Negation* into the semantics. Additionally, the exact verification clause for conjunction has been elaborated from what was given in both the possible worlds semantics and possibility semantics in order to respect relevance in the manner outlined above, where something similar may be said for the exact falsification clause for disjunction. Lastly, the exact verification clause for disjunction has returned to the simple form included in the possible worlds semantics for disjunction such that the exact verifiers for $A \vee \neg A$ may differ from $B \vee \neg B$, where something similar may be said for the exact falsifiers for $A \wedge \neg A$ and $B \wedge \neg B$ given the falsification clause for conjunction. It remains to assess whether the exact semantics can provide an adequate theory of propositions given our present aims.

Without imposing any further constraints on the space of models under consideration, we may observe with Fine (2017d, §6) that $A \wedge A$ and A need not have the same exact verifiers. For instance, supposing that $p \in \mathbb{L}$ is only exactly verified by s and t where neither $s \sqsubseteq t$ nor $t \sqsubseteq s$, it follows that $p \wedge p$ is exactly verified by $s.t$ in addition to both s and t. Insofar as sentences which express the same proposition are to have the same exact verifiers and falsifiers, it follows that $p \wedge p$ and p express distinct propositions. However, insofar as A and $A \wedge A$ agree in subject-matter and modal profile for any A, a conservative methodology recommends drawing fewer rather than more distinctions without clear motivation. In the absence of systematic reasons for rejecting $A \wedge A \equiv A$, I will take there to be a *de facto* presumption in favour of adopting $A \equiv A \wedge A$, and so A and $A \wedge A$ must have exactly the same exact verifiers and falsifiers. In order to achieve this result, I will follow Fine in requiring that whenever s and t exactly verify some $p \in \mathbb{L}$, the fusion $s.t$ is also an exact verifier for p. Imposing such a constraint rules out the case considered above where p is exactly verified by just s and t but not their fusion $s.t$. The same may also be said for the exact falsifiers for $p \vee p$, where a fusion $s.t$ of exact falsifiers s and t for p may also be said to exactly falsify $p \vee p$. More generally, I will require the exact verifiers and falsifiers for any sentence letter to be closed under fusion by restricting attention to the closed models of $\mathcal{L}$.

Despite requiring states to be wholly relevant to the sentences which they exactly verify or falsify, the considerations above admit of exact verifiers and falsifiers for sentences which contain more than what is needed to make those sentences true or false, respectively. For instance, if the fusion of two exact verifiers for p is also an exact verifier for p, then this fusion contains a proper part which is an exact verifier for p. Accordingly, it is worth elucidating the manner in which the verifiers and falsifiers for a sentence may be said to be exact by means of the following constraint, leaving the $\mathcal{S}$-model $\mathcal{M}$ implicit:

Exactness: A state s *exactly verifies* (*falsifies*) A *iff* $s = \bigsqcup X$ for some nonempty $X \subseteq S$ where every $x \in X$ exactly verifies (falsifies) A.

Rather than providing a definition of exact verification and falsification, the constraint above aims to characterise the nature of exactness in a manner which both allows for the kind of overdetermination brought out above, while also requiring that every part of an exact verifier or falsifier contribute to verifying of falsifying the sentence in question. However, insofar as the exact verifiers (falsifiers) for a sentence are permitted to overdetermine the truth (falsity) of that sentence, it is not clear what would motivate a restriction to finite degrees of overdetermination. For instance, given any prime number $x \in \mathbb{P}$, we may take s_x to be the state of x being a prime number, where $X_{\mathbb{P}} = \{s_x : x \in \mathbb{P}\}$. Insofar as each $s_x \in X_{\mathbb{P}}$ is an exact verifier for 'There is at least one prime number', we may admit that $\bigsqcup Y$ exactly verifies 'There is at least one prime number' for any nonempty $Y \subseteq X_{\mathbb{P}}$.[16] However, if infinite fusions of exact verifiers (falsifiers) for a sentence are also exact verifiers (falsifiers) for that sentence, then such infinite fusions must belong to S. Accordingly, I will follow Fine in restricting attention throughout what follows to closed full models of $\mathcal{L}$ such that the fusion of every set of states is also a state in the model, and the fusion of any nonempty set of exact verifiers (falsifiers) for a sentence letter is also an exact verifier (falsifier) for that sentence.

Having provided motivation for limiting consideration to models which are both closed and full, we find a corresponding reason to revise the exclusive semantics given above. In particular, consider the following property:

Uniformity: A class K of models of $\mathcal{L}$ is *uniform iff* for any $\mathcal{S}$-model $\mathcal{M} \in K$ and $A \in \mathtt{ext}(\mathbb{L})$, there is a $\mathcal{S}$-model $\mathcal{M}_\star \in K$ and $p \in \mathbb{L}$ where both $|p|^+_\star = \{s \in \mathcal{M} : \mathcal{M}, s \Vdash A\}$ and $|p|^-_\star = \{s \in \mathcal{M} : \mathcal{M}, s \dashv\Vdash A\}$.

[16] Considerations of gunk also motivate taking fusions of sets of exact verifiers of any non-zero cardinality to be exact verifiers. If a gunky ice cube is entirely pink, then the state of any part of it being pink will exactly verify the sentence 'Part of the ice cube is pink', where if a is a part of b, then the state of a being pink is also part of the state of b being pink.

It is natural to expect the class of models over which $\mathcal{L}$ is to be interpreted to be uniform since nothing about the sentence letters should prevent them from expressing the same propositions expressed by the complex sentences in the language. However, as it stands the class of models of $\mathcal{L}$ which are both closed and full fails to be uniform on account of the exact verification clause for disjunction and exact falsification clause for conjunction. For instance, consider a closed full $\mathcal{S}$-model $\mathcal{M}$ where s is the only exact verifier for p_1, and that t is the only exact verifier for p_2, and neither $s \sqsubseteq t$ nor $t \sqsubseteq s$. Given $(\vee)^+_{\text{E}}$, both s and t are exact verifiers for $p_1 \vee p_2$, but $s.t$ fails to be an exact verifier for $p_1 \vee p_2$. It follows that $\{s \in \mathcal{M} : \mathcal{M}, s \Vdash p_1 \vee p_2\}$ is not closed under nonempty fusion, and so there is no closed $\mathcal{S}$-model $\mathcal{M}_\star$ where $|p|^+_\star = \{s \in \mathcal{M} : \mathcal{M}, s \Vdash p_1 \vee p_2\}$ for $p \in \mathbb{L}$. Thus the class of closed full models fails to be uniform, where a similar counterexample may equally be derived from $(\wedge)^-_{\text{E}}$. Given a closed full $\mathcal{S}$-model $\mathcal{M}$ where s is the only exact falsifier for p_1, and t is the only exact falsifier for p_2, and neither $s \sqsubseteq t$ nor $t \sqsubseteq s$, it follows from $(\wedge)^-_{\text{E}}$ that both s and t are exact falsifiers for $p_1 \wedge p_2$, but $s.t$ fails to be an exact falsifier for $p_1 \wedge p_2$. Thus $\{s \in \mathcal{M} : \mathcal{M}, s \dashv\Vdash p_1 \wedge p_2\}$ is not closed under nonempty fusion, and so there is no closed $\mathcal{S}$-model $\mathcal{M}_\star$ where $|p|^-_\star = \{s \in \mathcal{M} : \mathcal{M}, s \dashv\Vdash p_1 \wedge p_2\}$ for any $p \in \mathbb{L}$. On these two counts, we may conclude that the class of closed full models fails to be uniform given the exclusive semantics.

In order to maintain uniformity of the class of closed full models, I will replace $(\wedge)^-_{\text{E}}$ and $(\vee)^+_{\text{E}}$ in the exclusive semantics with the alternative below, following Fine (2016, 2017d) in referring to the result as the *inclusive semantics*:

$(\wedge)^-$ $\quad \mathcal{M}, s \dashv\Vdash A \wedge B$ *iff* $\mathcal{M}, s \dashv\Vdash A$ or $\mathcal{M}, s \dashv\Vdash B$ or $\mathcal{M}, s \dashv\Vdash A \vee B$.

$(\vee)^+$ $\quad \mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$ or $\mathcal{M}, s \Vdash A \wedge B$.

Whereas disjunction and conjunction failed to preserve closure under fusion on the exclusive semantics, adopting the inclusive semantics extends uniformity

to the class of closed full models of $\mathcal{L}$.[17] Additionally, the inclusive semantics respects the claims that: (T) any exact verifier for a conjunction $A \wedge B$ will also be an exact verifier for the disjunction $A \vee B$; and, (F) any exact falsifier for $A \vee B$ will also be an exact falsifier for $A \wedge B$. Although by no means beyond dispute, such claims are extremely natural, providing further reason to amend the exclusive semantics originally presented.

The considerations above provide reason to restrict attention to the class of closed full models of $\mathcal{L}$, adopting the inclusive semantics in place of the exclusive semantics originally considered above. It remains to provide a semantics for propositional identity. As in the possibility semantics, I will take the proposition that $A \in \texttt{ext}(\mathbb{L})$ expresses in $\mathcal{M}$ to be defined as $|A| = \langle |A|^+, |A|^- \rangle$ where $|A|^+ = \{s \in S : \mathcal{M}, s \Vdash A\}$ and $|A|^- = \{s \in S : \mathcal{M}, s \dashv\Vdash A\}$. Given these definitions, we may provide the following semantics for propositional identity:

($\equiv$) $\quad \mathcal{M} \Vdash A \equiv B$ *iff* $|A| = |B|$.

Given the above, we may define falsity on a model in terms of truth at a model, obviating the need to articulate a distinct falsification clause for propositional identity. For any $\{\varphi\} \cup \Gamma \subseteq \texttt{id}(\mathbb{L})$ and class of models $\mathcal{K}$, φ is a *$\mathcal{K}$-logical consequence* of Γ, i.e. $\Gamma \Vdash_{\mathcal{K}} \varphi$, just in case for any $\mathcal{M} \in \mathcal{K}$, if $\mathcal{M} \Vdash \chi$ for all $\chi \in \Gamma$, then $\mathcal{M}, w \Vdash \varphi$. Additionally, φ is *$\mathcal{K}$-valid* just in case $\Vdash_{\mathcal{K}} \varphi$.

Given the inclusive semantics and letting $\mathcal{C}$ be the class of all closed full models of $\mathcal{L}$, we may observe that far from an obscure artefact of the framework, impossibilities do important work in drawing hyperintensional distinctions. In particular, if v is an exact verifier for A, and u is an exact falsifier for A, it follows from $(\wedge)^+$ that $u.v$ is an exact verifier for $A \wedge \neg A$, and $u.v$ is an exact falsifier for $A \vee \neg A$. However, assuming that A and B do not share exact

[17] *Sketch:* The proof goes by induction on complexity, where the main cases to consider are $(\wedge)^+_s$ and $(\vee)^-_s$. Given some nonempty $X = \{x_i : i \in I\}$ where $X \subseteq |A \wedge B|^+$, it follows that $x_i = t_i.d_i$ for all $x_i \in X$ with $t_i \in |A|^+$ and $d_i \in |B|^+$. By the induction hypothesis, $t = \bigsqcup\{t_i : i \in I\} \in |A|^+$ and $d = \bigsqcup\{d_i : i \in I\} \in |B|^+$. It is readily shown that $x = t.d$ where $t.d \in |A \wedge B|^+$ by $(\wedge)^+_s$, and so $|A \wedge B|^+$ is closed. Something similar holds for $|A \wedge B|^-$.

verifiers and falsifiers in common, $u.v$ will neither exactly verify $B \wedge \neg B$, nor exactly falsify $B \vee \neg B$. Thus the exact verifiers for $A \wedge \neg A$ and $B \wedge \neg B$ may diverge, where the same may be said of the exact falsifiers for $A \vee \neg A$ and $B \vee \neg B$, and so both **#Imps** and **#Necs** admit of counterexamples. Additionally, we may observe that insofar as we are to accept the inclusive semantics, the class of disjoint models fails to be uniform, providing decisive reason to consider the wider range of models in $\mathcal{C}$.[18]

In addition to admitting counterexamples to **#Imps** and **#Necs**, neither **#Abs1** nor **#Abs2** are $\mathcal{C}$-valid. In particular, we may consider the model $\mathcal{M}_A = \langle S_A, \subseteq, |\cdot|_A \rangle$ where $S_A = \mathcal{P}(\{a, b, c, d\})$ with $|p_1|_A = \langle \{\{a\}\}, \{\{b\}\} \rangle$ and $|p_2|_A = \langle \{\{c\}\}, \{\{d\}\} \rangle$, for pairwise distinct a, b, c, and d. We may then derive:

$$\begin{aligned}
|p_1 \wedge p_2|_A &= \langle \{\{a, c\}\}, \{\{b\}, \{d\}, \{b, d\}\} \rangle \\
|p_1 \vee p_2|_A &= \langle \{\{a\}, \{c\}, \{a, c\}\}, \{\{b, d\}\} \rangle \\
|p_1 \vee (p_1 \wedge p_2)|_A &= \langle \{\{a\}, \{a, c\}\}, \{\{b\}, \{b, d\}\} \rangle \\
|p_1 \wedge (p_2 \vee p_2)|_A &= \langle \{\{a\}, \{a, c\}\}, \{\{b\}, \{b, d\}\} \rangle.
\end{aligned}$$

Since $\{\{a\}\} \neq \{\{a\}, \{a, c\}\}$ and $\{\{b\}\} \neq \{\{b\}, \{b, d\}\}$, it follows from $(\equiv)$ that both $\mathcal{M}_A \not\Vdash p_1 \equiv p_1 \vee (p_1 \wedge p_2)$ and $\mathcal{M}_A \not\Vdash p_1 \equiv p_1 \wedge (p_1 \vee p_2)$. Since $\mathcal{M}_A \in \mathcal{C}$, it follows that neither **#Abs1** nor **#Abs2** are $\mathcal{C}$-valid as claimed above. We may add texture to the present counterexample by taking p_1 to be 'It is raining' and p_2 to be 'It is snowing', where $\{a\}$ is a rainy weather state and $\{c\}$ is a snowy weather state. It follows by the inclusive semantics that although $\{a, c\}$ exactly verifies 'It is raining and snowing', and so $\{a, c\}$ also exactly verifies 'It is raining or both raining and snowing', the same cannot be said for 'It is raining', thereby indicating a discrepancy between the exact verifiers for 'It is

[18] *Proof:* Letting $\mathcal{M} \in \mathcal{C}$ and $A \in \texttt{ext}(\mathbb{L})$ where there is some $s \in |A|^+ \cup |A|^-$, it follows that either $|A \wedge \neg A|^+ \cap |A \wedge \neg A|^- \neq \varnothing$ or $|A \vee \neg A|^+ \cap |A \vee \neg A|^- \neq \varnothing$, and so if $\mathcal{C}$ is uniform, there is some model $\mathcal{M}^*$ where $|p|^+ \cap |p|^- \neq \varnothing$. Thus $\mathcal{M}^*$ is non-disjoint. □

raining' and 'It is raining or both raining and snowing'. Similar considerations show that 'It is raining' and 'It is raining and either raining or snowing' have different verifiers, and so do not express the same proposition.

In contrast to both the possible worlds semantics and possibility semantics considered above, the inclusive semantics provides the resources needed to distinguish between the propositions expressed by A and both $A \vee (A \wedge B)$ and $A \wedge (A \vee B)$. The counterexamples to the absorption laws are in accordance with the aim set out in §3.2 to provide a theory of propositional identity which respects sameness of subject-matter in addition to modal profile without drawing any extraneous distinctions. However, the inclusive semantics also makes the following distribution laws invalid over $\mathcal{C}$:

#Dist1: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

#Dist2: $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

As I show in §3.8, the distribution laws cannot be ruled out by considerations of subject-matter, and of course do not contravene necessary equivalence. Thus I will take there to be a *de facto* presumption in favour of admitting **#Dist1** and **#Dist2** just as with $A \wedge A \equiv A$ considered above. Nevertheless, I will show in the following section that there are strong abductive reasons for maintaining the inclusive semantics, and so rejecting both of the distribution laws.

3.6 Distribution Laws

The inclusive semantics given above has been motivated by the aim to admit counterexamples to **#Necs**, **#Imps**, **#Abs1**, and **#Abs2**. Nevertheless, the definition of $\mathcal{C}$-logical consequence is perfectly general, yielding judgements about the $\mathcal{C}$-validity of each identity sentence in $\mathcal{L}$. Accordingly, we may ask whether $\mathcal{C}$-validity yields an extensionally adequate theory of propositional identity given the aim to track subject-matter.

Instead of attempting to provide a determinate answer for all identity sentences in $\mathtt{id}(\mathbb{L})$, I will devote the present section to the distribution laws indicated above. In particular, we may begin by observing that the inclusive semantics provides the following counterexamples to **#Dist1** and **#Dist2**. Letting $\mathcal{M}_D = \langle S_D, \subseteq, |\cdot|_D \rangle$ with $S_D = \mathcal{P}(\{a, b, c, d, e, f\})$, we may suppose that $|p_1|_D = \langle \{\{a\}\}, \{\{b\}\} \rangle$, $|p_2|_D = \langle \{\{c\}\}, \{\{d\}\} \rangle$, and $|p_3|_D = \langle \{\{e\}\}, \{\{f\}\} \rangle$, for pairwise distinct a, b, c, d, e, and f. We may derive the following identities:

$$\begin{aligned}
|p_1 \vee (p_2 \wedge p_3)|_D &= \langle \underline{\{\{a\}, \{c, e\}, \{a, c, e\}\}}, \{\{b, d\}, \{b, f\}, \{b, d, f\}\} \rangle \\
|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_D &= \langle \underline{\{\{a\}, \{a, c\}, \{a, e\}, \{c, e\}, \{a, c, e\}\}}, \{\{b, d\}, \{b, f\}, \{b, d, f\}\} \rangle \\
|p_1 \wedge (p_2 \vee p_3)|_D &= \langle \{\{a, c\}, \{a, e\}, \{a, c, e\}\}, \underline{\{\{b\}, \{d, f\}, \{b, d, f\}\}} \rangle \\
|(p_1 \wedge p_2) \vee (p_1 \wedge p_3)|_D &= \langle \{\{a, c\}, \{a, e\}, \{a, c, e\}\}, \underline{\{\{b\}, \{b, d\}, \{b, f\}, \{d, f\}, \{b, d, f\}\}} \rangle.
\end{aligned}$$

Given that the underlined sets of exact verifiers (respectively falsifiers) are not identical, we may conclude by $(\equiv)$ that $\mathcal{M}_D \nVdash A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ and $\mathcal{M}_D \nVdash A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$, thereby proving that neither **#Dist1** nor **#Dist2** is $\mathcal{C}$-valid. However, in contrast to the absorption laws, it cannot be shown that **#Dist1** and **#Dist2** ought to be excluded from a logic of propositional identity on account of considerations of subject-matter. This raises a question about what broader systematic considerations might be brought to bear upon the present issue, motivating either the inclusion or exclusion of the distribution laws from the logic of propositional identity.

We may begin by observing that, *modulo* simplifying assumptions, the counterexample given above commands a degree of intuitive appeal. For instance, let $\{a\}$ be the state of the ball being black, $\{c\}$ be the state of the ball being round, and $\{e\}$ be the state of the ball being iron, where p_1 is 'The ball is coloured', p_2 is 'The ball is shaped', and p_3 is 'The ball is metal'. Whereas the sentences 'The ball is coloured', 'The ball is shaped', and 'The ball is metal'

were each interpreted as only having one exact verifier and one exact falsifier, this assumption may be lifted while yielding the same non-identity results brought out above. Setting such complications aside, we may nevertheless observe that the state $\{a, c\}$ of the ball being black and round fails to exactly verify 'The ball is coloured or both shaped and metal'. This conclusion follows from the observations that: (1) $\{a, c\}$ includes $\{c\}$ as a part, and so does not exactly verify 'The ball is coloured' since $\{c\}$ is irrelevant; (2) $\{a, c\}$ is not a fusion of exact verifiers for 'The ball is shaped' and 'The ball is metal', and so does not exactly verify 'The ball is shaped and metal'; and (3) $\{a, c\}$ is not a fusion of exact verifiers for 'The ball is coloured' and 'The ball is shaped and metal', and so does not exactly verify 'The ball is coloured and both shaped and metal'. Without changing the inclusive semantic clauses given above, $\{a, c\}$ does not exactly verify 'The ball is coloured or both shaped and metal'. Nevertheless, it is easy to see that $\{a, c\}$ will exactly verify 'The ball is coloured or shaped' on account of exactly verifying 'The ball is coloured and shaped', and so $\{a, c\}$ exactly verifies 'The ball is both coloured or shaped, and coloured or metal'. Given these considerations, we may conclude that $\mathcal{M}_D$ corresponds to an intuitively compelling counterexample to **Dist1**, where something similar may be shown for **Dist2**, making it unnatural to exclude these cases.

Insofar as the present notion of validity may claim to be reasonably natural, arbitrary models of $\mathcal{L}$ should not be excluded from consideration by fiat alone. Rather than attempting to hand pick the models of $\mathcal{L}$ which are to be considered in evaluating the validity of identity sentences, one may seek instead to provide motivation for adopting a further model constraint to those defended above. In particular, Fine (2016, 2017b) considers the following restriction:

> *Convex:* $X \subseteq S$ is *convex iff* $y \in X$ whenever $x \sqsubseteq y \sqsubseteq z$ for some $x, z \in X$.
>
> *Convexity:* An $\mathcal{S}$-model is *convex iff* both $|p|^{\pm}$ are convex for all $p \in \mathbb{L}$.

Let $\mathcal{R}$ be the class of *regular* models $\mathcal{M}$ of $\mathcal{L}$ which are closed, full, and convex,

where propositions whose sets of exact verifiers and falsifiers are both closed and convex may also be said to be *regular*. In order to motivate *Convexity*, Fine (2017b) appeals to the simplicity of the resulting space of regular propositions:

> Regular propositions have an especially simple form. For each such proposition P (if non-empty) will have a maximal verifier p, the fusion of all its verifiers, which we identify with its subject-matter— the agglomeration of the facts, so to speak, from which its verifiers are drawn; and it will also have various low lying verifiers, with all other verifiers lying above them. The proposition itself will then consist of all the states that lie between the low lying verifiers and the maximal verifier. Regular propositions are therefore subject to a limited form of monotonicity; given that a state verifies a regular proposition then so does any extension of the state as long as it lies within the subject-matter of the proposition. (p. 628-9)

It is worth considering the manner in which the limited form of monotonicity to which Fine refers does not hold in the counterexamples to distribution presented above. In particular, observe that the outlier states in $|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|^+_D$ all lie between states which are included in $|p_1 \vee (p_2 \wedge p_3)|^+_D$, despite failing to be members of $|p_1 \vee (p_2 \wedge p_3)|^+_D$. Accordingly, $|p_1 \vee (p_2 \wedge p_3)|^+_D$ fails to be convex, where the same may be shown for $|p_1 \wedge (p_2 \vee p_3)|^-_D$.

Despite the fact that both $|p_1 \vee (p_2 \wedge p_3)|^\pm_D$ and $|p_1 \wedge (p_2 \vee p_3)|^\pm_D$ fail to be convex, we may nevertheless observe that $\mathcal{M}_D$ is a regular model, assuming every sentence letter is assigned to a regular proposition. Rather, it is only with respect to complex sentences such as $p_1 \vee (p_2 \wedge p_3)$ and $p_1 \wedge (p_2 \vee p_3)$ that convexity breaks down. Put otherwise, the class of regular models $\mathcal{R}$ fails to be uniform on the inclusive semantics, and so Fine's restriction to regular models fails to extend regularity to the propositions expressed by complex sentences. Rather than giving up convexity, one may maintain uniformity by replacing the inclusive semantics for conjunction and disjunction with the following:

$(\wedge)^-_{\mathrm{S}}$ $\mathcal{M}, s \dashv\!\shortmid A \wedge B$ *iff* $t \sqsubseteq s \sqsubseteq d$ for some t and d where $\mathcal{M}, d \dashv\!\shortmid A \vee B$ and either $\mathcal{M}, t \dashv\!\shortmid A$ or $\mathcal{M}, t \dashv\!\shortmid B$.

$(\vee)^+_{\mathrm{S}}$ $\mathcal{M}, s \shortmid\!\vdash A \vee B$ *iff* $t \sqsubseteq s \sqsubseteq d$ for some t and d where $\mathcal{M}, d \shortmid\!\vdash A \wedge B$ and either $\mathcal{M}, t \shortmid\!\vdash A$ or $\mathcal{M}, t \shortmid\!\vdash B$.

I will refer to the result of replacing $(\wedge)^-$ and $(\vee)^+$ in the inclusive semantics with the clauses above as the *super-inclusive semantics*. Whereas the inclusive semantics permits exact verifiers for $A \wedge B$ to be exact verifiers for $A \vee B$, and permits exact falsifiers for $A \vee B$ to be exact falsifiers for $A \wedge B$, the super-inclusive semantics lets s exactly verify $A \vee B$ so long as some t exactly verifies either A or B where $t \sqsubseteq s \sqsubseteq d$ and d exactly verifies $A \wedge B$. Whereas the state $\{a, c\}$ of the ball being black and round in $\mathcal{M}_D$ failed to exactly verify 'The ball is coloured or both shaped and metal' on the inclusive semantics, the opposite conclusion may be maintained on the super-inclusive semantics. After all $\{a\}$ exactly verifies 'The ball is coloured' and $\{c, e\}$ exactly verifies 'The ball is shaped and metal', and so $\{a, c, e\}$ exactly verifies 'The ball is coloured and both shaped and metal'. Since $\{a\} \subseteq \{a, c\} \subseteq \{a, c, e\}$, it follows by the super-inclusive semantics that $\{a, c\}$ exactly verifies 'The ball is coloured or both shaped and metal', thereby avoiding the counterexample above.

Given the super-inclusive semantics, we may show that $\mathcal{R}$ is uniform so that for any $\mathcal{M} \in \mathcal{R}$ and $A \in \mathtt{ext}(\mathbb{L})$, the proposition $|A|$ is regular.[19] In particular, we may show that with respect to the model $\mathcal{M}_D$, both of the following identities hold in contrast to the identities given above:

$$\begin{aligned} |p_1 \vee (p_2 \wedge p_3)|^+_D &= \{\{a\}, \{a, c\}, \{a, e\}, \{c, e\}, \{a, c, e\}\} \\ |p_1 \wedge (p_2 \vee p_3)|^-_D &= \{\{b\}, \{b, d\}, \{b, f\}, \{d, f\}, \{b, d, f\}\}. \end{aligned}$$

It follows from the above that $|p_1 \vee (p_2 \wedge p_3)|_D = |(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_D$ and $|p_1 \wedge (p_2 \vee p_3)|_D = |(p_1 \wedge p_2) \vee (p_1 \wedge p_3)|_D$, and so we may conclude that:

[19] *Sketch:* The proof goes by induction on complexity, where the base case is given by assumption. Since conjunction is readily shown to preserve convexity, the main cases to consider are $(\vee)^+_s$ and $(\wedge)^-_s$. Assuming that $x, z \in |A \vee B|^+$ where $x \sqsubseteq y \sqsubseteq z$, it follows for both $v \in \{x, z\}$ that $t_v \sqsubseteq v \sqsubseteq d_v$ for some $t_v \in |A|^+ \cup |B|^+$ and $d_v \in |A \wedge B|^+$. Thus $t_x \in |A|^+ \cup |B|^+$ and $d_z \in |A \wedge B|^+$ where $t_x \sqsubseteq x \sqsubseteq y \sqsubseteq z \sqsubseteq d_z$, and so $y \in |A \vee B|^+$. It follows that $|A \vee B|^+$ is convex, where something similar may be shown for $|A \wedge B|^-$.

$$\mathcal{M}_D \Vdash\ p_1 \vee (p_2 \wedge p_3)\ \equiv\ (p_1 \vee p_2) \wedge (p_1 \vee p_3)$$
$$\mathcal{M}_D \Vdash\ p_1 \wedge (p_2 \vee p_3)\ \equiv\ (p_1 \wedge p_2) \vee (p_1 \wedge p_3).$$

More generally, **#Dist1** and **#Dist2** are $\mathcal{R}$-valid over the super-inclusive semantics. These validities come at the cost of a significant increase in the complexity of the semantics for conjunction and disjunction. Even if the space of regular propositions could be shown to be simpler than the space of closed propositions, we must nevertheless weigh such gains in simplicity against the increase in complexity of the super-inclusive semantics.

Beyond complexity, one may object that the super-inclusive semantics does not capture the natural falsity-conditions for conjunction and truth-conditions for disjunction, abandoning the spirit of *Exactness* above. In particular, an exact verifier (falsifier) s for a sentence A was required to be identical to a fusion $\bigsqcup X$ for some nonempty X where every $x \in X$ exactly verifies (falsifies) A. For instance, recall the state of the ball being black $\{a\}$ and the state of the ball being round $\{c\}$. On the inclusive semantics, the fusion $\{a, c\}$ failed to exactly verify 'The ball is coloured or both shaped and metal' even though $\{a\}$ exactly verifies 'The ball is coloured' and $\{c\}$ is a part of $\{c, e\}$ which exactly verifies 'The ball is shaped and metal'. In defence of this conclusion, one may point out that, by itself, $\{c\}$ does not contribute to $\{a, c\}$ exactly verifying 'The ball is coloured or both shaped and metal'. Rather, it is only when fused with $\{e\}$ that the result $\{c, e\}$ may be said to exactly verify 'The ball is coloured or both shaped and metal'. For instance, when considering the state $\{a, c, e\}$, both $\{a\}$ and $\{c, e\}$ may be said to contribute to $\{a, c, e\}$ exactly verifying 'The ball is coloured or both shaped and metal'. However, given that $\{e\} \not\sqsubseteq \{a, c\}$, the same cannot be said for $\{a, c\}$, since $\{c\}$ does not contribute to $\{a, c\}$ by exactly verifying 'The ball is coloured or both shaped and metal'.

Since *Exactness* was only intended to sharpen the informal gloss of exact verifiers (falsifiers) as "wholly relevant" to the sentence in question, a proponent of the super-inclusive semantics may advocate the following alternative:

> *Inclusiveness:* A state s *inclusively verifies* (*falsifies*) A *iff* $s = \bigsqcup X$ for some $X \subseteq S$ where some $x \in X$ inclusively verifies (falsifies) A, and every $x \in X$ has some $y \sqsupseteq x$ where y inclusively verifies (falsifies) A.

In contrast to exact verification (falsification), *Inclusiveness* explicitly permits inclusive verifiers (falsifiers) to contain parts which without supplement do not contribute to inclusively verifying (falsifying) the sentence in question. For instance, in the example above, $\{a\}$ and $\{c, e\}$ inclusively verify 'The ball is coloured or both shaped and metal', but the same cannot be said for $\{c\}$ on its own. However, given that $\{c\} \subseteq \{c, e\}$, it follows by *Inclusiveness* that $\{a, c\}$ inclusively verifies 'The ball is coloured or both shaped and metal'.

Even in adopting the super-inclusive semantics together with *Inclusiveness*, one must restrict attention to the class $\mathcal{R}^+$ of nonvacuous regular models of $\mathcal{L}$. Otherwise, we may let $\mathcal{M}_E = \langle S_E, \subseteq, |\cdot|_E \rangle$ where $S_E = \mathcal{P}(\{a, b, c\})$ and both $|p_1|_E = \langle \varnothing, \{a\} \rangle$ and $|p_2|_E = \langle \{b\}, \{c\} \rangle$ for pairwise distinct a, b, and c, and so:

$$\begin{aligned} |p_1 \wedge p_2|_E &= \langle \varnothing, \varnothing \rangle \\ |p_1 \vee p_2|_E &= \langle \varnothing, \varnothing \rangle. \end{aligned}$$

Even for a proposition with no verifiers, the results above are far from natural. For instance, given that $\mathcal{M}, b \Vdash p_2$, it is natural to conclude with the inclusive semantics that $\mathcal{M}, b \Vdash p_1 \vee p_2$. By contrast, the super-inclusive semantics requires that $t \sqsubseteq b \sqsubseteq d$ for some t and d where $\mathcal{M}, d \Vdash p_1 \wedge p_2$ and either $\mathcal{M}, t \Vdash p_1$ or $\mathcal{M}, t \Vdash p_2$. Given that there is no d where $\mathcal{M}, d \Vdash p_1 \wedge p_2$, it follows that there is no b where $\mathcal{M}, b \Vdash p_1 \vee p_2$. Analogous reasoning applies to conjunction, providing reason to require every proposition to have at least one inclusive verifier and one inclusive falsifier. Nevertheless, a proponent of

the super-inclusive semantics may follow Fine in ruling out the counterexample above by restricting attention to the class of nonvacuous models in $\mathcal{R}^+$.

Given the considerations above, I will take the relative simplicity of the inclusive semantics interpreted over the much wider class of models in $\mathcal{C}$ to provide an advantage over the super-inclusive semantics restricted to $\mathcal{R}^+$. Although, the super-inclusive semantics remains a live alternative, the only motivation cited so far for restricting consideration to regular models of $\mathcal{L}$ and complicating the semantics is Fine's claim that the resulting space of regular propositions takes an especially simple form. In Chapter 4 and Chapter 5, I contest this claim, showing that the space of bilateral propositions closed under nonempty fusion forms a bilattice which is neither distributive nor interlaced. By contrast, the space of nonvacuous propositions is not bounded below despite being bounded above, and so does not form a bilattice.[20] Although one may show that the space of regular propositions forms a bilattice which is both distributive and interlaced, admitting vacuous propositions still faces the troubles brought out above for the super-inclusive semantics. Accordingly, I will restrict attention to the inclusive semantics throughout what follows, excluding **#Dist1** and **#Dist2** from the logic of propositional identity.

3.7 A Logic of Propositional Identity

Whereas §3.3 considered which theorems ought to be included in the logic of propositional identity, the syntax and semantic presented in §§3.4−3.6 restricted consideration to just the propositional identity sentences in $\mathtt{id}(\mathbb{L})$. Imposing this restriction simplifies the semantics, while still providing a systematic means of evaluating the propositional identities in $\mathtt{id}(\mathbb{L})$ for validity. In particular, the inclusive semantics interpreted over $\mathcal{C}$ included counterexamples

[20] A *bilattice* consists of two complete lattices whose orders $\leqslant$ and $\sqsubseteq$ together with an inversion operator $\neg$ satisfy: (1) $\neg\neg X = X$; (2) $X \leqslant Y = \neg X \sqsubseteq \neg Y$; and (3) $X \sqsubseteq Y = \neg X \leqslant \neg Y$.

to **#Necs**, **#Imps**, **#Abs1**, **#Abs2**, **#Dist1**, and **#Dist2**. By contrast, the super-inclusive semantics restricted to $\mathcal{R}^+$ includes counterexamples to **#Necs**, **#Imps**, **#Abs1**, and **#Abs2** while excluding the counterexamples to **#Dist1** and **#Dist2**. Although the distribution laws will be shown in §3.8 to preserve subject-matter in addition to modal profile, we nevertheless found strong abductive reasons for preferring the inclusive semantics to the super-inclusive semantics. For these reasons, I will exclude the distribution laws from the logic presented below, though the resulting logic may be consistently extended to include **#Dist1** and **#Dist2** if desired.

Given the syntactic restrictions above, I will refer to the following axiom system as *The Restricted Logic for Propositional Identity* (LPI$^-$):

A1	$A \wedge A \equiv A$.	**A2**	$A \vee A \equiv A$.
A3	$A \wedge B \equiv B \wedge A$.	**A4**	$A \vee B \equiv B \vee A$.
A5	$(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$.	**A6**	$(A \vee B) \vee C \equiv A \vee (B \vee C)$.
A7	$\neg(A \wedge B) \equiv (\neg A \vee \neg B)$.	**A8**	$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$.
A9	$A \equiv \neg\neg A$.	**R1**	$A \equiv B \vdash B \equiv A$.
R2	$A \equiv B \vdash (A \wedge C) \equiv (B \wedge C)$.	**R3**	$A \equiv B \vdash (A \vee C) \equiv (B \vee C)$.
R4	$A \equiv B,\ B \equiv C \vdash A \equiv C$.	**R5**	$A \equiv B \vdash \neg A \equiv \neg B$.

I will take $\vdash^-_{\text{LPI}}$ to be the smallest relation to satisfy the axioms and rules of inference given above which is closed under the standard structural rules. It is straightforward to show that LPI$^-$ is sound with respect to the inclusive semantics and class $\mathcal{C}$ of closed full models of $\mathcal{L}$. Accordingly, the motivations provided above for adopting the inclusive semantics and class of models $\mathcal{C}$ extend to each of the theorems of LPI$^-$. By contrast, Correia (2016b) and Correia and Skiles (2019) present a logic of generalised identity (GI), which is the result of both excluding **R5** from LPI$^-$ while also including **#Dist2**. Although Correia and Skiles (2019, p. 647) explicitly exclude **#Dist1** from GI,

they do not offer any motivation for leaving **#Dist1** out when **#Dist2** has been included in GI, deferring instead to the semantics provided in Correia (2016b). In what follows, I will review the motivations Correia provides for his semantics in contrast to the inclusive semantics defended above. In §3.8, I will draw a further connection between LPI$^-$ and Angell's (2002) *Logic of Analytic Containment* (AC) which Fine (2016) proves to be sound and complete on a state semantic over a class of nonvacuous regular models.

By contrast to the Finean state semantics which I have defended above, Correia (2016b) provides a *supersentence semantics* by which to interpret the sentences in $\mathtt{id}(\mathbb{L})$, showing that the semantics is equivalent to a Finean state semantics.[21] Given this equivalence, I will focus on the state semantics which Correia also considers so as to facilitate comparison with the semantics given above. We may begin by observing that in place of the semantics for propositional identity ($\equiv$) given above, Correia (2016b, p. 108) assumes:

$(\equiv)_{\mathrm{C}}$ $\quad \mathcal{M} \Vdash A \equiv B$ *iff* $|A|^+ = |B|^+$.

Whereas ($\equiv$) required that A and B have the same exact verifiers and falsifiers in $\mathcal{M}$, Correia only requires A and B to have the same exact verifiers in $\mathcal{M}$, taking $\varphi \in \mathtt{id}(\mathbb{L})$ to be valid just in case $\mathcal{M} \Vdash \varphi$ for all $\mathcal{M} \in \mathcal{C}^+$— i.e., $\Vdash_{\mathcal{C}^+} \varphi$— where $\mathcal{C}^+$ is the class of closed full nonvacuous models of $\mathcal{L}$. As mentioned above, the nonvacuous propositions defined over a state space fail to be bounded below despite being bounded above, and so do not form a bilattice. Correia does not provide any explicit motivation for adopting $(\equiv)_{\mathrm{C}}$ over the semantics given in ($\equiv$), despite how much turns on this choice as I will draw out below. Setting these demerits aside, I will focus on the results of adopting Correia's semantics for propositional identity.

[21] Correia (2016b, p. 104) refers to the sentences in $\mathtt{id}(\mathbb{L})$ as *factual equivalences*, writing, "Two sentences A and B are factually equivalent just in case the situations described by A are the same as the situations described by B," encouraging the assumption that the difference between factual equivalences and propositional identities is purely terminological.

Given Correia's semantics for propositional identity, it is easy to see why $\mathcal{M}_D$ is a counterexample to the Correia-validity of **#Dist1** but not to **#Dist2**. For instance, both $p_1 \wedge (p_2 \vee p_3)$ and $(p_1 \wedge p_2) \vee (p_1 \wedge p_3)$ have the very same exact verifiers on $\mathcal{M}_D$, where we may show more generally that $A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$ have the same exact verifiers for any model $\mathcal{M} \in \mathcal{C}^+$. Rather, I have excluded **#Dist2** from LPI^- on account of the divergence in the exact falsifiers for the sentences flanking the identity operator. Thus we find that the asymmetry in Correia's treatment of **#Dist1** and **#Dist2** is a direct result of the asymmetry in the role that the truth-conditions and falsity-conditions play in evaluating the truth of a propositional identities. For instance, recalling $\mathcal{M}_D$ from before, we may derive the following:

$$
\begin{aligned}
|\neg[p_1 \wedge (p_2 \vee p_3)]|_D &= \langle \underline{\{\{b\}, \{d, f\}, \{b, d, f\}\}}, \{\{a, c\}, \{a, e\}, \{a, c, e\}\}\rangle \\
|\neg[(p_1 \wedge p_2) \vee (p_1 \wedge p_3)]|_D &= \langle \underline{\{\{b\}, \{b, d\}, \{b, f\}, \{d, f\}, \{b, d, f\}\}}, \{\{a, c\}, \{a, e\}, \{a, c, e\}\}\rangle.
\end{aligned}
$$

Given the difference in exact verifiers, Correia's semantics distinguishes between the propositions expressed by the sentences above despite identifying the propositions expressed by their un-negated analogues. More generally, we may observe that $\neg[A \wedge (B \vee C)] \equiv \neg[(A \wedge B) \vee (A \wedge C)]$ admits of counterexamples even though $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ comes out valid on Correia's semantics. It is for this reason that Correia (2016b) and Correia and Skiles (2019) are forced to exclude **R5** from GI, citing Fine's (2016, p. 204) derivation of **#Dist1** from **#Dist2** were **R5** to be included in the logic.

By excluding **#Dist1** from GI while nevertheless including **#Dist2** as an axiom, conjunction and disjunction fail to be dual operators. In order to evaluate the true cost of following Correia (2016b) and Correia and Skiles (2019)

in admitting such asymmetries into the semantics, consider the following:

$(N)_D \quad ([p_1 \wedge (p_2 \vee p_3)] \equiv [(p_1 \wedge p_2) \vee (p_1 \wedge p_3)]) \rightarrow (\neg[p_1 \wedge (p_2 \vee p_3)] \equiv \neg[(p_1 \wedge p_2) \vee (p_1 \wedge p_3)]).$

Whereas the antecedent is an instance of **#Dist2** and so valid on the inclusive semantics over $\mathcal{C}^+$, the consequent admits of counterexamples as brought out by $\mathcal{M}_D$ above. Even though $(N)_D$ is not a sentence belonging to $\texttt{id}(\mathbb{L})$ on account of relating two identity sentences by the material conditional, this is no more than an artefact of the simplifying restriction to $\texttt{id}(\mathbb{L})$ in attempting to provide a logic of propositional identity. As Correia (2016b) puts the point:

> What is the logic of factual equivalence? This chapter attempts to give a partial answer to this question, by addressing the following, narrower yet important question: Given a standard propositional language with negation, conjunction and disjunction as primitive operators, which sentences of the language should be taken to be factually equivalent by virtue of their logical form? (p. 103)

The quote above indicates that Correia intends for his logic to endure extension beyond the sentences in $\texttt{id}(\mathbb{L})$. Given that $\mathcal{M}_D$ makes the antecedent of $(N)_D$ true while making the consequent of $(N)_D$ false, it is safe to conclude that $(N)_D$ will admit of counterexamples on any reasonable extension of $\mathcal{L}$ which includes extensional combinations of the sentences in $\texttt{id}(\mathbb{L})$.

Provided that $(N)_D$ admits of counterexamples, we may observe that the following more general claim must also admit of counterexamples:

N: $(A \equiv B) \rightarrow (\neg A \equiv \neg B)$.

Not only is $(N)_D$ an instance of **N**, but **N** is an instance of **Func** given in §3.3. Insofar as any instance of **N** admits of counterexamples, one must deny that negation is a transparent operator. If A and B express the same proposition as is conveyed by $A \equiv B$, and if negation operates solely on the proposition expressed by the sentence being negated, then the proposition expressed by $\neg A$ must be the same as the proposition expressed by $\neg B$. However, given

the counterexample to $(N)_D$, this is just what Correia and Skiles (2019) deny. Since Correia (2016b) has shown that GI is complete over his semantics, the point may be made even more general. Insofar as **R5** is to be excluded from GI, it follows by completeness that **R5** admits of counterexamples. However, if the sentences of a logic are to admit of extensional combination in at least some extension of the language, then any counterexample to **R5** will equally be a counterexample to **N**. But insofar as **N** admits of counterexamples, it follows that negation is not a transparent operator.

Although the present argument has relied on the definition of transparency given in §3.3, the same conclusion follows from Correia and Skiles' (2019) explicit reference to **LL** as presented by Dorr, commenting as follows:

> We also take it that some (interesting) restriction of Leibniz's Law for $\equiv$ must be correct. [...] We say "some restriction", because, just like its objectual counterpart, the law fails if understood unrestrictedly. [...] Determining exactly which linguistic contexts are opaque for objectual identity is no easy task, and the situation is no different in the case of generalized identity. Fortunately, we won't need to provide a detailed discussion of the issue, and we trust the reader will agree with us that our applications of **LL** will always involve contexts that are clear candidates to be transparent (or at least not clear candidates to be opaque). (p. 645)

It is extremely natural to assume that neither the extensional operators nor propositional identity operator generate opaque contexts. However, even in restricting **LL** to sentences which only include the extensional operators and propositional identity operator, it is straightforward to derive **N** from **Ref** by **LL**.[22] Insofar as **N** is to admit of counterexamples where **Ref** is beyond dispute, Correia and Skiles must entertain a restriction of **LL** to transparent contexts which blocks the derivation of **N** from **LL** and **Ref**.

The fact that Correia and Skiles cannot claim that **LL** holds without exception in a language with operators for propositional identity and the extensional operators is a direct result of the asymmetry between truth and

[22] *Proof:* $(A \equiv B) \to [(\neg A \equiv \neg A) \to (\neg A \equiv \neg B)]$ is an instance of **LL**, where $\neg A \equiv \neg A$ is an instance of **Ref**, and so **N** follows by propositional logic. □

falsity which Correia (2016b) includes in his semantics. By contrast, it is easy to see why **R5** would come out valid were one to take both the truth-conditions and falsity-conditions into consideration when evaluating propositional identity claims as in (≡) from §5.3. Assuming that $A \equiv B$ is true on an arbitrary model $\mathcal{M}$, it follows from (≡) that A and B have the same exact verifiers and falsifiers. Given that negation amounts to no more than inverting the exact verifiers and falsifiers, it follows that $\neg A$ and $\neg B$ will also have the same exact verifiers and falsifiers, and so $\neg A \equiv \neg B$ will also come out true on $\mathcal{M}$. Of course, this argument breaks down if the truth of $A \equiv B$ only requires sameness of truth-conditions. As brought out by $\mathcal{M}_D$, two sentences may share the same truth-conditions without sharing the same falsity-conditions, and so the fact that A and B share the same truth-conditions says nothing of whether $\neg A$ and $\neg B$ share the same truth-conditions. Despite limiting consideration to truth-conditions in evaluating propositional identity claims, Correia maintains consideration of both truth-conditions and falsity-conditions in evaluating the extensional sentences in $\texttt{ext}(\mathbb{L})$, making it all the more mysterious what motivates the sudden asymmetry late in his semantics.

Rather than attempting to maintain the asymmetry between truth-conditions and falsity-conditions included in Correia's semantics, I will take the validity of **R5** to be an immediate consequence of (≡), ruling out any suspicion that negation might be an opaque operator. It is for this reason that I have included **R5** in LPI$^-$, despite excluding both **#Dist1** and **#Dist2**.[23] In the following section, I will extend $\mathcal{L}$ to include an operator σ for subject-matter, tracing out the beginnings of a more substantial theory of subject-matter than what was provided in §3.2 by way of motivating LPI$^-$.

[23] As Fine (2016) shows, **R5** may be derived as an admissible rule given **#Dist1** and **#Dist2**.

3.8 Subject-Matter Revisited

Recall from §3.2 the manner in which differences in subject-matter were taken to indicate differences between propositions. In particular, **#Necs**, **#Imps**, **#Abs1**, and **#Abs2** were all found to admit of exceptions, motivating the state semantics defended above. As I argued in §3.6, there are strong abductive reasons for also taking exception to **#Dist1** and **#Dist2** despite the fact that both of the distribution laws respect sameness in subject-matter in addition to necessary equivalence. In what follows, I will make this latter observation precise by including a subject-matter operator σ in the language and contrasting three different proposals for the semantics of σ along with corresponding extensions of LPI^-, thereby providing more substantial theories of subject-matter than the broad outline sketched in §3.2. As we will see, neither **#Dist1** nor **#Dist2** can be rejected on account of failing to preserve subject-matter on any of the three theories of subject-matter given below.

As above, I will assume that the subject-matter of A is what A is about. Whereas $\dot{\sigma}A$ was taken to be a singular term in §3.2, it will be convenient in what follows to take 'σ' to be a unary sentential operator given that first-order resources have not been included in LPI^-. Accordingly, σA is a proposition.[24] Rather than beginning with a target reading of σA, I will provide three different semantics for σ, where the manner in which σA is to be read will depend on one's preferred semantics. We may begin by letting $\mathcal{L}^\sigma = \langle \mathbb{L}, \neg, \sigma, \vee, \wedge, \equiv \rangle$ be an extension of $\mathcal{L}$, where the non-identity sentences of $\mathcal{L}^\sigma$ are defined:

$$A ::= p \mid \neg A \mid \sigma A \mid A \wedge A \mid A \vee A.$$

Letting $\texttt{nid}(\mathbb{L})$ be the set of non-identity sentences of $\mathcal{L}^\sigma$, I will take $A \equiv B$

[24] Although it is not altogether clear how to read σA in English, such a predicament is not without precedent. See for instance Yablo (2014).

to be an *identity sentence* in $\mathcal{L}^\sigma$ for any $A, B \in \mathtt{nid}(\mathbb{L})$, where $\mathtt{id}^\sigma(\mathbb{L})$ is the set of all identity sentences in $\mathcal{L}^\sigma$. Given these expressive resources, we may adopt $\ulcorner A \leftrightarrow B \urcorner$ as an abbreviation of $\ulcorner \sigma A \equiv \sigma B \urcorner$. Given any $A, B \in \mathtt{nid}(\mathbb{L})$, it will be convenient to refer to sentences of the form $\ulcorner A \leftrightarrow B \urcorner$ as *equivalences* in $\mathcal{L}$, letting $\mathtt{eq}(\mathbb{L})$ be the set of all equivalences in $\mathcal{L}$.

Rather than relying on intuitions about general principles articulated in $\mathcal{L}^\sigma$, I will proceed semantically. In particular, I will maintain the inclusive semantics defended in §5.3 together with the clause for propositional identity ($\equiv$). We may then consider the following clauses for subject-matter:

$(\sigma)^+_{\mathrm{C}}$ $\quad \mathcal{M}, s \Vdash \sigma A$ *iff* $x \sqsubseteq s \sqsubseteq y$ for some $x, y \in S$ where $\mathcal{M}, x \Vdash A$ and $\mathcal{M}, y \Vdash A$.

$(\sigma)^-_{\mathrm{C}}$ $\quad \mathcal{M}, s \dashv\!\Vdash \sigma A$ *iff* $x \sqsubseteq s \sqsubseteq y$ for some $x, y \in S$ where $\mathcal{M}, x \dashv\!\Vdash A$ and $\mathcal{M}, y \dashv\!\Vdash A$.

I will refer to the above as the *convex semantics* for σ since the exact verifiers (falsifiers) for σA is the convex closure of the exact verifiers (falsifiers) for A. Letting LPI^-_σ be the result of including the following axioms and inference rule in LPI^-, we may show that LPI^-_σ is sound over $\mathcal{C}$ on the convex semantics:

It is worth noting that **R6** makes σ a transparent operator such that σ operates solely on the propositions expressed by the sentences to which it is appended, where this is in accordance with *Objectivity* given in §3.2.

Although the convex semantics explains why the distribution laws cannot be rejected on account of offending sameness of subject-matter, there is little else to be said in defence of the convex semantics. It is perhaps worth mentioning that $\leftrightarrow$ has the same extension in LPI^- as analytic equivalence does in Angell's (2002) logic AC, despite the distinct theoretical roles for analytic equivalence and sameness of subject-matter. Setting these connections aside, we may observe that neither of the following are valid over the convex semantics:

S1 $A \wedge B \leftrightarrow A \vee B$ **S2** $\neg A \leftrightarrow A$

Both of the principles above have some intuitive appeal. In order to validate **S1** and **S2**, one may adopt the *inclusive semantics* for σ given below:

$(\sigma)^+$ $\mathcal{M}, s \Vdash \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \Vdash A \vee \neg A$.

$(\sigma)^-$ $\mathcal{M}, s \dashv \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \dashv A \wedge \neg A$.

On the inclusive semantics, σA is the proposition whose exact verifiers and falsifiers are the parts of the exact verifiers and falsifiers for A. By contrast, Fine (2016, 2017c, 2020) identifies the subject-matter of a proposition with the fusion of all exact verifiers and falsifiers for that proposition. Accordingly, the subject-matter of A is not a proposition at all for Fine, but rather a single state $\dot{\sigma}A$ which I will refer to as the *Finean subject-matter* for A. Despite the disparity in kind between Fine's objectual account of subject-matter and the propositional account assumed above, the inclusive semantics validates the same equivalences in $\mathtt{eq}(\mathbb{L})$ as Fine's account of subject-matter. In particular, we may show that $\mathcal{M} \Vdash A \leftrightarrow B$ holds on the inclusive semantics just in case $\dot{\sigma}A = \dot{\sigma}B$ where A and B are both interpreted in $\mathcal{M}$.[25]

Given the tight correspondence between the Finean subject-matter $\dot{\sigma}$ and the inclusive semantics for σ, we may shed light on the contrast between the inclusive semantics and convex semantics for subject-matter by considering a third alternative, intermediate between the convex and inclusive semantics:

$(\sigma)^+_{\mathrm{D}}$ $\mathcal{M}, s \Vdash \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \Vdash A$.

$(\sigma)^-_{\mathrm{D}}$ $\mathcal{M}, s \dashv \sigma A$ *iff* $s \sqsubseteq t$ for some $t \in S$ where $\mathcal{M}, t \dashv A$.

I will refer to the clauses above as the *directed semantics* for subject-matter. Whereas neither **S1** nor **S2** are valid on the convex semantics, and both are

[25] *Sketch:* Assuming that $\mathcal{M} \Vdash A \leftrightarrow B$, it follows that $\mathcal{M}, \Vdash \sigma A \equiv \sigma B$, and so $|\sigma A| = |\sigma B|$. Thus the parts of the exact verifiers and falsifiers for $A \vee \neg A$ are the same as the parts of the exact verifiers and falsifiers for $B \vee \neg B$. In particular $|A \vee \neg A|^+ = |B \vee \neg B|^+$, and so $|A|^+ \cup |A|^- \cup |A \wedge \neg A|^+ = |B|^+ \cup |B|^- \cup |B \wedge \neg B|^+$ by the semantics. We may then show that $\bigsqcup(|A|^+ \cup |A|^-) = \bigsqcup(|B|^+ \cup |B|^-)$, and so $\dot{\sigma}A = \dot{\sigma}B$. The reverse direction is similar.

valid on the inclusive semantics, the directed semantics validate **S1** but not **S2**. Were one to adopt the directed semantics, LPI^{-}_{σ} may be extended to include **S1** but not **S2**— call this system $\text{LPI}^{-}_{\sigma^1}$— whereas by adopting the inclusive semantics, one may adopt both **S1** and **S2**— call this system $\text{LPI}^{-}_{\sigma^2}$. Despite their differences, each extension of LPI^{-} provides an account of which sentences have the same subject-matter in virtue of their logical form.

Without saying more about the intended theoretical role of subject-matter, it is difficult to assess which of the extensions of LPI^{-} is to be preferred over the others. Nevertheless, in the absence of countervailing considerations, I take both **S1** and **S2** to be extremely plausible, providing some reason to adopt $\text{LPI}^{-}_{\sigma^2}$ which is sound over the inclusive semantics. Additionally, the inclusive semantics yields a coarser-grained theory of subject-matter than either the convex or directed semantics, where in the absence of clear motivation otherwise, conservative methodology recommends drawing fewer distinctions rather than more. Lastly, we may observe that given the inclusive semantics, σA may be read ⌜It is partially the case that $A \vee \neg A$⌝. By contrast, it is not clear what σA expresses on either the convex or directed semantics, providing further reason to prefer the inclusive semantics for σ.

Setting aside the differences between the convex, directed, and inclusive semantics for subject-matter, we may observe that each semantic theory for σ admits counterexamples to the following principles:

$\texttt{\#Necs}^{\sigma}\texttt{:}\quad (A \vee \neg A) \leftrightarrow (B \vee \neg B).$ $\qquad$ $\texttt{\#Abs1}^{\sigma}\texttt{:}\quad A \leftrightarrow A \vee (A \wedge B).$

$\texttt{\#Imps}^{\sigma}\texttt{:}\quad (A \wedge \neg A) \leftrightarrow (B \wedge \neg B).$ $\qquad$ $\texttt{\#Abs2}^{\sigma}\texttt{:}\quad A \leftrightarrow A \wedge (A \vee B).$

Given **R6**, it follows from the invalidity of the above that **#Necs**, **#Imps**, **#Abs1**, and **#Abs2** all admit of counterexamples. A similar argument does not hold for the distribution laws since **A10** and **A11** are valid over $\mathcal{C}$ no matter which of the three semantic theories one adopts for subject-matter.

That the left and right sides of the distribution laws have the same subject-matter explains why we were unable to appeal to considerations of relevance or subject-matter in order to exclude **#Dist1** and **#Dist2** from LPI$^-$.

In contrast to the rough outline of a theory presented in §3.2, the extensions of LPI$^-$ to include axioms and rules for subject-matter provide much richer theories of subject-matter. Nevertheless, the σ-extensions of LPI$^-$ considered above remain restricted to the propositional identity sentences in $\mathtt{id}^\sigma(\mathbb{L})$. Accordingly, it is desirable to provide a much more general semantics in which these syntactic restriction may be dropped, interpreting sentences which contain any combination of sentential operators included in the language. Moreover, one may wish to include modal operators so that principles such as *Necessary Equivalence* may also be included in the logic. I provide a modal semantics in the following chapter for a language with primitive operators for ground, metaphysical necessity, and the extensional operators, showing that essence, reduction, propositions identity, and relevance may all be defined in these terms. Such a language may easily be extended to include a subject-matter operator as above. Nevertheless, maintaining the restriction to $\mathtt{id}^\sigma(\mathbb{L})$ provides a first step towards a more general logic of propositional identity, where LPI$^-$ may be seen to provide an elegant theory of propositional identity with a well motivated semantics which is able to individuate propositions according to their subject-matter rather than their modal profile alone. In this respect, I will take LPI$^-$ to have satisfied the aims set out above.

3.9 Appendix

I will begin by considering propositional languages of the form $\mathcal{L} = \langle \mathbb{L}, \vec{\mathcal{Q}} \rangle$ where each $\mathcal{Q}_i^n \in \vec{\mathcal{Q}}$ is an n-ary sentential operator for some $n \in \mathbb{N}$, and $\vec{\mathcal{Q}}$ includes the extensional connectives '$\neg$', '$\wedge$', and '$\vee$' along with the propositional identity operator '$\equiv$'. The *well-formed sentences* in $\mathtt{wfs}(\mathbb{L})$ consist of the sentence

letters in $\mathbb{L}$ along with any $\mathcal{Q}_i^n(\vec{A})$, where $\mathcal{Q}_i^n \in \vec{\mathcal{Q}}$ and $\vec{A}$ is a sequence of n well-formed sentences. Assuming a background classical logic along with standard conventions for '$\rightarrow$' and '$\leftrightarrow$', consider the following:

Ref: $A \equiv A$.

Sym: $(A \equiv B) \rightarrow (B \equiv A)$.

LL: $(A \equiv B) \rightarrow (C \rightarrow C_{(B/A)})$.

Trans: $(A \equiv B) \rightarrow [(B \equiv C) \rightarrow (A \equiv C)]$.

Imp: $(A \equiv B) \rightarrow (A \rightarrow B)$.

Func: $(A \equiv B) \rightarrow [\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A}_{(B/A)})]$.

As above, $\mathcal{L}$ is *transparent* just in case all instances of **Func** hold, where $\mathcal{Q}(\vec{A}_{(B/A)})$ is the result of replacing one or more instances of A in $\vec{A}$ with B. It will also be convenient to let $\mathcal{Q}(\vec{A}_{[B/A]})$ be the result of replacing all instances of A in $\vec{A}$ with B. Similarly, whereas $C_{(B/A)}$ is the result of replacing one or more instances of A in C with B, I will take $C_{[B/A]}$ be the result of replacing all instances of A in C with B. We may now prove the following propositions.

P1 If $\mathcal{L}$ is transparent, then **Ref** and **Imps** entail both **Sym** and **Trans**.

Proof. Assuming that $\mathcal{L}$ is transparent, $(A \equiv B) \rightarrow [(A \equiv A) \equiv (B \equiv A)]$ follows, where $[(A \equiv A) \equiv (B \equiv A)] \rightarrow [(A \equiv A) \rightarrow (B \equiv A)]$ holds by **Imps**, and so $(A \equiv B) \rightarrow (B \equiv A)$ follows from **Ref** by propositional logic. Again by transparency, $(A \equiv B) \rightarrow ([(A \equiv C) \rightarrow (A \equiv C)] \equiv [(B \equiv C) \rightarrow (A \equiv C)])$, and so $(A \equiv B) \rightarrow ([(A \equiv C) \rightarrow (A \equiv C)] \rightarrow [(B \equiv C) \rightarrow (A \equiv C)])$ by **Imps**. Since $(A \equiv C) \rightarrow (A \equiv C)$ holds by propositional logic, we may conclude as desired that $(A \equiv B) \rightarrow [(B \equiv C) \rightarrow (A \equiv C)]$. □

L9.1 If $\mathcal{L}$ is transparent, then **Ref** and **Imps** entail $(A \equiv B) \rightarrow (C \equiv C_{[B/A]})$.

Proof. Assuming that $\mathcal{L}$ is transparent, the proof proceeds by induction on the complexity of $C \in \mathtt{wfs}(\mathbb{L})$. Of course, if $C \in \mathbb{L}$, then $(A \equiv B) \rightarrow (C \equiv C_{[B/A]})$ holds by propositional logic if A occurs in C, and $C \equiv C_{[B/A]}$ holds by **Ref** otherwise, where $(A \equiv B) \rightarrow (C \equiv C_{[B/A]})$ follows by propositional logic.

Assume for induction that $(A \equiv B) \to (C \equiv C_{[B/A]})$ holds whenever $\mathtt{comp}(C) \leqslant n$, and let $\mathtt{comp}(C) = n + 1$. Assuming that $C = \mathcal{Q}^n(\vec{D})$, we may observe that for all $1 \leqslant i \leqslant n$ that $(A \equiv B) \to (D_i \equiv D_{i[B/A]})$ follows by hypothesis, where $(D_i \equiv D_{i[B/A]}) \to [\mathcal{Q}^n(\vec{E}) \equiv \mathcal{Q}^n(\vec{E}_{[D_{i[B/A]}/D_i]})]$ by the transparency of $\mathcal{L}$ for any $\vec{E}$. Assuming $A \equiv B$, it follows for all $1 \leqslant m \leqslant n$ that $\mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\dots[D_{m[B/A]}/D_m]}) \equiv \mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\dots[D_{m+1[B/A]}/D_{m+1}]})$. Given that **Ref** and **Imps**, it follows by **P1** that **Trans** holds, and so by $n - 1$ applications of **Trans**, $\mathcal{Q}^n(\vec{D}) \equiv \mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\dots[D_{n[B/A]}/D_n]})$. We may then observe that $\mathcal{Q}^n(\vec{D}_{[D_{1[B/A]}/D_1]\dots[D_{n[B/A]}/D_n]} = \mathcal{Q}^n(\vec{D})_{[B/A]}$, and so it follows by discharging our assumption that $(A \equiv B) \to (\mathcal{Q}^n(\vec{D}) \equiv \mathcal{Q}^n(\vec{D})_{[B/A]})$. Since $C = \mathcal{Q}^n(\vec{D})$, we may conclude that $(A \equiv B) \to (C \equiv C_{[B/A]})$. □

P2 Assuming **Ref** and **Imp**, then $\mathcal{L}$ is transparent just in case **LL** holds.

Proof. Assume **Ref** and **Imp**. Letting **LL** hold in $\mathcal{L}$ where $\mathcal{Q}$ is an operator in $\mathcal{L}$, it follows that $(A \equiv B) \to ([\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A})] \to [\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A})_{(B/A)}])$. However, given **Ref**, $\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A})$, and so $(A \equiv B) \to [\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A})_{(B/A)}]$. In particular, $(A \equiv B) \to [\mathcal{Q}(\vec{A}) \equiv \mathcal{Q}(\vec{A}_{(B/A)})]$ as in **Func**. Generalising on $A, B, \vec{A}$, and $\mathcal{Q}$, we may conclude that $\mathcal{L}$ is transparent.

Assume instead that $\mathcal{L}$ is transparent. Letting $p \in \mathbb{L}$ where p does not occur in B or C, it follows by **L9.1** that $(p \equiv B) \to (C_{(p/A)} \equiv C_{(p/A)[B/p]})$. However, $C_{(p/A)[B/p]} = C_{(B/A)}$, and so $(p \equiv B) \to (C_{(p/A)} \equiv C_{(B/A)})$. Thus $(A \equiv B) \to (C \equiv C_{(B/A)})$ in particular, and so **LL** follows by **Imp**. □

Chapter 4

A Modal Semantics for Essence and Ground

Chapter Abstract

Kit Fine (2012c, 2017b,c) has provided a state semantics for a unilateral grounding operator, as well as a state semantics the extensional operators, but does not incorporate the two. Although Fine (2017d, Draft) has devised the resources to construct possible worlds within a modalised state space, he does not employ these resources to interpret modal operators. This chapter draws on the resources which Fine has provided in order to construct a unified semantics by which to interpret a propositional language with primitive operators for ground, metaphysical necessity, and the extensional operators, where essence, relevance, propositional identity, and metaphysical possibility may then be defined in these terms. I conclude by presenting *The Modalised Logic of Essence and Ground* (EGM) which I show to be sound over the semantics.

4.1 Introduction

In an exploratory paper, Kit Fine (2012c) provides a taxonomy of primitive grounding operators, investigating their relationships in the absence of other operators in his *pure* logic of ground. However important such investigations are for first surveying the conceptual landscape in the attempt to identify which notions of ground are the most theoretically viable, Fine's pure logic includes a greater number of primitives than necessary in an expressively richer language. As shown in Chapter 1, all but the weak-full grounding operator '$\leqslant$' can be defined in a language which includes the extensional operators '$\neg$', '$\wedge$', and '$\vee$' for negation, conjunction, and disjunction, respectively.[1] Given this expressively more powerful language, Fine's *impure* logic of ground investigates the grounding relationships between propositions expressed by sentences in which only the extensional operators occur. Accordingly, neither the principle $(A \leqslant B) \rightarrow (A \rightarrow B)$ nor the rule $A, A \leqslant B \vdash B$ can be

[1] By contrast, Fine (2012b, §1.5) employs quantification into sentence position in stating his definitions. Fine also considers defining weak-full ground in terms of strict-full ground which does require further expressive resources: $A \leqslant B \coloneqq \forall X[(B \wedge C < X) \rightarrow (A \wedge C < X)]$.

articulated in Fine's impure logic of ground. Additionally, the metaphysical necessity operator '$\Box$' has not been included in the languages which Fine considers. Although Fine's pure and impure logics of ground make for a natural place to begin, such logics are no place for a satisfactory logic of ground to end. Accordingly, it is the ambition of the present chapter to take the next steps, providing both a semantics and logic for a language with operators for grounding and metaphysical necessity in addition to the extensional operators without imposing artificial restrictions on the set of well-formed sentences. As brought out below, operators for essence, relevance, weak reduction, and propositional identity may then be defined in terms of ground and the extensional operators.

In order to accurately frame the present pursuit, it is worth observing that restricting consideration to the impure logic of ground corresponds to the limitations in the existing semantic resources available for interpreting such grounding languages. Whereas the pure logic of ground only requires the semantics to be capable of interpreting grounding sentences in which two sentence letters flank a grounding operator, the impure logic also requires the semantics to interpret grounding sentences in which two sentences with any number of extensional operators flank a grounding operator. In both cases, Fine draws upon the resources of his *state semantics* where in place of the possible worlds employed in Kripke semantics, Fine considers a range of states-of-affairs— or *states*— ordered by a parthood relation. Although Fine's semantics goes some way towards providing an adequate semantics for his pure and impure logics of ground, certain important lacunas remain in interpreting a wider space of sentences. Specifically, Fine takes sentences which only include extensional operators to be interpreted at model-state pairs, whereas grounding sentences are to be interpreted at models alone. Accordingly, Fine's semantics does not provide the resources needed to interpret sentences such as $A \vee (A \leqslant B)$, motivating the restriction to grounding sentences of the form $\ulcorner A \leqslant B \urcorner$ where $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$ only include extensional operators. Although one could spurn such semantic limitations, attempting to develop a logic for ground independently of the development of a corresponding semantics, there are strong methodological reasons for resisting this temptation.

Developing a logic without knowledge of a corresponding space of models not only risks ending up in inconsistency, but relinquishes use of a powerful tool for systematically exploring the connections between the operators included in the language. This is not, however, to claim that a semantics ought to be developed independently of considering which principles ought to hold in full generality. Rather, it is by gathering together a stock of candidate principles that can be articulated in the language that one may provide a target which a successful semantics may aim to validate. By attempting to provide a definition of the models of the language along with semantic clauses that validate the stock of candidate principles, one may find reason to revise the initial stock of candidate principles adopted on a hunch, clarifying which principles hold without exception. Indeed, the space of models may be thought of as charting

the range of possible counterexamples to the sentences of the language, where it is only by exploring this space that we may arrive with any confidence at principles that may be taken to be free from counterexamples. Additionally, once a semantics has been provided for the language which validates some substantial set principles, one may begin to survey the range of sentences validated by the semantics in order to locate further principles to be included in the logic. Thus it is by trading between the parallel ambitions to develop a semantics and logic for a language that one may gain greater insight into the patterns of valid inference which can be articulated in the language.

Rather than attempting to provide a logic for ground in the absence of a corresponding semantics, Fine's pure and impure logics have been developed in lock-step with associated semantic theories over which each logic may be shown to be sound. Insofar as we are to maintain this methodology, lifting Fine's restriction on the set of well-formed sentences requires that the semantics not only be capable of interpreting a broader space of well-formed sentences, but that a theoretically viable set of sentences come out valid.

Given these ambitions, we may begin by taking the *well-formed sentences* generated from $\mathbb{L} = \{p_i : i \in \mathbb{N}\}$ by the sentential operators '$\neg$','$\wedge$','$\vee$', '$\Box$', and '$\leqslant$' to be defined as follows, where $p \in \mathbb{L}$ is arbitrary:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A \mid \Box A \mid A \leqslant A.$$

Letting $\mathcal{L} = \langle \mathtt{wfs}(\mathbb{L}), \neg, \wedge, \vee, \Box, \leqslant \rangle$ where $\mathtt{wfs}(\mathbb{L})$ is the set of all well-formed sentences, we may define a range of further operators in $\mathcal{L}$ as follows:

Possibility: $\ulcorner \Diamond A \urcorner$ abbreviates $\ulcorner \neg \Box \neg A \urcorner$.

Essence: $\ulcorner A \sqsubseteq B \urcorner$ abbreviates $\ulcorner \neg A \leqslant \neg B \urcorner$.

Relevance: $\ulcorner A \preceq B \urcorner$ abbreviates $\ulcorner A \wedge B \leqslant B \urcorner$.

Weak Reduction: $\ulcorner A \Rightarrow B \urcorner$ abbreviates $\ulcorner (A \sqsubseteq B) \wedge (A \leqslant B) \urcorner$.

Identity: $\ulcorner A \equiv B \urcorner$ abbreviates $\ulcorner (A \Rightarrow B) \wedge (B \Rightarrow A) \urcorner$.

Interdetermination: $\ulcorner A \approx B \urcorner$ abbreviates $\ulcorner (A \leqslant B) \wedge (B \leqslant A) \urcorner$.

Interdependence: $\ulcorner A \asymp B \urcorner$ abbreviates $\ulcorner (A \sqsubseteq B) \wedge (B \sqsubseteq A) \urcorner$.

Rather than beginning with the definitions above, one could begin with an excess of primitive operators in the language, establishing interdefinability results once a semantics and logic have been provided for $\mathcal{L}$. By contrast, I will adopt the metalinguistic abbreviations above for the sake of simplicity, drawing abductive support for these definitions by appealing to both their derived semantic clauses as well as the resulting space of theorems.

Given the primitive symbols of the language together with the definition given in *Relevance*, we may seek to provide a semantics which validates the following core principles, taking $\ulcorner A \rightarrow B \urcorner$ to abbreviate $\ulcorner \neg A \vee B \urcorner$ as usual:

Ref: $A \leqslant A$. **Trans:** $A \leqslant B, B \leqslant C \vdash A \leqslant C$.

DisL: $A \leqslant A \vee B$. **DisR:** $B \leqslant A \vee B$.

Necs: $A \leqslant B \vdash \Box(A \to B)$. **Rel:** $A \leqslant B \vdash A \preceq B$.

Whereas **Ref**, **Trans**, and **Rel** are pure principles of ground, both **DisjL** and **DisR** are impure principles, and **Necs** is an interaction principle. In addition to the tautologies of classical logic and the axioms and rules of an S5 modal logic, I will take the principles above to provide a target which an adequate semantics for $\mathcal{L}$ may aim to validate.[2]

Although by no means exhaustive, the principles above are worth beginning with on account of being especially difficult to reject without changing the object of study. For instance, **Ref** characterises the *weakness* of the intended notion of ground. Although there are *strict* notions of ground in which an analogue of **Ref** will fail to hold for all A, a strict notion of ground is not what '$\leqslant$' is intended to express. Similarly, **Trans** accords with the intended *mediate* understanding of ground, upon which it is not assumed that any given proposition has a "closest" ground as required by *immediate* ground. Additionally, we find that both **DisL** and **DisR** witness the *fullness* of the intended notion of ground, where it being the case that A is enough for it to be the case that $A \vee B$, requiring that nothing in addition to A must obtain. By contrast, neither $A \leqslant A \wedge B$ nor $B \leqslant A \wedge B$ have been included in the stock of principles above, for it being the case that A (similarly B) may fail to be enough for it to be the case that $A \wedge B$. This is not to deny that there are *partial* notions of ground which express that A (similarly B) participates in fully grounding $A \wedge B$. Rather, adopting **DistL** and **DistR** specifies that a partial notion of ground is not the present object of study.

In addition to these requirements, adopting **Necs** and **Rel** provide some indication of the intended reading of '$\leqslant$', where the proposition expressed by the antecedent of a grounding claim must both strictly imply and be wholly relevant to the proposition expressed by the consequent. These requirements indicate the manner in which the weak-full grounding operator '$\leqslant$' is meant to provide a regimentation of constitutive uses of the locution 'sufficient for'. However, this is not to deny that there are perfectly natural modal readings whereby snow being blue is sufficient for 2+2=4 on account of it being necessary that 2+2=4. Rather, it is by adopting **Rel** that we may specify that a modal regimentation of 'sufficient for' is not what '$\leqslant$' is intended to express.

Without providing a full defence of the definitions above, it is nevertheless worth considering the relationship between essence and ground, where essence may be taken to regiment constitutive uses of the locution 'necessary for'. In particular, consider the following characteristic example:

`Gold:` α having 79 protons is *necessary for* α to be gold.

[2] Given the T axiom for metaphysical necessity, it follows from **Necs** that $A, A \leqslant B \vdash B$.

`Gold*`: α failing to have 79 protons is *sufficient for* α to not be gold.

Assuming that essence and ground regiment 'necessary for' and 'sufficient for' respectively, `Gold*` follows from `Gold` by *Essence*. Moreover, assuming that $A \equiv \neg\neg A$, the following are interderivable given *Essence*:

`Red`: The ball being crimson is *sufficient for* the ball to be red.

`Red*`: The ball not being crimson is *necessary for* the ball to not be red.

I will take the plausibility of the claims above to provide evidence not only that essence and ground provide adequate regimentations of constitutive uses of 'necessary for' and 'sufficient for', but that *Essence* accurately describes their relationship. However, this is not to claim that ground is more basic than essence, for $\ulcorner A \leqslant B \urcorner$ may equally be take to abbreviate $\ulcorner \neg A \sqsubseteq \neg B \urcorner$ were we to include '$\sqsubseteq$' in the primitive operators of $\mathcal{L}$ instead of '$\leqslant$'. Accordingly, I will take the choice to include a primitive operator in $\mathcal{L}$ for ground rather than for essence to be a matter of convention alone, and so will refer to the proof system developed below as the logic of both essence and ground.

In the following sections, I will draw on the resources of Fine's state semantics in order to interpret the full range of well-formed sentences of $\mathcal{L}$. Although the state semantics provides the necessary resources for distinguishing hyperintensional contents, validating all of the principles defended above, the semantics does not provide high enough resolution power for distinguishing the propositions expressed by true grounding sentences as well as true modal sentences. The result is comparable to the way in which intensional frameworks can distinguish the propositions expressed by true extensional sentences, but make all true metaphysical modal sentences such as $\ulcorner \Diamond A \urcorner$ express the single necessary proposition modelled by the set of all possible worlds.[3] Whereas the coarse-graining of the semantic values assigned to modal sentences may be tolerated in modal logics by accepting that every modal sentence expresses a non-contingent proposition, there is greater reason to distinguish true grounding sentences in the present context. In particular, it is natural to deny that every true grounding sentence grounds every other true grounding sentence, though this result follows trivially on the semantics provided below. For instance, one might hope to deny that $(A \leqslant A \vee \neg A) \leqslant (B \leqslant B \vee \neg B)$ is valid. Perhaps even more striking, the present account grounds every true modal sentence in any other true modal sentence, despite the fact that it is natural to reject the validity of such claims as $A \rightarrow [\Diamond A \leqslant \Box(B \vee \neg B)]$.

In response, one could restrict the set of well-formed sentences so as to avoid interpreting sentences in which only extensional operators may occur within the scope of a grounding operator. However, in addition to being *ad hoc*, such restriction significantly complicate the semantics and logic. Accordingly, I have left the set of well-formed sentence unrestricted, providing a fully

[3] I am assuming that all worlds see all others. The *modal sentences* must be defined recursively, where any extensional combination of sentences of the form $\Box A$ is modal.

recursive semantics by which any well-formed sentence may be interpreted. Nevertheless, I have not attempted to capture every validity in the logic given below since nothing good is gained by including such erroneous principles as those considered above in the logic. Although the semantics and logic provided below succeeds in its ambition to study a wider range of principles than those which belong to Fine's pure and impure logics of ground by including interaction principles in a language with both modal and grounding operators, a full account of what grounds grounding claims or modal claims is left wanting. Rather than providing a final semantics and logic for $\mathcal{L}$, what is developed below amounts to the next step towards a fully satisfactory theory.

