Short-Time Structural Stability of Compressible Vortex Sheets with Surface Tension

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A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2014

Acknowledgements

I would like to thank my supervisor, Professor Gui-Qiang Chen, for guiding me towards the problem considered in my thesis. I would also like to thank the OxPDE support staff for their help with administrative matters. Finally, I would like to thank the Engineering and Physical Sciences Research Council for the funding that has allowed me to complete my thesis.

Abstract

The main purpose of this work is to prove short-time structural stability of compressible vortex sheets with surface tension.

The main result can be summarised as follows. Assume we start with an initial vortex-sheet configuration which consists of two inviscid fluids with density bounded below flowing smoothly past each other, where a strictly positive fixed coefficient of surface tension produces a surface tension force across the common interface, balanced by the pressure jump. We assume the fluids are modelled by the compressible Euler equations in three space dimensions with a very general equation of state relating the pressure, entropy and density in each fluid such that the sound speed is positive. Then, for a short time, which may depend on the initial configuration, there exists a unique solution of the equations with the same structure, that is, two fluids with density bounded below flowing smoothly past each other, where the surface tension force across the common interface balances the pressure jump.

The mathematical approach consists of introducing a carefully chosen artificial viscosity-type regularisation which allows one to linearise the system so as to obtain a collection of transport equations for the entropy, pressure and curl together with a parabolic-type equation for the velocity. We prove a high order energy estimate for the non-linear equations that is independent of the artificial viscosity parameter which allows us to send it to zero. This approach loosely follows that introduced by Shkoller et al in the setting of a compressible liquid-vacuum interface.

Although already considered by Shkoller et al, we also make some brief comments on the case of a compressible liquid-vacuum interface, which is obtained from the vortex sheets problem by replacing one of the fluids by vacuum, where it is possible to obtain a structural stability result even without surface tension.

Contents

1. Introduction

1.1 Description of a Vortex Sheet and the Stability Problem

In this work, a vortex sheet refers to a virtual surface separating two inviscid fluids or two phases of the same inviscid fluid across which there is no fluid flow. We consider the three-dimensional vortex-sheet problem, which means that the fluids occupy three space dimensions and the surface is two-dimensional, but our results can easily be adapted to the two-dimensional problem. In general the tangential velocity is discontinuous across the surface, although we do not require this, and the density and entropy (depending on the equations of state satisfied by the two fluid phases) may jump. The fluid flow is modelled by the conversation of mass, momentum and energy, which leads to the use of the compressible Euler equations for the fluid either side of the surface. Conservation of momentum implies that the jump in pressure across the surface is not arbitrary, but must balance any surface force. Such structures are often seen in nature such as when two currents of air at different temperatures meet in the atmosphere, or when oil is spilt on top of water. Vortex sheets can also develop in certain situations starting with smooth initial data for the Euler equations of gas dynamics. In this case clearly the same equations of state are satisfied by the two fluid phases, but here we will allow different equations of state which enables us to model the flow of two different fluids past each other.

We note that there are trivial vortex-sheet solutions to the Euler equations consisting of two constant states separated by a flat surface with zero normal velocity and zero pressure jump. For the rest of this section we denote the velocity of these constant states by $\bar{u}_+$ and $\bar{u}_-$, the density by $\bar{\rho}_+$ and $\bar{\rho}_-$, and the sound speed by $\bar{c}_+$ and $\bar{c}_-$. We are interested in whether such structures are stable, at least for a short time. By short-time stability here we really mean the short-time existence of a unique solution with a vortex-sheet structure given initial data with a vortex-sheet structure - i.e. short-time structural stability. However, in the course of the proof of uniqueness we do prove a kind of stability result whereby a certain energy of the difference of two solutions is controlled in terms of the corresponding energy of the difference of its initial values. In this work we do not require the initial data to be a perturbation of the flat vortex sheet described, but it is a good example to have in mind.

In the absence of a surface tension force which acts to flatten the surface of discontinuity, vortex sheets with a jump in tangential velocity are unstable in general. If the fluids are incompressible then we always have instability - this is the well-known Kelvin-Helmholtz instability. Miles in [17] showed by normal modes analysis that compressible vortex sheets in two dimensions are linearly unstable unless the following stability criterion is satisfied.

$$|[\bar{u}]| \geq (\bar{c}_+^{\frac{2}{3}} + \bar{c}_-^{\frac{2}{3}})^{\frac{3}{2}}$$

Flat Vortex Sheet

	Fluid in Ω_+	
	Density	$\bar{\rho}_+$
	Velocity	$\bar{u}_+ = (\bar{u}'_+, 0)$
$\Omega_+ = \{x_3 > 0\}$	Pressure	$\bar{p}_+ = \bar{p}$
	Internal Energy	$\bar{e}_+$
	Entropy	$\bar{s}_+$
	Sound Speed	$\bar{c}_+$

$\longrightarrow \bar{u}_+$

$\Gamma = \{x_3 = 0\}$

$\rightarrow \bar{u}_-$

	Fluid in Ω_-	
	Density	$\bar{\rho}_-$
	Velocity	$\bar{u}_- = (\bar{u}'_-, 0)$
$\Omega_- = \{x_3 < 0\}$	Pressure	$\bar{p}_- = \bar{p}$
	Internal Energy	$\bar{e}_-$
	Entropy	$\bar{s}_-$
	Sound Speed	$\bar{c}_-$

where $[\bar{u}] = \bar{u}_+ - \bar{u}_-$, under the simplifying assumption

$$\bar{\rho}_+\bar{c}_+^2 = \bar{\rho}_-\bar{c}_-^2$$

In this case they are linearly stable. In fact Coulombel and Secchi in [9] and [8] have shown using Nash-Moser iteration that (isentropic) vortex sheets in two dimensions are stable for a short time under this condition (as a strict inequality) if the initial perturbation of the flat vortex sheet is small enough in a high-order Sobolev norm. Miles and Fejer in [16] also showed, using a simple extension of the two dimensional argument, that vortex sheets in three dimensions are always linearly unstable.

Still in the absence of a surface tension, if the vortex sheet has no jump in tangential velocity initially, but the density jumps, then Rayleigh-Taylor instability will occur. Even if the Rayleigh-Taylor condition is satisfied (loosely speaking the pressure at the surface of discontinuity increases in the direction of the denser fluid), so that Rayleigh-Taylor instability does not occur, it seems that for general perturbations preserving the continuity of the velocity initially, the tangential velocity will immediately become discontinuous and then we will have the same instability as described above.

We therefore consider here vortex sheets in three dimensions with surface tension, in the hope that surface tension will provide a stabilising effect. A brief overview of surface tension is presented below.

1.1.1 Surface Tension

Surface tension is a surface force between two fluids or two fluid phases which acts to flatten the fluid interface. From a physical point of view it usually models an imbalance of electrostatic forces on the molecules in the surface, thus it is most often applied to liquids rather than gases, although mathematically this distinction will not be important here. In our case we are modelling general inviscid compressible fluids, which could be either gases or liquids.

Surface tension is most easily visualised as its lower dimensional equivalent, 'line' tension, acting on a curve in two dimensional space. In this case it gives rise to a pressure force per unit length which is proportional to the curvature and acts to flatten the curve, thus its magnitude is

$$\sigma\frac{1}{R}$$

where R is the radius of curvature of the curve and we call σ the coefficient of line tension.

For a surface in three dimensional space, this generalises as follows. Surface tension gives rise to a pressure force per unit area acting to flatten the surface which is proportional to the sum of the curvatures in two orthogonal directions, thus its magnitude is

$$\sigma(\frac{1}{R_1}+\frac{1}{R_2})$$

where R_1 and R_2 are the principle radii of curvature of the surface and we call σ the coefficient of surface tension, which has units of force per unit length. Note we can also write this as

$$2\sigma H$$

or

$$\sigma\nabla\cdot\hat{n}$$

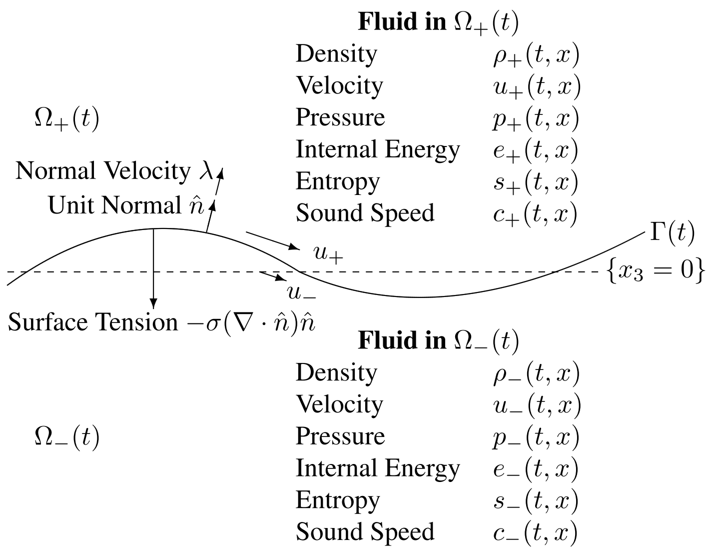
where H is the mean curvature, and $\hat{n}$ is a unit normal to the surface. The resulting surface force is given by

$$-\sigma(\nabla\cdot\hat{n})\hat{n}$$

Note that although $\hat{n}$ is defined only on the surface, we compute $\nabla\cdot\hat{n}$ as $\nabla\cdot\hat{h}$, where $\hat{h}$ is any (sufficiently smooth) unit vector field on $\mathbb{R}^3$ which coincides with $\hat{n}$ on the surface, which one can check is independent of the choice of $\hat{h}$.

In this work the coefficient of surface tension $\sigma > 0$ will be a constant for each pair of fluid phases considered.

Perturbation of a Flat Vortex Sheet



1.2 Summary of the Mathematical Problem

The main goal of this work is to prove theorem 2.4.1, which we summarise in words in its physical context as follows. Assume we start with an initial configuration which consists of two inviscid fluids with density bounded below flowing smoothly past each other, where a strictly positive fixed coefficient of surface tension produces a surface tension force across the common interface, balanced by the pressure jump. We assume the fluids are modelled by the compressible Euler equations with a very general equation of state relating the pressure, entropy and density in each fluid such that the sound speed is positive. Then, for a short time, which may depend on the size of the initial data, there exists a solution of the equations with the same structure, that is, two fluids with density bounded below flowing smoothly past each other, where the surface tension force across the common interface balances the pressure jump. The solution is unique within the class of solutions of the same regularity. In fact we also prove a stability statement which says that a certain energy of the difference of two solutions is estimated by the corresponding energy of the difference of their initial values multiplied by a time dependent factor, where the factor depends on a higher order energy of the initial data. This stability result is stated in 4.6.1 after fixing the domains.

1.3 Existing Results and Methodology for Similar Problems

The case of incompressible vortex sheets with surface tension has been studied by Shkoller, Coutand and Cheng in [6], where they succeed in proving short-time structural stability. Their paper [6] is based on the methodology of their previous work [12] and [11] for the incompressible fluid-vacuum interface problem. Here, we consider the case of compressible vortex sheets with a very general pressure law. All we will need is that the sound speed is well-defined - that is $\frac{\partial p}{\partial \rho}\Big|_s > 0$. We will also assume we are away from vacuum, ie the density is bounded below away from zero. We have already mentioned the short-time structural stability result proved by Coulombel and Secchi in [9] and [8] for compressible (isentropic) vortex sheets in two space dimensions without surface tension, which does not hold in three space dimensions as shown by the linear stability analysis of Miles and Fejer in [16]. The approach of Coulombel and Secchi involves a direct linearisation followed by the use of Kreiss' symmetrisers to obtain an energy estimate with loss of derivatives, then Nash-Moser iteration to overcome this loss of derivatives. This approach is very technical and complicated modifications have to be made to the standard Nash-Moser scheme. We use completely different methodology more similar to that of Shkoller et al, which involves approximating the equations before linearisation so that the linearised energy estimate for the approximate problem does not involve a loss of derivatives with respect to the coefficients.

The proof of our main result ultimately relies on being able to derive an a priori high-order energy estimate for the non-linear equations. Our approach in deriving an energy estimate is similar in parts to the approach of Shkoller et al in [6], except that they use Lagrangian coordinates on one side of the vortex sheet with an extension of the change of coordinate map to the other side whereas we simply add a lift of the graph of the free the surface to the third coordinate in order to fix the surface. However, since Shkoller et al in [6] are dealing with the incompressible case and we consider here the compressible case, some crucial differences arise.

In order to construct approximate solutions which preserve the energy estimate, we follow the artificial viscosity type regularisation of Shkoller, Coutand and Hole from [10], where they consider the free-surface problem for inviscid compressible liquids with surface tension, which they also use in their work on the physical vacuum boundary problem for gases, [13].

Another possible stabilisation effect on vortex sheets other than surface tension is the existence of a background magnetic field, assuming the fluid is electrically conducting, and this gives rise to current-vortex sheets in magnetohydrodynamics (MHD). Trakhinin in [18] and Chen and Wang in [4], [5] have succeeded in proving short-time existence of compressible current-vortex sheets under certain stability criteria involving the non-parallel jump in the magnetic field. Coulombel, Secchi et al in [7] have obtained an a priori energy estimate for incompressible current-vortex sheets under equivalent stability criteria, and the derivation of the energy estimate here is similar in some ways to their approach. We remark that in fact by combining the curl and divergence estimates together, we can avoid the smallness condition that they impose on the initial perturbation of the interface.

1.4 The Euler Equations for Inviscid Compressible Fluids

1.4.1 Physical Derivation of the Equations

Although a fluid is made up of many colliding particles at a microscopic level, we make the assumption that the macroscopic behaviour of a fluid can be described in terms of only a few quantities, called the state variables, which will be the dependent variables in our system of equations. These can be thought of as obtained by averaging over a large number of particles but on a sufficiently small scale for us to be able to justify defining them at every point in space. These quantities are the density, or mass per unit volume, ρ, the velocity, u, the pressure, or normal force per unit area experienced by a virtual surface in the fluid, p, and the internal energy per unit mass, e, which allows for the energy due to the random motion of the fluid particles which is not captured by the bulk fluid velocity u when averaged out.

To determine the conservation laws which should be satisfied by these quantities, we fix an arbitrary region of fluid, Ω, with smooth boundary $\partial\Omega$ and outward unit normal ν, and consider the rate of change of mass, momentum and energy within this region, which must be consistent with the principles of conservation of mass, Newton's second law, and conservation of energy. Note that locally the volume of fluid leaving Ω through an infinitesimal surface element $\mathrm{d}S$ is given by $u\cdot\nu\mathrm{d}S$. We make the assumption of no viscosity so that the only force on a virtual surface in the fluid is a normal force due to the pressure, or a surface tension force along a surface of discontinuity, and we also assume there is no heat conduction. The assumptions of no viscosity and no heat conduction usually make sense for gases but may also make sense for other inviscid fluids where there is no significant heat conduction over the time scale being considered.

We assume that surface tension acts only on a surface of discontinuity $\Gamma(t)$. We will assume $\Gamma(t)$ is smooth with a well defined unit normal $\hat{n}(t,x)$, and moves with normal velocity $\lambda(t,x)$, for $x \in \Gamma(t)$. We allow the coefficient of surface tension σ to be zero in the current section, although later we will require it to be strictly positive.

From conservation of mass, we obtain the following equation, where the term on the right hand side is the rate at which mass enters Ω due to the flow.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho \mathrm{d}x = -\int_{\partial\Omega}\rho u\cdot\nu \mathrm{d}S \tag{1.4.1}$$

From Newton's second law, we obtain the following equation, where the first term on the right hand side is the rate at which momentum enters Ω due to the flow, the second term is the force exerted on the region Ω due to the pressure, and the third term is the force exerted on the fluid in Ω due to surface tension on the surface $\Gamma(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho u \mathrm{d}x = -\int_{\partial\Omega}(\rho u)u\cdot\nu \mathrm{d}S - \int_{\partial\Omega}p\nu \mathrm{d}S - \int_{\Omega\cap\Gamma(t)}\sigma(\nabla\cdot\hat{n})\hat{n}\mathrm{d}S \tag{1.4.2}$$

From conservation of energy, which is given per unit volume by the sum of internal energy ρe and kinetic energy due to the bulk motion, $\frac{1}{2}\rho u^2$, we obtain the following equation, where the first term on the right hand side is the rate at which energy enters Ω due to the flow, the second term is the work done by the pressure-induced force on the fluid, and the third term is the work done by the surface tension-induced force on the fluid.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho e + \frac{1}{2}\rho u^2 \mathrm{d}x = -\int_{\partial\Omega}(\rho e + \frac{1}{2}\rho u^2)u\cdot\nu \mathrm{d}S - \int_{\partial\Omega}pu\cdot\nu \mathrm{d}S - \int_{\Omega\cap\Gamma(t)}\sigma(\nabla\cdot\hat{n})\lambda \mathrm{d}S \tag{1.4.3}$$

If we assume that the flow is smooth inside some region of fluid of which Ω is a subset, so that in particular Ω does not intersect a surface of discontinuity $\Gamma(t)$, then we may use the divergence theorem applied to the terms on the right hand side, together with the fact that Ω is arbitrary to obtain the following system of differential equations which must be satisfied in regions of smooth flow.

$$\partial_t\rho + \nabla\cdot(\rho u) = 0 \tag{1.4.4}$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u + pI) = 0 \tag{1.4.5}$$

$$\partial_t(\rho e + \rho\frac{|u|^2}{2}) + \nabla\cdot(u(\rho e + \rho\frac{|u|^2}{2} + p)) = 0 \tag{1.4.6}$$

Here, we have used the notation $\nabla\cdot$ for divergence, which is defined to act row-wise on matrices. The matrix $u\otimes u$ has (i,j)-entry u_iu_j and I is the identity matrix, so the i-entry of $\nabla\cdot(\rho u\otimes u + pI)$ is $\sum_{j=1}^{3}\partial_{x_j}(\rho u_iu_j) + \partial_{x_i}p$.

1.4.2 Physical Derivation of the Jump Conditions

Now let us consider the case where Ω intersects $\Gamma(t)$, which divides our arbitrary region Ω into two halves, $\Omega_+(t)$ and $\Omega_-(t)$, with outward normals ν_+ and ν_-, where the unit normal of $\Gamma(t)$, $\hat{n}$, points into $\Omega_+(t)$. We let $\rho_\pm$, $u_\pm$, $p_\pm$, $e_\pm$ denote the state variables in $\Omega_\pm(t)$, which will assume are smooth inside $\Omega_\pm(t)$ and continuous up to the boundary.

From the conservation of mass equation (1.4.1) applied to $\Omega = \Omega_+(t)\cup\Gamma(t)\cup\Omega_-(t)$, we have the following.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_+(t)}\rho_+\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_-(t)}\rho_-\mathrm{d}x\\ &= -\int_{\partial\Omega_+(t)}\rho_+u_+\cdot\nu_+\mathrm{d}S - \int_{\partial\Omega_-(t)}\rho_-u_-\cdot\nu_-\mathrm{d}S - \int_{\Gamma(t)\cap\Omega}\rho_+u_+\cdot\hat{n}\mathrm{d}S + \int_{\Gamma(t)\cap\Omega}\rho_-u_-\cdot\hat{n}\mathrm{d}S\end{aligned}$$

Using the formula for differentiating moving regions, we obtain

$$\int_{\Omega_+(t)} \partial_t \rho_+ \mathrm{d}x + \int_{\Omega_-(t)} \partial_t \rho_- \mathrm{d}x - \int_{\Gamma(t)\cap\Omega} \rho_+ \lambda \mathrm{d}S + \int_{\Gamma(t)\cap\Omega} \rho_- \lambda \mathrm{d}S$$
$$= - \int_{\partial\Omega_+(t)} \rho_+ u_+ \cdot \nu_+ \mathrm{d}S - \int_{\partial\Omega_-(t)} \rho_- u_- \cdot \nu_- \mathrm{d}S - \int_{\Gamma(t)\cap\Omega} \rho_+ u_+ \cdot \hat{n} \mathrm{d}S + \int_{\Gamma(t)\cap\Omega} \rho_- u_- \cdot \hat{n} \mathrm{d}S$$

Using the divergence theorem in $\Omega_\pm(t)$ together with the above equations (1.4.4)-(1.4.6) for smooth flow, we are left with only the surface terms on Γ.

$$\int_{\Gamma(t)\cap\Omega} \rho_+(u_+ \cdot \hat{n} - \lambda) - \rho_-(u_- \cdot \hat{n} - \lambda) \mathrm{d}S = 0$$

Since Ω is arbitrary, we obtain the jump condition

$$[\rho(u \cdot \hat{n} - \lambda)] = 0$$

where $[a] = a_+ - a_-$ denotes the jump in the quantity a across $\Gamma(t)$. Physically this means that the mass flux $m_{\hat{n}} := \rho(u \cdot \hat{n} - \lambda)$ across $\Gamma(t)$ must be continuous.

From the conservation of momentum equation (1.4.2) applied to $\Omega = \Omega_+(t) \cup \Gamma(t) \cup \Omega_-(t)$, we have the following.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_+(t)} \rho_+ u_+ \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_-(t)} \rho_- u_- \mathrm{d}x$$
$$= - \int_{\partial\Omega_+} (\rho_+ u_+) u_+ \cdot \nu_+ + p_+ \nu_+ \mathrm{d}S - \int_{\partial\Omega_-} (\rho_- u_-) u_- \cdot \nu_- + p_- \nu_- \mathrm{d}S$$
$$- \int_{\Gamma(t)\cap\Omega} (\rho_+ u_+) u_+ \cdot \hat{n} + p_+ \hat{n} \mathrm{d}S + \int_{\Gamma(t)\cap\Omega} (\rho_- u_-) u_- \cdot \hat{n} + p_- \hat{n} \mathrm{d}S - \int_{\Gamma(t)\cap\Omega} \sigma(\nabla \cdot \hat{n}) \hat{n} \mathrm{d}S$$

Using the formula for differentiating moving regions, we obtain

$$\int_{\Omega_+(t)} \partial_t(\rho_+ u_+) \mathrm{d}x + \int_{\Omega_-(t)} \partial_t(\rho_- u_-) \mathrm{d}x - \int_{\Gamma(t)\cap\Omega} \rho_+ u_+ \lambda \mathrm{d}S + \int_{\Gamma(t)\cap\Omega} \rho_- u_- \lambda \mathrm{d}S$$
$$= - \int_{\partial\Omega_+(t)} (\rho_+ u_+) u_+ \cdot \nu_+ + p_+ \nu_+ \mathrm{d}S - \int_{\partial\Omega_-(t)} (\rho_- u_-) u_- \cdot \nu_- + p_- \nu_- \mathrm{d}S$$
$$- \int_{\Gamma(t)\cap\Omega} (\rho_+ u_+) u_+ \cdot \hat{n} + p_+ \hat{n} \mathrm{d}S + \int_{\Gamma(t)\cap\Omega} (\rho_- u_-) u_- \cdot \hat{n} + p_- \hat{n} \mathrm{d}S - \int_{\Gamma(t)\cap\Omega} \sigma(\nabla \cdot \hat{n}) \hat{n} \mathrm{d}S$$

Using the divergence theorem in $\Omega_\pm(t)$ together with the above equations (1.4.4)-(1.4.6) for smooth flow, we are left with only the surface terms on Γ.

$$\int_{\Gamma(t)\cap\Omega} \rho_+ u_+ (u_+ \cdot \hat{n} - \lambda) - \rho_- u_- (u_- \cdot \hat{n} - \lambda) + (p_+ - p_- + \sigma \nabla \cdot \hat{n}) \hat{n} \mathrm{d}S = 0$$

Since Ω is arbitrary, we obtain the jump condition

$$[\rho(u \cdot \hat{n} - \lambda) u + p \hat{n}] = -\sigma(\nabla \cdot \hat{n}) \hat{n}$$

Similarly, we obtain the following jump condition from the energy equation.

$$[\rho(u \cdot \hat{n} - \lambda)(e + \frac{1}{2} u^2) + p u \cdot \hat{n}] = -\sigma(\nabla \cdot \hat{n}) \lambda$$

Rewriting these jump conditions in terms of $m_{\hat{n}}$ we obtain

$$
\begin{aligned}
[m_{\hat{n}}] &= 0 \\
m_{\hat{n}}[u_{\hat{n}}] + [p] &= -\sigma \nabla \cdot \hat{n} \\
m_{\hat{n}}[u_{\hat{\tau}}] &= 0 \\
m_{\hat{n}}[(e + \frac{1}{2}u^2) + \frac{p}{\rho}] + \lambda[p] &= -\sigma(\nabla \cdot \hat{n})\lambda
\end{aligned}
$$

where $u_{\hat{n}} = u \cdot \hat{n}$ is the normal component of u and $u_{\hat{\tau}}$ is the tangential part of u.

For vortex sheets, where by definition $m_{\hat{n}} = 0$, the above conditions simplify to the following jump conditions.

$$
\begin{aligned}
(u_{\hat{n}})_{\pm} &= \lambda \\
[p] &= -\sigma \nabla \cdot \hat{n}
\end{aligned}
$$

where the first line is equivalent to $(m_{\hat{n}})_{\pm} = 0$.

1.4.3 Equation of State

Note that the unknowns, or state variables, ρ, u, p, e consist of six scalar quantities (in three space dimensions), whereas there are only five scalar equations in the above system (1.4.4)-(1.4.6), hence we do not expect to be able to solve the system uniquely given some initial data without relating the state variables through an equation, known as an equation of state.

We consider here a compressible fluid in which the density may vary, a model for which comes from considering what occurs at a microscopic level. A virtual surface in a fluid will experience a force on each side due to particles colliding with it, and this force per unit area is what we have called the pressure. The frequency and impact speed of collisions will increase with e, and the frequency of collisions will also increase with ρ. Hence we expect $p = p(\rho, e)$ to be a positive function of ρ and e which increases with e and ρ (although we will not make exactly this assumption). The above equations (1.4.4)-(1.4.6) with this equation of state are known as the compressible Euler equations.

1.4.4 Entropy and a Symmetric Form of the Equations

The Euler equations as we have written them are not symmetric, which makes it impossible to obtain high-order energy estimates by differentiating the equations and integrating by parts. Therefore, we seek to rewrite the equations in symmetric form. In the general framework systems of conservation laws, this can be achieved by multiplying the system by the Hessian of a convex (mathematical) entropy. However, it turns out in this case that the equations can be made symmetric in regions of smooth flow by a change of dependent variables involving the physical entropy.

The physical quantity, entropy per unit mass, s, can be defined as a function of ρ and e through the fundamental thermodynamic relation

$$
T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}V = \mathrm{d}e - \frac{p}{\rho^2}\mathrm{d}\rho
$$

where $V = \frac{1}{\rho}$ is the volume per unit mass and this equation also defines the state variable temperature $T = T(\rho, e)$, which must be strictly positive.

There are various ways of justifying this definition and explaining why it might be useful. Very briefly, we see that for a smooth reversible process the right hand side is the gain in internal energy of the system minus the work done on the system by its surroundings due to local volume expansion, which should equal all the energy gained by the system via the microscopic motion of its molecules, which we call heat transfer. We introduce $\frac{1}{T}$ as an integrating factor so that we can write this heat transfer as T times a differential, which we call $\mathrm{d}s$.

If we wish to explicitly calculate $s(\rho, e)$ given $p(\rho, e)$, we combine the equations

$$\left.\frac{\partial s}{\partial \rho}\right|_e = -\frac{1}{T}\frac{p}{\rho^2}$$
$$\left.\frac{\partial s}{\partial e}\right|_\rho = \frac{1}{T}$$

to obtain

$$\left.\frac{\partial s}{\partial \rho}\right|_e + \frac{p}{\rho^2}\left.\frac{\partial s}{\partial e}\right|_\rho = 0 \tag{1.4.7}$$

where the requirement $T > 0$ becomes $\left.\frac{\partial s}{\partial e}\right|_\rho > 0$. Of course this gives us some degree of freedom in defining s as a function of ρ and e. For example, we may think of the above equation as a scalar linear transport equation for s which has smooth coefficients for $\rho > 0$, where ρ takes the place of the usual time variable. Thus if we fix some $\rho^0 > 0$ and a smooth function $s^0 = s^0(e)$ with $\frac{\mathrm{d}s^0}{\mathrm{d}e} > 0$, then by standard theory there is a unique solution $s(\rho, e)$ with $s(\rho^0, e) = s^0(e)$, which also satisfies $\left.\frac{\partial s}{\partial e}\right|_\rho > 0$. Note that we have the freedom to choose the function s^0 with $\frac{\mathrm{d}s^0}{\mathrm{d}e} > 0$. However, any smooth function $s(\rho, e)$ that satisfies the above equation with $\left.\frac{\partial s}{\partial e}\right|_\rho > 0$ for $\rho > 0$ will be fine for our purposes.

Now we see the justification for the introduction of entropy. Since there is no heat conduction term in the Euler equations, we expect the change in internal energy e moving with the flow to be due only to the work done due to local volume change, assuming smooth flow (reversibility), hence, via the fundamental thermodynamic relation, we expect entropy s to be constant moving with the flow. The transport of the bulk energy is already encoded in the momentum equation. Indeed, using the first two equations to simplify the energy equation we obtain

$$(\partial_t + u \cdot \nabla)e = -\frac{p}{\rho}\nabla \cdot u$$

Combining this with the first equation we obtain

$$(\partial_t + u \cdot \nabla)e = \frac{p}{\rho^2}(\partial_t + u \cdot \nabla)\rho$$

Now we calculate the change in entropy moving with the flow.

$$\begin{aligned}(\partial_t + u \cdot \nabla)s &= \left.\frac{\partial s}{\partial \rho}\right|_e (\partial_t + u \cdot \nabla)\rho + \left.\frac{\partial s}{\partial e}\right|_\rho (\partial_t + u \cdot \nabla)e \\ &= (\partial_t + u \cdot \nabla)\rho(\left.\frac{\partial s}{\partial \rho}\right|_e + \frac{p}{\rho^2}\left.\frac{\partial s}{\partial e}\right|_\rho) \\ &= 0\end{aligned}$$

where we have used the above equations. Using the density equation, we may also write this in divergence form as

$$\partial_t(\rho s) + \nabla \cdot (\rho u s) = 0 \tag{1.4.8}$$

Hence, assuming s is a smooth solution of (1.4.7) with $\frac{\partial s}{\partial e}\big|_\rho > 0$, we may replace the energy equation (1.4.6) in regions of smooth flow by the above equation for s, and we may write $p = p(\rho, s)$. Thus equations (1.4.4)-(1.4.6) become

$$(\partial_t + u \cdot \nabla)\rho + \rho \nabla \cdot u = 0 \tag{1.4.9}$$

$$\rho(\partial_t + u \cdot \nabla)u + \nabla p = 0 \tag{1.4.10}$$

$$(\partial_t + u \cdot \nabla)s = 0 \tag{1.4.11}$$

for the dependent variables ρ, u and s.

This is still not symmetric, but, assuming $\frac{\partial p}{\partial \rho}\Big|_s > 0$, we may write $\rho = \rho(p, s)$ and replace the dependent variable ρ with p. The system then becomes

$$\begin{aligned} \frac{1}{\rho c^2}(\partial_t + u \cdot \nabla)p + \nabla \cdot u &= 0 \\ \rho(\partial_t + u \cdot \nabla)u + \nabla p &= 0 \\ (\partial_t + u \cdot \nabla)s &= 0 \end{aligned}$$

where $c = \sqrt{\frac{\partial p}{\partial \rho}\Big|_s} > 0$ is called the sound speed. This system is symmetric, and makes sense provided $\rho > 0$ and $c > 0$. We note that these two conditions are physically reasonable. The condition $\rho > 0$ requires that the flow is away from vacuum, which we will assume throughout. If we examine the condition $\frac{\partial p}{\partial \rho}\Big|_s > 0$ in terms of the original pressure law $p = p(\rho, e)$, we see that

$$\begin{aligned} \frac{\partial p}{\partial \rho}\bigg|_s &= \frac{\partial p}{\partial \rho}\bigg|_e + \frac{\partial p}{\partial e}\bigg|_\rho \frac{\partial e}{\partial \rho}\bigg|_s \\ &= \frac{\partial p}{\partial \rho}\bigg|_e + \frac{p}{\rho^2}\frac{\partial p}{\partial e}\bigg|_\rho \end{aligned} \tag{1.4.12}$$

so this condition is fulfilled for example under the physically reasonable assumption that p strictly increases with density and internal energy per unit mass, assuming $p > 0$ and $\rho > 0$.

Finally, we make a remark on entropy and non-smooth solutions, where for the purposes of this remark we will neglect surface tension. The second law of thermodynamics requires that physical entropy should either remain constant or increase. We have seen that for smooth solutions of the Euler equations it remains constant moving with the fluid, but we will be considering solutions with a surface of discontinuity. We can write (1.4.8) in integral form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho s \mathrm{d}x = -\int_{\partial\Omega} \rho u s \cdot \nu \mathrm{d}S$$

where Ω is a fixed domain. This should hold for smooth flows. For non-smooth flows we should have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho s \mathrm{d}x \geq -\int_{\partial\Omega} \rho u s \cdot \nu \mathrm{d}S$$

Applying the same arguments as we did for the mass, momentum and energy equations, we obtain that across a surface of discontinuity $\Gamma(t)$, we should have

$$m_{\hat{n}}[s] \geq 0$$

In the case of vortex sheets, since $m_{\hat{n}} = 0$, this is automatically satisfied as an equality and the total entropy is conserved.

1.5 The Final Form of the Equations for Vortex Sheets

To summarise, we have shown that two compressible inviscid fluid phases which flow past each other without mixing, and experience a surface tension force with coefficient σ along their common interface, satisfy the following equations on either side of the interface

$$\frac{1}{\rho c^2}(\partial_t + u \cdot \nabla)p + \nabla \cdot u = 0 \tag{1.5.1}$$
$$\rho(\partial_t + u \cdot \nabla)u + \nabla p = 0 \tag{1.5.2}$$
$$(\partial_t + u \cdot \nabla)s = 0 \tag{1.5.3}$$

for the dependent variables pressure, p, velocity, u and entropy, s, where the density, $\rho = \rho(p, s) > 0$, satisfies $\left(\frac{\partial \rho}{\partial p}\Big|_s \right)^{-1} =: c^2 > 0$. We also have the following conditions which are satisfied on the interface, which has unit normal $\hat{n}$ and normal velocity λ in that direction.

$$(u_{\hat{n}})_+ = (u_{\hat{n}})_- = \lambda$$
$$[p] = -\sigma \nabla \cdot \hat{n}$$

where $[p]$ denotes the pressure on the side of the interface into which $\hat{n}$ points minus the pressure on the other side and $(u_{\hat{n}})_\pm$ denotes the normal velocity of the fluid on the $\pm$-side of the interface.

From now on, we will assume the coefficient of surface tension $\sigma > 0$ is fixed.

We also fix two functions ρ_+ and ρ_-, which we call the equations of state for the density in the $+$ and $-$ regions, such that

$$\rho_\pm = \rho_\pm(p, s) : \mathbb{R}^2 \to \mathbb{R}$$

are smooth functions of p and s in the region where $\rho_\pm > 0$. We also assume

$$\frac{\partial \rho_\pm}{\partial p}\Big|_s > 0$$

for $\rho_\pm > 0$. We define the equation of state for the sound speed $c_\pm$ as

$$c_\pm = \left(\frac{\partial \rho_\pm}{\partial p}\Big|_s \right)^{-\frac{1}{2}}$$

so that $c_\pm > 0$ for $\rho_\pm > 0$.

To simplify the analysis, we will make the assumption that the interface $\Gamma(t)$ can be written as the graph of a function $f : (0, T) \times \mathbb{R}^2 \to \mathbb{R}$. This assumption is fine if we start with an interface which can be written as a graph initially and we are only interested in short-time existence of solutions with the same structure. In this case, the normal $\hat{n}$ pointing into the region above the graph is given by

$$\hat{n} = \frac{n}{|n|}$$

where we have defined

$$n = (-\partial_{x_1} f, -\partial_{x_2} f, 1)$$

and the normal velocity λ is given by

$$\lambda = \frac{\partial_t f}{|n|}$$

The jump conditions then become

$$u_{\pm}|_{\Gamma(t)} \cdot n = \partial_t f$$
$$[p] = -\sigma \nabla \cdot \hat{n}$$

This is the form of the equations which we will start with in the analysis which follows.

1.5.1 Recovering Solutions to the Original Conservation Laws

Note that, as stated in the previous section, if we are given $p = p(\rho, s)$, then, provided that $\left.\frac{\partial p}{\partial \rho}\right|_s > 0$ for $\rho > 0$, we may invert the function $p = p(\rho, s)$ to obtain $\rho = \rho(p, s)$. The calculations in the previous section show that, supposing we can obtain a smooth solution $(p, u, s)(t, x)$ of the system (1.5.1)-(1.5.3), we may set $\rho(t, x) = \rho(p(t, x), s(t, x))$ to obtain a smooth solution $(\rho, u, s)(t, x)$ of the system (1.4.9)-(1.4.11). So solving the system (1.5.1)-(1.5.3) for smooth $(p, u, s)(t, x)$ given $\rho = \rho(p, s)$ with $\left.\frac{\partial \rho}{\partial p}\right|_s > 0$ for $\rho > 0$ is equivalent to solving the system (1.4.9)-(1.4.11) for smooth $(\rho, u, s)(t, x)$ given $p = p(\rho, s)$ with $\left.\frac{\partial p}{\partial \rho}\right|_s > 0$ for $\rho > 0$. Also note that any energy estimates for (p, u, s) can be converted to energy estimates for (ρ, u, s) using the chain rule.

Similarly, the calculations in the previous section show that if we are given $p = p(\rho, e)$, then we can solve the equation (1.4.7) for $s(\rho, e)$, in addition ensuring that $\left.\frac{\partial s}{\partial e}\right|_\rho > 0$ so we can write $e = e(\rho, s)$. Reversing the calculations in the previous section by setting $s = s(\rho, e)$ in (1.4.11) and using the density equation (1.4.9) and the velocity equation (1.4.10) multiplied by u, we see that, supposing we can obtain a smooth solution $(\rho, u, s)(t, x)$ of the system (1.4.9)-(1.4.11), we may set $e(t, x) = e(\rho(t, x), s(t, x))$ to obtain a smooth solution $(\rho, u, e)(t, x)$ of the system (1.4.4)-(1.4.6). Combining this with the previous deduction, provided that $\left.\frac{\partial p}{\partial \rho}\right|_s > 0$ for $\rho > 0$, which can be written in terms of ρ and e as in (1.4.12), solving the system (1.5.1)-(1.5.3) for smooth $(p, u, s)(t, x)$ given $\rho = \rho(p, s)$ with $\left.\frac{\partial \rho}{\partial p}\right|_s > 0$ for $\rho > 0$ is equivalent to solving the system (1.4.4)-(1.4.6) for smooth $(\rho, u, e)(t, x)$ given $p = p(\rho, e)$ which satisfies (1.4.12) for $\rho > 0$. Also note that any energy estimates for (p, u, s) can be converted to energy estimates for (ρ, u, e) using the chain rule.

Thus for smooth solutions with $\rho > 0$ and $c > 0$ there is no loss of generality in considering the system (1.5.1)-(1.5.3) instead of the system (1.4.4)-(1.4.6).

1.6 Notation

In this section we introduce the notation used throughout what follows.

Notation. *Let Ω be an open set in $\mathbb{R}^d$ and let $k \geq 0$. Then $C_b^k(\Omega)$ will denote the space of k-times continuously differentiable bounded functions on Ω with bounded derivatives up to order k. Note we write $C_b(\Omega) := C_b^0(\Omega)$.*

We write $C_c^k(\Omega)$ for the space of k-times continuously differentiable functions on Ω with compact support in Ω and we write $C_c^\infty(\Omega)$ for the space of infinitely differentiable functions on Ω with compact support in Ω.

Notation. *We shall often need to refer to a quantity, say a, defined in two regions, denoted by $+$ and $-$. We will write a_+ for the quantity represented by a in the $+$ region and a_- for the quantity represented by a in the $-$ region. Sometimes for brevity we will use $a_\pm$ in formulae which hold for both a_+ and a_- and sometimes for ease of reading we will omit the subscripts $_\pm$ altogether in such formulae, but the meaning should be clear from the context.*

Notation. *For a vector $w \in \mathbb{R}^3$, we will write w' for the first 2 components of w, so that $w = (w', w_3)$. For example, $x = (x', x_3)$, so that x' denotes horizontal position, $u = (u', u_3)$, so that u' denotes horizontal velocity, and $\nabla = (\nabla', \partial_{x_3})$, so that ∇' denotes horizontal gradient.*

We will also write $\Delta' a$ for the two-dimensional Laplacian $\partial_{x_1}^2 a + \partial_{x_2}^2 a$ for any function $a(x_1, x_2, x_3)$.

We will refer to the x_1 and x_2 directions as the horizontal or tangential directions and the x_3 direction as the vertical or normal direction. It will be important to distinguish these directions throughout what follows.

To avoid confusion, we will never use a prime $'$ to denote a derivative. However, we will sometimes use T' to denote a time interval when we have already used T to represent a different time interval and this has nothing to do with the notational convention introduced above.

Notation. *Suppose $a \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^d)$, where $d = 2$ or $d = 3$ in the main body of this work, has weak derivatives up to order m. Then we write*

$$\partial^\alpha a = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} a$$

where $\alpha \in \mathbb{N}_0^{1+d}$ is a multi-index with $|\alpha| \leq m$.

It is important to note that our definition of ∂^α includes time derivatives.

If $a \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a function of space only or if $a \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^d)$ as above, and has weak spatial derivatives up to order m, then we write

$$\nabla^\alpha a = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} a$$

where $\alpha \in \mathbb{N}_0^d$ is a multi-index with $|\alpha| \leq m$.

We also write

$$\partial a = (\partial_t a, \partial_{x_1} a, \dots, \partial_{x_k} a)$$

and

$$\nabla a = (\partial_{x_1} a, \ldots, \partial_{x_k} a)$$

Note that for convenience, we will sometimes write $\partial^\alpha a$ where $\alpha \in \mathbb{N}_0^{1+3}$ but $a \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^2)$. Clearly in this case derivatives with respect to x_3 are zero. If there is a possibility of confusion, then we will either explicitly write $\alpha_3 = 0$ or use the notation α' to denote a multi-index $\alpha \in \mathbb{N}_0^{1+2}$, so that $\partial^{\alpha'}$ involves no derivatives with respect to x_3. We use the same convention for $\nabla^\alpha a$. Explicitly,

$$\begin{aligned}
\partial^{\alpha'} a &= \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} a \\
\nabla^{\alpha'} a &= \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} a \\
\partial' a &= (\partial_t a, \partial_{x_1} a, \partial_{x_2} a) \\
\nabla' a &= (\partial_{x_1} a, \partial_{x_2} a)
\end{aligned}$$

Sometimes we will also write $\partial^l a$ for the collection of all the time and space derivatives of a of order l, and the product of this with another such quantity should be interpreted as sums of products of elements of the two quantities, and may be scalar or vector-valued as appropriate.

Notation. *If we write $w \in X$ where X usually denotes a space of scalar-valued functions, but $w = (w_1, \ldots, w_k)$ is a vector-valued function, then we mean that $w_i \in X$ for each i. Similarly, if we write $\partial^\alpha w$ then we mean $(\partial^\alpha w_1, \ldots, \partial^\alpha w_k)$.*

Notation. *Let $\Omega_\pm$ be two C^1 domains with common C^1 boundary Γ. If $a_\pm$ are functions defined on $\Omega_\pm$ which are uniformly continuous, so can be defined on Γ by continuity, write*

$$[a] = a_+|_\Gamma - a_-|_\Gamma$$

If $a_\pm \in H^1(\Omega_\pm)$ then we also use

$$a_\pm|_\Gamma$$

to denote the trace of $a_\pm$ on Γ and we recall that

$$||a_\pm|_\Gamma||_{H^{0.5}(\Gamma)} \leq C\, ||a_\pm||_{H^1(\Omega_\pm)}$$

See eg Adams and Fournier [1] for a proof.

It will be useful to have the following definition for smoothing later on.

Notation. *For $d = 2$ or 3, let η be the standard mollifier on $\mathbb{R}^d$. In particular $\eta \in C_c^\infty(\mathbb{R}^d)$ is supported in $\{|x| \leq 1\}$ with $0 \leq \eta \leq C$ for some constant C and $\int_{\mathbb{R}^d} \eta(x)\mathrm{d}x = 1$. For $\epsilon \in (0,1)$, set $\eta_\epsilon(x) = \epsilon^{-d}\eta(\frac{x}{\epsilon})$ so that $\eta_\epsilon \in C_c^\infty(\mathbb{R}^d)$ is supported in $\{|x| \leq \epsilon\}$ with $0 \leq \eta_\epsilon \leq C\epsilon^{-d}$ and $\int_{\mathbb{R}^d} \eta_\epsilon(x)\mathrm{d}x = 1$.*

It will also be useful to have the following definition of Sobolev extension.

Notation. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. We let $\mathrm{Ext}_\Omega : L^1_{\mathrm{loc}}(\Omega) \to L^1_{\mathrm{loc}}(\mathbb{R}^3)$ denote a Sobolev extension operator such that*

$$||\mathrm{Ext}_\Omega\, a||_{W^{k,p}(\mathbb{R}^3)} \leq C_k\, ||a||_{W^{k,p}(\Omega)}$$

for all $a \in L^1_{\mathrm{loc}}(\Omega)$, all $p \in [1,\infty]$ and all $k \geq 0$, where C_k denotes a constant depending on k. For example we could take Stein's extension operator. In fact we really only need the extension operator to satisfy the above inequality for k up to a certain order - for example, $k \leq 30$ would do.

2. The Main Theorem

2.1 Definition of Vortex Sheets with Surface Tension

Definition 2.1.1. Fix $T \in (0,\infty)$. Let $f \in C_b^1((0,T') \times \mathbb{R}^2)$ with $\nabla' f \in C_b^1((0,T') \times \mathbb{R}^2)$ for all $T' \in (0,T)$, which we call the (graph of) the front or interface.

Define the moving domains $\Omega_\pm(t)$ as

$$\Omega_+(t) = \{x \in \mathbb{R}^3 : x_3 > f(t,x')\}$$
$$\Omega_-(t) = \{x \in \mathbb{R}^3 : x_3 < f(t,x')\}$$

We also write

$$\Omega_+^T = \{(t,x) \in (0,T) \times \mathbb{R}^3 : x_3 > f(t,x')\}$$
$$\Omega_-^T = \{(t,x) \in (0,T) \times \mathbb{R}^3 : x_3 < f(t,x')\}$$

Define the moving boundary (or front or interface) $\Gamma(t)$ as

$$\Gamma(t) = \{x \in \mathbb{R}^3 : x_3 = f(t,x')\}$$

We also write

$$\Gamma^T = \{(t,x) \in (0,T) \times \mathbb{R}^3 : x_3 = f(t,x')\}$$

Define

$$n := (-\nabla' f, 1)$$
$$\hat{n} := \frac{n}{|n|}$$

as a spatial normal and the spatial unit normal to the interface pointing into $\Omega_+(t)$. Note that since these are independent of x_3, we can think of them as being defined on $(0,T) \times \mathbb{R}^2$ or on Γ^T via $n(t,x) = (-\nabla' f(t,x'),1)$ for $(t,x) \in \Gamma^T$.

Let $p_\pm, (u_i)_\pm \ (i = 1,2,3), s_\pm \in C_b^1(\Omega_\pm^{T'})$ for all $T' \in (0,T)$. Write $u_\pm = ((u_1)_\pm, (u_2)_\pm, (u_3)_\pm)$, $U_\pm = (p_\pm, u_\pm, s_\pm)$. Note that $U_\pm|_{t=0}$ and $f|_{t=0}$ are defined by uniform continuity.

Define $\rho_\pm(t,x) := \rho_\pm(p_\pm(t,x), s_\pm(t,x))$ and $c_\pm(t,x) := c_\pm(p_\pm(t,x), s_\pm(t,x))$, where $\rho_\pm(p,s)$ and $c_\pm(p,s)$ are the equations of state defined in 1.5.

We will say that (U_+, U_-, f) is a vortex sheet solution of the Euler equations with surface tension on the time interval $(0,T)$ with initial data $(U_+^0, U_-^0, f^0) := (U_+|_{t=0}, U_-|_{t=0}, f|_{t=0})$ provided that $\rho_\pm(t,x) > 0$ in $\Omega_\pm^T$, and $p_\pm, u_\pm, s_\pm, f$ satisfy the following system of PDEs.

$$\frac{1}{\rho c^2}(\partial_t + u \cdot \nabla)p + \nabla \cdot u = 0 \tag{2.1.1}$$

$$\rho(\partial_t + u \cdot \nabla)u + \nabla p = 0 \tag{2.1.2}$$

$$(\partial_t + u \cdot \nabla)s = 0 \tag{2.1.3}$$

in $\Omega_\pm^T$,

$$\partial_t f = u_\pm \cdot n \tag{2.1.4}$$

$$[p] = -\sigma \nabla' \cdot \hat{n} \tag{2.1.5}$$

on Γ^T.

2.2 The Initial Data and the Compatibility Conditions

Definition 2.2.1. We say that $(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ are initial data for the system (2.1.1)-(2.1.5) if the following holds. We require that $f^0 \in C_b^1(\mathbb{R}^2)$. We define the initial domains $\Omega_\pm^0$ as

$$\Omega_+^0 = \{x \in \mathbb{R}^3 : x_3 > f^0(x')\}$$
$$\Omega_-^0 = \{x \in \mathbb{R}^3 : x_3 < f^0(x')\}$$

and the initial boundary (or front or interface) Γ^0 as

$$\Gamma^0 = \{x \in \mathbb{R}^3 : x_3 = f(t, x')\}$$

and the initial normal and unit normal as

$$n^0 := (-\nabla' f^0, 1)$$
$$\hat{n}^0 := \frac{n^0}{|n^0|}$$

We also require that $p_\pm^0, u_\pm^0, s_\pm^0 \in C_b(\Omega_\pm^0)$ are uniformly continuous functions (where $u_\pm^0$ are vectors representing the initial velocity), so that we may define them on the interface Γ^0 by uniform continuity. Finally, we require

$$\rho_\pm^0(x) := \rho_\pm(p_\pm^0(x), s_\pm^0(x)) > 0$$

for all $x \in \Omega_\pm^0$.

Definition 2.2.2. For integer $j \geq 0$, we define the following differential operators ∂_0^j associated with the system (2.1.1)-(2.1.5) which act on differentiable expressions involving the initial data and its derivatives (including compositions). We call the operator ∂_0^j the initial time derivative operator of order j associated to the system (2.1.1)-(2.1.5).

To start with, we define $\partial_0^j p_\pm^0, \partial_0^j u_\pm^0, \partial_0^j s_\pm^0, \partial_0^j f^0$ as follows. We set

$$\partial_0^0 = \mathrm{id}$$

then for $j \geq 0$ we inductively define

$$\partial_0^{j+1} f^0 := \partial_t^j (u_+(t)\big|_{\Gamma(t)} \cdot n(t))\big|_{t=0} := \partial_t^j (u_+(t,x',f(t,x')) \cdot n(t))\big|_{t=0} \tag{2.2.1}$$

$$\partial_0^{j+1} p_\pm^0 := \partial_t^j (-(u_\pm(t)\cdot\nabla)p_\pm(t) - \rho_\pm(t)c_\pm^2(t)\nabla\cdot u_\pm(t))\big|_{t=0} \tag{2.2.2}$$

$$\partial_0^{j+1} u_\pm^0 := \partial_t^j (-(u_\pm(t)\cdot\nabla)u_\pm(t) - \frac{1}{\rho_\pm(t)}\nabla p_\pm(t))\big|_{t=0} \tag{2.2.3}$$

$$\partial_0^{j+1} s_\pm^0 := \partial_t^j (-(u_\pm(t)\cdot\nabla)s_\pm(t))\big|_{t=0} \tag{2.2.4}$$

where the right hand side means we think of $u_\pm(t)$ etc as functions of t and formally differentiate with respect to t j times according to the chain and Leibniz rules, then replace $\partial_t^l u_\pm(t)$ for $0 \leq l \leq j$ by $\partial_0^l u_\pm^0$ etc. Note that it is necessary for us to pick either $+$ or $-$ in the above definition for $\partial_0^{j+1} f^0$ since we have not yet imposed any compatibility conditions on the initial data. Note we require sufficient regularity on the initial data for the above to make sense.

We can now generalise the operators ∂_0^j to general differentiable expressions involving the initial data and its derivatives (including compositions) in the obvious way. Suppose $H(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ is such an expression. For $j \geq 0$, we set

$$\partial_0^j (H(p_\pm^0, u_\pm^0, s_\pm^0, f^0)) := \partial_t^j (H(p_\pm(t), u_\pm(t), s_\pm(t), f(t)))\big|_{t=0}$$

where, as above, the right hand side means we think of $u_\pm(t)$ etc as functions of t and formally differentiate with respect to t j times according to the chain and Leibniz rules, then replace $\partial_t^l u_\pm(t)$ for $0 \leq l \leq j$ by $\partial_0^l u_\pm^0$ etc.

Definition 2.2.3. Let $(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ be initial data for the system (2.1.1)-(2.1.5). We say that the initial data $(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ satisfy the compatibility conditions for the system (2.1.1)-(2.1.5) up to order k, for integer $k \geq 0$, if the following holds for all $x' \in \mathbb{R}^2$.

$$\partial_0^j [u^0\big|_{\Gamma^0}(x') \cdot n^0(x')] = 0 \tag{2.2.5}$$

for $0 \leq j \leq k$ and

$$[\partial_0^j (p^0\big|_{\Gamma^0}(x'))] = -\sigma \partial_0^j (\nabla' \cdot \hat{n}^0)(x') \tag{2.2.6}$$

for $0 \leq j \leq k$, where

$$u^0\big|_{\Gamma^0}(x') := u^0(x', f^0(x'))$$
$$p^0\big|_{\Gamma^0}(x') := p^0(x', f^0(x'))$$

2.3 The Energy Functional

Definition 2.3.1. Let $f \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^2)$, $U_\pm \in L^1_{\mathrm{loc}}(\Omega_\pm^T)$. We define the energy E for the vortex sheet equations, associated to (U_+, U_-, f), as follows.

Define the energy $E : (0,T] \to [0,\infty]$ by

$$E(t) = \sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} ||U_\pm||^2_{L^\infty(\Omega_\pm(\tau))} + \sum_\pm \sum_{1\leq|\alpha|\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha U_\pm||^2_{L^2(\Omega_\pm(\tau))}$$
$$+ \operatorname*{ess\,sup}_{\tau\in(0,t)} ||f||^2_{L^\infty(\mathbb{R}^2)} + \sum_{1\leq|\alpha|\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha f||^2_{H^1(\mathbb{R}^2)} + \sum_{j=0}^{2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\nabla' \partial_t^j f\right|\right|^2_{H^{3.5-j}(\mathbb{R}^2)}$$

Note we adopt the convention that if the solution does not have sufficiently many weak derivatives in $\Omega_\pm^T$ to define the energy then it is infinite, and if we write $E(T') < \infty$ then this implies that all the weak derivatives in $\Omega_\pm^{T'}$ present in the energy exist. Note also that if $E(T') < \infty$ then E is uniformly continuous on $(0, T']$.

Similarly, let (U_+^0, U_-^0, f^0) be initial data for the system (2.1.1)-(2.1.5). We define the energy $E^0 \in [0, \infty]$ of the initial data as

$$E^0 = \sum_\pm \left|\left|U_\pm^0\right|\right|^2_{L^\infty(\Omega_\pm^0)} + \sum_\pm \sum_{1\le|\beta|+j\le 3} \left|\left|\nabla^\beta \partial_0^j U_\pm^0\right|\right|^2_{L^2(\Omega_\pm^0)}$$

$$+ \left|\left|f^0\right|\right|^2_{L^\infty(\mathbb{R}^2)} + \sum_{1\le|\beta|+j\le 3} \left|\left|\nabla^\beta \partial_0^j f^0\right|\right|^2_{H^1(\mathbb{R}^2)} + \sum_{j=0}^{2} \left|\left|\nabla' \partial_0^j f^0\right|\right|^2_{H^{3.5-j}(\mathbb{R}^2)}$$

Note that we may consider the class of initial data such that $E^0 < \infty$, which has the obvious meaning that $U^0 \in L^\infty(\Omega_\pm^0)$, $f^0 \in L^\infty(\Gamma^0)$, $\nabla U^0 \in H^2(\Omega_\pm^0)$, $\nabla' f^0 \in H^{3.5}(\Gamma^0)$, which allows us to define $\partial_0^j(U_+^0, U_-^0, f^0)$ for $0 \le j \le 3$ using 2.2.2, and we then require $\partial_0^j f^0 \in H^{4-j}(\Gamma^0)$ for $1 \le j \le 3$ and $\partial_0^j f^0 \in H^{4.5-j}(\Gamma^0)$ for $1 \le j \le 2$. (Note that the required regularity of $\partial_0^j U_\pm^0$ for $1 \le j \le 3$ automatically follows from the definition 2.2.2 and the spatial regularity).

2.4 Statement of the Main Theorem

Theorem 2.4.1. *Let (U_+^0, U_-^0, f^0) be initial data as described in 2.2.1 with energy $E^0 < \infty$. Assume that this initial data satisfies the compatibility conditions (2.2.5)-(2.2.6) up to order 2, and note that these conditions make sense up to order 2 since we have $E^0 < \infty$. Assume also that the initial density satisfies*

$$\inf_{x\in\Omega_\pm^0} \rho_\pm^0(x) =: \delta^0 > 0 \tag{2.4.1}$$

Then there exists a time $T^0 > 0$ and a vortex sheet solution (U_+, U_-, f) of the Euler equations with surface tension on the time interval $(0, T^0)$, as in definition 2.1.1. Moreover,

$$E(T^0) \le C^0$$

$$\rho_\pm \ge \frac{\delta^0}{2} \text{ in } \Omega_\pm^{T^0}$$

The time $T^0 > 0$ is bounded below as follows.

$$T^0 \ge T(E^0, \frac{1}{\delta^0}) > 0$$

where $T(\cdot)$ is a smooth decreasing function. Similarly, the constant $C^0 > 0$ is bounded above as follows.

$$C^0 \le C(E^0, \frac{1}{\delta^0}) < \infty$$

where $C(\cdot)$ is a smooth increasing function.

In addition, the solution (U_+, U_-, f) is unique on the time interval $(0, T^0)$ in the sense that it is the only solution of the equations given in 2.1.1 on the time interval $(0, T^0)$ with initial data (U_+^0, U_-^0, f^0) satisfying the following properties

$$E(T^0) < \infty$$

$$\inf_{(t,x)\in\Omega_\pm^{T^0}} \rho_\pm > 0$$

Proof. See chapters 3-15. □

Remark 2.4.2. In fact a statement of stability also holds in addition to uniqueness, but this is more easily stated after fixing the domains, so we leave this statement to 4.6.1. Note that the statement in 4.6.1 seems to require an extra condition, namely that the Jacobian J^{ψ} of the change of coordinates used to fix the domains is bounded below, but given two solutions (U_+^1, U_-^1, f^1) and (U_+^2, U_-^2, f^2) on a time interval $(0, T^0)$ satisfying the energy estimate above, we have $\operatorname{ess\,sup}_{t\in(0,T^0)} ||f^i||^2_{L^\infty(\mathbb{R}^2)} \leq E^i(T^0) < \infty$ for $i = 1, 2$, from which it follows that we may choose our change of coordinates so that Jacobians are indeed bounded below. (Specifically, we set $a = \max_{i=1,2} \operatorname{ess\,sup}_{t\in(0,T^0)} ||f^i||_{L^\infty(\mathbb{R}^2)}$ in the construction of the lift in 4.2.2).

Remark 2.4.3. Note the requirement that the initial data satisfies the compatibility conditions (2.2.5)-(2.2.6) up to order 2 is necessary in order to obtain a solution with $E(t) < \infty$ for some $t > 0$. Indeed, given a solution with $E(t) < \infty$, there is enough regularity to define the trace at the time $t = 0$ of the quantities

$$\begin{aligned}
&\partial_t^j [u \cdot n] \\
&\partial_t^j ([p] + \sigma \nabla' \cdot \hat{n})
\end{aligned}$$

for $0 \leq j \leq 2$. But of course for a solution these are zero by (2.1.4)-(2.1.5), and we may also write these quantities at time $t = 0$ in terms of the initial time derivatives, hence we obtain

$$\begin{aligned}
\partial_0^j [u^0 \cdot n^0] &= 0 \\
\partial_0^j ([p^0] + \sigma \nabla' \cdot \hat{n}^0) &= 0
\end{aligned}$$

for $0 \leq j \leq 2$.

3. Summary of the Proof of the Main Theorem

Here we summarise the main steps in the proof of the main theorem 2.4.1.

3.1 Fixing the Domains

The graph of the interface f is part of the solution of the equations (2.1.1)-(2.1.5), thus the domains $\Omega_\pm^T$ and Γ^T in which the equations (2.1.1)-(2.1.5) should hold are unknown before we have solved these equations. To apply any sort of standard theory, we must know the domains on which the equations are posed in advance. Therefore we must introduce a change of coordinates $\theta : (0,T) \times \Omega_\pm \to \Omega_\pm^T$, $(t,x) \mapsto \theta(t,x)$ depending on the solution to obtain a new system of equations for the unknowns $\tilde{p} = p \circ \theta$ etc on fixed known domains $(0,T) \times \Omega_\pm$ with $\partial\Omega_+ = \partial\Omega_- =: \Gamma$ and $\theta((0,T) \times \Gamma) = \Gamma^T$. In order that the pressure interface condition (2.1.5) should transform to a reasonable condition in the fixed domains, we require that θ should be continuous across Γ. We then intend to solve the new system of equations, and show that solutions of this new system of equations on the fixed domains correspond in a one-to-one manner to solutions of the original system (2.1.1)-(2.1.5) on the moving domains.

There are several obvious choices for the change of coordinates $\theta(t,x)$. One is Lagrangian coordinates, in which (the spatial component of) θ is defined as the solution to $\partial_t \theta = u \circ \theta$ with $\theta(0,x) = x$, so that $\Gamma = \Gamma^0$. However, since the velocity u is discontinuous across Γ^T, we cannot apply a global Lagrangian coordinate transformation which is continuous across Γ, and hence we would have to pick a region Ω_+ or Ω_- in which to apply Lagrangian coordinates, and then extend the change of coordinates to $\mathbb{R}^3$ by Sobolev extension. This approach is used by Shkoller et al in [6], but has the disadvantage of producing a different set of equations in Ω_+ and Ω_-.

Other obvious approaches involve constructing a lift ψ of f defined on $(0,T)\times\Omega_\pm$ such that $\psi|_\Gamma = f$, where $\Omega_+ = \mathbb{R}^3_{>0}, \Omega_- = \mathbb{R}^3_{<0}$ and $\Gamma = \{x \in \mathbb{R}^3 : x_3 = 0\} = \mathbb{R}^2$. We would then set $\theta(t,x) = (t, x', x_3 + \psi(t,x))$. One way of constructing ψ is to require that $\psi_\pm$ satisfy the equations $\partial_t \psi_\pm = u_{3\pm} - u'_\pm(t, x_3 + \psi(t,x)) \cdot \nabla' \psi_\pm$ in $(0,T) \times \Omega_\pm$, which is an extension of the equations (2.1.4) to $(0,T) \times \Omega_\pm$, as in [9] and [4]. However, this has the major disadvantage that ψ does not inherit the high spatial regularity of f gained due to the presence of surface tension. The simplest method of defining ψ is to specify ψ directly in terms of f. This can be done in such a way that ψ gains half a degree of regularity from f - see A.2.3 for the construction, and this method is used in [7]. However, since it appears to be sufficient, we have chosen to use the simplest possible lift of f - we merely multiply f by a smooth cut-off function depending on x_3. This has the advantage that ψ inherits the regularity of f and is smooth in the x_3-direction. It is a little awkward that $\partial_{x_3}^k \psi$ is not in L^2, but since this always appears as a coefficient in the equations this is not a problem - in fact it should make things better from a regularity point of view, the only problem being that the high-order horizontal and time derivatives of ψ will have to be estimated in a different space - L^2 rather than L^∞.

Thus we obtain the system of equations (4.3.1)-(4.3.5) on the fixed domains $(0,T) \times \Omega_\pm$ and the analogous theorem to 2.4.1 corresponding to these new equations is given by 4.6.1.

Full details of the reduction of the theorem 2.4.1 in the moving domains to the theorem 4.6.1 in the fixed domains are given in 5.

3.2 Smoothing the Initial Data

Later on, we will need to introduce an approximation to the equations in the fixed domains (4.3.1)-(4.3.5). To maintain compatibility of the initial data, we will need to add a combination of the initial data and its derivatives to the equations, which will then be treated like a zero order term from the point of view of energy estimates. We will also solve these new equations with slightly higher regularity than the original equations, which will require that the compatibility condition (2.2.5) is satisfied to order 3 rather than 2. Also, in our fixed point scheme later on we will need to construct a approximate solution to start the iteration which at zero order (in terms of the energy) involves initial time derivatives of the initial data. For all these reasons, it is necessary for us to be able to reduce to the case of smooth compatible initial data. This means that given some initial data (U_+^0, U_-^0, f^0) which satisfies the compatibility conditions 2.2.3 (or 4.4.3 if we are considering the problem in the fixed domains) up to order 2 and given $m \geq 1$, we need to be able to construct some smooth initial data $(U_{m+}^0, U_{m-}^0, f_m^0)$ satisfying the compatibility condition (4.4.7) up to order 2 and the compatibility condition (4.4.6) up to order 3 such that $(U_{m+}^0, U_{m-}^0, f_m^0)$ converges (in the sense of the norms making up the energy E^0) to (U_+^0, U_-^0, f^0) as $m \to \infty$.

The details of the construction of this initial data are given in 6. We summarise the main ideas here. Firstly we smooth the initial data in a standard way, by extending $U_\pm^0$ to $\mathbb{R}^3$ by Sobolev extension (and doing nothing to f^0), then convolving with the standard mollifier. We then add a correction term $w_{\epsilon+}^0$ to the smoothed velocity in Ω_+ and a correction term $q_{\epsilon+}^0$ to the smoothed pressure in Ω_+ which ensures that the initial time derivatives of the smoothed front, $\partial_0^{\epsilon,j} f_\epsilon^0$, satisfy $\partial_0^{\epsilon,j} f_\epsilon^0 = (\partial_0^j f^0) * \eta_\epsilon$ for $0 \leq j \leq 3$, which is achieved by specifying the normal derivatives of $w_{\epsilon+}^0$ and $q_{\epsilon+}^0$ on the boundary in terms of the difference of these two quantities. We use the fact that we can derive an expression relating normal derivatives and initial time derivatives of u^0 and p^0. The the construction of $w_{\epsilon+}^0$ and $q_{\epsilon+}^0$ from specified values of their normal derivatives on the boundary uses the lifting lemma proved in A.2.1. Similarly, we add a correction term $w_{\epsilon-}^0$ to the smoothed velocity in Ω_- and a correction term $q_{\epsilon-}^0$ to the smoothed pressure in Ω_- which together ensure that the compatibility conditions (4.4.6)-(4.4.7) are satisfied up to order 2. Then we check that the smoothed initial data converges to the original initial data as required.

Having smoothed the initial data, we add another correction term to the smoothed initial pressure in Ω_- so that the compatibility condition (4.4.6) is satisfied up to order 3.

We then show that we can reduce to the case of smooth compatible initial data using the fact that the bound on the energy of solutions depends only on the size of the initial data and the lower bound on time interval of existence depends only on the size of the initial data.

3.3 Motivating a Strategy for Solving the Equations

Note that in the introduction we originally started with a system of equations in integral form (1.4.1)-(1.4.3). In particular the integral energy equation (1.4.3) gives rise to a conserved zero order energy. However, there is no good theory of weak solutions of systems of conservation laws in multi-space dimensions and even if we were able to obtain a sequence of approximate solutions with bounded zero order energy which by extracting a subsequence would converge weakly-$*$ in zero order norms, it would be difficult or impossible to prove that the limit was a solution of the equations due to the lack of convergence of the nonlinear terms. Furthermore, it is unlikely that this solution would be unique. Thus our approach will involve classical solutions. Since it is possible shocks and other singularities may form in finite time, we will restrict to the case of short-time solutions.

Since we are working with a complex system of nonlinear equations, any approach will involve an approximation of the equations at some point. To obtain a classical solution of the original equations, we must be able to show that a sufficiently high-order energy (by which we mean sum of Sobolev norms) of the approximate solutions is bounded uniformly with respect to the approximation or without loss of derivatives with respect to the linearised coefficients in the case that the approximation is a linearisation. Then we can use weak-$*$ compactness and the Sobolev compact embedding theorem or a fixed point scheme in the case of linearisation to obtain a strongly convergent subsequence on compact subsets of $(0,T) \times \Omega_\pm$ which will converge to a solution of the original equations. Thus the first step from a motivating point of view is to obtain a closed a priori high-order energy estimate for the system of equations on the fixed domains (4.3.1)-(4.3.5). One must then choose an approximation to the equations which makes them easier to solve but which preserves the energy estimate. Since one needs the energy estimate to apply to the approximate equations, we introduce the approximate equations before deriving the energy estimate even though motivation-wise one would obtain an energy estimate first.

There are two obvious ways in which to approximate the system of equations on the fixed domains (4.3.1)-(4.3.5). One approach would be to directly linearise the equations, try to obtain an energy estimate without losing regularity with respect to the coefficients, then use an iteration scheme to obtain a sequence of solutions to the linearised equations which converges a solution to the nonlinear equations. This is one of the standard approaches for hyperbolic conservation laws - see eg Serre and Benzoni-Gavage, [3]. However, a direct linearisation of the equations destroys the structure of the equations and it seems impossible to obtain an energy estimate without losing regularity with respect to the coefficients. For example, the fact that $u_3^\psi|_\Gamma = 0$, where u^ψ is defined in 4.2.3, is crucial to avoid generating additional boundary terms in the energy estimate. This relies on the fact that $u_3^\psi|_\Gamma = u \cdot n - \partial_t f = 0$ by virtue of the equation (4.3.4) for f, but linearising the equation (4.3.4) as well as replacing the coefficient u^ψ with something from a previous iteration step which preserves the key property $u_3^\psi|_\Gamma = 0$ seems impossible unless we first solve $\partial_t f = \bar{u} \cdot \bar{n}$ where $\bar{u}$ and $\bar{n}$ are from the previous iteration step. But then we would not be able to write $\nabla' \partial^\alpha f \cdot \nabla' \partial^\alpha (u \cdot n) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\nabla' \partial^\alpha f|^2$, which is a key part of the energy estimate. We may overcome a loss of derivatives in the energy estimate with respect to the coefficients caused by direct linearisation by using a Nash-Moser type iteration as in [9] and [4]. However, the Nash-Moser iteration scheme is very technical and is hampered by the fact that certain nonlinear constraints such as

$[u \cdot n] = 0$ should be satisfied at the start of each step of the iteration procedure which do not fit into a linear framework very well.

The second approach is to introduce an artificial viscosity or parabolic-type regularisation before linearisation. We must obtain an energy estimate which is independent of the artificial viscosity parameter, but having done so we may fix the artificial viscosity parameter and then solve the equations using a method which depends crucially on the artificial viscosity parameter. This is also one of the standard approaches for hyperbolic conservation laws - see eg Dafermos, [14]. This is the approach we will follow, except that our artificial viscosity will be degenerate because adding a full parabolic regularisation would not preserve the energy estimate. We follow the choice of artificial viscosity used by Shkoller et al in [10] and [13].

3.4 The μ-Approximate Equations - a Degenerate Parabolic-Type Regularisation

Motivated by the above discussion, we introduce the μ-approximate equations (7.1.2)-(7.1.6) in 7 which reduce to the equations in the fixed domains (4.3.1)-(4.3.5) in the case $\mu = 0$. There are four main points to note. Firstly, we regularise the divergence of the velocity by adding the term $\mu\nabla^{\psi}(\rho c^2 \nabla^{\psi} \cdot u)$ to the right hand side of the velocity equation. It is very important to note that by using the pressure equation, we have $\rho c^2 \nabla^{\psi} \cdot u = -(\partial_t + u^{\psi} \cdot \nabla)p =: D_t^u p =: \dot{p}$ which is the material derivative of the pressure, so that when moved to the other side we have effectively replaced ∇p with $\nabla p + \mu \nabla \dot{p}$, where $\dot{p}$ behaves like a time derivative of p, which will allow us to preserve our estimate of $\partial_{x3} p$ independent of μ. Since $\nabla \times \nabla(\cdot) = 0$, the curl estimate is also preserved. The addition of a full parabolic regularising term would not preserve these estimates. We replace $[p]$ with $[p - \mu\rho c^2 \nabla^{\psi} \cdot u]$ in the pressure interface condition (4.3.5) which naturally corresponds to the addition of the term $\mu\nabla^{\psi}(\rho c^2 \nabla^{\psi} \cdot u)$ to the right hand side of the velocity equation.

Secondly, we regularise $u \cdot n = \partial_t f$ by adding the term $\mu(\Delta' - 1)(u \cdot n)$ to the right hand side of the pressure interface condition (4.3.5). This has a number of benefits. First of all, since $u \cdot n = \partial_t f$, it regularises the time derivative of f, whereas surface tension only regularises in space. Secondly, for fixed μ, it means that good regularity of f follows from good regularity of the 'normal' component of u on the boundary, so that when we come to linearise we may in fact solve $\partial_t f = \bar{u} \cdot \bar{n}$ before solving the velocity equation. Thirdly, we may combine regularity of $u \cdot n$ with regularity of $\nabla^{\psi} \cdot u$ and $\nabla^{\psi} \times u$ using the Hodge decomposition estimate A.1.2 to obtain regularity of ∇u.

Thirdly, we smooth the coefficient ρ in the velocity equation (4.3.2). This is harmless for the μ-independent estimate since the coefficient ρ is already lower order. However, for the μ-approximate equations we will be estimating u to one order higher in space than p and s, hence ρ is no longer lower order for fixed μ, but the mollified density ρ_μ will be lower order.

Fourthly, we add some known smooth functions g_1 and g_2 to the right hand side of the velocity equation and pressure interface condition respectively. Eventually we will choose these to be functions of the initial data. The function g_1 will be chosen in terms of some given smooth initial data for the equations in the fixed domains so that the initial time derivatives associated with this initial data for the μ-approximate equations are the same as those for the equations in the fixed domains. Since the

μ-approximate equations reduce to the equations in the fixed domains in the case $\mu = 0$, we will have $g_1 \to 0$ as $\mu \to 0$. The function g_2 will be chosen in terms of some given smooth initial data for the equations in the fixed domains so that the modified compatibility condition (7.2.7) holds up to order 2, and since the compatibility condition (4.4.7) holds up to order 2 we will have $g_2 \to 0$ as $\mu \to 0$.

3.5 The μ-Independent Energy Estimate and Reduction to Solving the μ-Approximate Equations

As discussed above, we need to prove that some high-order energy associated to solutions of the μ-approximate equations, which we will call the μ-independent energy, remains bounded independent of μ on some time interval which has a lower bound independent of μ, provided that the solution exists for that length of time. The problem that the solution might not exist for long enough will be overcome if we can prove existence of solutions to the μ-approximate equations on a time interval which is maximal in the sense that the energy blows up as the t approaches this maximal time. Then in fact the maximal time interval must be larger than the μ-independent lower bound on the time interval from our energy estimate to avoid contradiction.

Assuming we can prove this energy estimate, we may use weak-$*$ compactness and the Sobolev compact embedding theorem to obtain a subsequence $\mu_j \to 0$ as $j \to \infty$ such that solutions of the μ_j-approximate equations converge to a solution of our original equations on the fixed domains, which proves the existence part of the main theorem 2.4.1. See 7 for the details of this reduction.

The μ-independent energy is chosen to include roughly space and time derivatives of the solution up to order 3, L^2 in space and L^∞ in time. We do not require zero order derivatives to be in L^2 with respect to space since we want in particular the density (thus pressure) to be bounded below. In addition there is one extra space derivative on f with up to 3 time derivatives, and 1.5 extra space derivatives on f with up to 2 time derivatives, which is an automatic consequence of the elliptic-type equation for f given by the surface tension term in the pressure interface condition (2.1.5). We need 3 derivatives on the solution in order to be able to estimate terms involving one derivative of the solution in L^∞ using the Sobolev embedding theorem. The extra terms involving μ are added so that we can close the estimate without constants depending on μ. In fact we can close the estimate with slightly fewer μ-dependent terms, but it is useful to have an estimate for some higher-order terms in order to be able to maximise the time interval of existence of solutions of the μ-approximate equations for fixed μ as described above.

The energy estimate is split into several parts which we briefly describe as follows. Firstly, we estimate the tangential space derivatives of the solution by differentiating the equations and multiplying by the differentiated solution, adding the equations and integrating by parts, using the symmetry in the standard manner. Integration by parts generates a boundary term, which after adding the estimates in $\Omega_\pm$ together can be written as the jump of a certain product of tangential derivatives. Since we are considering tangential derivatives, we may apply these derivatives to the boundary conditions in order to estimate this boundary term. Here we use crucially the extra spatial regularity of f provided by the elliptic-type equation for f on the interface contained in (2.1.5).

Secondly, we estimate the time derivatives of the solution. In order to remove the highest order time derivative of ψ from the equations which cannot be treated as a lower order term, we change dependent variables to a nonlinear version of the 'good' unknowns which were introduced by Alinhac in [2]. We differentiate the equations with respect to time, replacing the last time derivative by a 'material' derivative $D_t^{u_-}$ where in fact we extend u_- to Ω_+ by Sobolev extension so that it is a material derivative only in Ω_- and is continuous across Γ. We then proceed as in the estimate of the tangential derivatives, except that the estimate of the boundary term is more complicated. In fact we will need to split the jump of a product of two quantities into two terms each of which involves the jump of only one quantity, which allows us to use the boundary conditions but also generates a term which we must estimate by converting it to an integral over the interior of Ω_- and using the interior equations, which is why we needed to use the material derivative operator $D_t^{u_-}$ instead of a straightforward time derivative.

Thirdly, we must estimate the normal derivatives of the solution. Note that the normal derivatives of the entropy can be estimated in a standard manner. We may estimate the curl of the velocity by taking the curl of the velocity equation then differentiating and performing a standard energy estimate. In the standard way, we may also obtain an estimate weighted by u_3^ψ for the normal derivatives of the solution using the fact that $u_3^\psi|_\Gamma = 0$ which means no boundary terms are generated. Using this and the existing time and tangential estimates, we may estimate the divergence of the velocity and the normal derivative of the pressure by directly rearranging the pressure and velocity equations. We actually obtain an estimate for $\partial_{x_3}p + \mu\partial_{x_3}\dot{p}$ rather than $\partial_{x_3}p$, but thinking of $\dot{p}$ as behaving like $\partial_t p$, it is easy to see by considering a first-order ODE that this gives us a μ-independent estimate for $\partial_{x_3}p$. A combination of the divergence, curl and tangential derivative estimates of u gives estimates for all the derivatives of u.

The high-order space derivative estimate of f is obtained from the pressure interface condition using the fact that $\nabla' \cdot \hat{n}$ behaves like $-\Delta' f$ so that this is an elliptic equation for f, combined with existing estimates for the other terms in this equation. The high-order boundary artificial viscosity term involving $u \cdot n = \partial_t f$ is dealt with in exactly the same way as we dealt with the presence of the term involving $\dot{p}$ above.

The extra velocity term in the energy involving μ is estimated using a Hodge decomposition argument and the high-order $u \cdot n$ term involving μ is estimated from the elliptic-type equation given by the pressure interface condition. This completes the proof of the energy estimate. See 8 for the detailed proof.

3.6 Linearisation of the μ-Approximate Equations

We have designed the μ-approximate equations so that for fixed μ the equations have a nice linearised structure when linearised carefully. In fact the linearisation reduces to solving the following equations in order, which are given in 9. Firstly, we solve the very simple ODE $\partial_t f = \bar{u} \cdot \bar{n}$ for f. Next, we solve scalar linear transport equations where the transport is parallel to the interface for the entropy, the pressure, and (the components of) the curl, ω. We carefully replace u^ψ with $\overline{u^\psi}$ (see 9 for the definition) which is chosen so that $\overline{u^\psi}_3|_\Gamma = \bar{u} \cdot \bar{n} - \partial_t f = 0$ by virtue of the equation just given for f. Finally, we solve a parabolic-type equation for the velocity, where the full parabolicity comes from the fact that we may add the curl of the curl of the velocity to one side of the equation and the curl of the 'curl' ω just obtained from the transport equation for the curl, ω, to the other side of the equation.

3.7 Solving the Linearised Equations

The details of the solution of the transport equations, which is standard, are given in 10, and the details of the solution of the parabolic-type equation for the velocity are given in 11. Note that we solve the parabolic-type equation for the velocity in the domains $\Omega_\pm$ simultaneously whilst maintaining the boundary condition $[u \cdot n] = 0$. It seems very difficult to preserve an appropriate energy estimate without maintaining the condition $[u \cdot n] = 0$. This contrasts with the incompressible case considered by Shkoller et al in [6] where the equations are solved separately in $\Omega_\pm$ in a two-step iteration process.

To solve the parabolic-type equation, we first rotate the velocity and work with the rotated velocity $\widehat{u}$ so that the condition $[u \cdot n] = 0$ becomes $[\widehat{u}_3] = 0$ which can be incorporated in the definition of a vector space which is independent of time. Once we have done this, we formulate a weak version of the problem and construct a Galerkin basis etc, proceeding much as in the solution of a simple linear parabolic equation as described in Evans, [15]. However, a technical complication arises from the fact that we are working on the unbounded domains $\Omega_\pm$. Note that the standard Galerkin scheme method requires that the Galerkin basis is constructed so that the L^2 orthogonal projection onto the linear subspace formed by the span of the first m basis elements is the same as H^1 orthogonal projection, which is required in order to estimate the initial value of the approximate solution in terms of the given initial data. This requires that the Galerkin basis functions are eigenvalues of an elliptic operator which is required to be compact, but this only holds on compact domains. There are two ways of overcoming this problem.

One is to subtract the background state to reduce to the case of zero initial data, in which case we do not need the above property of the Galerkin basis and may use an abstract Hilbert space basis instead. However, this requires additional regularity on the background state, which would involve smoothing the background state, which would in turn involve smoothing the initial data for the μ-approximate equations. (It is not enough to solve the μ-approximate equations only for some fixed smoothed initial data for the equations on the fixed domains since we need to maximise the time interval of existence in order to eventually obtain a time interval of existence independent of μ). This would be a significant amount of work since we would need to preserve the compatibility conditions.

The other method, which we follow, is to solve the equations on a ball of radius $r \geq 1$ with zero boundary conditions on the boundary of the ball and obtain an energy estimate independent of r allowing us to pass to the limit $r \to \infty$ to obtain a weak solution of the equations on the unbounded domains. Note that we have to be careful of the order in which we prove the existence and regularity results. We first prove existence of a weak solution in the bounded domains, then pass to the limit $r \to \infty$. We then prove higher time regularity in the bounded domains (in fact this is done via the reuse of the approximate solutions formed from the Galerkin basis), which requires the compatibility conditions to be satisfied, then we pass to the limit $r \to \infty$. Finally we prove higher spatial regularity directly on the unbounded domains to avoid any problems on the boundary of the ball of radius r.

3.8 Obtaining a Fixed Point of the Linearised Equations

We use the contraction mapping theorem to prove the existence of a fixed point of the linearised equations, which is a solution of the μ-approximate equations by the construction of the linearised equations.

This requires three components.

Firstly, we define an energy for the μ-approximate equations which we will use for fixed μ which is slightly higher order than the μ-independent energy, and we also define a slightly modified linearised version of this which applies to the linearised equations. The reason that we make this energy higher order than the μ-independent energy is partly because it is convenient given the structure of the μ-dependent equations, and partly because we want to be able to differentiate the equations when obtaining our μ-independent energy estimate without worrying about whether the solution is regular enough. We need to show that given a background state with energy bounded by a sufficiently large constant, the solution of the linearised equations linearised about that state has energy bounded by the same constant provided we take the solution time interval sufficiently small. (In fact, we want to be slightly more careful in the energy estimate so that later we can use Gronwall's lemma to show that the higher-order μ-approximate energy remains bounded provided the μ-independent energy remains bounded.) The fact that the energy remains bounded by the same constant proves that we have a well-defined map from an appropriate metric space to itself which sends a background state to the solution of the linearised equations linearised about that state. The metric is defined using the norms which make up the energy, but two orders of derivatives lower, and the completeness of the metric space follows from weak-$*$ compactness and weak-$*$ lower semi-continuity of norms.

Secondly, we have to prove this map is a contraction in this metric, which is very similar to the proof of the energy estimate except that we take the difference of the differentiated equations for two solutions with two different background states.

The third point to prove is that the metric space is non-empty. This requires constructing a background state whose time derivatives at $t = 0$ match the initial time derivatives of some given initial data. To do this we simply form a truncated Taylor series using the initial time derivatives of the initial data, but note that this requires the initial data to be smooth. Since we have smoothed the initial data for the equations on the fixed domains, this, combined with the well-defined contraction map, allows us to obtain a solution on a μ-dependent time interval. However, in order to maximise the time interval of existence, which combined with our μ-independent energy estimate and the control of the higher order μ-approximate energy in terms of the μ-independent energy mentioned above allows us to obtain a time interval of existence independent of μ, we need to be able to take 'final' data of the equations as initial data, which will not be smoothed. We overcome this by constructing a background state on a larger time interval of existence than the solution just obtained by using Sobolev extension to extend the solution in time (which produces a background state, but not a solution, on a larger time interval). Thus we do not require the 'final' data to be smooth.

Hence we obtain a fixed point, which solves the μ-approximate equations on a maximal time interval - see 13 for the details. Since we have already reduced to the solution of the μ-approximate equations on a maximal time interval, this completes the proof of the existence part of the main theorem 2.4.1.

3.9 Proof of Uniqueness

The proof of uniqueness is very similar to the μ-independent energy estimate with μ set to zero. Given two solutions satisfying the conclusions of the existence part of theorem on the fixed domains 4.6.1, we define the 'difference' energy by replacing the solution by the difference of two solutions in the terms which make up the energy for the fixed equations as defined in 4.5.1, and decreasing the number of derivatives by one. We then show that this energy is bounded by a factor depending on the individual energies of the solutions as defined by 4.5.1 multiplying the sum of the difference energy at time zero and a time integral of the difference energy. Applying Gronwall's lemma gives a stability result for the equations in the fixed domains, from which uniqueness follows, both for the equations in the fixed and moving domains.

The strategy of the proof is just to follow the proof of the μ-independent energy estimate with $\mu = 0$, where we take the difference of the equations after differentiating. We write terms of the form $G(U^1,\psi^1)\partial^\alpha\partial U^1 - G(U^2,\psi^2)\partial^\alpha\partial U^2$ as $G(U^1,\psi^1)\partial^\alpha\partial(U^1-U^2)+(G(U^1,\psi^1)-G(U^2,\psi^2))\partial^\alpha\partial U^2$ and estimate the first term as in the μ-independent energy estimate. The second term is estimated by applying the mean value theorem to $G(U^1,\psi^1) - G(U^2,\psi^2)$ and using the fact that this is a lower order difference term, and using the higher order energy bound to bound the factor $\partial^\alpha\partial U^2$. Thus in fact the proof is almost exactly the same as that of the μ-independent energy estimate.

3.10 A List of the Intermediate Results Combined to Prove the Main Theorem

Now we list how all the results which make up the proof of the main theorem 2.4.1 fit together. First we apply 5.2.1 to reduce the proof of the main theorem in the original moving domains, 2.4.1, to the proof of the main theorem in the fixed domains, 4.6.1. Then we apply 6.2.1 to reduce to the case of smooth initial data which satisfy the compatibility condition (4.4.6) to one order higher than the original initial data.

Then we define the μ-approximate equations in 7.1.2. We prove a μ-independent a priori estimate (stated in 7.5.1 and proved in section 8) for these equations, and in 7.6.3 we reduce the existence part of the proof of 4.6.1 to the proof of existence of solutions to the μ-approximate solutions on a maximal time interval as stated in 7.4.2. To prove 7.4.2, we define a linearisation of the μ-approximate equations in 9.1.1 and show that there exists a unique solution of these equations (stated in 9.5.1 and proved in sections 10-11) which satisfies the energy estimate given by 9.4.1 (proved in section 12).

We then show in section 13 that the map that sends a state about which the equations are linearised to the solution of the equations is a well-defined (see 13.2.1) contraction (see 13.2.2) in an appropriate metric space (see 13.1) - this uses the linearised energy estimate 9.4.1 and the contraction property which is proved in section 14 in a similar manner to the linearised energy estimate. We also show in 13.3.2 that this metric space is non-empty for smooth initial data. Then in section 13.3.1 we show that there exists a fixed point, which we showed in lemma 13.3.1 solves the μ-approximate equations, which proves the existence of a solution on a non-maximised time interval as stated in 7.4.1. We then maximise the time interval in 13.3.2, using an extension of the solution just found to construct an element of our metric space without the need for any smoothing. This proves 7.4.2, which completes the proof of the existence part of the main theorem since we have already reduced to the case of proving 7.4.2. We then prove the uniqueness and stability part of 4.6.1 in 15 and we have already shown in 5.2.1 that this is sufficient to prove the uniqueness part of 2.4.1.

4. The Theorem in the Fixed Domains

In this section we present the equations we obtain from the system (2.1.1)-(2.1.5) rewritten on fixed domains after a change of variables. The change of variables will be detailed in section 5, and the motivation for this change of variables is summarised in 3.1.

4.1 Important Notational Convention

Everything that follows after section 5 will be devoted to the proof of the theorem in the fixed domains. Hence, to keep the notation simple, we will now redefine everything in the fixed domains, which will involve using some of the same symbols as in the moving domains for corresponding quantities, such as (U_+, U_-, f) for the solution, E for the energy, ∂_0^j for the initial time derivatives associated with the system etc. In section 5, where we show how the theorem in the fixed domains implies the main theorem, we will place a tilde $\tilde{}$ over variables in the fixed domains in order to distinguish corresponding quantities, but otherwise we will reserve the tilde $\tilde{}$ for other purposes.

4.2 The Fixed Domains, the Lifting Operator and the Transformed Derivatives

Definition 4.2.1. Define the fixed domains $\Omega_\pm$ as

$$\Omega_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$$
$$\Omega_- = \{x \in \mathbb{R}^3 : x_3 < 0\}$$

Define the fixed boundary (or front or interface) Γ as

$$\Gamma = \{x \in \mathbb{R}^3 : x_3 = 0\}$$

Definition 4.2.2. Let $a \in \mathbb{R}_{\geq 0}$. We define the lifting operator $L^a : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^3)$ by

$$(L^a f)(x) := \chi(\frac{x_3}{3(1+a)}) f(x')$$

where $\chi \in C_c^\infty(\mathbb{R})$ is a smooth cut-off function with $0 \leq \chi \leq 1$, $\chi(x_3) = 1$ for $|x_3| \leq 1$, $\chi(x_3) = 0$ for $|x_3| \geq 3$ and $|\partial_{x_3}\chi(x_3)| \leq 1$ for all $x_3 \in \mathbb{R}$.

Note that this implies the following.

$$(L^a f)(x', 0) = f(x')$$
$$\partial_{x_3}(L^a f)(x', 0) = 0$$
$$|\partial_{x_3}(L^a f)(x)| \leq \frac{1}{3(1+a)} \left|f(x')\right|$$

Clearly for $T > 0$ we may also define $L^a : L^2((0,T) \times \mathbb{R}^2) \to L^2((0,T) \times \mathbb{R}^3)$ by

$$(L^a f)(t,x) := \chi(\frac{x_3}{3(1+a)})f(t,x')$$

Note that $L^a f$ inherits the regularity of f and derivatives of $L^a f$ can be bounded by those of f independently of a. This means that all constants will be independent of a unless explicitly stated otherwise.

Definition 4.2.3. Given $\psi \in C^1((0,T) \times \mathbb{R}^3)$ bounded with bounded derivatives, and a vector $u \in \mathbb{R}^3$ (which we later take as a function of t and x), we define

$$J^\psi := 1 + \partial_{x_3}\psi \tag{4.2.1}$$

$$\partial^\psi := \partial - \frac{1}{J^\psi}\partial\psi\partial_{x_3} \tag{4.2.2}$$

$$\nabla^\psi := \nabla - \frac{1}{J^\psi}\nabla\psi\partial_{x_3} = \nabla' + \frac{1}{J^\psi}(-\nabla'\psi, 1)\partial_{x_3} \tag{4.2.3}$$

$$u^\psi := (u', \frac{1}{J^\psi}(u_3 - u' \cdot \nabla'\psi - \partial_t\psi)) \tag{4.2.4}$$

The quantity J^ψ will be the Jacobian of the change of coordinates introduced later and ∂ acting on the fluid variables in the moving domains will transform to ∂^ψ acting on the new fluid variables in the fixed domains after the change of coordinates introduced later.

4.3 Definition of the Equations in the Fixed Domains

Definition 4.3.1. Fix $T \in (0, \infty)$.

Let $f \in C_b^1((0,T') \times \mathbb{R}^2)$ with $\nabla' f \in C_b^1((0,T') \times \mathbb{R}^2)$ for all $T' \in (0,T)$, which we call the (graph of) the front or interface (although it is not the graph of the fixed interface $\Gamma = \mathbb{R}^2$). Define

$$n := (-\nabla' f, 1)$$
$$\hat{n} := \frac{n}{|n|}$$

which we call a spatial normal and the spatial unit normal to the interface pointing into Ω_+ (although they are not actually normals to the fixed interface $\Gamma = \mathbb{R}^2$).

Let $p_\pm, (u_i)_\pm\ (i = 1,2,3), s_\pm \in C_b^1((0,T') \times \Omega_\pm)$ for all $T' \in (0,T)$. For convenience we write $u_\pm = ((u_1)_\pm, (u_2)_\pm, (u_3)_\pm)$, $U_\pm = (p_\pm, u_\pm, s_\pm)$. Note that $U_\pm|_{t=0}$ and $f|_{t=0}$ are defined by uniform continuity.

Define $\rho_\pm(t,x) := \rho_\pm(p_\pm(t,x), s_\pm(t,x))$ and $c_\pm(t,x) := c_\pm(p_\pm(t,x), s_\pm(t,x))$, where $\rho_\pm(p,s)$ and $c_\pm(p,s)$ are the equations of state defined in 1.5.

We will say that (U_+, U_-, f) is a solution of the equations in the fixed domains on the time interval $(0,T)$ with lifting operator L^a and initial data $(U_+^0, U_-^0, f^0) := (U_+|_{t=0}, U_-|_{t=0}, f|_{t=0})$ provided that $\rho_\pm(t,x) > 0$ and $J^\psi(t,x) > 0$ in $(0,T) \times \Omega_\pm$, and $p_\pm, u_\pm, s_\pm, f$ satisfy the following system of PDEs.

$$\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)p + \nabla^\psi \cdot u = 0 \tag{4.3.1}$$

$$\rho(\partial_t + u^\psi \cdot \nabla)u + \nabla^\psi p = 0 \tag{4.3.2}$$

$$(\partial_t + u^\psi \cdot \nabla)s = 0 \tag{4.3.3}$$

in $(0,T)\times\Omega_\pm$,

$$\partial_t f = u_\pm \cdot n \tag{4.3.4}$$

$$[p] = -\sigma\nabla' \cdot \hat{n} \tag{4.3.5}$$

on $(0,T)\times\Gamma$, where ψ is a lift of f defined by

$$\psi := L^a f \tag{4.3.6}$$

Remark 4.3.2. Note that for ψ defined by (4.3.6), because of the properties of the lifting operator, we have the following on the interface Γ.

$$J^\psi = 1$$
$$\nabla^\psi = \nabla' + n\partial_{x_3}$$

on Γ. Also, if u and f satisfy the equation (4.3.4), then we have the following on the interface Γ.

$$u^\psi = (u', 0)$$

on Γ.

4.4 The Initial Data and the Compatibility Conditions

Definition 4.4.1. We say that $(p^0_\pm, u^0_\pm, s^0_\pm, f^0)$ are initial data for the system (4.3.1)-(4.3.6) if the following holds. We require that $f^0 \in C^1_b(\mathbb{R}^2)$. We define the initial normal and unit normal as

$$n^0 := (-\nabla' f^0, 1)$$
$$\hat{n}^0 := \frac{n^0}{|n^0|}$$

For convenience we use the obvious notation

$$\psi^0 := L^a f^0$$

We also require that $p^0_\pm, u^0_\pm, s^0_\pm \in C_b(\Omega_\pm)$ are uniformly continuous functions (where $u^0_\pm$ are vectors representing the initial velocity), so that we may define them on the interface Γ by uniform continuity. Finally, we require

$$\rho^0_\pm(x) := \rho_\pm(p^0_\pm(x), s^0_\pm(x)) > 0 \text{ for all } x \in \Omega_\pm$$
$$J^{\psi^0} := 1 + \partial_{x_3}\psi^0 > 0 \text{ for all } x \in \Omega_\pm$$

Definition 4.4.2. For integer $j \geq 0$, we define the following differential operators ∂^j_0 associated with the system (4.3.1)-(4.3.6) which act on differentiable expressions involving the initial data and its derivatives (including compositions). We call the operator ∂^j_0 the initial time derivative operator of order j associated to the system (4.3.1)-(4.3.6).

To start with, we define $\partial^j_0 p^0_\pm, \partial^j_0 u^0_\pm, \partial^j_0 s^0_\pm, \partial^j_0 f^0, \partial^j_0 \psi^0$ as follows. We set

$$\partial^0_0 = \mathrm{id}$$

then for $j \geq 0$ we inductively define

$$\partial_0^{j+1} f^0 := \partial_t^j (u_+(t)\big|_\Gamma \cdot n(t))\big|_{t=0} := \partial_t^j (u_+(t, x', 0) \cdot n(t))\big|_{t=0} \tag{4.4.1}$$

$$\partial_0^{j+1} \psi^0 := L^a \partial_0^{j+1} f^0 \tag{4.4.2}$$

$$\partial_0^{j+1} p_\pm^0 := \partial_t^j (-(u_\pm^{\psi(t)}(t) \cdot \nabla) p_\pm(t) - \rho_\pm(t) c_\pm^2(t) \nabla^{\psi(t)} \cdot u_\pm(t))\big|_{t=0} \tag{4.4.3}$$

$$\partial_0^{j+1} u_\pm^0 := \partial_t^j (-(u_\pm^{\psi(t)}(t) \cdot \nabla) u_\pm(t) - \frac{1}{\rho_\pm(t)} \nabla^{\psi(t)} p_\pm(t))\big|_{t=0} \tag{4.4.4}$$

$$\partial_0^{j+1} s_\pm^0 := \partial_t^j (-(u_\pm^{\psi(t)}(t) \cdot \nabla) s_\pm(t))\big|_{t=0} \tag{4.4.5}$$

where the right hand side in the first line and last three lines means we think of $u_\pm(t)$, $\psi(t)$ etc as functions of t and formally differentiate with respect to t j times according to the chain and Leibniz rules, then replace $\partial_t^l u_\pm(t)$ for $0 \leq l \leq j$ by $\partial_0^l u_\pm^0$ etc. Note that it is necessary for us to pick either $+$ or $-$ in the above definition for $\partial_0^{j+1} f^0$ since we have not yet imposed any compatibility conditions on the initial data. Note we require sufficient regularity on the initial data for the above to make sense.

We can now generalise the operators ∂_0^j to general differentiable expressions involving the initial data and its derivatives (including compositions) in the obvious way. Suppose $H(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ is such an expression. For $j \geq 0$, we set

$$\partial_0^j (H(p_\pm^0, u_\pm^0, s_\pm^0, f^0, \psi^0)) := \partial_t^j (H(p_\pm(t), u_\pm(t), s_\pm(t), f(t), \psi(t)))\big|_{t=0}$$

where, as above, the right hand side means we think of $u_\pm(t)$ etc as functions of t and formally differentiate with respect to t j times according to the chain and Leibniz rules, then replace $\partial_t^l u_\pm(t)$ for $0 \leq l \leq j$ by $\partial_0^l u_\pm^0$ etc.

For convenience later on, we define the notation

$$(u^\psi)^0 = ((u^0)', \frac{1}{J\psi^0}(u_3^0 - (u^0)' \cdot \nabla' \psi^0 - \partial_0^1 \psi^0))$$

Definition 4.4.3. Let $(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ be initial data for the system (4.3.1)-(4.3.6). We say that the initial data $(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ satisfy the compatibility conditions for the system (4.3.1)-(4.3.6) up to order k, for integer $k \geq 0$, if the following holds.

$$\partial_0^j [u^0 \cdot n^0] = 0 \tag{4.4.6}$$

for $0 \leq j \leq k$ and

$$[\partial_0^j p^0] = -\sigma \partial_0^j (\nabla' \cdot \hat{n}^0) \tag{4.4.7}$$

on Γ for $0 \leq j \leq k$.

4.5 The Energy Functional

Definition 4.5.1. Let $f \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^2)$, $U_\pm \in L^1_{\text{loc}}((0,T) \times \Omega_\pm)$. We define the energy E for the fixed equations, associated to (U_+, U_-, f), as follows.

Define the energy $E : (0,T] \to [0,\infty]$ by

$$E(t) = \sum_{\pm} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||U_\pm||^2_{L^\infty(\Omega_\pm)} + \sum_{\pm}\sum_{1\le|\alpha|\le 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha U_\pm||^2_{L^2(\Omega_\pm)}$$
$$+ \operatorname*{ess\,sup}_{\tau\in(0,t)} ||f||^2_{L^\infty(\mathbb{R}^2)} + \sum_{1\le|\alpha|\le 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha f||^2_{H^1(\mathbb{R}^2)} + \sum_{j=0}^{2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\nabla'\partial_t^j f\right|\right|^2_{H^{3.5-j}(\mathbb{R}^2)}$$

Note we adopt the convention that if the solution does not have sufficiently many weak derivatives in $(0,T)\times\Omega_\pm$ to define the energy then it is infinite, and if we write $E(T') < \infty$ then this implies that all the weak derivatives in $(0,T')\times\Omega_\pm$ present in the energy exist. Note also that if $E(T')<\infty$ then E is uniformly continuous on $(0,T']$.

Similarly, let (U_+^0, U_-^0, f^0) be initial data for the system (4.3.1)-(4.3.6). We define the energy $E^0 \in [0,\infty]$ of the initial data as

$$E^0 = \sum_{\pm} \left|\left|U_\pm^0\right|\right|^2_{L^\infty(\Omega_\pm)} + \sum_{\pm}\sum_{1\le|\beta|+j\le 3} \left|\left|\nabla^\beta\partial_0^j U_\pm^0\right|\right|^2_{L^2(\Omega_\pm)}$$
$$+ \left|\left|f^0\right|\right|^2_{L^\infty(\mathbb{R}^2)} + \sum_{1\le|\beta|+j\le 3} \left|\left|\nabla^\beta\partial_0^j f^0\right|\right|^2_{H^1(\mathbb{R}^2)} + \sum_{j=0}^{2} \left|\left|\nabla'\partial_0^j f^0\right|\right|^2_{H^{3.5-j}(\mathbb{R}^2)} \tag{4.5.1}$$

Note that we may consider the class of initial data such that $E^0 < \infty$, which has the obvious meaning that $U^0 \in L^\infty(\Omega_\pm)$, $f^0 \in L^\infty(\Gamma)$, $\nabla U^0 \in H^2(\Omega_\pm)$, $\nabla' f^0 \in H^{3.5}(\Gamma)$, which allows us to define $\partial_0^j(U_+^0, U_-^0, f^0)$ for $0 \le j \le 3$ using 4.4.2, and we then require $\partial_0^j f^0 \in H^{4-j}(\Gamma)$ for $1 \le j \le 3$ and $\partial_0^j f^0 \in H^{4.5-j}(\Gamma)$ for $1 \le j \le 2$. (Note that the required regularity of $\partial_0^j U_\pm^0$ for $1 \le j \le 3$ automatically follows from the definition 4.4.2 and the spatial regularity).

We also define the following lower order energy corresponding to the difference of two states which will appear in a statement of stability.

Definition 4.5.2. We define the energy $E_\Delta : (0,T] \to \mathbb{R}_{\ge 0}$ of the difference of the states (U_+^i, U_-^i, f^i) for $i = 1,2$ defined on the time interval $(0,T)$ as

$$E_\Delta(t) = \sum_{\pm}\sum_{0\le|\alpha|\le 2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\alpha(U^1-U^2)\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{0\le|\alpha|\le 2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\alpha(f^1-f^2)\right|\right|^2_{H^1(\Gamma)}$$
$$+ \sum_{0\le|\alpha|\le 1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\alpha(f^1-f^2)\right|\right|^2_{H^{2.5}(\Gamma)}$$

Note that this is effectively the energy $E(t)$ reduced by one order of regularity evaluated at the difference of two solutions. For simplicity we have chosen to require $U^1-U^2 \in L^\infty((0,T);L^2(\Omega_\pm))$ and $f^1-f^2 \in L^\infty((0,T);L^2(\Gamma))$ for $E_\Delta(T) < \infty$ so that we no longer have to deal with the zero order L^∞-in-space terms separately.

Similarly, we define the initial energy E_Δ^0 of the difference of two sets of initial data $(U_+^{0,i}, U_-^{0,i}, f^{0,i})$ for $i = 1,2$ as

$$E_\Delta^0 = \sum_{\pm}\sum_{0\le|\beta|+j\le 2} \left|\left|\nabla^\beta(\partial_0^{j,1}U^{0,1} - \partial_0^{j,2}U^{0,2})\right|\right|^2_{L^2(\Omega_\pm)}$$
$$+ \sum_{0\le|\beta|+j\le 2} \left|\left|\nabla^\beta(\partial_0^{j,1}f^{0,1} - \partial_0^{j,2}f^{0,2})\right|\right|^2_{H^1(\Gamma)} + \sum_{0\le|\beta|+j\le 1} \left|\left|\nabla^\beta(\partial_0^{j,1}f^{0,1} - \partial_0^{j,2}f^{0,2})\right|\right|^2_{H^{2.5}(\Gamma)}$$

where $\partial_0^{j,i}$ denotes the initial time derivatives of order j as defined in 4.4.2 acting on the initial data $(U_+^{0,i}, U_-^{0,i}, f^{0,i})$ for $i = 1, 2$.

4.6 Statement of the Theorem in the Fixed Domains

Theorem 4.6.1. *Let (U_+^0, U_-^0, f^0) be initial data as described in 4.4.1 with energy $E^0 < \infty$. Assume that this initial data satisfies the compatibility conditions* (4.4.6)-(4.4.7) *up to order* 2*, and note that these conditions make sense up to order* 2 *since we have $E^0 < \infty$. Assume also that the initial data satisfies*

$$\inf_{x\in\Omega_\pm} \rho_\pm^0(x) =: \delta^0 > 0 \tag{4.6.1}$$

$$\inf_{x\in\mathbb{R}^3} J^{\psi^0} =: \kappa^0 > 0 \tag{4.6.2}$$

where

$$\psi^0 := L^a f^0$$

Then there exists a time $T^0 > 0$ and a solution (U_+, U_-, f) of equations in the fixed domains on the time interval $(0, T^0)$ with lifting operator L^a, as in definition 4.3.1. Moreover,

$$\begin{aligned} E(T^0) &\le C^0 \\ \rho_\pm &\ge \frac{\delta^0}{2} \text{ in } (0,T^0)\times\Omega_\pm \\ J^\psi &\ge \frac{1}{2}\kappa^0 \text{ in } (0,T^0)\times\mathbb{R}^3 \end{aligned}$$

The time $T^0 > 0$ is bounded below as follows.

$$T^0 \ge T(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) > 0$$

where $T(\cdot)$ is a smooth decreasing function. Similarly, the constant $C^0 > 0$ is bounded above as follows.

$$C^0 \le C(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) < \infty$$

where $C(\cdot)$ is a smooth increasing function. Note that these bounds are independent of the constant a in the lifting operator L^a.

In addition, the solution (U_+, U_-, f) is unique on the time interval $(0, T^0)$ in the sense that it is the only solution of the equations given in 4.3.1 on the time interval $(0, T^0)$ with initial data (U_+^0, U_-^0, f^0) satisfying the following properties

$$\begin{aligned} E(T^0) &< \infty \\ \inf_{t\in(0,T^0)} \inf_{x\in\Omega_\pm} \rho_\pm &> 0 \\ \inf_{t\in(0,T^0)} \inf_{x\in\mathbb{R}^3} J^\psi &> 0 \end{aligned}$$

In fact, the following stability statement is satisfied. Given two sets of initial data $(U_+^{0,i}, U_-^{0,i}, f^{0,i})$ for $i = 1, 2$ satisfying the hypotheses above, let $T^{0,1}$ and $T^{0,2}$ be the associated existence times as given above and set $T^0 = \min\{T^{0,1}, T^{0,2}\}$. Let (U_+^1, U_-^1, f^1) and (U_+^2, U_-^2, f^2) be two solutions of *the equations in the fixed domains as defined in 4.3.1 on the time interval* $(0, T^0)$ *with initial data* $(U_+^{0,i}, U_-^{0,i}, f^{0,i})$ *respectively, with the properties*

$$E^i(T^0) =: C^i < \infty$$
$$\inf_{t\in(0,T^0)} \inf_{x\in\Omega_\pm} \rho_\pm^i =: \delta^i > 0$$
$$\inf_{t\in(0,T^0)} \inf_{x\in\mathbb{R}^3} J^{\psi^i} =: \kappa^i > 0$$

for $i = 1, 2$*, where the superscript* ^{i} *for* $i = 1, 2$ *is used to denote quantities associated with the solutions* (U_+^i, U_-^i, f^i)*. Assume also that* $U^{0,1} - U^{0,2} \in L^2(\Omega_\pm)$ *and* $f^{0,1} - f^{0,2} \in L^2(\Gamma)$ *so that*

$$E_\Delta^0 < \infty$$

where the initial difference energy E_Δ^0 *is defined in 4.5.2. Then*

$$E_\Delta(t) \leq C^{1,2} E_\Delta^0 \exp(C^{1,2} t)$$

for $t \in (0, T^0)$ *where* $E_\Delta(t)$ *is defined in 4.5.2 and the constant* $C^{1,2}$ *is bounded above as follows.*

$$C^{1,2} \leq F(C^1, C^2, \frac{1}{\delta^1}, \frac{1}{\delta^2}, \frac{1}{\kappa^1}, \frac{1}{\kappa^2})$$

for some smooth increasing function $F(\cdot)$.

Proof. See chapters 6-15. □

5. Fixing the Domains

In this section we show how the equations and theorem in the fixed domains relate to the equations and theorem in the original moving domains. See 3.1 for the motivation of this change of coordinates.

5.1 The Change of Coordinates

Lemma 5.1.1. *Let $f \in C^1((0,T) \times \mathbb{R}^2)$ and let $a \geq 0$ be a constant. Assume that*

$$\inf_{t \in (0,T)} \inf_{x \in \mathbb{R}^3} J^\psi \geq \kappa > 0$$

where where $\psi = L^a f$, $J^\psi := 1 + \partial_{x_3}\psi$ and the lifting operator L^a is defined in 4.2.2. Define $\theta_\pm \in C^1((0,T) \times \Omega_\pm)$ by

$$\theta_\pm(t,x) = (t, x', x_3 + \psi(t,x))$$

Then $\theta_\pm : (0,T) \times \Omega_\pm \to \Omega_\pm^T$ are diffeomorphisms, where the domains $\Omega_\pm^T$ are defined as in 2.1.1 in terms of f.

Moreover, if $w \in C^1(\Omega_\pm^T)$ and we set $\tilde{w} = w \circ \theta$, then

$$(\partial w) \circ \theta = \left(\partial - \frac{1}{J^\psi} \partial\psi \partial_{x_3} \right) \tilde{w} =: \partial^\psi \tilde{w} \tag{5.1.1}$$

In particular, note that if in addition $v \in C^0(\Omega_\pm^T)$ is a vector-valued function and $\tilde{v} := v \circ \theta$, then

$$((\partial_t + v \cdot \nabla) w) \circ \theta = (\partial_t + \tilde{v}^\psi \cdot \nabla) \tilde{w} \tag{5.1.2}$$

where

$$\tilde{v}^\psi := (\tilde{v}', \frac{1}{J^\psi}(\tilde{v}_3 - \tilde{v}' \cdot \nabla' \psi - \partial_t \psi))$$

Proof. Note that $J^\psi := 1 + \partial_{x_3}\psi \geq \kappa > 0$ implies $x_3 \mapsto x_3 + \psi(t,x)$ is a strictly increasing function of x_3 for fixed (t,x'). Hence, since $(t,x') \mapsto (t,x')$, we have that $\theta_\pm$ biject with their images. Now, since $\psi|_{x_3=0} = f$, we have that $\theta_\pm|_{x_3=0} : (0,T) \times \Gamma \to \Gamma^T$ bijectively, and since $x_3 \mapsto x_3 + \psi(t,x)$ is strictly increasing, $\theta_\pm : (0,T) \times \Omega_\pm \to \Omega_\pm^T$ bijectively. From the chain rule we have

$$\partial\theta_\pm = I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_t\psi & \partial_{x_1}\psi & \partial_{x_2}\psi & \partial_{x_3}\psi \end{pmatrix}$$

Note that the determinant is $J^\psi > 0$, so $\partial\theta_\pm$ is invertible, and $\theta_\pm$ are diffeomorphisms as claimed.

By computing $(\partial\theta_\pm)^{-1}$, we obtain

$$\partial(\theta_\pm^{-1}) = I - \frac{1}{J^\psi}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \partial_t\psi & \partial_{x_1}\psi & \partial_{x_2}\psi & \partial_{x_3}\psi \end{pmatrix} \circ \theta_\pm^{-1}$$

The coordinate change relation (5.1.1) follows from applying the chain rule to $w = \tilde{w} \circ \theta^{-1}$. Using this, we compute

$$\begin{aligned} ((\partial_t + v\cdot\nabla)w)\circ\theta &= (\partial_t + \tilde{v}\cdot\nabla)\tilde{w} - \frac{1}{J^\psi}(\partial_t\psi + \tilde{v}\cdot\nabla\psi)\partial_{x_3}\tilde{w} \\ &= (\partial_t + \tilde{v}'\cdot\nabla')\tilde{w} + \frac{1}{J^\psi}(\tilde{v}_3 - \partial_t\psi - \tilde{v}'\cdot\nabla'\psi)\partial_{x_3}\tilde{w} \\ &= (\partial_t + \tilde{v}^\psi\cdot\nabla)\tilde{w} \end{aligned}$$

where

$$\tilde{v}^\psi := (\tilde{v}', \frac{1}{J^\psi}(\tilde{v}_3 - \tilde{v}'\cdot\nabla'\psi - \partial_t\psi))$$

which is (5.1.2). □

Proposition 5.1.2. *Let $T > 0$, $a \geq 0$ and let $f \in C_b^1((0,T')\times\mathbb{R}^2)$ with $\nabla' f \in C_b^1((0,T')\times\mathbb{R}^2)$ for all $T' \in (0,T)$. Assume that*

$$\inf_{t\in(0,T)}\inf_{x\in\mathbb{R}^3} J^\psi \geq \kappa > 0$$

where $\psi = L^a f$. Let $\theta_\pm : (0,T)\times\Omega_\pm \to \Omega_\pm^T$ be defined in terms of f as in 5.1.1. Let $\tilde{U}_\pm \in C_b^1((0,T')\times\Omega_\pm)$ for all $T' \in (0,T)$ and let

$$U_\pm = \tilde{U}_\pm \circ \theta^{-1}$$

Then $(\tilde{U}_+, \tilde{U}_-, f)$ is a solution of the equations in the fixed domains $\Omega_\pm$ with lifting operator L^a on the time interval $(0,T)$, as in definition 4.3.1, if and only if (U_+, U_-, f) is a solution of the equations in the moving domains $\Omega_\pm^T$ on the time interval $(0,T)$, as in definition 2.1.1.

Proof. Firstly, note that $\rho_\pm > 0$ if and only if $\tilde{\rho}_\pm > 0$ since they have the same image.

Using the coordinate change relations (5.1.1)-(5.1.2), we immediately see that $(\tilde{U}_+, \tilde{U}_-, f)$ satisfies the equations (4.3.2)-(4.3.3) in $(0,T)\times\Omega_\pm$ if and only if (U_+, U_-, f) satisfies the equations (2.1.2)-(2.1.3) in $\Omega_\pm^T$. We note that $U_\pm|_{\Gamma(t)} = \tilde{U}_\pm|_\Gamma$, which immediately implies that $(\tilde{U}_+, \tilde{U}_-, f)$ satisfies the equations (4.3.4)-(4.3.5) on $(0,T)\times\Gamma$ if and only if (U_+, U_-, f) satisfies the equations (2.1.4)-(2.1.5) on Γ^T.

This completes the proof. □

Proposition 5.1.3. *Let (U_+^0, U_-^0, f^0) be initial data for the main theorem 2.4.1.*

Let $\psi^0 := L^a f^0$ as defined by (4.2.2) and assume $J^{\psi^0} > 0$. Define $\tilde{U}_\pm^0 := U_\pm^0 \circ \theta_\pm^0$ where $\theta_\pm^0 : \Omega_\pm \to \Omega_\pm^0$ are defined by $x \mapsto (x', x_3 + \psi^0(x))$ and are diffeomorphisms as shown in 5.1.2.

Then $(\tilde{U}_+^0, \tilde{U}_-^0, f^0)$ are initial data for the theorem in the fixed domains 4.6.1.

Moreover, let us write ∂_0^j for the initial time derivatives associated with the equations in the moving domains as defined in 2.2.2 and $\tilde{\partial}_0^j$ for the initial time derivatives associated with the equations in the fixed domains as defined in 4.4.2.

Let $H(p_\pm^0, u_\pm^0, s_\pm^0, f^0)$ be a differentiable expression involving the initial data (U_+^0, U_-^0, f^0) and suppose we can write

$$H(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0 = \tilde{H}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0)$$

where $\tilde{H}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0)$ is a differentiable expression involving the initial data $(\tilde{U}_+^0, \tilde{U}_-^0, f^0)$. Then for $j \geq 0$, we have

$$\tilde{\partial}_0^j(\tilde{H}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0)) = \partial_0^j(H(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0) \tag{5.1.3}$$

Proof. Since we have assumed $\nabla' f^0$ is continuous and bounded, it is clear that $\theta_\pm^0 : \Omega_\pm \to \Omega_\pm^0$ are diffeomorphisms and that $(\tilde{U}_+^0, \tilde{U}_-^0, f^0)$ are initial data for the theorem in the fixed domains.

It is clear that (5.1.3) holds for $j = 0$. Now for $j = 1$ we have

$$\partial_0^1 f^0 = u_+^0\big|_{\Gamma^0} \cdot n^0 = \tilde{u}_+^0\big|_{\Gamma} \cdot n^0 = \tilde{\partial}_0^1 f^0$$

Also

$$\begin{aligned}
\partial_0^1(p_\pm^0 \circ \theta_\pm^0) &= (\partial_0^1 p_\pm^0) \circ \theta_\pm^0 + \partial_0^1 \psi^0 (\partial_{x_3} p_\pm^0) \circ \theta_\pm^0 \\
&= (-(u_\pm^0 \cdot \nabla) p_\pm^0 - \rho_\pm^0 (c_\pm^0)^2 \nabla \cdot u_\pm^0) \circ \theta_\pm^0 + \frac{1}{J^{\psi^0}} \partial_0^1 \psi^0 (\partial_{x_3} p_\pm^0 \circ \theta_\pm^0) \\
&= -((\tilde{u}_\pm^0)^{\psi^0} \cdot \nabla) \tilde{p}_\pm^0 - \tilde{\rho}_\pm^0 (\tilde{c}_\pm^0)^2 \nabla^{\psi^0} \tilde{u}_\pm^0 \\
&= \tilde{\partial}_0^1 \tilde{p}_\pm^0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\partial_0^1(u_\pm^0 \circ \theta_\pm^0) &= \tilde{\partial}_0^1 \tilde{u}_\pm^0 \\
\partial_0^1(s_\pm^0 \circ \theta_\pm^0) &= \tilde{\partial}_0^1 \tilde{s}_\pm^0
\end{aligned}$$

Hence by definition we have that (5.1.3) holds for $j = 1$. Now from the definition of initial time derivatives, they have the property that $\partial_0^{j+1} \cdot = \partial_0^1(\partial_0^j \cdot)$ and $\tilde{\partial}_0^{j+1} \cdot = \tilde{\partial}_0^1(\tilde{\partial}_0^j \cdot)$ provided we write $\partial_0^j \cdot$ as a differentiable expression involving the initial data (U_+^0, U_-^0, f^0) and $\tilde{\partial}_0^j \cdot$ as a differentiable expression involving the initial data $(\tilde{U}_+^0, \tilde{U}_-^0, f^0)$. We assume inductively that (5.1.3) holds true for some $j \geq 1$ and try to prove it holds for $j + 1$. Indeed,

$$\begin{aligned}
\partial_0^{j+1}(H(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0) &= \partial_0^1(\partial_0^j(H(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0)) \\
&= \partial_0^1(G(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0)
\end{aligned}$$

where

$$\begin{aligned}
G(p_\pm^0, u_\pm^0, s_\pm^0, f^0) &= \partial_0^j(H(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0) \circ (\theta_\pm^0)^{-1} \\
&= \tilde{\partial}_0^j(\tilde{H}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0)) \circ (\theta_\pm^0)^{-1} \\
&=: \tilde{G}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0) \circ (\theta_\pm^0)^{-1}
\end{aligned}$$

where we have used the inductive assumption. Hence by the above

$$\begin{aligned}\partial_0^{j+1}(H(p_\pm^0, u_\pm^0, s_\pm^0, f^0) \circ \theta_\pm^0) &= \tilde{\partial}_0^1(\tilde{G}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0)) \\ &= \tilde{\partial}_0^1(\tilde{\partial}_0^j(\tilde{H}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0))) \\ &= \tilde{\partial}_0^{j+1}(\tilde{H}(\tilde{p}_\pm^0, \tilde{u}_\pm^0, \tilde{s}_\pm^0, f^0))\end{aligned}$$

as required. □

5.2 Reduction to the Case of Fixed Domains

Proposition 5.2.1. *The theorem 2.4.1 in the moving domains follows from the theorem 4.6.1 in the fixed domains.*

Proof. Assume that 4.6.1 holds. We aim to show that 2.4.1 holds.

Let (U_+^0, U_-^0, f^0) be initial data for the main theorem 2.4.1, which satisfy the hypotheses of the main theorem 2.4.1.

Let $\psi^0 := L^a f^0$ as defined by (4.2.2), where we set

$$a = ||f^0||_{L^\infty(\Gamma)}$$

Note from the definition of L^a this automatically implies

$$||\partial_{x_3}\psi^0||_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{3}$$

so

$$J^{\psi^0} := 1 + \partial_{x_3}\psi^0 \geq \kappa^0 > 0 \text{ in } \mathbb{R}^3$$

where in this case we can take $\kappa^0 = \frac{2}{3}$. Define $\tilde{U}_\pm^0 := U_\pm^0 \circ \theta_\pm^0$ where $\theta_\pm^0 : \Omega_\pm \to \Omega_\pm^0$ are defined by $x \mapsto (x', x_3 + \psi^0(x))$ and are diffeomorphisms as shown in 5.1.1.

We claim that $(\tilde{U}_+^0, \tilde{U}_-^0, f^0)$ are initial data for the theorem on the fixed domains 4.6.1 which satisfy the hypotheses of the theorem on the fixed domains 4.6.1.

Let us write ∂_0^j for the initial time derivatives associated with the equations in the moving domains as defined in 2.2.2 and $\tilde{\partial}_0^j$ for the initial time derivatives associated with the equations in the fixed domains as defined in 4.4.2. Note that, using 5.1.3,

$$\begin{aligned}\tilde{\partial}_0^j \tilde{U}_\pm^0|_\Gamma &= \partial_0^j(U_\pm^0 \circ \theta_\pm^0)|_{x_3=0} \\ &= \partial_0^j((U_\pm^0 \circ \theta_\pm^0)|_{x_3=0}) \\ &= \partial_0^j(U_\pm^0|_{\Gamma^0})\end{aligned}$$

Using this, we see that the compatibility conditions (4.4.6)- (4.4.7) up to order 2 follow immediately from the compatibility conditions (2.2.5)- (2.2.6) up to order 2.

Now, note that

$$\left|\left|\nabla'\psi^0\right|\right|^2_{H^2(\mathbb{R}^3)} \leq C\left|\left|\nabla' f^0\right|\right|^2_{H^2(\mathbb{R}^2)} \leq CE^0$$
$$\sum_{k=0}^{2}\left|\left|\partial_{x_3}^{k+1}\psi^0\right|\right|^2_{L^\infty(\mathbb{R}^3)} \leq C\left|\left|f^0\right|\right|^2_{L^\infty(\mathbb{R}^2)} \leq CE^0$$
$$\det(\nabla\theta^0_\pm) = J^{\psi^0} \geq \frac{2}{3}$$

Using the definition 4.4.2 to write the initial time derivatives of $\tilde{U}^0_\pm$ in terms of space derivatives, then applying the change of coordinates estimate A.3.7 with $m = 2 > \frac{d}{p} = \frac{3}{2}$, we obtain

$$\sum_{\pm}\sum_{1\leq|\beta|+j\leq 3}\left|\left|\nabla^\beta\tilde{\partial}_0^j\tilde{U}^0\right|\right|^2_{\Omega_\pm} \leq \tilde{F}(E^0) < \infty$$

where E^0 denotes the energy of (U^0_+, U^0_-, f^0) as defined in 2.3.1 and $\tilde{F}$ is a smooth increasing function. We also have by 5.1.3 that $\tilde{\partial}_0^j f^0 = \partial_0^j f^0$ for $0 \leq j \leq 3$, hence we obtain

$$\tilde{E}^0 \leq \tilde{F}(E^0) < \infty$$

where $\tilde{F}$ is a smooth increasing function.

Clearly, we have $\inf_{x\in\Omega_\pm}\tilde{\rho}^0_\pm = \delta^0 > 0$ since $\tilde{\rho}^0_\pm$ and $\rho^0_\pm$ have the same image. Now we let $T^0 > 0$ be given by 4.6.1 depending on this initial data.

Firstly, we prove the existence part of 2.4.1. Using 4.6.1 we obtain a solution $(\tilde{U}_+, \tilde{U}_-, f)$ on a time interval $(0, T^0)$ of the system (4.3.1)-(4.3.6) with initial data $(\tilde{U}^0_+, \tilde{U}^0_-, f^0)$ and moreover

$$\begin{aligned}\tilde{E}(T^0) &\leq \tilde{C}^0\\ \tilde{\rho}_\pm &\geq \frac{\delta^0}{2} \text{ in } (0,T^0)\times\Omega_\pm\\ J^\psi &\geq \frac{1}{2}\kappa^0 = \frac{1}{3} \text{ in } (0,T^0)\times\mathbb{R}^3\end{aligned}$$

where $\tilde{E}$ denotes the energy defined in 4.5.1.

The time $T^0 > 0$ is bounded below as follows.

$$T^0 \geq \tilde{T}(\tilde{E}^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) > 0$$

where $\tilde{T}(\cdot)$ is a smooth decreasing function. Similarly, the constant $\tilde{C}^0 > 0$ is bounded above as follows.

$$\tilde{C}^0 \leq \tilde{C}(\tilde{E}^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) < \infty$$

where $\tilde{C}(\cdot)$ is a smooth increasing function.

Now, we define $\theta_\pm : (0,T^0)\times\Omega_\pm \to \Omega^{T^0}_\pm$ by $(t,x) \mapsto (t, x', x_3 + \psi(x))$, which are diffeomorphisms according to 5.1.1. We set $U_\pm = \tilde{U}_\pm \circ \theta_\pm^{-1}$ and according to 5.1.2, we obtain that (U_+, U_-, f) is a vortex sheet solution of the Euler equations with surface tension (2.1.1)-(2.1.5) on the time interval $(0, T^0)$.

We note that since $f|_{t=0} = f^0$, we have that $\theta^0_\pm = \theta_\pm|_{t=0}$. Hence $(U_+|_{t=0}, U_-|_{t=0}, f|_{t=0}) = (\tilde{U}^0_+\circ(\theta^0_+)^{-1}, \tilde{U}^0_-\circ(\theta^0_-)^{-1}, f^0) = (U^0_+, U^0_-, f^0)$. Thus (U_+, U_-, f) is a vortex sheet solution of the Euler equations with surface tension (2.1.1)-(2.1.5) on the time interval $(0, T^0)$ with initial data (U^0_+, U^0_-, f^0).

Clearly, we have $\rho_\pm \geq \frac{\delta^0}{2}$ in $\Omega_\pm^{T^0}$ as $\rho_\pm$ and $\tilde{\rho}_\pm$ have the same image. Finally, we note that

$$J^\psi := 1 + \partial_{x_3}\psi \geq \frac{1}{3} \text{ in } (0, T^0) \times \mathbb{R}^3$$
$$\left|\left|\partial^j \partial' \psi\right|\right|^2_{H^{2-j}(\mathbb{R}^3)} \leq C \left|\left|\partial^j \partial' f\right|\right|^2_{H^{2-j}(\mathbb{R}^2)} \leq C\tilde{C}^0 \text{ for } 0 \leq j \leq 2$$
$$\sum_{k=0}^{2} \left|\left|\partial_{x_3}^{k+1}\psi\right|\right|^2_{L^\infty(\mathbb{R}^3)} \leq C \left|\left|f\right|\right|^2_{L^\infty(\mathbb{R}^2)} \leq C\tilde{C}^0$$

hence we may apply the change of coordinates estimate A.3.7 to obtain

$$E(T^0) \leq F(\tilde{C}^0) =: C^0$$

where E denotes the energy defined in 2.3.1 and F is a smooth increasing function.

We now note that the time $T^0 > 0$ is bounded below as follows.

$$T^0 \geq \tilde{T}(\tilde{E}^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) \geq \tilde{T}(\tilde{F}(E^0), \frac{1}{\delta^0}, \frac{3}{2}) =: T(E^0, \frac{1}{\delta^0}) > 0$$

and, from its definition, $T(\cdot)$ is a smooth decreasing function. Similarly, the constant $C^0 > 0$ is bounded above as follows.

$$C^0 = F(\tilde{C}^0) \leq F(\tilde{C}(\tilde{E}^0, \frac{1}{\delta^0}, \frac{3}{2})) \leq F(\tilde{C}(\tilde{F}(E^0), \frac{1}{\delta^0}, \frac{3}{2})) =: C(E^0, \frac{1}{\delta^0}) < \infty$$

and, from its definition, $C(\cdot)$ is a smooth increasing function. This completes the proof of the existence part of 2.4.1.

Now we intend to prove the uniqueness part of 2.4.1. We let (U_+^1, U_-^1, f^1) and (U_+^2, U_-^2, f^2) be two solutions of the equations in the moving domains as defined in 2.1.1 on the time interval $(0, T^0)$ with initial data (U_+^0, U_-^0, f^0), with the properties

$$E^i(T^0) =: C^i < \infty$$
$$\inf_{(t,x) \in \Omega_\pm^{T^0, i}} \rho_\pm^i =: \delta^i > 0$$

for $i = 1, 2$, where the superscript ^{i} for $i = 1, 2$ is used to denote quantities associated with the solutions (U_+^i, U_-^i, f^i), and the energy $E(t)$ is defined in 2.3.1. In particular, we have that $U_\pm^i$ are defined on the moving domains

$$\Omega_+^i(t) = \{x \in \mathbb{R}^3 : x_3 > f^i(t, x')\}$$
$$\Omega_-^i(t) = \{x \in \mathbb{R}^3 : x_3 < f^i(t, x')\}$$
$$\Omega_\pm^{T^0, i} = \{(t, x) \in (0, T^0) \times \mathbb{R}^3 : x \in \Omega_\pm^i(t)\}$$

To prove the statement of uniqueness, we need to show that $(U_+^1, U_-^1, f^1) = (U_+^2, U_-^2, f^2)$ on the time interval $(0, T^0)$.

We set $\psi^i = L^a f^i$ where

$$a = \max_{i \in \{1,2\}} \sup_{t \in (0, T^0)} \left|\left|f^i\right|\right|_{L^\infty(\Gamma)} < \infty$$

Note from the definition of L^a this automatically implies

$$\begin{aligned}||\partial_{x_3}\psi^i||_{L^\infty(\mathbb{R}^3)} &\leq \frac{1}{3(1+a)}\,||f^i||_{L^\infty(\Gamma)}\\ &\leq \frac{1}{3}\end{aligned}$$

Thus we have

$$\inf_{t\in(0,T^0)}\inf_{x\in\mathbb{R}^3} J^{\psi^i} =: \kappa^i \geq \frac{2}{3} > 0$$

where $J^{\psi^i} = 1+\partial_{x_3}\psi^i$. Thus we know that $\theta^i_\pm : (0,T^0)\times\Omega_\pm \to \Omega^{T^0,i}_\pm$ defined by $(t,x)\mapsto(t,x',x_3+\psi^i(t,x))$ are diffeomorphisms as shown in 5.1.1. Now we set

$$\tilde{U}^i_\pm = U^i_\pm \circ \theta^i_\pm$$

so that $\tilde{U}^i_\pm$ are defined on the fixed domains $(0,T^0)\times\Omega_\pm$. Note that this implies that

$$U^i_\pm = \tilde{U}^i_\pm \circ (\theta^i_\pm)^{-1}$$

so using 5.1.2 we may conclude that $(\tilde{U}^i_+,\tilde{U}^i_-,f^i)$ are solutions of the equations on the fixed domains as defined in 4.3.1 with initial data $(\tilde{U}^0_+,\tilde{U}^0_-,f^0)$. Clearly we have

$$\begin{gathered}\inf_{(t,x)\in\Omega^{T^0,i}_\pm}\tilde{\rho}^i_\pm > 0\\ \inf_{t\in(0,T^0)}\inf_{x\in\mathbb{R}^3} J^{\psi^i} > 0\end{gathered}$$

and applying the change of coordinates estimate A.3.7 we have

$$\tilde{E}^i(T^0) < \infty$$

where $\tilde{E}^i$ denotes the energy associated with $(\tilde{U}^i_+,\tilde{U}^i_-,f^i)$ as defined in 4.5.1. Thus we may apply the uniqueness part of 4.6.1 to obtain that

$$\begin{aligned}\tilde{U}^1_\pm &= \tilde{U}^2_\pm \text{ in } (0,T^0)\times\Omega_\pm\\ f^1 &= f^2 \text{ on } (0,T^0)\times\Gamma\end{aligned}$$

Note that $f^1 = f^2$ implies $\theta^1_\pm = \theta^2_\pm =: \theta_\pm$, hence we may evaluate the first line at $\theta^{-1}_\pm(t,x)$ to obtain that

$$\begin{aligned}U^1_\pm &= U^2_\pm \text{ in } \Omega^{T^0}_\pm\\ f^1 &= f^2 \text{ on } (0,T^0)\times\mathbb{R}^2\end{aligned}$$

which proves the uniqueness part of 2.4.1. □

The sections which follow will be devoted to proving the theorem 4.6.1 on the fixed domains, from which we have shown the main theorem 2.4.1 follows.

6. Smoothing the Initial Data

In this section we aim to show that it is sufficient to prove the theorem 4.6.1 for smooth initial data. The motivation behind smoothing the initial data and a sketch of how to smooth the initial data whilst preserving the compatibility conditions are given in 3.2. The full details are presented below.

6.1 Constructing Smooth Compatible Initial Data

Definition 6.1.1. For convenience, in this section we will write

$$V^0_\pm = (U^0_\pm, f^0, \chi(\frac{x_3}{3(1+a)}), u^0_+|_\Gamma, \nabla' U^0_\pm, \nabla(u^0_\pm)', \nabla' f^0, \nabla\chi(\frac{x_3}{3(1+a)}))$$

where we recall $\psi^0 = L^a f^0 := \chi(\frac{x_3}{3(1+a)}) f^0$. Note that we have included the terms $f^0, u^0_+|_\Gamma, \chi(\frac{x_3}{3(1+a)})$ whose derivatives are part of the make-up of the initial time derivatives of ψ^0 (and the terms involving χ are obviously harmless).

Lemma 6.1.2. *Let $j \geq 0$. If j is even then*

$$\begin{aligned}
\partial_0^j p^0_\pm &= \frac{F^j_p(\rho^0_\pm, c^0_\pm, 1 + |\nabla'\psi^0|^2)}{(J^{\psi^0})^j} \partial^j_{x_3} p^0_\pm + G^j_p(\{\nabla^\alpha V^0_\pm : 0 \leq |\alpha| \leq j-1\}) \\
&\quad + ((u^0_\pm)^{\psi^0})_3 H^j_p(\{\nabla^\alpha (V^0_\pm, \nabla U^0_\pm) : 0 \leq |\alpha| \leq j-1\}) \\
\partial_0^j u^0_\pm &= \frac{F^j_u(\rho^0_\pm, c^0_\pm, 1 + |\nabla'\psi^0|^2)}{(J^{\psi^0})^j} (-\nabla'\psi^0, 1) \partial^j_{x_3} (u^0)_{3\pm} + G^j_u(\{\nabla^\alpha V^0_\pm : 0 \leq |\alpha| \leq j-1\}) \\
&\quad + ((u^0_\pm)^{\psi^0})_3 H^j_u(\{\nabla^\alpha (V^0_\pm, \nabla U^0_\pm) : 0 \leq |\alpha| \leq j-1\})
\end{aligned}$$

where F^j_p, F^j_u, G^j_p, G^j_u and H^j_p, H^j_u are some functions which are smooth in their arguments for $\rho^0_\pm, c^0_\pm >$ 0, and moreover F^j_p, F^j_u are both the quotient of two polynomials with positive coefficients. If j is odd then

$$\begin{aligned}
\partial_0^j p^0_\pm &= -\frac{F^j_p(\rho^0_\pm, c^0_\pm, 1 + |\nabla'\psi^0|^2)}{(J^{\psi^0})^j} \partial^j_{x_3} (u^0)_{3\pm} + G^j_p(\{\nabla^\alpha V^0_\pm : 0 \leq |\alpha| \leq j-1\}) \\
&\quad + ((u^0_\pm)^{\psi^0})_3 H^j_p(\{\nabla^\alpha (V^0_\pm, \nabla U^0_\pm) : 0 \leq |\alpha| \leq j-1\}) \\
\partial_0^j u^0_- &= -\frac{F^j_u(\rho^0_\pm, c^0_\pm, 1 + |\nabla'\psi^0|^2)}{(J^{\psi^0})^j} (-\nabla'\psi^0, 1) \partial^j_{x_3} p^0_\pm + G^j_u(\{\nabla^\alpha V^0_\pm : 0 \leq |\alpha| \leq j-1\}) \\
&\quad + ((u^0_\pm)^{\psi^0})_3 H^j_u(\{\nabla^\alpha (V^0_\pm, \nabla U^0_\pm) : 0 \leq |\alpha| \leq j-1\})
\end{aligned}$$

Proof. Let us use F, G and H to denote generic functions of the type described above for $F^j_p, F^j_u, G^j_p, G^j_u$ and H^j_p, H^j_u. We proceed by induction on $j \geq 0$. The case $j = 0$ is clear. Now we assume the formulas are true up to order $j-1$ for some $j \geq 1$. Using the definition 4.4.2 of the initial time derivatives, we have

$$\partial_0^j p_\pm^0 = -\partial_0^{j-1}((u_\pm^{\psi^0}\cdot\nabla)p_\pm^0 + (\nabla^{\psi^0})'\cdot(u_\pm^0)' + \frac{\rho_\pm^0(c_\pm^0)^2}{J^{\psi^0}}(-\nabla'\psi^0,1)\cdot\partial_{x_3}u_\pm^0)$$
$$= -(u_\pm^{\psi^0}\cdot\nabla)\partial_0^{j-1}p_\pm^0 - \frac{\rho_\pm^0(c_\pm^0)^2}{J^{\psi^0}}(-\nabla'\psi^0,1)\cdot\partial_{x_3}\partial_0^{j-1}u_\pm^0 + G(\{\nabla^\alpha V_\pm^0 : 0\leq|\alpha|\leq j-1\})$$

Applying the induction hypothesis to $\partial_0^{j-1}p_\pm^0$ and $\partial_0^{j-1}(u^0)_{3\pm}$ we immediately obtain the claimed expression for $\partial_0^j p_\pm^0$. Similarly, we have

$$\partial_0^j u_\pm^0 = -\partial_0^{j-1}((u_\pm^{\psi^0}\cdot\nabla)u_\pm^0 + \frac{1}{\rho_\pm^0}\nabla' p_\pm^0 + \frac{1}{J^{\psi^0}\rho_\pm^0}(-\nabla'\psi^0,1)\partial_{x_3}p_\pm^0)$$
$$= -(u_\pm^{\psi^0}\cdot\nabla)\partial_0^{j-1}u_\pm^0 - \frac{1}{J^{\psi^0}\rho_\pm^0}(-\nabla'\psi^0,1)\partial_{x_3}\partial_0^{j-1}p_\pm^0 + G(\{\nabla^\alpha V_\pm^0 : 0\leq|\alpha|\leq j-1\})$$

Applying the induction hypothesis to $\partial_0^{j-1}p_\pm^0$ and $\partial_0^{j-1}u_\pm^0$ we immediately obtain the claimed expression for $\partial_0^j u_\pm^0$. This completes the induction and proves the result. □

Lemma 6.1.3. *For $j \geq 0$ even, we have*

$$\partial_0^j(u_+^0|_\Gamma\cdot n^0) = \frac{F_u^j(\rho_+^0, c_+^0, 1+\left|\nabla'\psi^0\right|^2)\left|n^0\right|^2}{(J^{\psi^0})^j}\partial_{x_3}^j(u^0)_{3+}|_\Gamma + \sum_{k+l=j,k\neq j}\partial_0^k(u_\pm^0|_\Gamma)'\cdot\partial_0^l n^0$$
$$+ G_u^j(\{\nabla^\alpha V_+^0 : 0\leq|\alpha|\leq j-1\})|_\Gamma\cdot n^0$$

and for $j \geq 0$ odd we have

$$\partial_0^j(u_+^0|_\Gamma\cdot n^0) = \frac{F_u^j(\rho_+^0, c_+^0, 1+\left|\nabla'\psi^0\right|^2)\left|n^0\right|^2}{(J^{\psi^0})^j}\partial_{x_3}^j(p^0)_\pm|_\Gamma + \sum_{k+l=j,k\neq j}\partial_0^k(u_+^0|_\Gamma)'\cdot\partial_0^l n^0$$
$$+ G_u^j(\{\nabla^\alpha V_+^0 : 0\leq|\alpha|\leq j-1\})|_\Gamma\cdot n^0$$

Moreover, if the initial data satisfies the compatibility condition $[u^0\cdot n^0] = 0$ then the above also holds with $+$ replaced by $-$.

Proof. Using the chain rule and the definition of the initial time derivatives,

$$\partial_0^j(u_\pm^0\cdot n^0) = \partial_0^j u_\pm^0\cdot n^0 + \sum_{k+l=j,k\neq j}\partial_0^k(u_\pm^0)'\cdot\partial_0^l n^0$$

Now we simply use the above lemma 6.1.2 to write the highest order initial time derivative of u^0 on the right hand side in terms of $\partial_{x_3}^j u_{3\pm}^0$ or $\partial_{x_3}^j p_\pm^0$. We use the fact that when restricting to Γ we have $((u_\pm^0)^{\psi^0})_{3\pm} = u_\pm^0\cdot n^0 - \partial_0^1 f^0$ which by definition of $\partial_0^1 f^0$ is equal to $u_\pm^0\cdot n^0 - u_+^0\cdot n^0$. □

Proposition 6.1.4. *Let (U_+^0, U_-^0, f^0) be initial data which satisfy the hypotheses of theorem 4.6.1, so in particular satisfy the compatibility conditions 4.4.3 up to order 2. Then for $\epsilon > 0$ there are functions $U_{\epsilon\pm}^0 = (p_{\epsilon\pm}^0, u_{\epsilon\pm}^0, s_{\epsilon\pm}^0) \in C^\infty(\Omega_\pm), f_\epsilon^0 \in C^\infty(\mathbb{R}^2)$ which form a set of initial data for the system* (4.3.1)-(4.3.6).

We also have

$$\partial_0^{\epsilon,j}\nabla^\beta U_{\epsilon\pm}^0 \in L^2(\Omega_\pm)$$
$$\partial_0^{\epsilon,j}\nabla^\beta f_\epsilon^0 \in L^2(\Gamma)$$

for all $j + |\beta| \geq 1$, where we write ∂_0^j for the initial time derivatives associated with the initial data (U_+^0, U_-^0, f^0) and $\partial_0^{\epsilon,j}$ for the initial time derivatives associated with the initial data $(U_{\epsilon+}^0, U_{\epsilon-}^0, f_\epsilon^0)$.

Additionally, this set of initial data satisfies the compatibility conditions 4.4.3 up to order 2. Also, it converges to the original initial data as $\epsilon \to 0$ in the following sense.

$$\begin{aligned}
\left|\left|U_\epsilon^0 - U^0\right|\right|_{L^\infty(\Omega_\pm)} &\to 0 \textit{ as } \epsilon \to 0 \\
\left|\left|\partial_0^{\epsilon,j}\nabla^\beta U_\epsilon^0 - \partial_0^j \nabla^\beta U^0\right|\right|_{L^2(\Omega_\pm)} &\to 0 \textit{ as } \epsilon \to 0 \textit{ for } 0 \leq |\beta| + j \leq 3 \\
\left|\left|f_\epsilon^0 - f^0\right|\right|_{L^\infty(\mathbb{R}^2)} &\to 0 \textit{ as } \epsilon \to 0 \\
\left|\left|\partial_0^{\epsilon,j}\nabla^\beta f_\epsilon^0 - \partial_0^j \nabla^\beta f^0\right|\right|_{H^1(\mathbb{R}^2)} &\to 0 \textit{ as } \epsilon \to 0 \textit{ for } 0 \leq |\beta| + j \leq 3 \\
\left|\left|\partial_0^{\epsilon,j}\nabla^\beta f_\epsilon^0 - \partial_0^j \nabla^\beta f^0\right|\right|_{H^{2.5}(\mathbb{R}^2)} &\to 0 \textit{ as } \epsilon \to 0 \textit{ for } 0 \leq |\beta| + j \leq 2
\end{aligned}$$

In particular,

$$E_\epsilon^0 \to E^0 \textit{ as } \epsilon \to 0$$

Using the obvious notation

$$\begin{aligned}
\rho_{\epsilon\pm}^0(x) &:= \rho_\pm(p_{\epsilon\pm}^0(x), s_{\epsilon\pm}^0(x)) \\
\psi_\epsilon^0 &:= L^a f_\epsilon^0
\end{aligned}$$

we have

$$\begin{aligned}
\inf_{x\in\Omega_\pm} \rho_{\epsilon\pm}^0 &=: \delta_\epsilon^0 \geq \frac{\delta^0}{2} \textit{ in } \Omega_\pm \\
\inf_{x\in\Omega_\pm} J^{\psi_\epsilon^0} &=: \kappa_\epsilon^0 \geq \frac{1}{2}\kappa^0
\end{aligned}$$

for all $\epsilon > 0$, where $\delta^0 > 0$ and $\kappa^0 \in (0,1)$ are defined as

$$\begin{aligned}
\inf_{x\in\Omega_\pm} \rho_\pm^0 &=: \delta^0 \\
\inf_{x\in\Omega_\pm} J^{\psi^0} &=: \kappa^0
\end{aligned}$$

and we have

$$\begin{aligned}
\delta_\epsilon^0 &\to \delta^0 \textit{ as } \epsilon \to 0 \\
\kappa_\epsilon^0 &\to \kappa^0 \textit{ as } \epsilon \to 0
\end{aligned}$$

Proof. We define $\tilde{p}_{\epsilon\pm}^0, \tilde{u}_{\epsilon\pm}^0, \tilde{s}_{\epsilon\pm}^0, \tilde{f}_\epsilon^0$ by convolution with the standard mollifier (see 1.6 for the notation) as follows.

$$\begin{aligned}
\tilde{p}_{\epsilon\pm}^0 &= (\mathrm{Ext}_{\Omega_\pm} p_\pm^0) * \eta_\epsilon\big|_{\Omega_\pm} \\
\tilde{u}_{\epsilon\pm}^0 &= (\mathrm{Ext}_{\Omega_\pm} u_\pm^0) * \eta_\epsilon\big|_{\Omega_\pm} \\
\tilde{s}_{\epsilon\pm}^0 &= (\mathrm{Ext}_{\Omega_\pm} s_\pm^0) * \eta_\epsilon\big|_{\Omega_\pm} \\
\tilde{f}_\epsilon^0 &= f^0 * \eta_\epsilon
\end{aligned}$$

where in the first three lines η_ϵ denotes the mollifier in three space dimensions and $\text{Ext}_{\Omega_\pm}$ Sobolev extension from $\Omega_\pm$ to $\mathbb{R}^3$, and in the last line η_ϵ denotes the mollifier in two space dimensions.

We now set

$$\begin{aligned} s^0_{\epsilon\pm} &= \tilde{s}^0_{\epsilon\pm} \\ f^0_\epsilon &= \tilde{f}^0_\epsilon \\ (u^0_{\epsilon+})' &= (\tilde{u}^0_{\epsilon+})' \\ (u^0_{\epsilon-})' &= (\tilde{u}^0_{\epsilon-})' \end{aligned}$$

Note that by the properties of mollifiers and the fact that $U^0_\pm \in W^{1,\infty}(\Omega_\pm)$, we have

$$\left|\left|\tilde{U}^0_\epsilon - U^0\right|\right|_{L^\infty(\Omega_\pm)} \to 0 \text{ as } \epsilon \to 0$$

We also use that fact that $f^0 \in W^{2,\infty}(\Gamma)$ and the standard properties of mollifiers to obtain

$$\begin{aligned} ||f^0_\epsilon - f^0||_{W^{1,\infty}(\Gamma)} &\to 0 \text{ as } \epsilon \to 0 \\ ||\psi^0_\epsilon - \psi^0||_{W^{1,\infty}(\mathbb{R}^3)} &\to 0 \text{ as } \epsilon \to 0 \end{aligned}$$

In particular, this implies we may take $\epsilon^0 > 0$ sufficiently small so that

$$J^{\psi^0_\epsilon} \geq \frac{1}{2}\kappa^0$$

for all $\epsilon \in (0, \epsilon^0]$ and clearly $\kappa^0_\epsilon \to \kappa^0$ as $\epsilon \to 0$. Note that we may simply take $(U^0_{\epsilon+}, U^0_{\epsilon-}, f^0_\epsilon) = (U^0_{\epsilon^0+}, U^0_{\epsilon^0-}, f^0_{\epsilon^0})$ for $\epsilon > \epsilon_0$.

To construct $p^0_{\epsilon\pm}$ and $(u^0_\epsilon)_{3\pm}$ we will need to use the following result, which is proved in A.2.1 in the appendix. Let $k \geq 0$ and let $h^j \in H^{l-j+0.5}(\Gamma)$ be given for $0 \leq j \leq k$ for some $l \geq k$. Then there exists a function $g \in H^{l+1}(\Omega_-)$ such that

$$\begin{aligned} \partial^j_{x_3} g|_\Gamma &= h^j \text{ for } 0 \leq j \leq k \\ ||g||_{H^{l+1}(\Omega-)} &\leq C_l \sum_{j=0}^k \left|\left|h^j\right|\right|_{H^{l-j+0.5}(\Gamma)} \end{aligned}$$

for some constant $C_l > 0$ depending on l. See A.2.1 for a proof, noting that to fit this proof we set $h^j = 0$ for $k+1 \leq j \leq l$. We will use this result with $k = 2$ and arbitrary $l \geq 2$.

We use this result to construct $q^0_{\epsilon\pm} \in H^{l+1}(\Omega_\pm)$ such that

$$\begin{aligned} \partial^j_{x_3} q^0_{\epsilon\pm}|_\Gamma &= Q^j_\epsilon - \partial^j_{x_3}\tilde{p}^0_{\epsilon\pm}|_\Gamma \text{ for } 0 \leq j \leq k \\ ||q^0_{\epsilon\pm}||_{H^{l+1}(\Omega-)} &\leq C_l \sum_{j=0}^k \left|\left|Q^j_{\epsilon\pm} - \partial^j_{x_3}\tilde{p}^0_{\epsilon\pm}\right|\right|_{H^{l-j+0.5}(\Gamma)} \end{aligned}$$

where the functions $Q^j_{\epsilon\pm} \in H^{l-j+0.5}(\Gamma)$ will be determined below and we have used the Sobolev trace theorem for the last term in the estimate. Then we set

$$p^0_{\epsilon\pm} = \tilde{p}^0_{\epsilon\pm} + q^0_{\epsilon\pm}$$

so that

$$\partial^j_{x_3} p^0_{\epsilon\pm}|_\Gamma = Q^j_{\epsilon\pm} \text{ for } 0 \le j \le k$$
$$||p^0_{\epsilon\pm} - \tilde{p}^0_{\epsilon\pm}||_{H^{l+1}(\Omega_\pm)} \le C_l \sum_{j=0}^{k} \left|\left|Q^j_{\epsilon\pm} - \partial^j_{x_3}\tilde{p}^0_{\epsilon\pm}\right|\right|_{H^{l-j+0.5}(\Gamma)}$$

Similarly, we construct $w^0_{\epsilon\pm} \in H^{l+1}(\Omega_\pm)$ such that

$$\partial^j_{x_3} w^0_{\epsilon\pm}|_\Gamma = W^j_{\epsilon\pm} - \partial^j_{x_3}(\tilde{u}^0_\epsilon)_{3\pm}|_\Gamma \text{ for } 0 \le j \le k$$
$$||w^0_\epsilon||_{H^{l+1}(\Omega_\pm)} \le C_l \sum_{j=0}^{k} \left|\left|W^j_{\epsilon\pm} - \partial^j_{x_3}(\tilde{u}^0_\epsilon)_{3\pm}\right|\right|_{H^{l-j+0.5}(\Gamma)}$$

where the functions $W^j_{\epsilon\pm} \in H^{l-j+0.5}(\Gamma)$ will be determined below. Then we set

$$(u^0_\epsilon)_{3\pm} = (\tilde{u}^0_\epsilon)_{3\pm} + w^0_{\epsilon\pm}$$

so that

$$\partial^j_{x_3}(u^0_\epsilon)_{3\pm}|_\Gamma = W^j_{\epsilon\pm} \text{ for } 0 \le j \le k$$
$$||(u^0_\epsilon)_{3\pm} - (\tilde{u}^0_\epsilon)_{3\pm}||_{H^{l+1}(\Omega_\pm)} \le C_l \sum_{j=0}^{k} \left|\left|W^j_{\epsilon\pm} - \partial^j_{x_3}(\tilde{u}^0_\epsilon)_{3\pm}\right|\right|_{H^{l-j+0.5}(\Gamma)}$$

Firstly, we aim to define $Q^j_{\epsilon+}$ and $W^j_{\epsilon+}$ in such a way that $\partial^{\epsilon,j}_0(u^0_{\epsilon+} \cdot n^0_\epsilon) = \partial^j_0(u^0 \cdot n^0) * \eta_\epsilon$ for $0 \le j \le 2$, where here η_ϵ is the mollifier in two space dimensions. From lemma 6.1.3, for $j \ge 0$ even, we have

$$\partial^{j,\epsilon}_0(u^0_{\epsilon+}|_\Gamma \cdot n^0_\epsilon) = \frac{F^j_u(\rho^0_{\epsilon+}, c^0_{\epsilon+}, 1 + \left|\nabla'\psi^0_\epsilon\right|^2)\left|n^0_\epsilon\right|^2}{(J^{\psi^0_\epsilon})^j}\partial^j_{x_3}(u^0_\epsilon)_{3+}|_\Gamma + \sum_{k+l=j, k\ne j}\partial^{\epsilon,k}_0(u^0_{\epsilon+}|_\Gamma)' \cdot \partial^{\epsilon,l}_0 n^0_\epsilon$$
$$+ G^j_u(\{\nabla^\alpha V^0_{\epsilon+} : 0 \le |\alpha| \le j-1\})|_\Gamma \cdot n^0_\epsilon$$

and for j odd we have

$$\partial^{j,\epsilon}_0(u^0_{\epsilon+}|_\Gamma \cdot n^0_\epsilon) = \frac{F^j_u(\rho^0_{\epsilon+}, c^0_{\epsilon+}, 1 + \left|\nabla'\psi^0_\epsilon\right|^2)\left|n^0_\epsilon\right|^2}{(J^{\psi^0_\epsilon})^j}\partial^j_{x_3} p^0_{\epsilon+}|_\Gamma + \sum_{k+l=j, k\ne j}\partial^{\epsilon,k}_0(u^0_{\epsilon+}|_\Gamma)' \cdot \partial^{\epsilon,l}_0 n^0_\epsilon$$
$$+ G^j_u(\{\nabla^\alpha V^0_{\epsilon+} : 0 \le |\alpha| \le j-1\})|_\Gamma \cdot n^0_\epsilon$$

Note that the initial time derivatives of n^0_ϵ on the right hand side only involve initial time derivatives of $u^0_{\epsilon+}$ up to order $j-1$.

Thus, inductively for $0 \le j \le 2$, we define $W^j_{\epsilon+}$ and $Q^j_{\epsilon+}$ as follows. For j even we define $W^j_{\epsilon+}$ by

$$\partial^j_{x_3}(u^0_\epsilon)_{3+}|_\Gamma =$$
$$\frac{(J^{\psi^0_\epsilon})^j}{F^j_u(\rho^0_{\epsilon+}, c^0_{\epsilon+}, 1 + |\nabla'\psi^0_\epsilon|^2)\,|n^0_\epsilon|^2}(\partial^j_0(u^0 \cdot n^0) * \eta_\epsilon - \sum_{k+l=j, k\ne j}\partial^{\epsilon,k}_0(u^0_{\epsilon+}|_\Gamma)' \cdot \partial^{\epsilon,l}_0 n^0_\epsilon$$
$$- G^j_u(\{\nabla^\alpha V^0_{\epsilon+} : 0 \le |\alpha| \le j-1\})|_\Gamma \cdot n^0_\epsilon)$$

where whenever we see $\partial_{x_3}^i (u_\epsilon^0)_{3+}$ for $0 \leq i \leq j$, we replace it with $W_{\epsilon+}^i$. We define $Q_{\epsilon+}^j$ by $Q_{\epsilon+}^j = \partial_{x_3}^j \tilde{p}_{\epsilon+}^0|_\Gamma$. Similarly, for j odd we define $Q_{\epsilon+}^j$ by

$$\begin{aligned}
&\partial_{x_3}^j p_{\epsilon+}^0|_\Gamma = \\
&\frac{(J^{\psi_\epsilon^0})^j}{F_u^j(\rho_{\epsilon+}^0, c_{\epsilon+}^0, 1 + |\nabla' \psi_\epsilon^0|^2)\,|n_\epsilon^0|^2}(\partial_0^j (u^0 \cdot n^0) * \eta_\epsilon - \sum_{k+l=j, k\neq j} \partial_0^{\epsilon,k}(u_{\epsilon+}^0|_\Gamma)' \cdot \partial_0^{\epsilon,l} n_\epsilon^0 \\
&- G_u^j(\{\nabla^\alpha V_{\epsilon+}^0 : 0 \leq |\alpha| \leq j-1\})|_\Gamma \cdot n_\epsilon^0)
\end{aligned}$$

where whenever we see $\partial_{x_3}^i p_{\epsilon+}^0$ for $0 \leq i \leq j$, we replace it with $Q_{\epsilon+}^i$. We define $W_{\epsilon+}^j$ by $W_{\epsilon+}^j = \partial_{x_3}^j (\tilde{u}_\epsilon^0)_{3+}|_\Gamma$.

Note that in the case $j = 0$, we have set $Q_{\epsilon+}^0 = \tilde{p}_{\epsilon+}^0|_\Gamma$ hence by the properties of mollification and the fact that $p_+^0, s_+^0 \in W^{1,\infty}(\Omega_+)$, we may choose ϵ^0 sufficiently small so that

$$\rho_+(Q_{\epsilon+}^0, s_{\epsilon+}^0|_\Gamma) \geq \frac{\delta^0}{2}$$

for all $\epsilon \in (0, \epsilon^0]$, so there is no problem with the terms involving $\frac{1}{\rho_{\epsilon+}^0}$.

This completes our definition of $Q_{\epsilon+}^j$ and $W_{\epsilon+}^j$ for $0 \leq j \leq 2$ and allows us to construct $(u_\epsilon^0)_{3+}$ and $p_{\epsilon+}^0$ as described above. We have defined $Q_{\epsilon+}^j$ and $W_{\epsilon+}^j$ in such a way that $\partial_0^{\epsilon,j}(u_{\epsilon+}^0 \cdot n_\epsilon^0) = \partial_0^j (u^0 \cdot n^0) * \eta_\epsilon$ for $0 \leq j \leq 2$.

Now, we aim to define $Q_{\epsilon-}^j$ and $W_{\epsilon-}^j$ in such a way that the compatibility conditions 4.4.3 hold for $(U_{\epsilon+}^0, U_{\epsilon-}^0, f_\epsilon^0)$ for $0 \leq j \leq 2$.

In the case $j = 0$, we set

$$W_{\epsilon-}^0 = u_{\epsilon+}^0|_\Gamma \cdot n_\epsilon^0 - (u_{\epsilon-}^0)' \cdot n_\epsilon^0$$

which ensures that $[u_\epsilon^0 \cdot n_\epsilon^0] = 0$ (replacing $(u_\epsilon^0)_{3-}|_\Gamma$ with $W_{\epsilon-}^0$). We also set

$$Q_{\epsilon-}^0 = p_{\epsilon+}^0|_\Gamma + \sigma \nabla' \cdot \hat{n}_\epsilon^0$$

which ensures that $[p_\epsilon^0] = -\sigma \nabla' \cdot \hat{n}_\epsilon^0$ (replacing $p_{\epsilon-}^0|_\Gamma$ with $Q_{\epsilon-}^0$).

Now inductively for $1 \leq j \leq 2$, we define $W_{\epsilon+}^j$ and $Q_{\epsilon+}^j$ as follows. For j even we define $W_{\epsilon-}^j$ by

$$\begin{aligned}
\partial_{x_3}^j (u_\epsilon^0)_{3-}|_\Gamma = &\frac{(J^{\psi_\epsilon^0})^j}{F_u^j(\rho_{\epsilon-}^0, c_{\epsilon-}^0, 1 + |\nabla' \psi_\epsilon^0|^2)\,|n_\epsilon^0|^2}(\partial_0^{\epsilon,j}(u_{\epsilon+}^0 \cdot n_\epsilon^0) - \sum_{k+l=j, k\neq j} \partial_0^{\epsilon,k}(u_{\epsilon-}^0)' \cdot \partial_0^{\epsilon,l} n_\epsilon^0 \\
&- G_u^j(\{\nabla^\alpha V_{\epsilon-}^0 : 0 \leq |\alpha| \leq j-1\}) \cdot n_\epsilon^0)|_\Gamma
\end{aligned}$$

and $Q_{\epsilon-}^j$ by

$$\begin{aligned}
\partial_{x_3}^j p_{\epsilon-}^0|_\Gamma = &\frac{(J^{\psi_\epsilon^0})^j}{F_p^j(\rho_{\epsilon-}^0, c_{\epsilon-}^0, 1 + |\nabla' \psi_\epsilon^0|^2)}(\partial_0^{\epsilon,j}(p_{\epsilon+}^0 + \sigma \nabla' \cdot \hat{n}_\epsilon^0) \\
&- G_p^j(\{\nabla^\alpha V_{\epsilon-}^0 : 0 \leq |\alpha| \leq j-1\}))|_\Gamma
\end{aligned}$$

where whenever we see $\partial_{x_3}^i p_{\epsilon-}^0$ for $0 \leq i \leq j$, we replace it with $Q_{\epsilon-}^i$ and whenever we see $\partial_{x_3}^i (u_\epsilon^0)_{3-}$ for $0 \leq i \leq j$, we replace it with $W_{\epsilon-}^i$.

Similarly, for j odd, we define $Q^j_{\epsilon-}$ by

$$\partial^j_{x_3} p^0_{\epsilon-}|_\Gamma = -\frac{1}{|n^0_\epsilon|^2} \frac{(J^{\psi^0_\epsilon})^j}{F^j_u(\rho^0_{\epsilon-}, c^0_{\epsilon-}, 1+|\nabla'\psi^0_\epsilon|^2)} (\partial^{\epsilon,j}_0(u^0_{\epsilon+}\cdot n^0_\epsilon) - \sum_{k+l=j, k\neq j} \partial^{\epsilon,k}_0 (u^0_{\epsilon-})' \cdot \partial^{\epsilon,l}_0 n^0_\epsilon$$
$$- G^j_u(\{\nabla^\alpha V^0_{\epsilon-} : 0 \leq |\alpha| \leq j-1\}) \cdot n^0_\epsilon)|_\Gamma$$

and $W^j_{\epsilon-}$ by

$$\partial^j_{x_3}(u^0_\epsilon)_{3-}|_\Gamma = -\frac{(J^{\psi^0_\epsilon})^j}{F^j_p(\rho^0_{\epsilon-}, c^0_{\epsilon-}, 1+|\nabla'\psi^0_\epsilon|^2)} (\partial^{\epsilon,j}_0 (p^0_{\epsilon+} + \sigma\nabla'\cdot \hat{n}^0_\epsilon)$$
$$- G^j_p(\{\nabla^\alpha V^0_{\epsilon-} : 0 \leq |\alpha| \leq j-1\}))|_\Gamma$$

where whenever we see $\partial^i_{x_3} p^0_{\epsilon-}$ for $0 \leq i \leq j$, we replace it with $Q^i_{\epsilon-}$ and whenever we see $\partial^i_{x_3}(u^0_\epsilon)_{3-}$ for $0 \leq i \leq j$, we replace it with $W^i_{\epsilon-}$.

Using lemma 6.1.2 we see that these definitions ensure that the compatibility conditions

$$\partial^{\epsilon,j}_0 [u^0_\epsilon \cdot n^0_\epsilon]$$
$$\partial^{\epsilon,j}_0 [p^0_\epsilon] = -\sigma \partial^{\epsilon,j}_0 \nabla'\cdot\hat{n}^0_\epsilon$$

are satisfied for $0 \leq j \leq 2$ (replacing $\partial^{\epsilon,i}_0 p^0_{\epsilon-}|_\Gamma$ with $Q^i_{\epsilon-}$ and $\partial^{\epsilon,i}_0 (u^0_\epsilon)_{3-}|_\Gamma$ with $W^i_{\epsilon-}$ for $0 \leq i \leq j$). Note we have used the zero order compatibility condition $[u^0_\epsilon \cdot n^0_\epsilon] = 0$ (replacing $(u^0_\epsilon)_{3-}|_\Gamma$ with $W^0_{\epsilon-}$) to ensure that $((u^0_-)^{\psi^0})_3|_\Gamma = 0$ (replacing $(u^0_\epsilon)_{3-}|_\Gamma$ with $W^0_{\epsilon-}$) when applying lemma 6.1.2 to remove the terms involving the functions H^j_u and H^j_p.

Note that since (U^0_+, U^0_-, f^0) satisfies the compatibility conditions up to order 2, we have the same formulas for $\partial^j_{x_3} p^0_-|_\Gamma$ and $\partial^j_{x_3} u^0_{3-}|_\Gamma$ without the ϵ. Hence we may write $Q^j_{\epsilon-} - \partial^j_{x_3} p^0_-|_\Gamma$ and $W^j_{\epsilon-} - \partial^j_{x_3} u^0_{3-}|_\Gamma$ as the difference of the expressions on the right hand side of the above formulas with and without ϵ. In particular, with $j = 0$, we may obtain

$$\left|\left|Q^0_{\epsilon-} - p^0_-|_\Gamma\right|\right|_{L^\infty(\Gamma)} \leq \left|\left|p^0_{\epsilon+}|_\Gamma - p^0_+|_\Gamma\right|\right|_{L^\infty(\Gamma)} + \sigma \left|\left|\nabla'\cdot(\hat{n}^0_\epsilon - \hat{n}^0)\right|\right|_{L^\infty(\Gamma)} \to 0 \text{ as } \epsilon \to 0$$

where we have used the fact that $f^0 \in W^{3,\infty}(\Gamma)$ together with the properties of mollifiers. Hence we may choose ϵ^0 sufficiently small so that

$$\rho_-(Q^0_{\epsilon-}, s^0_\epsilon|_\Gamma) \geq \frac{\delta^0}{2}$$

for all $\epsilon \in (0, \epsilon^0]$. Thus in the above formulas the factor of $\frac{1}{\rho^0_{\epsilon-}}$ does not cause a problem.

Using the estimates from the beginning of the proof and the smoothness of the quantities in the formulae for $Q^j_{\epsilon\pm}$ and $W^j_{\epsilon\pm}$, we have

$$\partial^j_0 \nabla^\beta U^0_{\epsilon\pm} \in L^2(\Omega_\pm)$$
$$\partial^j_0 \nabla^\beta f^0_\epsilon \in L^2(\Gamma)$$

for all $j + |\beta| \geq 1$.

Now we need to show the desired convergence properties. Firstly, we note that we have constructed $(u^0_\epsilon)_{3+}$, $p^0_{\epsilon+}$ and f^0_ϵ so that for $1 \leq j + |\beta| \leq 3$,

$$\left|\left|\nabla^\beta \partial^{\epsilon,j}_0 f^0_\epsilon - \nabla^\beta \partial^j_0 f^0\right|\right|_{H^1(\Gamma)} = \left|\left|\nabla^\beta \partial^j_0 f^0 * \eta_\epsilon - \nabla^\beta \partial^j_0 f^0\right|\right|_{H^1(\Gamma)}$$
$$\to 0 \text{ as } \epsilon \to 0$$

where we have used the properties of mollifiers. Similarly, for $0 \leq j \leq 2$,

$$\left|\left|\partial_0^{\epsilon,j} f_\epsilon^0 - \partial_0^j f^0\right|\right|_{H^{4.5-j}(\Gamma)} = \left|\left|\partial_0^j f^0 * \eta_\epsilon - \partial_0^j f^0\right|\right|_{H^{4.5-j}(\Gamma)}$$
$$\to 0 \text{ as } \epsilon \to 0$$

To show the desired convergence properties of $(u_\epsilon^0)_{3+}$ and $p_{\epsilon+}^0$ we proceed as follows. We note that the formulas for $\partial_{x_3}^j (u_\epsilon^0)_{3+}$ and $\partial_{x_3}^j p_{\epsilon+}^0$ which we used to define $Q_{\epsilon+}^j$ and $W_{\epsilon+}^j$ hold in the case $\epsilon = 0$ for (U_+^0, U_-^0, f^0). Thus we may subtract the formulas for the case $\epsilon = 0$ from the formulas for $Q_{\epsilon+}^j$ and $W_{\epsilon+}^j$ and use the mean value theorem to obtain

$$\begin{aligned}
&\left|\left|Q_{\epsilon+}^j - \partial_{x_3}^j p_+^0|_\Gamma\right|\right|_{H^{2.5-j}(\Gamma)} \\
&\leq (\left|\left|\tilde{U}_{\epsilon+}^0 - U_+^0\right|\right|_{H^{2.5}(\Gamma)} + ||\nabla\psi_\epsilon^0 - \nabla\psi^0||_{H^{2.5}(\Gamma)} \\
&+ \sum_{0\leq i\leq j} ||(\nabla'\partial_0^i f^0) * \eta_\epsilon - \nabla'\partial_0^i f^0||_{H^{2.5-i}(\Gamma)} + \left|\left|\partial_0^{j+1} f^0 * \eta_\epsilon - \partial_0^{j+1} f^0\right|\right|_{H^{2.5-j}(\Gamma)} \\
&+ \sum_{0\leq i\leq j-1} ||Q_{\epsilon+}^i - \partial_{x_3}^i p_+^0|_\Gamma||_{H^{2.5-i}(\Gamma)} + \sum_{0\leq i\leq j-1} ||W_{\epsilon+}^i - \partial_{x_3}^i u_{3+}^0|_\Gamma||_{H^{2.5-i}(\Gamma)})F(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) \\
&\left|\left|W_{\epsilon+}^j - \partial_{x_3}^j u_{3+}^0|_\Gamma\right|\right|_{H^{2.5-j}(\Gamma)} \\
&\leq (\left|\left|\tilde{U}_{\epsilon+}^0 - U_+^0\right|\right|_{H^{2.5}(\Gamma)} + ||\nabla\psi_\epsilon^0 - \nabla\psi^0||_{H^{2.5}(\Gamma)} \\
&+ \sum_{0\leq i\leq j} ||(\nabla'\partial_0^i f^0) * \eta_\epsilon - \nabla'\partial_0^i f^0||_{H^{2.5-i}(\Gamma)} + \left|\left|\partial_0^{j+1} f^0 * \eta_\epsilon - \partial_0^{j+1} f^0\right|\right|_{H^{2.5-j}(\Gamma)} \\
&+ \sum_{0\leq i\leq j-1} ||Q_{\epsilon+}^i - \partial_{x_3}^i p_+^0|_\Gamma||_{H^{2.5-i}(\Gamma)} + \sum_{0\leq i\leq j-1} ||W_{\epsilon+}^i - \partial_{x_3}^i u_{3+}^0|_\Gamma||_{H^{2.5-i}(\Gamma)})F(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0})
\end{aligned}$$

Thus using the properties of mollifiers we inductively obtain

$$\left|\left|Q_{\epsilon+}^j - \partial_{x_3}^j p_+^0|_\Gamma\right|\right|_{H^{2.5-j}(\Gamma)} \to 0 \text{ as } \epsilon \to 0$$
$$\left|\left|W_{\epsilon+}^j - \partial_{x_3}^j u_{3+}^0|_\Gamma\right|\right|_{H^{2.5-j}(\Gamma)} \to 0 \text{ as } \epsilon \to 0$$

Now we show the desired convergence properties of $(u_\epsilon^0)_{3-}$ and $p_{\epsilon-}^0$. Taking the difference of the formulas for $Q_{\epsilon-}^j$ and $W_{\epsilon-}^j$ and their counterparts without the ϵ satisfied by $\partial_{x_3}^j p_-^0|_\Gamma$ and $\partial_{x_3}^j u_{3-}^0|_\Gamma$ up to $j = 2$, along with the mean value theorem, we obtain

$$\begin{aligned}
&||Q_\epsilon^j - \partial_{x_3}^j p_-^0|_\Gamma||_{H^{2.5-j}(\Gamma)} \\
&\leq (\left|\left|\tilde{U}_{\epsilon-}^0 - U_-^0\right|\right|_{H^{2.5}(\Gamma)} + ||\nabla\psi_\epsilon^0 - \nabla\psi^0||_{H^{2.5}(\Gamma)} + ||U_{\epsilon+}^0 - U_+^0||_{H^{2.5}(\Gamma)} \\
&+ \sum_{0\leq i\leq j} \left|\left|\nabla'\partial_0^{\epsilon,i} f_\epsilon^0 - \nabla'\partial_0^i f^0\right|\right|_{H^{3.5-i}(\Gamma)} \\
&+ \sum_{0\leq i\leq j-1} ||Q_{\epsilon-}^i - \partial_{x_3}^i p_-^0|_\Gamma||_{H^{2.5-i}(\Gamma)} + \sum_{0\leq i\leq j-1} ||W_{\epsilon-}^i - \partial_{x_3}^i u_{3-}^0|_\Gamma||_{H^{2.5-i}(\Gamma)})F(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0})
\end{aligned}$$

and

$$
\begin{aligned}
&\left|\left|W_{\epsilon}^{j}-\partial_{x_3}^{j}u_{3-}^{0}|_{\Gamma}\right|\right|_{H^{2.5-j}(\Gamma)}\\
&\leq (\left|\left|\tilde{U}_{\epsilon-}^{0}-U_{-}^{0}\right|\right|_{H^{2.5}(\Gamma)}+\left|\left|\nabla\psi_{\epsilon}^{0}-\nabla\psi^{0}\right|\right|_{H^{2.5}(\Gamma)}+\left|\left|U_{\epsilon+}^{0}-U_{+}^{0}\right|\right|_{H^{2.5}(\Gamma)}\\
&+\sum_{0\leq i\leq j}\left|\left|\nabla'\partial_{0}^{\epsilon,i}f_{\epsilon}^{0}-\nabla'\partial_{0}^{i}f^{0}\right|\right|_{H^{3.5-i}(\Gamma)}\\
&+\sum_{0\leq i\leq j-1}\left|\left|Q_{\epsilon-}^{i}-\partial_{x_3}^{i}p_{-}^{0}|_{\Gamma}\right|\right|_{H^{2.5-i}(\Gamma)}+\sum_{0\leq i\leq j-1}\left|\left|W_{\epsilon-}^{i}-\partial_{x_3}^{i}u_{3-}^{0}|_{\Gamma}\right|\right|_{H^{2.5-i}(\Gamma)})F(E^{0},\frac{1}{\delta^{0}},\frac{1}{\kappa^{0}})
\end{aligned}
$$

Thus using the properties of mollifiers and existing estimates we inductively obtain

$$
\begin{aligned}
\left|\left|Q_{\epsilon-}^{j}-\partial_{x_3}^{j}p_{-}^{0}|_{\Gamma}\right|\right|_{H^{2.5-j}(\Gamma)}&\to 0 \text{ as } \epsilon\to 0\\
\left|\left|W_{\epsilon-}^{j}-\partial_{x_3}^{j}u_{3-}^{0}|_{\Gamma}\right|\right|_{H^{2.5-j}(\Gamma)}&\to 0 \text{ as } \epsilon\to 0
\end{aligned}
$$

Hence, using the estimate from the beginning of the proof, we we have

$$
\begin{aligned}
\left|\left|U_{\epsilon\pm}^{0}-\tilde{U}_{\epsilon\pm}^{0}\right|\right|_{H^{3}(\Omega_{\pm})}&\leq C\sum_{j=0}^{2}\left|\left|Q_{\epsilon\pm}^{j}-\partial_{x_3}^{j}\tilde{p}_{\epsilon\pm}^{0}\right|\right|_{H^{2.5-j}(\Gamma)}+C\sum_{j=0}^{2}\left|\left|W_{\epsilon\pm}^{j}-\partial_{x_3}^{j}(\tilde{u}_{\epsilon}^{0})_{3\pm}\right|\right|_{H^{2.5-j}(\Gamma)}\\
&\leq C\sum_{j=0}^{2}\left|\left|Q_{\epsilon\pm}^{j}-\partial_{x_3}^{j}p_{\pm}^{0}\right|\right|_{H^{2.5-j}(\Gamma)}+C\sum_{j=0}^{2}\left|\left|W_{\epsilon\pm}^{j}-\partial_{x_3}^{j}u_{3\pm}^{0}\right|\right|_{H^{2.5-j}(\Gamma)}\\
&+C\sum_{j=0}^{2}\left|\left|\partial_{x_3}^{j}\tilde{p}_{\epsilon\pm}^{0}-\partial_{x_3}^{j}p_{\pm}^{0}\right|\right|_{H^{2.5-j}(\Gamma)}+C\sum_{j=0}^{2}\left|\left|\partial_{x_3}^{j}(\tilde{u}_{\epsilon}^{0})_{3\pm}-\partial_{x_3}^{j}u_{3\pm}^{0}\right|\right|_{H^{2.5-j}(\Gamma)}\\
&\to 0 \text{ as } \epsilon\to 0
\end{aligned}
$$

where we have used the above estimates and the properties of mollifiers. Hence we have

$$
\begin{aligned}
\left|\left|U_{\epsilon\pm}^{0}-U_{\pm}^{0}\right|\right|_{H^{3}(\Omega_{\pm})}&\leq\left|\left|U_{\epsilon\pm}^{0}-\tilde{U}_{\epsilon\pm}^{0}\right|\right|_{H^{3}(\Omega_{\pm})}+\left|\left|\tilde{U}_{\epsilon\pm}^{0}-U_{\pm}^{0}\right|\right|_{H^{3}(\Omega_{\pm})}\\
&\to 0 \text{ as } \epsilon\to 0
\end{aligned}
$$

where we have used the above estimates and the properties of mollifiers again. Note that by the Sobolev embedding theorem this also implies

$$
\left|\left|U_{\epsilon\pm}^{0}-U_{\pm}^{0}\right|\right|_{L^{\infty}(\Omega_{\pm})}\to 0 \text{ as } \epsilon\to 0
$$

Now we note using the chain rule, mean value theorem and definition of the initial time derivatives that, for $0\leq j\leq 3$,

$$
\begin{aligned}
&\left|\left|\partial_{0}^{\epsilon,j}U_{\epsilon\pm}^{0}-\partial_{0}^{j}U_{\pm}^{0}\right|\right|_{H^{3-j}(\Omega_{\pm})}\\
&\leq F(E^{0},\frac{1}{\delta^{0}},\frac{1}{\kappa^{0}})(\left|\left|U_{\epsilon\pm}^{0}-U_{\pm}^{0}\right|\right|_{H^{3}(\Omega_{\pm})}+\sum_{i=0}^{j}\left|\left|\partial_{0}^{\epsilon,i}f_{\epsilon}^{0}-\partial_{0}^{i}f^{0}\right|\right|_{H^{3-i}(\Gamma)})\\
&\to 0 \text{ as } \epsilon\to 0
\end{aligned}
$$

where we have used the above estimates.

Finally, note that since

$$
\left|\left|U_{\epsilon}^{0}-U^{0}\right|\right|_{L^{\infty}(\Omega_{\pm})}\to 0 \text{ as } \epsilon\to 0
$$

we may take $\epsilon^0 > 0$ sufficiently small so that

$$\rho^0_{\epsilon\pm} \geq \frac{1}{2}\delta^0$$

for all $\epsilon \in (0, \epsilon^0]$ and clearly $\delta^0_\epsilon \to \delta^0$ as $\epsilon \to 0$.

This completes the proof. □

Proposition 6.1.5. *Let (U^0_+, U^0_-, f^0) be initial data which satisfy the hypotheses of theorem 4.6.1, so in particular satisfy the compatibility conditions up to order 2. Then there exists a sequence of initial data $(U^0_{m+}, U^0_{m-}, f^0_m)$, $m \geq 1$, which satisfies all the conclusions of 6.1.4 (replacing $\epsilon \to 0$ with $m \to \infty$) and in addition each element of the sequence satisfies the compatibility condition* (4.4.6) *up to order* 3.

Proof. We let $\epsilon > 0$ and $\eta > 0$. We define the initial data $(U^0_{\epsilon,\eta+}, U^0_{\epsilon,\eta-}, f^0_{\epsilon,\eta})$ as follows.

$$\begin{aligned}
p^0_{\epsilon,\eta+} &= p^0_{\epsilon+} \\
u^0_{\epsilon,\eta\pm} &= u^0_{\epsilon\pm} \\
s^0_{\epsilon,\eta\pm} &= s^0_{\epsilon\pm} \\
f^0_{\epsilon,\eta} &= f^0_\epsilon \\
p^0_{\epsilon,\eta-} &= p^0_{\epsilon-} + \frac{1}{3!}x_3^3\chi(\frac{x_3}{\eta})q^0_{\epsilon-}
\end{aligned}$$

where $p^0_{\epsilon\pm}$ etc. are given by the previous proposition 6.1.4 and where $\chi \in C^\infty_c(\mathbb{R})$ is a smooth cut-off function with $\chi(x_3) = 0$ for $|x_3| \geq 1$ and $\chi(x_3) = 1$ for $|x_3| \leq \frac{1}{2}$ and $q^0_{\epsilon-}$ is defined below.

Note that, after using the definition of the initial time derivatives, the compatibility conditions (4.4.6)-(4.4.7) up to order 2 only involve spatial derivatives of p^0_- up to order 2. Now note that

$$\nabla^\alpha(\frac{1}{3!}x_3^3\chi(\frac{x_3}{\eta})q^0_{\epsilon-})|_\Gamma = 0$$

for $0 \leq |\alpha| \leq 2$, so the compatibility conditions (4.4.6)-(4.4.7) up to order 2 are unaffected by the addition of the extra term, and still hold with $(U^0_{\epsilon+}, U^0_{\epsilon-}, f^0_\epsilon)$ replaced by $(U^0_{\epsilon,\eta+}, U^0_{\epsilon,\eta-}, f^0_{\epsilon,\eta})$.

The calculations from the proof of 6.1.4 show that the compatibility condition (4.4.6) holds at order 3 for the initial data $(U^0_{\epsilon,\eta+}, U^0_{\epsilon,\eta-}, f^0_{\epsilon,\eta})$ if and only if

$$\begin{aligned}
\partial^3_{x_3}p^0_{\epsilon,\eta-}|_\Gamma = \frac{1}{|n^0_\epsilon|}\frac{(J^{\psi^0_\epsilon})^3}{F^3_u(\rho^0_{\epsilon-}, c^0_{\epsilon-})}&(\partial_0^{\epsilon,3}(u^0_{\epsilon+}\cdot n^0_\epsilon) - \sum_{k+l=3,k\neq 3}\partial_0^{\epsilon,k}(u^0_{\epsilon-})'\cdot\partial_0^{\epsilon,l}n^0_\epsilon \\
&- G^3_u(\{\nabla^\alpha V^0_{\epsilon-} : 0 \leq |\alpha| \leq 2\})\cdot n^0_\epsilon)|_\Gamma
\end{aligned}$$

Using the definition of $(U^0_{\epsilon,\eta+}, U^0_{\epsilon,\eta-}, f^0_{\epsilon,\eta})$, this is equivalent to

$$\begin{aligned}
q^0_{\epsilon-}|_\Gamma = \frac{1}{|n^0_\epsilon|}\frac{(J^{\psi^0_\epsilon})^3}{F^3_u(\rho^0_{\epsilon-}, c^0_{\epsilon-})}&(\partial_0^{\epsilon,3}(u^0_{\epsilon+}\cdot n^0_\epsilon) - \sum_{k+l=3,k\neq 3}\partial_0^{\epsilon,k}(u^0_{\epsilon-})'\cdot\partial_0^{\epsilon,l}n^0_\epsilon \\
&- G^3_u(\{\nabla^\alpha V^0_{\epsilon-} : 0 \leq |\alpha| \leq 2\})\cdot n^0_\epsilon)|_\Gamma - \partial^3_{x_3}p^0_{\epsilon-}|_\Gamma
\end{aligned}$$

which is how we define $q^0_{\epsilon-}|_\Gamma$, which is extended to Ω_- by being independent of x_3.

We now claim that, for fixed $\epsilon > 0$,

$$\left|\left|\partial_0^{\eta,\epsilon,j} U^0_{\epsilon,\eta\pm} - \partial_0^{\epsilon,j} U^0_{\epsilon\pm}\right|\right|_{H^{3-j}(\Omega_\pm)} \to 0 \text{ as } \eta \to 0 \text{ for } 0 \le j \le 3$$
$$\left|\left|\partial_0^{\eta,\epsilon,j} f^0_{\epsilon,\eta} - \partial_0^{\epsilon,j} f^0_{\epsilon}\right|\right|_{H^{4-j}(\Gamma)} \to 0 \text{ as } \eta \to 0 \text{ for } 0 \le j \le 3$$
$$\left|\left|\partial_0^{\eta,\epsilon,j} f^0_{\epsilon,\eta} - \partial_0^{\epsilon,j} f^0_{\epsilon}\right|\right|_{H^{4.5-j}(\Gamma)} \to 0 \text{ as } \eta \to 0 \text{ for } 0 \le j \le 2$$

where we use $\partial_0^{\eta,\epsilon,j}$ to denote the initial time derivatives corresponding to the doubly approximating initial data $(U^0_{\epsilon,\eta+}, U^0_{\epsilon,\eta-}, f^0_{\epsilon,\eta})$. Indeed, note that the initial time derivatives of $U^0_{\epsilon,\eta+}$ and $f^0_{\epsilon,\eta}$ depend only on spatial derivatives of $U^0_{\epsilon,\eta+}$ and $f^0_{\epsilon,\eta}$ which are exactly equal to $U^0_{\epsilon+}$ and f^0_{ϵ}. Also, we note that $\partial_0^{\eta,\epsilon,j} U^0_{\epsilon,\eta-}$ for $0 \le j \le 3$ depends only on spatial derivatives of $U^0_{\epsilon,\eta\pm}$ and $f^0_{\epsilon,\eta}$ up to order 3. Since these are all equal to those for $\eta = 0$ except the spatial derivatives of $(p^0_{\epsilon,\eta})_{3-}$, it suffices, by using the mean value theorem, to prove that

$$\left|\left|(p^0_{\epsilon,\eta})_{3-} - (p^0_{\epsilon})_{3-}\right|\right|_{H^3(\Omega_-)} = \left|\left|\frac{1}{3!} x_3^3 \chi(\frac{x_3}{\eta}) q^0_{\epsilon-}\right|\right|_{H^3(\Omega_-)} \to 0 \text{ as } \eta \to 0$$

Indeed, we have

$$\begin{aligned}\left|\left|\frac{1}{3!} x_3^3 \chi(\frac{x_3}{\eta}) q^0_{\epsilon-}\right|\right|_{H^3(\Omega_-)} &\le C \sum_{k=0}^{3} \left|\left|\partial^k_{x_3}(x_3^3 \chi(\frac{x_3}{\eta}))\right|\right|_{L^\infty(\Omega_-)} \left|\left|q^0_{\epsilon-}\right|\right|_{H^3(\Omega_- \cap \{|x_3| \le \eta\})} \\ &\le C \left|\left|q^0_{\epsilon-}\right|\right|_{H^3(\Omega_- \cap \{|x_3| \le \eta\})}\end{aligned}$$

From the formula for $q^0_{\epsilon-}$ and writing the initial time derivatives on the right hand side of this formula in terms of spatial derivatives, we see that

$$\begin{aligned}\left|\left|q^0_{\epsilon-}\right|\right|_{H^3(\Omega_-)} &\le F(E^0_\epsilon + \sum_{\pm} \left|\left|\nabla U^0_{\epsilon\pm}\right|\right|_{H^{10}(\Omega_-)} + \left|\left|\nabla' f^0_\epsilon\right|\right|_{H^{10}(\Gamma)}) \\ &\le C_\epsilon\end{aligned}$$

for some increasing function F. Thus,

$$\begin{aligned}\left|\left|\frac{1}{3!} x_3^3 \chi(\frac{x_3}{\eta}) q^0_{\epsilon-}\right|\right|_{H^3(\Omega_-)} &\le C \left|\left|q^0_{\epsilon-}\right|\right|_{H^3(\Omega_- \cap \{|x_3| \le \eta\})} \\ &\to 0 \text{ as } \eta \to 0\end{aligned}$$

as claimed.

Now given $m \ge 1$, we pick $\epsilon_m > 0$ such that

$$\left|\left|U^0_{\epsilon_m} - U^0\right|\right|_{L^\infty(\Omega_\pm)} \le \frac{1}{m}$$
$$\left|\left|\partial_0^{\epsilon_m,j} \nabla^\beta U^0_{\epsilon_m} - \partial_0^j \nabla^\beta U^0\right|\right|_{L^2(\Omega_\pm)} \le \frac{1}{m} \text{ for } 0 \le |\beta| + j \le 3$$
$$\left|\left|f^0_{\epsilon_m} - f^0\right|\right|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{m}$$
$$\left|\left|\partial_0^{\epsilon_m,j} \nabla^\beta f^0_{\epsilon_m} - \partial_0^j \nabla^\beta f^0\right|\right|_{H^1(\mathbb{R}^2)} \le \frac{1}{m} \text{ for } 0 \le |\beta| + j \le 3$$
$$\left|\left|\partial_0^{\epsilon_m,j} \nabla^\beta f^0_{\epsilon_m} - \partial_0^j \nabla^\beta f^0\right|\right|_{H^{2.5}(\mathbb{R}^2)} \le \frac{1}{m} \text{ for } 0 \le |\beta| + j \le 2$$

We then pick η_m such that

$$\begin{aligned}
||U^0_{\epsilon_m,\eta_m} - U^0_{\epsilon_m}||_{L^\infty(\Omega_\pm)} &\leq \frac{1}{m} \\
\left|\left|\partial_0^{\epsilon_m,\eta_m,j}\nabla^\beta U^0_{\epsilon_m,\eta_m} - \partial_0^{\epsilon_m,j}\nabla^\beta U^0_{\epsilon_m}\right|\right|_{L^2(\Omega_\pm)} &\leq \frac{1}{m} \text{ for } 0 \leq |\beta| + j \leq 3 \\
||f^0_{\epsilon_m,\eta_m} - f^0_{\epsilon_m}||_{L^\infty(\mathbb{R}^2)} &\leq \frac{1}{m} \\
\left|\left|\partial_0^{\epsilon_m,\eta_m,j}\nabla^\beta f^0_{\epsilon_m,\eta_m} - \partial_0^{\epsilon_m,j}\nabla^\beta f^0_{\epsilon_m}\right|\right|_{H^1(\mathbb{R}^2)} &\leq \frac{1}{m} \text{ for } 0 \leq |\beta| + j \leq 3 \\
\left|\left|\partial_0^{\epsilon_m,\eta_m,j}\nabla^\beta f^0_{\epsilon_m,\eta_m} - \partial_0^{\epsilon_m,j}\nabla^\beta f^0_{\epsilon_m}\right|\right|_{H^{2.5}(\mathbb{R}^2)} &\leq \frac{1}{m} \text{ for } 0 \leq |\beta| + j \leq 2
\end{aligned}$$

Then we set $(U^0_{m+}, U^0_{m-}, f^0_m) = (U^0_{\epsilon_m,\eta_m+}, U^0_{\epsilon_m,\eta_m-}, f^0_{\epsilon_m,\eta_m})$. This completes the proof. □

6.2 Reduction to the Case of Smooth Initial Data

Proposition 6.2.1. *Suppose we can prove the theorem 4.6.1 under the additional assumption that the initial data (U^0_+, U^0_-, f^0) are smooth, ie $\nabla^\alpha U^0_\pm \in L^2(\Omega_\pm)$ and $\nabla^{\alpha'} f^0 \in L^2(\mathbb{R}^2)$ for all $|\alpha| \geq 1$, and that the compatibility condition* (4.4.6) *holds up to order* 3. *Then the theorem 4.6.1 follows (without the smoothness assumption).*

Proof. Let (U^0_+, U^0_-, f^0) be initial data which satisfy the hypotheses of theorem 4.6.1.