The chapter begins by setting up a modalised state space in §4.2 in the manner of Fine (2017d) in which worlds are constructed out of states. Whereas Fine (2012b) interprets atomic grounding claims at models alone, §4.3 will present a bilateral semantics for $\mathcal{L}$ which relativises the truth and falsity of all sentences of $\mathcal{L}$ to model-state pairs. After showing that every model of the language assigns each $A \in \mathtt{wfs}(\mathcal{L})$ to a corresponding proposition, as well as providing a definition of logical consequence in §4.4, I will compute a range of validities in §4.5. I will then present countermodels in §4.6 which show that the absorption and distribution laws for propositional identity are invalid, and so the resulting space of propositions cannot be said to form a Boolean algebra. Additionally, conjunction fails to be monotonic over ground, and relevance is not transitive. Considering the underlying reasons for these invalidities will shed light on the particular form of hyperintensionality in play, where it will turn out that the space of propositions forms a non-interlaced bilattice. I will then conclude in §5.9 by providing a logic of essence and ground which is sound over the semantics, presenting a range of theorems and admissible rules. Not only is the result natural to reason with, but provides a deeper understanding of the interaction between the operators in the language.

4.2 State Spaces

Having specified the formation rules for $\mathcal{L}$ along with a range of abbreviations, we may now turn to provide the definition of a state space along with a number of elementary results. Following Fine (2017d), I will take a *state space* to be any ordered triple $\mathcal{S} = \langle S, P, \sqsubseteq \rangle$ where S is the *carrier set*, $P \subseteq S$ is a nonempty set, and $\sqsubseteq$ is a partial order on S which satisfies the following:

Upper Bound: s is an *upper bound* of $X \subseteq S$ *iff* $x \sqsubseteq s$ for every $x \in X$.

Least Upper Bound: s is a *least upper bound* of $X \subseteq S$ *iff* s is an upper bound of X, and $s \sqsubseteq y$ for every upper bound y of X.

Completeness: For all $X \subseteq S$, there is a *least upper bound* s of X.

Stability: For all $s \in S$ and $t \in P$, if $s \sqsubseteq t$, then $s \in P$.

As uniqueness is readily proven, we may refer to *the* least upper bound of any $X \subseteq S$ as the *fusion* $\bigsqcup X$, taking $\ulcorner d.t \urcorner$ to abbreviate $\ulcorner \bigsqcup\{d, t\} \urcorner$, and letting '$\sqsubset$' designate the *null state* $\bigsqcup \varnothing$, and '$\blacksquare$' designate the *full state* $\bigsqcup S$.

It will often be convenient to write $\ulcorner s \in \mathcal{S} \urcorner$ and $\ulcorner X \subseteq \mathcal{S} \urcorner$ in place of $\ulcorner s \in S \urcorner$ and $\ulcorner X \subseteq S \urcorner$, respectively. Given this notation, I will follow Fine (2017d) in restricting consideration to world spaces which may be defined as follows:

Compatibility: $s \circ t$ *iff* $s.t \in P$.

Worlds: A state w is an $\mathcal{S}$*-world iff* $w \in P$ and $t \sqsubseteq w$ for every $t \in \mathcal{S}$ where $t \circ w$.

World Space: A state space $\mathcal{S}$ is a *world space iff* for every $t \in P$, there is some $\mathcal{S}$-world w where $t \sqsubseteq w$.

Given any state space $\mathcal{S}$, it will be convenient in what follows to let $\mathbb{W}_{\mathcal{S}}$ be the set of all $\mathcal{S}$-worlds. As brought out below, worlds will play a critical role in defining the logical consequence relation for $\mathcal{L}$. Having provided the definitions above, I will close the section with the following elementary results.

L2.1 $\sqsubset \sqsubseteq s$ for any state space $\mathcal{S}$ and $s \in \mathcal{S}$.

Proof. Letting $s \in \mathcal{S}$ be arbitrary, it follows from *Completeness* that $\bigsqcup \varnothing \sqsubseteq y$ for every y where $x \sqsubseteq y$ for every $x \in \varnothing$. Since $x \sqsubseteq s$ for every $x \in \varnothing$, we may conclude that $\bigsqcup \varnothing \sqsubseteq s$, and so $\sqsubset \sqsubseteq s$ as desired. □

L2.2 $\bigsqcup\{s\} = s$ for all $s \in \mathcal{S}$.

Proof. Let $s \in \mathcal{S}$ be arbitrary. By the definition of fusion, $s \sqsubseteq \bigsqcup\{s\}$, where $\bigsqcup\{s\} \sqsubseteq y$ for every y such that $s \sqsubseteq y$. By *Reflexivity*, $s \sqsubseteq s$, and so $\bigsqcup\{s\} \sqsubseteq s$. Thus $\bigsqcup\{s\} = s$ follows by *Antisymmetry*. □

L2.3 If $x_i \sqsubseteq y_i$ for all $i \in I$, then $\bigsqcup\{x_i : i \in I\} \sqsubseteq \bigsqcup\{y_i : i \in I\}$.

Proof. Assume $x_i \sqsubseteq y_i$ for all $i \in I$. It follows by the definition of fusion that $y_i \sqsubseteq \bigsqcup\{y_i : i \in I\}$ for all $i \in I$, and so $x_i \sqsubseteq \bigsqcup\{y_i : i \in I\}$ for all $i \in I$ by *Transitivity*. Again by the definition of fusion, we may conclude that $\bigsqcup\{x_i : i \in I\} \sqsubseteq \bigsqcup\{y_i : i \in I\}$ as desired. □

L2.4 If $x, y \sqsubseteq z$, then $x.y \sqsubseteq z$.

Proof. Assuming that $x, y \sqsubseteq z$, it follows from **L2.3** that $x.y \sqsubseteq \bigsqcup\{z\}$, and so $x.y \sqsubseteq z$ by **L2.2**. □

L2.5 $\bigsqcup\{\bigsqcup E_i : i \in I\} = \bigsqcup\bigcup\{E_i : i \in I\}$ where $E_i \subseteq S$ for each $i \in I$.

Proof. Given that $s_i \sqsubseteq \bigsqcup E_i$ for any $s_i \in E_i$ and $i \in I$, it follows from **L2.3** that $\bigsqcup\bigcup\{E_i : i \in I\} \sqsubseteq \bigsqcup\{\bigsqcup E_i : i \in I\}$. We may next observe that by the definition of fusion $s_i \sqsubseteq \bigsqcup\bigcup\{E_i : i \in I\}$ for any $s_i \in E_i$ and $i \in I$, and so $\bigsqcup E_i \sqsubseteq \bigsqcup\{\bigsqcup\bigcup\{E_i : i \in I\}\}$ for all $i \in I$ by **L2.3**. By **L2.2**, it follows that $\bigsqcup E_i \sqsubseteq \bigsqcup\bigcup\{E_i : i \in I\}$ for all $i \in I$, and so by the definition of fusion $\bigsqcup\{\bigsqcup E_i : i \in I\} \sqsubseteq \bigsqcup\bigcup\{E_i : i \in I\}$. Thus $\bigsqcup\{\bigsqcup E_i : i \in I\} = \bigsqcup\bigcup\{E_i : i \in I\}$ follows by *Antisymmetry*. □

L2.6 For any state space $\mathcal{S}$ and $u, v \in \mathbb{W}_{\mathcal{S}}$, if $u \circ v$, then $u = v$.

Proof. Let $\mathcal{S}$ be a state space and $u, v \in \mathbb{W}_{\mathcal{S}}$ where $u \circ v$. It follows that $u \sqsubseteq v$ and $v \sqsubseteq u$, and so $u = v$ by anti-symmetry. □

L2.7 $\mathbb{W}_{\mathcal{S}} \neq \varnothing$ for any world space $\mathcal{S}$.

Proof. Letting $\mathcal{S}$ be a world space, there is some $t \in P$, and so some $w \in \mathbb{W}_{\mathcal{S}}$ by *World Space*. Thus $\mathbb{W}_{\mathcal{S}} \neq \varnothing$ as desired. □

4.3 State Semantics

In order to interpret $\mathcal{L}$, we must first specify the range of propositions to which sentence letters may be assigned. Consider the following definitions:

Fusion Closure: $\lceil X \rceil = \{\bigsqcup Y : \varnothing \neq Y \subseteq X\}$.

Simple Contents: $\mathcal{C}_{\mathcal{S}} = \{X \subseteq \mathcal{S} : X = \lceil X \rceil\}$.

Exclusivity: $\langle V, F \rangle$ is *exclusive iff* there is no $t \in V$ and $d \in F$ where $t \circ d$.

Exhaustivity: $\langle V, F \rangle$ is *exhaustive iff* for every $t \in P$, there is some $d \in S$ where $d \circ t$ and either $d \in V$ or $d \in F$.

Propositions: $\langle V, F \rangle$ is an *$\mathcal{S}$-proposition iff* $\langle V, F \rangle \in \mathcal{C}^2_{\mathcal{S}}$ is exclusive and exhaustive.

It will be convenient to let $\mathbb{P}_{\mathcal{S}}$ be the set of all *$\mathcal{S}$-propositions*, where $\mathcal{S}$ is any state space. An *$\mathcal{S}$-model of $\mathcal{L}$* is any ordered quadruple $\mathcal{M}_v = \langle S, P, \sqsubseteq, |\cdot|_v \rangle$ where $\mathcal{S} = \langle S, P, \sqsubseteq \rangle$ is a world space and $|p|_v \in \mathbb{P}_{\mathcal{S}}$ for all $p \in \mathbb{L}$. For any $p \in \mathbb{L}$ and model $\mathcal{M}_v$, I will take $|p|_v = \langle |p|^+_v, |p|^-_v \rangle$, where $|p|^+_v$ are the *exact verifiers* for p in $\mathcal{M}_v$, and $|p|^-_v$ are the *exact falsifiers* for p in $\mathcal{M}_v$.

As a useful heuristic, we may consider an intended state space $\mathcal{S}_I$, where the states in $\mathcal{S}_I$ are taken to be ordered tuples $\langle \tau^n, \vec{\alpha} \rangle$ where τ^n is an n-place property instance— or *trope*— and $\vec{\alpha}$ is a sequence of n entities. For instance, we may consider the state of me sitting in the particular manner in which I am currently sitting. Additionally, P_I may be taken to include the *possible states* which could obtain, where S_I/P_I includes all other *impossible states*. By the lights of the intended state space, *worlds* are maximal possible states which include every state with which they are compatible as a part. Although worlds will play an important role in the definition of logical consequence for the language $\mathcal{L}$, the sentences of $\mathcal{L}$ will nevertheless be evaluated at states, whether possible or not, thereby admitting of many more distinctions between propositions than can be drawn in an intensional framework.

It remains to extend the interpretation of the sentence letters in $\mathbb{L}$ specified by any $\mathcal{S}$-model of $\mathcal{L}$ to every well-formed sentence in $\texttt{wfs}(\mathbb{L})$. Accordingly, I will now turn to present a state semantics for $\mathcal{L}$. Building on Fine's (2017b,c,d) state semantics for the extensional operators and ground, we may define the exact verification $\Vdash$ and exact falsification $\dashv\Vdash$ relations for any $\mathcal{S}$-model $\mathcal{M}_v$, state $s \in \mathcal{S}$, and $A, B \in \texttt{wfs}(\mathbb{L})$ as follows:

$(p_i)^+$ $\mathcal{M}_v, s \Vdash p_i$ *iff* $s \in |p_i|_v^+$. $(\neg)^+$ $\mathcal{M}_v, s \Vdash \neg A$ *iff* $\mathcal{M}_v, s \dashv\Vdash A$.

$(p_i)^-$ $\mathcal{M}_v, s \dashv\Vdash p_i$ *iff* $s \in |p_i|_v^-$. $(\neg)^-$ $\mathcal{M}_v, s \dashv\Vdash \neg A$ *iff* $\mathcal{M}_v, s \Vdash A$.

$(\wedge)^+$ $\mathcal{M}_v, s \Vdash A \wedge B$ *iff* $s = d.t$ where $\mathcal{M}_v, d \Vdash A$ and $\mathcal{M}_v, t \Vdash B$.

$(\wedge)^-$ $\mathcal{M}_v, s \dashv\Vdash A \wedge B$ *iff* $\mathcal{M}_v, s \dashv\Vdash A$ or $\mathcal{M}_v, s \dashv\Vdash B$ or $\mathcal{M}_v, s \dashv\Vdash A \vee B$.

$(\vee)^+$ $\mathcal{M}_v, s \Vdash A \vee B$ *iff* $\mathcal{M}_v, s \Vdash A$ or $\mathcal{M}_v, s \Vdash B$ or $\mathcal{M}_v, s \Vdash A \wedge B$.

$(\vee)^-$ $\mathcal{M}_v, s \dashv\Vdash A \vee B$ *iff* $s = d.t$ where $\mathcal{M}_v, d \dashv\Vdash A$ and $\mathcal{M}_v, t \dashv\Vdash B$.

$(\Box)^+$ $\mathcal{M}_v, s \Vdash \Box A$ *iff* for all $t \in P$ there is some $d \circ t$ where $\mathcal{M}_v, d \Vdash A$.

$(\Box)^-$ $\mathcal{M}_v, s \dashv\Vdash \Box A$ *iff* there is some $t \in P$ where $\mathcal{M}_v, t \dashv\Vdash A$.

$(\leqslant)^+$ $\mathcal{M}_v, s \Vdash A \leqslant B$ *iff* each of the following conditions hold:
(1) for all $t \in \mathcal{S}$, if $\mathcal{M}_v, t \Vdash A$, then $\mathcal{M}_v, t \Vdash B$;
(2) for all $t \in \mathcal{S}$, if $\mathcal{M}_v, t \dashv\Vdash B$, then $\mathcal{M}_v, d \dashv\Vdash A$ for some $d \sqsubseteq t$; and
(3) for all $t, d \in \mathcal{S}$, if $\mathcal{M}_v, t \dashv\Vdash A$ and $\mathcal{M}_v, d \dashv\Vdash B$, then $\mathcal{M}_v, t.d \dashv\Vdash B$.

$(\leqslant)^-$ $\mathcal{M}_v, s \dashv\Vdash A \leqslant B$ *iff* any of the following conditions hold:
(1) $\mathcal{M}_v, t \Vdash A$ and $\mathcal{M}_v, t \nVdash B$;
(2) $\mathcal{M}_v, t \dashv\Vdash B$ and $\mathcal{M}_v, d \not\dashv\Vdash A$ for all $d \sqsubseteq t$; or
(3) $t = d.r$ where $\mathcal{M}_v, d \dashv\Vdash A$ and $\mathcal{M}_v, r \dashv\Vdash B$ but $\mathcal{M}_v, t \not\dashv\Vdash B$.

Fine (2016, 2017b,c,d) provides the semantics for the extensional operators above, crediting Van Fraassen (1969) for developing an early account in this vicinity. Whereas the clauses for negation specify that negation inverts the exact verifiers and falsifiers for the sentence negated, the clause for conjunction verification requires that every exact verifier for a conjunction is a fusion of exact verifiers for each of its conjuncts. Additionally, the exact falsifiers for a conjunction include the exact falsifiers for either conjunct together with any fusion of exact falsifiers for both conjuncts. The clauses for disjunction are the result of exchanging exact verification and exact falsification in the clauses for conjunction. Although we may derive the semantics for disjunction from the semantics for negation and conjunction by taking $\ulcorner A \vee B \urcorner$ to abbreviate $\ulcorner \neg(\neg A \wedge \neg B) \urcorner$, I have maintained disjunction as a primitive operator so as to highlight the systematic connections between the operators. In particular, the De Morgan laws for propositional identity turn out to be valid.

Despite developing the resources to construct worlds in the manner given above, Fine (2017d, Draft) does not apply these resources to interpret the metaphysical modal operators. Accordingly, the semantics for $\mathcal{L}$ departs from Fine's constructions by providing semantic clauses for metaphysical necessity. In particular, the verifier clause requires that every possibility is compatible with an exact verifier for the operand, and so since every world is possible, every world is compatible with an exact verifier for the operand. Thus it follows by the definition given in *Worlds* that every world has a part which exactly verifies the operand. Additionally, it follows from the falsifier clause that there is some possibility which exactly falsifies the operand, and so by *World Space*,

there is some world with a part which exactly falsifies the operand. However natural these consequences may be, the exact verifier and falsifier states for a modal sentence do not play any role in the conditions provided by the semantic clauses. It follows that if a modal sentence is exactly verified by any state, then it is exactly verified by every state, and exactly falsified by no states, and so every true modal sentence expresses the same proposition. This failure of the present framework to distinguish the propositions expressed by true modal sentences, although regrettable, does not cause any distortion in interpreting the target range of interaction principles in the language.

By contrast with the semantics for metaphysical necessity, elements of the semantics for ground are present throughout Fine's work, where the semantic relation corresponding to (1) in the verification clause for ground appears in Fine (2012b,c, 2017c) under the names 'entailment' and 'disjunctive part'. The intuitive idea underlying the verification clause for ground is that if $A \leqslant B$, then whatever state exactly verifies A, must also exactly verify B. In this respect, (1) is similar to the semantic clauses for the material and strict conditionals, only that truth-values and worlds have been replaced with states. Just as the rules $A, A \to B \vdash B$ and $\Box A, \Box(A \to B) \vdash \Box B$ are valid for the material and strict conditional on an extensional and intensional semantics respectively, (1) ensures that $A, A \leqslant B \vdash B$ is valid for ground over a hyperintensional semantics. Looking ahead, the validity of $A, A \leqslant B \vdash B$ requires that every world with parts which exactly verify A and $A \leqslant B$ will also have a part which exactly verifies B. However, given any world w with parts d and t which exactly verify A and $A \leqslant B$ respectively, it follows from (1) in the semantics that d exactly verifies B. Thus w has a part which exactly verifies B.

In providing a semantics for ground, Fine restricts consideration to *unilateral propositions* such as given by the members of $\mathcal{C}_{\mathcal{S}}$ which only specify a set of exact verifiers. In a unilateral setting, the semantic relation given by (1) in the verification clause for ground will suffice to provide a semantics for ground taken on its own. However, a unilateral semantics cannot interpret negation by inverting the exact verifiers and falsifiers for the operand as in the semantics above. Since the exact verifiers for a sentence do not determine the exact verifiers for its negation, there is little hope of interpreting negation in a unilateral semantics. For instance, the set of exact verifiers for 'I am sitting' do not specify which states exactly verify 'I am not sitting'. In particular, the states which are incompatible with an exact verifier for 'I am sitting' need not exactly verify 'I am not sitting', diverging sharply from a possible worlds semantics in which negation is interpreted as mapping propositions to their complements within the set of worlds. It is by drawing on such considerations that Fine motivates the bilateral semantics for the extensional operators, but does not go on to provide a bilateral semantics for ground.

In defence of the bilateral semantics for ground given above, we may begin by observing that just as a grounding version of *modus ponens* is valid, one might equally expect a grounding version of *modus tollens* to be valid. Assuming a world w has parts d and t which exactly verify $\neg B$ and $A \leqslant B$

respectively, it follows by the semantics for negation that d exactly falsifies B. However, taken on its own, (1) in the semantics for ground does not specify any relationship between the exact falsifiers for the antecedent and consequent of a true grounding claim, and so we cannot conclude that w has a part which exactly verifies $\neg A$. Rather, it is (2) in the verification clause for ground which requires that every exact falsifier for the consequent include a part which exactly falsifies the antecedent. Given that t exactly verifies $A \leqslant B$ and d has been shown to exactly falsify B, it follows that d has a part r which exactly falsifies A, and so r exactly verifies $\neg A$ by the semantics for negation. By the transitivity of parthood, there is a part of w which exactly verifies $\neg A$, and so $\neg B, A \leqslant B \vdash \neg A$ comes out valid as one might expect.

By contrast with (2), the inclusion of (3) may be motivated by appeal to the relevance requirement imposed by **Rel**, where A must be wholly relevant to B if $A \leqslant B$. In order to bring this out, suppose that '$\preceq$' were to have been included as a primitive operator of $\mathcal{L}$ with the following semantic clauses:

$(\preceq)^+$ $\mathcal{M}_v, s \Vdash A \preceq B$ *iff* both of the following conditions hold:
(a) for all $t, d \in \mathcal{S}$, if $\mathcal{M}_v, t \Vdash A$ and $\mathcal{M}_v, d \Vdash B$, then $\mathcal{M}_v, t.d \Vdash B$.
(b) for all $t, d \in \mathcal{S}$, if $\mathcal{M}_v, t \dashv\Vdash A$ and $\mathcal{M}_v, d \dashv\Vdash B$, then $\mathcal{M}_v, t.d \dashv\Vdash B$.

$(\preceq)^-$ $\mathcal{M}_v, s \Vdash A \preceq B$ *iff* either of the following conditions hold:
(a) $\mathcal{M}_v, t.d \nVdash B$ for some t and d where $\mathcal{M}_v, t \Vdash A$ and $\mathcal{M}_v, d \Vdash B$.
(b) $\mathcal{M}_v, t.d \not\dashv\Vdash B$ for some t and d where $\mathcal{M}_v, t \dashv\Vdash A$ and $\mathcal{M}_v, d \dashv\Vdash B$.

Intuitively, the semantics for relevance excludes the scenario where some exact verifier for the antecedent is irrelevant to the consequent on account of failing to be a part of any exact verifier for the consequent, where something similar may be said for the exact falsifiers. This constraint is achieved by requiring any fusion of exact verifiers for the antecedent and consequent to exactly verify the consequent, and any fusion of exact falsifiers for the antecedent and consequent to exactly falsify the consequent, where the falsification clause for relevance is just the negation of the verification clause for relevance.

Given the semantics for relevance, we may observe that (1) in the verification clause for ground entails (a) in the verification clause for relevance. Assuming that $A \leqslant B$ where t and d exactly verify $A, B \in \mathbb{L}$ respectively, it follows from (1) that t exactly verifies B, and so by *Closure*, $t.d$ exactly verifies B in compliance with (a).[4] Although this goes some ways towards validating **Ref**, neither (1) nor (2) entail (b), and so (b) has been included in the semantics for ground as (3) so that the semantics for ground requires the antecedent to be wholly relevant to the consequent, thereby validating **Ref**.

A further manner of justifying the semantics for ground extends the thought that ground is to be likened to a disjunctive parthood relation within a bilateral semantics. In particular, suppose that one were to take propositional identity to be primitive in place of ground, where $\ulcorner A \leqslant B \urcorner$ is adopted as an abbreviation

[4] Extending this argument to all $A, B \in \mathtt{wfs}(\mathbb{L})$ requires showing that the exact verifiers for any $A \in \mathtt{wfs}(\mathbb{L})$ are closed under fusion as in **P2** below.

for $\ulcorner A \vee B \equiv B \urcorner$. On this supposition, we may derive the semantics for ground given above from the following semantics for propositional identity:

$(\equiv)^+$ $\mathcal{M}_v, s \Vdash A \equiv B$ *iff* for all $t \in \mathcal{S}$ both of the following conditions hold:
(1) $\mathcal{M}_v, t \Vdash A$ if and only if $\mathcal{M}_v, t \Vdash B$; and
(2) $\mathcal{M}_v, t \dashv\Vdash A$ if and only if $\mathcal{M}_v, t \dashv\Vdash B$.

$(\equiv)^-$ $\mathcal{M}_v, s \dashv\Vdash A \equiv B$ *iff* for some $t \in \mathcal{S}$ any of the following conditions hold:
(1) $\mathcal{M}_v, t \Vdash A$ and $\mathcal{M}_v, t \nVdash B$, or
(2) $\mathcal{M}_v, t \Vdash B$ and $\mathcal{M}_v, t \nVdash A$; or
(3) $\mathcal{M}_v, t \dashv\Vdash A$ and $\mathcal{M}_v, t \not\dashv\Vdash B$; or
(4) $\mathcal{M}_v, t \dashv\Vdash B$ and $\mathcal{M}_v, t \not\dashv\Vdash A$.

The semantics for propositional identity takes $A \equiv B$ to be exactly verified by all states just in case A and B have the same exact verifiers and falsifiers. Were one to take both '$\leqslant$' and '$\equiv$' to be primitive operators of $\mathcal{L}$ along with their respective semantic clauses given above, we may show that the following principles are exactly verified by all states in any model of $\mathcal{L}$:

$\vee$-Part: $(A \leqslant B) \leftrightarrow (A \vee B \equiv B)$.

$\wedge$-Part: $(A \sqsubseteq B) \leftrightarrow (A \wedge B \equiv B)$.

These principles bear witness to the idea that ground is interdefinable with disjunctive part, and similarly that essence is interdefinable with conjunctive part. That the semantics warrants these connections provides auxiliary support for adopting the semantics for $\mathcal{L}$ given above.

It is worth observing that whereas (1) in the verification clause for ground accords with what Fine calls disjunctive part, the combination of (2) and (3) diverge from the semantic relation that Fine (2017b, p. 649) considers under the names 'containment' and 'conjunctive part'. In order to draw out the contrast, consider the following definitions that Fine (2016, pp. 206-7) provides:

Subsumption: $X \gg Y$ *iff* for every $x \in X$, there is some $y \in Y$ where $y \sqsubseteq x$.

Subservience: $X \ll Y$ *iff* for every $x \in X$, there is some $y \in Y$ where $x \sqsubseteq y$.

Containment: $X \sqsubseteq\!\!\!\!\sqsubseteq Y$ *iff* $(X \ll Y) \wedge (Y \gg X)$.

Given this notation, (2) in the verification clause for ground requires that $\{s : \mathcal{M}_v, s \dashv\Vdash B\} \gg \{s : \mathcal{M}_v, s \dashv\Vdash A\}$. However, the conjunction of (2) and (3) does not amount to requiring $\{s : \mathcal{M}_v, s \dashv\Vdash A\} \sqsubseteq\!\!\!\!\sqsubseteq \{s : \mathcal{M}_v, s \dashv\Vdash B\}$ for (3) differs from the requirement that $\{s : \mathcal{M}_v, s \dashv\Vdash A\} \ll \{s : \mathcal{M}_v, s \dashv\Vdash B\}$.

We may further sharpen the contrast by supposing that one were to replace (3) in the verification clause for ground with the subsumption requirement that $\{s : \mathcal{M}_v, s \dashv\Vdash A\} \ll \{s : \mathcal{M}_v, s \dashv\Vdash B\}$. By then letting $\mathcal{M}_x$ be an $\mathcal{S}$-model where $\{s : \mathcal{M}_x, s \dashv\Vdash p_1\} \neq \varnothing$ and $\{s : \mathcal{M}_x, s \dashv\Vdash p_2\} = \varnothing$ for $p_1, p_2 \in \mathbb{L}$, it follows from

the semantics for disjunction that $\{s : \mathcal{M}_x, s \dashv\Vdash p_1 \vee p_2\} = \varnothing$. Since $X \nleftarrow \varnothing$ for any nonempty X, it follows that $\{s : \mathcal{M}_x, s \dashv\Vdash p_1\} \nleftarrow \{s : \mathcal{M}_x, s \dashv\Vdash p_1 \vee p_2\}$, and so $\mathcal{M}_x, s \nVdash p_1 \leqslant p_1 \vee p_2$ for any $s \in \mathcal{S}$. By similar reasoning, we may also show that $A \sqsubseteq A \wedge B$ admits of counterexamples. By contrast, maintaining (3) in the verification clause for ground makes $\mathcal{M}_v, s \Vdash p_1 \leqslant p_1 \vee p_2$ for any $s \in \mathcal{S}$ on account of the fact that (3) holds trivially whenever there are no exact falsifiers for either the antecedent or consequent of the grounding sentence in question. Given that $A \leqslant A \vee B$ was included among the core grounding principles, we find an initial source of motivation for adopting (3) in the verification clause for ground rather than the subsumption requirement.

Despite these considerations in favour of adopting (3), one may choose instead to restrict consideration to the *non-vacuous* $\mathcal{S}$-propositions in which neither content is empty.[5] Although this avoids the troubles brought out above, it is natural to complain that such a response is *ad hoc*. As I show in §5.9, the non-vacuity constraint also distort the natural symmetry between the upper and lower bounds within the space of propositions. Additionally, the non-vacuity constraint is incompatible with the present semantics for metaphysical necessity and ground, for true modal and grounding sentences do not have any falsifiers. However, even if one were to adapt the semantics to accommodate the non-vacuity of propositions, these restrictions fail to address the fact that maintaining the subsumption requirement in place of (3) admits counterexamples to both **$\vee$-Part** and **$\wedge$-Part** given above.[6,7]

The counterexamples to **$\vee$-Part** and **$\wedge$-Part** turn on the non-convexity of the propositions included in $\mathbb{P}_{\mathcal{S}}$.[8] In order to defend **$\vee$-Part** and **$\wedge$-Part** while replacing (3) in the semantics for ground with the subsumption condition, one may require propositions to be convex and non-vacuous as follows:

Convex Closure: $\lfloor X \rfloor = \{y : x, z \in X \text{ and } x \sqsubseteq y \sqsubseteq z\}$.

Regular Contents: $\mathcal{R}_{\mathcal{S}} = \{X \subseteq \mathcal{S} : \varnothing \neq X = \lceil X \rceil = \lfloor X \rfloor\}$.

Regular Propositions: $\mathbb{P}^R_{\mathcal{S}} = \mathbb{P}_{\mathcal{S}} \cap \mathcal{R}^2_{\mathcal{S}}$.

[5] For instance, see Fine (2017d, p. 649) for this restriction.

[6] *Countermodel:* Let $\mathcal{M}_* = \langle S, P, \subseteq, |\cdot| \rangle$ where $S = P = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c, d\}\}$ as well as $\{s \in S : \mathcal{M}_*, s \dashv\Vdash p_1\} = \{\{a\}, \{a, b\}\}$ and $\{s \in S : \mathcal{M}_*, s \dashv\Vdash p_2\} = \{\{a, c\}, \{a, b, c, d\}\}$. It follows by the semantics that $\{s \in S : \mathcal{M}_*, s \dashv\Vdash p_1 \vee p_2\} = \{\{a, c\}, \{a, b, c\}, \{a, b, c, d\}\}$. Since $\mathcal{M}_*, \{a, b, c\} \dashv\Vdash p_1 \vee p_2$ and $\mathcal{M}_*, \{a, b, c\} \not\dashv\Vdash p_2$, and so $\mathcal{M}_*, \{a, b, c\} \Vdash \neg(p_1 \vee p_2)$ but $\mathcal{M}_*, \{a, b, c\} \nVdash \neg p_2$ by the semantics, it follows that $\mathcal{M}_*, s \nVdash \neg(p_1 \vee p_2) \leqslant \neg p_2$ for any $s \in S$, and so $\mathcal{M}_*, s \nVdash p_1 \vee p_2 \sqsubseteq p_2$ for any $s \in S$ by *Essence*. Thus $\mathcal{M}_*, s \nVdash p_1 \vee p_2 \equiv p_2$ for any $s \in S$ by the semantics. Nevertheless, $\mathcal{M}_*, s \Vdash p_1 \leqslant p_2$ holds for any $s \in S$ if (3) is replaced with subsumption, and so $\mathcal{M}_*, s \nVdash (p_1 \leqslant p_2) \leftrightarrow (p_1 \vee p_2 \equiv p_2)$ again by the semantics. $\diamond$

[7] *Sketch:* If **$\wedge$-Part** holds for all $A, B \in \mathtt{wfs}(\mathbb{L})$, then $(\neg A \sqsubseteq \neg B) \leftrightarrow (\neg A \wedge \neg B \equiv \neg B)$, and so $(\neg\neg A \leqslant \neg\neg B) \leftrightarrow (\neg A \wedge \neg B \equiv \neg B)$ by *Essence*. However, we may also show that $(\neg A \wedge \neg B \equiv \neg B) \leftrightarrow [\neg(A \vee B) \equiv \neg B]$ and $[\neg(A \vee B) \equiv \neg B] \leftrightarrow (A \vee B \equiv B)$, as well as $(A \leqslant B) \leftrightarrow (\neg\neg A \leqslant \neg\neg B)$. Thus it follows that $(A \leqslant B) \leftrightarrow (A \vee B \equiv B)$. Accordingly, if **$\vee$-Part** admits of counterexamples, then so will **$\wedge$-Part**.

[8] See Fine (2017b, p. 657) for a proof of semantic analogues of **$\vee$-Part** and **$\wedge$-Part** which makes essential use of a convexity constraint on propositions.

Even if every sentence letter is assigned to a regular proposition, convexity is not preserved by the semantics. Consider the model $\mathcal{M}_{\#} = \{\mathcal{P}(\mathbb{Z}), \{\varnothing\}, \subseteq, |\cdot|_{\#}\}$ where $|p_i|_{\#} = \langle\{\{i\}\}, \{\{-i\}\}\rangle$ for all $p_i \in \mathbb{L}$. Although $|p_i|_{\#} \in \mathbb{P}^{R}_{\mathcal{S}_{\#}}$ for every $p_i \in \mathbb{L}$, it follows from the semantics that both $\mathcal{M}_{\#}, \{1\} \Vdash p_1 \vee (p_1 \wedge p_2 \wedge p_3)$ and $\mathcal{M}_{\#}, \{1,2,3\} \Vdash p_1 \vee (p_1 \wedge p_2 \wedge p_3)$ despite the fact that by the semantics $\mathcal{M}_{\#}, \{1,2\} \nVdash p_1 \vee (p_1 \wedge p_2 \wedge p_3)$ even though $\{1\} \subseteq \{1,2\} \subseteq \{1,2,3\}$. Thus the set of exact verifiers for $p_1 \vee (p_1 \wedge p_2 \wedge p_3)$ is not convex, and so $p_1 \vee (p_1 \wedge p_2 \wedge p_3)$ does not express a regular proposition. Rather than revising the semantics for conjunction and disjunction so as to preserve convexity, or else denying the validity of **$\vee$-Part** and **$\wedge$-Part**, I will maintain (3) in the semantics for ground in place of the subsumption condition.[9]

Given this defence of the verification clause for ground, the falsification clause amounts to the negation of the verification clause. As brought out in **L4.16**, we may derive the semantics for relevance and propositional identity given above from their definitions, and so there is no need to take either '$\preceq$' or '$\equiv$' to be primitives of the language. Additionally, as in the semantics for metaphysical necessity, we may observe that if a grounding sentence is exactly verified by any state, then it is exactly verified by every state, and exactly falsified by no states. Similarly, if a grounding sentence is exactly falsified by any state, then it is exactly falsified by every state and exactly verified by no states. This completes an initial defence of the semantics above.

Given the semantics for the primitive operators included in $\mathcal{L}$, we may derive corresponding semantic clauses for the range of abbreviations indicated above. In particular, the semantics for essence will exchange exact falsification with exact verification in the conditions included in the semantics for ground. Were one to include '$\sqsubseteq$' in addition to '$\leqslant$' in the primitive operators in $\mathcal{L}$, where each was given its respective semantic clauses, we may show that the following principles are exactly verified by every state in any model of $\mathcal{L}$:

GE: $(A \sqsubseteq B) \leftrightarrow (\neg A \leqslant \neg B)$.

EG: $(A \leqslant B) \leftrightarrow (\neg A \sqsubseteq \neg B)$.

Although (Fine 2017c, pp. 700-1) does not provide a bilateral semantics for ground, Fine (2017b, p. 661) nevertheless establishes semantic analogues of the principles above. However, since Fine adopts subsumption instead of the relation given by (3) in the verification clause for ground, Fine must restrict consideration to the space of non-vacuous propositions. By contrast, no similar restriction is required by the semantic relations included above.

4.4 Logical Consequence

Having defended the semantic clauses for the primitive operators in $\mathcal{L}$, we may now turn to provide a definition of logical consequence for $\mathcal{L}$. I will begin by

[9] See §6 in Chapter 3 for further arguments against such alternative semantic clauses.

showing that every $\mathcal{S}$-model of $\mathcal{L}$ assigns each $A \in \texttt{wfs}(\mathbb{L})$ to a proposition in $\mathbb{P}_{\mathcal{S}}$. Accordingly, it will help to expand the domain of the interpretation function $|\cdot|_v$ specified by a model $\mathcal{M}_v$ to include all $A \in \texttt{wfs}(\mathbb{L})$ of $\mathcal{L}$ as follows:

Valuation: $\mathcal{S} \in |A|_v^+$ *iff* $\mathcal{M}_v, \mathcal{S} \Vdash A$.
$\mathcal{S} \in |A|_v^-$ *iff* $\mathcal{M}_v, \mathcal{S} \dashv\Vdash A$.

Adopting $|A|_v = \langle |A|_v^+, |A|_v^- \rangle$ as standard notation for the proposition expressed by A in $\mathcal{M}_v$, it will often be convenient to leave off the subscript when ambiguity does not threaten. In order to show that $|A| \in \mathbb{P}_{\mathcal{S}}$ for every $A \in \texttt{wfs}(\mathbb{L})$ and $\mathcal{S}$-model $\mathcal{M}$, I will define semantic analogues of the syntactic operators in $\mathcal{L}$ which, given a state space $\mathcal{S}$, operate on the propositions in $\mathbb{P}_{\mathcal{S}}$ as follows:

Negation: $\neg\langle V, F\rangle = \langle F, V\rangle$.

Product: $X \wedge Y = \{x.y : x \in X,\ y \in Y\}$.

Sum: $X \vee Y = X \cup Y \cup (X \wedge Y)$.

Conjunction: $\langle V, F\rangle \wedge \langle V', F'\rangle = \langle V \wedge V', F \vee F'\rangle$.

Disjunction: $\langle V, F\rangle \vee \langle V', F'\rangle = \langle V \vee V', F \wedge F'\rangle$.

Positive Necessity: $\Box^+\langle V, F\rangle = \{s \in \mathcal{S} : \forall_{t \in P} \exists_{d \in V} (t \circ d)\}$.

Negative Necessity: $\Box^-\langle V, F\rangle = \{s \in \mathcal{S} : P \cap F \neq \varnothing\}$.

Necessity: $\Box\langle V, F\rangle = \langle \Box^+\langle V, F\rangle, \Box^-\langle V, F\rangle\rangle$.

Entailment: $\langle V, F\rangle \leqslant^+ \langle V', F'\rangle = \{s \in \mathcal{S} : \langle V, F\rangle \vee \langle V', F'\rangle = \langle V', F'\rangle\}$

Containment: $\langle V, F\rangle \leqslant^- \langle V', F'\rangle = \{s \in \mathcal{S} : \langle V, F\rangle \vee \langle V', F'\rangle \neq \langle V', F'\rangle\}$

Grounding: $\langle V, F\rangle \leqslant \langle V', F'\rangle = \langle \langle V, F\rangle \leqslant^+ \langle V', F'\rangle,\ \langle V, F\rangle \leqslant^- \langle V', F'\rangle\rangle$.

By the formation rules, $\mathcal{L} = \langle \texttt{wfs}(\mathbb{L}), \neg, \wedge, \vee, \Box, \leqslant\rangle$ is an algebra with the same signature as $\mathcal{A}_{\mathcal{S}} = \langle \mathbb{P}_{\mathcal{S}}, \neg, \wedge, \vee, \Box, \leqslant\rangle$, where the valuation function induced by any $\mathcal{S}$-model $\mathcal{M}$ is an $\mathcal{L}$-homomorphism $|\cdot| : \mathcal{L} \to \mathcal{A}_{\mathcal{S}}$ insofar as:

Homomorphism: For any state space $\mathcal{S}$ and $\mathcal{S}$-model $\mathcal{M}$, the valuation function $|\cdot| : \mathcal{L} \to \mathcal{A}_{\mathcal{S}}$ is an *$\mathcal{L}$-homomorphism iff* for any $A, B \in \texttt{wfs}(\mathbb{L})$: (1) $|\neg A| = \neg|A|$; (2) $|A \wedge B| = |A| \wedge |B|$; (2) $|A \vee B| = |A| \vee |B|$; (4) $|\Box A| = \Box|A|$; and (5) $|A \leqslant B| = |A| \leqslant |B|$.

P1 $\mathcal{A}_{\mathcal{S}}$ is an algebra for any state space $\mathcal{S}$.

P2 $|\cdot| : \mathcal{L} \to \mathcal{A}_{\mathcal{S}}$ is an $\mathcal{L}$-homomorphism for any $\mathcal{S}$-model $\mathcal{M}$.

Given that every $\mathcal{S}$-model assigns each $A \in \texttt{wfs}(\mathbb{L})$ to a proposition in $\mathbb{P}_{\mathcal{S}}$, we may now move to define validity in $\mathcal{L}$. However, instead of evaluating $A \in \texttt{wfs}(\mathbb{L})$ for validity by requiring a sentence to be exactly verified by all model-state pairs, we may begin by defining the broader notions of the truth and falsity of $A \in \texttt{wfs}(\mathbb{L})$ at an $\mathcal{S}$-model $\mathcal{M}$ and world $w \in \mathbb{W}_{\mathcal{S}}$— i.e., $\mathcal{M}, w \models A$ and $\mathcal{M}, w \dashv A$— by requiring w to contain a part s which exactly verifies or falsifies for A, respectively. More specifically, consider the following definitions:

Truth: $\mathcal{M}, w \models A$ *iff* there is some $s \sqsubseteq w$ where $\mathcal{M}, s \Vdash A$.[10]

Falsity: $\mathcal{M}, w \dashv A$ *iff* there is some $s \sqsubseteq w$ where $\mathcal{M}, s \dashv\!\Vdash A$.

Whereas classical tautologies such as $A \vee \neg A$ may be shown to be true at all worlds $w \in \mathbb{W}_{\mathcal{S}}$ for any $\mathcal{S}$-model $\mathcal{M}$, $A \vee \neg A$ will typically not be exactly verified by ever state in $\mathcal{S}$, thereby justifying the introduction of the broader notions of truth and falsity at a world in a model. In contrast to the verifier and falsifier clauses in the exact semantics, the following lemma shows that truth and falsity at a world in a model are interdefinable such that further consideration of falsity may be left out entirely.

L4.9 For any $\mathcal{S}$-models $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$: $\mathcal{M}, w \dashv A$ *iff* $\mathcal{M}, w \not\models A$.

Whereas in the bilateral semantics, the exact verifiers for a sentence A do not determine the exact falsifiers for A, the definitions of truth and falsity given above restores a degree of classicality to the semantics. Accordingly, we may avoid further use of '$\dashv$'— employing '$\not\models$' in place of '$\dashv$'— thereby justifying the focus on truth over falsity in defining logical consequence:

Logical Consequence: $\Gamma \models A$ *iff* for any $\mathcal{S}$-model $\mathcal{M}$ and $s \in \mathbb{W}_{\mathcal{S}}$, if $\mathcal{M}, s \models B$ for all $B \in \Gamma$, then $\mathcal{M}, s \models A$.

As usual, we may say that A is *valid* just in case $\models A$, where $\Gamma \not\models A$ just in case $\Gamma \models A$ does not hold on account of there being some $\mathcal{S}$-model $\mathcal{M}$ and $s \in \mathbb{W}_{\mathcal{S}}$ where $\mathcal{M}, s \models B$ for all $B \in \Gamma$, but $\mathcal{M}, s \not\models A$. Given these definition, I will compute a range of valid principles in the following section, followed by a range of invalidities in §4.6, providing greater insight into the general principles for essence and ground. As a first step towards establishing these results, I will conclude the section by establishing the following results.

[10] The notions of truth and falsity may be contrasted with Fine's (2017b,d) notions of both loose and inexact verification and falsification, where focusing on verification, s *inexactly verifies* a sentence A in a model $\mathcal{M}$ just in case there is some $t \sqsubseteq s$ where t exactly verifies A in $\mathcal{M}$, and s *loosely verifies* a sentence A in a model $\mathcal{M}$ just in case s is incompatible with any t which exactly falsifies A in $\mathcal{M}$. The definition of inexact verification is only adequate in an $\mathcal{S}$-model if *Weak Supplementation* holds such that for all $s, t \in \mathcal{S}$, if $s \sqsubseteq t$ and $t \not\sqsubseteq s$, then there is some $r \in \mathcal{S}$ where $r \sqsubseteq t$ but no $d \in \mathcal{S}$ where both $d \sqsubseteq s$ and $d \sqsubseteq r$. See Deigan (2019) for an example of how things unravel if *Weak Supplementation* is overlooked.

L4.1 $X \subseteq \lceil X \rceil$ for any $X \subseteq \mathcal{S}$ and state space $\mathcal{S}$.

Proof. Immediate from **L2.2**. □

L4.2 $\lceil\lceil X\rceil\rceil = \lceil X \rceil$ for any $X \subseteq \mathcal{S}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space and $X \subseteq \mathcal{S}$. If $X = \varnothing$, then $\lceil\lceil X\rceil\rceil = \lceil X \rceil$ since $\lceil\varnothing\rceil = \varnothing$ follows immediately from *Closure*. Assume instead that X is nonempty. Given **L4.1**, we need only show that $\lceil\lceil X\rceil\rceil \subseteq \lceil X \rceil$. Letting $s \in \lceil\lceil X\rceil\rceil$ there is some $Y \subseteq \lceil X \rceil$ where $s = \bigsqcup Y$. Letting $Y = \{t_i : i \in I\}$, it follows for each $i \in I$ that $t_i = \bigsqcup Z_i$ for some $Z_i \subseteq X$. Consider:

$$\begin{aligned} s &= \bigsqcup\{t_i : i \in I\} \\ &= \bigsqcup\{\bigsqcup Z_i : i \in I\} \\ &= \bigsqcup\bigcup\{Z_i : i \in I\} \end{aligned}$$

The latter identity holds by **L2.5**, where the former are given above. Since $Z_i \subseteq X$ for all $i \in I$, it follows that $\bigcup\{Z_i : i \in I\} \subseteq X$, and so $\bigsqcup\bigcup\{Z_i : i \in I\} \in \lceil X \rceil$. Thus $s \in \lceil X \rceil$, and so $\lceil\lceil X\rceil\rceil \subseteq \lceil X \rceil$ as desired. □

L4.3 $X \wedge Y \in \mathbb{C}_{\mathcal{S}}$ for any $X, Y \in \mathbb{C}_{\mathcal{S}}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space, and $X, Y \in \mathbb{C}_{\mathcal{S}}$. Since $X \wedge Y \subseteq \lceil X \wedge Y \rceil$ by **L4.1**, assume that $t \in \lceil X \wedge Y \rceil$. It follows that $t = \bigsqcup\Gamma$ for some nonempty $\Gamma \subseteq X \wedge Y$. Letting $\Gamma = \{\gamma_i : i \in I\}$, we may observe for each $i \in I$ that $\gamma_i = u_i.v_i$ for some $u_i \in X$ and $v_i \in Y$. Thus we have:

$$\begin{aligned} t &= \bigsqcup\{\bigsqcup\{u_i, v_i\} : i \in I\} \\ &= \bigsqcup\bigcup\{\{u_i, v_i\} : i \in I\} \\ &= \bigsqcup\bigcup\{\{u_i : i \in I\}, \{v_i : i \in I\}\} \\ &= \bigsqcup\{\bigsqcup\{u_i : i \in I\}, \bigsqcup\{v_i : i \in I\}\} \end{aligned}$$

Since $u_i \in X$ and $v_i \in Y$ for all $i \in I$, and both $X, Y \in \mathbb{C}_{\mathcal{S}}$, it follows that $\bigsqcup\{u_i : i \in I\} \in X$ and $\bigsqcup\{v_i : i \in I\} \in Y$, and so $t \in X \wedge Y$. Given the above, we may conclude that $X \wedge Y = \lceil X \wedge Y \rceil$, and so $X \wedge Y \in \mathbb{C}_{\mathcal{S}}$. □

L4.4 $X \vee Y \in \mathbb{C}_{\mathcal{S}}$ for any $X, Y \in \mathbb{C}_{\mathcal{S}}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space, and $X, Y \in \mathbb{C}_{\mathcal{S}}$. Since $X \vee Y \subseteq \lceil X \vee Y \rceil$ by **L4.1**, assume that $t \in \lceil X \vee Y \rceil$, and so $t = \bigsqcup\Gamma$ for some nonempty $Z \subseteq X \vee Y$. Letting $Z = \{d_i : i \in I\}$, it follows that $d_i \in X \cup Y \cup (X \wedge Y)$ for each $i \in I$. Thus $Z = (Z \cap X) \cup (Z \cap Y) \cup (Z \cap (X \wedge Y))$, where we may assume to start that neither $Z \cap X$, $Z \cap Y$, nor $Z \cap (X \wedge Y)$ is empty. By reasoning similar to that given in **L4.3**, it follows that

$\bigsqcup(Z \cap (X \wedge Y)) = u.v$ for some $u \in X$ and $v \in Y$. Now consider:

$$\begin{aligned} s &= \bigsqcup\bigcup\{Z \cap X, Z \cap Y, Z \cap (X \wedge Y)\} \\ &= \bigsqcup\{\bigsqcup(Z \cap X), \bigsqcup(Z \cap Y), \bigsqcup(Z \cap (X \wedge Y))\} \\ &= \bigsqcup\{\bigsqcup(Z \cap X), \bigsqcup(Z \cap Y), \bigsqcup\{u, v\}\} \\ &= \bigsqcup\bigcup\{Z \cap X, Z \cap Y, \{u, v\}\} \\ &= \bigsqcup\bigcup\{(Z \cap X) \cup \{u\}, (Z \cap Y) \cup \{v\}\} \\ &= \bigsqcup\{\bigsqcup((Z \cap X) \cup \{u\}), \bigsqcup((Z \cap Y) \cup \{v\})\} \end{aligned}$$

The identities above hold by *Associativity* together with what was already shown. Since $(Z \cap X) \cup \{u\} \subseteq X$ and $(Z \cap Y) \cup \{v\} \subseteq Y$, it follows that $\bigsqcup((Z \cap X) \cup \{u\}) \in X$ and $\bigsqcup((Z \cap Y) \cup \{v\}) \in Y$, and so $t \in X \wedge Y$. By *Sum*, $t \in X \vee Y$, where the same result is achieved more easily if one or at most two of the sets $Z \cap X$, $Z \cap Y$, and $Z \cap (X \wedge Y)$ are empty, given that Z is nonempty. Thus $X \vee Y = \lceil X \vee Y \rceil$, and so $X \vee Y \in \mathbb{C}_{\mathcal{S}}$. □

L4.5 $X \wedge X = X$ for any $X \in \mathbb{C}_{\mathcal{S}}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space, $X \in \mathbb{C}_{\mathcal{S}}$, and $x \in X$. Thus $x.x \in X \wedge X$. Since $x = x.x$ by **L2.2**, it follows that $X \subseteq X \wedge X$. Assuming instead that $x \in X \wedge X$, it follows that $x = y.z$ for some $y \in X$ and $z \in X$, and so $x \in X$. Thus $X \wedge X = X$ as desired. □

L4.6 $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z)$ for any $X, Y, Z \subseteq \mathcal{S}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space, and $X, Y, Z \subseteq \mathcal{S}$. We may then observe that the following identities hold:

$$\begin{aligned} X \wedge (Y \cup Z) &= \{\bigsqcup\{x, u\} : x \in X, u \in Y \cup Z\} \\ &= \{\bigsqcup\{x, u\} : x \in X, u \in Y\} \cup \{\bigsqcup\{x, u\} : x \in X, u \in Z\} \\ &= (X \wedge Y) \cup (X \wedge Z) \end{aligned}$$

The identities above all hold by definition. Generalising on $\mathcal{S}$ and $X, Y, Z \subseteq \mathcal{S}$ concludes the proof. □

L4.7 $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$ for any $X, Y, Z \subseteq \mathcal{S}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space, and $X, Y, Z \subseteq \mathcal{S}$. We may then consider the following identities:

$$\begin{aligned}
X \wedge (Y \wedge Z) &=_1 \{\bigsqcup\{x,u\} : x \in X, u \in Y \wedge Z\}\\
&=_2 \{\bigsqcup\{x,u\} : x \in X, u \in \{\bigsqcup\{y,z\} : y \in Y, z \in Z\}\}\\
&=_3 \{\bigsqcup\{x,\bigsqcup\{y,z\}\} : x \in X, y \in Y, z \in Z\}\\
&=_4 \{\bigsqcup\{\bigsqcup\{x\},\bigsqcup\{y,z\}\} : x \in X, y \in Y, z \in Z\}\\
&=_5 \{\bigsqcup\bigcup\{\{x\},\{y,z\}\} : x \in X, y \in Y, z \in Z\}\\
&=_6 \{\bigsqcup\bigcup\{\{x,y\},\{z\}\} : x \in X, y \in Y, z \in Z\}\\
&=_7 \{\bigsqcup\{\bigsqcup\{x,y\},\bigsqcup\{z\}\} : x \in X, y \in Y, z \in Z\}\\
&=_8 \{\bigsqcup\{\bigsqcup\{x,y\},z\} : x \in X, y \in Y, z \in Z\}\\
&=_9 \{\bigsqcup\{v,z\} : v \in \{\bigsqcup\{x,y\} : x \in X, y \in Y\}, z \in Z\}\\
&=_{10} \{\bigsqcup\{v,z\} : v \in X \wedge Y, z \in Z\}\\
&=_{11} (X \wedge Y) \wedge Z.
\end{aligned}$$

Whereas (4) and (8) follow from **L2.2**, and (5) and (7) follow from **L2.5**, the remaining identities are given by the definitions. Generalising on $\mathcal{S}$ and $X, Y, Z \subseteq \mathcal{S}$ concludes the proof. □

L4.8 $X \vee (Y \vee Z) = (X \vee Y) \vee Z$ for any $X, Y, Z \subseteq \mathcal{S}$ and state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space, and $X, Y, Z \subseteq \mathcal{S}$. We may then observe the following identities:

$$\begin{aligned}
X \vee (Y \vee Z) &=_1 X \vee [Y \cup Z \cup (Y \wedge Z)]\\
&=_2 X \cup Y \cup Z \cup (Y \wedge Z) \cup (X \wedge [Y \cup Z \cup (Y \wedge Z)])\\
&=_3 X \cup Y \cup Z \cup (Y \wedge Z) \cup (X \wedge Y) \cup (X \wedge Z) \cup [X \wedge (Y \wedge Z)]\\
&=_4 X \cup Y \cup (X \wedge Y) \cup Z \cup (X \wedge Z) \cup (Y \wedge Z) \cup [(X \wedge Y) \wedge Z]\\
&=_5 [X \cup Y \cup (X \wedge Y)] \cup Z \cup ([X \cup Y \cup (X \wedge Y)] \wedge Z)\\
&=_6 [X \cup Y \cup (X \wedge Y)] \vee Z\\
&=_7 (X \vee Y) \vee Z.
\end{aligned}$$

Whereas (3) and (5) follow from **L4.6**, (4) follows from **L4.7**, and the remaining identities hold by the definitions. Generalising on $\mathcal{S}$ and $X, Y, Z \subseteq \mathcal{S}$ concludes the proof. □

P1 $\mathcal{A}_{\mathcal{S}}$ is an algebra for any state space $\mathcal{S}$.

Proof. Let $\mathcal{S}$ be a state space where $\langle V, F\rangle, \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$, and so by *Propositions* both $\langle V, F\rangle, \langle V', F'\rangle \in \mathcal{C}^2_{\mathcal{S}}$ are exclusive and exhaustive. We may then consider each operator in turn.

Negation: Given that $\langle V, F\rangle \in \mathbb{P}_{\mathcal{S}}$ it follows that $V, F \in \mathcal{C}_{\mathcal{S}}$ where: (1) there is no $t \in V$ and $d \in F$ where $t \circ d$; and (2) for every $t \in P$, there is some $d \in S$ where $t \circ d$ and either $d \in V$ or $d \in F$. By swapping V and F in (1) and (2), it follows that $\langle F, V\rangle \in \mathcal{C}^2_{\mathcal{S}}$ is exclusive and exhaustive, and so $\langle F, V\rangle \in \mathbb{P}_{\mathcal{S}}$. Thus $\neg\langle V, F\rangle \in \mathbb{P}_{\mathcal{S}}$ by *Negation*.

Conjunction: Given that $\langle V, F\rangle, \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$, we may observe that $V, F, V', F' \in \mathcal{C}_{\mathcal{S}}$. It follows that $V \wedge V', F \vee F' \in \mathbb{C}_{\mathcal{S}}$ by **L4.3** and **L4.4**, and so we may conclude that $\langle V \wedge V', F \vee F'\rangle \in \mathbb{C}^2_{\mathcal{S}}$.

Next assume that $t \in V \wedge V'$ and $d \in F \vee F'$. It follows that $t = u.v$ for some $u \in V$ and $v \in V'$, where either $d \in F$, $d \in F'$, or $d = x.y$ for some $x \in F$ and $y \in F'$. Since $\langle V, F\rangle$ and $\langle V', F'\rangle$ are both exclusive, it follows that either: $u \phi d$, $v \phi d$, or both $u \phi x$ and $v \phi y$. If $t \circ d$, then $t.d \in P$ by *Compatibility*, and so either $u.d \in P$, $v.d \in P$, or both $u.x \in P$ and $v.y \in P$ by *Stability*. Thus either $u \circ d$, $v \circ d$, or both $u \circ x$ and $v \circ y$, contradicting the above, and so $t \phi d$. Generalising on t and d, it follows that $\langle V \wedge V', F \vee F'\rangle$ is exclusive.

Letting $t \in P$, we know by hypothesis that $\langle V, F\rangle$ and $\langle V', F'\rangle$ are exhaustive, and so there are some $u, v \in \mathcal{S}$ where $t \circ u$ and $t \circ v$, and $u \in V \cup F$ and $v \in V' \cup F'$. If $u \in F$ or $v \in F'$, then there is some $r \in \mathcal{S}$ where $r \in F \vee F'$. If $u \notin F$ and $v \notin F'$, then $u \in V$ and $v \in V'$, and so $u.v \in V \wedge V'$. Thus in either case, there is some $r \in \mathcal{S}$ where either $r \in F \vee F'$ or $r \in V \vee V'$, and so $\langle V \wedge V', F \vee F'\rangle$ is exhaustive.

Having shown that $\langle V \wedge V', F \vee F'\rangle \in \mathbb{C}^2_{\mathcal{S}}$ is exclusive and exhaustive, it follows that $\langle V \wedge V', F \vee F'\rangle \in \mathbb{P}_{\mathcal{S}}$, and so we may conclude by *Conjunction* that $\langle V, F\rangle \wedge \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$.

Disjunction: As above $V, F, V', F' \in \mathcal{C}_{\mathcal{S}}$ since $\langle V, F\rangle, \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$. It follows that $V \vee V', F \wedge F' \in \mathbb{C}_{\mathcal{S}}$ by **L4.4** and **L4.3**, and so we may conclude that $\langle V \vee V', F \wedge F'\rangle \in \mathbb{C}^2_{\mathcal{S}}$.

Next assume that $t \in V \vee V'$ and $d \in F \wedge F'$. It follows that $d = u.v$ for some $u \in F$ and $v \in F'$, where either $t \in V$, $t \in V'$, or $t = x.y$ for some $x \in V$ and $y \in V'$. Since $\langle V, F\rangle$ and $\langle V', F'\rangle$ are both exclusive, it follows that either: $u \phi t$, $v \phi t$, or both $u \phi x$ and $v \phi y$. If $t \circ d$, then $t.d \in P$ by *Compatibility*, and so either $u.t \in P$, $v.t \in P$, or both $u.x \in P$ and $v.y \in P$ by *Stability*. Thus either $u \circ t$, $v \circ t$, or both $u \circ x$ and $v \circ y$, contradicting the above, and so $t \phi d$. Generalising on t and d, it follows that $\langle V \vee V', F \wedge F'\rangle$ is exclusive.

Letting $t \in P$, we know by hypothesis that $\langle V, F\rangle$ and $\langle V', F'\rangle$ are exhaustive, and so there are some $u, v \in \mathcal{S}$ where $t \circ u$ and $t \circ v$, and $u \in V \cup F$ and $v \in V' \cup F'$. If $u \in V$ or $v \in V'$, then there is some $r \in \mathcal{S}$ where $r \in V \vee V'$. If $u \notin V$ and $v \notin V'$, then $u \in F$ and $v \in F'$, and so $u.v \in F \wedge F'$. Thus in either case, there is some $r \in \mathcal{S}$ where either $r \in V \vee V'$ or $r \in F \wedge F'$, and so $\langle V \vee V', F \wedge F'\rangle$ is exhaustive.

Having shown that $\langle V \vee V', F \wedge F'\rangle \in \mathbb{C}^2_{\mathcal{S}}$ is exclusive and exhaustive, it follows that $\langle V \vee V', F \wedge F'\rangle \in \mathbb{P}_{\mathcal{S}}$, and so we may conclude by *Conjunction* that $\langle V, F\rangle \vee \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$.

Necessity: Given that $\langle V, F\rangle \in \mathbb{P}_{\mathcal{S}}$, it follows from the semantics that if for every $t \in P$ there is some $d \circ t$ where $d \in V$, then $\Box\langle V, F\rangle = \langle S, \varnothing\rangle$, where otherwise we may conclude that $\Box\langle V, F\rangle = \langle \varnothing, S\rangle$. Thus in either case $\Box\langle V, F\rangle \in \mathbb{C}^2_{\mathcal{S}}$ is exclusive and exhaustive.

Grounding: Given that $\langle V, F\rangle, \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$, it follows that either $\langle V, F\rangle \leqslant \langle V', F'\rangle = \langle S, \varnothing\rangle$ or $\langle V, F\rangle \leqslant \langle V', F'\rangle = \langle \varnothing, S\rangle$. In either case, $\langle V, F\rangle \leqslant \langle V', F'\rangle \in \mathbb{C}^2_{\mathcal{S}}$ is exclusive and exhaustive.

By generalising on $\langle V, F\rangle, \langle V', F'\rangle \in \mathbb{P}_{\mathcal{S}}$ and $\mathcal{S}$, we may conclude that $\mathcal{A}_{\mathcal{S}}$ is an algebra for any state space $\mathcal{S}$. □

P2 $|\cdot| : \mathcal{L} \to \mathcal{A}_{\mathcal{S}}$ is an $\mathcal{L}$-homomorphism for any $\mathcal{S}$-model $\mathcal{M}$.

Proof. Letting $\mathcal{M} = \langle S, P, \sqsubseteq, |\cdot| \rangle$ be an $\mathcal{S}$-model for a state space $\mathcal{S}$, we know that $|p| \in \mathbb{P}_{\mathcal{S}}$ for all $p \in \mathbb{L}$. Letting $\mathcal{L}^n = \langle \mathtt{wfs}^n(\mathbb{L}), \neg, \wedge, \vee, \Box, \leqslant \rangle$ where $\mathtt{wfs}^n(\mathbb{L}) = \{A \in \mathtt{wfs}(\mathbb{L}) : \mathtt{comp}(A) \leqslant n\}$, assume for induction that $|\cdot| : \mathcal{L}^n \to \mathcal{A}_{\mathcal{S}}$ is an $\mathcal{L}$-homomorphism. Given some $A \in \mathtt{wfs}(\mathbb{L})$ where $\mathtt{comp}(A) = n + 1$, we may then consider the following cases:

Case 1: Assuming that $A = \neg B$, it follows by hypothesis that $|B| \in \mathbb{P}_{\mathcal{S}}$ where $|B| = \langle |B|^+, |B|^- \rangle$. By *Negation*, $\neg|B| = \langle |B|^-, |B|^+ \rangle$, and so $\neg|B| = \langle |\neg B|^+, |\neg B|^- \rangle$ by the semantics. Thus we may conclude that $\neg|B| = |\neg B|$ where $\neg|B| \in \mathbb{P}_{\mathcal{S}}$ by **P1**.

Case 2: Assuming that $A = B \wedge C$, both $|B|, |C| \in \mathbb{P}_{\mathcal{S}}$ by hypothesis, where $|B| = \langle |B|^+, |B|^- \rangle$ and $|C| = \langle |C|^+, |C|^- \rangle$. Letting $t \in |B \wedge C|^+$, it follows that $t = d.r$ for some $d \in |B|^+$ and $r \in |C|^+$, and so by *Product*, $t \in |B|^+ \wedge |C|^+$. Supposing instead that $t \in |B|^+ \wedge |C|^+$, it follows that $t = d.r$ for some $d \in |B|^+$ and $r \in |C|^+$, and so $t \in |B \wedge C|^+$. Thus $|B \wedge C|^+ = |B|^+ \wedge |C|^+$. Assume next that $t \in |B \wedge C|^-$, we know that either $t \in |B|^-$, or $t \in |C|^-$, or $t = d.r$ for some $d \in |B|^-$ and $r \in |C|^-$, and so $t \in |B|^- \vee |C|^-$ by *Sum*. Assuming instead that $t \in |B|^- \vee |C|^-$, it follows by similar reasoning that $t \in |B \wedge C|^-$, and so we may conclude that $|B \wedge C| = \langle |B|^+ \wedge |C|^+, |B|^- \vee |C|^- \rangle$. Thus $|B \wedge C| = |B| \wedge |C|$ follows by *Conjunction*, where $|B| \wedge |C| \in \mathbb{P}_{\mathcal{S}}$ by **P1**.

Case 3: Assuming that $A = B \vee C$, both $|B|, |C| \in \mathbb{P}_{\mathcal{S}}$ by hypothesis, where $|B| = \langle |B|^+, |B|^- \rangle$ and $|C| = \langle |C|^+, |C|^- \rangle$. Letting $t \in |B \vee C|^+$, we know that either $t \in |B|^+$, or $t \in |C|^+$, or $t = d.r$ for some $d \in |B|^+$ and $r \in |C|^+$, and so $t \in |B|^+ \vee |C|^+$ by *Sum*. By assuming instead that $t \in |B|^+ \vee |C|^+$, it follows similarly that $t \in |B \vee C|^+$, and so $|B \vee C|^+ = |B|^+ \vee |C|^+$. Next we may assume that $t \in |B \vee C|^-$, it follows that $t = d.r$ for some $d \in |B|^-$ and $r \in |C|^-$, and so $t \in |B|^- \wedge |C|^-$ by *Product*. Supposing instead that $t \in |B|^- \wedge |C|^-$, it follows that $t = d.r$ for some $d \in |B|^-$ and $r \in |C|^-$, and so $t \in |B \vee C|^-$. Thus it follows that $|B \vee C| = \langle |B|^+ \vee |C|^+, |B|^- \wedge |C|^- \rangle$. We may conclude by *Conjunction* that $|B \vee C| = |B| \vee |C|$ where $|B| \vee |C| \in \mathbb{P}_{\mathcal{S}}$ by **P1**.

Case 4: Assume $A = \Box B$. It follows by hypothesis that $|B| \in \mathbb{P}_{\mathcal{S}}$, where $|B| = \langle |B|^+, |B|^- \rangle$. If for every $t \in P$ there is some $d \circ t$ where $d \in |B|^+$, then $\Box^+|B| = S = |\Box B|^+$ and $\Box^-|B| = \varnothing = |\Box B|^-$. If instead there is some $t \in P$ where $d \notin |B|^+$ for any $d \circ t$, then $\Box^+|B| = \varnothing = |\Box B|^+$ and $\Box^-|B| = S = |\Box B|^-$. Thus in either case $\Box|B| = \langle |\Box B|^+, |\Box B|^- \rangle$, and so $\Box|B| = |\Box B|$ by *Necessity* where $\Box|B| \in \mathbb{P}_{\mathcal{S}}$ by **P1**.

Case 5: Assuming that $A = B \leqslant C$, it follows by hypothesis that $|B|, |C| \in \mathbb{P}_{\mathcal{S}}$, where $|B| = \langle |B|^+, |B|^- \rangle$ and $|C| = \langle |C|^+, |C|^- \rangle$. Letting $s \in |B| \leqslant^+ |C|$, it follows that $|B| \leqslant^+ |C| = S$ and $|B| \leqslant^- |C| = \varnothing$ where $|B| \vee |C| = |C|$ by *Entailment* and *Containment*. It follows that $|B|^+ \vee |C|^+ = |C|^+$ and $|B|^- \wedge |C|^- = |C|^-$ by *Disjunction*, and so $|B|^+ \cup |C|^+ \cup (|B|^+ \wedge |C|^+) = |C|^+$ by *Sum*. Thus $|B|^+ \subseteq |C|^+$, thereby satisfying (1) in the verification clause for ground. Letting $x \in |C|^-$, it follows that $x = u.v$ for some $u \in |B|^-$ and $v \in |C|^-$ by *Product*, and so there is some $u \in |B|^-$ where $u \sqsubseteq x$, thereby satisfying (2) in the verification clause for ground. By instead letting $t \in |B|^-$ and $d \in |C|^-$, it follows that $t.d \in |B|^- \wedge |C|^-$ by *Product*, and so $t.d \in |C|^-$ by the above, thereby satisfying (3) in the verification clause for ground. Accordingly,

$|B \leqslant C|^+ = S$ and $|B \leqslant C|^- = \varnothing$ by the semantics for ground, and so it follows by *Grounding* that $|B| \leqslant |C| = |B \leqslant C|$ as desired.

Assume that $s \notin |B| \leqslant^+ |C|$, it follows that $|B| \leqslant^+ |C| = \varnothing$ and $|B| \leqslant^- |C| = S$ where $|B| \vee |C| \neq |C|$. Thus either $|B|^+ \vee |C|^+ \neq |C|^+$ or $|B|^- \wedge |C|^- \neq |C|^-$ by *Disjunction*. If $|B|^+ \vee |C|^+ \neq |C|^+$, then $|B|^+ \cup |C|^+ \cup (|B|^+ \wedge |C|^+) \not\subseteq |C|^+$ by *Sum*, and so $|B|^+ \not\subseteq |C|^+$ thereby satisfying (1) in the falsification clause for ground. If instead $|B|^- \wedge |C|^- \neq |C|^-$, then either $|C|^- \not\subseteq |B|^- \wedge |C|^-$ or $|B|^- \wedge |C|^- \not\subseteq |C|^-$, thereby satisfying either (2) or (3) in the falsification clause for ground, respectively. Thus in each case $|B \leqslant C|^+ = \varnothing$ and $|B \leqslant C|^- = S$, and so $|B| \leqslant^+ |C| = |B \leqslant C|^+$ and $|B| \leqslant^- |C| = |B \leqslant C|^-$ by the above. We may then conclude by *Grounding* that $|B| \leqslant |C| = |B \leqslant C|$ whether $s \in |B| \leqslant^+ |C|$ or not, where $|B| \leqslant |C| \in \mathbb{P}_{\mathcal{S}}$ by **P1**.

Given the cases above, it follows $|\cdot| : \mathcal{L}^{n+1} \to \mathcal{A}_{\mathcal{S}}$ is an $\mathcal{L}$-homomorphism, and so $|\cdot| : \mathcal{L} \to \mathcal{A}_{\mathcal{S}}$ is an $\mathcal{L}$-homomorphism by induction. □

L4.9 For any $\mathcal{S}$-models $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$: $\mathcal{M}, w \dashv A$ *iff* $\mathcal{M}, w \nVdash A$.

Proof. Letting $\mathcal{M}$ be an $\mathcal{S}$-model, $w \in \mathbb{W}_{\mathcal{S}}$, and $A \in \mathtt{wfs}(\mathbb{L})$, assume that $\mathcal{M}, w \dashv A$. It follows that there is some $t \sqsubseteq w$ where $\mathcal{M}, t \dashv\!\shortmid A$. Assume for contradiction that there is some $d \sqsubseteq w$ where $\mathcal{M}, d \Vdash A$. Since $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, it follows that $t \diamond d$ by *Exclusivity*. However, $t.d \sqsubseteq w$ by **L2.4**, and so $t.d \in P$ by *Stability*. By *Composition* $t \circ d$, contradicting the above, and so there is no $d \sqsubseteq w$ where $\mathcal{M}, d \Vdash A$. Thus we may conclude that $\mathcal{M}, w \nVdash A$.

Assuming instead that $\mathcal{M}, w \nVdash A$, it follows that there is no $d \sqsubseteq w$ where $\mathcal{M}, d \Vdash A$. Since $w \in P$ and $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, there is some $t \circ w$ where either $\mathcal{M}, t \Vdash A$ or $\mathcal{M}, t \dashv\!\shortmid A$ by *Exhaustivity*. Since $\mathcal{M}, t \nVdash A$, it follows that $\mathcal{M}, t \dashv\!\shortmid A$ where $t \sqsubseteq w$ by *Worlds*. Thus $\mathcal{M}, w \dashv A$. □

L4.10 For any $\mathcal{S}$-models $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$: $\mathcal{M}, w \vDash \neg A$ *iff* $\mathcal{M}, w \nVdash A$.

Proof. Assuming that $\mathcal{M}, w \vDash \neg A$, it follows that $\mathcal{M}, t \Vdash \neg A$ for some $t \sqsubseteq w$, and so $\mathcal{M}, t \dashv\!\shortmid A$ by the semantics. Thus $\mathcal{M}, w \dashv A$, and so $\mathcal{M}, w \nVdash A$ by **L4.9**, where the same reasoning holds in reverse. □

L4.11 For any $\mathcal{S}$-model $\mathcal{M}$, $|\Box A|^+ \neq \varnothing$ just in case $|\Box A|^- = \varnothing$.

Proof. Let $\mathcal{M}$ be an $\mathcal{S}$-model and $A \in \mathtt{wfs}(\mathbb{L})$ where $|\Box A|^- = \varnothing$. By **P2**, $\Box^- |A| = \varnothing$, and so there is no $t \in P$ where $t \in |A|^-$. Since $P \neq \varnothing$, we may choose an arbitrary $t \in P$. Given that $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, there is some $d \circ t$ where either $d \in |A|^+$ or $d \in |A|^-$ by *Exhaustivity*. Since $d \notin |A|^-$, it follows that $d \in |A|^+$. By generalising on $t \in P$, it follows that $s \in \Box^+ |A|$ for any $s \in \mathcal{S}$, and so $\Box^+ |A| \neq \varnothing$ given that $\mathcal{S} \neq \varnothing$. Thus we may conclude that $|\Box A|^+ \neq \varnothing$ again by **P2**.

Assuming instead that $|\Box A|^+ \neq \varnothing$, it follows that $\Box^+ |A| \neq \varnothing$ by **P2**, and so for every $t \in P$, there is some $d \circ t$ where $d \in |A|^+$. Assuming for contradiction that $|\Box A|^- \neq \varnothing$, it follows that $\Box^- |A| \neq \varnothing$ by **P2**, and so $\mathcal{M}, r \dashv\!\shortmid A$ for some $r \in P$. By the above, there is some $d \circ r$ where $d \in |A|^+$. However, $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, and so $d \diamond r$ by *Exclusivity*, contradicting the above. Thus $|\Box A|^- = \varnothing$ as desired. □

L4.12 For any $\mathcal{S}$-model $\mathcal{M}$ and $A, B \in \mathtt{wfs}(\mathbb{L})$: $|A \leqslant B|^+ \neq \varnothing$ *iff* $|A \leqslant B|^- = \varnothing$.

Proof. Letting $\mathcal{M}$ be an $\mathcal{S}$-model and $A, B \in \mathtt{wfs}(\mathbb{L})$, assume that $|A \leqslant B|^+ \neq \varnothing$. Assume for contradiction that $|A \leqslant B|^- \neq \varnothing$. By **P2**, $|A \leqslant B|^+ = |A| \leqslant^+ |B|$ and $|A \leqslant B|^- = |A| \leqslant^- |B|$. Thus there is some $u \in |A| \leqslant^+ |B|$ and some $v \in |A| \leqslant^- |B|$, and so by the latter there is some $t \in \mathcal{S}$ which satisfies either: (1) $t \in |A|^+$ but $t \notin |B|^+$; (2) $t \in |B|^-$ but $d \notin |A|^-$ for any $d \sqsubseteq t$; or (3) $t = d.r$ where $d \in |A|^-$ and $r \in |B|^-$ but $t \notin |B|^-$. However, if $u \in |A| \leqslant^+ |B|$, then: (1)′ $d \in |B|^+$ for every $d \in |A|^+$, contradicting (1); (2)′ for every $d \in |B|^-$ there is some $r \sqsubseteq d$ where $r \in |A|^-$, contradicting (2); and (3)′ $d.r \in |B|^-$ whenever $d \in |A|^-$ and $r \in |B|^-$, contradicting (3). Having arrived at a contradiction in each case, we may conclude that $|A \leqslant B|^- = \varnothing$.

Assume instead that $|A \leqslant B|^+ = \varnothing$. By **P2**, $|A| \leqslant^+ |B| = \varnothing$, and so by *Entailment*, either: (1) there is some $t \in |A|^+$ where $t \notin |B|^+$; (2) there is some $t \in |B|^-$ where $d \notin |A|^-$ for any $d \sqsubseteq t$; or (3) there is some $t = d.r$ where $d \in |A|^-$ and $r \in |B|^-$ but $t \notin |B|^-$. Thus $|A| \leqslant^- |B| \neq \varnothing$ by *Containment*, and so $|A \leqslant B|^- \neq \varnothing$ by **P2** as desired. □

L4.13 For any $\mathcal{S}$-model $\mathcal{M}$, $w \in \mathbb{W}_{\mathcal{S}}$, and $A \in \mathtt{wfs}(\mathbb{L})$: $\mathcal{M}, w \vDash \Box A$ *iff* $|\Box A|^- = \varnothing$.

Proof. Assuming that $\mathcal{M}, w \vDash \Box A$ for an $\mathcal{S}$-model $\mathcal{M}$, $w \in \mathbb{W}_{\mathcal{S}}$, and $A \in \mathtt{wfs}(\mathbb{L})$, it follows that $\mathcal{M}, t \Vdash \Box A$ for some $t \sqsubseteq w$, and so $|\Box A|^+ \neq \varnothing$. We may then conclude by **L4.11** that $|\Box A|^- = \varnothing$. Running the reasoning above in reverse establishes the biconditional. □

L4.14 For any $\mathcal{S}$-model $\mathcal{M}$, $w \in \mathbb{W}_{\mathcal{S}}$, and $A, B \in \mathtt{wfs}(\mathbb{L})$: $\mathcal{M}, w \vDash A \leqslant B$ *iff* $|A \leqslant B|^- = \varnothing$.

Proof. Assuming that $\mathcal{M}, w \vDash A \leqslant B$ for an $\mathcal{S}$-model $\mathcal{M}$, $w \in \mathbb{W}_{\mathcal{S}}$, and $A, B \in \mathtt{wfs}(\mathbb{L})$, it follows that $\mathcal{M}, t \Vdash A \leqslant B$ for some $t \sqsubseteq w$, and so $|A \leqslant B|^+ \neq \varnothing$. We may then conclude by **L4.12** that $|A \leqslant B|^- = \varnothing$. Running the reasoning above in reverse establishes the biconditional. □

L4.15 For any $\mathcal{S}$-model $\mathcal{M}$, $w \in \mathbb{W}_{\mathcal{S}}$, and $A, B \in \mathtt{wfs}(\mathbb{L})$, if $|A| = |B|$, then $\mathcal{M}, w \vDash A \leqslant B$.

Proof. Assume that $|A| = |B|$ for an $\mathcal{S}$-model $\mathcal{M}$ and $A, B \in \mathtt{wfs}(\mathbb{L})$. Assume for contradiction that there is some $t \in |A \leqslant B|^-$, and so $t \in |A| \leqslant^- |B|$ by **P2**. It follows that either: (1) there is some $d \in |A|^+$ where $d \notin |B|^+$; (2) there is some $d \in |A|^-$ where there is no $r \sqsubseteq d$ where $r \in |B|^-$; or (3) there is some $d \in |A|^-$ and $r \in |B|^-$ where $d.r \notin |B|^-$. However, given our assumption, neither (1), (2), now (3) hold. Thus $|A \leqslant B|^- = \varnothing$, and so $\mathcal{M}, w \vDash A \leqslant B$ follows by **L4.14**. □

L4.16 For all $\mathcal{S}$-models $\mathcal{M}$, states $s \in \mathcal{S}$, and $A, B \in \mathtt{wfs}(\mathbb{L})$:

$(\Diamond)^+$ $\mathcal{M}, s \Vdash \Diamond A$ *iff* there is some $t \in P$ where $\mathcal{M}, t \Vdash A$.