Let $(U^0_{m+}, U^0_{m-}, f^0_m)$ be the smoothed set of initial data as defined in 6.1.5 for $m \geq 1$. Let $l \geq 1$ be an integer. Note that since $E^0_m \to E^0$ as $m \to \infty$ we have

$$E^0_m \leq E^0_l$$

for some constant E^0_l independent of m for all $m \geq l$ with $E^0_l \to E^0$ as $l \to \infty$. From the construction of $(U^0_{m+}, U^0_{m-}, f^0_m)$, we may also bound the difference

$$\sum_\pm ||U^0_m - U^0||^2_{L^2(\Omega_\pm)} + ||f^0_m - f^0||^2_{L^2(\Gamma)} \leq CE^0_l$$

We also have

$$\begin{aligned}
\inf_{x\in\Omega_\pm} \rho^0_{m\pm} &=: \delta^0_m \geq \delta^0_l \\
\inf_{x\in\mathbb{R}^3} \psi^0_m &=: \kappa^0_m \geq \kappa^0_l
\end{aligned}$$

for some constants $\delta^0_l > 0$ and $\kappa^0_l > 0$ for all $m \geq l$ with $\delta^0_l \to \delta^0$ and $\kappa^0_l \to \kappa^0$ as $l \to \infty$. Note that $(U^0_{m+}, U^0_{m-}, f^0_m)$ also satisfy the compatibility conditions up to order 2 and the compatibility condition (4.4.6) up to order 3, hence the initial data $(U^0_{m+}, U^0_{m-}, f^0_m)$ satisfies the hypotheses of theorem 4.6.1 with E^0 replaced by E^0_m, δ^0 replaced by δ^0_m and κ^0 replaced by κ^0_m, with the additional smoothness and compatibility assumptions.

Now we apply the theorem 4.6.1 with initial data $(U^0_{m+}, U^0_{m-}, f^0_m)$ for each $m \geq l$ to obtain a solution (U_{m+}, U_{m-}, f_m) on the time interval $(0, T^0_m)$, for $T^0_m > 0$. We also have

$$\begin{aligned}
E_m(T^0_m) &\leq C^0_m \\
\rho_{m\pm} &\geq \frac{\delta^0_m}{2} \text{ in } (0, T^0_m) \times \Omega_\pm \\
J^{\psi_m} &\geq \frac{1}{2}\kappa^0 \text{ in } (0, T^0_m) \times \mathbb{R}^3
\end{aligned}$$

(Note we use the obvious notation E_m for the energy of the solution (U_{m+}, U_{m-}, f_m) etc).

The time $T_m^0 > 0$ is bounded below as follows.

$$T_m^0 \geq T(E_m^0, \frac{1}{\delta_m^0}, \frac{1}{\kappa_m^0}) > 0$$

where $T(\cdot)$ a fixed smooth decreasing function given in theorem 4.6.1. Similarly, the constant $C_m^0 > 0$ is bounded above as follows.

$$C_m^0 \leq C(E_m^0, \frac{1}{\delta_m^0}, \frac{1}{\kappa_m^0}) < \infty$$

where $C(\cdot)$ is a fixed smooth increasing function given in theorem 4.6.1. Thus

$$T_m^0 \geq T(E_l^0, \frac{1}{\delta_l^0}, \frac{1}{\kappa_l^0}) =: T_l^0 > 0$$
$$C_m^0 \leq C(E_l^0, \frac{1}{\delta_l^0}, \frac{1}{\kappa_l^0}) =: C_l^0 < \infty$$

Let us also note that

$$\lim_{l\to\infty} T_l^0 = T(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) =: T^0 > 0$$
$$\lim_{l\to\infty} C_l^0 = C(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) =: C^0 < \infty$$

Since for all $m \geq l$ we have

$$E_m(T_l^0) \leq C_l^0$$

we may use weak-$*$ compactness in $L^\infty((0,T_l^0); L^2(\Omega_\pm))$ and $L^\infty((0,T_l^0) \times \Omega_\pm)$ to obtain a subsequence m_k and elements $U_\pm \in L^\infty((0,T_l^0) \times \Omega_\pm)$, $f \in L^\infty((0,T_l^0) \times \mathbb{R}^2)$ such that

$$\partial^\alpha U \in L^\infty((0,T_l^0); L^2(\Omega_\pm)) \text{ for all } 1 \leq |\alpha| \leq 3$$
$$\partial^\alpha f \in L^\infty((0,T_l^0); H^1(\mathbb{R}^2)) \text{ for all } 1 \leq |\alpha| \leq 3$$
$$\nabla' \partial_t^j f \in L^\infty((0,T_l^0); H^{3.5-j}(\mathbb{R}^2)) \text{ for all } 0 \leq j \leq 2$$

and

$$U_{m_k} - U \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T_l^0) \times \Omega_\pm)$$
$$f_{m_k} - f \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T_l^0) \times \mathbb{R}^2)$$
$$\partial^\alpha U_{m_k} - \partial^\alpha U \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T_l^0); L^2(\Omega_\pm)) \text{ for all } 0 \leq |\alpha| \leq 3$$
$$\partial^\alpha f_{m_k} - \partial^\alpha f \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T_l^0); H^1(\mathbb{R}^2)) \text{ for all } 0 \leq |\alpha| \leq 3$$
$$\nabla' \partial_t^j f_{m_k} - \nabla' \partial_t^j f \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T_l^0); H^{3.5-j}(\mathbb{R}^2)) \text{ for all } 0 \leq j \leq 2$$

By weak-$*$ lower semi-continuity of norms, we have

$$E(T_l^0) \leq \liminf_{k\to\infty} C_{m_k}^0 = C^0$$

where E denotes the energy associated to (U_+, U_-, f). Note that for the zero order convergence we have used the fact that we may bound the difference

$$\begin{aligned}
&\sum_{\pm} ||U_m - U^0||^2_{L^2(\Omega_\pm)} + ||f_m - f^0||^2_{L^2(\Gamma)} \\
&\leq C \sum_{\pm} ||U^0_m - U^0||^2_{L^2(\Omega_\pm)} + C \sum_{\pm} ||\partial_t U_m||^2_{L^2(\Omega_\pm)} + C\,||f^0_m - f^0||^2_{L^2(\Gamma)} + C\,||\partial_t f_m||^2_{L^2(\Gamma)} \\
&\leq C E^0_l
\end{aligned}$$

Note that, using the Sobolev embedding theorem, since $\partial U \in H^2(\Omega_\pm)$ and $2 > \frac{d}{p} = \frac{3}{2}$, $U_\pm \in C^1_b((0, T^0_l) \times \Omega_\pm)$, and similarly $f \in C^1_b((0, T^0_l) \times \mathbb{R}^2)$, $\nabla' f \in C^1_b((0, T^0_l) \times \mathbb{R}^2)$. Note in particular this implies that $U_\pm$ and f are uniformly continuous (in fact Lipschitz).

Using Sobolev compact embedding, we have the following strong convergences, where $\tilde{\Omega}_\pm$ are any smooth bounded subsets of $\Omega_\pm$ and $\tilde{\Gamma}$ is any smooth bounded subset of $\mathbb{R}^2$.

$$\begin{aligned}
\partial^\alpha U_{m_k} - \partial^\alpha U \to 0 \text{ in } L^\infty((0, T^0_l); L^2(\tilde{\Omega}_\pm)) \text{ for all } 1 \leq |\alpha| \leq 2 \\
\partial^\alpha f_{m_k} - \partial^\alpha f \to 0 \text{ in } L^\infty((0, T^0_l); H^1(\tilde{\Gamma})) \text{ for all } 0 \leq |\alpha| \leq 2
\end{aligned}$$

In particular, using the Sobolev embedding theorem again, we have $U_{m_k} \to U$ uniformly on smooth bounded subsets of $(0, T^0_l) \times \Omega_\pm$ and $f_{m_k} \to f$, $\partial f_{m_k} \to \partial f$ uniformly on smooth bounded subsets of $(0, T^0_l) \times \mathbb{R}^2$. This also implies $U_{m_k}|_\Gamma \to U|_\Gamma$ uniformly on smooth bounded subsets of $(0, T^0_l) \times \mathbb{R}^2$. Since $\rho_{m\pm}$ are smooth functions of p and s, we also have $\rho_{m_k\pm} \to \rho_\pm$ uniformly on smooth bounded subsets of $(0, T^0_l) \times \Omega_\pm$. Hence, we have pointwise in $(0, T^0_l) \times \Omega_\pm$,

$$\rho_\pm = \lim_{k\to\infty} \rho_{m_k\pm} \geq \frac{\delta^0_{m_k}}{2} \to \frac{\delta^0}{2}$$

Similarly, we have $\partial_{x_3} \psi_{m_k} \to \partial_{x_3} \psi$ uniformly on smooth bounded subsets of $(0, T^0_l) \times \mathbb{R}^3$. Hence,

$$J^\psi = \lim_{k\to\infty} J^{\psi_{m_k}} \geq \frac{\kappa^0_{m_k}}{2} \to \frac{\kappa^0}{2}$$

Note that by passing to a subsequence if necessary we may assume additionally that $\partial U_{m_k} \to \partial U$ almost everywhere in $(0, T^0_l) \times \Omega_\pm$ and $\nabla' \partial f_{m_k} \to \nabla' \partial f$ almost everywhere on $(0, T^0_l) \times \Gamma$. We use this convergence almost everywhere to let $k \to \infty$ for almost every $(t, x) \in (0, T^0_l) \times \Omega_\pm$ and almost every $(t, x') \in (0, T^0_l) \times \mathbb{R}^2$ in the system of equations

$$\begin{aligned}
\frac{1}{\rho_{m_k} c^2_{m_k}} (\partial_t + u^{\psi_{m_k}}_{m_k} \cdot \nabla) p_{m_k} + \nabla^{\psi_{m_k}} \cdot u_{m_k} &= 0 \\
\rho_{m_k} (\partial_t + u^{\psi_{m_k}}_{m_k} \cdot \nabla) u_{m_k} + \nabla^{\psi_{m_k}} p_{m_k} &= 0 \\
(\partial_t + u^{\psi_{m_k}} \cdot \nabla) s_{m_k} &= 0
\end{aligned}$$

in $(0, T_l) \times \Omega_\pm$,

$$\begin{aligned}
\partial_t f_{m_k} &= u_{m_k\pm} \cdot n \\
[p_{m_k}] &= -\sigma \nabla' \cdot \hat{n}_{m_k}
\end{aligned}$$

on $(0, T_l) \times \Gamma$,

$$\psi_{m_k} := L^a f_{m_k}$$

to obtain that (U_+, U_-, f) is a solution of the system (4.3.1)-(4.3.6) on the time interval $(0, T_l^0)$ as defined in 4.3.1 with $\psi = L^a f$. Note that the continuity of U, ∂U and f, ∂f, $\nabla' \partial f$ imply that these equations are in fact satisfied everywhere in the classical sense.

We now claim that $(U_+|_{t=0}, U_-|_{t=0}, f_{t=0}) = (U_+^0, U_-^0, f^0)$. Indeed, we note that using the uniform continuity and convergence which we have shown, we have

$$\begin{aligned} U_{m_k \pm}|_{t=0} &\to U_\pm|_{t=0} \text{ in } \Omega_\pm \\ f_{m_k}|_{t=0} &\to f|_{t=0} \text{ in } \mathbb{R}^2 \end{aligned}$$

But we have from 6.1.4 that

$$\begin{aligned} U_{m_k \pm}^0 &\to U_\pm^0 \text{ in } \Omega_\pm \\ f_{m_k}^0 &\to f^0 \text{ in } \mathbb{R}^2 \end{aligned}$$

hence $(U_+|_{t=0}, U_-|_{t=0}, f_{t=0}) = (U_+^0, U_-^0, f^0)$ as claimed.

Now for each l we may apply the above compactness results successively to $(U_{m_k+}, U_{m_k-}, f_{m_k})$ to obtain further subsequences $m_{l,k}$ and limit functions $U_{l\pm}, f_l$ such that the above convergences etc hold on $(0, T_l^0)$ and $(U_{l\pm}, f_l) = (U_{m\pm}, f_m)$ whenever both are defined. Since $T_l^0 \to T^0$ as $l \to \infty$, we may define $(U_\pm, f)$ on $(0, T^0)$ by defining it in the manner described above on on $(0, T_l^0)$ for each l, and by the above it is a solution to the equations on the fixed domains on the interval $(0, T^0)$ with the desired properties. This completes the proof. $\square$

7. The μ-Approximate Equations

Here we define a sort of viscous regularisation of the equations (4.3.1)-(4.3.6) on the fixed domains. See 3.3 for a motivation of the general strategy of introducing such a viscous regularisation, and see 3.4 for the motivation behind the specific regularisation chosen.

For brevity we do not redefine quantities whose definition is obvious given notation we have introduced previously.

Also, we will not write explicitly the dependence on μ of a solution to the μ-approximate equations else the notation would be too messy.

7.1 Definition of the μ-Approximate Equations

Definition 7.1.1. For $\mu > 0$, given the density functions $\rho_\pm(t) \in C^1_b(\Omega_\pm)$ for each t, we define the smoothed density $\rho_{\mu\pm}$ as

$$\rho_{\mu\pm} = (\operatorname{Ext}_{\Omega_\pm} \rho_\pm) * \eta_\mu \big|_{\Omega_\pm}$$

where the Sobolev extension operators $\operatorname{Ext}_{\Omega_\pm}$ are defined in 1.6 and the standard mollifier η_μ is defined in 1.6. Note that we mollify only in space, not in time. We also apply this definition to the initial data $\rho^0_\pm$.

Note that by properties of mollification and Sobolev extension,

$$||\rho_\mu - \rho||_{L^\infty(\Omega_\pm)} \leq \mu C \, ||\nabla\rho||_{L^\infty(\Omega_\pm)}$$

hence

$$||\rho_\mu - \rho||_{L^\infty(\Omega_\pm)} \to 0 \text{ as } \mu \to 0 \tag{7.1.1}$$

provided $||\nabla\rho||_{L^\infty(\Omega_\pm)}$ is bounded.

Definition 7.1.2. Let $\mu \in (0, 1]$.

Fix $T \in (0, \infty)$. Also fix $(g_1)_\pm \in C_b((0,T) \times \Omega_\pm; \mathbb{R}^3)$, $g_2 \in C_b((0,T) \times \Gamma)$ and write $g = (g_1, g_2)$ which we will call the right hand side of the equations.

Let $f \in C^1_b((0,T') \times \mathbb{R}^2)$ with $\nabla' f \in C^1_b((0,T') \times \mathbb{R}^2)$ for all $T' \in (0,T)$. Let $U_\pm = (p_\pm, u_\pm, s_\pm) \in C^1_b((0,T') \times \Omega_\pm)$ for all $T' \in (0,T)$. In addition, assume $u_\pm \in C^2_b(\Omega_\pm)$, $u_\pm|_\Gamma \cdot n \in C^2_b(\mathbb{R}^2)$ for each $t \in (0,T)$.

We will say that (U_+, U_-, f) is a solution of the μ-approximate equations on the time interval $(0,T)$ with lifting operator L^a and right hand side g, with initial data

$$(U^0_+, U^0_-, f^0) := (U_+|_{t=0}, U_-|_{t=0}, f|_{t=0})$$

if $\rho(t,x) > 0$, $\rho_\mu(t,x) > 0$ and $J^\psi(t,x) > 0$ in $(0,T) \times \Omega_\pm$, and $p_\pm, u_\pm, s_\pm, f$ satisfy the following system of PDEs.

$$\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)p + \nabla^\psi \cdot u = 0 \tag{7.1.2}$$

$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)u + \nabla^\psi p = \mu \nabla^\psi(\rho c^2 \nabla^\psi \cdot u) + g_1 \tag{7.1.3}$$

$$(\partial_t + u^\psi \cdot \nabla)s = 0 \tag{7.1.4}$$

in $(0,T) \times \Omega_\pm$,

$$\partial_t f = u_\pm \cdot n \tag{7.1.5}$$

$$[p - \mu\rho c^2 \nabla^\psi \cdot u] = -\sigma \nabla' \cdot \hat{n} + \mu(\Delta' - 1)(u \cdot n) + g_2 \tag{7.1.6}$$

on $(0,T) \times \Gamma$,

$$\psi := L^a f \tag{7.1.7}$$

7.2 Initial Data and the Compatibility Conditions

Definition 7.2.1. We say that $(p^0_\pm, u^0_\pm, s^0_\pm, f^0)$ are initial data for the system (7.1.2)-(7.1.7) if the following holds. We require that $f^0 \in C^1_b(\mathbb{R}^2)$. We continue to use notation from before such as $\psi^0 := L^a f^0$. We also require that $p^0_\pm, u^0_\pm, s^0_\pm \in C_b(\Omega_\pm)$ are uniformly continuous functions (where $u^0_\pm$ are vectors representing the initial velocity), so that we may define them on the interface Γ by uniform continuity and we require

$$[u^0 \cdot n^0] = 0$$

Finally, we require

$$\inf_{x\in\Omega_\pm} \rho^0 =: \delta^0 > 0$$

$$\inf_{x\in\Omega_\pm} \rho^0_\mu =: \delta^0_\mu > 0$$

$$\inf_{x\in\mathbb{R}^3} J^{\psi^0} =: \kappa^0 > 0$$

Definition 7.2.2. In the same way as we defined the initial time derivatives in 4.4.2, we may define the initial time derivatives $\partial_0^{\mu,g,j}$ associated with the μ-approximate equations with lifting operator L^a and right hand side g, which act on differentiable expressions involving the initial data (U^0_+, U^0_-, f^0) as follows.

To start with, we define $\partial_0^{\mu,g,j} p^0_\pm, \partial_0^{\mu,g,j} u^0_\pm, \partial_0^{\mu,g,j} s^0_\pm, \partial_0^{\mu,g,j} f^0, \partial_0^{\mu,g,j} \psi^0$ as follows. We set

$$\partial_0^{\mu,g,0} = \mathrm{id}$$

then for $j \geq 0$ we inductively define

$$\partial_0^{\mu,g,j+1} f^0 := \partial_t^j (u_+(t)\big|_\Gamma \cdot n(t))\big|_{t=0} := \partial_t^j (u_+(t,x',0) \cdot n(t))\big|_{t=0} \tag{7.2.1}$$

$$\partial_0^{\mu,g,j+1} \psi^0 := L^a \partial_0^{\mu,g,j+1} f^0 \tag{7.2.2}$$

$$\partial_0^{\mu,g,j+1} p_\pm^0 := \partial_t^j (-(u_\pm^{\psi(t)}(t) \cdot \nabla) p_\pm(t) - \rho_\pm(t) c_\pm^2(t) \nabla^{\psi(t)} \cdot u_\pm(t))\big|_{t=0} \tag{7.2.3}$$

$$\partial_0^{\mu,g,j+1} u_\pm^0 :=$$

$$\partial_t^j (-(u_\pm^{\psi(t)}(t) \cdot \nabla) u_\pm(t) - \frac{1}{\rho_{\mu\pm}(t)} \nabla^{\psi(t)} (p_\pm(t) - \mu \rho_\pm(t) c_\pm^2(t) \nabla^{\psi(t)} \cdot u_\pm(t)) + g_1(t))\big|_{t=0} \tag{7.2.4}$$

$$\partial_0^{\mu,g,j+1} s_\pm^0 := \partial_t^j (-(u_\pm^{\psi(t)}(t) \cdot \nabla) s_\pm(t))\big|_{t=0} \tag{7.2.5}$$

where the expression $\partial_t^j (\cdot)|_{t=0}$ on the right hand side in the first line and last three lines means we think of $u_\pm(t)$, $\psi(t)$ etc as functions of t and formally differentiate with respect to t j times according to the chain and Leibniz rules, then replace $\partial_t^l u_\pm(t)$ for $0 \leq l \leq j$ by $\partial_0^{\mu,g,l} u_\pm^0$ etc.

As before we can extend this definition so that $\partial_0^{\mu,g,j}$ act on differentiable expressions involving the initial data.

Definition 7.2.3. Let (U_+^0, U_-^0, f^0) be initial data for the μ-approximate equations with lifting operator L^a and right hand side g. We say that (U_+^0, U_-^0, f^0) satisfies the normal velocity jump compatibility condition up to order k if

$$\partial_0^{\mu,g,j} [u^0 \cdot n^0] = 0 \tag{7.2.6}$$

for $0 \leq j \leq k$. We say that (U_+^0, U_-^0, f^0) satisfies the pressure interface compatibility condition up to order k if

$$\partial_0^{\mu,g,j} [p^0 - \mu \rho^0 (c^0)^2 \nabla^{\psi^0} \cdot u^0] = \partial_0^{\mu,g,j} (-\sigma (\nabla' \cdot \hat{n}^0) + \mu (\Delta' - 1)(u^0|_\Gamma \cdot n^0)) + \partial_t^j g_2(t)|_{t=0} \tag{7.2.7}$$

for $0 \leq j \leq k$.

Note we continue to use the notation $\psi^0 := L^a f^0$.

7.3 Energy for the μ-Approximate Equations

Here, we introduce two separate energies, E_μ and $\mathcal{E}_\mu$, associated with the μ-approximate equations. We will call the energy E_μ the μ-independent energy since it will be used in a μ-independent energy estimate and should be thought of as being like the energy E for the equations in the fixed domains as defined in 4.5.1 but with some extra terms multiplied by powers of μ added which correspond to the addition of the terms involving μ to the equations in the fixed domains, allowing us to close the energy estimate for the μ-approximate equations. We will call the energy $\mathcal{E}_\mu$ the higher order μ-approximate energy and we will use this for solutions of the μ-approximate equations with μ fixed. The advantage of introducing this energy is that we may perform the μ-independent energy estimate on solutions of the μ-approximate equations with $\mathcal{E}_\mu(T) < \infty$ without having to worry about whether the solution has enough derivatives for us to differentiate the equations in the standard manner.

Definition 7.3.1. Let $f \in L^1_{\text{loc}}((0,T)\times\mathbb{R}^2)$, $U_\pm \in L^1_{\text{loc}}((0,T)\times\Omega_\pm)$. We define the lower-order energy $E_\mu : (0,T] \to [0,\infty]$ for the μ-approximate equations associated to (U_+, U_-, f), as follows.

$$E_\mu(t) = E(t) + \mu^2 \sum_\pm \sum_{0\le j\le 2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla u\right|\right|^2_{H^{3-j}(\Omega_\pm)} + \mu^2 \sum_\pm \int_0^t \left|\left|\partial_t^3 u\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau$$
$$+ \mu \sum_{0\le j\le 3} \int_0^t \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{4-j}(\mathbb{R}^2)} \mathrm{d}\tau + \mu^2 \sum_{0\le j\le 2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{4.5-j}(\mathbb{R}^2)}$$

where $E(t)$ denotes the energy for the fixed equations associated to (U_+, U_-, f), as defined in 4.5.1. Note we adopt the convention that if the solution does not have sufficiently many weak derivatives in $(0,T')\times\Omega_\pm$ to define the energy then it is infinite, and if we write $\mathcal{E}(T') < \infty$ then this implies that all the weak derivatives in $(0,T')\times\Omega_\pm$ present in the energy exist. Note that if $E_\mu(T') < \infty$ then E_μ is uniformly continuous on $(0,T']$.

We define the corresponding initial lower-order energy E^0_μ for the μ-approximate equations with lifting operator L^a and right hand side g associated to the initial data (U^0_+, U^0_-, f^0) as follows.

$$E^0_\mu = E^0 + \mu^2 \sum_\pm \sum_{j=0}^{2} \left|\left|\partial_0^{\mu,g,j} \nabla u^0\right|\right|_{H^{3-j}(\Omega_\pm)} + \mu^2 \sum_{0\le j\le 2} \left|\left|\partial_0^{\mu,g,j} (u^0\cdot n^0)\right|\right|^2_{H^{4.5-j}(\mathbb{R}^2)}$$

where in the definition of E^0 we replace ∂_0^j with $\partial_0^{\mu,g,j}$.

Note that this is the energy we will use in proving a μ-independent energy estimate for the μ-approximate equations, hence the exact dependence of the terms on μ is important.

Definition 7.3.2. Let $f \in L^1_{\text{loc}}((0,T)\times\mathbb{R}^2)$, $U_\pm \in L^1_{\text{loc}}((0,T)\times\Omega_\pm)$. We define the higher order energy $\mathcal{E}_\mu : (0,T] \to [0,\infty]$ associated with (U_+, U_-, f) as follows.

$$\mathcal{E}_\mu(t) = E_\mu(t) + \operatorname*{ess\,sup}_{\tau\in(0,t)} \sum_\pm \sum_{0\le j\le 3} \left|\left|\partial_t^j \nabla U\right|\right|^2_{H^{3-j}(\Omega_\pm)} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \sum_\pm \sum_{0\le j\le 2} \left|\left|\partial_t^j (\nabla^\psi \times u)\right|\right|^2_{H^{4-j}(\Omega_\pm)}$$
$$+ \sum_\pm \int_0^t \left|\left|\partial_t^4 U\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \sum_\pm \sum_{0\le j\le 3} \int_0^t \left|\left|\partial_t^j \nabla u\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau$$
$$+ \sum_{0\le j\le 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)} + \int_0^t \left|\left|\partial_t^4 f\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau$$
$$+ \operatorname*{ess\,sup}_{\tau\in(0,t)} \sum_{0\le j\le 3} \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{4-j}(\Gamma)} + \sum_{0\le j\le 3} \int_0^t \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau$$

Note that if $\mathcal{E}_\mu(T') < \infty$ then $\mathcal{E}_\mu$ is uniformly continuous on $(0,T']$.

We define the corresponding higher-order initial energy $\mathcal{E}^0_\mu$ for the μ-approximate equations with lifting operator L^a and right hand side g associated to the initial data (U^0_+, U^0_-, f^0) as follows.

$$\mathcal{E}^0_\mu = E^0_\mu + \sum_\pm \sum_{0\le j\le 3} \left|\left|\partial_0^{\mu,g,j} \nabla U^0\right|\right|^2_{H^{3-j}(\Omega_\pm)} + \sum_\pm \sum_{0\le j\le 2} \left|\left|\partial_0^{\mu,g,j} (\nabla^{\psi^0} \times u^0)\right|\right|^2_{H^{4-j}(\Omega_\pm)}$$
$$+ \sum_{0\le j\le 3} \left|\left|\partial_0^{\mu,g,j} \nabla' f^0\right|\right|^2_{H^{4.5-j}(\Gamma)} + \sum_{0\le j\le 3} \left|\left|\partial_0^{\mu,g,j} (u^0\cdot n^0)\right|\right|^2_{H^{4-j}(\Gamma)}$$

Note that we will use this energy for fixed μ when proving existence of solutions to the μ-approximate equations and estimates involving it will always depend on μ.

Definition 7.3.3. Given a right hand side function g for shorthand we define the energy associated with g as

$$E_g(t) := \sum_{\pm}\sum_{j=0}^{3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j g_1\right|\right|^2_{H^{5-j}(\Omega_\pm)} + \sum_{j=0}^{4} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j g_2\right|\right|^2_{H^{5-j}(\Gamma)} \tag{7.3.1}$$

for $t \in (0,1]$. Note that if $E_g(T') < \infty$ then E_g is uniformly continuous on $(0,T']$.

Note that for simplicity we have not attempted to make the order of $E_g(t)$ in space as low as possible since eventually g will depend on the smoothed initial data for the original problem in the fixed domains.

Definition 7.3.4. We say that initial data (U_+^0, U_-^0, f^0) for the μ-approximate equations is smooth if

$$\mathcal{E}^0_{\max} < \infty$$

where the high-order initial energy $\mathcal{E}^0_{\max}$ is defined below.

$$\mathcal{E}^0_{\max} = \sum_{\pm} \left|\left|U^0\right|\right|^2_{L^\infty(\Omega_\pm)} + \left|\left|f^0\right|\right|^2_{L^\infty(\Gamma)} + \sum_{\pm} \left|\left|\nabla U^0\right|\right|_{H^{20}(\Omega_\pm)} + \left|\left|\nabla' f^0\right|\right|^2_{H^{20}(\Gamma)} \tag{7.3.2}$$

7.4 Statement of Existence of Solutions to the μ-Approximate Equations

Theorem 7.4.1. *Let $\mu \in (0,1]$ and let (U_+^0, U_-^0, f^0) be smooth initial data for the μ-approximate equations with lifting operator L^a and right hand side g, with $\mathcal{E}^0_{\max} < \infty$. Assume that this initial data satisfies the compatibility condition* (7.2.6) *up to order* 3 *and the condition* (7.2.7) *up to order* 2. *Assume that the right hand side g satisfies $E_g(1) < \infty$. Assume also that the initial data satisfy*

$$\inf_{x\in\Omega_\pm} \rho^0 =: \delta^0 > 0 \tag{7.4.1}$$

$$\inf_{x\in\Omega_\pm} \rho^0_\mu =: \delta^0_\mu > 0 \tag{7.4.2}$$

$$\inf_{x\in\mathbb{R}^3} J^{\psi^0} =: \kappa^0 > 0 \tag{7.4.3}$$

Then there exists a time $T^0_\mu \in (0,1]$ and a solution (U_+, U_-, f) of the μ-approximate equations with lifting operator L^a and right hand side g on the time interval $(0, T^0_\mu)$, as in definition 7.1.2. Moreover,

$$\begin{aligned} \mathcal{E}_\mu(T^0_\mu) &\le C^0_\mu < \infty \\ \rho &\ge \frac{\delta^0}{2} \text{ in } (0,T^0_\mu)\times\Omega_\pm \\ \rho_\mu &\ge \frac{\delta^0_\mu}{2} \text{ in } (0,T^0_\mu)\times\Omega_\pm \\ J^\psi &\ge \frac{1}{2}\kappa^0 \text{ in } (0,T^0_\mu)\times\mathbb{R}^3 \end{aligned}$$

The constant $C^0_\mu > 0$ is bounded above as follows.

$$C^0_\mu \le C_\mu(\mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) < \infty$$

where $C_\mu(\cdot)$ is a smooth increasing function of its arguments, and may depend on μ.

The time $T^0_\mu > 0$ is bounded below as follows.

$$T^0_\mu \ge T_\mu(\mathcal{E}^0_{\max}, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) > 0$$

where $T_\mu(\cdot)$ is a smooth decreasing function of its arguments, and may depend on μ. Note that the time interval of existence here depends on the high-order energy $\mathcal{E}^0_{\max}$ of the initial data.

Proof. See sections 9-14. $\square$

Proposition 7.4.2. *Let $\mu \in (0,1]$ and let (U^0_+, U^0_-, f^0) and g satisfy the hypotheses of 7.4.1. Then there exists $T_\mu \in (0,1]$ and a solution (U_+, U_-, f) of the μ-approximate equations with lifting operator L^a and right hand side g on the time interval $(0, T_\mu)$, satisfying the following additional properties.*

$$\mathcal{E}_\mu(t) < \infty$$
$$\inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho > 0$$
$$\inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho_\mu > 0$$
$$\inf_{\tau\in(0,t)} \inf_{x\in\mathbb{R}} J^\psi > 0$$

for all $t \in (0, T_\mu)$ and if $T_\mu < 1$ then one of the following holds as $t \uparrow T_\mu$.

$$E_\mu(t) \to \infty$$
$$\inf_{x\in\Omega_\pm} \rho \to 0$$
$$\inf_{x\in\Omega_\pm} \rho_\mu \to 0$$
$$\inf_{x\in\Omega_\pm} J^\psi \to 0$$

Note carefully that the blow-up condition is on the lower-order energy $E_\mu(t)$.

Proof. See section 13.3. Note that, due to the smoothness requirement on the initial data in 7.4.1, this proposition is not a directly corollary of 7.4.1 in the usual way via taking final data as initial data to extend the time interval of the solution. We use a slight variant of this idea, hence we record the proof in 13.3.2 where it can be written more conveniently. $\square$

7.5 Statement of the μ-Independent Energy Estimate for the μ-Approximate Equations

Proposition 7.5.1. *Let (U_+, U_-, f) be a solution of the μ-approximate equations with lifting operator L^a and right hand side g as defined in 7.1.2 on the time interval $(0, T)$ with initial data (U^0_+, U^0_-, f^0), where $\mu \in (0,1]$. Suppose in addition that $\mathcal{E}_\mu(T) < \infty$ and that the initial data satisfies the conditions* (7.4.1)-(7.4.3).

Then there exists a time $0 < T^0 \le 1$ and a constant $C^0 > 0$ such that

$$E_\mu(\min\{T^0, T\}) \le C^0$$
$$\rho_\pm \ge \frac{\delta^0}{2} \text{ in } (0, \min\{T^0, T\}) \times \Omega_\pm$$
$$\rho_{\mu\pm} \ge \frac{\delta^0_\mu}{2} \text{ in } (0, \min\{T^0, T\}) \times \Omega_\pm$$
$$J^\psi \ge \frac{1}{2}\kappa^0 \text{ in } (0, \min\{T^0, T\}) \times \mathbb{R}^3$$

The time $T^0 > 0$ is bounded below as follows.

$$T^0 \geq T(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, \frac{1}{\delta^0_\mu}, E_g(1)) > 0$$

where $T(\cdot)$ is a smooth descreasing function of its arguments. Similarly, the constant $C^0 > 0$ is bounded above as follows.

$$C^0 \leq C(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) < \infty$$

where $C(\cdot)$ is a smooth increasing function of its arguments.

Proof. See section 8. □

7.6 Reduction to Solving the μ-Approximate Equations

Proposition 7.6.1. *Let $\mu \in (0,1]$, let (U^0_+, U^0_-, f^0) and g satisfy the hypotheses of 7.4.1, and suppose that propositions 7.4.2 and 7.5.1 hold. Then there exists $T^0 \in (0,1]$ and a solution (U_+, U_-, f) of the μ-approximate equations with lifting operator L^a and right hand side g on the time interval $(0, T^0)$, satisfying the following additional properties.*

$$\begin{aligned} E_\mu(T^0) &\leq C^0 \\ \rho_\pm &\geq \frac{\delta^0}{2} \text{ in } (0,T^0) \times \Omega_\pm \\ \rho_{\mu\pm} &\geq \frac{\delta^0_\mu}{2} \text{ in } (0,T^0) \times \Omega_\pm \\ J^\psi &\geq \frac{1}{2}\kappa^0 \text{ in } (0,T^0) \times \mathbb{R}^3 \end{aligned}$$

The time $T^0 > 0$ is bounded below as follows.

$$T^0 \geq T(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, \frac{1}{\delta^0_\mu}, E_g(1)) > 0$$

where $T(\cdot)$ is a smooth descreasing function of its arguments. Similarly, the constant $C^0 > 0$ is bounded above as follows.

$$C^0 \leq C(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) < \infty$$

where $C(\cdot)$ is a smooth increasing function of its arguments.

Proof. We know by 7.4.2 that there exists $T_\mu \in (0,1]$ and a solution (U_+, U_-, f) of the μ-approximate equations with lifting operator L^a, right hand side g on the time interval $(0, T_\mu)$, satisfying the following additional properties.

$$\begin{aligned} \mathcal{E}_\mu(t) &< \infty \\ \inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho^\mu &> 0 \\ \inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho^\mu_\mu &> 0 \\ \inf_{\tau\in(0,t)} \inf_{x\in\mathbb{R}} J^{\psi^\mu} &> 0 \end{aligned}$$

for all $t \in (0, T_\mu)$ and if $T_\mu < 1$ then one of the following holds as $t \uparrow T_\mu$.

$$\begin{aligned} E_\mu(t) &\to \infty \\ \inf_{x \in \Omega_\pm} \rho^\mu &\to 0 \\ \inf_{x \in \Omega_\pm} \rho_\mu &\to 0 \\ \inf_{x \in \Omega_\pm} J^\psi &\to 0 \end{aligned}$$

Now the solution (U_+, U_-, f) satisfies the hypotheses of the energy estimate 7.5.1 on the interval $(0, \tilde{T})$ for every $\tilde{T} < T_\mu$. Hence we may apply the energy estimate 7.5.1 to obtain a $T^0 \in (0, 1]$ and $C^0 > 0$ such that

$$\begin{aligned} E_\mu(\min\{T^0, \tilde{T}\}) &\leq C^0 \\ \rho_\pm &\geq \frac{\delta^0}{2} \text{ in } (0, \min\{T^0, \tilde{T}\}) \times \Omega_\pm \\ \rho_{\mu\pm} &\geq \frac{\delta^0_\mu}{2} \text{ in } (0, \min\{T^0, \tilde{T}\}) \times \Omega_\pm \\ J^\psi &\geq \frac{\kappa^0}{2} \text{ in } (0, \min\{T^0, \tilde{T}\}) \times \mathbb{R}^3 \end{aligned}$$

for every $\tilde{T} < T_\mu$. But if $T_\mu \leq T^0$ and $T_\mu < 1$, then this contradicts the assumption that one of the above singular convergences holds as $t \to T_\mu$, hence we obtain $T_\mu > T^0$ or $T_\mu = T^0 = 1$ and

$$\begin{aligned} E_\mu(T^0) &\leq C^0 \\ \rho_\pm &\geq \frac{\delta^0}{2} \text{ in } (0, T^0) \times \Omega_\pm \\ \rho_{\mu\pm} &\geq \frac{\delta^0_\mu}{2} \text{ in } (0, T^0) \times \Omega_\pm \\ J^\psi &\geq \frac{\kappa^0}{2} \text{ in } (0, T^0) \times \mathbb{R}^3 \end{aligned}$$

Note that the energy estimate provides the following bounds on T^0 and C^0.

$$T^0 \geq T(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, \frac{1}{\delta^0_\mu}, E_g(1)) > 0$$

where $T(\cdot)$ is a smooth descreasing function of its arguments and

$$C^0 \leq C(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) < \infty$$

where $C(\cdot)$ is a smooth increasing function of its arguments.

This completes the proof. □

Proposition 7.6.2. *Let (U^0_+, U^0_-, f^0) be smooth initial data for the original equations in the fixed domains satisfying the compatibility condition 4.4.7 up to order 2, and the compatibility condition 4.4.6 up to order 3, plus the conditions*

$$\begin{aligned} \inf_{x \in \Omega_\pm} \rho^0 &=: \delta^0 > 0 \\ \inf_{x \in \mathbb{R}^3} J^{\psi^0} &=: \kappa^0 > 0 \end{aligned}$$

Note that by being smooth we mean $\mathcal{E}^0_{\max} < \infty$ where $\mathcal{E}^0_{\max}$ is given in terms of (U^0_+, U^0_-, f^0) by (7.3.2)*. Then there exists a $\mu^0 \in (0,1]$ depending only on the initial density $\rho^0 = \rho(p^0(x), s^0(x))$ and functions $c_1(t,x), c_2(t,x)$ depending only on μ and the initial data (U^0_+, U^0_-, f^0) which are smooth for $\rho^0, \rho^0_\mu, J^{\psi^0} > 0$ such that the following is true. Firstly,*

$$\inf_{x\in\Omega_\pm} \rho^0_\mu =: \delta^0_\mu \geq \frac{\delta^0}{2}$$

for all $\mu \in (0, \mu^0]$. Secondly

$$E_c(1) \to 0 \text{ as } \mu \to 0$$

where $c = (c_1, c_2)$ and E_c is defined in (7.3.1) *with c replacing g. Thirdly, the initial data (U^0_+, U^0_-, f^0) satisfy the compatibility condition 7.2.7 up to order* 2*, and the compatibility condition 7.2.6 up to order* 3 *for the μ-approximate equations with right hand side $c = (c_1, c_2)$.*

Proof. First of all, fix $\mu^0 \in (0,1]$ such that

$$\rho^0_\mu \geq \frac{\delta^0}{2}$$

for all $\mu \in (0, \mu^0]$. Note this is possible by (7.1.1).

We define the functions c_1 and c_2 to make the initial data (U^0_+, U^0_-, f^0) compatible with the μ-approximate equations with right hand side $c = (c_1, c_2)$. We set

$$c_{1\pm}(t) = \sum_{k=0}^{2} \frac{t^k}{k!} c^k_{1\pm}$$
$$c_2(t) = \sum_{k=0}^{2} \frac{t^k}{k!} c^k_2$$

where the coefficients $c^k_{1\pm}, c^k_2$ are defined below. Note that

$$\partial^j_t c_{1\pm}(t)|_{t=0} = c^j_{1\pm} \text{ for } 0 \leq j \leq 2$$
$$\partial^j_t c_2(t)|_{t=0} = c^j_2 \text{ for } 0 \leq j \leq 2$$

We define the coefficients $c^k_{1\pm}, c^k_2$ as follows.

We want to have, for $0 \leq j \leq 3$,

$$\partial^{\mu,c,j}_0 U^0_\pm = \partial^j_0 U^0_\pm \tag{7.6.1}$$
$$\partial^{\mu,c,j}_0 f^0 = \partial^j_0 f^0 \tag{7.6.2}$$

where ∂^j_0 are the initial time derivatives associated with the equations on the fixed domains as defined in (4.4.2). We see that the equations defining $\partial^{\mu,c,j}_0$ are all the same as the equations defining ∂^j_0 except for the velocity equation (7.2.4). Let us assume inductively that (7.6.1)-(7.6.2) hold for $0 \leq k \leq j$ (and note the case $j = 0$ is trivial). This means it makes sense for us to subtract (4.4.4) from (7.2.4) to obtain

$$\partial^{\mu,c,j+1}_0 u^0_\pm - \partial^{j+1}_0 u^0_\pm = \partial^j_0\big(\big(\frac{1}{\rho^0_\pm} - \frac{1}{\rho^0_{\mu\pm}}\big)\nabla^{\psi^0} p^0_\pm + \frac{\mu}{\rho^0_{\mu\pm}}\nabla^{\psi^0}(\rho^0_\pm (c^0_\pm)^2 \nabla^{\psi^0}\cdot u^0_\pm)\big)$$
$$+ \sum_{k+l=j, l\neq j} \partial^k_0\big(\frac{1}{\rho^0_{\mu\pm}}\big) c^l_{1\pm} + \frac{1}{\rho^0_{\mu\pm}} c^j_{1\pm}$$

Thus we define inductively, for $0 \le j \le 2$,

$$c^j_{1\pm} = -\rho^0_{\mu\pm} \sum_{k+l=j, l\neq j} \partial^k_0(\frac{1}{\rho^0_{\mu\pm}})c^l_{1\pm} - \rho^0_{\mu\pm}\partial^j_0((\frac{1}{\rho^0_\pm} - \frac{1}{\rho^0_{\mu\pm}})\nabla^{\psi^0} p^0_\pm + \frac{\mu}{\rho^0_{\mu\pm}}\nabla^{\psi^0}(\rho^0_\pm(c^0_\pm)^2\nabla^{\psi^0}\cdot u^0_\pm))$$

Hence (7.6.1)-(7.6.2) hold for $0 \le j \le 3$. Having established this, we will now write ∂^j_0 instead of $\partial^{\mu,c,j}_0$ for the rest of this proof where appropriate, when acting on this specific initial data (U^0_+, U^0_-, f^0).

We want the initial data (U^0_+, U^0_-, f^0) to be compatible with the μ-approximate equations, that is, to satisfy the compatibility condition (7.2.7) up to order 2 and the compatibility condition (7.2.6) up to order 3.

The fact that (7.2.6) holds up to order 3 for (U^0_+, U^0_-, f^0) follows directly from the fact that (4.4.6) holds up to order 3 and (7.6.1)-(7.6.2) hold for $0 \le j \le 3$. This means $\partial^j_0(u^0_+ \cdot n^0) = \partial^j_0(u^0_- \cdot n^0)$ for $0 \le j \le 3$ so we can write simply $\partial^j_0(u^0 \cdot n^0)$. We define

$$c^j_2 := -\mu(\Delta' - 1)\partial^j_0(u^0 \cdot n^0) - \mu[\partial^j_0(\rho^0(c^0)^2\nabla^{\psi^0} \cdot u^0)]$$

Using the compatibility condition (4.4.7) for $0 \le j \le 2$, we have

$$\begin{aligned}
&\partial^j_0[p^0 - \mu\rho^0(c^0)^2\nabla^{\psi^0}\cdot u^0] - \partial^j_0(-\sigma(\nabla'\cdot\hat{n}^0) + \mu(\Delta' - 1)(u^0|_\Gamma \cdot n^0)) \\
&= -\partial^j_0[\mu\rho^0(c^0)^2\nabla^{\psi^0}\cdot u^0] - \partial^j_0(\mu(\Delta' - 1)(u^0 \cdot n^0)) \\
&= c^j_2
\end{aligned}$$

which is the compatibility condition (7.2.7). Thus (7.2.7) holds up to order 2.

It remains to show that

$$E_c(1) \to 0 \text{ as } \mu \to 0$$

We note from the definition of c_2 that

$$\begin{aligned}
\left|\left|c^j_2\right|\right|_{H^5(\Gamma)} &\le \mu(\left|\left|(\Delta' - 1)\partial^j_0(u^0 \cdot n^0)\right|\right|_{H^5(\Gamma)} + \left|\left|[\partial^j_0(\rho^0(c^0)^2\nabla^{\psi^0}\cdot u^0)]\right|\right|_{H^5(\Gamma)}) \\
&\le \mu F(\mathcal{E}^0_{\max}, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}) \\
&\to 0 \text{ as } \mu \to 0
\end{aligned}$$

where F is some smooth increasing function. We estimate c_1 as follows. Write $\tilde{\rho}^0_\pm = \mathrm{Ext}_{\Omega_\pm}\rho^0_\pm$ where $\mathrm{Ext}_{\Omega_\pm}$ denotes the Sobolev extension operator as defined in 1.6, and let η_μ denote the standard scaled mollifier as defined in 1.6. First we use the mean value inequity to estimate

$$\begin{aligned}
\left|\partial^k_0\rho^0_\mu - \partial^k_0\rho^0\right| &= \left|\int_{|y-x|\le\mu}(\partial^k_0\tilde{\rho}^0(y) - \partial^k_0\tilde{\rho}^0(x))\eta_\mu(x-y)\mathrm{d}y\right| \\
&\le \int_{|y-x|\le\mu}\left|\left|\partial^k_0\nabla\tilde{\rho}^0\right|\right|_{L^\infty(\mathbb{R}^3)}|y-x|\,\eta_\mu(x-y)\mathrm{d}y \\
&\le C\mu\left|\left|\partial^k_0\nabla\rho^0\right|\right|_{L^\infty(\Omega_\pm)}\int_{\mathbb{R}^3}\eta_\mu(x-y)\mathrm{d}y \\
&\le C\mu\left|\left|\partial^k_0\nabla\rho^0\right|\right|_{L^\infty(\Omega_\pm)}
\end{aligned}$$

Now we note that

$$\begin{aligned}
\left|\left|\partial_0^j((\frac{1}{\rho^0} - \frac{1}{\rho_\mu^0})\nabla^{\psi^0} p^0)\right|\right|_{H^5(\Omega_\pm)} &= \left|\left|\partial_0^j((\rho_\mu^0 - \rho^0)\frac{\nabla^{\psi^0} p^0}{\rho^0 \rho_\mu^0})\right|\right|_{H^5(\Omega_\pm)} \\
&= \left|\left|\sum_{k+l=j} (\partial_0^k \rho_\mu^0 - \partial_0^k \rho^0)\partial_0^l(\frac{\nabla^{\psi^0} p^0}{\rho^0 \rho_\mu^0})\right|\right|_{H^5(\Omega_\pm)}
\end{aligned}$$

Hence, from the definition of c_1 we have

$$\begin{aligned}
\left|\left|c_1^j\right|\right|_{H^5(\Omega_\pm)} \leq& \left|\left|\rho_{\mu\pm}^0 \sum_{k+l=j, l\neq j} \partial_0^k(\frac{1}{\rho_{\mu\pm}^0})c_{1\pm}^l\right|\right|_{H^5(\Omega_\pm)} + \left|\left|\rho_{\mu\pm}^0 \partial_0^j((\frac{1}{\rho_\pm^0} - \frac{1}{\rho_{\mu\pm}^0})\nabla^{\psi^0} p_\pm^0\right|\right|_{H^5(\Omega_\pm)} \\
&+ \mu \left|\left|\frac{1}{\rho_{\mu\pm}^0}\nabla^{\psi^0}(\rho_\pm^0 (c_\pm^0)^2 \nabla^{\psi^0} \cdot u_\pm^0))\right|\right|_{H^5(\Omega_\pm)} \\
\leq& \left|\left|\rho_{\mu\pm}^0 \sum_{k+l=j, l\neq j} \partial_0^k(\frac{1}{\rho_{\mu\pm}^0})c_{1\pm}^l\right|\right|_{H^5(\Omega_\pm)} + \mu F(\mathcal{E}_{\max}^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0})
\end{aligned}$$

where F is some smooth increasing function. Note in the case $j = 0$ the first term on the right is zero, hence we obtain $||c_1^0||_{H^5(\Omega_\pm)} \to 0$ as $\mu \to 0$. Then we apply the above estimate inductively to obtain $\left|\left|c_1^j\right|\right|_{H^5(\Omega_\pm)} \to 0$ as $\mu \to 0$ for $0 \leq j \leq 3$. Putting these estimates together we have shown that $E_c(1) \to 0$ as $\mu \to 0$.

This completes the proof. □

Proposition 7.6.3. *Suppose that proposition 7.4.2 and proposition 7.5.1 are true.*

Then the main theorem in the fixed domains, 4.6.1, follows, excluding the statement of uniqueness.

Proof. We let (U_+^0, U_-^0, f^0) be smooth initial data for the original equations in the fixed domains, satisfying the compatibility condition 4.4.7 up to order 2, and the compatibility condition 4.4.6 up to order 3, plus the conditions

$$\inf_{x\in\Omega_\pm} \rho^0 =: \delta^0 > 0$$
$$\inf_{x\in\mathbb{R}^3} J^{\psi^0} =: \kappa^0 > 0$$

Note that by being smooth we mean $\mathcal{E}_{\max}^0 < \infty$ where $\mathcal{E}_{\max}^0$ is given in terms of (U_+^0, U_-^0, f^0) by (7.3.2). By 6.2.1, it suffices to consider such initial data. Let $\mu^0 \in (0, 1]$ be given by 7.6.2 so that

$$\inf_{x\in\Omega_\pm} \rho_\mu^0 =: \delta_\mu^0 \geq \frac{\delta^0}{2}$$

for all $\mu \in (0, \mu^0]$. For each $\mu \in (0, \mu^0]$ let $c^\mu = (c_1^\mu, c_2^\mu)$ be given by 7.6.2 so that (U_+^0, U_-^0, f^0) satisfy the compatibility condition 7.2.7 up to order 2, and the compatibility condition 7.2.6 up to order 3 for the μ-approximate equations with right hand side c^μ. Then we apply 7.6.1 for each $\mu \in (0, \mu^0]$, to obtain $T^{0,\mu} \in (0, 1]$ and a solution $(U_+^\mu, U_-^\mu, f^\mu)$ of the μ-approximate equations with lifting operator L^a, right hand side c^μ and initial data (U_+^0, U_-^0, f^0) on the time interval $(0, T^{0,\mu})$, satisfying the following additional properties.

$$
\begin{aligned}
E_\mu(T^{0,\mu}) &\leq C^{0,\mu} \\
\rho_\pm &\geq \frac{\delta^0}{2} \text{ in } (0, T^{0,\mu}) \times \Omega_\pm \\
\rho^\mu_{\mu\pm} &\geq \frac{\delta^0_\mu}{2} \text{ in } (0, T^{0,\mu}) \times \Omega_\pm \\
J^{\psi^\mu} &\geq \frac{1}{2}\kappa^0 \text{ in } (0, T^{0,\mu}) \times \mathbb{R}^3
\end{aligned}
$$

where we use the superscript $^\mu$ to denote quantities corresponding to the solution $(U^\mu_+, U^\mu_-, f^\mu)$ where necessary.

The time $T^{0,\mu} > 0$ is bounded below as follows.

$$
T^{0,\mu} \geq T(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, \frac{1}{\delta^0_\mu}, E_{c^\mu}(1)) > 0
$$

where $T(\cdot)$ is a smooth descreasing function of its arguments. Similarly, the constant $C^{0,\mu} > 0$ is bounded above as follows.

$$
C^{0,\mu} \leq C(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_{c^\mu}(1)) < \infty
$$

where $C(\cdot)$ is a smooth increasing function of its arguments.

For the sake of simplicity let us redefine $E_{c^\mu}(1)$ to be $\sup_{\tilde{\mu}\in[0,\mu]} E_{c^{\tilde{\mu}}}(1)$ so that $E_{c^\mu}(1)$ is increasing with μ. We do the same with δ^0_μ. We recall that $E_{c^\mu}(1) \to 0$ as $\mu \to 0$ and $\delta^0_\mu \to \delta^0$ as $\mu \to 0$.

Thus we have

$$
T^{0,\mu} \geq T^{0,\mu^0}
$$

and

$$
T^{0,\mu} \uparrow T^0 := T(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, \frac{1}{\delta^0}, 0) \text{ as } \mu \downarrow 0
$$

Similarly, we have

$$
C^{0,\mu} \leq C^{0,\mu^0}
$$

and

$$
C^{0,\mu} \downarrow C^0 := C(E^0, \frac{1}{\delta^0}, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, 0) \text{ as } \mu \downarrow 0
$$

For each $n \in \mathbb{N}$ we consider $\mu \in (0, \mu^n]$ where $\mu^n := \frac{1}{n}\mu^0$. We have

$$
E_\mu(T^{0,\mu^n}) \leq C^{0,\mu^n}
$$

Thus, using weak-$*$ compactness there exists a subsequence μ_k and functions (U_+, U_-, f) with energy $E(T^{0,\mu^n}) < \infty$, where $E(t)$ is the energy associated with (U_+, U_-, f) as defined in 4.3.1, such that the following holds.

$$U^{\mu_k} - U \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T^{0,\mu^n}) \times \Omega_\pm)$$
$$f^{\mu_k} - f \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T^{0,\mu^n}) \times \mathbb{R}^2)$$
$$\partial^\alpha U^{\mu_k} - \partial^\alpha U \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T^{0,\mu^n}); L^2(\Omega_\pm)) \text{ for all } 0 \le |\alpha| \le 3$$
$$\partial^\alpha f^{\mu_k} - \partial^\alpha f \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T^{0,\mu^n}); H^1(\mathbb{R}^2)) \text{ for all } 0 \le |\alpha| \le 3$$
$$\nabla'\partial_t^j f^{\mu_k} - \nabla'\partial_t^j f \overset{*}{\rightharpoonup} 0 \text{ in } L^\infty((0,T^{0,\mu^n}); H^{4.5-j}(\mathbb{R}^2)) \text{ for all } 0 \le j \le 2$$

By weak-$*$ lower semi-continuity of norms, we have

$$E(T^{0,\mu^n}) \le \liminf_{k\to\infty} C^{0,\mu_k} = C^0$$

Note that for the zero order convergence we have used the fact that we may bound the difference

$$\begin{aligned}\sum_\pm \left|\left|U^\mu - U^0\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|f^\mu - f^0\right|\right|^2_{L^2(\Gamma)} &\le C\sum_\pm \left|\left|\partial_t U^\mu\right|\right|^2_{L^2(\Omega_\pm)} + C\left|\left|\partial_t f^\mu\right|\right|^2_{L^2(\Gamma)} \\ &\le CC^{0,\mu^0}\end{aligned}$$

Note that, using the Sobolev embedding theorem, $U_\pm \in C^1_b((0,T^{0,\mu^n}) \times \Omega_\pm)$, $f \in C^1_b((0,T^{0,\mu^n}) \times \mathbb{R}^2)$ and $\nabla' f \in C^1_b((0,T^{0,\mu^n}) \times \mathbb{R}^2)$. Note in particular this implies that $U_\pm$ and f are uniformly continuous (in fact Lipschitz).

Using Sobolev compact embedding, we have the following strong convergences, where $\tilde{\Omega}_\pm$ are any smooth bounded subsets of $\Omega_\pm$ and $\tilde{\Gamma}$ is any smooth bounded subset of $\mathbb{R}^2$.

$$\partial^\alpha U^{\mu_k} - \partial^\alpha U \to 0 \text{ in } L^\infty((0,T^{0,\mu^n}); L^2(\tilde{\Omega}_\pm)) \text{ for all } 0 \le |\alpha| \le 2$$
$$\partial^\alpha f^{\mu_k} - \partial^\alpha f \to 0 \text{ in } L^\infty((0,T^{0,\mu^n}); H^1(\tilde{\Gamma})) \text{ for all } 0 \le |\alpha| \le 2$$

In particular, using the Sobolev embedding theorem again, we have $U^{\mu_k} \to U$ uniformly on smooth bounded subsets of $(0,T^{0,\mu^n}) \times \Omega_\pm$ and $f^{\mu_k} \to f$, $\partial f^{\mu_k} \to \partial f$ uniformly on smooth bounded subsets of $(0,T^{0,\mu^n}) \times \mathbb{R}^2$. This also implies $U^{\mu_k}|_\Gamma \to U|_\Gamma$ uniformly on smooth bounded subsets of $(0,T^{0,\mu^n}) \times \mathbb{R}^2$. Since $\rho^\mu_\pm$ are smooth functions of p and s, we also have $\rho^{\mu_k} \to \rho$ uniformly on smooth bounded subsets of $(0,T^{0,\mu^n}) \times \Omega_\pm$. Hence, we have pointwise in $(0,T^0_{\mu^n}) \times \Omega_\pm$,

$$\rho_\pm = \lim_{k\to\infty} \rho^{\mu_k}_\pm \ge \frac{\delta^0}{2}$$
$$J^\psi = \lim_{k\to\infty} J^{\psi^{\mu_k}} \ge \frac{\kappa^0}{2}$$

Using the standard properties of mollifiers, we also obtain $\rho^{\mu_k}_{\mu_k} \to \rho$ uniformly on smooth bounded subsets of $(0,T^{0,\mu^n}) \times \Omega_\pm$.

Note that by passing to a subsequence if necessary we may assume additionally that $\partial U_{m_k} \to \partial U$ almost everywhere in $(0,T^0_l) \times \Omega_\pm$ and $\nabla'\partial f_{m_k} \to \nabla'\partial f$ almost everywhere on $(0,T^0_l) \times \Gamma$. We use this convergence almost everywhere to let $k \to \infty$ for almost every $(t,x) \in (0,T^{0,\mu^n}) \times \Omega_\pm$ and almost every $(t,x') \in (0,T^{0,\mu^n}) \times \mathbb{R}^2$ in the system of equations

$$\frac{1}{\rho^{\mu_k}(c^{\mu_k})^2}(\partial_t + (u^{\mu_k})^{\psi^{\mu_k}} \cdot \nabla)p^{\mu_k} + \nabla^{\psi^{\mu_k}} \cdot u^{\mu_k} = 0$$
$$\rho^{\mu_k}_{\mu_k}(\partial_t + (u^{\mu_k})^{\psi^{\mu_k}} \cdot \nabla)u^{\mu_k} + \nabla^{\psi^{\mu_k}} p^{\mu_k} = \mu_k \nabla^{\psi^{\mu_k}}(\rho^{\mu_k}(c^{\mu_k})^2 \nabla^{\psi^{\mu_k}} \cdot u^{\mu_k}) + \rho^{\mu_k}_{\mu_k} c^{\mu_k}_1(t)$$
$$(\partial_t + (u^{\mu_k})^{\psi^{\mu_k}} \cdot \nabla)s^{\mu_k} = 0$$

in $(0, T^{0,\mu^n}) \times \Omega_\pm$,

$$\partial_t f^{\mu_k} = u_\pm^{\mu_k} \cdot n^{\mu_k}$$
$$[p^{\mu_k} - \mu_k \rho^{\mu_k} (c^{\mu_k})^2 \nabla^{\psi^{\mu_k}} \cdot u^{\mu_k}] = -\sigma \nabla' \cdot \hat{n}^{\mu_k} + \mu_k (\Delta' - 1)(u^{\mu_k} \cdot n^{\mu_k}) + c_2^{\mu_k}(t)$$

on $(0, T^{0,\mu^n}) \times \Gamma$,

$$\psi^{\mu_k} := L^{f^0} f^{\mu_k}$$

In particular, note $\mu_k \to 0$. Note also that

$$\begin{aligned}\left|\left|\nabla^{\psi^{\mu_k}} (\rho^{\mu_k} (c^{\mu_k})^2 \nabla^{\psi^{\mu_k}} \cdot u^{\mu_k})\right|\right|^2_{L^2(\Omega_\pm)} &\leq E_{\mu_k}(t) \\ &\leq C^{0,\mu^n}\end{aligned}$$

so that, passing to a further subsequence if necessary, we may assume that

$$\mu_k \nabla^{\psi^{\mu_k}} (\rho^{\mu_k} (c^{\mu_k})^2 \nabla^{\psi^{\mu_k}} \cdot u^{\mu_k}) \to 0$$

almost everywhere in $(0, T^{0,\mu^n}) \times \Omega_\pm$. Similarly,

$$\begin{aligned}\left|\left|(\Delta' - 1)(u^{\mu_k} \cdot n^{\mu_k})\right|\right|^2_{L^2(\Gamma)} &\leq E_{\mu_k}(t) \\ &\leq C^{0,\mu^n}\end{aligned}$$

so we may assume that

$$\mu_k (\Delta' - 1)(u^{\mu_k} \cdot n^{\mu_k}) \to 0$$

almost everywhere on $(0, T^{0,\mu^n}) \times \Gamma$.

We also note that $E_{c^\mu}(1) \to 0$ clearly implies that $c_1^{\mu_k} \to 0$ and $c_2^{\mu_k} \to 0$ as $k \to 0$ almost everywhere in $(0, T^{0,\mu^n}) \times \Omega_\pm$ and on $(0, T^{0,\mu^n}) \times \Gamma$ respectively.

Hence in the limit $k \to \infty$ we obtain that (U_+, U_-, f) is a solution of the system (4.3.1)-(4.3.6) on the time interval $(0, T^{0,\mu^n})$ as defined in 4.3.1.

We now claim that $(U_+|_{t=0}, U_-|_{t=0}, f_{t=0}) = (U_+^0, U_-^0, f^0)$. Indeed, we note that using the uniform continuity and convergence which we have shown, we have

$$U_\pm^0 = U_\pm^{\mu_k}|_{t=0} \to U_\pm|_{t=0} \text{ in } \Omega_\pm$$
$$f^0 = f^{\mu_k}|_{t=0} \to f|_{t=0} \text{ in } \mathbb{R}^2$$

hence $(U_+|_{t=0}, U_-|_{t=0}, f|_{t=0}) = (U_+^0, U_-^0, f^0)$ as claimed.

By taking successive subsequences of $(U_+^\mu, U_-^\mu, f^\mu)$ for each n and repeating the above process, we can define (U_+, U_-, f) on $(0, T^{0,\mu^n})$ for all n, and hence on the entire interval $(0, T^0)$. We know that (U_+, U_-, f) is a solution of the system (4.3.1)-(4.3.6) on the time interval $(0, T^0)$ as defined in 4.3.1 with initial data (U_+^0, U_-^0, f^0). Moreover, the following inequalities hold.

$$E(T^0) \leq C^0$$
$$\rho_\pm \geq \frac{\delta^0}{2} \text{ in } (0, T^0) \times \Omega_\pm$$
$$J^\psi \geq \frac{\kappa^0}{2} \text{ in } (0, T^0) \times \mathbb{R}^3$$

This completes the proof. □

7.7 The Curl of the Velocity Equation

It will be necessary during the μ-independent energy estimate to take the curl of the velocity equation. We will also introduce a linearised equation for the curl later, which is motivated by taking the curl of the velocity equation before linearisation. For these reasons we present the curl of the velocity equation, written as an equation for the curl of the velocity, $\nabla^\psi \times u$, below.

Lemma 7.7.1. *Suppose that u satisfies the μ-approximate velocity equation* (7.1.3). *Suppose that in addition to the regularity stated in 7.1.2, we have $\partial_t\psi, \partial_t u, g_1 \in H^1_{\text{loc}}(\Omega_\pm)$, $p, s, u, \nabla\psi, \nabla u \in H^2_{\text{loc}}(\Omega_\pm)$. Then $\nabla^\psi \times u$ satisfies the following equation.*

$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)(\nabla^\psi \times u) = -\nabla^\psi \rho_\mu \times (\partial_t + u^\psi \cdot \nabla)u - \rho_\mu(\epsilon_{ijk}\partial^\psi_j u_l \partial^\psi_l u_k) + \nabla^\psi \times g_1 \tag{7.7.1}$$

in $(0,T) \times \Omega_\pm$, where $(\epsilon_{ijk}\partial^\psi_j u_l \partial^\psi_l u_k)$ denotes the vector with i-th component $\epsilon_{ijk}\partial^\psi_j u_l \partial^\psi_l u_k$ and we have used summation convention for repeated indices.

In particular, our definition of initial time derivatives allows us to restrict the above equation to the time $t = 0$ and replace ∂_t with $\partial_0^{\mu,g,1}$ etc.

Proof. We could apply $\nabla^\psi\times$ to the μ-approximate velocity equation (7.1.3) directly, but it is easiest first to transform back to the original domains $\Omega_\pm(t)$. We recall the velocity equation

$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)u + \nabla^\psi p = \mu\nabla^\psi(\rho c^2 \nabla^\psi \cdot u) + g_1$$

Setting $\check{p} = p\circ\theta^{-1}$, $\check{u} = u\circ\theta^{-1}$ and $\check{s} = s\circ\theta^{-1}$, $\check{\rho}_\mu = \rho_\mu\circ\theta^{-1}$, $\check{\rho} = \rho\circ\theta^{-1}$, $\check{c} = c\circ\theta^{-1}$, $\check{g}_1 = g_1\circ\theta^{-1}$, where the diffeomorphism θ is defined in 5.1.1, and using the coordinate change relations (5.1.1)-(5.1.2), we obtain

$$\check{\rho}_\mu(\partial_t + \check{u}\cdot\nabla)\check{u} + \nabla\check{p} = \mu\nabla(\check{\rho}\check{c}^2\nabla\cdot\check{u}) + \check{g}_1$$

in $\Omega^T_\pm$. Now we apply $\nabla\times$ to obtain

$$\check{\rho}_\mu(\partial_t + \check{u}\cdot\nabla)(\nabla\times\check{u}) + (\nabla\check{\rho}_\mu)\times(\partial_t + \check{u}\cdot\nabla)\check{u} + \check{\rho}_\mu(\epsilon_{ijk}\partial_{x_j}\check{u}_l\partial_{x_l}\check{u}_k) = \nabla\times\check{g}_1$$

Then we use the coordinate change relations (5.1.1)-(5.1.2) again to obtain

$$\rho_\mu(\partial_t + u^\psi\cdot\nabla)(\nabla^\psi\times u) + (\nabla^\psi\rho_\mu)\times(\partial_t + u^\psi\cdot\nabla)u + \rho_\mu(\epsilon_{ijk}\partial^\psi_j u_l\partial^\psi_l u_k) = \nabla^\psi\times g_1$$

as required. $\square$

8. Proof of the μ-Independent Energy Estimate for the μ-Approximate Equations

In this section we assume we are given a solution (U_+, U_-, f) of the μ-approximate equations (7.1.2)-(7.1.7) satisfying the hypotheses of 7.5.1 and we proceed to show that the conclusions of 7.5.1 hold. See 3.5 for a summary of the main steps in the proof of the energy estimate.

8.1 Notation for Constants and Functions of the Energy and Initial Data

Definition 8.1.1. It will be convenient to define some generic constants/functions depending on the energy and initial data in a given way.

Firstly, we will always write F for a generic smooth increasing function of its arguments which is independent of μ.

We will write C for a generic constant which is independent of the solution, including being independent of the initial data, and independent of μ.

We define M^0 as a generic constant of the form

$$M^0 := F(E^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1))$$

We define $K(t)$ as a generic function of t of the form

$$K(t) := F(E_\mu(t), M^0)$$

Remark 8.1.2. Note that for t bounded above, we may bound the square of the expression

$$M^0 + tK(t)$$

by an expression of the same form. In fact, since we are considering short-time existence only, we have chosen to always bound t above by 1, where 1 is chosen as a convenient strictly positive constant. Note there is nothing special about this choice of 1, and if we wish the time interval of existence as stated in the main theorem 2.4.1 may always be maximised using the energy estimate given in 2.4.1 and taking final data as initial data.

Note that throughout this section we will group lower order terms into generic remainder terms, denoted by R with appropriate superscripts and subscripts to denote the order of the remainder. We define these remainder terms and prove estimates for them in 8.18 in order to try and focus on the more important estimates in the main body of the proof of the μ-independent energy estimate.

8.2 The Material Derivative Operators

Definition 8.2.1. We define the material derivative operators $D_t^{u\pm}$ by

$$D_t^{u\pm} = \partial_t + u_\pm^\psi \cdot \nabla$$

It is important to note that we allow D_t^{u-} to act on functions on Ω_+ by defining u_- on Ω_+ by Sobolev extension, and vice-versa, in which case D_t^{u-} (or D_t^{u+}) is no longer a material derivative in Ω_+ (or Ω_-), but it has the important advantage of being continuous across the interface Γ.

Note that $D_t^{u\pm}$ acts along Γ since

$$D_t^{u\pm} = \partial_t + u'_\pm \cdot \nabla' \tag{8.2.1}$$

on Γ as $u_{3\pm}^\psi = 0$ on Γ.

We also define the material derivative of the pressure $\dot{p}$ by

$$\dot{p}_\pm = D_t^{u\pm} p_\pm$$

Note that, by virtue of the pressure equation (7.1.2) which is the same as (4.3.1), we have the important relation

$$\dot{p} = -\rho c^2 \nabla^\psi \cdot u$$

for solutions of the equations.

8.3 Restriction of the Time Interval

Note that by uniform continuity and the conditions (7.4.1)-(7.4.3), there exists a $0 < \tilde{T} \leq \min\{T, 1\}$, which may depend on the solution, such that

$$\rho \geq \frac{1}{2}\delta^0 \tag{8.3.1}$$

$$\rho_\mu \geq \frac{1}{2}\delta_\mu^0 \tag{8.3.2}$$

$$J^\psi \geq \frac{1}{2}\kappa^0 \tag{8.3.3}$$

for $t \in (0, \tilde{T})$.

We define T' to be the supremum over all such $\tilde{T}$.

In the estimates below will will assume that $t \in (0, T')$ until stated otherwise. Part of our goal will be to prove that T' can be taken bigger than T^0, which depends only on the initial data.

Our main aim will be to show that for $t \in (0, T')$ we have

$$E_\mu(t) \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

where we are free to choose $\epsilon \in (0, 1]$. We could absorb the term $\epsilon^{\frac{1}{2}} tK(t)E_\mu(t)$ into the term $\frac{1}{\epsilon^{\frac{3}{2}}} tK(t)$ but because this is how the estimate naturally comes out, and also to maintain consistency with later similar notation, we leave it like this.

8.4 Using the Energy to Extend the Assumptions on the Initial Data

We claim that the following inequalities hold.

$$\left|\left|\rho - \rho^0\right|\right|_{L^\infty(\Omega_\pm)} \leq tK(t) \tag{8.4.1}$$

$$\left|\left|\rho_\mu - \rho_\mu^0\right|\right|_{L^\infty(\Omega_\pm)} \leq tK(t) \tag{8.4.2}$$

$$\left|\left|\partial_{x_3}\psi - \partial_{x_3}\psi^0\right|\right|_{L^\infty(\Omega_\pm)} \leq tK(t) \tag{8.4.3}$$

To prove this, we use the Sobolev embedding theorem $H^2(\Omega_\pm) \subset L^\infty(\Omega_\pm)$.

Indeed, we have

$$\left|\partial_t(\rho - \rho^0)\right| = |\partial_t \rho| \leq C\,||\partial_t\rho||_{H^2(\Omega_\pm)} \leq K(t)$$

where we have used the chain rule. Hence

$$-K(t) \leq \partial_t(\rho - \rho^0) \leq K(t)$$

and integrating from 0 to t we obtain

$$-tK(t) \leq \rho - \rho^0 \leq tK(t)$$

which proves the first inequality. The estimate for ρ_μ is exactly similar, and we merely need to note that $|\partial_t\rho_\mu| \leq C\,||\partial_t\rho||_{L^\infty(\Omega_\pm)}$. The third inequality is also similar.

8.5 Estimate of the Lower Order Terms in the Energy

We claim that

$$||U, \psi||_{L^\infty(\Omega_\pm)} \leq M^0 + tK(t) \tag{8.5.1}$$

$$||f||_{W^{1,\infty}(\Gamma)} \leq M^0 + tK(t) \tag{8.5.2}$$

$$||\partial^\alpha U||^2_{L^2(\Omega_\pm)} \leq M^0 + tK(t) \tag{8.5.3}$$

$$||\partial^\alpha f||^2_{H^1(\Gamma)} + \left|\left|\partial^\beta f\right|\right|^2_{H^{2.5}(\Gamma)} \leq M^0 + tK(t) \tag{8.5.4}$$

where $1 \leq |\alpha| \leq 2$ and $|\beta| = 1$.

Indeed, the proof is very similar to the proof of the inequalities in the previous section. Let a be a component of (U, ψ). We have

$$|a| \leq \left|a^0\right| + \int_0^t |\partial_t a|\,\mathrm{d}\tau \leq M^0 + Ct\,||\partial_t a||_{H^2(\Omega_\pm)} \leq M^0 + tK(t)$$

where we have used the fact that $\left|\left|(U^0, \psi^0)\right|\right|^2_{L^\infty(\Omega_\pm)} \leq E^0_\mu$ and $||\partial_t(U, \psi)||^2_{H^2(\Omega_\pm)} \leq E_\mu(t)$. The estimate of $||f||_{W^{1,\infty}(\Gamma)}$ is exactly the same with $a = f$ or $a = \nabla' f$.

Now let a be an entry of $\partial^\alpha U$ where $1 \leq |\alpha| \leq 2$. We have

$$||a||_{L^2(\Omega_\pm)} \leq \left|\left|a^0\right|\right|_{L^2(\Omega_\pm)} + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\,||a||_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \leq M^0 + \int_0^t ||\partial_t a||_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \leq M^0 + tK(t)$$

After squaring, we obtain (8.5.3). Note that the inequality $\frac{\mathrm{d}}{\mathrm{d}t}\,||a||_{L^2(\Omega_\pm)} \leq ||\partial_t a||_{L^2(\Omega_\pm)}$ is true in general for the derivative with respect to t of an inner-product-induced norm of a quantity depending on the parameter t in a differentiable way. The estimate (8.5.4) for f is exactly similar.

8.6 The Differentiated Equations and the Corrected Pressure, Velocity and Entropy

Once we have differentiated the equations enough times, the lower order terms may all be estimated in terms of the energy by the Sobolev embedding theorem, and hence we may try to obtain energy estimates in the same way as for constant-coefficient linear equations. One problem is that the pressure and velocity equations involve a time derivative of the lift of the front, ψ, whose time derivatives can only be estimated to the same order as the velocity etc, so it cannot be treated as a coefficient. Treating ψ like the other unknowns U makes the equations non-symmetric so that we cannot obtain energy estimates directly. To rectify this, we introduce corrected pressure, velocity and entropy functions - introduced in the linear setting by Alinhac in [2] and referred to as the good unknowns - which absorbs the highest order derivatives of ψ into the unknowns. Note that we only need to use these when estimating the highest order time derivatives, although we may use them for estimating other derivatives as well simply for convenience.

8.6.1 The Differentiated Interior Equations

We apply ∂^α to the equations (7.1.2)-(7.1.4), where $1 \leq |\alpha| \leq 3$, and use the Leibniz and chain rules to obtain the following equations. Note that, prior to differentiation, we have replaced $\rho c^2 \nabla^\psi \cdot u$ with $-\dot{p}$ which is more convenient to work with.

$$\begin{aligned}
&\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)\partial^\alpha p - \frac{1}{\rho c^2 J^\psi}(\partial_t \partial^\alpha \psi + u \cdot \nabla \partial^\alpha \psi - \frac{1}{J^\psi}(\partial_{x_3}\partial^\alpha \psi)(\partial_t \psi + u \cdot \nabla \psi))\partial_{x_3} p \\
&+ \nabla^\psi \cdot \partial^\alpha u - \frac{\nabla \partial^\alpha \psi}{J^\psi} \cdot \partial_{x_3} u + \frac{\nabla \psi}{(J^\psi)^2}\partial_{x_3}\partial^\alpha \psi \cdot \partial_{x_3} u = R^{|\alpha|}(U, \psi) \qquad (8.6.1) \\
&\rho(\partial_t + u^\psi \cdot \nabla)\partial^\alpha u - \frac{\rho}{J^\psi}(\partial_t \partial^\alpha \psi + u \cdot \nabla \partial^\alpha \psi - \frac{1}{J^\psi}(\partial_{x_3}\partial^\alpha \psi)(\partial_t \psi + u \cdot \nabla \psi))\partial_{x_3} u \\
&+ \nabla^\psi \partial^\alpha p - \frac{\nabla \partial^\alpha \psi}{J^\psi}\partial_{x_3} p + \frac{\nabla \psi}{(J^\psi)^2}\partial_{x_3}\partial^\alpha \psi \partial_{x_3} p = -\mu \nabla^\psi \partial^\alpha \dot{p} + R^{|\alpha|}(U, \psi) + \mu R^{|\alpha|}(U, \psi, \dot{p}) \quad (8.6.2) \\
&(\partial_t + u^\psi \cdot \nabla)\partial^\alpha s - \frac{1}{J^\psi}(\partial_t \partial^\alpha \psi + u \cdot \nabla \partial^\alpha \psi - \frac{1}{J^\psi}(\partial_{x_3}\partial^\alpha \psi)(\partial_t \psi + u \cdot \nabla \psi))\partial_{x_3} s = R^{|\alpha|}(U, \psi) \\
&\qquad (8.6.3)
\end{aligned}$$

in $(0, T') \times \Omega_\pm$. The lower order remainder terms on the right hand side are defined in 8.18.5 and 8.18.8. Note that in fact, we could absorb the terms involving $\nabla \partial^\alpha \psi$ into the remainder, but not the terms involving $\partial_t \partial^\alpha \psi$ if ∂^α consists of all time derivatives.

Indeed, we now assume that $\alpha_0 \leq 2$. Because ∂^α contains a maximum of 2 time derivatives, we may absorb the terms involving high-order derivatives of ψ in the equations (8.6.1)-(8.6.2) into the remainder terms. Hence we obtain the following equations.

$$\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)\partial^\alpha p + \nabla^\psi \cdot \partial^\alpha u = R^{|\alpha|}(U, \psi) \qquad (8.6.4)$$

$$\rho(\partial_t + u^\psi \cdot \nabla)\partial^\alpha u + \nabla^\psi \partial^\alpha p = -\mu \nabla^\psi \partial^\alpha \dot{p} + R^{|\alpha|}(U, \psi) + \mu R^{|\alpha|}(U, \psi, \dot{p}) \qquad (8.6.5)$$

$$(\partial_t + u^\psi \cdot \nabla)\partial^\alpha s = R^{|\alpha|}(U, \psi) \qquad (8.6.6)$$

in $(0, T') \times \Omega_\pm$.

8.6.2 The Corrected Pressure, Velocity and Entropy

To remove the terms involving high order time derivatives of ψ from the differentiated pressure and velocity equations, we introduce the corrected pressure, velocity and entropy. These were first introduced by Alinhac in [2] in the linearised setting and called the 'good unknowns'.

For α a multi-index with $2 \leq |\alpha| \leq 3$, we define the corrected unknowns (pressure, velocity and entropy) as

$$U^\alpha = \partial^\alpha U - \frac{\partial^\alpha \psi}{J^\psi} \partial_{x_3} U \tag{8.6.7}$$

We write

$$U^\alpha = (p^\alpha, u^\alpha, s^\alpha)$$

Note that, provided $|\alpha|$ is large enough, $\partial U^\alpha = \partial \partial^\alpha U - \frac{\partial \partial^\alpha \psi}{J^\psi} \partial_{x_3} U$ plus lower order terms, and we see exactly this term in the differentiated pressure equations. This is the motivation for the definition of U^α.

Lemma 8.6.1. *For $2 \leq |\alpha| \leq 3$ and $|\beta| = 2$, we have*

$$||\partial^\alpha U||^2_{L^2(\Omega_\pm)} \leq C\,||U^\alpha||^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))\,||\partial^\alpha f||^2_{H^1(\Gamma)} \tag{8.6.8}$$

$$\left|\left|D_t^{u_-} \partial^\beta U\right|\right|^2_{L^2(\Omega_\pm)} \leq C\left|\left|D_t^{u_-} U^\beta\right|\right|^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))\left|\left|(\partial^\beta f, \partial' \partial^\beta f)\right|\right|^2_{H^1(\Gamma)} \tag{8.6.9}$$

$$||U^\alpha||^2_{L^2(\Omega_\pm)} \leq C\,||\partial^\alpha U||^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))\,||\partial^\alpha f||^2_{H^1(\Gamma)} \tag{8.6.10}$$

Proof. Let $2 \leq |\alpha| \leq 3$ and $0 \leq |\gamma| \leq 3 - |\alpha|$. Using the above definition, we have

$$
\begin{aligned}
&\frac{1}{C}\,||\partial^\gamma(\partial^\alpha U)||^2_{L^2(\Omega_\pm)} \leq ||\partial^\gamma(U^\alpha)||^2_{L^2(\Omega_\pm)} + \left|\left|\partial^\gamma(\frac{\partial^\alpha \psi}{J^\psi}\partial_{x_3} U)\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq ||\partial^\gamma(U^\alpha)||^2_{L^2(\Omega_\pm)} + \left|\left|\frac{1}{J^\psi}\right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial_{x_3} U||^2_{H^1(\Omega_\pm)}\,||\partial^\gamma \partial^\alpha \psi||^2_{H^1(\Omega_\pm)} \\
&+ \left|\left|\frac{1}{J^\psi}\right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial^\gamma \partial_{x_3} U||^2_{L^2(\Omega_\pm)}\,||\partial^\alpha \psi||^2_{H^2(\Omega_\pm)} + \left|\left|\frac{\partial^\gamma \partial_{x_3}\psi}{(J^\psi)^2}\right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial_{x_3} U||^2_{H^1(\Omega_\pm)}\,||\partial^\alpha \psi||^2_{H^1(\Omega_\pm)} \\
&\leq ||\partial^\gamma U^\alpha||^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))(||\partial^\gamma \partial^\alpha f||^2_{H^1(\Gamma)} + ||\partial^\alpha f||^2_{H^{1+|\gamma|}(\Gamma)})
\end{aligned}
$$

where we have used the lower order estimates (8.5.1)-(8.5.3) and the Sobolev embedding theorem to estimate the L^2 norm of a product in terms of the product of H^1 norms, and the last two terms in the second to last line do not exist in the case $|\alpha| = 3$. Along with the lower order estimate (8.5.1) for u_-^ψ, this proves (8.6.8)-(8.6.9). Similarly, we have

$$
\begin{aligned}
\frac{1}{C}\,||U^\alpha||^2_{L^2(\Omega_\pm)} &\leq ||\partial^\alpha U||^2_{L^2(\Omega_\pm)} + \left|\left|\frac{\partial^\alpha \psi}{J^\psi}\partial_{x_3} U\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq ||\partial^\alpha U||^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))\,||\partial^\alpha f||^2_{H^1(\Gamma)}
\end{aligned}
$$

□

Lemma 8.6.2. *For $2 \leq |\alpha| \leq 3$, we have*

$$(\partial_t + u^\psi \cdot \nabla)\partial^\alpha U = (\partial_t + u^\psi \cdot \nabla)U^\alpha + \frac{(\partial_t + u^\psi \cdot \nabla)\partial^\alpha \psi}{J^\psi}\partial_{x_3} U + R^{|\alpha|}(U, \psi)$$
$$= (\partial_t + u^\psi \cdot \nabla)U^\alpha + \frac{1}{J^\psi}((\partial_t + u \cdot \nabla)\partial^\alpha \psi - \frac{1}{J^\psi}(\partial_t \psi + u \cdot \nabla \psi)\partial_{x_3}\partial^\alpha \psi)\partial_{x_3} U + R^{|\alpha|}(U, \psi)$$

and

$$\nabla^\psi \partial^\alpha U = \nabla^\psi U^\alpha + \frac{\nabla^\psi \partial^\alpha \psi}{J^\psi}\partial_{x_3} U + R^{|\alpha|}(U, \psi)$$
$$= \nabla^\psi U^\alpha + \frac{1}{J^\psi}(\nabla \partial^\alpha \psi - \frac{\nabla \psi}{J^\psi}\partial_{x_3}\partial^\alpha \psi)\partial_{x_3} U + R^{|\alpha|}(U, \psi)$$

where $R^k(U, \psi)$ denotes the same generic remainder term given in 8.18.5.

Proof. We may simply differentiate the formula (8.6.7) to find

$$\partial \partial^\alpha U = \partial U^\alpha + \frac{\partial \partial^\alpha \psi}{J^\psi}\partial_{x_3} U + \frac{\partial^\alpha \psi}{J^\psi}\partial \partial_{x_3} U - \frac{\partial^\alpha \psi}{(J^\psi)^2}\partial \partial_{x_3}\psi \partial_{x_3} U$$

and we note that the last two terms on the right have the same form as the remainder $R^{|\alpha|}(U, \psi)$, provided that $2 \leq |\alpha| \leq 3$. Then we simply use the definitions of ∇^ψ and u^ψ. ☐

Remark 8.6.3. Note in the case $|\alpha| = 1$, the remainder term in the above lemma is not quite of the correct form, so for convenience we exclude this case. However, estimating such lower order terms is very easy since their time derivatives are controlled by the energy.

We now substitute

$$\partial^\alpha U = U^\alpha + \frac{\partial^\alpha \psi}{J^\psi}\partial_{x_3} U$$

in the equations (8.6.1)-(8.6.3) for $2 \leq |\alpha| \leq 3$ and use the above lemma to obtain

$$\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)p^\alpha + \nabla^\psi \cdot u^\alpha = R^{|\alpha|}(U, \psi) \tag{8.6.11}$$
$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)u^\alpha + \nabla^\psi p^\alpha = -\mu \nabla^\psi \partial^\alpha \dot{p} + R^{|\alpha|}(U, \psi) + \mu R^{|\alpha|}(U, \psi, \dot{p}) \tag{8.6.12}$$
$$(\partial_t + u^\psi \cdot \nabla)s^\alpha = R^{|\alpha|}(U, \psi) \tag{8.6.13}$$

in $(0, T') \times \Omega_\pm$.

8.6.3 Tangential and Time Derivatives of the Jump Conditions

We may differentiate the jump conditions (7.1.5)-(7.1.6) with respect to time and the horizontal directions to obtain

$$\partial_t \partial^\alpha f = \partial^\alpha u_\pm \cdot n - u'_\pm \cdot \nabla' \partial^\alpha f + R_\Gamma^{|\alpha|-1}(U, f) \tag{8.6.14}$$
$$[\partial^\alpha p + \mu \partial^\alpha \dot{p}] = -\sigma \nabla' \cdot \partial^\alpha \hat{n} + \mu(\Delta' - 1)\partial^\alpha(u \cdot n) + \partial^\alpha g_2 \tag{8.6.15}$$

on $(0, T') \times \Gamma$, where α is a multi-index with $|\alpha| \leq 3$ and $\alpha_3 = 0$. We find it convenient not to try and write these in terms of the corrected pressure and velocity for the time being. The lower order remainder terms on the right hand side are defined in 8.18.14.

8.6.4 Differentiating the Unit Normal

Lemma 8.6.4. *Let $1 \leq |\alpha| \leq 3$ with $\alpha_3 = 0$. Then*

$$\partial^\alpha(\frac{\nabla' f}{|n|}) = \frac{\partial^\alpha \nabla' f}{|n|} - \frac{(\nabla' f \cdot \partial^\alpha \nabla' f)\nabla' f}{|n|^3} + R_\Gamma^{|\alpha|-1}(f)$$

where the remainder $R_\Gamma^{|\alpha|-1}(f)$ is defined in 8.18.19.

Proof. This is a simple application of the product rule. Let $\beta \leq \alpha$ with $|\beta| = 1$. Then, using the product rule,

$$\begin{aligned}\partial^\beta(\frac{\nabla' f}{|n|}) &= \frac{\partial^\beta \nabla' f}{|n|} + \nabla' f \partial^\beta(\frac{1}{|n|}) \\ &= \frac{\partial^\beta \nabla' f}{|n|} - \frac{(\nabla' f \cdot \partial^\beta \nabla' f)\nabla' f}{|n|^3}\end{aligned}$$

where we have used the fact that $n = (-\nabla' f, 1)$. Applying more derivatives, they either act on the terms $\partial^\beta \nabla' f$, or give lower order terms which can be absorbed into the remainder $R_\Gamma^{|\alpha|-1}(f)$. $\square$

Lemma 8.6.5. *Let $w \in \mathbb{R}^2$ be any vector (in particular we are interested in $w = \partial^\alpha \nabla' f$). Then*

$$\frac{1}{|n|^3}|w|^2 \leq \frac{1}{|n|}|w|^2 - \frac{1}{|n|^3}\left|\nabla' f \cdot w\right|^2$$

Proof. Using Cauchy-Schwartz,

$$\begin{aligned}&\frac{1}{|n|}|w|^2 - \frac{1}{|n|^3}\left|\nabla' f \cdot w\right|^2 \\ &\geq \frac{1}{|n|}|w|^2 - \frac{1}{|n|^3}\left|\nabla' f\right|^2|w|^2 \\ &= \frac{1}{|n|^3}(|w|^2(1 + \left|\nabla' f\right|^2) - \left|\nabla' f\right|^2|w|^2) \\ &= \frac{1}{|n|^3}|w|^2\end{aligned}$$

$\square$

8.7 A Slightly Lower Order Estimate for the Front

Let $|\alpha| = 3$ with $\alpha_3 = 0$. In this section we wish to estimate $\partial^\alpha f$ as opposed to $\partial^\alpha \nabla' f$.

Suppose $\partial^\alpha = \partial_{x_i}\partial^\beta$ for $|\beta| = 2$. Then in fact

$$\begin{aligned}||\partial^\alpha f||^2_{L^2(\Gamma)} &\leq \left|\left|\partial^\beta \nabla' f\right|\right|^2_{L^2(\Gamma)} \\ &\leq M^0 + tK(t)\end{aligned}$$

where we have used the lower order estimate (8.5.4).

Otherwise, $\partial^\alpha = \partial_t \partial^\beta$ for $|\beta| = 2$. Then the equation (8.6.14) implies

$$\partial_t \partial^\beta f = \partial^\beta u \cdot n - u' \cdot \partial^\beta \nabla' f + R_\Gamma^1(U, f)$$

Hence, using the lower order estimates (8.5.1)-(8.5.4) and the lower order remainder estimate (8.18.12), we have

$$\begin{aligned}\frac{1}{C}\left|\left|\partial_t\partial^\beta f\right|\right|^2_{L^2(\Gamma)} &\le \left|\left|\partial^\beta u\right|\right|^2_{L^2(\Gamma)}||n||^2_{L^\infty(\Gamma)} + ||u'||^2_{L^\infty(\Gamma)}\left|\left|\partial^\beta\nabla' f\right|\right|^2_{L^2(\Gamma)} + ||R^1_\Gamma(U,f)||^2_{L^2(\Gamma)} \\ &\le (M^0 + tK(t))(1 + \left|\left|\partial^\beta u\right|\right|^2_{L^2(\Gamma)})\end{aligned}$$

We estimate $||\partial^\beta u||^2_{L^2(\Gamma)}$ as follows, by converting to an estimate over the interior - either Ω_+ or Ω_- - then using Cauchy's inequality.

$$\begin{aligned}\left|\left|\partial^\beta u\right|\right|^2_{L^2(\Gamma)} &= \int_\Gamma \partial^\beta u\cdot\partial^\beta u \mathrm{d}x' \\ &= \mp\int_{\Omega_\pm}\partial_{x_3}(\partial^\beta u\cdot\partial^\beta u)\mathrm{d}x \\ &= \mp 2\int_{\Omega_\pm}(\partial_{x_3}\partial^\beta u)\cdot\partial^\beta u\mathrm{d}x \\ &\le \epsilon\left|\left|\partial_{x_3}\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \frac{1}{\epsilon}\left|\left|\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} \\ &\le \epsilon E_\mu(t) + \frac{1}{\epsilon}(M^0 + tK(t))\end{aligned}$$

where we have used the lower order estimate (8.5.3) in the last line. Hence

$$\left|\left|\partial_t\partial^\beta f\right|\right|^2_{L^2(\Gamma)} \le (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))$$

Hence we have shown that

$$\sum_{|\alpha|=3}||\partial^\alpha f||^2_{L^2(\Gamma)} \le (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t)) \tag{8.7.1}$$

8.8 Estimate of the Entropy

We multiply equation (8.6.13) by s^α, where $2 \le |\alpha| \le 3$, and use the product rule to obtain

$$\begin{aligned}\frac{1}{2}\partial_t|s^\alpha|^2 + \frac{1}{2}\nabla\cdot(u^\psi|s^\alpha|^2) &= \frac{1}{2}(\nabla\cdot u^\psi)|s^\alpha|^2 + R^{|\alpha|}(U,\psi)s^\alpha \\ &= R^{|\alpha|}(U,\psi)s^\alpha\end{aligned}$$

in $(0,T')\times\Omega_\pm$. Note that we have absorbed the lower-order term generated by the product rule into the remainder $R^{|\alpha|}(U,\psi)$ on the right hand side. Then we integrate over $\Omega_\pm$ and apply the divergence theorem for Sobolev functions to the second term on the left. Note that since $(u^\psi)_{3\pm} = 0$ on Γ we are left with no boundary terms. Hence we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}|s^\alpha|^2\,\mathrm{d}x = \int_{\Omega_\pm}R^{|\alpha|}(U,\psi)s^\alpha\mathrm{d}x$$

Applying Cauchy's inequality to the right hand side, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}||s^\alpha||^2_{L^2(\Omega_\pm)} \le \left|\left|R^{|\alpha|}(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} + ||s^\alpha||^2_{L^2(\Omega_\pm)}$$

Then using the estimate of the remainder given by lemma (8.18.3), and the estimate of s^α from (8.6.8) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,||s^\alpha||^2_{L^2(\Omega_\pm)} \leq K(t)$$

Integrating from 0 to t, we obtain

$$||s^\alpha||^2_{L^2(\Omega_\pm)} \leq M^0 + tK(t) \tag{8.8.1}$$

8.9 Estimate of the Tangential Derivatives of the Pressure, Velocity and Front

Here we come to the one of the main parts of the energy estimate. Let α be a multi-index with $|\alpha| = 3$ and $\alpha_3 = 0$, so that ∂^α contains no normal derivatives. Assume also that $\alpha_0 \leq 2$, so that we may, if we wish, write $\partial^\alpha = \partial_{x_i}\partial^\beta$ for some $1 \leq i \leq 2$ and $|\beta| = 2$.

We intend to multiply the differentiated pressure and velocity equations (8.6.4)-(8.6.5) by $\partial^\alpha p$ and $\partial^\alpha u$ respectively, then integrate by parts to obtain an energy estimate as for the entropy. Unfortunately, since there are normal derivatives in these equations, we will generate a boundary term on the interface. To deal with this boundary term we will need to use the pressure interface condition (7.1.6) and the equation (7.1.5) for the front. Note that we have differentiated these in the tangential and time directions to obtain (8.6.15) and (8.6.14), but we cannot do the same for the normal derivatives. Hence we will have to estimate these separately. In this section we exclude the estimate of the highest order time derivatives since the surface tension regularises the front in space, but not in time.