$(\Diamond)^-$ $\mathcal{M}, s \dashv\Vdash \Diamond A$ *iff* for every $t \in P$ there is some $d \circ t$ where $\mathcal{M}, d \dashv\Vdash A$.

$(\sqsubseteq)^+$ $\mathcal{M}, s \Vdash A \sqsubseteq B$ *iff* each of the following conditions hold:
(1) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash A$, then $t \sqsubseteq s$ and $\mathcal{M}, t \dashv\Vdash B$;
(2) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \Vdash B$, then $t \sqsubseteq s$ where $\mathcal{M}, d \Vdash A$ for some $d \sqsubseteq t$; and
(3) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, then $t.d \sqsubseteq s$ where $\mathcal{M}, t.d \Vdash B$.

$(\sqsubseteq)^-$ $\mathcal{M}, s \dashv A \sqsubseteq B$ *iff* any of the following conditions hold:
(1) $\mathcal{M}, t \dashv A$ and $\mathcal{M}, t \not\dashv B$;
(2) $\mathcal{M}, t \Vdash B$ and $\mathcal{M}, d \not\Vdash A$ for all $d \sqsubseteq t$; or
(3) $t = d.r$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, r \Vdash B$ but $\mathcal{M}, t \not\Vdash B$.

$(\equiv)^+$ $\mathcal{M}, s \Vdash A \equiv B$ *iff* for all $t \in \mathcal{S}$ both of the following conditions hold:
(1) $\mathcal{M}, t \Vdash A$ if and only if $\mathcal{M}, t \Vdash B$; and
(2) $\mathcal{M}, t \dashv A$ if and only if $\mathcal{M}, t \dashv B$.

$(\equiv)^-$ $\mathcal{M}, s \dashv A \equiv B$ *iff* for some $t \in \mathcal{S}$ any of the following conditions hold:
(1) $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, t \not\Vdash B$, or
(2) $\mathcal{M}, t \Vdash B$ and $\mathcal{M}, t \not\Vdash A$; or
(3) $\mathcal{M}, t \dashv A$ and $\mathcal{M}, t \not\dashv B$; or
(4) $\mathcal{M}, t \dashv B$ and $\mathcal{M}, t \not\dashv A$.

$(\preceq)^+$ $\mathcal{M}, s \Vdash A \preceq B$ *iff* for all $t, d \in \mathcal{S}$, both of the following conditions hold:
(1) if $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, then $\mathcal{M}, t.d \Vdash B$.
(2) if $\mathcal{M}, t \dashv A$ and $\mathcal{M}, d \dashv B$, then $\mathcal{M}, t.d \dashv B$.

$(\preceq)^-$ $\mathcal{M}, s \dashv A \preceq B$ *iff* for some $t, d \in \mathcal{S}$, either of the following conditions hold:
(1) $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$ but $\mathcal{M}, t.d \not\Vdash B$.
(2) $\mathcal{M}, t \dashv A$ and $\mathcal{M}, d \dashv B$ but $\mathcal{M}, t.d \not\dashv B$.

Proof. Let $\mathcal{M}$ be an $\mathcal{S}$-model, $s \in \mathcal{S}$, and $A, B \in \mathtt{wfs}(\mathbb{L})$. We may then consider the following cases:

Case $(\Diamond)^+$*:* Assume $\mathcal{M}, s \Vdash \Diamond A$. By *Possibility*, $\mathcal{M}, s \Vdash \neg\Box\neg A$, and so $\mathcal{M}, s \dashv \Box\neg A$ by the semantics. Again by the semantics, there is some $t \in P$ where $\mathcal{M}, t \dashv \neg A$, and so there is some $t \in P$ where $\mathcal{M}, t \Vdash A$. Identical considerations may be run in reverse.

Case $(\Diamond)^-$*:* Assume $\mathcal{M}, s \dashv \Diamond A$. By *Possibility*, $\mathcal{M}, s \dashv \neg\Box\neg A$, and so $\mathcal{M}, s \Vdash \Box\neg A$ by the semantics. Thus for every $t \in P$ there is some $d \circ t$ where $\mathcal{M}, d \Vdash \neg A$, and so for every $t \in P$ there is some $d \circ t$ where $\mathcal{M}, d \dashv A$. Identical considerations hold in reverse.

Case $(\sqsubseteq)^+$*:* Assume $\mathcal{M}, s \Vdash A \sqsubseteq B$. Thus $\mathcal{M}, s \Vdash \neg A \leqslant \neg B$ by *Essence*, and so by the semantics, the following conditions hold: (1) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \Vdash \neg A$, then $t \sqsubseteq s$ and $\mathcal{M}, t \Vdash \neg B$; (2) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \dashv \neg B$, then $t \sqsubseteq s$ where $\mathcal{M}, d \dashv \neg A$ for some $d \sqsubseteq t$; and (3) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \dashv \neg A$ and $\mathcal{M}, d \dashv \neg B$, then $t.d \sqsubseteq s$ where $\mathcal{M}, t.d \dashv \neg B$. Again by the semantics, the following conditions hold: $(1)'$ for all $t \in \mathcal{S}$, if $\mathcal{M}, t \dashv A$, then $t \sqsubseteq s$ and $\mathcal{M}, t \dashv B$; $(2)'$ for all $t \in \mathcal{S}$, if $\mathcal{M}, t \Vdash B$, then $t \sqsubseteq s$ where $\mathcal{M}, d \Vdash A$ for some $d \sqsubseteq t$; and $(3)'$ for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, then $t.d \sqsubseteq s$ where $\mathcal{M}, t.d \Vdash B$. Identical considerations may be run in reverse.

Case $(\sqsubseteq)^-$*:* Assume $\mathcal{M}, s \dashv A \sqsubseteq B$. By *Essence*, $\mathcal{M}, s \dashv \neg A \leqslant \neg B$, and so by the semantics $s \sqsubseteq \bigsqcup\{r \in \mathcal{S} : \mathcal{M}, r \dashv A \wedge \neg A \wedge \neg B\}$ and either: (1) there is some $t \sqsubseteq s$ where $\mathcal{M}, t \dashv A$ and $\mathcal{M}, t \not\dashv B$; (2) there is some $t \sqsubseteq s$ where $\mathcal{M}, t \Vdash B$ and $\mathcal{M}, d \not\Vdash A$ for all $d \sqsubseteq t$; or (3) there is some $t, d \sqsubseteq s$ where $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$ but $\mathcal{M}, t.d \not\Vdash B$. Identical considerations may be run in reverse.

Case $(\equiv)^+$*:* Assume $\mathcal{M}, s \Vdash A \equiv B$. It follows by *Identity* and *Weak Reduction* that $\mathcal{M}, s \Vdash [(A \leqslant B) \wedge (A \sqsubseteq B)] \wedge [(B \leqslant A) \wedge (B \sqsubseteq A)]$,

and so we may conclude by the semantics that $s = d.t.u.v$ where:

$$\mathcal{M}, d \Vdash A \leqslant B \qquad \mathcal{M}, t \Vdash A \sqsubseteq B$$
$$\mathcal{M}, u \Vdash B \leqslant A \qquad \mathcal{M}, v \Vdash B \sqsubseteq A.$$

It follows from the semantics that for all $t \in \mathcal{S}$: (1) if $\mathcal{M}, t \Vdash A$, then $\mathcal{M}, t \Vdash B$; (2) if $\mathcal{M}, t \dashv\Vdash A$, then $\mathcal{M}, t \dashv\Vdash B$; (3) if $\mathcal{M}, t \Vdash B$, then $\mathcal{M}, t \Vdash A$; and (4) if $\mathcal{M}, t \dashv\Vdash B$, then $\mathcal{M}, t \dashv\Vdash A$. Thus we may conclude that for all $t \in \mathcal{S}$ both: (a) $\mathcal{M}, t \Vdash A$ just in case $\mathcal{M}, t \Vdash B$; and (b) $\mathcal{M}, t \dashv\Vdash A$ just in case $\mathcal{M}, t \dashv\Vdash B$.

Assume instead that for all $t \in \mathcal{S}$ both: (a) $\mathcal{M}, t \Vdash A$ just in case $\mathcal{M}, t \Vdash B$; and (b) $\mathcal{M}, t \dashv\Vdash A$ just in case $\mathcal{M}, t \dashv\Vdash B$. It follows by the semantics that $\mathcal{M}, s' \Vdash A \leqslant B$, $\mathcal{M}, s' \Vdash A \sqsubseteq B$, $\mathcal{M}, s' \Vdash A \leqslant B$, and $\mathcal{M}, s' \Vdash A \sqsubseteq B$ for any $s' \in \mathcal{S}$. Given **L2.2**, it follows from the semantics that $\mathcal{M}, s' \Vdash [(A \leqslant B) \wedge (A \sqsubseteq B)] \wedge [(B \leqslant A) \wedge (B \sqsubseteq A)]$ for any $s' \in \mathcal{S}$, and so for $s \in \mathcal{S}$ in particular. Thus we may conclude that $\mathcal{M}, s \Vdash A \equiv B$ by *Weak Reduction* and *Identity*.

Case $(\equiv)^-$*:* Assume $\mathcal{M}, s \dashv\Vdash A \equiv B$. It follows by *Identity* and *Weak Reduction* that $\mathcal{M}, s \dashv\Vdash [(A \leqslant B) \wedge (A \sqsubseteq B)] \wedge [(B \leqslant A) \wedge (B \sqsubseteq A)]$, and so given the semantics we know that there is some $t \in \mathcal{S}$ where either: (1) $\mathcal{M}, t \dashv\Vdash A \leqslant B$; (2) $\mathcal{M}, t \dashv\Vdash A \sqsubseteq B$; (3) $\mathcal{M}, t \dashv\Vdash B \leqslant A$; or (4) $\mathcal{M}, t \dashv\Vdash B \sqsubseteq A$. Assuming (1), it follows that there is some $d \in \mathcal{S}$ where either: (*a*) $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, d \nVdash B$; (*b*) $\mathcal{M}, d \dashv\Vdash B$ and $\mathcal{M}, r \not\dashv\Vdash A$ for all $r \sqsubseteq d$; or (*c*) $d = u.v$ where $\mathcal{M}, u \dashv\Vdash A$ and $\mathcal{M}, v \dashv\Vdash B$ but $\mathcal{M}, d \not\dashv\Vdash B$. It follows from (*b*) by reflexivity that $\mathcal{M}, d \dashv\Vdash B$ and $\mathcal{M}, d \not\dashv\Vdash A$. Additionally, it follows from (*c*) that there is some u where $\mathcal{M}, u \dashv\Vdash A$ but $\mathcal{M}, u \not\dashv\Vdash B$ given **P2** and *Closure*. Thus it follows from (1) that there is some $d \in \mathcal{S}$ where either: (i) $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, d \nVdash B$; (ii) $\mathcal{M}, d \Vdash B$ and $\mathcal{M}, d \nVdash A$; (iii) $\mathcal{M}, d \dashv\Vdash A$ and $\mathcal{M}, d \not\dashv\Vdash B$; or (iv) $\mathcal{M}, d \dashv\Vdash B$ and $\mathcal{M}, d \not\dashv\Vdash A$. The same results may be derived by similar reasoning from (2) – (4).

Assume instead that there is some $d \in \mathcal{S}$ where either: (i) $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, d \nVdash B$; (ii) $\mathcal{M}, d \Vdash B$ and $\mathcal{M}, d \nVdash A$; (iii) $\mathcal{M}, d \dashv\Vdash A$ and $\mathcal{M}, d \not\dashv\Vdash B$; or (iv) $\mathcal{M}, d \dashv\Vdash B$ and $\mathcal{M}, d \not\dashv\Vdash A$. It follows from each of the cases above that $\mathcal{M}, t \dashv\Vdash [(A \leqslant B) \wedge (A \sqsubseteq B)] \wedge [(B \leqslant A) \wedge (B \sqsubseteq A)]$ for any $t \in \mathcal{S}$, and so for $s \in \mathcal{S}$ in particular. Thus we may conclude that $\mathcal{M}, s \dashv\Vdash A \equiv B$ by *Weak Reduction* and *Identity*.

Case $(\preceq)^+$*:* Assuming $\mathcal{M}, s \Vdash A \preceq B$, it follows by *Relevance* that $\mathcal{M}, s \Vdash A \wedge B \leqslant B$, and so by the semantics it follows that: (a) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \Vdash A \wedge B$, then $\mathcal{M}, t \Vdash B$; (b) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash B$, then $\mathcal{M}, d \dashv\Vdash A \wedge B$ for some $d \sqsubseteq t$; and (c) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash A \wedge B$ and $\mathcal{M}, d \dashv\Vdash B$, then $\mathcal{M}, t.d \dashv\Vdash B$. Assuming that $t, d \in \mathcal{S}$ where $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, it follows by the semantics that $\mathcal{M}, t.d \Vdash A \wedge B$, and so $\mathcal{M}, t.d \Vdash B$ by (a). Assume instead that $t, d \in \mathcal{S}$ where $\mathcal{M}, t \dashv\Vdash A$ and $\mathcal{M}, d \dashv\Vdash B$. It follows that $\mathcal{M}, t.d \dashv\Vdash A \wedge B$ by the semantics, and so $\mathcal{M}, t.d \dashv\Vdash B$ by (c). Thus we may conclude that: (1) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, then $t.d \sqsubseteq s$ where $\mathcal{M}, t.d \Vdash B$; and (2) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash A$ and $\mathcal{M}, d \dashv\Vdash B$, then $t.d \sqsubseteq s$ where $\mathcal{M}, t.d \dashv\Vdash B$.

Assume instead that: (1) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, then $\mathcal{M}, t.d \Vdash B$; and (2) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash A$ and $\mathcal{M}, d \dashv\Vdash B$, then $\mathcal{M}, t.d \dashv\Vdash B$. Letting $t' \in \mathcal{S}$ be such that $\mathcal{M}, t' \Vdash A \wedge B$, it follows by the semantics that $t' = t.d$ where $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash B$, and so it follows from (1) that $\mathcal{M}, t' \Vdash B$ as in (a). Next assume that $\mathcal{M}, t' \dashv\Vdash B$.

It follows immediately by the semantics that $\mathcal{M}, t' \dashv\Vdash A \wedge B$, and since $t' \sqsubseteq t'$, there is some $d \sqsubseteq t'$ where $\mathcal{M}, d \dashv\Vdash A \wedge B$ as in (b). Supposing that $\mathcal{M}, t' \dashv\Vdash A \wedge B$ and $\mathcal{M}, d' \dashv\Vdash B$, it follows from the former that either: (i) $\mathcal{M}, t' \dashv\Vdash A$; (ii) $\mathcal{M}, t' \dashv\Vdash B$; or (iii) $\mathcal{M}, t' \dashv\Vdash A \vee B$. If (i) holds, then $\mathcal{M}, t'.d' \dashv\Vdash B$ by (2) above. If (ii) holds, then $\mathcal{M}, t'.d' \dashv\Vdash B$ by **P2** and *Closure*. If (iii) holds, then $t' = u'.v'$ where $\mathcal{M}, u' \dashv\Vdash A$ and $\mathcal{M}, v' \dashv\Vdash B$, and so $\mathcal{M}, u'.d' \dashv\Vdash B$ by (2), where $\mathcal{M}, u'.v'.d' \dashv\Vdash B$ again by **P2** and *Closure*. Thus in each case, $\mathcal{M}, t'.d' \dashv\Vdash B$ as in (c) above. By the semantics $\mathcal{M}, s \Vdash A \wedge B \leqslant B$, and so $\mathcal{M}, s \Vdash A \preceq B$ by *Relevance*.

Case $(\preceq)^-$*:* Assume $\mathcal{M}, s \dashv\Vdash A \preceq B$. It follows by *Relevance* that $\mathcal{M}, s \dashv\Vdash A \wedge B \leqslant B$, and so by the semantics there is some $t \in \mathcal{S}$ where either: (1) $\mathcal{M}, t \Vdash A \wedge B$ and $\mathcal{M}, t \nVdash B$; (2) $\mathcal{M}, t \dashv\Vdash B$ but $\mathcal{M}, d \not\dashv\Vdash A \wedge B$ for any $d \sqsubseteq t$; and (3) $t = u.v$ where $\mathcal{M}, u \dashv\Vdash A \wedge B$ and $\mathcal{M}, v \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$. If (1) holds, then $t = d'.r'$ where $\mathcal{M}, d' \Vdash A$ and $\mathcal{M}, r' \Vdash B$, but $\mathcal{M}, t \nVdash B$. Assuming (2), it follows that $\mathcal{M}, t \dashv\Vdash B$, and so $\mathcal{M}, t \dashv\Vdash A \wedge B$. Since $t \sqsubseteq t$, it follows that there is some $d \sqsubseteq t$ where $\mathcal{M}, t \dashv\Vdash A \wedge B$, contradicting (2). Thus (2) cannot hold. Assuming (3) holds, $t = u.v$ where $\mathcal{M}, u \dashv\Vdash A \wedge B$ and $\mathcal{M}, v \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$. From the former, either: (a) $\mathcal{M}, u \dashv\Vdash A$; (b) $\mathcal{M}, u \dashv\Vdash B$; or (c) $\mathcal{M}, u \dashv\Vdash A \vee B$. If (a) holds, $t = u.v$ where $\mathcal{M}, u \dashv\Vdash A$ and $\mathcal{M}, v \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$. Assuming (b) holds, $t = u.v$ where $\mathcal{M}, u \dashv\Vdash B$ and $\mathcal{M}, v \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$. However, it then follows that $\mathcal{M}, t \dashv\Vdash B$, and so (b) cannot hold. If instead (c) holds, then $u = x.y$ where $\mathcal{M}, x \dashv\Vdash A$ and $\mathcal{M}, y \dashv\Vdash B$, and so $\mathcal{M}, y.v \dashv\Vdash B$ by **P2** and *Closure*. Thus $\mathcal{M}, x \dashv\Vdash A$ and $\mathcal{M}, y \dashv\Vdash B$, but $\mathcal{M}, x.y.v \not\dashv\Vdash B$ since $x.y.v = u.v = t$ and $\mathcal{M}, t \not\dashv\Vdash B$. In each case, there is some $t \in \mathcal{S}$ where either: (1) $t = d.r$ for some $d, r \in \mathcal{S}$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, r \Vdash B$ but $\mathcal{M}, t \nVdash B$; or (2) $t = d.r$ for some $d, r \in \mathcal{S}$ where $\mathcal{M}, d \dashv\Vdash A$ and $\mathcal{M}, r \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$.

Assume that there is some $t \in \mathcal{S}$ where either: (1) $t = d.r$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, r \Vdash B$ but $\mathcal{M}, t \nVdash B$; or (2) $t = d.r$ where $\mathcal{M}, d \dashv\Vdash A$ and $\mathcal{M}, r \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$. Assuming (1), it follows that $\mathcal{M}, t \Vdash A \wedge B$ but $\mathcal{M}, t \nVdash B$, and so $\mathcal{M}, s' \dashv\Vdash A \wedge B \leqslant B$ for all $s' \in \mathcal{S}$. Assuming (2) instead, it follows that $t = d.r$ where $\mathcal{M}, d \dashv\Vdash A \wedge B$ and $\mathcal{M}, r \dashv\Vdash B$ but $\mathcal{M}, t \not\dashv\Vdash B$, and so $\mathcal{M}, s' \dashv\Vdash A \wedge B \leqslant B$ for all $s' \in \mathcal{S}$. By *Relevance*, $\mathcal{M}, s' \dashv\Vdash A \preceq B$ for all $s' \in \mathcal{S}$, and so $\mathcal{M}, s \dashv\Vdash A \preceq B$ in particular. □

4.5 Towards Axiomatisation

Rather than attempting to construct a logic for $\mathcal{L}$ by appealing to intuition without crossing over into inconsistency, we may draw on the semantics above in order to compute a range of validities. In particular, the present section establishes the validity of the principles, rules of inference, and meta-rules which figure in the proof system provided in §5.9, where it follows from the results given below that the proof system is sound over the semantics. As many of the proof are similar, I will omit details for a number of results.

L5.1 If $\Gamma, A \vDash B$, then $\Gamma \vDash A \rightarrow B$.

Proof. Assume $\Gamma, A \vDash B$, and let $\mathcal{M}$ be an $\mathcal{S}$-model and $w \in \mathbb{W}_{\mathcal{S}}$ where

$\mathcal{M}, w \vDash C$ for all $C \in \Gamma$. Assume for contradiction that $\mathcal{M}, w \nvDash A \to B$, and so $\mathcal{M}, w \dashv A \to B$ by **L4.9**. Thus there is some $t \sqsubseteq w$ where $\mathcal{M}, t \dashv\!\vdash A \to B$. It follows by the definition of '$\to$' and the semantics that $t = u.v$ for some $\mathcal{M}, u \Vdash A$ and $\mathcal{M}, v \dashv\!\vdash B$. Since $u, v \sqsubseteq t$, it follows that $u, v \sqsubseteq w$, and so $\mathcal{M}, w \vDash A$ and $\mathcal{M}, w \dashv B$, where $\mathcal{M}, w \nvDash B$ follows from the latter by **L4.9**. Given that $\mathcal{M}, w \vDash C$ for all $C \in \Gamma$ where $\Gamma, A \vDash B$, we know that $\mathcal{M}, w \vDash B$ contradicting the above. Thus $\mathcal{M}, w \vDash A \to B$, and so $\Gamma \vDash A \to B$ by generalisation. □

L5.2 If $\Gamma, \neg A \vDash B$ and $\Gamma, \neg A \vDash \neg B$, then $\Gamma \vDash A$.

L5.3 If $\Gamma \vDash A$ and $\Gamma \vDash B$, then $\Gamma \vDash A \wedge B$.

Proof. Assuming that both $\Gamma \vDash A$ and $\Gamma \vDash B$ for $\Gamma \cup \{A, B\} \subseteq \texttt{wfs}(\mathbb{L})$, let $\mathcal{M}$ be an $\mathcal{S}$-model and $w \in \mathbb{W}_{\mathcal{S}}$ where $\mathcal{M}, w \vDash C$ for all $C \in \Gamma$. It follows that $\mathcal{M}, w \vDash A$ and $\mathcal{M}, w \vDash B$, and so there is some $d, t \sqsubseteq w$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$. By the semantics, $\mathcal{M}, d.t \Vdash A \wedge B$, where $d.t \sqsubseteq w$ by **L2.4**. Thus $\mathcal{M}, w \vDash A \wedge B$, and so $\Gamma \vDash A \wedge B$ follows from generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$. □

L5.4 If $\Gamma \vDash A$, then $\Box\Gamma \vDash \Box A$.

Proof. Assume for contraposition that $\Box\Gamma \nvDash \Box A$ for $\Gamma \cup \{A\} \subseteq \texttt{wfs}(\mathbb{L})$. Thus for some $\mathcal{S}$-model $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$ it follows that $\mathcal{M}, w \vDash \Box B$ for all $B \in \Gamma$ but $\mathcal{M}, w \nvDash \Box A$, and so $\mathcal{M}, w \dashv \Box A$ by **L4.9**. It follows that $\mathcal{M}, t \dashv\!\vdash \Box A$ for some $t \sqsubseteq w$, and so $\mathcal{M}, d \dashv\!\vdash A$ for some $d \in P$. By *World Space*, $d \sqsubseteq w'$ for some $w' \in \mathbb{W}_{\mathcal{S}}$, and so $\mathcal{M}, w' \dashv A$. Thus we may conclude that $\mathcal{M}, w' \nvDash A$ again by **L4.9**.

Letting $B \in \Gamma$, it follows from the assumptions above that $|\Box B|^- = \varnothing$ by **L4.13**, and so $\mathcal{M}, w'' \vDash \Box B$ for any $w'' \in \mathbb{W}_{\mathcal{S}}$ again by **L4.13**. In particular, $\mathcal{M}, w' \vDash \Box B$, and so more generally, $\mathcal{M}, w' \vDash B$ for all $B \in \Gamma$. Given that $\mathcal{M}, w' \nvDash A$, we may conclude that $\Gamma \nvDash A$ as desired. □

L5.5 $\vDash A \approx \neg\neg A$.

Proof. Letting $\mathcal{M}$ be an $\mathcal{S}$-model and $A \in \texttt{wfs}(\mathbb{L})$, it follows from **P2** that $|A| \in \mathbb{P}_{\mathcal{S}}$ where $|A| = \langle |A|^+, |A|^- \rangle$. Given the semantics, we may conclude that $|\neg\neg A| = \langle |A|^+, |A|^- \rangle$, and so $|A| = |\neg\neg A|$. Thus $\mathcal{M}, w \vDash A \leqslant \neg\neg A$ and $\mathcal{M}, w \vDash \neg\neg A \leqslant A$ follow by **L4.15**, and so $\mathcal{M}, w \vDash A \approx \neg\neg A$ follows from **L5.3**. By then generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$, we may conclude that $\vDash A \approx \neg\neg A$. □

L5.6 $\vDash A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$.

L5.7 $\vDash \neg(A \wedge B) \approx \neg A \vee \neg B$.

L5.8 $\vDash \neg(A \vee B) \approx \neg A \wedge \neg B$.

L5.9 $\vDash A \wedge B \leqslant B \wedge A$.

L5.10 $\vDash A \wedge B \leqslant A \vee B$.

Proof. Let $\mathcal{M}$ be an $\mathcal{S}$-model and both $A, B \in \mathtt{wfs}(\mathbb{L})$. Assume for contradiction that there is some $t \in |A \wedge B \leqslant A \vee B|^-$. It follows that $t \in |A \wedge B| \leqslant^- |A \vee B|$, and so we know by definition that either: (1) there is some $d \in |A \wedge B|^+$ where $d \notin |A \vee B|^+$; (2) there is some $d \in |A \vee B|^-$ where $r \notin |A \wedge B|^-$ for any $r \sqsubseteq d$; or (3) there is some $d \in |A \wedge B|^-$ and $r \in |A \vee B|^-$ where $d.r \notin |A \vee B|^-$.

Case 1: Since $|A \wedge B|^+ = |A|^+ \wedge |B|^+$ and $|A \vee B|^+ = |A|^+ \vee |B|^+$ hold by definition, and $|A|^+ \vee |B|^+ = |A|^+ \cup |B|^+ \cup (|A|^+ \wedge |B|^+)$, it follows that $|A \wedge B|^+ \subseteq |A \vee B|^+$, thereby contradicting (1).

Case 2: Letting $d \in |A \vee B|^-$, it follows that $d = u.v$ for some $u \in |A|^-$ and $v \in |B|^-$. Since we know that both $|A \wedge B|^- = |A|^- \vee |B|^-$ and $|A|^- \vee |B|^- = |A|^- \cup |B|^- \cup (|A|^- \wedge |B|^-)$, it follows by the definition of fusion that $u \sqsubseteq d$ where $u \in |A \wedge B|^-$, contradicting (2).

Case 3: Letting $d \in |A \wedge B|^-$ and $r \in |A \vee B|^-$, we know that $d \in |A \vee B|^-$ given the identities observed in *Case 2*. Since $|A \vee B|^- \in \mathbb{C}_{\mathcal{S}}$ by **P2**, we know that $d.r \in |A \vee B|^-$, contradicting (3).

Given the contradictions above, it follows that $|A \wedge B \leqslant A \vee B|^- = \varnothing$, and so $\mathcal{M}, w \vDash A \wedge B \leqslant A \vee B$ by **L4.14**. By then generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$, we may conclude that $\vDash A \wedge B \leqslant A \vee B$ as needed. □

L5.11 $\vDash A \leqslant A \vee B$ and $\vDash B \leqslant A \vee B$.

Proof. Letting $\mathcal{M}$ be an $\mathcal{S}$-model and $A, B \in \mathtt{wfs}(\mathbb{L})$, we may assume for contradiction that there is some $t \in |A \leqslant A \vee B|^-$. It follows that $t \in |A| \leqslant^- |A \vee B|$ by **P2**. Thus either: (1) there is some $d \in |A|^+$ where $d \sqsubseteq t$ and $d \notin |A \vee B|^+$; (2) there is some $d \in |A \vee B|^-$ where both $d \sqsubseteq t$ and $r \notin |A|^-$ for any $r \sqsubseteq d$; or (3) there is some $d \in |A|^-$ and some $r \in |A \vee B|^-$ where $d.r \sqsubseteq t$ and $d.r \notin |A \vee B|^-$. We may then observe that $|A \vee B|^+ = |A|^+ \vee |B|^+$ by **P2**, and so $|A|^+ \subseteq |A \vee B|^+$ follows by *Sum*, thereby contradicting (1). Additionally, $|A \vee B|^- = |A|^- \wedge |B|^-$ by **P2**, and so if $d \in |A \vee B|^-$, then $d = u.v$ for some $u \in |A|^-$ and $v \in |B|^-$. Since $u \sqsubseteq d$ by the definition of fusion, it follows that for any $d \in |A \vee B|^-$ both $d \sqsubseteq t$ and $r \in |A|^-$ for some $r \sqsubseteq d$, contradicting (2). Lastly, choosing any $d \in |A|^-$ and $r \in |A \vee B|^-$, we know that $r = u.v$ for some $u \in |A|^-$ and $v \in |B|^-$, and so $d.u \in |A|^-$ by **P2**. Accordingly, $(d.u).v \in |A|^- \wedge |B|^-$ by *Product*, and so $d.r \in |A \vee B|^-$ by **L2.5** and **P2**, thereby contradicting (3). It follows that $|A \leqslant A \vee B|^- = \varnothing$, and so $\mathcal{M}, w \vDash A \leqslant A \vee B$ by **L4.14**. Thus $\vDash A \leqslant A \vee B$ follows by generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$. Analogous reasoning shows that $\vDash B \leqslant A \vee B$. □

L5.12 $\vDash A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)$.

Proof. Letting $\mathcal{M}$ be an $\mathcal{S}$-model and $A, B, C \in \mathtt{wfs}(\mathbb{L})$, we may assume for contradiction that there is some $t \in |A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)|^-$. It follows that $t \in |A \wedge (B \vee C)| \leqslant^- |(A \wedge B) \vee (A \wedge C)|$, and so we know by definition that either: (1) there is some $d \in |A \wedge (B \vee C)|^+$ where $d \notin |(A \wedge B) \vee (A \wedge C)|^+$; (2) there is some $d \in |(A \wedge B) \vee (A \wedge C)|^-$ where $r \notin |A \wedge (B \vee C)|^-$ for any $r \sqsubseteq d$; or (3) there is some $d \in |A \wedge (B \vee C)|^-$ and $r \in |(A \wedge B) \vee (A \wedge C)|^-$ where $d.r \notin |(A \wedge B) \vee (A \wedge C)|^-$.

Case 1: Given that $|A \wedge (B \vee C)|^+ = |A|^+ \wedge (|B|^+ \vee |C|^+)$, and that $|B|^+ \vee |C|^+ = |B|^+ \cup |C|^+ \cup (|B|^+ \wedge |C|^+)$, it follows from **L4.6** that

$|A \wedge (B \vee C)|^+ = (|A|^+ \wedge |B|^+) \cup (|A|^+ \wedge |C|^+) \cup [|A|^+ \wedge (|B|^+ \wedge |C|^+)]$. We may then observe the following identities:

$$\begin{aligned}
(|A|^+ \wedge |B|^+) \wedge (|A|^+ \wedge |C|^+) &=_1 (|B|^+ \wedge |A|^+) \wedge (|A|^+ \wedge |C|^+) \\
&=_2 |B|^+ \wedge [|A|^+ \wedge (|A|^+ \wedge |C|^+)] \\
&=_3 |B|^+ \wedge [(|A|^+ \wedge |A|^+) \wedge |C|^+] \\
&=_4 |B|^+ \wedge (|A|^+ \wedge |C|^+) \\
&=_5 (|B|^+ \wedge |A|^+) \wedge |C|^+ \\
&=_6 (|A|^+ \wedge |B|^+) \wedge |C|^+ \\
&=_7 |A|^+ \wedge (|B|^+ \wedge |C|^+).
\end{aligned}$$

Since $|(A \wedge B) \vee (A \wedge C)|^+ = (|A|^+ \wedge |B|^+) \vee (|A|^+ \wedge |C|^+)$, it follows from the definitions together with the above that:

$$|(A \wedge B) \vee (A \wedge C)|^+ = (|A|^+ \wedge |B|^+) \cup (|A|^+ \wedge |C|^+) \cup [|A|^+ \wedge (|B|^+ \wedge |C|^+)].$$

Thus $|A \wedge (B \vee C)|^+ = |(A \wedge B) \vee (A \wedge C)|^+$, contradicting (1).

Case 2: Letting $d \in |(A \wedge B) \vee (A \wedge C)|^-$, it follows that $d = u.v$ for some $u \in |A \wedge B|^-$ and $v \in |A \wedge C|^-$, and so $u \in |A|^- \vee |B|^-$ and $v \in |A|^- \vee |C|^-$. If either $u \in |A|^-$ or $v \in |A|^-$, it follows by the definition of fusion that there is some $r \sqsubseteq d$ where $r \in |A|^- \vee |(B \vee C)|^-$, and so $r \in |A \wedge (B \vee C)|^-$. Supposing instead that both $u \notin |A|^-$ and $v \notin |A|^-$, it follows that there is some $u' \sqsubseteq u$ and $v' \sqsubseteq v$ where $u' \in |B|^-$ and $v' \in |C|^-$, and so $u'.v' \in |B|^- \wedge |C|^-$. It follows that both $u' \sqsubseteq d$ and $v' \sqsubseteq d$, and so $u'.v' \sqsubseteq d$ by **L2.4**. Thus there is some $r \sqsubseteq d$ where $r \in |B|^- \wedge |C|^-$, and so $r \in |A \wedge (B \vee C)|^-$, contradicting (2).

Case 3: Let $d \in |A \wedge (B \vee C)|^-$ and $r \in |(A \wedge B) \vee (A \wedge C)|^-$. It follows that $d \in |A|^- \vee (|B|^- \wedge |C|^-)$, and so we may conclude by definition that $d \in |A|^- \cup (|B|^- \wedge |C|^-) \cup [|A|^- \wedge (|B|^- \wedge |C|^-)]$. Additionally, $r = u.v$ for some $u \in |A \wedge B|^-$ and $v \in |A \wedge C|^-$, and so $u \in |A|^- \cup |B|^- \cup (|A|^- \wedge |B|^-)$ and $v \in |A|^- \cup |C|^- \cup (|A|^- \wedge |C|^-)$. It follows that if $d \in |A|^-$, then both $d \in |A \wedge B|^-$ and $d \in |A \wedge C|^-$, and so $d \in |(A \wedge B) \vee (A \wedge C)|^-$ such that $d.r \in |(A \wedge B) \vee (A \wedge C)|^-$ by **P2**. Supposing instead that $d \in |B|^- \wedge |C|^-$, it follows that $d = u'.v'$ where $u' \in |B|^-$ and $v' \in |C|^-$. It follows that $u' \in |A \wedge B|^-$ and $v' \in |A \wedge C|^-$, and so $u.u' \in |A \wedge B|^-$ and $v.v' \in |A \wedge C|^-$ again by **P2**. Thus $u.u'.v.v' \in |(A \wedge B) \vee (A \wedge C)|^-$, and so $d.r \in |(A \wedge B) \vee (A \wedge C)|^-$. Lastly, suppose that $d \in |A|^- \wedge (|B|^- \wedge |C|^-)$. We may then conclude that $d = u'.v'.r'$ where $u' \in |A|^-$, $v' \in |B|^-$, and $r' \in |C|^-$. Accordingly, $u'.v'.u \in |A \wedge B|^-$ and $r'.v' \in |A \wedge C|^-$, and so it follows that $u'.v'.u.r'.v' \in |(A \wedge B) \vee (A \wedge C)|^-$ where $u'.v'.u.r'.v' = d.r$. Thus $d.r \in |(A \wedge B) \vee (A \wedge C)|^-$ holds for any $d \in |A \wedge (B \vee C)|^-$ contradicting (3), and so each case leads to a contradiction.

We may then conclude that $|A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)|^- = \varnothing$, and so $\mathcal{M}, w \vDash A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)$ follows by **L4.14**, where generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$ completes the proof. □

L5.13 $\vDash A \vee (B \wedge C) \leqslant (A \vee B) \wedge (A \vee C)$.

L5.14 $\vDash \Box A \to A$.

Proof. Assume for contradiction that $\not\models \Box A \rightarrow A$ for some $A \in \mathtt{wfs}(\mathbb{L})$, and so $\mathcal{M}, w \not\models \Box A \rightarrow A$ for some $\mathcal{S}$-model $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$. By **L4.9**, $\mathcal{M}, w \dashv\Vdash \Box A \rightarrow A$, and so $\mathcal{M}, t \dashv\Vdash \Box A \rightarrow A$ for some $t \sqsubseteq w$. By the definition of '$\rightarrow$' and the semantics, $t = u.v$ for some $\mathcal{M}, u \Vdash \Box A$ and $\mathcal{M}, v \dashv\Vdash A$. It follows from the former that for every $d \in P$, there is some $r \circ d$ where $\mathcal{M}, r \Vdash A$. Since $v \sqsubseteq t$ and $t \sqsubseteq w$, it follows that $v \sqsubseteq w$. Given that $w \in P$, it follows that $v \in P$ by *Stability*, and so there is some $r \circ v$ where $\mathcal{M}, r \Vdash A$. However, $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, and so $r \diamond v$ follows by *Exclusivity* from $\mathcal{M}, r \Vdash A$ and $\mathcal{M}, v \dashv\Vdash A$, thereby contradicting the above. Thus we may conclude that $\models \Box A \rightarrow A$ as desired. □

L5.15 $\models \Diamond A \rightarrow \Box\Diamond A$.

Proof. Assume for contradiction that $\not\models \Diamond A \rightarrow \Box\Diamond A$ where $A \in \mathtt{wfs}(\mathbb{L})$, and so $\mathcal{M}, w \not\models \Diamond A \rightarrow \Box\Diamond A$ for some $\mathcal{S}$-model $\mathcal{M}$. It follows by **L4.9** that $\mathcal{M}, w \dashv\Vdash \Diamond A \rightarrow \Box\Diamond A$, and so $\mathcal{M}, t \dashv\Vdash \Diamond A \rightarrow \Box\Diamond A$ for some $t \sqsubseteq w$. By the definition of '$\rightarrow$' and the semantics, $t = u.v$ for some $\mathcal{M}, u \Vdash \Diamond A$ and $\mathcal{M}, v \dashv\Vdash \Box\Diamond A$. It follows by **L4.16** that there is some $d \in P$ where $\mathcal{M}, d \Vdash A$, and by the semantics, there is some $r \in P$ where $\mathcal{M}, r \dashv\Vdash \Diamond A$. Again by **L4.16**, it follows that for every $x \in P$ there is some $y \circ x$ where $\mathcal{M}, y \dashv\Vdash A$. In particular, there is some $y \circ d$ where $\mathcal{M}, y \dashv\Vdash A$, since $d \in P$. However, given that $\mathcal{M}, d \Vdash A$ and $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, it follows from *Exclusivity* that $y \diamond d$, contradicting the above. Thus $\models \Diamond A \rightarrow \Box\Diamond A$. □

L5.16 $A, A \rightarrow B \models B$.

Proof. Letting $\mathcal{M}$ be an $\mathcal{S}$-model and $A, B \in \mathtt{wfs}(\mathbb{L})$, assume that both $\mathcal{M}, w \models A$ and $\mathcal{M}, w \models A \rightarrow B$. It follows that there is some $t, d \sqsubseteq w$ where $\mathcal{M}, t \Vdash A$ and $\mathcal{M}, d \Vdash A \rightarrow B$. Given the semantics and the definition of '$\rightarrow$', it follows from the latter that there is some $d \sqsubseteq w$ where either $\mathcal{M}, d \dashv\Vdash A$ or $\mathcal{M}, d \Vdash B$. Assume for contradiction that there is some $d \sqsubseteq w$ where $\mathcal{M}, d \dashv\Vdash A$. Since $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, and $t \in |A|^+$ and $d \in |A|^-$, it follows that $t \diamond d$ by *Exclusivity*. However, $t.d \sqsubseteq w$ by **L2.4**, and so $t.d \in P$ by *Stability*. Thus $t \circ d$ holds by *Composition*, contradicting the above. It follows that there is no $d \sqsubseteq w$ where $\mathcal{M}, d \dashv\Vdash A$, and so instead there is some $d \sqsubseteq w$ where $\mathcal{M}, d \Vdash B$. Thus $\mathcal{M}, w \models B$, and so $A, A \rightarrow B \models B$ follows from generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$. □

L5.17 $A \leqslant B, B \leqslant C \models A \leqslant C$.

Proof. Assuming for contradiction that $A \leqslant B, B \leqslant C \not\models A \leqslant C$ for some $A, B, C \in \mathtt{wfs}(\mathbb{L})$, there is some $\mathcal{S}$-model $\mathcal{M}$ where $\mathcal{M}, w \models A \leqslant B$ and $\mathcal{M}, w \models B \leqslant C$, while $\mathcal{M}, w \not\models A \leqslant C$. It follows from the former that $\mathcal{M}, s' \Vdash A \leqslant B$ and $\mathcal{M}, s'' \Vdash B \leqslant C$ for some $s', s'' \sqsubseteq w$, where $\mathcal{M}, s''' \dashv\Vdash A \leqslant C$ follows from the latter by **L4.14**. Thus either: (1) there is some $d \in |A|^+$ where $d \notin |C|^+$; (2) there is some $d \in |C|^-$ where $r \notin |A|^-$ for any $r \sqsubseteq d$; or else (3) there is some $d \in |A|^-$ and $r \in |C|^-$ where $d.r \notin |C|^-$. Consider the following cases:

Case 1: Assuming (1), there is some $d \in |A|^+$, and so $d \in |B|^+$ given $\mathcal{M}, s' \Vdash A \leqslant B$. It then follows from $\mathcal{M}, s'' \Vdash B \leqslant C$ that $d \in |C|^+$, thereby contradicting (1).

Case 2: Assuming (2), there is some $d \in |C|^-$, and so there is some $r \sqsubseteq d$ where $r \in |B|^-$ by $\mathcal{M}, s'' \Vdash B \leqslant C$. Thus there is some $u \sqsubseteq r$

where $u \in |A|^-$ by $\mathcal{M}, s' \Vdash A \leqslant B$. It follows that $u \sqsubseteq d$, and so some $u \sqsubseteq d$ where $u \in |A|^-$, thereby contradicting (2).

Case 3: Assuming (3), there is some $d \in |A|^-$ and $r \in |C|^-$. It follows from $\mathcal{M}, s'' \Vdash B \leqslant C$ that there is some $u \sqsubseteq r$ where $u \in |B|^-$, and so $d.u \in |B|^-$ by $\mathcal{M}, s' \Vdash A \leqslant B$. It then follows by $\mathcal{M}, s'' \Vdash B \leqslant C$ that $d.u.r \in |C|^-$, where $d.u.r = d.r$ since $u \sqsubseteq r$. Thus we may conclude that $d.r \in |C|^-$, thereby contradicting (3).

Thus we may conclude that $A \leqslant B, B \leqslant C \vDash A \leqslant C$. □

L5.18 $A \leqslant C, B \leqslant C \vDash A \vee B \leqslant C$.

Proof. Assuming for contradiction that $A \leqslant C, B \leqslant C \nvDash A \vee B \leqslant C$ for some $A, B, C \in \mathtt{wfs}(\mathbb{L})$, there is some $\mathcal{S}$-model $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$ where both $\mathcal{M}, w \vDash A \leqslant C$ and $\mathcal{M}, w \vDash B \leqslant C$, while $\mathcal{M}, w \nvDash A \vee B \leqslant C$. It follows from **L4.14** and **L4.12** that $\mathcal{M}, s \Vdash A \leqslant C$ and $\mathcal{M}, s' \Vdash A \leqslant C$ for some $s, s' \in \mathcal{S}$, where $|A \vee B \leqslant C|^- \neq \varnothing$. By **P2**, $|A \vee B| \leqslant^- |C| \neq \varnothing$, and so either: (1) there is some $d \in |A \vee B|^+$ where $d \notin |C|^+$; (2) there is some $d \in |C|^-$ where $r \notin |A \vee B|^-$ for any $r \sqsubseteq d$; or (3) there is some $d \in |A \vee B|^-$ and $r \in |C|^-$ where $d.r \notin |C|^-$. Consider the cases:

Case 1: Assuming (1), there is some $d \in |A \vee B|^+$, and so by **P2** and *Sum*, $d \in |A|^+ \cup |B|^+ \cup (|A|^+ \wedge |B|^+)$. However, if $d \in |A|^+$, then $d \in |C|^+$ since $\mathcal{M}, s \vDash A \leqslant C$, and similarly, if $d \in |B|^+$, then $d \in |C|^+$ since $\mathcal{M}, s' \vDash B \leqslant C$. Assuming instead that $d \in |A|^+ \wedge |B|^+$, it follows that $d = u.v$ for some $u \in |A|^+$ and $v \in |B|^+$, and so $u, v \in |C|^+$ since $\mathcal{M}, s \vDash A \leqslant C$ and $\mathcal{M}, s' \vDash B \leqslant C$. Given that $|C|^+ \in \mathbb{C}_{\mathcal{S}}$ by **P2**, $d \in |C|^+$ in every case, thereby contradicting (1).

Case 2: Assuming (2), there is some $d \in |C|^-$. Since $\mathcal{M}, s \vDash A \leqslant C$ and $\mathcal{M}, s \vDash B \leqslant C$, it follows there are some $u, v \sqsubseteq d$ where $u \in |A|^-$ and $v \in |B|^-$, and so $u.v \in |A \vee B|^-$ by **P2** and *Product*. However, $u.v \sqsubseteq d$ follows from **L2.4**, it follows that there is some $r \sqsubseteq d$ where $r \in |A \vee B|^-$, thereby contradicting (2).

Case 3: Assuming (3), there is some $d \in |A \vee B|^-$ and $r \in |C|^-$. It follows by **P2** that $d \in |A|^- \wedge |B|^-$, and so $d = u.v$ for some $u \in |A|^-$ and $v \in |B|^-$ by *Product*. Since $\mathcal{M}, s \vDash A \leqslant C$ and $\mathcal{M}, s \vDash B \leqslant C$, it follows that $u, v \in |C|^-$, and so $u.v.r \in |C|^-$ given that $|C| \in \mathbb{P}_{\mathcal{S}}$ by **P2**. Thus $d.r \in |C|^-$, thereby contradicting (3).

Given that a contradiction results from each of the cases above, we may conclude that $A \leqslant C, B \leqslant C \vDash A \vee B \leqslant C$ as desired. □

L5.19 $A \leqslant B, A \leqslant C \vDash A \leqslant B \wedge C$.

L5.20 $A \leqslant B \vDash A \vee B \sqsubseteq B$.

Proof. Assume that $\mathcal{M}, w \vDash A \leqslant B$ for an $\mathcal{S}$-model $\mathcal{M}$, $w \in \mathbb{W}_{\mathcal{S}}$, and $A, B \in \mathtt{wfs}(\mathbb{L})$. It follows that $|A \leqslant B|^+ \neq \varnothing$ for some $s \in \mathcal{S}$ by **L4.12** and **L4.14**, and so: (1) $|A|^+ \subseteq |B|^+$; (2) for all $t \in |B|^-$, there is some $d \sqsubseteq t$ where $d \in |A|^-$; and (3) $t.d \in |B|^-$ for all $t \in |A|^-$ and $d \in |B|^-$.

Assume $\mathcal{M}, w \nvDash A \vee B \sqsubseteq B$ for contradiction. It follows by **L4.9** that $\mathcal{M}, s \dashv\Vdash A \vee B \sqsubseteq B$ for some $s \in \mathcal{S}$, and so by **L4.16**, either: $(1)'$ there is some $t \in |A \vee B|^+$ and $d \in |B|^+$ where $t.d \notin |B|^+$; $(2)'$ $|A \vee B|^- \nsubseteq |B|^-$; or $(3)'$ there is some $t \in |B|^+$ where no $d \sqsubseteq t$ is such that $d \in |A \vee B|^+$. Consider the following cases:

Case 1: Assuming $(1)'$, there is some $t \in |A \vee B|^-$ where $t \notin |B|^-$, and so by the semantics, $t = u.v$ for some $u \in |A|^-$ and $v \in |B|^-$. However, $t \in |B|^-$ follows from (3), contradicting the above.

Case 2: Assuming $(2)'$, there is some $t \in |B|^+$ where no $d \sqsubseteq t$ is such that $d \in |A \vee B|^+$. However, $t \notin |A \vee B|^+$ since $t \sqsubseteq t$. Thus it follows by the semantics that $t \notin |B|^+$, contradicting the above.

Case 3: Assuming $(3)'$, there is some $t \in |A \vee B|^+$ and $d \in |B|^+$ where $t.d \notin |B|^+$. By the semantics, either $t \in |A|^+$, $t \in |B|^+$, or $t \in |A \wedge B|^+$. Given that $|A|^+ \subseteq |B|^+$ by (1), we need only consider the case where $t \in |A \wedge B|^+$, and so $t = u.v$ for some $u \in |A|^+$ and $v \in |B|^+$. Thus $u \in |B|^+$ by (1), and so given that $|B| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, it follows that $t \in |B|^+$. Thus it follows from $t \in |A \vee B|^+$ that $t \in |B|^+$, and so $t.d \in |B|^+$ again by **P2**, contradicting the above.

Having reached a contradiction in each of the cases above, we may conclude that $\mathcal{M}, w \vDash A \vee B \sqsubseteq B$, and so $A \leqslant B \vDash A \vee B \sqsubseteq B$ follows from generalising on $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$. □

L5.21 $A \sqsubseteq B \vDash A \wedge B \leqslant B$.

L5.22 $A \leqslant B \vDash \Box(A \to B)$.

Proof. Assuming for contradiction that $A \leqslant B \nvDash \Box(A \to B)$ for some $A, B \in \mathtt{wfs}(\mathbb{L})$, it follows that $\mathcal{M}, w \vDash A \leqslant B$ and $\mathcal{M}, w \nvDash \Box(A \to B)$ for some $\mathcal{S}$-model $\mathcal{M}$ and $w \in \mathbb{W}_{\mathcal{S}}$. It follows from the former that $\mathcal{M}, s \Vdash A \leqslant B$ for some $s \in \mathcal{S}$, and so by the semantics: (a) $|A|^+ \subseteq |B|^+$; (b) for all $t \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash B$, then $\mathcal{M}, d \dashv\Vdash A$ for some $d \sqsubseteq t$; and (c) for all $t, d \in \mathcal{S}$, if $\mathcal{M}, t \dashv\Vdash A$ and $\mathcal{M}, d \dashv\Vdash B$, then $\mathcal{M}, t.d \dashv\Vdash B$. Additionally, given that $\mathcal{M}, w \nvDash \Box(A \to B)$, it follows from **L4.9** that $\mathcal{M}, w \dashv\Vdash \Box(A \to B)$, and so $\mathcal{M}, d \dashv\Vdash A \to B$ for some $d \sqsubseteq w$. By the definition of '$\to$' and the semantics, $d = u.v$ for some $\mathcal{M}, u \Vdash A$ and $\mathcal{M}, v \dashv\Vdash B$. However, it follows from (a) that $\mathcal{M}, u \Vdash B$, and since $|B| \in \mathbb{P}_{\mathcal{S}}$ by **P2**, we may conclude that $u \circ v$ by *Exclusivity*, thereby contradicting the above. Thus it follows that $A \leqslant B \vDash \Box(A \to B)$ as desired. □

L5.23 $A \leqslant B \vDash \Box(A \leqslant B)$.

Proof. Assuming for contradiction that $A \leqslant B \nvDash \Box(A \leqslant B)$ for some $A, B \in \mathtt{wfs}(\mathbb{L})$, it follows that $\mathcal{M}, w \vDash A \leqslant B$ but $\mathcal{M}, w \nvDash \Box(A \leqslant B)$ for some $\mathcal{S}$-model $\mathcal{M}$. It follows that $\mathcal{M}, w \dashv\Vdash \Box(A \leqslant B)$ from the latter by **L4.9**, and so $\mathcal{M}, s \dashv\Vdash \Box(A \leqslant B)$ for some $s \sqsubseteq w$. By the semantics, there is some $t \in P$ where $\mathcal{M}, t \dashv\Vdash A \leqslant B$, and so $|A \leqslant B|^- \neq \varnothing$. It follows that $\mathcal{M}, w \nvDash A \leqslant B$ by **L4.14**, contradicting the above. Thus we may conclude that $A \leqslant B \vDash \Box(A \leqslant B)$ as desired. □

4.6 Countermodels

In order to characterise the hyperintensionality of the theory of propositions presented above, we may contrast propositional identity with an intensional notion of modal equivalence which may be defined as follows:

Equivalence: $\ulcorner A \cong B \urcorner$ abbreviates $\ulcorner \Box(A \leftrightarrow B) \urcorner$.

It is easy to show that all of the standard Boolean equivalences are valid over the semantics. In particular, the following principles may be shown to be valid:

In contrast to the equivalences above, both the absorption laws and distribution laws for propositional identity admit of counterexamples:

L6.1 $\not\models A \equiv [A \wedge (A \vee B)]$ and $\not\models A \equiv [A \vee (A \wedge B)]$.

L6.2 $\not\models [A \vee (B \wedge C)] \equiv [(A \vee B) \wedge (A \vee C)]$ and
$\not\models [A \wedge (B \vee C)] \equiv [(A \wedge B) \vee (A \wedge C)]$.

The counterexamples to the absorption and distribution laws demonstrate the fineness of grain of the present theory of proposition in contrast to standard intensional theories of propositions where propositions are sets of possible worlds. Since neither the absorption nor distribution laws hold without exception for propositional identity, the space of propositions $\mathbb{P}_{\mathcal{S}}$ cannot be said to form a Boolean algebra for any state space $\mathcal{S}$. Rather, by first defining $X \lesssim Y \coloneqq (X \vee Y = Y)$ and $X \sqsubseteq Y \coloneqq (X \wedge Y = Y)$ for $X, Y \in \mathbb{P}_{\mathcal{S}}$, we may show that $\mathcal{B}_{\mathcal{S}} = \langle \mathbb{P}_{\mathcal{S}}, \lesssim, \sqsubseteq, \neg \rangle$ forms a *bilattice* since $\langle \mathbb{P}_{\mathcal{S}}, \lesssim \rangle$ and $\langle \mathbb{P}_{\mathcal{S}}, \sqsubseteq \rangle$ are both complete lattices where the following conditions hold for all $X, Y \in \mathbb{P}_{\mathcal{S}}$:

NN: $\neg\neg X = X$.

GE: If $X \lesssim Y$, then $\neg X \sqsubseteq \neg Y$.

EG: If $X \sqsubseteq Y$, then $\neg X \lesssim \neg Y$.

Given the countermodel to the distribution laws, it follows that $\mathcal{B}_{\mathcal{S}}$ need not be *distributive* for any state space $\mathcal{S}$ since $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$ and $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ may fail to hold for some $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$. In particular, we may choose $|p_1|_u, |p_2|_u, |p_3|_u \in \mathbb{P}_{\mathcal{S}}$ as in **L6.2** given above. Additionally, the following lemma shows that $\mathcal{B}_{\mathcal{S}}$ need not be *interlaced* for any state space $\mathcal{S}$ on account of the fact that it may fail to be the case that $\wedge$ and $\vee$ are monotonic with respect to both $\lesssim$ and $\sqsubseteq$.

L6.3 $A \leqslant B \not\models A \wedge C \leqslant B \wedge C$.

It follows from the countermodel given above that there is a state space $\mathcal{S}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$ where $X \lesssim Y$ and yet $X \wedge Z \not\lesssim Y \wedge Z$, and so $\wedge$ is not monotonic with respect to $\lesssim$. In particular, we may choose $|p_1|_u, |p_2|_u, |p_3|_u \in \mathbb{P}_{\mathcal{S}}$ as in **L6.3**. Thus $\mathcal{B}_{\mathcal{S}}$ need not be interlaced for every state space $\mathcal{S}$.

In addition to these findings, we may observe that **L6.3** is a special case of the following supplementation law for conjunction defended by Lovett (2020):

$\wedge$-Sup: $A \leqslant B, C \leqslant D \vdash A \wedge C \leqslant B \wedge D$.

In defence of the **∧-Sup**, Lovett begins by saying that the principle, "seem[s] to me pretty clearly valid" (p. 19), but does not provide a semantics by which to confirm the validity of **∧-Sup**. Rather, Lovett goes on to defend the principle by way of the *subsumption characterisation* of ground that, "A weakly grounds B iff A explains everything B explains" (p. 15). Lovett then argues as follows:

> It seems plausible to me that if A explains everything B explains, and C explains everything D explains, then $A \wedge C$ explains everything $B \wedge D$ explains. This is because otherwise there would have to be something which A, C explained and $B \wedge D$ explained, but $A \wedge C$ did not explain. I think it is implausible that there is such a thing. So, by the explanatory subsumption characterization of weak ground, Supplementation-$\wedge$ must follow. (p. 20)

In order to refute **∧-Sup** by the lights of Lovett's subsumption characterisation, $B \wedge D$ must explain something which $A \wedge C$ does not explain despite the fact that A explains everything that B explains, and C explains everything that D explains. It remains, however, to say what explanation amounts to, describing its relationship to ground. Here Lovett suggests that 'explains' is be understood as strict-full ground, contemplating the definition, "A weakly grounds B iff A strictly grounds everything B strictly grounds" (p. 15).[11] Accordingly, a counterexample which refutes **∧-Sup** by Lovett's lights must indicate a proposition which $B \wedge D$ strictly grounds but that $A \wedge C$ does not strictly ground, while admitting that A strictly grounds everything that B strictly grounds, and C strictly ground everything that D strictly grounds.

Despite Lovett's doubts, **L6.3** points the way to such a counterexample. In order to add texture, let p_i be the sentence 'Line l_i is not exactly one inch long' for each $i \in \{1, 2, 3, 4\}$, where $\{i\}$ is the state of line l_i being exactly one inch long. We may then observe the following identities, where fusion is set union:

$$\begin{aligned} |p_1 \wedge p_3|_l^- &= \{\{1\}, \{3\}, \{1, 3\}\} \\ |p_1 \vee p_2|_l^- &= \{\{1, 2\}\} \\ |(p_1 \vee p_2) \wedge p_3|_l^- &= \{\{1, 2\}, \{3\}, \{1, 2, 3\}\} \\ |((p_1 \vee p_2) \wedge p_3) \vee p_4|_l^- &= \{\{1, 2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}. \end{aligned}$$

Here we may observe that the fusion $\{1, 3\}$ of $\{1\}$ and $\{3\}$ does not belong to $|(p_1 \vee p_2) \wedge p_3|_l^-$ despite the fact that $\{1\} \in |p_1 \wedge p_3|_l^-$ and $\{3\} \in |(p_1 \vee p_2) \wedge p_3|_l^-$. By the semantics for ground, we may conclude that $p_1 \wedge p_3 \leqslant (p_1 \vee p_2) \wedge p_3$ does not hold on its present interpretation. Moreover, we may observe that $(p_1 \vee p_2) \wedge p_3 < ((p_1 \vee p_2) \wedge p_3) \vee p_4$, but that $p_1 \wedge p_3 \not< ((p_1 \vee p_2) \wedge p_3) \vee p_4$, and so $p_1 \wedge p_3 \leqslant (p_1 \vee p_2) \wedge p_3$ also fails to hold by Lovett's subsumption characterisation of weak-full ground. At the same time, both $A \leqslant A \vee B$ and

[11] See Fine (2012b, p. 52) for this characterisation of weak-full ground. Lovett (2020, p.14) also adopts the definition, "A strictly grounds B iff A weakly grounds B and B doesn't even help weakly ground A" given by Fine (2012b, p. 52). On pain of circularity, one cannot take Lovett's definition of weak ground in terms of strict ground as more than a characterisation.

$C \leqslant C$ hold for all $A, B, C \in \mathtt{wfs}(\mathbb{L})$, and so both $p_1 \leqslant p_1 \vee p_2$ and $p_3 \leqslant p_3$ in particular, thereby providing a counterexample to **∧-Sup**. Although the present example does not amount to a full formal model of $\mathcal{L}$, it is nevertheless straightforward to specify exact verifiers for each sentence in a manner which respects the intended grounding relationships.

In attempt to resist the counterexample given above, one may seek to adapt the semantics so as to validate **∧-Sup**. One way to achieve this result is to require propositions to be convex. However, as observed in §4.3, this restriction comes with considerable costs, ruling out the semantic clauses for conjunction and disjunction. Although it remains open to proponents of **∧-Sup** to devise an alternative to the semantics given above, any increases in complexity or decrease in intuitive appeal must be weighed against whatever is gained by including **∧-Sup**. Alternatively, one could maintain the validity of **∧-Sup** by dropping (3) from the semantics for ground.[12,13] However, as brought out in §4.3, (3) was included specifically to validate **Rel**, and so excluding (3) from the semantics comes at the cost of admitting countermodels to **Rel**.[14]

Rather than attempting to adapt the semantics for $\mathcal{L}$ so as to avoid the counterexamples above, we may seek to account for the invalidity of **∧-Sup** as well as the non-monotonicity of conjunction over ground demonstrated in **L6.3** by appealing to the non-monotonicity of conjunction over relevance. In particular, we may show that the following principle is invalid:

≤-Mon: $(A \preceq B) \rightarrow (A \wedge C \preceq B \wedge C)$.[15]

Just because A is wholly relevant to B, we find that $A \wedge C$ may fail to be wholly relevant to $B \wedge C$. As brought out by the counterexamples above, for states u and v which exactly falsify A and C respectively, $u.v$ need not exactly falsify $B \wedge C$ despite the fact that u exactly falsifies $A \wedge C$ and v exactly

[12] *Proof:* Let $\mathcal{M}$ be an $\mathcal{S}$-model, $w \in \mathbb{W}_{\mathcal{S}}$, and $A, B, C, D \in \mathtt{wfs}(\mathbb{L})$ with $\mathcal{M}, w \vDash A \leqslant^{\star} B$ and $\mathcal{M}, w \vDash C \leqslant^{\star} D$, where '$\leqslant^{\star}$' only includes (1) and (2) in its semantics. It follows from **L4.14** that $\mathcal{M}, s \Vdash A \leqslant C$ and $\mathcal{M}, s' \Vdash A \leqslant C$ for some $s, s' \in \mathcal{S}$. If $x \in |A \wedge C|^{+}$, then $x = u.v$ for some $u \in |A|^{+}$ and $v \in |C|^{+}$. Given the above, $u \in |B|^{+}$ and $v \in |D|^{+}$, and so $x \in |B \wedge D|^{+}$. Thus $|A \wedge C|^{+} \subseteq |B \wedge D|^{+}$. If $y \in |B \wedge D|^{-}$, it follows that either: (a) $d \in |B|^{-}$; (b) $d \in |D|^{-}$; or (c) $d = u'.v'$ for some $u' \in |B|^{-}$ and $v' \in |D|^{-}$. If (a), then there is some $t \sqsubseteq d$ where $t \in |A|^{-}$, and so $t \in |A \wedge C|^{-}$. If (b), then there is some $t \sqsubseteq d$ where $t \in |C|^{-}$, and so $t \in |A \wedge C|^{-}$. If (c), then there is some $t \sqsubseteq u'$ and $r \sqsubseteq v'$ with $t \in |A|^{-}$ and $r \in |C|^{-}$, and so $t.r \in |A \wedge C|^{-}$ where $t.r \sqsubseteq d$. Thus in either case, there is some $z \sqsubseteq d$ where $z \in |A \wedge C|^{-}$, and so together with the above $|A \wedge C \leqslant B \wedge D|^{+} \neq \varnothing$. Thus we may conclude that $\mathcal{M}, w \vDash A \wedge C \leqslant^{\star} B \wedge D$ by **L4.12** and **L4.14**. □

[13] The result of dropping (3) from the semantics for ground may be referred to as *consequential ground*, and is comparable to the contrast Fine (1995b, p. 276) develops between consequential and constitutive essence, despite the disparities in logical form.

[14] *Sketch:* Letting $\mathcal{M}_l$ be a model which assigns falsifiers to p_i for $i \in \{1, 2, 3, 4\}$ as in the example above, we may observe that $\mathcal{M}, w \vDash p_1 \wedge p_3 \leqslant^{\star} (p_1 \vee p_2) \wedge p_3$ for any world w in $\mathcal{M}_l$, where the semantics for '$\leqslant^{\star}$' excludes the condition given in (3). Nevertheless, $\mathcal{M}_l, w \nvDash p_1 \wedge p_3 \preceq (p_1 \vee p_2) \wedge p_3$, thereby providing a countermodel to **Rel**.

[15] *Countermodel*: Let $\mathcal{M}_r$ be as in **L6.3** and $s \in \mathcal{S}_r$. However, $\mathcal{M}_r, s \Vdash p_1 \preceq p_2$ despite the fact that $\mathcal{M}_r, s \nVdash p_1 \wedge p_3 \preceq p_2 \wedge p_3$, and so for any $w \in \mathbb{W}_{\mathcal{S}_r}$ both $\mathcal{M}_r, w \vDash p_1 \preceq p_2$ and $\mathcal{M}_r, w \nvDash p_1 \wedge p_3 \preceq p_2 \wedge p_3$. Thus $\nvDash (p_1 \preceq p_2) \rightarrow (p_1 \wedge p_2 \preceq p_2 \wedge p_3)$. ◇

falsifies $B \wedge C$. What the invalidity of $\wedge$**-Mon** brings out is the strength of the connection required by relevance, and included in the semantics for ground by way of (1) and (3). It is worth noting that the same degree of strength is included in the semantics by defining ground in terms of propositional identity as in $\vee$**-Part** from §4.3. Of course, one could insist upon dropping (3) from the semantics of ground and rejecting both **Rel** and $\vee$**-Part**. By contrast, I will take the countermodels above to show that despite the intuitive appeal which $\wedge$**-Sup**, $\preceq$**-Mon**, and the monotonicity of conjunction over ground may hold, such intuitions do not outweigh the validity of **Rel**.

However compelling $\wedge$**-Sup** may first appear, intuitions about the validity of the principles and rules that can be articulated in an object language cannot provide more than a flimsy basis of support. This is not to claim that intuitions have no role to play in developing a logic and semantics for ground. Indeed, the stock of candidate principles presented in §5.1 relied on a similar basis of intuitive judgements. However, rather than taking such intuitions to be the theoretical bedrock upon which the logic is to be erected, I have tested the candidate principles presented in §5.1 by providing a semantics with which we may survey the space of all possible counterexamples to such principles. Whereas all of the principles included in §5.1 have been shown to be valid, the same cannot be said for $\wedge$**-Sup** on account of admitting counterexamples. Additionally, in gathering together a set of candidate principles for weak-full ground, I was careful to restrict attention to principles that would be difficult to reject without changing the target notion under consideration. In particular, I did not include any initial judgement about whether conjunction or disjunction are monotonic with respect to either essence or ground.

With these considerations in mind, we may turn to provide a countermodel to the transitivity of relevance as given by the following lemma:

L6.4 $A \preceq B, B \preceq C \not\models A \preceq C$.

In contrast to the principles articulated in terms of ground, the relevance operator has been defined rather than taken to be primitive. Accordingly, our semantic findings about relevance will not only depend on the semantics for ground along with the extensional operators, but also on the definition of relevance in terms of ground given in *Relevance*. Nevertheless, we find some support in defence of *Relevance* by looking to the derived semantic clauses given in **L4.16**. Additionally, we may observe that nothing in our pre-theoretic intuitive grasp of the notion of relevance appears to force transitivity upon us. In this respect, relevance may be taken to resemble similarity, where x can be similar to y, and y can be similar to z, without x also being similar to z. Although such intuitive determinations should not be taken to be indefeasible, alignment with our pre-theoretic conception of relevance— at least insofar as there is such a pre-theoretic conception— may nevertheless provide some support in favour of rejecting the transitivity of relevance.

L6.1 $\not\models A \equiv [A \wedge (A \vee B)]$ and $\not\models A \equiv [A \vee (A \wedge B)]$.

Proof. Let $\mathcal{M}_v = \langle S_v, P_v, \bigsqcup, |\cdot|_v\rangle$ where $S_v = \mathcal{P}(\{a,b,c,d\})$, and $P_v = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}\}$ where:

$$\begin{aligned} |p_1|_v &= \langle\{\{a\}\}, \{\{b\}\}\rangle, \\ |p_2|_v &= \langle\{\{c\}\}, \{\{d\}\}\rangle. \end{aligned}$$

Observe that $\mathcal{M}_v$ is a model of $\mathcal{L}$. We may then consider:

$$\begin{aligned} |p_1 \wedge p_2|_v &= \langle\{\{a,c\}\}, \{\{b\},\{d\},\{b,d\}\}\rangle \\ |p_1 \vee p_2|_v &= \langle\{\{a\},\{c\},\{a,c\}\}, \{\{b,d\}\}\rangle \\ |p_1 \vee (p_1 \wedge p_2)|_v &= \langle\{\{a\},\{a,c\}\}, \{\{b\},\{b,d\}\}\rangle \\ |p_1 \wedge (p_2 \vee p_2)|_v &= \langle\{\{a\},\{a,c\}\}, \{\{b\},\{b,d\}\}\rangle. \end{aligned}$$

Since $\{\{a\},\{a,c\}\} \not\subseteq \{\{a\}\}$ and $\{\{b\},\{b,d\}\} \not\subseteq \{\{b\}\}$, it follows that $\mathcal{M}_v, s \dashv\Vdash p_1 \equiv p_1 \vee (p_1 \wedge p_2)$ and $\mathcal{M}_v, s \dashv\Vdash p_1 \equiv p_1 \wedge (p_1 \vee p_2)$ for all $s \in \mathcal{S}_v$ by **L4.16**, and so both $\mathcal{M}_v, w \not\Vdash p_1 \equiv p_1 \vee (p_1 \wedge p_2)$ and $\mathcal{M}_v, w \not\Vdash p_1 \equiv p_1 \wedge (p_1 \vee p_2)$ for all $w \in \mathbb{W}_{\mathcal{S}_v}$. Given that $\mathbb{W}_{\mathcal{S}} \neq \varnothing$ by **L2.7**, we may conclude that both $\not\Vdash p_1 \equiv [p_1 \wedge (p_1 \vee p_2)]$ and $\not\Vdash p_1 \equiv [p_1 \vee (p_1 \wedge p_2)]$, and so there are some $A, B \in \mathtt{wfs}(\mathbb{L})$ where $\not\Vdash A \equiv A \vee (A \wedge B)$ and $\not\Vdash A \equiv A \wedge (A \vee B)$. □

L6.2 $\not\Vdash [A \vee (B \wedge C)] \equiv [(A \vee B) \wedge (A \vee C)]$ and $\not\Vdash [A \wedge (B \vee C)] \equiv [(A \wedge B) \vee (A \wedge C)]$.

Proof. Let $\mathcal{M}_u = \langle S_u, P_u, \bigsqcup, |\cdot|_u\rangle$ where $S_u = \mathcal{P}(\{a,b,c,d,e,f\})$, and P_u includes all subsets of members of the following set:

$$\mathbb{W}_{\mathcal{S}_u} = \{\{a,c,e\},\{b,c,e\},\{a,d,e\},\{a,c,f\},\{b,d,e\},\{b,c,f\},\{a,d,f\},\{b,d,e\}\}$$

Letting $\ulcorner\overline{x}\urcorner$ abbreviate $\{x\}$, and $\ulcorner x.y\urcorner$ abbreviate $\ulcorner x \cup y\urcorner$, we may set $|p_1|_u = \langle\{\overline{a}\},\{\overline{b}\}\rangle$, $|p_2|_u = \langle\{\overline{c}\},\{\overline{d}\}\rangle$, and $|P_3|_u = \langle\{\overline{e}\},\{\overline{f}\}\rangle$, observing that $\mathcal{M}_u$ is a model of $\mathcal{L}$. Consider the following:

$$\begin{aligned} |p_1 \wedge p_2|_u &= \langle\{\overline{a}.\overline{c}\}, \{\overline{b}, \overline{d}, \overline{b}.\overline{d}\}\rangle \\ |p_1 \wedge p_3|_u &= \langle\{\overline{a}.\overline{e}\}, \{\overline{b}, \overline{f}, \overline{b}.\overline{f}\}\rangle \\ |p_2 \vee p_3|_u &= \langle\{\overline{c}, \overline{e}, \overline{c}.\overline{e}\}, \{\overline{d}.\overline{f}\}\rangle \\ |p_2 \wedge p_3|_u &= \langle\{\overline{c}.\overline{e}\}, \{\overline{d}, \overline{f}, \overline{d}.\overline{f}\}\rangle \\ |p_1 \vee (p_2 \wedge p_3)|_u &= \langle\underline{\{\overline{a}, \overline{c}.\overline{e}, \overline{a}.\overline{c}.\overline{e}\}}, \{\overline{b}.\overline{d}, \overline{b}.\overline{f}, \overline{b}.\overline{d}.\overline{f}\}\rangle \\ |(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_u &= \langle\{\overline{a}, (\overline{a}.\overline{c}), (\overline{a}.\overline{e}), \overline{c}.\overline{e}, \overline{a}.\overline{c}.\overline{e}\}, \{\overline{b}.\overline{d}, \overline{b}.\overline{f}, \overline{b}.\overline{d}.\overline{f}\}\rangle \\ |p_1 \wedge (p_2 \vee p_3)|_u &= \langle\{\overline{a}.\overline{c}, \overline{a}.\overline{e}, \overline{a}.\overline{c}.\overline{e}\}, \underline{\{\overline{b}, \overline{d}.\overline{f}, \overline{b}.\overline{d}.\overline{f}\}}\rangle \\ |(p_1 \wedge p_2) \vee (p_1 \wedge p_3)|_u &= \langle\{\overline{a}.\overline{c}, \overline{a}.\overline{e}, \overline{a}.\overline{c}.\overline{e}\}, \{\overline{b}, (\overline{b}.\overline{d}), (\overline{b}.\overline{f}), \overline{d}.\overline{f}, \overline{b}.\overline{d}.\overline{f}\}\rangle. \end{aligned}$$

Since the exact verifiers (falsifiers) enclosed in parentheses do not belong to the underlined set of exact verifiers (falsifiers), every $s \in \mathcal{S}_u$ is such that $\mathcal{M}_v, s \dashv\Vdash [p_1 \vee (p_2 \wedge p_3)] \equiv [(p_1 \vee p_2) \wedge (p_1 \vee p_3)]$ and $\mathcal{M}_v, s \dashv\Vdash [p_1 \wedge (p_2 \vee p_3)] \equiv [(p_1 \wedge p_2) \vee (p_1 \wedge p_3)]$ by **L4.16**. It

follows that both $\mathcal{M}_v, w \not\Vdash [p_1 \vee (p_2 \wedge p_3)] \equiv [(p_1 \vee p_2) \wedge (p_1 \vee p_3)]$ and $\mathcal{M}_v, w \not\Vdash [p_1 \wedge (p_2 \vee p_3)] \equiv [(p_1 \wedge p_2) \vee (p_1 \wedge p_3)]$ for all $w \in \mathbb{W}_{\mathcal{S}_u}$, and so $\not\Vdash [p_1 \vee (p_2 \wedge p_3)] \equiv [(p_1 \vee p_2) \wedge (p_1 \vee p_3)]$ and $\not\Vdash [p_1 \wedge (p_2 \vee p_3)] \equiv [(p_1 \wedge p_2) \vee (p_1 \wedge p_3)]$ given that $\mathbb{W}_{\mathcal{S}} \neq \varnothing$ by **L2.7**. Thus it follows that both $\not\Vdash [A \vee (B \wedge C)] \equiv [(A \vee B) \wedge (A \vee C)]$ and $\not\Vdash [A \wedge (B \vee C)] \equiv [(A \wedge B) \vee (A \wedge C)]$ for $A, B, C \in \mathtt{wfs}(\mathbb{L})$. □

L6.3 $A \leqslant B \not\Vdash A \wedge C \leqslant B \wedge C$.

Proof. Let $\mathcal{M}_r = \langle S_r, P_r, \bigcup, |\cdot|_r \rangle$ where $S_r = S_u$, and $P_r = P_u$ from **L6.2**. Assuming the same conventions given in **L6.2**, we may now set $|p_1|_r = \langle \{\overline{a}\}, \{\overline{b}\} \rangle$, $|p_2|_r = \langle \{\overline{a}, \overline{c}, \overline{a}.\overline{c}\}, \{\overline{b}.\overline{d}\} \rangle$, and $|P_3|_r = \langle \{\overline{e}\}, \{\overline{f}\} \rangle$, observing that $\mathcal{M}_r$ is a model of $\mathcal{L}$. Consider:

$$\begin{aligned} |p_1 \wedge p_3|_r &= \langle \{\overline{a}.\overline{e}\}, \{(\overline{a}), \overline{f}, \overline{a}.\overline{f}\} \rangle \\ |p_2 \wedge p_3|_r &= \langle \{\overline{a}.\overline{e}, \overline{c}.\overline{e}, \overline{a}.\overline{c}.\overline{e}\}, \{\underline{\overline{b}.\overline{d}, (\overline{f}), \overline{b}.\overline{d}.\overline{f}}\} \rangle. \end{aligned}$$

Since the fusion of the exact falsifiers enclosed in parentheses do not belong to the underlined set of exact falsifiers, every $s \in \mathcal{S}_r$ is such that $\mathcal{M}_v, s \dashv\Vdash (p_1 \wedge p_3) \leqslant (p_2 \wedge p_3)$, despite the fact that $\mathcal{M}, s \Vdash p_1 \leqslant p_2$ for all $s \in \mathcal{S}_r$. Thus for any $w \in \mathbb{W}_{\mathcal{S}_r}$, it follows that $\mathcal{M}_v, w \dashv\Vdash (p_1 \wedge p_3) \leqslant (p_2 \wedge p_3)$ as well as $\mathcal{M}, w \vDash p_1 \leqslant p_2$, and so $p_1 \leqslant p_2 \not\Vdash (p_1 \wedge p_3) \leqslant (p_2 \wedge p_3)$ by **L4.9**. Since $\mathbb{W}_{\mathcal{S}} \neq \varnothing$ by **L2.7**, $A \leqslant B \not\Vdash (A \wedge C) \leqslant (B \wedge C)$ for some $A, B, C \in \mathtt{wfs}(\mathbb{L})$. □

L6.4 $A \preceq B, B \preceq C \not\Vdash A \preceq C$.

Proof. Let $\mathcal{M}_u$ be as in **L6.2**, adopting the same abbreviation conventions. We may then consider the following:

$$\begin{aligned} |p_1|_u &= \langle \{(\overline{a})\}, \{\overline{b}\} \rangle \\ |p_1 \wedge p_2|_u &= \langle \{\overline{a}.\overline{c}\}, \{\overline{b}, \overline{d}, \overline{b}.\overline{d}\} \rangle \\ |p_1 \wedge p_2 \wedge p_3|_u &= \langle \{\overline{a}.\overline{c}.\overline{e}\}, \{\overline{b}, \overline{d}, \overline{f}, \overline{b}.\overline{d}, \overline{b}.\overline{f}, \overline{d}.\overline{f}, \overline{b}.\overline{d}.\overline{f}\} \rangle \\ |(p_1 \wedge p_2 \wedge p_3) \vee p_3|_u &= \langle \{\underline{\overline{a}.\overline{c}.\overline{e}, (\overline{e})}\}, \{\overline{f}, \overline{b}.\overline{f}, \overline{d}.\overline{f}, \overline{b}.\overline{d}.\overline{f}\} \rangle. \end{aligned}$$

Since the fusion of exact verifiers enclosed in parentheses do not belong to the underlined set of exact verifiers, every $s \in \mathcal{S}_u$ is such that $\mathcal{M}_v, s \dashv\Vdash p_1 \preceq (p_1 \wedge p_2 \wedge p_3) \vee p_3$ despite the fact that both $\mathcal{M}_v, s \Vdash p_1 \preceq p_1 \wedge p_2$ and $\mathcal{M}_v, s \Vdash p_1 \wedge p_2 \preceq (p_1 \wedge p_2 \wedge p_3) \vee p_3$ for every $s \in \mathcal{S}_u$. Thus $\mathcal{M}_v, w \dashv\Vdash p_1 \preceq (p_1 \wedge p_2 \wedge p_3) \vee p_3$, and so $\mathcal{M}_v, w \not\Vdash p_1 \preceq (p_1 \wedge p_2 \wedge p_3) \vee p_3$ by **L4.9**, while at the same time both $\mathcal{M}_v, w \vDash p_1 \preceq p_1 \wedge p_2$ and $\mathcal{M}_v, w \vDash p_1 \wedge p_2 \preceq (p_1 \wedge p_2 \wedge p_3) \vee p_3$ for all $w \in \mathbb{W}_{\mathcal{S}_u}$. Since $\mathbb{W}_{\mathcal{S}} \neq \varnothing$ by **L2.7**, it follows that there are some $A, B, C \in \mathtt{wfs}(\mathbb{L})$ where $A \preceq B, B \preceq C \not\Vdash A \preceq C$. □

4.7 A Modal Logic of Essence and Ground

Given the semantics above, we may now turn to provide a proof system by which to derive a range of theorems and rules of inference which may be articulated in $\mathcal{L}$, studying the interaction between the both the primitive and defined operators in the language. In particular, I will refer to the following axiom system as *The Modalised Logic of Essence and Ground* (EGM), where $A, B, C \in \mathtt{wfs}(\mathbb{L})$ and $\ulcorner\Box\Gamma\urcorner$ abbreviates $\ulcorner\{\Box A : A \in \Gamma\}\urcorner$ for any $\Gamma \subseteq \mathtt{wfs}(\mathbb{L})$:

A1 $A \approx \neg\neg A$.

A2 $A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$.

A3 $\neg(A \wedge B) \approx \neg A \vee \neg B$.

A4 $\neg(A \vee B) \approx \neg A \wedge \neg B$.

A5 $A \leqslant A \vee B$.

A6 $B \leqslant A \vee B$.

A7 $A \wedge B \leqslant B \wedge A$.

A8 $A \wedge B \leqslant A \vee B$.

A9 $A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C)$.

A10 $A \vee (B \wedge C) \leqslant (A \vee B) \wedge (A \vee C)$.

A11 $\Box A \to A$.

A12 $\Diamond A \to \Box\Diamond A$.

R1 $A \to B,\ A \vdash B$.

R2 $A \leqslant B,\ B \leqslant C \vdash A \leqslant C$.

R3 $A \leqslant C,\ B \leqslant C \vdash A \vee B \leqslant C$.

R4 $A \leqslant B,\ A \leqslant C \vdash A \leqslant B \wedge C$.

R5 $A \leqslant B \vdash A \vee B \sqsubseteq B$.

R6 $A \leqslant B \vdash B \sqsubseteq A \vee B$.

R7 $A \leqslant B \vdash \Box(A \to B)$.

R8 $A \leqslant B \vdash \Box(A \leqslant B)$.

M1 If $\Gamma,\ A \vdash B$, then $\Gamma \vdash A \to B$.

M2 If $\Gamma,\ \neg A \vdash B$ and $\Gamma,\ \neg A \vdash \neg B$, then $\Gamma \vdash A$.

M3 If $\Gamma \vdash A$ and $\Gamma \vdash B$, then $\Gamma \vdash A \wedge B$.

M4 If $\Gamma \vdash A$, then $\Box\Gamma \vdash \Box A$.

I will take $\vdash_{\text{MEG}}$ to be the smallest relation which satisfies the axioms and rules given above and is closed under the standard structural rules, dropping '$_{\text{MEG}}$' from '$\vdash_{\text{MEG}}$' when ambiguity does no threaten. By drawing of the results given in §4.5, we may now show that EGM is sound over the class of models of $\mathcal{L}$.

P3 If $\Gamma \vdash A$, then $\Gamma \models A$.

> *Proof.* The proof proceeds by induction where the base case is established by **L5.5**–**L5.23**, and the induction step is given by **L5.1**–**L5.4**. It follows that EGM is sound over the class of all models of $\mathcal{L}$. □

Given that EGM is sound over the semantics, we may draw the following conclusions from the countermodels **L6.1**–**L6.4** in §4.6 in order to show:

≡-Abs1: $\not\vdash A \equiv A \wedge (A \vee B)$.

≡-Abs2: $\not\vdash A \equiv A \vee (A \wedge B)$.

≡-Dist1: $\not\vdash A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

≡-Dist2: $\not\vdash A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

∧-Mon: $A \leqslant B \not\vdash A \wedge C \leqslant B \wedge C$.

≤-Trans: $A \preceq B, B \preceq C \not\vdash A \preceq C$.

The first four results shed light on the manner in which EGM diverges from Boolean logics, requiring a much more fine-grained theory of propositions than an intensional semantics can provide. Additionally, we may show that $A \sqsubseteq B \not\vdash A \vee C \sqsubseteq B \vee C$, for otherwise **∧-Mon** would be derivable. However, many of the Boolean identities remain theorems of EGM. As brought out below, EGM includes analogues of the absorption and distribution laws where propositional identity has been replaced with reduction.

In order to provide a better sense of the deductive power of EGM, I will conclude by asserting without proof a range of admissible rules of inferences and theorems of EGM, taking $\ulcorner C_{[A/B]} \urcorner$ to be the result of uniformly replacing all occurrences of $\ulcorner B \urcorner$ in $\ulcorner C \urcorner$ with $\ulcorner A \urcorner$, and letting $\ulcorner C_{(A/B)} \urcorner$ be the result of replacing one or more occurrences of $\ulcorner B \urcorner$ in $\ulcorner C \urcorner$ with $\ulcorner A \urcorner$.

Admissible Rules

AR1 $A,\ A \leqslant B\ \vdash B$.

AR2 $\neg B,\ A \leqslant B\ \vdash \neg A$.

AR3 $B,\ A \sqsubseteq B\ \vdash A$.

AR4 $\neg A,\ A \sqsubseteq B\ \vdash \neg B$.

AR5 $A \sqsubseteq B,\ A \sqsubseteq C\ \vdash A \sqsubseteq B \vee C$.

AR6 $A \sqsubseteq B,\ C \sqsubseteq D\ \vdash A \wedge C \sqsubseteq B \wedge D$.

AR7 $A \sqsubseteq C,\ B \sqsubseteq C\ \vdash A \wedge B \sqsubseteq C$.

AR8 $A \leqslant B,\ C \leqslant D\ \vdash A \vee C \leqslant B \vee D$.

AR9 $A \sqsubseteq B\ \vdash\ A \wedge B \leqslant B$.

AR10 $A \sqsubseteq B\ \vdash\ B \leqslant A \wedge B$.

AR11 $A \sqsubseteq B\ \vdash\ \Box(B \to A)$.

AR12 $A \sqsubseteq B\ \vdash\ \Box(A \sqsubseteq B)$.

AR13 $A \leqslant B\ \vdash A \preceq B$.

AR14 $A \sqsubseteq B\ \vdash A \preceq B$.

AR15 $A \sqsubseteq B,\ B \sqsubseteq C\ \vdash A \sqsubseteq C$.

AR16 $A \equiv B,\ C\ \vdash C_{(A/B)}$.

Essence Theorems

E1 $A \sqsubseteq A \wedge B$.

E3 $A \vee B \sqsubseteq A \wedge B$.

E2 $B \sqsubseteq A \wedge B$.

Identity Theorems

ID1 $A \wedge A \equiv A$.

ID2 $A \vee A \equiv A$.

ID3 $A \wedge B \equiv B \wedge A$.

ID4 $A \vee B \equiv B \vee A$.

ID5 $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$.

ID6 $A \vee (B \vee C) \equiv (A \vee B) \vee C$.

ID7 $\neg(A \wedge B) \equiv \neg A \vee \neg B$.

ID8 $\neg(A \vee B) \equiv \neg A \wedge \neg B$.

ID9 $(A \sqsubseteq B) \to (A \wedge B \equiv B)$.

ID10 $(A \leqslant B) \to (A \vee B \equiv B)$.

ID11 $A \equiv \neg\neg A$.

IR1 $A \equiv B \vdash B \equiv A$.

IR2 $A \equiv B \vdash (A \wedge C) \equiv (B \wedge C)$.

IR3 $A \equiv B \vdash (A \vee C) \equiv (B \vee C)$.

IR4 $A \equiv B,\ B \equiv C \vdash A \equiv C$.

IR5 $A \equiv B \vdash \neg A \equiv \neg B$.

Reduction Theorems

RT1 $A \Rightarrow A \wedge (A \vee B)$.

RT2 $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$.

RT3 $A \Rightarrow A \vee (A \wedge B)$.

RT4 $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.

RT5 $A \Rightarrow A$.

RT6 $(A \Rightarrow B) \wedge (B \Rightarrow A)] \rightarrow (A \equiv B)]$.

RT7 $\Box(A \leftrightarrow B) \sqsubseteq (A \Rightarrow B)$.

RT8 $(A \preceq B) \sqsubseteq (A \Rightarrow B)$.

RT9 $(A \Mapsto B) \rightarrow (B \not\Mapsto A)$.

RT10 $(A \Mapsto B) \rightarrow [(B \Mapsto C) \rightarrow (A \Mapsto C)]$.

RT10 $\Box(A \leftrightarrow B) \sqsubseteq (A \Mapsto B)$.

RT12 $(A \preceq B) \sqsubseteq (A \Mapsto B)$.

That the occurrences of '$\sqsubseteq$' and '$\leqslant$' in the theorems of EGM may naturally be read as 'necessary for' and 'sufficient for' provides abductive support for adopting the definition given in *Essence* whereby $\ulcorner A \sqsubseteq B \urcorner$ has been taken to abbreviate $\ulcorner \neg A \leqslant \neg B \urcorner$. We also find abductive support for *Identity* by considering the symmetries among the identity theorems given above which—with the exception of **ID9** and **ID10** on account of including essence and grounding operators— all belong to LPI$^-$ defended in Chapter 3.

Whereas the semantics for $\mathcal{L}$ was devised in attempt to validate the core principles defended in §5.1, EGM aims to provide a useful description of the logical consequence relation defined by way of the semantics for $\mathcal{L}$. Although the valid principles that can be articulated in $\mathcal{L}$ go some ways towards characterising the abstract structure of the space of $\mathcal{S}$-propositions for any state space $\mathcal{S}$, one may nevertheless ask if anything has been left out. In particular, we may observe that $\mathbb{P}_{\mathcal{S}}$ is *bounded* by the following four *extremal propositions*:

$\lesssim$*-Top:* $\langle S, \varnothing \rangle$.

$\sqsubseteq$*-Top:* $\langle \varnothing, S \rangle$.

$\lesssim$*-Bot:* $\langle \varnothing, \{\square\} \rangle$.

$\sqsubseteq$*-Bot:* $\langle \{\square\}, \varnothing \rangle$.

Whereas Boolean lattices are ordered by a single entailment relation, the bilattice $\mathcal{B}_{\mathcal{S}}$ includes two orders, neither of which is the converse of the other. Nevertheless, these orders are related by the principles given in **NN**, **GE**, and **EG** in §4.6. With respect to $\lesssim$, the space of propositions $\mathbb{P}_{\mathcal{S}}$ is *bounded above* by $\lesssim$*-Top* since $X \lesssim \langle S, \varnothing \rangle$ for all $X \in \mathbb{P}_{\mathcal{S}}$, and *bounded below* by $\lesssim$*-Bot* since $\langle \varnothing, \{\square\} \rangle \lesssim X$ for all $X \in \mathbb{P}_{\mathcal{S}}$. Similarly, $\mathbb{P}_{\mathcal{S}}$ is bounded above and below with respect to $\sqsubseteq$ by $\sqsubseteq$*-Top* and $\sqsubseteq$*-Bot*, respectively. However, given only the expressive resources included in $\mathcal{L}$, these facts cannot be articulated, leaving an important feature of $\mathcal{B}_{\mathcal{S}}$ undescribed.

By letting $\mathbb{L}^+ = \mathbb{L} \cup \{\bot, \top\}$, we may take $\mathcal{L}^+ = \langle \mathtt{wfs}(\mathbb{L}^+), \neg, \wedge, \vee, \Box, \leqslant \rangle$ to be the extension of $\mathcal{L}$ which includes the extremal terms '$\bot$' and '$\top$', where an $\mathcal{S}$*-model* $\mathcal{M}_v$ *of* $\mathcal{L}^+$ is any ordered quadruple $\mathcal{M}_v = \langle S, P, \sqsubseteq, |\cdot|_v \rangle$ where $\mathcal{S} = \langle S, P, \sqsubseteq \rangle$ is a world space, $|p|_v \in \mathbb{P}_{\mathcal{S}}$ for all $p \in \mathbb{L}$, and both $|\bot|_v = \langle \varnothing, \{\square\} \rangle$ and $|\top|_v = \langle S, \varnothing \rangle$. Letting '$\perp$' abbreviate '$\neg\bot$', where '$\top$' abbreviates '$\neg\top$', we may articulate the following extremal principles:

A13 $A \leqslant \top$. **E11** $A \sqsubseteq \overline{\top}$.

A14 $\bot \leqslant A$. **E12** $\underline{\bot} \sqsubseteq A$.

By letting EGM$^+$ be the result of extending EGM to include **A13** and **A14** among its axioms, **E11** and **E12** follow immediately from *Essence* and the definitions above. Additionally, we may derive the following identities:

ID12 $A \vee \top \equiv \top$. **ID13** $A \wedge \overline{\top} \equiv \overline{\top}$.

ID14 $A \vee \bot \equiv A$. **ID15** $A \wedge \underline{\bot} \equiv A$.

Given the above, we may think of $\top$ and $\overline{\top}$ as the zero elements for disjunction and conjunction respectively, where $\bot$ and $\underline{\bot}$ may be thought of as the unit elements for disjunction and conjunction respectively.

The axioms and theorems for the extremal elements bring out symmetries in the manner in which the set of propositions for a given state space $\mathcal{S}$ is bounded above and below with respect to both essence and ground. By imposing a non-vacuity constraint on propositions, Fine (2017b, p. 649) excludes the semantic values that I have assigned to '$\bot$' and '$\underline{\bot}$' from the space of propositions. Accordingly, Fine's space of propositions fails to be bounded below with respect to either essence or ground despite being bounded above with respect to both orders, distorting the otherwise natural symmetries brought out by the extremal theorems above. By contrast, I have not imposed Fine's non-vacuity constraint on propositions, and so may maintain that the space of propositions is bounded above and below with respect to both essence and ground.

By extending $\mathcal{L}$ to include extremal terms, we were able to articulate the boundedness of the space of propositions. Rather than relying on pre-theoretic intuitions, I take such claims about upper and lower bounds to find support in wholly systematic considerations about the abstract structure of the space of propositions. In contemplating the critical role which the semantics plays in support of adopting the extremal axioms and resulting theorems, it is natural to ask if there are any other important features encoded into the semantics which EGM$^+$ has not yet managed to capture.

Whereas including the extremal terms in $\mathcal{L}^+$ permitted the expression of the boundedness of the space of propositions in EGM$^+$, we may observe that $\mathcal{L}^+$ also includes the resources to articulate features of the semantics which are not described by EGM$^+$. In particular, it follows from the semantics that $|\Box A| = |B \leqslant C| = |\top|$ for any $\mathcal{S}$-model $\mathcal{M}$ and $A, B, C \in \mathtt{wfs}(\mathbb{L}^+)$ where $\mathcal{M}, w \vDash \Box A$ and $\mathcal{M}, w' \vDash B \leqslant C$ for some $w, w' \in \mathbb{W}_{\mathcal{S}}$. Thus we may conclude that $\Box A \vDash \Box A \equiv \top$ and $B \leqslant C \vDash (B \leqslant C) \equiv \top$ despite the fact that neither $\Box A \vdash \Box A \equiv \top$ nor $B \leqslant C \vdash (B \leqslant C) \equiv \top$ are derivable in EGM$^+$.

Letting a *hyper-sentence* be any $A \in \mathtt{wfs}(\mathbb{L})$ in which either intensional or hyperintensional operators occur within the scope of a hyperintensional operator, one could attempt to complete the logic by including valid axioms strong enough to derive all valid hyper-sentences. However, given that no theoretical target has been provided which may guide the ambition to construct

a semantics by which to interpret the set of hyper-sentences, there is no saying whether the semantics has strayed off course in validating the principles mentioned above. Thus I have opted to leave the logic incomplete.

Supposing that one were to find reason to reject $\Box A \vDash \Box A \equiv \top$ and $B \leqslant C \vDash (B \leqslant C) \equiv \top$, thereby motivating an attempt to revise the semantics for $\mathcal{L}^+$ given above, it is worth considering the significant pitfalls which stand in the way. In particular, it is not clear that any individual state can exactly verify a modal sentence or grounding sentence in the same way that the state of my sitting may be said to exactly verify 'I am sitting'. Whereas the truth or falsity of an extensional sentence within the semantics can be determined by a single state, both grounding and modal sentences concern global features of the entire state space provided by the model in question. For instance, the truth of $\ulcorner A \leqslant B \urcorner$ requires, among other things, that every exact verifier for A be an exact verifier for B. However, even when every exact verifier for A is an exact verifier for B, no single state within the space determines this fact about the state space on its own. Similarly, the truth of $\ulcorner \Box A \urcorner$ requires that every possible state be compatible with an exact verifier for A, but even if this is so, no single state within the space can claim responsibility.

Given that modal and grounding operators quantify over all states within the space, one might attempt to take the exact verifiers for grounding or modal sentences to be the fusion of all states within the space, or all possible states, or else all states which exactly verify or falsify the operands. Despite encompassing all states which play a role in the truth of the grounding or modal sentence in question, such fusions will typically be impossible. For instance, let s_n be the state of the ball only reflecting light with a wavelength of n nanometres, where $\lceil \{s_n : 625 \leqslant n \leqslant 630\} \rceil$ is the set of exact verifiers for 'The ball is crimson'. Given that s_n and s_m are incompatible for $n \neq m$, any fusion of more than one of the exact verifiers for 'The ball is crimson' will be impossible. Accordingly, one cannot expect to construct an exact verifier for grounding claims such as 'The ball being crimson grounds the ball being red' by fusing together the various exact verifiers for the antecedent without crossing over into impossibility, not to mention the exact verifiers for the consequent. Similarly, the fusion of all possible states, or else all possible exact verifiers for the operand of a modal sentence, will also typically be impossible.

Although there is room for impossible states within the state semantic framework, no world will contain an impossible state as a part. If grounding and modal sentences were only exactly verified by impossible states, then instead of being necessary, all grounding and modal sentences would be impossible. Rather than attempting to find a state within a given state space to exactly verify modal and grounding sentences, one might attempt to extend a given state space $\mathcal{S}$ by introducing a range of additional *grounding states* of the form $\langle X, Y \rangle$ where $X, Y \in \mathbb{P}_{\mathcal{S}}$ and $X \lesssim Y$. Accordingly, $\langle X, Y \rangle$ may be said to exactly verify $\ulcorner A \leqslant B \urcorner$ in an $\mathcal{S}$-model $\mathcal{M}$ just in case $|A| = X$, $|B| = Y$, and $X \lesssim Y$. Even so, it remains to say what it is for a grounding state to be a part of another, not to mention what fusions of grounding and non-grounding states

might look like. Moreover, the construction would have to take an iterative form in order to interpret sentences such as $\ulcorner A \leqslant (B \leqslant (C \leqslant D))\urcorner$ or $\ulcorner \Box\Box\Box A\urcorner$, where $\mathcal{S}$ was only the first stage in an infinite hierarchy of new state space constructions. Although a similar strategy could be employed to also introduce *necessary states*, the result would take on an astronomical level of complexity, with no guarantee of shedding light on the language.

Rather than attempting to capture the undescribed validities brought out above in an extension of EGM^+, or else modify the semantics so as to avoid validating such sentences, one could prevent the articulation of hyper-sentences in the first place by restricting the well-formed sentences so that at most the extensional operators occur within the scope of a grounding operator. However, in addition to being *ad hoc*, such restrictions complicate both the syntax and semantics for the language. In order to avoid imposing such restriction on the well-formed sentences of $\mathcal{L}^+$, I have permitted the propositions expressed by the intensional and hyperintensional sentences to be maximally coarse-grained. More precisely, we may take the *purely non-extensional sentences* to include any sentence of the form $\ulcorner \Box A\urcorner$ or $\ulcorner A \leqslant B\urcorner$, as well as any sentence of the form $\ulcorner \neg\varphi\urcorner$, $\ulcorner \varphi \wedge \psi\urcorner$, or $\ulcorner \varphi \vee \psi\urcorner$ where $\ulcorner \varphi\urcorner$ and $\ulcorner \psi\urcorner$ are purely non-extensional sentences. Whereas the propositions expressed by extensional sentences admit of hyperintensional distinctions, every purely non-extensional sentences is assigned to only one of two different propositions, namely $\top$ or $\mp$.

Given that the purely non-extensional sentences will be assigned to one of two values, we may refer to the interpretation of such sentences as *Boolean*. The present scenario is comparable to the manner in which modal sentences are interpreted in intensional frameworks, assuming for simplicity that every world is accessible to every other. Although non-modal sentences may assume a wide range of semantic values, modal sentences will be assigned to either the set of all worlds, or else the set of no worlds. However, flattening the range of semantic values which modal sentences may assume does not cause any distortion among the patterns of valid inference described by an S5 modal logic. In a similar spirit, we may take the semantics for $\mathcal{L}^+$ to succeed in shedding a great deal of light on the relationships between the operators included in $\mathcal{L}^+$. In particular, we may observe that all of the principles included in the theoretical target set out for the logic of essence and ground belong to EGM^+. Thus I will take the semantics and logic for $\mathcal{L}^+$ to capture a natural means of interpreting and reasoning with the notions of essence, ground, metaphysical necessity, relevance, propositional identity, and the extensional operators.

Chapter 5

A Complete Logic of Ground

Chapter Abstract

Drawing on Kit Fine's (2012c, 2017b,c) state semantics, this chapter establishes soundness and completeness for *The Specific Logic of Unilateral Ground* (UGS). After framing the project in §5.1, I will restrict consideration in §5.2 to a proof system UGSN in which a specificity operator has been included and the negation operator does not occur within the scope of a grounding operator, providing a semantics for UGSN in §5.3 where propositions are closed under finite fusion. After proving soundness and completeness for UGSN in §5.4 and §5.5, I will extend these results in §5.6 to a class of models where propositions are closed under infinite fusion. In §5.7, I will extend UGSN to UGS which permits negation to occur within the scope of the unilateral grounding operator, showing that UGS is sound and complete over the class of bilateral infinite fusion models. By contrast with the Boolean lattices of extensional and intensional logics, I will show in §5.8 that the space of propositions form a non-interlaced bilattice. I will then conclude in §5.9 by defining bilateral essence and ground in terms of unilateral ground, deriving a range of theorems and admissible rules by which to reason with the bilateral notions of essence and ground.

5.1 Introduction

In a number of publications, Kit Fine (2012c, 2016, 2017a,b,c) has developed a hyperintensional theory of propositions in which propositions are *exactly verified* and *exactly falsified* by *states*.[1] Whereas worlds are understood to be maximal on account of determining the truth-value of every proposition whatsoever, states only determine the truth values of propositions to which they

[1] Fine has developed applications of the state semantic framework to a wide variety of topics, reaching well beyond the semantics of ground. See Fine (2012a, 2014, 2013, 2018a,b, 2020).

are wholly relevant, and so is able to draw distinctions which possible worlds cannot. In addition to abandoning the maximality required of worlds, states need not be possible. Although a primitive distinction between possible and impossible states plays a critical role in developing a state semantics for modal logic, no such distinction will be needed for the present pursuit. Accordingly, both the semantic machinery used to study validity in the logic of ground, as well as the intended model used to guide our interpretation of the semantics may be said to be free of any dependence on primitive modality.

Given the requirement that states be wholly relevant to the propositions which they exactly verify or falsify, the exact verifiers for A and $\neg A$ will not typically partition the set of states. Rather, many states may exactly verify neither A nor $\neg A$, where similar may be said of the exact falsifier states. Moreover, the exact verifiers for A will in no way determine the exact verifiers for $\neg A$, and so without recourse to exact falsification in addition to exact verification, there is no systematic way to relate the propositions expressed by a sentence and its negations. Accordingly, I will follow Fine in taking the exact verifiers for $\neg A$ to be the exact falsifiers for A, and the exact falsifiers for $\neg A$ to be the exact verifiers for A. In this way the *bilateral proposition* consisting of both exact verifiers and falsifiers assigned to a sentence will determine the bilateral proposition assigned to its negation.[2]

It also follows from the requirement that states be wholly relevant to the propositions which they exactly verify or falsify that conjunction cannot be interpreted by set intersection as in extensional and intensional semantics. For instance, the exact verifiers for A may be disjoint from the exact verifiers for B, and yet $A \wedge B$ may have exact verifiers. Whereas possible worlds are related by an accessibility relation, states are partially ordered by a *parthood* relation, where *fusion* is defined as least upper bound with respect to parthood. Given

[2] See Fine (2017d,b) for further discussion.

a notion of state fusion, I will follow Fine in taking $A \wedge B$ to be exactly verified by any fusion of an exact verifier for A and an exact verifier for B. Similarly, given an exact falsifier for A, and an exact falsifier for B, their fusion will be an exact falsifier for $A \vee B$. Additionally, I will assume what Fine calls an *inclusive semantics* for disjunction and conjunction where any exact verifier for A, B, or $A \wedge B$ is an exact verifier for $A \vee B$, and any exact falsifier for A, B, or $A \vee B$ is an exact falsifier for $A \wedge B$. As brought out below, the sets of exact verifiers and falsifiers for propositions will be closed under fusion. Building on Fine's work, it remains to develop a logic for the most natural entailment relations holding between propositions in this hyperintensional setting.

We may begin by observing that in any theory of propositions where propositional identity satisfies the the Boolean identities, disjunctive part may be shown to be the converse of conjunctive part, where entailment may be defined in terms of either. More specifically, letting $A \leqslant B \coloneqq A \vee B \equiv B$ and $A \sqsubseteq B \coloneqq A \wedge B \equiv B$ where '$\equiv$' is taken to express propositional identity, we may show that if the theory of propositions in question is Boolean, then $(A \leqslant B) \leftrightarrow (B \sqsubseteq A)$.[3] For instance, assuming an extensional theory of propositions, '$\leqslant$' expresses the material conditional and '$\sqsubseteq$' expresses its converse. Similarly, in an intensional theory of propositions, '$\leqslant$' expresses the strict conditional whereas '$\sqsubseteq$' expresses its converse. On both extensional and intensional theories of propositions, there is just one entailment relation since nothing new is added by also including converse relations.

Whereas in extensional and intensional logics, conjunctive and disjunctive part are converse relations, these definitions correspond to distinct entailment relations in the present setting. As brought out in §5.8, the theory of bilateral propositions elaborated below does not satisfy the Boolean absorption laws, where as a result $(A \leqslant B) \leftrightarrow (B \sqsubseteq A)$ does not hold in general. Accordingly,

[3] *Proof:* If $A \leqslant B$, it follows that $A \vee B \equiv B$, and so by substitution $A \wedge (A \vee B) \equiv A \wedge B$. By absorption, $A \equiv A \wedge B$, and so $B \sqsubseteq A$. The reverse derivation is similar. □

the state semantic framework admits of two natural entailment relations $\sqsubseteq$ and $\leqslant$ which I will call by the names *essence* and *ground*, respectively. Informally, I will take '$\sqsubseteq$' and '$\leqslant$' to regiment 'necessary for' and 'sufficient for' respectively, where essence and ground may be shown to track relevance in addition to modal profile. Additionally, I will take both essence and ground to be "worldly" insofar as they are to relate propositions understood as *things being certain ways* rather than representations of things being some way or other. Moreover, I will assume essence and ground are reflexive and non-factive so that every proposition grounds itself independent of whether that proposition obtains or not. Both the irreflexive and factive analogues may then be defined in terms of the reflexive non-factive notions of essence and ground.[4]

Although essence and ground are not converses, essence and ground are nevertheless interdefinable in a language with a negation operator which satisfies the involution law $\neg\neg A \equiv A$. As shown in §5.8, both $(A \leqslant B) \leftrightarrow (\neg A \sqsubseteq \neg B)$ and $(A \sqsubseteq B) \leftrightarrow (\neg A \leqslant \neg B)$ are valid, where the space of bilateral propositions may be shown to form a bilattice ordered by essence and ground. Given that essence and ground are interdefinable, I will take $\ulcorner A \sqsubseteq B \urcorner$ to abbreviate $\ulcorner \neg A \leqslant \neg B \urcorner$ purely as a matter of convention, defining propositional identity in terms of essence and ground rather than *vice versa* in §5.9.[5] In order to articulate such definitions, negation must be permitted to occur within the scope of a grounding operator. However, we may observe that sentences of this kind are excluded from Fine's (2012c) *Pure Logic of Ground* (PLG) which aims to study the atomic sentences which can be articulated by a range of primitive grounding operators in the absence of any other operators.[6]

Instead of admitting distinct primitives for each notion of ground that Fine includes in PLG while excluding all other operators from the logic, I will begin

[4] See Chapter 2 as well as Fine (2001, 2012b, 2015b) for related discussion.
[5] See §5 in Chapter 2 and §1 in Chapter 4 for related discussion of these definitions.
[6] See Chapter 1 for further discussion.

with a single primitive grounding operator '$\trianglelefteq$' which I will refer to as *unilateral ground*. In contrast to $\leqslant$ which imposes constraints on both the exact verifiers and falsifiers for the propositions involved, $A \trianglelefteq B$ only requires that every exact verifier for A is also an exact verifier for B. We may observe that the semantics for unilateral ground is similar to the semantics for the material and strict conditionals, only that bilateral propositions have been substituted for extensional and intensional propositions. Nevertheless, we may define $\leqslant$ and $\sqsubseteq$ in terms of $\trianglelefteq$ given a language which includes the extensional operators:

Unilateral Equivalence: Let $\ulcorner A \approx B \urcorner$ abbreviate $\ulcorner (A \trianglelefteq B) \wedge (B \trianglelefteq A) \urcorner$.

Unilateral Essence: Let $\ulcorner A \trianglerighteq B \urcorner$ abbreviate $\ulcorner A \wedge B \approx B \urcorner$.

Ground: Let $\ulcorner A \leqslant B \urcorner$ abbreviate $\ulcorner (A \trianglelefteq B) \wedge (\neg A \trianglerighteq \neg B) \urcorner$.

Essence: Let $\ulcorner A \sqsubseteq B \urcorner$ abbreviate $\ulcorner (A \trianglerighteq B) \wedge (\neg A \trianglelefteq \neg B) \urcorner$.

Restricting attention to a language which includes '$\trianglelefteq$' rather than '$\leqslant$' among its primitive operators has the effect of greatly simplifying the present attempt to provide a complete logic for ground.[7] In order to define ground in terms of unilateral ground, the operator for negation must be permitted to occur within the scope of a grounding operator, where conjunction is permitted to occur between atomic grounding sentences. Accordingly, I will lift the restrictions that Fine imposes in developing PLG, permitting any extensional combination of atomic unilateral grounding sentences of the form $\ulcorner A \trianglelefteq B \urcorner$ where at most the extensional operators may occur within $\ulcorner A \urcorner$ and $\ulcorner B \urcorner$.

Despite admitting of a much wider range of well-formed sentences than is included in PLG, I will nevertheless exclude consideration of sentences in which grounding operators occur within the scope of a grounding operator, as well as purely extensional sentences and extensional combinations on purely extensional and non-extensional sentences such as $\ulcorner A \wedge (A \leqslant B) \urcorner$. Although

[7] See Chapter 1 and Fine (2012b) for discussion of a range of further grounding operators which may be defined in terms of ground in a language which includes extensional operators.

one might hope to provide a logic which admits of a wider range of sentences, such ambitions reach beyond the scope of the present pursuit.[8]

By contrast with the attempt to establish completeness for a logic with a primitive grounding operator, Fine and Jago (2019) provide a complete logic for their *exact entailment* relation $\Vdash$ which holds between a set of sentences in a purely extensional language and a further sentence of that language.[9] Whereas grounding operates on propositions, exact entailment is a logical consequence relation which quantifies over both models and states. Leaving aside the details of their definition of a model, Fine and Jago state that $\Gamma \Vdash \varphi$ just in case for every model of the language, and every state s in that model, if s exactly verifies every $\gamma \in \Gamma$, then s exactly verifies φ. Fine and Jago also require the exact verifiers and falsifiers for a proposition to be *convex* insofar as any state between two exact verifiers (falsifiers) in mereological order for a proposition must also be an exact verifier (falsifier) for that proposition. However, imposing convexity requires making some alteration to the inclusive semantics for conjunction and disjunction.[10] Nevertheless, were Fine and Jago to abandon their convexity constraint, we could show that $A \Vdash B$ just in case $A \trianglelefteq B$ is valid on the definition of validity that I will go on to provide. Despite this connection, Fine and Jago do not include a grounding operator in their language, and cannot articulate conjunctions of atomic grounding sentences such as $\ulcorner (A \trianglelefteq B) \wedge (B \trianglelefteq A) \urcorner$, and so cannot define $\leqslant$, $\sqsubseteq$, nor $\equiv$ in their language. Thus I will not further consider exact entailment in what follows.

In addition to taking '$\trianglelefteq$' to be primitive and '$\leqslant$' to be defined, I will also include a *specificity operator* '\$' in the language presented below. Let a proposition be called *specific* just in case exactly one state verifies that

[8] See §7 of Chapter 4 for discussion of the semantics limitations the present framework faces in attempting to interpret sentences with nested grounding operators.

[9] I have adapted their notation to avoid confusion with what is to come below.

[10] See §6 in Chapter 3 and §3 in Chapter 4 for discussion.

proposition.[11] The specificity operator may be compared to the *singularity predicate* '$\mathtt{S}$' included in plural logics where $\ulcorner\mathtt{S}(aa)\urcorner$ reads $\ulcorner$There is exactly one of the aas$\urcorner$, as well as to the *numerical possibility operator* '$\Box_1$' where $\ulcorner\Box_1 A\urcorner$ reads $\ulcorner$There is exactly one possibility in which $A\urcorner$. As brought out in §5.5, the specificity operator will play a crucial role in the completeness proof, where states in the Henkin model are ground-theoretic equivalence classes generated from specific sentences. Attempts to establish completeness without a specificity operator faced numerous difficulties. Although the motivation for including the specificity operator in the logic takes a purely technical form, specificity operators are nevertheless of interest in their own right.

The completeness proof will be structured as follows. I will begin in §5.2 by introducing *The Specific Logic of Unilateral Ground without Negation* (UGSN) in which negation does not occur within the scope of a unilateral grounding operator, providing a state semantics for UGSN in which the exact verifiers for a proposition are closed under finite fusion in §5.3. I will then prove that UGSN is both sound and complete over the class $\mathcal{C}$ of finite fusion models in §5.4 and §5.5, respectively. These results will then be extended to a class of models $\mathcal{C}^\infty$ in which exact verifiers are also closed under infinite fusion in §5.6, proving in §5.7 that *The Specific Logic of Unilateral Ground* (UGS) in which negation is permitted to occur within the scope of a unilateral grounding operator is sound and complete over the class $\mathcal{C}^\pm$ of bilateral infinite fusion models. Instead of forming a Boolean lattice, I will I show in §5.8 that the space of propositions forms a non-distributive bilattice. I will conclude in §5.9 by deriving a subsystem of UGS which I will call *The Logic of Essence and Ground* (EG) which includes a range of theorems and admissible rules for reasoning with the notions of essence and ground.

[11] Fine (2017c, p. 695) employs the label 'determinate' rather than 'specific'.

5.2 Proof Theory

We may begin by restricting consideration to a simplified propositional language $\mathcal{L}^-$ where the well formed sentences are built up in two separate stages. Given a set of sentence letters $\mathbb{L} = \{p_i : i \in \mathbb{N}\}$ together with a set of *extremal constants* $\mathbb{E} = \{\mathcal{T}, \bot, \mathcal{V}, \bot\!\!\!\bot\}$, we may define the *pre-formed sentences* (pfs) of $\mathcal{L}^-$ as follows, where $p \in \mathbb{L}$ and $e \in \mathbb{E}$ are both arbitrary:

$$A ::= p \mid e \mid A \wedge A \mid A \vee A.$$

I will refer to '$\mathcal{T}$' as the *top* and '$\bot$' as the *bottom*, reading '$\mathcal{V}$' as the *verum* and '$\bot\!\!\!\bot$' as the *falsum*. As brought out in §5.3, $\mathcal{T}$ is the proposition which is exactly verified by any state whatsoever, whereas $\mathcal{V}$ is only exactly verified by the fusion of all states. Additionally, $\bot$ is exactly verified by no states, whereas $\bot\!\!\!\bot$ is only exactly verified by the state which trivially obtains.

Let $\texttt{pfs}(\mathcal{L}^-)$ be the set of pfs of $\mathcal{L}^-$, where $A, B, \ldots \in \texttt{pfs}(\mathcal{L}^-)$, and $\texttt{comp}(A)$ is the number of occurrences of '$\vee$' and '$\wedge$' in A. If $A, B \in \texttt{pfs}(\mathcal{L}^-)$, then both $\ulcorner\$A\urcorner$ and $\ulcorner A \trianglelefteq B\urcorner$ are *well formed atomic sentence* (wfas) of $\mathcal{L}^-$, where $\texttt{atoms}(\mathcal{L}^-)$ is the set of all wfas of $\mathcal{L}^-$. We may then define the *well formed sentences* (wfs) of $\mathcal{L}^-$, letting $\alpha \in \texttt{atoms}(\mathcal{L}^-)$ be arbitrary:

$$\varphi ::= \alpha \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

Let $\texttt{wfs}(\mathcal{L}^-)$ be the set of wfs of $\mathcal{L}^-$, where $\varphi, \psi, \ldots \in \texttt{wfs}(\mathcal{L}^-)$, and let $\texttt{comp}^+(\varphi)$ be the number of occurrences of '$\neg$', '$\vee$', and '$\wedge$' in φ which do not occur in any preformed subsentences (sub-pfs) of φ. For any sentences $A, B \in \texttt{pfs}(\mathcal{L}^-)$, $\ulcorner A \trianglelefteq B\urcorner$ reads 'It being the case that A grounds it being the case that B', or more simply, 'A grounds B', indicating that A obtaining is sufficient for B to obtain. We may also read $\ulcorner\$A\urcorner$ as 'There is exactly one way

for it to be the case that A⌝, or more simply, ⌜A is specific⌝, indicating that there is only one state which exactly verifies A. As we will see, the theorems of the logic belong to $\mathtt{wfs}(\mathcal{L}^-)$ and not to $\mathtt{pfs}(\mathcal{L}^-)$.

As usual, '$\rightarrow$' and '$\leftrightarrow$' may be introduced as metalinguistic abbreviations, where I will also take ⌜$A \not\trianglelefteq B$⌝ to abbreviate ⌜$\neg(A \trianglelefteq B)$⌝, letting ⌜$A \approx B$⌝ abbreviate ⌜$(A \trianglelefteq B) \wedge (B \trianglelefteq A)$⌝ which expresses ground-theoretic equivalence. Given the formation rules for $\mathcal{L}^-$, we may define syntactic consequence $\vdash_{\text{UGSN}}$ for *The Specific Logic of Unilateral Ground without Negation* (UGSN) to be the smallest relation closed under truth-functional consequence which satisfies the following, where $A, B, C, D \in \mathtt{pfs}(\mathbb{L})$ and $\Gamma \cup \{\varphi\} \subseteq \mathtt{wfs}(\mathbb{L})$:

Grounding Axioms and Rules

GA1 $A \trianglelefteq A \vee B$.

GA2 $B \trianglelefteq A \vee B$.

GA3 $A \trianglelefteq A \wedge A$.

GA4 $A \wedge A \trianglelefteq A$.

GA5 $A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge C$.

GA6 $(A \wedge B) \wedge C \trianglelefteq A \wedge (B \wedge C)$.

GA7 $A \wedge B \trianglelefteq B \wedge A$.

GA8 $A \trianglelefteq B, C \trianglelefteq D \vdash A \wedge C \trianglelefteq B \wedge D$.

GA9 $A \trianglelefteq B, B \trianglelefteq C \vdash A \trianglelefteq C$.

Extremal Axioms

VF1 $\$\bot$.

VF2 $\$\mathcal{V}$.

VF3 $\bot \wedge A \trianglelefteq A$.

VF4 $A \trianglelefteq \bot \wedge A$.

VF5 $\mathcal{V} \wedge A \trianglelefteq \mathcal{V}$.

VF6 $\mathcal{V} \trianglelefteq \mathcal{V} \wedge A$.

VF7 $A \trianglelefteq \mathcal{T}$.

Specificity Rules

SP5 $\$A, A \trianglelefteq C \vee D \vdash (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$.

SP6 If $\Gamma \vdash \$p \rightarrow [(p \trianglelefteq A) \rightarrow (p \trianglelefteq B)]$ where $p \in \mathbb{L}$ does not occur in A, B, or in any $\gamma \in \Gamma$, then $\Gamma \vdash A \trianglelefteq B$.

SP7 If $\Gamma, [(A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)] \vdash \varphi$ where distinct $p, q \in \mathbb{L}$ do not occur in φ or in any $\gamma \in \Gamma$, then $\Gamma \vdash \varphi$.

A wfs φ of $\mathcal{L}^-$ is a *theorem* of UGSN just in case $\vdash_{\text{UGSN}} \varphi$. Letting $\ulcorner\varphi_{[A/B]}\urcorner$ be the result of uniformly replacing all occurrences of $\ulcorner B\urcorner$ in $\ulcorner\varphi\urcorner$ with $\ulcorner A\urcorner$, and taking $\ulcorner\varphi_{(A/B)}\urcorner$ to be the result of replacing zero or more occurrences of $\ulcorner B\urcorner$ in $\ulcorner\varphi\urcorner$ with $\ulcorner A\urcorner$, we may derive the following admissible rules and theorems:

Admissible Rules

AR1 $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \vee B \trianglelefteq C$.

AR2 If $\Gamma \vdash_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGSN}} \varphi_{[A/p]}$.

AR3 $A \approx B \vdash_{\text{UGSN}} C \approx C_{(A/B)}$.

AR4 $A \approx B, \varphi \vdash_{\text{UGSN}} \varphi_{(A/B)}$.

Grounding Theorems

T1 $A \wedge \bot \trianglelefteq B$.

T2 $A \trianglelefteq B \vee \mathcal{T}$.

T3 $A \trianglelefteq A \wedge (A \vee B)$.

T4 $A \trianglelefteq A \vee (A \wedge B)$.

T5 $A \wedge B \trianglelefteq A \vee B$.

T6 $A \vee (B \wedge C) \trianglelefteq (A \vee B) \wedge (A \vee C)$.

Equivalence Theorems

E1 $A \approx A$.

E2 $A \wedge \bot \approx \bot$.

E3 $A \vee \mathcal{T} \approx \mathcal{T}$.

E4 $A \wedge \mathcal{V} \approx \mathcal{V}$

E5 $A \vee \bot \approx A$.

E6 $A \wedge \perp\!\!\!\perp \approx A$.

E7 $A \vee A \approx A$.

E8 $A \wedge A \approx A$.

E9 $A \vee B \approx B \vee A$.

E10 $A \wedge B \approx B \wedge A$.

E11 $A \vee (B \vee C) \approx (A \vee B) \vee C$.

E12 $A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$.

E13 $A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$.

E14 $A \wedge (B \wedge C) \approx (A \wedge B) \wedge (A \wedge C)$.

It is worth observing that of the equivalences for complete distributive lattices, the following four principles are conspicuously absent:

#Abs1 $A \wedge (A \vee B) \approx A$.

#Dist $A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$.

#Abs2 $A \vee (A \wedge B) \approx A$.

The following section will present countermodels to each of the above, and so the corresponding space of propositions does not form a distributive lattice.

We may also note that although **E13** is a theorems of UGSN, its dual **#Dist** is not. As I will bring out in §5.7, these apparent asymmetries may be shown to be an artefact of our present concern with unilateral propositions. Once negation is permitted to occur within the scope of the grounding operator, we may derive a range of theorems in §5.8 which maintains duality.

I will now provide proofs for selected theorems and admissible rules which are novel to UGSN, assuming standard results from classical logic.

AR1 $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \vee B \trianglelefteq C$.

Proof. Let $A, B, C \in \mathtt{pfs}(\mathcal{L}^-)$ and choose some $p \in \mathbb{L}$ which does not occur in A, B, or C. Thus $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \wedge B \trianglelefteq C$ follows from **GA8** and **GA4** by **GA9**. Additionally, we know by **SP5** that $\$p, p \trianglelefteq A \vee B \vdash_{\text{UGSN}} (p \trianglelefteq A) \vee (p \trianglelefteq B) \vee (p \trianglelefteq A \wedge B)$. Consider:

$$p \trianglelefteq A, A \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq C \tag{1}$$

$$p \trianglelefteq B, B \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq C \tag{2}$$

$$p \trianglelefteq A \wedge B, A \wedge B \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq C \tag{3}$$

It follows that $A \trianglelefteq C, B \trianglelefteq C, \$p, p \trianglelefteq A \vee B \vdash_{\text{UGSN}} p \trianglelefteq C$. Thus we know that $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} \$p \to [(p \trianglelefteq A \vee B) \to (p \trianglelefteq C)]$, and so by **SP6** $A \trianglelefteq C, B \trianglelefteq C \vdash_{\text{UGSN}} A \vee B \trianglelefteq C$ since p does not occur in A, B, or C. □

AR2 If $\Gamma \vdash_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGSN}} \varphi_{[A/p]}$. (*Uniform Substitution*)

Proof. The proof goes by induction on the number of applications of the metarules in UGSN, where the only novel cases are given below:

Case SP6: Assume $\Gamma \vdash^{n}_{\text{UGSN}} \varphi$ follows by **SP6**. Thus $\varphi = A \trianglelefteq B$ where $\Gamma \vdash^{n-1}_{\text{UGSN}} \$q \to [(q \trianglelefteq A) \to (q \trianglelefteq B)]$ for some $q \in \mathbb{L}$ which does not occur in A, B, or any $\gamma \in \Gamma$. Thus we know by hypothesis that $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q_{[A/p]} \to [(q_{[A/p]} \trianglelefteq A_{[A/p]}) \to (q_{[A/p]} \trianglelefteq B_{[A/p]})]$. If $p = q$, then it follows trivially that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$ since q does not occur in A, B, or any $\gamma \in \Gamma$. If instead $p \neq q$, then it follows from the above that

$\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q \to [(q \trianglelefteq A_{[A/p]}) \to (q \trianglelefteq B_{[A/p]})]$, for some $q \in \mathbb{L}$ which does not occur in A, B, or in any $\gamma \in \Gamma$. Again by hypothesis, we know that $\Gamma_{[A/p],[q^\star/q]} \vdash^{n-1}_{\text{UGSN}} \$q_{[q^\star/q]} \to [(q_{[q^\star/q]} \trianglelefteq A_{[A/p]}) \to (q_{[q^\star/q]} \trianglelefteq B_{[A/p]})]$, where we may choose $q^\star \in \mathbb{L}$ to be the sentence letter with the lowest index which does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Equivalently, $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q^\star \to [(q^\star \trianglelefteq A_{[A/p]}) \to (q^\star \trianglelefteq B_{[A/p]})]$ where $q^\star$ does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Thus it follows by **SP6** that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} A_{[A/p]} \trianglelefteq B_{[A/p]}$, and so $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$.

Case SP7: Assume $\Gamma \vdash^{n}_{\text{UGSN}} \varphi$ follows by **SP7**. Thus it follows that $\Gamma, [\$A \wedge (A \trianglelefteq B \wedge C)] \to [\$r \wedge \$q \wedge (r \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (r \wedge q \trianglelefteq A)] \vdash_{\text{UGSN}} \varphi$ where distinct $r, q \in \mathbb{L}$ do not occur in φ. By hypothesis we know that:

$$\Gamma_{[A/p]}, [\$A_{[A/p]} \wedge (A_{[A/p]} \trianglelefteq B_{[A/p]} \wedge C_{[A/p]})] \to$$
$$[\$r_{[A/p]} \wedge \$q_{[A/p]} \wedge (r_{[A/p]} \trianglelefteq B_{[A/p]}) \wedge (q_{[A/p]} \trianglelefteq C_{[A/p]}) \wedge (r_{[A/p]} \wedge q_{[A/p]} \trianglelefteq A_{[A/p]})]$$
$$\vdash^{n-1}_{\text{UGSN}} \varphi_{[A/p]}$$

$\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q_{[A/p]} \to [(q_{[A/p]} \trianglelefteq A_{[A/p]}) \to (q_{[A/p]} \trianglelefteq B_{[A/p]})]$. If $p = q$, then it follows trivially that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$ since q does not occur in A, B, or any $\gamma \in \Gamma$. If instead $p \neq q$, then it follows from the above that $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q \to [(q \trianglelefteq A_{[A/p]}) \to (q \trianglelefteq B_{[A/p]})]$, for some $q \in \mathbb{L}$ which does not occur in A, B, or in any $\gamma \in \Gamma$. Again by hypothesis, we know that $\Gamma_{[A/p],[q^\star/q]} \vdash^{n-1}_{\text{UGSN}} \$q_{[q^\star/q]} \to [(q_{[q^\star/q]} \trianglelefteq A_{[A/p]}) \to (q_{[q^\star/q]} \trianglelefteq B_{[A/p]})]$, where we may choose $q^\star \in \mathbb{L}$ to be the sentence letter with the lowest index which does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Equivalently, $\Gamma_{[A/p]} \vdash^{n-1}_{\text{UGSN}} \$q^\star \to [(q^\star \trianglelefteq A_{[A/p]}) \to (q^\star \trianglelefteq B_{[A/p]})]$ where $q^\star$ does not occur in $A_{[A/p]}, B_{[A/p]}$, or in any $\gamma \in \Gamma_{[A/p]}$. Thus it follows by **SP7** that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} A_{[A/p]} \trianglelefteq B_{[A/p]}$, and so $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$.

Given that $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$ holds in each of the cases above, we may conclude by discharge that if $\Gamma \vdash^{n}_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash^{n}_{\text{UGSN}} \varphi_{[A/p]}$, and so by induction that if $\Gamma \vdash_{\text{UGSN}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGSN}} \varphi_{[A/p]}$. □

T1 $\vdash_{\text{UGSN}} A \wedge \bot \trianglelefteq B$.

Proof. Let $A, B \in \mathtt{pfs}(\mathcal{L}^-)$ and choose some $p \in \mathbb{L}$ where p does not occur in either A or B. Observe that $\$p \vdash_{\text{UGSN}} p \not\trianglelefteq \bot$ by **SP1**, and so by propositional logic we know that $\$p, p \trianglelefteq A \wedge \bot \vdash_{\text{UGSN}} (p \not\trianglelefteq \bot) \wedge (p \trianglelefteq A \wedge \bot)$. Choose distinct $r, q \in \mathbb{L}$ which do not occur in A, B, or p. Consider:

$$[(p \not\trianglelefteq \bot) \wedge (p \trianglelefteq A \wedge \bot)] \to [\$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)]. \quad (*)$$

Thus $\$p, p \trianglelefteq A \wedge \bot, (*) \vdash_{\text{UGSN}} \$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)$, follows by propositional logic. However, we know that $\vdash_{\text{UGSN}} \$r \to (r \not\trianglelefteq \bot)$ by **SP1**, and so $\vdash_{\text{UGSN}} \neg\$r \vee (r \not\trianglelefteq \bot)$ by abbreviation. It follows that $\vdash_{\text{UGSN}} \neg\$r \vee \neg\$q \vee (r \not\trianglelefteq \bot) \vee (q \not\trianglelefteq A) \vee (r \wedge q \not\trianglelefteq p)$ by propositional logic, and so $\vdash_{\text{UGSN}} \neg[\$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)]$. It follows that $\$p, p \trianglelefteq A \wedge \bot, (*) \vdash_{\text{UGSN}} \neg[\$r \wedge \$q \wedge (r \trianglelefteq \bot) \wedge (q \trianglelefteq A) \wedge (r \wedge q \trianglelefteq p)]$. Thus $\$p, p \trianglelefteq A \wedge \bot, (*) \vdash_{\text{UGSN}} p \trianglelefteq B$ by *ex falso quodlibet*, and so we may conclude that $(*) \vdash_{\text{UGSN}} \$p \to [(p \trianglelefteq A \wedge \bot) \to (p \trianglelefteq B)]$.

Given the above, $\vdash_{\text{UGSN}} \$p \to [(p \trianglelefteq A \wedge \bot) \to (p \trianglelefteq B)]$ follows by **SP7**. Since $p \in \mathbb{L}$ where p does not occur in either A or B, we may conclude that $\vdash_{\text{UGSN}} A \wedge \bot \trianglelefteq B$ by **SP6** as desired. □

T6 $\vdash_{\text{UGSN}} A \vee (B \wedge C) \trianglelefteq (A \vee B) \wedge (A \vee C)$.

Proof. Let $A, B, C \in \mathtt{pfs}(\mathcal{L}^-)$, and choose some $p \in \mathbb{L}$ where p does not occur in A, B, or C. Observe that $\$p \vdash_{\text{UGSN}} p \not\trianglelefteq \bot$ by **SP1**, and so by propositional logic $\$p, p \trianglelefteq A \wedge (B \vee C) \vdash_{\text{UGSN}} (p \not\trianglelefteq \bot) \wedge (p \trianglelefteq A \wedge (B \vee C))$. Choose distinct $r, q \in \mathbb{L}$ which do not occur in A, B, C, or p. Consider:

$$[(p \not\trianglelefteq \bot) \wedge (p \trianglelefteq A \wedge (B \vee C))] \to [\$r \wedge \$q \wedge (r \trianglelefteq A) \wedge (q \trianglelefteq B \vee C) \wedge (r \wedge q \trianglelefteq p)]. \quad (*)$$

So $\$p, p \trianglelefteq A \wedge (B \vee C), (*) \vdash_{\text{UGSN}} \$r \wedge \$q \wedge (r \trianglelefteq A) \wedge (q \trianglelefteq B \vee C) \wedge (r \wedge q \trianglelefteq p)$, and so by **SP4** that $\$p, r \wedge q \trianglelefteq p \vdash_{\text{UGSN}} (p \trianglelefteq r \wedge q) \wedge (r \wedge q \trianglelefteq \bot)$, where it follows from **SP3** that $\$r, \$q \vdash_{\text{UGSN}} \$(r \wedge q)$, and from **SP1** that $\$(r \wedge q) \vdash_{\text{UGSN}} r \wedge q \not\trianglelefteq \bot$. Thus $\$p, p \trianglelefteq A \wedge (B \vee C), (*) \vdash_{\text{UGSN}} p \trianglelefteq r \wedge q$

by propositional logic. We may now observe that it follows by **SP5** that $\$q, q \trianglelefteq B \vee C \vdash_{\text{UGSN}} (q \trianglelefteq B) \vee (q \trianglelefteq C) \vee (q \trianglelefteq B \wedge C)$. Consider:

$$p \trianglelefteq r \wedge q, r \trianglelefteq A, q \trianglelefteq B \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq (A \wedge B) \vee (A \wedge C) \tag{1}$$

$$p \trianglelefteq r \wedge q, r \trianglelefteq A, q \trianglelefteq C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq (A \wedge B) \vee (A \wedge C) \tag{2}$$

$$p \trianglelefteq r \wedge q, r \trianglelefteq A, q \trianglelefteq B \wedge C \quad \vdash_{\text{UGSN}} \quad p \trianglelefteq (A \wedge B) \vee (A \wedge C). \tag{3}$$

Here, (1) follows by **GA8**, **GA1**, and **GA9**, where (2) is similar but draws on **GA2** in place of **GA1**. In order to justify (3), it is enough to observe that $\vdash_{\text{UGSN}} A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge (A \wedge C)$ by **E14**, and so together with **T5** we may conclude that $\vdash_{\text{UGSN}} A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \vee (A \wedge C)$. Together with **GA8** and **GA9**, (3) follows. Thus we may conclude that $\$p, p \trianglelefteq A \wedge (B \vee C), (*) \vdash_{\text{UGSN}} p \trianglelefteq (A \wedge B) \vee (A \wedge C)$. Since r and q are distinct and do not occur in A, B, C, or p, we know by **SP7** that $\$p, p \trianglelefteq A \wedge (B \vee C) \vdash_{\text{UGSN}} p \trianglelefteq (A \wedge B) \vee (A \wedge C)$, and so it follows that $\vdash_{\text{UGSN}} \$p \rightarrow [p \trianglelefteq A \wedge (B \vee C) \rightarrow p \trianglelefteq (A \wedge B) \vee (A \wedge C)]$. Thus $\vdash_{\text{UGSN}} A \wedge (B \vee C) \trianglelefteq (A \wedge B) \vee (A \wedge C)$ follows by **SP6**. □

E13 $\vdash_{\text{UGSN}} A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$.

Proof. Given that $\vdash_{\text{UGSN}} A \trianglelefteq A$ by **E1**, and $\vdash_{\text{UGSN}} B \trianglelefteq B \vee C$ by **GA1**, we know that $\vdash_{\text{UGSN}} A \wedge B \trianglelefteq A \wedge (B \vee C)$ follows by **GA8**. Similarly, $\vdash_{\text{UGSN}} A \wedge C \trianglelefteq A \wedge (B \vee C)$ follows by **E1**, **GA2**, and **GA8**. Thus $\vdash_{\text{UGSN}} (A \wedge B) \vee (A \wedge C) \trianglelefteq A \wedge (B \vee C)$ follows by **AR1**, and so together with **T6** we may conclude that $\vdash_{\text{UGSN}} A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$. □

5.3 Semantics

This section will draw on Kit Fine's (2017b,c) recent work in order to provide a semantics for $\mathcal{L}^-$. Rather than working over a space of possible worlds W with an accessibility relation R, I will follow Fine in taking S to be a set states of affairs, or *states* for short.[12] Let a *state space* $\mathcal{S}$ be any ordered pair $\langle S, \star \rangle$

[12] See Fine (2017a, Draft) for discussion.

where S is any set closed under the binary operation $\star$, mapping any two states to their *fusion*. A state space $\langle S, \star \rangle$ is *mereological* just in case it satisfies:

Null State: There is a *null state* $\sqsubset \in S$ such that $\sqsubset \star s = s$ for all $s \in S$.

Full State: There is a *full state* $\blacksquare \in S$ such that $\blacksquare \star s = \blacksquare$ for all $s \in S$.

Idempotency: $s \star s = s$ for all $s \in S$.

Commutativity: $s \star t = t \star s$ for all $s, t \in S$.

Associativity: $(s \star t) \star r = s \star (t \star r)$ for all $s, t, r \in S$.

Assuming a "no class"-theory of classes, we may let $\mathbb{M}$ be the class of all mereological state spaces, employing set notation where convenient. In order to define the relevant classes of models for $\mathcal{L}^-$, consider the following definitions:

Fusion: $\overline{X} = \{x \star y : x, y \in X\}$.

$\mathcal{S}$-Propositions: $\mathbb{P}_{\mathcal{S}} = \{X \subseteq S : X = \overline{X}\}$.

Given any state space $\mathcal{S} \in \mathbb{M}$, a *unilateral $\mathcal{S}$-model of $\mathcal{L}^-$* is an ordered triple $\mathcal{M}_v = \langle S, \star, |\cdot|_v \rangle$ where $\mathcal{S} = \langle S, \star \rangle$ and $|p|_v \in \mathbb{P}_{\mathcal{S}}$ for all $p \in \mathbb{L}$. For ease of exposition, it will often be convenient to drop the subscript which names the model. We may then let $\mathcal{C}_{\mathcal{S}}$ be the set of all $\mathcal{S}$-models, and $\mathcal{C} = \bigcup\{\mathcal{C}_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$ be the proper class of all models of $\mathcal{L}^-$ whatsoever.

We may now provide a Finean state semantics for exact verification $\Vdash$ for all pfs $\mathcal{L}^-$ by means of the following recursive clauses:

Unilateral Pre-Semantics:

(p_i) $\mathcal{M}, s \Vdash p_i$ *iff* $s \in |p_i|$.

($\mathcal{T}$) $\mathcal{M}, s \Vdash \mathcal{T}$ *iff* $s = s$.

($\mathcal{V}$) $\mathcal{M}, s \Vdash \mathcal{V}$ *iff* $s = \blacksquare$.

($\bot$) $\mathcal{M}, s \Vdash \bot$ *iff* $s \neq s$.

($\perp\!\!\!\perp$) $\mathcal{M}, s \Vdash \perp\!\!\!\perp$ *iff* $s = \sqsubset$.

($\wedge$) $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = d \star t$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$.

($\vee$) $\mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$ or $\mathcal{M}, s \Vdash A \wedge B$.

Since there is no threat of ambiguity in what follows, I will often drop 'exact' from 'exact verification'. The semantics may be called *unilateral* on account of only including clauses for verification as opposed to both verification and falsification as will be given below.[13] Whereas only the null state $\sqsubset$ verifies $\dot{\bot}$, no state verifies $\bot$. By contrast, every state verifies the $\mathcal{T}$, whereas only the full state $\blacksquare$ verifies $\mathcal{V}$. The conjunction clause formalises the idea that only a fusion of verifiers for each of the conjuncts will verify the conjunction as a whole. In the case of disjunction, a verifier for either disjunct will verify the disjunction, as will a fusion of verifiers for each of the disjuncts.

Given that each $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ assigns every $p \in \mathcal{L}^-$ to a proposition $|p| \in \mathbb{P}_{\mathcal{S}}$ consisting of the verifiers for p in $\mathcal{M}$, we may extend $|\cdot|$ to all $A \in \mathtt{pfs}(\mathcal{L}^-)$:

Unilateral Valuation: $s \in |A|$ *iff* $\mathcal{M}, s \Vdash A$.

Letting $|A|$ be the proposition that $A \in \mathtt{pfs}(\mathcal{L}^-)$ expresses in $\mathcal{M}$, we may define the *extremal propositions* in $\mathbb{P}_{\mathcal{S}}$ as follows:

Unilateral Extremal Propositions:

Top: $\mathcal{T}_{\mathcal{S}} = S$.	*Verum:* $\mathcal{V}_{\mathcal{S}} = \{\blacksquare\}$.
Bottom: $\bot_{\mathcal{S}} = \varnothing$.	*Falsum:* $\dot{\bot}_{\mathcal{S}} = \{\sqsubset\}$.

When ambiguity does not threaten, I will drop the subscript, letting context determine the corresponding state space.

Having extended the valuation function to all $A \in \mathtt{pfs}(\mathcal{L}^-)$ in **Unilateral Valuation**, we may now state the semantics for the wfs of $\mathcal{L}^-$ as follows:

Semantics:

($\trianglelefteq$) $\mathcal{M} \vDash A \trianglelefteq B$ *iff* $|A| \subseteq |B|$.

($\$$) $\mathcal{M} \vDash \$A$ *iff* there is exactly one $s \in |A|$.

($\neg$) $\mathcal{M} \vDash \neg\varphi$ *iff* $\mathcal{M} \nvDash \varphi$.

[13] See Fine (2016, 2017a,b,c) for this usage. Fine (2017b) provides a semantics for sentences which includes negation, crediting Van Fraassen (1969) for providing a related construction.

($\wedge$) $\mathcal{M} \vDash \varphi \wedge \psi$ *iff* $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \psi$.

($\vee$) $\mathcal{M} \vDash \varphi \vee \psi$ *iff* $\mathcal{M} \vDash \varphi$ or $\mathcal{M} \vDash \psi$.

The semantic clause for grounding holds that $A \trianglelefteq B$ is true in $\mathcal{M}$ just in case every exact verifier for A is an exact verifier for B, where $\$A$ is true in $\mathcal{M}$ just in case A is verified by only one state.[14] Let φ be a *logical consequence* of Γ over $\mathcal{C}$— symbolised by '$\Gamma \vDash_{\mathcal{C}} \varphi$'— just in case for all $\mathcal{M} \in \mathcal{C}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$. A wfs φ is *$\mathcal{C}$-valid* just in case $\vDash_{\mathcal{C}} \varphi$.

We may now define algebraic analogues of conjunction and disjunction which are defined over the space of propositions $\mathbb{P}_{\mathcal{S}}$:

Product: $X \wedge Y = \{x \star y : x \in X,\ y \in Y\}$.

Sum: $X \vee Y = X \cup Y \cup (X \wedge Y)$.

Given any $\mathcal{S} \in \mathbb{M}$, we may let $\mathcal{A}_{\mathcal{S}} = \langle \mathbb{P}_{\mathcal{S}}, \wedge, \vee, \mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\perp \rangle$ where not only are $\mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\perp \in \mathbb{P}_{\mathcal{S}}$, but we may show that $\mathbb{P}_{\mathcal{S}}$ is closed under $\wedge$ and $\vee$ as follows:

L3.2 $X \wedge Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

L3.3 $X \vee Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

By the formation rules, $\mathcal{A}_{\mathcal{L}^-} = \langle \mathtt{pfs}(\mathcal{L}^-), \wedge, \vee, \mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\perp \rangle$ is an algebra with the same signature $\sigma_{\mathcal{L}^-} = \langle \{\wedge, \vee\}, \mathbb{E} \rangle$ as $\mathcal{A}_{\mathcal{S}}$, where the valuation function induced by any $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ is an $\mathcal{L}^-$-homomorphism $|\cdot| : \mathcal{A}_{\mathcal{L}^-} \to \mathcal{A}_{\mathcal{S}}$ as below:

$\mathcal{L}^-$-Homomorphism: For any $\mathcal{S} \in \mathbb{M}$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$, the function $|\cdot| : \mathcal{L}^- \to \mathcal{A}_{\mathcal{S}}$ is an *$\mathcal{L}^-$-homomorphism iff* for every $A, B \in \mathtt{pfs}(\mathbb{L})$ both: (1) $|A \wedge B| = |A| \wedge |B|$; and (2) $|A \vee B| = |A| \vee |B|$.

L3.4 $|A \wedge B| = |A| \wedge |B|$.

L3.5 $|A \vee B| = |A| \vee |B|$.

P3.1 $|A| \in \mathbb{P}_{\mathcal{S}}$.

The results above will be of use throughout what follows, where similar results may be shown to hold in §5.7 once negation is included in the language.

[14] Fine (2012b,c) gives a similar semantics for $\trianglelefteq$ in a language without $\$, \neg, \wedge$, and $\vee$.

We may then observe that $\langle \mathbb{P}_{\mathcal{S}}, \wedge \rangle$ and $\langle \mathbb{P}_{\mathcal{S}}, \vee \rangle$ are semilattices on account of satisfying idempotency, commutativity, and associativity. In particular, the following identities hold for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$:

L3.6 $X \wedge X = X$.

L3.7 $X \wedge Y = Y \wedge X$.

L3.8 $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$.

L3.9 $X \vee X = X$.

L3.10 $X \vee Y = Y \vee X$.

L3.11 $(X \vee Y) \vee Z = X \vee (Y \vee Z)$.

Despite these results, $\mathcal{A}_{\mathcal{S}}$ need not form a lattice on account of failing to satisfy the absorption laws, since there is some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$ such that:

L3.18 $X \wedge (X \vee Y) \neq X$.

L3.19 $X \vee (X \wedge Y) \neq X$.

In addition to failing to form a lattice, we may show that $\mathcal{A}_{\mathcal{S}}$ is non-distributive since there is some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$ such that:

L3.21 $X \vee (Y \wedge Z) \neq (X \vee Y) \wedge (X \vee Z)$.

These results correspond to the absence of **#Abs1**, **#Abs2**, and **#Dist** from UGSN, where the following section shows that **#Abs1**, **#Abs2**, and **#Dist** are not theorems of UGSN. Nevertheless, for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$:

L3.22 $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$.

Although $\vee$ does not distributes over $\wedge$, we may distribute $\wedge$ over $\vee$. As we will see in §5.7, the apparent asymmetry above will disappear once negation has been included in the language. The remainder of the present section will be devoted to establishing a selection of the claims enumerated above, drawing on these results to prove that UGSN is sound over $\mathcal{C}$ in the following section.

L3.1 $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$, letting $s \in X \wedge (Y \cup Z)$. It follows that $s = x \star u$ for some $x \in X$ and $u \in Y \cup Z$. If $u \in Y$, then

$s \in X \wedge Y$, and so $s \in (X \wedge Y) \cup (X \wedge Z)$. Similarly, if $u \in Z$, then $s \in X \wedge Z$, and so $s \in (X \wedge Y) \cup (X \wedge Z)$. Thus $X \wedge (Y \cup Z) \subseteq (X \wedge Y) \cup (X \wedge Z)$.

Assume instead that $s \in (X \wedge Y) \cup (X \wedge Z)$. If $s \in X \wedge Y$, then $s = x \star y$ for some $x \in X$ and $y \in Y$, and so $y \in Y \cup Z$. Thus $s \in X \wedge (Y \cup Z)$. Similarly, if $s \in X \wedge Z$, then $s = x \star z$ for some $x \in X$ and $z \in Z$, and so $z \in Y \cup Z$. Thus $s \in X \wedge (Y \cup Z)$, and so $(X \wedge Y) \cup (X \wedge Z) \subseteq X \wedge (Y \cup Z)$. Together with the above, $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z)$ as needed. □

L3.2 $X \wedge Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ where $X, Y \in \mathbb{P}_{\mathcal{S}}$, and choose some $s \in X \wedge Y$. By *Idempotency*, $s \in \overline{X \wedge Y}$, and so $X \wedge Y \subseteq \overline{X \wedge Y}$. To establish the converse inclusion, assume that $s \in \overline{X \wedge Y}$. It follows that $s = x \star y$ for some $x, y \in X \wedge Y$, and so $x = u \star v$ and $y = w \star z$ for some $u, w \in X$ and $v, z \in Y$. It follows that $u \star w \in \overline{X}$ and $v \star z \in \overline{Y}$. Given that $X, Y \in \mathbb{P}_{\mathcal{S}}$, we know that $\overline{X} = X$ and $\overline{Y} = Y$, and so $u \star w \in X$ and $v \star z \in Y$. Thus $(u \star w) \star (v \star z) \in X \wedge Y$, where $(u \star w) \star (v \star z) = s$ by *Associativity* and *Commutativity*. It follows that $s \in X \wedge Y$, and so $\overline{X \wedge Y} \subseteq X \wedge Y$. Thus we may conclude that $X \wedge Y = \overline{X \wedge Y}$, and so $X \wedge Y \in \mathbb{P}_{\mathcal{S}}$. □

L3.3 $X \vee Y \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Follows from the definition of $\mathbb{P}_{\mathcal{S}}$ together with the *Idempotency*, *Commutativity*, and *Associativity* of all $\mathcal{S} \in \mathbb{M}$ as in **L3.2**. □

L3.4 $|A \wedge B| = |A| \wedge |B|$ for all $A, B \in \mathtt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$.

Proof. Let $A, B \in \mathtt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$. By **Unilateral Valuation**, we know that $|A|, |B| \subseteq S$. We may then consider the following:

$$
\begin{aligned}
s \in |A \wedge B| \;\; & \textit{iff} \;\; \mathcal{M}, s \Vdash A \wedge B \\
& \textit{iff} \;\; s = a \star b \text{ where } \mathcal{M}, a \Vdash A \text{ and } \mathcal{M}, b \Vdash B \\
& \textit{iff} \;\; s = a \star b \text{ where } a \in |A| \text{ and } b \in |B| \\
& \textit{iff} \;\; s \in |A| \wedge |B|.
\end{aligned}
$$

All of the biconditionals above hold by definition. Thus we may conclude that $|A \wedge B| = |A| \wedge |B|$ as desired. □

L3.5 $|A \vee B| = |A| \vee |B|$ for all $A, B \in \mathtt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_\mathcal{S}$.

Proof. Let $A, B \in \mathtt{pfs}(\mathcal{L}^-)$ and $\mathcal{M} \in \mathcal{C}_\mathcal{S}$. By **Unilateral Valuation**, we know that $|A|, |B| \subseteq S$. We may then consider the following:

$$\begin{aligned}
s \in |A \vee B| \quad & \textit{iff} \quad \mathcal{M}, s \Vdash A \vee B \\
& \textit{iff} \quad \mathcal{M}, s \Vdash A, \text{ or } \mathcal{M}, s \Vdash B, \text{ or } \mathcal{M}, s \Vdash A \wedge B \\
& \textit{iff} \quad s \in |A|, \text{ or } s \in |B|, \text{ or } s \in |A \wedge B| \\
(\dagger) \quad & \textit{iff} \quad s \in |A|, \text{ or } s \in |B|, \text{ or } s \in |A| \wedge |B| \\
& \textit{iff} \quad s \in |A| \cup |B| \cup (|A| \wedge |B|) \\
(\ddagger) \quad & \textit{iff} \quad s \in |A| \vee |B|.
\end{aligned}$$

The biconditionals above all hold by definition with the exception of (†) which follows by **L3.4**, and (‡) which follows by *Sum*. Thus we may conclude that $|A \vee B| = |A| \vee |B|$. □

L3.6 $X \wedge X = X$ for all $\mathcal{S} \in \mathbb{M}$ and $X \in \mathbb{P}_\mathcal{S}$ or $X \in \mathbb{P}_\mathcal{S}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X \in \mathbb{P}_\mathcal{S}$, and let $s \in X \wedge X$. It follows that $s = x \star x$ for some $x \in X$, and so $s \in \overline{X}$. Since $X \in \mathbb{P}_\mathcal{S}$ or $X \in \mathbb{P}_\mathcal{S}$, we know that $\overline{X} = X$, and so $s \in X$. Thus $X \wedge X \subseteq X$.

Assume instead that $s \in X$. By definition, $s \star s \in X \wedge X$, and so $s \in X \wedge X$ by *Idempotency*. Thus $X \subseteq X \wedge X$, and so $X \wedge X = X$. □

L3.7 $X \wedge Y = Y \wedge X$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_\mathcal{S}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_\mathcal{S}$, letting $s \in X \wedge Y$. It follows that $s = x \star y$ for some $x \in X$ and $y \in Y$. Thus $y \star x \in Y \wedge X$, and so $s \in Y \wedge X$ by *Commutativity*. We may then conclude that $X \wedge Y \subseteq Y \wedge X$, and so $X \wedge Y = Y \wedge X$ by symmetry of reasoning. □

L3.8 $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$, we may then reason as follows:

$$\begin{aligned}
s \in (X \wedge Y) \wedge Z \quad & \textit{iff} \quad s = u \star z \text{ for some } u \in X \wedge Y \text{ and } z \in Z \\
& \textit{iff} \quad s = (x \star y) \star z \text{ for some } x \in X, y \in Y, \text{ and } z \in Z \\
(*) \quad & \textit{iff} \quad s = x \star (y \star z) \text{ for some } x \in X, y \in Y, \text{ and } z \in Z \\
& \textit{iff} \quad s = x \star v \text{ for some } x \in X, \text{ and } v \in Y \wedge Z \\
& \textit{iff} \quad s \in X \wedge (Y \wedge Z).
\end{aligned}$$

The biconditionals above all hold by definition with the exception of $(*)$ which is given by *Associativity*. Thus $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$. □

L3.9 $X \vee X = X$ for all $\mathcal{S} \in \mathbb{M}$ where $X \in \mathbb{P}_{\mathcal{S}}$ or $X \in \mathbb{P}_{\mathcal{S}}$.

Proof. By definition, $X \vee X = \overline{X \cup X} = \overline{X}$. Given that $X \in \mathbb{P}_{\mathcal{S}}$ or $X \in \mathbb{P}_{\mathcal{S}}$, we know that $\overline{X} = X$, and so $X \vee X = X$ as desired. □

L3.10 $X \vee Y = Y \vee X$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. By definition, $X \vee Y = \overline{X \cup Y} = Y \vee X$ since $X \cup Y = Y \cup X$. □

L3.11 $(X \vee Y) \vee Z = X \vee (Y \vee Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$. We may then reason as follows:

$$\begin{aligned}
(X \vee Y) \vee Z &= (X \vee Y) \cup Z \cup [(X \vee Y) \wedge Z] \\
&= X \cup Y \cup Z \cup (X \wedge Y) \cup [(X \cup Y \cup [X \wedge Y]) \wedge Z] \\
(1) \quad &= X \cup Y \cup Z \cup (X \wedge Y) \cup (X \wedge Z) \cup (Y \wedge Z) \cup [(X \wedge Y) \wedge Z] \\
(2) \quad &= X \cup Y \cup Z \cup (Y \wedge Z) \cup (X \wedge Y) \cup (X \wedge Z) \cup [X \wedge (Y \wedge Z)] \\
(3) \quad &= X \cup Y \cup Z \cup (Y \wedge Z) \cup [X \wedge (Y \cup Z \cup [Y \wedge Z])] \\
&= X \cup (Y \vee Z) \cup [X \wedge (Y \vee Z)] \\
&= X \vee (Y \vee Z).
\end{aligned}$$

Whereas both (1) and (3) follow by **L3.1**, (2) is given by *Sum*. Thus we may conclude that $(X \vee Y) \vee Z = X \vee (Y \vee Z)$, as needed. □

P3.1 $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$. The base case is immediate from the definitions. Assume for induction that $|A|, |B| \in \mathbb{P}_{\mathcal{S}}$. We know by **L3.4** that $|A \wedge B| = |A| \wedge |B|$ and by **L3.5** that $|A \vee B| = |A| \vee |B|$, where both $|A| \wedge |B|, |A| \vee |B| \in \mathbb{P}_{\mathcal{S}}$ by **L3.2** and **L3.3**. It follows that $|A \vee B|, |A \wedge B| \in \mathbb{P}_{\mathcal{S}}$. By induction, $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$. □

L3.12 $\not\models_{\mathcal{C}} (A \vee B) \wedge (A \vee C) \trianglelefteq A \vee (B \wedge C)$.