We multiply (8.6.4) by $\partial^\alpha p$ and (8.6.5) by $\partial^\alpha u$. We then add the resulting equations to obtain

$$\begin{aligned}
&\frac{1}{\rho c^2}((\partial_t + u^\psi\cdot\nabla)\partial^\alpha p)\partial^\alpha p + \rho_\mu((\partial_t + u^\psi\cdot\nabla)\partial^\alpha u)\cdot\partial^\alpha u + (\nabla^\psi\cdot\partial^\alpha u)\partial^\alpha p + (\nabla^\psi\partial^\alpha p)\cdot\partial^\alpha u\\
&= -\mu\nabla^\psi\partial^\alpha\dot{p}\cdot\partial^\alpha u + R^3(U,\psi)\partial^\alpha p + (R^3(U,\psi) + \mu R^3(U,\psi,\dot{p}))\cdot\partial^\alpha u
\end{aligned}$$

in $(0,T')\times\Omega_\pm$. Now we use the product rule to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t(\frac{1}{\rho c^2}\,|\partial^\alpha p|^2) + \frac{1}{2}\partial_t(\rho_\mu\,|\partial^\alpha u|^2) + \frac{1}{2}\nabla\cdot(\frac{1}{\rho c^2}u^\psi\,|\partial^\alpha p|^2) + \frac{1}{2}\nabla\cdot(\rho_\mu u^\psi\,|\partial^\alpha u|^2)\\
&+\nabla'\cdot(\partial^\alpha p\partial^\alpha u) + \partial_{x_3}(\frac{(1,-\nabla'\psi)}{J^\psi}\partial^\alpha p\cdot\partial^\alpha u) + \mu(\nabla'\partial^\alpha\dot{p} + \partial_{x_3}(\frac{(1,-\nabla'\psi)}{J^\psi}\partial^\alpha\dot{p}))\cdot\partial^\alpha u\\
&= R^3(U,\psi)\partial^\alpha p + R^3(U,\psi)\cdot\partial^\alpha u + \mu R^3(U,\psi,\dot{p})\cdot\partial^\alpha u
\end{aligned}$$

where we have absorbed some lower order terms into the remainder on the right hand side. Now we integrate over $\Omega_\pm$ and apply the divergence theorem. Note that the outward unit normal to Γ in $\Omega_\pm$ is $(0,0,\mp 1)$. The terms which are tangential divergences vanish, and $(u^\psi)_{3\pm} = 0$ on Γ hence the third and fourth terms vanish. We obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\frac{1}{\rho c^2}\,|\partial^\alpha p|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\rho_\mu\,|\partial^\alpha u|^2\,\mathrm{d}x \mp 2\int_\Gamma(\partial^\alpha p + \mu\partial^\alpha\dot{p})\partial^\alpha u\cdot n\mathrm{d}x'\\
&-\mu\int_{\Omega_\pm}\partial^\alpha\dot{p}\nabla^\psi\cdot\partial^\alpha u\mathrm{d}x = \int_{\Omega_\pm}R^3(U,\psi)\partial^\alpha p + R^3(U,\psi)\cdot\partial^\alpha u + \mu R^3(U,\psi,\dot{p})\cdot\partial^\alpha u\mathrm{d}x
\end{aligned}$$

We note that

$$\begin{aligned}\partial^\alpha \dot{p} &= -\partial^\alpha(\rho c^2 \nabla^\psi \cdot u) \\ &= -\rho c^2 \nabla^\psi \cdot \partial^\alpha u + R^3(U,\psi)\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \frac{1}{\rho c^2}\left|\partial^\alpha p\right|^2 \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \rho_\mu \left|\partial^\alpha u\right|^2 \mathrm{d}x \mp 2\int_\Gamma (\partial^\alpha p + \mu \partial^\alpha \dot{p})\partial^\alpha u \cdot n \mathrm{d}x' \\ &+ \mu \int_{\Omega_\pm} \rho c^2 \left|\nabla^\psi \cdot \partial^\alpha u\right|^2 \mathrm{d}x \\ &= \int_{\Omega_\pm} R^3(U,\psi)\partial^\alpha p + R^3(U,\psi)\cdot \partial^\alpha u + \mu R^3(U,\psi,\dot{p})\cdot \partial^\alpha u + \mu R^3(U,\psi)\nabla^\psi \cdot \partial^\alpha u \mathrm{d}x\end{aligned}$$

We integrate in time from 0 to t, apply Cauchy's inequality and use the estimates (8.18.3) and (8.18.7) for the terms on the right hand side. Hence we obtain

$$\begin{aligned}&\int_{\Omega_\pm} \frac{1}{\rho c^2}\left|\partial^\alpha p\right|^2 \mathrm{d}x + \int_{\Omega_\pm} \rho_\mu \left|\partial^\alpha u\right|^2 \mathrm{d}x + \mu \int_0^t \int_{\Omega_\pm} \rho c^2 \left|\nabla^\psi \cdot \partial^\alpha u\right|^2 \mathrm{d}x \mathrm{d}\tau \\ &\mp 2\int_0^t \int_\Gamma (\partial^\alpha p + \mu \partial^\alpha \dot{p})\partial^\alpha u \cdot n \mathrm{d}x' \mathrm{d}\tau \leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))\end{aligned}$$

To deal with the boundary integral over Γ, which is a key point in the estimates and is where we need to use the presence of surface tension, we must sum over $\pm$ to obtain a jump condition. Doing this we obtain

$$\begin{aligned}&\sum_\pm \int_{\Omega_\pm} \frac{1}{\rho c^2}\left|\partial^\alpha p\right|^2 \mathrm{d}x + \sum_\pm \int_{\Omega_\pm} \rho_\mu \left|\partial^\alpha u\right|^2 \mathrm{d}x + \mu \sum_\pm \int_0^t \int_{\Omega_\pm} \rho c^2 \left|\nabla^\psi \cdot \partial^\alpha u\right|^2 \mathrm{d}x \mathrm{d}\tau - 2I^\alpha \\ &\leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))\end{aligned}$$

where

$$I^\alpha = \int_0^t \int_\Gamma [n \cdot \partial^\alpha u(\partial^\alpha p + \mu \partial^\alpha \dot{p})] \mathrm{d}x' \mathrm{d}\tau$$

To estimate I^α, firstly, we use the differentiated equation for the front, (8.6.14), to replace $n \cdot \partial^\alpha u$, and we obtain the following.

$$\begin{aligned}I^\alpha &= \int_0^t \int_\Gamma [n \cdot \partial^\alpha u \partial^\alpha (p + \mu \dot{p})] \mathrm{d}x' \mathrm{d}\tau \\ &= \int_0^t \int_\Gamma \partial_t \partial^\alpha f [\partial^\alpha (p + \mu \dot{p})] \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_\Gamma [u' \cdot \partial^\alpha \nabla' f \partial^\alpha (p + \mu \dot{p})] \mathrm{d}x' \mathrm{d}\tau \\ &+ \int_0^t \int_\Gamma [R^2_\Gamma(U,f)\partial^\alpha (p + \mu \dot{p})] \mathrm{d}x' \mathrm{d}\tau \\ &= I^\alpha_1 + I^\alpha_2 + I^\alpha_3\end{aligned}$$

where, as before, $R^2_\Gamma(U,f)$ denotes a remainder term given in lemma 8.18.14 and we feel free to absorb lower order terms of the correct form into this remainder.

Firstly, we note that, using the remainder estimate (8.18.11), we have

$$|I^\alpha_3| \leq \frac{1}{\epsilon}M^0 + \frac{1}{\epsilon}tK(t) + \epsilon(M^0 + tK(t))E_\mu(t)$$

Let us consider I_1^α. Using the differentiated pressure interface condition (8.6.15), we have

$$
\begin{aligned}
&I_1^\alpha = -\sigma\int_0^t\int_\Gamma \partial_t\partial^\alpha f\partial^\alpha\nabla'\cdot\hat{n}\mathrm{d}x'\mathrm{d}\tau + \mu\int_0^t\int_\Gamma \partial_t\partial^\alpha f\partial^\alpha(\Delta'-1)(u\cdot n)\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma \partial_t\partial^\alpha f\partial^\alpha g_2\mathrm{d}x'\mathrm{d}\tau\\
&=\sigma\int_0^t\int_\Gamma \partial_t\partial^\alpha f\nabla'\cdot\partial^\alpha(\frac{\nabla' f}{|n|})\mathrm{d}x'\mathrm{d}\tau + \mu\int_0^t\int_\Gamma \partial_t\partial^\alpha f\partial^\alpha(\Delta'-1)\partial_t f\mathrm{d}x'\mathrm{d}\tau + \int_0^t\int_\Gamma R_\Gamma^3(f)^2\mathrm{d}x'\mathrm{d}\tau\\
&=-\sigma\int_0^t\int_\Gamma \partial_t\partial^\alpha\nabla' f\cdot\partial^\alpha(\frac{\nabla' f}{|n|})\mathrm{d}x'\mathrm{d}\tau - \mu\int_0^t\int_\Gamma \left|\nabla'\partial^\alpha\partial_t f\right|^2\mathrm{d}x'\mathrm{d}\tau - \mu\int_0^t\int_\Gamma \left|\partial^\alpha\partial_t f\right|^2\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma R_\Gamma^3(f)^2\mathrm{d}x'\mathrm{d}\tau\\
&=I_{11}^\alpha - \mu\int_0^t\int_\Gamma \left|\nabla'\partial^\alpha\partial_t f\right|^2\mathrm{d}x'\mathrm{d}\tau - \mu\int_0^t\int_\Gamma \left|\partial^\alpha\partial_t f\right|^2\mathrm{d}x'\mathrm{d}\tau + \int_0^t\int_\Gamma R_\Gamma^3(f)^2\mathrm{d}x'\mathrm{d}\tau
\end{aligned}
$$

where we have used integration by parts in the second to last line. Assuming there was no factor of $\frac{1}{|n|}$ in I_{11}^α, we could use the product rule to convert the integrand to $\frac{1}{2}\partial_t\left|\partial^\alpha\nabla' f\right|^2$. We intend to do almost the same thing, but to deal with the factor of $\frac{1}{|n|}$ we will need to use lemma 8.6.4.

We have

$$
\begin{aligned}
-I_{11}^\alpha &= \sigma\int_0^t\int_\Gamma \partial_t\partial^\alpha\nabla' f\cdot(\frac{\partial^\alpha\nabla' f}{|n|} - \frac{(\nabla' f\cdot\partial^\alpha\nabla' f)\nabla' f}{|n|^3} + R_\Gamma^2(f))\mathrm{d}x'\mathrm{d}\tau\\
&= \sigma\int_0^t\int_\Gamma \partial_t\partial^\alpha\nabla' f\cdot\frac{\partial^\alpha\nabla' f}{|n|} - \partial_t(\nabla' f\cdot\partial^\alpha\nabla' f)\frac{(\nabla' f\cdot\partial^\alpha\nabla' f)}{|n|^3}\mathrm{d}x'\mathrm{d}\tau\\
&+\sigma\int_0^t\int_\Gamma \partial_t\partial^\alpha\nabla' f\cdot R_\Gamma^2(f)\mathrm{d}x'\mathrm{d}\tau + \sigma\int_0^t\int_\Gamma R_\Gamma^3(f)\cdot R_\Gamma^3(f)\mathrm{d}x'\mathrm{d}\tau\\
&= \frac{1}{2}\sigma\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\frac{1}{|n|}\left|\partial^\alpha\nabla' f\right|^2 - \frac{1}{|n|^3}\left|\nabla' f\cdot\partial^\alpha\nabla' f\right|^2\mathrm{d}x'\mathrm{d}\tau\\
&-\sigma\int_0^t\int_\Gamma \partial_t\partial^\beta\nabla' f\cdot\partial_{x_i}R_\Gamma^2(f)\mathrm{d}x'\mathrm{d}\tau + \sigma\int_0^t\int_\Gamma R_\Gamma^3(f)\cdot R_\Gamma^3(f)\mathrm{d}x'\mathrm{d}\tau\\
&\geq \frac{1}{2}\sigma\int_\Gamma\frac{1}{|n|}\left|\partial^\alpha\nabla' f\right|^2 - \frac{1}{|n|^3}\left|\nabla' f\cdot\partial^\alpha\nabla' f\right|^2\mathrm{d}x' - M^0\\
&-\sigma\int_0^t\left|\left|\partial_t\partial^\beta f\right|\right|_{H^1(\Gamma)}\left|\left|R_\Gamma^2(f)\right|\right|_{H^1(\Gamma)}\mathrm{d}\tau - \sigma\int_0^t\left|\left|R_\Gamma^3(f)\right|\right|_{L^2(\Gamma)}^2\mathrm{d}\tau
\end{aligned}
$$

where we have used the fact that $\partial^\alpha = \partial_{x_i}\partial^\beta$ together with integration by parts for the second term on the right. We now apply lemma 8.6.5 to the first term on the right hand side.

Hence

$$
\begin{aligned}
-I_{11}^\alpha &\geq \frac{1}{2}\sigma\int_\Gamma\frac{1}{|n|^3}\left|\partial^\alpha\nabla' f\right|^2\mathrm{d}x' - M^0 - \sigma\int_0^t\left|\left|\partial_t\partial^\beta f\right|\right|_{H^1(\Gamma)}\left|\left|R_\Gamma^2(f)\right|\right|_{H^1(\Gamma)}\mathrm{d}\tau\\
&-\sigma\int_0^t\left|\left|R_\Gamma^3(f)\right|\right|_{L^2(\Gamma)}^2\mathrm{d}\tau\\
&\geq \frac{1}{2}\sigma\int_\Gamma\frac{1}{|n|^3}\left|\partial^\alpha\nabla' f\right|^2\mathrm{d}x' - M^0 - tK(t)
\end{aligned}
$$

where we have used the remainder estimate (8.18.14).

Hence we obtain

$$
\begin{aligned}
-I_1^\alpha &\geq \frac{1}{2}\sigma\int_\Gamma\frac{1}{|n|^3}\left|\partial^\alpha\nabla' f\right|^2\mathrm{d}x' + \mu\int_0^t\int_\Gamma\left|\nabla'\partial^\alpha(u\cdot n)\right|^2\mathrm{d}x'\mathrm{d}\tau + \mu\int_0^t\int_\Gamma\left|\partial^\alpha(u\cdot n)\right|^2\mathrm{d}x'\mathrm{d}\tau\\
&- M^0 - tK(t)
\end{aligned}
$$

Now we proceed to estimate I_2^α. We have

$$\begin{aligned}
|I_2^\alpha| &\le \sum_\pm \left| \int_0^t \int_\Gamma u'_\pm \cdot \partial^\alpha \nabla' f \partial^\alpha (p_\pm + \mu \dot{p}_\pm) \mathrm{d}x' \mathrm{d}\tau \right| \\
&= \sum_\pm \left| \int_0^t \int_\Gamma u'_\pm \cdot \partial_{x_i} \partial^\beta \nabla' f \partial_{x_i} \partial^\beta (p_\pm + \mu \dot{p}_\pm) \mathrm{d}x' \mathrm{d}\tau \right| \\
&\le \sum_\pm \int_0^t \left|\left| u'_\pm \cdot \partial_{x_i} \partial^\beta \nabla' f \right|\right|_{H^{0.5}(\Gamma)} \left|\left| \partial_{x_i} \partial^\beta (p_\pm + \mu \dot{p}_\pm) \right|\right|_{H^{-0.5}(\Gamma)} \mathrm{d}\tau \\
&\le \sum_\pm \int_0^t ||u_\pm||_{W^{1,\infty}(\Gamma)} \left|\left| \partial_{x_i} \partial^\beta \nabla' f \right|\right|_{H^{0.5}(\Gamma)} \left|\left| \partial^\beta (p_\pm + \mu \dot{p}_\pm) \right|\right|_{H^{0.5}(\Gamma)} \mathrm{d}\tau \\
&\le \sum_\pm \int_0^t ||u_\pm||^2_{W^{1,\infty}(\Gamma)} \left|\left| \partial_{x_i} \partial^\beta \nabla' f \right|\right|^2_{H^{0.5}(\Gamma)} \mathrm{d}t + \sum_\pm \int_0^t \left|\left| \partial^\beta p_\pm \right|\right|^2_{H^{0.5}(\Gamma)} \mathrm{d}\tau \\
&+ \mu^2 \sum_\pm \int_0^t \left|\left| \partial^\beta \dot{p}_\pm \right|\right|^2_{H^{0.5}(\Gamma)} \mathrm{d}\tau \\
&\le tK(t)
\end{aligned}$$

where we have again used the Sobolev trace estimate and the estimate (8.18.5) for $\dot{p}$.

Putting what we have so far together, we have shown that

$$\begin{aligned}
&\sum_\pm \int_{\Omega_\pm} \frac{1}{\rho c^2} |\partial^\alpha p|^2 \mathrm{d}x + \sum_\pm \int_{\Omega_\pm} \rho_\mu |\partial^\alpha u|^2 \mathrm{d}x + \mu \sum_\pm \int_0^t \int_{\Omega_\pm} \rho c^2 \left| \nabla^\psi \cdot \partial^\alpha u \right|^2 \mathrm{d}x \mathrm{d}\tau \\
&+ \int_\Gamma \frac{\sigma}{|n|^3} \left| \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' + \mu \int_0^t \int_\Gamma \left| \nabla' \partial^\alpha (u \cdot n) \right|^2 \mathrm{d}x' \mathrm{d}\tau + \mu \int_0^t \int_\Gamma |\partial^\alpha (u \cdot n)|^2 \mathrm{d}x' \mathrm{d}\tau \\
&\le (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))
\end{aligned}$$

Thus, using the inequalities (8.3.1)-(8.3.2) and the lower order estimates (8.5.1)-(8.5.2), we have

$$\begin{aligned}
&\sum_\pm ||\partial^\alpha p||^2_{L^2(\Omega_\pm)} + \sum_\pm ||\partial^\alpha u||^2_{L^2(\Omega_\pm)} + \mu \sum_\pm \int_0^t \left|\left| \nabla^\psi \cdot \partial^\alpha u \right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ \left|\left| \partial^\alpha \nabla' f \right|\right|^2_{L^2(\Gamma)} + \mu \int_0^t ||\partial^\alpha (u \cdot n)||^2_{H^1(\Gamma)} \mathrm{d}\tau \\
&\le (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))
\end{aligned} \tag{8.9.1}$$

8.10 Estimate of the Time Derivatives of the Corrected Pressure and Velocity and the Front

We cannot apply the elliptic-type estimate for the front to estimate the highest order time derivatives, although part of the approach will be similar to the previous section. Instead we will need to use the structure of the equations. We also need to use the corrected unknowns in order to deal with the highest order time derivative of ψ.

We let β be a multi-index with $|\beta| = 2$, $\beta_3 = 0$ and $\beta_0 \ge 1$, then we apply the operator $D_t^{u_-} := \partial_t + u_-^\psi \cdot \nabla$ (with u_- defined on Ω_+ by Sobolev extension) to the corrected pressure and velocity equations (8.6.11)-(8.6.12) with index β to obtain the following equations.

$$\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla) D_t^{u_-} p^\beta + \nabla^\psi D_t^{u_-} u^\beta = R^3(U, \psi) \tag{8.10.1}$$

$$\rho_\mu (\partial_t + u^\psi \cdot \nabla) D_t^{u_-} u^\beta + \nabla^\psi D_t^{u_-} p^\beta + \mu \nabla^\psi D_t^{u_-} \partial^\beta \dot{p} = R^3(U, \psi) + \mu R^3(U, \psi, \dot{p}) \tag{8.10.2}$$

We now proceed as for the tangential derivatives. Multiplying (8.10.1) by $D_t^{u-}p^\beta$ and (8.10.2) by $D_t^{u-}u^\beta$, integrating over $\Omega_\pm$ and summing over $\pm$ and integrating from 0 to t, we obtain

$$\sum_\pm \int_{\Omega_\pm} \frac{1}{\rho c^2}\left|D_t^{u-}p^\beta\right|^2 \mathrm{d}x + \sum_\pm \int_{\Omega_\pm} \rho_\mu \left|D_t^{u-}u^\beta\right|^2 \mathrm{d}x + \mu \sum_\pm \int_0^t \int_{\Omega_\pm} \rho c^2 \left|\nabla^\psi \cdot (D_t^{u-}\partial^\beta u)\right|^2 \mathrm{d}x\mathrm{d}\tau$$
$$- 2\int_0^t \int_\Gamma [n \cdot D_t^{u-}u^\beta (D_t^{u-}p^\beta + \mu D_t^{u-}\partial^\beta \dot{p})]\mathrm{d}x'\mathrm{d}\tau$$
$$\leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))$$

where we have used the remainder estimates (8.18.3) and (8.18.7).

Define

$$K^\beta = \int_0^t \int_\Gamma [n \cdot D_t^{u-}u^\beta (D_t^{u-}p^\beta + \mu D_t^{u-}\partial^\beta \dot{p})]\mathrm{d}x'\mathrm{d}\tau$$

We must write out $[n \cdot D_t^{u-}u^\beta (D_t^{u-}p^\beta + \mu D_t^{u-}\partial^\beta \dot{p})]$ in terms of the quantities p and u for which we have boundary conditions.

$$K^\beta = \int_0^t \int_\Gamma [n \cdot D_t^{u-}u^\beta (D_t^{u-}p^\beta + \mu D_t^{u-}\partial^\beta \dot{p})]\mathrm{d}x'\mathrm{d}\tau = K_1^\beta - K_2^\beta - K_3^\beta + K_4^\beta$$

where

$$K_1^\beta = \int_0^t \int_\Gamma [n \cdot D_t^{u-}\partial^\beta u D_t^{u-}\partial^\beta (p + \mu \dot{p})]\mathrm{d}x'\mathrm{d}\tau$$
$$K_2^\beta = \int_0^t \int_\Gamma [n \cdot D_t^{u-}\partial^\beta u D_t^{u-}(\partial^\beta f \partial_3 p)]\mathrm{d}x'\mathrm{d}\tau$$
$$K_3^\beta = \int_0^t \int_\Gamma [n \cdot D_t^{u-}\partial^\beta (p + \mu \dot{p}) D_t^{u-}(\partial^\beta f \partial_3 u)]\mathrm{d}x'\mathrm{d}\tau$$
$$K_4^\beta = \int_0^t \int_\Gamma [n \cdot D_t^{u-}(\partial^\beta f \partial_3 p) D_t^{u-}(\partial^\beta f \partial_3 u)]\mathrm{d}x'\mathrm{d}\tau$$

To estimate K_4^β, note that

$$\left|K_4^\beta\right| = \left|\int_0^t \int_\Gamma [n \cdot D_t^{u-}(\partial^\beta f \partial_3 p) D_t^{u-}(\partial^\beta f \partial_3 u)]\mathrm{d}x'\mathrm{d}\tau\right|$$
$$\leq \int_0^t \left|\left|R_\Gamma^2(U,f)\right|\right|_{L^2(\Gamma)}^2 \mathrm{d}\tau \leq M^0 + tK(t)$$

where we have used (8.18.10).

To estimate K_2^β, we note that

$$\left|K_2^\beta\right| \leq \left|\int_0^t \int_\Gamma [\partial\partial^\beta u \cdot R_\Gamma^2(U,f)]\mathrm{d}x'\mathrm{d}\tau\right|$$
$$\leq \sum_\pm \left|\int_0^t \int_\Gamma \partial\partial^\beta u_\pm \cdot R_\Gamma^2(U,f)\mathrm{d}x'\mathrm{d}\tau\right| \leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))$$

where we have used the remainder estimate (8.18.11).

Similarly,

$$\left|K_3^\beta\right| \leq \left|\int_0^t \int_\Gamma [\partial\partial^\beta (p + \mu \dot{p}) \cdot R_\Gamma^2(U,f)]\mathrm{d}x'\mathrm{d}\tau\right|$$
$$\leq \sum_\pm \left|\int_0^t \int_\Gamma \partial\partial^\beta (p_\pm + \mu \dot{p}_\pm) \cdot R_\Gamma^2(U,f)\mathrm{d}x'\mathrm{d}\tau\right| \leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))$$

where we have used the remainder estimate (8.18.11).

We now consider K_1^β. We use the differentiated equation for the front, (8.6.14), to replace $n \cdot D_t^{u-}\partial^\beta u$, and we obtain the following.

$$\begin{aligned}
K_1^\beta &= \int_0^t \int_\Gamma [n \cdot D_t^{u-}\partial^\beta u D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau \\
&= \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f[D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau + \int_0^t \int_\Gamma [u' \cdot D_t^{u-}\partial^\beta \nabla' f D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma [R_\Gamma^2(U,f) D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau \\
&= \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f[D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau + \int_0^t \int_\Gamma (u'_+ \cdot \nabla')D_t^{u-}\partial^\beta f[D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma ([u']\cdot\nabla')D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta(p_- + \mu\dot{p}_-)\mathrm{d}x'\mathrm{d}\tau + \int_0^t \int_\Gamma [R_\Gamma^2(U,f)D_t^{u-}\partial^\beta(p+\mu\dot{p})]\mathrm{d}x'\mathrm{d}\tau \\
&= K_{11}^\beta + K_{12}^\beta + K_{13}^\beta + K_{14}^\beta
\end{aligned}$$

Applying the remainder estimate lemma (8.18.11), we have

$$\left|K_{14}^\beta\right| \le (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))$$

The estimate of K_{11}^β is similar to the estimate of I_1^α. Using the differentiated pressure interface condition (8.6.15) with ∂^α replaced by $D_t^{u-}\partial^\beta$ and integrating by parts, we have

$$\begin{aligned}
K_{11}^\beta &= -\sigma \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta \nabla' \cdot \hat{n}\mathrm{d}x'\mathrm{d}\tau + \mu \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta(\Delta' - 1)\partial_t f \mathrm{d}x'\mathrm{d}\tau \\
&+ K_{112}^\beta \\
&= \sigma \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f \nabla' \cdot D_t^{u-}\partial^\beta(\frac{\nabla' f}{|n|})\mathrm{d}x'\mathrm{d}\tau + \mu \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f \nabla' \cdot D_t^{u-}\partial^\beta \nabla' \partial_t f \mathrm{d}x'\mathrm{d}\tau \\
&- \mu \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta \partial_t f \mathrm{d}x'\mathrm{d}\tau + K_{112}^\beta + K_{113}^\beta + K_{114}^\beta \\
&= -\sigma \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta \nabla' f \cdot D_t^{u-}\partial^\beta(\frac{\nabla' f}{|n|})\mathrm{d}x'\mathrm{d}\tau - \mu \int_0^t \int_\Gamma \nabla' \partial_t D_t^{u-}\partial^\beta f \cdot D_t^{u-}\partial^\beta \nabla' \partial_t f \mathrm{d}x'\mathrm{d}\tau \\
&- \mu \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta \partial_t f \mathrm{d}x'\mathrm{d}\tau + K_{112}^\beta + K_{113}^\beta + K_{114}^\beta + K_{115}^\beta \\
&= -\sigma \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta \nabla' f \cdot D_t^{u-}\partial^\beta(\frac{\nabla' f}{|n|})\mathrm{d}x'\mathrm{d}\tau - \mu \int_0^t \int_\Gamma \left|\nabla' D_t^{u-}\partial^\beta \partial_t f\right|^2 + \left|D_t^{u-}\partial^\beta \partial_t f\right|^2 \mathrm{d}x'\mathrm{d}\tau \\
&+ K_{112}^\beta + K_{113}^\beta + K_{114}^\beta + K_{115}^\beta + K_{116}^\beta + K_{117}^\beta
\end{aligned}$$

where all of

$$\begin{aligned}
K_{112}^\beta &= \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta g_2 \mathrm{d}x'\mathrm{d}\tau \\
K_{113}^\beta &= -\sigma \sum_{j=1}^2 \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f(\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta (\frac{\partial_{x_j} f}{|n|})\mathrm{d}x'\mathrm{d}\tau \\
K_{114}^\beta &= -\mu \sum_{j=1}^2 \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta f(\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta \partial_{x_j} \partial_t f \mathrm{d}x'\mathrm{d}\tau \\
K_{115}^\beta &= -\sigma \sum_{j=1}^2 \int_0^t \int_\Gamma \partial_t((\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta f) D_t^{u-}\partial^\beta (\frac{\partial_{x_j} f}{|n|})\mathrm{d}x'\mathrm{d}\tau
\end{aligned}$$

and

$$K_{116}^{\beta} = -\mu \int_0^t \int_\Gamma \nabla'((\partial_t u'_- \cdot \nabla')\partial^\beta f) \cdot D_t^{u-}\partial^\beta \nabla' \partial_t f \mathrm{d}x' \mathrm{d}\tau$$

$$K_{117}^{\beta} = \mu \sum_{j=1}^2 \int_0^t \int_\Gamma \partial_{x_j}(D_t^{u-}\partial^\beta \partial_t f)(\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta \partial_t f \mathrm{d}x' \mathrm{d}\tau$$

can be thought of as lower order terms. Let us also label the first term on the right as

$$K_{111}^{\beta} = -\sigma \int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta \nabla' f \cdot D_t^{u-}\partial^\beta (\frac{\nabla' f}{|n|}) \mathrm{d}x' \mathrm{d}\tau$$

Let us estimate the lower order terms. These estimates can be skipped on first reading. Integrating by parts in time then in space, and using the fractional Sobolev product estimate A.3.9, we have

$$\begin{aligned}
\left|K_{113}^{\beta}\right| &\leq \sigma \sum_{j=1}^2 \left| \int_\Gamma D_t^{u-}\partial^\beta f(\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \right| \Big|_0^t \\
&+ \sigma \sum_{j=1}^2 \left| \int_0^t \int_\Gamma D_t^{u-}\partial^\beta f(\partial_t \partial_{x_j} u'_-) \cdot \nabla' \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&+ \sigma \sum_{j=1}^2 \left| \int_0^t \int_\Gamma D_t^{u-}\partial^\beta f(\partial_{x_j} u'_-) \cdot \partial_t \nabla' \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq M^0 + \epsilon^{\frac{1}{2}} \sum_{j=1}^2 \left|\left| D_t^{u-}\partial^\beta f(\partial_{x_j} u'_-) \right|\right|_{L^2(\Gamma)}^2 + \frac{1}{\epsilon^{\frac{1}{2}}} C \sum_{j=1}^2 \left|\left| \nabla' \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \right|\right|_{L^2(\Gamma)}^2 \\
&+ \sigma \sum_{j=1}^2 \left| \int_0^t \int_\Gamma D_t^{u-}\partial^\beta f(\partial_t \partial_{x_j} u'_-) \cdot \nabla' \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&+ \sigma \sum_{j=1}^2 \left| \int_0^t \int_\Gamma \nabla' D_t^{u-}\partial^\beta f \cdot (\partial_{x_j} u'_-) \partial_t \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&+ \sigma \sum_{j=1}^2 \left| \int_0^t \int_\Gamma D_t^{u-}\partial^\beta f(\nabla' \cdot \partial_{x_j} u'_-) \partial_t \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq M^0 + \epsilon^{\frac{1}{2}} \left|\left| D_t^{u-}\partial^\beta f \right|\right|_{H^1(\Gamma)}^2 \left|\left|\nabla' u\right|\right|_{H^{0.5}(\Gamma)}^2 + \frac{1}{\epsilon^{\frac{1}{2}}} \left|\left| \nabla' \partial^\beta (\frac{\nabla' f}{|n|}) \right|\right|_{L^2(\Gamma)}^2 \\
&+ C \int_0^t \left|\left| D_t^{u-}\partial^\beta f \right|\right|_{H^1(\Gamma)}^2 \left|\left|\partial_t \nabla' u\right|\right|_{H^{0.5}(\Gamma)}^2 \mathrm{d}\tau + C \int_0^t \left|\left| \nabla' \partial^\beta (\frac{\nabla' f}{|n|}) \right|\right|_{L^2(\Gamma)} \mathrm{d}\tau \\
&+ C \int_0^t \left|\left| \nabla' D_t^{u-}\partial^\beta f \right|\right|_{L^2(\Gamma)}^2 \left|\left|\nabla' u\right|\right|_{L^\infty(\Gamma)} \mathrm{d}\tau + C \int_0^t \left|\left| \partial_t \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \right|\right|_{L^2(\Gamma)}^2 \mathrm{d}\tau \\
&+ C \int_0^t \left|\left| D_t^{u-}\partial^\beta f \right|\right|_{H^1(\Gamma)}^2 \left|\left|\nabla' u\right|\right|_{H^{1.5}(\Gamma)}^2 \mathrm{d}\tau + C \left|\left| \partial_t \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \right|\right|_{L^2(\Gamma)}^2 \mathrm{d}\tau \\
&\leq M^0 + \epsilon^{\frac{1}{2}} (M^0 + tK(t)) E_\mu(t) + \frac{1}{\epsilon^{\frac{1}{2}}} (M^0 + tK(t)) \left|\left| \nabla' \partial^\beta \nabla' f \right|\right|_{L^2(\Gamma)}^2 + tK(t) \\
&\leq (M^0 + tK(t)) (\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}$$

where we have used the Sobolev trace estimate, the chain rule together with the lower order estimates (8.5.1)-(8.5.4), and the existing tangential estimate for $\nabla' f$ given in (8.9.1).

We now estimate K^{β}_{115} using fractional Sobolev product estimate A.3.9.

$$
\begin{aligned}
\left|K^{\beta}_{115}\right| &\leq \sigma \sum_{j=1}^{2} \left| \int_0^t \int_\Gamma (\partial_t \partial_{x_j} u'_-) \cdot \nabla' \partial^\beta f D_t^{u-} \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&+ \sigma \sum_{j=1}^{2} \left| \int_0^t \int_\Gamma (\partial_{x_j} u'_-) \cdot \partial_t \nabla' \partial^\beta f D_t^{u-} \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq C \int_0^t ||\partial_t \nabla' u||^2_{H^{0.5}(\Gamma)} \left|\left|\nabla' \partial^\beta f\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau + \int_0^t \left|\left| D_t^{u-} \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&+ C \int_0^t ||\nabla' u||^2_{L^\infty(\Gamma)} \left|\left|\partial_t \nabla' \partial^\beta f\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau + \int_0^t \left|\left| D_t^{u-} \partial^\beta (\frac{\partial_{x_j} f}{|n|}) \right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&\leq tK(t)
\end{aligned}
$$

where we have used the Sobolev trace estimate and the chain rule.

We estimate K^{β}_{114} using the fractional Sobolev product estimate A.3.9 as follows.

$$
\begin{aligned}
\left|K^{\beta}_{114}\right| &\leq \mu \sum_{j=1}^{2} \left| \int_0^t \int_\Gamma D_t^{u-} \partial^\beta \partial_t f (\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta \partial_{x_j} \partial_t f \mathrm{d}x' \mathrm{d}\tau \right| \\
&+ \mu \sum_{j=1}^{2} \left| \int_0^t \int_\Gamma (\partial_t u'_- \cdot \nabla') \partial^\beta f (\partial_{x_j} u'_-) \cdot \nabla' \partial^\beta \partial_{x_j} \partial_t f \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq \epsilon^{\frac{1}{2}} \mu \int_0^t \left|\left| D_t^{u-} \partial^\beta \partial_t f \right|\right|^2_{H^1(\Gamma)} ||\nabla' u||^2_{H^{0.5}(\Gamma)} \mathrm{d}\tau + \frac{1}{\epsilon^{\frac{1}{2}}} C\mu \int_0^t \left|\left|\nabla' \partial^\beta \partial_{x_j} \partial_t f\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&+ \int_0^t ||\partial_t u||^2_{L^\infty(\Gamma)} \left|\left|\nabla' \partial^\beta f\right|\right|^2 ||\nabla' u||^2_{L^\infty(\Gamma)} \mathrm{d}\tau + \mu^2 \int_0^t \left|\left|\nabla' \partial^\beta \nabla' \partial_t f\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&\leq \epsilon^{\frac{1}{2}} (M^0 + tK(t)) E_\mu(t) + C \frac{1}{\epsilon^{\frac{1}{2}}} \mu \int_0^t \left|\left|\nabla' \partial^\beta \partial_{x_j} \partial_t f\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau + tK(t) \\
&\leq (M^0 + tK(t)) (\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}
$$

where we have used the Sobolev trace estimate, the chain rule together with the lower order estimates (8.5.1)-(8.5.4), and the existing tangential estimate for $\partial_t f = u \cdot n$ given in (8.9.1).

We estimate K^{β}_{116} using the fractional Sobolev product estimate A.3.9 as follows.

$$
\begin{aligned}
\left|K^{\beta}_{116}\right| &\leq \mu \left| \int_0^t \int_\Gamma (\nabla' (\partial_t u'_- \cdot \nabla')) \partial^\beta f \cdot D_t^{u-} \partial^\beta \nabla' \partial_t f \mathrm{d}x' \mathrm{d}\tau \right| \\
&+ \mu \left| \int_0^t \int_\Gamma (\partial_t u'_- \cdot \nabla') \nabla' \partial^\beta f \cdot D_t^{u-} \partial^\beta \nabla' \partial_t f \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq \frac{1}{\epsilon} \mu \int_0^t ||\nabla' \partial_t u||^2_{H^{0.5}(\Gamma)} \left|\left|\partial^\beta f\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau + \frac{1}{\epsilon} \mu \int_0^t ||\partial_t u||^2_{L^\infty(\Gamma)} \left|\left|\nabla' \partial^\beta f\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \\
&+ \epsilon \mu \int_0^t \left|\left| D_t^{u-} \partial^\beta \nabla' \partial_t f \right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&\leq \frac{1}{\epsilon} tK(t) + \epsilon (M^0 + tK(t)) E_\mu(t)
\end{aligned}
$$

We estimate K^β_{117} using the fractional Sobolev product estimate A.3.9 as follows.

$$\begin{aligned}
\left|K^\beta_{117}\right| &\leq \mu\sum_{j=1}^{2}\left|\int_0^t\int_\Gamma \partial_{x_j}(D_t^{u_-}\partial^\beta\partial_t f)(\partial_{x_j}u'_-)\cdot\nabla'\partial^\beta\partial_t f\mathrm{d}x'\mathrm{d}\tau\right| \\
&\leq \epsilon\mu\int_0^t \left|\left|\partial\partial^\beta\partial_t f\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau + \frac{1}{\epsilon}\mu\int_0^t(1+||u||^4_{W^{1,\infty}(\Gamma)})\left|\left|\partial^\beta\partial_t f\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}tK(t)+\epsilon E_\mu(t)
\end{aligned}$$

To estimate K^β_{112}, we integrate by parts in time. We have

$$\begin{aligned}
\left|K^\beta_{112}\right| &= \left|\int_0^t\int_\Gamma \partial_t D_t^{u_-}\partial^\beta f D_t^{u_-}\partial^\beta g_2\mathrm{d}x'\mathrm{d}\tau\right| \\
&= \left|\int_\Gamma D_t^{u_-}\partial^\beta f D_t^{u_-}\partial^\beta g_2\mathrm{d}x' - M^0 + \int_0^t\int_\Gamma D_t^{u_-}\partial^\beta f\partial_t D_t^{u_-}\partial^\beta g_2\mathrm{d}x'\mathrm{d}\tau\right| \\
&\leq \epsilon(1+||u||^4_{L^\infty(\Gamma)})\left|\left|\partial\partial^\beta f\right|\right|^2_{L^2(\Gamma)} + M^0 + \frac{1}{\epsilon}\left|\left|\partial\partial^\beta g_2\right|\right|^2_{L^2(\Gamma)} + \int_0^t\left|\left|\partial\partial^\beta f\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau \\
&+ \int_0^t(1+||(u,\partial_t u)||^4_{W^{1,\infty}(\Gamma)})\left|\left|\partial_t\partial\partial^\beta g_2\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}M^0 + tK(t) + \epsilon(M^0+tK(t))E_\mu(t)
\end{aligned}$$

Having estimated the lower order terms, we return to the more important estimate of K^β_{111}. We use lemma 8.6.4 to obtain the following.

$$\begin{aligned}
&- K^\beta_{111} \\
&= \sigma\int_0^t\int_\Gamma \partial_t D_t^{u_-}\partial^\beta\nabla' f\cdot(\frac{D_t^{u_-}\partial^\beta\nabla' f}{|n|} - \frac{(\nabla' f\cdot D_t^{u_-}\partial^\beta\nabla' f)\nabla' f}{|n|^3} + R^0_{\infty,\Gamma}(U,f)R^2_\Gamma(f))\mathrm{d}x'\mathrm{d}\tau \\
&= \sigma\int_0^t\int_\Gamma \partial_t D_t^{u_-}\partial^\beta\nabla' f\cdot\frac{D_t^{u_-}\partial^\beta\nabla' f}{|n|} - \partial_t(\nabla' f\cdot D_t^{u_-}\partial^\beta\nabla' f)\frac{(\nabla' f\cdot D_t^{u_-}\partial^\beta\nabla' f)}{|n|^3}\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t\int_\Gamma \partial_t D_t^{u_-}\partial^\beta\nabla' f\cdot R^0_{\infty,\Gamma}(U,f)R^2_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t\int_\Gamma R^0_{\infty,\Gamma}(U,f)R^3_\Gamma(f)R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^t\sigma\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma\frac{1}{|n|}\left|D_t^{u_-}\partial^\beta\nabla' f\right|^2 - \frac{1}{|n|^3}\left|\nabla' f\cdot D_t^{u_-}\partial^\beta\nabla' f\right|^2\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t\int_\Gamma \partial_t D_t^{u_-}\partial^\beta\nabla' f\cdot R^0_{\infty,\Gamma}(U,f)R^2_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t\int_\Gamma R^0_{\infty,\Gamma}(U,f)R^3_\Gamma(f)R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau
\end{aligned}$$

We now apply lemma 8.6.5 to the first term on the right hand side and we apply the remainder estimates

(8.18.8) and (8.18.14) to the last term. The other term is estimated as follows.

$$
\begin{aligned}
&\left|\int_0^t \int_\Gamma \partial_t D_t^{u-}\partial^\beta \nabla' f \cdot R^0_{\infty,\Gamma}(U,f) R^2_\Gamma(f) \mathrm{d}x' \mathrm{d}\tau\right| \\
&= \left|\int_\Gamma D_t^{u-}\partial^\beta \nabla' f \cdot R^0_{\infty,\Gamma}(U,f) R^2_\Gamma(f) \mathrm{d}x' - M^0 + \int_0^t \int_\Gamma D_t^{u-}\partial^\beta \nabla' f \cdot \partial_t (R^0_{\infty,\Gamma}(U,f) R^2_\Gamma(f)) \mathrm{d}x' \mathrm{d}\tau\right| \\
&\leq \epsilon \left|\left|R^0_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)} \left|\left|\partial\partial^\beta f\right|\right|^2_{H^1(\Gamma)} + \frac{1}{\epsilon}\left|\left|R^0_{\infty,\Gamma}(U,f)R^2_\Gamma(f)\right|\right|^2_{L^2(\Gamma)} + \int_0^t \left|\left|\partial\partial^\beta f\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \\
&+ \int_0^t \left|\left|R^0_{\infty,\Gamma}(U,f)R^1_\Gamma(U,f)R^2_\Gamma(f)\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau + \int_0^t \left|\left|R^0_{\infty,\Gamma}(U,f)R^3_\Gamma(f)\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau + M^0 \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))
\end{aligned}
$$

where we have used the remainder estimates (8.18.8), (8.18.9), (8.18.14) and (8.18.15). Hence

$$
-K^\beta_{111} \geq \frac{1}{2}\sigma \int_\Gamma \frac{1}{|n|^3}\left|D_t^{u-}\partial^\beta \nabla' f\right|^2 \mathrm{d}x' - (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))
$$

Thus we have shown that

$$
-K^\beta_{11} \geq \frac{1}{2}\sigma \int_\Gamma \frac{1}{|n|^3}\left|D_t^{u-}\partial^\beta \nabla' f\right|^2 \mathrm{d}x' - (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
$$

The estimate of K^β_{12} is similar to the estimate of K^β_{11}, except this time the remainder terms are easier to deal with because we have replaced a time derivative with a tangential derivative. Using the differentiated pressure interface condition (8.6.15) with ∂^α replaced by $D_t^{u-}\partial^\beta$, we have

$$
\begin{aligned}
K^\beta_{12} &= -\sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla') D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta \nabla' \cdot \hat{n} \mathrm{d}x' \mathrm{d}\tau \\
&+ \mu \int_0^t \int_\Gamma (u_+ \cdot \nabla') D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta (\Delta' - 1)\partial_t f \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_\Gamma (u_+ \cdot \nabla') D_t^{u-}\partial^\beta f D_t^{u-}\partial^\beta g_2 \mathrm{d}x' \mathrm{d}\tau \\
&= \sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla') D_t^{u-}\partial^\beta f \nabla' \cdot D_t^{u-}\partial^\beta (\frac{\nabla' f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \\
&+ \mu \int_0^t \int_\Gamma (u_+ \cdot \nabla') D_t^{u-}\partial^\beta f \nabla' \cdot D_t^{u-}\partial^\beta \nabla' \partial_t f \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f) R^3_\Gamma(f) \cdot R^3_\Gamma(f) \mathrm{d}x' \mathrm{d}\tau + \mu \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f) R^3_\Gamma(f) \cdot \partial\partial^\beta \partial_t f \mathrm{d}x' \mathrm{d}\tau \\
&= -\sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla') D_t^{u-}\partial^\beta \nabla' f \cdot D_t^{u-}\partial^\beta (\frac{\nabla' f}{|n|}) \mathrm{d}x' \mathrm{d}\tau \\
&- \mu \int_0^t \int_\Gamma (u_+ \cdot \nabla') \nabla' D_t^{u-}\partial^\beta f \cdot D_t^{u-}\partial^\beta \nabla' \partial_t f \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f) R^3_\Gamma(f) \cdot R^3_\Gamma(f) \mathrm{d}x' \mathrm{d}\tau \\
&+ \mu \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f) R^3_\Gamma(f) \cdot \partial\partial^\beta \partial_t f \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_\Gamma (\nabla' \nabla' u) R^2_\Gamma(f) \cdot R^1_{\infty,\Gamma}(U,f) R^3_\Gamma(f) \mathrm{d}x' \mathrm{d}\tau \\
&= K^\beta_{121} + K^\beta_{122} + K^\beta_{123} + K^\beta_{124} + K^\beta_{125}
\end{aligned}
$$

where we have used integration by parts.

We now use lemma 8.6.4 to estimate K^β_{121}.

$$\begin{aligned}
-K^\beta_{121} &= \sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla')D_t^{u-}\partial^\beta \nabla' f \cdot (\frac{D_t^{u-}\partial^\beta \nabla' f}{|n|} - \frac{(\nabla' f \cdot D_t^{u-}\partial^\beta \nabla' f)\nabla' f}{|n|^3})\mathrm{d}x'\mathrm{d}\tau \\
&+ \sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla')D_t^{u-}\partial^\beta \nabla' f \cdot R^0_{\infty,\Gamma}(U,f)R^2_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau \\
&= \sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla')D_t^{u-}\partial^\beta \nabla' f \cdot \frac{D_t^{u-}\partial^\beta \nabla' f}{|n|}\mathrm{d}x'\mathrm{d}\tau \\
&- \sigma \int_0^t \int_\Gamma (u_+ \cdot \nabla')(\nabla' f \cdot D_t^{u-}\partial^\beta \nabla' f)\frac{(\nabla' f \cdot D_t^{u-}\partial^\beta \nabla' f)}{|n|^3}\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma (u_+ \cdot \nabla')D_t^{u-}\partial^\beta \nabla' f \cdot R^0_{\infty,\Gamma}(U,f)R^2_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma R^0_{\infty,\Gamma}(U,f)R^3_\Gamma(f) \cdot R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau \\
&= \frac{1}{2}\sigma \int_0^t \int_\Gamma \nabla' \cdot (u_+(\frac{1}{|n|}\left|D_t^{u-}\partial^\beta \nabla' f\right|^2 - \frac{1}{|n|^3}\left|\nabla' f \cdot D_t^{u-}\partial^\beta \nabla' f\right|^2))\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f)R^3_\Gamma \nabla' \cdot R^2_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau + \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f)R^3_\Gamma(f) \cdot R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau
\end{aligned}$$

We now apply the divergence theorem and the remainder estimates (8.18.8) and (8.18.14) to obtain

$$\begin{aligned}
\left|K^\beta_{121}\right| &\leq \left|\int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f)R^3_\Gamma \nabla' \cdot R^2_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau\right| + \left|\int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f)R^3_\Gamma(f) \cdot R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau\right| \\
&\leq tK(t)
\end{aligned}$$

Clearly, applying the remainder estimates (8.18.8) and (8.18.14), we have

$$\begin{aligned}
\left|K^\beta_{123}\right| &\leq \left|\int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f)R^3_\Gamma(f)R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau\right| \\
&\leq M^0 + tK(t)
\end{aligned}$$

Similarly, applying the the remainder estimates (8.18.8) and (8.18.14), we have

$$\begin{aligned}
\left|K^\beta_{124}\right| &\leq \mu \left|\int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U,f)R^3_\Gamma(f) \cdot \partial\partial^\beta \partial_t f \mathrm{d}x'\mathrm{d}\tau\right| \\
&\leq \frac{1}{\epsilon}\mu \int_0^t \left|\left|R^1_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)} \left|\left|R^3_\Gamma(f)\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau + \epsilon \int_0^t \left|\left|\partial\partial^\beta \partial_t f\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&\leq \frac{1}{\epsilon} tK(t) + \epsilon E_\mu(t)
\end{aligned}$$

We estimate K^β_{125} using the fractional Sobolev product estimate A.3.9 and the remainder estimates (8.18.8) and (8.18.14) as follows.

$$\begin{aligned}
\left|K^\beta_{125}\right| &\leq \left|\int_0^t \int_\Gamma (\nabla'\nabla' u)R^2_\Gamma(f) \cdot R^1_{\infty,\Gamma}(U,f)R^3_\Gamma(f)\mathrm{d}x'\mathrm{d}\tau\right| \\
&\leq \int_0^t \left|\left|\nabla'\nabla' u\right|\right|^2_{H^{0.5}(\Gamma)} \left|\left|R^2_\Gamma(f)\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau + \int_0^t \left|\left|R^1_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)} \left|\left|R^3_\Gamma(f)\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\
&\leq tK(t)
\end{aligned}$$

Finally, we estimate K^β_{122}, which requires some care. Recall we have assumed that $\beta_0 \geq 1$, ie $\partial^\beta = \partial^\gamma\partial_t$ for some $|\gamma| = 1$. We note that, from the tangential estimate (8.9.1) we already know

$$\begin{aligned}\mu\int_0^t ||\partial^\alpha\partial_t f||^2_{H^1(\Gamma)}\,\mathrm{d}\tau = \mu\int_0^t ||\partial^\alpha(u\cdot n)||^2_{H^1(\Gamma)}\,\mathrm{d}\tau\\ \leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))\end{aligned}$$

provided ∂^α contains at least one tangential space derivative, for $|\alpha| = 3$. Using Cauchy's inequality, we have

$$\begin{aligned}\left|K^\beta_{122}\right| &= \mu\left|\int_0^t\int_\Gamma (u_+\cdot\nabla')\nabla' D_t^{u_-}\partial^\beta\nabla' f\cdot D_t^{u_-}\partial^\beta\nabla'\partial_t f\mathrm{d}x'\mathrm{d}\tau\right|\\ &\leq \mu\frac{1}{\epsilon^{\frac{1}{2}}}\int_0^t \left|\left|(u_+\cdot\nabla')\nabla' D_t^{u_-}\partial^\gamma\partial_t f\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau + \mu\epsilon^{\frac{1}{2}}\int_0^t\left|\left|D_t^{u_-}\partial^\beta\nabla'\partial_t f\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau\\ &\leq \mu\frac{1}{\epsilon^{\frac{1}{2}}}((M^0+tK(t))\int_0^t ||\partial\partial^\gamma\nabla'\partial_t f||^2_{H^1(\Gamma)}\mathrm{d}\tau + K(t)\int_0^t\left|\left|\partial\partial^\beta f\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau\\ &+ K(t)\int_0^t ||\nabla'\nabla' u||^2_{H^{0.5}(\Gamma)}\left|\left|\partial\partial^\beta f\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau) + \mu\epsilon^{\frac{1}{2}}(M^0+tK(t))\int_0^t\left|\left|\partial\partial^\beta\nabla'\partial_t f\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau\end{aligned}$$

where we have used the lower order estimate (8.5.1) and the fractional Sobolev product estimate A.3.9. Using the above tangential estimate for the first term and the definition of the energy for others, we have

$$\left|K^\beta_{122}\right| \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t))$$

Thus we have shown that

$$\left|K^\beta_{12}\right| \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t))$$

We proceed to estimate K^β_{13}. Note that, by rearranging the differentiated equation (8.6.14) for the front, we obtain

$$D_t^{u_-}\partial^\beta f = \partial^\beta u_-\cdot n + R_\Gamma^{|\beta|-1}(U,f) \tag{8.10.3}$$

Using the the equation (8.10.3), we have

$$\begin{aligned}K^\beta_{13} = \int_0^t\int_\Gamma n\cdot([u']\cdot\nabla')\partial^\beta u_- D_t^{u_-}\partial^\beta(p_- + \mu\dot{p}_-)\mathrm{d}x'\mathrm{d}\tau\\ + \int_0^t\int_\Gamma R^2_\Gamma(U,f)\cdot\partial\partial^\beta(p_- + \mu\dot{p}_-)\mathrm{d}x'\mathrm{d}\tau\end{aligned}$$

We want to convert this into an integral over the interior Ω_-. First of all we need to go back to using the corrected pressure and velocity. We have

$$
\begin{aligned}
K_{13}^{\beta} &= \int_0^t \int_{\Gamma} n \cdot ([u'] \cdot \nabla') u_-^{\beta} D_t^{u_-} (p_-^{\beta} + \mu \partial^{\beta} \dot{p}_-) \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_{\Gamma} n \cdot ([u'] \cdot \nabla') (\partial^{\beta} f \partial_{x_3} u_-) D_t^{u_-} \partial^{\beta} (p_- + \mu \dot{p}_-) \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_{\Gamma} n \cdot ([u'] \cdot \nabla') \partial^{\beta} u_- D_t^{u_-} (\partial^{\beta} f \partial_{x_3} p_-) \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_{\Gamma} n \cdot ([u'] \cdot \nabla') (\partial^{\beta} f \partial_{x_3} u_-) D_t^{u_-} (\partial^{\beta} f \partial_{x_3} p_-) \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_{\Gamma} R_{\Gamma}^2(U, f) \cdot \partial \partial^{\beta} (p_- + \mu \dot{p}_-) \mathrm{d}x' \mathrm{d}\tau \\
&= \int_0^t \int_{\Gamma} n \cdot ([u'] \cdot \nabla') u_-^{\beta} D_t^{u_-} (p_-^{\beta} + \mu \partial^{\beta} \dot{p}_-) \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_{\Gamma} R_{\Gamma}^2(U, f) \cdot \partial \partial^{\beta} (p_- + \mu \dot{p}_-) \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_{\Gamma} \nabla' \partial^{\beta} u_- \cdot R_{\Gamma}^2(U, f) \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_{\Gamma} R_{\Gamma}^2(U, f) \cdot R_{\Gamma}^2(U, f) \mathrm{d}x' \mathrm{d}\tau \\
&= K_{131}^{\beta} + K_{132}^{\beta} + K_{133}^{\beta} + K_{134}^{\beta}
\end{aligned}
$$

Using the remainder estimates (8.18.10) and (8.18.11), we have

$$
\left| K_{132}^{\beta} + K_{133}^{\beta} + K_{134}^{\beta} \right| \leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_{\mu}(t))
$$

To estimate K_{131}^{β} we convert the boundary integral into an integral over the interior Ω_-. Note that we may add the integral of derivatives in the tangential directions because these integrate to zero. We also note that $(-\nabla'\psi, 1)|_{\Gamma} = n$, and $J^{\psi}|_{\Gamma} = 1$. Note that by extending u_+ to Ω_- by Sobolev extension we obtain a natural definition for $[u'] = u_+ - u_-$ in Ω_-. We obtain

$$
\begin{aligned}
K_{131}^{\beta} = &\int_0^t \int_{\Omega_-} \nabla' \cdot (([u'] \cdot \nabla') u^{\beta} D_t^{u_-} (p^{\beta} + \mu \partial^{\beta} \dot{p})) \\
&+ \partial_{x_3} (\frac{1}{J^{\psi}} (-\nabla'\psi, 1) \cdot ([u'] \cdot \nabla') u^{\beta} D_t^{u_-} (p^{\beta} + \mu \partial^{\beta} \dot{p})) \mathrm{d}x \mathrm{d}\tau \\
= &\int_0^t \int_{\Omega_-} \nabla' \cdot (([u'] \cdot \nabla') u^{\beta} D_t^{u_-} (p^{\beta} + \mu \partial^{\beta} \dot{p})) \\
&+ \frac{(-\nabla'\psi, 1)}{J^{\psi}} \cdot \partial_{x_3} (([u'] \cdot \nabla') u^{\beta} D_t^{u_-} (p^{\beta} + \mu \partial^{\beta} \dot{p})) \mathrm{d}x \mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} R^3(U, \psi) \cdot (R^3(U, \psi) + \mu R^3(U, \psi, \dot{p})) \mathrm{d}x \mathrm{d}\tau \\
= &\int_0^t \int_{\Omega_-} ([u'] \cdot \nabla') (\nabla' \cdot u^{\beta} + \frac{(-\nabla'\psi, 1)}{J^{\psi}} \cdot \partial_{x_3} u^{\beta}) D_t^{u_-} (p^{\beta} + \mu \partial^{\beta} \dot{p}) \mathrm{d}x \mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} ([u'] \cdot \nabla') u^{\beta} \cdot D_t^{u_-} (\nabla' (p^{\beta} + \mu \partial^{\beta} \dot{p}) + \frac{(-\nabla'\psi, 1)}{J^{\psi}} \partial_{x_3} (p^{\beta} + \mu \partial^{\beta} \dot{p})) \mathrm{d}x \mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} R^3(U, \psi) \cdot (R^3(U, \psi) + \mu R^3(U, \psi, \dot{p})) \mathrm{d}x \mathrm{d}\tau
\end{aligned}
$$

Using the corrected pressure and velocity equations (8.6.11)-(8.6.12), we obtain

$$
\begin{aligned}
K_{131}^{\beta} = &-\int_0^t\int_{\Omega_-}\frac{1}{\rho c^2}(([u']\cdot\nabla')D_t^{u-}p^{\beta}D_t^{u-}p^{\beta})\mathrm{d}x\mathrm{d}\tau \\
&-\int_0^t\int_{\Omega_-}\frac{1}{\rho c^2}(([u']\cdot\nabla')D_t^{u-}p^{\beta}D_t^{u-}\mu\partial^{\beta}\dot{p})\mathrm{d}x\mathrm{d}\tau \\
&-\int_0^t\int_{\Omega_-}\rho_{\mu}(([u']\cdot\nabla')u^{\beta}\cdot D_t^{u-}D_t^{u-}u^{\beta})\mathrm{d}x\mathrm{d}\tau \\
&+\int_0^t\int_{\Omega_-}R^3(U,\psi)\cdot(R^3(U,\psi)+\mu R^3(U,\psi,\dot{p}))\mathrm{d}x\mathrm{d}\tau \\
&= K_{1311}^{\beta}+K_{1312}^{\beta}+K_{1313}^{\beta}+K_{1314}^{\beta}
\end{aligned}
$$

The remainder estimates (8.18.3) and (8.18.7) together with Cauchy's inequality imply

$$
\left|K_{1314}^{\beta}\right| \le \frac{1}{\epsilon}tK(t)+\epsilon(M^0+K(t))E_{\mu}(t)
$$

Using the divergence theorem and the remainder estimate (8.18.3), we have

$$
\begin{aligned}
\left|K_{1311}^{\beta}\right| &\le \frac{1}{2}\left|\int_0^t\int_{\Omega_-}\nabla'\cdot(\frac{1}{\rho c^2}[u']D_t^{u-}p^{\beta}D_t^{u-}p^{\beta})\mathrm{d}x\mathrm{d}\tau\right| + \left|\int_0^t\int_{\Omega_-}R^3(U,\psi)\cdot R^3(U,\psi)\mathrm{d}x\mathrm{d}\tau\right| \\
&= \left|\int_0^t\int_{\Omega_-}R^3(U,\psi)\cdot R^3(U,\psi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\le M^0+tK(t)
\end{aligned}
$$

Using the definition of $\dot{p}$ and p^{β}, we have

$$
\begin{aligned}
&\left|K_{1312}^{\beta}\right| = \mu\left|\int_0^t\int_{\Omega_-}\frac{1}{\rho c^2}([u']\cdot\nabla')D_t^{u-}p^{\beta}D_t^{u-}\partial^{\beta}\dot{p}\mathrm{d}x\mathrm{d}\tau\right| \\
&\le \mu\left|\int_0^t\int_{\Omega_-}\frac{1}{\rho c^2}([u']\cdot\nabla')\partial^{\beta}\dot{p}D_t^{u-}\partial^{\beta}\dot{p}\mathrm{d}x\mathrm{d}\tau\right| + \mu\left|\int_0^t\int_{\Omega_-}R^3(U,\psi)D_t^{u-}\partial^{\beta}\dot{p}\mathrm{d}x\mathrm{d}\tau\right| \\
&\le \frac{1}{\tilde{\epsilon}}(M^0+tK(t))\mu\int_0^t\left|\left|\nabla'\partial^{\beta}\dot{p}\right|\right|_{L^2(\Omega_-)}^2\mathrm{d}\tau + \tilde{\epsilon}\mu\int_0^t\left|\left|D_t^{u-}\partial^{\beta}\dot{p}\right|\right|_{L^2(\Omega_-)}^2\mathrm{d}\tau \\
&+ \frac{1}{\epsilon}tK(t)+\epsilon(M^0+tK(t))E_{\mu}(t) \\
&\le \frac{1}{\tilde{\epsilon}}(M^0+tK(t))\mu\int_0^t(\left|\left|\rho c^2\nabla^{\psi}\cdot\nabla'\partial^{\beta}u\right|\right|_{L^2(\Omega_-)}^2 + \left|\left|R^3(U,\psi)\right|\right|_{L^2(\Omega_-)}^2)\mathrm{d}\tau \\
&+ \tilde{\epsilon}\mu\int_0^t\left|\left|\rho c^2\nabla^{\psi}\cdot(D_t^{u-}\partial^{\beta}u)\right|\right|_{L^2(\Omega_-)}^2\mathrm{d}\tau + \frac{1}{\epsilon}tK(t)+\epsilon(M^0+tK(t))E_{\mu}(t) \\
&\le \frac{1}{\tilde{\epsilon}}(M^0+tK(t))(\frac{1}{\epsilon}+\epsilon E_{\mu}(t)) + \tilde{\epsilon}\mu\int_0^t\left|\left|\rho c^2\nabla^{\psi}\cdot(D_t^{u-}\partial^{\beta}u)\right|\right|_{L^2(\Omega_-)}^2\mathrm{d}\tau
\end{aligned}
$$

where we have used the lower order estimates (8.5.1), (8.5.3) and the remainder estimate (8.18.7). We have also used crucially the tangential estimate (8.9.1) already obtained to estimate the term involving $\mu\int_0^t\left|\left|\nabla^{\psi}\cdot\nabla'\partial^{\beta}u\right|\right|_{L^2(\Omega_-)}^2\mathrm{d}\tau$.

To estimate K^β_{1313}, we use integration by parts in time and space and the divergence theorem.

$$\begin{aligned}
\left|K^\beta_{1313}\right| &\leq \left|\int_0^t\int_{\Omega_-} D_t^{u_-}(\rho_\mu([u']\cdot\nabla')u^\beta\cdot D_t^{u_-}u^\beta)\mathrm{d}x\mathrm{d}\tau\right| \\
&+\left|\int_0^t\int_{\Omega_-}\rho_\mu([u']\cdot\nabla')D_t^{u_-}u^\beta\cdot D_t^{u_-}u^\beta\mathrm{d}x\mathrm{d}\tau\right| + \left|\int_0^t\int_{\Omega_-}R^3(U,\psi)\cdot R^3(U,\psi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\leq \left|\int_{\Omega_-}\rho_\mu([u']\cdot\nabla')u^\beta\cdot D_t^{u_-}u^\beta\mathrm{d}x\Big|_0^t\right| + \frac{1}{2}\left|\int_0^t\int_{\Omega_-}\nabla\cdot(u_-^\psi\rho_\mu([u']\cdot\nabla')u^\beta\cdot D_t^{u_-}u^\beta)\mathrm{d}x\mathrm{d}\tau\right| \\
&+\frac{1}{2}\left|\int_0^t\int_{\Omega_-}\nabla'\cdot(\rho_\mu[u']D_t^{u_-}u^\beta\cdot D_t^{u_-}u^\beta)\mathrm{d}x\mathrm{d}\tau\right| + \left|\int_0^t\int_{\Omega_-}R^3(U,\psi)\cdot R^3(U,\psi)\mathrm{d}x\mathrm{d}\tau\right| \\
&\leq \left|\int_{\Omega_-}R^0_\infty(U,\psi)\nabla'u^\beta\cdot\partial u^\beta\mathrm{d}x\Big|_0^t\right| + \int_0^t K(t)\mathrm{d}\tau
\end{aligned}$$

where we have used the remainder estimate (8.18.3) and the fact that $(u_-^\psi)_3|_\Gamma = 0$. We now apply Cauchy's lemma to the first term on the right, followed by the estimate we already have for the tangential derivative $\nabla' u^\beta$ given by (8.9.1) together with (8.6.10) and the lower order remainder estimate (8.18.2), to obtain

$$\begin{aligned}
\left|K^\beta_{1313}\right| &\leq \frac{1}{\epsilon^{\frac{1}{2}}}\left|\left|R^0_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_-)}\left|\left|\nabla'u^\beta\right|\right|^2_{L^2(\Omega_-)} + \epsilon^{\frac{1}{2}}\left|\left|\partial u^\beta\right|\right|^2_{L^2(\Omega_-)} + tK(t) \\
&\leq (M^0+tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}}+\epsilon^{\frac{1}{2}}E_\mu(t))
\end{aligned}$$

Thus we have eventually shown that

$$\begin{aligned}
-K^\beta \geq\ &\frac{1}{2}\sigma\int_\Gamma\frac{1}{|n|^3}\left|D_t^{u_-}\partial^\beta\nabla'f\right|^2\mathrm{d}x' \\
&+\mu\int_0^t\int_\Gamma\left|\nabla'D_t^{u_-}\partial^\beta(u\cdot n)\right|^2+\left|D_t^{u_-}\partial^\beta(u\cdot n)\right|^2\mathrm{d}x'\mathrm{d}\tau - (M^0+tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}}+\epsilon^{\frac{1}{2}}E_\mu(t)) \\
&-\frac{1}{\tilde\epsilon}(M^0+tK(t))(\frac{1}{\epsilon}+\epsilon E_\mu(t)) - \tilde\epsilon\mu\int_0^t\left|\left|\rho c^2\nabla^\psi\cdot(D_t^{u_-}\partial^\beta u)\right|\right|^2_{L^2(\Omega_-)}\mathrm{d}\tau
\end{aligned}$$

Hence we have shown

$$\begin{aligned}
&\sum_\pm\int_{\Omega_\pm}\frac{1}{\rho c^2}\left|D_t^{u_-}p^\beta\right|^2\mathrm{d}x + \sum_\pm\int_{\Omega_\pm}\rho_\mu\left|D_t^{u_-}u^\beta\right|^2\mathrm{d}x + \mu\sum_\pm\int_0^t\rho c^2\left|\nabla^\psi\cdot(D_t^{u_-}\partial^\beta u)\right|^2\mathrm{d}x\mathrm{d}\tau \\
&+\sigma\int_\Gamma\frac{1}{|n|^3}\left|D_t^{u_-}\partial^\beta\nabla'f\right|^2\mathrm{d}x' + \mu\int_0^t\left|\left|D_t^{u_-}\partial^\beta(u\cdot n)\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau \\
&\leq (M^0+tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}}+\epsilon^{\frac{1}{2}}E_\mu(t)) \\
&+\frac{1}{\tilde\epsilon}(M^0+tK(t))(\frac{1}{\epsilon}+\epsilon E_\mu(t)) + \tilde\epsilon\mu\int_0^t\left|\left|\rho c^2\nabla^\psi\cdot(D_t^{u_-}\partial^\beta u)\right|\right|^2_{L^2(\Omega_-)}\mathrm{d}\tau
\end{aligned}$$

Choosing $\tilde\epsilon$ sufficiently small, we obtain

$$\begin{aligned}
&\sum_\pm\int_{\Omega_\pm}\frac{1}{\rho c^2}\left|D_t^{u_-}p^\beta\right|^2\mathrm{d}x + \sum_\pm\int_{\Omega_\pm}\rho_\mu\left|D_t^{u_-}u^\beta\right|^2\mathrm{d}x + \mu\sum_\pm\int_0^t\rho c^2\left|\nabla^\psi\cdot(D_t^{u_-}\partial^\beta u)\right|^2\mathrm{d}x\mathrm{d}\tau \\
&+\sigma\int_\Gamma\frac{1}{|n|^3}\left|D_t^{u_-}\partial^\beta\nabla'f\right|^2\mathrm{d}x' + \mu\int_0^t\left|\left|D_t^{u_-}\partial^\beta(u\cdot n)\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau \\
&\leq (M^0+tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}}+\epsilon^{\frac{1}{2}}E_\mu(t))
\end{aligned}$$

Thus, using the inequalities (8.3.1)-(8.3.2) and the lower order estimates (8.5.1)-(8.5.2), we have

$$\begin{aligned}
&\sum_{\pm} \left|\left|D_t^{u_-} p^\beta\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{\pm} \left|\left|D_t^{u_-} u^\beta\right|\right|^2_{L^2(\Omega_\pm)} + \mu \sum_{\pm} \int_0^t \left|\left|\nabla^\psi \cdot (D_t^{u_-} \partial^\beta u)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ \left|\left|D_t^{u_-} \partial^\beta \nabla' f\right|\right|^2_{L^2(\Gamma)} + \mu \int_0^t \left|\left|D_t^{u_-} \partial^\beta (u \cdot n)\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \\
&\le (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \qquad (8.10.4)
\end{aligned}$$

In particular, we obtain

$$\left|\left|D_t^{u_-} \partial^\beta \nabla' f\right|\right|^2_{L^2(\Gamma)} + \mu \int_0^t \left|\left|D_t^{u_-} \partial^\beta (u \cdot n)\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \le (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

Combining this with the tangential estimate for $\left|\left|\nabla' \partial^\beta \nabla' f\right|\right|^2_{L^2(\Gamma)} + \mu \int_0^t \left|\left|\nabla' \partial^\beta (u \cdot n)\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau$ and the lower order estimate (8.5.1), we obtain

$$\left|\left|\partial' \partial^\beta \nabla' f\right|\right|^2_{L^2(\Gamma)} + \mu \int_0^t \left|\left|\partial' \partial^\beta (u \cdot n)\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \le (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \qquad (8.10.5)$$

Note that if we combine this estimate with the estimate (8.8.1) for the corrected entropy and the estimate (8.6.8), we obtain the estimate for the entropy

$$||\partial^\alpha s||^2_{L^2(\Omega_\pm)} \le (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \qquad (8.10.6)$$

for $|\alpha| = 3$.

Finally, we convert the estimate (8.10.4) into an estimate for the non-corrected pressure and velocity using the estimate (8.6.9) together with the estimate (8.10.5) for the front. We obtain

$$\begin{aligned}
&\sum_{\pm} \left|\left|D_t^{u_-} \partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{\pm} \left|\left|D_t^{u_-} \partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \mu \sum_{\pm} \int_0^t \left|\left|\nabla^\psi \cdot (D_t^{u_-} \partial^\beta u)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\le (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \qquad (8.10.7)
\end{aligned}$$

8.11 A Weighted Normal Derivative Estimate

It is useful to think of the operator $D_t^{u\pm}$ as like a time plus a tangential derivative, which is true on the boundary, since $D_t^{u\pm}|_\Gamma = \partial_t + (u^\psi_\pm)' \cdot \nabla'$ because $(u^\psi)_{3\pm}|_\Gamma = 0$. But in the interior in fact we have

$$D_t^u = \partial_t + (u^\psi)' \cdot \nabla' + (u^\psi)_3 \partial_{x_3}$$

Hence it will be useful to have an estimate for $(u^\psi)_3 \partial_{x_3} \partial^\beta p$ and $(u^\psi)_3 \partial_{x_3} \partial^\beta u$ where $|\beta| = 2$, and this will be a straightforward energy estimate since $(u^\psi)_3|_\Gamma = 0$.

Let w be given by either $w = \mathrm{Ext}_{\Omega_-} (u^\psi)_{3+}$ in $\mathbb{R}^3$ or $w = \mathrm{Ext}_{\Omega_+} (u^\psi)_{3-}$ in $\mathbb{R}^3$ or $w_\pm = (u^\psi)_{3\pm}$ in $\Omega_\pm$.

We let β be a multi-index with $|\beta| = 2$. We let $\partial^\alpha = \partial_{x_3} \partial^\beta$ in the differentiated equations (8.6.4)-(8.6.5). We then multiply by w to obtain the following.

$$\begin{aligned}
&\frac{1}{\rho c^2} (\partial_t + u^\psi \cdot \nabla)(w \partial^\alpha p) + \nabla^\psi \cdot (w \partial^\alpha u) = R^{|\alpha|}(U, \psi) \\
&\rho (\partial_t + u^\psi \cdot \nabla)(w \partial^\alpha u) + \nabla^\psi (w \partial^\alpha p) = -\mu \nabla^\psi (w \partial^\alpha \dot{p}) + R^{|\alpha|}(U, \psi) + \mu R^{|\alpha|}(U, \psi, \dot{p})
\end{aligned}$$

We multiply the first equation by $w\partial^\alpha p$ and the second by $w\partial^\alpha u$ then sum the equations to obtain the following.

$$
\begin{aligned}
&\frac{1}{2}\partial_t(\frac{1}{\rho c^2}\left|w\partial^\alpha p\right|^2) + \frac{1}{2}\partial_t(\rho_\mu \left|w\partial^\alpha u\right|^2) + \frac{1}{2}\nabla\cdot(\frac{1}{\rho c^2}u^\psi \left|w\partial^\alpha p\right|^2) + \frac{1}{2}\nabla\cdot(\rho_\mu u^\psi\left|w\partial^\alpha u\right|^2)\\
&+\nabla^\psi\cdot(w\partial^\alpha u w\partial^\alpha p) + \mu\nabla^\psi\cdot(w\partial^\alpha \dot{p} w\partial^\alpha u) - \mu w\partial^\alpha\dot{p}\nabla^\psi\cdot(w\partial^\alpha u)\\
&= R^{|\alpha|}(U,\psi)\cdot\partial^\alpha U + \mu R^{|\alpha|}(U,\psi,\dot{p})\cdot\partial^\alpha u
\end{aligned}
$$

We integrate over $\Omega_\pm$ and use the divergence theorem together with the fact that $w=0$ on Γ. We also replace $w\partial^\alpha\dot{p}$ with $-w\rho c^2\nabla^\psi\cdot\partial^\alpha u$ up to lower order terms. We obtain the following.

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\frac{1}{\rho c^2}\left|w\partial^\alpha p\right|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\rho_\mu\left|w\partial^\alpha u\right|^2\mathrm{d}x + \mu\int_{\Omega_\pm}\rho c^2\left|w\nabla^\psi\cdot\partial^\alpha u\right|^2\mathrm{d}x\\
&=\int_{\Omega_\pm}R^{|\alpha|}(U,\psi)\cdot\partial^\alpha U\mathrm{d}x + \mu\int_{\Omega_\pm}R^{|\alpha|}(U,\psi,\dot{p})\cdot\partial^\alpha u\mathrm{d}x
\end{aligned}
$$

Integrating from 0 to t, using the remainder estimates (8.18.3) and (8.18.7) and the conditions (8.3.1), (8.3.2) and the estimate (8.5.1), we obtain

$$
\begin{aligned}
&\int_{\Omega_\pm}\left|w\partial^\alpha p\right|^2\mathrm{d}x + \int_{\Omega_\pm}\left|w\partial^\alpha u\right|^2\mathrm{d}x + \mu\int_0^t\int_{\Omega_\pm}\left|w\nabla^\psi\cdot\partial^\alpha u\right|^2\mathrm{d}x\mathrm{d}\tau\\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon} + \epsilon E_\mu(t))
\end{aligned}
\tag{8.11.1}
$$

Assume now β is a multi-index with $|\beta|=2$ and $\beta_3=0$. Combining the above estimate with the high-order time derivative estimate (8.10.7), and using the fact that

$$
D_t^{u-} = \partial_t + (u_-^\psi)'\cdot\nabla' + (u^\psi)_{3-}\partial_{x_3}
$$

we obtain the following simpler time derivative estimate.

$$
\begin{aligned}
&\sum_\pm\left|\left|\partial_t\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm\left|\left|\partial_t\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \mu\sum_\pm\int_0^t\left|\left|\nabla^\psi\cdot\partial_t\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\
&\leq (M^0+tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t))
\end{aligned}
\tag{8.11.2}
$$

for $|\beta|=2$ with $\beta_3=0$.

8.12 Estimate of the Curl

We know that the curl of the velocity, $\nabla^\psi\times u$, satisfies the equation (7.7.1). We apply ∂^β to this equation for $1\leq|\beta|\leq 2$ and gather the lower order terms on the right hand side, noting that the high order terms in ψ can be treated as remainder terms since they all involve at most three time derivatives. We obtain

$$
\rho_\mu(\partial_t + u^\psi\cdot\nabla)\nabla^\psi\times\partial^\beta u = R^{|\beta|+1}(U,\psi) \tag{8.12.1}
$$

in $(0,T')\times\Omega_\pm$.

For β a multi-index with $1\leq|\beta|\leq 2$, we define the corrected curl as

$$
\omega^\beta = \nabla^\psi\times\partial^\beta u \tag{8.12.2}
$$

Thus the corrected curl satisfies the equation

$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)\omega^\beta = R^{|\beta|+1}(U, \psi) \tag{8.12.3}$$

in $(0, T') \times \Omega_\pm$.

Now it is straightforward to estimate the corrected curl in the standard manner, since $(u^\psi)_{3\pm} = 0$ on Γ so there is no boundary term. We multiply equation (8.12.3) by ω^β and use the product rule to obtain

$$\frac{1}{2}\partial_t(\rho_\mu \left|\omega^\beta\right|^2) + \frac{1}{2}\nabla \cdot (\rho_\mu u^\psi \left|\omega^\beta\right|^2) = R^3(U, \psi)\omega^\beta$$

in $(0, T') \times \Omega_\pm$. Note that we have absorbed lower-order terms generated by the product rule into the remainder $R^3(U, \psi)$ on the right hand side.

Then we integrate over $\Omega_\pm$ and apply the divergence theorem for Sobolev functions to the second term on the left. Note that since $(u^\psi)_{3\pm} = 0$ on Γ we are left with no boundary terms. Hence we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \rho_\mu \left|\omega^\beta\right|^2 \mathrm{d}x = \int_{\Omega_\pm} R^3(U, \psi)\omega^\beta \mathrm{d}x$$

Applying Cauchy's inequality to the right hand side, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\sqrt{\rho_\mu}\omega^\beta\right|\right|^2_{L^2(\Omega_\pm)} \leq \left|\left|R^3(U, \psi)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\omega^\beta\right|\right|^2_{L^2(\Omega_\pm)}$$

Then using the estimate (8.18.6) of the remainder, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\sqrt{\rho_\mu}\omega^\beta\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)$$

Integrating from 0 to t, we obtain

$$\left|\left|\sqrt{\rho_\mu}\omega^\beta\right|\right|^2_{L^2(\Omega_\pm)} \leq M^0 + tK(t)$$

Since $\rho_\mu \geq \frac{1}{2}\delta^0_\mu$ by (8.3.2), we obtain

$$\left|\left|\omega^\beta\right|\right|^2_{L^2(\Omega_\pm)} \leq M^0 + tK(t) \tag{8.12.4}$$

8.13 Estimate of the Normal Derivatives of the Pressure and Velocity

For $1 \leq |\beta| \leq 2$, we define the corrected divergence as

$$\eta^\beta = \nabla^\psi \cdot \partial^\beta u \tag{8.13.1}$$

We aim to combine our estimate of the corrected curl along with an estimate of the corrected divergence with $\beta_3 = 0$ to obtain an estimate for $\partial_{x_3}\partial^\beta u$. We also aim to obtain an estimate of the normal derivative of the pressure directly from the equations. We then hope to use an inductive argument to estimate all the normal derivatives. This is detailed below.

To estimate the normal derivative of the pressure, we will need the following lemma, as used by Shkoller et al in [10] and [13].

Lemma 8.13.1. *Let $w \in W^{1,1}(0,T)$ and let $g \in L^1(0,T)$. Let $\mu \geq 0$ and suppose that*

$$w + \mu \frac{\mathrm{d}w}{\mathrm{d}t} \leq g$$

for almost every $t \in (0,T)$. Then, for any $t \in (0,T)$,

$$\sup_{\tau \in (0,t)} w(\tau) \leq w(0) + \operatorname*{ess\,sup}_{\tau \in (0,t)} |g(\tau)|$$

Proof. The case $\mu = 0$ is obvious, so assume $\mu > 0$. Multiplying by $\frac{1}{\mu} e^{\frac{1}{\mu}t}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(w e^{\frac{1}{\mu}t}) \leq \frac{1}{\mu} e^{\frac{1}{\mu}t} g$$

Integrating from 0 to t, we obtain

$$\begin{aligned} w(t) e^{\frac{1}{\mu}t} &\leq w(0) + \int_0^t \frac{1}{\mu} e^{\frac{1}{\mu}\tau} g \mathrm{d}\tau \\ &\leq w(0) + \operatorname*{ess\,sup}_{\tau \in (0,t)} |g(\tau)| \int_0^t \frac{1}{\mu} e^{\frac{1}{\mu}\tau} \mathrm{d}\tau \\ &= w(0) + \operatorname*{ess\,sup}_{\tau \in (0,t)} |g(\tau)| \, (e^{\frac{1}{\mu}t} - 1) \end{aligned}$$

Hence

$$\begin{aligned} w(t) &\leq w(0) e^{-\frac{1}{\mu}t} + \operatorname*{ess\,sup}_{\tau \in (0,t)} |g(\tau)| \, e^{-\frac{1}{\mu}t} (e^{\frac{1}{\mu}t} - 1) \\ &\leq w(0) + \operatorname*{ess\,sup}_{\tau \in (0,t)} |g(\tau)| \end{aligned}$$

as required. □

We also need the following lemma, which says that the normal derivative of $\partial^\beta u$ can be expressed in terms of the corrected divergence, the corrected curl, the tangential derivatives and the derivatives of ψ.

Lemma 8.13.2. *Let $v \in H^1(\Omega_\pm; \mathbb{R}^3)$. Then*

$$\partial_{x_3} v_i = G_i(\nabla \psi) \cdot (\partial_{x_1} v, \partial_{x_2} v, \nabla^\psi \cdot v, \nabla^\psi \times v)$$

where $G_i(\nabla \psi)$ is a smooth vector-valued function for each $i = 1, 2, 3$.

Proof. First we note that we can write $\nabla^\psi \cdot v$ as

$$\nabla^\psi \cdot v = \nabla' \cdot v' - \frac{\nabla' \psi}{J^\psi} \cdot \partial_{x_3} v' + \frac{1}{J^\psi} \partial_{x_3} v_3$$

Hence, rearranging,

$$\partial_{x_3} v_3 = J^\psi (\nabla^\psi \cdot v) - J^\psi \nabla' \cdot v' + \nabla' \psi \cdot \partial_{x_3} v'$$

From the definition of $\nabla^\psi \times$, we have

$$\begin{aligned} (\nabla^\psi \times v)_1 &= \partial_{x_2} v_3 - \partial_{x_3} v_2 - \frac{1}{J^\psi} \partial_{x_2} \psi \partial_{x_3} v_3 + \frac{1}{J^\psi} \partial_{x_3} \psi \partial_{x_3} v_2 \\ (\nabla^\psi \times v)_2 &= \partial_{x_3} v_1 - \partial_{x_1} v_3 - \frac{1}{J^\psi} \partial_{x_3} \psi \partial_{x_3} v_1 + \frac{1}{J^\psi} \partial_{x_1} \psi \partial_{x_3} v_3 \end{aligned}$$

Rearranging, we obtain

$$\partial_{x_3} v_2 = -J^\psi (\nabla^\psi \times v)_1 + J^\psi \partial_{x_2} v_3 - \partial_{x_2}\psi \partial_{x_3} v_3$$
$$\partial_{x_3} v_1 = J(\nabla^\psi \times v)_2 + J^\psi \partial_{x_1} v_3 - \partial_{x_1}\psi \partial_{x_3} v_3$$

Using these two identities in the equation for $\partial_{x_3} v_3$, we obtain

$$\begin{aligned}\partial_{x_3} v_3 &= J^\psi(\nabla^\psi \cdot v) - J^\psi \nabla' \cdot v' + \partial_{x_1}\psi(J^\psi(\nabla^\psi \times v)_2 + J^\psi \partial_{x_1} v_3 - \partial_{x_1}\psi \partial_{x_3} v_3) \\ &\quad + \partial_{x_2}\psi(-J^\psi(\nabla^\psi \times v)_1 + J^\psi \partial_{x_2} v_3 - \partial_{x_2}\psi \partial_{x_3} v_3)\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}\partial_{x_3} v_3(1 + \left|\nabla'\psi\right|^2) &= J^\psi(\nabla^\psi \cdot v) - J^\psi \nabla' \cdot v' + \partial_{x_1}\psi(J^\psi(\nabla^\psi \times v)_2 + J^\psi \partial_{x_1} v_3) \\ &\quad + \partial_{x_2}\psi(-J^\psi(\nabla^\psi \times v)_1 + J^\psi \partial_{x_2} v_3)\end{aligned}$$

Dividing by $1 + |\nabla'\psi|^2$, we obtain the desired result for $\partial_{x_3} v_3$. Putting this expression back into the identities for $\partial_{x_3} v_2$ and $\partial_{x_3} v_1$, we obtain the result. $\square$

Corollary 8.13.3. *Let* $1 \leq |\beta| \leq 2$. *Then*

$$\left|\left|\partial_{x_3}\partial^\beta u\right|\right|_{L^2(\Omega_\pm)} \leq \left|\left|R^0_\infty(U,\psi)\right|\right|_{L^\infty(\Omega_\pm)}(\left|\left|\nabla'\partial^\beta u\right|\right|_{L^2(\Omega_\pm)} + \left|\left|\eta^\beta\right|\right|_{L^2(\Omega_\pm)} + \left|\left|\omega^\beta\right|\right|_{L^2(\Omega_\pm)})$$

Proof. This is immediate from the above lemma applied to $v = \partial^\beta u$, where we have replaced $G_i(\nabla\psi)$ by the remainder $R^0_\infty(U,\psi)$. $\square$

Now, let $|\beta| = 2$. Note that by (8.6.1), and the fact that $\eta^\beta = \nabla^\psi \cdot \partial^\beta u$, we have

$$\eta^\beta = -\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)\partial^\beta p + R^{|\beta|}(U,\psi) + R^0_\infty(U,\psi) \cdot \nabla p \partial_t \partial^\beta \psi$$

where we have absorbed all the high-order terms in ψ involving a spatial derivative into the remainder $R^{|\beta|}(U,\psi)$. Note also that $\left|\left|\partial_t\partial^\beta\psi\right|\right|_{H^1(\Omega_\pm)}$ can be estimated by the estimates (8.7.1) and (8.10.5). Thus

$$\begin{aligned}&\frac{1}{C}\left|\left|\eta^\beta\right|\right|^2_{L^2(\Omega_\pm)} \leq \qquad (8.13.2)\\ &\left|\left|\frac{1}{\rho c^2}\right|\right|^2_{L^\infty(\Omega_\pm)}(1 + \left|\left|u^\psi\right|\right|_{L^\infty(\Omega_\pm)})\left(\left|\left|\partial_t\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|(u^\psi)_3\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)}\right) \\ &+ \left|\left|R^{|\beta|}(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))\left|\left|\nabla p\right|\right|^2_{H^1(\Omega_\pm)}\left|\left|\partial_t\partial^\beta\psi\right|\right|^2_{H^1(\Omega_\pm)} \\ &\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t) + \left|\left|\partial_t\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)}) \qquad (8.13.3)\end{aligned}$$

where we have used the estimates (8.3.1), (8.5.1), (8.5.3) and the lower order remainder estimate (8.18.4), together with the weighted normal derivative estimate (8.11.1).

Also, from the third component of (8.6.2), we have

$$\partial_{x_3}\partial^\beta(p + \mu\dot{p}) = -J^\psi\rho_\mu(\partial_t + u^\psi \cdot \nabla)\partial^\beta u_3 + R^{|\beta|}(U,\psi) + R^0_\infty(U,\psi) \cdot \nabla u_3 \partial_t \partial^\beta \psi$$

where again we have absorbed all the high-order terms in ψ involving a spatial derivative into the remainder $R^{|\beta|}(U,\psi)$ and we note also that $\left|\left|\partial_t\partial^\beta\psi\right|\right|_{H^1(\Omega_\pm)}$ can be estimated by the estimates (8.7.1) and (8.10.5). Now we must take some care to deal with the term involving $\mu\dot{p}$. In fact we chose this approximation to the original equations, as originally done by Shkoller et al in [10] and [13], specifically so that this term can be easily estimated. We multiply by $\partial_{x_3}\partial^\beta p$ and integrate in space to obtain

$$\begin{aligned}
&\int_{\Omega_\pm} \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x \\
&= -\mu \int_{\Omega_\pm} \partial_{x_3}\partial^\beta \partial_t p \partial_{x_3}\partial^\beta p \mathrm{d}x - \mu \int_{\Omega_\pm} \partial_{x_3}\partial^\beta (u^\psi \cdot \nabla) p \partial_{x_3}\partial^\beta p \mathrm{d}x - \int_{\Omega_\pm} J^\psi \rho_\mu D_t^{u\pm} \partial^\beta u_3 \partial_{x_3}\partial^\beta p \mathrm{d}x \\
&+ \int_{\Omega_\pm} R^{|\beta|}(U,\psi) \partial_{x_3}\partial^\beta p \mathrm{d}x + \int_{\Omega_\pm} R^0_\infty(U,\psi) \cdot \nabla u_3 \partial_t \partial^\beta \psi \partial_{x_3}\partial^\beta p \mathrm{d}x \\
&\leq -\frac{1}{2}\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\pm} \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x - \mu \frac{1}{2} \int_{\Omega_\pm} \nabla \cdot (u^\psi \left|\partial_{x_3}\partial^\beta p\right|^2) \mathrm{d}x \\
&- \mu \int_{\Omega_\pm} (\partial_{x_3}\partial^\beta u^\psi) \cdot \nabla p \partial_{x_3}\partial^\beta p \mathrm{d}x + \frac{1}{2}\mu \int_{\Omega_\pm} (\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x \\
&- \int_{\Omega_\pm} J^\psi \rho_\mu D_t^{u\pm} \partial^\beta u_3 \partial_{x_3}\partial^\beta p \mathrm{d}x + \int_{\Omega_\pm} R^{|\beta|}(U,\psi) \partial_{x_3}\partial^\beta p \mathrm{d}x \\
&+ \left|\left|R^0_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} \left|\left|\nabla u_3\right|\right|^2_{H^1(\Omega_\pm)} \left|\left|\partial_t \partial^\beta \psi\right|\right|^2_{H^1(\Omega_\pm)} + \frac{1}{4}\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq -\frac{1}{2}\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\pm} \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x + \mu^2 \left|\left|(\partial_{x_3}\partial^\beta u^\psi)\cdot \nabla p\right|\right|^2_{L^2(\Omega_\pm)} + \frac{1}{4}\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \\
&+ \mu \int_{\Omega_\pm} (\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x + C \int_{\Omega_\pm} \left|J^\psi \rho_\mu D_t^{u\pm} u_3^\beta\right|^2 \mathrm{d}x + \frac{1}{4}\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \\
&+ (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}$$

where we have used the divergence theorem together with the fact that $(u^\psi)_{3\pm} = 0$ on Γ, the lower order estimate (8.5.3), the lower order remainder estimate (8.18.4) and Cauchy's inequality.

Using the fact that $(u^\psi_-)_3|_\Gamma = 0$ together with the divergence theorem, we have

$$\begin{aligned}
&\mu \int_{\Omega_\pm} (\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x \\
&\leq M^0 + \mu \int_0^t \int_{\Omega_\pm} \partial_t((\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2) \mathrm{d}x\mathrm{d}\tau \\
&= M^0 + \mu \int_0^t \int_{\Omega_\pm} \partial_t((\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2) \mathrm{d}x\mathrm{d}\tau + \mu \int_0^t \int_{\Omega_\pm} \nabla \cdot (u^\psi_-(\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2) \mathrm{d}x\mathrm{d}\tau \\
&= M^0 + \mu \int_0^t \int_{\Omega_\pm} D_t^{u-}((\nabla \cdot u^\psi) \left|\partial_{x_3}\partial^\beta p\right|^2) \mathrm{d}x\mathrm{d}\tau + \mu \int_0^t \int_{\Omega_\pm} (\nabla \cdot u^\psi)^2 \left|\partial_{x_3}\partial^\beta p\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\leq M^0 + C \int_0^t \mu \left|\left|D_t^{u-} \nabla \cdot u^\psi\right|\right|_{H^2(\Omega_\pm)} \left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ C \int_0^t \mu \left|\left|\nabla \cdot u^\psi\right|\right|_{H^2(\Omega_\pm)} \left|\left|D_t^{u-} \partial_{x_3}\partial^\beta p\right|\right|_{L^2(\Omega_\pm)} \left|\left|\partial_{x_3}\partial^\beta p\right|\right|_{L^2(\Omega_\pm)} \mathrm{d}\tau + tK(t) \\
&\leq M^0 + tK(t) + \mu^2 \int_0^t \left|\left|D_t^{u-} \nabla \cdot u^\psi\right|\right|^2_{H^2(\Omega_\pm)} \mathrm{d}\tau + \mu^2 \int_0^t \left|\left|\partial_{x_3}\partial^\beta \dot{p}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ \int_0^t \left|\left|R^3(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq M^0 + tK(t)
\end{aligned}$$

where we have used the remainder estimate (8.18.3) and the estimate (8.18.5) for $\dot{p}$. Also, we have

$$
\begin{aligned}
&\mu^2 \left|\left|(\partial_{x_3}\partial^\beta u^\psi)\cdot\nabla p\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq M^0 + \int_0^t \mu^2 \left|\left|(\partial_t\partial_{x_3}\partial^\beta u^\psi)\cdot\nabla p\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \int_0^t \mu^2 \left|\left|\partial_{x_3}\partial^\beta u^\psi \cdot \partial_t\nabla p\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq M^0 + K(t)\int_0^t \mu^2 \left|\left|\partial_t\partial_{x_3}\partial^\beta u^\psi\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \int_0^t \mu^2 \left|\left|\partial_{x_3}\partial^\beta u^\psi\right|\right|^2_{H^1(\Omega_\pm)} ||\partial_t\nabla p||^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\
&\leq M^0 + tK(t)
\end{aligned}
$$

Combining these estimates, we obtain

$$
\begin{aligned}
&\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} + \mu\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t) + \left|\left|D_t^{u_\pm}\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)}) \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t) + \left|\left|\partial_t\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|(u^\psi)_3\partial_{x_3}\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)}) \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t) + \left|\left|\partial_t\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)})
\end{aligned}
$$

where we have used the lower order estimates (8.5.1), (8.5.3) and the weighted normal derivative estimate (8.11.1). Applying lemma 8.13.1, we obtain

$$
\begin{aligned}
&\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t) + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\nabla'\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)})
\end{aligned} \tag{8.13.4}
$$

Now we claim that

$$
\left|\left|\partial_{x_3}\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t)) \tag{8.13.5}
$$

$$
\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t)) \tag{8.13.6}
$$

We proceed by induction on β_3. In the case that $\beta_3 = 0$, we note that we may apply (8.9.1) and (8.11.2) to the right hand side of (8.13.3) and (8.13.4) to obtain

$$
\begin{aligned}
\left|\left|\eta^\beta\right|\right|^2_{L^2(\Omega_\pm)} &\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t)) \\
\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} &\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t))
\end{aligned}
$$

Then we apply the estimate from corollary 8.13.3 together with the lower order remainder estimate (8.18.4), the estimate (8.9.1) for the tangential derivatives, the estimate (8.12.4) for the corrected curl and the estimate just obtained for the corrected divergence to conclude that

$$
\left|\left|\partial_{x_3}\partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t))
$$

which proves the result in the case $\beta_3 = 0$.

Now we suppose the result is true for $\beta_3 \leq k$ for some $0 \leq k \leq |\beta| - 1$ and try to show it is true for $\beta_3 = k + 1$. But this is straightforward and is done in the same way. Note that the previous induction step gives us the estimate for

$$||\partial^\gamma u||^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$
$$||\partial^\gamma p||^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

where $|\gamma| = 3$ and $\gamma_3 \leq k + 1$. We apply these estimates to the right hand side of (8.13.3) and (8.13.4) to obtain

$$\left|\left|\eta^\beta\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$
$$\left|\left|\partial_{x_3}\partial^\beta p\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

Then we apply the estimate from corollary 8.13.3 together with the lower order remainder estimate (8.18.4), the previous induction step for the term $\nabla' \partial^\beta u$, the estimate (8.12.4) for the corrected curl and the estimate just obtained for the corrected divergence to conclude the result for the case $\beta_3 = k + 1$. This proves the claim.

8.14 Estimate of the Artificial Viscosity Term

Let us observe that we may rearrange the differentiated velocity equation (8.6.2) to obtain

$$-\mu \nabla^\psi \partial^\beta \dot{p} = \rho_\mu (\partial_t + u^\psi \cdot \nabla)\partial^\beta u + \nabla^\psi \partial^\beta p + R^{|\beta|}(U, \psi) + R^0_\infty(U, \psi)\nabla \cdot u \partial \partial^\beta \psi + \mu R^{|\beta|}(U, \psi, \dot{p})$$

where $|\beta| \leq 2$. We use the fact that we have already obtained estimates for everything on the right hand side in $L^2(\Omega_\pm)$ to obtain

$$\mu^2 \left|\left|\nabla^\psi \partial^\beta \dot{p}\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

Using the fact that

$$\partial_{x_3}\partial^\beta \dot{p} = J^\psi (\nabla^\psi)_3 \partial^\beta \dot{p}$$

and the lower order estimate for $\nabla \psi$, we easily obtain

$$\mu^2 \left|\left|\nabla \partial^\beta \dot{p}\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \tag{8.14.1}$$

Using the fact that $\dot{p} = -\rho c^2 \nabla^\psi \cdot u$ together with the chain rule and existing estimates we also obtain

$$\mu^2 \left|\left|\nabla^\psi \cdot \nabla \partial^\beta u\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \tag{8.14.2}$$

8.15 Elliptic-Type Estimate for the Front

In this section we derive an elliptic type estimate for the front using the pressure interface condition (7.1.6).

Lemma 8.15.1.

$$||\partial^\alpha \nabla' f||^2_{H^{0.5}(\Gamma)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \tag{8.15.1}$$

for $0 \leq |\alpha| \leq 3$*,* $\alpha_0 \leq 2$ *and* $\alpha_3 = 0$*.*

Proof. Let α be a multi-index with $0 \leq |\alpha| \leq 3$, $\alpha_0 \leq 2$ and $\alpha_3 = 0$, so that we can write $\partial^\alpha = \partial^\beta \partial_{x_i}$ for some $i \in \{1, 2\}$. Let $\langle D' \rangle^{0.5}$ denote the half-order derivative operator which corresponds to multiplying by $\langle \xi' \rangle^{0.5}$ in Fourier space - see the notation in the appendix.

Applying $\langle D' \rangle^{0.5}$ to the differentiated pressure interface condition (8.6.15), we have

$$\langle D' \rangle^{0.5} [\partial^\alpha (p + \mu \dot{p})] = \sigma \langle D' \rangle^{0.5} \nabla' \cdot \partial^\alpha (\frac{\nabla' f}{|n|}) + \mu(\Delta' - 1) \langle D' \rangle^{0.5} \partial^\alpha \partial_t f + \langle D' \rangle^{0.5} \partial^\alpha g_2$$

We multiply both sides by $\langle D' \rangle^{0.5} \partial^\alpha f$ and integrate by parts to obtain

$$\begin{aligned}
&\int_\Gamma (\langle D' \rangle^{0.5} [\partial^\beta (p + \mu \dot{p})])(\langle D' \rangle^{0.5} \partial_{x_i} \partial^\alpha f) \mathrm{d}x' = \sigma \int_\Gamma (\langle D' \rangle^{0.5} \partial^\alpha (\frac{\nabla' f}{|n|})) \cdot (\langle D' \rangle^{0.5} \partial^\alpha \nabla' f) \mathrm{d}x' \\
&+ \mu \int_\Gamma (\langle D' \rangle^{0.5} \partial^\alpha \nabla' \partial_t f)(\langle D' \rangle^{0.5} \partial^\alpha \nabla' f) \mathrm{d}x' + \mu \int_\Gamma (\langle D' \rangle^{0.5} \partial^\alpha \partial_t f)(\langle D' \rangle^{0.5} \partial^\alpha f) \mathrm{d}x' \\
&+ \int_\Gamma (\langle D' \rangle^{0.5} R^2_\Gamma(f))^2 \mathrm{d}x'
\end{aligned}$$

where the remainder $R^2_\Gamma(f)$ is defined in 8.18.19. We now use lemma 8.6.4 to differentiate $\frac{\nabla' f}{|n|}$, together with the fractional product rule A.3.8 from the appendix, and following the notation of this lemma, we denote the remainder term by $R(G(\nabla' f), \partial^\alpha \nabla' f)$, where G is a smooth function. We obtain

$$\begin{aligned}
&\int_\Gamma (\langle D' \rangle^{0.5} [\partial^\beta (p + \mu \dot{p})])(\langle D' \rangle^{0.5} \partial^\alpha \partial_{x_i} f) \mathrm{d}x' \\
&= \sigma \int_\Gamma (\langle D' \rangle^{0.5} \partial^\alpha (\frac{\nabla' f}{|n|})) \cdot (\langle D' \rangle^{0.5} \partial^\alpha \nabla' f) \mathrm{d}x' \\
&+ \mu \int_\Gamma (\langle D' \rangle^{0.5} \partial^\alpha \nabla' \partial_t f)(\langle D' \rangle^{0.5} \partial^\alpha \nabla' f) \mathrm{d}x' \\
&+ \int_\Gamma (\langle D' \rangle^{0.5} \partial_t R^2_\Gamma(f)) \langle D' \rangle^{0.5} R^2_\Gamma(f) \mathrm{d}x' + \int_\Gamma (\langle D' \rangle^{0.5} R^2_\Gamma(f))^2 \mathrm{d}x' \\
&= \sigma \int_\Gamma \frac{1}{|n|} \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \right|^2 - \frac{1}{|n|^3} \left| \nabla' f \cdot \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&+ \sigma \int_\Gamma \langle D' \rangle^{0.5} R^2_\Gamma(f) \cdot \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \mathrm{d}x' \\
&+ \sigma \int_\Gamma R(G(\nabla' f), \partial^\alpha \nabla' f) \cdot \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \mathrm{d}x' + \mu \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&+ \int_\Gamma (\langle D' \rangle^{0.5} \partial_t R^2_\Gamma(f)) \langle D' \rangle^{0.5} R^2_\Gamma(f) \mathrm{d}x' + \int_\Gamma (\langle D' \rangle^{0.5} R^2_\Gamma(f))^2 \mathrm{d}x' \\
&\geq \sigma \int_\Gamma \frac{1}{|n|^3} \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&- \frac{1}{\delta} ||R(G(\nabla' f), \partial^\alpha \nabla' f)||^2_{L^2(\Gamma)} - \delta \int_\Gamma \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&+ \mu \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' - ||R^2_\Gamma(f)||^2_{H^1(\Gamma)} - ||\partial_t R^2_\Gamma(f)||^2_{L^2(\Gamma)} - C ||\partial^\alpha \nabla' f||^2_{L^2(\Gamma)}
\end{aligned}$$

where we have used lemma 8.6.5. We have also used Cauchy's inequality in the last step, and we are free to choose $0 < \delta \leq 1$. Note that $\partial R^2_\Gamma(f)$ is a sum of terms of the form $G(\nabla' f)(\partial\partial^\gamma(\nabla' f, g_2) + \partial\nabla' f R^2(f))$ where $|\gamma| \leq 2$ and G is a smooth function. We recall that we have already estimated $\partial\partial^\gamma\nabla' f$ in (8.10.5). Hence, using Cauchy's inequality, the remainder estimate (8.18.15), and the estimate of the remainder from A.3.8, we obtain

$$\begin{aligned}
&\sigma \int_\Gamma \frac{1}{|n|^3} \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' + \frac{1}{2}\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&\leq \frac{C}{\delta} \left|\left| \left\langle D' \right\rangle^{0.5} [\partial^\beta (p + \mu \dot{p})] \right|\right|^2_{L^2(\Gamma)} + \delta \left|\left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|\right|^2_{L^2(\Gamma)} + \frac{1}{\delta}(M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}$$

We choose $\delta \leq \frac{\sigma}{2||n||^3_{L^\infty(\Gamma)}}$ and rearrange and use the Sobolev trace theorem to obtain

$$\begin{aligned}
&\sigma \int_\Gamma \frac{1}{|n|^3} \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' + \mu \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&\leq C \sum_\pm \frac{||n||^3_{L^\infty(\Gamma)}}{\sigma} \left|\left| \partial^\beta (p + \mu \dot{p}) \right|\right|^2_{H^1(\Omega_\pm)} + (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}$$

where we have used the lower order estimate (8.5.3) and the existing energy estimates (8.9.1), (8.13.6) and (8.14.1). Hence

$$\begin{aligned}
&\int_\Gamma \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' + \mu \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}$$

Now we apply the lemma 8.13.1 to obtain

$$\int_\Gamma \left| \left\langle D' \right\rangle^{0.5} \partial^\alpha \nabla' f \right|^2 \mathrm{d}x' \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

This completes the proof. □

8.16 Estimate of the Additional Terms in the Energy Involving μ

Applying ∂^α where $|\alpha| = 3$ and $\alpha_0 \leq 2$ to the curl equation (7.7.1), we obtain

$$\begin{aligned}
&\rho_\mu(\partial_t + u^\psi \cdot \nabla)\partial^\alpha(\nabla^\psi \times u) = -\nabla^\psi \partial^\alpha \rho_\mu \times (\partial_t + u^\psi \cdot \nabla) u - \nabla^\psi \rho_\mu \times (\partial_t + u^\psi \cdot \nabla)\partial^\alpha u \\
&- \rho_\mu(\epsilon_{ijk} \partial^\psi_j \partial^\alpha u_l \partial^\psi_l u_k) - \rho_\mu(\epsilon_{ijk} \partial^\psi_j u_l \partial^\psi_l \partial^\alpha u_k) + \nabla^\psi \times \partial^\alpha g_1 \\
&+ R^1_\infty(U, \psi)(R^3(U, \psi) + \partial^3(\rho_\mu, u)\partial^2(\rho_\mu, u, \nabla\psi)) \\
&= R^1_\infty(U, \psi)(\nabla\partial^\alpha \rho_\mu + \partial\partial^\alpha u + \nabla\partial^\alpha g_1 + R^3(U, \psi) + \partial^3(\rho_\mu, u)\partial^2(\rho_\mu, u, \partial'\psi))
\end{aligned}$$

Now we multiply thus equation by $\mu^2 \partial^\alpha(\nabla^\psi \times u)$ and use the product rule to obtain

$$\begin{aligned}
&\frac{1}{2}\mu^2 \partial_t(\rho_\mu \left| \partial^\alpha(\nabla^\psi \times u) \right|^2) + \mu^2 \frac{1}{2} \nabla \cdot (\rho_\mu u^\psi \left| \partial^\alpha(\nabla^\psi \times u) \right|^2) \\
&= \mu^2 R^1_\infty(U, \psi)(\nabla\partial^\alpha \rho_\mu + \partial\partial^\alpha u + \nabla\partial^\alpha g_1 + R^3(U, \psi) + \partial^3(\rho_\mu, u)\partial^2(\rho_\mu, u, \partial'\psi))\partial^\alpha(\nabla^\psi \times u)
\end{aligned}$$

in $(0, T') \times \Omega_\pm$ where we have absorbed some lower order terms generated by the product rule into the right hand side.

Then we integrate over $\Omega_\pm$ and apply the divergence theorem for Sobolev functions to the second term on the left. Note that since $(u^\psi)_{3\pm} = 0$ on Γ we are left with no boundary terms. Hence we obtain

$$\begin{aligned}
&\mu^2 \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \rho_\mu \left|\partial^\alpha(\nabla^\psi \times u)\right|^2 \mathrm{d}x \\
&\leq \mu^2 \left|\left|R^1_\infty(U,\psi)(\nabla\partial^\alpha\rho_\mu + \partial\partial^\alpha u + \nabla\partial^\alpha g_1 + R^3(U,\psi) + \partial^3(\rho_\mu,u)\partial^2(\rho_\mu,u,\partial'\psi))\right|\right|^2_{L^2(\Omega_\pm)} \\
&+ \mu^2 \left|\left|\partial^\alpha(\nabla^\psi \times u)\right|\right|^2_{L^2(\Omega_\pm)}
\end{aligned}$$

We claim

$$\mu^2 \left|\left|R^1_\infty(U,\psi)(\nabla\partial^\alpha\rho_\mu + \partial\partial^\alpha u + \nabla\partial^\alpha g_1 + R^3(U,\psi) + \partial^3(\rho_\mu,u)\partial^2(\rho_\mu,u,\partial'\psi))\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)$$

Indeed, note that by the properties of mollifiers,

$$\mu^2 \left|\left|\nabla\partial^\alpha\rho_\mu\right|\right|^2_{L^2(\Omega_\pm)} \leq C\left|\left|\partial^\alpha\rho_\mu\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)$$

Since $\alpha_0 \leq 2$ we have

$$\mu^2 \left|\left|\partial\partial^\alpha u\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)$$

We also note that, using the properties of mollifiers again for the highest order term involving ρ_μ,

$$\mu^2 \left|\left|\partial^3(\rho_\mu,u)\partial^2(\rho_\mu,u,\partial'\psi)\right|\right|^2_{L^2(\Omega_\pm)} \leq \mu^2 \left|\left|\partial^3(\rho_\mu,u)\right|\right|^2_{H^1(\Omega_\pm)} \left|\left|\partial^2(\rho_\mu,u,\partial'\psi)\right|\right|^2_{H^1(\Omega_\pm)} \leq K(t)$$

Clearly

$$\left|\left|\nabla\partial^\alpha g_1\right|\right|^2_{L^2(\Omega_\pm)} \leq E_g(t) \leq K(t)$$

Along with the remainder estimates (8.18.1) and (8.18.3), this proves the claim. Hence

$$\mu^2 \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \sqrt{\rho_\mu} \left|\partial^\alpha(\nabla^\psi \times u)\right|^2 \mathrm{d}x \leq K(t)$$

Integrating from 0 to t, we obtain

$$\mu^2 \left|\left|\rho_\mu\partial^\alpha(\nabla^\psi \times u)\right|\right|^2_{L^2(\Omega_\pm)} \leq M^0 + tK(t)$$

Since $\rho_\mu \geq \frac{1}{2}\delta^0_\mu$ by (8.3.2), and we already have an estimate for $\partial^\alpha\nabla\psi$ given by (8.10.5), we obtain

$$\mu^2 \left|\left|\nabla^\psi \times \partial^\alpha u\right|\right|^2 \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}}E_\mu(t)) \tag{8.16.1}$$

Applying ∂^α where $|\alpha| = 2$ to the curl equation (7.7.1), we obtain

$$\rho_\mu\nabla^\psi \times \partial_t\partial^\alpha u = -(u^\psi \cdot \nabla)\nabla^\psi \times \partial^\alpha u + R^3(U,\psi)$$

Multiplying by μ, squaring, integrating in time and using the remainder estimate (8.18.3), we obtain

$$\mu^2 \int_0^t \left|\left|\nabla^\psi \times \partial_t\partial^\alpha u\right|\right|^2 \mathrm{d}\tau \leq tK(t) \tag{8.16.2}$$

for $|\alpha| = 2$.

Now we obtain an elliptic-type estimate for $u \cdot n$ similar to the elliptic estimate we obtained for f. First we prove a very simple lemma which we will reuse later.

Lemma 8.16.1. *Let $z \in H^1(\Gamma)$ and $h \in H^s(\Gamma)$ for $s \in \mathbb{R}_{\geq 0}$ with*

$$(1 - \Delta')z = h$$

on Γ in the sense of distributions. Then $z \in H^{s+2}(\Gamma)$ with

$$||z||_{H^{s+2}(\Gamma)} \leq C\,||h||_{H^s(\Gamma)}$$

Proof. Taking Fourier transforms with respect to x', we have

$$\langle \xi' \rangle^2 \hat{z} = \hat{h}$$

Multiplying by $\langle \xi' \rangle^s$, then squaring, we obtain

$$\left| \langle \xi' \rangle^{s+2} \hat{z} \right|^2 = \left| \langle \xi' \rangle^s \hat{h} \right|^2$$

Integrating over $\mathbb{R}^2$ and using the definition of the Sobolev norm in terms of Fourier transforms, we obtain $z \in H^{s+2}(\Gamma)$ with

$$||z||^2_{H^{s+2}(\Gamma)} = ||h||^2_{H^s(\Gamma)}$$

as required. □

Now we let $0 \leq j \leq 2$ and apply ∂_t^j to the pressure interface condition (7.1.6). We obtain

$$\mu(1 - \Delta')\partial_t^j(u \cdot n) = \partial_t^j(-[p] - \sigma\nabla' \cdot \hat{n} + g_2 - \mu[\dot{p}])$$

We apply the above lemma with

$$\begin{aligned} z &= \mu\partial_t^j(u \cdot n) \\ h &= \partial_t^j(-[p] - \sigma\nabla' \cdot \hat{n} + g_2 - \mu[\dot{p}]) \end{aligned}$$

and the Sobolev trace estimate to conclude that

$$\begin{aligned} &\mu^2 \left|\left| \partial_t^j(u \cdot n) \right|\right|^2_{H^{4.5-j}(\Gamma)} \leq C \left|\left| \partial_t^j(-[p] - \sigma\nabla' \cdot \hat{n} + g_2 - \mu[\dot{p}]) \right|\right|^2_{H^{2.5-j}(\Gamma)} \\ &\leq C\sum_{\pm} \left|\left| \partial_t^j p \right|\right|^2_{H^{3-j}(\Omega_\pm)} + C\mu^2 \sum_{\pm} \left|\left| \partial_t^j \dot{p} \right|\right|^2_{H^{3-j}(\Omega_\pm)} + C\left|\left| \partial_t^j g_2 \right|\right|^2_{H^{2.5-j}(\Gamma)} + C\,||\nabla' \cdot \hat{n}||^2_{H^{2.5-j}(\Gamma)} \\ &\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \end{aligned} \tag{8.16.3}$$

where we have used the chain and product rules, the lower order estimates (8.5.1), (8.5.3), (8.5.4) and the existing estimates (8.9.1), (8.10.5), (8.11.2), (8.13.6), (8.13.5), (8.10.6) and (8.14.2).

Let $1 \leq |\alpha| \leq 3$ with $\alpha_3 = 0$. We apply the Hodge decomposition estimate A.1.2 along with the lower order estimate (8.5.3) to obtain

$$\begin{aligned} &||\partial^\alpha u||^2_{H^1(\Omega_\pm)} \\ &\leq (M^0 + tK(t))(||\partial^\alpha u||^2_{L^2(\Omega_\pm)} + \left|\left| \nabla^\psi \cdot \partial^\alpha u \right|\right|^2_{L^2(\Omega_\pm)} + \left|\left| \nabla^\psi \times \partial^\alpha u \right|\right|^2_{L^2(\Omega_\pm)} + ||\partial^\alpha u \cdot n||^2_{H^{0.5}(\Gamma)}) \\ &\leq (M^0 + tK(t))(||\partial^\alpha u||^2_{L^2(\Omega_\pm)} + \left|\left| \nabla^\psi \cdot \partial^\alpha u \right|\right|^2_{L^2(\Omega_\pm)} + \left|\left| \nabla^\psi \times \partial^\alpha u \right|\right|^2_{L^2(\Omega_\pm)} \\ &+ ||\partial^\alpha (u \cdot n)||^2_{H^{0.5}(\Gamma)} + ||u \cdot \partial^\alpha n||^2_{H^{0.5}(\Gamma)} + \sum_{\beta+\gamma=\alpha, \beta\neq\alpha, \gamma\neq\alpha} \left|\left| \partial^\beta u \cdot \partial^\gamma n \right|\right|^2_{H^{0.5}(\Gamma)}) \end{aligned}$$

Now in the case $\alpha_0 \leq 2$, we multiply by μ^2 apply the existing estimates (8.14.2), (8.16.1), (8.9.1), (8.11.2), (8.13.5), (8.15.1) and the Sobolev embedding theorem to obtain

$$\mu^2 \left|\left|\partial^\alpha u\right|\right|^2_{H^1(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

In the case $\alpha_0 = 3$, we multiply by μ^2, integrate in time, apply the existing estimates (8.16.2), (8.9.1), (8.11.2), (8.13.5), (8.15.1) and the Sobolev embedding theorem, and rewrite the term $||u \cdot \partial^\alpha n||^2_{H^{0.5}(\Gamma)}$ as $\left|\left|u \cdot \partial_t^2 \nabla'(u \cdot n)\right|\right|^2_{H^{0.5}(\Gamma)}$ to obtain

$$\mu^2 \int_0^t \left|\left|\partial^\alpha u\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \tag{8.16.4}$$

Now it remains to estimate $||\partial^\alpha u||_{H^1(\Omega_\pm)}$ for $0 \leq |\alpha| \leq 3$ where $\alpha_3 \geq 1$. But we may apply lemma 8.13.2 to obtain for $|\alpha| = 3$ and $\alpha_0 \leq 2$,

$$\partial_{x_3} \partial^\alpha u_i = G_i(\nabla\psi) \cdot (\partial_{x_1}\partial^\alpha u, \partial_{x_2}\partial^\alpha u, \nabla^\psi \cdot \partial^\alpha u, \nabla^\psi \times \partial^\alpha u)$$

where $G_i(\nabla\psi)$ is a smooth vector-valued function for each $i = 1, 2, 3$. We claim inductively on α_3 for $0 \leq \alpha_3 \leq 3$ that

$$\mu^2 \left|\left|\partial_{x_3}\partial^\alpha u_i\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

Indeed, we have already proved the case $\alpha_3 = 0$. So we assume it is true for $\alpha_3 \leq k$ where $0 \leq k \leq 2$ and try to prove it is true for $\alpha_3 = k + 1$. Using the above formula we have

$$\begin{aligned}
&\mu^2 \left|\left|\partial_{x_3}\partial^\alpha u_i\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq \mu^2 (M^0 + tK(t))(\left|\left|\nabla'\partial^\alpha u\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla^\psi \cdot \partial^\alpha u\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla^\psi \times \partial^\alpha u\right|\right|^2_{L^2(\Omega_\pm)}) \\
&\leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))
\end{aligned}$$

where we have used the induction hypothesis for the first term on the right, the estimate (8.14.2) for the second term on the right and the estimate (8.16.1) for the third term on the right. This completes the proof of the induction step. Hence we have finally proved that

$$\mu^2 \left|\left|\partial^\alpha u\right|\right|^2_{H^1(\Omega_\pm)} \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t)) \tag{8.16.5}$$

for $1 \leq |\alpha| \leq 3$ with $\alpha_0 \leq 2$.

8.17 Conclusion of the Energy Estimate

Combining the estimates (8.5.1), (8.5.3), (8.5.4), (8.7.1), (8.9.1), (8.11.2), (8.10.5), (8.10.6), (8.13.5), (8.13.6), (8.15.1), (8.16.5), (8.16.4) and (8.16.3) we obtain

$$E_\mu(t) \leq (M^0 + tK(t))(\frac{1}{\epsilon^{\frac{3}{2}}} + \epsilon^{\frac{1}{2}} E_\mu(t))$$

Now for each t we may choose

$$\epsilon^{\frac{1}{2}} \leq \frac{1}{2(M^0 + tK(t))}$$

and rearrange to obtain

$$E_\mu(t) \leq M^0 + tK(t) \tag{8.17.1}$$

where we have redefined M^0 and $K(t)$. We claim that the result of proposition 7.5.1 follows from this inequality.

Indeed, we set

$$T^0 = \sup\{t \in (0, T') : E_\mu(t) \leq 2M^0, \inf_{\tau\in(0,t)} \inf_{\Omega_\pm} \rho \geq \frac{2\delta^0}{3}, \inf_{\tau\in(0,t)} \inf_{\Omega_\pm} \rho_\mu \geq \frac{2\delta^0_\mu}{3}, \inf_{\tau\in(0,t)} \inf_{\mathbb{R}^3} J^\psi \geq \frac{2\kappa^0}{3}\}$$

Note that, by uniform continuity, the set is non-empty and $T^0 > 0$. Suppose that $T^0 = T'$. If in addition $T' = T$ or $T' = 1$ then we have

$$E_\mu(t) \leq 2M^0$$

for $t \in (0, \min\{T, 1\})$. Otherwise, by continuity, we have

$$\begin{aligned} \rho &\geq \frac{1}{2}\delta^0 \\ \rho_\mu &\geq \frac{1}{2}\delta^0_\mu \\ J^\psi &\geq \frac{1}{2}\kappa^0 \end{aligned}$$

for $t \in (0, T' + \epsilon)$, for some $\epsilon > 0$, which contradicts the definition of T'.

Thus we may suppose that $T^0 < T'$. Again using continuity and the definition of the supremum, this implies that one of the following equalities hold.

$$\begin{aligned} E_\mu(T^0) &= 2M^0 \\ \inf_{\Omega_\pm} \rho(T^0) &= \frac{2\delta^0}{3} \\ \inf_{\Omega_\pm} \rho_\mu(T^0) &= \frac{2\delta^0_\mu}{3} \\ \inf_{\mathbb{R}^3} J^\psi(T^0) &= \frac{2\kappa^0}{3} \end{aligned}$$

From the inequalities (8.4.1)-(8.4.3) and (8.17.1) evaluated at $t = T^0$ we have

$$\begin{aligned} E_\mu(T^0) &\leq M^0 + T^0 F_1(E_\mu(T^0), M^0_1) \\ \inf_{\Omega_\pm} \rho(T^0) &\geq \delta^0 - T^0 F_2(E_\mu(T^0), M^0_2) \\ \inf_{\Omega_\pm} \rho_\mu(T^0) &\geq \delta^0_\mu - T^0 F_3(E_\mu(T^0), M^0_3) \\ \inf_{\mathbb{R}^3} J^\psi(T^0) &\geq \kappa^0 - T^0 F_4(E_\mu(T^0), M^0_4) \end{aligned}$$

where F_i are smooth increasing functions and M^0_i has the same form as M^0. Hence, noting that $E_\mu(T^0) \leq 2M^0$, one of the following inequalities holds.

$$\begin{aligned} 2M^0 &\leq M^0 + T^0 F_1(2M^0, M^0_1) \\ \frac{2\delta^0}{3} &\geq \delta^0 - T^0 F_2(2M^0, M^0_2) \\ \frac{2\delta^0_\mu}{3} &\geq \delta^0_\mu - T^0 F_3(2M^0, M^0_3) \\ \frac{2\kappa^0}{3} &\geq \kappa^0 - T^0 F_4(2M^0, M^0_4) \end{aligned}$$

Thus, rearranging (and simplifying slightly), one of the following inequalities holds.

$$T^0 \geq \frac{M^0}{F_1(2M^0, M_1^0)}$$
$$T^0 \geq \frac{\delta^0}{3F_2(2M^0, M_2^0)}$$
$$T^0 \geq \frac{\delta_\mu^0}{3F_3(2M^0, M_3^0)}$$
$$T^0 \geq \frac{\kappa^0}{3F_4(2M^0, M_4^0)}$$

Hence T^0 is bounded below by the minimum of the quantities on the right hand side and we have

$$E_\mu(t) \leq 2M^0$$
$$\rho \geq \frac{1}{2}\delta^0$$
$$\rho_\mu \geq \frac{1}{2}\delta_\mu^0$$
$$J^\psi \geq \frac{1}{2}\kappa^0$$

for $t \in (0, T^0)$.

This completes the proof of proposition 7.5.1.

8.18 Remainder Estimates

One of the key observations in deriving high-order energy estimates is that once we have differentiated the equations enough times, the lower order terms may all be estimated in terms of the energy by the Sobolev embedding theorem, and hence we may try and obtain energy estimates in the same way as for constant-coefficient linear equations. To make the analysis as clear as possible, in this section we define the types of generic remainder term which are generated and prove estimates for them, so that we can concentrate on dealing with the important higher-order terms in the main body of the proof of the energy estimate. This section or the proofs in it may be skipped on first reading.

Remark 8.18.1. We note that U_+ can be extended to $(0,T) \times \Omega_-$ in such a way that the Sobolev norm of the extended function is controlled by a constant times the Sobolev norm of the original function, and similarly U_- can be extended to $(0,T) \times \Omega_+$. Hence in the following section on remainders, we will write U when we could be referring to either U_+ or U_-, or their extensions to $(0,T)\times\Omega_-$ or $(0,T)\times\Omega_+$. We can do this because the estimates only depend on a control of their Sobolev norms, and our energy functional is symmetric with respect to $\pm$ terms.

Definition 8.18.2. Let $0 \leq k \leq 1$. We will write $R_\infty^k(U, \psi)$ for a generic remainder term of the form

$$R_\infty^k(U, \psi) = G(\{\partial^\beta(U, \rho_\mu, \psi, \partial\psi, g_1) : |\beta| \leq k\})$$

where G is a smooth function of its arguments in the region $\rho > 0, \rho_\mu > 0, J^\psi > 0$.

Lemma 8.18.3. *Given $t \in (0, T')$, the remainder term $R_\infty^k(U, \psi)$ satisfies*

$$\left|\left|R_\infty^k(U, \psi)\right|\right|^2_{L^\infty(\Omega_\pm)} \leq K(t) \tag{8.18.1}$$

Proof. For $|\beta| \leq 1$, the Sobolev embedding theorem and the definition of ψ as a lift of f implies

$$\begin{aligned}\left|\left|\partial^\beta(U,\psi,\partial\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} &\leq C\,||U||^2_{L^\infty(\Omega_\pm)} + C\,||f||^2_{L^\infty(\Gamma)} + C\,||\partial U||^2_{H^2(\Omega_\pm)} + C\,\left|\left|(\partial f,\partial^2 f)\right|\right|^2_{H^2(\Gamma)} \\ &\leq CE_\mu(t)\end{aligned}$$

From the definition of the energy E_g for g and the Sobolev embedding theorem, we have

$$\left|\left|\partial^\beta g_1\right|\right|^2_{L^\infty(\Omega_\pm)} \leq M^0$$

Hence, since G is a smooth function of its arguments in the region $\rho > 0, \rho_\mu > 0, J^\psi > 0$, and we have $\rho \geq \frac{\delta^0}{2}$, $\rho_\mu \geq \frac{\delta^0_\mu}{2}$, $J^\psi \geq \frac{\kappa^0}{2}$ on $(0,T')$, we have

$$\left|\left|G(\{\partial^\beta(U,\rho_\mu,\psi,\partial\psi,g_1) : |\beta| \leq k\})\right|\right|^2_{L^\infty(\Omega_\pm)} \leq K(t)$$

□

Lemma 8.18.4. *Let $k = 0$. Then the remainder $R^k_\infty(U,\psi)$ satisfies*

$$\left|\left|R^k_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} \leq M^0 + tK(t) \tag{8.18.2}$$

Proof. From the way we have defined $R^k_\infty(U,\psi)$, we have

$$\partial_t R^k_\infty(U,\psi) = R^{k+1}_\infty(U,\psi)$$

Thus, by the estimate (8.18.1),

$$\begin{aligned}\left|\left|R^k_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} &\leq C\left|\left|R^k_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_\pm)}\Big|_{t=0} + C\int_0^t \left|\left|\partial_t R^k_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} \mathrm{d}\tau \\ &\leq M^0 + tK(t)\end{aligned}$$

□

Definition 8.18.5. Let $0 \leq k \leq 3$. We will write $R^k(U,\psi)$ for a generic remainder term which consists of sums of terms of the form

$$G(\{\partial^j_{x_3}\psi : 0 \leq j \leq k+1\})R^0_\infty(U,\psi)\prod_{i=1}^n \partial^{\alpha^i}(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)}$$

(which is not allowed in the case $k = 0$) and

$$G(\{\partial^j_{x_3}\psi : 0 \leq j \leq k+1\})R^0_\infty(U,\psi)\cdot\partial^\alpha g_1$$

Here, n is an integer with $1 \leq n \leq k+1$, α^i are multi-indices with $0 \leq \left|\alpha^i\right| \leq k$ and $1 \leq \sum_{i=1}^n \left|\alpha^i\right| \leq k+1$, $(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)}$ denotes an entry of $(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)$ depending on i, $R^0_\infty(U,\psi)$ is defined in 8.18.2, G is a smooth function of its arguments, and $|\alpha| \leq k$. Finally, we impose the important constraint that if $(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)} = \partial_t\psi$ then $\left|\alpha^i\right| \leq k-1$ or ∂^{α^i} contains at least one space derivative (so it is not all time derivatives).

Lemma 8.18.6. *Given $t \in (0,T')$, the remainder term $R^k(U,\psi)$ satisfies*

$$\left|\left|R^k(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t) \tag{8.18.3}$$

Proof. By (8.18.1), we have

$$||R^0_\infty(U,\psi)||^2_{L^\infty(\Omega_\pm)} \le K(t)$$

Using the definition of ψ as a lift of f which involves multiplying by a smooth cut-off function, we have

$$||G(\{\partial^j_{x_3}\psi : 0 \le j \le k+1\})||^2_{L^\infty(\Omega_\pm)} \le F(||f||^2_{L^\infty(\Gamma)}) \le K(t)$$

Now we intend to apply the Sobolev product lemma A.3.1 to the product which occurs in the remainder, $\prod_{i=1}^n \partial^{\alpha^i}(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)}$. Suppose $|\alpha_i| = 0$ for some i. Then we may estimate the term $\partial^{\alpha^i}(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)}$ in $L^\infty(\Omega_\pm)$ and remove it from the product, so we may assume that $|\alpha_i| > 0$ for all i.

Set $m_i = 3 - |\alpha^i|$ and $m = 2 > \frac{3}{2} = \frac{d}{p}$. Note that since $1 \le |\alpha^i| \le k \le 3$, we have $0 \le m_i \le m$. Note also that

$$\sum_{i=1}^n m_i = 3n - \sum_{i=1}^n |\alpha^i| \ge 3n - 4 \ge (n-1)2 = (n-1)m$$

provided $n \ge 2$, and if $n = 1$ then $\sum_{i=1}^n |\alpha^i| \le k \le 3$ so we still have the necessary inequality. Hence we may apply the lemma A.3.1 to obtain

$$\left|\left|\prod_{i=1}^n \partial^{\alpha^i}(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)}\right|\right|_{L^2(\Omega_\pm)} \le C\prod_{i=1}^n \left|\left|\partial^{\alpha^i}(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)\right|\right|_{H^{3-|\alpha^i|}(\Omega_\pm)}$$

Clearly, we have

$$\left|\left|\partial^{\alpha^i}(U,\rho_\mu,\nabla'\psi,g_1)\right|\right|^2_{H^{3-|\alpha^i|}(\Omega_\pm)} \le K(t)$$

where we have used the definition of E_g to estimate g_1, which also allows us the estimate $\partial^\alpha g$. Also, we note that by definition if $(U,\rho_\mu,\nabla'\psi,\partial_t\psi,g_1)_{e(i)} = \partial_t\psi$ then $|\alpha^i| \le k-1$ or ∂^{α^i} contains at least one space derivative, so

$$\left|\left|\partial^{\alpha^i}\partial_t\psi\right|\right|^2_{H^{3-|\alpha^i|}(\Omega_\pm)} \le K(t)$$

Combining this with the estimates for $R^0_\infty(U,\psi)$ and G, we obtain the result. □

Lemma 8.18.7. *Let $0 \le k \le 2$. Then the remainder $R^k(U,\psi)$ satisfies*

$$\left|\left|R^k(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \le M^0 + tK(t) \tag{8.18.4}$$

Proof. Note that that by the definition of $R^k(U,\psi)$,

$$\partial_t R^k(U,\psi) = R^{k+1}(U,\psi)$$

Thus by the estimate (8.18.3),

$$\left|\left|R^k(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \le C\left|\left|R^k(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)}|_{t=0} + C\int_0^t \left|\left|\partial_t R^k(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \le M^0 + tK(t)$$

□

Definition 8.18.8. Let $0 \le k \le 3$. We will write $R^k(U, \psi, \dot{p})$ for a generic remainder term which consists of sums of terms of the form

$$R^k(U, \psi)$$

or

$$G(\{\partial_{x_3}^j \psi : 0 \le j \le k+1\}) R^0_\infty(U, \psi) \cdot (\prod_{i=1}^n \partial^{\alpha^i}(U, \rho_\mu, \nabla' \psi, g)_{e(i)}) \partial^\beta \dot{p}$$

Here, n is an integer with $0 \le n \le k+1$, α^i are multi-indices with $0 \le \left|\alpha^i\right| \le k$, β is a multi-index with $0 \le |\beta| \le k$ and $0 \le \sum_{i=1}^n \left|\alpha^i\right| + |\beta| \le k$, $(U, \rho_\mu, \nabla'\psi, g)_{e(i)}$ denotes an entry of $(U, \rho_\mu, \nabla'\psi, g)$ depending on i, $R^0_\infty(U, \psi)$ is defined in 8.18.2, and G is a smooth function of its arguments.

Lemma 8.18.9. *Let* $0 \le |\beta| \le 2$ *and* $0 \le |\alpha| \le 3$. *We have*

$$\mu^2 \left|\left|\partial^\beta \dot{p}\right|\right|^2_{H^1(\Omega_\pm)} \le (M^0 + tK(t)) E_\mu(t) \tag{8.18.5}$$

$$\mu^2 \int_0^t ||\partial^\alpha \dot{p}||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau \le (M^0 + tK(t)) E_\mu(t) \tag{8.18.6}$$

Proof. We have

$$\begin{aligned} \mu \dot{p} &= -\mu \rho c^2 \nabla^\psi \cdot u \\ &= \mu \tilde{G}(\partial_{x_3}\psi) G(v) \nabla u \end{aligned}$$

where $\tilde{G}$ is a smooth function, G is a smooth vector-valued function and $v = (s, p, \nabla'\psi)$. Note that using (8.18.2) we have

$$\begin{aligned} \left|\left|\tilde{G}(\partial_{x_3}\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} &\le M^0 + tK(t) \\ ||G(v)||^2_{L^\infty(\Omega_\pm)} &\le M^0 + tK(t) \end{aligned}$$

Similarly,

$$\begin{aligned} \mu^2 \left|\left|\nabla u\right|\right|^2_{H^2(\Omega_\pm)} &\le M^0 + C\mu^2 \int_0^t ||\partial_t \nabla u||^2_{H^2(\Omega_\pm)} \,\mathrm{d}\tau \\ &\le M^0 + tE_\mu(t) \end{aligned}$$

Let $1 \le |\alpha| \le 3$. Applying the chain rule, we have that $\partial^\alpha(\tilde{G}(\partial_{x_3}\psi)G(v))$ is a sum of terms of the form

$$\tilde{G}(\{\partial_{x_3}^{j+1}\psi : 0 \le j \le 3\}) G(v) \prod_{i=1}^n \partial^{\alpha_i} v_{e(i)}$$

or

$$\tilde{G}(\partial_{x_3}\psi) G(v) \partial^\alpha v_i$$

where $1 \le n \le |\alpha|$, $\sum_{i=1}^n \alpha_i \le \alpha$ and $1 \le |\alpha_i| \le |\alpha| - 1$. Using the Sobolev product rule,

$$\begin{aligned} \left|\left|\prod_{i=1}^n \partial^{\alpha_i} v_{e(i)}\right|\right|^2_{L^2(\Omega_\pm)} &\le M^0 + C\int_0^t \left|\left|\partial_t \prod_{i=1}^n \partial^{\alpha_i} v_{e(i)}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\ &\le M^0 + tK(t) \end{aligned}$$

We also have

$$\left|\left|\tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha v_i\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK(t))E_\mu(t)$$

Using these estimates we have

$$\begin{aligned}
&\frac{1}{C}\left|\left|\partial^\alpha(\mu\tilde{G}(\partial_{x_3}\psi)G(v)\nabla u)\right|\right|^2_{L^2(\Omega_\pm)} \leq \mu^2\left|\left|\tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha\nabla u\right|\right|^2_{L^2(\Omega_\pm)} \\
&+\mu^2\left|\left|\partial^\alpha(\tilde{G}(\partial_{x_3}\psi)G(v))\nabla u\right|\right|^2_{L^2(\Omega_\pm)} + \mu^2\sum_{\beta+\gamma=\alpha,|\beta|=1}\left|\left|\partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\partial^\gamma\nabla u\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq (M^0 + tK(t))\mu^2\left|\left|\partial^\alpha\nabla u\right|\right|^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))E_\mu(t) \\
&+\mu^2\sum_{\beta+\gamma=\alpha,1\leq|\beta|\leq 2}\left|\left|\partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\right|\right|^2_{H^1(\Omega_\pm)}\left|\left|\partial^\gamma\nabla u\right|\right|^2_{H^1(\Omega_\pm)} \\
&\leq (M^0 + tK(t))\mu^2\left|\left|\partial^\alpha\nabla u\right|\right|^2_{L^2(\Omega_\pm)} + (M^0 + tK(t))E_\mu(t)
\end{aligned}$$

If $\partial^\alpha = \partial_{x_j}\partial^\beta$ or $|\alpha| \leq 2$ then by combining the above estimates we immediately obtain the first estimate. Else, integrating in time we obtain the second estimate. □

Lemma 8.18.10. *Given $t \in (0, T')$, the remainder term $R^k(U, \psi, \dot{p})$ satisfies*

$$\left|\left|R^k(U, \psi, \dot{p})\right|\right|_{L^2(\Omega_\pm)} \leq K(t)\sum_\pm\sum_{j=0}^{3}\left|\left|\partial_t^j\dot{p}\right|\right|_{H^{3-j}(\Omega_\pm)}$$

$$\mu\int_0^t\int_{\Omega_\pm}\left|R^k(U, \psi, \dot{p})\cdot\partial^\alpha U\right|\mathrm{d}x\mathrm{d}\tau \leq \frac{1}{\epsilon}tK(t) + \epsilon(M^0 + tK(t))E_\mu(t) \tag{8.18.7}$$

for $1 \leq |\alpha| \leq 3$ where we are free to choose ϵ in the range $0 < \epsilon \leq 1$.

Proof. As before, by (8.18.1), we have

$$\left|\left|R^0_\infty(U, \psi)\right|\right|^2_{L^\infty(\Omega_\pm)} \leq K(t)$$

Using the definition of ψ as a lift of f, which involves multiplying by a smooth cut-off function, we have

$$\left|\left|G(\{\partial^j_{x_3}\psi : 0 \leq j \leq k+1\})\right|\right|_{L^\infty(\Omega_\pm)} \leq F(||f||_{L^\infty(\Gamma)}) \leq K(t)$$

Now we intend to apply the Sobolev product lemma A.3.1 to the product which occurs in the remainder, $(\prod_{i=1}^n\partial^{\alpha^i}(U, \rho_\mu, \nabla'\psi, g)_{e(i)})\partial^\beta\dot{p}$. Note if $\left|\alpha^i\right| = 0$ for some i then we may estimate this term in $L^\infty(\Omega_\pm)$ using the Sobolev embedding theorem as above and remove it from the product, so we may assume $\left|\alpha^i\right| > 0$ for all i. Similarly, we may assume $|\beta| > 0$ else we estimate $\mu\dot{p}$ in L^∞ using the Sobolev embedding theorem and obtain the remainder $R^k(U, \psi)$ which we have already estimated.

For $1 \leq i \leq n$, set $m_i = 3 - \left|\alpha^i\right|$, set $m_0 = 3 - |\beta|$ and $m = 2 > \frac{3}{2} = \frac{d}{p}$. Note that since $1 \leq \left|\alpha^i\right|, |\beta| \leq k \leq 3$, we have $0 \leq m_i \leq m$. Note also that

$$\sum_{i=0}^n m_i = 3(n+1) - \sum_{i=1}^n\left|\alpha^i\right| - |\beta| \geq 3(n+1) - 4 \geq 2n = nm$$

Hence we may apply the lemma to obtain

$$\left|\left|\partial^\beta\dot{p}\prod_{i=1}^n\partial^{\alpha^i}(U, \rho_\mu, \nabla'\psi, g_1)_{e(i)}\right|\right|_{L^2(\Omega_\pm)} \leq C\left|\left|\partial^\beta\dot{p}\right|\right|_{H^{3-|\beta|}(\Omega_\pm)}\prod_{i=1}^n\left|\left|\partial^{\alpha^i}(U, \rho_\mu, \nabla'\psi, g)\right|\right|_{H^{3-|\alpha^i|}(\Omega_\pm)}$$

Clearly, as before, we have

$$\left|\left|\partial^{\alpha^i}(U, \rho_\mu, \nabla'\psi, g)\right|\right|_{H^{3-|\alpha^i|}(\Omega_\pm)} \leq K(t)$$

Combining this with the estimates for $R^0_\infty(U,\psi)$ and G, and using Cauchy's inequality, we obtain

$$\left|\left|R^k(U,\psi,\dot{p})\right|\right|_{L^2(\Omega_\pm)} \leq K(t)\left|\left|\partial^\beta \dot{p}\right|\right|_{H^{3-|\beta|}(\Omega_\pm)} \leq K(t)\sum_\pm\sum_{j=0}^3\left|\left|\partial_t^j\dot{p}\right|\right|_{H^{3-j}(\Omega_\pm)}$$

Using this we obtain

$$\begin{aligned}
&\mu\int_0^t\int_{\Omega_\pm}\left|R^k(U,\psi,\dot{p})\cdot\partial^\alpha U\right|\mathrm{d}x\mathrm{d}\tau \leq \mu\int_0^t||\partial^\alpha U||_{L^2(\Omega_\pm)}K(t)\sum_\pm\sum_{j=0}^3\left|\left|\partial_t^j\dot{p}\right|\right|_{H^{3-j}(\Omega_\pm)}\mathrm{d}\tau\\
&\leq \frac{1}{\epsilon}\int_0^t K(t)\mathrm{d}\tau + \epsilon\mu^2\sum_\pm\sum_{j=0}^3\int_0^t\left|\left|\partial_t^j\dot{p}\right|\right|^2_{H^{3-j}(\Omega_\pm)}\mathrm{d}\tau\\
&\leq \frac{1}{\epsilon}tK(t) + \epsilon(M^0 + tK(t))E_\mu(t)
\end{aligned}$$

as required, where we have used the estimates (8.18.5)-(8.18.6) for $\dot{p}$. □

Definition 8.18.11. Let $0 \leq k \leq 1$. We will write $R^k_{\infty,\Gamma}(U,f)$ for a generic remainder term of the form

$$R^k_{\infty,\Gamma}(U,\psi) = G(\{\partial^\beta(U|_\Gamma, f, \partial' f) : |\beta| \leq k\})$$

where G is a smooth function of its arguments.

Lemma 8.18.12. *Given $t \in (0, T')$, the remainder term $R^k_{\infty,\Gamma}(U,f)$ satisfies*

$$\left|\left|R^k_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)} \leq K(t) \tag{8.18.8}$$

Proof. Note that, for $|\beta| \leq 1$, by uniform continuity and the Sobolev embedding theorem,

$$\left|\left|\partial^\beta(U|_\Gamma, f, \partial' f)\right|\right|^2_{L^\infty(\Gamma)} \leq C\left|\left|\partial^\beta U\right|\right|^2_{L^\infty(\Omega_\pm)} + C\left|\left|\partial^\beta(f, \partial' f)\right|\right|^2_{L^\infty(\Gamma)} \leq CE_\mu(t)$$

Hence we have

$$\left|\left|G(\{\partial^\beta(U|_\Gamma, f, \partial' f) : |\beta| \leq k)\right|\right|^2_{L^\infty(\Gamma)} \leq K(t)$$

□

Lemma 8.18.13. *Let $k = 0$. Then the remainder $R^k_{\infty,\Gamma}(U,f)$ satisfies*

$$\left|\left|R^k_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)} \leq M^0 + tK(t) \tag{8.18.9}$$

Proof. From the way we have defined $R^k_{\infty,\Gamma}(U,f)$, we have

$$\partial_t R^k_{\infty,\Gamma}(U,\psi) = R^{k+1}_{\infty,\Gamma}(U,f)$$

Thus, by the estimate (8.18.8),

$$\begin{aligned}
\left|\left|R^k_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_\pm)} &\leq C\left|\left|R^k_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)}\Big|_{t=0} + C\int_0^t\left|\left|\partial_t R^k_{\infty,\Gamma}(U,f)\right|\right|^2_{L^\infty(\Gamma)}\mathrm{d}\tau\\
&\leq M^0 + tK(t)
\end{aligned}$$

□

Definition 8.18.14. Let $1 \leq k \leq 2$. We will write $R^k_\Gamma(U,f)$ for a generic remainder term which consists of sums of terms of the form

$$R^0_{\infty,\Gamma}(U,f)\prod_{i=1}^{n}\partial^{\alpha^i}(U|_\Gamma,\partial' f)_{e(i)}$$

Here, n is an integer with $1 \leq n \leq k+1$, α^i are multi-indices with $\alpha^i_3 = 0$, $1 \leq \left|\alpha^i\right| \leq k$ and $\sum_{i=1}^n \left|\alpha^i\right| \leq k+1$, $(U|_\Gamma,\partial' f)_{e(i)}$ denotes an entry of $(U|_\Gamma,\partial' f)$ depending on i, and $R^0_{\infty,\Gamma}(U,f)$ is defined in 8.18.11.

We may sometimes write $R^0_\Gamma(U,f)$, which is defined to be zero.

Lemma 8.18.15. *Let $t \in (0,T')$. The remainder term $R^k_\Gamma(U,f)$ satisfies the following estimate.*

$$\left|\left|R^k_\Gamma(U,f)\right|\right|^2_{L^2(\Gamma)} \leq K(t) \tag{8.18.10}$$

Let α be a multi-index with $\alpha_3 \leq |\alpha|-1$ and $1 \leq |\alpha| \leq 3$. Thus we have either $\partial^\alpha = \partial_{x_j}\partial^\beta$ for some $1 \leq j \leq 2$ and $|\beta| = |\alpha|-1$, or $\partial^\alpha = \partial_t\partial^\beta$ for some $|\beta| = |\alpha|-1$. Then

$$\left|\int_0^t\int_\Gamma R^k_\Gamma(U,f)\partial^\alpha(U,\mu\dot{p})\mathrm{d}x'\mathrm{d}\tau\right| \leq (M^0+tK(t))(\frac{1}{\epsilon}+\epsilon E_\mu(t)) \tag{8.18.11}$$

where we are free to choose ϵ in the range $0 < \epsilon \leq 1$.

Proof. Firstly, we note that since $\partial_t f = u\cdot n = u_3 - u'\cdot\nabla' f$ by (7.1.5), we can in fact write $R^k_\Gamma(U,f)$ as a sum of terms of the form

$$R^0_{\infty,\Gamma}(U,f)\prod_{i=1}^{n}\partial^{\alpha^i}(U|_\Gamma,\nabla' f)_{e(i)}$$

We note that in fact this has the form

$$R^k(U,\psi)|_\Gamma$$

where $R^k(U,\psi)$ is defined in 8.18.5. Therefore, using the Sobolev trace lemma,

$$\left|\left|R^k_\Gamma(U,f)\right|\right|^2_{L^2(\Gamma)} \leq \left|\left|R^k(U,f)\right|\right|^2_{H^1(\Omega_\pm)} \leq \left|\left|R^{k+1}(U,f)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)$$

where we have used (8.18.3).

For the second inequality we convert the above to an integral over the interior. We have

$$\begin{aligned}
&\left|\int_0^t\int_\Gamma R^k_\Gamma(U,f)\partial^\alpha(U,\mu\dot{p})\mathrm{d}x'\mathrm{d}\tau\right| \leq \sum_\pm\left|\int_0^t\int_{\Omega_\pm}\partial_{x_3}(R^k(U,\psi)\partial^\alpha(U,\mu\dot{p}))\mathrm{d}x\mathrm{d}\tau\right| \\
&\leq \sum_\pm\left|\int_0^t\int_{\Omega_\pm}R^{k+1}(U,\psi)\partial^\alpha(U,\mu\dot{p})\mathrm{d}x\mathrm{d}\tau\right| + \sum_\pm\left|\int_0^t\int_{\Omega_\pm}R^k(U,\psi)\partial^\alpha\partial_{x_3}(U,\mu\dot{p})\mathrm{d}x\mathrm{d}\tau\right| \\
&\leq \sum_\pm\left|\int_0^t\int_{\Omega_\pm}R^k(U,\psi)\partial^\alpha\partial_{x_3}(U,\mu\dot{p})\mathrm{d}x\mathrm{d}\tau\right| + \frac{1}{\epsilon}\left|\left|R^{k+1}(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \\
&+\epsilon\sum_\pm\mu^2\int_0^t||\partial^\alpha\dot{p}||^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \\
&\leq \sum_\pm\left|\int_0^t\int_{\Omega_\pm}R^k(U,\psi)\partial^\alpha\partial_{x_3}(U,\mu\dot{p})\mathrm{d}x\mathrm{d}\tau\right| + \frac{1}{\epsilon}tK(t)+\epsilon(M^0+tK(t))E_\mu(t)
\end{aligned}$$

where we have used the estimates (8.18.5)-(8.18.6) for $\dot{p}$.

Suppose $\partial^\alpha = \partial_{x_j}\partial^\beta$. Then, using integration by parts,

$$\sum_\pm \left| \int_0^t \int_{\Omega_\pm} R^k(U,\psi)\partial^\alpha \partial_{x_3}(U,\mu\dot{p}) \mathrm{d}x \mathrm{d}\tau \right| \le \sum_\pm \left| \int_0^t \int_{\Omega_\pm} \partial_{x_j} R^k(U,\psi)\partial^\beta \partial_{x_3}(U,\mu\dot{p}) \mathrm{d}x \mathrm{d}\tau \right|$$
$$\le tK(t)$$

where we have used the estimate (8.18.3) for $\partial_{x_j} R^k(U,\psi) = R^{k+1}(U,\psi)$ and the estimate (8.18.5) for $\dot{p}$.

Now suppose $\partial^\alpha = \partial_t \partial^\beta$. We must integrate by parts in time, but otherwise we proceed in a similar manner to the above. We have

$$\sum_\pm \left| \int_0^t \int_{\Omega_\pm} R^k(U,\psi)\partial^\alpha \partial_{x_3}(U,\mu\dot{p}) \mathrm{d}x \mathrm{d}\tau \right| \le M^0 + \sum_\pm \left| \int_{\Omega_\pm} R^k(U,\psi)\partial^\beta \partial_{x_3}(U,\mu\dot{p}) \mathrm{d}x \right|$$
$$+ \sum_\pm \left| \int_0^t \int_{\Omega_\pm} \partial_t R^k(U,\psi)\partial^\beta \partial_{x_3}(U,\mu\dot{p}) \mathrm{d}x \mathrm{d}\tau \right|$$
$$\le M^0 + \frac{1}{\epsilon}\sum_\pm \left|\left| R^k(U,\psi) \right|\right|^2_{L^2(\Omega_\pm)} + \epsilon \left|\left| \partial^\beta \partial_{x_3}(U,\mu\dot{p}) \right|\right|^2_{L^2(\Omega_\pm)} + tK(t)$$
$$\le \frac{1}{\epsilon}(M^0 + tK(t)) + \epsilon(M^0 + tK(t))E_\mu(t)$$

where we have used the above estimate (8.18.3) for $\partial_t R^k(U,\psi) = R^{k+1}(U,\psi)$, the estimate (8.18.6) for $\dot{p}$ and the lower order remainder estimate (8.18.4).

Combining the above estimates proves the result. □

Lemma 8.18.16. *Let $k = 1$. Then the remainder $R^k_\Gamma(U,f)$ satisfies*

$$\left|\left| R^k_\Gamma(U,f) \right|\right|^2_{L^2(\Gamma)} \le M^0 + tK(t) \tag{8.18.12}$$

Proof. Note that that by the definition of $R^k_\Gamma(U,f)$,

$$\partial_t R^k_\Gamma(U,f) = R^{k+1}_\Gamma(U,f)$$

Thus, by the estimate (8.18.10),

$$\left|\left| R^k_\Gamma(U,f) \right|\right|^2_{L^2(\Gamma)} \le C \left|\left| R^k_\Gamma(U,f) \right|\right|^2_{L^2(\Gamma)} |_{t=0} + C\int_0^t \left|\left| \partial_t R^k_\Gamma(U,f) \right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \le M^0 + tK(t)$$

□

Definition 8.18.17. We will write $R_{\infty,\Gamma}(f)$ for a generic remainder term of the form

$$G(\nabla' f, g_2)$$

where G is a smooth function of its arguments.

Lemma 8.18.18. *Given $t \in (0,T')$, the remainder term $R_{\infty,\Gamma}(f)$ satisfies*

$$||R_{\infty,\Gamma}(f)||^2_{L^\infty(\Gamma)} \le K(t) \tag{8.18.13}$$

Proof. We have

$$\left|\left|G(\nabla' f, g_2)\right|\right|_{L^\infty(\Gamma)}^2 \le F(\left|\left|\nabla' f\right|\right|_{H^2(\Gamma)}^2, ||g_2||_{H^2(\Gamma)}^2) \le K(t)$$

where F is a smooth increasing function. This completes the proof. $\square$

Definition 8.18.19. Let $1 \le k \le 3$. We will write $R_\Gamma^k(f)$ for a generic remainder term which consists of sums of terms of the form

$$R_{\infty,\Gamma}(f) \prod_{i=1}^n \partial^{\alpha^i} (\nabla' f, g_2)_{e(i)}$$

Here, n is an integer with $1 \le n \le k+1$, α^i are multi-indices with $\alpha_3^i = 0$, $1 \le \left|\alpha^i\right| \le k$ and $\sum_{i=1}^n \left|\alpha^i\right| \le k+1$, $(\nabla' f, g_2)_{e(i)}$ denotes an entry of $(\nabla' f, g_2)$ depending on i, and $R_{\infty,\Gamma}(f)$ is defined in 8.18.17.

We may sometimes write $R_\Gamma^0(f)$, which is defined to be zero.

Lemma 8.18.20. *Given $t \in (0, T')$, the remainder term $R_\Gamma^k(f)$ satisfies*

$$\left|\left|R_\Gamma^k(f)\right|\right|_{L^2(\Gamma)}^2 \le K(t) \tag{8.18.14}$$

Proof. We intend to apply the Sobolev product lemma A.3.1 to the product $\prod_{i=1}^n \partial^{\alpha^i}(\nabla' f, g_2)_{e(i)}$. Indeed, set $m_i = 3 - \left|\alpha^i\right|$ and $m = 2 > 1 = \frac{d}{p}$ (in this case the dimension is 2). Note that since $1 \le \left|\alpha^i\right| \le k \le 3$, we have $0 \le m_i \le m$. Note also that

$$\sum_{i=1}^n m_i = 3n - \sum_{i=1}^n \left|\alpha^i\right| \ge 3n - 4 \ge (n-1)2 = (n-1)m$$

provided $n \ge 2$, and if $n = 1$ then $\sum_{i=1}^n \left|\alpha^i\right| \le k \le 3$ so we still have the necessary inequality. Hence we may apply the lemma to obtain

$$\left|\left|\prod_{i=1}^n \partial^{\alpha^i}(\nabla' f, g_2)_{e(i)}\right|\right|_{L^2(\Gamma)} \le C \prod_{i=1}^n \left|\left|\partial^{\alpha^i}(\nabla' f, g_2)\right|\right|_{H^{3-\left|\alpha^i\right|}(\Gamma)}$$

Clearly, by definition of E_μ, we have

$$\left|\left|\partial^{\alpha^i} \nabla' f\right|\right|_{H^{3-\left|\alpha^i\right|}(\Gamma)}^2 \le K(t)$$

Also, using the definition of E_g we have

$$\left|\left|\partial^{\alpha^i} g_2\right|\right|_{H^{3-\left|\alpha^i\right|}(\Gamma)}^2 \le K(t)$$

Combining this with the estimate for $R_{\infty,\Gamma}(f)$, we obtain the result. $\square$

Lemma 8.18.21. *Let $1 \le k \le 2$. Then the remainder $R_\Gamma^k(f)$ satisfies*

$$\left|\left|R_\Gamma^k(f)\right|\right|_{L^2(\Gamma)}^2 \le M^0 + tK(t) \tag{8.18.15}$$

Proof. Note that that by the definition of $R_\Gamma^k(f)$,

$$\partial_t R_\Gamma^k(f) = R_\Gamma^{k+1}(f)$$

Thus, by the estimate (8.18.14),

$$\left|\left|R_\Gamma^k(f)\right|\right|_{L^2(\Gamma)}^2 \le C \left|\left|R_\Gamma^k(f)\right|\right|_{L^2(\Gamma)}^2 |_{t=0} + C \int_0^t \left|\left|\partial_t R_\Gamma^k(f)\right|\right|_{L^2(\Gamma)}^2 \mathrm{d}\tau \le M^0 + tK(t)$$

$\square$

9. The Linearised Equations

In this section we introduce a linearisation of the μ-approximate equations which is carefully chosen so that it admits an energy estimate without loss of regularity with respect to the coefficients and hence fits nicely into the contraction-mapping framework which we intend to use to prove the existence of a fixed point which solves the μ-approximate equations. See 3.6 for a brief motivation of the choice of linearisation.

9.1 Definition of the Linearised Equations

Definition 9.1.1. Let $\bar{T} \in (0, \infty)$. Let $\bar{f} \in C_b^1((0, \bar{T}) \times \mathbb{R}^2)$, $\bar{U}_\pm = (\bar{p}_\pm, \bar{u}_\pm, \bar{s}_\pm) \in C_b^1((0, \bar{T}) \times \Omega_\pm)$. Assume that

$$[\bar{u} \cdot \bar{n}] = 0$$

We say that (U_+, U_-, f), where $f \in C_b^1((0, T) \times \mathbb{R}^2)$, $U_\pm = (p_\pm, u_\pm, s_\pm) \in C_b^1((0, T) \times \Omega_\pm)$, is a solution of the linearised equations with lifting operator L^a and right hand side g on the time interval $(0, T)$ (where $T \in (0, \bar{T}]$), linearised about the state $(\bar{U}_+, \bar{U}_-, \bar{f})$, if the following holds.

$$\begin{aligned} u \in C_b^2(\Omega_\pm) &\text{ for each } t \in (0, T) \\ f, u \cdot n \in C_b^2(\Gamma) &\text{ for each } t \in (0, T) \\ (U_+|_{t=0}, U_-|_{t=0}, f|_{t=0}) &= (\bar{U}_+|_{t=0}, \bar{U}_-|_{t=0}, \bar{f}|_{t=0}) \\ \rho &> 0 \\ \rho_\mu &> 0 \\ J^\psi := 1 + \partial_{x_3}\psi &> 0 \end{aligned}$$

where

$$\psi := L^a f \tag{9.1.1}$$

and $p_\pm, u_\pm, s_\pm, f$ satisfy the following system of PDEs.

Firstly,

$$\partial_t f = \bar{u} \cdot \bar{n} \text{ on } (0, T) \times \Gamma \tag{9.1.2}$$

In the following equations, we have defined

$$\overline{u^\psi} := (\bar{u}', \frac{1}{J^\psi}(\bar{u}_3 - \bar{u}' \cdot \nabla' \bar{\psi} - \partial_t \psi))$$

Note carefully that the term $\nabla'\bar{\psi}$, where $\bar{\psi} := L^a \bar{f}$, instead of $\nabla'\psi$ is used so that $(\overline{u^\psi})_3|_\Gamma = 0$.

Next, the following (decoupled) transport equations are to be satisfied in $(0,T) \times \Omega_\pm$.

$$(\partial_t + \overline{u^\psi} \cdot \nabla)s = 0 \tag{9.1.3}$$

$$(\partial_t + \overline{u^\psi} \cdot \nabla)p + \bar{\rho}\bar{c}^2 \nabla^\psi \cdot \bar{u} = 0 \tag{9.1.4}$$

We define ρ and ρ_μ from p and s in the obvious manner following previous notation.

Next, as an intermediate step, we require that there exist functions $\omega_\pm \in C^1_b((0,T) \times \Omega_\pm)$ which we call the linearised curl, solving the following transport equation in $(0,T) \times \Omega_\pm$.

$$\rho_\mu(\partial_t + \overline{u^\psi} \cdot \nabla)\omega = -\nabla^\psi \rho_\mu \times (\partial_t + \overline{u^\psi} \cdot \nabla)\bar{u} - \rho_\mu(\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi \bar{u}_k) + \nabla^\psi \times g_1 \tag{9.1.5}$$

with

$$\omega|_{t=0} = (\nabla^{\bar{\psi}} \times \bar{u})|_{t=0} \tag{9.1.6}$$

Where $(\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi \bar{u}_k)$ denotes the vector with i-th component $\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi \bar{u}_k$ and we have used summation convention for repeated indices. We call this the linearised curl equation.

Finally, we require that the velocity $u_\pm$ solves the following parabolic-type equation.

$$\rho_\mu(\partial_t + \overline{u^\psi} \cdot \nabla)u - \mu\nabla^\psi(\rho c^2 \nabla^\psi \cdot u) + \mu\nabla^\psi \times (\rho c^2 \nabla^\psi \times u) = -\nabla^\psi p + \mu\nabla^\psi \times (\rho c^2 \omega) + g_1 \tag{9.1.7}$$

in $(0,T) \times \Omega_\pm$ and

$$[u \cdot n] = 0 \tag{9.1.8}$$

$$\mu(\Delta' - 1)(u \cdot n) + \mu[\rho c^2 \nabla^\psi \cdot u] = [p] + \sigma\nabla' \cdot \hat{n} - g_2 \tag{9.1.9}$$

$$n \times (\nabla^\psi \times u_\pm) = n \times \omega_\pm \tag{9.1.10}$$

on $(0,T) \times \Gamma$.

9.2 Definition of the Linearised Energy

Definition 9.2.1. We define the linearised energy $\mathcal{E}_L : (0,T] \to [0,\infty]$ associated to (U_+, U_-, f) as being equal to the μ-approximate energy $\mathcal{E}_\mu(t)$ associated to (U_+, U_-, f) defined in 7.3.1, but without the term involving $\nabla^\psi \times u$.

We define the energy $\mathcal{E}_L(\omega) : (0,T] \to [0,\infty]$ associated with the curl ω as

$$\mathcal{E}_L(\omega)(t) := \sum_\pm \sum_{j=0}^{2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \omega\right|\right|^2_{H^{4-j}(\Omega_\pm)} + \sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^3 \omega\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm \int_0^t \left|\left|\partial_t^3 \omega\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau$$

We also define the slightly lower-order energy $E_\mu(\omega) : (0,T] \to [0,\infty]$ associated with ω as

$$E_\mu(\omega)(t) := \sum_\pm \sum_{j=0}^{2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \omega\right|\right|^2_{H^{2-j}(\Omega_\pm)} + \mu^2 \sum_\pm \sum_{j=0}^{2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \omega\right|\right|^2_{H^{3-j}(\Omega_\pm)}$$

9.3 Notation for Constants and Functions of the Energy and Initial Data

Definition 9.3.1. For fixed $\mu > 0$ it will be convenient to define some generic constants/functions depending on the energy and initial data in a given way which are also allowed to depend on μ.

We will write F_μ for a generic smooth increasing function of its arguments which depends on μ.

We will write C_μ for a generic constant which is independent of the solution, including being independent of the initial data, which depends on μ.

We define M_μ^0 as generic a constant of the form

$$M_\mu^0 := F_\mu(\mathcal{E}_\mu^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1))$$

We write $\bar{E}_\mu$ to denote the energy E_μ with (U_+, U_-, f) replaced by $(\bar{U}_+, \bar{U}_-, \bar{f})$, and we define $\bar{\mathcal{E}}_L$ similarly.

We define $K_\mu(t)$ as a generic function of t of the form

$$K_\mu(t) := F_\mu(\bar{E}_\mu(t), E_\mu(t), E_\mu(\omega)(t), M_\mu^0)$$

Similarly, we define $\mathcal{K}_L(t)$ as a generic function of t of the form

$$\mathcal{K}_L(t) := F_\mu(\bar{\mathcal{E}}_L(t), \mathcal{E}_L(t), \mathcal{E}_L(\omega)(t), M_\mu^0)$$

Remark 9.3.2. Note that in the following sections, constants and generic functions will be independent of T (for $0 < T \leq 1$). In particular, we will only use the Sobolev embedding theorem with respect to space and not with respect to time.

Lemma 9.3.3. *Suppose $\omega_\pm = \nabla^\psi \times u_\pm$. Then*

$$\begin{aligned}
E_\mu(\omega)(t) &\leq F_\mu(E_\mu(t)) \\
K_\mu(t) &\leq F_\mu(\bar{E}_\mu(t), E_\mu(t), M_\mu^0) \\
\mathcal{E}_\mu(t) &\leq \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \\
\mathcal{E}_L(t) &\leq \mathcal{E}_\mu(t) \\
\mathcal{E}_L(\omega)(t) &\leq F_\mu(E_\mu(t))\mathcal{E}_\mu(t)
\end{aligned}$$

Proof. The first inequality follows from the definitions of $E_\mu(\omega)(t)$ and $E_\mu(t)$ and the product rule. The second inequality follows from the first. The third and fourth inequalities are immediate from the definitions of $\mathcal{E}_\mu(t)$, $\mathcal{E}_L(t)$ and $\mathcal{E}_L(\omega)(t)$. The fifth inequality follows from the definitions of $\mathcal{E}_L(\omega)(t)$ and $\mathcal{E}_\mu(t)$ and the product rule. □

9.4 Statement of the Energy Estimate for the Linearised Equations

Proposition 9.4.1. *Let (U_+^0, U_-^0, f^0) be initial data for the μ-approximate equations with initial energy $\mathcal{E}_\mu^0 < \infty$. Assume that*

$$\begin{aligned}
\inf_{\Omega_\pm} \rho^0 &=: \delta^0 > 0 \\
\inf_{\Omega_\pm} \rho_\mu^0 &=: \delta_\mu^0 > 0 \\
\inf_{\mathbb{R}^3} J^{\psi^0} &=: \kappa^0 > 0
\end{aligned}$$

Let $(\bar{U}_+, \bar{U}_-, \bar{f})$ be a background state satisfying the conditions stated in the definition 9.1.1 of the linearised equations above defined on the interval $(0, \bar{T})$. Suppose that in addition the following conditions hold.

$$\bar{\mathcal{E}}_\mu(\bar{T}) < \infty$$
$$\partial_t^j(\bar{U}_+, \bar{U}_-, \bar{f})|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \leq j \leq 3$$

Let $(U_+, U_-, \omega_+, \omega_-, f)$ be a solution of the linearised equations with background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ as defined in in the definition 9.1.1 on some time interval $(0, T)$ where $T \in (0, \bar{T})$. Suppose in addition that

$$\begin{aligned}
\mathcal{E}_L(T) + \mathcal{E}_L(\omega)(T) &< \infty \\
\rho &\geq \frac{\delta^0}{4} \\
\rho_\mu &\geq \frac{\delta_\mu^0}{4} \\
J^\psi &\geq \frac{1}{4}\kappa^0
\end{aligned}$$

for $t \in (0, T)$.

Then in fact the energy satisfies the following bound.

$$\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq M_\mu^0 + K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) + \frac{1}{4}\bar{\mathcal{E}}_L(t)$$

for $t \in (0, T)$. Moreover, we have

$$\begin{aligned}
\rho &\geq \delta^0 - tK_\mu(t) \\
\rho_\mu &\geq \delta_\mu^0 - tK_\mu(t) \\
J^\psi &\geq \kappa^0 - tK_\mu(t)
\end{aligned}$$

on the time interval $(0, T)$.

Proof. See section 12. □

9.5 Statement of Existence and Uniqueness of Solutions to the Linearised Equations

Proposition 9.5.1. *Let (U_+^0, U_-^0, f^0) be initial data for the μ-approximate equations with initial energy $\mathcal{E}_\mu^0 < \infty$ satisfying the compatibility condition* (7.2.6) *up to order* 3 *and the condition* (7.2.7) *up to order* 2. *Assume that*

$$\begin{aligned}
\inf_{\Omega_\pm} \rho^0 &=: \delta^0 > 0 \\
\inf_{\Omega_\pm} \rho_\mu^0 &=: \delta_\mu^0 > 0 \\
\inf_{\mathbb{R}^3} J^{\psi^0} &=: \kappa^0 > 0
\end{aligned}$$

Let $(\bar{U}_+, \bar{U}_-, \bar{f})$ be a background state satisfying the conditions stated in the definition 9.1.1 of the linearised equations above defined on the interval $(0, \bar{T})$. Suppose that in addition the following conditions hold.

$$\begin{aligned} \bar{\mathcal{E}}_L(\bar{T}) &< \infty \\ \partial_t^j(\bar{U}_+, \bar{U}_-, \bar{f})|_{t=0} &= \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \leq j \leq 3 \end{aligned} \tag{9.5.1}$$

Then there exists a $T \in (0, \bar{T}]$ and a solution $(U_+, U_-, \omega_+, \omega_-, f)$ of the linearised equations on the time interval $(0, T)$, as defined in 9.1.1, satisfying the following additional properties.

$$\begin{aligned} \mathcal{E}_L(T) + \mathcal{E}_L(\omega)(T) &< \infty \\ \rho &\geq \frac{\delta^0}{4} \\ \rho_\mu &\geq \frac{\delta_\mu^0}{4} \\ J^\psi &\geq \frac{1}{4}\kappa^0 \end{aligned}$$

for all $t \in (0, T)$ and

$$\partial_t^j(U_+, U_-, \omega_+, \omega_-, f)|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, \nabla^{\psi^0} \times u_+^0, \nabla^{\psi^0} \times u_-^0, f^0) \text{ for } 0 \leq j \leq 3$$

Moreover if $T < \bar{T}$ then one of the following holds as $t \uparrow T$.

$$\begin{aligned} &\inf_{x \in \Omega_\pm} \rho \to \frac{\delta^0}{4} \\ &\inf_{x \in \Omega_\pm} \rho_\mu \to \frac{\delta_\mu^0}{4} \\ &\inf_{x \in \mathbb{R}^3} J^\psi \to \frac{1}{4}\kappa^0 \end{aligned}$$

Also, the solution is unique on the time interval $(0, T)$ amongst solutions with the above properties.

Proof. See sections 10-11 for the proof. □

9.6 The Curl of the Linearised Velocity Equation

We record here the result of taking the curl of the linearised velocity equation, although we will only need this when it comes to proving a fixed point of the linearised equations solves the μ-approximate equations.

Lemma 9.6.1. *Suppose that u satisfies the linearised velocity equation* (9.1.7). *Suppose that in addition to the regularity stated in 9.1.1, we have $\partial_t \psi, \partial_t u, g_1, \bar{u} \in H^1_{\mathrm{loc}}(\Omega_\pm)$, $p, s, u, \nabla\psi, \bar{\psi}, \nabla u \in H^2_{\mathrm{loc}}(\Omega_\pm)$. Then $\nabla^\psi \times u$ satisfies the following equation.*

$$\rho_\mu(\partial_t + \bar{u}^\psi \cdot \nabla)(\nabla^\psi \times u) = -\nabla^\psi \rho_\mu \times (\partial_t + \bar{u}^\psi \cdot \nabla)u - \rho_\mu(\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi u_k) + \nabla^\psi \times g_1 \tag{9.6.1}$$

$$- \mu\nabla^\psi \times (\nabla^\psi \times (\nabla^\psi \times u)) + \mu\nabla^\psi \times (\nabla^\psi \times \omega) - \nabla^\psi \times (\rho_\mu \frac{1}{J^\psi}\bar{u}' \cdot \nabla'(\psi - \bar{\psi})\partial_{x_3} u) \tag{9.6.2}$$

in $(0, T) \times \Omega_\pm$, where $(\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi u_k)$ denotes the vector with i-th component $\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi u_k$ and we have used summation convention for repeated indices.

Proof. We could apply $\nabla^\psi\times$ to the linearised velocity equation (9.1.7) directly, but it is easiest first to transform back to the original domains $\Omega_\pm(t)$. We rewrite the velocity equation as

$$\begin{aligned}&\rho_\mu(\partial_t+\bar{u}^\psi\cdot\nabla)u-\mu\nabla^\psi(\rho c^2\nabla^\psi\cdot u)+\mu\nabla^\psi\times(\rho c^2\nabla^\psi\times u)\\&=-\nabla^\psi p+\mu\nabla^\psi\times(\rho c^2\omega)+g_1-\rho_\mu\frac{1}{J\psi}\bar{u}'\cdot\nabla'(\psi-\bar{\psi})\partial_{x_3}u\end{aligned}$$

Setting $\check{p}=p\circ\theta^{-1}$, $\check{\bar{u}}=\bar{u}\circ\theta^{-1}$, $\check{u}=u\circ\theta^{-1}$, $\check{s}=s\circ\theta^{-1}$, $\check{\rho}_\mu=\rho_\mu\circ\theta^{-1}$, $\check{\rho}=\rho\circ\theta^{-1}$, $\check{c}=c\circ\theta^{-1}$, where the diffeomorphism θ is defined in 5.1.1 in terms of ψ, and using the coordinate change relations (5.1.1)-(5.1.2), we obtain

$$\begin{aligned}&\check{\rho}_\mu(\partial_t+\check{\bar{u}}\cdot\nabla)\check{u}-\mu\nabla(\check{\rho}\check{c}^2\nabla\cdot\check{u})+\mu\nabla\times(\check{\rho}\check{c}^2\nabla\times\check{u})\\&=-\nabla\check{p}+(\mu\nabla^\psi\times(\rho c^2\omega)+g_1-\rho_\mu\frac{1}{J\psi}\bar{u}'\cdot\nabla'(\psi-\bar{\psi})\partial_{x_3}u)\circ\theta^{-1}\end{aligned}$$

in $\Omega^T_\pm$. Now we apply $\nabla\times$ to obtain

$$\begin{aligned}&\check{\rho}_\mu(\partial_t+\check{\bar{u}}\cdot\nabla)(\nabla\times\check{u})+(\nabla\check{\rho}_\mu)\times(\partial_t+\check{\bar{u}}\cdot\nabla)\check{u}+\check{\rho}_\mu(\epsilon_{ijk}\partial_{x_j}\check{\bar{u}}_l\partial_{x_l}\check{u}_k)+\mu\nabla\times(\nabla\times(\check{\rho}\check{c}^2\nabla\times\check{u}))\\&=\nabla\times((\mu\nabla^\psi\times(\rho c^2\omega)+g_1-\rho_\mu\frac{1}{J\psi}\bar{u}'\cdot\nabla'(\psi-\bar{\psi})\partial_{x_3}u)\circ\theta^{-1})\end{aligned}$$

Then we use the coordinate change relations (5.1.1)-(5.1.2) again to obtain

$$\begin{aligned}&\rho_\mu(\partial_t+\bar{u}^\psi\cdot\nabla)(\nabla^\psi\times u)+\nabla^\psi\rho_\mu\times(\partial_t+\bar{u}^\psi\cdot\nabla)u+\rho_\mu(\epsilon_{ijk}\partial^\psi_j\bar{u}_l\partial^\psi_l u_k)\\&+\mu\nabla^\psi\times(\nabla^\psi\times(\rho c^2\nabla^\psi\times u))\\&=\nabla^\psi\times g_1+\mu\nabla^\psi\times(\nabla^\psi\times\omega)-\nabla^\psi\times(\rho_\mu\frac{1}{J\psi}\bar{u}'\cdot\nabla'(\psi-\bar{\psi})\partial_{x_3}u)\end{aligned}$$

as required. $\square$

10. Solving the Linearised Equations for the Front, Entropy, Pressure and Curl

In this section we assume we are given the background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ on the time interval $(0, \bar{T})$ and initial data (U^0_+, U^0_-, f^0) such that $(\bar{U}_+, \bar{U}_-, \bar{f})$ and (U^0_+, U^0_-, f^0) satisfy the hypotheses of proposition 9.5.1.

10.1 Solving the Linearised Equation for the Front by Direct Integration

We start with the very simple construction of the front f and the lift ψ of the front f.

Proposition 10.1.1. *There exists a $\tilde{T} \in (0, \bar{T}]$ and $f \in L^\infty((0, \tilde{T}) \times \Gamma)$ such that*

$$\partial_t f = \bar{u} \cdot \bar{n}$$

on $(0, \tilde{T}) \times \Gamma$ and

$$\partial_t^j f|_{t=0} = \partial_0^{\mu,g,j} f^0 \text{ for } 0 \leq j \leq 3$$

Moreover, f has the following regularity

$$\begin{aligned} \partial_t^j \nabla' f &\in L^\infty((0, \tilde{T}); H^{4.5-j}(\Gamma)) \text{ for } 0 \leq j \leq 3 \\ \partial_t^{j+1} f &\in L^2((0, \tilde{T}); H^{5.5-j}(\Gamma)) \text{ for } 0 \leq j \leq 3 \\ \partial_t^{j+1} f &\in L^\infty((0, \tilde{T}); H^{4-j}(\Gamma)) \text{ for } 0 \leq j \leq 3 \end{aligned}$$

and such an f is unique.

Also, defining

$$\psi = L^a f$$

we have

$$\inf_{x \in \mathbb{R}^3} J^\psi \geq \frac{1}{4} \kappa^0$$

for all $t \in (0, \tilde{T})$ and if $\tilde{T} < \bar{T}$ then

$$\inf_{x \in \mathbb{R}^3} J^\psi \to \frac{1}{4} \kappa^0 \text{ as } t \uparrow \tilde{T}$$

Proof. For $t \in (0, \bar{T})$, we set

$$f = f^0 + \int_0^t \bar{u} \cdot \bar{n} \mathrm{d}\tau$$

Clearly then f satisfies the desired equation and $f|_{t=0} = f^0$. Applying ∂_t^{j+1} for $0 \le j \le 2$, we see that

$$\begin{aligned} \partial_t^{j+1} f|_{t=0} &= \partial_t^j (\bar{u} \cdot \bar{n})|_{t=0} \\ &= \partial_0^{j,\mu,g} (u^0 \cdot n^0) \end{aligned}$$

since

$$\begin{aligned} \partial_t^j \bar{u}|_{t=0} &= \partial_0^{j,\mu,g} u^0 \\ \partial_t^j \bar{f}|_{t=0} &= \partial_0^{j,\mu,g} f^0 \end{aligned}$$

for $0 \le j \le 3$. Now by definition,

$$\partial_0^{j+1,\mu,g} f^0 = \partial_0^{j,\mu,g} (u^0 \cdot n^0)$$

and hence

$$\partial_t^{j+1} f|_{t=0} = \partial_0^{j+1,\mu,g} f^0$$

for $0 \le j \le 2$.

The regularity of f follows easily from its definition together with the regularity of f^0 and $\bar{u} \cdot \bar{n}$, given that $\bar{\mathcal{E}}_L(\bar{T}) < \infty$.

It is clear that f is unique by integrating the equation to obtain the integral formula for f as given above.

Now we let $\tilde{T} = \sup\{t \in (0, \bar{T}) : \inf_{\tau \in (0,t)} \inf_{x \in \mathbb{R}^3} J^\psi \ge \frac{\kappa^0}{4}\}$. Using uniform continuity and the fact that $J^\psi|_{t=0} \ge \kappa^0$, we know $\tilde{T} > 0$. We also know from the definition of $\tilde{T}$ and continuity that if $\tilde{T} < \bar{T}$ then $\inf_{x \in \mathbb{R}^3} J^\psi \to \frac{1}{4}\kappa^0$ as $t \uparrow \tilde{T}$. □

10.2 Solving a General Linear Scalar Transport Equation of the Appropriate Form

The solution of the following transport equations is quite standard. However, for completeness we briefly recall the main steps in the proof. We start with the following lemma about solutions to first order ODEs.

Lemma 10.2.1. *Let $b \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$. Then for each $(t,x) \in [0,T] \times \mathbb{R}^d$, there exists a unique solution $X \in W^{1,\infty}((0,T); \mathbb{R}^d)$ to the equation*

$$\frac{\mathrm{d}X}{\mathrm{d}\tau}(\tau) = b(\tau, X(\tau)) \tag{10.2.1}$$

with

$$X(t) = x$$

which we write as $X(\tau; t, x)$ to denote the dependence on the point (t,x) through which the solution passes.

Moreover, let us define $\Psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *by*

$$\Psi(t,x) = X(t;0,x)$$

Then $\Psi \in W^{1,\infty}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ *and for each* $t \in [0,T]$ *the function* $\Psi(t) : \mathbb{R}^d \to \mathbb{R}^d$*,* $x \mapsto \Psi(t,x)$ *is a Lipschitz-diffeomorphism with inverse* $\Phi(t) := \Psi(t)^{-1}$*. We have* $\Phi \in W^{1,\infty}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ *and in fact* Φ *is given by* $\Phi(t,x) = X(0;t,x)$*.*

Finally, suppose for some $k \geq 1$*,* $l \geq 0$ *with* $k + l - 1 > \frac{d}{2}$*, and all* $1 \leq j \leq k$*, we have*

$$\partial_t^j b \in L^\infty((0,T); H^{k-j+l}(\mathbb{R}^d))$$
$$\nabla b \in L^\infty((0,T); H^{k-1+l}(\mathbb{R}^d))$$

Then

$$\partial_t^{j+1}\Psi, \partial_t^{j+1}\Phi \in L^\infty((0,T); H^{k-j+l}(\mathbb{R}^d))$$
$$\partial_t \nabla \Psi, \partial_t \nabla \Phi \in L^\infty((0,T); H^{k-1+l}(\mathbb{R}^d))$$

for all $1 \leq j \leq k$*.*

Proof. The existence and uniqueness of such an X is standard since b is Lipschitz in its second argument, uniformly in time. The regularity of $\Psi = X(t;0,x)$ and $\Phi = X(0;t,x)$ follow from smooth dependence of the solution on initial conditions (which can be proved by applying Gronwall's inequality to difference quotients).

By uniqueness of solutions, we have $X(0;t,X(t;0,x)) = x$, since if $X(\tau;0,x)$ is the unique solution to 10.2.1 passing through $(0,x)$, then it passes through $(t, X(t;0,x))$, so is the unique solution to 10.2.1 passing through $(t, X(t;0,x))$ and hence by definition $X(\tau;t,X(t;0,x)) = X(\tau;0,x)$ so $X(0;t,X(t;0,x)) = X(0;0,x) = x$. Similarly $X(t;0,X(0;t,x)) = x$. This shows that Ψ has inverse Φ.

The additional regularity comes from differentiating the equation 10.2.1 with respect to time and then invoking smooth dependence on initial data. The requirement $k + l - 1 > \frac{d}{2}$ comes from the Sobolev embedding theorem which is needed to estimate the lower order terms. $\square$

Lemma 10.2.2. *Let* $b \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$*,* $h \in L^2((0,T); H^1(\mathbb{R}^d))$ *and* $w^0 \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^d)$ *with* $\nabla w^0 \in L^2(\mathbb{R}^d)$*. Then there exists a unique solution* $w \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^d)$ *with* $\nabla w \in L^2((0,T); L^2(\mathbb{R}^d))$ *and* $\partial_t w \in L^2((0,T); L^2(\mathbb{R}^d))$ *of the equation*

$$(\partial_t + b \cdot \nabla) w = h \tag{10.2.2}$$

with

$$w|_{t=0} = w^0$$

Moreover, if for some $k \geq 1$*,* $l \geq 0$ *with* $k + l - 1 > \frac{d}{2}$ *and all* $1 \leq j \leq k$*, we have*

$$\partial_t^j b \in L^\infty((0,T); H^{k-j+l}(\mathbb{R}^d))$$
$$\partial_t^{j-1} h \in L^\infty((0,T); H^{k-j+l}(\mathbb{R}^d))$$
$$\nabla b \in L^\infty((0,T); H^{k-1+l}(\mathbb{R}^d))$$
$$\nabla h \in L^2((0,T); H^{k-1+l}(\mathbb{R}^d))$$
$$\nabla w^0 \in H^{k-1+l}(\mathbb{R}^d)$$

then

$$\partial_t^j w \in L^\infty((0,T); H^{k-j+l}(\mathbb{R}^d))$$
$$\nabla w \in L^\infty((0,T); H^{k-1+l}(\mathbb{R}^d))$$

for all $1 \leq j \leq k$.

Note that we can allow w*,* w^0 *and* h *to be vector valued since then the equation holds for each of the components of* w*.*

Proof. We let Ψ and Φ be as in the previous lemma with b given above. We claim that the solution we seek is given by

$$w(t,x) := w^0(\Phi(t,x)) + \int_0^t h(\tau, \Psi(\tau, \Phi(t,x)))\mathrm{d}\tau$$

Clearly from the regularity of Ψ, Φ and h we have $w \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ with $\nabla w \in L^2((0,T); L^2(\mathbb{R}^d))$ and $\partial_t w \in L^2((0,T); L^2(\mathbb{R}^d))$. Using the fact that $\Psi(t)$ and $\Phi(t)$ are inverses, we have

$$w(t, \Psi(t,x)) = w^0(x) + \int_0^t h(\tau, \Psi(\tau, x))\mathrm{d}\tau$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t,\Psi(t,x))) = h(t, \Psi(t,x))$$

But differentiating the left hand side we obtain

$$\partial_t w(t,\Psi(t,x)) + (\partial_t \Psi(t,x)) \cdot \nabla w(t, \Psi(t,x)) = h(t,\Psi(t,x))$$

and using the fact that

$$\partial_t \Psi(t,x) = b(t, \Psi(t,x))$$

we obtain

$$\partial_t w(t,\Psi(t,x)) + b(t,\Psi(t,x)) \cdot \nabla w(t,\Psi(t,x)) = h(t,\Psi(t,x))$$

for all $(t,x) \in (0,T) \times \mathbb{R}^d$. Replacing x by $\Phi(t,x)$ we obtain

$$\partial_t w(t,x) + b(t,x) \cdot \nabla w(t,x) = h(t,x)$$

as required.

In fact we have also shown uniqueness, since for any solution $v \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ with $\nabla v \in L^2((0,T); L^2(\mathbb{R}^d))$, $\partial_t v \in L^2((0,T); L^2(\mathbb{R}^d))$ and $v|_{t=0} = w^0$ of 10.2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(v(t,\Psi(t,x))) = h(t,\Psi(t,x))$$

and so

$$v(t,\Psi(t,x)) = w^0(x) + \int_0^t h(\tau, \Psi(\tau,x))\mathrm{d}\tau$$

and using the fact that $\Psi(t)$ and $\Phi(t)$ are inverses, we have

$$v(t,x) = w^0(\Phi(t,x)) + \int_0^t h(\tau, \Psi(\tau, \Phi(t,x)))\mathrm{d}\tau = w(t,x)$$

The regularity follows from the formula

$$w(t,x) := w^0(\Phi(t,x)) + \int_0^t h(\tau, \Psi(\tau, \Phi(t,x)))\mathrm{d}\tau$$

together with the chain rule for Sobolev functions. □

Lemma 10.2.3. *In this lemma functions will be defined either on Ω_+ or Ω_-.*

Let $b \in L^\infty((0,T); W^{1,\infty}(\Omega_\pm;\mathbb{R}^3))$, $h \in L^2((0,T); H^1(\Omega_\pm))$ and $w^0 \in L^1_{\mathrm{loc}}((0,T)\times\Omega_\pm)$ with $\nabla w^0 \in L^2(\Omega_\pm)$. Suppose also that $b_3|_\Gamma = 0$. Then there exists a unique solution $w \in L^1_{\mathrm{loc}}((0,T)\times\Omega_\pm)$ with $\nabla w \in L^2((0,T); L^2(\Omega_\pm))$ and $\partial_t w \in L^2((0,T); L^2(\Omega_\pm))$ of the equation

$$(\partial_t + b\cdot\nabla)w = h$$

in $(0,T)\times\Omega_\pm$ with

$$w|_{t=0} = w^0$$

Moreover, if for some $k \geq 1$, $l \geq 0$ with $k + l \geq 3$ and all $1 \leq j \leq k$

$$\begin{aligned}
\partial_t^j b &\in L^\infty((0,T); H^{k-j+l}(\Omega_\pm)) \\
\partial_t^{j-1} h &\in L^\infty((0,T); H^{k-j+l}(\Omega_\pm)) \\
\nabla b &\in L^\infty((0,T); H^{k-1+l}(\Omega_\pm)) \\
\nabla h &\in L^2((0,T); H^{k-1+l}(\Omega_\pm)) \\
\nabla w^0 &\in H^{k-1+l}(\mathbb{R}^d)
\end{aligned}$$

then

$$\begin{aligned}
\partial_t^j w &\in L^\infty((0,T); H^{k-j+l}(\Omega_\pm)) \\
\nabla w &\in L^\infty((0,T); H^{k-1+l}(\Omega_\pm))
\end{aligned}$$

for all $1 \leq j \leq k$.

Proof. First we prove uniqueness. Suppose there are two solutions w^1 and w^2 of the equation. Write $\tilde{w} = w^1 - w^2$. Note that since $\partial_t w^i \in L^2((0,T)\times\Omega_\pm)$ for $i = 1, 2$, and $\tilde{w}(0) = 0$, we have $\tilde{w} \in L^2((0,T)\times\Omega_\pm)$. Subtracting the equation for w^2 from that for w^1 we obtain

$$(\partial_t + b\cdot\nabla)\tilde{w} = 0$$

Multiplying by $\tilde{w}$ and using the product rule we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{w}|^2 + \nabla\cdot(b\,|\tilde{w}|^2) = (\nabla\cdot b)\,|\tilde{w}|^2$$

Integrating over $\Omega_\pm$ and using the divergence theorem we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\,||\tilde{w}||^2_{L^2(\Omega_\pm)} \mp \int_\Gamma b_3\,|\tilde{w}|^2\,\mathrm{d}x' = \int_{\Omega_\pm} (\nabla\cdot b)\,|\tilde{w}|^2\,\mathrm{d}x$$

Using the fact that $b_3|_\Gamma = 0$ we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,||\tilde{w}||^2_{L^2(\Omega_\pm)} &= \int_{\Omega_\pm} (\nabla\cdot b)\,|\tilde{w}|^2\,\mathrm{d}x \\ &\leq ||b||_{W^{1,\infty}(\Omega_\pm)}\,||\tilde{w}||^2_{L^2(\Omega_\pm)}\end{aligned}$$

Applying Gronwall's lemma together with the fact that $\tilde{w}|_{t=0} = 0$ we conclude that $\tilde{w} = 0$. This proves the uniqueness.

To prove the existence we simply apply the Sobolev extension operator $\mathrm{Ext}_{\Omega_\pm}$ in space to extend b, h and w^0 to be defined on $\mathbb{R}^3$, noting that we have the regularity required to apply the above result 10.2.2 to obtain $w \in L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$ with $\nabla w \in L^2((0,T);L^2(\mathbb{R}^3))$ and $\partial_t w \in L^2((0,T);L^2(\mathbb{R}^3))$ satisfying the equation in $(0,T)\times\mathbb{R}^3$. Then $w|_{\Omega_\pm}$ satisfies the equation in $\Omega_\pm$ and is the unique solution by the above.

Also, the regularity of $w|_{\Omega_\pm}$ follows from the regularity of w given in the above result 10.2.2. □

10.3 Solving the Linear Transport Equation for the Entropy

Proposition 10.3.1. *Let $\tilde{T}$, f and ψ be given by 10.1.1. Then there exists $s \in L^\infty((0,\tilde{T})\times\Omega_\pm)$ such that*

$$(\partial_t + \overline{u^\psi}\cdot\nabla)s = 0$$

in $(0,\tilde{T})\times\Omega_\pm$ and

$$\partial_t^j s|_{t=0} = \partial_0^{\mu,g,j} s^0 \text{ for } 0 \leq j \leq 3 \tag{10.3.1}$$

Moreover, s has the following regularity

$$\begin{aligned}&\partial_t^j s \in L^\infty((0,\tilde{T});H^{4-j}(\Omega_\pm)) \text{ for } 1\leq j\leq 3\\ &\nabla s \in L^\infty((0,\tilde{T});H^3(\Omega_\pm))\\ &\partial_t^4 s \in L^\infty((0,\tilde{T});L^2(\Omega_\pm))\end{aligned}$$

and such an s is unique.

Proof. Apply lemma 10.2.3 with $b = \overline{u^\psi}$, $h = 0$ and $w^0 = s^0$ to obtain a unique solution $s \in L^1_{\mathrm{loc}}((0,T)\times\Omega_\pm)$ with $\nabla s \in L^2((0,T);L^2(\Omega_\pm))$ and $\partial_t s \in L^2((0,T);L^2(\Omega_\pm))$ to the equation with $s(0) = s^0$. Note that from the regularity of $\bar{u}$, $\bar{\psi}$, ψ and s^0, we have

$$\begin{aligned}\partial_t^j b &\in L^\infty((0,\tilde{T});H^{k-j+l}(\Omega_\pm))\\ \partial_t^{j-1} h &\in L^\infty((0,\tilde{T});H^{k-j+l}(\Omega_\pm))\\ \nabla b &\in L^\infty((0,\tilde{T});H^{k-1+l}(\Omega_\pm))\\ \nabla h &\in L^2((0,\tilde{T});H^{k-1+l}(\Omega_\pm))\\ \nabla w^0 &\in H^{k-1+l}(\mathbb{R}^d)\end{aligned}$$

for $1 \le j \le k$, where $k = 3$ and $l = 1$. Hence lemma 10.2.3 implies

$$\partial_t^j s \in L^\infty((0,\tilde{T}); H^{4-j}(\Omega_\pm))$$
$$\nabla s \in L^\infty((0,\tilde{T}); H^3(\Omega_\pm))$$

for all $1 \le j \le 3$. Using this regularity and the equation for s it is also clear that

$$\partial_t^4 s \in L^\infty((0,\tilde{T}); L^2(\Omega_\pm))$$

To see that the initial time derivatives match, note that the background state satisfies (9.5.1), that is

$$\partial_t^j(\bar{U}_+, \bar{U}_-, \bar{f})|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \le j \le 3$$

by assumption, and note that $s(0) = s^0$. Using these two facts we see that the linearised equation for s is the same as the μ-approximate equation for s when restricted to $t = 0$, which shows that (10.3.1) holds for $j = 1$. Now this allows us to differentiate the equations with respect to time and obtain the case $j = 2$. We continue until we reach $j = 3$. (Note we use the fact that the background state satisfies (9.5.1) up to $j = 3$). □

10.4 Solving the Linear Transport Equation for the Pressure

Proposition 10.4.1. *Let $\tilde{T}$, f and ψ be given by 10.1.1. Then there exists $p \in L^\infty((0,\tilde{T}) \times \Omega_\pm)$ such that*

$$(\partial_t + \overline{u^\psi} \cdot \nabla)p + \bar{\rho}\bar{c}^2 \nabla^\psi \cdot \bar{u} = 0$$

in $(0,\tilde{T}) \times \Omega_\pm$ and

$$\partial_t^j p|_{t=0} = \partial_0^{\mu,g,j} p^0 \text{ for } 0 \le j \le 3 \tag{10.4.1}$$

Moreover, p has the following regularity

$$\partial_t^j p \in L^\infty((0,\tilde{T}); H^{4-j}(\Omega_\pm)) \text{ for } 1 \le j \le 3$$
$$\nabla p \in L^\infty((0,\tilde{T}); H^3(\Omega_\pm))$$
$$\partial_t^4 p \in L^\infty((0,\tilde{T}); L^2(\Omega_\pm))$$

and such a p is unique.