Proof. Let $\mathcal{S}_3 = \langle S, \cup \rangle$ where $S = \mathcal{P}(\{a, b, c\})$. Since $x \cup y \in S$ for all $x, y \in S$, we may conclude that $\mathcal{S}_3$ is a state space. Additionally, both $\varnothing \cup s = s$ for all $s \in S$, and $S \cup s = S$ for all $s \in S$, and so $\mathcal{S}_3$ satisfies *Null State* and *Full State*. Observe that $x \cup x = x$, $x \cup y = y \cup x$, and $x \cup (y \cup z) = (x \cup y) \cup z$ for arbitrary $x, y, z \in S$, and so $\mathcal{S}_3$ also satisfies *Idempotency*, *Commutativity*, and *Associativity*. Thus $\mathcal{S}_3 \in \mathbb{M}$.

Let $\mathcal{M}_3$ be an $\mathcal{S}_3$-model where $|p_1|_3 = \{\{a\}\}$, $|p_2|_3 = \{\{b\}\}$, and $|p_3|_3 = \{\{c\}\}$, and all other assignments are arbitrary singletons. Given that $\mathcal{S}_3$ satisfies *Idempotency*, it follows immediately that $|p_i|_3 = \overline{|p_i|_3}$ for all $i \in \mathbb{N}$, and so $|p_i|_3 \in \mathbb{P}_{\mathcal{S}_3}$ for all $i \in \mathbb{N}$. Thus $\mathcal{M}_3 \in \mathcal{C}$.

It follows that $|p_1 \vee p_2|_3 = |p_1|_3 \vee |p_2|_3$ and $|p_1 \vee p_3|_3 = |p_1|_3 \vee |p_3|_3$ by **L3.5**, and so it follows by **L3.4** both $|p_2 \wedge p_3|_3 = |p_2|_3 \wedge |p_3|_3$ and $|(p_1 \vee p_2) \wedge (p_1 \vee p_3)|_3 = (|p_1|_3 \vee |p_2|_3) \wedge (|p_1|_3 \vee |p_3|_3)$. Again by **L3.5**, we know that $|p_1 \vee (p_2 \wedge p_3)|_3 = |p_1|_3 \vee (|p_2|_3 \wedge |p_3|_3)$.

By definition of $\vee$ and $\wedge$, we know that $|p_1|_3 \vee |p_2|_3 = \{\{a\}, \{b\}, \{a, b\}\}$, $|p_1|_3 \vee |p_3|_3 = \{\{a\}, \{c\}, \{a, c\}\}$, and $|p_2|_3 \wedge |p_3|_3 = \{\{b, c\}\}$, and so we know $(|p_1|_3 \vee |p_2|_3) \wedge (|p_1|_3 \vee |p_3|_3) = \{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $|p_1|_3 \vee (|p_2|_3 \wedge |p_3|_3) = \{\{a\}, \{b, c\}, \{a, b, c\}\}$. We may then observe that $(|p_1|_3 \vee |p_2|_3) \wedge (|p_1|_3 \vee |p_3|_3) \nsubseteq |p_1|_3 \vee (|p_2|_3 \wedge |p_3|_3)$. Given the **Unilateral Semantics**, $\mathcal{M}_3 \not\models (p_1 \vee p_2) \wedge (p_1 \vee p_3) \trianglelefteq p_1 \vee (p_2 \wedge p_3)$, and so $\not\models_{\mathcal{C}} (A \vee B) \wedge (A \vee C) \trianglelefteq A \vee (B \wedge C)$. □

L3.13 $\not\models_{\mathcal{C}} A \wedge (A \vee B) \trianglelefteq A$.

Proof. Let $\mathcal{M}_3$ be as in **L3.12**. Since $|p_1 \vee p_2|_3 = |p_1|_3 \vee |p_2|_3$ by **L3.5**, we know that $|p_1 \wedge (p_1 \vee p_2)|_3 = |p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3)$ by **L3.4**. However, $|p_1|_3 \vee |p_2|_3 = \{\{a\}, \{b\}, \{a, b\}\}$ where $|p_1|_3 = \{\{a\}\}$, and so it follows that $|p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3) = \{\{a\}, \{a, b\}\}$. Since $|p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3) \nsubseteq |p_1|_3$, we know that $\mathcal{M}_3 \not\models p_1 \wedge (p_1 \vee p_2) \trianglelefteq p_1$ by the **Unilateral Semantics**, and so may conclude that $\not\models_{\mathcal{C}} A \wedge (A \vee B) \trianglelefteq A$ as desired. □

L3.14 $\not\models_{\mathcal{C}} A \vee (A \wedge B) \trianglelefteq A$.

Proof. Let $\mathcal{M}_3$ be defined as in **L3.12**. Thus $|p_1 \wedge p_2|_3 = |p_1|_3 \wedge |p_2|_3$ by **L3.4**, and so $|p_1 \vee (p_1 \wedge p_2)|_3 = |p_1|_3 \vee (|p_1|_3 \wedge |p_2|_3)$ by **L3.5**. However, $|p_1|_3 \wedge |p_2|_3 = \{\{a, b\}\}$, and since $|p_1|_3 = \{\{a\}\}$, we may conclude that $|p_1|_3 \vee (|p_1|_3 \wedge |p_2|_3) = \{\{a\}, \{a, b\}\}$. Thus $|p_1|_3 \wedge (|p_1|_3 \vee |p_2|_3) \nsubseteq |p_1|_3$, and so we know that $\mathcal{M}_3 \not\models p_1 \vee (p_1 \wedge p_2) \trianglelefteq p_1$ by the **Unilateral Semantics**. Thus it follows that $\not\models_{\mathcal{C}} A \vee (A \wedge B) \trianglelefteq A$ as desired. □

L3.15 $\not\models_{\mathcal{C}} A \wedge (A \vee B) \approx A$.

Proof. Follows from **L3.13**. □

L3.16 $\not\models_{\mathcal{C}} A \vee (A \wedge B) \approx A$

Proof. Follows from **L3.14**. □

L3.17 $\not\models_{\mathcal{C}} A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$.

Proof. Follows from **L3.12**. □

L3.18 $X \wedge (X \vee Y) \neq X$ for some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Given **L3.15**, $\mathcal{M} \not\models p_1 \wedge (p_1 \vee p_2) \approx p_1$ for some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$, and so $|p_1 \wedge (p_1 \vee p_2)| \neq |p_1|$ by the **Unilateral Semantics**. Thus it follows by **L3.4** and **L3.5** that $|p_1| \wedge (|p_1| \vee |p_2|) \neq |p_1|$. Since $p_1, p_2 \in \mathtt{pfs}(\mathcal{L}^-)$, we know that $|p_1|, |p_2| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so we may conclude the proof by existentially generalising on $|p_1|, |p_2|$ and $\mathcal{S} \in \mathbb{M}$. □

L3.19 $X \vee (X \wedge Y) \neq X$ for some $\mathcal{S} \in \mathbb{M}$ and $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Given **L3.16**, $\mathcal{M} \not\models p_1 \vee (p_1 \wedge p_2) \approx p_1$ for some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$, and so $|p_1 \vee (p_1 \wedge p_2)| \neq |p_1|$ by the **Unilateral Semantics**. Thus it follows by **L3.4** and **L3.5** that $|p_1| \vee (|p_1| \wedge |p_2|) \neq |p_1|$. Since $p_1, p_2 \in \mathtt{pfs}(\mathcal{L}^-)$, we know that $|p_1|, |p_2| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so we may conclude the proof by existentially generalising on $|p_1|, |p_2|$ and $\mathcal{S} \in \mathbb{M}$. □

L3.20 For all $\mathcal{S} \in \mathbb{M}$ and $X, Y, U, V \in \mathbb{P}_{\mathcal{S}}$, if $X \subseteq Y$ and $U \subseteq V$, then $X \wedge U \subseteq Y \wedge V$.

Proof. Let $\mathcal{S} \in \mathbb{M}$ and $X, Y, U, V \in \mathbb{P}_{\mathcal{S}}$, and assume $X \subseteq Y$ and $U \subseteq V$. Choose some $s \in X \vee U$. By *Sum*, $s \in X \cup U \cup (X \wedge U)$. If $s \in X \cup U$, then $s \in Y \cup V$, and so $s \in Y \cup V \cup (X \wedge V)$. If instead $s \in X \wedge U$, then $s = x \star u$ for some $x \in X$ and $u \in U$. It follows that $x \in Y$ and $u \in V$, and so $s \in Y \wedge V$. Thus $s \in Y \cup V \cup (Y \wedge V)$ in either case, and so $s \in Y \vee V$ by *Sum*. We may then conclude that $X \vee U \subseteq Y \vee V$. □

L3.21 $X \vee (Y \wedge Z) \neq (X \vee Y) \wedge (X \vee Z)$ for some $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Given **L3.17** above, we know that there is some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ where $\mathcal{M} \not\models p_1 \vee (p_2 \wedge p_3) \approx (p_1 \vee p_2) \wedge (p_1 \vee p_3)$, and so it follows from the **Unilateral Semantics** that $|p_1 \vee (p_2 \wedge p_3)| \neq |(p_1 \vee p_2) \wedge (p_1 \vee p_3)|$. Thus $|p_1| \vee (|p_2| \wedge |p_3|) \neq (|p_1| \vee |p_2|) \wedge (|p_1| \vee |p_3|)$ by **L3.4** and **L3.5**. Since $p_1, p_2, p_3 \in \mathtt{pfs}(\mathcal{L}^-)$, we know that $|p_1|, |p_2|, |p_3| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so we may conclude the proof by existential generalisation. □

L3.22 $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}$ and $X, Y, Z \in \mathbb{P}_{\mathcal{S}}$. We may then reason as follows:

$$\begin{aligned}
X \wedge (Y \vee Z) &=_1 X \wedge [Y \cup Z \cup (Y \wedge Z)] \\
&=_2 (X \wedge Y) \cup (X \wedge Z) \cup [X \wedge (Y \wedge Z)] \\
&=_3 (X \wedge Y) \cup (X \wedge Z) \cup [(X \wedge X) \wedge (Y \wedge Z)] \\
&=_4 (X \wedge Y) \cup (X \wedge Z) \cup [(X \wedge Y) \wedge (X \wedge Z)] \\
&=_5 (X \wedge Y) \vee (X \wedge Z).
\end{aligned}$$

Whereas both (1) and (5) are given by *Sum*, (2) follows from **L3.1** and (3) holds by **L2.13**. We may then justify (4) by appeal to **L2.12** and **L2.13**. Thus $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$. □

5.4 Soundness

Given the semantics above, we may now prove that UGSN is sound over $\mathcal{C}$ by induction on the number of applications of the meta-rules as follows:

T1 (*Soundness*) If $\Sigma \vdash_{\text{UGSN}} \varphi$, then $\Sigma \models_{\mathcal{C}} \varphi$.

Proof. The proof goes by a routine induction, drawing on **L4.1** – **L4.18** in addition to standard validities for propositional logic. □

It follows from **L3.15** –**L3.17** and *Soundness* that **#Abs1**, **#Abs2**, and **#Dist** are not theorems of UGSN. It remains to prove **L4.1** – **L4.18**.

L4.1 $\models_{\mathcal{C}} A \trianglelefteq A \vee B$ and $\models_{\mathcal{C}} B \trianglelefteq A \vee B$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A, B \in \mathtt{pfs}(\mathcal{L}^-)$. Thus $|A \vee B| = |A| \vee |B|$ by **L3.5**, where $|A| \vee |B| = |A| \cup |B| \cup |A \wedge B|$ by *Sum*. It follows that $|A| \subseteq |A \vee B|$ and $|B| \subseteq |A \vee B|$, and so both $\mathcal{M} \models A \trianglelefteq A \vee B$ and $\mathcal{M} \models B \trianglelefteq A \vee B$ by the semantics. □

L4.2 $\models_{\mathcal{C}} A \trianglelefteq A \wedge A$ and $\models_{\mathcal{C}} A \wedge A \trianglelefteq A$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$. By **L3.4**, $|A \wedge A| = |A| \wedge |A|$, where $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so $|A| \wedge |A| = |A|$ by **L2.13**. Thus $|A \wedge A| = |A|$, and so $\mathcal{M} \models A \trianglelefteq A \wedge A$ and $\mathcal{M} \models A \wedge A \trianglelefteq A$. □

L4.3 $\models_{\mathcal{C}} A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge C$ and $\models_{\mathcal{C}} (A \wedge B) \wedge C \trianglelefteq A \wedge (B \wedge C)$.

Proof. Letting $\mathcal{M} \in \mathcal{C}$ and $A, B, C \in \mathtt{pfs}(\mathcal{L}^-)$, it follows by **P3.1** that $|A|, |B|, |C| \in \mathbb{P}_{\mathcal{S}}$. Thus we may argue as follows:

$$\begin{aligned} |A \wedge (B \wedge C)| &=_1 |A| \wedge (|B| \wedge |C|) \\ &=_2 (|A| \wedge |B|) \wedge |C| \\ &=_3 |(A \wedge B) \wedge C|. \end{aligned}$$

Whereas both (1) and (3) follow from **L3.4**, (2) is given by **L3.8**. Thus $|A \wedge (B \wedge C)| = |(A \wedge B) \wedge C|$, and so $\mathcal{M} \vDash A \wedge (B \wedge C) \trianglelefteq (A \wedge B) \wedge C$ and $\mathcal{M} \vDash (A \wedge B) \wedge C \trianglelefteq A \wedge (B \wedge C)$. □

L4.4 $\vDash_{\mathcal{C}} A \wedge B \trianglelefteq B \wedge A$.

Proof. Letting $\mathcal{M} \in \mathcal{C}$ and $A, B \in \mathtt{pfs}(\mathcal{L}^-)$, it follows by **P3.1** that $|A|, |B| \in \mathbb{P}_{\mathcal{S}}$. Thus we may argue as follows:

$$\begin{aligned} |A \wedge B| &=_1 |A| \wedge |B| \\ &=_2 |B| \wedge |A| \\ &=_3 |B \wedge A|. \end{aligned}$$

Whereas both (1) and (3) follow from **L3.4**, (2) is given by **L3.7**. Thus $|A \wedge B| = |B \wedge A|$, and so $\mathcal{M} \vDash A \wedge B \trianglelefteq B \wedge A$. □

L4.5 $A \trianglelefteq B, C \trianglelefteq D \vDash_{\mathcal{C}} A \wedge C \trianglelefteq B \wedge D$.

Proof. Assume $\mathcal{M} \vDash A \trianglelefteq B$ and $\mathcal{M} \vDash C \trianglelefteq D$ for some $\mathcal{M} \in \mathcal{C}$ and $A, B, C, D \in \mathtt{pfs}(\mathcal{L}^-)$. Thus $|A| \subseteq |B|$ and $|C| \subseteq |D|$, where we know by **P3.1** that $|A|, |B|, |C|, |D| \in \mathbb{P}_{\mathcal{S}}$. By **L3.20**, $|A| \wedge |C| \subseteq |B| \wedge |D|$, and so $|A \wedge C| \subseteq |B \wedge D|$ by **L3.4**. Thus $\mathcal{M} \vDash A \wedge C \trianglelefteq B \wedge D$. □

L4.6 $A \trianglelefteq B, B \trianglelefteq C \vDash_{\mathcal{C}} A \trianglelefteq C$.

Proof. Assume $\mathcal{M} \vDash A \trianglelefteq B$ and $\mathcal{M} \vDash B \trianglelefteq C$ for some $\mathcal{M} \in \mathcal{C}$ and $A, B, C \in \mathtt{pfs}(\mathcal{L}^-)$. Thus $|A| \subseteq |B|$ and $|B| \subseteq |C|$, and so $|A| \subseteq |C|$. We may then conclude that $\mathcal{M} \vDash A \trianglelefteq B$. □

L4.7 $\models_{\mathcal{C}} A \trianglelefteq \mathcal{T}$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^{-})$. By **P3.1**, $|A| \in \mathbb{P}_{\mathcal{S}}$, and so $|A| \subseteq S$. Thus, $|A| \subseteq |\mathcal{T}|$ since $|\mathcal{T}| = S$, and so $\mathcal{M} \vDash A \trianglelefteq \mathcal{T}$. □

L4.8 $\models_{\mathcal{C}} \$\bot$.

Proof. Immediate from the semantics. □

L4.9 $\models_{\mathcal{C}} \$\mathcal{V}$.

Proof. Immediate from the semantics. □

L4.10 $\models_{\mathcal{C}} \bot \wedge A \trianglelefteq A$ and $\models_{\mathcal{C}} A \trianglelefteq \bot \wedge A$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^{-})$. We may then argue as follows:

$$\begin{aligned} s \in |\bot \wedge A| \quad & \textit{iff} \quad s = \square \star y \text{ for some } y \in |A| \\ (*) \quad & \textit{iff} \quad s = y \text{ for some } y \in |A| \\ & \textit{iff} \quad s \in |A|. \end{aligned}$$

Given that $(*)$ holds by *Null State*, it follows that $|\bot \wedge A| = |A|$, and so both $\mathcal{M} \vDash \bot \wedge A \trianglelefteq A$ and $\mathcal{M} \vDash A \trianglelefteq \bot \wedge A$. □

L4.11 $\models_{\mathcal{C}} \mathcal{V} \wedge A \trianglelefteq \mathcal{V}$ and $\models_{\mathcal{C}} \mathcal{V} \trianglelefteq \mathcal{V} \wedge A$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^{-})$. We may then argue as follows:

$$\begin{aligned} s \in |\mathcal{V} \wedge A| \quad & \textit{iff} \quad s = \blacksquare \star y \text{ for some } y \in |A| \\ (*) \quad & \textit{iff} \quad s = \blacksquare \\ & \textit{iff} \quad s \in |\mathcal{V}|. \end{aligned}$$

Given that $(*)$ holds by *Full State*, it follows that $|\mathcal{V} \wedge A| = |\mathcal{V}|$, and so both $\mathcal{M} \vDash \mathcal{V} \wedge A \trianglelefteq \mathcal{V}$ and $\mathcal{M} \vDash \mathcal{V} \trianglelefteq \mathcal{V} \wedge A$. □

L4.12 $\$A \models_{\mathcal{C}} A \trianglelefteq \bot$.

Proof. Assume that $\mathcal{M} \vDash \$A$ for some $\mathcal{M} \in \mathcal{C}$. It follows that $|A| = \{s\}$ for some $s \in S$, where $|\bot| = \varnothing$, and so $|A| \not\subseteq |\bot|$. Thus $\mathcal{M} \nvDash A \trianglelefteq \bot$, and so $\mathcal{M} \vDash A \not\trianglelefteq \bot$. □

L4.13 $A \approx B \models_{\mathcal{C}} \$A \leftrightarrow \$B$.

Proof. Let $\mathcal{M} \in \mathcal{C}$ and assume for contraposition that $\mathcal{M} \vDash \$A$ and $\mathcal{M} \nvDash \$B$. It follows that $|A| = \{s\}$ for some $s \in S$ where $|B| \neq \{s\}$, and so $|A| \neq |B|$. Thus either $|A| \not\subseteq |B|$ or $|B| \not\subseteq |A|$, and so either $\mathcal{M} \nvDash A \trianglelefteq B$ or $\mathcal{M} \nvDash B \trianglelefteq B$. We may then conclude that $\mathcal{M} \nvDash A \approx B$, where the same holds if $\mathcal{M} \nvDash \$A$ and $\mathcal{M} \vDash \$B$. □

L4.14 $\$A, \$B \models_{\mathcal{C}} \$(A \wedge B)$.

Proof. Assume $\mathcal{M} \vDash \$A$ and $\mathcal{M} \vDash \$B$ for some $\mathcal{M} \in \mathcal{C}$. It follows that $|A| = \{s\}$ and $|B| = \{t\}$ for some $s, t \in S$, where we know by **L3.4** that $|A \wedge B| = \{x \star y : x \in |A| \text{ and } y \in |B|\}$. Thus $|A \wedge B| = \{s \star t\}$, and so $\mathcal{M} \vDash \$(A \wedge B)$. □

L4.15 $\$A, B \trianglelefteq A \models_{\mathcal{C}} (A \trianglelefteq B) \vee (B \trianglelefteq \bot)$.

Proof. Let $\mathcal{M} \in \mathcal{C}$, and assume $\mathcal{M} \vDash \$A$ and $\mathcal{M} \vDash B \trianglelefteq A$. It follows that $|A| = \{s\}$ for some $s \in S$, and $|B| \subseteq |A|$, and so $|B| \subseteq \{s\}$. Thus either $|B| = \{s\}$ or $|B| = \varnothing$, and so either $|A| \subseteq |B|$ or $|B| \subseteq |\bot|$. In either case, $\mathcal{M} \vDash (A \trianglelefteq B) \vee (B \trianglelefteq \bot)$. □

L4.16 $\$A, A \trianglelefteq C \vee D \models_{\mathcal{C}} (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$.

Proof. Let $\mathcal{M} \in \mathcal{C}$, and assume $\mathcal{M} \vDash \$A$ and $\mathcal{M} \vDash A \trianglelefteq C \vee D$. Thus $|A| = \{s\}$ for some $s \in S$, and $|A| \subseteq |C \vee D|$, where we know by **L3.5** and *Sum* that $|C \vee D| = |C| \cup |D| \cup |C \wedge D|$, and so $s \in |C| \cup |D| \cup |C \wedge D|$. Given that $|A| = \{s\}$, either $|A| \subseteq |C|$ or $|A| \subseteq |D|$ or $|A| \subseteq |C \wedge D|$, and so $\mathcal{M} \vDash A \trianglelefteq C$ or $\mathcal{M} \vDash A \trianglelefteq D$ or $\mathcal{M} \vDash A \trianglelefteq C \wedge D$. Thus $\mathcal{M} \vDash (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$. □

L4.17 If $\Gamma \models_{\mathcal{C}} \$p \to [(p \trianglelefteq A) \to (p \trianglelefteq B)]$ where $p \in \mathbb{L}$ does not occur in A, B or any $\gamma \in \Gamma$, then $\Gamma \models_{\mathcal{C}} A \trianglelefteq B$.

Proof. Let $p \in \mathbb{L}$ be an arbitrary sentence letter which does not occur in A, B or any $\gamma \in \Gamma$. Assume for contraposition that $\Gamma \not\models_{\mathcal{C}} A \trianglelefteq B$. Thus there is some $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, but $\mathcal{M} \not\models A \trianglelefteq B$. It follows that $|A| \not\subseteq |B|$, and so there is some $s \in |A|$ such that $s \notin |B|$. Let $\mathcal{M}_u$ differ at most from $\mathcal{M}$ by setting $|p|_u = \{s\}$. Given that p does not occur in A or B, we may conclude that $|A| = |A|_u$ and $|B| = |B|_u$, and so both $|p|_u \subseteq |A|_u$ but $|p|_u \not\subseteq |B|_u$ given the above. Thus $\mathcal{M}_u \models \$p$ where $\mathcal{M}_u \models p \trianglelefteq A$ but $\mathcal{M}_u \not\models p \trianglelefteq B$, and so $\mathcal{M}_u \not\models (p \trianglelefteq A) \to (p \trianglelefteq B)$ and $\mathcal{M}_u \not\models \$p \to [(p \trianglelefteq A) \to (p \trianglelefteq B)]$. Since p does not occur in any $\gamma \in \Gamma$, and $\mathcal{M}_u$ differs from $\mathcal{M}$ at most in p, it follows that $\mathcal{M}_u \models \gamma$ for all $\gamma \in \Gamma$. Thus $\Gamma \not\models_{\mathcal{C}} \$p \to [(p \trianglelefteq A) \to (p \trianglelefteq B)]$. □

L4.18 If $\Gamma, [(A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \to [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)] \models_{\mathcal{C}} \varphi$ for distinct $p, q \in \mathbb{L}$ which do not occur in φ, A, B or any $\gamma \in \Gamma$, then $\Gamma \models_{\mathcal{C}} \varphi$.

Proof. Choose some distinct $p, q \in \mathbb{L}$ which do not occur in φ, A, B or any $\gamma \in \Gamma$, and assume $\Gamma \not\models_{\mathcal{C}} \varphi$ for contraposition. It follows that there is some $\mathcal{M} \in \mathcal{C}$ where $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$ but $\mathcal{M} \not\models \varphi$. Of course, either: (a) $\mathcal{M} \models (A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$; or (b) $\mathcal{M} \not\models (A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$. Assume (b) to start. It follows that:

$$\mathcal{M} \models [(A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \to [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)].$$

Given that $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, the antecedent of the claim to be proven is false. Assume (a) instead: $\mathcal{M} \models (A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$. Thus $\mathcal{M} \not\models A \trianglelefteq \bot$ and $\mathcal{M} \models A \trianglelefteq B \wedge C$, and so $|A| \not\subseteq \varnothing$ and $|A| \subseteq |B \wedge C|$. It follows that there is some $a \in |A|$, where $a \in |B \wedge C|$, and so $a = b \star c$ for some $b \in |B|$ and $c \in |C|$. Let $\mathcal{M}_u$ differ at most from $\mathcal{M}$ by setting $|p|_u = \{b\}$ and $|q|_u = \{c\}$, and so $\mathcal{M}_u \models \$p$ and $\mathcal{M}_u \models \$q$. Since p and q do not occur in A, B, φ or any $\gamma \in \Gamma$, we know that $\mathcal{M}_u \not\models \varphi$ and $\mathcal{M}_u \models \gamma$ for all $\gamma \in \Gamma$. Additionally, $|A| = |A|_u$, $|B| = |B|_u$, and $|C| = |C|_u$, and

so $|p|_u \subseteq |B|_u$ and $|q|_u \subseteq |C|_u$. Thus $\mathcal{M}_u \vDash p \trianglelefteq B$ and $\mathcal{M}_u \vDash q \trianglelefteq C$. Given **L3.4**, we may argue as follows:

$$\begin{aligned} |p \wedge q|_u &= \{x \star y : x \in |p|_u \text{ and } y \in |q|_u\} \\ &= \{b \star c\} \\ &= \{a\} \end{aligned}$$

Since $a \in |A| = |A|_u$, we know $|p \wedge q|_u \subseteq |A|_u$, and so $\mathcal{M}_u \vDash p \wedge q \trianglelefteq A$. Thus $\mathcal{M}_u \vDash \$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)$, and so trivially:

$$\mathcal{M}_u \vDash [(A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$p \wedge \$q \wedge (p \trianglelefteq B) \wedge (q \trianglelefteq C) \wedge (p \wedge q \trianglelefteq A)].$$

Given that $\mathcal{M}_u \nvDash \varphi$ where $\mathcal{M}_u \vDash \gamma$ for all $\gamma \in \Gamma$, it follows that the antecedent of the claim to be proven is false. Thus the antecedent is false whether $\mathcal{M} \vDash (A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$ or $\mathcal{M} \nvDash (A \not\trianglelefteq \bot) \wedge (A \trianglelefteq B \wedge C)$. The claim to be proven then follows by discharge and contraposition. □

5.5 Completeness

Let $\mathcal{L}^+$ be a language like $\mathcal{L}^-$ but with $\mathbb{L}^+ = \mathbb{L} \cup \mathbb{Q} \cup \mathbb{W}$ in place of $\mathbb{L}$ where $\mathbb{Q} = \{q_i : i \in \mathbb{N}\}$ and $\mathbb{W} = \bigcup_{i \in \mathbb{N}} \{r_i, s_i\}$. Keeping the formation rules the same as before, let $\mathtt{pfs}(\mathcal{L}^+)$ be the set of pfs recursively generated from $\mathbb{L}^+$ rather than $\mathbb{L}$, where $\mathtt{wfs}(\mathcal{L}^+)$ is then generated from $\mathtt{pfs}(\mathcal{L}^+)$ *via* $\mathtt{atoms}(\mathcal{L}^+)$ as above. In defining $\vdash_{\text{UGSN}}$, we may then permit instances of the axioms and rules of inference included in UGSN to draw upon both $\mathtt{pfs}(\mathcal{L}^+)$ and $\mathtt{wfs}(\mathcal{L}^+)$. Whereas in §5.3 and §5.4 we were concerned to evaluate truth relative to models in $\mathcal{C}$, we must now extend consideration to all $\mathtt{wfs}(\mathcal{L}^+)$. Letting $\mathcal{M} = \langle S, \star, |\cdot| \rangle$ be a *unilateral $\mathcal{S}$-model of* $\mathcal{L}^+$ just in case $\mathcal{S} \in \mathbb{M}$ where $\mathcal{S} = \langle S, \star \rangle$ and $|p| \in \mathbb{P}_{\mathcal{S}}$ for all $p \in \mathbb{L}^+$, we may take $\mathcal{C}^+_{\mathcal{S}}$ to be the class of all $\mathcal{S}$-models of $\mathcal{L}^+$, where $\mathcal{C}^+ = \bigcup\{\mathcal{C}^+_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$. We are now in a position to prove that UGSN is complete over $\mathcal{C}$ by first showing that UGSN is complete over $\mathcal{C}^+$.

T2 (*Completeness*) If $\Sigma \models_{\mathcal{C}} \varphi$, then $\Sigma \vdash_{\text{UGSN}} \varphi$.

Proof. The proof that UGSN is complete over $\mathcal{C}$ goes by contraposition. Assume for discharge that $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$. In order to prove that $\Sigma \not\models_{\mathcal{C}} \theta$, I will construct a Henkin model $\mathcal{M}_{\Gamma_{\Sigma,\theta}}$ where $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \not\models \theta$ for a carefully chosen set of wfss $\Gamma_{\Sigma,\theta} \subseteq \mathtt{wfs}(\mathcal{L}^+)$. With this aim in mind, consider the following definitions:

Maximality: Γ is *maximal iff* for all $\varphi \in \mathtt{wfs}(\mathcal{L}^+)$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Consistency: Γ is *consistent iff* $\nvdash_{\text{UGSN}} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n)$ for any $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$.

Saturated: Γ is *saturated iff* for all $A, B \in \mathtt{pfs}(\mathcal{L}^-)$, there is some $q \in \mathbb{Q}$ not occurring in A or B where $[(q \trianglelefteq A) \to (q \trianglelefteq B)] \to (A \trianglelefteq B) \in \Gamma$.

Conjunctive: Γ is *conjunctive iff* whenever $A \ntrianglelefteq \bot, A \trianglelefteq B \wedge C \in \Gamma$, then there are some $X, Y \in \mathtt{pfs}(\mathcal{L}^+)$ where $\$X, \$Y, X \trianglelefteq B, Y \trianglelefteq C, X \wedge Y \trianglelefteq A \in \Gamma$.

$\trianglelefteq$*-Consistent:* Γ is $\trianglelefteq$*-consistent iff* Γ is saturated, conjunctive, and consistent.

Given the definitions above and any $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$ where $\Sigma \nvdash_{\text{UGSN}} \theta$, we may construct a maximal $\trianglelefteq$-consistent set $\Gamma_{\Sigma,\theta}$, proving the following:

L5.6: For all $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Gamma_{\Sigma,\theta} \subseteq \mathtt{wfs}(\mathcal{L}^+)$ is maximal $\trianglelefteq$-consistent where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$.

Given any maximal $\trianglelefteq$-consistent set $\Gamma_{\Sigma,\theta}$, I will show how to construct a Henkin model $\mathcal{M}_{\Gamma_{\Sigma,\theta}}$. Letting $\mathcal{M}^{\mathfrak{R}}$ be the $\mathcal{L}^-$-reduct of $\mathcal{M} \in \mathcal{C}^+$, we may prove:

L5.10: If $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent, then $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \in \mathcal{C}^+$.

L5.11: If $\mathcal{M} \in \mathcal{C}^+$, then $\mathcal{M}^{\mathfrak{R}} \in \mathcal{C}$ where $\mathcal{M}^{\mathfrak{R}} \models \varphi$ iff $\mathcal{M} \models \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

P5.1: If $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$, then $\chi \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \chi$ for all $\chi \in \mathtt{wfs}(\mathcal{L}^-)$.

Given the assumption that $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$, it follows by **L5.6** that $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$. By **P5.1**, $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \sigma$ for all $\sigma \in \Sigma$ where $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \not\models \theta$. Given that $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \in \mathcal{C}^+$ by **L5.10**, it follows from **L5.11** that $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \in \mathcal{C}$ where $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \models \varphi$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$. Thus $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \models \sigma$ for all $\sigma \in \Sigma$ where $\mathcal{M}^{\mathfrak{R}}_{\Gamma_{\Sigma,\theta}} \not\models \theta$, and so

$\Sigma \not\models_{\mathcal{C}} \theta$. By discharge and contraposition, we may conclude that if $\Sigma \models_{\mathcal{C}} \theta$ then $\Sigma \vdash_{\text{UGSN}} \theta$. It remains to establish each of the results stated above. □

In what follows, I will prove a number of preliminary results, culminating in proofs of the lemmas and propositions cited above.

L5.1 If $\Gamma \vdash_{\text{UGSN}} \varphi$ and $\Gamma \vdash_{\text{UGSN}} \neg\varphi$, then Γ is inconsistent.

Proof. Standard. □

L5.2 If Γ is a maximal consistent subset of $\mathtt{wfs}(\mathcal{L}^+)$, then for all $\varphi \in \mathtt{wfs}(\mathcal{L}^+)$:

(a) $\varphi \in \Gamma$ if and only if $\neg\varphi \notin \Gamma$;

(b) $\Gamma \vdash_{\text{UGSN}} \varphi$ if and only if $\varphi \in \Gamma$;

(c) $\varphi \vee \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ or $\psi \in \Gamma$;

(d) $\varphi \wedge \psi \in \Gamma$ if and only if $\varphi \in \Gamma$ and $\psi \in \Gamma$.

Proof. Standard. □

L5.3 If $X \cup Y$ is inconsistent for $Y \neq \varnothing$, then $X \vdash_{\text{UGSN}} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n)$ for some $\{\gamma_1, \ldots, \gamma_n\} \subseteq Y$.

Proof. Standard. □

Given any set of sentences $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$ for which $\Sigma \nvdash_{\text{UGSN}} \theta$, we may construct a maximal consistent set $\Gamma_{\Sigma,\theta}$ where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$. The construction will proceed in three stages. First we define a saturated set $\Delta_{\Sigma,\theta}$ where $\Sigma \cup \{\neg\theta\} \subseteq \Delta_{\Sigma,\theta}$, showing that $\Delta_{\Sigma,\theta}$ is consistent. We then move to extend $\Delta_{\Sigma,\theta}$ to a conjunctive set $\Omega_{\Sigma,\theta}$, showing that $\Omega_{\Sigma,\theta}$ is also consistent. Lastly, we define a maximal consistent extension $\Gamma_{\Sigma,\theta}$ of $\Delta_{\Sigma,\theta}$ in the usual manner. More specifically, consider the following definitions:

α-Ordering: Let $\mathtt{atoms}(\mathcal{L}^+) = \{\alpha_i : i \in \mathbb{N}\}$ fix an ordering.

α-Witnesses: $\delta_i = [(q_i^\star \trianglelefteq A) \to (q_i^\star \trianglelefteq B)] \to (A \trianglelefteq B)$ where $q_i^\star$ is the lowest indexed member of $\mathbb{Q}$ not occurring in A, B, or δ_j for any $j < i$.

Saturation: Let $\Delta^0_{\Sigma,\theta} = \Sigma \cup \{\neg\theta\}$, $\Delta^{n+1}_{\Sigma,\theta} = \Delta^n_{\Sigma,\theta} \cup \{\delta_{n+1}\} \cup \{\$q^\star_{n+1}\}$, and $\Delta_{\Sigma,\theta} = \bigcup_{n\in\mathbb{N}} \Delta^n_{\Sigma,\theta}$.

Given these definitions, we may move to establish the first consistency proof.

L5.4 For all $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Delta_{\Sigma,\theta}$ is consistent.

Proof. The proof goes by induction, showing that $\Delta^n_{\Sigma,\theta}$ is consistent for all $n \in \mathbb{N}$. Let $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$ and assume for discharge that $\Sigma \nvdash_{\text{UGSN}} \theta$. Assume for *reductio* that $\Delta^0_{\Sigma,\theta}$ is inconsistent. It follows that $\Sigma \vdash_{\text{UGSN}} \neg\theta$ by **L5.3**, contradicting the above. Thus $\Delta^0_{\Sigma,\theta}$ is consistent by *reductio*.

Assume for induction that $\Delta^n_{\Sigma,\theta}$ is consistent. Assume for *reductio* that $\Delta^{n+1}_{\Sigma,\theta}$ is inconsistent. Since $\Delta^{n+1}_{\Sigma,\theta} = \Delta^n_{\Sigma,\theta} \cup \{\delta_{n+1}\} \cup \{\$q^\star_{n+1}\}$, it follows by **L5.3** that $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg(\delta_{n+1} \wedge \$q^\star_{n+1})$. Consider the following:

$$
\begin{array}{rll}
\Delta^n_{\Sigma,\theta} & \vdash_{\text{UGSN}} & \neg(\delta_{n+1} \wedge \$q^\star_{n+1}) \\
& \vdash_{\text{UGSN}} & \neg[([(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)] \to [A \trianglelefteq B]) \wedge \$q^\star_{n+1}] \\
& \vdash_{\text{UGSN}} & \neg\$q^\star_{n+1} \vee ([(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)] \wedge [A \ntrianglelefteq B]) \\
& \vdash_{\text{UGSN}} & (\$q^\star_{n+1} \to [(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)]) \wedge (\$q^\star_{n+1} \to [A \ntrianglelefteq B]) \\
& \vdash_{\text{UGSN}} & \$q^\star_{n+1} \to [A \ntrianglelefteq B] \\
& \vdash_{\text{UGSN}} & \$q^\star_{n+1} \to [(q^\star_{n+1} \trianglelefteq A) \to (q^\star_{n+1} \trianglelefteq B)] \\
(*) & \vdash_{\text{UGSN}} & A \trianglelefteq B \\
& \vdash_{\text{UGSN}} & \neg\$q^\star_{n+1}
\end{array}
$$

The above follows by propositional logic with the exception of $(*)$ which follows by the **SP6**. Thus we may conclude that $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\$q^\star_{n+1}$ where $q^\star_{n+1}$ is the lowest indexed member of $\mathbb{Q}$ not occurring in A, B, or δ_m for any $m < n+1$. It follows by **AR2** that $\Delta^n_{\Sigma,\theta[q^\star_n/q^\star_{n+1}]} \vdash_{\text{UGSN}} \neg\$q^\star_n$, and so $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\$q^\star_n$ since $q^\star_{n+1}$ does not occur in $\Delta^n_{\Sigma,\theta}$. However, by construction $\$q^\star_n \in \Delta^n_{\Sigma,\theta}$, and so $\Delta^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \$q^\star_n$. Thus $\Delta^n_{\Sigma,\theta}$ is inconsistent by **L5.1**, contradicting the above. By *reductio*, $\Delta^{n+1}_{\Sigma,\theta}$ is consistent, and so $\Delta_{\Sigma,\theta}$ is consistent by induction. □

Having proven that $\Delta_{\Sigma,\theta}$ is consistent for any non-theorem θ, we may now proceed to construct $\Omega_{\Sigma,\theta}$ from $\Delta_{\Sigma,\theta}$ as follows:

$\wedge$-Atoms: Let $\texttt{atoms}^\wedge(\mathcal{L}^+) = \{A \trianglelefteq B \wedge C : A, B, C \in \texttt{pfs}(\mathcal{L}^+)\}$.

β-Ordering: Let $\texttt{atoms}^\wedge(\mathcal{L}^+) = \{\beta_i : i \in \mathbb{N}\}$ fix an ordering.

β-Witnesses: $\omega_i = [\$A \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$r_i^\star \wedge \$s_i^\star \wedge (r_i^\star \trianglelefteq B) \wedge (s_i^\star \trianglelefteq C) \wedge (r_i^\star \wedge s_i^\star \approx A)]$ where $r_i^\star$ and $s_i^\star$ are the lowest indexed member of $\mathbb{W}$ not occurring in A, B, C, or ω_j for any $j < i$.

Completion: Let $\Omega^0_{\Sigma,\theta} = \Delta_{\Sigma,\theta}$, $\Omega^{n+1}_{\Sigma,\theta} = \Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\}$, and $\Omega_{\Sigma,\theta} = \bigcup_{n\in\mathbb{N}} \Omega^n_{\Sigma,\theta}$.

We may now move to show that $\Omega_{\Sigma,\theta}$ is also consistent.

L5.5 For all $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Omega_{\Sigma,\theta}$ is consistent.

Proof. Let $\Sigma \cup \{\theta\} \subseteq \texttt{wfs}(\mathcal{L}^-)$ and assume for discharge that $\Sigma \nvdash_{\text{UGSN}} \theta$. It follows by **L5.4** that $\Delta_{\Sigma,\theta}$ is consistent, establishing the base case.

Assume for induction that $\Omega^n_{\Sigma,\theta}$ is consistent. Assume for *reductio* that $\Omega^{n+1}_{\Sigma,\theta}$ is inconsistent. Since $\Omega^{n+1}_{\Sigma,\theta} = \Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\}$, it follows by **L5.3** that $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\omega_{n+1}$, and so $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\theta \rightarrow \neg\omega_{n+1}$. By contraposition, $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \omega_{n+1} \rightarrow \theta$, and so $\Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\} \vdash_{\text{UGSN}} \theta$. Since $\theta \in \texttt{wfs}(\mathcal{L}^-)$, neither $r^\star_{n+1}$ nor $s^\star_{n+1}$ occur in θ. Additionally, $\mathbb{W} \cap \Delta_{\Sigma,\theta} = \varnothing$, and so neither $r^\star_{n+1}$ nor $s^\star_{n+1}$ are in $\Omega^0_{\Sigma,\theta}$. Since by construction $r^\star_{n+1}$ and $s^\star_{n+1}$ are the lowest indexed member of $\mathbb{W}$ not occurring in ω_j for any $j < i$, we may conclude that neither $r^\star_{n+1}$ nor $s^\star_{n+1}$ are in $\Omega^n_{\Sigma,\theta}$. It follows from $\Omega^n_{\Sigma,\theta} \cup \{\omega_{n+1}\} \vdash_{\text{UGSN}} \theta$ that $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \theta$ by **SP7**. However, we also know that $\neg\theta \in \Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta}$, and so $\Omega^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\theta$. Thus by **L5.1**, $\Omega^n_{\Sigma,\theta}$ is inconsistent, contradicting the above. By *reductio*, $\Omega^{n+1}_{\Sigma,\theta}$ is consistent, and so $\Omega_{\Sigma,\theta}$ is consistent by induction. □

It remains to identify a maximal consistent extension of $\Omega_{\Sigma,\theta}$. In particular, consider the following Henkin construction:

φ-Ordering: Let $\texttt{wfs}(\mathcal{L}^+) = \{\varphi_n : n \in \mathbb{N}\}$ fix an ordering.

$$\begin{aligned}
\Gamma^0_{\Sigma,\theta} &= \Omega_{\Sigma,\theta} \\
\Gamma^{n+1}_{\Sigma,\theta} &= \begin{cases} \Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\} & \text{if } \Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\} \text{ is consistent} \\ \Gamma^n_{\Sigma,\theta} \cup \{\neg\varphi_{n+1}\} & \text{otherwise.} \end{cases} \\
\Gamma_{\Sigma,\theta} &= \bigcup_{n\in\mathbb{N}} \Gamma^n_{\Sigma,\theta}.
\end{aligned}$$

We now move to show that $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent and includes $\Sigma \cup \{\neg\theta\}$.

L5.6 For all $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$, if $\Sigma \nvdash_{\text{UGSN}} \theta$, then $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent where $\Sigma \subseteq \Gamma_{\Sigma,\theta}$ but $\theta \notin \Gamma_{\Sigma,\theta}$.

Proof. Let $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$ and assume $\Sigma \nvdash_{\text{UGSN}} \theta$ for discharge. By **L5.5**, $\Omega_{\Sigma,\theta}$ is consistent, and so $\Gamma^0_{\Sigma,\theta}$ is consistent. Assume for induction that $\Gamma^n_{\Sigma,\theta}$ is consistent. If $\Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\}$ is consistent, then $\Gamma^{n+1}_{\Sigma,\theta}$ is consistent. Assume instead that $\Gamma^n_{\Sigma,\theta} \cup \{\varphi_{n+1}\}$ is inconsistent. Thus $\Gamma^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\varphi_{n+1}$ follows by **L5.3**, where $\Gamma^{n+1}_{\Sigma,\theta} = \Gamma_{\Sigma,\theta} \cup \{\neg\varphi_{n+1}\}$. Assume for *reductio* that $\Gamma^{n+1}_{\Sigma,\theta}$ is inconsistent. Again by **L5.3** it follows that $\Gamma^n_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg\neg\varphi_{n+1}$, and so $\Gamma^n_{\Sigma,\theta}$ is inconsistent by **L5.1**. By *reductio*, $\Gamma^{n+1}_{\Sigma,\theta}$ is consistent. Thus $\Gamma_{\Sigma,\theta}$ is consistent by induction.

In order to show that $\Gamma_{\Sigma,\theta}$ is maximal, let $\varphi \in \mathtt{wfs}(\mathcal{L}^+)$ be arbitrary. It follows that $\varphi = \varphi_n$ for some $n \in \mathbb{N}$. Thus either $\Gamma^n_{\Sigma,\theta} = \Gamma^{n-1}_{\Sigma,\theta} \cup \{\varphi_n\}$ or $\Gamma^n_{\Sigma,\theta} = \Gamma^{n-1}_{\Sigma,\theta} \cup \{\neg\varphi_n\}$, and so either $\varphi \in \Gamma^n_{\Sigma,\theta}$ or $\neg\varphi \in \Gamma^n_{\Sigma,\theta}$. Since $\Gamma^n_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$ where φ was arbitrary, $\Gamma_{\Sigma,\theta}$ is maximal.

We next show that $\Gamma_{\Sigma,\theta}$ is saturated. Let $A, B \in \mathtt{pfs}(\mathcal{L}^-)$. It follows that $A \trianglelefteq B = \alpha$ for some $\alpha \in \mathtt{atoms}(\mathcal{L}^+)$. By construction, there is some $\delta \in \Delta_{\Sigma,\theta}$ where $\delta = [(q \trianglelefteq A) \to (q \trianglelefteq B)] \to (A \trianglelefteq B)$ and q is the lowest indexed member of $\mathbb{Q}$ not occurring in A or B. Since $\Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$ and $A, B \in \mathtt{pfs}(\mathcal{L}^-)$ were arbitrary, it follows that $\Gamma_{\Sigma,\theta}$ is saturated.

Lastly, we show that $\Gamma_{\Sigma,\theta}$ is conjunctive. Assume for discharge that $\$A, A \trianglelefteq B \wedge C \in \Gamma_{\Sigma,\theta}$. It follows that $\$A \wedge (A \trianglelefteq B \wedge C) \in \Gamma_{\Sigma,\theta}$ by **L5.2d**, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$A \wedge (A \trianglelefteq B \wedge C)$. We also know that $A \trianglelefteq B \wedge C = \beta_i$

for some $\beta_i \in \mathtt{atoms}^\wedge(\mathcal{L}^+)$. By construction, $\omega_i \in \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$, where:

$$\omega_i = [\$A \wedge (A \trianglelefteq B \wedge C)] \rightarrow [\$r^\star \wedge \$s^\star \wedge (r^\star \trianglelefteq B) \wedge (s^\star \trianglelefteq C) \wedge (r^\star \wedge s^\star \trianglelefteq A)].$$

Thus it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \omega_i$. Given the above, we may then conclude that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$r^\star \wedge \$s^\star \wedge (r^\star \trianglelefteq B) \wedge (s^\star \trianglelefteq C) \wedge (r^\star \wedge s^\star \trianglelefteq A)$. By existentially generalising on $r^\star$ and $s^\star$, there are some $X, Y \in \mathtt{pfs}(\mathcal{L}^+)$ where $\Gamma \vdash_{\text{UGSN}} \$X$, $\Gamma \vdash_{\text{UGSN}} \$Y$, $\Gamma \vdash_{\text{UGSN}} X \trianglelefteq B$, $\Gamma \vdash_{\text{UGSN}} Y \trianglelefteq C$, as well as $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq A$. By **L5.2b**, there are some $X, Y \in \mathtt{pfs}(\mathcal{L}^+)$ where $\$X$, $\$Y$, $X \trianglelefteq B$, $Y \trianglelefteq C$, $X \wedge Y \trianglelefteq A \in \Gamma$. Thus $\Gamma_{\Sigma,\theta}$ is conjunctive.

Having established that $\Gamma_{\Sigma,\theta}$ is maximal, consistent, saturated, and conjunctive, we may conclude that $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent as needed. Moreover, $\Sigma \cup \{\neg\theta\} \subseteq \Gamma_{\Sigma,\theta}$ since $\Sigma \cup \{\neg\theta\} \subseteq \Delta^0_{\Sigma,\theta} \subseteq \Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$. Thus by **L5.2a**, $\theta \notin \Gamma_{\Sigma,\theta}$, and so we may conclude by discharge. □

We are now in a position to construct a Henkin model $\mathcal{M}_\Gamma \in \mathcal{C}^+$, where Γ is any maximal $\trianglelefteq$-consistent subset of $\mathtt{wfs}(\mathcal{L}^+)$. Consider the following:

Γ-*Class:* $[A]_\Gamma = \{X : A \approx X \in \Gamma\}$.

Γ-*States:* $S_\Gamma = \{[A]_\Gamma : \$A \in \Gamma\}$.

Γ-*Fusion:* $[A]_\Gamma \star [B]_\Gamma = [A \wedge B]_\Gamma$.

Γ-*Valuation:* $|p|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq p \in \Gamma\}$ for all $p \in \mathbb{L}^+$.

Γ-*Model:* $\mathcal{M}_\Gamma = \langle S_\Gamma, \star, |\cdot|_\Gamma \rangle$.

Given this construction, I will show that $\mathcal{M}_\Gamma$ is indeed a $\mathcal{S}$-model of $\mathcal{L}^+$, beginning with a few preliminary results.

L5.7 If Γ is maximal $\trianglelefteq$-consistent and $A, B \in \mathtt{pfs}(\mathcal{L}^+)$, then $[A]_\Gamma = [B]_\Gamma$ iff $A \approx B \in \Gamma$.

Proof. Let $A, B \in \mathtt{pfs}(\mathcal{L}^+)$, and assume $[A]_\Gamma = [B]_\Gamma$. By **E1** we know that $\vdash_{\text{UGSN}} B \approx B$, and so $B \approx B \in \Gamma$ by **L5.2b**. Thus $B \in [B]_\Gamma$. Given the identity above, $B \in [A]_\Gamma$, and so $A \approx B \in \Gamma$ by definition.

Assume instead that $A \approx B \in \Gamma$. We may then argue as follows:

$$\begin{aligned} X \in [A]_\Gamma \quad & \textit{iff} \quad A \approx X \in \Gamma \\ & \textit{iff} \quad \Gamma \vdash_{\text{UGSN}} A \approx X \\ (*) \quad & \textit{iff} \quad \Gamma \vdash_{\text{UGSN}} B \approx X \\ & \textit{iff} \quad B \approx X \in \Gamma \\ & \textit{iff} \quad X \in [B]_\Gamma. \end{aligned}$$

The identities above all hold by definition and **L5.2b**, with the exception of $(*)$ which follows from **GA9**. Thus $[A]_\Gamma = [B]_\Gamma$. Given the above, we may conclude that $[A]_\Gamma = [B]_\Gamma$ just in case $A \approx B \in \Gamma$. □

L5.8 If Γ is maximal $\trianglelefteq$-consistent, then $x \star y \in S_\Gamma$ for all $x, y \in S_\Gamma$.

Proof. Assume that Γ is maximal $\trianglelefteq$-consistent. In order to show that $\star$ is well-defined over S_Γ, let $a_1, a_2, b_1, b_2 \in S_\Gamma$, and assume that both $a_1 = a_2$ and $b_1 = b_2$. By definition, $a_1 = [A_1]_\Gamma$, $a_2 = [A_2]_\Gamma$, $b_1 = [B_1]_\Gamma$, and $b_2 = [B_2]_\Gamma$ for some $A_1, A_2, B_1, B_2 \in \mathtt{pfs}(\mathcal{L}^+)$. Thus it follows that $[A_1]_\Gamma = [A_2]_\Gamma$ and $[B_1]_\Gamma = [B_2]_\Gamma$, and so $A_1 \approx A_2 \in \Gamma$ and $B_1 \approx B_2 \in \Gamma$ by **L5.7**. By **GA8**, we know that $\Gamma \vdash_{\text{UGSN}} A_1 \wedge B_1 \approx A_2 \wedge B_2$, and so $A_1 \wedge B_1 \approx A_2 \wedge B_2 \in \Gamma$ by **L5.2b**. Again by **L5.7**, we know that $[A_1 \wedge B_1]_\Gamma = [A_2 \wedge B_2]_\Gamma$, and so $[A_1]_\Gamma \star [B_1]_\Gamma = [A_2]_\Gamma \star [B_2]_\Gamma$ by definition. Thus $a_1 \star b_1 = a_2 \star b_2$, and so $\star$ is well-defined.

Let $x, y \in S_\Gamma$ by arbitrary. It follows that $x = [A]_\Gamma$ where $\$A \in \Gamma$, and similarly $y = [B]_\Gamma$ where $\$B \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} \$(A \wedge B)$ follows from **SP3**, and so $\$(A \wedge B) \in \Gamma$ by **L5.2b**. We may then conclude that $[A \wedge B]_\Gamma \in S_\Gamma$. However, $[A \wedge B]_\Gamma = [A]_\Gamma \star [B]_\Gamma$, and so $x \star y \in S_\Gamma$. Thus we may conclude that $x \star y \in S_\Gamma$ for all $x, y \in S_\Gamma$. □

L5.9 If Γ is maximal $\trianglelefteq$-consistent, then $\mathcal{M}_\Gamma \in \mathcal{C}^+$.

Proof. Assume Γ is maximal $\trianglelefteq$-consistent. It follows by **L5.8** that $\langle S_\Gamma, \star \rangle$ is a state space. We must show that $\langle S_\Gamma, \star \rangle \in \mathbb{M}$ by satisfying *Null State*, *Full State*, *Idempotency*, *Commutativity*, and *Associativity*.

Observe that $\vdash_{\text{UGSN}} \$\bot$ and $\vdash_{\text{UGSN}} \$\mathcal{V}$ follow from **VF1** and **VF2**, respectively. Thus $\Gamma \vdash_{\text{UGSN}} \$\bot$ and $\Gamma \vdash_{\text{UGSN}} \$\mathcal{V}$, and so $\$\bot, \$\mathcal{V} \in \Gamma$ by **L5.2b**. By Γ-*States*, both $[\bot]_\Gamma, [\mathcal{V}]_\Gamma \in S_\Gamma$. Letting $x \in S_\Gamma$, it follows by Γ-*States* that $x = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^+)$ where $\$A \in \Gamma$, and so:

$$\begin{aligned} [\bot]_\Gamma \star x &= [\bot]_\Gamma \star [A]_\Gamma \\ &= [\bot \wedge A]_\Gamma \\ (\dagger) \quad &= [A]_\Gamma \\ &= x. \end{aligned} \qquad \begin{aligned} [\mathcal{V}]_\Gamma \star x &= [\mathcal{V}]_\Gamma \star [A]_\Gamma \\ &= [\mathcal{V} \wedge A]_\Gamma \\ (\ddagger) \quad &= [\mathcal{V}]_\Gamma. \end{aligned}$$

The identities above hold by definition or assumption with the exception of (†) and (‡) which follow by **L5.7** from **E4** and **E6**, respectively. Since $x \in S_\Gamma$ was arbitrary, we may conclude that there is some $\boxdot \in S_\Gamma$ where $\boxdot \star x = x$ for all $x \in S_\Gamma$, and some $\blacksquare \in S_\Gamma$ where $\blacksquare \star x = \blacksquare$ for all $x \in S_\Gamma$. Thus $\langle S_\Gamma, \star \rangle$ satisfies both *Null State* and *Full State*.

Letting $x \in S_\Gamma$. It follows that $x = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^+)$. Recall that $\Gamma \vdash_{\text{UGSN}} A \wedge A \approx A$ by **E4**, and so $A \wedge A \approx A \in \Gamma$ by **L5.2b**. Thus $[A \wedge A]_\Gamma = [A]_\Gamma$ by **L5.7**, and so $[A]_\Gamma \star [A]_\Gamma = [A]_\Gamma$. Equivalently, $x \star x = x$. Since $x \in S_\Gamma$ was arbitrary, $\langle S_\Gamma, \star \rangle$ satisfies *Idempotency*.

Let $x, y \in S_\Gamma$. It follows that $x = [A]_\Gamma$ and $y = [B]_\Gamma$ for some $A, B \in \mathtt{pfs}(\mathcal{L}^+)$. Recall that $\Gamma \vdash_{\text{UGSN}} A \wedge B \approx B \wedge A$ by **E10**, and so $A \wedge B \approx B \wedge A \in \Gamma$ by **L5.2b**. Thus $[A \wedge B]_\Gamma = [B \wedge A]_\Gamma$ by **L5.7**. By definition, $[A]_\Gamma \star [B]_\Gamma = [B]_\Gamma \star [A]_\Gamma$, and so $x \star y = y \star x$. Since $x, y \in S_\Gamma$ were arbitrary, it follows that $\langle S_\Gamma, \star \rangle$ satisfies *Commutativity*.

Let $x, y, z \in S_\Gamma$. Thus $x = [A]_\Gamma$, $y = [B]_\Gamma$, and $z = [C]_\Gamma$ for some $A, B, C \in \mathtt{pfs}(\mathcal{L}^+)$. Recall that $\Gamma \vdash_{\text{UGSN}} A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$ by **E12**, and so $A \wedge (B \wedge C) \approx (A \wedge B) \wedge C \in \Gamma$ by **L5.2b**. Thus

$[A \wedge (B \wedge C)]_\Gamma = [(A \wedge B) \wedge C]_\Gamma$ by **L5.7**. Consider the following:

$$\begin{aligned} x \star (y \star z) &= [A]_\Gamma \star ([B]_\Gamma \star [C]_\Gamma) \\ &= [A \wedge (B \wedge C)]_\Gamma \\ (*) &= [(A \wedge B) \wedge C]_\Gamma \\ &= ([A]_\Gamma \star [B]_\Gamma) \star [C]_\Gamma. \\ &= (x \star y) \star z. \end{aligned}$$

The identities above all follow by definition or assumption with the exception of $(*)$ which was already established. Since $x, y, z \in S_\Gamma$ were arbitrary, $\langle S_\Gamma, \star \rangle$ satisfies *Associativity* as desired.

Given the results above, $\langle S_\Gamma, \star \rangle \in \mathbb{M}$. We now show that $|p|_\Gamma \in \mathbb{P}_{\mathcal{S}_\Gamma}$ for all $p \in \mathbb{L}^+$. Letting $p \in \mathbb{L}^+$, we know that $|p|_\Gamma \subseteq S_\Gamma$, and so $|p|_\Gamma \subseteq \overline{|p|_\Gamma}$. In order to establish the converse inclusion, let $s \in \overline{|p|_\Gamma}$. It follows that $s = x \star y$ for some $x, y \in |p|_\Gamma$, and so $x = [X]_\Gamma$ and $y = [Y]_\Gamma$ for some $X, Y \in \mathtt{pfs}(\mathcal{L}^+)$ where both $\$X, \$Y \in \Gamma$ and $X \trianglelefteq p, Y \trianglelefteq p \in \Gamma$. By **SP3**, $\Gamma \vdash_{\text{UGSN}} \$(X \wedge Y)$, and so $\$(X \wedge Y) \in \Gamma$ by **L5.2b**. Since we also know that $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq p \wedge p$ by **GA8**, it follows that $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq p$ by **GA4**. Again by **L5.2b**, $X \wedge Y \trianglelefteq p \in \Gamma$, and so $[X \wedge Y]_\Gamma \in |p|_\Gamma$ by Γ-*Valuation*. By definition, $[X \wedge Y]_\Gamma = [X]_\Gamma \star [Y]_\Gamma$. It follows that $x \star y \in |p|_\Gamma$, and so $s \in |p|_\Gamma$. Since $s \in \overline{|p|_\Gamma}$ was arbitrary, $\overline{|p|_\Gamma} \subseteq |p|_\Gamma$. Given the above, $|p|_\Gamma = \overline{|p|_\Gamma}$, and so $|p|_\Gamma \in \mathbb{P}_{\mathcal{S}_\Gamma}$ where $\mathcal{S}_\Gamma = \langle S_\Gamma, \star \rangle$. Since $p \in \mathbb{L}^+$ was arbitrary and $\langle S_\Gamma, \star \rangle \in \mathbb{M}$, and so $\mathcal{M}_\Gamma \in \mathcal{C}^+$. □

L5.10 If Γ is maximal $\trianglelefteq$-consistent, then for all $B \in \mathtt{pfs}(\mathcal{L}^+)$ $|B|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq B \in \Gamma\}$.

Proof. Assuming that Γ is maximal $\trianglelefteq$-consistent, the proof proceeds by induction on complexity. Assume to start that $B \in \mathtt{pfs}(\mathcal{L}^+)$ where $\mathtt{comp}(B) = 0$. It follows that either: (1) $B = \bot$; (2) $B = \bot$; (3) $B = \mathcal{T}$; (4) $B = \mathcal{V}$; or (5) $B = p$ for some $p \in \mathbb{L}^+$. Since (5) is given immediately by Γ-*Valuation*, we may restrict consideration to (1) – (4).

Case 1: Assume $B = \bot\!\!\!\bot$. We know that $|\bot\!\!\!\bot|_\Gamma = \{\square\}$, where $\vdash_{\text{UGSN}} \$\bot\!\!\!\bot$ by **VF1** and $\vdash_{\text{UGSN}} \bot\!\!\!\bot \trianglelefteq \bot\!\!\!\bot$ by **E1**, and so both $\$\bot\!\!\!\bot \in \Gamma$ and $\bot\!\!\!\bot \trianglelefteq \bot\!\!\!\bot \in \Gamma$ by **L5.2b**. Since $\square = [\bot\!\!\!\bot]_\Gamma$, it follows that $\square \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot\!\!\!\bot \in \Gamma\}$. Let $x \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot\!\!\!\bot \in \Gamma\}$. By definition, $x = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^+)$ where both $\$A \in \Gamma$ and $A \trianglelefteq \bot\!\!\!\bot \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} A \not\trianglelefteq \bot$ and $\Gamma \vdash_{\text{UGSN}} (\bot\!\!\!\bot \trianglelefteq A) \vee (A \trianglelefteq \bot)$ follow by **SP1** and **SP4**, respectively. It follows by **L5.2** that $A \trianglelefteq \bot \notin \Gamma$ and either $\bot\!\!\!\bot \trianglelefteq A \in \Gamma$ or $A \trianglelefteq \bot \in \Gamma$, and so $\bot\!\!\!\bot \trianglelefteq A \in \Gamma$. Having already shown that $A \trianglelefteq \bot\!\!\!\bot \in \Gamma$, it follows by **L5.2b** that $A \approx \bot\!\!\!\bot \in \Gamma$, and so $[A]_\Gamma = [\bot\!\!\!\bot]_\Gamma$ by **L5.7**. Thus $x = \square$, and so $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot\!\!\!\bot \in \Gamma\} = \{\square\}$ since x was arbitrary. Given the above, $|\bot\!\!\!\bot|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot\!\!\!\bot \in \Gamma\}$ as desired.

Case 2: Assuming $B = \bot$, it follows that $|\bot|_\Gamma = \varnothing$. Assume for *reducto* that there is some $x \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot \in \Gamma\}$. Thus $x = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^+)$ where $\$A \in \Gamma$ and $A \trianglelefteq \bot \in \Gamma$. It follows that $\Gamma \vdash_{\text{UGSN}} A \not\trianglelefteq \bot$ by **SP1**, and so $A \trianglelefteq \bot \notin \Gamma$ by **L5.2**. Thus $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot \in \Gamma\} = \varnothing$ by *reductio*, and so we may conclude that $|\bot|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \bot \in \Gamma\}$.

Case 3: Assume $B = \mathcal{T}$. Thus $|\mathcal{T}|_\Gamma = S_\Gamma$. Letting $s \in S_\Gamma$, it follows that $s = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^+)$ where $\$A \in \Gamma$. By **VF7**, $\vdash_{\text{UGSN}} A \trianglelefteq \mathcal{T}$, and so $A \trianglelefteq \mathcal{T} \in \Gamma$ by **L5.2b**. Thus $s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{T} \in \Gamma\}$. Since $s \in S_\Gamma$ was arbitrary, $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{T} \in \Gamma\} = S_\Gamma$. It follows that $|\mathcal{T}|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{T} \in \Gamma\}$ by the above.

Case 4: Assuming $B = \mathcal{V}$, it follows that $|\mathcal{V}|_\Gamma = \{\blacksquare\}$, where $\vdash_{\text{UGSN}} \$\mathcal{V}$ by **VF1** and $\vdash_{\text{UGSN}} \mathcal{V} \trianglelefteq \mathcal{V}$ by **E1**, and so both $\$\mathcal{V} \in \Gamma$ and $\mathcal{V} \trianglelefteq \mathcal{V} \in \Gamma$ by **L5.2b**. Since $\blacksquare = [\mathcal{V}]_\Gamma$, we know that $\blacksquare \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\}$. Let $x \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\}$. By definition, $x = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^+)$ where both $\$A \in \Gamma$ and $A \trianglelefteq \mathcal{V} \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} A \not\trianglelefteq \bot$ and $\Gamma \vdash_{\text{UGSN}} (\mathcal{V} \trianglelefteq A) \vee (A \trianglelefteq \bot)$ follow by **SP1** and **SP4**, respectively. It follows by **L5.2** that $A \trianglelefteq \bot \notin \Gamma$ and either $\mathcal{V} \trianglelefteq A \in \Gamma$ or $A \trianglelefteq \bot \in \Gamma$, and so $\mathcal{V} \trianglelefteq A \in \Gamma$. Having already shown that $A \trianglelefteq \mathcal{V} \in \Gamma$, it follows by **L5.2b** that $A \approx \mathcal{V} \in \Gamma$, and so $[A]_\Gamma = [\mathcal{V}]_\Gamma$ by **L5.7**. Thus $x = \blacksquare$, and so $\{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\} = \{\blacksquare\}$ since x was arbitrary. Given the

above, $|\mathcal{V}|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq \mathcal{V} \in \Gamma\}$.

Having established the base cases, we may assume for induction that $|B|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq B \in \Gamma\}$ for all $B \in \mathtt{pfs}(\mathcal{L}^+)$ such that $\mathtt{comp}(B) < n$. Letting $B \in \mathtt{pfs}(\mathcal{L}^+)$ be such that $\mathtt{comp}(B) = n$, it follows that either $B = C \wedge D$ or $B = C \vee D$. Consider the following:

Case 1: Assume $B = C \wedge D$. Since both $\mathtt{comp}(C), \mathtt{comp}(D) < n$, we know by hypothesis that $|C|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \in \Gamma\}$ and $|D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq D \in \Gamma\}$. Consider the following:

$$
\begin{aligned}
s \in |C \wedge D|_\Gamma \quad & \textit{iff} \quad \mathcal{M}_\Gamma, s \Vdash C \wedge D \\
& \textit{iff} \quad s = c \star d \text{ for some } c, d \text{ where } \mathcal{M}_\Gamma, c \Vdash C \text{ and } \mathcal{M}_\Gamma, d \Vdash D \\
& \textit{iff} \quad s = c \star d \text{ for some } c \in |C|_\Gamma \text{ and } d \in |D|_\Gamma \\
& \textit{iff} \quad s = c \star d \text{ for some } c \in \{[X]_\Gamma : \$X \in \Gamma \text{ and } X \trianglelefteq C \in \Gamma\} \\
& \qquad\qquad \text{and } d \in \{[Y]_\Gamma : \$Y \in \Gamma \text{ and } Y \trianglelefteq D \in \Gamma\} \\
& \textit{iff} \quad s = [X]_\Gamma \star [Y]_\Gamma \text{ for some } X, Y \in \mathtt{pfs}(\mathcal{L}^+) \text{ such that } \$X, \$Y \in \Gamma \\
& \qquad\qquad \text{where both } X \trianglelefteq C, Y \trianglelefteq D \in \Gamma \\
(\dagger) \quad & \textit{iff} \quad s = [X \wedge Y]_\Gamma \text{ such that } \$(X \wedge Y) \in \Gamma \text{ where } X \wedge Y \trianglelefteq C \wedge D \in \Gamma \\
(\ddagger) \quad & \textit{iff} \quad s \in \{[Z]_\Gamma : \$Z \in \Gamma \text{ and } Z \trianglelefteq C \wedge D \in \Gamma\}. \qquad [\wedge]
\end{aligned}
$$

The biconditionals above hold by definition or assumption with the exception of (†) and (‡). Starting with (†), assume that $s = [X]_\Gamma \star [Y]_\Gamma$ for some X and Y such that $\$X, \$Y \in \Gamma$ where $X \trianglelefteq C \in \Gamma$ and $Y \trianglelefteq D \in \Gamma$. It follows by **SP3** that $\Gamma \vdash_{\text{UGSN}} \$(X \wedge Y)$, and so $\$(X \wedge Y) \in \Gamma$ by **L5.2b**. We also know that $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq C \wedge D$ by **GA8**, and so $X \wedge Y \trianglelefteq C \wedge D \in \Gamma$ by **L5.2b**. Additionally, it follows that $s = [X \wedge Y]_\Gamma$ by Γ-*Fusion*. Altogether, $s = [X \wedge Y]_\Gamma$ such that $\$(X \wedge Y) \in \Gamma$ where $X \wedge Y \trianglelefteq C \wedge D \in \Gamma$. Existentially, generalising on $X \wedge Y$, we may conclude that there is some $Z \in \mathtt{pfs}(\mathcal{L}^+)$ such that $s = [Z]_\Gamma$ where $\$Z \in \Gamma$ and $Z \trianglelefteq C \wedge D \in \Gamma$, and so $s \in \{[Z]_\Gamma : \$Z \in \Gamma \text{ and } Z \trianglelefteq C \wedge D \in \Gamma\}$.

Assume instead that $s \in \{[Z]_\Gamma : \$Z \in \Gamma \text{ and } Z \trianglelefteq C \wedge D \in \Gamma\}$, and so for some $Z \in \mathtt{pfs}(\mathcal{L}^+)$, $s = [Z]_\Gamma$ where $\$Z \in \Gamma$ and $Z \trianglelefteq C \wedge D \in \Gamma$.

Since Γ is $\trianglelefteq$-consistent, we know Γ is conjunctive, and so there are some $X, Y \in \mathtt{pfs}(\mathcal{L}^+)$ where $\$X, \$Y, X \trianglelefteq C, Y \trianglelefteq D, X \wedge Y \trianglelefteq Z \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} X \wedge Y \trianglelefteq C \wedge D$ by **GA8**, and $\Gamma \vdash_{\text{UGSN}} \$(X \wedge Y)$ by **SP3**. We may then conclude that $\Gamma \vdash_{\text{UGSN}} (Z \trianglelefteq X \wedge Y) \vee (X \wedge Y \trianglelefteq \bot)$ by **SP4**, and so either $Z \trianglelefteq X \wedge Y \in \Gamma$ or $X \wedge Y \trianglelefteq \bot \in \Gamma$ by **L5.2**. We also know that $\$(X \wedge Y) \vdash_{\text{UGSN}} X \wedge Y \not\trianglelefteq \bot$ by **SP1**, and so $\Gamma \vdash_{\text{UGSN}} X \wedge Y \not\trianglelefteq \bot$. Thus $X \wedge Y \trianglelefteq \bot \notin \Gamma$ by **L5.2**, and so $Z \trianglelefteq X \wedge Y \in \Gamma$ given the above. Given that $X \wedge Y \trianglelefteq Z \in \Gamma$, it follows that $\Gamma \vdash_{\text{UGSN}} Z \approx X \wedge Y$, and so $Z \approx X \wedge Y \in \Gamma$ by **L5.2b**. Thus $[Z]_\Gamma = [X]_\Gamma \star [Y]_\Gamma$ by **L5.7**, and so $s = [X]_\Gamma \star [Y]_\Gamma$ for some X and Y where $\$X, \$Y \in \Gamma$ and $X \trianglelefteq C, Y \trianglelefteq D \in \Gamma$ since $s = [Z]_\Gamma$. Together with the forward direction, it follows that $|C \wedge D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \wedge D \in \Gamma\}$.

Case 2: Assume $B = C \vee D$. Since both $\mathtt{comp}(C), \mathtt{comp}(D) < n$, we know by hypothesis that $|C|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \in \Gamma\}$ and $|D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq D \in \Gamma\}$. Consider the following:

$$
\begin{aligned}
s \in |C \vee D|_\Gamma \quad & \textit{iff} \quad \mathcal{M}_\Gamma, s \Vdash C \vee D \\
& \textit{iff} \quad \mathcal{M}_\Gamma, s \Vdash C, \text{ or } \mathcal{M}_\Gamma, s \Vdash D, \text{ or } \mathcal{M}_\Gamma, s \Vdash C \wedge D \\
& \textit{iff} \quad s \in |C|_\Gamma, \text{ or } s \in |D|_\Gamma, \text{ or } s \in |C \wedge D|_\Gamma \\
& \textit{iff} \quad s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \in \Gamma\} \text{ or } s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq D \in \Gamma\} \\
& \qquad\qquad \text{or } s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \wedge D \in \Gamma\} \\
& \textit{iff} \quad s = [A]_\Gamma \text{ for some } A \in \mathtt{pfs}(\mathcal{L}^+) \text{ where } \$A \in \Gamma \text{ and} \\
& \qquad\qquad \text{and either } A \trianglelefteq C \in \Gamma \text{ or } A \trianglelefteq D \in \Gamma \text{ or } A \trianglelefteq C \wedge D \in \Gamma \\
(*) \quad & \textit{iff} \quad s = [A]_\Gamma \text{ for some } A \in \mathtt{pfs}(\mathcal{L}^+) \text{ where } \$A \in \Gamma \text{ and } A \trianglelefteq C \vee D \in \Gamma \\
& \textit{iff} \quad s \in \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \vee D \in \Gamma\}. \qquad [\vee]
\end{aligned}
$$

All of the biconditionals above follow by assumption or definition, with the exception of $(*)$. For the forward direction, let $s = [A]_\Gamma$ for some $A \in \mathtt{pfs}(\mathcal{L}^-)$ where $\$A \in \Gamma$ and either $A \trianglelefteq C \in \Gamma$ or $A \trianglelefteq D \in \Gamma$ or $A \trianglelefteq C \wedge D \in \Gamma$. Given **GA1**, **GA2**, and **T5**, we know by **L5.2b** that $C \trianglelefteq C \vee D \in \Gamma$, $D \trianglelefteq C \vee D \in \Gamma$ and $C \wedge D \trianglelefteq C \vee D \in \Gamma$. Thus it follows

by **GA9** that $\Gamma \vdash_{\text{UGSN}} A \trianglelefteq C \vee D$ in each case, and so $A \trianglelefteq C \vee D \in \Gamma$ by **L5.2b**. Together with the above, $s = [A]_\Gamma$ for some A where $\$A \in \Gamma$ and $A \trianglelefteq C \vee D \in \Gamma$, thereby establishing the forward direction.

Assume instead that $s = [A]_\Gamma$ for some A where both $\$A \in \Gamma$ and $A \trianglelefteq C \vee D \in \Gamma$. Thus $\Gamma \vdash_{\text{UGSN}} (A \trianglelefteq C) \vee (A \trianglelefteq D) \vee (A \trianglelefteq C \wedge D)$ follows by **SP5**, and so either $A \trianglelefteq C \in \Gamma$, $A \trianglelefteq D \in \Gamma$, or $A \trianglelefteq C \wedge D \in \Gamma$ by **L5.2**. Thus it follows that $s = [A]_\Gamma$ for some A where $\$A \in \Gamma$ and either $A \trianglelefteq C \in \Gamma$, $A \trianglelefteq D \in \Gamma$, or $A \trianglelefteq C \wedge D \in \Gamma$. Together with the above, it follows that $|C \vee D|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq C \vee D \in \Gamma\}$.

Given the cases proven above, it follows by induction on complexity that $|B|_\Gamma = \{[A]_\Gamma : \$A \in \Gamma \text{ and } A \trianglelefteq B \in \Gamma\}$ for all $B \in \mathtt{pfs}(\mathcal{L}^+)$. □

It remains to show that every model $\mathcal{M} \in \mathcal{C}^+$ of the expanded language $\mathcal{L}^+$ has a *reduct* $\mathcal{M}^\mathfrak{R} \in \mathcal{C}$ such that $\mathcal{M}^\mathfrak{R} \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$. Given any $\mathcal{S}$-model $\mathcal{M} = \langle S, \star, |\cdot| \rangle \in \mathcal{C}^+$, consider the following:

Restriction: Let $|\cdot|^\mathfrak{R} : \mathbb{L} \to \mathbb{P}_\mathcal{S}$ where $|p|^\mathfrak{R} = |p|$ for all $p \in \mathbb{L}$.

$\mathfrak{R}$*-Map:* Let $\mathcal{M}^\mathfrak{R} = \langle S, \star, |\cdot|^\mathfrak{R} \rangle$ where $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle \in \mathcal{C}^+$.

The following lemma proves that $\mathcal{M}^\mathfrak{R} \in \mathcal{C}$ for any $\mathcal{M} \in \mathcal{C}^+$, where $\mathcal{M}^\mathfrak{R}$ makes the same wfs of $\mathcal{L}^-$ true as $\mathcal{M}$.

L5.11 If $\mathcal{M} \in \mathcal{C}^+$, then $\mathcal{M}^\mathfrak{R} \in \mathcal{C}$ where $\mathcal{M}^\mathfrak{R} \vDash \varphi$ iff $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

Proof. Let $\mathcal{M} \in \mathcal{C}^+$. By definition, $\mathcal{M} = \langle S, \star, |\cdot| \rangle$ where $\langle S, \star \rangle \in \mathbb{M}$ and $|p_i|, |q_i|, |r_i|, |s_i| \in \mathbb{P}_\mathcal{S}$ for all $i \in \mathbb{N}$. Thus $\mathcal{M}^\mathfrak{R} = \langle S, \star, |\cdot| \rangle$ where $\langle S, \star \rangle \in \mathbb{M}$ and $|p_i| \in \mathbb{P}_\mathcal{S}$ for all $i \in \mathbb{N}$, and so $\mathcal{M}^\mathfrak{R} \in \mathcal{C}$.

By construction, $|p_i|^\mathfrak{R} = |p_i|$ for all $p_i \in \mathbb{L}$. Thus it follows by two routine induction proofs that: (1) $|A|^\mathfrak{R} = |A|$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$; and (2) $\mathcal{M}^\mathfrak{R} \vDash \varphi$ just in case $\mathcal{M} \vDash \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$ as desired. □

L5.12 $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$ for all $A, B \in \mathtt{pfs}(\mathcal{L}^+)$.