Proof. Apply lemma 10.2.3 with $b = \overline{u^\psi}$, $h = -\bar{\rho}\bar{c}^2 \nabla^\psi \cdot \bar{u}$ and $w^0 = p^0$ to obtain a unique solution $p \in L^1_{\text{loc}}((0,T) \times \Omega_\pm)$ with $\nabla p \in L^2((0,T); L^2(\Omega_\pm))$ and $\partial_t p \in L^2((0,T); L^2(\Omega_\pm))$ to the equation with $p(0) = p^0$. Note that from the regularity of $\bar{u}$, $\bar{p}$, $\bar{s}$, $\bar{\psi}$, ψ and p^0, we have

$$\partial_t^j b \in L^\infty((0,\tilde{T}); H^{k-j+l}(\Omega_\pm))$$
$$\partial_t^{j-1} h \in L^\infty((0,\tilde{T}); H^{k-j+l}(\Omega_\pm))$$
$$\nabla b \in L^\infty((0,\tilde{T}); H^{k-1+l}(\Omega_\pm))$$
$$\nabla h \in L^2((0,\tilde{T}); H^{k-1+l}(\Omega_\pm))$$
$$\nabla w^0 \in H^{k-1+l}(\mathbb{R}^d)$$

for $1 \leq j \leq k$, where $k = 3$ and $l = 1$. Hence lemma 10.2.3 implies

$$\partial_t^j p \in L^\infty((0,\tilde{T}); H^{4-j}(\Omega_\pm))$$
$$\nabla p \in L^\infty((0,\tilde{T}); H^3(\Omega_\pm))$$

for all $1 \leq j \leq 3$. Using this regularity and the equation for p it is also clear that

$$\partial_t^4 p \in L^\infty((0,\tilde{T}); L^2(\Omega_\pm))$$

To see that the initial time derivatives match, note that the background state satisfies (9.5.1), that is

$$\partial_t^j(\bar{U}_+, \bar{U}_-, \bar{f})|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \leq j \leq 3$$

by assumption, and note that $p(0) = p^0$. Using these two facts we see that the linearised equation for p is the same as the μ-approximate equation for p when restricted to $t = 0$, which shows that (10.4.1) holds for $j = 1$. Now this allows us to differentiate the equations with respect to time and obtain the case $j = 2$. We continue until we reach $j = 3$. (Note we use the fact that the background state satisfies (9.5.1) up to $j = 3$). □

10.5 Constructing the Density

Proposition 10.5.1. *Let $\rho_\pm(t,x) := \rho_\pm(p_\pm(t,x), s_\pm(t,x))$ in $(0,\tilde{T}) \times \Omega_\pm$ where $\tilde{T}$ is given by 10.1.1, s is given by 10.3.1 and p is given by 10.4.1. Then there exists a $T \in (0,\tilde{T}) \subset (0,\bar{T})$ such that the following holds.*

$$\rho \geq \frac{\delta^0}{4} \textit{ in } (0,T) \times \Omega_\pm$$
$$\rho_\mu \geq \frac{\delta_\mu^0}{4} \textit{ in } (0,T) \times \Omega_\pm$$

and if $T < \tilde{T}$ then one of the following holds.

$$\inf_{x\in\Omega_\pm} \rho \to \frac{\delta^0}{4} \textit{ as } t \uparrow T$$
$$\inf_{x\in\Omega_\pm} \rho_\mu \to \frac{\delta_\mu^0}{4} \textit{ as } t \uparrow T$$

Proof. Let $T = \sup\{t \in (0,\tilde{T}) : \inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho \geq \frac{\delta^0}{4}, \inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho_\mu \geq \frac{\delta_\mu^0}{4}\}$. Note that by uniform continuity $T > 0$ since $\rho^0 \geq \delta^0$ and $\rho_\mu^0 \geq \delta_\mu^0$. We also know from the definition of T and continuity that if $T < \tilde{T}$ then one of the above two convergences holds as $t \uparrow T$. □

10.6 Solving the Linear Transport Equation for the Curl

Proposition 10.6.1. *Let f and ψ be given by 10.1.1 and T, ρ_μ be given by 10.5.1. Then there exists $\omega \in L^\infty((0,T) \times \Omega_\pm)$ such that*

$$(\partial_t + \overline{u^\psi} \cdot \nabla)\omega = \frac{1}{\rho_\mu}(-\nabla^\psi \rho_\mu \times (\partial_t + \overline{u^\psi} \cdot \nabla)\bar{u} - \rho_\mu(\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi \bar{u}_k) + \nabla^\psi \times g_1)$$

in $(0,T) \times \Omega_\pm$ *and*

$$\partial_t^j \omega|_{t=0} = \partial_0^{\mu,g,j}(\nabla^{\psi^0} \times u^0) \textit{ for } 0 \le j \le 3 \tag{10.6.1}$$

Moreover, ω *has the following regularity*

$$\begin{aligned} &\partial_t^j \omega \in L^\infty((0,T); H^{4-j}(\Omega_\pm)) \textit{ for } 1 \le j \le 3 \\ &\nabla \omega \in L^\infty((0,T); H^3(\Omega_\pm)) \\ &\partial_t^4 \omega \in L^2((0,T); L^2(\Omega_\pm)) \end{aligned}$$

and such an ω *is unique.*

Proof. Apply lemma 10.2.3 with $b = \overline{u^\psi}$, h equal to the right hand side of the above equation and $w^0 = \nabla^{\psi^0} \times u^0$ to obtain a unique solution $\omega \in L^1_{\text{loc}}((0,T) \times \Omega_\pm)$ with $\nabla\omega \in L^2((0,T); L^2(\Omega_\pm))$ and $\partial_t \omega \in L^2((0,T); L^2(\Omega_\pm))$ to the equation with $\omega(0) = \nabla^{\psi^0} \times u^0$. Note that from the regularity of $\bar{u}$, ρ_μ, $\bar{\psi}$, ψ and $\nabla^{\psi^0} \times u^0$, we have

$$\begin{aligned} \partial_t^j b &\in L^\infty((0,T); H^{k-j+l}(\Omega_\pm)) \\ \partial_t^{j-1} h &\in L^\infty((0,T); H^{k-j+l}(\Omega_\pm)) \\ \nabla b &\in L^\infty((0,T); H^{k-1+l}(\Omega_\pm)) \\ \nabla h &\in L^2((0,T); H^{k-1+l}(\Omega_\pm)) \\ \nabla w^0 &\in H^{k-1+l}(\mathbb{R}^d) \end{aligned}$$

for $1 \le j \le k$, where $k = 3$ and $l = 1$. Hence lemma 10.2.3 implies

$$\begin{aligned} \partial_t^j \omega &\in L^\infty((0,T); H^{4-j}(\Omega_\pm)) \\ \nabla \omega &\in L^\infty((0,T); H^3(\Omega_\pm)) \end{aligned}$$

for all $1 \le j \le 3$. Using this regularity and the equation for ω it is also clear that

$$\partial_t^4 \omega \in L^2((0,T); L^2(\Omega_\pm))$$

To see that the initial time derivatives match, note that the background state satisfies (9.5.1), that is

$$\partial_t^j (\bar{U}_+, \bar{U}_-, \bar{f})|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \le j \le 3$$

by assumption, and note that $\omega(0) = \nabla^{\psi^0} \times u^0$. Using these two facts we see that the linearised equation for ω is the same as the μ-approximate equation (7.7.1) for $\nabla^\psi \times u$ when restricted to $t = 0$, which shows that (10.6.1) holds for $j = 1$. Now this allows us to differentiate the equations with respect to time and obtain the case $j = 2$. We continue until we reach $j = 3$. (Note we use the fact that the background state satisfies (9.5.1) up to $j = 3$). $\square$

11. Solving the Linearised Parabolic-Type Equation for the Velocity

The basic approach to solving the linearised equation for the velocity uses a weak form of the equation and a Galerkin approximation with energy estimates for the Galerkin approximation providing additional regularity - see eg Evans [15] for a simple version of this method. See 3.7 for a brief outline of the proof.

In this section we assume we are given the background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ on the time interval $(0, \bar{T})$ and initial data (U_+^0, U_-^0, f^0) such that $(\bar{U}_+, \bar{U}_-, \bar{f})$ and (U_+^0, U_-^0, f^0) satisfy the hypotheses of proposition 9.5.1. Given this initial data and background state, we assume that f and ψ are given by 10.1.1, s is given by 10.3.1, p is given by 10.4.1, T, ρ, ρ_μ are given by 10.5.1, and ω is given by 10.6.1.

11.1 Notation for Constants Depending on the Background State and the Solutions to Preceding Equations

Definition 11.1.1. It will be useful to define a constant depending on the background state and f, s, p and ω which we will use as a bound on a high order energy of solutions of the linearised equations. We define $C_{\max}$ as a generic constant of the form

$$\begin{aligned} C_{\max} = F_\mu(M_\mu^0, \bar{\mathcal{E}}_L(\bar{T}), &\sum_\pm \operatorname*{ess\,sup}_{t\in(0,T)} ||(p,s,\omega)||^2_{L^\infty(\Omega_\pm)} + \operatorname*{ess\,sup}_{t\in(0,T)} ||f||^2_{L^\infty(\Gamma)} \\ &+ \sum_\pm \sum_{1\le|\alpha|\le 3} \operatorname*{ess\,sup}_{t\in(0,T)} ||\partial^\alpha(p,s,\omega)||^2_{H^1(\Omega_\pm)} + \sum_\pm \int_0^T \left|\left|\partial_t^4(p,s,\omega)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}t \\ &+ \sum_{0\le j\le 3} \operatorname*{ess\,sup}_{t\in(0,T)} \left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)} + \sum_{0\le j\le 3} \int_0^T \left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}t) \end{aligned}$$

for some smooth increasing function F_μ depending on μ.

Note that, using the regularity for p, s, ω, f, we have

$$C_{\max} < \infty$$

It will also be useful to define constants depending on $\nabla' f$, ρ_μ, ρc^2 which we will use as a bound on coefficients of the equations. For $0 \le j \le 1$, we define $C_{\infty,j} > 0$ as a generic constant which is bounded above as follows

$$\begin{aligned} C_{\infty,j}(t) \le F_\mu(&\sum_\pm \sum_{k=0}^{j} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^k(\rho c^2)\right|\right|^2_{W^{j-k,\infty}(\Omega_\pm)}, \sum_\pm \sum_{k=0}^{j} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^k(\rho_\mu, \partial_t\rho_\mu)\right|\right|^2_{W^{j-k,\infty}(\Omega_\pm)}, \\ &\sum_{k=0}^{j+1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^k \nabla' f\right|\right|^2_{H^{2.5+j-k}(\Gamma)}, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}) \end{aligned}$$

for some smooth increasing function F_μ depending on μ.

11.2 Statement of Existence and Uniqueness of Solutions to the Parabolic-Type Linearised Equation for the Velocity

Proposition 11.2.1. *Let f and ψ be given by 10.1.1, s be given by 10.3.1, p be given by 10.4.1, T, ρ, ρ_μ be given by 10.5.1, and ω be given by 10.6.1. Then there exists $u \in L^\infty((0,T) \times \Omega_\pm)$ such that*

$$\rho_\mu(\partial_t + \overline{u^\psi} \cdot \nabla)u - \mu\nabla^\psi(\rho c^2 \nabla^\psi \cdot u) + \mu\nabla^\psi \times (\rho c^2 \nabla^\psi \times u) = -\nabla^\psi p + \mu\nabla^\psi \times (\rho c^2 \omega) + g_1$$

in $(0,T) \times \Omega_\pm$ and

$$\begin{gathered}[u \cdot n] = 0 \\ \mu(\Delta' - 1)(u \cdot n) + \mu[\rho c^2 \nabla^\psi \cdot u] = [p] + \sigma\nabla' \cdot \hat{n} - g_2 \\ n \times (\nabla^\psi \times u_\pm) = n \times \omega_\pm\end{gathered}$$

on $(0,T) \times \Gamma$ and

$$\partial_t^j u|_{t=0} = \partial_0^{\mu,g,j} u^0 \text{ for } 0 \le j \le 3$$

Moreover

$$\begin{aligned}&\sum_\pm \sum_{0\le j\le 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla u\right|\right|^2_{H^{3-j}(\Omega_\pm)} \mathrm{d}\tau + \sum_\pm \sum_{0\le j\le 3} \int_0^t \left|\left|\partial_t^j \nabla u\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau \\ &+ \sum_\pm \int_0^t \left|\left|\partial_t^4 u\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \sum_{j=0}^3 \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{4-j}(\Gamma)} + \sum_{j=0}^3 \int_0^t \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau \\ &\le C_{\max}\end{aligned}$$

Proof. See the remainder of this chapter for the proof. ☐

11.3 Rotating the Velocity to Fix the Normal Jump Condition

Definition 11.3.1. For the purposes of this section we will extend $\nabla' f$ from Γ to $\underset{\sim}{\nabla' f}$ defined on $\mathbb{R}^3$ via a linear lifting operator which gains half a derivative, so that

$$\left|\left|\underset{\sim}{\nabla' f}\right|\right|_{H^{l+1}(\mathbb{R}^3)} \le C \left|\left|\nabla' f\right|\right|_{H^{l+0.5}(\Gamma)}$$

for $l \ge 0$. See lemma A.2.3 in the appendix for a construction of such a lifting operator.

We define the matrix N depending on $\underset{\sim}{\nabla' f}$ as a scalar multiple of an orthogonal matrix, with columns as follows.

$$N = \frac{1}{|\underset{\sim}{n}|} \begin{pmatrix} \underset{\sim}{\hat{\tau}}^1 & \underset{\sim}{\hat{\tau}}^2 & \underset{\sim}{\hat{n}} \end{pmatrix}$$

where

$$\begin{aligned}\underset{\sim}{n} &= (-\underset{\sim}{\nabla' f}, 1) \\ \underset{\sim}{\tau}^1 &= (1, 0, \underset{\sim}{\partial_{x_1} f}) \\ \underset{\sim}{\tau}^2 &= \underset{\sim}{n} \times \underset{\sim}{\tau}^1 = (-\underset{\sim}{\partial_{x_1} f}\underset{\sim}{\partial_{x_2} f}, 1 + (\underset{\sim}{\partial_{x_1} f})^2, \underset{\sim}{\partial_{x_2} f})\end{aligned}$$

and

$$\hat{\underset{\sim}{\tau}}^1 = \frac{\underset{\sim}{\tau}^1}{\left|\underset{\sim}{\tau}^1\right|}$$
$$\hat{\underset{\sim}{\tau}}^2 = \frac{\underset{\sim}{\tau}^2}{\left|\underset{\sim}{\tau}^2\right|}$$
$$\hat{\underset{\sim}{n}} = \frac{\underset{\sim}{n}}{\left|\underset{\sim}{n}\right|}$$

are orthogonal each other, so that $(\hat{\underset{\sim}{\tau}}^1, \hat{\underset{\sim}{\tau}}^2, \hat{\underset{\sim}{n}})$ form a right-handed orthonormal basis of $\mathbb{R}^3$ for each $(t, x) \in (0, T) \times \mathbb{R}^3$.

We write N^T for the transpose of N and N^{-1} for its inverse. Note that by orthogonality we have

$$N^{-1} = \left|\underset{\sim}{n}\right|^2 N^T$$

Note that orthogonality also implies that, for any vector w,

$$\begin{aligned} Nw &= \frac{1}{\left|\underset{\sim}{n}\right|}(w_1\hat{\underset{\sim}{\tau}}^1 + w_2\hat{\underset{\sim}{\tau}}^2 + w_3\hat{\underset{\sim}{n}}) \\ (Nw) \cdot \underset{\sim}{n} &= w_3 \\ (N^{-1}w)_3 &= w \cdot \underset{\sim}{n} \end{aligned}$$

We also write $\bar{N}$ to denote the matrix given by replacing f with $\bar{f}$ in the definition of N.

Definition 11.3.2. We define the rotated initial velocity

$$\widehat{u}^0 := N^{-1}|_{t=0}u^0$$

The initial time derivatives act on $\widehat{u}^0$ in the obvious way. For $0 \leq j \leq 3$,

$$\partial_0^{\mu,g,j}\ \widehat{u}^0 := \partial_t^j(N^{-1}(t)u(t))|_{t=0}$$

where on the right hand side we take the formal derivative with respect to t then replace $\partial_t^l u|_{t=0}$ with $\partial_0^{\mu,g,l}u^0$ for $0 \leq l \leq j$. Note that we have $\partial_t^j f|_{t=0} = \partial_0^{\mu,g,j} f^0$ for $0 \leq j \leq 3$, so if we write N^0 for the matrix defined by replacing $\nabla' f$ with $\nabla' f^0$ in the definition of N, then we may write $\partial_t^j N|_{t=0} = \partial_0^{\mu,g,j} N^0$.

Definition 11.3.3. We say that the rotated velocity $\widehat{u}_\pm \in C_b^2((0, T) \times \Omega_\pm)$ with $\widehat{u}_3 \in C_b^2((0, T) \times \Gamma)$ is a solution of the rotated linearised equations for the velocity if the following holds.

$$\widehat{u}\ (0) = \widehat{u}^0$$

and $\widehat{u}$ satisfies the following equations.

$$\begin{aligned} &\rho_\mu \frac{1}{\left|\underset{\sim}{n}\right|^2}\partial_t\ \widehat{u} - \mu N^T\nabla^\psi(\rho c^2\nabla^\psi \cdot (N\ \widehat{u})) + \mu N^T\nabla^\psi \times (\rho c^2\nabla^\psi \times (N\ \widehat{u})) + \rho_\mu N^T(\overline{u^\psi} \cdot \nabla)(N\ \widehat{u}) \\ &+ \rho_\mu N^T(\partial_t N)\ \widehat{u} + N^T(\nabla^\psi p - \mu\nabla^\psi \times (\rho c^2\omega)) = N^T g_1 \end{aligned} \tag{11.3.1}$$

in $(0, T) \times \Omega_\pm$ and

$$[\widehat{u}_3] = 0 \tag{11.3.2}$$

$$\mu(\Delta' - 1)\ \widehat{u}_3 + \mu[\rho c^2\nabla^\psi \cdot (N\ \widehat{u})] = [p] + \sigma\nabla' \cdot \hat{n} - g_2 \tag{11.3.3}$$

$$n \times (\nabla^\psi \times (N\ \widehat{u}_\pm)) = n \times \omega_\pm \tag{11.3.4}$$

on $(0, T) \times \Gamma$.

Proposition 11.3.4. *Suppose $\widehat{u}$ is a solution of the rotated linearised equations for the velocity as defined in 11.3.3 satisfying*

$$\begin{aligned}
&\partial_t^j \widehat{u}\,|_{t=0} = \partial_0^{\mu,g,j}\,\widehat{u}^0 \text{ for } 0 \leq j \leq 3\\
&\sum_{\pm}\sum_{0\leq j\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla\,\widehat{u}\right|\right|^2_{H^{3-j}(\Omega_\pm)} \mathrm{d}\tau + \sum_{\pm}\sum_{0\leq j\leq 3} \int_0^t \left|\left|\partial_t^j \nabla\,\widehat{u}\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau\\
&+ \sum_{\pm}\int_0^t \left|\left|\partial_t^4\,\widehat{u}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \sum_{j=0}^3 \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j\,\widehat{u}_3\right|\right|^2_{H^{4-j}(\Gamma)} + \sum_{j=0}^3 \int_0^t \left|\left|\partial_t^j\,\widehat{u}_3\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau\\
&\leq C_{\max}
\end{aligned}$$

Then $u := N\,\widehat{u}$ is a solution of the linearised equations for the velocity (9.1.7)-(9.1.10) *with*

$$\begin{aligned}
&\partial_t^j u|_{t=0} = \partial_0^{\mu,g,j} u^0 \text{ for } 0 \leq j \leq 3\\
&\sum_{\pm}\sum_{0\leq j\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla u\right|\right|^2_{H^{3-j}(\Omega_\pm)} \mathrm{d}\tau + \sum_{\pm}\sum_{0\leq j\leq 3} \int_0^t \left|\left|\partial_t^j \nabla u\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau\\
&+ \sum_{\pm}\int_0^t \left|\left|\partial_t^4 u\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \sum_{j=0}^3 \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{4-j}(\Gamma)} + \sum_{j=0}^3 \int_0^t \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau\\
&\leq C_{\max}
\end{aligned}$$

Proof. Let $0 \leq j \leq 3$. Using the fact that $NN^{-1} = \mathrm{id}$ together with the definition of $\partial_0^{\mu,g,j}\,\widehat{u}^0$, the definition of $u := N\,\widehat{u}$ and the fact that $\partial_t^j\,\widehat{u}\,|_{t=0} = \partial_0^{\mu,g,j}\,\widehat{u}^0$, we see that $\partial_t^j u|_{t=0} = \partial_t^j(N(t)\,\widehat{u}\,(t))|_{t=0} = \partial_0^{\mu,g,j} u^0$.

We set $\widehat{u}= N^{-1}u$ in the equations (11.3.1)-(11.3.4). In (11.3.1), we obtain

$$\begin{aligned}
&\rho_\mu \frac{1}{|\underset{\sim}{n}|^2}\partial_t(N^{-1}u) - \mu N^T\nabla^\psi(\rho c^2\nabla^\psi\cdot u) + \mu N^T\nabla^\psi\times(\rho c^2\nabla^\psi\times u) + \rho_\mu N^T(\overline{u^\psi}\cdot\nabla)u\\
&+ \rho_\mu N^T(\partial_t N)N^{-1}u + N^T(\nabla^\psi p - \mu\nabla^\psi\times(\rho c^2\omega)) = N^T g_1
\end{aligned}$$

We multiply by $|\underset{\sim}{n}|^2\,N = (N^T)^{-1}$ to obtain

$$\begin{aligned}
&\rho_\mu\partial_t u - \mu\nabla^\psi(\rho c^2\nabla^\psi\cdot u) + \mu\nabla^\psi\times(\rho c^2\nabla^\psi\times u) + \rho_\mu(\overline{u^\psi}\cdot\nabla)u + \rho_\mu N\partial_t(N^{-1})u + \rho_\mu(\partial_t N)N^{-1}u\\
&+ \nabla^\psi p - \mu\nabla^\psi\times(\rho c^2\omega) = g_1
\end{aligned}$$

Using the fact that

$$0 = \partial_t(NN^{-1}) = N\partial_t N^{-1} + (\partial_t N)N^{-1}$$

we obtain the equation (9.1.7).

In (11.3.2)-(11.3.4), we obtain

$$\begin{aligned}
[(N^{-1}u)_3] &= 0\\
\mu(\Delta' - 1)(N^{-1}u)_3 + \mu[\rho c^2\nabla^\psi\cdot u] &= [p] + \sigma\nabla'\cdot\hat{n} - g_2\\
n\times(\nabla^\psi\times u_\pm) &= n\times\omega_\pm
\end{aligned}$$

Using the fact that $(N^{-1}u)_3 = u\cdot n$, we obtain the equations (9.1.8)-(9.1.10).

Note that we have

$$\sum_{\pm}\sum_{0\le j\le 3}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j\nabla N\right|\right|^2_{H^{3-j}(\Omega_\pm)}\mathrm{d}\tau+\sum_{\pm}\sum_{0\le j\le 3}\int_0^t\left|\left|\partial_t^j\nabla N\right|\right|^2_{H^{4-j}(\Omega_\pm)}\mathrm{d}\tau$$
$$+\sum_{\pm}\int_0^t\left|\left|\partial_t^4 N\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau$$
$$\le C_{\max}$$

where we have used the fact that by the lifting property,

$$\int_0^t ||N||_{H^5(\Omega_\pm)}\,\mathrm{d}\tau\le C\int_0^t ||\nabla' f||_{H^{4.5}(\Gamma)}\,\mathrm{d}\tau$$
$$\le C_{\max}$$

We combine this estimate with the estimate for $\widehat{u}$ and the fact that $\widehat{u}_3 = u\cdot n$ on Γ, together with the product rule for Sobolev functions, to obtain the desired energy estimate for u.

This completes the proof. □

11.4 The Spaces for a Weak Form of the Equations

Definition 11.4.1. For $r\in[1,\infty]$, we define the domains $\Omega_{r\pm}$ and interface Γ_r as follows.

$$\Omega_{r\pm}:=\Omega_\pm\cap B_r(0)$$
$$\Gamma_r:=\Gamma\cap B_r(0)$$

where $B_r(0)$ denotes the open ball of radius r about 0, and is equal to $\mathbb{R}^3$ in the case $r=\infty$. For $r<\infty$ we refer to these as the bounded domains.

Definition 11.4.2. For $r\in[1,\infty]$, we define the Hilbert spaces B_r^0 and B_r^1 as follows.

$$B_r^0:=\{v=(v_+,v_-)\in L^2(\Omega_{r+};\mathbb{R}^3)\times L^2(\Omega_{r-};\mathbb{R}^3)\}$$

For each $t\in[0,T]$ (at the end points we use uniform continuity for the following definition) we define the inner product $(\cdot,\cdot)_{B_r^0(t)}$ on B_r^0 as follows, for $v,w\in B_r^0$.

$$(v,w)_{B_r^0(t)}=\sum_{\pm}(\rho_\mu(t)\frac{1}{\left|\underset{\sim}{n}(t)\right|^2}v,w)_{L^2(\Omega_{r\pm})}$$

We define the inner product $(\cdot,\cdot)_{B_r^0}$ as

$$(v,w)_{B_r^0}=(v,w)_{B_r^0(0)}$$

which is independent of time. We will use this inner product on B_r^0 when applying any results from functional analysis. The other inner products are defined for notational convenience. Note that for all $t\in[0,T]$ the inner product $(\cdot,\cdot)_{B_r^0(t)}$ is equivalent to the standard inner product on $L^2(\Omega_{r+})\times L^2(\Omega_{r-})$ since there exists a constant $C_{\infty,0}>0$ such that

$$\frac{1}{C_{\infty,0}}\le\rho_{\mu\pm}(t,x)\le C_{\infty,0}$$
$$\frac{1}{C_{\infty,0}}\le\left|\underset{\sim}{n}(t,x)\right|\le C_{\infty,0}$$

for all (t, x).

We define

$$B_r^1 := \{v = (v_+, v_-) \in H^1(\Omega_{r+}; \mathbb{R}^3) \times H^1(\Omega_{r-}; \mathbb{R}^3) : [v_3] = 0, v_3|_{\Gamma_r} \in H^1(\Gamma_r), v|_{|x|=r} = 0\}$$

Note that by $v|_{|x|=r} = 0$ we mean that the $H^1(\Omega_{r\pm})$-trace of v on $\{|x| = r\} \cap \Omega_\pm$ is zero and the $H^1(\Gamma_r)$-trace of v_3 on $\{|x| = r\} \cap \Gamma$ is zero, and this condition should be dropped entirely in the case $r = \infty$. For each $t \in [0, T]$ (at the end points we use uniform continuity for the following definition) we define the semi-norm-inducing-inner product $\{\cdot, \cdot\}_{B_r^1(t)}$ on B_r^1 as follows, for $v, w \in B_r^1$.

$$\begin{aligned}
\{v, w\}_{B_r^1(t)} &= \sum_\pm (\rho(t) c^2(t) \nabla^{\psi(t)} \cdot (N(t)v), \nabla^{\psi(t)} \cdot (N(t)w))_{L^2(\Omega_{r\pm})} \\
&+ \sum_\pm (\rho(t) c^2(t) \nabla^{\psi(t)} \times (N(t)v), \nabla^{\psi(t)} \times (N(t)w))_{L^2(\Omega_{r\pm})} \\
&+ (\nabla' v_3, \nabla' w_3)_{L^2(\Gamma_r)} + (v_3, w_3)_{L^2(\Gamma_r)}
\end{aligned}$$

We define the semi-norm-inducing-inner product $\{\cdot, \cdot\}_{B_r^1}$ as

$$\{v, w\}_{B_r^1} = \{v, w\}_{B_r^1(0)}$$

which is independent of time.

We will write the induced semi-norm as

$$|v|^2_{B_r^1(t)} := \{v, v\}_{B_r^1(t)}$$

We then define the time-dependent full inner product and induced norm on B_r^1 as

$$\begin{aligned}
(v, w)_{B_r^1(t)} &= (v, w)_{B_r^0(t)} + \{v, w\}_{B_r^1(t)} \\
||v||^2_{B_r^1(t)} &= (v, v)_{B_r^0(t)} + \{v, v\}_{B_r^1(t)}
\end{aligned}$$

and the time-independent full inner product

$$(v, w)_{B_r^1} = (v, w)_{B_r^0} + \{v, w\}_{B_r^1}$$

We will use this inner product on B_r^1 when applying any results from functional analysis. The other inner products are defined for notational convenience.

We will sometimes write B^0 and B^1 in the case $r = \infty$.

Proposition 11.4.3. *The map $\Phi : B_r^1 \to \Phi(B_r^1) \leq H^1(\Omega_{r+}; \mathbb{R}^3) \times H^1(\Omega_{r-}; \mathbb{R}^3) \times H^1(\Gamma_r)$ given by*

$$v \mapsto (v_+, v_-, v_3|_\Gamma)$$

is an isomorphism between Hilbert spaces, where the target space is equipped with the obvious product inner product, and there exists $C_{\infty,0} > 0$ independent of time such that

$$\frac{1}{C_{\infty,0}} ||v||^2_{B_r^1(t)} \leq ||\Phi(v)||^2 := ||v_+||^2_{H^1(\Omega_{r+})} + ||v_-||^2_{H^1(\Omega_{r-})} + ||v_3||^2_{H^1(\Gamma_r)} \leq C_{\infty,0} ||v||^2_{B_r^1(t)}$$

for all $t \in [0, T]$.

Proof. First note that since $v_{\pm}|_{|x|=r} = 0$, we may extend $v_{\pm}$ by zero to obtain $v_{\pm} \in H^1(\Omega_{\pm})$, $v_3 \in H^1(\Gamma)$.

Note that there exists a constant $C_{\infty,0} > 0$ such that

$$\frac{1}{C_{\infty,0}} \leq \rho_{\mu\pm}(t,x) \leq C_{\infty,0}$$
$$\left|\left|\nabla' f\right|\right|_{H^{2.5}(\Gamma)} \leq C_{\infty,0}$$

for all (t,x). Using this and the definition of $||\cdot||_{B^1_r(t)}$, together with the lifting property in the definition of N, the left-hand inequality is clear.

Note that

$$v_3 = (N(t)v) \cdot \underset{\sim}{n}(t)$$

Note also that, by orthogonality of $\left|\underset{\sim}{n}\right| N(t)$

$$||N(t)v||_{L^2(\Omega_{\pm})} = \left|\left|\frac{1}{\left|\underset{\sim}{n}\right|} v\right|\right|_{L^2(\Omega_{\pm})} \leq C_{\infty,0} \, ||v||_{B^1_r(t)}$$

Applying the Hodge decomposition estimate A.1.2 to $N(t)v$ we have

$$\begin{aligned}
||N(t)v||_{H^1(\Omega_{\pm})} &\leq C(1 + \left|\left|\nabla' f(t)\right|\right|^3_{W^{1,\infty}(\Gamma)}) \bigg(||N(t)v||_{L^2(\Omega_{\pm})} + \left|\left|\nabla^{\psi} \cdot (N(t)v)\right|\right|_{L^2(\Omega_{\pm})} \\
&\quad + \left|\left|\nabla^{\psi} \times (N(t)v)\right|\right|_{L^2(\Omega_{r\pm})} + ||(N(t)v) \cdot n(t)||_{H^{0.5}(\Gamma)} \bigg) \\
&\leq C_{\infty,0}(||v||_{B^0(t)} + ||v||_{B^1(t)})
\end{aligned}$$

Now, using the orthogonality of $\left|\underset{\sim}{n}\right| N(t)$, we have

$$\begin{aligned}
||N(t)v||^2_{H^1(\Omega_{\pm})} &= ||N(t)v||^2_{L^2(\Omega_{\pm})} + \sum_{j=1}^{3} ||(\partial_j N(t))v||^2_{L^2(\Omega_{\pm})} + \sum_{j=1}^{3} ||N(t)\partial_j v||^2_{L^2(\Omega_{\pm})} \\
&= \left|\left|\frac{1}{\left|\underset{\sim}{n}\right|} v\right|\right|^2_{L^2(\Omega_{\pm})} + \sum_{j=1}^{3} \left|\left|\frac{1}{\left|\underset{\sim}{n}\right|} \partial_{x_j} v\right|\right|^2_{L^2(\Omega_{\pm})} + \sum_{j=1}^{3} \left|\left|(\partial_{x_j} N(t))v\right|\right|^2_{L^2(\Omega_{\pm})} \\
&\geq C_{\infty,0} \, ||v||^2_{H^1(\Omega_{\pm})} - C_{\infty,0} \left|\left|\nabla' f\right|\right|^2_{W^{1,\infty}(\Gamma)} ||v||^2_{L^2(\Omega_{\pm})}
\end{aligned}$$

Since we have already estimated

$$||v||_{L^2(\Omega_{\pm})} \leq C_{\infty,0} \, ||v||_{B^0_r(t)}$$

The right-hand inequality follows. □

Corollary 11.4.4. *For $r < \infty$, B^1_r is compactly embedded in B^0_r (with norms $||\cdot||_{B^1_r(t)}$ and $||\cdot||_{B^0_r(t)}$ respectively).*

Proof. Using the above result and the standard Sobolev compact embedding theorem, we have $B^1_r \subset H^1(\Omega_{r+};\mathbb{R}^3) \times H^1(\Omega_{r-};\mathbb{R}^3) \subset\subset L^2(\Omega_{r+};\mathbb{R}^3) \times L^2(\Omega_{r-};\mathbb{R}^3)$, where the symbol $\subset\subset$ means compactly contained. Now we have already shown that $L^2(\Omega_{r+};\mathbb{R}^3) \times L^2(\Omega_{r-};\mathbb{R}^3)$ is isomorphic to B^0_r with equivalent norms, and this proves the result. □

Definition 11.4.5. For $r \in [1, \infty]$, we define the space X_r as follows.

$$X_r = \{v = (v_+, v_-) : v_\pm \in L^1_{\text{loc}}((0,T) \times \Omega_{r\pm}), v \in L^\infty((0,T); B^1_r), \partial_t v \in L^2((0,T); B^0_r)\}$$

We will sometimes write X for the case $r = \infty$.

Note that if $v \in X$ then $v \in C([0,T]; B^0_r)$, since $\partial_t v \in L^2((0,T); B^0_r)$.

11.5 Construction of a Galerkin Basis

Lemma 11.5.1. *Let $m \geq 1$ be an integer, and let $z_1, \dots, z_m \in B^0_r$ be orthonormal with respect to the inner product $(\cdot,\cdot)_{B^0_r}$. Define the matrix $M(t)$ depending on $t \in (0,T)$ by*

$$(M(t))_{kl} = (z_k, z_l)_{B^0_r(t)}$$

Then $M(t)$ is uniformly positive definite on $(0,T)$, ie there exists a constant $C_{\infty,0} > 0$ independent of t such that

$$b^T M(t) b \geq \frac{1}{C_{\infty,0}} |b|^2$$

for all $b \in \mathbb{R}^m$, and hence $M(t)$ is invertible with

$$\left|M(t)^{-1}\right| \leq C_{\infty,0}$$

for some $C_\infty > 0$ independent of t. Hence, since $M \in W^{1,\infty}(0,T)$ with

$$|\partial_t M| \leq C_{\infty,0}$$

we have that $M^{-1} \in W^{1,\infty}(0,T)$ with

$$\left|\partial_t M^{-1}\right| \leq C_{\infty,0}$$

Proof. Let $b \in \mathbb{R}^m$. Then

$$\begin{aligned}
b^T M(t) b &= \sum_{k,l=1}^m b_k (z_k, z_l)_{B^0_r(t)} b_l \\
&= (\sum_{k=1}^m b_k z_k, \sum_{l=1}^m b_l z_l)_{B^0_r(t)} \\
&= (v_m, v_m)_{B^0_r(t)}
\end{aligned}$$

where $v_m = \sum_{k=1}^m b_k$. Using the fact that $C_{\infty,0} \geq \rho_\mu \frac{1}{|\underset{\sim}{n}|^2} \geq \frac{\delta^0_\mu}{C_{\infty,0}}$, we obtain

$$\begin{aligned}
b^T M(t) b &= ||v_m||^2_{B^0_r(t)} \\
&= \left|\left|\left(\frac{\rho_\mu \left|\underset{\sim}{n}^0\right|^2}{\rho^0_\mu \left|\underset{\sim}{n}\right|^2}\right)^{\frac{1}{2}} v_m\right|\right|^2_{B^0_r} \\
&\geq \inf_{\Omega_\pm}\left(\frac{\rho_\mu \left|\underset{\sim}{n}^0\right|^2}{\rho^0_\mu \left|\underset{\sim}{n}\right|^2}\right) ||v_m||^2_{B^0_r} \\
&\geq \frac{1}{C_{\infty,0}} \sum_{k,l=1}^m b_k (z_k, z_l)_{B^0_r} b_l \\
&\geq \frac{1}{C_{\infty,0}} |b|^2
\end{aligned}$$

where in the last line we have used the fact that $z_1, \dots, z_m$ are orthonormal with respect to the inner product $(\cdot,\cdot)_{B_r^0}$. This completes the proof. $\square$

Lemma 11.5.2. *For $r < \infty$, there exists a sequence of functions $(z_k)_{k=1}^{\infty} \subset B_r^1$ which form an orthogonal Hilbert space basis for B_r^1 (with norm $||\cdot||_{B_r^1}$) and an orthonormal Hilbert space basis for B_r^0 (with norm $||\cdot||_{B_r^0}$). We call these the Galerkin basis functions. Moreover, the functions z_k satisfy the following eigenvalue equation*

$$(z_k, w)_{B_r^1} = \lambda_k (z_k, w)_{B_r^0} \textit{ for all } w \in B_r^1 \tag{11.5.1}$$

where $\lambda_k > 0$ for each k.

Proof. We define the bounded linear operator $S : B_r^0 \to B_r^0$ as follows.

$$Sf = u$$

where $u \in B_r^1 \subset B_r^0$ is the unique solution to the equation

$$(u, w)_{B_r^1} = (f, w)_{B_r^0} \text{ for all } w \in B_r^1 \tag{11.5.2}$$

We claim S is well-defined. Indeed, given $f \in B_r^0$, we can define a continuous linear functional ϕ_f on B_r^1 by $\phi_f(w) = (f, w)_{B_r^0}$ for $w \in B_r^1$. Now the Riesz representation theorem says there exists a unique element $u \in B_r^1$ such that $(u, w)_{B_r^1} = \phi_f(w) = (f, w)_{B_r^0}$ for all $w \in B_r^1$. We define $Sf = u$. It is clear that S is linear and

$$||Sf||_{B_r^0} \leq ||Sf||_{B_r^1} = ||u||_{B_r^1} \leq ||f||_{B_r^0}$$

Hence S is bounded. What is more, this last line shows that S is a compact operator, since $B_r^1 \subset\subset B_r^0$.

We claim that S is also symmetric and positive definite. Indeed, note that, if $Sf = u$, then

$$(Sf, f)_{B_r^0} = (u, f)_{B_r^0} = (f, u)_{B_r^0} = (u, u)_{B_r^0} \geq 0$$

with equality if and only if $u = 0$ if and only if $f = 0$. Note that we have used the symmetry of the inner product followed by the equation (11.5.2) choosing $w = u$. Also, suppose $Sf_1 = u_1$ and $Sf_2 = u_2$. Then, using the same trick,

$$\begin{aligned}
(Sf_1, f_2)_{B_r^0} &= (u_1, f_2)_{B_r^0} = (f_2, u_1)_{B_r^0} = (u_2, u_1)_{B_r^0} \\
(Sf_2, f_1)_{B_r^0} &= (u_2, f_1)_{B_r^0} = (f_1, u_2)_{B_r^0} = (u_1, u_2)_{B_r^0}
\end{aligned}$$

But since $(u_1, u_2)_{B_r^0} = (u_2, u_1)_{B_r^0}$, we have $(Sf_1, f_2)_{B_r^0} = (Sf_2, f_1)_{B_r^0} = (f_1, Sf_2)_{B_r^0}$, hence S is symmetric.

Thus, by the spectral decomposition theorem for compact self-adjoint operators, there exists an orthonormal basis $(z_k)_{k=1}^{\infty}$ of B_r^0 consisting of eigenfunctions of S with real strictly positive eigenvalues $(\mu_k)_{k=1}^{\infty}$. In particular, this means $z_k \in B_r^1$ for all k and, defining $\lambda_k := \mu_k^{-1}$,

$$(z_k, w)_{B_r^1} = \lambda_k (z_k, w)_{B_r^0} \text{ for all } w \in B_r^1$$

Note that this implies

$$(z_k, z_l)_{B_r^1} = \lambda_k (z_k, z_l)_{B_r^0} = \lambda_k \delta_{kl}$$

for all k, l, hence $(z_k)_{k=1}^{\infty}$ is an orthogonal set in B_r^1.

We claim the span of $(z_k)_{k=1}^{\infty}$ is dense in B_r^1. Indeed, let $u \in B_r^1$. Suppose $(u, z_k)_{B_r^1} = 0$ for all k. It suffices to prove that $u = 0$. But we know from (11.5.1) that

$$(z_k, u)_{B_r^1} = \lambda_k (z_k, u)_{B_r^0}$$

for all k, hence

$$(z_k, u)_{B_r^0} = 0$$

for all k. This implies $u = 0$ in B_r^0 since $(z_k)_{k=1}^{\infty}$ is a basis of B_r^0, and this implies $u = 0$ in B_r^1. This proves the claim. □

11.6 Existence, Uniqueness and Regularity Theory for Weak Equations of the Appropriate Form

Notation. *If v is a vector, we will write $(v, \nabla v)$ as a matrix containing all the entries of v and all the derivatives $\partial_{x_j} v_i$ for $0 \leq i, j \leq 3$. We write just ∇v for the matrix with entries $\partial_{x_j} v_i$ for $0 \leq i, j \leq 3$.*

For simplicity, we will denote a linear function of $v, \nabla v$ or $(v, \nabla v)$ whose output may be a scalar, vector, or matrix, by an expression of the form

$$Av$$

$$A\nabla v$$

$$A(v, \nabla v)$$

respectively, where A is a tensor of the appropriate rank.

We will also be using G to denote a generic scalar, vector or matrix as appropriate, but note in particular that if G is a matrix and $(\cdot, \cdot)$ denotes an inner product then

$$(G, \nabla v) = \sum_{i,j=1}^{3} G_{ij} \partial_{x_j} v_i$$

Also, if G is a matrix then G_j will denote the j-th column of G and $\nabla \cdot$ acts on the rows of G.

Notation. *If we write $G \in H$ or $G(t) \in H$ for some normed space H and then we refer to a constant C_G or $C_G(t)$ it will be implicitly assumed that*

$$C_G \leq F_\mu(||G||_H)$$

or

$$C_G(t) \leq F_\mu(||G(t)||_H)$$

for some smooth increasing function F_μ which may depend on μ, and we extend the notation to include more norms if we write that G belongs to more spaces.

Definition 11.6.1. We will sometimes find it convenient to write

$$(\partial_t v, w)_{B^0_r(t)} = \sum_{\pm}(A^0\partial_t v, w)_{L^2(\Omega_{r\pm})}$$
$$\{v, w\}_{B^1_r(t)} = \sum_{\pm}(A^1(v, \nabla v), (w, \nabla w))_{L^2(\Omega_{r\pm})} + (v_3, w_3)_{H^1(\Gamma_r)}$$

for a scalar function A^0 and a tensor of appropriate rank A^1.

We also define the tensor A^2 such that

$$\begin{aligned}A^2(v, \nabla v) &= \mu N^T\partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi})\rho c^2\nabla^\psi \cdot (Nv) - \mu N^T\partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi}) \times (\nabla^\psi \times (Nv))\\ &+ \rho_\mu N^T(\overline{u^\psi} \cdot \nabla)(Nv) + \rho_\mu N^T(\partial_t N)v\end{aligned}$$

Note that A^i have the form

$$\begin{aligned}A^0 &= A(\rho_\mu, N)\\ A^1 &= A(\rho c^2, \nabla\psi, \nabla N, N)\\ A^2 &= A(\rho_\mu, \rho c^2, \bar{u}, \nabla\psi, \nabla\bar{\psi}, \partial_t\psi, \nabla N, \partial_t N, N)\end{aligned}$$

where A denotes a generic smooth function.

Definition 11.6.2. We write C_A for a constant bounded above as follows.

$$\begin{aligned}C_A(t) \le F_\mu(&\sum_{\pm}(\operatorname*{ess\,sup}_{\tau\in(0,t)} ||z||^2_{L^\infty(\Omega_\pm)} + \sum_{1\le|\alpha|\le3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha z||^2_{H^1(\Omega_\pm)} + \int_0^t \left|\left|\partial_t^4 A^1\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau),\\ &\sum_{k=0}^{4} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_{x_3}^{k+1}\psi\right|\right|^2_{L^\infty(\mathbb{R}^3)}, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0})\end{aligned}$$

for some smooth increasing function F_μ, where

$$z = (\rho_\mu, \rho c^2, \bar{u}, \nabla'\psi, \nabla'\bar{\psi}, \partial_t\psi, \nabla N, \partial_t N, N)$$

Note that, given the regularity of $f, s, p, \bar{u}, \bar{f}$, we have

$$C_A(T) < \infty$$

Note that we have used the fact that $||\nabla N||_{H^4(\Omega_\pm)} \le C\,||\nabla' f||_{H^{4.5}(\Gamma)}$ due to the lifting property.

Proposition 11.6.3. *Let $r \in [1, \infty]$ and let $W_r \le B^1_r$. Let $G^1 \in L^2((0, T) \times \Omega_{r\pm})$ be a vector, $G^2 \in L^2((0, T) \times \Omega_{r\pm})$ be a matrix, $G^3 \in L^2((0, T) \times \Gamma_r)$ be a scalar and $G^4 \in L^2((0, T) \times \Gamma_r)$ be a 2-vector. Suppose that $v \in X \cap W_r$ is such that*

$$\begin{aligned}&(\partial_t v, w)_{B^0_r(t)} + \mu\{v, w\}_{B^1_r(t)} + \sum_{\pm}(A^2(v, \nabla v), w)_{L^2(\Omega_{r\pm})}\\ &= \sum_{\pm}(G^1, w)_{L^2(\Omega_{r\pm})} + \sum_{\pm}(G^2, \nabla w)_{L^2(\Omega_{r\pm})} + (G^3, w_3)_{L^2(\Gamma_r)} + (G^4, \nabla' w_3)_{L^2(\Gamma_r)}\end{aligned}$$

for all $w \in W_r$ for almost every $t \in (0,T)$. Then we have the estimates

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||v||^2_{B^0_r(\tau)} \leq C_{\infty,0}(||v(0)||^2_{B^0_r}$$
$$+ \int_0^t \sum_{\pm}(\left|\left|G^1\right|\right|^2_{L^2(\Omega_{r\pm})} + \left|\left|G^2\right|\right|^2_{L^2(\Omega_{r\pm})}) + \left|\left|G^3\right|\right|^2_{L^2(\Gamma_r)} + \left|\left|G^4\right|\right|^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau) \tag{11.6.1}$$
$$\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau \leq$$
$$C_{\infty,0}\int_0^t ||v(0)||^2_{B^0_r} + \sum_{\pm}(\left|\left|G^1\right|\right|^2_{L^2(\Omega_{r\pm})} + \left|\left|G^2\right|\right|^2_{L^2(\Omega_{r\pm})}) + \left|\left|G^3\right|\right|^2_{L^2(\Gamma_r)} + \left|\left|G^4\right|\right|^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau \tag{11.6.2}$$

Moreover, suppose that we have $G^2 \in L^\infty((0,T);L^2(\Omega_{r\pm}))$, $\partial_t G^2 \in L^2((0,T)\times\Omega_{r\pm})$, $G^3, G^4 \in L^\infty((0,T);L^2(\Gamma_r))$, $\partial_t G^3, \partial_t G^4 \in L^2((0,T)\times\Gamma_r)$, $\partial_t v \in L^2((0,T);W_r)$, so that in particular $v \in C([0,T];B^1_r)$. Then

$$\int_0^t ||\partial_t v||^2_{B^0_r(\tau)}\,\mathrm{d}\tau + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||v||^2_{B^1_r(t)}$$
$$\leq \frac{C_{\infty,1}}{\epsilon}(\int_0^t ||v(0)||^2_{B^0_r} + \sum_{\pm}\left|\left|G^1\right|\right|^2_{L^2(\Omega_{r\pm})} + \sum_{\pm}\left|\left|G^2\right|\right|^2_{L^2(\Omega_{r\pm})} + \left|\left|G^3\right|\right|^2_{L^2(\Gamma_r)} + \left|\left|G^4\right|\right|^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau)$$
$$+ C_{\infty,0}(||v(0)||^2_{B^1_r} + \sum_{\pm}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|G^2\right|\right|^2_{L^2(\Omega_{r\pm})} + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|G^3\right|\right|^2_{L^2(\Gamma_r)} + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|G^4\right|\right|^2_{L^2(\Gamma_r)})$$
$$+ \epsilon\int_0^t \sum_{\pm}\left|\left|\partial_t G^2\right|\right|^2_{L^2(\Omega_{r\pm})} + \left|\left|\partial_t G^3\right|\right|^2_{L^2(\Gamma_r)} + \left|\left|\partial_t G^4\right|\right|^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau \tag{11.6.3}$$

where we are free to choose $\epsilon \in (0,1]$.

Proof. Firstly, we note that since the equation holds for all $w \in W_r$, and we have $v \in W_r$, we may set $w = v$ in the equation. Doing this and using the product rule in time we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||v||^2_{B^0_r(t)} - \frac{1}{2}\sum_{\pm}(\partial_t A^0 v, v)_{L^2(\Omega_{r\pm})} + \mu\,|v|^2_{B^1_r(t)} + \sum_{\pm}(A^2(v,\nabla v), v)_{L^2(\Omega_{r\pm})}$$
$$= \sum_{\pm}(G^1, v)_{L^2(\Omega_{r\pm})} + \sum_{\pm}(G^2, \nabla v)_{L^2(\Omega_{r\pm})} + (G^3, v_3)_{L^2(\Gamma_r)} + (G^4, \nabla' v_3)_{L^2(\Gamma_r)}$$

Integrating in time and using Cauchy's inequality, we obtain

$$\frac{1}{2}||v||^2_{B^0_r(t)} + \mu\int_0^t |v|^2_{B^1_r(\tau)}\,\mathrm{d}\tau$$
$$\leq \frac{1}{2}||v(0)||^2_{B^0_r} + \frac{C_{\infty,0}}{\delta}\int_0^t ||v||^2_{L^2(\Omega_{r\pm})}\,\mathrm{d}\tau + \delta\int_0^t ||v||^2_{H^1(\Omega_{r\pm})}\,\mathrm{d}\tau + \delta\int_0^t ||v_3||^2_{H^1(\Gamma_r)}\,\mathrm{d}\tau$$
$$+ \frac{1}{\delta}\int_0^t \sum_{\pm}\left|\left|G^1\right|\right|^2_{L^2(\Omega_{r\pm})} + \sum_{\pm}\left|\left|G^2\right|\right|^2_{L^2(\Omega_{r\pm})} + \left|\left|G^3\right|\right|^2_{L^2(\Gamma_r)} + \left|\left|G^4\right|\right|^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau$$

Using the fact that $||v||_{L^2(\Omega_{r\pm})} \leq C_{\infty,0}\,||v||_{B^0_r(t)}$ and $||v||_{H^1(\Omega_{r\pm})} + ||v_3||_{H^1(\Gamma_r)} \leq C_{\infty,0}\,||v||_{B^1_r(t)}$ as proved in 11.4.3, we obtain

$$\frac{1}{2}||v||^2_{B^0_r(t)} + \mu\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau$$
$$\leq \frac{1}{2}||v(0)||^2_{B^0_r} + \frac{C_{\infty,0}}{\delta}\int_0^t ||v||_{B^0_r(\tau)}\,\mathrm{d}\tau + C_{\infty,0}\delta\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau$$
$$+ \frac{1}{\delta}\int_0^t \sum_{\pm}\left|\left|G^1\right|\right|^2_{L^2(\Omega_{r\pm})} + \sum_{\pm}\left|\left|G^2\right|\right|^2_{L^2(\Omega_{r\pm})} + \left|\left|G^3\right|\right|^2_{L^2(\Gamma_r)} + \left|\left|G^4\right|\right|^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau$$

Choosing δ sufficiently small depending on $C_{\infty,0}$ we obtain

$$||v||^2_{B^0_r(t)} + 2\mu\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau \le C_{\infty,0}\int_0^t ||v||_{B^0_r(\tau)}\,\mathrm{d}\tau$$
$$+ ||v(0)||^2_{B^0_r} + C_{\infty,0}\int_0^t \sum_\pm ||G^1||^2_{L^2(\Omega_{r\pm})} + \sum_\pm ||G^2||^2_{L^2(\Omega_{r\pm})} + ||G^3||^2_{L^2(\Gamma_r)} + ||G^4||^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau$$

Applying Gronwall's lemma, we obtain

$$||v||^2_{B^0_r(t)} \le C_{\infty,0}(||v(0)||^2_{B^0_r}$$
$$+ \int_0^t \sum_\pm(||G^1||^2_{L^2(\Omega_{r\pm})} + ||G^2||^2_{L^2(\Omega_{r\pm})}) + ||G^3||^2_{L^2(\Gamma_r)} + ||G^4||^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau)$$

which proves (11.6.1). Returning to the previous inequality and taking the supremum over $\tau \in (0,t)$ which allows us to absorb the term $||v(0)||^2_{B^0_r}$ into the left hand side, we obtain

$$\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau \le C_{\infty,0}\int_0^t ||v||_{B^0_r(\tau)}\,\mathrm{d}\tau$$
$$+ C_{\infty,0}\int_0^t \sum_\pm ||G^1||^2_{L^2(\Omega_{r\pm})} + \sum_\pm ||G^2||^2_{L^2(\Omega_{r\pm})} + ||G^3||^2_{L^2(\Gamma_r)} + ||G^4||^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau$$

Using the estimate (11.6.1) for $||v||_{B^0_r(\tau)}$, we obtain (11.6.2). This proves the first part of the result.

We now assume the additional regularity as stated in the second part of the proposition. Since we have $\partial_t v \in W_r$ for almost every $t \in (0,T)$, we may set $w = \partial_t v$ in the equation. We obtain

$$||\partial_t v||^2_{B^0_r(t)} + \frac{1}{2}\mu\frac{\mathrm{d}}{\mathrm{d}t}\,|v|^2_{B^1_r(t)} - \frac{1}{2}\mu\sum_\pm(\partial_t A^1(v,\nabla v),(v,\nabla v))^2_{L^2(\Omega_{r\pm})} + \sum_\pm(A^2(v,\nabla v),\partial_t v)_{L^2(\Omega_{r\pm})}$$
$$= \sum_\pm(G^1,\partial_t v)_{L^2(\Omega_{r\pm})} + \sum_\pm \frac{\mathrm{d}}{\mathrm{d}t}(G^2,\nabla v)_{L^2(\Omega_{r\pm})} - (\partial_t G^2,\nabla v)_{L^2(\Omega_{r\pm})}$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}(G^3,v_3)_{L^2(\Gamma_r)} - (\partial_t G^3,v_3)_{L^2(\Gamma_r)} + \frac{\mathrm{d}}{\mathrm{d}t}(G^4,\nabla' v_3)_{L^2(\Gamma_r)} - (\partial_t G^4,\nabla' v_3)_{L^2(\Gamma_r)}$$

Integrating from 0 to t for $t \in (0,T)$ and using Cauchy's inequality, and again using the equivalence of norms as proved in (11.4.3), we obtain

$$\int_0^t ||\partial_t v||^2_{B^0_r(\tau)}\,\mathrm{d}\tau + \frac{1}{2}\mu\,|v|^2_{B^1_r(t)}$$
$$= \frac{1}{2}\mu\,|v(0)|^2_{B^1_r} + \frac{1}{2}\mu\sum_\pm\int_0^t(\partial_t A^1(v,\nabla v),(v,\nabla v))_{L^2(\Omega_{r\pm})}\mathrm{d}\tau - \sum_\pm\int_0^t(A^2(v,\nabla v),\partial_t v)_{L^2(\Omega_{r\pm})}\mathrm{d}\tau$$
$$+ \sum_\pm\int_0^t(G^1,\partial_t v)_{L^2(\Omega_{r\pm})}\mathrm{d}\tau + \sum_\pm(G^2,\nabla v)_{L^2(\Omega_{r\pm})} - \sum_\pm(G^2|_{t=0},\nabla v|_{t=0})_{L^2(\Omega_{r\pm})}$$
$$- \int_0^t(\partial_t G^2,\nabla v)_{L^2(\Omega_{r\pm})}\mathrm{d}\tau + (G^3,v_3)_{L^2(\Gamma_r)} - (G^3|_{t=0},v_3|_{t=0})_{L^2(\Gamma_r)} - \int_0^t(\partial_t G^3,v_3)_{L^2(\Gamma_r)}\mathrm{d}\tau$$
$$+ (G^4,\nabla' v_3)_{L^2(\Gamma_r)} - (G^4|_{t=0},\nabla' v_3|_{t=0})_{L^2(\Gamma_r)} - \int_0^t(\partial_t G^4,\nabla' v_3)_{L^2(\Gamma_r)}\mathrm{d}\tau$$
$$\le \frac{\mu}{2}\,||v(0)||^2_{B^1_r} + \frac{C_{\infty,1}}{\delta}\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau + \frac{C_{\infty,0}}{\epsilon}\int_0^t ||v||^2_{B^1_r(\tau)}\,\mathrm{d}\tau + \delta\int_0^t ||\partial_t v||^2_{B^0_r(\tau)}\,\mathrm{d}\tau + \delta\,||v||^2_{B^1_r(t)}$$
$$+ \frac{C_{\infty,0}}{\delta}(\int_0^t \sum_\pm ||G^1||^2_{L^2(\Omega_{r\pm})}\,\mathrm{d}\tau + \operatorname*{ess\,sup}_{\tau\in(0,t)}(\sum_\pm ||G^2||^2_{L^2(\Omega_{r\pm})} + ||G^3||^2_{L^2(\Gamma_r)} + ||G^4||^2_{L^2(\Gamma_r)}))$$
$$+ \epsilon\int_0^t \sum_\pm ||\partial_t G^2||^2_{L^2(\Omega_{r\pm})} + ||\partial_t G^3||^2_{L^2(\Gamma_r)} + ||\partial_t G^4||^2_{L^2(\Gamma_r)}\,\mathrm{d}\tau$$

Now we choose $\delta > 0$ sufficiently small and use the existing estimate for $||v||_{B_r^0(t)}$ and $\int_0^t ||v||^2_{B_r^1(\tau)} \,\mathrm{d}\tau$ to obtain

$$\begin{aligned}
&\int_0^t ||\partial_t v||^2_{B_r^0(t)} \,\mathrm{d}\tau + ||v||^2_{B_r^1(t)} \\
&\leq \frac{C_{\infty,1}}{\epsilon}\Big(\int_0^t ||v(0)||^2_{B_r^0} + \sum_{\pm} \big|\big|G^1\big|\big|^2_{L^2(\Omega_{r\pm})} + \sum_{\pm} \big|\big|G^2\big|\big|^2_{L^2(\Omega_{r\pm})} + \big|\big|G^3\big|\big|^2_{L^2(\Gamma_r)} + \big|\big|G^4\big|\big|^2_{L^2(\Gamma_r)} \,\mathrm{d}\tau\Big) \\
&+ C_{\infty,0}(||v(0)||^2_{B_r^1} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \big|\big|G^2\big|\big|^2_{L^2(\Omega_{r\pm})} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \big|\big|G^3\big|\big|^2_{L^2(\Gamma_r)} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \big|\big|G^4\big|\big|^2_{L^2(\Gamma_r)}) \\
&+ \epsilon \int_0^t \sum_{\pm} \big|\big|\partial_t G^2\big|\big|^2_{L^2(\Omega_{r\pm})} + \big|\big|\partial_t G^3\big|\big|^2_{L^2(\Gamma_r)} + \big|\big|\partial_t G^4\big|\big|^2_{L^2(\Gamma_r)} \,\mathrm{d}\tau
\end{aligned}$$

This completes the proof. □

Proposition 11.6.4. *Let* $1 \leq r < \infty$.

Let $G^1 \in L^2((0,T) \times \Omega_{r\pm})$ *be a vector,* $G^2 \in L^\infty((0,T); L^2(\Omega_{r\pm}))$ *with* $\partial_t G^2 \in L^2((0,T) \times \Omega_{r\pm})$ *be a matrix,* $G^3 \in L^\infty((0,T); L^2(\Gamma_r))$ *with* $\partial_t G^3 \in L^2((0,T) \times \Gamma_r)$ *be a scalar and* $G^4 \in L^\infty((0,T); L^2(\Gamma_r))$ *with* $\partial_t G^4 \in L^2((0,T) \times \Gamma_r)$ *be a 2-vector. Then there exists a unique* $v \in X_r$ *with*

$$v(0) = 0$$

such that

$$\begin{aligned}
&(\partial_t v, w)_{B_r^0(t)} + \mu\{v, w\}_{B_r^1(t)} + \sum_{\pm}(A^2(v, \nabla v), w)_{L^2(\Omega_{r\pm})} \\
&= \sum_{\pm}(G^1, w)_{L^2(\Omega_{r\pm})} + \sum_{\pm}(G^2, \nabla w)_{L^2(\Omega_{r\pm})} + (G^3, w_3)_{L^2(\Gamma_r)} + (G^4, \nabla' w_3)_{L^2(\Gamma_r)}
\end{aligned} \tag{11.6.4}$$

for all $w \in B_r^1$ *for almost every* $t \in (0,T)$*, and* v *satisfies the estimate* (11.6.3).

Proof. Firstly, we claim that for each integer $m \geq 1$, there exist functions $d_m^1, \ldots, d_m^m \in H^1(0,T)$ such that

$$v_m := \sum_{k=1}^m d_m^k(t) z_k \in H^1((0,T); B_r^1)$$

satisfies the equation (11.6.4) for all test functions $w \in W_r^m := \langle z_1, \ldots, z_m \rangle$, and also $v_m|_{t=0} = 0$, where $(z_k)_{k=1}^\infty$ is the Galerkin basis defined in 11.5.2. Explicitly,

$$v_m|_{t=0} = 0$$

and the following holds for almost every $t \in (0,T)$.

$$\begin{aligned}
&(\partial_t v_m, w)_{B_r^0(t)} + \mu\{v_m, w\}_{B_r^1(t)} + \sum_{\pm}(A^2(v_m, \nabla v_m), w)_{L^2(\Omega_{r\pm})} \\
&= \sum_{\pm}(G^1, w)_{L^2(\Omega_{r\pm})} + \sum_{\pm}(G^2, \nabla w)_{L^2(\Omega_{r\pm})} + (G^3, w_3)_{L^2(\Gamma_r)} + (G^4, \nabla' w_3)_{L^2(\Gamma_r)}
\end{aligned} \tag{11.6.5}$$

for all $w \in W_r^m := \langle z_1, \ldots, z_m \rangle$.

Indeed, we define $(d_m^1, \ldots, d_m^m)$ as the solution of the following system.

$$\sum_{k=1}^m \frac{\mathrm{d}d_m^k}{\mathrm{d}t}(t)(z_k, z_l)_{B_r^0(t)} + \sum_{k=1}^m d_m^k(t) \left(\mu\{z_k, z_l\}_{B_r^1(t)} + \sum_{\pm} (A^2(z_k, \nabla z_k), z_l)_{L^2(\Omega_{r\pm})} \right)$$
$$= \sum_{\pm} (G^1, z_l)_{L^2(\Omega_{r\pm})} + \sum_{\pm} (G^2, \nabla z_l)_{L^2(\Omega_{r\pm})} + (G^3, (z_l)_3)_{L^2(\Gamma_r)} + (G^4, \nabla'(z_l)_3)_{L^2(\Gamma_r)}$$

for all $1 \leq l \leq m$ with $d_m^k(0) = 0$ for all k. Note that we have shown in lemma 11.5.1 that the matrix $(M(t))_{kl} := (z_k, z_l)_{B_r^0(t)}$ multiplying the vector d_m is invertible with inverse in $W^{1,\infty}(0,T)$, hence we can write this as a standard first-order linear system of ODEs of the form

$$\frac{\mathrm{d}d_m}{\mathrm{d}t} = A(t)d_m + b(t)$$

where $A(t) \in W^{1,\infty}(0,T)$, $b(t) \in L^2(0,T)$. Hence, by standard theory, a solution $d_m = (d_m^1, \ldots, d_m^m)$ exists on the interval $(0,T)$ with $d_m^1, \ldots, d_m^m \in H^1(0,T)$.

We then define

$$v_m := \sum_{k=1}^m d_m^k(t) z_k$$

It is clear by the construction of d_m^k and linearity that v_m satisfies the equation (11.6.5) for all test functions $w \in \langle z_1, \ldots, z_m \rangle$, and also $v_m|_{t=0} = 0$.

This proves the claim. We also note that, if, in addition, we have $\partial_t G^1 \in L^2((0,T) \times \Omega_\pm)$, then $b(t) \in H^1(0,T)$ and hence $d_m^1, \ldots, d_m^m \in H^2(0,T)$ and $v_m \in H^2((0,T); B_r^1)$.

Note that since $v_m, \partial_t v_m \in W_r^m$, and given the regularity of the G^i, we see that v_m satisfies the estimate (11.6.3). Using this energy estimate, we can now prove the existence of a weak solution. Indeed, using weak-$*$ compactness, there exists a $v \in X$ and a subsequence v_{m_l} such that

$$v_{m_l} \overset{*}{\rightharpoonup} v \text{ in } L^\infty((0,T); B^1)$$
$$\partial_t v_{m_l} \overset{*}{\rightharpoonup} \partial_t v \text{ in } L^2((0,T); B^0)$$

We also have by weak-$*$ lower semi-continuity of norms that v satisfies the same estimate. We claim that v is a weak solution of the equation. Indeed, let $w \in C^1([0,T]; B_r^1)$ be a function of the form

$$w = \sum_{k=1}^M a^k(t) z_k$$

Pick L such that $m_l \geq M$ for all $l \geq L$. Then, by linearity, we have that the equation (11.6.5) holds for v_{m_l} with test function $w(t)$, for all $l \geq L$ and almost every t. We integrate from 0 to T, then let $l \to \infty$ and use the weak/weak-$*$ convergences above. We obtain that the time-integrated equation holds for v with test function w. Since $(z_k)_{k=1}^\infty$ are dense in B_r^1, we may approximate any test function $w \in L^2([0,T]; B_r^1)$ with functions of this form, thus obtaining that the time-integrated equation holds for v with any test function $w \in L^2([0,T]; B_r^1)$. In particular, we have that the the equation (11.6.4) holds for v for all $w \in B_r^1$ for almost every t. This proves that v is a weak solution of the equation.

It remains to show that $v(0) = 0$. We note that $\partial_t v \in L^2((0,T); B_r^0)$. Let $w_\pm \in C^1([0,T]; H_0^1(\Omega_\pm))$ with $w(T) = 0$. Then

$$\int_0^T (\partial_t v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (v(0), w(0))_{L^2(\Omega_\pm)}$$

But we also know that

$$\int_0^T (\partial_t v_{m_l}, w)_{L^2(\Omega_\pm)}\mathrm{d}t = -\int_0^T (v_{m_l}, \partial_t w)_{L^2(\Omega_\pm)}\mathrm{d}t - (v_{m_l}(0), w(0))_{L^2(\Omega_\pm)}$$
$$= -\int_0^T (v_{m_l}, \partial_t w)_{L^2(\Omega_\pm)}\mathrm{d}t$$

since $v_{m_l}(0) = 0$ for all l. Passing to the limit as $l \to \infty$ using weak convergence, we find

$$\int_0^T (\partial_t v, w)_{L^2(\Omega_\pm)}\mathrm{d}t = -\int_0^T (v, \partial_t w)_{L^2(\Omega_\pm)}\mathrm{d}t$$

Comparing this with the above equation, we find that

$$(v(0), w(0))_{L^2(\Omega_\pm)} = 0$$

for all such w, hence $v(0) = 0$. This completes the proof of the existence of a weak solution.

The proof of uniqueness is straightforward using the energy estimate. Suppose v^1 and v^2 are two weak solutions. We subtract the equation (11.6.4) for v_2 from that for v_1 and write $\tilde{v} = v_1 - v_2$ to obtain

$$(\partial_t \tilde{v}, w)_{B_r^0(t)} + \mu\{\tilde{v}, w\}_{B_r^1(t)} + \sum_\pm (A^2(\tilde{v}, \nabla\tilde{v}), w)_{L^2(\Omega_{r\pm})} = 0$$

for all $w \in B_r^1$ and almost every t. Applying the energy estimate (11.6.1) with $G^1 = G^2 = G^3 = G^4 = 0$ and $v(0) = 0$, we obtain

$$\operatorname*{ess\,sup}_{t\in(0,T)} ||\tilde{v}||_{B_r^0(t)} = 0$$

and hence $v_1 = v_2$ almost everywhere in $(0,T) \times \Omega_{r\pm}$ as required. $\square$

Proposition 11.6.5. *The previous proposition holds in the case $r = \infty$.*

Proof. Firstly, for each $1 \le r < \infty$, we let $\phi_r \in C_c^\infty(B_r(0))$ with $\phi_r = 1$ on $B_{\frac{r}{2}}(0)$ and $|\nabla\phi_r| \le C$ where C is independent of r.

We then define $G_r^1, G_r^2, G_r^3, G_r^4$ as follows.

$$G_r^1 = \phi_r G^1 + G^2\nabla\phi_r$$
$$G_r^2 = \phi_r G^2$$
$$G_r^3 = \phi_r G^3 + \nabla'\psi_r \cdot G^4$$
$$G_r^4 = \phi_r G^4$$

which implies that for all $w \in B^1$ we have

$$\sum_\pm (G_r^1, w)_{L^2(\Omega_{r\pm})} + \sum_\pm (G_r^2, \nabla w)_{L^2(\Omega_{r\pm})} + (G_r^3, w_3)_{L^2(\Gamma_r)} + (G_r^4, \nabla' w_3)_{L^2(\Gamma_r)}$$
$$= \sum_\pm (G^1, \phi_r w)_{L^2(\Omega_\pm)} + \sum_\pm (G^2, \nabla(\phi_r w))_{L^2(\Omega_\pm)} + (G^3, \phi_r w_3)_{L^2(\Gamma)} + (G^4, \nabla'(\phi_r w_3))_{L^2(\Gamma)}$$

We may apply the previous result to obtain a weak solution $v_r \in X_r$ to the equation

$$(\partial_t v_r, w)_{B_r^0(t)} + \mu\{v_r, w\}_{B_r^1(t)} + \sum_\pm (A^2(v_r, \nabla v_r), w)_{L^2(\Omega_{r\pm})}$$
$$= \sum_\pm (G_r^1, w)_{L^2(\Omega_{r\pm})} + \sum_\pm (G_r^2, \nabla w)_{L^2(\Omega_{r\pm})} + (G_r^3, w_3)_{L^2(\Gamma_r)} + (G_r^4, \nabla' w_3)_{L^2(\Gamma_r)}$$

for all $w \in B_r^1$ for almost every $t \in (0,T)$ with

$$v_r(0) = 0$$

We also have the energy estimate (11.6.3) for v_r. Now let us define $v_{r\pm}$ on the whole of $\Omega_\pm$ by extension by zero, using the fact that $v_{r\pm}|_{|x|=r} = 0$ to ensure $v_{r\pm}(t) \in B^1$ for almost every t. We thus have

$$\begin{aligned} v_r &\in L^\infty((0,T); B^1) \\ \partial_t v_r &\in L^2((0,T); B^0) \end{aligned}$$

with the same energy estimate. Note the right-hand side of this estimate is independent of r. Then we may use weak-$*$ compactness to pass to a limit along a subsequence r_l as $l \to \infty$. We obtain a function $v \in L^\infty((0,T); B^1)$ with $\partial_t v \in L^2((0,T); B^0)$ such that

$$\begin{aligned} v_{r_l} &\overset{*}{\rightharpoonup} v \text{ in } L^\infty((0,T); B^1) \\ \partial_t v_{r_l} &\overset{*}{\rightharpoonup} \partial_t v \text{ in } L^2((0,T); B^0) \end{aligned}$$

and by weak-$*$ lower semi-continuity of norms the same estimate holds for v.

Now let $w \in L^2((0,T); B^1)$ with compact support in $B_s(0)$ for almost every t. We see that if $r > 2s$ then, since $\phi_r = 1$ on $B_{\frac{r}{2}}(0)$, we have

$$\begin{aligned} &\int_0^T (\partial_t v_r, w)_{B^0(t)} + \mu\{v_r, w\}_{B^1(t)} + \sum_\pm (A^2(v_r, \nabla v_r), w)_{L^2(\Omega_\pm)} \mathrm{d}t \\ &= \int_0^T \sum_\pm (G^1, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^2, \nabla w)_{L^2(\Omega_\pm)} + (G^3, w_3)_{L^2(\Gamma)} \mathrm{d}t + (G^4, \nabla' w_3)_{L^2(\Gamma)} \mathrm{d}t \end{aligned}$$

Using the above weak-$*$ convergence and passing to the limit along the subsequence r_l, we obtain

$$\begin{aligned} &(\partial_t v, w(t))_{B^0(t)} + \mu\{v, w\}_{B^1(t)} + \sum_\pm (A^2(v, \nabla v), w(t))_{L^2(\Omega_\pm)} \\ &= \sum_\pm (G^1, w(t))_{L^2(\Omega_\pm)} + \sum_\pm (G^2, \nabla w(t))_{L^2(\Omega_\pm)} + (G^3, w_3(t))_{L^2(\Gamma)} + (G^4, \nabla' w_3(t))_{L^2(\Gamma)} \end{aligned}$$

for almost every $t \in (0,T)$. Now given $w \in B^1$ with compact support in $B_s(0)$ for each such t we may pick $\tilde{w} \in L^2((0,T); B^1)$ with $w(t) = w$ and hence the above holds for all $w \in B^1$ with compact support in $B_s(0)$ for almost every t. But since such w are dense in B^1, the above equation is satisfied for all $w \in B^1$.

Now we show that $v(0) = 0$. We note that $\partial_t v \in L^2((0,T); B^0)$. Let $w_\pm \in C^1([0,T]; H_0^1(\Omega_\pm))$ with $w(T) = 0$. Suppose also $w(t)$ has compact support in $B_s(0)$ for all t and let $r > 2s$. We have

$$\int_0^T (\partial_t v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (v(0), w(0))_{L^2(\Omega_\pm)}$$

But we also know that

$$\begin{aligned} \int_0^T (\partial_t v_r, w)_{L^2(\Omega_\pm)} \mathrm{d}t &= -\int_0^T (v_r, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (v_r(0), w(0))_{L^2(\Omega_\pm)} \\ &= -\int_0^T (v_r, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t \end{aligned}$$

since $v_r(0) = 0$ for all r. Passing to the limit along the subsequence r_l as $l \to \infty$ using weak convergence, we find

$$\int_0^T (\partial_t v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t$$

Comparing this with the above equation, we find that

$$(v(0), w(0))_{L^2(\Omega_\pm)} = 0$$

for all such w, hence $v(0) = 0$ by density of compactly supported functions in L^2.

As in the previous proposition, uniqueness of the solution follows from applying the energy estimate (11.6.1) (which holds even in the case $r = \infty$) to the difference of two solutions v^1 and v^2.

This completes the proof. □

Proposition 11.6.6. *Let $1 \le r < \infty$. Suppose the assumptions of proposition 11.6.4 hold, and let $v \in X_r$ be the unique weak solution given by this proposition.*

Suppose in addition that $G^1 \in L^\infty((0,T); L^2(\Omega_{r\pm}))$, $\partial_t G^1 \in L^2((0,T) \times \Omega_{r\pm})$ and that $G^2|_{t=0} \in H^1(\Omega_{r\pm})$. Let us write

$$\sum_\pm (G^1|_{t=0}, w)_{L^2(\Omega_{r\pm})} - \sum_\pm (\nabla \cdot G^2|_{t=0}, w)_{L^2(\Omega_{r\pm})} = (h, w)_{B_r^0}$$

so that $h \in B_r^0$ is given by

$$h = \frac{|\underset{\sim}{n}^0|^2}{\rho_\mu^0}(G^1|_{t=0} - \nabla \cdot G^2|_{t=0})$$

and suppose that

$$-\sum_\pm (\pm)(G_3^2|_{t=0}, w)_{L^2(\Gamma_r)} + (G^3|_{t=0}, w_3)_{L^2(\Gamma_r)} + (G^4|_{t=0}, \nabla' w_3)_{L^2(\Gamma_r)} = 0$$

for all $w \in B_r^1$.

Then in fact we have

$$\partial_t v \in L^\infty((0,T); B_r^0)$$
$$\partial_t v \in L^2((0,T); B_r^1)$$

with the estimate

$$\operatorname*{ess\,sup}_{\tau \in (0,t)} ||\partial_t v||_{B_r^0(\tau)} + \int_0^t ||\partial_t v||_{B_r^1(\tau)} \,\mathrm{d}\tau \le C_{\infty,1} C_G(t)$$

where $C_G(t)$ denotes a constant depending linearly on the appropriate norms of G^i.

Finally, suppose in addition that we have $\partial_t G^2 \in L^\infty((0,T); L^2(\Omega_{r\pm}))$, $\partial_t^2 G^2 \in L^2((0,T) \times \Omega_{r\pm})$, $\partial_t G^3, \partial_t G^4 \in L^\infty((0,T); L^2(\Gamma_r))$, $\partial_t^2 G^3, \partial_t^2 G^4 \in L^2((0,T) \times \Gamma_r)$. Suppose also that $h \in B_r^1$.

Then in fact we have

$$\partial_t v \in L^\infty((0,T); B_r^1)$$
$$\partial_t^2 v \in L^2((0,T); B_r^0)$$
$$\partial_t v|_{t=0} = h$$

with the estimate

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial_t v||_{B_r^1(\tau)} + \int_0^t \left|\left|\partial_t^2 v\right|\right|_{B_r^0(\tau)} \mathrm{d}\tau \leq C_A(t)(C_G(t) + ||h||_{B_r^1}^2)$$

where $C_G(t)$ denotes a constant depending linearly on the appropriate norms of G^i. Moreover, $\partial_t v$ satisfies the equation

$$\begin{aligned}
&(\partial_t^2 v, w)_{B_r^0(t)} + \mu\{\partial_t v, w\}_{B_r^1(t)} + \sum_{\pm}(A^2(\partial_t v, \nabla\partial_t v), w)_{L^2(\Omega_{r\pm})} \\
&= \sum_{\pm}(G^{1,1}, w)_{L^2(\Omega_{r\pm})} + \sum_{\pm}(G^{2,1}, \nabla w)_{L^2(\Omega_{r\pm})} + (G^{3,1}, w_3)_{L^2(\Gamma_r)} + (G^{4,1}, \nabla' w_3)_{L^2(\Gamma_r)}
\end{aligned}$$

for all $w \in B_r^1$ for almost every $t \in (0,T)$, where we have defined

$$\begin{aligned}
G^{1,1} &:= \partial_t G^1 - \partial_t A^0 \partial_t v - \mu\partial_t A^1(v, \nabla v) - \partial_t A^2(v, \nabla v) \\
G^{2,1} &:= \partial_t G^3 - \mu\partial_t A^1(v, \nabla v) \\
G^{3,1} &: = \partial_t G^3 \\
G^{4,1} &: = \partial_t G^4
\end{aligned} \tag{11.6.6}$$

Note that in the first two lines above $\partial_t A^1(v, \nabla v)$ should be understood to mean the appropriate part or row of the full tensor $\partial_t A^1(v, \nabla v)$.

Proof. We let v_m be the same sequence of approximate solutions as in 11.6.4.

We evaluate the approximate equation (11.6.5) at time $t = 0$ which is possible because of the given regularity. We use the fact that $v_m(0) = 0$ to see that the second and third terms on the left hand side disappear, the we integrate by parts in the term involving G^2, using the fact that $w|_{|x|=r} = 0$, to obtain the following.

$$\begin{aligned}
(\partial_t v_m|_{t=0}, w)_{B_r^0} &= \sum_{\pm}(G^1|_{t=0}, w)_{L^2(\Omega_{r\pm})} - \sum_{\pm}(\nabla\cdot G^2|_{t=0}, w)_{L^2(\Omega_{r\pm})} \\
&\quad - \sum_{\pm}(\pm)(G_3^2|_{t=0}, w)_{L^2(\Gamma_r)} + (G^3|_{t=0}, w_3)_{L^2(\Gamma_r)} + (G^4|_{t=0}, \nabla' w_3)_{L^2(\Gamma_r)}
\end{aligned}$$

By assumption, the boundary terms cancel and we are left with

$$(\partial_t v_m|_{t=0}, w)_{B_r^0} = (h, w)_{B_r^0}$$

for all $w \in W_r^m$. In particular, setting $w = \partial_t v_m|_{t=0}$, we obtain

$$||\partial_t v_m|_{t=0}||_{B_r^0} \leq ||h||_{B_r^0}$$

Note that since $\partial_t G^1 \in L^2((0,T) \times \Omega_{r\pm})$, we know from the construction of the Galerkin basis in 11.6.4 that $v_m \in H^2((0,T); B_r^1)$. This allows us to differentiate the equation (11.6.5) with respect to t. We obtain

$$\begin{aligned}
&(\partial_t^2 v_m, w)_{B_r^0(t)} + \mu\{\partial_t v_m, w\}_{B_r^1(t)} + \sum_{\pm}(A^2(\partial_t v_m, \nabla\partial_t v_m), w)_{L^2(\Omega_{r\pm})} \\
&= \sum_{\pm}(G^{1,1,m}, w)_{L^2(\Omega_{r\pm})} + \sum_{\pm}(G^{2,1,m}, \nabla w)_{L^2(\Omega_{r\pm})} + (G^{3,1,m}, w_3)_{L^2(\Gamma_r)} + (G^{4,1,m}, \nabla' w_3)_{L^2(\Gamma_r)}
\end{aligned}$$

where $G^{i,1,m}$ are as defined in (11.6.6), with v_m replacing v. Note that given the regularity of $\partial_t G^i$ for $1 \leq i \leq 4$ and the regularity of v_m and $\partial_t v_m$ already obtained, we have $G^{1,1,m} \in L^2((0,T) \times \Omega_{r\pm})$, $G^{2,1,m} \in L^2((0,T) \times \Omega_{r\pm})$, $G^{3,1,m}, G^{4,1,m} \in L^2((0,T) \times \Gamma_r)$ with corresponding norms bounded independently of m. We also know that $\partial_t v_m \in W_r^m$ for each t, hence we may apply the energy estimates (11.6.1)-(11.6.2) to $\partial_t v_m$ to obtain that $\partial_t v_m$ satisfies the estimates (11.6.1)- (11.6.2) with G^i replaced by $G^{i,1,m}$ for $1 \leq i \leq 4$ and $v(0)$ replaced by h, which gives

$$\operatorname*{ess\,sup}_{\tau \in (0,t)} ||\partial_t v_m||^2_{B_r^0(\tau)} + \int_0^t ||\partial_t v_m||^2_{B_r^1(\tau)} \, \mathrm{d}\tau \leq C_{\infty,1} C_G(t)$$

where we use $C_G(t)$ to denote a constant depending linearly on the norms of G^i. Now we may pass to the limit $m \to \infty$ along (a subsequence of) the subsequence m_l and use weak-$*$ compactness to obtain $\partial_t v \in L^\infty((0,T); B_r^0)$, $\partial_t v \in L^2((0,T); B_r^1)$ with v satisfying the same estimate by weak-$*$ lower semi-continuity of norms. This proves the second part of the proposition.

We now move on to the second part. Given the additional regularity of G^i and the estimates proved for v_m and $\partial_t v_m$ above, we may conclude that $G^{2,1,m} \in L^\infty((0,T); L^2(\Omega_{r\pm}))$, $\partial_t G^{2,1,m} \in L^2((0,T) \times \Omega_{r\pm})$, $G^{3,1,m}, G^{4,1,m} \in L^\infty((0,T); L^2(\Gamma_r))$, $\partial_t G^{3,1,m}, \partial_t G^{4,1,m} \in L^2((0,T) \times \Gamma_r)$ with corresponding norms bounded independent of m. We also know that $\partial_t^2 v_m \in L^2((0,T); W_r^m)$. Note that, using the properties of our Galerkin basis, we have

$$0 = (\partial_t v_m|_{t=0} - h, z_k)_{B_r^0} = \lambda_k^{-1} (\partial_t v_m|_{t=0} - h, z_k)_{B_r^1}$$

for all $1 \leq k \leq m$ (note here we need to use the fact that $h \in B_r^1$) and hence

$$(\partial_t v_m|_{t=0}, w)_{B_r^1} = (h, w)_{B_r^1}$$

for all $w \in W_r^m$, so in particular, setting $w = \partial_t v_m|_{t=0}$, we obtain

$$||\partial_t v_m|_{t=0}||_{B_r^1} \leq ||h||_{B_r^1}$$

Thus we may apply the energy estimate 11.6.3 to $\partial_t v_m$ to obtain

$$\operatorname*{ess\,sup}_{\tau \in (0,t)} ||\partial_t v_m||^2_{B_r^1(\tau)} + \int_0^t \left|\left|\partial_t^2 v_m\right|\right|^2_{B_r^0(\tau)} \, \mathrm{d}\tau \leq C_A(t)(C_G(t) + ||h||^2_{B_r^1})$$

where we use $C_G(t)$ to denote a constant depending linearly on the norms of G^i. Now we may pass to the limit $m \to \infty$ along (a subsequence of) the subsequence m_l and use weak-$*$ compactness to obtain $\partial_t v \in L^\infty((0,T); B_r^1)$, $\partial_t^2 v \in L^2((0,T); B_r^0)$ with

$$\operatorname*{ess\,sup}_{t \in (0,T)} ||\partial_t v||^2_{B_r^1(t)} + \int_0^T \left|\left|\partial_t^2 v\right|\right|^2_{B_r^0(t)} \, \mathrm{d}t \leq C_A(t)(C_G(t) + ||h||^2_{B_r^1})$$

by weak-$*$ lower semi-continuity of norms.

Finally, we show that

$$\partial_t v|_{t=0} = h$$

Indeed, we note that $\partial_t^2 v \in L^2((0,T);B_r^0)$. Let $w_\pm \in C^1([0,T];H_0^1(\Omega_\pm))$ with $w(T)=0$. In fact let us assume that w is of the form

$$w = \sum_{k=1}^{M} a^k(t) z_k$$

for some $a^k \in C^1([0,T])$. Then

$$\int_0^T (\partial_t^2 v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (\partial_t v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (\partial_t v(0), w(0))_{L^2(\Omega_\pm)}$$

But we also know that for $m \geq M$,

$$\begin{aligned}\int_0^T (\partial_t^2 v_m, w)_{L^2(\Omega_\pm)} \mathrm{d}t &= -\int_0^T (\partial_t v_m, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (\partial_t v_m(0), w(0))_{L^2(\Omega_\pm)} \\ &= -\int_0^T (\partial_t v_m, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (h, w(0))_{L^2(\Omega_\pm)}\end{aligned}$$

since

$$(\partial_t v_m|_{t=0}, w)_{B_r^0} = (h, w)_{B_r^0}$$

for all $w \in W_r^M$. Passing to the limit along a subsequence of m_l as $m \to \infty$ using weak convergence, we find

$$\int_0^T (\partial_t^2 v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (\partial_t v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (h, w(0))_{L^2(\Omega_\pm)}$$

Comparing this with the above equation, we find that

$$(\partial_t v(0), w(0))_{L^2(\Omega_\pm)} = (h, w(0))_{L^2(\Omega_\pm)}$$

for all such w. But by density of functions of the form $\sum_{k=1}^K b^k z_k$ in B_r^0 we obtain that

$$(\partial_t v(0), w)_{L^2(\Omega_\pm)} = (h, w)_{L^2(\Omega_\pm)}$$

for all $w \in L^2(\Omega_\pm)$, which implies

$$\partial_t v|_{t=0} = h$$

This completes the proof of the proposition. □

Proposition 11.6.7. *The previous proposition holds in the case $r = \infty$.*

Proof. We define ϕ_r and G_r^i as in 11.6.5, and as before we extend the weak solution $v_r \in X_r$ of the equation 11.6.4 by zero to the whole of $\Omega_\pm$.

Now let us add the assumptions corresponding to the first part of the previous proposition. We note that

$$\begin{aligned}&-\sum_\pm (\pm)(((G_r^2)_3|_{t=0}, w)_{L^2(\Gamma_r)} + (G_r^3|_{t=0}, w_3)_{L^2(\Gamma_r)} + (G_r^4|_{t=0}, \nabla' w_3)_{L^2(\Gamma_r)} \\ &= -\sum_\pm (\pm)(G_3^2|_{t=0}, \phi_r w)_{L^2(\Gamma)} + (G^3|_{t=0} - \nabla' \cdot G^4|_{t=0}, \phi_r w_3)_{L^2(\Gamma)} + (G^4|_{t=0}, \phi_r \nabla' w_3)_{L^2(\Gamma)} \\ &= -\sum_\pm (\pm)(G_3^2|_{t=0}, \phi_r w)_{L^2(\Gamma)} + (G^3|_{t=0}, \phi_r w_3)_{L^2(\Gamma)} + (G^4|_{t=0}, \nabla'(\phi_r w_3))_{L^2(\Gamma)} \\ &= 0\end{aligned}$$

Also,

$$
\begin{aligned}
&\sum_{\pm}(G_r^1|_{t=0}, w)_{L^2(\Omega_{r\pm})} - \sum_{\pm}(\nabla\cdot G_r^2|_{t=0}, w)_{L^2(\Omega_{r\pm})} \\
&= \sum_{\pm}(\phi_r(G^1|_{t=0} - \nabla\cdot G^2|_{t=0}), w)_{L^2(\Omega_{r\pm})} - \sum_{\pm}((\nabla\phi_r)^T G^2|_{t=0}, w)_{L^2(\Omega_{r\pm})} \\
&+ \sum_{\pm}((\nabla\phi_r)^T G^2|_{t=0}, w)_{L^2(\Omega_{r\pm})} \\
&= (\phi_r h, w)_{B_r^0} \\
&=: (h_r, w)_{B_r^0}
\end{aligned}
$$

where

$$h_r := \phi_r h$$

Thus we may apply the first part of the previous proposition to obtain

$$
\begin{aligned}
\partial_t v_r &\in L^\infty((0,T); B_r^0) \\
\partial_t v_r &\in L^2((0,T); B_r^1)
\end{aligned}
$$

with the estimates

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial_t v_r||^2_{B_r^0(\tau)} + \int_0^t ||\partial_t v_r||^2_{B_r^1(t)}\,\mathrm{d}\tau \le C_{\infty,1}C_G(t)$$

where the constant $C_{\infty,1}C_G(t)$ is independent of r. We thus have, after extension by zero,

$$
\begin{aligned}
\partial_t v_r &\in L^\infty((0,T); B^0) \\
\partial_t v_r &\in L^2((0,T); B^1)
\end{aligned}
$$

with the estimate

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial_t v_r||^2_{B^0(\tau)} + \int_0^t ||\partial_t v_r||^2_{B^1(\tau)}\,\mathrm{d}\tau \le C_{\infty,1}C_G(t)$$

where the constant $C_{\infty,1}C_G(t)$ is independent of r. Then we may use weak-$*$ compactness to pass to a limit along a subsequence r_l as $l\to\infty$ and by uniqueness of weak-$*$ limits we know that v_{r_l} converges to v. We obtain that $\partial_t v \in L^\infty((0,T); B^0)$ with $\partial_t v \in L^2((0,T); B^1)$ and that the same estimate holds for v.

Now let us add the assumptions corresponding to the second part of the previous proposition. We see from the definition of h_r that $h_r \in B_r^1$ with

$$||h_r||_{B_r^1} \le C\,||h||_{B^1}$$

Thus we may apply the second part of the previous proposition to obtain

$$
\begin{aligned}
\partial_t v_r &\in L^\infty((0,T); B_r^1) \\
\partial_t^2 v_r &\in L^2((0,T); B_r^0) \\
\partial_t v_r|_{t=0} &= h_r
\end{aligned}
$$

with the estimate

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial_t v_r||^2_{B^1_r(\tau)} + \int_0^t \left|\left|\partial_t^2 v_r\right|\right|^2_{B^0_r(\tau)} \mathrm{d}\tau \le C_A(t)(C_G(t) + ||h||^2_{B^1})$$

where the constants $C_A(t)$ and $C_G(t)$ are independent of r. We thus have, after extension by zero,

$$\partial_t v_r \in L^\infty((0,T); B^1)$$
$$\partial_t^2 v_r \in L^2((0,T); B^0)$$

with the estimate

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial_t v_r||^2_{B^1} + \int_0^t \left|\left|\partial_t^2 v_r\right|\right|^2_{B^0} \mathrm{d}\tau \le C_A(t)(C_G(t) + ||h||^2_{B^1})$$

Then we may use weak-$*$ compactness to pass to a limit along a subsequence r_l as $l \to \infty$ and by uniqueness of weak-$*$ limits we know that v_{r_l} converges to v. We obtain that $\partial_t v \in L^\infty((0,T); B^1)$ and $\partial_t^2 v_r \in L^2((0,T); B^0)$ with the estimate as above.

We now show that

$$\partial_t v|_{t=0} = h$$

Indeed, we note that $\partial_t^2 v \in L^2((0,T); B^0)$. Let $w_\pm \in C^1([0,T]; H^1_0(\Omega_\pm))$ with $w(T) = 0$. Suppose also $w(t)$ has compact support in $B_s(0)$ for all t and let $r > 2s$. We have

$$\int_0^T (\partial_t^2 v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (\partial_t v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (\partial_t v(0), w(0))_{L^2(\Omega_\pm)}$$

But we also know that

$$\begin{aligned}\int_0^T (\partial_t^2 v_r, w)_{L^2(\Omega_\pm)} \mathrm{d}t &= -\int_0^T (\partial_t v_r, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (\partial_t v_r(0), w(0))_{L^2(\Omega_\pm)} \\ &= -\int_0^T (\partial_t v_r, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (h, w(0))_{L^2(\Omega_\pm)}\end{aligned}$$

since

$$(\partial_t v_r|_{t=0}, w)_{B^0} = (h, w)_{B^0}$$

for all such w (because $\phi = 0$ on $B_{\frac{r}{2}}(0)$ and so $h_r = h$). Passing to the limit along the subsequence r_l as $l \to \infty$ using weak convergence, we find

$$\int_0^T (\partial_t^2 v, w)_{L^2(\Omega_\pm)} \mathrm{d}t = -\int_0^T (\partial_t v, \partial_t w)_{L^2(\Omega_\pm)} \mathrm{d}t - (h, w(0))_{L^2(\Omega_\pm)}$$

Comparing this with the above equation, we find that

$$(\partial_t v(0), w(0))_{L^2(\Omega_\pm)} = (h, w(0))_{L^2(\Omega_\pm)}$$

for all such w. But by density of H^1_0 functions in B^0 we obtain that

$$(\partial_t v(0), w)_{L^2(\Omega_\pm)} = (h, w)_{L^2(\Omega_\pm)}$$

for all $w \in L^2(\Omega_\pm)$, which implies

$$\partial_t v|_{t=0} = h$$

Finally, we may use the regularity of v to differentiate the equation for v with respect to time to obtain that $\partial_t v$ satisfies the equation

$$\begin{aligned}
&(\partial_t v, w)_{B^0(t)} + \mu\{v, w\}_{B^1(t)} + \sum_\pm (A^2(v, \nabla v), w)_{L^2(\Omega_\pm)} \\
&= \sum_\pm (G^{1,1}, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^{2,1}, \nabla w)_{L^2(\Omega_\pm)} + (G^{3,1}, w_3)_{L^2(\Gamma)} + (G^{4,1}, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}$$

for all $w \in B^1$ for almost every $t \in (0, T)$, where

$$\begin{aligned}
G^{1,1} &:= \partial_t G^1 - \partial_t A^0 \partial_t v - \mu \partial_t A^1(v, \nabla v) - \partial_t A^2(v, \nabla v) \\
G^{2,1} &:= \partial_t G^3 - \mu \partial_t A^1(v, \nabla v) \\
G^{3,1} &:= \partial_t G^3 \\
G^{4,1} &:= \partial_t G^4
\end{aligned}$$

This completes the proof. □

Definition 11.6.8. Let $w \in L^2(\Omega_\pm)$ or $w \in L^2(\Gamma)$. For $h \in \mathbb{R}$, $h \neq 0$ and $1 \leq i \leq 2$, we define the translation in direction i and difference quotient in direction i respectively as

$$\begin{aligned}
w^h(x) &:= w(x + he_i) \\
D_i^h w(x) &:= \frac{w(x + he_i) - w(x)}{h}
\end{aligned}$$

for $x \in \Omega_\pm$ or $x \in \Gamma$, where e_i denotes the i-th coordinate vector. For $w \in L^2(\Omega_\pm)$ and $i = 3$ we define

$$\begin{aligned}
w^h(x) &:= w(x + he_i) \\
D_i^h w(x) &:= \frac{w(x + he_i) - w(x)}{h}
\end{aligned}$$

for $x \in \Omega_\pm$ and $|h| \neq 0$ sufficiently small such that $x + he_i \in \Omega_\pm$ (where we mean Ω_+ if $x \in \Omega_+$ and Ω_- if $x \in \Omega_-$).

To deal with the case $i = 3$ efficiently, we set $\zeta_{1\pm} \equiv \zeta_{2\pm} \equiv 1$ and let $\zeta_{3\pm} \in C_c^\infty(\Omega_\pm)$.

We recall the following standard properties of difference quotients.

$$\begin{aligned}
\zeta_i D_i^h(wz) &= \zeta_i (w^h D_i^h z + z D_i^h w) \\
\int_{\Omega_\pm} w D_i^{-h}(\zeta_i z) \mathrm{d}x &= -\int_{\Omega_\pm} (D_i^h w) \zeta_i z \mathrm{d}x \\
\int_\Gamma w D_i^{-h} z \mathrm{d}x' &= -\int_\Gamma (D_i^h w) z \mathrm{d}x' \text{ for } 1 \leq i \leq 2
\end{aligned}$$

assuming $|h|$ is sufficiently small.

We also recall the following standard estimate. Let $1 \leq p \leq \infty$ and suppose $w \in W^{1,p}(\Omega_\pm)$ or $w \in W^{1,p}(\Gamma)$. Then

$$\begin{aligned}
\left|\left|\zeta_i D_i^h w\right|\right|_{L^p(\Omega_\pm)} &\leq C \left|\left|\partial_{x_i} w\right|\right|_{L^p(\Omega_\pm)} \\
\left|\left|D_i^h w\right|\right|_{L^p(\Gamma)} &\leq C \left|\left|\partial_{x_i} w\right|\right|_{L^p(\Gamma)} \text{ for } 1 \leq i \leq 2
\end{aligned}$$

Moreover, suppose we know that $w \in L^q((0,T); L^p(\Omega_\pm))$ or $w \in L^q((0,T); L^p(\Gamma))$ for $1 < p, q \leq \infty$ and there exists a constant C such that

$$\left|\left|\zeta_i D_i^h w\right|\right|_{L^q((0,T);L^p(\Omega_\pm))} \leq C$$

or

$$\left|\left|D_i^h w\right|\right|_{L^q((0,T);L^p(\Gamma))} \leq C \text{ for } 1 \leq i \leq 2$$

Then we have, for $1 \leq i \leq 2$, $\partial_{x_i} w \in L^q((0,T); L^p(\Omega_\pm))$ or $\partial_{x_i} w \in L^q((0,T); L^p(\Gamma))$ with

$$||\partial_{x_i} w||_{L^q((0,T);L^p(\Omega_\pm))} \leq C$$

or

$$||\partial_{x_i} w||_{L^q((0,T);L^p(\Gamma))} \leq C$$

Suppose that $\zeta_3 \equiv 1$ on a compact set containing $\Omega_\pm^\epsilon \subset\subset \Omega_\pm$. Then $\partial_{x_3} w \in L^q((0,T); L^p(\Omega_\pm^\epsilon))$ and

$$||\partial_{x_3} w||_{L^q((0,T);L^p(\Omega_\pm^\epsilon))} \leq C_\epsilon$$

See eg Evans [15] for a proof of all these properties.

Proposition 11.6.9. *Suppose $v \in X$ satisfies*

$$\begin{aligned}
&(\partial_t v, w)_{B^0(t)} + \mu\{v, w\}_{B^1(t)} + \sum_\pm (A^2(v, \nabla v), w)_{L^2(\Omega_\pm)} \\
&= \sum_\pm (G^1, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^2, \nabla w)_{L^2(\Omega_\pm)} + (G^3, w_3)_{L^2(\Gamma)} + (G^4, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}$$

for all $w \in B^1$ for almost every $t \in (0,T)$, where G^1, G^2, G^3, G^4 have the following regularity.

$$\begin{aligned}
G^1 &\in L^2((0,T); L^2(\Omega_\pm)) \\
G^2 &\in L^2((0,T); H^1(\Omega_\pm)) \\
G^3 &\in L^2((0,T); L^2(\Gamma)) \\
G^4 &\in L^2((0,T); H^1(\Gamma))
\end{aligned}$$

Assume also

$$v(0) \in B^1$$

Then in fact

$$\begin{aligned}
v &\in L^2((0,T); H^2(\Omega_\pm)) \\
v_3 &\in L^2((0,T); H^2(\Gamma))
\end{aligned}$$

with corresponding norms bounded by $C_A(t)(C_G(t) + ||v(0)||_{B^1})$. *In particular, the following estimate holds for any* $\epsilon \in (0, 1]$.

$$\left|\left|\nabla' v\right|\right|^2_{B^0(t)} + 2\mu \int_0^t \left|\left|\nabla' v\right|\right|^2_{B^1(\tau)} \mathrm{d}\tau \leq C \left|\left|v(0)\right|\right|^2_{B^1(t)}$$
$$+ C_{\infty,1} \int_0^t \sum_{\pm} \left|\left|G^1\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{\pm} \left|\left|G^2\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|G^3\right|\right|^2_{L^2(\Gamma)} + \left|\left|G^4\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau$$
$$+ \frac{C_{\infty,1}}{\epsilon} \int_0^t ||v||^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \epsilon \int_0^t ||\partial_t v||^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \tag{11.6.7}$$

In addition, the following equations are satisfied.

$$\rho_\mu \frac{1}{|\underset{\sim}{n}|^2} \partial_t v - \mu N^T \nabla^\psi (\rho c^2 \nabla^\psi \cdot (Nv)) + \mu N^T \nabla^\psi \times (\rho c^2 \nabla^\psi \times (Nv)) + \rho_\mu N^T (\overline{u^\psi} \cdot \nabla)(Nv)$$
$$+ \rho_\mu N^T (\partial_t N) v = G^1 - \nabla \cdot G^2$$

almost everywhere in $(0,T) \times \Omega_\pm$ *and*

$$\mu(\Delta' - 1)v_3 + \mu[\rho c^2 \nabla^\psi \cdot (Nv)] = -[G^2_{33}] + G^3 - \nabla' \cdot G^4$$
$$\mu N^T (\rho_\pm c_\pm^2 n \times (\nabla^\psi \times (N v_\pm))) = -(G^2_{13\pm}, G^2_{23\pm}, 0)$$

almost everywhere on $(0,T) \times \Gamma$.

Moreover, suppose for some $1 \leq k \leq 3$ *that*

$$G^1 \in L^2((0,T); H^k(\Omega_\pm))$$
$$G^2 \in L^2((0,T); H^{k+1}(\Omega_\pm))$$
$$G^3 \in L^2((0,T); H^k(\Gamma))$$
$$G^4 \in L^2((0,T); H^{k+1}(\Gamma))$$
$$v(0) \in H^{k+1}(\Omega_\pm)$$
$$v_3(0) \in H^{k+1}(\Gamma)$$

and

$$\partial_t v \in L^2((0,T); H^k(\Omega_\pm)) \cap L^\infty((0,T); H^{k-1}(\Omega_\pm))$$

Then $v \in L^2((0,T); H^{k+1}(\Omega_\pm)) \cap L^\infty((0,T); H^k(\Omega_\pm))$, $v_3 \in L^2((0,T); H^{k+1}(\Gamma))$ *with corresponding norms bounded by* $C_A(t)(C_G(t) + ||v(0)||_{H^{k+1}(\Omega_\pm)} + ||v_3(0)||_{H^{k+1}(\Gamma)} + \int_0^T ||\partial_t v||_{H^k(\Omega_\pm)} \mathrm{d}t)$.

Proof. We let

$$w = -D_i^{-h}(\zeta_i^2 D_i^h v)$$

for $|h| \neq 0$ sufficiently small and $1 \leq i \leq 3$. Applying the integration by parts formula for difference quotients we obtain

$$(\partial_t \zeta_i D_i^h v, \zeta_i D_i^h v)_{B^0(t)} + \mu\{\zeta_i D_i^h v, \zeta_i D_i^h v\}_{B^1(t)} + \sum_{\pm} (A^2(\zeta_i D_i^h v, \nabla(\zeta_i D_i^h v)), \zeta_i D_i^h v)_{L^2(\Omega_\pm)}$$
$$= -\sum_{\pm} (G^1, D_i^{-h}(\zeta_i^2 D_i^h v))_{L^2(\Omega_\pm)} + \sum_{\pm} (\zeta_i D_i^h G^2, \nabla \zeta_i D_i^h v)_{L^2(\Omega_\pm)} - (G^3, (D_i^{-h}(\zeta_i^2 D_i^h v_3))_{L^2(\Gamma)}$$
$$+ (\zeta_i D_i^h G^4, \nabla'(\zeta_i D_i^h v_3))_{L^2(\Gamma)} - \sum_{\pm} ((\zeta_i D_i^h A^0) \partial_t v^h, \zeta_i D_i^h v)_{L^2(\Omega_\pm)}$$
$$- \mu \sum_{\pm} ((\zeta_i D_i^h A^1)(v^h, \nabla v^h), \zeta_i D_i^h (v, \nabla v))_{L^2(\Omega_\pm)} - \sum_{\pm} ((\zeta_i D_i^h A^2)(v^h, \nabla v^h), \zeta_i D_i^h v)_{L^2(\Omega_\pm)}$$

where the boundary terms are zero in the case $i = 3$ since $\zeta_3 = 0$ in a neighbourhood of Γ. Thus we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\zeta_i D_i^h v\right|\right|^2_{B^0(t)} + \mu\left|\zeta_i D_i^h v\right|^2_{B^1(t)} \\
&\leq \frac{1}{\delta}(\sum_{\pm}\left|\left|G^1\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{\pm}\left|\left|G^2\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|G^3\right|\right|^2_{L^2(\Gamma)} + \left|\left|G^4\right|\right|^2_{H^1(\Gamma)}) \\
&+ \frac{C_{\infty,1}}{\delta\epsilon}\sum_{\pm}\left|\left|v\right|\right|^2_{H^1(\Omega_\pm)} + \delta\epsilon\left|\left|\partial_t v\right|\right|^2_{B^0} + \delta\left|\left|\zeta_i D_i^h v\right|\right|^2_{B^1(t)}
\end{aligned}$$

Choosing $\delta > 0$ sufficiently small and integrating from 0 to t we obtain the estimate

$$\begin{aligned}
&\left|\left|\zeta_i D_i^h v\right|\right|^2_{B^0(t)} + 2\mu\int_0^t\left|\left|\zeta_i D_i^h v\right|\right|^2_{B^1(t)}\mathrm{d}\tau \leq C\left|\left|\nabla v(0)\right|\right|^2_{B^0(t)} \\
&+ C_{\infty,1}\int_0^t\sum_{\pm}\left|\left|G^1\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{\pm}\left|\left|G^2\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|G^3\right|\right|^2_{L^2(\Gamma)} + \left|\left|G^4\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau \\
&+ \frac{C_{\infty,1}}{\epsilon}\sum_{\pm}\int_0^t\left|\left|v\right|\right|^2_{H^1(\Omega_\pm)}\mathrm{d}\tau + \epsilon\int_0^t\left|\left|\partial_t v\right|\right|^2_{B^0}\mathrm{d}\tau
\end{aligned}$$

Since the right hand side is independent of h, using the properties of difference quotients as stated in 11.6.8, we obtain the estimate (11.6.7). Setting $\epsilon = 1$, then using the estimate (11.6.3) with $\epsilon = 1$ for $\left|\left|v\right|\right|^2_{B^1(t)}$ and $\left|\left|\partial_t v\right|\right|^2_{B^0(t)}$, we obtain

$$\left|\left|\zeta_i D_i^h v\right|\right|^2_{B^0(t)} + \int_0^t\left|\left|\zeta_i D_i^h v\right|\right|^2_{B^1(\tau)}\mathrm{d}\tau \leq C_A(t)(C_G(t) + \left|\left|v(0)\right|\right|_{B^1})$$

Thus we see that $\partial_{x_i} v \in B^1$ for almost every t for $1 \leq i \leq 2$ and

$$\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\nabla' v\right|\right|^2_{B^0(\tau)} + \int_0^t\left|\left|\nabla' v\right|\right|^2_{B^1(\tau)}\mathrm{d}\tau \leq C_A(t)(C_G(t) + \left|\left|v(0)\right|\right|_{B^1}) \tag{11.6.8}$$

Also, we may take $\zeta_3 \equiv 1$ on a compact set containing $\Omega^\epsilon_\pm \subset\subset \Omega_\pm$ for arbitrary open subsets $\Omega^\epsilon_\pm \subset\subset \Omega_\pm$ and use the properties of difference quotients with the h-independent estimate to obtain $\partial_{x_3} v \in L^2((0,T); H^1_{\mathrm{loc}}(\Omega_\pm))$.

Now let $w \in C_c^\infty(\Omega_\pm)$ and test the weak equation with w, noting that there are no boundary terms since $w = 0$ in a neighbourhood of Γ. We intend to integrate by parts to remove the derivatives from w. Note that since $v_\pm \in L^2((0,T); H^2_{\mathrm{loc}}(\Omega_\pm))$ this is possible for almost every $t \in (0,T)$. Indeed, we have

$$\begin{aligned}
&\{v, w\}_{B^1(t)} = \sum_{\pm}(\rho c^2\nabla^\psi\cdot(Nv), \nabla^\psi\cdot Nw)_{L^2(\Omega_\pm)} + \sum_{\pm}(\rho c^2\nabla^\psi\times(Nv), \nabla^\psi\times Nw)_{L^2(\Omega_\pm)} \\
&= -\sum_{\pm}(N^T\nabla^\psi(\rho c^2\nabla^\psi\cdot(Nv)), w)_{L^2(\Omega_\pm)} - \sum_{\pm}(N^T\partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J\psi})\rho c^2\nabla^\psi\cdot(Nv), w)_{L^2(\Omega_\pm)} \\
&+ \sum_{\pm}(N^T\nabla^\psi\times(\rho c^2\nabla^\psi\times(Nv)), w)_{L^2(\Omega_\pm)} \\
&+ \sum_{\pm}(\rho c^2 N^T\partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi})\times(\nabla^\psi\times(Nv)), w)_{L^2(\Omega_\pm)}
\end{aligned}$$

In the weak equation we obtain

$$
\begin{aligned}
&(\partial_t v, w)_{B^0(t)} \\
&-\mu\sum_{\pm}(N^T\nabla^\psi(\rho c^2\nabla^\psi\cdot(Nv)),w)_{L^2(\Omega_\pm)}-\mu\sum_{\pm}(N^T\partial_{x_3}(\frac{(-\nabla'\psi,1)}{J^\psi})\rho c^2\nabla^\psi\cdot(Nv),w)_{L^2(\Omega_\pm)} \\
&+\mu\sum_{\pm}(N^T\nabla^\psi\times(\rho c^2\nabla^\psi\times(Nv)),w)_{L^2(\Omega_\pm)} \\
&+\mu\sum_{\pm}(\rho c^2 N^T\partial_{x_3}(\frac{(-\nabla'\psi,1)}{J^\psi})\times(\nabla^\psi\times(Nv)),w)_{L^2(\Omega_\pm)}+\sum_{\pm}(A^2(v,\nabla v),w)_{L^2(\Omega_\pm)} \\
&=\sum_{\pm}(G^1,w)_{L^2(\Omega_\pm)}-\sum_{\pm}(\nabla\cdot G^2,w)_{L^2(\Omega_\pm)}
\end{aligned}
$$

Using the definition of $A^2(v,\nabla v)$, we obtain

$$
\begin{aligned}
&\sum_{\pm}(\rho_\mu\frac{1}{|\underset{\sim}{n}|^2}\partial_t v,w)_{L^2(\Omega_\pm)}+\sum_{\pm}(\rho_\mu N^T(\overline{u^\psi}\cdot\nabla)(Nv),w)_{L^2(\Omega_\pm)}+\sum_{\pm}(\rho_\mu N^T(\partial_t N)v,w)_{L^2(\Omega_\pm)} \\
&-\mu\sum_{\pm}(N^T\nabla^\psi(\rho c^2\nabla^\psi\cdot(Nv)),w)_{L^2(\Omega_\pm)}+\mu\sum_{\pm}(N^T\nabla^\psi\times(\rho c^2\nabla^\psi\times(Nv)),w)_{L^2(\Omega_\pm)} \\
&=\sum_{\pm}(G^1,w)_{L^2(\Omega_\pm)}-\sum_{\pm}(\nabla\cdot G^2,w)_{L^2(\Omega_\pm)}
\end{aligned}
$$

Choosing either $w_\pm=0$ in turn, we obtain

$$
\begin{aligned}
&\rho_\mu\frac{1}{|\underset{\sim}{n}|^2}\partial_t v+\rho_\mu N^T(\overline{u^\psi}\cdot\nabla)(Nv)+\rho_\mu N^T(\partial_t N)v \\
&-\mu N^T\nabla^\psi(\rho c^2\nabla^\psi\cdot(Nv)+\mu N^T\nabla^\psi\times(\rho c^2\nabla^\psi\times(Nv))=G^1-\nabla\cdot G^2
\end{aligned}
$$

in $(0,T)\times\Omega_\pm$. This is the interior equation above. Note that the Hodge decomposition A.1.1 implies

$$
\begin{aligned}
&\mu N^T\nabla^\psi(\rho c^2\nabla^\psi\cdot(Nv))-\mu N^T\nabla^\psi\times(\rho c^2\nabla^\psi\times(Nv)) \\
&=\mu\frac{1}{|\underset{\sim}{n}|^2}\rho c^2(\nabla^\psi\nabla^\psi\cdot v-\nabla^\psi\times(\nabla^\psi\times v)) \\
&+A(\rho c^2,\nabla(\rho c^2),\nabla\psi,\nabla\nabla\psi,\nabla N)((\nabla N)v,(\nabla\nabla N)v,\nabla v) \\
&=\mu\frac{1}{|\underset{\sim}{n}|^2}\rho c^2\Delta^\psi v+A(\rho c^2,\nabla(\rho c^2),\nabla\psi,\nabla\nabla\psi,\nabla N)((\nabla N)v,(\nabla\nabla N)v,\nabla v) \\
&=\mu\frac{\rho c^2}{|\underset{\sim}{n}|^2}(\frac{1+|\nabla'\psi|^2}{(J^\psi)^2}\partial_3^2 v+\Delta' v-2\frac{\nabla'\psi}{J^\psi}\cdot\nabla'\partial_{x_3}v) \\
&+A(\rho c^2,\nabla(\rho c^2),\nabla\psi,\nabla\nabla\psi,\nabla N)((\nabla N)v,(\nabla\nabla N)v,\nabla v)
\end{aligned}
$$

for some generic tensor-valued A (not the same throughout) which is a smooth function of its arguments. Thus, rearranging the interior equation, we obtain

$$
\begin{aligned}
&\mu\frac{\rho c^2(1+|\nabla'\psi|^2)}{|\underset{\sim}{n}|^2(J^\psi)^2}\partial_3^2 v \\
&=\mu\frac{\rho c^2}{|\underset{\sim}{n}|^2}(-\Delta' v+2\frac{\nabla'\psi}{J^\psi}\cdot\nabla'\partial_{x_3}v)-A(\rho c^2,\nabla(\rho c^2),\nabla\psi,\nabla\nabla\psi,\nabla N)((\nabla N)v,(\nabla\nabla N)v,\nabla v) \\
&+\rho_\mu\frac{1}{|\underset{\sim}{n}|^2}\partial_t v+\rho_\mu N^T(\overline{u^\psi}\cdot\nabla)(Nv)+\rho_\mu N^T(\partial_t N)v-G^1+\nabla\cdot G^2
\end{aligned}
\tag{11.6.9}
$$

squaring both sides, integrating over $\Omega_\pm$ and integrating from 0 to t, we obtain

$$\int_0^t \left|\left|\partial_{x_3}^2 v\right|\right|_{L^2(\Omega_\pm)}^2 \,\mathrm{d}\tau \leq C_A(t)(C_G(t) + ||v(0)||_{B^1}^2)$$

where we have used the existing estimates (11.6.8) for the right hand side. Combining this with the existing estimates (11.6.8), we obtain

$$\sum_\pm \int_0^t ||v||_{H^2(\Omega_\pm)}^2 \,\mathrm{d}\tau + \int_0^t ||v_3||_{H^2(\Gamma)}^2 \,\mathrm{d}\tau \leq C_A(t)(C_G(t) + ||v(0)||_{B^1}^2)$$

as required.

We now return to the weak equation for $w \in B^1$. Given that $v_\pm \in H^2(\Omega_\pm)$ and $v_3 \in H^2(\Gamma)$ for almost every t, we may integrate by parts in the term $(v, w)_{B^1(t)}$ to obtain

$$\begin{aligned}
&\{v, w\}_{B^1(t)} = \sum_\pm (\rho c^2 \nabla^\psi \cdot (Nv), \nabla^\psi \cdot Nw)_{L^2(\Omega_\pm)} \\
&+ \sum_\pm (\rho c^2 \nabla^\psi \times (Nv), \nabla^\psi \times Nw)_{L^2(\Omega_\pm)} + (v_3, w_3)_{H^1(\Gamma)} \\
&= (v, w)_{B^0(t)} - \sum_\pm (N^T \nabla^\psi (\rho c^2 \nabla^\psi \cdot (Nv)), w)_{L^2(\Omega_\pm)} - ([\rho c^2 \nabla^\psi \cdot (Nv)], w_3)_{L^2(\Gamma)} \\
&- \sum_\pm (N^T \partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi}) \rho c^2 \nabla^\psi \cdot (Nv), w)_{L^2(\Omega_\pm)} + \sum_\pm (N^T \nabla^\psi \times (\rho c^2 \nabla^\psi \times (Nv)), w)_{L^2(\Omega_\pm)} \\
&+ \sum_\pm (\pm \rho c^2 n \times (\nabla^\psi \times (Nv)), Nw)_{L^2(\Gamma)} + \sum_\pm (\rho c^2 N^T \partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi}) \times (\nabla^\psi \times (Nv)), w)_{L^2(\Omega_\pm)} \\
&+ ((1 - \Delta')v_3, w_3)_{L^2(\Gamma)}
\end{aligned}$$

Integrating by parts in the term $(G^2, \nabla w)_{L^2(\Omega_\pm)}$ we obtain

$$\begin{aligned}
\sum_\pm (G^2, \nabla w)_{L^2(\Omega_\pm)} &= \sum_\pm (-\nabla \cdot G^2, w)_{L^2(\Omega_\pm)} - \sum_\pm (\pm G_3^2, w_\pm)_{L^2(\Gamma)} \\
&= \sum_\pm (-\nabla \cdot G^2, w)_{L^2(\Omega_\pm)} - ([G_{33}^2], w_3)_{L^2(\Gamma)} - \sum_\pm (\pm (G_3^3)', w_\pm')_{L^2(\Gamma)}
\end{aligned}$$

Integrating by parts in the G^4 term we obtain

$$(G^4, \nabla' w_3)_{L^2(\Gamma)} = (-\nabla' \cdot G^4, w_3)_{L^2(\Gamma)}$$

By replacing w with $w_\epsilon(x) = \chi(\frac{x_3}{\epsilon}) z(x')$ in the above equation, where χ is a smooth cut-off function and $z_\pm \in C_c^\infty(\Gamma; \mathbb{R}^3)$ with $(z_+)_3 = (z_-)_3$ for $\epsilon > 0$, and letting $\epsilon \to 0$, we obtain

$$\begin{aligned}
&- ([\rho c^2 \nabla^\psi \cdot (Nv)], z_3)_{L^2(\Gamma)} + \sum_\pm (\pm \rho_\pm c_\pm^2 n \times (\nabla^\psi \times (Nv_\pm)), Nz_\pm)_{L^2(\Gamma)} \\
&+ ((1 - \Delta')(v_3, z_3)_{L^2(\Gamma)} \\
&= -([G_{33}^2], z_3)_{L^2(\Gamma)} - \sum_\pm (\pm (G_{3\pm}^2)', z_\pm')_{L^2(\Gamma)} + (G^3 - \nabla' \cdot G^4, z_3)_{L^2(\Gamma)}
\end{aligned}$$

for all $z_\pm \in C_c^\infty(\Gamma; \mathbb{R}^3)$ with $(z_+)_3 = (z_-)_3$. Now let us take $z_\pm = (0, 0, z_3)$ for $z_3 \in C_c^\infty(\Gamma)$. Note that this implies Nz is parallel to n, so we obtain

$$\begin{aligned}
&- ([\rho c^2 \nabla^\psi \cdot (Nv)], z_3)_{L^2(\Gamma)} + ((1 - \Delta')(v_3, z_3)_{L^2(\Gamma)} \\
&= -([G_{33}^2], z_3)_{L^2(\Gamma)} + (G^3 - \nabla' \cdot G^4, z_3)_{L^2(\Gamma)}
\end{aligned}$$

for all $z_3 \in C_c^\infty(\Gamma)$. Hence

$$[\rho c^2 \nabla^\psi \cdot (Nv)] + (\Delta' - 1)v_3 = -[G^2_{33}] + G^3 - \nabla' \cdot G^4$$

which is the first interface condition.

Now let us take $z_\pm = (z_{1\pm}, z_{2\pm}, 0)$ for $z_{1\pm}, z_{2\pm} \in C_c^\infty(\Gamma)$. We obtain

$$\sum_\pm (\pm N^T(\rho_\pm c_\pm^2 n \times (\nabla^\psi \times (Nv_\pm))), z_\pm)_{L^2(\Gamma)} = -\sum_\pm (\pm (G^2_{3\pm})', z'_\pm)_{L^2(\Gamma)}$$

for all $z_\pm \in C_c^\infty(\Gamma; \mathbb{R}^3)$ with $z_{3\pm} = 0$. In fact since $(N^T(\rho c^2 n \times (\nabla^\psi \times (Nv)))_3 = 0$ the formula holds for all $z_\pm \in C_c^\infty(\Gamma; \mathbb{R}^3)$. Now, taking one of $z_\pm$ to be zero in turn, we obtain

$$(N^T(\rho_\pm c_\pm^2 n \times (\nabla^\psi \times (Nv_\pm))), z_\pm)_{L^2(\Gamma)} = -\sum_\pm (\pm (G^2_{3\pm})', z'_\pm)_{L^2(\Gamma)}$$

for all $z_\pm \in C_c^\infty(\Gamma; \mathbb{R}^3)$. Hence

$$N^T(\rho_\pm c_\pm^2 n \times (\nabla^\psi \times (Nv_\pm))) = -(G^2_{13\pm}, G^2_{23\pm}, 0)$$

on $(0, T) \times \Gamma$, which is the second interface condition.

Now suppose for some $1 \le k \le 3$ that

$$\begin{aligned}
G^1 &\in L^2((0,T); H^k(\Omega_\pm)) \\
G^2 &\in L^2((0,T); H^{k+1}(\Omega_\pm)) \\
G^3 &\in L^2((0,T); H^k(\Gamma)) \\
G^4 &\in L^2((0,T); H^{k+1}(\Gamma)) \\
v(0) &\in H^{k+1}(\Omega_\pm) \\
v_3(0) &\in H^{k+1}(\Omega_\pm)
\end{aligned}$$

Assume inductively on k that we have $\nabla^\alpha v \in L^2((0,T); H^1(\Omega_\pm)) \cap L^\infty((0,T); L^2(\Omega_\pm))$ and $\nabla^{\alpha'} v_3 \in L^2((0,T); H^1(\Gamma))$ for $1 \le |\alpha| \le k+1$, with

$$\begin{aligned}
&\sum_\pm \operatorname*{ess\,sup}_{\tau \in (0,t)} ||\nabla^\alpha v||^2_{L^2(\Omega_\pm)} + \sum_\pm \int_0^t ||\nabla^\alpha v||^2_{H^1(\Omega_\pm)} \, d\tau + \int_0^t \left|\left|\nabla^{\alpha'} v_3\right|\right|^2_{H^1(\Gamma)} d\tau \\
&\le C_A(t)(C_G(t) + ||v(0)||^2_{H^{k+1}(\Omega_\pm)} + ||v_3(0)||^2_{H^{k+1}(\Gamma)} + \int_0^t ||\nabla^\alpha \partial_t v||^2_{B^0(\tau)} \, d\tau)
\end{aligned}$$

Note that the case $k = 0$ is covered above. If $0 \le k \le 2$, we claim that the same holds true for $k+1$ replacing k. Indeed, let $1 \le |\alpha| \le k+1$. Assume first $\alpha_3 = 0$ and test the equation with $(-1)^{|\alpha|} \nabla^\alpha w$ instead of w, where $w \in B^1$ is smooth in the horizontal directions. Then we integrate by parts horizontally to obtain

$$\begin{aligned}
&(\partial_t \nabla^\alpha v, w)_{B^0(t)} + \mu \{\nabla^\alpha v, w\}_{B^1(t)} + \sum_\pm (A^2(\nabla^\alpha v, \nabla \nabla^\alpha v), w)_{L^2(\Omega_\pm)} \\
&= \sum_\pm (\nabla^\alpha G^1, w)_{L^2(\Omega_\pm)} + \sum_\pm (\nabla^\alpha G^2, \nabla w)_{L^2(\Omega_\pm)} + (\nabla^\alpha G^3, w_3)_{L^2(\Gamma)} + (\nabla^\alpha G^4, \nabla' w_3)_{L^2(\Gamma)} \\
&- \sum_\pm \sum_{\beta+\gamma=\alpha, \beta<\alpha} (\nabla^\gamma A^0 \nabla^\beta \partial_t v, w)_{L^2(\Omega_\pm)} - \mu \sum_\pm \sum_{\beta+\gamma=\alpha, \beta<\alpha} (\nabla^\gamma A^1(\nabla^\beta v, \nabla \nabla^\beta v), (w, \nabla w))_{L^2(\Omega_\pm)} \\
&- \sum_\pm \sum_{\beta+\gamma=\alpha, \beta<\alpha} (\nabla^\gamma A^2(\nabla^\beta v, \nabla \nabla^\beta v), w)_{L^2(\Omega_\pm)} \\
&=: \sum_\pm (G^{1,\alpha}, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^{2,\alpha}, \nabla w)_{L^2(\Omega_\pm)} + (G^{3,\alpha}, w_3)_{L^2(\Gamma)} + (G^{4,\alpha}, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}$$

where

$$
\begin{aligned}
G^{1,\alpha} &:= \nabla^\alpha G^1 - \sum_{\beta+\gamma=\alpha,\beta\neq\alpha} (\nabla^\gamma A^0 \nabla^\beta \partial_t v + \mu \nabla^\gamma A^1(\nabla^\beta v, \nabla\nabla^\beta v) + \nabla^\gamma A^2(\nabla^\beta v, \nabla\nabla^\beta v)) \\
G^{2,\alpha} &:= \nabla^\alpha G^2 - \mu \sum_{\beta+\gamma=\alpha,\beta\neq\alpha} \nabla^\gamma A^1(\nabla^\beta v, \nabla\nabla^\beta v) \\
G^{3,\alpha} &:= \nabla^\alpha G^3 \\
G^{4,\alpha} &:= \nabla^\alpha G^4
\end{aligned}
$$

Since such w are clearly dense in B^1 (eg consider horizontal mollification) we have that this holds for all $w \in B^1$. Now, noting that we can use the Sobolev embedding theorem in the case $|\alpha| \geq 2$ together with the existing estimates for $\nabla^\beta v$ in $L^2((0,T); H^2(\Omega_\pm))$ for $|\beta| < |\alpha|$, we see that

$$
\begin{aligned}
G^{1,\alpha} &\in L^2((0,T); L^2(\Omega_\pm)) \\
G^{2,\alpha} &\in L^2((0,T); H^2(\Omega_\pm)) \\
G^{3,\alpha} &\in L^2((0,T); L^2(\Gamma)) \\
G^{4,\alpha} &\in L^2((0,T); H^1(\Gamma)) \\
\nabla^\alpha \nabla v(0) &\in L^2(\Omega_\pm) \\
\nabla^{\alpha'} v_3(0) &\in H^1(\Gamma)
\end{aligned}
$$

with norms bounded by $C_A(t)(C_G(t) + ||v(0)||_{H^{k+2}(\Omega_\pm)} + ||v_3(0)||_{H^{k+2}(\Gamma)} + \int_0^t ||\nabla^\alpha \partial_t v||^2_{B^0(\tau)} \,\mathrm{d}\tau)$.

We may now apply the first part of the proof again to $\nabla^\alpha v$ to obtain $\nabla' \nabla^\alpha v \in L^2((0,T); B^1) \cap L^\infty((0,T); B^0)$ with

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\nabla' \nabla^\alpha v||^2_{B^0(\tau)} + \int_0^t ||\nabla' \nabla^\alpha v||^2_{B^1(\tau)} \,\mathrm{d}\tau \\
&\leq C_A(t)(C_G + ||v(0)||^2_{H^{k+2}(\Omega_\pm)} + ||v_3(0)||^2_{H^{k+2}(\Gamma)} + \int_0^t ||\nabla^\alpha \partial_t v||^2_{B^0(\tau)} \,\mathrm{d}\tau)
\end{aligned}
$$

Now we claim by a secondary induction on α_3 for $0 \leq \alpha_3 \leq |\alpha|$ that

$$
\nabla^\alpha v \in L^2((0,T); H^2(\Omega_\pm))
$$

with corresponding norm bounded by

$$
C_A(t)(C_G(t) + ||v(0)||_{H^{k+2}(\Omega_\pm)} + ||v_3(0)||_{H^{k+2}(\Gamma)} + \int_0^t ||\nabla^\alpha \partial_t v||^2_{B^0(\tau)} \,\mathrm{d}\tau)
$$

The case $\alpha_3 = 0$ is covered above, so we suppose it holds true for some $0 \leq \alpha_3 \leq |\alpha| - 1$ and attempt to prove it holds for $\alpha_3 + 1$. Indeed, applying ∇^β to the rearranged interior equation where $\beta_3 = \alpha_3 + 1$ and $1 \leq |\beta| \leq |\alpha|$, we have

$$
\begin{aligned}
\partial_3^2 \nabla^\beta v = \nabla^\beta (&\frac{|\underset{\sim}{n}|^2 (J^\psi)^2}{\mu\rho c^2 (1 + |\nabla'\psi|^2)} (\mu \frac{\rho c^2}{|\underset{\sim}{n}|^2} (-\Delta' v + 2\frac{\nabla'\psi}{J^\psi} \cdot \nabla' \partial_{x_3} v) \\
&- A(\rho c^2, \nabla(\rho c^2), \nabla\psi, \nabla\nabla\psi, \nabla N, \nabla\nabla N)(v, \nabla v) \\
&+ \rho_\mu \frac{1}{|\underset{\sim}{n}|^2} \partial_t v + \rho_\mu N^T (\overline{u^\psi} \cdot \nabla)(N v) + \rho_\mu N^T (\partial_t N) v - G^1 + \nabla \cdot G^2))
\end{aligned}
$$

squaring both sides, integrating over $\Omega_\pm$ and integrating from 0 to t, we obtain

$$\int_0^t ||\partial_{x_3}^2 v||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau \le C_A(t)(C_G(t) + ||v(0)||^2_{H^{k+2}(\Omega_\pm)} + ||v_3(0)||^2_{H^{k+2}(\Gamma)} + \int_0^t \left|\left|\nabla^\beta \partial_t v\right|\right|^2_{B^0(\tau)} \,\mathrm{d}\tau)$$

where we have used the previous induction step and the assumed time derivative estimate for the right hand side. Combining this with the existing tangential estimates, we obtain

$$\begin{aligned}
&\sum_\pm \int_0^t \left|\left|\nabla^\beta v\right|\right|^2_{H^2(\Omega_\pm)} \,\mathrm{d}\tau \\
&\le C_A(t)(C_G(t) + ||v(0)||^2_{H^{k+2}(\Omega_\pm)} + ||v_3(0)||^2_{H^{k+2}(\Gamma)} + \int_0^t \left|\left|\nabla^\beta \partial_t v\right|\right|^2_{B^0(\tau)} \,\mathrm{d}\tau)
\end{aligned}$$

as required. The L^∞ in time estimate is proved in exactly the same way, but applying one less derivative. This completes the secondary induction step, and thus the original induction step and thus proves the claim.

This completes the proof. □

11.7 Proof of Existence, Uniqueness and Regularity of Solutions of the Equations for the Rotated Velocity

Definition 11.7.1. We say $\widehat{u}$ is a weak solution of the rotated equations for the velocity if $\widehat{u} - \widehat{u}^0 \in X$,

$$\widehat{u}\ |_{t=0} = \widehat{u}^0$$

and the following holds for almost every $t \in (0, T)$.

$$\begin{aligned}
&(\partial_t \widehat{u}, w)_{B^0(t)} + \mu\{\widehat{u}, w\}_{B^1(t)} + \sum_\pm (A^1(\widehat{u}, \nabla \widehat{u}), w)_{L^2(\Omega_\pm)} \\
&= \sum_\pm (G^1, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^2, \nabla w)_{L^2(\Omega_\pm)} + (G^3, w_3)_{L^2(\Gamma)} + (G^4, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned} \tag{11.7.1}$$

for all $w \in B^1$. Note that we rely on the fact that $[\widehat{u}_3^0] = 0$, which is true due to the compatibility condition 7.2.6 at order zero, for the term $\{\widehat{u}, w\}_{B^1(t)}$ to be well-defined.

The functions G^1, G^2, G^3 and G^4 are defined such that

$$\begin{aligned}
&\sum_\pm (G^1, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^2, \nabla w)_{L^2(\Omega_\pm)} \\
&= (N^T(g_1 + \partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi})p - \mu\partial_{x_3}(\frac{(-\nabla'\psi, 1)}{J^\psi}) \times \rho c^2 \omega), w)_{L^2(\Omega_\pm)} \\
&+ (p, \nabla^\psi \cdot (Nw))_{L^2(\Omega_\pm)} + \mu(\rho c^2 \omega, \nabla^\psi \times (Nw))_{L^2(\Omega_\pm)} \\
&G^3 = g_2 \\
&G^4 = \sigma \hat{n}
\end{aligned}$$

Note that G^i have the form

$$\begin{aligned}
G^1 &= G(N, \nabla N, \nabla\psi, p, \omega, \rho c^2, g_1) \\
G^2 &= G(N, \nabla\psi, p, \omega, \rho c^2) \\
G^3 &= g_2 \\
G^4 &= G(\nabla' f)
\end{aligned}$$

for G a generic smooth function which may be scalar, vector, matrix valued etc. Using the product rule for Sobolev functions, we note that

$$\partial_t^k G^1 \in L^2((0,T); H^{3-k}(\Omega_\pm))$$

for $0 \leq k \leq 3$ and

$$\begin{aligned}
\partial_t^k G^2 &\in L^2((0,T); H^{4-k}(\Omega_\pm)) \\
\partial_t^k G^3 &\in L^2((0,T); H^{4-k}(\Gamma)) \\
\partial_t^k G^4 &\in L^2((0,T); H^{4-k}(\Gamma))
\end{aligned}$$

for $0 \leq k \leq 4$ with norms bounded by $C_{\max}$ provided

$$\begin{aligned}
\partial_t^k (p, s, \omega, g_1, N, \nabla\psi) &\in L^2((0,T); H^{4-k}(\Omega_\pm)) \\
\partial_t^k \nabla' f &\in L^2((0,T); L^{4-k}(\Gamma))
\end{aligned}$$

for $0 \leq k \leq 4$ with norms bounded by $C_{\max}$, which is indeed the case.

Proposition 11.7.2. *There exists unique solution $\widehat{u}$ with $\widehat{u} - \widehat{u}^0 \in X$ and $\widehat{u}\,(0) = \widehat{u}^0$ to the weak velocity equation above.*

In fact, for $0 \leq j \leq 3$, we have $\partial_t^j \nabla\, \widehat{u} \in L^\infty((0,T); H^{3-j}(\Omega_\pm)) \cap L^2((0,T); H^{4-j}(\Omega_\pm))$, $\partial_t^4\, \widehat{u} \in L^2((0,T); L^2(\Omega_\pm))$, $\partial_t^j\, \widehat{u}_3 \in L^2((0,T); H^{5-j}(\Gamma)) \cap L^\infty((0,T); H^{4-j}(\Gamma))$, with corresponding norms bounded by $C_{\max}$ and, for $0 \leq j \leq 3$,

$$\partial_t^j\, \widehat{u}\,|_{t=0} = \partial_0^{\mu,g,j}\, \widehat{u}^0$$

Moreover, $\widehat{u}$ is the unique solution to the equations for the rotated velocity with $\widehat{u}\,(0) = \widehat{u}^0$ as defined in 11.3.3.

Proof. Let us define

$$\begin{aligned}
\tilde{G}^1 &:= G^1 - \mu A^1(\widehat{u}^0, \nabla\, \widehat{u}^0) - A^2(\widehat{u}^0, \nabla\, \widehat{u}^0) \\
\tilde{G}^2 &:= G^2 - \mu A^1(\widehat{u}^0, \nabla\, \widehat{u}^0) \\
\tilde{G}^3 &:= G^3 - \mu\, \widehat{u}_3^0 \\
\tilde{G}^3 &:= G^4 - \mu \nabla'\, \widehat{u}_3^0
\end{aligned}$$

We claim that there exists a unique solution $v \in X$ of the equation

$$\begin{aligned}
&(\partial_t v, w)_{B^0(t)} + \mu\{v, w\}_{B^1(t)} + \sum_\pm (A^1(v, \nabla v), w)_{L^2(\Omega_\pm)} \\
&= \sum_\pm (\tilde{G}^1, w)_{L^2(\Omega_\pm)} + \sum_\pm (\tilde{G}^2, \nabla w)_{L^2(\Omega_\pm)} + (\tilde{G}^3, w_3)_{L^2(\Gamma)} + (\tilde{G}^4, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}$$

for all $w \in B^1$ for almost every $t \in (0,T)$ with

$$v(0) = 0$$

Indeed, applying proposition 11.6.5, we immediately obtain such a solution. Now we may apply proposition 11.6.9 to obtain immediately that $v \in L^2((0,T); H^2(\Omega_\pm))$, $v_3 \in L^2((0,T); H^2(\Gamma))$ and that v satisfies the corresponding strong form of the equations and boundary conditions almost everywhere in $(0,T) \times \Omega_\pm$ and on $(0,T) \times \Gamma$. We then define $\widehat{u} = v + \widehat{u}^0$ to find that $\widehat{u}$ satisfies the corresponding strong form of the equations with G^i replacing $\tilde{G}^i$ using the definition of $\tilde{G}^i$. Note that G^1, G^2, G^3 and G^4 were chosen so that these are exactly the equations for the rotated velocity, hence $\widehat{u}$ satisfies the strong form of equations for the rotated velocity with $\widehat{u}(0) = \widehat{u}^0$. Note also that, from the definition of $\tilde{G}^1$, $\tilde{G}^2$, $\tilde{G}^3$ and $\tilde{G}^4$ we have that $\widehat{u}$ with $\widehat{u} - \widehat{u}^0 \in X$ satisfies the weak velocity equation if and only if v satisfies the above weak equation with $v(0) = 0$, thus we immediately obtain a unique solution $\widehat{u}$ to the weak velocity equation with $\widehat{u}(0) = \widehat{u}^0$. This implies that $\widehat{u}$ with the regularity we have just inferred must be a unique solution to the strong form of equations for the rotated velocity since by going backwards we obtain a solution to the weak equation which we know is unique.

We remark that we applied proposition 11.6.9 to $\widehat{u} - \widehat{u}^0$ here rather than $\widehat{u}$ because without applying any derivatives $\widehat{u}^0 \notin L^2(\Omega_\pm)$. However, for the higher order spatial regularity we will work directly with derivatives of $\widehat{u}$ else there will be too many derivatives on $\widehat{u}^0$.

Now by induction on j for $0 \le j \le 3$ we claim that $\widehat{u}$ satisfies

$$\begin{aligned}
\nabla \partial_t^k \widehat{u} &\in L^\infty((0,T); H^{j-k}(\Omega_\pm)) \text{ for } 0 \le k \le j \\
\nabla \partial_t^k \widehat{u} &\in L^2((0,T); H^{j+1-k}(\Omega_\pm)) \text{ for } 0 \le k \le j \\
\partial_t^k \widehat{u}_3 &\in L^\infty((0,T); H^{j+1-k}(\Gamma)) \text{ for } 0 \le k \le j \\
\partial_t^k \widehat{u}_3 &\in L^2((0,T); H^{j+2-k}(\Gamma)) \text{ for } 0 \le k \le j \\
\partial_t^{j+1} \widehat{u} &\in L^2((0,T); L^2(\Omega_\pm)) \\
\partial_t^j \widehat{u}\,|_{t=0} &= \partial_0^{\mu,g,j} \widehat{u}^0
\end{aligned}$$

with corresponding norms bounded by $C_{\max}$, and also $\partial_t^j \widehat{u}$ satisfies the equation

$$\begin{aligned}
&(\partial_t^{j+1} \widehat{u}, w)_{B^0(t)} + \mu\{\partial_t^j \widehat{u}, w\}_{B^1(t)} + \sum_\pm (A^2(\partial_t^j \widehat{u}, \nabla \partial_t^j \widehat{u}), w)_{L^2(\Omega_\pm)} \\
&= \sum_\pm (G^{1,j}, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^{2,j}, \nabla w)_{L^2(\Omega_\pm)} + (G^{3,j}, w_3)_{L^2(\Gamma)} + (G^{4,j}, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}$$

for all $w \in B^1$ for almost every $t \in (0,T)$, where we have defined

$$\begin{aligned}
G^{1,j} &:= \partial_t^j G^1 - \sum_{k+l=j, k<j} (\partial_t^l A^0 \partial_t^{k+1} \widehat{u} + \mu \partial_t^l A^1(\partial_t^k \widehat{u}, \nabla \partial_t^k \widehat{u}) + \partial_t^l A^2(\partial_t^k \widehat{u}, \nabla \partial_t^k \widehat{u})) \\
G^{2,j} &:= \partial_t^j G^2 - \sum_{k+l=j, k<j} \mu \partial_t^l A^1(\partial_t^k \widehat{u}, \nabla \partial_t^k \widehat{u}) \\
G^{3,j} &:= \partial_t^j G^3 \\
G^{4,j} &:= \partial_t^j G^4
\end{aligned}$$

Note the case $j = 0$ is covered above. We assume that this holds for some $0 \le j \le 2$ and try to show it holds for $j + 1$.

Firstly, we define

$$\begin{aligned}
\tilde{G}^{1,j} &:= G^{1,j} - \mu A^1(\partial_0^{\mu,g,j}\ \widehat{u}^0, \nabla\partial_0^{\mu,g,j}\ \widehat{u}^0) - A^2(\partial_0^{\mu,g,j}\ \widehat{u}^0, \nabla\partial_0^{\mu,g,j}\ \widehat{u}^0)\\
\tilde{G}^{2,j} &:= G^{2,j} - \mu A^1(\partial_0^{\mu,g,j}\ \widehat{u}^0, \nabla\partial_0^{\mu,g,j}\ \widehat{u}^0)\\
\tilde{G}^{3,j} &: = G^{3,j} - \mu\partial_0^{\mu,g,j}\ \widehat{u}_3^0\\
\tilde{G}^{4,j} &: = G^{4,j} - \mu\nabla'\partial_0^{\mu,g,j}\ \widehat{u}_3^0
\end{aligned}$$

Defining $v^j = \partial_t^j\ \widehat{u}\ -\partial_0^{\mu,g,j}\ \widehat{u}^0$, we see that v^j satisfies the equation

$$\begin{aligned}
&(\partial_t v^j, w)_{B^0(t)} + \mu\{v^j, w\}_{B^1(t)} + \sum_{\pm}(A^2(v^j, \nabla v^j), w)_{L^2(\Omega_\pm)}\\
&= \sum_{\pm}(\tilde{G}^{1,j}, w)_{L^2(\Omega_\pm)} + \sum_{\pm}(\tilde{G}^{2,j}, \nabla w)_{L^2(\Omega_\pm)} + (\tilde{G}^{3,j}, w_3)_{L^2(\Gamma)} + (\tilde{G}^{4,j}, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}$$

where we have used the fact that $\partial_0^{\mu,g,j}\ \widehat{u}^0 \in B^1$ due to the regularity of the initial data and the compatibility condition (7.2.6) at order j.

Now note that the background solution satisfies the initial conditions in the sense

$$\partial_t^k(s, p, \bar{u}, f, \bar{f}, \omega)|_{t=0} = \partial_0^{\mu,g,k}(s^0, p^0, u^0, f^0, f^0, \nabla^{\psi^0} \times u^0) \text{ and } \partial_t^{k+1} f|_{t=0} = \partial_0^{\mu,g,k+1} f^0$$

for $0 \le k \le j$ and also recall the definition of $\widehat{u}^0$ and its initial time derivatives from 11.3.2.

Using this, and the definition of $\tilde{G}^{1,j}$, $\tilde{G}^{2,j}$, $\tilde{G}^{3,j}$ and $\tilde{G}^{4,j}$, we have (integrating by parts) that

$$\begin{aligned}
&-\sum_{\pm}(\pm)(\tilde{G}_3^{2,j}|_{t=0}, w)_{L^2(\Gamma)} + (\tilde{G}^{3,j}|_{t=0}, w_3)_{L^2(\Gamma)} + (\tilde{G}^{4,j}|_{t=0}, \nabla' w_3)_{L^2(\Gamma)}\\
&= -\mu\sum_{\pm}(\pm\partial_0^{\mu,g,j}(N^T(n \times \rho c^2(\nabla^\psi \times (N\ \widehat{u}_\pm^0)) - n \times \rho c^2\omega_\pm)), w_\pm)_{L^2(\Gamma)}\\
&+ (\partial_0^{\mu,g,j}(\mu[\rho c^2\nabla^\psi\cdot\ \widehat{u}^0] - [p] + \mu(\Delta' - 1)\ \widehat{u}_3^0\ -\sigma\nabla'\cdot\hat{n} + g_2), w_3)_{L^2(\Gamma)}
\end{aligned}$$

Using the compatibility condition 7.2.7 at order j, and the fact that $\widehat{u}_3^0 = u^0\cdot n^0$, we have

$$\partial_0^{\mu,g,j}\left(\mu(\Delta' - 1)\ \widehat{u}_3^0\ +\mu[\rho c^2\nabla^\psi\cdot(N\ \widehat{u}^0)] - [p] - \sigma\nabla'\cdot\hat{n} + g_2\right) = 0$$

Also, using $\partial_t^k\omega|_{t=0} = \partial_0^{\mu,g,k}(\nabla^{\psi^0} \times u^0)$, we have

$$\partial_0^{\mu,g,j}\left(n \times \rho c^2(\nabla^\psi \times (N\ \widehat{u}_\pm^0)) - n \times \rho c^2\omega_\pm\right) = 0$$

Hence

$$-\sum_{\pm}(\pm)(\tilde{G}_3^{2,j}|_{t=0}, w)_{L^2(\Gamma)} + (\tilde{G}^{3,j}|_{t=0}, w_3)_{L^2(\Gamma)} + (\tilde{G}^{4,j}|_{t=0}, \nabla' w_3)_{L^2(\Gamma)} = 0$$

We also have

$$\begin{aligned}
&\tilde{G}^{1,j} - \nabla\cdot\tilde{G}^{2,j} = \partial_0^{\mu,g,j}(N^T(g_1 - \nabla^\psi p + \mu\nabla^\psi \times (\rho c^2\omega)\\
&-\mu\nabla^\psi(\rho c^2\nabla^\psi\cdot(N\ \widehat{u}^0)) + \mu\nabla^\psi \times (\rho c^2\nabla^\psi \times (N\ \widehat{u}^0)) - \rho_\mu(\overline{u^\psi}\cdot\nabla)(N\ \widehat{u}^0) - \rho_\mu(\partial_t N)\ \widehat{u}^0))\\
&-\sum_{k+l=j, k<j}\partial_0^{\mu,g,l}(\frac{\rho_\mu^0}{|n^0|^2})\partial_0^{\mu,g,k+1}\ \widehat{u}^0\\
&= \partial_0^{\mu,g,j+1}(\frac{\rho_\mu^0}{|n^0|^2}\ \widehat{u}^0) - \sum_{k+l=j, k<j}\partial_0^{\mu,g,l}(\frac{\rho_\mu^0}{|n^0|^2})\partial_0^{\mu,g,k+1}\ \widehat{u}^0\\
&= \frac{\rho_\mu^0}{|n^0|^2}\partial_0^{\mu,g,j+1}\ \widehat{u}^0
\end{aligned}$$

Hence

$$\sum_{\pm}(\tilde{G}^{1,j}|_{t=0}, w)_{L^2(\Omega_\pm)} - \sum_{\pm}(\nabla \cdot \tilde{G}^{2,j}|_{t=0}, w)_{L^2(\Omega_\pm)} = (\partial_0^{\mu,g,j+1}\, \widehat{u}^0, w)_{B^0}$$

The compatibility condition (7.2.6) at order $j+1$ implies that $\partial_0^{\mu,g,j+1}\, \widehat{u}^0 \in B^1$. Now we check that the regularity on G^i and A implies that

$$\sum_{l=0}^{1} \partial_t^l \tilde{G}^{1,j} \in L^2((0,T); L^2(\Omega_\pm))$$
$$\sum_{l=0}^{2} \partial_t^l \tilde{G}^{2,j} \in L^2((0,T); L^2(\Omega_\pm))$$
$$\sum_{l=0}^{2} \partial_t^l \tilde{G}^{3,j} \in L^2((0,T); L^2(\Gamma))$$
$$\sum_{l=0}^{2} \partial_t^l \tilde{G}^{4,j} \in L^2((0,T); L^2(\Gamma))$$

Using the Sobolev embedding theorem for the terms involving A^i and $\widehat{u}$, we require that

$$\sum_{l=0}^{1} \partial_t^{l+j} G^1 \in L^2((0,T); L^2(\Omega_\pm))$$
$$\sum_{l=0}^{2} \partial_t^{l+j} G^2 \in L^2((0,T); L^2(\Omega_\pm))$$
$$\sum_{l=0}^{2} \partial_t^{l+j} G^3 \in L^2((0,T); L^2(\Gamma))$$
$$\sum_{l=0}^{2} \partial_t^{l+j} G^4 \in L^2((0,T); L^2(\Gamma))$$

which is indeed the case. Thus we may apply proposition 11.6.7 with $h = \partial_0^{\mu,g,j}\, \widehat{u}^0$ to obtain that

$$\partial_t v^j = \partial_t^j\, \widehat{u} \in L^\infty((0,T); B^1)$$
$$\partial_t^2 v^j = \partial_t^{j+2}\, \widehat{u} \in L^2((0,T); B^0)$$
$$\partial_t v^j|_{t=0} = \partial_t^{j+1}\, \widehat{u}\,|_{t=0} = \partial_0^{\mu,g,j}\, \widehat{u}^0$$

Moreover, differentiating the weak equation for $\partial_t^j\, \widehat{u}$ with respect to t we have that $\partial_t^{j+1}\, \widehat{u}$ satisfies the equation

$$\begin{aligned}(\partial_t^{j+2}\, \widehat{u}, w)_{B^0(t)} + \mu\{\partial_t^{j+1}\, \widehat{u}, w\}_{B^1(t)} + \sum_{\pm}(A^2(\partial_t^{j+1}\, \widehat{u}, \nabla\partial_t^{j+1}\, \widehat{u}), w)_{L^2(\Omega_\pm)} \\ = \sum_{\pm}(G^{1,j+1}, w)_{L^2(\Omega_\pm)} + \sum_{\pm}(G^{2,j+1}, \nabla w)_{L^2(\Omega_\pm)} + (G^{3,j+1}, w_3)_{L^2(\Gamma)} + (G^{4,j+1}, \nabla' w_3)_{L^2(\Gamma)}\end{aligned}$$

for all $w \in B^1$ for almost every $t \in (0,T)$, where $G^{i,j+1}$ are of the claimed form.

Given that $\partial_t^{j+2}\nabla\, \widehat{u} \in L^2((0,T); L^2(\Omega_\pm))$, we can successively apply proposition 11.6.9 to the equations for $\partial_t^{j+1-k}\, \widehat{u}$ for $0 \le k \le j+1$ to obtain the spatial regularity $\partial_t^{j+1-k}\nabla\, \widehat{u} \in L^2((0,T); H^k(\Omega_\pm))$,

$\partial_t^{j+1-k}\ \widehat{u}_3 \in L^2((0,T);H^{k+2}(\Gamma))$ with corresponding norms bounded by $C_{\max}$. Note that this requires the regularity

$$\begin{aligned}
\partial_t^k G^1 &\in L^2((0,T);H^{j+1-k}(\Omega_\pm)) \\
\partial_t^k G^2 &\in L^2((0,T);H^{j+2-k}(\Omega_\pm)) \\
\partial_t^k G^3 &\in L^2((0,T);H^{j+1-k}(\Gamma)) \\
\partial_t^k G^4 &\in L^2((0,T);H^{j+2-k}(\Gamma))
\end{aligned}$$

for $0 \leq k \leq j+1$, which is indeed satisfied, since $0 \leq j \leq 2$.

Note that we have shown in the course of this proof that $\partial_t^{j+1}\ \widehat{u}_3 \in L^\infty((0,T);H^1(\Gamma))$, which came from applying 11.6.6 to the equation for $\partial_t^j\ \widehat{u}$. We also have from the above paragraph that $\partial_t^k\ \widehat{u}_3 \in L^2((0,T);H^2(\Gamma))$ for $0 \leq k \leq j+1$. To show that $\partial_t^k\ \widehat{u}_3 \in L^\infty((0,T);H^{j+2-k}(\Gamma))$ for $0 \leq k \leq j$, we simply use the estimate

$$\begin{aligned}
\left|\left|\partial_t^k\ \widehat{u}_3\right|\right|^2_{H^{j+2-k}(\Gamma)} &\leq C \left|\left|\partial_0^{\mu,g,k}\ \widehat{u}_3^0\right|\right|^2_{H^{j+2-k}(\Gamma)} + \int_0^t \left|\left|\partial_t^{k+1}\ \widehat{u}_3\right|\right|^2_{H^{j+2-k}(\Gamma)} \mathrm{d}\tau \\
&\leq C_{\max}
\end{aligned}$$

by the existing estimate for $\left|\left|\partial_t^{k+1}\ \widehat{u}_3\right|\right|_{H^{j+2-k}(\Gamma)}$.

This completes the proof of the induction step and completes the proof of the proposition. □

We now show the additional boundary regularity. We will use the elliptic estimate lemma 8.16.1.

Lemma 11.7.3. *For $0 \leq j \leq 3$, we have $\partial_t^j\ \widehat{u}_3 \in L^2((0,T);H^{5.5-j}(\Gamma))$ with corresponding norms bounded by $C_{\max}$.*

Proof. Applying ∂_t^j to the pressure interface condition (11.3.3) we have

$$(1-\Delta')\partial_t^j\ \widehat{u}_3 = \partial_t^j(\frac{1}{\mu}(-[p] - \sigma\nabla' \cdot \hat{n} + g_2) + [\rho c^2 \nabla^\psi \cdot (N\ \widehat{u})])$$

We apply the lemma 8.16.1 with

$$\begin{aligned}
z &= \partial_t^j\ \widehat{u}_3 \\
h &= \partial_t^j(\frac{1}{\mu}(-[p] - \sigma\nabla' \cdot \hat{n} + g_2) + [\rho c^2 \nabla^\psi \cdot (N\ \widehat{u})])
\end{aligned}$$

and the Sobolev trace estimate to conclude that for almost every t, $\partial_t^j\ \widehat{u}_3 \in H^{5.5-j}(\Gamma)$ with

$$\left|\left|\partial_t^j\ \widehat{u}_3\right|\right|^2_{H^{5.5-j}(\Gamma)} \leq C \left|\left|\partial_t^j(\frac{1}{\mu}(-[p] - \sigma\nabla' \cdot \hat{n} + g_2) + [\rho c^2 \nabla^\psi \cdot (N\ \widehat{u})])\right|\right|^2_{H^{3.5-j}(\Gamma)}$$

Integrating from 0 to t and using existing estimates together with the product rule, we obtain

$$\int_0^t \left|\left|\partial_t^j\ \widehat{u}_3\right|\right|^2_{H^{5.5-j}(\Gamma)} \leq C_{\max}$$

This completes the proof. □

Applying proposition 11.7.2, lemma 11.7.3 and proposition 11.3.4, we prove proposition 11.2.1. Note that together with the results in the previous section, we have now proved 9.5.1.

12. Proof of the Energy Estimate for the Linearised Equations

In this section we prove the energy estimate for the linearised equations stated in 9.4.1. The energy estimate is necessary as part of the construction of a well-defined contraction map sending a background state to a solution of the linearised equations linearised about that state. See 3.8 for a summary of the construction of the contraction map. The main steps in the energy estimate are quite clear. The front is estimated by direct integration and the entropy, pressure and curl are estimated in the standard manner of energy estimates for transport equations, with some minor differences in the curl estimate. The velocity is then estimated in a manner which is fairly standard for parabolic equations. We then obtain a slightly higher order estimate for $u \cdot n$ from the elliptic equation provided by the interface condition (9.1.9).

Our main task will be to show that

$$\begin{aligned}\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq \frac{1}{\epsilon} M_\mu^0 + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau) \mathrm{d}\tau) \\ + \epsilon(M_\mu^0 + tK_\mu(t))(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t))\end{aligned}$$

where we are free to choose $\epsilon \in (0, 1]$.

12.1 A Chain Rule Lemma

Lemma 12.1.1. *Let G be a smooth, possibly matrix-valued, function, $v : (0, T) \times \Omega_\pm \to \mathbb{R}^d$ for some $d \geq 1$ and let β be a multi-index with $1 \leq |\beta| \leq 4$. Then*

$$\begin{aligned}\left|\left|\partial^\beta G(v)\right|\right|_{L^2(\Omega_\pm)} \leq F(||v||_{L^\infty(\Omega_\pm)}, \sum_{k=1}^{|\beta|-2} \left|\left|\partial^k v\right|\right|_{H^1(\Omega_\pm)})(||\partial^2 v||_{H^1(\Omega_\pm)} + \sum_{\gamma \leq \beta, \gamma \neq \beta} ||\partial^\gamma v||_{L^2(\Omega_\pm)}) \\ + F(||v||_{L^\infty(\Omega_\pm)}) \left|\left|\partial^\beta v\right|\right|_{L^2(\Omega_\pm)} \quad (12.1.1)\end{aligned}$$

for some smooth increasing function F, provided that v is regular enough for the right hand side to be finite. In particular, if $v = (\bar{U}, \rho_\mu, U, \partial\bar{\psi}, \partial\psi)$ and $1 \leq |\alpha| \leq 3$ then

$$||\partial^\alpha G(v)||^2_{H^1(\Omega_\pm)} \leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \quad (12.1.2)$$

Now let $w : (0, T) \times \Omega_\pm \to \mathbb{R}^{d'}$ for some $d' \geq 1$ and let β be a multi-index with $1 \leq |\beta| \leq 4$. Then

$$\begin{aligned}\left|\left|\sum_{\gamma+\delta=\beta, \delta\neq\beta} \partial^\gamma G(v) \partial^\delta w\right|\right|_{L^2(\Omega_\pm)} \leq F(||v||_{L^\infty(\Omega_\pm)}, \sum_{k=1}^{|\beta|-2} \left|\left|\partial^k v\right|\right|_{H^1(\Omega_\pm)}) \sum_{\gamma \leq \beta, \gamma \neq \beta} ||\partial^\gamma w||_{L^2(\Omega_\pm)} \\ + ||\partial w||_{H^1(\Omega_\pm)} F(||v||_{L^\infty(\Omega_\pm)}) \left|\left|\partial^{|\beta|-1} v\right|\right|_{H^1(\Omega_\pm)} + ||w||_{L^\infty(\Omega_\pm)} F(||v||_{L^\infty(\Omega_\pm)}) \left|\left|\partial^\beta v\right|\right|_{L^2(\Omega_\pm)} \\ (12.1.3)\end{aligned}$$

for some smooth increasing function F, provided that v and w are regular enough for the right hand side to be finite. In particular, if $v = (\bar{U}, U, \partial\bar{\psi}, \partial\psi)$, $||w||^2_{H^2(\Omega_\pm)} \le K_\mu(t)$ and $1 \le |\alpha| \le 3$, then

$$\left|\left|\sum_{\beta+\gamma=\alpha,\gamma\neq\alpha} \partial^\beta G(v)\partial^\gamma w\right|\right|^2_{H^1(\Omega_\pm)} \le K_\mu(t)(\sum_{k=0}^{2} \left|\left|\partial^k w\right|\right|^2_{H^1(\Omega_\pm)} + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \tag{12.1.4}$$

Proof. For $0 \le |\gamma| \le 4$, the chain rule implies that $\partial^\gamma G(v)$ is a sum of terms of the form

$$G_k(v)\prod_{i=1}^{l}\partial^{\gamma^i} v_{e(i)}$$

for some smooth function G_k, where $\sum_{i=1}^{l}\gamma^i = \gamma$ and $e(i)$ denotes an entry of v depending on i. Note that if $\left|\gamma^i\right| \ge 3$ for some i or $|\gamma| \le 3$ and $\left|\gamma^i\right| \ge 2$ for some i, then $\left|\gamma^j\right| \le 1$ for all $j \ne i$ and if $\left|\gamma^i\right| = 2$ for some i, then in fact $\left|\gamma^j\right| \le 2$ for all j. Using these observations and the product rule for Sobolev functions and the Sobolev embedding theorem where necessary, we obtain the result. □

12.2 Matching Initial Time Derivatives

We claim that

$$\partial_t^j(U_+, U_-, \omega_+, \omega_-, f)|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, \nabla^{\psi^0} \times u_+^0, \nabla^{\psi^0} \times u_-^0, f^0) \text{ for } 0 \le j \le 3 \tag{12.2.1}$$

Indeed, note that the background state satisfies

$$\partial_t^j(\bar{U}_+, \bar{U}_-, \bar{f})|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \le j \le 3 \tag{12.2.2}$$

by assumption, and note that (12.2.1) holds for $j = 0$ by assumption (this was part of the definition of a solution of the linearised equations). Using these two facts we see that the linearised equations for f, p, s and u are the same as the μ-approximate equations when restricted to $t = 0$ and the linearised equation for ω is the same as the μ-approximate equation for $\nabla^\psi \times u$ when restricted to $t = 0$, which shows that (12.2.1) holds for $j = 1$. Now this allows us to differentiate the equations with respect to time and obtain the case $j = 2$. We continue until we reach $j = 3$. (Note we use the fact that the background state satisfies (12.2.2) up to $j = 3$).

12.3 Lower Order Estimate

We start with the following simple lower order estimate.

Lemma 12.3.1. *We have the following lower order estimates.*

$$\sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} ||G(p,s,u)||^2_{L^\infty(\Omega_\pm)} + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||G(f)||^2_{W^{1,\infty}(\Gamma)} \le M_\mu^0 + tK_\mu(t) \tag{12.3.1}$$

$$\sum_{1\le|\beta|\le 2} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\beta(p,s,u)\right|\right|^2_{L^2(\Omega_\pm)} + \sum_{0\le|\beta|\le 1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\beta \nabla u\right|\right|^2_{H^1(\Omega_\pm)} \le M_\mu^0 + tK_\mu(t) \tag{12.3.2}$$

$$\sum_{0\le|\beta|\le 2} \left|\left|\partial^\beta \nabla' f\right|\right|^2_{H^{1.5}(\Gamma)} \le M_\mu^0 + tK_\mu(t) \tag{12.3.3}$$

where G is a smooth function of its arguments.

Proof. For the first estimate, we have

$$\begin{aligned}|G(p,s,u)|^2 &\le |G(p,s,u)|_{t=0}|^2 + C\int_0^t |\partial_t G(p,s,u)|^2 \,\mathrm{d}\tau \\ &\le F(||(p,s,u)|_{t=0}||^2_{L^\infty(\Omega_\pm)}) + C\int_0^t F(||(p,s,u)||^2_{L^\infty(\Omega_\pm)})\,||\partial_t(p,s,u)||^2_{H^2(\Omega_\pm)}\,\mathrm{d}\tau \\ &\le M^0_\mu + tK_\mu(t)\end{aligned}$$

Similarly have have

$$\begin{aligned}&\left|(G(f),\nabla' G(f))\right|^2 \le \left|(G(f),\nabla' G(f))_{t=0}\right|^2 + C\int_0^t \left|\partial_t(G(f),\nabla' G(f))\right|^2 \mathrm{d}\tau \\ &\le F(\left|\left|(f^0,\nabla' f^0)\right|\right|^2_{L^\infty(\Gamma)}) + C\int_0^t F(\left|\left|(f,\nabla' f)\right|\right|^2_{L^\infty(\Gamma)})\,\left|\left|\partial_t(f,\nabla' f)\right|\right|^2_{H^2(\Gamma)}\,\mathrm{d}\tau \\ &\le M^0_\mu + tK_\mu(t)\end{aligned}$$

Taking the supremum over (τ,x) or (τ,x') for $\tau\in(0,t)$ on the left hand side, this proves the first estimate.

For the second estimate, we have

$$\begin{aligned}&\sum_\pm\sum_{1\le|\beta|\le2}\left|\left|\partial^\beta(p,s,u)\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm\sum_{0\le|\beta|\le1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial^\beta\nabla u\right|\right|^2_{H^1(\Omega_\pm)} \\ &\le M^0_\mu + C\sum_\pm\sum_{1\le|\beta|\le2}\int_0^t\left|\left|\partial_t\partial^\beta(p,s,u)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau + C_\mu\mu^2\sum_\pm\sum_{0\le|\beta|\le1}\int_0^t\left|\left|\partial_t\partial^\beta\nabla u\right|\right|^2_{H^1(\Omega_\pm)}\mathrm{d}\tau \\ &\le M^0_\mu + C_\mu\int_0^t E_\mu(\tau)\mathrm{d}\tau\end{aligned}$$

Finally, let us note that

$$\begin{aligned}\sum_{0\le|\beta|\le2}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial^\beta\nabla' f\right|\right|^2_{H^{1.5}(\Gamma)} &\le M^0_\mu + C_\mu\mu^2\sum_{0\le|\beta|\le2}\int_0^t\left|\left|\partial^\beta(\bar u\cdot\bar n)\right|\right|^2_{H^{2.5}(\Gamma)}\mathrm{d}\tau \\ &\le M^0_\mu + C_\mu\int_0^t\bar E_\mu(\tau)\mathrm{d}\tau\end{aligned}$$

□

12.4 Estimate of the Front

Proposition 12.4.1. *Define*

$$\mathcal{E}_L(f)(t) := \operatorname*{ess\,sup}_{\tau\in(0,t)}\sum_{0\le j\le3}\left|\left|\partial_t^j\nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)} + \int_0^t\left|\left|\partial_t^4 f\right|\right|^2_{H^1(\Gamma)}\mathrm{d}\tau$$

The front satisfies the following estimates for $t\in(0,T)$.

$$\mathcal{E}_L(f)(t) \le M^0_\mu + \frac{C}{\epsilon}\int_0^t\mathcal{E}_L(\tau)\mathrm{d}\tau + \epsilon\bar{\mathcal{E}}_L(t) \tag{12.4.1}$$

$$\operatorname*{ess\,sup}_{\tau\in(0,t)}\sum_{0\le j\le3}\left|\left|\partial_t^j\nabla' f\right|\right|^2_{H^{3-j}(\Gamma)} + \sum_{0\le j\le3}\int_0^t\left|\left|\partial_t^{j+1}\nabla' f\right|\right|^2_{H^{3-j}(\Gamma)}\mathrm{d}\tau \le M^0_\mu + C\int_0^t\bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau \tag{12.4.2}$$

$$\sum_{0\le j\le3}\int_0^t\left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{5.5-j}(\Gamma)}\mathrm{d}\tau + \sum_{0\le j\le3}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{4-j}(\Gamma)} \le C\bar{\mathcal{E}}_L(t) \tag{12.4.3}$$

Moreover, we have

$$J^{\psi} \geq \kappa^0 - tK_\mu(t) \tag{12.4.4}$$

Proof. We note that

$$\begin{aligned}
&\sum_{0\le j\le 3}\int_0^t \left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau + \sum_{0\le j\le 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{4-j}(\Gamma)} \\
&= \sum_{0\le j\le 3}\int_0^t \left|\left|\partial_t^{j}(\bar{u}\cdot\bar{n})\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau + \sum_{0\le j\le 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^{j}(\bar{u}\cdot\bar{n})\right|\right|^2_{H^{4-j}(\Gamma)} \\
&\le C\bar{\mathcal{E}}_L(t)
\end{aligned}$$

by definition of $\bar{\mathcal{E}}_L(t)$. This proves the third estimate.

Now note that, for $0 \le j \le 3$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)} &= 2(\partial_t^{j+1}\nabla' f, \partial_t^j \nabla' f)_{H^{4.5-j}(\Gamma)} \\
&\le \epsilon \left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{5.5-j}(\Gamma)} + \frac{1}{\epsilon}\left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)}
\end{aligned}$$

Integrating in time we obtain

$$\begin{aligned}
\left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)} &\le M^0 + \epsilon\int_0^t \left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau + \frac{1}{\epsilon}\int_0^t \left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{4.5-j}(\Gamma)} \mathrm{d}\tau \\
&\le M_\mu^0 + C\epsilon\bar{\mathcal{E}}_L(t) + \frac{1}{\epsilon}\int_0^t \mathcal{E}_L(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the third estimate for the second term. Together with a direct integrating in time of the third estimate, this proves the first estimate.

To prove the second estimate we note that for $0 \le j \le 3$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla' f\right|\right|^2_{H^{3-j}(\Gamma)} + \int_0^t \left|\left|\partial_t^{j+1}\nabla' f\right|\right|^2_{H^{3-j}(\Gamma)} \mathrm{d}\tau \\
&\le M_\mu^0 + C\int_0^t \left|\left|\partial_t^{j+1}\nabla' f\right|\right|^2_{H^{3-j}(\Gamma)} \mathrm{d}\tau + \int_0^t \left|\left|\partial_t^{j+1} f\right|\right|^2_{H^{4-j}(\Gamma)} \mathrm{d}\tau \\
&\le M_\mu^0 + C\int_0^t \bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the third estimate.

We have by Sobolev embedding and the definition of ψ that

$$\begin{aligned}
J^{\psi} &\ge J^{\psi^0} - \int_0^t ||\partial_t \partial_{x_3}\psi||_{H^2(\Gamma)} \mathrm{d}\tau \\
&\ge \kappa^0 - C\int_0^t (1 + ||\partial_t f||^2_{H^2(\Gamma)})\mathrm{d}\tau \\
&\ge \kappa^0 - tK_\mu(t)
\end{aligned}$$

This completes the proof. $\square$

12.5 Estimate of the Entropy, Pressure and Curl

We start with an estimate for general transport equations of the correct form.

Lemma 12.5.1. *In this lemma functions will be defined either on Ω_+ or Ω_-.*

Let $b \in L^\infty((0,T); W^{1,\infty}(\Omega_\pm;\mathbb{R}^3))$, $h \in L^2((0,T); L^2(\Omega_\pm))$ and $w \in L^2((0,T); H^1(\Omega_\pm))$ with $\partial_t w \in L^2((0,T); L^2(\Omega_\pm))$. Suppose also that $b_3|_\Gamma = 0$ and that w satisfies the equation

$$(\partial_t + b\cdot\nabla)w = h$$

in $(0,T)\times\Omega_\pm$. Then w satisfies the following estimates, where we are free to choose $\epsilon \in (0,1]$.

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau\in(0,t)} ||w||^2_{L^2(\Omega_\pm)} \leq ||w(0)||^2_{L^2(\Omega_\pm)} + (\frac{1}{\epsilon} + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)}) \int_0^t ||w||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \\ + \epsilon\int_0^t ||h||^2_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau\end{aligned} \tag{12.5.1}$$

$$\int_0^t ||\partial_t w||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \leq C \operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} \int_0^t ||\nabla w||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau + C\int_0^t ||h||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \tag{12.5.2}$$

Moreover, if $h \in L^2((0,T); H^1(\Omega_\pm))$ and $w(0) \in H^1(\Omega_\pm)$ then

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau\in(0,t)} ||w||^2_{H^1(\Omega_\pm)} \leq ||w(0)||^2_{H^1(\Omega_\pm)} + (\frac{1}{\epsilon} + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)}) \int_0^t ||w||^2_{H^1(\mathbb{R}^d)}\,\mathrm{d}\tau \\ + \epsilon\int_0^t ||h||^2_{H^1(\Omega_\pm)}\,\mathrm{d}\tau\end{aligned} \tag{12.5.3}$$

where we are free to choose $\epsilon \in (0,1]$.

Proof. The second estimate is obtained easily by rearranging the equation to give

$$\partial_t w = -b\cdot\nabla w + h$$

then squaring both sides and integrating in space and time.

For the first estimate we multiply the equation by w, then use the product rule to obtain

$$\frac{1}{2}\partial_t |w|^2 + \frac{1}{2}\nabla\cdot(b\,|w|^2) = \frac{1}{2}(\nabla\cdot b)\,|w|^2 + hw$$

We then integrate in space and time, using the divergence theorem for the second term, together with the fact that $b_3|_\Gamma = 0$ which implies the boundary term vanishes, to obtain

$$\begin{aligned}||w||^2_{L^2(\Omega_\pm)} &= ||w(0)||^2_{L^2(\Omega_\pm)} + \int_0^t\int_{\Omega_\pm} (\nabla\cdot b)\,|w|^2\,\mathrm{d}x\mathrm{d}\tau + 2\int_0^t\int_{\Omega_\pm} hw\mathrm{d}x\mathrm{d}\tau \\ &\leq ||w(0)||^2_{L^2(\Omega_\pm)} + (\operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} + \frac{1}{\epsilon}) \int_0^t ||w||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau + \epsilon\int_0^t ||h||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\end{aligned}$$

where we have used Cauchy's inequality. This proves the first estimate.

To prove the third estimate, we would like to differentiate with respect to x_i, but w is not regular enough. To overcome this, we let $\eta \in C_c^\infty(\mathbb{R}^3)$ be the standard mollifier, and for $\nu \in (0,1]$ we define $\eta_\nu(x) := \nu^{-3}\eta(\frac{x}{\nu})$ in the standard manner. We apply the Sobolev extension operator $\mathrm{Ext}_{\Omega_\pm}$ in space to extend b, h and w^0 to be defined on $\mathbb{R}^3$. We then define $\tilde{w}$ to be the unique solution to the equation

$$(\partial_t + b\cdot\nabla)\tilde{w} = h$$

in $(0,T)\times\mathbb{R}^3$ with $\tilde{w}(0)=w^0$, as given by 10.2.2. Note that the additional regularity we have assumed on w^0 and h allow us to apply this result. Note also that by the uniqueness of the solution to the equation in $\Omega_\pm$ stated in 10.2.3, we have that $\tilde{w}|_{\Omega_\pm}=w$. We fix $\delta>0$, then take the convolution of the equation with η_ν, where $\nu\le\frac{\delta}{2}$, to obtain

$$\partial_t\tilde{w}_\nu+(b\cdot\nabla\tilde{w})*\eta_\nu=h_\nu$$

where $\tilde{w}_\nu=\tilde{w}*\eta_\nu$ and $h_\nu=h*\eta_\nu$. We now apply ∂_{x_i} where $1\le i\le 3$ to obtain

$$\partial_t\partial_{x_i}\tilde{w}_\nu+(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu=\partial_{x_i}h_\nu$$

which we rewrite as

$$\partial_t\partial_{x_i}\tilde{w}_\nu+b\cdot\nabla\partial_{x_i}\tilde{w}_\nu=\partial_{x_i}h_\nu-(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu+b\cdot\nabla\partial_{x_i}\tilde{w}_\nu$$

Now we apply the first estimate to $\partial_{x_i}\tilde{w}_\nu$ to obtain

$$\begin{aligned}||\partial_{x_i}\tilde{w}_\nu||^2_{L^2(\Omega_\pm)}\le{}&||\partial_{x_i}\tilde{w}_\nu(0)||^2_{L^2(\Omega_\pm)}+(\operatorname*{ess\,sup}_{\tau\in(0,t)}||b||^2_{W^{1,\infty}(\Omega_\pm)}+\frac{1}{\epsilon})\int_0^t||\partial_{x_i}\tilde{w}_\nu||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\&+\epsilon\int_0^t||\partial_{x_i}h_\nu||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau+\int_0^t||(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu-b\cdot\nabla\partial_{x_i}\tilde{w}_\nu||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\end{aligned}$$

We claim that

$$||(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu-b\cdot\nabla\partial_{x_i}\tilde{w}_\nu||_{L^2(\Omega_\pm)}\le C\,||b||_{W^{1,\infty}(\Omega_\pm)}\,||\nabla\tilde{w}||_{L^2(\Omega^\delta_\pm)}$$

where

$$\begin{aligned}\Omega^\delta_+&=\{x\in\mathbb{R}^3:x_3>-\delta\}\\\Omega^\delta_-&=\{x\in\mathbb{R}^3:x_3<\delta\}\end{aligned}$$

Indeed, note that

$$\begin{aligned}(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu-b\cdot\nabla\partial_{x_i}\tilde{w}_\nu&=(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu-b\cdot(\nabla\tilde{w}*\partial_{x_i}\eta_\nu)\\&=\int_{\mathbb{R}^3}(b(y)-b(x))\cdot\nabla\tilde{w}(y)\partial_{x_i}\eta_\nu(x-y)\mathrm{d}y\\&=\frac{1}{\nu}\int_{|y-x|\le\nu}(b(y)-b(x))\cdot\nabla\tilde{w}(y)(\partial_{x_i}\eta)_\nu(x-y)\mathrm{d}y\end{aligned}$$

where $(\partial_{x_i}\eta)_\nu(x):=\nu^{-3}(\partial_{x_i}\eta)(\frac{x}{\nu})$. Hence, applying the mean value theorem to $b(y)-b(x)$ and using $|y-x|\le\nu$ to cancel the factor $\frac{1}{\nu}$, we have

$$\begin{aligned}|(b\cdot\nabla\tilde{w})*\partial_{x_i}\eta_\nu-b\cdot\nabla\partial_{x_i}\tilde{w}_\nu|&\le||b||_{W^{1,\infty}(\mathbb{R}^3)}\int_{\mathbb{R}^3}|(\partial_{x_i}\eta)_\nu(x-y)|\,|\nabla\tilde{w}(y)|\,\mathrm{d}y\\&\le C\,||b||_{W^{1,\infty}(\Omega_\pm)}\,|(\partial_{x_i}\eta)_\nu|*|\nabla w(y)|\end{aligned}$$

But, applying Young's inequality for convolutions we have

$$\begin{aligned}&|||(\partial_{x_i}\eta)_\nu|*|\nabla\tilde{w}(y)|||_{L^2(\Omega_\pm)}\le\left|\left||(\partial_{x_i}\eta)_\nu|*\left|\nabla\tilde{w}(y)\mathbb{1}_{\Omega^\delta_\pm}(y)\right|\right|\right|_{L^2(\mathbb{R}^3)}\\&\le||(\partial_{x_i}\eta)_\nu||_{L^1(\mathbb{R}^3)}\left|\left|\nabla\tilde{w}\mathbb{1}_{\Omega^\delta_\pm}\right|\right|_{L^2(\mathbb{R}^3)}\\&\le C\,||\nabla\tilde{w}||_{L^2(\Omega^\delta_\pm)}\end{aligned}$$

which completes the proof of the claim.

Thus, summing over $1 \leq i \leq 3$, we obtain

$$\begin{aligned}||\nabla \tilde{w}_\nu||^2_{L^2(\Omega_\pm)} &\leq ||\nabla \tilde{w}_\nu(0)||^2_{L^2(\Omega_\pm)} + (\operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} + \frac{1}{\epsilon}) \int_0^t ||\nabla \tilde{w}_\nu||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau \\ &+ \epsilon \int_0^t ||\nabla h_\nu||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau + C \operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} \int_0^t ||\nabla \tilde{w}||_{L^2(\Omega^\delta_\pm)} \,\mathrm{d}\tau \\ &\leq ||\nabla \tilde{w}_\nu(0)||^2_{L^2(\Omega_\pm)} + C(\operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} + \frac{1}{\epsilon}) \int_0^t ||\nabla \tilde{w}||^2_{L^2(\Omega^\delta_\pm)} \,\mathrm{d}\tau \\ &+ \epsilon \int_0^t ||\nabla h_\nu||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau\end{aligned}$$

Letting $\nu \to 0$ and using the properties of mollifiers, we obtain

$$\begin{aligned}||\nabla w||^2_{L^2(\Omega_\pm)} &\leq ||\nabla w(0)||^2_{L^2(\Omega_\pm)} + C(\operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} + \frac{1}{\epsilon}) \int_0^t ||\nabla \tilde{w}||^2_{L^2(\Omega^\delta_\pm)} \,\mathrm{d}\tau \\ &+ \epsilon \int_0^t ||\nabla h||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\begin{aligned}||\nabla w||^2_{L^2(\Omega_\pm)} &\leq ||\nabla w(0)||^2_{L^2(\Omega_\pm)} + C(\operatorname*{ess\,sup}_{\tau\in(0,t)} ||b||^2_{W^{1,\infty}(\Omega_\pm)} + \frac{1}{\epsilon}) \int_0^t ||\nabla w||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau \\ &+ \epsilon \int_0^t ||\nabla h||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau\end{aligned}$$

as required. ☐

Lemma 12.5.2. *We have that $\overline{u^\psi}$ satisfies the following estimate.*

$$\left|\left|\partial^\gamma \overline{u^\psi}\right|\right|^2_{L^\infty(\Omega_\pm)} \leq K_\mu(t) \textit{ for } 0 \leq |\gamma| \leq 1$$

Proof. Let us note that

$$\overline{u^\psi} = G(\partial_t\psi, \nabla\psi, \nabla'\bar{\psi}, \bar{u})$$

for some smooth function G (at least in the region where $J^\psi > 0$). The estimate follows immediately using the chain rule and the Sobolev embedding theorem. ☐

Proposition 12.5.3. *Define*

$$\mathcal{E}_L(s)(t) := \sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} ||s||^2_{L^\infty(\Omega_\pm)} + \sum_\pm \sum_{1\leq|\alpha|\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha s||^2_{H^1(\Omega_\pm)} + \sum_\pm \int_0^t \left|\left|\partial_t^4 s\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau$$

The entropy s satisfies the following energy estimate.

$$\mathcal{E}_L(s)(t) \leq M^0_\mu + K_\mu(t)(\int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau) \tag{12.5.4}$$

for $t \in (0, T)$.

Proof. Let $1 \leq |\alpha| \leq 3$ and apply ∂^α to the linearised entropy equation (9.1.3) to obtain

$$\partial_t \partial^\alpha s + \overline{u^\psi} \cdot \nabla \partial^\alpha s = R_L^\alpha(s)$$

where

$$R_L^\alpha(s) = -\sum_{\beta+\gamma=\alpha, \gamma\neq\alpha} \partial^\beta \overline{u^\psi} \cdot \partial^\gamma \nabla s$$

We claim that

$$||R_L^\alpha(s)||^2_{H^1(\Omega_\pm)} \leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

Indeed, writing $\overline{u^\psi} = \tilde{G}(\partial_{x_3}\psi)G(v)$, where $v = (\partial_t\psi, \nabla'\psi, \nabla'\bar{\psi}, \bar{u})$, and writing $w = \nabla s$, it follows immediately from (12.1.4) that

$$\begin{aligned}
||R_L^\alpha(s)||^2_{H^1(\Omega_\pm)} &\leq K_\mu(t)(\sum_{k=0}^{2} \left|\left|\partial^k \nabla s\right|\right|^2_{H^1(\Omega_\pm)} + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \\
&\leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))
\end{aligned}$$

as required.

Applying the estimate (12.5.3) with $w = \partial^\alpha s$, $b = \overline{u^\psi}$ and $h = R_L^\alpha(s)$ we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha s||^2_{H^1(\Omega_\pm)} &\leq ||\partial^\alpha s(0)||^2_{H^1(\Omega_\pm)} + (1 + \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\overline{u^\psi}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}) \int_0^t ||\partial^\alpha s||^2_{H^1(\mathbb{R}^d)} \,\mathrm{d}\tau \\
&\quad + \int_0^t ||R_L^\alpha(s)||^2_{H^1(\Omega_\pm)} \,\mathrm{d}\tau \\
&\leq M_\mu^0 + K_\mu(t)(\int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau)
\end{aligned}$$

where we have used the above estimate for $R_L^\alpha(s)$.

We now apply the estimate (12.5.2) to the differentiated equation to obtain

$$\begin{aligned}
\int_0^t ||\partial_t\partial^\alpha s||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau &\leq C \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\overline{u^\psi}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)} \int_0^t ||\nabla\partial^\alpha s||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau \\
&\quad + C \int_0^t ||R_L^\alpha(s)||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau \\
&\leq K_\mu(t) \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the above estimate for $R_L^\alpha(s)$. □

Proposition 12.5.4. *Define*

$$\mathcal{E}_L(p)(t) := \sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} ||p||^2_{L^\infty(\Omega_\pm)} + \sum_\pm \sum_{1\leq|\alpha|\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha p||^2_{H^1(\Omega_\pm)} + \sum_\pm \int_0^t \left|\left|\partial_t^4 p\right|\right|^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau$$

The pressure p satisfies the following energy estimate.

$$\mathcal{E}_L(p)(t) \leq M_\mu^0 + \frac{1}{\epsilon} K_\mu(t)(\int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau) + \epsilon(M_\mu^0 + tK_\mu(t))(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \qquad (12.5.5)$$

for $t \in (0, T)$, where we are free to choose $\epsilon > 0$.

Proof. Let $1 \leq |\alpha| \leq 3$ and apply ∂^α to the linearised pressure equation (9.1.4) to obtain

$$\partial_t \partial^\alpha p + \overline{u^\psi} \cdot \nabla \partial^\alpha p = \partial^\alpha G_p + R_L^\alpha(p)$$

where

$$G_p := -\bar{\rho}\bar{c}^2 \nabla^\psi \cdot \bar{u}$$

Note that we have

$$||R_L^\alpha(p)||^2_{H^1(\Omega_\pm)} \leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

exactly as in the estimate of the entropy. We claim that

$$\int_0^t ||\partial^\alpha G_p||^2_{H^1(\Omega_\pm)} \, \mathrm{d}\tau \leq (M_\mu^0 + tK_\mu(t))(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

and

$$\int_0^t ||\partial^\alpha G_p||^2_{L^2(\Omega_\pm)} \, \mathrm{d}\tau \leq K_\mu(t) \int_0^t \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) \mathrm{d}\tau$$

Indeed, we write $G_p = \tilde{G}(\partial_{x_3}\psi)G(v)w$ where $v = (\bar{p}, \bar{s}, \nabla'\psi)$, $\tilde{G}$ is a smooth function and G is a smooth vector-valued function and $w = \nabla \bar{u}$. Hence

$$\partial^\alpha G_p = \sum_{\beta+\gamma=\alpha, \gamma\neq\alpha} \partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\partial^\gamma w + \tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w$$

Applying (12.1.4) and estimating $\partial_{x_3}^{k+1}\psi$ in terms of $||f||_{L^\infty(\Gamma)}$ for $0 \leq k \leq 4$ we immediately obtain

$$\left|\left| \sum_{\beta+\gamma=\alpha, \gamma\neq\alpha} \partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\partial^\gamma w \right|\right|^2_{H^1(\Omega_\pm)} \leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

We also note that

$$\left|\left| \tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w \right|\right|_{L^2(\Omega_\pm)} \leq K_\mu(t)\bar{\mathcal{E}}_L(t)$$

Thus integrating with respect to time proves the second estimate in the claim. Now note that

$$\begin{aligned} \left|\left| \tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w \right|\right|^2_{H^1(\Omega_\pm)} &\leq K_\mu(t)\bar{\mathcal{E}}_L(t) + C \left|\left| \tilde{G}(\partial_{x_3}\psi)G(v) \right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial^\alpha \nabla \bar{u}||^2_{H^1(\Omega_\pm)} \\ &\leq K_\mu(t)\bar{\mathcal{E}}_L(t) + (M^0 + tK_\mu(t)) \, ||\partial^\alpha \nabla \bar{u}||^2_{H^1(\Omega_\pm)} \end{aligned}$$

where we have used the lower order estimate (12.3.1). Integrating with respect to time proves the claim.

Applying the estimate (12.5.3) with $w = \partial^\alpha p$, $b = \overline{u^\psi}$ and $h = \partial^\alpha G_p + R_L^\alpha(p)$ we obtain

$$\begin{aligned} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha p||^2_{H^1(\Omega_\pm)} &\leq ||\partial^\alpha p(0)||^2_{H^1(\Omega_\pm)} + (\frac{1}{\epsilon} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\overline{u^\psi}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}) \int_0^t ||\partial^\alpha p||^2_{H^1(\mathbb{R}^d)} \, \mathrm{d}\tau \\ &+ \int_0^t ||R_L^\alpha(p)||^2_{H^1(\Omega_\pm)} \, \mathrm{d}\tau + \epsilon \int_0^t ||\partial^\alpha G_p||^2_{H^1(\Omega_\pm)} \, \mathrm{d}\tau \\ &\leq M_\mu^0 + \frac{1}{\epsilon} K_\mu(t) (\int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) \mathrm{d}\tau) + \epsilon(M_\mu^0 + tK_\mu(t))(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \end{aligned}$$

where we have used the above estimates for $R_L^\alpha(p)$ and $\partial^\alpha G_p$.

We now apply the estimate (12.5.2) to the differentiated equation to obtain

$$\begin{aligned}\int_0^t ||\partial_t\partial^\alpha p||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau &\leq C\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\overline{u^\psi}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}\int_0^t ||\nabla\partial^\alpha p||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\ &\quad + C\int_0^t ||R_L^\alpha(p)||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau + C\int_0^t ||\partial^\alpha G_p||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\ &\leq K_\mu(t)\int_0^t \mathcal{E}_L(\tau)+\bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau\end{aligned}$$

where we have used the above estimates for $R_L^\alpha(p)$ and $\partial^\alpha G_p$. □

Proposition 12.5.5. *We have*

$$\rho_\mu(t,x)\geq\delta^0 - tK_\mu(t) \tag{12.5.6}$$

$$\rho_\mu(t,x)\geq\delta^0_\mu - tK_\mu(t) \tag{12.5.7}$$

Proof. Note that by Sobolev embedding, Holder's inequality and the chain rule for Sobolev functions, we have

$$\begin{aligned}\rho(t,x) &\geq \rho^0(x) - C\int_0^t ||\partial_t\rho||_{H^2(\Omega_\pm)}\,\mathrm{d}t\\ &\geq \rho^0(x) - \int_0^t F(\sum_\pm ||(p,s)||^2_{L^\infty(\Omega_\pm)}, \sum_\pm ||\partial_t(p,s)||^2_{H^2(\Omega_\pm)}, \frac{1}{\delta^0})\mathrm{d}\tau\\ &\geq \delta^0 - tK_\mu(t)\end{aligned}$$

In exactly the same way we obtain

$$\rho_\mu(t,x)\geq\delta^0_\mu - tK_\mu(t)$$

This completes the proof. □

Proposition 12.5.6. *The curl ω satisfies the following energy estimates.*

$$\begin{aligned}\mathcal{E}_L(\omega)(t) &\leq M^0_\mu + \frac{1}{\epsilon}K_\mu(t)(t+\int_0^t \bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t)+\mathcal{E}_L(\omega)(t)\mathrm{d}\tau\\ &\quad + \epsilon(M^0_\mu + tK_\mu(t))(\bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t)+\mathcal{E}_L(\omega)(t))\end{aligned} \tag{12.5.8}$$

$$\int_0^t \left|\left|\partial_t^3\omega\right|\right|^2_{H^1(\Omega_\pm)}\mathrm{d}\tau \leq K_\mu(t)(t+\int_0^t \bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t)+\mathcal{E}_L(\omega)(t)\mathrm{d}\tau) \tag{12.5.9}$$

$$\int_0^t \left|\left|\partial_t^4\omega\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \leq (M^0_\mu + tK_\mu(t))(1+\bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t)+\mathcal{E}_L(\omega)(t)) \tag{12.5.10}$$

for $t\in(0,T)$, where we are free to choose $\epsilon>0$.

Proof. Let $1\leq|\alpha|\leq 3$ and apply ∂^α to the linearised curl equation (9.1.5) to obtain

$$(\partial_t + \overline{u^\psi}\cdot\nabla)\partial^\alpha\omega = \partial^\alpha G_\omega + R_L^\alpha(\omega)$$

where

$$R_L^\alpha(\omega) = -\sum_{\gamma+\delta=\alpha,\delta\neq\alpha}\partial^\gamma\overline{u^\psi}\cdot\partial^\delta\nabla\omega$$

and

$$G_\omega := \frac{1}{\rho_\mu}(-\nabla^\psi \rho_\mu \times (\partial_t + \overline{u^\psi} \cdot \nabla)\bar{u} - \rho_\mu(\epsilon_{ijk}\partial_j^\psi \bar{u}_l \partial_l^\psi \bar{u}_k) + \nabla^\psi \times g_1)$$

We claim that

$$||R_L^\alpha(\omega)||^2_{H^1(\Omega_\pm)} \leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t))$$

Indeed, writing $\overline{u^\psi} = \tilde{G}(\partial_{x_3}\psi)G(v)$, where $v = (\partial_t\psi, \nabla'\psi, \nabla'\bar{\psi}, \bar{u})$, writing $w = \nabla\omega$, and using the fact that $||\nabla\omega||^2_{H^2(\Omega_\pm)} \leq K_\mu(t)$, it follows immediately from (12.1.4) that

$$\begin{aligned} ||R_L^\alpha(\omega)||^2_{H^1(\Omega_\pm)} &\leq K_\mu(t)(\sum_{k=0}^{2}\left|\left|\partial^k \nabla\omega\right|\right|^2_{H^1(\Omega_\pm)} + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \\ &\leq K_\mu(t)(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \end{aligned}$$

as required.