Proof. Let $A, B \in \mathtt{pfs}(\mathcal{L}^+)$, and assume $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$. Let $s \in |A|_{\Gamma_{\Sigma,\theta}}$. By **L5.10**, $s = [X]_{\Gamma_{\Sigma,\theta}}$ where $\$X \in \Gamma_{\Sigma,\theta}$ and $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$, and so

$\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} X \trianglelefteq B$ and $X \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ by **GA9** and **L5.2b**, respectively. Given that $\$X \in \Gamma_{\Sigma,\theta}$, it follows that $[X]_{\Gamma_{\Sigma,\theta}} \in |B|_{\Gamma_{\Sigma,\theta}}$ by **L5.10**, and so $s \in |B|_{\Gamma_{\Sigma,\theta}}$. Thus $|A|_{\Gamma_{\Sigma,\theta}} \subseteq |B|_{\Gamma_{\Sigma,\theta}}$, and so $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$. It follows that if $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$, then $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$.

Assume instead that $A \trianglelefteq B \notin \Gamma_{\Sigma,\theta}$. It follows that $A \ntrianglelefteq B \in \Gamma_{\Sigma,\theta}$ and $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} A \ntrianglelefteq B$ by **L5.2a** and **L5.2b**, respectively. We also know that $A \trianglelefteq B \in \mathtt{atoms}(\mathcal{L}^-)$, and so $A \trianglelefteq B = \alpha_i$ for some $i \in \mathbb{N}$. By construction, $\omega_i, \$q_i \in \Delta^i_{\Sigma,\theta}$ where $\omega_i = [(q_i \trianglelefteq A) \to (q_i \trianglelefteq B)] \to (A \trianglelefteq B)$. Since $\Delta^i_{\Sigma,\theta} \subseteq \Delta_{\Sigma,\theta} \subseteq \Omega_{\Sigma,\theta} \subseteq \Gamma_{\Sigma,\theta}$, it follows that $\omega_i, \$q_i \in \Gamma_{\Sigma,\theta}$. We may then conclude that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} [(q_i \trianglelefteq A) \to (q_i \trianglelefteq B)] \to (A \trianglelefteq B)$, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \neg[(q_i \trianglelefteq A) \to (q_i \trianglelefteq B)]$ since $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} A \ntrianglelefteq B$. Thus $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} (q_i \trianglelefteq A) \wedge (q_i \ntrianglelefteq B)$, and so $q_i \trianglelefteq A \in \Gamma_{\Sigma,\theta}$ and $q_i \trianglelefteq B \notin \Gamma_{\Sigma,\theta}$ by **L5.2**. Since $\$q_i \in \Gamma_{\Sigma,\theta}$ by construction, we know that $[q_i]_{\Gamma_{\Sigma,\theta}} \in S_{\Gamma_{\Sigma,\theta}}$. Given that $q_i \trianglelefteq A \in \Gamma_{\Sigma,\theta}$ but $q_i \trianglelefteq B \notin \Gamma_{\Sigma,\theta}$, it follows by **L5.10** that $[q_i]_{\Gamma_{\Sigma,\theta}} \in |A|_{\Gamma_{\Sigma,\theta}}$ but $[q_i]_{\Gamma_{\Sigma,\theta}} \notin |B|_{\Gamma_{\Sigma,\theta}}$, and so $|A|_{\Gamma_{\Sigma,\theta}} \not\subseteq |B|_{\Gamma_{\Sigma,\theta}}$. Thus $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \nvDash A \trianglelefteq B$, and so if $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$, then $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$. We may then conclude that $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash A \trianglelefteq B$. □

L5.13 $\$A \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$ for all $A \in \mathtt{pfs}(\mathcal{L}^+)$.

Proof. Let $A \in \mathtt{pfs}(\mathcal{L}^+)$, and assume $\$A \in \Gamma_{\Sigma,\theta}$. By **E1**, $\vdash_{\text{UGSN}} A \trianglelefteq A$, and so $A \trianglelefteq A \in \Gamma_{\Sigma,\theta}$ by **L5.2b**. Thus $[A]_{\Gamma_{\Sigma,\theta}} \in |A|_{\Gamma_{\Sigma,\theta}}$, and so it follows that $[A]_{\Gamma_{\Sigma,\theta}} \in |A|_{\Gamma_{\Sigma,\theta}}$ by **L5.10**. Now assume $x \in |A|_{\Gamma_{\Sigma,\theta}}$. By **L5.10**, there is some $X \in \mathtt{pfs}(\mathcal{L}^+)$ such that $x = [X]_{\Gamma_{\Sigma,\theta}}$ where $\$X \in \Gamma_{\Sigma,\theta}$ and $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$. By **SP4**, $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} (A \trianglelefteq X) \vee (X \trianglelefteq \bot)$, and so either $A \trianglelefteq X \in \Gamma_{\Sigma,\theta}$ or $X \trianglelefteq \bot \in \Gamma_{\Sigma,\theta}$ by **L5.2**. Since $\$X \in \Gamma_{\Sigma,\theta}$, it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} X \ntrianglelefteq \bot$ by **SP1**, and so $X \trianglelefteq \bot \notin \Gamma_{\Sigma,\theta}$ by **L5.2**. Thus $A \trianglelefteq X \in \Gamma_{\Sigma,\theta}$ given the disjunction above. Having already shown that $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$, it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} X \approx A$, and so $X \approx A \in \Gamma_{\Sigma,\theta}$ by **L5.2b**. Thus $[X]_{\Gamma_{\Sigma,\theta}} = [A]_{\Gamma_{\Sigma,\theta}}$ by **L5.7**, and so $x = [A]_{\Gamma_{\Sigma,\theta}}$. Since x was arbitrary, $|A|_{\Gamma_{\Sigma,\theta}} = \{[A]_{\Gamma_{\Sigma,\theta}}\}$, and so $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$. We may then conclude that if $\$A \in \Gamma_{\Sigma,\theta}$, then $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$.

Assuming instead that $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \vDash \$A$, we know that $|A|_{\Gamma_{\Sigma,\theta}} = \{x\}$

for some $x \in S_{\Gamma_{\Sigma,\theta}}$. By **L5.10**, there is some $X \in \mathtt{pfs}(\mathcal{L}^+)$ such that $x = [X]_{\Gamma_{\Sigma,\theta}}$ where both $\$X \in \Gamma_{\Sigma,\theta}$ and $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$. Since $\vdash_{\text{UGSN}} X \trianglelefteq X$ by **E1**, it follows that $X \trianglelefteq X \in \Gamma$ by **L5.2b**, and so $x \in |X|_{\Gamma_{\Sigma,\theta}}$ by **L5.10**. Assume for *reductio* that $A \trianglelefteq X \notin \Gamma_{\Sigma,\theta}$. It follows by **L5.12** that $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \not\models A \trianglelefteq X$, and so $|A|_{\Gamma_{\Sigma,\theta}} \nsubseteq |X|_{\Gamma_{\Sigma,\theta}}$. Thus some $y \in |A|_{\Gamma_{\Sigma,\theta}}$ where $y \notin |X|_{\Gamma_{\Sigma,\theta}}$. Given that $|A|_{\Gamma_{\Sigma,\theta}} = \{x\}$, we know that $y = x$, and so $x \notin |X|_{\Gamma_{\Sigma,\theta}}$, contradicting the above. By *reductio*, $A \trianglelefteq X \in \Gamma_{\Sigma,\theta}$, and so $\Gamma_{\Sigma,\theta} \vdash_{\text{UGS}} X \approx A$ given that $X \trianglelefteq A \in \Gamma_{\Sigma,\theta}$. Since $X \approx A \vdash_{\text{UGSN}} \$X \equiv \$A$ by **SP2**, we know $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$X \equiv \$A$. Given that $\$X \in \Gamma_{\Sigma,\theta}$, it follows that $\Gamma_{\Sigma,\theta} \vdash_{\text{UGSN}} \$A$, and so $\$A \in \Gamma_{\Sigma,\theta}$ by **L5.2b**. Thus if $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \$A$, then $\$A \in \Gamma_{\Sigma,\theta}$, and so $\$A \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \$A$. □

P5.1 If $\Sigma \nvdash_{\text{UGSN}} \theta$ for $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$, then $\chi \in \Gamma_{\Sigma,\theta}$ iff $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \chi$ for all $\chi \in \mathtt{wfs}(\mathcal{L}^+)$.

Proof. Let $\Sigma \cup \{\theta\} \subseteq \mathtt{wfs}(\mathcal{L}^-)$ and assume $\Sigma \nvdash_{\text{UGSN}} \theta$. Thus $\Gamma_{\Sigma,\theta}$ is maximal $\trianglelefteq$-consistent. Assuming $\mathtt{comp}^+(\chi) = 0$, either $\chi = A \trianglelefteq B$ or $\chi = \$A$ for some $A, B \in \mathtt{pfs}(\mathcal{L}^+)$. If $\chi = A \trianglelefteq B$, then $A \trianglelefteq B \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models A \trianglelefteq B$ by **L5.12**, and so $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \chi$. If instead $\chi = \$A$, then $\$A \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \$A$ by **L5.13**, and so $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \chi$. Thus $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \chi$ for any $\chi \in \mathtt{wfs}(\mathcal{L}^+)$ where $\mathtt{comp}^+(\chi) = 0$. By induction we may show for all $\chi \in \mathtt{wfs}(\mathcal{L}^+)$ that $\chi \in \Gamma_{\Sigma,\theta}$ just in case $\mathcal{M}_{\Gamma_{\Sigma,\theta}} \models \chi$. □

5.6 Infinite Fusion

Given any mereological state space $\langle S, \star \rangle \in \mathbb{M}$ and nonempty finite $X \subseteq S$, there is a unique fusion of the states which belong to X. However, if S is infinite, we may also consider the fusion of infinite subsets of $X \subseteq S$. Accordingly, I will restrict attention to the class of state spaces which is closed under both finite and infinite fusion. More specifically, let an *infinite state space* be any ordered pair $\langle S, \bigsqcup \rangle$ where S is a set closed under the fusion operator $\bigsqcup$ which maps subsets of S to members of S, where $\bigsqcup \varnothing = \square$ is the designated *null*

state, and $\bigsqcup S = \blacksquare$ is the designated *full state*. An infinite state space $\langle S, \bigsqcup \rangle$ is *mereological* just in case it also satisfies the following:

Collapse: $\bigsqcup\{s\} = s$ for all $s \in S$.

Associativity: $\bigsqcup\{\bigsqcup E_i : i \in I\} = \bigsqcup\bigcup\{E_i : i \in I\}$ where I indexes each $E_i \subseteq S$.[15]

Let $\mathbb{M}_\infty$ be the class of all infinite mereological state spaces. Now consider:

∞*-Closure:* $\lceil X \rceil = \{\bigsqcup Y : \varnothing \neq Y \subseteq X\}$.

$\mathcal{S}_\infty$*-Propositions:* $\mathbb{P}^\infty_{\mathcal{S}} = \{X \subseteq S : X = \lceil X \rceil\}$.

Given any infinite state space $\mathcal{S} \in \mathbb{M}_\infty$ where $\mathcal{S} = \langle S, \bigsqcup \rangle$, an *infinite unilateral* $\mathcal{S}$*-model* $\mathcal{M}$ *of* $\mathcal{L}^-$ is any ordered triple $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle$ where $|p| \in \mathbb{P}^\infty_{\mathcal{S}}$ for all $p \in \mathbb{L}$. Let $\mathcal{C}^\infty_{\mathcal{S}}$ be the class of infinite $\mathcal{S}$-models, and $\mathcal{C}^\infty = \bigcup\{\mathcal{C}^\infty_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$. We may then make the following amendment to the **Unilateral Pre-Semantics**:

$(\dot\wedge)$ $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = \bigsqcup\{d, t\}$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$.

With the exception of the clause given above, the definition of exact verification $\Vdash$ is otherwise unchanged. As before, we may extend the domain of $|\cdot|$:

Infinitary Valuation: $s \in |A|$ *iff* $\mathcal{M}, s \Vdash A$.

Given this definition, the **Unilateral Semantics** may be preserved, thereby defining $\models$. As before, $\Gamma \models_{\mathcal{C}^\infty} \varphi$ just in case for all $\mathcal{M} \in \mathcal{C}^\infty$, if $\mathcal{M} \models \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$, where a wfs φ is $\mathcal{C}^\infty$*-valid* just in case $\models_{\mathcal{C}^\infty} \varphi$.

In §5.3, the space of propositions $\mathbb{P}_{\mathcal{S}}$ was shown to be closed under the operators $\wedge$ and $\vee$. We may now show that the space of infinite propositions $\mathbb{P}^\infty_{\mathcal{S}}$ is closed under the infinitary analogues which we may define as follows:

Cartesian Product: Let $\Pi\{X_i : i \in I\}$ be the set of functions $f : I \to \bigcup\{X_i : i \in I\}$ such that $f(i) \in X_i$ for all $i \in I$.

Infinite Product: Let $\bigwedge\{X_i : i \in I\} = \{\bigsqcup\{f(i) : i \in I\} : f \in \Pi\{X_i : i \in I\}\}$.

[15] I will let context determine which definition of associativity is intended.

Infinite Sum: Let $\bigvee\{X_i : i \in I\} = \lceil\bigcup\{X_i : i \in I\}\rceil$.

Given any $\mathcal{S} \in \mathbb{M}$ and indexed family of propositions $\{X_i : i \in I\} \in \mathbb{P}^\infty_\mathcal{S}$, both:

L6.4 $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^\infty_\mathcal{S}$. **L6.5** $\bigvee\{X_i : i \in I\} \in \mathbb{P}^\infty_\mathcal{S}$.

Whereas $\mathbb{P}_\mathcal{S}$ was only shown to be closed under finitary produce and sum, the results above prove that $\mathbb{P}^\infty_\mathcal{S}$ is closed under infinite product and sum.

We may then define finitary analogues of infinite product and sum by letting $X \wedge Y = \bigwedge\{X, Y\}$ and $X \vee Y = \bigvee\{X, Y\}$. Given any $\mathcal{S} \in \mathbb{M}_\infty$, it follows from **L6.6** and **L6.7** that $\mathcal{A}^\infty_\mathcal{S} = \langle \mathbb{P}^\infty_\mathcal{S}, \wedge, \vee, \mathcal{T}, \bot, \mathcal{V}, \perp\!\!\!\perp \rangle$ is an algebra with the same signature as $\mathcal{A}_{\mathcal{L}^-}$. Moreover, we may show that for any $\mathcal{M} \in \mathcal{C}^\infty_\mathcal{S}$, the valuation function $|\cdot| : \mathcal{A}_{\mathcal{L}^-} \to \mathcal{A}^\infty_\mathcal{S}$ is a $\mathcal{L}^-$-homomorphism:

L6.9 $|A \wedge B| = |A| \wedge |B|$. **P6.1** $|A| \in \mathbb{P}^\infty_\mathcal{S}$.

L6.10 $|A \vee B| = |A| \vee |B|$.

Instead of forming a lattice, $\mathcal{A}^\infty_\mathcal{S}$ may be re-described as a *pre-bilattice* consisting of two complete lattices.[16] In particular, we may let $\mathcal{B}^\infty_\mathcal{S} = \langle \mathbb{P}_\mathcal{S}, \subseteq, \propto \rangle$ where $\subseteq$ is subset inclusion, and $\propto$ is defined by means of the following:

Parthood: $x \sqsubseteq y$ *iff* $\bigsqcup\{x, y\} = y$.

Subsumption: $X \gg Y$ *iff* for all $y \in Y$, there is some $x \in X$ where $x \sqsubseteq y$.

Subservience: $X \ll Y$ *iff* $x \star y \in Y$ for all $x \in X$ and $y \in Y$.[17]

Containment: $X \propto Y$ *iff* $X \gg Y$ and $X \ll Y$.

We may then prove that for any indexed family of sets $\{X_i : i \in I\} \subseteq \mathbb{P}^\infty_\mathcal{S}$, both:

L6.6 $\bigwedge\{X_i : i \in I\} = \mathtt{lub}^\propto\{X_i : i \in I\}$.

L6.7 $\bigvee\{X_i : i \in I\} = \mathtt{lub}^\subseteq\{X_i : i \in I\}$.

[16] In §5.7, bilattices are defined in terms of pre-bilattices. See also Ginsberg (1988), Fitting (1991, 2002), and Arieli and Avron (1996). Fine (2017b) also draws this connection.

[17] One could define $X \ll Y$ as for all $x \in X$, there is some $y \in Y$ where $x \sqsubseteq y$. These definitions are equivalent if and only if X and Y are convex. Compare Fine (2016, p. 207).

Together with **L6.4** and **L6.5**, it follows from the above that both $\langle \mathbb{P}^\infty_\mathcal{S}, \propto \rangle$ and $\langle \mathbb{P}^\infty_\mathcal{S}, \subseteq \rangle$ are complete lattices. Accordingly, $\mathcal{B}^\infty_\mathcal{S} = \langle \mathbb{P}^\infty_\mathcal{S}, \propto, \subseteq \rangle$ is a pre-bilattice, paving the way for the introduction of negation in §5.7.

Given these results, we may extend the *Soundness* and *Completeness* results proven above by restricting consideration to the class of models $\mathcal{C}_\infty$ defined over the infinite state spaces in $\mathcal{M}_\infty$. In particular:

T3 (*Infinite Fusion*) $\quad \Sigma \models_{\mathcal{C}^\infty} \varphi$ iff $\Sigma \vdash_{\text{UGSN}} \varphi$.

Proof. We begin by showing that there are two class functions $\mathfrak{F} : \mathcal{C} \to \mathcal{C}^\infty$ and $\mathfrak{B} : \mathcal{C}^\infty \to \mathcal{C}$ which preserve logical consequence. More specifically:

P6.3: $\mathcal{M} \models \varphi$ iff $\mathcal{M}^\mathfrak{F} \models \varphi$ for all $\mathcal{M} \in \mathcal{C}$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

P6.4: $\mathcal{M} \models \varphi$ iff $\mathcal{M}^\mathfrak{B} \models \varphi$ for all $\mathcal{M} \in \mathcal{C}^\infty$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

Given that $\Sigma \not\models_\mathcal{C} \varphi$, there is some $\mathcal{M} \in \mathcal{C}$ where $\mathcal{M} \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M} \not\models \varphi$. By **P6.3**, $\mathcal{M}^\mathfrak{F} \in \mathcal{C}^\infty$ where $\mathcal{M}^\mathfrak{F} \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}^\mathfrak{F} \not\models \varphi$, and so $\Sigma \not\models_{\mathcal{C}^\infty} \varphi$. Thus by contraposition, if $\Sigma \models_{\mathcal{C}^\infty} \varphi$, then $\Sigma \models_\mathcal{C} \varphi$. Assuming instead $\Sigma \not\models_{\mathcal{C}^\infty} \varphi$, there is some $\mathcal{M} \in \mathcal{C}^\infty$ where $\mathcal{M} \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M} \not\models \varphi$. By **P6.4**, $\mathcal{M}^\mathfrak{B} \in \mathcal{C}$ where $\mathcal{M}^\mathfrak{B} \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}^\mathfrak{B} \not\models \varphi$, and so $\Sigma \not\models_\mathcal{C} \varphi$. Thus $\Sigma \not\models_\mathcal{C} \varphi$, and so by contraposition, if $\Sigma \models_\mathcal{C} \varphi$, then $\Sigma \models_{\mathcal{C}^\infty} \varphi$. Together, $\Sigma \models_{\mathcal{C}^\infty} \varphi$ iff $\Sigma \models_\mathcal{C} \varphi$. By **Theorem T1** and **Theorem T2**, we know that $\Sigma \models_\mathcal{C} \varphi$ iff $\Sigma \vdash_{\text{UGSN}} \varphi$, and so $\Sigma \models_{\mathcal{C}^\infty} \varphi$ iff $\Sigma \vdash_{\text{UGSN}} \varphi$. □

The remainder of the present section will be devoted to proving the results above, many of which will play an important role in the following sections.

L6.1 For any $\mathcal{S} \in \mathbb{M}_\mathcal{S}$ and $X \subseteq S$, if $x \in X$ and $\bigsqcup X = y$, then $x \sqsubseteq y$.

Proof. Follows from *Collapse* and *Associativity*. □

L6.2 $x \sqsubseteq x$ for all $\mathcal{S} \in \mathbb{M}_\mathcal{S}$ and $x \in S$.

Proof. Immediate from *Collapse*. □

L6.3 For any $\mathcal{S} \in \mathbb{M}_{\mathcal{S}}$ and $x, y, z \in S$, if $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$.

Proof. Follows from *Collapse* and *Associativity*. □

L6.4 $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $Y \subseteq \bigwedge\{X_i : i \in I\}$, where $Y = \{y_j : j \in J\}$. By definition, $y_j = \bigsqcup\{f_j(i) : i \in I\}$ where $f_j \in \Pi\{X_i : i \in I\}$ for each $j \in J$. Letting $z_i = \bigsqcup\{f_j(i) : j \in J\}$ for each $i \in I$, it follows that $z_i \in X_i$ since $X_i \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $\bigsqcup\{z_i : i \in I\} \in \bigwedge\{X_i : i \in I\}$. We may then observe that:

$$\begin{aligned}
\bigsqcup\{z_i : i \in I\} &= \bigsqcup\{\bigsqcup\{f_j(i) : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\{f_j(i) : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\{f_j(i) : i \in I\} : j \in J\} \\
&= \bigsqcup\{\bigsqcup\{f_j(i) : i \in I\} : j \in J\} \\
&= \bigsqcup\{y_j : j \in J\} \\
&= \bigsqcup Y.
\end{aligned}$$

Thus $\bigsqcup Y \in \bigwedge\{X_i : i \in I\}$, and so $\lceil\bigwedge\{X_i : i \in I\}\rceil \subseteq \bigwedge\{X_i : i \in I\}$, where the converse follows by *Collapse*. Thus $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. □

L6.5 $\bigvee\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{M}$ and $\{X_i : i \in I\} \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $Y \subseteq \bigvee\{X_i : i \in I\}$ be nonempty, setting $Y = \{y_j : j \in J\}$. By definition, $y_j = \bigsqcup Z_j$ where $Z_j \subseteq \bigcup\{X_i : i \in I\}$ for each $j \in J$. For each $i \in I$, we may let $x_i = \bigsqcup\bigcup\{Z_j \cap X_i : j \in J\}$, observing that $x_i \in X_i$ since $X_i \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $\bigsqcup\{x_i : i \in I\} \in \bigvee\{X_i : i \in I\}$.

We may then argue as follows:

$$\begin{aligned}
\bigsqcup\{x_i : i \in I\} &= \bigsqcup\{\bigsqcup\bigcup\{Z_j \cap X_i : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\bigcup\{Z_j \cap X_i : j \in J\} : i \in I\} \\
&= \bigsqcup\bigcup\{\bigcup\{Z_j \cap X_i : i \in I\} : j \in J\} \\
&= \bigsqcup\bigcup\{Z_j : j \in J\} \\
&= \bigsqcup\{\bigsqcup Z_j : j \in J\} \\
&= \bigsqcup\{y_j : j \in J\} \\
&= \bigsqcup Y.
\end{aligned}$$

Thus $\bigsqcup Y \in \bigvee\{X_i : i \in I\}$, and so $\lceil\bigvee\{X_i : i \in I\}\rceil \subseteq \bigvee\{X_i : i \in I\}$, where the converse holds by *Collapse*. Thus $\bigwedge\{X_i : i \in I\} \in \mathbb{P}^\infty_{\mathcal{S}}$. □

L6.6 $\bigwedge\{X_i : i \in I\} = \mathtt{lub}^\propto\{X_i : i \in I\}$ where $\mathcal{S} \in \mathcal{M}_\infty$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}_\infty$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^\infty_{\mathcal{S}}$, and let $s \in \bigwedge\{X_i : i \in I\}$. It follows that $s = \bigsqcup\{g(i) : i \in I\}$ for some $g : I \to \bigcup\{X_i : i \in I\}$ such that $g(i) \in X_i$ for all $i \in I$. Choose some $i \in I$. If follows that $g(i) \in X_i$, and so $g(i) \sqsubseteq s$. Generalising on s, it follows that $X_i \gg \bigwedge\{X_i : i \in I\}$. Now choose some $x \in X_i$ and $z \in Z$. It follows that $z = \bigsqcup\{f(i) : i \in I\}$ for some $f : I \to \bigcup\{X_i : i \in I\}$ such that $f(i) \in X_i$ for all $i \in I$. Since $X_i \in \mathbb{P}^\infty_{\mathcal{S}}$, we know that $\bigsqcup\{x, f(i)\} \in X_i$. Consider the definition:

$$f'(j) = \begin{cases} f(j) & \text{if } \ j \neq i \\ \bigsqcup\{x, f(i)\} & \text{otherwise.} \end{cases}$$

It follows that $\bigsqcup\{f'(i) : i \in I\} \in Z$. We may then observe the following:

$$\begin{aligned}
\bigsqcup\{x, z\} &= \bigsqcup\{\bigsqcup\{x\}, \bigsqcup\{f(j) : j \in I\}\} \\
&= \bigsqcup\bigcup\{\{x\}, \{f(j) : j \in I\}\} \\
&= \bigsqcup\bigcup\{\{x, f(i)\}, \{f(j) : j \in I \text{ where } j \neq i\}\} \\
&= \bigsqcup\{\bigsqcup\{x, f(i)\}, \bigsqcup\{f(j) : j \in I \text{ where } j \neq i\}\} \\
&= \bigsqcup\bigcup\{\{\bigsqcup\{x, f(i)\}\}, \{f(j) : j \in I \text{ where } j \neq i\}\} \\
&= \bigsqcup\{f'(j) : j \in I\}.
\end{aligned}$$

Thus $\bigsqcup\{x, z\} \in Z$, and so $X_i \ll \bigwedge\{X_i : i \in I\}$. Together with the above $X_i \propto \bigwedge\{X_i : i \in I\}$. Generalising on $i \in I$, we may conclude that $\bigwedge\{X_i : i \in I\}$ is an upper bound of $\{X_i : i \in I\}$ with respect to $\propto$.

Let $Z \in \mathbb{P}^\infty_S$ be an upper bound of $\{X_i : i \in I\}$ with respect to $\propto$. Accordingly, $X_i \propto Z$ for all $i \in I$, and so both $X_i \gg Z$ and $X_i \ll Z$ for all $i \in I$. Choose some $z \in Z$. It follows that for each $i \in I$, there is some $x_i \in X_i$ where $x_i \sqsubseteq z$, and so $\bigsqcup\{x_i, z\} = z$ for each $i \in I$. Accordingly, we may let $f : I \to \bigcup\{X_i : i \in I\}$ be such that $f(i) \in X_i$ for all $i \in I$ where $\bigsqcup\{x_i, z\} = z$. By definition, $\bigsqcup\{f(i) : i \in I\} \in \bigwedge\{X_i : i \in I\}$. Consider:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{f(i) : i \in I\}, z\} &= \bigsqcup\{\bigsqcup\{f(i) : i \in I\}, \bigsqcup\{z\}\} \\
&= \bigsqcup\bigcup\{\{f(i) : i \in I\}, \{z\}\} \\
&= \bigsqcup\bigcup\{\{f(i), z\} : i \in I\} \\
&= \bigsqcup\{\bigsqcup\{f(i), z\} : i \in I\} \\
&= \bigsqcup\{z : i \in I\} \\
&= z.
\end{aligned}$$

By definition, $\bigsqcup\{f(i) : i \in I\} \sqsubseteq z$, and so $\bigwedge\{X_i : i \in I\} \gg Z$ as desired.

Choose instead some $x \in \bigwedge\{X_i : i \in I\}$ and $z \in Z$. It follows that $x = \bigsqcup\{h(i) : i \in I\}$, where $h(i) \in X_i$ for all $i \in I$. Since $X_i \ll Z$ for all $i \in I$, we know that $\bigsqcup\{h(i), z\} \in Z$ for all $i \in I$, and so it follows that

$\bigsqcup\{\bigsqcup\{h(i), z\} : i \in I\} \in Z$ given that $Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Consider the following:

$$\begin{aligned}
\bigsqcup\{x, z\} &= \bigsqcup\{\bigsqcup\{h(i) : i \in I\}, \bigsqcup\{z\}\} \\
&= \bigsqcup\bigcup\{\{h(i) : i \in I\}, \{z\}\} \\
&= \bigsqcup\bigcup\{\{h(i), z\} : i \in I\} \\
&= \bigsqcup\{\bigsqcup\{h(i), z\} : i \in I\}.
\end{aligned}$$

Thus $\bigsqcup\{x, z\} \in Z$, and so $\bigwedge\{X_i : i \in I\} \ll Z$. Together with the above, $\bigwedge\{X_i : i \in I\} \propto Z$, and so $\bigwedge\{X_i : i \in I\} = \mathtt{lub}^{\propto}\{X, Y\}$ as desired. □

L6.7 $\bigvee\{X_i : i \in I\} = \mathtt{lub}^{\subseteq}\{X_i : i \in I\}$ where $\mathcal{S} \in \mathcal{M}_{\infty}$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Assume $\mathcal{S} \in \mathbb{M}_{\infty}$ and $\{X_i : i \in I\} \subseteq \mathbb{P}^{\infty}_{\mathcal{S}}$. Choose some $i \in I$, and $x \in X_i$. It follows that $x \in \bigcup\{X_i : i \in I\}$, and so $\bigsqcup\{x\} \in \lceil\bigcup\{X_i : i \in I\}\rceil$. Thus $x \in \bigvee\{X_i : i \in I\}$ by *Collapse*, and so $X_i \subseteq \bigvee\{X_i : i \in I\}$. Since $i \in I$ was arbitrary, we may conclude that $\bigvee\{X_i : i \in I\}$ is an upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$.

Let $Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$ be an upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$. Choose some $x \in \bigvee\{X_i : i \in I\}$. It follows that $x = \bigsqcup Y$ for some nonempty $Y \subseteq \bigcup\{X_i : i \in I\}$. Letting $Y_i = Y \cap X_i$ for each $i \in I$, set $W = \{\bigsqcup Y_i : \varnothing \neq Y_i\}$. Thus for any $w \in W$, we know that $w = \bigsqcup Y_i$ for some $i \in I$, where $Y_i \subseteq X_i$. Since $X_i \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $i \in I$, it follows that $w \in X_i$, and so $w \in Z$ since Z is an upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$. Thus $W \subseteq Z$, and so $\bigsqcup W \in Z$ since $Z \in \mathbb{P}^{\infty}_{\mathcal{S}}$. However:

$$\begin{aligned}
\bigsqcup W &= \bigsqcup\{\bigsqcup Y_i : \varnothing \neq Y_i\} \\
&= \bigsqcup\bigcup\{Y_i : \varnothing \neq Y_i\} \\
&= \bigsqcup\bigcup\{Y \cap X_i : i \in I\} \\
&= \bigsqcup Y \\
&= x.
\end{aligned}$$

Thus $x \in Z$, and so $\bigvee\{X_i : i \in I\} \subseteq Z$ more generally. Given that Z was an arbitrary upper bound of $\{X_i : i \in I\}$ with respect to $\subseteq$, we may

conclude that $\bigvee\{X_i : i \in I\} = \texttt{lub}^{\subseteq} X, Y$ as needed. □

L6.8 $U \vee V = U \cup V \cup (U \wedge V)$ for all $U, V \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Similar to *Sum*. □

L6.9 $|A \wedge B| = |A| \wedge |B|$ for all $\mathcal{M} \in \mathcal{C}^{\infty}$ and $A, B \in \texttt{pfs}(\mathcal{L}^{-})$.

Proof. Similar to **L3.4**. □

L6.10 $|A \vee B| = |A| \vee |B|$ for all $\mathcal{M} \in \mathcal{C}^{\infty}$ and A, B.

Proof. Similar to **L3.5**. □

P6.1 $|A| \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}^{\infty}_{\mathcal{S}}$ and $A \in \texttt{pfs}(\mathcal{L}^{-})$.

Proof. Assume $\mathcal{M} \in \mathcal{C}^{\infty}$. By definition, $|p_i| \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for every $p_i \in \mathbb{L}$, where $|e| \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $e \in \mathbb{E}$. Assume for induction that $|A|, |B| \in \mathbb{P}^{\infty}_{\mathcal{S}}$. We know that $|A \wedge B| = |A| \wedge |B|$ by **L6.9**, and $|A \vee B| = |A| \vee |B|$ by **L6.10**, and so both $|A| \wedge |B|, |A| \vee |B| \in \mathbb{P}^{\infty}_{\mathcal{S}}$ by **L6.6** and **L6.7**. It follows by induction that $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $A \in \texttt{pfs}(\mathcal{L}^{-})$. □

L6.11 $X \wedge \perp\!\!\!\perp = X$ for all $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Consider the following biconditionals:

$$\begin{aligned}
s \in X \wedge \perp\!\!\!\perp \ \ \textit{iff}\ \ & s = \bigsqcup\{x, \square\} \text{ for some } x \in X \\
\textit{iff}\ \ & s = \bigsqcup\{\bigsqcup\{x\}, \bigsqcup\varnothing\} \text{ for some } x \in X \\
\textit{iff}\ \ & s = \bigsqcup\bigcup\{\{x\}, \varnothing\} \text{ for some } x \in X \\
\textit{iff}\ \ & s = \bigsqcup\{x\} \text{ for some } x \in X \\
\textit{iff}\ \ & s \in X.
\end{aligned}$$

The above hold by *Collapse* and *Associativity*, and so $X \wedge \perp\!\!\!\perp = X$. □

L6.12 $X \vee \perp = X$ for all $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$. By **L6.8**, $X \vee \bot = X \cup \varnothing \cup (X \wedge \varnothing)$. Since $\varnothing \wedge X = \varnothing$, it follows that $X \vee \varnothing = X$. □

L6.13 $X \wedge \varnothing = \varnothing$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Immediate from *Infinite Product*. □

L6.14 $X \vee S = S$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X \in \mathbb{P}^\infty_\mathcal{S}$. By **L6.8**, $X \vee S = X \cup S \cup (X \wedge S)$. Since $X, S \in \mathbb{P}^\infty_\mathcal{S}$, it follows by **L6.4** that $X \wedge S \in \mathbb{P}^\infty_\mathcal{S}$, so $X \wedge S \subseteq S$. Of course, we also know that $X \subseteq S$, and so $S \cup X \cup (S \wedge X) = S$. Thus we may conclude that $S \vee X = X$ a needed. □

L6.15 $(X \wedge Y) \vee (X \wedge Z) \subseteq X \wedge (Y \vee Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}_\mathcal{S}\infty$.

In order to define the function $\mathfrak{F}$, we introduce the following definitions where $\mathcal{S} = \langle S, \star \rangle$ is an arbitrary mereological state space in $\mathbb{M}$:

Ideal: $X \subseteq S$ is an *ideal* in $\langle S, \star \rangle$ *iff* all x, y are such that $x, y \in X$ just in case $x \star y \in X$.

$\mathcal{S}$-Ideals: Let $\mathbb{I}_\mathcal{S}$ be the set of all ideals in $\mathcal{S}$.

X-Ideal: Let $\texttt{ideal}_\mathcal{S}(X) = \bigcap\{Y \in \mathbb{I}_\mathcal{S} : X \subseteq Y\}$.

Parts: Let $\texttt{parts}_\mathcal{S}(x) = \{y \in S : y \star x = x\}$.

We may now prove the following lemmas for an arbitrary state space $\mathcal{S} \in \mathbb{M}$:

L6.16 If $X \subseteq S$, then $\texttt{ideal}_\mathcal{S}(X) \in \mathbb{I}_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}$ where $\mathcal{S} = \langle S, \star \rangle$, and choose some $J \subseteq \mathbb{I}_\mathcal{S}$. We may then let $x, y \in \texttt{ideal}_\mathcal{S}(X)$, choosing some $Y \in \mathbb{I}_\mathcal{S}$ where $X \subseteq Y$. Thus $x, y \in Y$, and so $x \star y \in Y$. Generalising on Y, it follows that $x \star y \in \texttt{ideal}_\mathcal{S}(X)$. Assume instead that $x \star y \in \texttt{ideal}_\mathcal{S}(X)$, choosing an arbitrary $Y \in \mathbb{I}_\mathcal{S}$ where $X \subseteq Y$. It follows that $x \star y \in Y$, and so $x, y \in Y$. Thus $x, y \in \texttt{ideal}_\mathcal{S}(X)$, and so $\texttt{ideal}_\mathcal{S}(X) \in \mathbb{I}_\mathcal{S}$. □

L6.17 For all $X, Y \subseteq S$, if $X \subseteq Y$, then $\texttt{ideal}_{\mathcal{S}}(X) \subseteq \texttt{ideal}_{\mathcal{S}}(Y)$.

Proof. Let $X, Y \subseteq S$ where $X \subseteq Y$, and assume $Z \in \mathbb{I}_{\mathcal{S}}$ where $Y \subseteq Z$. It follows that $X \subseteq Z$, and so $\{Z \in \mathbb{I}_{\mathcal{S}} : Y \subseteq Z\} \subseteq \{Z \in \mathbb{I}_{\mathcal{S}} : X \subseteq Z\}$. Thus $\bigcap\{Z \in \mathbb{I}_{\mathcal{S}} : X \subseteq Z\} \subseteq \bigcap\{Z \in \mathbb{I}_{\mathcal{S}} : Y \subseteq Z\}$, or equivalently $\texttt{ideal}_{\mathcal{S}}(X) \subseteq \texttt{ideal}_{\mathcal{S}}(Y)$. This concludes the proof. □

L6.18 If $x \in S$, then $\texttt{parts}_{\mathcal{S}}(x) = \texttt{ideal}_{\mathcal{S}}(\{x\})$.

Proof. Let $x \in S$, and assume $y \in \texttt{parts}_{\mathcal{S}}(x)$. It follows that $y \star x = x$. Choose an arbitrary $Y \in \mathbb{I}_{\mathcal{S}}$ where $\{x\} \subseteq Y$. It follows that $x \in Y$, and so $x \star y \in Y$. Given that Y is an ideal, we know that $x, y \in Y$, and so $y \in Y$. Since $Y \in \mathbb{I}_{\mathcal{S}}$ where $\{x\} \subseteq Y$ was arbitrary, it follows that $y \in \texttt{ideal}_{\mathcal{S}}(\{x\})$. Thus $\texttt{parts}_{\mathcal{S}}(x) \subseteq \texttt{ideal}_{\mathcal{S}}(\{x\})$ as desired.

In order to show that $\texttt{parts}_{\mathcal{S}}(x)$ is an ideal, let $y, z \in \texttt{parts}_{\mathcal{S}}(x)$. It follows that $y \star x = x$ and $z \star x = x$, and so $y \star (z \star x) = x$. By *Associativity*, $(y \star z) \star x = x$, and so $y \star z \in \texttt{parts}_{\mathcal{S}}(x)$. Assume instead that $y \star z \in \texttt{parts}_{\mathcal{S}}(x)$. It follows that $(y \star z) \star x = x$, and so both $y \star x = x$ and $z \star x = x$ by *Idempotency*, *Commutativity*, and *Associativity*, and so $y, z \in \texttt{parts}_{\mathcal{S}}(x)$. Since y and z were arbitrary, $\texttt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$.

It follows by *Idempotency* that $x \star x = x$, and so $x \in \texttt{parts}_{\mathcal{S}}(x)$. Letting $y \in \texttt{ideal}_{\mathcal{S}}(\{x\})$ be arbitrary, it follows from the above that $y \in \texttt{parts}_{\mathcal{S}}(x)$ since $\texttt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$ where $\{x\} \subseteq \texttt{parts}_{\mathcal{S}}(x)$. Thus $\texttt{ideal}_{\mathcal{S}}(\{x\}) \subseteq \texttt{parts}_{\mathcal{S}}(x)$, and so $\texttt{parts}_{\mathcal{S}}(x) = \texttt{ideal}_{\mathcal{S}}(\{x\})$. □

L6.19 If $x, y \in S$, then $\texttt{parts}_{\mathcal{S}}(x) \cup \texttt{parts}_{\mathcal{S}}(y) \subseteq \texttt{parts}_{\mathcal{S}}(x \star y)$.

Proof. Assume $x, y \in S$, and let $z \in \texttt{parts}_{\mathcal{S}}(x) \cup \texttt{parts}_{\mathcal{S}}(y)$. It follows that either $z \in \texttt{parts}_{\mathcal{S}}(x)$ or $z \in \texttt{parts}_{\mathcal{S}}(y)$, and so $z \star x = x$ or $z \star y = y$. In either case $z \star (x \star y) = x \star y$, and so $z \in \texttt{parts}_{\mathcal{S}}(x \star y)$. Thus we may conclude that $\texttt{parts}_{\mathcal{S}}(x) \cup \texttt{parts}_{\mathcal{S}}(y) \subseteq \texttt{parts}_{\mathcal{S}}(x \star y)$. □

L6.20 For all $x, y \in S$, if $\texttt{parts}_{\mathcal{S}}(x) = \texttt{parts}_{\mathcal{S}}(y)$, then $x = y$.

Proof. Let $x, y \in S$, and assume that $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{parts}_{\mathcal{S}}(y)$. It follows by *Idempotency*, $x \in \mathtt{parts}_{\mathcal{S}}(x)$ and $y \in \mathtt{parts}_{\mathcal{S}}(y)$, and so $x \in \mathtt{parts}_{\mathcal{S}}(y)$ and $y \in \mathtt{parts}_{\mathcal{S}}(x)$. Thus $x \star y = y$ and $y \star x = x$, and so $x = y$. □

L6.21 If $X \subseteq S$, then $X \subseteq \overline{X} \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$.

Proof. Assume $X \subseteq S$, and let $x \in X$. It follows that $x \star x \in \overline{X}$, and so $x \in \overline{X}$ by *Idempotency*. Thus $X \subseteq \overline{X}$.

Assume instead that $x \in \overline{X}$. It follows that $x = y \star z$ for some $y, z \in X$. Choose some $Z \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Z$. Thus $y, z \in Z$, and so $y \star z \in Z$. Generalising on Z, $y \star z \in \mathtt{ideal}_{\mathcal{S}}(X)$, and so $\overline{X} \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$. □

L6.22 If $X \in \mathbb{I}_{\mathcal{S}}$, then $\mathtt{ideal}_{\mathcal{S}}(X) = X$.

Proof. Assume $X \in \mathbb{I}_{\mathcal{S}}$, and let $x \in X$. Choose an arbitrary $Y \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$. It follows that $x \in Y$. Generalising on Y we know that $x \in \mathtt{ideal}_{\mathcal{S}}(X)$, and so it follows that $X \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$.

Assume instead that $x \in \mathtt{ideal}_{\mathcal{S}}(X)$. It follows that $x \in Y$ for all $Y \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$. In particular, $x \in X$ given that $X \in \mathbb{I}_{\mathcal{S}}$. Thus $\mathtt{ideal}_{\mathcal{S}}(X) \subseteq X$, and so $\mathtt{ideal}_{\mathcal{S}}(X) = X$. □

L6.23 If $X \subseteq S$, then $\mathtt{ideal}_{\mathcal{S}}(X) = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$.

Proof. We begin by showing that $\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \in \mathbb{I}_{\mathcal{S}}$. Assume $y, z \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. Thus $y \in \mathtt{parts}_{\mathcal{S}}(u)$ and $z \in \mathtt{parts}_{\mathcal{S}}(v)$ for some $u, v \in \overline{X}$, and so $y \star u = u$ and $z \star v = v$. Since $\overline{X}$ is closed under fusion, we know that $u \star v \in \overline{X}$, and so $(y \star u) \star (z \star v) \in \overline{X}$. However:

$$\begin{aligned} u \star v &= (y \star u) \star (z \star v) \\ &= (y \star z) \star (u \star v) \end{aligned}$$

Thus $y \star z \in \mathtt{parts}_{\mathcal{S}}(u \star v)$ where $u \star v \in \overline{X}$, and so we may conclude that $y \star z \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. In order to establish the converse, assume that $y \star z \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. Thus $y \star z \in \mathtt{parts}_{\mathcal{S}}(w)$ where $w \in \overline{X}$, and so $(y \star z) \star w = w$. As in **L6.18**, it follows that $y \star w = w$

and $z \star w = w$, and so both $y, z \in \mathtt{parts}_{\mathcal{S}}(w)$ where $w \in \overline{X}$. Thus both $y, z \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$, and so $\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \in \mathbb{I}_{\mathcal{S}}$.

Letting $x \in X$ be arbitrary, $x \star x \in \overline{X}$, and so $x \in \overline{X}$ and $x \in \mathtt{parts}_{\mathcal{S}}(x)$. Thus $x \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$, and so $X \subseteq \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. It follows that $\mathtt{ideal}_{\mathcal{S}} X \subseteq \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\})$ by **L6.17**, and so $\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}) = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$ given that $\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \in \mathbb{I}_{\mathcal{S}}$. Thus $\mathtt{ideal}_{\mathcal{S}} X \subseteq \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$.

Assume instead that $x \in \overline{X}$. By **L6.17**, $\mathtt{ideal}_{\mathcal{S}}(\{x\}) \subseteq \mathtt{ideal}_{\mathcal{S}}(\overline{X})$. Thus $\mathtt{parts}_{\mathcal{S}}(x) \subseteq \mathtt{ideal}_{\mathcal{S}}(\overline{X})$ by **L6.18**. Additionally, $\overline{X} \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$ by **L6.21**, and so $\mathtt{ideal}_{\mathcal{S}}(\overline{X}) \subseteq \mathtt{ideal}_{\mathcal{S}}(\mathtt{ideal}_{\mathcal{S}}(X))$ by **L6.17**. Thus $\mathtt{parts}_{\mathcal{S}}(x) \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$ since $\mathtt{ideal}_{\mathcal{S}}(\mathtt{ideal}_{\mathcal{S}}(X)) = \mathtt{ideal}_{\mathcal{S}}(X)$ by **L6.22**. Since $x \in \overline{X}$ was arbitrary, $\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\} \subseteq \mathtt{ideal}_{\mathcal{S}} X$. Together with the above, $\mathtt{ideal}_{\mathcal{S}} X = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. □

L6.24 If $X \subseteq S$, then $\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}) = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$.

Proof. We first show that $\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}) = \mathtt{ideal}_{\mathcal{S}}(X)$. Choose some $Y \in \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$, and let $y \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}$. It follows that $y \in \mathtt{parts}_{\mathcal{S}}(z)$ for some $z \in X$, and so $y \star z = z$ where $z \in Y$. Thus $y \star z \in Y$, and so $y \in Y$. More generally, $\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y$, and so $Y \in \{Y \in \mathbb{I}_{\mathcal{S}} : \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y\}$. Generalising on Y, we know that $\{Y \in \mathbb{I}_{\mathcal{S}} : X \subseteq Y\} \subseteq \{Y \in \mathbb{I}_{\mathcal{S}} : \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y\}$, and so $\bigcap\{Y \in \mathbb{I}_{\mathcal{S}} : \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq Y\} \subseteq \bigcap\{Y \in \mathbb{I}_{\mathcal{S}} : X \subseteq Y\}$. Equivalently, $\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}) \subseteq \mathtt{ideal}_{\mathcal{S}}(X)$.

Assume instead that $x \in X$. By *Idempotency*, $x \in \mathtt{parts}_{\mathcal{S}}(x)$, and so $x \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}$. Thus $X \subseteq \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}$, and so $\mathtt{ideal}_{\mathcal{S}}(X) \subseteq \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\})$ by **L6.17**. Given the above, $\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}) = \mathtt{ideal}_{\mathcal{S}}(X)$, and so by **L6.23**, $\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}) = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X}\}$. □

Given any finite state space $\mathcal{S} \in \mathbb{M}$, we may construct an infinite state space $\mathcal{S}_\infty = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup \rangle$ where $\bigsqcup$ is defined as follows:

Infinite Fusion: $\bigsqcup X = \mathtt{ideal}_{\mathcal{S}}(\bigcup X)$ for all $X \subseteq \mathbb{I}_{\mathcal{S}}$.

As I will go on to show, $\mathcal{S}_\infty \in \mathbb{M}_\infty$ for all $\mathcal{S} \in \mathbb{M}$, and so given the definition above, $\mathbb{P}^\infty_{\mathcal{S}_\infty} = \{X \subseteq \mathbb{I}_\mathcal{S} : X = \lceil X \rceil\}$, which I will write $\mathbb{P}^\infty_\mathcal{S}$ for simplicity. We may then define a function $\mathfrak{F} : \mathbb{P}_\mathcal{S} \to \mathbb{P}^\infty_\mathcal{S}$ as below:

Forward: $\mathfrak{F}(X) = \lceil \{\mathtt{parts}_\mathcal{S}(x) : x \in X\} \rceil$ for all $X \in \mathbb{P}_\mathcal{S}$.

In what follows, I will extend $\mathfrak{F}$ to a function which maps $\mathcal{C}$ into $\mathcal{C}^\infty$, where the lemmas given below build towards the construction of an infinite model $\mathcal{M}^\mathfrak{F} \in \mathcal{C}^\infty$ from any finite model $\mathcal{M} \in \mathcal{C}$, culminating in the proof of **P6.3**.

L6.25 $\bigsqcup U \in \mathbb{I}_\mathcal{S}$ for all $U \subseteq \mathbb{I}_\mathcal{S}$.

Proof. Let $U \subseteq \mathbb{I}_\mathcal{S}$, and assume $x, y \in \mathtt{ideal}_\mathcal{S}(\bigcup U)$. Choose some $Z \in \mathbb{I}_\mathcal{S}$ where $\bigcup U \subseteq Z$. Thus $x, y \in Z$, and so $x \star y \in Z$. More generally, $x \star y \in \mathtt{ideal}_\mathcal{S}(\bigcup U)$. Assume instead that $x \star y \in \mathtt{ideal}_\mathcal{S}(\bigcup U)$, letting $V \in \mathbb{I}_\mathcal{S}$ where $\bigcup U \subseteq V$. Thus $x \star y \in V$, and so $x, y \in V$. More generally, $x, y \in \mathtt{ideal}_\mathcal{S}(\bigcup U)$. Thus $\mathtt{ideal}_\mathcal{S}(\bigcup U) \in \mathbb{I}_\mathcal{S}$, and so $\bigsqcup U \in \mathbb{I}_\mathcal{S}$. □

L6.26 $\bigsqcup\{x\} = x$ for all $x \in \mathbb{I}_\mathcal{S}$.

Proof. Let $x \in \mathbb{I}_\mathcal{S}$. It follows that $\bigsqcup\{x\} = \mathtt{ideal}_\mathcal{S}(\bigcup\{x\}) = \mathtt{ideal}_\mathcal{S}(x)$, and so $\bigsqcup\{x\} = x$ by **L6.22**. □

L6.27 $\bigsqcup\{\bigsqcup U_i : i \in I\} = \bigsqcup\bigcup\{U_i : i \in I\}$ if $U_i \subseteq \mathbb{I}_\mathcal{S}$ for all $i \in I$.

Proof. Assume $U_i \subseteq \mathbb{I}_\mathcal{S}$ for all $i \in I$. We first show that every $Z \in \mathbb{I}_\mathcal{S}$ is such that $\bigcup\{\mathtt{ideal}_\mathcal{S}(\bigcup U_i) : i \in I\} \subseteq Z$ just in case $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$. Let $Z \in \mathbb{I}_\mathcal{S}$ where $\bigcup\{\mathtt{ideal}_\mathcal{S}(\bigcup U_i) : i \in I\} \subseteq Z$, and choose some $x \in \bigcup\bigcup\{U_i : i \in I\}$. Thus $x \in \bigcup U_i$ for some $i \in I$. Since $\bigcup U_i \subseteq S$, we know that $\bigcup U_i \subseteq \mathtt{ideal}_\mathcal{S}(\bigcup U_i)$ by **L6.21**, and so $x \in \mathtt{ideal}_\mathcal{S}(\bigcup U_i)$. Thus $x \in \bigcup\{\mathtt{ideal}_\mathcal{S}(\bigcup U_i) : i \in I\} \subseteq Z$, and so $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$. By discharge, if $\bigcup\{\mathtt{ideal}_\mathcal{S}(\bigcup U_i) : i \in I\} \subseteq Z$, then $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$.

Assume $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$, and let $x \in \bigcup\{\mathtt{ideal}_\mathcal{S}(\bigcup U_i) : i \in I\}$. Thus $x \in \mathtt{ideal}_\mathcal{S}(\bigcup U_i)$ for some $i \in I$, so $x \in \bigcup\{\mathtt{parts}_\mathcal{S}(y) : y \in \overline{\bigcup U_i}\}$ by **L6.23**. It follows that $x \in \mathtt{parts}_\mathcal{S}(y)$ for some $y \in \overline{\bigcup U_i}$, and so

$x \star y = y$ where $y = a \star b$ for some $a, b \in \bigcup U_i$. Thus $a, b \in \bigcup\bigcup\{U_i : i \in I\}$. By assumption, $a, b \in Z$, and so $a \star b \in Z$ since $Z \in \mathbb{I}_{\mathcal{S}}$. Thus $y \in Z$, and so $x \star y \in Z$, given the above. We may then conclude that $x \in Z$, and so $\bigcup\{\texttt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$. We know by discharge that if $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$, then $\bigcup\{\texttt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$. Thus $\bigcup\{\texttt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z$ just in case $\bigcup\bigcup\{U_i : i \in I\} \subseteq Z$ for all $Z \in \mathbb{I}_{\mathcal{S}}$ given that $Z \in \mathbb{I}_{\mathcal{S}}$ was arbitrary. Put otherwise:

$$\Big\{Z \in \mathbb{I}_s : \bigcup\{\texttt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\} \subseteq Z\Big\} = \Big\{Z \in \mathbb{I}_{\mathcal{S}} : \bigcup\bigcup\{U_i : i \in I\} \subseteq Z\Big\}.$$

By then taking the intersection of both sides of the identity above, we may observe that the identities below follow from the definitions:

$$\begin{aligned} \texttt{ideal}_{\mathcal{S}}(\bigcup\{\texttt{ideal}_{\mathcal{S}}(\bigcup U_i) : i \in I\}) &= \texttt{ideal}_{\mathcal{S}}(\bigcup\bigcup\{U_i : i \in I\}) \\ \bigsqcup\{\bigsqcup U_i : i \in I\} &= \bigsqcup\bigcup\{U_i : i \in I\}. \end{aligned}$$

Given this final identity, we may conclude the proof by generalising on the family of sets $U_i \subseteq \mathbb{I}_{\mathcal{S}}$ indexed by I. □

L6.28 For all $X, Y \subseteq \mathbb{I}_{\mathcal{S}}$, if $X \subseteq Y$, then $\lceil X \rceil \subseteq \lceil Y \rceil$.

Proof. Let $X, Y \subseteq \mathbb{I}_{\mathcal{S}}$ where $X \subseteq Y$, and let $x \in \lceil X \rceil$. Thus some $Z \subseteq X$ where $x = \bigsqcup Z$, and so $Z \subseteq Y$. Thus $x \in \lceil Y \rceil$, and so $\lceil X \rceil \subseteq \lceil Y \rceil$. □

L6.29 $\lceil \lfloor U \rfloor \rceil = \lceil U \rceil$ for all $U \subseteq \mathbb{I}_{\mathcal{S}}$.

Proof. Let $U \subseteq \mathbb{I}_{\mathcal{S}}$, and assume $u \in \lceil \lfloor U \rfloor \rceil$. It follows that $u = \bigsqcup V$ for some $V \subseteq \lceil U \rceil$. Let V be indexed by I such that $V = \{v_i : i \in I\}$, and choose some $v_i \in V$. It follows that $v_i = \bigsqcup V_i$ for some $V_i \subseteq U$. Thus:

$$\begin{aligned} u &= \bigsqcup V \\ &= \bigsqcup\{\bigsqcup V_i : i \in I\} \\ &= \bigsqcup\bigcup\{V_i : i \in I\} \end{aligned}$$

The latter identity holds by **L6.27**. Since $V_i \subseteq U$ for all $i \in I$, we know that $\bigcup\{V_i : i \in I\} \subseteq U$, and so $\bigsqcup\bigcup\{V_i : i \in I\} \in \lceil U \rceil$. Thus $u \in \lceil U \rceil$. Generalising on u, it follows that $\lceil\lceil U \rceil\rceil \subseteq \lceil U \rceil$ as desired.

Assume instead that $u \in U$. It follows that $\bigsqcup\{u\} \in \lceil U \rceil$, and so $u \in \lceil U \rceil$ by **L6.26**. Thus $U \subseteq \lceil U \rceil$, and so $\lceil U \rceil \subseteq \lceil\lceil U \rceil\rceil$ by **L6.28**. Together with the above, $\lceil\lceil U \rceil\rceil = \lceil U \rceil$ for all $U \subseteq \mathbb{I}_{\mathcal{S}}$. □

L6.30 $\mathfrak{F}(X) \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $X \subseteq S$.

Proof. Let $X \subseteq S$, and $x \in X$. Thus $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{ideal}_{\mathcal{S}}(\{x\})$ by **L6.18**, and $\mathtt{ideal}_{\mathcal{S}}(\{x\}) \in \mathbb{I}_{\mathcal{S}}$ by **L6.16**. More generally, $\mathtt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$ for all $x \in X$, and so $\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq \mathbb{I}_{\mathcal{S}}$, where we know by **L6.29** that $\lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil = \lceil\lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil\rceil$. By definition, $\mathfrak{F}(X) = \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$, and so $\mathfrak{F}(X) = \lceil\mathfrak{F}(X)\rceil$. Thus $\mathfrak{F}(X) \in \mathbb{P}^{\infty}_{\mathcal{S}}$, where generalising on $X \subseteq S$ concludes the proof. □

L6.31 $x \in X$ iff $\mathtt{parts}_{\mathcal{S}}(x) \in \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$ for all $x \in S$ and $X \in \mathbb{P}_{\mathcal{S}}$.

Proof. Let $x \in S$ and $X \in \mathbb{P}_{\mathcal{S}}$ where $x \in X$. Of course, it follows that $\mathtt{parts}_{\mathcal{S}}(x) \in \{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}$, where we know by definition that $\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x)\} \in \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$. Since $x \in S$, it follows that $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{ideal}_{\mathcal{S}}(\{x\})$ by **L6.18**, and $\mathtt{ideal}_{\mathcal{S}}(\{x\}) \in \mathbb{I}_{\mathcal{S}}$ by **L6.16**, and so $\mathtt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$. Thus $\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x)\} = \mathtt{parts}_{\mathcal{S}}(x)$ follows by **L6.26**, and so $\mathtt{parts}_{\mathcal{S}}(x) \in \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil$.

Assume instead that $\mathtt{parts}_{\mathcal{S}}(x) \in \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$. Thus there is some $Y \subseteq \{\mathtt{parts}_{\mathcal{S}}(y) : y \in X\}$ where $\mathtt{parts}_{\mathcal{S}}(x) = \bigsqcup Y$. By definition, $\bigsqcup Y = \mathtt{ideal}_{\mathcal{S}}(\bigcup Y)$, and so $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{ideal}_{\mathcal{S}}(\bigcup Y)$. We may then let $Z = \{z \in X : \mathtt{parts}_{\mathcal{S}}(z) \in Y\}$, and so $\bigcup Y = \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Z\}$. Accordingly, $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Z\})$, and so by **L6.24** we know that $\mathtt{parts}_{\mathcal{S}}(x) = \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in \overline{Z}\}$. Of course, $x \in \mathtt{parts}_{\mathcal{S}}(x)$ by *Idempotency*, and so $x \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in \overline{Z}\}$. Thus $x = \mathtt{parts}_{\mathcal{S}}(y)$ for some $y \in \overline{Z}$, and so $x \star y = y$. Letting $u \in \mathtt{parts}_{\mathcal{S}}(x)$,

it follows that $u \star x = x$, and so we may reason as follows:

$$\begin{aligned} u \star y &= u \star (x \star y) \\ &= (u \star x) \star y \\ &= x \star y \\ &= y \end{aligned}$$

Thus it follows that $u \in \mathtt{parts}_{\mathcal{S}}(y)$. Generalising on u, it follows that $\mathtt{parts}_{\mathcal{S}}(x) \subseteq \mathtt{parts}_{\mathcal{S}}(y)$. However, given that $y \in \overline{Z}$, we also know that $\mathtt{parts}_{\mathcal{S}}(y) \subseteq \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in \overline{Z}\} = \mathtt{parts}_{\mathcal{S}}(x)$, from which it follows that $\mathtt{parts}_{\mathcal{S}}(y) \subseteq \mathtt{parts}_{\mathcal{S}}(x)$. Given the above, $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{parts}_{\mathcal{S}}(y)$, and so $x = y$ by **L6.20**. Thus we may conclude that $x \in \overline{Z}$.

Recall from above that $Z = \{z \in X : \mathtt{parts}_{\mathcal{S}}(z) \in Y\}$. It follows that $Z \subseteq X$, and so $\overline{Z} \subseteq \overline{X}$. However, we know by assumption that $X \in \mathbb{P}_{\mathcal{S}}$, and so $\overline{X} = X$. Thus $\overline{Z} \subseteq X$, and so $x \in X$ given the above. Thus $x \in X$ just in case $\mathtt{parts}_{\mathcal{S}}(x) \in \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$. □

L6.32 $X \subseteq Y$ iff $\mathfrak{F}(X) \subseteq \mathfrak{F}(Y)$, $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Assume $X \subseteq Y$ for some $X, Y \in \mathbb{P}_{\mathcal{S}}$. We may then observe that $\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq \{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y\}$, and so it follows by **L6.28** that $\lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil \subseteq \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y\}\rceil$. Thus $\mathfrak{F}(X) \subseteq \mathfrak{F}(Y)$.

Assume $X \not\subseteq Y$ instead, and so $x \notin Y$ for some $x \in X$. Thus we may conclude by **L6.31** that $\mathtt{parts}_{\mathcal{S}}(x) \in \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in X\}\rceil$ but $\mathtt{parts}_{\mathcal{S}}(x) \in \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y\}\rceil$. It follows that $\mathfrak{F}(X) \not\subseteq \mathfrak{F}(Y)$. □

L6.33 $\mathfrak{F}(\{x\}) = \{\mathtt{parts}_{\mathcal{S}}(x)\}$ for all $x \in S$.

Proof. Let $x \in S$, so $\mathfrak{F}(\{x\}) = \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \{x\}\}\rceil = \lceil\{\mathtt{parts}_{\mathcal{S}}(x)\}\rceil$, where $\lceil\{\mathtt{parts}_{\mathcal{S}}(x)\}\rceil = \{\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x)\}\} = \{\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x)\})\}$. Of course, $\{\mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x)\})\} = \{\mathtt{ideal}_{\mathcal{S}}(\mathtt{parts}_{\mathcal{S}}(x))\}$. By **L6.18** and **L6.16**, we also know that $\mathtt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$, and so it follows by **L6.22** that $\mathtt{ideal}_{\mathcal{S}}(\mathtt{parts}_{\mathcal{S}}(x)) = \mathtt{parts}_{\mathcal{S}}(x)$. Thus $\mathfrak{F}(\{x\}) = \{\mathtt{parts}_{\mathcal{S}}(x)\}$. □

L6.34 $\mathfrak{F}(X) \wedge \mathfrak{F}(Y) = \mathfrak{F}(X \wedge Y)$ for all $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Let $X, Y \in \mathbb{P}_{\mathcal{S}}$, and assume $Z \in \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$. Thus $Z = \bigsqcup\{U, V\}$ for some $U \in \mathfrak{F}(X)$ and $V \in \mathfrak{F}(Y)$, and so $Z = \mathtt{ideal}_{\mathcal{S}}(U \cup V)$. We also know that $U = \bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}$ and $V = \bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in Y'\}$ for some $X' \subseteq X$ and $Y' \subseteq Y$. If follows by definition that both:

$$U = \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}),$$
$$V = \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in Y'\}).$$

Given these identities, we may observe that the identities below follow by definition or assumption with exception of $(*)$ which is given by **L6.24**:

$$\begin{aligned}
Z &= \mathtt{ideal}_{\mathcal{S}}(U \cup V) \\
&= \mathtt{ideal}_{\mathcal{S}}[\mathtt{ideal}_{\mathcal{S}}(\textstyle\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}) \cup \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in Y'\})] \\
(*) &= \mathtt{ideal}_{\mathcal{S}}[\textstyle\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X'}\} \cup \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{Y'}\}] \\
&= \mathtt{ideal}_{\mathcal{S}}[\textstyle\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) : x \in \overline{X'}, y \in \overline{Y'}\}] \\
&= \textstyle\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) : x \in \overline{X'}, y \in \overline{Y'}\}
\end{aligned}$$

Let $z \in \mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y)$ for some $x \in \overline{X'}$ and $y \in \overline{Y'}$. By **L6.19**, $z \in \mathtt{parts}_{\mathcal{S}}(x \star y)$, where $x \in X$ and $y \in Y$ given that $\overline{X'} \subseteq \overline{X} = X$ and $\overline{Y'} \subseteq \overline{Y} = Y$. Thus $z \in \mathtt{parts}_{\mathcal{S}}(x \star y)$ for $x \in X$ and $y \in Y$, and so $z \in \{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}$. Generalising on z, it follows that:

$$\{\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) : x \in X, y \in Y\} \subseteq \{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\},$$
$$\textstyle\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x) \cup \mathtt{parts}_{\mathcal{S}}(y) : x \in X, y \in Y\} \in \lceil\{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}\rceil.$$

Given the above, $Z \in \lceil\{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}\rceil$. However, by definition, $\lceil\{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}\rceil = \mathfrak{F}(X \wedge Y)$, and so $Z \in \mathfrak{F}(X \wedge Y)$. Thus it follows that $\mathfrak{F}(X) \wedge \mathfrak{F}(Y) \subseteq \mathfrak{F}(X \wedge Y)$.

Assume instead that $Z \in \mathfrak{F}(X \wedge Y)$. It follows that $Z = \bigsqcup J$ for some $J \subseteq \{\mathtt{parts}_{\mathcal{S}}(w) : w \in X \wedge Y\}$. Thus there is some $K \subseteq X \wedge Y$ where

$J = \{\mathtt{parts}_{\mathcal{S}}(w) : w \in K\}$. Hence it follows that:

$$\begin{aligned} Z &= \bigsqcup J \\ &= \bigsqcup\{\mathtt{parts}_{\mathcal{S}}(w) : w \in K\} \\ &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(w) : w \in K\}) \\ (\dagger) &= \bigcup\{\mathtt{parts}_{\mathcal{S}}(w) : w \in \overline{K}\} \\ (\ddagger) &= \mathtt{ideal}_{\mathcal{S}}(K) \end{aligned}$$

Whereas (†) is given by **L6.24**, (‡) holds by **L6.23**, where the remaining identities follow from the above by definition. Consider the definitions:

$$\begin{aligned} X' &= \{x \in X : \exists y \in Y \text{ where } x \star y \in K\}, \\ Y' &= \{y \in Y : \exists x \in X \text{ where } x \star y \in K\}, \\ U &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}), \\ V &= \mathtt{ideal}_{\mathcal{S}}(\bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y'\}). \end{aligned}$$

We may then show that $\mathtt{ideal}_{\mathcal{S}}(K) = \mathtt{ideal}_{\mathcal{S}}(U \cup V)$ where $U \subseteq X$ and $V \subseteq Y$. Let $k \in K$. Since $K \subseteq X \wedge Y$, we know that $k = x \star y$ where $x \in X$ and $y \in Y$, and so $x \in X'$ and $y \in Y'$. Given that $x \in \mathtt{parts}_{\mathcal{S}}(x)$ and $y \in \mathtt{parts}_{\mathcal{S}}(y)$ by *Idempotency*, both $x \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\}$ and $y \in \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y'\}$. By **L6.21**, $x \in U$ and $y \in V$. Letting $Z \in \mathbb{I}_{\mathcal{S}}$ where $U \cup V \subseteq Z$, it follows that $x, y \in Z$, and so $x \star y \in Z$. Generalising on Z, we may conclude that $x \star y \in \mathtt{ideal}_{\mathcal{S}}(U \cup V)$, and so $k \in \mathtt{ideal}_{\mathcal{S}}(U \cup V)$. Thus $K \subseteq \mathtt{ideal}_{\mathcal{S}}(U \cup V)$, and so by **L6.17** $\mathtt{ideal}_{\mathcal{S}}(K) \subseteq \mathtt{ideal}_{\mathcal{S}}(\mathtt{ideal}_{\mathcal{S}}(U \cup V))$. Since $\mathtt{ideal}_{\mathcal{S}}(U \cup V) \in \mathbb{I}_{\mathcal{S}}$ by **L6.16**, it follows that $\mathtt{ideal}_{\mathcal{S}}(\mathtt{ideal}_{\mathcal{S}}(U \cup V)) = \mathtt{ideal}_{\mathcal{S}}(U \cup V)$ by **L6.22**, and so $\mathtt{ideal}_{\mathcal{S}}(K) \subseteq \mathtt{ideal}_{\mathcal{S}}(U \cup V)$ given the above.

In order to establish the converse inclusion, choose some $W \in \mathbb{I}_{\mathcal{S}}$ where $K \subseteq W$, and let $w \in U \cup V$. Consider the following cases:

Case 1: Assume $w \in U$. By **L6.24**, $U = \bigcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in \overline{X'}\}$, and so $w \in \mathtt{parts}_{\mathcal{S}}(x)$ for some $x \in \overline{X'}$. It follows that $w \star x = x$ where

$x = a \star b$ for some $a, b \in X'$. Thus there are some u and v such that $a \star u \in K$ and $b \star v \in K$, and so both $a \star u, b \star v \in W$ given that $K \subseteq W$. Since $W \in \mathbb{I}_{\mathcal{S}}$, both $a, b \in W$, and so $a \star b \in W$. Given the above, we may conclude that $w \star x \in W$, and so $w \in W$ as desired. *Case 2:* Assume $w \in V$. By **L6.24**, $V = \bigcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in \overline{Y'}\}$, and so $w \in \mathtt{parts}_{\mathcal{S}}(y)$ for some $y \in \overline{Y'}$. It follows that $w \star y = y$ where $y = c \star d$ for some $c, d \in Y'$. Thus there are some u and v such that $c \star u \in K$ and $d \star v \in K$, and so both $c \star u, d \star v \in W$ given that $K \subseteq W$. Since $W \in \mathbb{I}_{\mathcal{S}}$, both $c, d \in W$, and so $c \star d \in W$. Given the above, we may conclude that $w \star y \in W$, and so $w \in W$ as desired.

Since $w \in W$ in both of the cases above, it follows that $U \cup V \subseteq W$, and so $\{W \in \mathbb{I}_{\mathcal{S}} : K \subseteq W\} \subseteq \{W \in \mathbb{I}_{\mathcal{S}} : U \cup V \subseteq W\}$. Thus we know that $\bigcap\{W \in \mathbb{I}_{\mathcal{S}} : U \cup V \subseteq W\} \subseteq \bigcap\{W \in \mathbb{I}_{\mathcal{S}} : K \subseteq W\}$, and so $\mathtt{ideal}_{\mathcal{S}}(U \cup V) \subseteq \mathtt{ideal}_{\mathcal{S}}(K)$ by definition. Together with the inclusion given above, it follow that $\mathtt{ideal}_{\mathcal{S}}(K) = \mathtt{ideal}_{\mathcal{S}}(U \cup V)$.

Thus we have $Z = \mathtt{ideal}_{\mathcal{S}}(K) = \mathtt{ideal}_{\mathcal{S}}(U \cup V) = \bigsqcup\{U, V\}$. Since $X' \subseteq X$ and $Y' \subseteq Y$, both $\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\} \subseteq \{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}$ and $\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y'\} \subseteq \{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y\}$. We may then observe:

$$U = \bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X'\} \in \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}\rceil = \mathfrak{F}(X),$$
$$V = \bigsqcup\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y'\} \in \lceil\{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y\}\rceil = \mathfrak{F}(Y).$$

Having shown that $Z = \bigsqcup\{U, V\}$ where $U \in \mathfrak{F}(X)$ and $V \in \mathfrak{F}(Y)$, it follows that $Z \in \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$. Since $Z \in \mathfrak{F}(X \wedge Y)$ was arbitrary, it follows that $\mathfrak{F}(X \wedge Y) \subseteq \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$, and so $\mathfrak{F}(X) \wedge \mathfrak{F}(Y) = \mathfrak{F}(X \wedge Y)$. □

L6.35 $\mathfrak{F}(X) \vee \mathfrak{F}(Y) = \mathfrak{F}(X \vee Y)$ for all $X, Y \in \mathbb{P}_{\mathcal{S}}$.

Proof. Letting $X, Y \in \mathbb{P}_{\mathcal{S}}$, the inclusions given on the left hold by *Sum*, where the inclusions on the right follow by **L6.32**:

$$\begin{aligned} X &\subseteq X \vee Y \\ Y &\subseteq X \vee Y \\ X \wedge Y &\subseteq X \vee Y \end{aligned}$$

$$\begin{aligned} \mathfrak{F}(X) &\subseteq \mathfrak{F}(X \vee Y) \\ \mathfrak{F}(Y) &\subseteq \mathfrak{F}(X \vee Y) \\ \mathfrak{F}(X \wedge Y) &\subseteq \mathfrak{F}(X \vee Y). \end{aligned}$$

Since $\mathfrak{F}(X \wedge Y) = \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$ by **L6.34**, we may conclude from the above that $\mathfrak{F}(X) \cup \mathfrak{F}(Y) \cup (\mathfrak{F}(X) \wedge \mathfrak{F}(Y)) \subseteq \mathfrak{F}(X \vee Y)$. Thus by **L6.8**, we may conclude that $\mathfrak{F}(X) \vee \mathfrak{F}(Y) \subseteq \mathfrak{F}(X \vee Y)$.

Choose some $x \in X$. Since $\{\mathtt{parts}_{\mathcal{S}}(x)\} \subseteq \{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\}$, we know by definiton that $\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x)\} \in \mathfrak{F}(X)$. We also know by **L6.16** that $\mathtt{parts}_{\mathcal{S}}(x) \in \mathbb{I}_{\mathcal{S}}$, and so $\bigsqcup\{\mathtt{parts}_{\mathcal{S}}(x)\} = \mathtt{parts}_{\mathcal{S}}(x)$ by **L6.26**. Thus $\mathtt{parts}_{\mathcal{S}}(x) \in \mathfrak{F}(X)$, and so $\{\mathtt{parts}_{\mathcal{S}}(x) : x \in X\} \subseteq \mathfrak{F}(X)$ follows by generalising on $x \in X$. By analogous arguments:

$$\begin{aligned} \{\mathtt{parts}_{\mathcal{S}}(y) : y \in Y\} &\subseteq \mathfrak{F}(Y) \\ \{\mathtt{parts}_{\mathcal{S}}(z) : z \in X \wedge Y\} &\subseteq \mathfrak{F}(X \wedge Y). \end{aligned}$$

However, $\mathfrak{F}(X \wedge Y) = \mathfrak{F}(X) \wedge \mathfrak{F}(Y)$ by **L6.34**, and so it follows that:

$$\{\mathtt{parts}_{\mathcal{S}}(z) : z \in X \cup Y \cup (X \wedge Y)\} \subseteq \mathfrak{F}(X) \cup \mathfrak{F}(Y) \cup (\mathfrak{F}(X) \wedge \mathfrak{F}(Y)).$$

Thus $\{\mathtt{parts}_{\mathcal{S}}(z) : z \in X \vee Y\} \subseteq \mathfrak{F}(X) \vee \mathfrak{F}(Y)$ by *Sum* and **L6.8**. By **L6.28**, $\lceil\{\mathtt{parts}_{\mathcal{S}}(z) : z \in X \vee Y\}\rceil \subseteq \lceil\mathfrak{F}(X) \vee \mathfrak{F}(Y)\rceil$, and so by definition $\mathfrak{F}(X \vee Y) \subseteq \lceil\mathfrak{F}(X) \vee \mathfrak{F}(Y)\rceil$. However, $\mathfrak{F}(X) \vee \mathfrak{F}(Y) = \lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil$, and so $\mathfrak{F}(X \vee Y) \subseteq \lceil\lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil\rceil$. Since $X, Y \in \mathbb{P}_{\mathcal{S}}$, we know by **L6.30** that both $\mathfrak{F}(X), \mathfrak{F}(Y) \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and so $\mathfrak{F}(X), \mathfrak{F}(Y) \subseteq \mathbb{I}_{\mathcal{S}}$. Thus $\mathfrak{F}(X) \cup \mathfrak{F}(Y) \subseteq \mathbb{I}_{\mathcal{S}}$, and so $\lceil\lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil\rceil = \lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil$ by **L6.29**. We may then conclude that $\mathfrak{F}(X \vee Y) \subseteq \lceil\mathfrak{F}(X) \cup \mathfrak{F}(Y)\rceil = \mathfrak{F}(X) \vee \mathfrak{F}(Y)$, and so by the inclusion established above, we know that $\mathfrak{F}(X) \vee \mathfrak{F}(Y) = \mathfrak{F}(X \vee Y)$. □

Given these results, we are now in a position to extend the mapping $\mathfrak{F}$ from propositions to models. In particular, consider the following:

$\mathfrak{F}$-*Map:* Let $\mathcal{M}^{\mathfrak{F}} = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup, |\cdot|^{\mathfrak{F}} \rangle$ where $\mathcal{M} = \langle S, \star, |\cdot| \rangle \in \mathcal{C}$ and $|\cdot|^{\mathfrak{F}} = \mathfrak{F}(|\cdot|)$.

I will then show that $\mathfrak{F} : \mathcal{C} \to \mathcal{C}^{\infty}$ is a strong homomorphism with respect to logical consequence. To begin with, consider the following lemmas.

L6.36 $\mathcal{M}^{\mathfrak{F}} \in \mathcal{C}^{\infty}$ for all $\mathcal{M} \in \mathcal{C}$.

Proof. Let $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$ where $\mathcal{S} = \langle S, \star \rangle$. By **L6.25**, $\mathcal{I} = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup \rangle$ is an infinite state space, where $\mathcal{I} \in \mathbb{M}_{\infty}$ by **L6.26** and **L6.27**. Letting $p \in \mathbb{L}$ be arbitrary, we may then observe that $|p| \subseteq S$, and so $\mathfrak{F}(|p|) \in \mathbb{P}^{\infty}_{\mathcal{S}}$ by **L6.30**. Generalising on $p \in \mathbb{L}$, it follows that $\mathcal{M}^{\mathfrak{F}} = \langle \mathbb{I}_{\mathcal{S}}, \bigsqcup, |\cdot|^{\mathfrak{F}} \rangle \in \mathcal{C}^{\infty}$. □

L6.37 $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ for all $A \in \mathtt{pfs}(\mathcal{L}^{-})$.

Proof. The proof goes by induction on complexity where the base case is given by $\mathfrak{F}$-*Map*. Assume for induction that $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ for all $A \in \mathtt{pfs}(\mathcal{L}^{-})$ where $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}(A)$ with $\mathtt{comp}(A) = n$. It follows that either $A = B \wedge C$ or $A = B \vee C$, where in either case $|B|^{\mathfrak{F}} = \mathfrak{F}(|B|)$ and $|C|^{\mathfrak{F}} = \mathfrak{F}(|C|)$ by hypothesis.

Case 1: Assume $A = B \wedge C$. We may then reason as follows:

$$
\begin{aligned}
s \in |B \wedge C|^{\mathfrak{F}} \quad & \textit{iff} \quad \mathcal{M}^{\mathfrak{F}}, s \Vdash B \wedge C \\
& \textit{iff} \quad s = \bigsqcup\{a, b\} \text{ where } \mathcal{M}^{\mathfrak{F}}, a \Vdash B \text{ and } \mathcal{M}^{\mathfrak{F}}, b \Vdash C \\
& \textit{iff} \quad s = \bigsqcup\{a, b\} \text{ where } a \in |B|^{\mathfrak{F}} \text{ and } b \in |C|^{\mathfrak{F}} \\
(1) \quad & \textit{iff} \quad s = \bigsqcup\{a, b\} \text{ where } a \in \mathfrak{F}(|B|) \text{ and } b \in \mathfrak{F}(|C|) \\
& \textit{iff} \quad s \in \mathfrak{F}(|B|) \wedge \mathfrak{F}(|C|). \\
(2) \quad & \textit{iff} \quad s \in \mathfrak{F}(|B| \wedge |C|). \\
(3) \quad & \textit{iff} \quad s \in \mathfrak{F}(|B \wedge C|).
\end{aligned}
$$

Whereas (1) follows by hypothesis, (2) is given by **L6.34**, and (3) follows from **L3.4**. All of the other biconditionals are immediate from the definitions. Thus $|B \wedge C|^{\mathfrak{F}} = \mathfrak{F}(|B \wedge C|)$ and so $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$.

Case 2: Assume $B = C \vee C$ instead. Thus if follows that:

$$
\begin{aligned}
s \in |B \vee C|^{\mathfrak{F}} \;\; & \textit{iff} \;\; \mathcal{M}^{\mathfrak{F}}, s \Vdash B \vee C \\
& \textit{iff} \;\; \mathcal{M}^{\mathfrak{F}}, s \Vdash B, \text{ or } \mathcal{M}^{\mathfrak{F}}, s \Vdash C, \text{ or } \mathcal{M}^{\mathfrak{F}}, s \Vdash B \wedge C \\
& \textit{iff} \;\; s \in |B|^{\mathfrak{F}}, \text{ or } s \in |C|^{\mathfrak{F}}, \text{ or } s \in |B \wedge C|^{\mathfrak{F}} \\
& (1) \;\; \textit{iff} \;\; s \in \mathfrak{F}(|B|), \text{ or } s \in \mathfrak{F}(|C|), \text{ or } s \in \mathfrak{F}(|B \wedge C|) \\
& (2) \;\; \textit{iff} \;\; s \in \mathfrak{F}(|B|) \cup \mathfrak{F}(|C|) \cup (\mathfrak{F}(|B|) \wedge \mathfrak{F}(|C|)) \\
& (3) \;\; \textit{iff} \;\; s \in \mathfrak{F}(|B|) \vee \mathfrak{F}(|C|). \\
& (4) \;\; \textit{iff} \;\; s \in \mathfrak{F}(|B| \vee |C|). \\
& (5) \;\; \textit{iff} \;\; s \in \mathfrak{F}(|B \vee C|).
\end{aligned}
$$

As above, (1) is given by hypothesis, whereas (2) follows by the argument given in *Case 1*, (3) holds by **L6.8**, (4) is given by **L6.35**, and (5) follows from **L3.3**. Since the other biconditionals all hold by definition, we may conclude that $|B \vee C|^{\mathfrak{F}} = \mathfrak{F}(|B \vee C|)$ and so $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$.

Since $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ holds in both of the cases above, we may conclude by induction that $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$ as desired. □

P6.2 $|A|^{\mathfrak{F}} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M} \in \mathcal{C}$. By **P3.1**, $|A| \in \mathbb{P}_{\mathcal{S}}$, and so $|A| \subseteq S$. Thus $\mathfrak{F}(|A|) \in \mathbb{P}^{\infty}_{\mathcal{S}}$ by **L6.30**, and so $|A|^{\mathfrak{F}} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ follows by **L6.37**. □

P6.3 $\mathcal{M} \vDash \varphi$ iff $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$, for all $\mathcal{M} \in \mathcal{C}$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

Proof. Let $\mathcal{M} = \langle S, \star, |\cdot| \rangle$ for some $\mathcal{M} \in \mathcal{C}_{\mathcal{S}}$. The proof goes by induction on the complexity. Assume $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$ where $\mathtt{comp}^+(\varphi) = 0$. Thus either $\varphi = \$A$ or $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}(\mathcal{L}^-)$.

Case (\$)*:* Assume $\varphi = \$A$ for $A \in \mathtt{pfs}(\mathcal{L}^-)$. Thus $|A| \in \mathbb{P}_{\mathcal{S}}$ by **P3.1**, and so $s \in |A|$ just in case $\mathtt{parts}_{\mathcal{S}}(s) \in \lceil \{\mathtt{parts}_{\mathcal{S}}(x) : x \in |A|\} \rceil$ for any $s \in S$ by **L6.31**. Since $\mathfrak{F}(|A|) = \lceil \{\mathtt{parts}_{\mathcal{S}}(x) : x \in |A|\} \rceil$ by definition,

and $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ by **L6.37**, we may conclude the following:

$$s \in |A| \;\; \textit{iff} \;\; \mathtt{parts}_{\mathcal{S}}(s) \in |A|^{\mathfrak{F}}. \tag{$*$}$$

Assume $\mathcal{M} \vDash \varphi$. It follows that $\mathcal{M} \vDash \$A$, and so $|A| = \{a\}$ for some $a \in S$. By **L6.33**, $\mathfrak{F}(\{a\}) = \{\mathtt{parts}_{\mathcal{S}}(a)\}$, and so $|A|^{\mathfrak{F}} = \{\mathtt{parts}_{\mathcal{S}}(a)\}$. Thus $\mathcal{M}^{\mathfrak{F}} \vDash \$A$, and so we may conclude that $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$.

Assume $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ instead. Thus $\mathcal{M}^{\mathfrak{F}} \vDash \$A$, and so $|A|^{\mathfrak{F}} = \{b\}$ for some $b \in \mathbb{I}_{\mathcal{S}}$. Since $|A|^{\mathfrak{F}} = \lceil\{\mathtt{parts}_{\mathcal{S}}(x) : x \in |A|\}\rceil$ by definition, we may conclude that $\{b\} = \{\bigsqcup Y : \varnothing \neq Y \subseteq \{\mathtt{parts}_{\mathcal{S}}(x) : x \in |A|\}\}$. It follows that $|A| \neq \varnothing$, for otherwise $\{b\} = \varnothing$, and so there is some $x \in |A|$. In order to prove uniqueness, assume $y \in |A|$. By $(*)$, both $\mathtt{parts}_{\mathcal{S}}(x), \mathtt{parts}_{\mathcal{S}}(y) \in |A|^{\mathfrak{F}}$, and so $\mathtt{parts}_{\mathcal{S}}(x) = \mathtt{parts}_{\mathcal{S}}(y)$ given that $|A|^{\mathfrak{F}} = \{b\}$. Thus $x = y$ by **L6.20**, and so every $y \in |A|$ is such that $x = y$. Thus $|A| = \{x\}$, and so $\mathcal{M} \vDash \$A$. It follows that $\mathcal{M} \vDash \varphi$, and so together with the above, $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$.

Case ($\trianglelefteq$)*:* Assume $\varphi = A \trianglelefteq B$ for $A, B \in \mathtt{pfs}(\mathcal{L}^-)$. By **P3.1** both $|A|, |B| \in \mathbb{P}_{\mathcal{S}}$, and so $|A| \subseteq |B|$ just in case $\mathfrak{F}(|A|) \subseteq \mathfrak{F}(|B|)$ by **L6.32**. By **L6.37**, $|A|^{\mathfrak{F}} = \mathfrak{F}(|A|)$ and $|B|^{\mathfrak{F}} = \mathfrak{F}(|B|)$, and so we know:

$$|A| \subseteq |B| \;\; \textit{iff} \;\; |A|^{\mathfrak{F}} \subseteq |B|^{\mathfrak{F}}. \tag{$*$}$$

Thus $\mathcal{M} \vDash A \trianglelefteq B$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash A \trianglelefteq B$, or equivalently, $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ as desired. Given the above, $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$ such that $\mathtt{comp}^+(\varphi) = 0$.

Since $\mathcal{M}$ agree $\mathcal{M}^{\mathfrak{F}}$ on all atomic sentences of $\mathcal{L}^-$, it follows by routine induction that $\mathcal{M} \vDash \varphi$ just in case $\mathcal{M}^{\mathfrak{F}} \vDash \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$. □

By **P6.3**, $\mathfrak{F} : \mathcal{C} \to \mathcal{C}^{\infty}$ is a strong homomorphism with respect to $\vDash$. It remains to construct a function $\mathfrak{B} : \mathcal{C}^{\infty} \to \mathcal{C}$ with the same structure preserving property. Given any $\mathcal{M}_u \in \mathcal{C}^{\infty}$ where $\mathcal{M}_u = \langle S, \bigsqcup, |\cdot|_u \rangle$ let:

Backwards: $\mathfrak{B}(X) = X$ for all $X \in \mathbb{P}^{\infty}_{\mathcal{S}}$.

$\mathfrak{B}$-*Map:* Let $\mathcal{M}_u^{\mathfrak{B}} = \langle S, \star, |\cdot|_u^{\mathfrak{B}} \rangle$ where $\mathcal{M}_u = \langle S, \bigsqcup, |\cdot|_u \rangle \in \mathcal{C}^\infty$ and $|\cdot|_u^{\mathfrak{B}} = \mathfrak{B}(|\cdot|_u)$.

The following lemma proves that $\mathfrak{B}$ preserves semantic entailment in evaluating wfs of $\mathcal{L}^-$ at finite models in $\mathcal{C}$ rather than infinite models in $\mathcal{C}^\infty$.

L6.38 $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ for all $\mathcal{M}_u \in \mathcal{C}^\infty$ and $A \in \mathtt{pfs}(\mathcal{L}^-)$.

Proof. The proof goes by induction on complexity where the base case is given by $\mathfrak{B}$-*Map*. Assume for induction that $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$ where $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}(A)$ with $\mathtt{comp}(A) = n$. It follows that either $A = B \wedge C$ or $A = B \vee C$, where in either case $|B|_u^{\mathfrak{B}} = \mathfrak{B}(|B|_u)$ and $|C|_u^{\mathfrak{B}} = \mathfrak{B}(|C|_u)$ by hypothesis.

Case 1: Assume $A = B \wedge C$. We may then reason as follows:

$$
\begin{aligned}
s \in |B \wedge C|_u^{\mathfrak{B}} \quad & \textit{iff} \quad \mathcal{M}_u^{\mathfrak{B}}, s \Vdash B \wedge C \\
& \textit{iff} \quad s = a \star b \text{ where } \mathcal{M}_u^{\mathfrak{B}}, a \Vdash B \text{ and } \mathcal{M}_u^{\mathfrak{B}}, b \Vdash C \\
& \textit{iff} \quad s = a \star b \text{ where } a \in |B|_u^{\mathfrak{B}} \text{ and } b \in |C|_u^{\mathfrak{B}} \\
(*) \quad & \textit{iff} \quad s = a \star b \text{ where } a \in \mathfrak{B}(|B|_u) \text{ and } b \in \mathfrak{B}(|C|_u) \\
& \textit{iff} \quad s = \bigsqcup\{a, b\} \text{ where } a \in |B|_u \text{ and } b \in |C|_u \\
& \textit{iff} \quad \mathcal{M}_u, s \Vdash B \wedge C \\
& \textit{iff} \quad s \in |B \wedge C|_u.
\end{aligned}
$$

Whereas $(*)$ follows by hypothesis, all of the other biconditionals are immediate from the definitions. Thus $|B \wedge C|_u^{\mathfrak{B}} = |B \wedge C|_u = \mathfrak{B}(|B \wedge C|_u)$ and so $|A|_u^{\mathfrak{B}} = \mathfrak{B}(|A|_u)$ as desired.

Case 2: Assume $B = C \vee C$ instead. Thus if follows that:

$$\begin{aligned}
s \in |B \vee C|^{\mathfrak{B}}_u \quad & \textit{iff} \quad \mathcal{M}^{\mathfrak{B}}_u, s \Vdash B \vee C \\
& \textit{iff} \quad \mathcal{M}^{\mathfrak{B}}_u, s \Vdash B, \text{ or } \mathcal{M}^{\mathfrak{B}}_u, s \Vdash C, \text{ or } \mathcal{M}^{\mathfrak{B}}_u, s \Vdash B \wedge C \\
& \textit{iff} \quad s \in |B|^{\mathfrak{B}}_u, \text{ or } s \in |C|^{\mathfrak{B}}_u, \text{ or } s \in |B \wedge C|^{\mathfrak{B}}_u \\
(*) \quad & \textit{iff} \quad s \in \mathfrak{B}(|B|_u), \text{ or } s \in \mathfrak{B}(|C|_u), \text{ or } s \in \mathfrak{B}(|B \wedge C|_u) \\
& \textit{iff} \quad \mathcal{M}_u, s \Vdash B, \text{ or } \mathcal{M}_u, s \Vdash C, \text{ or } \mathcal{M}_u, s \Vdash B \wedge C \\
& \textit{iff} \quad \mathcal{M}_u, s \Vdash B \vee C \\
& \textit{iff} \quad s \in |B \vee C|_u.
\end{aligned}$$

As above, (*) is given by hypothesis, whereas the other biconditionals follow by definition. Thus $|B \vee C|^{\mathfrak{B}}_u = |B \vee C|_u = \mathfrak{B}(|B \vee C|_u)$ and so $|A|^{\mathfrak{B}}_u = \mathfrak{B}(|A|_u)$. Since $|A|^{\mathfrak{B}}_u = \mathfrak{B}(|A|_u)$ holds in both of the cases above, it follows by induction that $|A|^{\mathfrak{B}}_u = \mathfrak{B}(|A|_u)$ for all $A \in \mathtt{pfs}(\mathcal{L}^-)$. □

P6.4 $\mathcal{M}_u \models \varphi$ iff $\mathcal{M}^{\mathfrak{B}}_u \models \varphi$, for all $\mathcal{M}_u \in \mathcal{C}^\infty$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$.

Proof. Assume $\mathcal{M}_u \in \mathcal{C}^\infty$ and $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$. The proof goes by induction on the complexity of $\varphi \in (\mathcal{L}^-)$. Assume to start that $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$ where $\mathtt{comp}^+(\varphi) = 0$. Thus either $\varphi = \$A$ or $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}(\mathcal{L}^-)$. I will consider these cases in order.

Case (\$)*:* Assume $\varphi = \$A$ for $A \in \mathtt{pfs}(\mathcal{L}^-)$. Since $|A|^{\mathfrak{B}}_u = \mathfrak{B}(|A|_u)$ by the **L6.38**, where $|A|_u = \mathfrak{B}(|A|_u)$ by $\mathfrak{B}$-*Map*, it follows that $|A|_u = \{s\}$ for some $s \in S$ just in case $|A|^{\mathfrak{B}}_u = \{s\}$ for some $s \in S$. Thus $\mathcal{M}_u \models \$A$ just in case $\mathcal{M}^{\mathfrak{B}}_u \models \$A$, and so $\mathcal{M}_u \models \varphi$ just in case $\mathcal{M}^{\mathfrak{B}}_u \models \varphi$.

Case ($\trianglelefteq$)*:* Assume $\varphi = A \trianglelefteq B$ for $A, B \in \mathtt{pfs}(\mathcal{L}^-)$. Thus by **L6.38**, both $|A|^{\mathfrak{B}}_u = \mathfrak{B}(|A|_u)$ and $|B|^{\mathfrak{B}}_u = \mathfrak{B}(|B|_u)$, where $|A|_u = \mathfrak{B}(|A|_u)$ and $|B|_u = \mathfrak{B}(|B|_u)$ by $\mathfrak{B}$-*Map*, we know $|A|_u \subseteq |B|_u$ just in case $|A|^{\mathfrak{B}}_u \subseteq |B|^{\mathfrak{B}}_u$. It follows that $\mathcal{M}_u \models A \trianglelefteq B$ just in case $\mathcal{M}^{\mathfrak{B}}_u \models A \trianglelefteq B$, and so we may conclude that $\mathcal{M}_u \models \varphi$ just in case $\mathcal{M}^{\mathfrak{B}}_u \models \varphi$ as desired.

Given that $\mathcal{M}$ agree $\mathcal{M}^{\mathfrak{B}}$ on all atomic sentences of $\mathcal{L}^-$, it follows by a standard induction that $\mathcal{M} \models \varphi$ iff $\mathcal{M}^{\mathfrak{B}} \models \varphi$ for all $\varphi \in \mathtt{wfs}(\mathcal{L}^-)$. □

5.7 Negation

Recall that negation was excluded from the `pfs` of $\mathcal{L}^{-}$. I will now extend the results proven above to a logic in which this simplification is dropped. Accordingly, I will take '$\mathcal{V}$' and '$\perp$' to abbreviate '$\neg\mathcal{T}$' and '$\neg\perp$' respectively, letting $\mathcal{L}$ be the result of excluding $\mathcal{V}$ and $\perp$ from the primitive symbols of $\mathcal{L}^{-}$ while including '$\neg$'. We may then amend the formation rules as follows:

($\neg$) If A is a pfs of $\mathcal{L}$, then $\neg A$ is a pfs of $\mathcal{L}$.

Let $\texttt{pfs}^{\neg}(\mathcal{L})$ be the set of all pfss of $\mathcal{L}$ once the clause for negation has been added to the formation rules given above, replacing '$\mathcal{L}^{-}$' with '$\mathcal{L}$' throughout. We may then let $\texttt{wfs}^{\neg}(\mathcal{L})$ be the set of all wfss of $\mathcal{L}$ generated recursively from $\texttt{pfs}^{\neg}(\mathcal{L})$ *via* $\texttt{atoms}^{\neg}(\mathcal{L})$ in the same manner as before. In addition to the axioms and rules of inference for UGSN, we may now include the following:

Negation Axioms

NA1 $A \trianglelefteq \neg\neg A$. **NA2** $\neg\neg A \trianglelefteq A$.

NA3 $\neg A \wedge \neg B \trianglelefteq \neg(A \vee B)$. **NA4** $\neg(A \vee B) \trianglelefteq \neg A \wedge \neg B$.

NA5 $\neg A \vee \neg B \trianglelefteq \neg(A \wedge B)$. **NA6** $\neg(A \wedge B) \trianglelefteq \neg A \vee \neg B$.

Let $\vdash_{\text{UGS}}$ be the smallest relation closed under the axioms and rules given both here and in §5.2. A *theorem* of *The Specific Logic of Unilateral Ground* (UGS) is any $\varphi \in \texttt{wfs}^{\neg}(\mathcal{L})$ where $\vdash_{\text{UGS}} \varphi$. We may then derive the following:

Negation Equivalences

E15 $\neg A \wedge \neg B \approx \neg(A \vee B)$. **E17** $A \approx \neg\neg A$.

E16 $\neg A \vee \neg B \approx \neg(A \wedge B)$.

It remains to adapt the semantics to accommodate negation. In particular, every proposition will be shown to have a unique inverse.

Following Fine (2017b,c), I will take propositions to be ordered pairs of sets of states, where the states belonging to the first set may be referred to as the *exact verifiers* and the second set as the *exact falsifiers*. More specifically, for any $\mathcal{S} \in \mathbb{M}_\infty$, we may define the space of *bilateral propositions* as follows:

Bilateral $\mathcal{S}$-Propositions: $\mathbb{P}^{\pm}_{\mathcal{S}} = \{\langle V, F\rangle : V, F \in \mathbb{P}^{\infty}_{\mathcal{S}}\}$.

Given any $\mathcal{S} \in \mathbb{M}_\infty$ where $\mathcal{S} = \langle S, \bigsqcup\rangle$, a *bilateral $\mathcal{S}$-model* is any ordered triple $\mathcal{M} = \langle S, \bigsqcup, |\cdot|\rangle$ where every $p_i \in \mathbb{L}$ is such that $|p_i| = \langle |p_i|^+, |p_i|^-\rangle$ for some $|p_i|^{\pm} \in \mathbb{P}_{\mathcal{S}}$. Letting $\mathcal{C}^{\pm}_{\mathcal{S}}$ be the class of all bilateral $\mathcal{S}$-models for any $\mathcal{S} \in \mathbb{M}_\infty$, and $\mathcal{C}^{\pm} = \bigcup\{\mathcal{C}^{\pm}_{\mathcal{S}} : \mathcal{S} \in \mathbb{M}\}$, I will adopt the following bilateral pre-semantics:

Bilateral Pre-Semantics:

$(\mathcal{T})^+$ $\mathcal{M}, s \Vdash \mathcal{T}$ *iff* $s = s$. $(\bot)^+$ $\mathcal{M}, s \Vdash \bot$ *iff* $s \neq s$.

$(\mathcal{T})^-$ $\mathcal{M}, s \dashv \mathcal{T}$ *iff* $s = \blacksquare$. $(\bot)^-$ $\mathcal{M}, s \dashv \bot$ *iff* $s = \square$.

$(p_i)^+$ $\mathcal{M}, s \Vdash p_i$ *iff* $s \in |p_i|^+$. $(\neg)^+$ $\mathcal{M}, s \Vdash \neg A$ *iff* $\mathcal{M}, s \dashv A$.

$(p_i)^-$ $\mathcal{M}, s \dashv p_i$ *iff* $s \in |p_i|^-$. $(\neg)^-$ $\mathcal{M}, s \dashv \neg A$ *iff* $\mathcal{M}, s \Vdash A$.

$(\wedge)^+$ $\mathcal{M}, s \Vdash A \wedge B$ *iff* $s = \bigsqcup\{d, t\}$ where $\mathcal{M}, d \Vdash A$ and $\mathcal{M}, t \Vdash B$.

$(\wedge)^-$ $\mathcal{M}, s \dashv A \wedge B$ *iff* $\mathcal{M}, s \dashv A$ or $\mathcal{M}, s \dashv B$ or $\mathcal{M}, s \dashv A \vee B$.

$(\vee)^+$ $\mathcal{M}, s \Vdash A \vee B$ *iff* $\mathcal{M}, s \Vdash A$ or $\mathcal{M}, s \Vdash B$ or $\mathcal{M}, s \Vdash A \wedge B$.

$(\vee)^-$ $\mathcal{M}, s \dashv A \vee B$ *iff* $s = \bigsqcup\{d, t\}$ where $\mathcal{M}, d \dashv A$ and $\mathcal{M}, t \dashv B$.

Whereas $\mathcal{T}$ is verified by every state and falsified by the full state which we may think of as the most impossible state, $\bot$ is verified by no state and falsified by the null state which may think of as obtaining trivially. We may then observe that the negation clauses formalise the idea that the exact verifiers for $\neg A$ are exact falsifiers for A, and the exact falsifiers for $\neg A$ are exact verifiers for A. For instance, the state of affairs of my sitting exactly verifies 'I am sitting' just as much as it exactly falsifies its negation 'I am not sitting'. Whereas $(\wedge)^+$ and $(\vee)^+$ are the same as the clauses $(\wedge)$ and $(\vee)$ given in the **Unilateral Pre-Semantics** above, we find in $(\vee)^-$ that only the fusion of falsifiers for

both disjuncts will falsify the disjunction as a whole, and in $(\wedge)^-$ an exact falsifier for either conjunct will exactly falsify the conjunction as a whole, as well as a fusion of exact falsifiers for each of the conjuncts.

Having stated the definition of $\Vdash$ and $\dashv\Vdash$, we may then expand the domain of the functions $|\cdot|^{\pm}$ to include all $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$ as follows:

Bilateral Valuation: $s \in |A|^+$ *iff* $\mathcal{M}, s \Vdash A$.
$s \in |A|^-$ *iff* $\mathcal{M}, s \dashv\Vdash A$.

We may then adopt $|A| = \langle |A|^+, |A|^- \rangle$ as standard notation for the proposition that $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$ expresses in $\mathcal{M} \in \mathcal{C}^{\pm}$. The semantic clauses for the wfss of $\mathcal{L}$ are identical to those given in §5.3 with the exception of replacing $|\cdot|$ in **Unilateral Semantics** with $|\cdot|^+$. The atomic clauses will then read:

Bilateral Semantics:

$(\trianglelefteq)^{\pm}$ $\mathcal{M} \vDash A \trianglelefteq B$ *iff* $|A|^+ \subseteq |B|^+$.
$(\$)^{\pm}$ $\mathcal{M}^+ \vDash \$A$ *iff* there is exactly one $s \in |A|^+$.

The semantic clauses for $\neg$, $\wedge$, and $\vee$ are the same as given before. Accordingly, we may let $\Gamma \vDash_{\mathcal{C}^{\pm}} \varphi$ just in case for all $\mathcal{M} \in \mathcal{C}^{\pm}$, if $\mathcal{M} \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathcal{M} \vDash \varphi$. As usual, a wfs φ is $\mathcal{C}^{\pm}$-*valid* just in case $\vDash_{\mathcal{C}^{\pm}} \varphi$.

Given the addition of falsifiers to the semantics, we are now in a position to introduce the following operations where $P_i = \langle P_i^+, P_i^- \rangle$ for all $i \in I$:

Bilateral Product: Let $\bigwedge\{P_i : i \in I\} = \langle \bigwedge\{P_i^+ : i \in I\}, \bigvee\{P_i^- : i \in I\} \rangle$.

Bilateral Sum: Let $\bigvee\{P_i : i \in I\} = \langle \bigvee\{P_i^+ : i \in I\}, \bigwedge\{P_i^- : i \in I\} \rangle$.

As before, it will be useful to consider binary analogues of bilateral product and sum, letting $P \wedge Q = \bigwedge\{P, Q\}$ and $P \vee Q = \bigvee\{P, Q\}$. We may then introduce the following unary inversion operator, where $X, Y \in \mathbb{P}_{\mathcal{S}}^{\infty}$ are arbitrary:

Inversion: $\neg\langle X, Y \rangle = \langle Y, X \rangle$.

The reason for introducing falsifiers in addition to verifiers is best exhibited by the definition given above, since a set of verifiers on their own cannot determine a unique set of verifiers for its inverse. By taking propositions to be ordered pairs of a set of verifiers and set of falsifiers, inverses are uniquely determined by permuting the sets of verifiers and falsifiers.

Given the expanded formation rules for the pfss of $\mathcal{L}$, we may observe that $\mathcal{A}_{\mathcal{L}} = \langle \mathtt{pfs}^{\neg}(\mathcal{L}), \neg, \wedge, \vee, \mathcal{T}, \bot \rangle$ and $\mathcal{A}^{\pm}_{\mathcal{S}} = \langle \mathbb{P}^{\pm}_{\mathcal{S}}, \neg, \wedge, \vee, \mathcal{T}, \bot \rangle$ are both algebras with the same signature for any $\mathcal{S} \in \mathbb{M}_{\infty}$, where every model $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ induces an $\mathcal{L}$-homomorphism $|\cdot| : \mathcal{A}_{\mathcal{L}} \to \mathcal{A}^{\pm}_{\mathcal{S}}$ since $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$:

P7.1 $|A| \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

L7.1 $|\neg A| = \neg |A|$.

L7.2 $|A \wedge B| = |A| \wedge |B|$.

L7.3 $|A \vee B| = |A| \vee |B|$.

In addition to the results given above, it is easy to show that for any $\mathcal{M} \in \mathcal{C}^{\pm}$, the bilateral extremal propositions are as follows:

Bilateral Extremal Propositions

L7.4: $|\mathcal{T}| = \langle S, \{\blacksquare\} \rangle$.

L7.5: $|\bot| = \langle \varnothing, \{\square\} \rangle$.

L7.6: $|\mathcal{V}| = \langle \{\blacksquare\}, S \rangle$.

L7.7: $|\perp\!\!\!\perp| = \langle \{\square\}, \varnothing \rangle$.

I will now turn to establish the following theorem, extending the *Soundness* and *Completeness* results proven above to the expanded system UGS.

T4 (*Negation Extension*) $\Sigma \models_{\mathcal{C}^{\pm}} \varphi$ iff $\Sigma \vdash_{\text{UGS}} \varphi$.

Proof. In order to extend both *Soundness* and *Completeness* to UGS, I will introduce the functions $\mathtt{neg} : \mathtt{wfs}^{\neg}(\mathcal{L}) \to \mathtt{wfs}(\mathcal{L}^{-})$ and $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$ where $\mathtt{neg}(\Sigma) = \{\mathtt{neg}(\sigma) : \sigma \in \Sigma\}$, establishing the following results:

P7.2: $\Sigma \vdash_{\text{UGS}} \varphi$ iff $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$ for all $\Sigma \cup \{\varphi\} \subseteq \mathtt{wfs}^{\neg}(\mathcal{L})$.

P7.3: $\mathcal{M} \models \mathtt{neg}(\varphi)$ iff $\mathcal{M}^{\mathfrak{N}} \models \varphi$ for all $\mathcal{M} \in \mathcal{C}^{\infty}$ and $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$.

P7.4: $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$ is a surjection.

Given the above, we may then establish both the *Soundness* and *Completeness* for UGS over $\mathcal{C}^{\pm}$ by means of the following argument:

$\Sigma \nvdash_{\text{UGS}} \varphi$ *iff*$_1$ $\mathtt{neg}(\Sigma) \nvdash_{\text{UGSN}} \mathtt{neg}(\varphi)$

iff$_2$ $\mathtt{neg}(\Sigma) \not\models_{\mathcal{C}^{\infty}} \mathtt{neg}(\varphi)$

iff$_3$ some $\mathcal{M} \in \mathcal{C}^{\infty}$ is such that $\mathcal{M} \models \mathtt{neg}(\sigma)$ for all $\sigma \in \Sigma$ but $\mathcal{M} \not\models \mathtt{neg}(\varphi)$

iff$_4$ some $\mathcal{M}_u \in \mathcal{C}^{\pm}$ is such that $\mathcal{M}_u \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}_u \not\models \varphi$

iff$_5$ $\Sigma \not\models_{\mathcal{C}^{\pm}} \varphi$.

Here (1) is given by **P7.2**, (2) follows from **Theorem T3**, both (3) and (5) hold by definition. It remains to establish (4).

Assume for discharge that there is some $\mathcal{M} \in \mathcal{C}^{\infty}$ such that $\mathcal{M} \models \mathtt{neg}(\sigma)$ for all $\sigma \in \Sigma$ but where $\mathcal{M} \not\models \mathtt{neg}(\varphi)$. It follows by **P7.3** that $\mathcal{M}^{\mathfrak{N}} \models \sigma$ for all $\sigma \in \Sigma$ but $\mathcal{M}^{\mathfrak{N}} \not\models \varphi$, where existentially generalising on $\mathcal{M}^{\mathfrak{N}} \in \mathcal{C}^{\pm}$ completes the forward direction. Assume instead that there is some $\mathcal{M}_u \in \mathcal{C}^{\pm}$ such that $\mathcal{M}_u \models \sigma$ for all $\sigma \in \Sigma$ but where $\mathcal{M}_u \not\models \varphi$. By **P7.4**, we know that there there is some $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M}^{\mathfrak{N}} = \mathcal{M}_u$. Thus it follows by **P7.3** that $\mathcal{M} \models \mathtt{neg}(\sigma)$ for all $\sigma \in \Sigma$ but where $\mathcal{M} \not\models \mathtt{neg}(\varphi)$, thereby completing the reverse direction. We may then conclude that $\Sigma \models_{\mathcal{C}^{\pm}} \varphi$ iff $\Sigma \vdash_{\text{UGS}} \varphi$. □

The remainder of the present section will be devoted to proving the results stated above. The following section will draw connections with bilattice theory.

R2$^{\pm}$ If $\Gamma \vdash_{\text{UGS}} \varphi$, then $\Gamma_{[A/p]} \vdash_{\text{UGS}} \varphi_{[A/p]}$. (*Uniform Substitution*)

Proof. The proof is similar to **AR2**, where identical reasoning applies to the axioms **NA2** – **NA6** and the axioms and rules in UGSN. □

E15 $\vdash_{\text{UGSN}} \neg A \wedge \neg B \approx \neg(A \vee B)$.

Proof. Follows from **NA3** and **NA4**. □

E16 $\vdash_{\text{UGSN}} \neg A \vee \neg B \approx \neg(A \wedge B)$.

Proof. Follows from **NA5** and **NA6**. □

E17 $\vdash_{\text{UGSN}} A \approx \neg\neg A$.

Proof. Follows from **NA1** and **NA2**. □

L7.1 $|\neg A| = \langle |A|^-, |A|^+ \rangle$ if $\mathcal{M} \in \mathcal{C}^{\pm}$ and $|A| \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Given the semantics for negation, we know that:

$$\begin{array}{lll} s \in |\neg A|^+ & \textit{iff} & \mathcal{M}, s \Vdash \neg A \\ & \textit{iff} & \mathcal{M}, s \dashv\Vdash A \\ & \textit{iff} & s \in |A|^-. \qquad [^+_\neg] \end{array} \qquad \begin{array}{lll} s \in |\neg A|^- & \textit{iff} & \mathcal{M}, s \dashv\Vdash \neg A \\ & \textit{iff} & \mathcal{M}, s \Vdash A \\ & \textit{iff} & s \in |A|^+. \qquad [^-_\neg] \end{array}$$

It follows that $|\neg A|^+ = |A|^-$ and $|\neg A|^- = |A|^+$, and so we may conclude that $|\neg A| = \langle |A|^-, |A|^+ \rangle$ as desired. □

L7.2 $|A \wedge B| = \langle |A|^+ \wedge |B|^+, |A|^- \vee |B|^- \rangle$ if $\mathcal{M} \in \mathcal{C}^{\pm}$ and $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Assume $\mathcal{M} \in \mathcal{C}^{\pm}$ and $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. We first demonstrate that $|A \wedge B|^+ = |A|^+ \wedge |B|^+$ and $|A \vee B|^- = |A|^- \wedge |B|^-$ as follows:

$$\begin{array}{lll} s \in |A \wedge B|^+ & \textit{iff} & \mathcal{M}, s \Vdash A \wedge B \\ & \textit{iff} & s = \bigsqcup\{d, t\} \text{ where } \mathcal{M}, d \Vdash A \text{ and } \mathcal{M}, t \Vdash B \\ & \textit{iff} & s = \bigsqcup\{d, t\} \text{ where } d \in |A|^+ \text{ and } t \in |B|^+ \\ & \textit{iff} & s \in |A|^+ \wedge |B|^+. \qquad [^+_\wedge] \end{array}$$

$$\begin{array}{lll} s \in |A \vee B|^- & \textit{iff} & \mathcal{M}, s \dashv\Vdash A \vee B \\ & \textit{iff} & s = \bigsqcup\{d, t\} \text{ where } \mathcal{M}, d \dashv\Vdash A \text{ and } \mathcal{M}, t \dashv\Vdash B \\ & \textit{iff} & s = \bigsqcup\{d, t\} \text{ where } d \in |A|^- \text{ and } t \in |B|^- \\ & \textit{iff} & s \in |A|^- \wedge |B|^-. \qquad [^-_\vee] \end{array}$$

The biconditionals in the arguments above hold by definition. Given that $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$, it follows that $|A|^-, |B|^- \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Consider the following:

$$\begin{aligned}
s_i \in |A \wedge B|^- \quad & \textit{iff} \quad \mathcal{M}, s_i \dashv\Vdash A \wedge B \\
& \textit{iff} \quad \mathcal{M}, s_i \dashv\Vdash A, \text{ or } \mathcal{M}, s_i \dashv\Vdash B, \text{ or } \mathcal{M}, s_i \dashv\Vdash A \vee B \\
& \textit{iff} \quad s_i \in |A|^-, \text{ or } s_i \in |B|^-, \text{ or } s_i \in |A \vee B|^- \\
& \textit{iff} \quad s_i \in |A|^- \cup |B|^- \cup |A \vee B|^- \\
(\dagger) \quad & \textit{iff} \quad s_i \in |A|^- \cup |B|^- \cup (|A|^- \wedge |B|^-) \\
(\ddagger) \quad & \textit{iff} \quad s \in |A|^- \vee |B|^-. \qquad [\overset{-}{\wedge}]
\end{aligned}$$

Each of the biconditionals above hold by definition with the exception of (†) which follows from $[\overset{-}{\vee}]$, and (‡) which is given by **L6.8**. Thus it follows that $|A \wedge B|^- = |A|^- \vee |B|^-$. Since $|A \wedge B| = \langle |A \wedge B|^+, |A \wedge B|^- \rangle$ by **Valuation**, we know that $|A \wedge B| = \langle |A|^+ \wedge |B|^+, |A|^- \vee |B|^- \rangle$. □

L7.3 $|A \vee B| = \langle |A|^+ \vee |B|^+, |A|^- \wedge |B|^- \rangle$ if $\mathcal{M} \in \mathcal{C}^{\pm}$ and $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Assume $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. We know $|A \vee B| = \langle |A \vee B|^+, |A \vee B|^- \rangle$ by **Valuation**, and $|A|^+, |B|^+ \in \mathbb{P}^{\infty}_{\mathcal{S}}$ given that $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. Thus:

$$\begin{aligned}
s \in |A \vee B|^+ \quad & \textit{iff} \quad \mathcal{M}, s \Vdash A \vee B \\
& \textit{iff} \quad \mathcal{M}, s \Vdash A, \text{ or } \mathcal{M}, s \Vdash B, \text{ or } \mathcal{M}, s \Vdash A \wedge B \\
& \textit{iff} \quad s \in |A|^+, \text{ or } s \in |B|^+, \text{ or } s \in |A \wedge B|^+ \\
& \textit{iff} \quad s \in |A|^+ \cup |B|^+ \cup |A \wedge B|^+ \\
(\dagger) \quad & \textit{iff} \quad s \in |A|^+ \cup |B|^+ \cup (|A|^+ \wedge |B|^+) \\
(\ddagger) \quad & \textit{iff} \quad s \in |A|^+ \vee |B|^+. \qquad [\overset{+}{\vee}]
\end{aligned}$$

The biconditionals above hold by definition with the exception of (†) which follows from $[\overset{+}{\wedge}]$, and (‡) which is given by **L6.8**. Thus it follows that $|A \vee B|^+ = |A|^+ \vee |B|^+$, and since $|A \vee B|^- = |A|^- \wedge |B|^-$ as shown by $[\overset{-}{\vee}]$ in **L7.2**, we know that $|A \vee B| = \langle |A|^+ \vee |B|^+, |A|^- \wedge |B|^- \rangle$. □

P7.1 $|A| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for all $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ and $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Proof. Assume $\mathcal{M} \in \mathcal{C}^{\pm}$. By definition, $|p| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for every $p \in \mathbb{L}$, where $|e| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for all $e \in \mathbb{E}$ by **Bilateral Extremal Propositions**, thereby establishing the base case. Assume $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for induction. Thus $|A|^{\pm}, |B|^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$, where we know by **L7.1**, **L7.2**, and **L7.3** that:

$$\begin{aligned} |\neg A| &= \langle |A|^{-}, |A|^{+} \rangle \\ |A \wedge B| &= \langle |A|^{+} \wedge |B|^{+}, |A|^{-} \vee |B|^{-} \rangle \\ |A \vee B| &= \langle |A|^{+} \vee |B|^{+}, |A|^{-} \wedge |B|^{-} \rangle. \end{aligned}$$

We know by **L6.4** and **L6.5** that $|A|^{+} \wedge |B|^{+}$, $|A|^{-} \vee |B|^{-}$, $|A|^{+} \vee |B|^{+}$, and $|A|^{-} \wedge |B|^{-}$ are members of $\mathbb{P}^{\infty}_{\mathcal{S}}$, and so $|\neg A|, |A \wedge B|, |A \vee B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$. It follows by induction that $|A| \in \mathbb{P}_{\mathcal{S}}$ for all $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$. □

L7.4 $|\mathcal{T}| = \langle S, \{\blacksquare\} \rangle$.

Proof. Follows from the **Bilateral Pre-Semantics**. □

L7.5 $|\bot| = \langle \varnothing, \{\square\} \rangle$.

Proof. Follows from the **Bilateral Pre-Semantics**. □

L7.6 $|\mathcal{V}| = \langle \{\blacksquare\}, S \rangle$.

Proof. By abbreviation, $|\mathcal{V}| = |\neg\mathcal{T}|$, where $|\neg\mathcal{T}| = \neg|\mathcal{T}|$ by **L7.1**. Thus $\neg|\mathcal{T}| = \neg\langle S, \{\blacksquare\} \rangle$ by **L7.4**, where $\neg\langle S, \{\blacksquare\} \rangle = \langle \{\blacksquare\}, S \rangle$ holds by definition. We may then conclude that $|\mathcal{V}| = \langle \{\blacksquare\}, S \rangle$ as needed. □

L7.7 $|\perp\!\!\!\perp| = \langle \{\square\}, \varnothing \rangle$.

Proof. By abbreviation, $|\perp\!\!\!\perp| = |\neg\bot|$, where $|\neg\bot| = \neg|\bot|$ by **L7.1**. Thus $\neg|\bot| = \neg\langle \varnothing, \{\square\} \rangle$ by **L7.5**, where $\neg\langle \varnothing, \{\square\} \rangle = \langle \{\square\}, \varnothing \rangle$ by definition. We may then conclude that $|\perp\!\!\!\perp| = \langle \{\square\}, \varnothing \rangle$ as needed. □

We may now introduce the function $\mathtt{neg} : \mathtt{wfs}^{\neg}(\mathcal{L}) \to \mathtt{wfs}(\mathcal{L}^{-})$ which works by first distributing negation over disjunction and conjunction in the sub-pfs

of any $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$, after which the literals which occur therein are mapped to unique sentence letters in $\mathbb{L}$. In particular, consider the following:

$$
\begin{aligned}
\mathtt{neg}(\mathcal{T}) &= \mathcal{T} \\
\mathtt{neg}(\bot) &= \bot \\
\mathtt{neg}(\neg\mathcal{T}) &= \mathcal{V} \\
\mathtt{neg}(\neg\bot) &= \bot\!\!\!\bot \\
\mathtt{neg}(p_i) &= p_{2i} \\
\mathtt{neg}(\neg p_i) &= p_{2i+1} \\
\mathtt{neg}(\neg\neg A) &= \mathtt{neg}(A) \\
\mathtt{neg}(A \wedge B) &= \mathtt{neg}(A) \wedge \mathtt{neg}(B) \\
\mathtt{neg}(A \vee B) &= \mathtt{neg}(A) \vee \mathtt{neg}(B) \\
\mathtt{neg}(\neg(A \vee B)) &= \mathtt{neg}(\neg A) \wedge \mathtt{neg}(\neg B) \\
\mathtt{neg}(\neg(A \wedge B)) &= \mathtt{neg}(\neg A) \vee \mathtt{neg}(\neg B) \\
\mathtt{neg}(A \trianglelefteq B) &= \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(B) \\
\mathtt{neg}(A \ntrianglelefteq B) &= \mathtt{neg}(A) \ntrianglelefteq \mathtt{neg}(B) \\
\mathtt{neg}(\$A) &= \$\,\mathtt{neg}(A) \\
\mathtt{neg}(\neg\$A) &= \neg\$\,\mathtt{neg}(A) \\
\mathtt{neg}(\neg\neg\varphi) &= \neg\neg\,\mathtt{neg}(\varphi) \\
\mathtt{neg}(\varphi \wedge \psi) &= \mathtt{neg}(\varphi) \wedge \mathtt{neg}(\psi) \\
\mathtt{neg}(\varphi \vee \psi) &= \mathtt{neg}(\varphi) \vee \mathtt{neg}(\psi) \\
\mathtt{neg}(\neg(\varphi \vee \psi)) &= \neg(\mathtt{neg}(\varphi) \vee \mathtt{neg}(\psi)) \\
\mathtt{neg}(\neg(\varphi \wedge \psi)) &= \neg(\mathtt{neg}(\varphi) \wedge \mathtt{neg}(\psi)).
\end{aligned}
$$

The function `neg` works by replacing the elements of $\mathtt{pfs}^{\neg}(\mathcal{L})$ which occur in any $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$ with $\mathtt{pfs}(\mathcal{L}^{-})$, thereby returning a sentence which belongs to

$\mathtt{wfs}(\mathcal{L}^-)$. We may then show that $\mathtt{neg} : \mathtt{wfs}^\neg(\mathcal{L}) \to \mathtt{wfs}(\mathcal{L}^-)$ as follows.

L7.8 $\mathtt{neg}(A) \in \mathtt{pfs}(\mathcal{L}^-)$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Follows by a routine induction proof. □

L7.9 $\mathtt{neg}(\varphi) \in \mathtt{wfs}(\mathcal{L}^-)$ for all $\varphi \in \mathtt{wfs}^\neg(\mathcal{L})$.

Proof. Follows from **L7.8** by a routine induction proof. □

Having defined the function $\mathtt{neg} : \mathtt{wfs}^\neg(\mathcal{L}) \to \mathtt{wfs}(\mathcal{L}^-)$, we may now draw on uniform substitution in order to define a function $\mathtt{gen} : \mathtt{wfs}(\mathcal{L}^-) \to \mathtt{wfs}^\neg(\mathcal{L})$:

Substitution: $\mathtt{gen}(\mathcal{A}) = \mathcal{A}_{[\neg\bot/\dot{\bot}][\neg\mathcal{T}/\mathcal{V}][p_i/p_{2i}][\neg p_i/p_{2i+1}]}$
$\mathtt{gen}(\varphi) = \varphi_{[\neg\bot/\dot{\bot}][\neg\mathcal{T}/\mathcal{V}][p_i/p_{2i}][\neg p_i/p_{2i+1}]}$.

The following two lemmas show that $\mathtt{gen}(\mathtt{neg}(\varphi))$ and φ are ground-theoretically equivalent for all $\varphi \in \mathtt{wfs}^\neg(\mathcal{L})$.

L7.10 $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Let $A \in \mathtt{pfs}^\neg(\mathcal{L})$. The proof goes by induction on complexity.

Assume to start that $\mathtt{comp}(A) < 2$. Thus either: (1) $A = \mathcal{T}$; (2) $A = \neg\mathcal{T}$; (3) $A = \bot$; (4) $A = \neg\bot$; (5) $A = p_i$; or (6) $A = \neg p_i$. Consider:

Case 1: Assume $A = \mathcal{T}$. Thus $\mathtt{neg}(A) = \mathcal{T}$, so $\mathtt{gen}(\mathtt{neg}(A)) = \mathcal{T}$. We may then conclude that $\mathtt{gen}(\mathtt{neg}(A)) = A$.

Case 2: Assume $A = \neg\mathcal{T}$. Thus $\mathtt{neg}(A) = \mathcal{V}$, and so $\mathtt{gen}(\mathtt{neg}(A)) = \neg\mathcal{T}$. We may then conclude that $\mathtt{gen}(\mathtt{neg}(A)) = A$.

Case 3: Assume $A = \bot$. Thus $\mathtt{neg}(A) = \bot$, so $\mathtt{gen}(\mathtt{neg}(A)) = \bot$. We may then conclude that $\mathtt{gen}(\mathtt{neg}(A)) = A$.

Case 4: Assume $A = \neg\bot$. Thus $\mathtt{neg}(A) = \dot{\bot}$, and so $\mathtt{gen}(\mathtt{neg}(A)) = \neg\bot$. We may then conclude that $\mathtt{gen}(\mathtt{neg}(A)) = A$.

Case 5: Assume $A = p_i$. Thus $\mathtt{neg}(A) = p_{2i}$, and so $\mathtt{gen}(\mathtt{neg}(A)) = A$. We may then conclude that $\mathtt{gen}(\mathtt{neg}(A)) = A$.

Case 6: Assume $A = \neg p_i$. Thus $\mathtt{neg}(A) = p_{2i+1}$, and so $\mathtt{gen}(\mathtt{neg}(A)) = A$. We may then conclude that $\mathtt{gen}(\mathtt{neg}(A)) = A$.

Since $\mathtt{gen}(\mathtt{neg}(A)) = A$ holds in each of the cases above, we know by **E1** that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$, thereby completing the base case.

Assume for induction that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ for all $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$ such that $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$ be such that $\mathtt{comp}(A) = n$. It follows that either: (1) $A = \neg\neg B$; (2) $A = B \wedge C$; (3) $A = B \vee C$; (4) $A = \neg(B \vee C)$; or (5) $A = \neg(B \wedge C)$. Consider the following:

Case 1: Assume $A = \neg\neg B$. By definition, $\mathtt{neg}(A) = \mathtt{neg}(B)$, and so $\mathtt{gen}(\mathtt{neg}(A)) = \mathtt{gen}(\mathtt{neg}(B))$ where $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \approx B$ by hypothesis. Thus if follows that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx B$. Since $\vdash_{\text{UGS}} B \approx \neg\neg B$ by **E17**, we may conclude that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx \neg\neg B$ by **GA9**, and so $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ follows by assumption.

Case 2: Assume $A = B \wedge C$. Thus $\mathtt{neg}(A) = \mathtt{neg}(B) \wedge \mathtt{neg}(C)$, and so $\mathtt{gen}(\mathtt{neg}(A)) = \mathtt{gen}(\mathtt{neg}(B) \wedge \mathtt{neg}(C))$. It follows by definition that $\mathtt{gen}(\mathtt{neg}(B) \wedge \mathtt{neg}(C)) = \mathtt{gen}(\mathtt{neg}(B)) \wedge \mathtt{gen}(\mathtt{neg}(C))$, where we know that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \approx B$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(C)) \approx C$ by hypothesis. It follows that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \wedge \mathtt{gen}(\mathtt{neg}(C)) \approx B \wedge C$ by **GA8**, and so $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ follows from the identities above.

Case 3: Assume $A = B \vee C$. Thus $\mathtt{neg}(A) = \mathtt{neg}(B) \vee \mathtt{neg}(C)$, and $\mathtt{gen}(\mathtt{neg}(A)) = \mathtt{gen}(\mathtt{neg}(B) \vee \mathtt{neg}(C))$. However, we know by definition that $\mathtt{gen}(\mathtt{neg}(B) \vee \mathtt{neg}(C)) = \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C))$, where both $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \approx B$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(C)) \approx C$ follow by hypothesis. Additionally, $\vdash_{\text{UGS}} B \trianglelefteq B \vee C$ and $\vdash_{\text{UGS}} C \trianglelefteq B \vee C$ by **GA1** and **GA2**, and so $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \trianglelefteq B \vee C$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(C)) \trianglelefteq B \vee C$ by **GA9**. Thus $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C)) \trianglelefteq B \vee C$ follows by **AR1**. Similarly, we know that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \trianglelefteq \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C))$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(C)) \trianglelefteq \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C))$ by **GA1** and **GA2**, and so it follows that both $\vdash_{\text{UGS}} B \trianglelefteq \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C))$ and $\vdash_{\text{UGS}} C \trianglelefteq \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C))$ by **GA9**. Thus we know by **AR1** that $\vdash_{\text{UGS}} B \vee C \trianglelefteq \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C))$, and so given the above, $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \vee \mathtt{gen}(\mathtt{neg}(C)) \approx B \vee C$. Thus we may conclude by the identities above that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ as desired.

Case 4: Assume $A = \neg(B \vee C)$. Thus $\mathtt{neg}(A) = \mathtt{neg}(\neg B) \wedge \mathtt{neg}(\neg C)$,

and so $\mathtt{gen}(\mathtt{neg}(A)) = \mathtt{gen}(\mathtt{neg}(\neg B)) \wedge \mathtt{gen}(\mathtt{neg}(\neg C))$. By hypothesis, $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \approx \neg B$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg C)) \approx \neg C$, and so by **GA8**, $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \wedge \mathtt{gen}(\mathtt{neg}(\neg C)) \approx \neg B \wedge \neg C$. Additionally, we know that $\vdash_{\text{UGS}} \neg B \wedge \neg C \approx \neg(B \wedge C)$ by **E15**, and so it follows by **GA9** that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \wedge \mathtt{gen}(\mathtt{neg}(\neg C)) \approx \neg(B \vee C)$. Given the identities above, we may conclude that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$.

Case 5: Assume $A = \neg(B \wedge C)$. Thus $\mathtt{neg}(A) = \mathtt{neg}(\neg B) \vee \mathtt{neg}(\neg C)$, and so $\mathtt{gen}(\mathtt{neg}(A)) = \mathtt{gen}(\mathtt{neg}(\neg B) \vee \mathtt{neg}(\neg C))$. It follows by definition that $\mathtt{gen}(\mathtt{neg}(\neg B) \vee \mathtt{neg}(\neg C)) = \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C))$, where $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \approx \neg B$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg C)) \approx \neg C$ by hypothesis. By **GA1** and **GA2**, $\vdash_{\text{UGS}} \neg B \trianglelefteq \neg B \vee \neg C$ and $\vdash_{\text{UGS}} \neg C \trianglelefteq \neg B \vee \neg C$, and so $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \trianglelefteq \neg B \vee \neg C$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg C)) \trianglelefteq \neg B \vee \neg C$ by **GA9**. Thus $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C)) \trianglelefteq \neg B \vee \neg C$ follows by **AR1**. Similarly, $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \trianglelefteq \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C))$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg C)) \trianglelefteq \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C))$ follow by **GA1** and **GA2**, and so by **GA9** $\vdash_{\text{UGS}} \neg B \trianglelefteq \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C))$ and $\vdash_{\text{UGS}} \neg C \trianglelefteq \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C))$. Thus we may conclude that $\vdash_{\text{UGS}} \neg B \vee \neg C \trianglelefteq \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C))$ by **AR1**, and so given the above, $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C)) \approx \neg B \vee \neg C$. However, we also know by **E16** $\vdash_{\text{UGS}} \neg B \vee \neg C \approx \neg(B \wedge C)$, and so again by **GA9** it follows that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\neg B)) \vee \mathtt{gen}(\mathtt{neg}(\neg C)) \approx \neg(B \wedge C)$. Thus by the identities above $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ as desired.

Thus $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ holds in each of the cases above, and so it follows by induction that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ for all $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$. □

L7.11 $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi)) \leftrightarrow \varphi$, for all $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$.

Proof. Assume $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$. The proof goes by induction where we may assume to start that $\mathtt{comp}^{+}(\varphi) = 0$. Thus either $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, or else $\varphi = \$A$ for some $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Case I: Assume $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. It follows that $\mathtt{gen}(\mathtt{neg}(\varphi)) = \mathtt{gen}(\mathtt{neg}(A)) \trianglelefteq \mathtt{gen}(\mathtt{neg}(B))$, and so by **L7.10** both $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ and $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(B)) \approx B$. Thus we know by

GA9 that $\vdash_{\text{UGS}} (\mathtt{gen}(\mathtt{neg}(A)) \trianglelefteq \mathtt{gen}(\mathtt{neg}(B))) \leftrightarrow (A \trianglelefteq B)$, and so it follows that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi)) \leftrightarrow \varphi$ given the identity above.

Case II: Assume $\varphi = \$A$ for some $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$. Thus it follows that $\mathtt{gen}(\mathtt{neg}(\varphi)) = \$\,\mathtt{gen}(\mathtt{neg}(A))$, where $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(A)) \approx A$ by **L7.10**. By **SP2**, $\mathtt{gen}(\mathtt{neg}(A)) \approx A \vdash_{\text{UGS}} \$\,\mathtt{gen}(\mathtt{neg}(A)) \leftrightarrow \$A$, and so we may conclude that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi)) \leftrightarrow \varphi$.

Since $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi)) \leftrightarrow \varphi$ holds in both of the base cases, we know that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi)) \leftrightarrow \varphi$. We may then conclude by a routine induction proof that $\vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi)) \leftrightarrow \varphi$ for all $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$. □

P7.2 $\Sigma \vdash_{\text{UGS}} \varphi$ iff $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$, for all $\Sigma \cup \{\varphi\} \subseteq \mathtt{wfs}^{\neg}(\mathcal{L})$.

Proof. Let $\Sigma \cup \{\varphi\} \subseteq \mathtt{wfs}^{\neg}(\mathcal{L})$, and assume $\Sigma \vdash_{\text{UGS}} \varphi$. We argue by induction on the length of proof, where we assume to start that $\Sigma \vdash^{0}_{\text{UGS}} \varphi$. It follows that $\Sigma \vdash_{\text{UGS}} \varphi$ holds by one of the rules or axioms of UG.

Case GA1: Assume $\Sigma \vdash_{\text{UGS}} \varphi$ follows by **GA1**, and so $\Sigma = \varnothing$ and $\varphi = A \trianglelefteq A \vee B$ for some $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. However, we know by **L7.8** that $\mathtt{neg}(A), \mathtt{neg}(B) \in \mathtt{pfs}(\mathcal{L}^{-})$, and so $\vdash_{\text{UGSN}} \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(A) \vee \mathtt{neg}(B)$ since **GA1** belongs to UGSN. Thus $\vdash_{\text{UGSN}} \mathtt{neg}(A \trianglelefteq A \vee B)$ as desired. We may then conclude that $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$. I will omit consideration of the other grounding rules and axioms, all of which are similar.

Case SP1: Assume $\Sigma \vdash_{\text{UGS}} \varphi$ follows by **SP1**, and so $\Sigma = \{\$A\}$ and $\varphi = A \ntrianglelefteq \bot$. We know by **L7.8** that $\mathtt{neg}(A) \in \mathtt{pfs}(\mathcal{L}^{-})$, and so $\$\,\mathtt{neg}(A) \vdash_{\text{UGSN}} \mathtt{neg}(A) \ntrianglelefteq \bot$ since **SP1** belongs to UGSN. It follows that $\mathtt{neg}(\$A) \vdash_{\text{UGSN}} \mathtt{neg}(A \ntrianglelefteq \bot)$, since by definition both $\mathtt{neg}(\$A) = \$\,\mathtt{neg}(A)$ and $\mathtt{neg}(A \ntrianglelefteq \bot) = \mathtt{neg}(A) \ntrianglelefteq \bot$. Thus $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$. I will omit consideration of **SP2** – **SP5**, all of which are similar to **SP1**.

Case NA1: Assume $\Sigma \vdash_{\text{UGS}} \varphi$ follows by **NA1**, and so $\Sigma = \varnothing$ and $\varphi = A \trianglelefteq \neg\neg A$ for some $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$. By **L7.8**, $\mathtt{neg}(A) \in \mathtt{pfs}(\mathcal{L}^{-})$, and so $\vdash_{\text{UGSN}} \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(A)$ follows by **E1**. However, we also know that $\mathtt{neg}(A \trianglelefteq \neg\neg A) = \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(A)$, and so $\vdash_{\text{UGSN}} \mathtt{neg}(A \trianglelefteq \neg\neg A)$. Thus $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$ as desired. I will omit consideration of **NA2**

– **NA6** which are similar.

Since $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$ holds in each of the cases above, we may conclude that if $\Sigma \vdash^{0}_{\text{UGS}} \varphi$, then $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$. Assume for induction that for all $k < n$, if $\Sigma \vdash^{k}_{\text{UGS}} \varphi$, then $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$. Assume for discharge that $\Sigma \vdash^{n}_{\text{UGS}} \varphi$. Thus $\Sigma \vdash^{n}_{\text{UGS}} \varphi$ follows from one of the metarules included in UG. Consider the following cases.

Case SP6: Assume $\Sigma \vdash^{n}_{\text{UGS}} \varphi$ follows by **SP6**. Thus $\varphi = A \trianglelefteq B$ where $\Sigma \vdash^{k}_{\text{UGS}} \$p_i \to [(p_i \trianglelefteq A) \to (p_i \trianglelefteq B)]$ for some $k < n$ and $p_i \in \mathbb{L}$ which does not occur in A, B, or in any $\sigma \in \Sigma$. We may then conclude by hypothesis that $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\$p_i \to [(p_i \trianglelefteq A) \to (p_i \trianglelefteq B)])$, and so $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \$p_{2i} \to ([p_{2i} \trianglelefteq \mathtt{neg}(A)] \to [p_{2i} \trianglelefteq \mathtt{neg}(B)])$. Given that p_i does not occur in A, B, or in any $\sigma \in \Sigma$, we may observe that p_{2i} does not occur in $\mathtt{neg}(A)$, $\mathtt{neg}(B)$, or in $\mathtt{neg}(\sigma)$ for any $\sigma \in \Sigma$. It follows that $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(B)$ by **SP6**, and so $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(A \trianglelefteq B)$. Thus we may then conclude that $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$ as desired. I will omit consideration of **SP7** which is similar.

Given the cases above, it follows by induction that if $\Sigma \vdash_{\text{UGS}} \varphi$, then $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$. In order to prove the converse, assume instead that $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$. It follows that $\mathtt{neg}(\Sigma) \vdash_{\text{UGS}} \mathtt{neg}(\varphi)$ since UG extends UGSN. By $\mathbf{R2}^{\pm}$, $\mathtt{gen}(\mathtt{neg}(\Sigma)) \vdash_{\text{UGS}} \mathtt{gen}(\mathtt{neg}(\varphi))$, and so $\Sigma \vdash_{\text{UGS}} \varphi$ by **L7.11**. Together with the above, we may conclude that $\Sigma \vdash_{\text{UGS}} \varphi$ just in case $\mathtt{neg}(\Sigma) \vdash_{\text{UGSN}} \mathtt{neg}(\varphi)$ for all $\Sigma \cup \{\varphi\} \subseteq \mathtt{wfs}^{\neg}(\mathcal{L})$. □

We are now in a position to introduce the function $\mathfrak{N} : \mathcal{C}^{\infty} \to \mathcal{C}^{\pm}$, showing every $\mathcal{M} \in \mathcal{C}^{\infty}$ is such that $\mathcal{M} \vDash \mathtt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \vDash \varphi$. Consider:

$\mathfrak{N}$-*Model:* Given any $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle$, let $|\cdot|^{\mathfrak{N}} = \langle |\cdot|^{\mathfrak{N}^+}, |\cdot|^{\mathfrak{N}^-} \rangle$ where $|p_i|^{\mathfrak{N}^+} = |p_{2i}|$ and $|p_i|^{\mathfrak{N}^-} = |p_{2i+1}|$ for all $p_i \in \mathbb{L}$.

$\mathfrak{N}$-*Map:* Given any $\mathcal{M} \in \mathcal{C}^{\infty}$ where $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle$, let $\mathcal{M}^{\mathfrak{N}} = \langle S, \bigsqcup, |\cdot|^{\mathfrak{N}} \rangle$.

We may then prove the following lemmas.

L7.12 $\mathcal{M}^{\mathfrak{N}} \in \mathcal{C}^{\pm}$ for all $\mathcal{M} \in \mathcal{C}^{\infty}$.

Proof. Let $\mathcal{M} \in \mathcal{C}^\infty$. It follows that $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle$ for some $\mathcal{S} \in \mathbb{M}_\infty$ where $\mathcal{S} = \langle S, \bigsqcup \rangle$. By definition, $|p| \in \mathbb{P}^\infty_\mathcal{S}$ for all $p \in \mathbb{L}$. Choose some $p_i \in \mathbb{L}$. It follows that both $|p_{2i}|, |p_{2i+1}| \in \mathbb{P}^\infty_\mathcal{S}$, and so $\langle |p_{2i}|, |p_{2i+1}| \rangle \in \mathbb{P}^\pm_\mathcal{S}$. Thus $|p_i|^\mathfrak{N} \in \mathbb{P}^\infty_\mathcal{S}$ since $|p_i|^\mathfrak{N} = \langle |p_i|^{\mathfrak{N}^+}, |p_i|^{\mathfrak{N}^-} \rangle$ where $|p_i|^{\mathfrak{N}^+} = |p_{2i}|$ and $|p_i|^{\mathfrak{N}^-} = |p_{2i+1}|$. Since $p_i \in \mathbb{L}$ was arbitrary, we may conclude that $\mathcal{M}^\mathfrak{N} \in \mathcal{C}^\pm$ where $\mathcal{M}^\mathfrak{N} = \langle S, \bigsqcup, |\cdot|^\mathfrak{N} \rangle$. □

L7.13 $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$ and $\mathcal{M} \in \mathcal{C}^\infty$.

Proof. The proof goes by induction on complexity. Assume to start that $A \in \mathtt{pfs}^\neg(\mathcal{L})$ where $\mathtt{comp}(A) < 2$. Thus either: (1) $A = \mathcal{T}$; (2) $A = \neg\mathcal{T}$; (3) $A = \bot$; (4) $A = \neg\bot$; (5) $A = p_i$; or (6) $A = \neg p_i$. Consider:

Case 1: Let $A = \mathcal{T}$, and so $\mathtt{neg}(A) = \mathcal{T}$. Thus $|\mathtt{neg}(A)| = |\mathcal{T}|$. Since $|\mathcal{V}|^{\mathfrak{N}^+} = |\mathcal{T}|$, we know that $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 2: Let $A = \neg\mathcal{T}$, and so $\mathtt{neg}(A) = \mathcal{V}$. Thus it follows that $|\mathtt{neg}(A)| = |\mathcal{V}|$. By **L7.12** and **L7.1**, $|\neg\mathcal{T}|^{\mathfrak{N}^+} = |\mathcal{T}|^{\mathfrak{N}^-}$, where $|\mathcal{T}|^{\mathfrak{N}^-} = |\mathcal{V}|$, and so $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 3: Let $A = \bot$, and so $\mathtt{neg}(A) = \bot$. Thus $|\mathtt{neg}(A)| = |\bot|$. Since $|\bot|^{\mathfrak{N}^+} = |\bot|$, we know that $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 4: Let $A = \neg\bot$, and so $\mathtt{neg}(A) = \perp\!\!\!\perp$. Thus it follows that $|\mathtt{neg}(A)| = |\perp\!\!\!\perp|$. By **L7.12** and **L7.1**, $|\neg\bot|^{\mathfrak{N}^+} = |\bot|^{\mathfrak{N}^-}$, where $|\bot|^{\mathfrak{N}^-} = |\perp\!\!\!\perp|$, and so $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 5: Let $A = p_i$ for some $p_i \in \mathbb{L}$. It follows that $\mathtt{neg}(A) = p_{2i}$, and so $|\mathtt{neg}(A)| = |p_{2i}|$. We also know that $|p_i|^{\mathfrak{N}^+} = |p_{2i}|$, and so $|A|^{\mathfrak{N}^+} = |p_{2i}|$. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case 6: Let $A = \neg p_i$ for some $p_i \in \mathbb{L}$. Thus $\mathtt{neg}(A) = p_{2i+1}$, and so it follows that $|\mathtt{neg}(A)| = |p_{2i+1}|$. By **L7.12** and **L7.1**, $|\neg p_i|^{\mathfrak{N}^+} = |p_i|^{\mathfrak{N}^-}$, where $|p_i|^{\mathfrak{N}^-} = |p_{2i+1}|$. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Given the cases above, $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$ where $\mathtt{comp}(A) < 2$. Assume for induction that $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$ such that $\mathtt{comp}(A) < n$. Let $A \in \mathtt{pfs}^\neg(\mathcal{L})$ be such that $\mathtt{comp}(A) = n$. Thus it follows that either: (*a*) $A = \neg\neg B$; (*b*) $A = B \wedge C$; (*c*) $A = B \vee C$; (*d*) $A = \neg(B \vee C)$; or (*e*) $A = \neg(B \wedge C)$.

Case (*a*)*:* Assume $A = \neg\neg B$. Thus $\mathtt{neg}(A) = \mathtt{neg}(B)$, where we know that $|\mathtt{neg}(B)| = |B|^{\mathfrak{N}^+}$ by hypothesis. By **L7.12** and **L7.1**, it follows that $|B|^{\mathfrak{N}^+} = |\neg B|^{\mathfrak{N}^-} = |\neg\neg B|^{\mathfrak{N}^+}$. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case (*b*)*:* Assume $A = B \wedge C$. Thus $\mathtt{neg}(A) = \mathtt{neg}(B) \wedge \mathtt{neg}(C)$, where both $|\mathtt{neg}(B)| = |B|^{\mathfrak{N}^+}$ and $|\mathtt{neg}(C)| = |C|^{\mathfrak{N}^+}$ by hypothesis. By **L7.12** and **L7.2**, $|B \wedge C|^{\mathfrak{N}^+} = |B|^{\mathfrak{N}^+} \wedge |C|^{\mathfrak{N}^+}$. It follows that:

$$\begin{aligned} |\mathtt{neg}(A)| &= |\mathtt{neg}(B) \wedge \mathtt{neg}(C)| \\ (*) &= |\mathtt{neg}(B)| \wedge |\mathtt{neg}(C)| \\ &= |B|^{\mathfrak{N}^+} \wedge |C|^{\mathfrak{N}^+} \\ &= |B \wedge C|^{\mathfrak{N}^+} \\ &= |A|^{\mathfrak{N}^+}. \end{aligned}$$

Here $(*)$ follows by an argument identical to $[^+_\wedge]$ given in **L7.2**, where the other identities follow from the above. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case (*c*)*:* Assume $A = B \vee C$. Thus $\mathtt{neg}(A) = \mathtt{neg}(B) \vee \mathtt{neg}(C)$, where both $|\mathtt{neg}(B)| = |B|^{\mathfrak{N}^+}$ and $|\mathtt{neg}(C)| = |C|^{\mathfrak{N}^+}$ by hypothesis. By **L7.12** and **L7.3**, $|B \vee C|^{\mathfrak{N}^+} = |B|^{\mathfrak{N}^+} \vee |C|^{\mathfrak{N}^+}$. It follows that:

$$\begin{aligned} |\mathtt{neg}(A)| &= |\mathtt{neg}(B) \vee \mathtt{neg}(C)| \\ (*) &= |\mathtt{neg}(B)| \vee |\mathtt{neg}(C)| \\ &= |B|^{\mathfrak{N}^+} \vee |C|^{\mathfrak{N}^+} \\ &= |B \vee C|^{\mathfrak{N}^+} \\ &= |A|^{\mathfrak{N}^+}. \end{aligned}$$

Here $(*)$ follows by an argument identical to $[^+_\vee]$ given in **L7.3**, where the other identities follow from the above. Thus $|\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case (*d*)*:* Assume $A = \neg(B \vee C)$, so $\mathtt{neg}(A) = \mathtt{neg}(\neg B) \wedge \mathtt{neg}(\neg C)$. By hypothesis, both $|\mathtt{neg}(\neg B)| = |\neg B|^{\mathfrak{N}^+}$ and $|\mathtt{neg}(\neg C)| = |\neg C|^{\mathfrak{N}^+}$. By **L7.12** and **L7.1**, $|\neg B|^{\mathfrak{N}^+} = |B|^{\mathfrak{N}^-}$ and $|\neg C|^{\mathfrak{N}^+} = |C|^{\mathfrak{N}^-}$, as well as $|\neg(B \vee C)|^{\mathfrak{N}^+} = |B \vee C|^{\mathfrak{N}^-}$, where $|B \vee C|^{\mathfrak{N}^-} = |B|^{\mathfrak{N}^-} \wedge |C|^{\mathfrak{N}^-}$ follows

by **L7.3**. We may then argue as follows:

$$\begin{aligned}
|\,\mathtt{neg}(A)| &= |\,\mathtt{neg}(\neg B) \wedge \mathtt{neg}(\neg C)| \\
(*) &= |\,\mathtt{neg}(\neg B)| \wedge |\,\mathtt{neg}(\neg C)| \\
&= |\neg B|^{\mathfrak{N}^+} \wedge |\neg C|^{\mathfrak{N}^+} \\
&= |B|^{\mathfrak{N}^-} \wedge |C|^{\mathfrak{N}^-} \\
&= |B \vee C|^{\mathfrak{N}^-} \\
&= |\neg(B \vee C)|^{\mathfrak{N}^+} \\
&= |A|^{\mathfrak{N}^+}.
\end{aligned}$$

Here $(*)$ follows by an argument identical to $[^+_\wedge]$ given in **L7.2**, where the other identities follow from the above. Thus $|\,\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Case (e)*:* Assume $A = \neg(B \wedge C)$, so $\mathtt{neg}(A) = \mathtt{neg}(\neg B) \vee \mathtt{neg}(\neg C)$. By hypothesis, $|\,\mathtt{neg}(\neg B)| = |\neg B|^{\mathfrak{N}^+}$ and $|\,\mathtt{neg}(\neg C)| = |\neg C|^{\mathfrak{N}^+}$. By **L7.12** and **L7.1**, we know that $|\neg B|^{\mathfrak{N}^+} = |B|^{\mathfrak{N}^-}$ and $|\neg C|^{\mathfrak{N}^+} = |C|^{\mathfrak{N}^-}$, as well as $|\neg(B \wedge C)|^{\mathfrak{N}^+} = |B \wedge C|^{\mathfrak{N}^-}$, where $|B \wedge C|^{\mathfrak{N}^-} = |B|^{\mathfrak{N}^-} \vee |C|^{\mathfrak{N}^-}$ follows by **L7.2**. We may then argue as follows:

$$\begin{aligned}
|\,\mathtt{neg}(A)| &= |\,\mathtt{neg}(\neg B) \vee \mathtt{neg}(\neg C)| \\
(*) &= |\,\mathtt{neg}(\neg B)| \vee |\,\mathtt{neg}(\neg C)| \\
&= |\neg B|^{\mathfrak{N}^+} \vee |\neg C|^{\mathfrak{N}^+} \\
&= |B|^{\mathfrak{N}^-} \vee |C|^{\mathfrak{N}^-} \\
&= |B \wedge C|^{\mathfrak{N}^-} \\
&= |\neg(B \wedge C)|^{\mathfrak{N}^+} \\
&= |A|^{\mathfrak{N}^+}.
\end{aligned}$$

Here $(*)$ follows by an argument identical to $[^+_\vee]$ given in **L7.3**, where the other identities follow from the above. Thus $|\,\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$.

Given that $|\,\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ in each of the cases above, we may conclude by induction that $|\,\mathtt{neg}(A)| = |A|^{\mathfrak{N}^+}$ for all $A \in \mathtt{pfs}^\neg(\mathcal{L})$. $\square$

P7.3 $\mathcal{M} \models \mathtt{neg}(\varphi)$ iff $\mathcal{M}^{\mathfrak{N}} \models \varphi$ for all $\mathcal{M} \in \mathcal{C}^{\infty}$ and $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\infty}$ and $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$. The proof goes by induction on complexity. Assume $\mathtt{comp}^{+}(\varphi) = 0$. It follows that either $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, or $\varphi = \$A$ for some $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Case I: Assume $\varphi = A \trianglelefteq B$ for some $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. It follows that $\mathtt{neg}(\varphi) = \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(B)$. We may then reason as follows:

$$\begin{aligned}
\mathcal{M} \models \mathtt{neg}(\varphi) \quad & \textit{iff} \quad \mathcal{M} \models \mathtt{neg}(A) \trianglelefteq \mathtt{neg}(B) \\
& \textit{iff} \quad |\mathtt{neg}(A)| \subseteq |\mathtt{neg}(B)| \\
(*) \quad & \textit{iff} \quad |A|^{\mathfrak{N}+} \subseteq |B|^{\mathfrak{N}+} \\
& \textit{iff} \quad \mathcal{M}^{\mathfrak{N}} \models A \trianglelefteq B \\
& \textit{iff} \quad \mathcal{M}^{\mathfrak{N}} \models \varphi.
\end{aligned}$$

The biconditionals above all follow by definition or assumption with the exception of $(*)$ which is given by **L7.13**. Thus we may conclude that $\mathcal{M} \models \mathtt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \models \varphi$ as desired.

Case II: Assume $\varphi = \$A$ for some $A \in \mathtt{pfs}^{\neg}(\mathcal{L})$. It follows that $\mathtt{neg}(\varphi) = \$\,\mathtt{neg}(A)$. We may then reason as follows:

$$\begin{aligned}
\mathcal{M} \models \mathtt{neg}(\varphi) \quad & \textit{iff} \quad \mathcal{M} \models \$\,\mathtt{neg}(A) \\
& \textit{iff} \quad |\mathtt{neg}(A)| = \{s\} \text{ for some } s \in S \\
(*) \quad & \textit{iff} \quad |A|^{\mathfrak{N}+} = \{s\} \text{ for some } s \in S \\
& \textit{iff} \quad \mathcal{M}^{\mathfrak{N}} \models \$A \\
& \textit{iff} \quad \mathcal{M}^{\mathfrak{N}} \models \varphi.
\end{aligned}$$

As before, the biconditionals above all follow by definition or assumption with the exception of $(*)$ which is given by **L7.13**. Thus we may conclude that $\mathcal{M} \models \mathtt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \models \varphi$.

Given the cases above, it follows by a routine induction proof that $\mathcal{M} \models \mathtt{neg}(\varphi)$ just in case $\mathcal{M}^{\mathfrak{N}} \models \varphi$ for any $\varphi \in \mathtt{wfs}^{\neg}(\mathcal{L})$. □

P7.4 $\mathfrak{N} : \mathcal{C}^\infty \to \mathcal{C}^\pm$ is a surjection.

Proof. Choose some $\mathcal{M}_u \in \mathcal{C}^\pm_\mathcal{S}$. It follows that $\mathcal{M}_u = \langle S, \bigsqcup, |\cdot|_u \rangle$, where $|p_i|_u = \langle |p_i|^+_u, |p_i|^-_u \rangle$ for all $p_i \in \mathbb{L}$. We may then define $|\cdot|$ as follows:

$$|p_i| = \begin{cases} |p_{\frac{i}{2}}|^+_u & \text{if } i \text{ is even} \\ |p_{\frac{i-1}{2}}|^-_u & \text{otherwise.} \end{cases}$$

Since $\mathcal{M}_u \in \mathcal{C}^\pm$, we know that $\langle |p_i|^+_u, |p_i|^-_u \rangle \in \mathbb{P}^\pm_\mathcal{S}$, and so $|p_i|^+_u, |p_i|^-_u \in \mathbb{P}^\infty_\mathcal{S}$. Thus $|p_{2i}|, |p_{2i+1}| \in \mathbb{P}^\infty_\mathcal{S}$ for all $i \in \mathbb{N}$, and so $|p_i| \in \mathbb{P}^\infty_\mathcal{S}$ for all $p_i \in \mathbb{L}$. By definition, $\mathcal{M} = \langle S, \bigsqcup, |\cdot| \rangle \in \mathcal{C}^\infty$. We may then consider the following:

$$\begin{aligned} |p_i|^\mathfrak{N} &= \langle |p_i|^{\mathfrak{N}^+}, |p_i|^{\mathfrak{N}^-} \rangle \\ &= \langle |p_{2i}|, |p_{2i+1}| \rangle \\ &= \langle |p_i|^+_u, |p_i|^-_u \rangle \\ &= |p_i|_u. \end{aligned}$$

The identities above all hold by definition or assumption. It follows that $\mathcal{M}^\mathfrak{N} = \mathcal{M}_u$, and so there is some $\mathcal{M} \in \mathcal{C}^\infty$ where $\mathcal{M}^\mathfrak{N} = \mathcal{M}_u$ for any $\mathcal{M}_u \in \mathcal{C}^\pm$. Thus we may conclude that $\mathfrak{N}$ is surjective. □

5.8 Bilattice Theory

Having extended *Soundness* and *Completeness* to UGS, we may now employ the resources above to define bilateral analogues of $\propto$ and $\subseteq$, showing that the resulting structure forms a bilattice. In particular, consider the following:

Bilateral Containment: $\langle X, Y \rangle \Subset \langle U, V \rangle$ *iff* $X \propto U$ and $Y \subseteq V$.

Bilateral Entailment: $\langle X, Y \rangle \Subset \langle U, V \rangle$ *iff* $X \subseteq U$ and $Y \propto V$.

In order to prove that both $\langle \mathbb{P}^\pm_\mathcal{S}, \Subset \rangle$ and $\langle \mathbb{P}^\pm_\mathcal{S}, \Subset \rangle$ are complete lattices, we may show that for any $\mathcal{S} \in \mathbb{M}_\infty$ and indexed family of propositions $\{P_i : i \in I\} \subseteq \mathbb{P}^\pm_\mathcal{S}$,

$\bigwedge\{P_i : i \in I\}$ is the least upper bound with respect to $\sqsubseteq$, and $\bigvee\{P_i : i \in I\}$ is the least upper bound with respect to $\Subset$ as follows:

L8.1 $\bigwedge\{P_i : i \in I\} = \mathtt{lub}^{\sqsubseteq}\{P_i : i \in I\}$.

L8.2 $\bigvee\{P_i : i \in I\} = \mathtt{lub}^{\Subset}\{P_i : i \in I\}$.

Since both $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset\rangle$ and $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \sqsubseteq\rangle$ are complete lattices, it follows that $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset, \sqsubseteq\rangle$ is a *pre-bilattice*. Accordingly, $\mathcal{B}^{\pm}_{\mathcal{S}} = \langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset, \sqsubseteq, \neg\rangle$ is a *bilattice* on account of the fact that $\langle \mathbb{P}^{\pm}_{\mathcal{S}}, \Subset, \sqsubseteq\rangle$ is a pre-bilattice where '$\neg$' is a unary operator on $\mathbb{P}^{\pm}_{\mathcal{S}}$ which obeys the following conditions for all $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$:

L8.3 $\neg\neg P = P$.

L8.4 If $P \Subset Q$, then $\neg P \sqsubseteq \neg Q$.

L8.5 If $P \sqsubseteq Q$, then $\neg P \Subset \neg Q$.

It remains to further characterise the properties which $\mathcal{B}^{\pm}_{\mathcal{S}}$ exhibits, marking important points of departure from Boolean theories in which the space of propositions forms a complemented distributive lattice.[18]

Given any $\mathcal{S} \in \mathcal{C}^{\pm}$, we may begin by showing that $\mathcal{B}^{\pm}_{\mathcal{S}}$ is bounded below by proving that the following identities hold for any $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$:

L8.6 $P \vee \bot = P$. **L8.7** $P \wedge \perp\!\!\!\perp = P$.

Observe that $\bigvee \varnothing = \bot$ and $\bigwedge \varnothing = \perp\!\!\!\perp$, and so $\bigvee \varnothing$ and $\bigwedge \varnothing$ provide lower bounds for $\mathbb{P}^{\pm}_{\mathcal{S}}$ with respect to $\Subset$ and $\sqsubseteq$. By contrast, $\bigvee \mathbb{P}^{\pm}_{\mathcal{S}} = \langle \mathcal{S}, \varnothing\rangle \neq \mathcal{T}$ and $\bigwedge \mathbb{P}^{\pm}_{\mathcal{S}} = \langle \varnothing, \mathcal{S}\rangle \neq \mathcal{V}$ given that $\bot, \perp\!\!\!\perp \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Rather we find that $\bigvee \mathbb{P}^{\pm}_{\mathcal{S}} = \mathcal{T} \vee \perp\!\!\!\perp$ and $\bigwedge \mathbb{P}^{\pm}_{\mathcal{S}} = \bot \wedge \mathcal{V}$. Letting $\top = \mathcal{T} \vee \perp\!\!\!\perp$ and $\overline{\top} = \bot \wedge \mathcal{V}$, it follows that:

L8.8 $P \vee \top = \top$. **L8.9** $P \wedge \overline{\top} = \overline{\top}$.

[18] Ginsberg (1988) first proposed the definition of a bilattice. Alternatively, one could drop the requirement that pre-bilattices consist of lattices which are *complete* as in Rivieccio (2010). See also Mobasher et al. (2000) for discussion of such discrepancies in usage.

It follows that $\mathcal{B}^{\pm}_{\mathcal{S}}$ is bounded above by $\top$ and $\overline{\top}$ with respect to $\Subset$ and $\sqsubseteq$ rather than $\mathcal{T}$ and $\mathcal{V}$.[19] Nevertheless, $\top$ and $\overline{\top}$ may be defined in terms of $\mathcal{T}, \bot, \mathcal{V}$, and $\underline{\bot}$, whereas the same cannot be said in reverse.[20]

We may now turn to observe that both $\wedge$ and $\vee$ are monotonic over their respective orders. More specifically, for any $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$:

L8.13 If $P \sqsubseteq Q$, then $P \wedge R \sqsubseteq Q \wedge R$.

L8.14 If $P \Subset Q$, then $P \vee R \Subset Q \vee R$.

By contrast with the above, $\wedge$ and $\vee$ are not monotonic over each other's orders given that there are some $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$ for which:

L8.15 $P \sqsubseteq Q$ and $P \vee R \not\sqsubseteq Q \vee R$.

L8.16 $P \Subset Q$ and $P \wedge R \not\Subset Q \wedge R$.

It follows that $\mathcal{B}^{\pm}_{\mathcal{S}}$ is not *interlaced* which requires $\wedge$ and $\vee$ to be monotonic over both orders $\sqsubseteq$ and $\Subset$.[21] We may also show that both absorption laws fail, where $\mathcal{B}^{\pm}_{\mathcal{S}}$ is distributive only in negation since conjunction does not distribute over disjunction, nor does disjunction distribute over conjunction:

L8.17 $P \wedge (P \vee Q) \neq P$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_{\mathcal{S}}$.

L8.18 $P \vee (P \wedge Q) \neq P$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_{\mathcal{S}}$.

L8.19 $\neg(P \wedge Q) = \neg P \vee \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}_{\mathcal{S}}$.

L8.20 $\neg(P \vee Q) = \neg P \wedge \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}_{\mathcal{S}}$.

L8.21 $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_{\mathcal{S}}$.

L8.22 $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_{\mathcal{S}}$.

The symmetry in distribution laws for conjunction over disjunction and *vice versa* may be contrasted with the asymmetry between **L3.21** and **L3.22**.

[19] See §7 of Chapter 4 and the Conclusion for related discussion.
[20] However, by letting $\mathbb{P}^{(\pm)}_{\mathcal{S}} = \{\langle V, F\rangle : V, F \in \mathbb{P}^{\infty}_{\mathcal{S}}/\varnothing\}$, we may show that both $\bigwedge(\mathbb{P}^{(\pm)}_{\mathcal{S}}) = \mathcal{V}$ and $\bigvee(\mathbb{P}^{(\pm)}_{\mathcal{S}}) = \mathcal{T}$. See Fine (2017b, p. 642) for related discussion.
[21] Restricting $\mathbb{P}^{\infty}_{\mathcal{S}}$ to convex propositions makes the resulting pre-bilattice interlaced.

Having offered a preliminary survey of the structure of $\mathbb{P}^{\pm}_{\mathcal{S}}$, I will turn to relate these results to the theorems of UGS in the following section.

L8.1 $\bigwedge\{P_i : i \in I\} = \mathtt{lub}^{\sqsubseteq}\{P_i : i \in I\}$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $\{P_i : i \in I\} \subseteq \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $\{P_i : i \in I\} \subseteq \mathbb{P}^{\pm}_{\mathcal{S}}$, where $P_i = \langle P_i^+, P_i^- \rangle$ for all $i \in I$. By definition, $\bigwedge\{P_i : i \in I\} = \langle \bigwedge\{P_i^+ : i \in I\}, \bigvee\{P_i^- : i \in I\} \rangle$. Since $P_i^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$ for all $i \in I$, both $\bigwedge\{P_i^+ : i \in I\} = \mathtt{lub}^{\trianglerighteq}\{P_i^+ : i \in I\}$ and $\bigvee\{P_i^- : i \in I\} = \mathtt{lub}^{\subseteq}\{P_i^- : i \in I\}$ follow from **L6.6** and **L6.7**, respectively. Thus $P_i^+ \trianglerighteq \bigwedge\{P_i^+ : i \in I\}$ and $P_i^- \subseteq \bigvee\{P_i^- : i \in I\}$ for all $i \in I$, and so $P_i \sqsubseteq \bigwedge\{P_i : i \in I\}$ for all $i \in I$. It follows that $\bigwedge\{P_i : i \in I\}$ is an upper bound of $\{P_i : i \in I\}$ with respect to $\sqsubseteq$.

Let $Z \in \mathbb{P}^{\pm}_{\mathcal{S}}$, and assume that $P_i \sqsubseteq Z$ for all $i \in I$. Thus $Z = \langle Z^+, Z^- \rangle$ and $P_i = \langle P_i^+, P_i^- \rangle$ for each $i \in I$, where $Z^{\pm}, P_i^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Again by **L6.6** and **L6.7**, we know that $\bigwedge\{P_i^+ : i \in I\} \trianglerighteq Z^+$ and $\bigvee\{P_i^- : i \in I\} \subseteq Z^-$, and so $\bigwedge\{P_i : i \in I\} \sqsubseteq Z$. Since Z was arbitrary, we may conclude from the above that $\bigwedge\{P_i : i \in I\} = \mathtt{lub}^{\sqsubseteq}\{P_i : i \in I\}$. □

L8.2 $\bigvee\{P_i : i \in I\} = \mathtt{lub}^{\sqsubseteq}\{P_i : i \in I\}$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $\{P_i : i \in I\} \subseteq \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Similar to **L8.1**. □

L8.3 $\neg\neg P = P$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$. It follows that $P = \langle P^+, P^- \rangle$ for some $P^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. By definition, $\neg\langle P^+, P^- \rangle = \langle P^-, P^+ \rangle$, and so it follows that $\neg\neg\langle P^+, P^- \rangle = \langle P^+, P^- \rangle$. Equivalently, $\neg\neg P = P$. □

L8.4 For all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^{\pm}_{\mathcal{S}}$, if $P \Subset Q$, then $\neg P \sqsubseteq \neg Q$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$, and assume $P \Subset Q$. Thus it follows that $P = \langle P^+, P^- \rangle$ and $Q = \langle Q^+, Q^- \rangle$ for some $P^{\pm}, Q^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$, where both $P^+ \subseteq Q^+$ and $P^- \trianglerighteq Q-$. Thus $\langle P^-, P^+ \rangle \sqsubseteq \langle Q^-, Q^+ \rangle$, and so $\neg\langle P^+, P^- \rangle \sqsubseteq \neg\langle Q^+, Q^- \rangle$. Equivalently, $\neg P \sqsubseteq \neg Q$ as desired. □

L8.5 For all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^{\pm}_{\mathcal{S}}$, if $P \sqsubseteq Q$, then $\neg P \Subset \neg Q$.

Proof. Similar to **L8.4**. □

L8.6 $P \vee \bot = P$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$. It follows that $P = \langle P^+, P^- \rangle$ for some $P^\pm \in \mathbb{P}^\infty_\mathcal{S}$. Since both $P^- \vee \varnothing = P^-$ and $P^+ \wedge \{\square\} = P^+$, it follows that $\langle P^+ \wedge \{\square\}, P^- \vee \varnothing \rangle = \langle P^+, P^- \rangle$, and so $P \wedge \bot = P$ by definition. □

L8.7 $P \wedge \bot\!\!\!\bot = P$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Similar to **L8.6**. □

L8.8 $P \vee \top = \top$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$. It follows that $P = \langle P^+, P^- \rangle$ for some $P^\pm \in \mathbb{P}^\infty_\mathcal{S}$, and so $P^+ \vee S = S$ by **L6.12**, and $P^- \wedge \varnothing = \varnothing$ by **L6.13**. Thus $\langle P^+ \vee S, P^- \wedge \varnothing \rangle = \langle S, \varnothing \rangle$, and so $P \vee \top = \top$ as desired. □

L8.9 $P \wedge \top\!\!\!\top = \top\!\!\!\top$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Similar to **L8.8**. □

L8.10 $\mathcal{S}^3 \in \mathbb{M}_\infty$ where $\mathcal{S}^3 = \langle \mathcal{P}(\{a, b, c\}), \bigcup \rangle$.

Proof. Let $\mathcal{S}^3 = \langle \mathcal{P}(\{a, b, c\}), \bigcup \rangle$. Since $\bigcup$ satisfies *Associativity* where $\bigcup\{x\} = x$ for all $x \in \mathcal{P}(\{a, b, c\})$, we know that $\mathcal{S}^3 \in \mathbb{M}_\infty$. □

L8.11 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$, if $X \trianglerighteq Y$, then $X \wedge Z \trianglerighteq Y \wedge Z$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_\mathcal{S}$, and assume $X \trianglerighteq Y$. It follows that $X \gg Y$ and $X \ll Y$. Letting $s \in Y \wedge Z$, we know that $s = \bigsqcup\{x, y\}$ for some $x \in Y$ and $y \in Z$, and so there is some $z \in X$ where $z \sqsubseteq x$. Thus

$\bigsqcup\{z, y\} \in Y \wedge Z$ where $\bigsqcup\{z, x\} = x$. We may then reason as follows:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{z, y\}, s\} &= \bigsqcup\{\bigsqcup\{z, y\}, \bigsqcup\{x, y\}\} \\
&= \bigsqcup\bigcup\{\{z, y\}, \{x, y\}\} \\
&= \bigsqcup\bigcup\{\{z, x\}, \{y\}\} \\
&= \bigsqcup\{\bigsqcup\{z, x\}, \bigsqcup\{y\}\} \\
&= \bigsqcup\{x, y\} \\
&= s.
\end{aligned}$$

The identities stated above follow by *Collapse* and *Associativity* given our assumptions. Thus $\bigsqcup\{\bigsqcup\{z, y\}, s\} = s$, and so $\bigsqcup\{z, y\} \sqsubseteq s$. Generalising on s, we may conclude that that $X \wedge Z \gg Y \wedge Z$.

Choose instead some $u \in X \wedge Z$ and $v \in Y \wedge Z$. Thus $u = \bigsqcup\{x, z\}$ and $v = \bigsqcup\{y, w\}$ for some $x \in X$, $y \in Y$, and $z, w \in Z$. Since $X \ll Y$, we know that $\bigsqcup\{x, y\} \in Y$, where $\bigsqcup\{z, w\} \in Z$ given that $Z \in \mathbb{P}_{\mathcal{S}}$. Accordingly, $\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{z, w\}\} \in Y \wedge Z$, and so $\bigsqcup\{\bigsqcup\{x, z\}, \bigsqcup\{y, w\}\} \in Y \wedge Z$, so $\bigsqcup\{u, v\} \in Y \wedge Z$. Thus $X \wedge Z \ll Y \wedge Z$, and so $X \wedge Z \trianglerighteq Y \wedge Z$. □

L8.12 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$, if $X \subseteq Y$, then $X \vee Z \subseteq Y \vee Z$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$, and assume $X \trianglerighteq Y$. Choose $s \in X \vee R^+$. By **L6.8**, $s \in X \cup Z \cup (X \wedge Z)$. If $s \in X \cup Z$, then $s \in Y \cup Z$, and so $s \in Y \cup Z \cup (Y \wedge Z)$. If $s \in X \wedge Z$, then $s = \bigsqcup\{x, y\}$ for some $x \in X$ and $y \in Z$, and so $s \in Y \wedge Z$ since $x \in Y$. Thus $s \in Y \cup Z \cup (Y \wedge Z)$, and so $s \in Y \vee Z$ by **L6.8**. It follows that $X \vee Z \subseteq Y \vee Z$. □

L8.13 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_{\mathcal{S}}$, if $P \sqsubseteq Q$, then $P \wedge R \sqsubseteq Q \wedge R$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_{\mathcal{S}}$, and assume $P \sqsubseteq Q$ for discharge. Thus $P^+ \trianglerighteq Q^+$ and $P^- \subseteq Q^-$, where $P^\pm, Q^\pm, R^\pm \in \mathbb{P}^\infty_{\mathcal{S}}$. It follows that $P^+ \wedge R^+ \trianglerighteq Q^+ \wedge R^+$ by **L8.11**, and $P^- \vee R^- \subseteq Q^- \vee R^-$ by **L8.12**. Accordingly, $\langle P^+ \wedge R^+, P^- \vee R^- \rangle \sqsubseteq \langle Q^+ \wedge R^+, Q^- \vee R^- \rangle$, and so by definition we may conclude that $P \wedge R \sqsubseteq Q \wedge R$. □

L8.14 For any $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$, if $P \sqsubset Q$, then $P \vee R \sqsubset Q \vee R$.

Proof. Similar to **L8.13**. □

L8.15 $P \sqsubseteq Q$ and $P \vee R \not\sqsubseteq Q \vee R$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S}_3 = \langle S, \cup \rangle$ where $S = \mathcal{P}(\{a, b, c\})$. By **L3.12**, we know that $\mathcal{S}_3 \in \mathbb{M}_\infty$. Let $P = \langle \{\{a\}\}, \{\{a\}\} \rangle$, $Q = \langle \{\{a\}, \{b\}, \{a, b\}\}, \{\{a\}, \{b\}, \{a, b\}\} \rangle$, and $R = \langle \{\{c\}\}, \{\{c\}\} \rangle$, and observe that $P, Q, R \in \mathbb{P}^\pm_{\mathcal{S}_3}$. For the sake of readability, let $\alpha = \{a\}$, $\beta = \{b\}$, and $\gamma = \{c\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. We may then observe the following:

$$\begin{aligned} P &= \langle \{\alpha\}, \{\alpha\} \rangle & P \vee R &= \langle \{\alpha, \gamma, \alpha.\gamma\}, \{\alpha.\gamma\} \rangle \\ Q &= \langle \{\alpha.\beta\}, \{\alpha.\beta\} \rangle & Q \vee R &= \langle \{\beta, \alpha.\beta, \alpha.\beta.\gamma\}, \{\alpha.\beta.\gamma\} \rangle. \\ R &= \langle \{\gamma\}, \{\gamma\} \rangle \end{aligned}$$

Given that $\gamma \in P \vee R$ and $\beta \in Q \vee R$, but $\gamma.\beta \notin Q \vee R$, we may conclude that $P \vee R \not\sqsubseteq Q \vee R$. However, $P \sqsubseteq Q$, concluding the proof. □

L8.16 $P \sqsubset Q$ and $P \wedge R \not\sqsubset Q \wedge R$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Similar to **L8.15**. □

L8.17 $P \neq P \wedge (P \vee Q)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

Proof. Let $\mathcal{S}_3 = \langle S, \cup \rangle$ where $S = \mathcal{P}(\{a, b, c\})$. By **L3.12**, we know that $\mathcal{S}_3 \in \mathbb{M}_\infty$. Let $P = \langle \{\{a\}\}, \{\{a\}\} \rangle$ and $Q = \langle \{\{b\}\}, \{\{b\}\} \rangle$, observing that $P, Q \in \mathbb{P}^\pm_{\mathcal{S}_3}$. For ease of exposition, let $\alpha = \{a\}$ and $\beta = \{b\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. Thus it follows that:

$$\begin{aligned} P \vee Q &= \langle \{\alpha, \beta, \alpha.\beta\}, \{\alpha.\beta\} \rangle \\ P \wedge (P \vee Q) &= \langle \{\alpha, \alpha.\beta\}, \{\alpha, \alpha.\beta\} \rangle. \end{aligned}$$

We may conclude the proof by observing that $P \neq P \wedge (P \vee Q)$. □

L8.18 $P \neq P \vee (P \wedge Q)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}_\mathcal{S}$.

Proof. Similar to **L8.17**. □

L8.19 $\neg(P \wedge Q) = \neg P \vee \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$. It follows that $P = \langle P^+, P^- \rangle$ and $Q = \langle Q^+, Q^- \rangle$ for some $P^\pm, Q^\pm \in \mathbb{P}^\infty_\mathcal{S}$. We may then argue as follows:

$$\begin{aligned}
\neg(P \wedge Q) &= \neg(\langle P^+, P^- \rangle \wedge \langle Q^+, Q^- \rangle) \\
&= \neg(\langle P^+ \wedge Q^+, P^- \vee Q^- \rangle) \\
&= \langle P^- \vee Q^-, P^+ \wedge Q^+ \rangle \\
&= \langle P^-, P^+ \rangle \vee \langle Q^-, Q^+ \rangle \\
&= \neg P \vee \neg Q.
\end{aligned}$$

The identities above all follow by definition or assumption. □

L8.20 $\neg(P \vee Q) = \neg P \wedge \neg Q$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Similar to **L8.19**. □

L8.21 $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$.

Proof. Let $P = \langle \{\{a\}\}, \{\{b\}\} \rangle$, $Q = \langle \{\{b\}\}, \{\{c\}\} \rangle$, and $R = \langle \{\{c\}\}, \{\{a\}\} \rangle$. For ease of exposition, let $\alpha = \{a\}$, $\beta = \{b\}$, and $\gamma = \{c\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. We may then observe the following:

$$\begin{aligned}
P &= \langle \{\alpha\}, \{\beta\} \rangle & \qquad Q \wedge R &= \langle \{\alpha.\gamma\}, \{\alpha, \beta, \alpha.\beta\} \rangle \\
Q &= \langle \{\beta\}, \{\gamma\} \rangle & \qquad P \vee Q &= \langle \{\alpha, \beta, \alpha.\beta\}, \{\beta.\gamma\} \rangle \\
R &= \langle \{\gamma\}, \{\alpha\} \rangle & \qquad P \vee R &= \langle \{\alpha, \gamma, \alpha.\gamma\}, \{\alpha.\gamma\} \rangle.
\end{aligned}$$

$$\begin{aligned}
P \vee (Q \wedge R) &= \langle \{\alpha, \alpha.\gamma\}, \{\beta, \alpha.\beta\} \rangle \\
(P \vee Q) \wedge (P \vee R) &= \langle \{\alpha, \alpha.\beta, \alpha.\gamma, \alpha.\beta.\gamma\}, \{\alpha.\gamma, \beta.\gamma, \alpha.\beta.\gamma\} \rangle.
\end{aligned}$$

By inspection, we may conclude that $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$, observing that $P, Q, R \in \mathbb{P}^\pm_{\mathcal{S}^3}$ and $s^3 \in \mathbb{M}_\infty$ by **L8.10**. □

L8.22 $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$.

Proof. Let $P = \langle\{\{a\}\}, \{\{b\}\}\rangle$, $Q = \langle\{\{b\}\}, \{\{c\}\}\rangle$, and $R = \langle\{\{c\}\}, \{\{a\}\}\rangle$. For ease of exposition, let $\alpha = \{a\}$, $\beta = \{b\}$, and $\gamma = \{c\}$, where '$x.y$' abbreviates '$\bigcup\{x, y\}$'. Given the identities in **L8.21**, it follows that:

$$\begin{aligned}
P \wedge Q &= \langle\{\beta.\gamma\}, \{\alpha, \beta, \alpha.\beta\}\rangle \\
P \wedge R &= \langle\{\alpha.\gamma\}, \{\alpha, \gamma, \alpha.\gamma\}\rangle \\
P \wedge (Q \vee R) &= \langle\{\alpha.\gamma\}, \{\alpha, \beta, \alpha.\beta\}\rangle \\
(P \wedge Q) \vee (P \wedge R) &= \langle\{\alpha.\beta, \alpha.\gamma, \alpha.\beta.\gamma\}, \{\beta, \alpha.\beta, \alpha.\gamma, \beta.\gamma, \alpha.\beta.\gamma\}\rangle.
\end{aligned}$$

By inspection, we may conclude that $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$, observing that $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}^3}$ and $s^3 \in \mathbb{M}_\infty$ by **L8.10**. □

5.9 The Logic of Essence and Ground

In order to make the connections between the propositional bilattice and UGS more perspicuous, it will help to introduce a number of further abbreviations, defining essence and ground in terms of unilateral ground as in §5.1.

Unilateral Essence: Let $\ulcorner A \trianglerighteq B \urcorner$ abbreviate $\ulcorner A \wedge B \approx B \urcorner$.

Ground: Let $\ulcorner A \leqslant B \urcorner$ abbreviate $\ulcorner (A \trianglelefteq B) \wedge (\neg A \trianglerighteq \neg B) \urcorner$.

Essence: Let $\ulcorner A \sqsubseteq B \urcorner$ abbreviate $\ulcorner (A \trianglerighteq B) \wedge (\neg A \trianglelefteq \neg B) \urcorner$.

Reduction: Let $\ulcorner A \Rightarrow B \urcorner$ abbreviate $\ulcorner (A \leqslant B) \wedge (A \sqsubseteq B) \urcorner$.

Identity: Let $\ulcorner A \equiv B \urcorner$ abbreviate $\ulcorner (A \Rightarrow B) \wedge (B \Rightarrow A) \urcorner$.

Whereas unilateral ground only describes the relationship between the exact verifiers for the antecedent and consequent, essence and ground constrain both the exact verifiers and falsifiers for the propositions involved.

The following results will help to bring out the manner in which the operators defined above are able to describe the propositional bilattice $\mathcal{B}^{\pm}_{\mathcal{S}}$

characterised above for any given $\mathcal{S} \in \mathbb{M}_\infty$. In particular, I will show that the following hold for any $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$:

L9.1 $\mathcal{M} \vDash A \trianglerighteq B$ *iff* $|A|^+ \trianglerighteq |B|^+$.

L9.2 $\mathcal{M} \vDash A \leqslant B$ *iff* $|A| \Subset |B|$.

L9.3 $\mathcal{M} \vDash A \sqsubseteq B$ *iff* $|A| \Subset\!\!\!\Subset |B|$.

L9.4 $\mathcal{M} \vDash A \Rightarrow B$ *iff* $|A| \Subset |B|$ and $|A| \Subset\!\!\!\Subset |B|$.

L9.5 $\mathcal{M} \vDash A \equiv B$ *iff* $|A| = |B|$.

Given these connections, we may show that each of the following Boolean identities admit of countermodels in the present setting:

These results help to distinguish UGS from extensional and intensional logics which describe spaces of propositions which form complemented distributive lattices. Additionally, we may show that analogues of the identities above hold when identity has been replaced with reduction, highlighting the distinctive role which reduction has to play in the present theory.

Rather than continuing to work over UGS, it will help to further exhibit the relationships between essence and ground and the extensional operators by considering the following range of theorems and admissible rules within UGS:

T1 $\bot \leqslant A$.

T2 $A \leqslant \top$.

T3 $\perp\!\!\!\bot \sqsubseteq A$.

T4 $A \sqsubseteq \top\!\!\!\top$.

T5 $A \leqslant A \vee B$.

T6 $B \leqslant A \vee B$.

T7 $A \sqsubseteq A \wedge B$.

T8 $B \sqsubseteq A \wedge B$.

T9 $A \wedge B \leqslant A \vee B$.

T10 $A \vee B \sqsubseteq A \wedge B$.

T11 $A \Rightarrow A \wedge (A \vee B)$.

T12 $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.

T13 $A \Rightarrow A \vee (A \wedge B)$.

T14 $A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$.

R1 $A \leqslant B \vdash B \sqsubseteq A \vee B.$ **R2** $A \leqslant B \vdash A \vee B \sqsubseteq B.$

R3 $A \sqsubseteq B \vdash B \leqslant A \wedge B.$ **R4** $A \sqsubseteq B \vdash A \wedge B \leqslant B.$

R5 $A \leqslant C,\ B \leqslant C \vdash A \vee B \leqslant C.$ **R6** $A \leqslant B,\ A \leqslant C \vdash A \leqslant B \wedge C.$

R7 $A \sqsubseteq C,\ B \sqsubseteq C \vdash A \wedge B \sqsubseteq C.$ **R8** $A \sqsubseteq B,\ A \sqsubseteq C \vdash A \sqsubseteq B \vee C.$

R9 $A \leqslant B,\ B \leqslant C \vdash A \leqslant C.$ **R10** $A \sqsubseteq B,\ C \sqsubseteq D \vdash A \wedge C \sqsubseteq B \wedge D.$

R11 $A \sqsubseteq B,\ B \sqsubseteq C \vdash A \sqsubseteq C.$ **R12** $A \leqslant B,\ C \leqslant D \vdash A \vee C \leqslant B \vee D.$

Let the syntactic consequent relation $\vdash_{\text{EG}}$ for *The Logic of Essence and Ground* (EG) be the smallest relation closed under truth-functional consequence which satisfies the above. In contrast to UGS, the subsystem EG aims to capture a natural means of reasoning about essence and ground.

Rather than deriving each of the axioms and rules for EG within UGS, the following lemmas establish **T11** – **T14** as characteristic examples:

L9.10 $\vdash_{\text{UGS}} A \Rightarrow A \wedge (A \vee B).$

L9.11 $\vdash_{\text{UGS}} A \Rightarrow A \vee (A \wedge B).$

L9.19 $\vdash_{\text{UGS}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C).$

L9.20 $\vdash_{\text{UGS}} A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C).$

In order to further characterise the granularity of the present theory of bilateral propositions, we may also derive the following with UGS, where I will establish the theoremhood of **ID9** and **ID11** by way of **T4** for convenience:

ID1 $A \vee \bot \equiv A.$ **ID2** $A \wedge \bot\!\!\!\bot \equiv A.$

ID3 $A \vee \top \equiv \top.$ **ID4** $A \wedge \mathbin{\overline{\top}} \equiv \mathbin{\overline{\top}}.$

ID5 $A \equiv A \wedge A$ **ID6** $A \equiv A \vee A$

ID7 $A \wedge B \equiv B \wedge A$ **ID8** $A \vee B \equiv B \vee A$

ID9 $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$ **ID10** $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$

ID11 $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$ **ID12** $(A \vee B) \vee C \equiv A \vee (B \vee C)$

ID13 $A \equiv \neg\neg A$ **IDR1** $A \equiv B \vdash B \equiv A$

IDR2 $A \equiv B \vdash (A \wedge C) \equiv (B \wedge C)$ **IDR3** $A \equiv B \vdash (A \vee C) \equiv (B \vee C)$

IDR4 $A \equiv B,\ B \equiv C \vdash A \equiv C$ **IDR5** $A \equiv B \vdash \neg A \equiv \neg B$

Given the results above, we may let the syntactic consequent relation $\vdash_{\text{LPI}}$ for *The Logic of Propositional Identity* (LPI) be the smallest relation closed under truth-functional consequence which satisfies the above. In contrast EG which streamlines reasoning about essence and ground, LPI aims to capture a natural means of reasoning about propositional identity in the present setting. These results help to further characterise the structure of the present theory of bilateral propositions in contrast to the traditional Boolean theories of propositions developed in extensional and intensional logics.

Despite having proven that UGS is sound and complete over $\mathcal{C}^{\pm}$, it remains to show whether the subsystems EG and LPI also admit of completeness results. In particular, one might aim to show that UGS is a conservative extension of both EG as LPI. Leaving these further pursuits for another day, I will conclude by presenting proofs of a selection of the claims asserted above.

L9.1 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \trianglerighteq B$ *iff* $|A|^{+} \trianglerighteq |B|^{+}$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned}
\mathcal{M} \vDash A \trianglerighteq B \ \ &\textit{iff}\ \ \mathcal{M} \vDash B \approx A \wedge B \\
&\textit{iff}\ \ |B|^{+} = |A \wedge B|^{+} \\
(\dagger)\ \ &\textit{iff}\ \ |B|^{+} = |A|^{+} \wedge |B|^{+} \\
(\ddagger)\ \ &\textit{iff}\ \ |A|^{+} \trianglerighteq |B|^{+}.
\end{aligned}$$

The biconditionals above hold by definition with the exception of (†) which is given by **L7.2**, and (‡) which requires further argument.

Assume $|B|^{+} = |A|^{+} \wedge |B|^{+}$, and let $s \in |B|^{+}$. Thus $s \in |A|^{+} \wedge |B|^{+}$, and so $s = \bigsqcup\{x, y\}$ for some $x \in |A|^{+}$ and $y \in |B|^{+}$. It follows that $x \sqsubseteq s$, and so $|A|^{+} \gg |B|^{+}$ since s was arbitrary. Assuming that $x \in |A|^{+}$ and

$y \in |B|^+$, it follows that $\bigsqcup\{x, y\} \in |A|^+ \wedge |B|^+$, and so $\bigsqcup\{x, y\} \in |B|^+$ by assumption. Thus $|A|^+ \ll |B|^+$, and so $|A|^+ \trianglerighteq |B|^+$.

Assume instead that $|A|^+ \trianglerighteq |B|^+$. Thus $|A|^+ \gg |B|^+$ and $|A|^+ \ll |B|^+$. Letting $s \in |B|^+$, it follows that there is some $y \in |A|^+$ where $y \sqsubseteq s$, and so $\bigsqcup\{y, s\} = s$. By definition, $\bigsqcup\{y, s\} \in |A|^+ \wedge |B|^+$, and so $s \in |A|^+ \wedge |B|^+$. Thus $|B|^+ \subseteq |A|^+ \wedge |B|^+$. Assume instead that $t \in |A|^+ \wedge |B|^+$. It follows that $t = \bigsqcup\{x, y\}$ for some $x \in |A|^+$ and $y \in |B|^+$. Since $|A|^+ \ll |B|^+$, it follows that $t \in |B|^+$, and so $|A|^+ \wedge |B|^+ \subseteq |B|^+$. Given the above, $|B|^+ = |A|^+ \wedge |B|^+$, thereby establishing (‡). □

L9.2 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \leqslant B$ *iff* $|A| \Subset |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned}
\mathcal{M} \vDash A \leqslant B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \trianglelefteq B) \wedge (\neg A \trianglerighteq \neg B) \\
& \textit{iff} \quad \mathcal{M} \vDash A \trianglelefteq B \text{ and } \mathcal{M} \vDash \neg A \trianglerighteq \neg B \\
(*) \quad & \textit{iff} \quad |A|^+ \subseteq |B|^+ \text{ and } |\neg A|^+ \trianglerighteq |\neg B|^+ \\
& \textit{iff} \quad |A|^+ \subseteq |B|^+ \text{ and } |A|^- \trianglerighteq |B|^- \\
& \textit{iff} \quad |A| \Subset |B|.
\end{aligned}$$

All of the biconditionals above hold by definition with the exception of $(*)$ which follows from the **Bilateral Semantics** together with **L9.1**. □

L9.3 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \sqsubseteq B$ *iff* $|A| \Subset\!\!\!\!= |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned}
\mathcal{M} \vDash A \sqsubseteq B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \trianglerighteq B) \wedge (\neg A \trianglelefteq \neg B) \\
& \textit{iff} \quad \mathcal{M} \vDash A \trianglerighteq B \text{ and } \mathcal{M} \vDash \neg A \trianglelefteq \neg B \\
(*) \quad & \textit{iff} \quad |A|^+ \trianglerighteq |B|^+ \text{ and } |\neg A|^+ \subseteq |\neg B|^+ \\
& \textit{iff} \quad |A|^+ \trianglerighteq |B|^+ \text{ and } |A|^- \subseteq |B|^- \\
& \textit{iff} \quad |A| \Subset\!\!\!\!= |B|.
\end{aligned}$$

All of the biconditionals above hold by definition with the exception of $(*)$ which follows from the **Bilateral Semantics** together with **L9.1**. $\square$

L9.4 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \Rightarrow B$ *iff* $|A| \Subset |B|$ and $|A| \sqsubseteq |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \Rightarrow B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \leqslant B) \wedge (A \sqsubseteq B) \\ & \textit{iff} \quad \mathcal{M} \vDash A \leqslant B \text{ and } \mathcal{M} \vDash A \sqsubseteq B \\ (*) \quad & \textit{iff} \quad |A| \Subset |B| \text{ and } |A| \sqsubseteq |B|. \end{aligned}$$

The biconditionals above follow by definition with the exception of $(*)$ which is given by **L9.2** and **L9.3**. $\square$

L9.5 For all $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, $\mathcal{M} \vDash A \equiv B$ *iff* $|A| = |B|$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. We may the argue as follows:

$$\begin{aligned} \mathcal{M} \vDash A \equiv B \quad & \textit{iff} \quad \mathcal{M} \vDash (A \Rightarrow B) \wedge (B \Rightarrow A) \\ & \textit{iff} \quad \mathcal{M} \vDash A \Rightarrow B \text{ and } \mathcal{M} \vDash B \Rightarrow A \\ (\dagger) \quad & \textit{iff} \quad |A| \Subset |B|, |A| \sqsubseteq |B|, |B| \Subset |A|, \text{ and } |B| \sqsubseteq |A| \\ (\ddagger) \quad & \textit{iff} \quad |A| = |B|. \end{aligned}$$

The biconditionals above all follow by definition with the exception of $(\dagger)$ which is given by **L9.4**, and $(\ddagger)$ which requires further support.

Assume that: (1) $|A| \Subset |B|$; (2) $|A| \sqsubseteq |B|$; (3) $|B| \Subset |A|$; and (4) $|B| \sqsubseteq |A|$. It follows that $|A|^{+} \subseteq |B|^{+}$ and $|B|^{+} \subseteq |A|^{+}$ from (1) and (3), where similarly $|A|^{-} \subseteq |B|^{-}$ and $|B|^{-} \subseteq |A|^{-}$ follow from (2) and (4). Thus we may conclude that $|A| = |B|$.

Assume instead that $|A| = |B|$. Thus $|A|^{\pm} = |B|^{\pm}$ and $|B|^{\pm} = |A|^{\pm}$, and so $|A|^{\pm} \subseteq |B|^{\pm}$ and $|B|^{\pm} \subseteq |A|^{\pm}$. Since $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$, both $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **P7.1**, and so $|A|^{\pm}, |B|^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Thus $|A|^{\pm} \trianglerighteq |B|^{\pm}$ and $|B|^{\pm} \trianglerighteq |A|^{\pm}$ follow by **L9.14**, and so (1) – (4) follow by definition. $\square$

L9.6 $\not\models_{\mathcal{C}^\pm} A \equiv A \wedge (A \vee B)$ for some $A, B \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Recall that $P \neq P \wedge (P \vee Q)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$ by **L8.17**. Let $\mathcal{M} \in \mathcal{C}^\pm_\mathcal{S}$ where $|p_1| = P$ and $|p_2| = Q$. It follows that $|p_1| \neq |p_1| \wedge (|p_1| \vee |p_2|)$. Thus we may reason as follows:

$$\begin{aligned} |p_1| \neq |p_1| \wedge (|p_1| \vee |p_2|) \;\; &\textit{iff} \;\; |p_1| \neq |p_1| \wedge |p_1 \vee p_2| \\ &\textit{iff} \;\; |p_1| \neq |p_1 \wedge (p_1 \vee p_2)| \\ (*) \;\; &\textit{iff} \;\; \mathcal{M} \not\models p_1 \equiv p_1 \wedge (p_1 \vee p_2). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \equiv [p_1 \wedge (p_2 \vee p_3)]$. □

L9.7 $\not\models_{\mathcal{C}^\pm} A \equiv A \vee (A \wedge B)$ for some $A, B \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Recall that $P \neq P \vee (P \wedge Q)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q \in \mathbb{P}^\pm_\mathcal{S}$ by **L8.18**. Let $\mathcal{M} \in \mathcal{C}^\pm_\mathcal{S}$ where $|p_1| = P$ and $|p_2| = Q$. It follows that $|p_1| \neq |p_1| \vee (|p_1| \wedge |p_2|)$. Thus we may reason as follows:

$$\begin{aligned} |p_1| \neq |p_1| \vee (|p_1| \wedge |p_2|) \;\; &\textit{iff} \;\; |p_1| \neq |p_1| \vee |p_1 \wedge p_2| \\ &\textit{iff} \;\; |p_1| \neq |p_1 \vee (p_1 \wedge p_2)| \\ (*) \;\; &\textit{iff} \;\; \mathcal{M} \not\models p_1 \equiv p_1 \vee (p_1 \wedge p_2). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \equiv [p_1 \vee (p_2 \wedge p_3)]$. □

L9.8 $\not\models_{\mathcal{C}^\pm} A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ for some $A, B, C \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Recall that $P \vee (Q \wedge R) \neq (P \vee Q) \wedge (P \vee R)$ for some $\mathcal{S} \in \mathbb{M}_\infty$ and $P, Q, R \in \mathbb{P}^\pm_\mathcal{S}$ by **L8.21**. Let $\mathcal{M} \in \mathcal{C}^\pm_\mathcal{S}$ where $|p_1| = P$, $|p_2| = Q$ and

$|p_3| = R$. Thus $|p_1| \vee (|p_2| \wedge |p_3|) \neq (|p_1| \vee |p_2|) \wedge (|p_1| \vee |p_3|)$, and so:

$$\begin{aligned} & |p_1| \vee (|p_2| \wedge |p_3|) \neq (|p_1| \vee |p_2|) \wedge (|p_1| \vee |p_3|) \\ & \quad \textit{iff} \ \ |p_1| \vee |p_2 \wedge p_3| \neq |p_1 \vee p_2| \wedge |p_1 \vee p_3| \\ & \quad \textit{iff} \ \ |p_1 \vee (p_2 \wedge p_3)| \neq |(p_1 \vee p_2) \wedge (p_1 \vee p_3)| \\ (*) \qquad & \quad \textit{iff} \ \ \mathcal{M} \not\models p_1 \vee (p_2 \wedge p_3) \equiv (p_1 \vee p_2) \wedge (p_1 \vee p_3). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \vee (p_2 \wedge p_3) \equiv (p_1 \vee p_2) \wedge (p_1 \vee p_3)$. $\square$

L9.9 $\not\models_{\mathcal{C}^{\pm}} A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ for some $A, B, C \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Proof. Recall that $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ for some $\mathcal{S} \in \mathbb{M}_{\infty}$ and $P, Q, R \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **L8.22**. Let $\mathcal{M} \in \mathcal{C}^{\pm}_{\mathcal{S}}$ where $|p_1| = P$, $|p_2| = Q$, and $|p_3| = R$. It follows that $|p_1| \wedge (|p_2| \vee |p_3|) \neq (|p_1| \wedge |p_2|) \vee (|p_1| \wedge |p_3|)$. Thus we may reason as follows:

$$\begin{aligned} & |p_1| \wedge (|p_2| \vee |p_3|) \neq (|p_1| \wedge |p_2|) \vee (|p_1| \wedge |p_3|) \\ & \quad \textit{iff} \ \ |p_1| \wedge |p_2 \vee p_3| \neq |p_1 \wedge p_2| \vee |p_1 \wedge p_3| \\ & \quad \textit{iff} \ \ |p_1 \wedge (p_2 \vee p_3)| \neq |(p_1 \wedge p_2) \vee (p_1 \wedge p_3)| \\ (*) \qquad & \quad \textit{iff} \ \ \mathcal{M} \not\models p_1 \wedge (p_2 \vee p_3) \equiv (p_1 \wedge p_2) \vee (p_1 \wedge p_3). \end{aligned}$$

The biconditionals above hold by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\mathcal{M} \not\models p_1 \wedge (p_2 \vee p_3) \equiv (p_1 \wedge p_2) \vee (p_1 \vee p_3)$. $\square$

L9.10 $\vdash_{\text{UGS}} A \Rightarrow A \wedge (A \vee B)$ for all $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Proof. We know that $\vdash_{\text{UGS}} A \wedge [A \vee (A \wedge B)] \trianglelefteq (A \wedge A) \vee [A \wedge (A \wedge B)]$ by **E13**, where $\vdash_{\text{UGS}} A \wedge A \trianglelefteq A \vee (A \wedge B)$ follows from **GA4** and **GA1**. Additionally, $\vdash_{\text{UGS}} A \wedge (A \wedge B) \trianglelefteq A \vee (A \wedge B)$ holds by **GA5**, **GA4**, **GA8**, and **GA2**. Thus $\vdash_{\text{UGS}} (A \wedge A) \vee [A \wedge (A \wedge B)] \trianglelefteq A \vee (A \wedge B)$ follows by **AR1**, and so $\vdash_{\text{UGS}} A \wedge [A \vee (A \wedge B)] \trianglelefteq A \vee (A \wedge B)$ holds by **GA9**. Next observe that $\vdash_{\text{UGS}} A \wedge (A \wedge B) \trianglelefteq A \wedge [A \vee (A \wedge B)]$ follows from **GA2**, **E1**, and **GA8**, and so $\vdash_{\text{UGS}} A \wedge B \trianglelefteq A \wedge [A \vee (A \wedge B)]$ holds by **GA6**,

GA3, **E1**, and **GA8**. Additionally, $\vdash_{\text{UGS}} A \trianglelefteq A \wedge [A \vee (A \wedge B)]$ follows from **E1**, **GA1**, **GA8**, **GA3**, and **GA9**, and so we may conclude that $\vdash_{\text{UGS}} A \vee (A \wedge B) \trianglelefteq A \wedge [A \vee (A \wedge B)]$ by **AR1**. Together with the above, it follows that $\vdash_{\text{UGS}} A \vee (A \wedge B) \approx A \wedge [A \vee (A \wedge B)]$, and so by $\mathbf{R2}^{\pm}$ $\vdash_{\text{UGS}} \neg A \vee (\neg A \wedge \neg B) \approx \neg A \wedge [\neg A \vee (\neg A \wedge \neg B)]$. By **NE12**, **NE13**, and **AR4**, it follows that $\vdash_{\text{UGS}} \neg A \vee (\neg A \wedge \neg B) \approx \neg A \wedge \neg[A \wedge (A \vee B)]$, and so $\vdash_{\text{UGS}} \neg A \trianglerighteq \neg[A \wedge (A \vee B)]$ by definition.

Given that $\vdash_{\text{UGS}} A \trianglelefteq A \wedge (A \vee B)$ by **T3**, it follows from the above that $\vdash_{\text{UGS}} A \leqslant A \wedge (A \vee B)$. Since we may show by a similar argument that $\vdash_{\text{UGS}} A \sqsubseteq A \wedge (A \vee B)$, we may conclude that $\vdash_{\text{UGS}} A \Rightarrow A \wedge (A \vee B)$. □

L9.11 $\vdash_{\text{UGS}} A \Rightarrow A \vee (A \wedge B)$ for all $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Proof. Similar to **L9.10**. □

L9.12 $\vdash_{\text{UGS}} \neg A \vee \neg B \equiv \neg(A \wedge B)$ for all $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$. By **P7.1**, $|A|, |B| \in \mathbb{P}^{\pm}_{\mathcal{S}}$, and so $\neg(|A| \wedge |B|) = \neg|A| \vee \neg|B|$ by **L8.19**. Consider the following:

$$\begin{aligned} \neg(|A| \wedge |B|) = \neg|A| \vee \neg|B| \quad &\textit{iff} \quad \neg|A \wedge B| = |\neg A| \wedge |\neg B| \\ &\textit{iff} \quad |\neg(A \wedge B)| = |\neg A \vee \neg B| \\ (*) \quad &\textit{iff} \quad \models_{\mathcal{C}^{\pm}} \neg(A \wedge B) \equiv \neg A \vee \neg B. \end{aligned}$$

The biconditionals above follow by definition with the exception of $(*)$ which is given by **L9.5**. Thus $\models_{\mathcal{C}^{\pm}} \neg(A \vee B) \equiv \neg A \wedge \neg B$, and so we may conclude that $\vdash_{\text{UGS}} \neg(A \vee B) \equiv \neg A \wedge \neg B$ by **Theorem T4**. □

L9.13 $\models_{\mathcal{C}^{\pm}} \neg(A \vee B) \equiv \neg A \wedge \neg B$ for all $A, B \in \mathtt{pfs}^{\neg}(\mathcal{L})$.

Proof. Similar to **L9.10**, drawing on **L8.20** in place of **L8.19**. □

L9.14 For all $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X, Y \in \mathbb{P}^{\infty}_{\mathcal{S}}$, if $X \subseteq Y$, then $Y \trianglerighteq X$.

Proof. Let $\mathcal{S} \in \mathbb{M}_{\infty}$ and $X, Y \in \mathbb{P}^{\infty}_{\mathcal{S}}$, and assume $X \subseteq Y$. Choose some $s \in X$. It follows that $s \in Y$, where $s \sqsubseteq s$ since $\bigsqcup\{s, s\} = s$ by *Collapse*. Generalising on s, we may conclude that $Y \trianglerighteq X$. □

L9.15 $X \wedge (Y \vee Z) \sqsubseteq (X \wedge Y) \vee (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$, and assume $s \in X \wedge (Y \vee Z)$. Thus $s = \bigsqcup\{x, u\}$ for some $x \in X$ and $u \in Y \vee Z$. By **L6.8**, $u \in Y \cup Z \cup (Y \wedge Z)$. If $u \in Y \cup Z$, then $s \in (X \wedge Y) \cup (X \wedge Z)$, and so $s \in (X \wedge Y) \vee (X \wedge Z)$ by **L6.8**. Assume instead $u \in Y \wedge Z$. It follows that $u = \bigsqcup\{y, z\}$ for some $y \in Y$ and $z \in Z$, and so $\bigsqcup\{x, y\} \in X \wedge Y$ and $\bigsqcup\{x, z\} \in X \wedge Z$. Thus $\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\} \in (X \wedge Y) \wedge (X \wedge Z)$. Observe the following:

$$\begin{aligned}
\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\} &= \bigsqcup\bigcup\{\{x, y\}, \{x, z\}\} \\
&= \bigsqcup\bigcup\{\{x\}, \{y, z\}\} \\
&= \bigsqcup\{\bigsqcup\{x\}, \bigsqcup\{y, z\}\} \\
&= \bigsqcup\{x, u\} \\
&= s.
\end{aligned}$$

Thus $s \in (X \wedge Y) \wedge (X \wedge Z)$, and so $s \in (X \wedge Y) \vee (X \wedge Z)$ by **L6.8**. Given the above, $X \wedge (Y \vee Z) \sqsubseteq (X \wedge Y) \vee (X \wedge Z)$ as needed. □

L9.16 $X \vee (Y \wedge Z) \sqsubseteq (X \vee Y) \wedge (X \vee Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Let $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$, and assume $s \in X \vee (Y \wedge Z)$. By **L6.8**, $s \in X \cup (Y \wedge Z) \cup [X \wedge (Y \wedge Z)]$. If $s \in X$, then $s \in X \vee Y$ and $s \in X \vee Z$, and so $s \in (X \vee Y) \wedge (X \vee Z)$. Assume instead that $s \in Y \wedge Z$, then $s = \bigsqcup\{y, z\}$ for some $y \in Y$ and $z \in Z$. By **L6.8**, $y \in X \vee Y$ and $z \in X \vee Z$, and so $s \in (X \vee Y) \wedge (X \vee Z)$. If instead $s \in X \wedge (Y \wedge Z)$, then $s = \bigsqcup\{x, u\}$ for some $x \in X$ and $u \in Y \wedge Z$, and so $u = \bigsqcup\{y, z\}$ for some $y \in Y$ and $z \in Z$. By **L6.8**, $\bigsqcup\{x, y\} \in X \vee Y$ and $\bigsqcup\{x, z\} \in X \vee Z$, and so $\bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\} \in (X \vee Y) \wedge (X \vee Z)$. By the same argument given in **L9.15**, $s = \bigsqcup\{\bigsqcup\{x, y\}, \bigsqcup\{x, z\}\}$, and so $s \in (X \vee Y) \wedge (X \vee Z)$. Thus we may conclude that $X \vee (Y \wedge Z) \sqsubseteq (X \vee Y) \wedge (X \vee Z)$. □

L9.17 $X \wedge (Y \vee Z) \sqsupseteq (X \wedge Y) \vee (X \wedge Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Immediate from **L9.15** and **L9.14**. □

L9.18 $X \vee (Y \wedge Z) \trianglerighteq (X \vee Y) \wedge (X \vee Z)$ for all $\mathcal{S} \in \mathbb{M}_\infty$ and $X, Y, Z \in \mathbb{P}^\infty_{\mathcal{S}}$.

Proof. Immediate from **L9.16** and **L9.14**. □

L9.19 $\vdash_{\text{UGS}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ for all $A, B, C \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^\pm$ and $A, B, C \in \mathtt{pfs}^\neg(\mathcal{L})$. Thus $|A|, |B|, |C| \in \mathbb{P}^\pm_{\mathcal{S}}$ by **P7.1**, and so $|A|^\pm, |B|^\pm, |C|^\pm \in \mathbb{P}^\infty_{\mathcal{S}}$. Consider the following:

$$
\begin{aligned}
&\mathcal{M} \vDash A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C) \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \wedge (B \vee C) \leqslant (A \wedge B) \vee (A \wedge C) \text{ and} \\
&\qquad \mathcal{M} \vDash A \wedge (B \vee C) \sqsubseteq (A \wedge B) \vee (A \wedge C) \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash A \wedge (B \vee C) \trianglelefteq (A \wedge B) \vee (A \wedge C), && (1) \\
&\qquad \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglerighteq \neg[(A \wedge B) \vee (A \wedge C)], && (2) \\
&\qquad \mathcal{M} \vDash A \wedge (B \vee C) \trianglerighteq (A \wedge B) \vee (A \wedge C), \text{ and} && (3) \\
&\qquad \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglelefteq \neg[(A \wedge B) \vee (A \wedge C)]. && (4)
\end{aligned}
$$

Whereas (1) follows immediately from **T6** by **Theorem T4**, each of the remaining conjuncts require further argument.

Conjunct 2: Since $|A|^- \wedge (|B|^- \vee |C|^-) \trianglerighteq (|A|^- \wedge |B|^-) \vee (|A|^- \wedge |C|^-)$ by **L9.17**, we may argue as follows:

$$
\begin{aligned}
&|A|^- \wedge (|B|^- \vee |C|^-) \trianglerighteq (|A|^- \wedge |B|^-) \vee (|A|^- \wedge |C|^-) \\
&\quad \textit{iff}\ \ |A|^- \wedge |B \vee C|^- \trianglerighteq |A \wedge B|^- \vee |A \wedge C|^- \\
&\quad \textit{iff}\ \ |A \wedge (B \vee C)|^- \trianglerighteq |(A \wedge B) \vee (A \wedge C)|^- \\
&\quad \textit{iff}\ \ |\neg[A \wedge (B \vee C)]|^+ \trianglerighteq |\neg[(A \wedge B) \vee (A \wedge C)]|^+ \\
&\quad \textit{iff}\ \ \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglerighteq \neg[(A \wedge B) \vee (A \wedge C)].
\end{aligned}
$$

The biconditionals above follow from **L7.1** – **L7.3** with the exception of the last which follows from **L9.1**. This establishes (2) as desired.

Conjunct 3: Since $|A|^+ \wedge (|B|^+ \vee |C|^+) \trianglerighteq (|A|^+ \wedge |B|^+) \vee (|A|^+ \wedge |C|^+)$

by **L9.17**, we may argue as follows:

$$
\begin{aligned}
&|A|^+ \wedge (|B|^+ \vee |C|^+) \trianglerighteq (|A|^+ \wedge |B|^+) \vee (|A|^+ \wedge |C|^+) \\
&\quad \textit{iff} \;\; |A|^+ \wedge |B \vee C|^+ \trianglerighteq |A \wedge B|^+ \vee |A \wedge C|^+ \\
&\quad \textit{iff} \;\; |A \wedge (B \vee C)|^+ \trianglerighteq |(A \wedge B) \vee (A \wedge C)|^+ \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash A \wedge (B \vee C) \trianglerighteq (A \wedge B) \vee (A \wedge C).
\end{aligned}
$$

The biconditionals above follow from **L7.2** and **L7.3** with the exception of the last which follows from **L9.1**. This establishes (3) as desired.

Conjunct 4: By **T6**, $\vdash_{\text{UGS}} \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)$, and so $\vDash_{\mathcal{C}^\pm} \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)$ by **Theorem T4**. Thus $\mathcal{M} \vDash \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)$, and so:

$$
\begin{aligned}
&\mathcal{M} \vDash \neg A \wedge (\neg B \vee \neg C) \trianglelefteq (\neg A \wedge \neg B) \vee (\neg A \wedge \neg C) \\
&\quad \textit{iff} \;\; |\neg A \wedge (\neg B \vee \neg C)|^+ \subseteq |(\neg A \wedge \neg B) \vee (\neg A \wedge \neg C)|^+ \\
&\quad \textit{iff} \;\; |\neg A|^+ \wedge |\neg B \vee \neg C|^+ \subseteq |\neg A \wedge \neg B|^+ \vee |\neg A \wedge \neg C|^+ \\
&\quad \textit{iff} \;\; |\neg A|^+ \wedge (|\neg B|^+ \vee |\neg C|^+) \subseteq (|\neg A|^+ \wedge |\neg B|^+) \vee (|\neg A|^+ \wedge |\neg C|^+) \\
&\quad \textit{iff} \;\; |A|^- \wedge (|B|^- \vee |C|^-) \subseteq (|A|^- \wedge |B|^-) \vee (|A|^- \wedge |C|^-) \\
&\quad \textit{iff} \;\; |A|^- \wedge |B \vee C|^- \subseteq |A \wedge B|^- \vee |A \wedge C|^- \\
&\quad \textit{iff} \;\; |A \wedge (B \vee C)|^- \subseteq |(A \wedge B) \vee (A \wedge C)|^- \\
&\quad \textit{iff} \;\; |\neg[A \wedge (B \vee C)]|^+ \subseteq |\neg[(A \wedge B) \vee (A \wedge C)]|^+ \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash \neg[A \wedge (B \vee C)] \trianglelefteq \neg[(A \wedge B) \vee (A \wedge C)].
\end{aligned}
$$

The biconditionals above follow from **L7.1** – **L7.3** with the exception of the last which follows by the **Unilateral Semantics**. This proves (4).

Since all of the conditions (1) – (4) are met, we may conclude that $\mathcal{M} \vDash A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$, and so by generalising on $\mathcal{M} \in \mathcal{C}^\pm$, we know that $\vDash_{\mathcal{C}^\pm} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$. Thus it follows by **Theorem T4** that $\vdash_{\text{UGS}} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$. □

L9.20 $\vDash_{\mathcal{C}^\pm} A \vee (B \wedge C) \Rightarrow (A \vee B) \wedge (A \vee C)$ for all $A, B, C \in \mathtt{pfs}^\neg(\mathcal{L})$.

Proof. Let $\mathcal{M} \in \mathcal{C}^{\pm}$ and $A, B, C \in \mathtt{pfs}^{\neg}(\mathcal{L})$. Thus $|A|, |B|, |C| \in \mathbb{P}^{\pm}_{\mathcal{S}}$ by **P7.1**, and so $|A|^{\pm}, |B|^{\pm}, |C|^{\pm} \in \mathbb{P}^{\infty}_{\mathcal{S}}$. Consider the following:

$$
\begin{aligned}
&\mathcal{M} \vDash A \vee (B \wedge C) \Rrightarrow (A \vee B) \wedge (A \vee C) \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash A \vee (B \wedge C) \leqslant (A \vee B) \wedge (A \vee C) \text{ and} \\
&\qquad \mathcal{M} \vDash A \vee (B \wedge C) \sqsubseteq (A \vee B) \wedge (A \vee C) \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash A \vee (B \wedge C) \trianglelefteq (A \vee B) \wedge (A \vee C), && (1) \\
&\qquad \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglerighteq \neg[(A \vee B) \wedge (A \vee C)], && (2) \\
&\qquad \mathcal{M} \vDash A \vee (B \wedge C) \trianglerighteq (A \vee B) \wedge (A \vee C), \text{ and} && (3) \\
&\qquad \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglelefteq \neg[(A \vee B) \wedge (A \vee C)]. && (4)
\end{aligned}
$$

Whereas (1) follows immediately from **T6** by **Theorem T4**, each of the remaining conjuncts require further argument.

Conjunct 2: As $|A|^{-} \vee (|B|^{-} \wedge |C|^{-}) \trianglerighteq (|A|^{-} \vee |B|^{-}) \wedge (|A|^{-} \vee |C|^{-})$ by **L9.18**, we may provide the following argument:

$$
\begin{aligned}
&|A|^{-} \vee (|B|^{-} \wedge |C|^{-}) \trianglerighteq (|A|^{-} \vee |B|^{-}) \wedge (|A|^{-} \vee |C|^{-}) \\
&\quad \textit{iff} \;\; |A|^{-} \vee |B \wedge C|^{-} \trianglerighteq |A \vee B|^{-} \wedge |A \vee C|^{-} \\
&\quad \textit{iff} \;\; |A \vee (B \wedge C)|^{-} \trianglerighteq |(A \vee B) \wedge (A \vee C)|^{-} \\
&\quad \textit{iff} \;\; |\neg[A \vee (B \wedge C)]|^{+} \trianglerighteq |\neg[(A \vee B) \wedge (A \vee C)]|^{+} \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglerighteq \neg[(A \vee B) \wedge (A \vee C)].
\end{aligned}
$$

Each of the biconditionals above hold by definition with the exception of the last which follows from **L9.1**. This establishes (2) as desired.

Conjunct 3: As $|A|^{+} \vee (|B|^{+} \wedge |C|^{+}) \trianglerighteq (|A|^{+} \vee |B|^{+}) \wedge (|A|^{+} \vee |C|^{+})$

by **L9.18**, we may provide the following argument:

$$
\begin{aligned}
&|A|^+ \vee (|B|^+ \wedge |C|^+) \trianglerighteq (|A|^+ \vee |B|^+) \wedge (|A|^+ \vee |C|^+) \\
&\quad \textit{iff} \;\; |A|^+ \vee |B \wedge C|^+ \trianglerighteq |A \vee B|^+ \wedge |A \vee C|^+ \\
&\quad \textit{iff} \;\; |A \vee (B \wedge C)|^+ \trianglerighteq |(A \vee B) \wedge (A \vee C)|^+ \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash A \vee (B \wedge C) \trianglerighteq (A \vee B) \wedge (A \vee C).
\end{aligned}
$$

Each of the biconditionals above hold by definition with the exception of the last which follows from **L9.1**. This establishes (3) as desired.

Conjunct 4: By **T6**, $\vdash_{\text{UGS}} \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$, and so $\vDash_{\mathcal{C}^\pm} \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$ by **Theorem T4**. Thus it follows that $\mathcal{M} \vDash \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C)$, and so we may reason as follows:

$$
\begin{aligned}
&\mathcal{M} \vDash \neg A \vee (\neg B \wedge \neg C) \trianglelefteq (\neg A \vee \neg B) \wedge (\neg A \vee \neg C) \\
&\quad \textit{iff} \;\; |\neg A \vee (\neg B \wedge \neg C)|^+ \subseteq |(\neg A \vee \neg B) \wedge (\neg A \vee \neg C)|^+ \\
&\quad \textit{iff} \;\; |\neg A|^+ \vee |\neg B \wedge \neg C|^+ \subseteq |\neg A \vee \neg B|^+ \wedge |\neg A \vee \neg C|^+ \\
&\quad \textit{iff} \;\; |\neg A|^+ \vee (|\neg B|^+ \wedge |\neg C|^+) \subseteq (|\neg A|^+ \vee |\neg B|^+) \wedge (|\neg A|^+ \vee |\neg C|^+) \\
&\quad \textit{iff} \;\; |A|^- \vee (|B|^- \wedge |C|^-) \subseteq (|A|^- \vee |B|^-) \wedge (|A|^- \vee |C|^-) \\
&\quad \textit{iff} \;\; |A|^- \vee |B \wedge C|^- \subseteq |A \vee B|^- \wedge |A \vee C|^- \\
&\quad \textit{iff} \;\; |A \vee (B \wedge C)|^- \subseteq |(A \vee B) \wedge (A \vee C)|^- \\
&\quad \textit{iff} \;\; |\neg[A \vee (B \wedge C)]|^+ \subseteq |\neg[(A \vee B) \wedge (A \vee C)]|^+ \\
&\quad \textit{iff} \;\; \mathcal{M} \vDash \neg[A \vee (B \wedge C)] \trianglelefteq \neg[(A \vee B) \wedge (A \vee C)].
\end{aligned}
$$

The biconditionals above all hold by definition with the exception of the last which follows by the **Unilateral Semantics**, thereby proving (4).

Since all of the conditions (1) – (4) are met, we may conclude that $\mathcal{M} \vDash A \vee (B \wedge C) \Rrightarrow (A \vee B) \wedge (A \vee C)$, and so by generalising on $\mathcal{M} \in \mathcal{C}^\pm$, we know that $\vDash_{\mathcal{C}^\pm} A \vee (B \wedge C) \Rrightarrow (A \vee B) \wedge (A \vee C)$. Thus it follows by **Theorem T4** that $\vdash_{\text{UGS}} A \vee (B \wedge C) \Rrightarrow (A \vee B) \wedge (A \vee C)$. $\square$

Conclusion

Kit Fine has built a foundation upon which to construct a theory of essence and ground. Whereas Fine (2001, 2012b, 2015b) begins to develop a philosophical account of the theoretical roles which essence and ground are to play, Fine (2012c, 2017d, 2016, 2017b,c) provides the semantic resources needed to evaluate essence and grounding claims for validity. In an exploratory spirit, this thesis aims to advance these pursuits, extending Fine's pure logic of ground in order to describe the interactions between essence, ground, propositional identity, relevance, reduction, necessity, and the extensional operators in an otherwise classical propositional logic. In conclusion of the preceding chapters, I will examine the points at which I have departed from Fine's account, explaining what motivates each departure as well as examining the relationships between the various proof systems that I have presented along the way.

Fine (2012b) provides the beginnings of a theory of metaphysical grounding as a form of constitutive explanation which is to be distinguished among explanatory forms by the modal force which it imparts, writing:

> Just as metaphysical necessity is the strictest form of necessity (at least as compared to natural and normative necessity), so it is natural to suppose that statements of metaphysical ground are the strictest form of in-virtue-of claim. (p. 38)

Given that $A \leqslant B$, it follows that $\Box(A \to B)$. By contrast, Fine takes normative and natural explanations to only impart modal connections with normative or natural force, respectively. Rather than amounting to nothing

over and above strict implication of the associated kind, Fine also writes:

> In addition to the modal connection, there would also appear to be an explanatory or determinative connection— a movement, so to speak, from antecedent to consequent; and what is most distinctive about the in-virtue-of claims is this element of movement or determination. (p. 38)

Rather than attempting to define ground in other terms, Fine studies the interactions between a host of primitive grounding operators and the extensional operators, ultimately focusing on a many-one weak-full grounding operator which gives expression to claims of the form $\ulcorner A_1, A_2, \ldots \leqslant B \urcorner$.

Rather than including infinitary expressive power in the grounding operator by permitting an infinite number of antecedents, Chapter 1 defends the restriction to a binary sentential weak-full grounding operator by noting that we may define Fine's many-one grounding operator in terms of the binary grounding operator with the help of infinitary conjunction. In Chapter 2, I go on to clarify the constitutive nature which Fine attributes to grounding by noting that in addition to its modal force, grounding requires the antecedent to be wholly relevant to the consequent. As I show, the constitutive nature of essence may also be characterised in this way, where essence entails that the consequent strictly implies the antecedent, while also requiring the antecedent to be wholly relevant to the consequent just as with ground.

Although Fine (2012b) draws preliminary connections between ground and essence, he does not offer a systematic account of their interaction. Fine (2015b) takes further steps towards providing a unified account of essence and ground, providing an operator notation for essence claims in parallel to ground, as well as drawing a connection between essence and ground and constitutive readings of the locutions 'necessary for' and 'sufficient for', writing as follows:

> But behind the notational parallel lies a deeper conceptual parallel. For in the case of essence, the statement on the left serves as a necessary condition for the statements on the right while, in the case of ground, the statements on the left serve as a sufficient condition for the statement on

> the right. In each case, the condition is a constitutive condition—either constitutively necessary or constitutively sufficient. (p. 306)

Fine also notes the distinctive role that essence and ground play in providing an account of reduction, or what he also calls real definition, writing:

> I have previously suggested that definitions, either nominal or real, might plausibly be taken to correspond to statements of essence (simply involving the reverse arrow '$\leftarrowtriangle$').What I would now like to suggest is that reductive definitions be taken to correspond to real definitions in which the arrow can be reversed, so that we have what is both a constitutively necessary and a constitutively sufficient condition for something to hold. (p. 308)

Whereas Fine uses '$\leftarrowtriangle$' and '$\rightarrowtriangle$' for essence and ground, I have adopted '$\sqsubseteq$' and '$\leqslant$' in their place. Nevertheless, Chapter 2 develops an account of reduction in terms of essence and ground in the manner which Fine suggests.

As Fine observes, essence and ground are not to be thought of as converses of each other, where Chapter 2 defends this claim. Despite congruency between Fine's account of essence and ground and my own, there are also a number of significant differences. To begin with, Fine (2015b) expresses his scepticism that essence and ground can be interdefined, writing:

> Not only are the relations of necessary and of sufficient condition not converses, it is hard to see how either could be defined in terms of the other. (p. 307)

By contrast, I defend the equivalence principles $(A \sqsubseteq B) \leftrightarrow (\neg A \leqslant \neg B)$ and $(A \leqslant B) \leftrightarrow (\neg A \leqslant \neg B)$ in Chapter 2, demonstrating their validity in Chapter 4 and Chapter 5. In addition to admitting intuitive informal readings in terms of the locutions 'necessary for' and 'sufficient for', I show in Chapter 4 and Chapter 5 that it is partly in virtue of these equivalences that essence and ground form a bilattice over the space of propositions. As evidenced by a mounting body of mathematical literature, bilattices are structures with an extremely elegant and natural form, comparable to the Boolean lattices employed by extensional and intensional theories of propositions.

In contrast to extensional logics which interpret sentences over a space consisting of just two propositions, Kripke's possible worlds semantics greatly expanded the domain of distinct propositions to which sentences may be assigned. However powerful such intensional theories of propositions are for interpreting modal logics, I argue in Chapter 3 that sets of possible worlds do not draw the distinctions needed to track relevance. Insofar as essence and ground both require their antecedents to be wholly relevant to their consequents in addition to the modal connections that they impose, an intensional theory of propositions does not suffice for interpreting essence and ground. Rather, Fine (2016, 2017b,c,d) has set about constructing a hyperintensional alternative whereby each sentence may be interpreted by an ordered pair of sets of states which exactly verify or falsify the sentence in question.

Among the applications which Fine (2012b, 2017c) considers for his state semantics is unilateral ground. Although, Fine does not provide a unilateral semantics for essence, nor a bilateral semantics for either essence or ground as developed in Chapter 4 and Chapter 5, Fine (2016, 2017b) presents a semantics for analytic containment which is closely related to what might be taken to be a unilateral semantics for essence. However, Chapter 4 shows that such a semantics produces counterexamples to the principles $A \leqslant A \vee B$ and $B \leqslant A \vee B$, as well as $A \sqsubseteq A \wedge B$ and $B \sqsubseteq A \wedge B$. In order to avoid these consequences, Fine contemplates the following:

> There are a number of ways of dealing with this problem. One is to ban the proposition and insist that each proposition have a verifier; another is to loosen the definition of conjunctive part and allow any proposition P whatever to be a conjunctive part of $\bot$, given that $\bot \equiv P \wedge \bot$. But each of these modifications comes at a price. If we restrict our attention to verifiable propositions then there will not exist a proposition which is a disjunctive part of every other proposition; and if we loosen the definition of conjunctive part then we will lose the principle that if R is part of Q then $P \vee R$ should be part of $P \vee Q$. For let Q be $\bot$. Then any proposition R will be part of $\bot$; and so, given the principle, $P \vee R$ will be apart of $P \vee \bot \equiv P$. I have found the most satisfactory solution to be one which countenances $\bot$ but treats it as a 'dead end' proposition which

> is not allowed to be conjoined with other propositions. We need not go so far as to deny that the conjunction of $\bot$ with any other proposition exists. It is just that there is no need to conjoin $\bot$ with any other proposition (it just gives us back F); and we get a smoother theory if the application of conjunction is disallowed in this case. (p. 642)

I take Fine's preferred solution of restricting the domain of the conjunction operator to exclude $\bot$ to be *ad hoc*. Additionally, as shown in Chapter 4 and Chapter 5, giving up $\bot$ altogether by insisting that every proposition have an exact verifier and falsifier makes the space of propositions fail to have any lower bounds despite admitting of upper bounds.

Rather than developing either of these alternatives, Chapter 4 defends a bilateral semantics for essence and ground which aligns with what I take to be the most natural definitions of conjunctive and disjunctive part, validating the principles $(A \sqsubseteq B) \leftrightarrow (A \wedge B \equiv B)$ and $(A \leqslant B) \leftrightarrow (A \vee B \equiv B)$. Although, as Fine observes above, every unilateral proposition is essential to $\bot$— or equivalently, a conjunctive part of $\bot$— the same cannot be said for bilateral propositions. In particular, we may show that $A \not\sqsubseteq \bot$ and $A \wedge \bot \not\equiv \bot$ whenever A has an exact falsifier which is distinct from the null state. Accordingly, including $\bot$ within the space of bilateral propositions does not yield the result that $P \vee R \sqsubseteq P$ for any P and R as Fine worries above.

In addition to the troubles which Fine brings out for unilateral containment, interpreting '$\sqsubseteq$' in terms of Fine's containment relation admits exceptions to the equivalence between essence and conjunctive part, unless propositions are required to be convex. However, as shown in Chapter 3 and Chapter 4, convexity is not preserved by the most natural semantic clauses for conjunction and disjunction. Insofar as we maintain the inclusive semantics for conjunction and disjunction while admitting the uniform substitution of complex sentences for sentence letters, convexity must be given up, and so containment cannot provide a semantics for essence which is equivalent to conjunctive part. For these reasons, I do not follow Fine in taking propositions to be convex.

Although the bilateral semantics which I provide for essence and ground does not validate the principles $A \sqsubseteq \bot$ and $A \wedge \bot \equiv \bot$ which Fine presents in the quote above, admitting of non-convex propositions nevertheless invalidates the the monotonicity of both disjunction over essence as well as the monotonicity of conjunction over ground. Although the monotonicity principles may at first appear to be intuitive, the counterexamples which I present in Chapter 4 help to dispel this illusion, showing that it is on account of the strength of the relevance requirement imposed by essence and ground that the monotonicity principles fail to hold. Rather it is only by weakening the semantics for essence, that disjunction and conjunction may be said to be monotonic over essence and ground, respectively. However, given this weaker semantics, one must prevent disjunction and conjunction from operating on $\bot$ in order to avoid admitting counterexamples to the principles $A \leqslant A \vee B$ and $B \leqslant A \vee B$ as well as $A \sqsubseteq A \wedge B$ and $B \sqsubseteq A \wedge B$ brought out above. For these reasons, I take the bilateral semantics which I provide for essence to be preferable to Fine's containment relation, where I have required propositions to be closed under fusion, without requiring them to be convex or non-vacuous.

By contrast with the points of divergence from the manner in which Fine has developed the semantics for ground, there is considerably less room for disagreement in considering the logic of ground. As brought out in the introduction, Fine begins by restricting consideration to a language which only includes atomic grounding sentences in which just the grounding operators occur in his *Pure Logic of Ground* (PLG). By contrast, I have lifted these restriction, investigating the interactions between essence, ground, reduction, propositional identity, relevance, metaphysical necessity, and the extensional operators. Accordingly, Chapter 2 provides a philosophical target for a logic of essence and ground, presenting *The Modal Logic of Essence and Ground* (EGM) which describes a wide range of interaction principles in such a language.

In Chapter 4, I go on to provides a semantics over which EGM may be shown to be sound. Rather than conflicting with PLG at specific points, EGM vastly outstrips the range of theorems that can be derived in PLG.

In contemplating the theorems of EGM, it is natural to wonder if any principles have been left out. Rather than attempting to show that EGM is complete over its semantics, I have taken steps in this direction by proving that *The Specific Logic of Unilateral Ground* (UGS) is sound and complete over a natural semantics and corresponding class of models. Whereas EGM includes a primitive bilateral grounding operator, UGS provides axioms and rules for a unilateral grounding operator and specificity operator, as well as the extensional operators. Additionally, I have restricted consideration to extensional combinations of atomic grounding sentences in which at most the extensional operators occur within the scope of a unilateral grounding operator. Although these restrictions greatly simply the completeness proof, they come at the cost of excluding a range of principles which belong to EGM. Not only is UGS a non-modal logic incapable of stating such rules as $A \leqslant B \vdash \Box(A \to B)$, but even rules such as $A, A \leqslant B \vdash B$ cannot be articulated in UGS.

Despite these shortcomings, I show that the bilateral notions of essence and ground may be defined within the language of UGS, deriving a wide range of theorems and admissible rules which I refer to as *The Logic of Essence and Ground* (EG). In contrast to UGS which does not include axioms and rules for bilateral essence or ground, EG provides a natural subsystem by which to reason with the notions of essence and ground in the presence of the extensional operators. By then defining propositional identity in terms of essence and ground, I derive a further range of propositional identities within UGS, referring to the result as *The Logic of Propositional Identity* (LPI).

By contrast with LPI derived within UGS, Chapter 3 provides a much more restricted *Logic for Propositional Identity* (LPI^-) which only includes

atomic identity sentences in which at most the extensional operators occur within the scope of a propositional identity operator. The syntactic constraints imposed on LPI^- facilitate comparison with logics such as Correia's (2016b) *Logic of Generalised Identity* (GI), as well as Angell's (2002) *Logic for Analytic Containment* (AC) which Fine (2016) proves to be sound and complete over his state semantics. Nevertheless, LPI may be shown to include LPI^- as a subsystem, amounting to little more than the closure of LPI^- under truth-functional consequence. The same cannot be said, however, of the extensions LPI^-_σ, $\text{LPI}^-_{\sigma^1}$, and $\text{LPI}^-_{\sigma^2}$ which include axioms and rules for a subject-matter operator which does not belong to the language of UGS. It remains to show whether or not UGS is a conservative extension of EG and LPI, thereby extending the completeness result to EG and LPI, respectively. Additionally, one might attempt to provide a completeness result for EGM which includes a primitive modal operator for metaphysical necessity.

In addition to establishing the completeness of stronger systems than UGS over the class of models that I have provided, one may also hope to adapt the semantics to admit of distinctions between the propositions expressed by true grounding and modal claims. As observed in Chapter 4, all true grounding and modal sentences express the same top element $\mathcal{T}$ within the space of propositions. Accordingly, no interesting theory of what grounds grounding claims, or what grounds modal claims, can be developed over the semantic frameworks that I have considered. However, prior to undertaking such a task, one must specify at least the broad outline of a philosophically well-motivated theoretical target for such a semantics, gathering together a stock of candidate principles which the semantics may aim to validate. The very same may be said with respect to the current semantics, where without a philosophical account of what grounds grounding or modal claims, it is impossible to assess whether or not we may reasonably accept that all true grounding and modal claims

express the same proposition as in the semantics in Chapter 4.

Beyond these ambitions to extend the propositional logics for essence and ground as well as the metalogical properties which such systems may be shown to enjoy, one may also seek to integrate the resources of first-order logic. Although Fine (2012b, 2015a) explores a range of first-order principles in the logic of essence and ground, it remains to provide a hyperintensional semantics by which to interpret a stock of predicates, constants, variables, and variable-binding operators. I hope to attend to this task in future work, as well as exploring applications of such logics within metaphysics. Although the logic of essence and ground remains in its infancy, it has been the ambition of this thesis to demonstrate that its development holds considerable promise.

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