Now we claim that

$$\int_0^t ||\partial^\alpha G_\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \leq (M_\mu^0 + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

and if $\alpha_0 \leq 2$, then

$$\begin{aligned} \int_0^t ||\partial^\alpha G_\omega||^2_{H^1(\Omega_\pm)}\,\mathrm{d}\tau &\leq (M_\mu^0 + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \\ ||\partial^\alpha G_\omega||^2_{L^2(\Omega_\pm)} &\leq K_\mu(t)(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) \end{aligned}$$

Indeed, we write $G_\omega = \tilde{G}(\partial_{x_3}\psi)G(v)w$ where $v = (\rho_\mu, \nabla\rho_\mu, \nabla'\psi, \nabla'\bar{\psi}, \partial_t\psi, \bar{u}, \nabla\bar{u}, \nabla g_1)$, $\tilde{G}$ is a smooth function and G is a smooth vector-valued function and $w = (v, \partial_t\bar{u})$. Hence

$$\partial^\alpha G_\omega = \sum_{\beta+\gamma=\alpha, \gamma\neq\alpha} \partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\partial^\gamma w + \tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w$$

Applying (12.1.3) and estimating $\partial_{x_3}^{k+1}\psi$ in terms of $||f||_{L^\infty(\Gamma)}$ for $0 \leq k \leq 4$ we immediately obtain

$$\left|\left|\sum_{\beta+\gamma=\alpha, \gamma\neq\alpha} \partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\partial^\gamma w\right|\right|^2_{H^1(\Omega_\pm)} \leq K_\mu(t)(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \left|\left|\partial^3\nabla\bar{u}\right|\right|^2_{H^1(\Omega_\pm)})$$

We also obtain

$$\left|\left|\sum_{\beta+\gamma=\alpha, \gamma\neq\alpha} \partial^\beta(\tilde{G}(\partial_{x_3}\psi)G(v))\partial^\gamma w\right|\right|^2_{L^2(\Omega_\pm)} \leq K_\mu(t)(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

We also note that

$$\left|\left|\tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w\right|\right|^2_{L^2(\Omega_\pm)} \leq (M^0 + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + ||\partial^\alpha\partial_t\bar{u}||^2_{L^2(\Omega_\pm)})$$

Thus integrating with respect to time proves the first estimate in the claim. Now if $\alpha_0 \leq 2$ then

$$\left|\left|\tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w\right|\right|^2_{L^2(\Omega_\pm)} \leq K_\mu(t)(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))$$

Combining estimates proves the last estimate in the claim. Finally, we note that if $\alpha_0 \le 2$ then

$$\left|\left|\tilde{G}(\partial_{x_3}\psi)G(v)\partial^\alpha w\right|\right|^2_{H^1(\Omega_\pm)} \le K_\mu(t)(1+\bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t)) + (M^0+tK_\mu(t))\,||\partial^\alpha\partial_t\bar{u}||^2_{H^1(\Omega_\pm)}$$

and integrating with respect to time we obtain the second estimate in the claim.

Let $\alpha_0 \le 2$. Applying the estimate (12.5.3) with $w = \partial^\alpha\omega$, $b = \overline{u^\psi}$ and $h = \partial^\alpha G_\omega + R^\alpha_L(\omega)$ we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha\omega||^2_{H^1(\Omega_\pm)} \le ||\partial^\alpha\omega(0)||^2_{H^1(\Omega_\pm)} + (\frac{1}{\epsilon} + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\overline{u^\psi}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)})\int_0^t ||\partial^\alpha\omega||^2_{H^1(\mathbb{R}^d)}\,\mathrm{d}\tau\\
&+\int_0^t ||R^\alpha_L(\omega)||^2_{H^1(\Omega_\pm)}\,\mathrm{d}\tau + \epsilon\int_0^t ||\partial^\alpha G_\omega||^2_{H^1(\Omega_\pm)}\,\mathrm{d}\tau\\
&\le M^0_\mu + \frac{1}{\epsilon}K_\mu(t)(t+\int_0^t \mathcal{E}_L(\tau)+\bar{\mathcal{E}}_L(\tau)+\mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) + \epsilon(M^0_\mu+tK_\mu(t))(\bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t))
\end{aligned}$$

where we have used the above estimates for $R^\alpha_L(\omega)$ and $\partial^\alpha G_\omega$ with $\alpha_0 \le 2$.

We now apply the estimate (12.5.2) to the differentiated equation to obtain

$$\begin{aligned}
\int_0^t ||\partial_t\partial^\alpha\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau &\le C\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\overline{u^\psi}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}\int_0^t ||\nabla\partial^\alpha\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&+C\int_0^t ||R^\alpha_L(\omega)||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau + C\int_0^t ||\partial^\alpha G_\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&\le K_\mu(t)\int_0^t ||\nabla\partial^\alpha\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau + C\int_0^t ||\partial^\alpha G_\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&+K_\mu(t)\int_0^t \mathcal{E}_L(\tau)+\bar{\mathcal{E}}_L(\tau)+\mathcal{E}_L(\omega)(t)\mathrm{d}\tau
\end{aligned}$$

where we have used the above estimate for $R^\alpha_L(\omega)$. In the case $\alpha_0 \le 2$, we obtain

$$\int_0^t ||\partial_t\partial^\alpha\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau \le K_\mu(t)(t+\int_0^t \mathcal{E}_L(\tau)+\bar{\mathcal{E}}_L(\tau)+\mathcal{E}_L(\omega)(t)\mathrm{d}\tau)$$

where we have used the above estimate for $\partial^\alpha G_\omega$. This completes the estimate of $\mathcal{E}_L(\omega)(t)$ and also proves the second estimate. Without the restriction on α_0, we obtain

$$\begin{aligned}
&\int_0^t ||\partial_t\partial^\alpha\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&\le K_\mu(t)\int_0^t ||\nabla\partial^\alpha\omega||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau + (M^0_\mu+tK_\mu(t))(1+\mathcal{E}_L(\omega)(t)+\bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t))\\
&\le (M^0_\mu+tK_\mu(t))(1+\mathcal{E}_L(\omega)(t)+\bar{\mathcal{E}}_L(t)+\mathcal{E}_L(t))
\end{aligned}$$

where we have used the above estimate for $\partial^\alpha G_\omega$ and the estimate just obtained for $\nabla\partial^\alpha\omega$.

This completes the proof. □

12.6 Estimate of the Velocity

Definition 12.6.1. We define the part of the energy $\mathcal{E}_L$ associated with the velocity u for $t \in (0,T)$ by

$$\begin{aligned}
\mathcal{E}_L(u)(t) &:= \sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} ||u||^2_{L^\infty(\Omega_\pm)} + \sum_\pm\sum_{0\le j\le 3}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j\nabla u\right|\right|^2_{H^{3-j}(\Omega_\pm)}\\
&+\sum_\pm\sum_{0\le j\le 3}\int_0^t\left|\left|\partial_t^j\nabla u\right|\right|^2_{H^{4-j}(\Omega_\pm)}\mathrm{d}\tau + \sum_\pm\int_0^t ||\partial_t^4 u||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&+\sum_{0\le j\le 3}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j(u\cdot n)\right|\right|^2_{H^{4-j}(\mathbb{R}^2)} + \sum_{0\le j\le 3}\int_0^t\left|\left|\partial_t^j(u\cdot n)\right|\right|^2_{H^{5-j}(\mathbb{R}^2)}\mathrm{d}\tau
\end{aligned}$$

Proposition 12.6.2. *The velocity u satisfies the following energy estimate.*

$$\begin{aligned}\mathcal{E}_L(u)(t) \leq \frac{1}{\epsilon}M_\mu^0 + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ + \epsilon(M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t)) \end{aligned} \tag{12.6.1}$$

for $t \in (0,T)$, where we are free to choose $\epsilon > 0$.

Proof. See the rest of this section for the proof $\square$

We use the weak form of the equations to avoid the difficulty of not having enough regularity on u. As in 11.3, we set

$$\widehat{u} = N^{-1}u$$

where the matrix N is defined in 11.3.1. We define the energy associated with $\widehat{u}$ as

$$\begin{aligned}\mathcal{E}_L(\widehat{u})(t) := \sum_{\pm} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\widehat{u}||^2_{L^\infty(\Omega_\pm)} + \sum_{\pm}\sum_{0\leq j\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j \nabla\ \widehat{u}\right|\right|^2_{H^{3-j}(\Omega_\pm)} \\ + \sum_{\pm}\sum_{0\leq j\leq 3} \int_0^t \left|\left|\partial_t^j \nabla\ \widehat{u}\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau + \sum_{\pm}\int_0^t ||\partial_t^4\ \widehat{u}||^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \\ + \sum_{0\leq j\leq 3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j\ \widehat{u}_3\right|\right|^2_{H^{4-j}(\mathbb{R}^2)} + \sum_{0\leq j\leq 3}\int_0^t \left|\left|\partial_t^j\ \widehat{u}_3\right|\right|^2_{H^{5-j}(\mathbb{R}^2)}\mathrm{d}\tau\end{aligned}$$

Lemma 12.6.3. *We claim that*

$$\mathcal{E}_L(u)(t) \leq (M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(\widehat{u})(t) + \mathcal{E}_L(f)(t)) \tag{12.6.2}$$

and conversely

$$\mathcal{E}_L(\widehat{u})(t) \leq (M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(u)(t) + \mathcal{E}_L(f)(t)) \tag{12.6.3}$$

Proof. Recall that $\widehat{u}_3 = (N^{-1}u)_3 = u \cdot n$, so the estimate of the boundary terms is immediate.

Now we note that for $1 \leq |\alpha| \leq k$, where $k = 4$ or $k = 5$, and $\alpha_0 \leq 3$, we have

$$||\partial^\alpha u||_{L^2(\Omega_\pm)} = \sum_{\beta+\gamma=\alpha} \left|\left|\partial^\beta N \partial^\gamma\ \widehat{u}\right|\right|_{L^2(\Omega_\pm)}$$

Note that the lower order estimate (12.3.3) and the lifting property in the definition of N implies that if $0 \leq |\beta| \leq 2$ then

$$\left|\left|\partial^\beta N\right|\right|^2_{L^\infty(\Omega_\pm)} \leq C\left|\left|\partial^\beta \nabla' f\right|\right|^2_{H^{1.5}(\Gamma)} \leq M_\mu^0 + tK_\mu(t)$$

The lower order estimate (12.3.1) implies

$$||\partial^\gamma\ \widehat{u}||^2_{L^\infty(\Omega_\pm)} \leq M_\mu^0 + tK_\mu(t) \text{ for } 0 \leq |\gamma| \leq 1$$

We note also that

$$||\partial^\gamma\ \widehat{u}||^2_{L^\infty(\Omega_\pm)} \leq K_\mu(t) \text{ for } 0 \leq |\gamma| \leq 2$$

Finally, let us note that for $3 \le |\beta| \le 5$ (with $\beta_0 \le 3$),

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\beta N\right|\right|^2_{L^2(\Omega_\pm)} \le C \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\beta f\right|\right|^2_{H^{5.5-|\beta|}(\Omega_\pm)} \le \mathcal{E}_L(f)(t)$$

Thus we obtain in the case $k = 4$,

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha u||^2_{L^2(\Omega_\pm)} &\le C \sum_{\beta+\gamma=\alpha} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\beta N \partial^\gamma \widehat{u}\right|\right|^2_{L^2(\Omega_\pm)} \\ &\le (M^0_\mu + tK_\mu(t))(\mathcal{E}_L(\widehat{u})(t) + \mathcal{E}_L(f)(t))\end{aligned}$$

In the case $k = 5$ we obtain

$$\begin{aligned}\int_0^t ||\partial^\alpha u||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau &\le C \sum_{\beta+\gamma=\alpha} \int_0^t \left|\left|\partial^\beta N \partial^\gamma \widehat{u}\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \\ &\le (M^0_\mu + tK_\mu(t))\mathcal{E}_L(\widehat{u})(t) + K_\mu(t)\int_0^t \mathcal{E}_L(f)(\tau)\mathrm{d}\tau\end{aligned}$$

Now we consider the case $\partial^\alpha = \partial_t^4$. Note that if $|\beta| \le 4$ then

$$\begin{aligned}\int_0^t \left|\left|\partial^\beta N\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau &\le C\int_0^t \left|\left|\partial^\beta \nabla' f\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau \\ &\le C\mathcal{E}_L(f)(t)\end{aligned}$$

Combining this with the estimates above we see that

$$\begin{aligned}\int_0^t ||\partial^\alpha u||^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau &\le C \sum_{\beta+\gamma=\alpha} \int_0^t \left|\left|\partial^\beta N \partial^\gamma \widehat{u}\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \\ &\le (M^0_\mu + tK_\mu(t))(\mathcal{E}_L(\widehat{u})(t) + \mathcal{E}_L(f)(t))\end{aligned}$$

This completes the proof of the first inequality. The second inequality follows simply by replacing N by N^{-1} and reversing u and $\widehat{u}$ in the above estimates. □

We recall that $\widehat{u}$ satisfies the weak equation (11.7.1). Let $0 \le |\alpha| \le 3$ with $\alpha_3 = 0$ be a multi-index. We may also write $\partial^\alpha = \partial_t^j \nabla^\beta$ where $\beta_3 = 0$ and $0 \le j + |\beta| \le 3$. We apply ∂_t^j to the weak equation (11.7.1), then test it with $(-1)^{|\beta|}\nabla^\beta w$ for $w \in B^1$ smooth in the horizontal directions, then integrate by parts horizontally and use the density of such w in B^1 to obtain that $\partial^\alpha\ \widehat{u}$ satisfies the following equation.

$$\begin{aligned}&(\partial_t\partial^\alpha\ \widehat{u}, w)_{B^0(t)} + \mu\{\partial^\alpha\ \widehat{u}, w\}_{B^1(t)} + \sum_\pm (A^2(\partial^\alpha\ \widehat{u}, \nabla\partial^\alpha\ \widehat{u}), w)_{L^2(\Omega_\pm)} \\ &= \sum_\pm (G^{1,\alpha}, w)_{L^2(\Omega_\pm)} + \sum_\pm (G^{2,\alpha}, \nabla w)_{L^2(\Omega_\pm)} + (G^{3,\alpha}, w_3)_{L^2(\Gamma)} + (G^{4,\alpha}, \nabla' w_3)_{L^2(\Gamma)} \qquad (12.6.4)\end{aligned}$$

for all $w \in B^1$ and almost every $t \in (0, T)$, where

$$G^{1,\alpha} := \partial^\alpha G^1 - \sum_{\beta+\gamma=\alpha,\beta\ne\alpha} (\partial^\gamma A^0 \partial^\beta \partial_t\ \widehat{u} + \mu\partial^\gamma A^1(\partial^\beta\ \widehat{u}, \partial\partial^\beta\ \widehat{u}) + \partial^\gamma A^2(\partial^\beta\ \widehat{u}, \nabla\partial^\beta\ \widehat{u})) \qquad (12.6.5)$$

$$G^{2,\alpha} := \partial^\alpha G^2 - \mu \sum_{\beta+\gamma=\alpha,\beta\ne\alpha} \partial^\gamma A^1(\partial^\beta\ \widehat{u}, \nabla\partial^\beta\ \widehat{u}) \qquad (12.6.6)$$

$$G^{3,\alpha} := \partial^\alpha G^3 \qquad (12.6.7)$$

$$G^{4,\alpha} := \partial^\alpha G^4 \qquad (12.6.8)$$

and A^i for $0 \le i \le 2$ are defined in 11.6.1 and G^i for $1 \le i \le 4$ are defined in 11.7.1.

Lemma 12.6.4. *We claim that the constant $C_{\infty,0}(t)$ and $C_{\infty,1}(t)$, as defined in 11.1.1, satisfy the following estimates.*

$$C_{\infty,0}(t) \leq M_\mu^0 + tK_\mu(t) \tag{12.6.9}$$

$$C_{\infty,1}(t) \leq K_\mu(t) \tag{12.6.10}$$

Proof. Firstly we recall that

$$\begin{aligned}C_{\infty,j}(t) \leq F_\mu(&\sum_\pm \sum_{k=0}^{j} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^k(\rho c^2)\right|\right|^2_{W^{j-k,\infty}(\Omega_\pm)}, \sum_\pm \sum_{k=0}^{j} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^k(\rho_\mu, \partial_t\rho_\mu)\right|\right|^2_{W^{j-k,\infty}(\Omega_\pm)},\\ &\sum_{k=0}^{j+1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^k \nabla' f\right|\right|^2_{H^{2.5+j-k}(\Gamma)}, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0})\end{aligned}$$

for some smooth increasing function F_μ depending on μ. Thus we have

$$\begin{aligned}C_{\infty,0}(t) &\leq M^0 + tC_{\infty,0}(t)(\sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t(\rho c^2)\right|\right|^2_{L^\infty(\Omega_\pm)} + \sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t(\rho_\mu, \partial_t\rho_\mu)\right|\right|^2_{L^\infty(\Omega_\pm)} +\\ &\sum_{k=0}^{1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^{k+1} \nabla' f\right|\right|^2_{H^{2.5-k}(\Gamma)})\\ &\leq M^0 + tK(t)\end{aligned}$$

It follows from the Sobolev embedding theorem and the definition of $K_\mu(t)$ that

$$C_{\infty,1}(t) \leq K_\mu(t)$$

□

Lemma 12.6.5. *Let $1 \leq |\alpha| \leq 3$. We claim that*

$$\begin{aligned}\int_0^t \left|\left|G^{1,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)\\ \int_0^t \left|\left|\partial_t G^{2,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq (M_\mu^0 + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t))\\ \int_0^t \left|\left|G^{2,\alpha}\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau &\leq K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)\\ \sum_{k=0}^{1} \int_0^t \left|\left|\partial^k G^{3,\alpha}\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau &\leq tK_\mu(t)\\ \sum_{k=0}^{1} \int_0^t \left|\left|\partial^k G^{4,\alpha}\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau &\leq K_\mu(t) \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau)\mathrm{d}\tau\end{aligned}$$

Furthermore, for $2 \leq i \leq 4$,

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} \leq M_\mu^0 &+ \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)\\ &+ \epsilon(M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))\end{aligned}$$

Proof. Firstly, we recall that G^i have the form

$$\begin{aligned}
G^1 &= G(N, \nabla N, \nabla\psi, p, s, \omega, g_1) \\
G^2 &= G(N, \nabla\psi, p, s, \omega) \\
G^3 &= g_2 \\
G^4 &= G(\nabla' f)
\end{aligned}$$

for G a generic smooth function which may be scalar, vector, matrix valued etc and A^i have the form

$$\begin{aligned}
A^0 &= A(p, s, N) \\
A^1 &= A(p, s, \nabla\psi, N, \nabla N) \\
A^2 &= A(p, s, \bar{u}, \nabla\psi, \nabla\bar{\psi}, \partial_t\psi, N, \nabla N, \partial_t N)
\end{aligned}$$

We start by estimating $\left|\left|\partial^\alpha G^2\right|\right|_{L^2(\Omega_\pm)}$ and $\left|\left|\partial\partial^\alpha G^2\right|\right|_{L^2(\Omega_\pm)}$. Applying (12.1.1), we see that

$$\begin{aligned}
&\left|\left|\partial\partial^\alpha G^2\right|\right|_{L^2(\Omega_\pm)} \leq F(||v||_{L^\infty(\Omega_\pm)}, \sum_{k=1}^{2}\left|\left|\partial^k v\right|\right|_{H^1(\Omega_\pm)})(\left|\left|\partial^2 v\right|\right|_{H^1(\Omega_\pm)} + \sum_{\beta\leq\alpha, \beta\neq\alpha}\left|\left|\partial\partial^\beta v\right|\right|_{L^2(\Omega_\pm)}) \\
&+ F(||v||_{L^\infty(\Omega_\pm)})\,||\partial\partial^\alpha v||_{L^2(\Omega_\pm)}
\end{aligned}$$

for some smooth increasing function F, where $v = (N, \nabla'\psi, p, s, \omega)$, and we have estimated $\partial_{x_3}^{k+1}\psi$ in terms of $||\psi||_{L^\infty(\Omega_\pm)}$ for $0 \leq k \leq 4$. Now we note that from the lower order estimate (12.3.1), we have

$$||v||^2_{L^\infty(\Omega_\pm)} \leq M^0_\mu + tK_\mu(t)$$

Also,

$$\begin{aligned}
||\partial v||^2_{L^\infty(\Omega_\pm)} &\leq K_\mu(t) \\
\left|\left|\partial^2 v\right|\right|^2_{H^1(\Omega_\pm)} &\leq K_\mu(t)
\end{aligned}$$

If $l = 3$ then

$$\begin{aligned}
&\int_0^t \left|\left|\partial^l v\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\
&\leq C\int_0^t \left|\left|\partial^l (p,s)\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|\partial^l N\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|\partial^l \nabla'\psi\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|\partial^l \omega\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\
&\leq K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)
\end{aligned}$$

where we have used the curl estimate (12.5.9) for the term involving ω in the case of 3 time derivatives. If $l = 4$, then

$$\begin{aligned}
&\int_0^t \left|\left|\partial^l v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq C\int_0^t \left|\left|\partial^l (p,s)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l N\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l \nabla\psi\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l \omega\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq C\mathcal{E}_L(t) + C\bar{\mathcal{E}}_L(t) + (M^0_\mu + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t)) \\
&\leq (M^0_\mu + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t))
\end{aligned}$$

where in the case of the highest order time derivatives we have used the estimate (12.4.3) for the front in the second and third terms and the estimate (12.5.10) for the curl in the fourth term. Combining these estimates, we see that

$$\int_0^t \left|\left|\partial_t\partial^\alpha G^2\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq (M^0_\mu + tK_\mu(t))(1 + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t))$$

and

$$\int_0^t \left|\left|\partial^\alpha G^2\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)$$

Now let us estimate $\left|\left|\partial^\alpha G^4\right|\right|_{L^2(\Gamma)}$ and $\left|\left|\partial\partial^\alpha G^4\right|\right|_{L^2(\Gamma)}$. Applying (12.1.1), we see that

$$\left|\left|\partial\partial^\alpha G^4\right|\right|_{L^2(\Gamma)} \leq F(\left|\left|v\right|\right|_{L^\infty(\Gamma)}, \sum_{k=1}^2 \left|\left|\partial^k v\right|\right|_{H^1(\Gamma)})(\left|\left|\partial^2 v\right|\right|_{H^1(\Gamma)} + \sum_{\beta\leq\alpha,\beta\neq\alpha} \left|\left|\partial\partial^\beta v\right|\right|_{L^2(\Gamma)})$$
$$+ F(\left|\left|v\right|\right|_{L^\infty(\Gamma)}) \left|\left|\partial\partial^\alpha v\right|\right|_{L^2(\Gamma)}$$

for some smooth increasing function F, where $v = \nabla' f$. Note that

$$\left|\left|(v, \partial v)\right|\right|^2_{L^\infty(\Omega_\pm)} \leq K_\mu(t)$$
$$\left|\left|\partial^2 v\right|\right|^2_{H^1(\Omega_\pm)} \leq K_\mu(t)$$

If $3 \leq l \leq 4$, then

$$\int_0^t \left|\left|\partial^l v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq C \int_0^t \left|\left|\partial^l \nabla' f\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau$$
$$\leq C \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau)\mathrm{d}\tau$$

where we have used the estimate (12.4.2) for the front in the case of the highest order time derivative. Combining these estimates, we see that

$$\int_0^t \left|\left|\partial^\alpha G^4\right|\right|^2_{L^2(\Gamma)} + \left|\left|\partial\partial^\alpha G^4\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \leq K_\mu(t) \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau)\mathrm{d}\tau$$

Clearly we have

$$\int_0^t \left|\left|\partial^\alpha G^3\right|\right|^2_{L^2(\Gamma)} + \left|\left|\partial\partial^\alpha G^3\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \leq tK_\mu(t)$$

Now we estimate $\left|\left|\partial^\alpha G^1\right|\right|_{L^2(\Omega_\pm)}$. Applying (12.1.1), we see that

$$\left|\left|\partial^\alpha G^1\right|\right|_{L^2(\Omega_\pm)} \leq F(\left|\left|v\right|\right|_{L^\infty(\Omega_\pm)}, \sum_{k=1}^2 \left|\left|\partial^k v\right|\right|_{H^1(\Omega_\pm)})(\left|\left|\partial^2 v\right|\right|_{H^1(\Omega_\pm)} + \sum_{\beta\leq\alpha,\beta\neq\alpha} \left|\left|\partial^\beta v\right|\right|_{L^2(\Omega_\pm)})$$
$$+ F(\left|\left|v\right|\right|_{L^\infty(\Omega_\pm)}) \left|\left|\partial^\alpha v\right|\right|_{L^2(\Omega_\pm)}$$

for some smooth increasing function F, where $v = (N, \nabla N, \nabla'\psi, p, s, \omega, g_1)$, and we have estimated $\partial^{k+1}_{x_3}\psi$ in terms of $\left|\left|\psi\right|\right|_{L^\infty(\Omega_\pm)}$ for $0 \leq k \leq 3$. Note

$$\left|\left|(v, \partial v)\right|\right|^2_{L^\infty(\Omega_\pm)} \leq K_\mu(t)$$
$$\left|\left|\partial^2 v\right|\right|^2_{H^1(\Omega_\pm)} \leq K_\mu(t)$$

where we have used the fact that $\left|\left|\partial^2 f\right|\right|^2_{H^{2.5}(\Gamma)} \leq K_\mu(t)$. If $2 \leq l \leq 3$, then

$$\begin{aligned}
&\int_0^t \left|\left|\partial^l v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq C \int_0^t \left|\left|\partial^l (p,s)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l \nabla' f\right|\right|^2_{H^{0.5}(\Omega_\pm)} + \left|\left|\partial^l g_1\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l \omega\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq tK_\mu(t) + C\int_0^t \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau
\end{aligned}$$

Combining these estimates, we see that

$$\int_0^t \left|\left|\partial^\alpha G^1\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq (M^0_\mu + tK_\mu(t)) \int_0^t \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau + tK_\mu(t)$$

Now we estimate the terms involving A^i. Firstly, we estimate $\sum_{\gamma+\delta=\beta,\delta\neq\beta} \partial^\gamma A^1(\partial^\delta\ \widehat{u}, \nabla\partial^\delta\ \widehat{u})$ where $|\beta| \leq |\alpha| + 1 \leq 4$. Set $v = (p, s, \nabla'\psi, N, \nabla N)$. Note that

$$\begin{aligned}
\left|\left|v\right|\right|^2_{L^\infty(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t) \\
\left|\left|\partial^k v\right|\right|^2_{L^\infty(\Omega_\pm)} &\leq K_\mu(t) \text{ for } k \leq 1 \\
\left|\left|\partial^k v\right|\right|^2_{H^1(\Omega_\pm)} &\leq K_\mu(t) \text{ for } k \leq 2
\end{aligned}$$

where we have used the fact that $\left|\left|\partial^2 \nabla' f\right|\right|^2_{H^{1.5}(\Gamma)} \leq K_\mu(t)$ and the lower order estimate (12.3.1). The lower order estimates (12.3.1)-(12.3.3) imply

$$\begin{aligned}
\left|\left|\widehat{u}\right|\right|^2_{W^{1,\infty}(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t) \\
\left|\left|\partial\ \widehat{u}\right|\right|^2_{H^2(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t)
\end{aligned}$$

For $l = 3$, we have

$$\begin{aligned}
\int_0^t \left|\left|\partial^l v\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau &\leq C\int_0^t \left|\left|\partial^l (p,s)\right|\right|^2_{H^1(\Omega_\pm)} + C\left|\left|\partial^l \nabla' f\right|\right|^2_{H^{1.5}(\Omega_\pm)} \mathrm{d}\tau \\
&\leq C\int_0^t \mathcal{E}_L(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the lifting property for N. In the case $l = 4$ we obtain

$$\begin{aligned}
\int_0^t \left|\left|\partial^l v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq C\int_0^t \left|\left|\partial^l (p,s)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l \nabla' f\right|\right|^2_{H^{0.5}(\Omega_\pm)} \mathrm{d}\tau \\
&\leq C(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t))
\end{aligned}$$

where we have used the estimate (12.4.3) for the front in the case of the highest order time derivative. Finally, if $2 \leq k \leq 3$, then

$$\left|\left|\partial^k\ \widehat{u}\right|\right|^2_{H^1(\Omega_\pm)} \leq \mathcal{E}_L(\widehat{u})(t)$$

Combining all these estimates, using the chain and product rule estimate (12.1.3), and estimating $\partial^{k+1}_{x_3}\psi$ in terms of $\left|\left|\psi\right|\right|_{L^\infty(\Omega_\pm)}$ for $0 \leq k \leq 4$, we have

$$\begin{aligned}
&\int_0^t \left|\left|\sum_{\gamma+\delta=\beta,\delta\neq\beta} \partial^\gamma A^1(\partial^\delta\ \widehat{u}, \nabla\partial^\delta\ \widehat{u})\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq (M^0_\mu + tK_\mu(t))(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t)) + K_\mu(t)\int_0^t \mathcal{E}_L(\widehat{u})(\tau)\mathrm{d}\tau \\
&\leq (M^0_\mu + tK_\mu(t))(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t))
\end{aligned}$$

where we have used the estimate (12.6.3). Note that we also obtain that if $\beta_0 \leq 3$ then

$$\int_0^t \left|\left|\sum_{\gamma+\delta=\beta,\delta\neq\beta} \partial^\gamma A^1(\partial^\delta\, \widehat{u}, \nabla\partial^\delta\, \widehat{u})\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq K_\mu(t)\int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau)\mathrm{d}\tau$$

Now, we estimate $\sum_{\gamma+\delta=\beta,\delta\neq\beta} \partial^\gamma A^0\partial^\delta\partial_t\, \widehat{u}$ where $|\beta| \leq |\alpha| \leq 3$. Set $v = (p, s, N)$. Note that

$$\begin{aligned} ||v||^2_{L^\infty(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t) \\ \left|\left|\partial^k v\right|\right|^2_{L^\infty(\Omega_\pm)} &\leq K_\mu(t) \text{ for } k \leq 1 \\ \left|\left|\partial^k v\right|\right|^2_{H^1(\Omega_\pm)} &\leq K_\mu(t) \text{ for } k \leq 2 \end{aligned}$$

If $0 \leq k \leq 2$ then

$$\begin{aligned} \left|\left|\partial^k\partial_t\, \widehat{u}\right|\right|^2_{H^1(\Omega_\pm)} &\leq F(\left|\left|\partial^k\partial_t(N, u)\right|\right|^2_{H^1(\Omega_\pm)}) \\ &\leq K_\mu(t) \end{aligned}$$

We have

$$||v||^2_{L^\infty(\Omega_\pm)} + \sum_{k=1}^{2}\left|\left|\partial^k v\right|\right|^2_{H^1(\Omega_\pm)} \leq K_\mu(t)$$

For $2 \leq l \leq 3$, we have

$$\begin{aligned} \int_0^t \left|\left|\partial^l v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq C\int_0^t \left|\left|\partial^l(p, s)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l\nabla' f\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\ &\leq C\int_0^t \mathcal{E}_L(\tau)\mathrm{d}\tau \end{aligned}$$

Combining all these estimates and using (12.1.3), we see that

$$\begin{aligned} \int_0^t \left|\left|\sum_{\gamma+\delta=\beta,\delta\neq\beta} \partial^\gamma A^0\partial^\delta\partial_t\, \widehat{u}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq K_\mu(t)\int_0^t \mathcal{E}_L(\tau) + \mathcal{E}_L(\widehat{u})(\tau)\mathrm{d}\tau \\ &\leq K_\mu(t)\int_0^t \mathcal{E}_L(\tau)\mathrm{d}\tau \end{aligned}$$

where we have used the estimate (12.6.3).

Finally, we estimate $\sum_{\gamma+\delta=\beta,\delta\neq\beta} \partial^\gamma A^2\partial^\delta(\widehat{u}, \nabla\, \widehat{u})$ where $|\beta| \leq |\alpha| \leq 3$. Set

$$v = (p, s, \bar{u}, \nabla'\psi, \nabla'\bar{\psi}, \partial_t\psi, N, \nabla N, \partial_t N)$$

If $1 \leq k \leq 2$ then

$$\left|\left|\partial^k\, \widehat{u}\right|\right|^2_{H^2(\Omega_\pm)} \leq F(\left|\left|\partial^k(N, u)\right|\right|^2_{H^2(\Omega_\pm)}) \leq K_\mu(t)$$

We have

$$||v||^2_{L^\infty(\Omega_\pm)} + \sum_{k=1}^{2}\left|\left|\partial^k v\right|\right|^2_{H^1(\Omega_\pm)} \leq K_\mu(t)$$

where we have used the fact that $\left|\left|\partial^2\nabla' f\right|\right|^2_{H^{1.5}(\Gamma)} \leq K_\mu(t)$. For $2 \leq l \leq 3$, we have

$$\begin{aligned}\int_0^t \left|\left|\partial^l v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq C\int_0^t \left|\left|\partial^l(p, s, \bar{u}, \partial'\psi, \nabla'\bar{\psi})\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial^l \partial f\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\ &\leq C\int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau)\mathrm{d}\tau\end{aligned}$$

where we have used the estimate (12.4.2) for f in the case of the highest order time derivative. Combining all these estimates, estimating $\partial^{k+1}_{x_3}\psi$ in terms of $||\psi||_{L^\infty(\Omega_\pm)}$ for $0 \leq k \leq 3$, and applying the chain and product rule estimate (12.1.3), we see that

$$\begin{aligned}\int_0^t \left|\left|\sum_{\gamma+\delta=\beta, \delta\neq\beta} \partial^\gamma A^2 \partial^\delta(\widehat{u}, \nabla\,\widehat{u})\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq K_\mu(t)\int_0^t \mathcal{E}_L(\tau) + \mathcal{E}_L(\widehat{u})(\tau)\mathrm{d}\tau \\ &\leq K_\mu(t)\int_0^t \mathcal{E}_L(\tau)\mathrm{d}\tau\end{aligned}$$

where we have used the estimate (12.6.3).

Combining all of the above, we have proved the first set of inequalities. Now note that for $2 \leq i \leq 4$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} &= 2(G^{i,\alpha}, \partial_t G^{i,\alpha})^2_{L^2(\Omega_\pm)} \\ &\leq \frac{1}{\epsilon}\left|\left|G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} + \epsilon\left|\left|\partial_t G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)}\end{aligned}$$

Thus, integrating in time,

$$\begin{aligned}\left|\left|G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} &\leq M^0_\mu + \frac{1}{\epsilon}\int_0^t \left|\left|G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \epsilon\int_0^t \left|\left|\partial_t G^{i,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\ &\leq M^0_\mu + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ &+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))\end{aligned}$$

This completes the proof. □

Now we apply the energy estimate (11.6.3) to the equation for $\partial^\alpha\,\widehat{u}$, where $1 \leq |\alpha| \leq 3$ and $\alpha_3 = 0$ to obtain the following estimate.

$$\begin{aligned}&\int_0^t ||\partial_t\partial^\alpha\,\widehat{u}||^2_{B^0(\tau)} \mathrm{d}\tau + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha\,\widehat{u}||^2_{B^1(t)} \leq \frac{C_{\infty,1}}{\epsilon}t\left|\left|\partial_0^{\mu,g,j}\nabla^\beta\,\widehat{u}^0\right|\right|^2_{B^0} \\ &+ \frac{C_{\infty,1}}{\epsilon}\int_0^t \sum_\pm \left|\left|G^{1,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm \left|\left|G^{2,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|G^{3,\alpha}\right|\right|^2_{L^2(\Gamma)} + \left|\left|G^{4,\alpha}\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \\ &+ C_{\infty,0}(\left|\left|\partial_0^{\mu,g,j}\nabla^\beta\,\widehat{u}^0\right|\right|^2_{B^1} + \operatorname*{ess\,sup}_{\tau\in(0,t)}(\sum_\pm \left|\left|G^{2,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|G^{3,\alpha}\right|\right|^2_{L^2(\Gamma)} + \left|\left|G^{4,\alpha}\right|\right|^2_{L^2(\Gamma)})) \\ &+ \epsilon\int_0^t \sum_\pm \left|\left|\partial_t G^{2,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial_t G^{3,\alpha}\right|\right|^2_{L^2(\Gamma)} + \left|\left|\partial_t G^{4,\alpha}\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau\end{aligned}$$

To estimate the right hand side we use the estimate 12.6.4 for $C_{\infty,0}$ and $C_{\infty,1}$ and the estimate 12.6.5 for the $G^{i,\alpha}$, thus we obtain

$$\begin{aligned}&\int_0^t ||\partial_t\partial^\alpha\,\widehat{u}||^2_{B^0(\tau)} \mathrm{d}\tau + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha\,\widehat{u}||^2_{B^1(t)} \\ &\leq M^0_\mu + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ &+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))\end{aligned}$$

Now to estimate the highest order space derivative, we apply the energy estimate (11.6.7) to the equation for $\partial^\alpha \widehat{u} = \partial_t^j \nabla^\beta$, where $1 \le |\alpha| \le 3$ and $\alpha_3 = 0$. We obtain

$$
\begin{aligned}
&\left|\left|\nabla'\partial^\alpha \widehat{u}\right|\right|^2_{B^0(t)} + \int_0^t \left|\left|\nabla'\partial^\alpha \widehat{u}\right|\right|^2_{B^1(\tau)} \mathrm{d}\tau \le C \left|\left|\partial_0^{\mu,g,j}\nabla^\beta \widehat{u}^0\right|\right|^2_{B^1} \\
&+ C_{\infty,1}\int_0^t \sum_\pm \left|\left|G^{1,\alpha}\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm \left|\left|G^{2,\alpha}\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \left|\left|G^{3,\alpha}\right|\right|^2_{L^2(\Gamma)} + \sum_\pm \left|\left|G^{4,\alpha}\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \\
&+ \frac{C_{\infty,1}}{\epsilon}\int_0^t ||\partial^\alpha \widehat{u}||^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \epsilon \int_0^t ||\partial_t\partial^\alpha \widehat{u}||^2_{L^2(\Omega_\pm)} \mathrm{d}\tau
\end{aligned}
$$

Applying the estimates 12.6.4 and 12.6.5 to the right hand side, we obtain

$$
\begin{aligned}
&\left|\left|\nabla'\partial^\alpha \widehat{u}\right|\right|^2_{B^0(t)} + \int_0^t \left|\left|\nabla'\partial^\alpha \widehat{u}\right|\right|^2_{B^1(\tau)} \mathrm{d}\tau \\
&\le M_\mu^0 + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) + \epsilon(M_\mu^0 + tK_\mu(t))\mathcal{E}_L(t)
\end{aligned}
$$

It remains to estimate the highest order normal derivatives. We proceed as in 11.6.9. Indeed, we see from (11.6.9) that

$$
\begin{aligned}
&\mu\frac{\rho c^2(1+|\nabla'\psi|^2)}{\left|\underset{\sim}{n}\right|^2 (J^\psi)^2}\partial_3^2 \widehat{u} \\
&= \mu\frac{\rho c^2}{\left|\underset{\sim}{n}\right|^2}(-\Delta' \widehat{u} + 2\frac{\nabla'\psi}{J^\psi}\cdot\nabla'\partial_{x_3} \widehat{u}) - A(\rho c^2, \nabla(\rho c^2), \nabla\psi, \nabla\nabla\psi, \nabla N)((\nabla N)v, (\nabla\nabla N)v, \nabla v) \\
&+ \rho_\mu\frac{1}{\left|\underset{\sim}{n}\right|^2}\partial_t \widehat{u} + \rho_\mu N^T(\overline{u^\psi}\cdot\nabla)(N \widehat{u}) + \rho_\mu N^T(\partial_t N) \widehat{u} - G^1 + \nabla\cdot G^2
\end{aligned}
$$

for some smooth function A.

Now let $1 \le |\alpha| \le 3$. We claim by induction on α_3 for $0 \le \alpha_3 \le |\alpha|$ that

$$
\begin{aligned}
\int_0^t ||\partial^\alpha \widehat{u}||^2_{H^2(\Omega_\pm)} \mathrm{d}\tau &\le \frac{1}{\epsilon}M_\mu^0 + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\
&+ \epsilon(M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))
\end{aligned}
$$

We suppose it holds true for some $0 \le \alpha_3 \le |\alpha| - 1$ and attempt to prove it holds for $\alpha_3 + 1$. Indeed, applying ∂^β to the rearranged interior equation where $\beta_3 \le \alpha_3 + 1$ and $1 \le |\beta| \le |\alpha|$, we have

$$
\begin{aligned}
\partial_3^2\partial^\beta \widehat{u} &= \partial^\beta\left(\frac{\left|\underset{\sim}{n}\right|^2 (J^\psi)^2}{\mu\rho c^2(1+|\nabla'\psi|^2)}\left(\mu\frac{\rho c^2}{\left|\underset{\sim}{n}\right|^2}(-\Delta' \widehat{u} + 2\frac{\nabla'\psi}{J^\psi}\cdot\nabla'\partial_{x_3} \widehat{u})\right.\right. \\
&- A(\rho c^2, \nabla(\rho c^2), \nabla\psi, \nabla\nabla\psi, \nabla N)((\nabla N) \widehat{u}, (\nabla\nabla N) \widehat{u}, \nabla \widehat{u}) \\
&\left.\left.+\rho_\mu\frac{1}{\left|\underset{\sim}{n}\right|^2}\partial_t \widehat{u} + \rho_\mu N^T(\overline{u^\psi}\cdot\nabla)(N \widehat{u}) + \rho_\mu N^T(\partial_t N) \widehat{u} - G^1 + \nabla\cdot G^2\right)\right) \\
&=: \partial^\beta(G(\partial_{x_3}\psi, \partial_{x_3}^2\psi, v)w) \\
&= \sum_{\gamma+\delta=\beta,\delta\ne\beta} \partial^\gamma G(\partial_{x_3}\psi, \partial_{x_3}^2\psi, v)\partial^\delta w + G(\partial_{x_3}\psi, \partial_{x_3}^2\psi, v)\partial^\beta w
\end{aligned}
\tag{12.6.11}
$$

for some smooth function G, where

$$
v = (\nabla'\psi, \nabla\nabla'\psi, p, s, \omega, \bar{u}, \nabla'\bar{\psi}, \partial_t\psi, N, \nabla N, \partial_t N, \widehat{u}, \nabla \widehat{u}) \tag{12.6.12}
$$

$$
w = (\nabla \widehat{u}, \nabla p, \nabla s, \nabla\omega, \nabla N, (\nabla N) \widehat{u}, (\nabla\nabla N) \widehat{u}, \nabla'\nabla \widehat{u}, \partial_t \widehat{u}) \tag{12.6.13}
$$

Now we note that, using the lifting property of N, and the lower order estimates (12.3.1)-(12.3.3), we have

$$\begin{aligned}
||v||^2_{L^\infty(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t)\\
||\partial v||^2_{H^1(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t)\\
\left|\left|\partial^2 v\right|\right|^2_{H^1(\Omega_\pm)} &\leq K_\mu(t)\\
||w||^2_{L^\infty(\Omega_\pm)} &\leq K_\mu(t)\\
\left|\left|(\partial w, \partial^2 w)\right|\right|^2_{L^2(\Omega_\pm)} &\leq K_\mu(t)
\end{aligned}$$

Using the estimate (12.4.2) for f in the case the highest order time derivative, the existing high-order time derivative and tangential space derivative estimates for $\widehat{u}$ and the previous induction step in the case $\beta_3 > 0$, and the lifting property of N we have

$$\int_0^t \left|\left|\partial^3 w\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + \int_0^t \left|\left|\partial^3 v\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \frac{1}{\epsilon} M^0_\mu$$
$$+ \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) + \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))$$

Combining these estimates and using (12.1.3), we obtain

$$\int_0^t \left|\left|\partial^2_{x_3}\partial^\beta \,\widehat{u}\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \frac{1}{\epsilon} M^0_\mu + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)$$
$$+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))$$

Combining this with the existing tangential estimates, we obtain

$$\sum_\pm \int_0^t \left|\left|\partial^\beta \,\widehat{u}\right|\right|^2_{H^2(\Omega_\pm)} \mathrm{d}\tau \leq \frac{1}{\epsilon} M^0_\mu + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)$$
$$+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))$$

This is the next step of the induction, and we have also proved the case $\beta_3 = 0$ which starts the induction. This completes the estimate of the normal derivatives.

Combining all the above estimates, we have shown that

$$\begin{aligned}
\mathcal{E}_L(\widehat{u})(t) \leq \frac{1}{\epsilon} M^0_\mu + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)\\
+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))
\end{aligned}$$

Now using the estimate (12.6.2) and the estimate (12.4.1) for the front, we obtain

$$\begin{aligned}
\mathcal{E}_L(u)(t) \leq \frac{1}{\epsilon} M^0_\mu + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)\\
+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))
\end{aligned}$$

This completes the proof of the estimate of the velocity 12.6.2.

Proposition 12.6.6. *The normal component of the velocity, $u \cdot n$, satisfies the following additional energy estimate.*

$$\sum_{j=0}^3 \int_0^t \left|\left|\partial_t^j (u\cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau \leq \frac{1}{\epsilon} M^0_\mu + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)$$
$$+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t))$$

for $t \in (0, T)$.

Proof. Let $0 \leq j \leq 3$. Applying ∂_t^j to the pressure interface condition (9.1.9) we have

$$(1-\Delta')\partial_t^j(u\cdot n) = \partial_t^j([\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p] - \frac{1}{\mu}\sigma\nabla'\cdot\hat{n} + \frac{1}{\mu}g_2)$$

We apply lemma 8.16.1 with

$$\begin{aligned} z &= \partial_t^j(u\cdot n) \\ h &= \partial_t^j([\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p] - \frac{1}{\mu}\sigma\nabla'\cdot\hat{n} + \frac{1}{\mu}g_2) \end{aligned}$$

to conclude that

$$\left|\left|\partial_t^j(u\cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \leq C\left|\left|\partial_t^j([\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p] - \frac{1}{\mu}\sigma\nabla'\cdot\hat{n} + \frac{1}{\mu}g_2)\right|\right|^2_{H^{3.5-j}(\Gamma)}$$

Now we integrate in time and us the Sobolev trace lemma to obtain

$$\begin{aligned} &\int_0^t \left|\left|\partial_t^j(u\cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau \leq C\int_0^t \left|\left|\partial_t^j([\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p] - \frac{1}{\mu}\sigma\nabla'\cdot\hat{n} + \frac{1}{\mu}g_2)\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau \\ &\leq C\sum_\pm \int_0^t \left|\left|\partial_t^j(\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p)\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau + C\int_0^t \left|\left|\partial_t^j(-\sigma\nabla'\cdot\hat{n} + g_2)\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau \end{aligned}$$

Note that

$$\int_0^t \left|\left|\partial_t^j(\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p)\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau = \int_0^t \left|\left|\partial_t^j(\tilde{G}(\partial_{x_3}\psi)G(v))\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau$$

where $v = (p, s, \nabla'\psi, \nabla u)$ and $\tilde{G}$,G are smooth functions. Using the lower order estimate (12.3.1), we have

$$\begin{aligned} ||v||^2_{L^\infty(\Omega_\pm)} &\leq M^0_\mu + tK_\mu(t) \\ ||\partial^\gamma v||^2_{L^\infty(\Omega_\pm)} &\leq K_\mu(t) \text{ for } 0 \leq |\gamma| \leq 1 \end{aligned}$$

Also, from the definition of $\mathcal{E}_L(t)$ and the estimate of the velocity 12.6.2,

$$\begin{aligned} ||\partial^\gamma v||^2_{L^2(\Omega_\pm)} &\leq C\mathcal{E}_L(t) \text{ for } 2 \leq |\gamma| \leq 3 \\ \int_0^t ||\partial^\gamma \nabla v||^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\leq \frac{1}{\epsilon}M^0_\mu + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ &\quad + \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t)) \text{ for } |\gamma| = 3 \end{aligned}$$

Combining these estimates and using the chain rule proves that

$$\begin{aligned} &\int_0^t \left|\left|\partial_t^j(\rho c^2 \nabla^\psi \cdot u - \frac{1}{\mu}p)\right|\right|^2_{H^{4-j}(\Omega_\pm)} \mathrm{d}\tau \\ &\leq \frac{1}{\epsilon}M^0_\mu + \frac{1}{\epsilon}K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ &+ \epsilon(M^0_\mu + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t)) \end{aligned}$$

Clearly,

$$\int_0^t \left|\left|\partial_t^j g_2\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau \leq tK_\mu(t)$$

Finally,

$$\int_0^t \left|\left|\partial_t^j \nabla' \cdot \hat{n}\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau = \int_0^t \left|\left|\partial_t^j \nabla' G(\nabla' f)\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau$$

for some smooth function G. Note that we have

$$\left|\left|\partial^\gamma \nabla' f\right|\right|^2_{W^{1,\infty}(\Omega_\pm)} \leq K_\mu(t) \text{ for } 0 \leq |\gamma| \leq 1$$

Also, from the definition of $\mathcal{E}_L(t)$,

$$\left|\left|\partial^\gamma \nabla' f\right|\right|^2_{H^{1.5}(\Omega_\pm)} \leq C\mathcal{E}_L(t) \text{ for } 2 \leq |\gamma| \leq 3$$

Combining these estimates and using the chain rule proves that

$$\int_0^t \left|\left|\partial_t^j \nabla' \cdot \hat{n}\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau \leq CK_\mu(t) \int_0^t \mathcal{E}_L(t) \mathrm{d}\tau$$

Using these estimates we obtain

$$\begin{aligned} \int_0^t \left|\left|\partial_t^j (u \cdot n)\right|\right|^2_{H^{5.5-j}(\Gamma)} \mathrm{d}\tau \leq \frac{1}{\epsilon} M_\mu^0 + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ + \epsilon(M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t)) \end{aligned}$$

as required. □

12.7 Conclusion of the Energy Estimate

Putting together what we have so far we have shown that

$$\begin{aligned} \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq \frac{1}{\epsilon} M_\mu^0 + \frac{1}{\epsilon} K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ + \epsilon(M_\mu^0 + tK_\mu(t))(\mathcal{E}_L(t) + \bar{\mathcal{E}}_L(t) + \mathcal{E}_L(\omega)(t)) \end{aligned}$$

for $t \in (0, T)$.

Moreover from (12.4.4), (12.5.6), and (12.5.7), we have

$$\begin{aligned} \rho &\geq \delta^0 - tK_\mu(t) \\ \rho_\mu &\geq \delta_\mu^0 - tK_\mu(t) \\ J^\psi &\geq \kappa^0 - tK_\mu(t) \end{aligned}$$

Choosing $\epsilon = \frac{1}{5(M_\mu^0 + tK_\mu(t))}$, we obtain

$$\begin{aligned} \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq M_\mu^0 + K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) \\ + \frac{1}{5}(\bar{\mathcal{E}}_L(t) + \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t)) \end{aligned}$$

for $t \in (0, T)$ where we have redefined M_μ^0 and $K_\mu(t)$. Rearranging we obtain

$$\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq M_\mu^0 + K_\mu(t)(t + \int_0^t \bar{\mathcal{E}}_L(\tau) + \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) + \frac{1}{4}\bar{\mathcal{E}}_L(t)$$

for $t \in (0, T)$.

This completes the proof.

13. The Fixed Point Scheme

Here we intend to prove the existence of a solution to the μ-approximate equations for fixed μ using the contraction mapping theorem. See 3.8 for a summary of the approach.

13.1 The Functional Framework

In this section we assume we are given initial data (U_+^0, U_-^0, f^0) for the μ-approximate equations with $\mathcal{E}_\mu^0 < \infty$ which satisfies the compatibility condition 7.2.6 up to order 3 and the compatibility condition 7.2.7 up to order 2 and

$$\begin{aligned}\inf_{\Omega_\pm} \rho^0 &=: \delta^0 > 0\\ \inf_{\Omega_\pm} \rho_\mu^0 &=: \delta_\mu^0 > 0\\ \inf_{\mathbb{R}^3} J^{\psi^0} &=: \kappa^0 > 0\end{aligned}$$

Definition 13.1.1. We define the set Y as follows.

$$\begin{aligned}Y =&\{(U_+, U_-, f) : U_\pm \in L^\infty((0,T)\times\Omega_\pm), f \in L^\infty((0,T)\times\Gamma) \text{ with } \mathcal{E}_L(T) \leq R^0,\\ &\partial_t^j (U_+, U_-, f)|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \leq j \leq 3, [u\cdot n] = 0\}\end{aligned}$$

where $R^0 < \infty$ is a constant which will be fixed later, sufficiently large depending the initial data, $T \in (0,1]$ will be fixed later, sufficiently small depending on the initial data, and $\mathcal{E}_L(t)$ is the energy associated to (U_+, U_-, f) as defined in 7.3.2. Note if sufficiently many weak derivatives do not exist to define $\mathcal{E}_L(t)$ then we adopt the convention $\mathcal{E}_L(t) = \infty$.

Definition 13.1.2. We define the Banach space Z as follows.

$$\begin{aligned}Z =&\{(U_+, U_-, f, w) : U_\pm \in L^1_{\text{loc}}((0,T)\times\Omega_\pm), f, w \in L^1_{\text{loc}}((0,T)\times\Gamma)\\ &\text{with } ||(U_+, U_-, f, w)||_Z < \infty\}\end{aligned}$$

where

$$||(U_+, U_-, f, w)||_Z := ||(U_+, U_-, f, w)||_{Z(T)}$$

and for each $t \in (0,T]$ we define

$$
\begin{aligned}
&||(U_+, U_-, f, w)||^2_{Z(t)} \\
&= \sum_{\pm} \sum_{0\le j\le 1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j U\right|\right|^2_{H^{2-j}(\Omega_\pm)} + \sum_{\pm} \int_0^t ||\partial_t^2 U||^2_{L^2(\Omega_\pm)} \,\mathrm{d}\tau + \sum_{\pm}\sum_{0\le j\le 1} \int_0^t \left|\left|\partial_t^j u\right|\right|^2_{H^{3-j}(\Omega_\pm)} \mathrm{d}\tau \\
&+ \sum_{0\le j\le 1} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||f||^2_{H^{3.5-j}(\mathbb{R}^2)} + \int_0^t ||\partial_t^2 f||^2_{H^1(\Gamma)} \,\mathrm{d}\tau \\
&+ \sum_{0\le j\le 1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t^j w\right|\right|^2_{H^{2-j}(\Gamma)} + \sum_{0\le j\le 1} \int_0^t \left|\left|\partial_t^j w\right|\right|^2_{H^{3.5-j}(\Gamma)} \mathrm{d}\tau
\end{aligned}
$$

We define the metric d_Y on Y as follows.

$$d_Y((U_+, U_-, f), (\tilde{U}_+, \tilde{U}_-, \tilde{f})) := \left|\left|(U_+, U_-, f, u\cdot n) - (\tilde{U}_+, \tilde{U}_-, \tilde{f}, \tilde{u}\cdot\tilde{n})\right|\right|_Z$$

Clearly Z is a Banach space with norm $||\cdot||_Z$, and if $(U_+, U_-, f) \in Y$ then $(U_+, U_-, f, u\cdot n) - (U_+^0, U_-^0, f^0, u^0\cdot n^0) \in Z$.

Note that $||\cdot||_{Z(t)}$ is defined so that $||(U_+, U_-, f, u\cdot n)||^2_{Z(t)}$ is effectively equal to the linearised energy $\mathcal{E}_L$ but two orders of derivatives lower and with L^2-in-space instead of L^∞-in-space norms at zero order.

Proposition 13.1.3. *Y is a complete metric space with respect to the metric d_Y.*

Proof. Let (U_+^n, U_-^n, f^n) be a sequence in Y which is Cauchy with respect to the metric d_Y. Since Z is a Banach space, there exists (U_+, U_-, f, w) with $(U_+, U_-, f, w) - (U_+^0, U_-^0, f^0, u^0\cdot n^0) \in Z$ such that $\left|\left|(U_+^n, U_-^n, f^n, u^n\cdot n^n) - (U_+, U_-, f, w)\right|\right|_Z \to 0$. In particular, passing to a subsequence if necessary, we know that $U_\pm^n$ converges almost everywhere to $U_\pm$ in $(0,T)\times\Omega_\pm$ and f^n converges almost everywhere to f on $(0,T)\times\Gamma$. We now show that $\mathcal{E}_L(T) \le R^0$, where we write $\mathcal{E}_L$ for the energy associated to (U_+, U_-, f). Let us write $\mathcal{E}_L^n$ for the energy associated to (U_+^n, U_-^n, f^n). We know that

$$\mathcal{E}_L^n(T) \le R^0$$

for all n. Thus we may use weak-$*$ compactness to obtain a subsequence n_k which converges weakly-$*$ with respect to the L^2 and L^∞ norms making up the energy $\mathcal{E}_L$. Using the weak-$*$ lower semi-continuity of norms, we have

$$\mathcal{E}_L(T) \le \liminf_{k\to\infty} \mathcal{E}_L^{n_k}(T) \le R^0$$

We may use Sobolev compact embedding theorems to obtain that

$$(U_+^{n_k}, U_-^{n_k}, f^{n_k}, \nabla' f^{n_k}) \to (U_+, U_-, f, \nabla' f)$$

uniformly on compact subsets of $(0,T)\times\Omega_\pm$ (or $(0,T)\times\Gamma$ as appropriate) and hence we have

$$(U_+, U_-, f)|_{t=0} = (U_+^0, U_-^0, f^0)$$
$$[u\cdot n] = 0$$

Now let $w \in C_c^\infty((-\infty, T) \times \mathbb{R}^3)$ and let $0 \le j \le 3$. Integrating by parts in time, we have

$$\int_0^T \int_{\Omega_\pm} \partial_t^{j+1} U^{n_k} w \mathrm{d}x\mathrm{d}t = -\int_0^T \int_{\Omega_\pm} \partial_t^j U^{n_k} \partial_t w \mathrm{d}x\mathrm{d}t - \int_{\Omega_\pm} \partial_0^{\mu,g,j} U^0 w(0) \mathrm{d}x$$

Using weak convergence and letting $k \to \infty$, we obtain

$$\int_0^T \int_{\Omega_\pm} \partial_t^{j+1} U w \mathrm{d}x\mathrm{d}t = -\int_0^T \int_{\Omega_\pm} \partial_t^j U \partial_t w \mathrm{d}x\mathrm{d}t - \int_{\Omega_\pm} \partial_0^{\mu,g,j} U^0 w(0) \mathrm{d}x$$

Integrating by parts in time again we obtain

$$\int_{\Omega_\pm} \partial_t^j U|_{t=0} w(0) \mathrm{d}x = \int_{\Omega_\pm} \partial_0^{\mu,g,j} U^0 w(0) \mathrm{d}x$$

Since $w(0)$ for such w forms a dense subset of $L^2(\mathbb{R}^3)$, we obtain that

$$\partial_t^j (U_+, U_-)|_{t=0} = \partial_0^{\mu,g,j} (U_+^0, U_-^0) \text{ for } 0 \le j \le 3$$

Doing exactly the same for f we obtain

$$\partial_t^j f|_{t=0} = \partial_0^{\mu,g,j} f^0 \text{ for } 0 \le j \le 3$$

This completes the proof. $\square$

13.2 The Contraction Mapping

Proposition 13.2.1. *Given a background state $(\bar{U}_+, \bar{U}_-, \bar{f}) \in Y$, there exists a solution $(U_+, U_-, f) \in Y$ of the linearised equations with background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ assuming we have chosen $T > 0$ sufficiently small and R^0 sufficiently large depending on the size of the initial data. In fact T and R^0 should satisfy the following inequalities.*

$$R^0 \ge R_\mu(\mathcal{E}_\mu^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1))$$
$$T \le T_\mu(R^0, \mathcal{E}_\mu^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1))$$

where R_μ is a smooth increasing function and T_μ is a smooth decreasing function. Moreover, if ω is the corresponding solution to the linearised curl equation 9.1.5, we may assume

$$\mathcal{E}_L(T) + \mathcal{E}_L(\omega)(T) \le R^0$$

Proof. Let us assume we are given a background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ defined on the interval $(0, \bar{T})$ with $\bar{\mathcal{E}}_L(\bar{T}) < \infty$ satisfying the conditions of 9.5.1. We let (U_+, U_-, f) be a solution of the linearised equations with background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ as given by proposition 9.5.1 on the time interval $(0, \tilde{T})$, where $\tilde{T} \in (0, \bar{T}]$ and if $\tilde{T} < \bar{T}$ then one of the following holds as $t \uparrow \tilde{T}$.

$$\inf_{x \in \Omega_\pm} \rho \to \frac{\delta^0}{4}$$
$$\inf_{x \in \Omega_\pm} \rho_\mu \to \frac{\delta_\mu^0}{4}$$
$$\inf_{x \in \mathbb{R}^3} J^\psi \to \frac{1}{4}\kappa^0$$

We note that from 9.4.1 we have the energy estimate

$$\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \le M^0_{\mu,0} + t\mathcal{K}_{L,1}(t) + \frac{1}{4}\bar{\mathcal{E}}_L(t)$$

for $t \in (0, \tilde{T})$, where $\mathcal{E}_L$ denotes the energy associated to (U_+, U_-, f). We also have

$$\begin{aligned} \rho &\ge \delta^0 - t\mathcal{K}_{L,2}(t) \\ \rho_\mu &\ge \delta^0_\mu - t\mathcal{K}_{L,3}(t) \\ J^\psi &\ge \kappa^0 - t\mathcal{K}_{L,4}(t) \end{aligned}$$

on the time interval $(0, \tilde{T})$ where

$$\mathcal{K}_{L,i}(t) = F_{\mu,i}(\bar{\mathcal{E}}_L(t), \mathcal{E}_L(t), \mathcal{E}_L(\omega)(t), M^0_{\mu,i})$$

for some smooth increasing function $F_{\mu,i}(\cdot)$ and where $M^0_{\mu,i}$ has the same form as M^0_μ.

Let us suppose $R^0 \ge 2M^0_{\mu,0}$ and assume for the moment $\bar{\mathcal{E}}_L(\bar{T}) \le R^0$.

Now we let $T^0 = \sup\{t \in (0, \tilde{T})\}$ such that the following hold.

$$\begin{aligned} \mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) &\le R^0 \\ \inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho(\tau) &\ge \frac{1}{2}\delta^0 \\ \inf_{\tau\in(0,t)} \inf_{x\in\Omega_\pm} \rho_\mu(\tau) &\ge \frac{1}{2}\delta^0_\mu \\ \inf_{\tau\in(0,t)} \inf_{x\in\mathbb{R}^3} J^\psi(\tau) &\ge \frac{1}{2}\kappa^0 \end{aligned}$$

Note that by uniform continuity we have $T^0 > 0$ and either $T^0 = \tilde{T}$ or one of the above inequalities holds as an equality. If $T^0 = \tilde{T}$ then to avoid contradiction we must have $T^0 = \tilde{T} = \bar{T}$, so clearly (together with the other properties of the solution stated in 9.5.1), we have $(U_+, U_-, f) \in Y$ if Y is equipped with the time $\bar{T}$ and the constant R^0. Otherwise, one of the following holds.

$$\begin{aligned} \mathcal{E}_L(T^0) + \mathcal{E}_L(\omega)(T^0) &= R^0 \\ \inf_{x\in\Omega_\pm} \rho(T^0) &= \frac{1}{2}\delta^0 \\ \inf_{x\in\Omega_\pm} \rho_\mu(T^0) &= \frac{1}{2}\delta^0_\mu \\ \inf_{x\in\mathbb{R}^3} J^\psi(T^0) &= \frac{1}{2}\kappa^0 \end{aligned}$$

Using the above inequalities, this means one of the following holds

$$\begin{aligned} &R^0 \le M^0_{\mu,0} + T^0 F_{\mu,1}(R^0, R^0, R^0, M^0_{\mu,1}) + \frac{1}{4}R^0 \le \frac{3}{4}R^0 + T^0 F_{\mu,1}(R^0, R^0, R^0, M^0_{\mu,1}) \\ &\frac{1}{2}\delta^0 \ge \delta^0 - T^0 F_{\mu,2}(R^0, R^0, R^0, M^0_{\mu,2}) \\ &\frac{1}{2}\delta^0_\mu \ge \delta^0_\mu - T^0 F_{\mu,3}(R^0, R^0, R^0, M^0_{\mu,3}) \\ &\frac{1}{2}\kappa^0 \ge \kappa^0 - T^0 F_{\mu,4}(R^0, R^0, R^0, M^0_{\mu,4}) \end{aligned}$$

Hence one of the following holds

$$T^0 \geq \frac{R^0}{4F_{\mu,1}(R^0, R^0, R^0, M^0_{\mu,1})}$$
$$T^0 \geq \frac{\delta^0}{2F_{\mu,2}(R^0, R^0, R^0, M^0_{\mu,2})}$$
$$T^0 \geq \frac{\delta^0_\mu}{2F_{\mu,3}(R^0, R^0, R^0, M^0_{\mu,3})}$$
$$T^0 \geq \frac{\kappa^0}{2F_{\mu,4}(R^0, R^0, R^0, M^0_{\mu,4})}$$

We set T to be the minimum of the quantities on the right-hand side of the above.

Thus, if $(\bar{U}_+, \bar{U}_-, \bar{f}) \in Y$ where Y is equipped with time time T and the constant R^0 as just defined, then we may take $\bar{T} = T$ in the above proof and the assumption $\bar{\mathcal{E}}_L(\bar{T}) \leq R^0$ automatically holds. Together with the other properties of the solution stated in 9.5.1, this implies $(U_+, U_-, f) \in Y$ if Y is equipped with the time T and the constant R^0. $\square$

Proposition 13.2.2. *For $i = 1, 2$, let $(U^i_+, U^i_-, f^i) \in Y$ be solutions of the linearised equations with background states $(\bar{U}^i_+, \bar{U}^i_-, \bar{f}^i) \in Y$, as described in 9.1.1. Then*

$$d_Y((U^1_+, U^1_-, f^1), (U^2_+, U^2_-, f^2)) \leq \lambda d_Y((\bar{U}^1_+, \bar{U}^1_-, \bar{f}^1), (\bar{U}^2_+, \bar{U}^2_-, \bar{f}^2))$$

for some $\lambda < 1$ which is fixed independent of the solutions and background states, provided we have chosen $T > 0$ sufficiently small and R^0 sufficiently large depending on the size of the initial data. In fact T and R^0 should satisfy the following inequalities.

$$R^0 \geq \tilde{R}_\mu(\mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1))$$
$$T \leq \tilde{T}_\mu(R^0, \mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1))$$

where $\tilde{R}_\mu$ is a smooth increasing function and $\tilde{T}_\mu$ is a smooth decreasing function.

Proof. See chapter 14. $\square$

Proposition 13.2.3. *Define the map $\Phi : Y \to Y$ by $(\bar{U}_+, \bar{U}_-, \bar{f}) \mapsto (U_+, U_-, f)$ where (U_+, U_-, f) is the unique solution of the linearised equations with background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ as given by 9.5.1. Assume that R^0 and T satisfy the inequalities in propositions 13.2.1 and 13.2.2. Assume also that the space Y is non-empty. Then Φ has a fixed point.*

Proof. By proposition 9.5.1 we know that

$$\mathcal{E}_L(T) < \infty$$
$$\partial^j_t(U_+, U_-, f)|_{t=0} = \partial^{\mu,g,j}_0(U^0_+, U^0_-, f^0) \text{ for } 0 \leq j \leq 3$$
$$[u \cdot n] = 0$$

By proposition 13.2.1 we have

$$\mathcal{E}_L(T) \leq R^0$$

and hence $(U_+, U_-, f) \in Y$. Now proposition 13.2.2 implies that the map $\Phi : Y \to Y$ is a contraction. Since Y is complete, we may apply the contraction mapping theorem to conclude that Φ has a unique fixed point in Y. $\square$

13.3 The Existence of a Solution to the μ-Approximate Equations

Lemma 13.3.1. *Suppose (U_+, U_-, f) is a fixed point of the linearised equations on the time interval $(0, T)$, by which we mean (U_+, U_-, f) together with (ω_+, ω_-) is a solution of the linearised equations on the time interval $(0, T)$ as defined in 9.1.1, linearised about the state (U_+, U_-, f), with initial value $(U_+, U_-, f)|_{t=0} = (U_+^0, U_-^0, f^0)$. Suppose also*

$$\begin{aligned} \mathcal{E}_L(T) &< \infty \\ \rho &\geq \frac{\delta^0}{4} \\ \rho_\mu &\geq \frac{\delta_\mu^0}{4} \\ J^\psi &\geq \frac{1}{4}\kappa^0 \end{aligned}$$

where $\mathcal{E}_L(t)$ is the linearised energy associated with (U_+, U_-, f). Then (U_+, U_-, f) is a solution of the μ-approximate equations on the time interval $(0, T)$ with initial data (U_+^0, U_-^0, f^0).

Moreover, there exists $T_\mu^0 \in (0, T]$ and $C_\mu^0 > 0$ such that

$$\begin{aligned} \mathcal{E}_\mu(\min\{T_\mu^0, T\}) &\leq C_\mu^0 < \infty \\ \rho &\geq \frac{\delta^0}{2} \text{ in } (0, \min\{T_\mu^0, T\}) \times \Omega_\pm \\ \rho_\mu &\geq \frac{\delta_\mu^0}{2} \text{ in } (0, \min\{T_\mu^0, T\}) \times \Omega_\pm \\ J^\psi &\geq \frac{1}{2}\kappa^0 \text{ in } (0, \min\{T_\mu^0, T\}) \times \mathbb{R}^3 \end{aligned}$$

The constant $C_\mu^0 > 0$ is bounded above as follows.

$$C_\mu^0 \leq C_\mu(\mathcal{E}_\mu^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1)) > 0$$

where $C_\mu(\cdot)$ is a smooth increasing function of its arguments, and may depend on μ.

The time $T_\mu^0 > 0$ is bounded below as follows.

$$T_\mu^0 \geq T_\mu(\mathcal{E}_\mu^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1)) > 0$$

where $T_\mu(\cdot)$ is a smooth decreasing function of its arguments, and may depend on μ.

Proof. Note that the definition of the linearised equations implies that

$$\frac{1}{\rho c^2}(\partial_t + u^\psi \cdot \nabla)p + \nabla^\psi \cdot u = 0 \tag{13.3.1}$$

$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)\omega = -\nabla^\psi \rho_\mu \times (\partial_t + u^\psi \cdot \nabla)u - \rho_\mu(\epsilon_{ijk}\partial_j^\psi u_l \partial_l^\psi u_k) + \nabla^\psi \times g_1 \tag{13.3.2}$$

$$\rho_\mu(\partial_t + u^\psi \cdot \nabla)u + \nabla^\psi p = \mu\nabla^\psi(\rho c^2 \nabla^\psi \cdot u) + g_1 + \mu\nabla^\psi \times (\rho c^2 \omega) - \mu\nabla^\psi \times (\rho c^2 \nabla^\psi \times u) \tag{13.3.3}$$

$$(\partial_t + u^\psi \cdot \nabla)s = 0 \tag{13.3.4}$$

in $(0,T)\times\Omega_\pm$ and

$$\partial_t f = u_\pm \cdot n \tag{13.3.5}$$

$$[p-\mu\rho c^2\nabla^\psi\cdot u] = -\sigma\nabla'\cdot\hat{n} - \mu(\Delta'-1)(u\cdot n) + g_2 \tag{13.3.6}$$

$$n\times(\nabla^\psi\times u_\pm) = n\times\omega_\pm \tag{13.3.7}$$

on $(0,T)\times\Gamma$, where

$$\begin{aligned}\psi &:= L^a f\\ u^\psi &:= (u', \frac{1}{J^\psi}(u_3 - u'\cdot\nabla'\psi - \partial_t\psi))\end{aligned}$$

Thus it remains to show that the velocity equation

$$\rho_\mu(\partial_t + u^\psi\cdot\nabla)u + \nabla^\psi p = \mu\nabla^\psi(\rho c^2\nabla^\psi\cdot u) + g_1$$

holds, which will follow from the above if we can show

$$\omega = \nabla^\psi \times u$$

To do this we take the curl of the equation (13.3.3) and compare it to the equation (13.3.2). Using lemma 9.6.1 and the fact that $\psi=\bar{\psi}$ and $u=\bar{u}$, we see that $\nabla^\psi\times u$ satisfies the following equation.

$$\begin{aligned}\rho_\mu(\partial_t + u^\psi\cdot\nabla)(\nabla^\psi\times u) = &-\nabla^\psi\rho_\mu\times(\partial_t+u^\psi\cdot\nabla)u - \rho_\mu(\epsilon_{ijk}\partial_j^\psi u_l\partial_l^\psi u_k) + \nabla^\psi\times g_1\\ &+\mu\nabla^\psi\times(\nabla^\psi\times(\rho c^2\omega)) - \mu\nabla^\psi\times(\nabla^\psi\times(\rho c^2\nabla^\psi\times u))\end{aligned}$$

Subtracting the curl equation (13.3.2) and writing $\xi = \rho c^2\nabla^\psi\times u - \rho c^2\omega$, we have

$$\rho_\mu(\partial_t + u^\psi\cdot\nabla)(\frac{\xi}{\rho c^2}) = -\mu\nabla^\psi\times(\nabla^\psi\times\xi)$$

Multiplying by ξ and integrating by parts separately in $\Omega_\pm$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\frac{\rho_\mu}{\rho c^2}|\xi|^2\,\mathrm{d}x \mp \frac{1}{2}\int_\Gamma\frac{\rho_\mu}{\rho c^2}(u^\psi)_3|\xi|^2\,\mathrm{d}x\\ &= \frac{1}{2}\int_{\Omega_\pm}\partial_t\rho_\mu|\xi|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Omega_\pm}\nabla\cdot(\rho_\mu u^\psi)|\xi|^2\,\mathrm{d}x\\ &-\frac{1}{2}\int_{\Omega_\pm}\rho_\mu\partial_t(\frac{1}{\rho c^2})|\xi|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_\pm}\rho_\mu u^\psi\cdot\nabla(\frac{1}{\rho c^2})|\xi|^2\,\mathrm{d}x\\ &\pm\mu\int_\Gamma n\times(\nabla^\psi\times\xi)\cdot\xi\mathrm{d}x' - \mu\int_{\Omega_\pm}\left|\nabla^\psi\times\xi\right|^2\mathrm{d}x - \mu\int_{\Omega_\pm}\partial_{x_3}(\frac{\nabla\psi}{J^\psi})\times(\nabla^\psi\times\xi)\cdot\xi\mathrm{d}x\end{aligned}$$

Note that using the basic vector identity $(a\times b)\cdot c = (c\times a)\cdot b$ we have $n\times(\nabla^\psi\times\xi)\cdot\xi = (n\times\xi)\cdot(\nabla^\psi\times\xi) = 0$ using the boundary condition (13.3.7). Hence, using this together with the fact that $(u^\psi)_3|_\Gamma = 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}|\xi|^2\,\mathrm{d}x \le \frac{1}{\delta^0}(C_{U_\pm,f}\int_{\Omega_\pm}|\xi|^2\,\mathrm{d}x)$$

where the constant $C_{U_\pm,f}<\infty$ depends on $||U_\pm||_{W^{1,\infty}((0,T)\times\Omega_\pm)}$ and $||f||_{W^{2,\infty}((0,T)\times\Gamma)}$. We used Cauchy's inequality to absorb the last term into the second to last term. Note that by definition

$$\xi_\pm|_{t=0} = 0$$

hence we may apply Gronwall's lemma to conclude that $\xi_\pm = 0$. This proves that (U_+, U_-, f) is a solution of the μ-approximate equations on the time interval $(0,T)$.

Since $(U_+, U_-, f)|_{t=0} = (U_+^0, U_-^0, f^0)$ and (U_+, U_-, f) solves the μ-approximate equations, it is now clear by definition of the initial time derivatives that

$$\partial_t^j (U_+, U_-, f)|_{t=0} = \partial_0^{\mu,g,j}(U_+^0, U_-^0, f^0) \text{ for } 0 \leq j \leq 3$$

Thus we may apply the linearised energy estimate 9.4.1 with background state (U_+, U_-, f) to conclude that

$$\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq M_\mu^0 + K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau) + \frac{1}{4}\mathcal{E}_L(t)$$

for $t \in (0,T)$ which we rearrange to obtain

$$\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t) \leq M_\mu^0 + K_\mu(t)(t + \int_0^t \mathcal{E}_L(\tau) + \mathcal{E}_L(\omega)(\tau)\mathrm{d}\tau)$$

Now we relate $\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t)$ and $\mathcal{E}_\mu(t)$ using 9.3.3 to obtain

$$\begin{aligned}\mathcal{E}_\mu(t) &\leq M_\mu^0 + K_\mu(t)(t + \int_0^t \mathcal{E}_\mu(\tau)\mathrm{d}\tau) \\ &\leq M_{\mu,0}^0 + tF_{\mu,1}(M_{\mu,1}^0, \mathcal{E}_\mu(t))\end{aligned}$$

Where $F_{\mu,1}$ is a smooth increasing function and $M_{\mu,i}^0$ has the same form as M_μ^0. Moreover, we have

$$\begin{aligned}\rho &\geq \delta^0 - tK_{\mu,2}(t) \\ \rho_\mu &\geq \delta_\mu^0 - tK_{\mu,3}(t) \\ J^\psi &\geq \kappa^0 - tK_{\mu,4}(t)\end{aligned}$$

on the time interval $(0,T)$ where $K_{\mu,i}$ have the same for as K_μ. Note that we can write $K_{\mu,i}(t) \leq F_{\mu,i}(M_{\mu,i}^0, \mathcal{E}_\mu(t))$ for some smooth increasing functions $F_{\mu,i}$.

We let $T_\mu^0 = \sup\{T' \in (0,T)\}$ such that the following hold for all $t \in (0,T')$.

$$\begin{aligned}\mathcal{E}_\mu(t) &\leq 2M_{\mu,0}^0 \\ \inf_{\Omega_\pm} \rho &\geq \frac{\delta^0}{2} \\ \inf_{\Omega_\pm} \rho_\mu &\geq \frac{\delta_\mu^0}{2} \\ \inf_{\mathbb{R}^3} J^\psi &\geq \frac{1}{2}\kappa^0\end{aligned}$$

Note by uniform continuity that $T_\mu^0 > 0$ and either $T_\mu^0 = T$ or one of the above inequalities holds as an equality. In the latter case one of the following holds.

$$\begin{aligned}2M_{\mu,0}^0 &\leq M_{\mu,0}^0 + T_\mu^0 F_{\mu,1}(M_{\mu,1}^0, 2M_{\mu,0}^0) \\ \frac{\delta^0}{2} &\geq \delta^0 - T_\mu^0 F_{\mu,2}(M_{\mu,2}^0, 2M_{\mu,0}^0) \\ \frac{\delta_\mu^0}{2} &\geq \delta_\mu^0 - T_\mu^0 F_{\mu,3}(M_{\mu,3}^0, 2M_{\mu,0}^0) \\ \frac{\kappa^0}{2} &\geq \kappa^0 - T_\mu^0 F_{\mu,4}(M_{\mu,4}^0, 2M_{\mu,0}^0)\end{aligned}$$

and hence

$$T_\mu^0 \geq \min\{\frac{M_{\mu,0}^0}{F_\mu(M_{\mu,1}^0, 2M_{\mu,0}^0)}, \frac{\delta^0}{2F_{\mu,2}(M_{\mu,2}^0, 2M_{\mu,0}^0)}, \frac{\delta_\mu^0}{2F_{\mu,3}(M_{\mu,3}^0, 2M_{\mu,0}^0)}, \frac{\kappa^0}{2F_{\mu,3}(M_{\mu,3}^0, 2M_{\mu,0}^0)}\}$$

This completes the proof. □

Lemma 13.3.2. *The metric space Y is non-empty for sufficiently large R^0 where R^0 depends on the high order energy $\mathcal{E}_{\max}^0$ of the initial data (U_+^0, U_-^0, f^0), provided we assume $\mathcal{E}_{\max}^0 < \infty$. More precisely, R^0 should satisfy*

$$R^0 \geq F_\mu(\mathcal{E}_{\max}^0)$$

where $F_\mu(\cdot)$ is a given smooth increasing function.

Proof. We define (U_+, U_-, f) in terms of the initial data as follows.

$$p_\pm = \sum_{j=0}^{3} \frac{1}{j!} t^j \partial_0^{\mu,g,j} p_\pm^0$$

$$s_\pm = \sum_{j=0}^{3} \frac{1}{j!} t^j \partial_0^{\mu,g,j} s_\pm^0$$

$$f = \sum_{j=0}^{3} \frac{1}{j!} t^j \partial_0^{\mu,g,j} f_\pm^0$$

Then clearly

$$\partial_t^j (p_\pm, s_\pm, f)|_{t=0} = \partial_0^{\mu,g,j} (p_\pm^0, s_\pm^0, f^0) \text{ for } 0 \leq j \leq 3$$

Note that having defined f we can now compute $n, \hat{n}$ etc in terms of f. Now, following the notation for the rotation matrix N defined in terms of f such that $(N^{-1}w)_3 = w \cdot n$ for any vector w, as defined in section 11.3, we set

$$\widehat{u}^0 = (N^0)^{-1} u^0$$

Then we define

$$u(t) = N(t) \sum_{j=0}^{3} \frac{1}{j!} t^j \partial_0^{\mu,g,j} \ \widehat{u}^0$$

Note that the definition implies that for $0 \leq j \leq 3$ we have

$$\begin{aligned}
\partial_t^j u|_{t=0} &= \sum_{k+l=j} \partial_t^k (N(t) N^{-1}(t))|_{t=0} \partial_0^{\mu,g,l} u^0 \\
&= \partial_0^{\mu,g,j} u^0
\end{aligned}$$

as required. Also,

$$\begin{aligned}
[u \cdot n] &= [(N^{-1}u)_3] \\
&= \sum_{j=0}^{3} \frac{1}{j!} t^j \partial_0^{\mu,g,j} [\widehat{u}_3^0] \\
&= \sum_{j=0}^{3} \frac{1}{j!} t^j \partial_0^{\mu,g,j} [u^0 \cdot n^0] \\
&= 0
\end{aligned}$$

since the initial data satisfies the compatibility condition 7.2.6 up to order 3.

Since the initial data is smooth, we have $\mathcal{E}_L(1) < \infty$, where $\mathcal{E}_L(t)$ is the linearised energy associated with the above (U_+, U_-, f), so we have $(U_+, U_-, f) \in Y$ provided

$$R^0 \geq \mathcal{E}_L(1)$$

We note that

$$\mathcal{E}_L(1) \leq F_\mu(\mathcal{E}^0_{\max})$$

for some smooth increasing function $F_\mu(\cdot)$. This completes the proof. □

13.3.1 Conclusion of the Proof of the Existence of Solutions to the μ-Approximate Equations

The proof of theorem 7.4.1. By lemma 13.3.2, we have that the space Y is non-empty for sufficiently large R^0 depending on the high-order energy $\mathcal{E}^0_{\max}$ of the initial data. Propositions 13.2.1 and 13.2.2 show that the map $\Phi : Y \to Y$ defined by $(\bar{U}_+, \bar{U}_-, \bar{f}) \mapsto (U_+, U_-, f)$ where (U_+, U_-, f) is the unique solution of the linearised equations with background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ as given by 9.5.1 is a well-defined contraction, provided that R^0 and T satisfy the following bounds.

$$\begin{aligned} R^0 &\geq R_\mu(\mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) \\ T &\leq T_\mu(R^0, \mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) \end{aligned}$$

where R_μ is a smooth increasing function and T_μ is a smooth decreasing function. Thus we set R^0 to be the maximum of $R_\mu(\mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1))$ and $F_\mu(\mathcal{E}^0_{\max})$ as given in lemma 13.3.2, then set

$$T = T_\mu(R^0, \mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1))$$

We apply proposition 13.2.3 to conclude that Φ has a fixed point $(U_+, U_-, f) \in Y$. We then apply lemma 13.3.1 to conclude that (U_+, U_-, f) is a solution of the μ-approximate equations on the time interval $(0, T)$. We also obtain a $T^0_\mu \in (0, T]$ and $C^0_\mu > 0$ such that

$$\begin{aligned} \mathcal{E}_\mu(\min\{T^0_\mu, T\}) \leq C^0_\mu &< \infty \\ \rho &\geq \frac{\delta^0}{2} \text{ in } (0, \min\{T^0_\mu, T\}) \times \Omega_\pm \\ \rho_\mu &\geq \frac{\delta^0_\mu}{2} \text{ in } (0, \min\{T^0_\mu, T\}) \times \Omega_\pm \\ J^\psi &\geq \frac{1}{2}\kappa^0 \text{ in } (0, \min\{T^0_\mu, T\}) \times \mathbb{R}^3 \end{aligned}$$

The constant $C^0_\mu > 0$ is bounded above as follows.

$$C^0_\mu \leq C_\mu(\mathcal{E}^0_\mu, \frac{1}{\delta^0}, \frac{1}{\delta^0_\mu}, \frac{1}{\kappa^0}, E_g(1)) > 0$$

where $C_\mu(\cdot)$ is a smooth increasing function of its arguments, and may depend on μ.

The time $T_\mu^0 > 0$ is bounded below as follows.

$$T_\mu^0 \geq T_\mu(\mathcal{E}_\mu^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1)) > 0$$

where $T_\mu(\cdot)$ is a smooth decreasing function of its arguments, and may depend on μ.

Redefining $T_\mu^0 = \min\{T_\mu^0, T\}$, so that the lower bound becomes

$$T_\mu^0 \geq T_\mu(\mathcal{E}_{\max}^0, \frac{1}{\delta^0}, \frac{1}{\delta_\mu^0}, \frac{1}{\kappa^0}, E_g(1)) > 0$$

where $T_\mu(\cdot)$ is a smooth decreasing function of its arguments, we complete the proof. □

13.3.2 Conclusion of the Proof of the Existence of Maximal-Time Solutions to the μ-Approximate Equations

The proof of theorem 7.4.2. We need to maximise the time interval of existence of a solution to the μ-approximate equations as given by 7.4.1. We wish to take the final data of the solution given in 7.4.1 and use it as initial data. The only problem is we needed to assume the initial data was smooth in order to prove the space Y was non-empty. To overcome this we will construct a background state using the solution given by 7.4.1 and note that we did not need to use the smoothness of the initial data elsewhere.

Let (U_+^0, U_-^0, f^0) be smooth initial data as given in the statement of 7.4.2. Let $\tilde{T} = \sup\{T' \in (0,1)\}$ such that there exists a solution (U_+, U_-, f) to the μ-approximate equations with initial data (U_+^0, U_-^0, f^0) on the time interval $(0, T')$ satisfying the inequalities

$$\mathcal{E}_\mu(t) < \infty$$
$$\inf_{\Omega_\pm} \rho > 0$$
$$\inf_{\Omega_\pm} \rho_\mu > 0$$
$$\inf_{\mathbb{R}^3} J^\psi > 0$$

for all $t \in (0, T')$. Note that by 7.4.1 we have $\tilde{T} > 0$. If $\tilde{T} = 1$ then we are done, so assume not. If one of

$$E_\mu(t) \to \infty$$
$$\inf_{\Omega_\pm} \rho \to 0$$
$$\inf_{\Omega_\pm} \rho_\mu \to 0$$
$$\inf_{\mathbb{R}^3} J^\psi \to 0$$

holds as $t \uparrow \tilde{T}$ then we are also done so suppose not. In particular, note this implies that

$$E_\mu(\tilde{T}) \leq \tilde{C} < \infty$$
$$\inf_{t\in(0,\tilde{T})} \inf_{\Omega_\pm} \rho \geq \frac{1}{\tilde{C}} > 0$$
$$\inf_{t\in(0,\tilde{T})} \inf_{\Omega_\pm} \rho_\mu \geq \frac{1}{\tilde{C}} > 0$$
$$\inf_{t\in(0,\tilde{T})} \inf_{\mathbb{R}^3} J^\psi \geq \frac{1}{\tilde{C}} > 0$$

for some constant $\tilde{C} > 0$. Note that (U_+, U_-, f) together with $\omega_\pm = \nabla^\psi \times u_\pm$ solves the linearised equations with background state (U_+, U_-, f), hence we may apply the energy estimate 9.4.1 together with the relation between $\mathcal{E}_L(t) + \mathcal{E}_L(\omega)(t)$ and $\mathcal{E}_\mu(t)$ given by 9.3.3 to obtain

$$\mathcal{E}_\mu(t) \le M_\mu^0 + K_\mu(t)(t + \int_0^t \mathcal{E}_\mu(\tau)\mathrm{d}\tau)$$

for all $t \in (0, \tilde{T})$, where $K_\mu(t) = F_\mu(M_\mu^0, E_\mu(t))$. Thus an application of Gronwall's lemma implies

$$\mathcal{E}_\mu(\tilde{T}) \le \tilde{C}$$

where we have redefined $\tilde{C}$, which may depend on M_μ^0. Note that for arbitrary $\epsilon \in (0, \tilde{T})$ we can pick $T^\epsilon \in (\tilde{T} - \epsilon, \tilde{T}]$ such that if we set

$$(U_+^{0,\epsilon}, U_-^{0,\epsilon}, f^{0,\epsilon}) := (U_+(T^\epsilon), U_-(T^\epsilon), f(T^\epsilon))$$

then we have

$$\begin{aligned}
&\mathcal{E}_\mu^{0,\epsilon} \le \tilde{C} < \infty \\
&\delta^{0,\epsilon} := \inf_{\Omega_\pm} \rho^{0,\epsilon} \ge \frac{1}{\tilde{C}} > 0 \\
&\delta_\mu^{0,\epsilon} := \inf_{\Omega_\pm} \rho_\mu^{0,\epsilon} \ge \frac{1}{\tilde{C}} > 0 \\
&\kappa^{0,\epsilon} := \inf_{\mathbb{R}^3} J^{\psi^{0,\epsilon}} \ge \frac{1}{\tilde{C}} > 0
\end{aligned}$$

where $\mathcal{E}_\mu^{0,\epsilon}$ denotes the initial energy associated with $(U_+^{0,\epsilon}, U_-^{0,\epsilon}, f^{0,\epsilon})$. Let us also define, for $t \in (0,1)$,

$$g^\epsilon(t) = g(t + T^\epsilon)$$

Technically, g is only defined on the time interval $(0,1)$ but we may as well extend it via Sobolev extension to be defined on the time interval $(0,2)$. Note that this implies

$$E_{g^\epsilon}(1) \le C E_g(1) < \infty$$

Note that since (U_+, U_-, f) is actually a solution of the μ-approximate equations with right hand side g, by differentiating with respect to time and evaluating at $t = T^\epsilon$ we see that the newly defined initial data $(U_+^{0,\epsilon}, U_-^{0,\epsilon}, f^{0,\epsilon})$ satisfies the compatibility condition 7.2.6 up to order 3 and the compatibility condition 7.2.7 up to order 2 for the μ-approximate equations with right hand side g^ϵ.

We define a background state $(\bar{U}_+, \bar{U}_-, \bar{f})$ on the time interval $(0, T^\epsilon)$ as follows. We extend (U_+, U_-, f) to the time interval $(0, 2T^\epsilon)$ in the following manner. Firstly, extend $p_\pm, s_\pm, f$ to the time interval $(0, 2T^\epsilon)$ by Sobolev extension. We set

$$\widehat{u} = N^{-1}u \text{ on } (0, T^\epsilon)$$

where N is the rotation matrix defined in terms of $\nabla' f$ in section 11.3 such that $(N^{-1}w)_3 = w \cdot n$ for any vector w. Then we define $\widehat{u}$ on $(0, 2T^\epsilon)$ by Sobolev extension, then we set $u = N\,\widehat{u}$ on $(0, 2T^\epsilon)$. Note since Sobolev extension is linear we have

$$[u \cdot n] = [(N^{-1}N\,\widehat{u})_3] = [\widehat{u}_3] = 0$$

on $(0, 2T^\epsilon)$ and $u|_{(0,T^\epsilon)} = NN^{-1}u = u$. Thus we have constructed (U_+, U_-, f) on $(0, 2T^\epsilon)$. Now we define $(\bar{U}_+(t), \bar{U}_-(t), \bar{f}(t)) = (U_+(t + T^\epsilon), U_-(t + T^\epsilon), f(t + T^\epsilon))$ for $t \in (0, T^\epsilon)$.

Now in the definition of our space Y we take $(U_+^{0,\epsilon}, U_-^{0,\epsilon}, f^{0,\epsilon})$ as the initial data and note that by construction $(\bar{U}_+, \bar{U}_-, \bar{f}) \in Y$ provided we take the time $T > 0$ in the definition of Y small enough such that $T \leq T^\epsilon$ and the constant R^0 large enough depending on the energy of $(\bar{U}_+, \bar{U}_-, \bar{f})$ (which in fact depends on $\mathcal{E}_\mu^{0,\epsilon}$, which is bounded by $\mathcal{E}_\mu(\tilde{T})$) such that $\bar{\mathcal{E}}_L(T) \leq R^0$.

We now run the argument from the previous proof again. Propositions 13.2.1 and 13.2.2 show that the map $\Phi : Y \to Y$ sending an element of Y to a solution of the linearised equations with background state given by that element is a well-defined contraction, provided that R^0 and T satisfy the following bounds.

$$
\begin{aligned}
R^0 &\geq R_\mu(\mathcal{E}_\mu^{0,\epsilon}, \frac{1}{\delta^{0,\epsilon}}, \frac{1}{\delta_\mu^{0,\epsilon}}, \frac{1}{\kappa^{0,\epsilon}}, E_{g^\epsilon}(1)) \\
T &\leq T_\mu(R^0, \mathcal{E}_\mu^{0,\epsilon}, \frac{1}{\delta^{0,\epsilon}}, \frac{1}{\delta_\mu^{0,\epsilon}}, \frac{1}{\kappa^{0,\epsilon}}, E_{g^\epsilon}(1))
\end{aligned}
$$

Let us increase R^0 and decrease T if necessary such that these inequalities are satisfied. Note that this implies we may choose R^0 and T such that $R^0$and T are bounded as follows.

$$
\begin{aligned}
R^0 &\leq R_\mu(\tilde{C}, E_g(1)) \\
T &\geq T_\mu(\tilde{C}, E_g(1))
\end{aligned}
$$

for some smooth increasing function R_μ and some smooth decreasing function T_μ.

We apply proposition 13.2.3 to conclude that Φ has a fixed point $(U_+^\epsilon, U_-^\epsilon, f^\epsilon) \in Y$. We then apply lemma 13.3.1 to conclude that $(U_+^\epsilon, U_-^\epsilon, f^\epsilon)$ is a solution of the μ-approximate equations on the time interval $(0, T)$ satisfying, for $t \in (0, T)$,

$$
\begin{aligned}
\mathcal{E}_\mu^\epsilon(t) &< \infty \\
\inf_{\Omega_\pm} \rho^\epsilon &> 0 \\
\inf_{\Omega_\pm} \rho_\mu^\epsilon &> 0 \\
\inf_{\mathbb{R}^3} J^{\psi^\epsilon} &> 0
\end{aligned}
$$

We now use uniform continuity of the solution and its derivatives to patch together the solution (U_+, U_-, f) on the time interval $(0, T^\epsilon)$ with the solution $(U_+^\epsilon, U_-^\epsilon, f^\epsilon)$ on the time interval $(0, T)$ to obtain a solution satisfying the above inequalities on the time interval $(0, T^\epsilon + T)$. Note that we have a lower bound on T independent of ϵ as $\epsilon \to 0$ and hence we may choose ϵ such that $T^\epsilon + T > \tilde{T}$. But this contradicts the supremum definition of $\tilde{T}$. Hence we must indeed have either $\tilde{T} = 1$ or one of the following must hold as $t \uparrow \tilde{T}$.

$$
\begin{aligned}
E_\mu(t) &\to \infty \\
\inf_{\Omega_\pm} \rho &\to 0 \\
\inf_{\Omega_\pm} \rho_\mu &\to 0 \\
\inf_{\mathbb{R}^3} J^\psi &\to 0
\end{aligned}
$$

This completes the proof. $\square$

14. Proof of the Contraction Property

In this section we aim to prove the contraction property as stated in 13.2.2. Note that the proof is very similar to that of the linearised energy estimate proved in 12. We are working two orders of regularity lower than the energy estimate so we do not need to be so careful about estimating coefficients. The only other change is that we take the difference of the equations for two solutions after differentiating before proceeding as in the linearised energy estimate. We write terms of the form $G(\bar{U}^1, \bar{\psi}^1)\partial^\alpha \partial U^1 - G(\bar{U}^2, \bar{\psi}^2)\partial^\alpha \partial U^2$ as $G(\bar{U}^1, \bar{\psi}^1)\partial^\alpha \partial(U^1 - U^2) + (G(\bar{U}^1, \bar{\psi}^1) - G(\bar{U}^2, \bar{\psi}^2))\partial^\alpha \partial U^2$ and estimate the first term as in the linearised energy estimate. The second term is estimated be applying the mean value theorem to $G(\bar{U}^1, \bar{\psi}^1) - G(\bar{U}^2, \bar{\psi}^2)$ and using the fact that this is a lower order difference term, and using the higher order energy bound to bound the factor $\partial^\alpha \partial U^2$.

14.1 Notation and Definitions

Let R^0 and T satisfy the inequalities given in 13.2.1.

For $i = 1, 2$, we let (U^i_+, U^i_-, f^i) be solutions of the linearised equations with background states $(\bar{U}^i_+, \bar{U}^i_-, \bar{f}^i) \in Y$, as given in the statement of 13.2.2. We let ω^i be the corresponding solutions of the linearised curl equation (9.1.5). Note that since R^0 and T satisfy the inequalities given in 13.2.1, we have $(U^i_+, U^i_-, f^i) \in Y$ and in particular

$$\mathcal{E}^i_L(T) + \mathcal{E}_L(\omega^i)(T) \leq R^0$$

where $\mathcal{E}^i_L(t)$ denotes the energy associated with (U^i_+, U^i_-, f^i).

Notation. *In general we will use the superscript* ^{i} *where* $i = 1, 2$ *to denote quantities associated with the solution* (U^i_+, U^i_-, f^i) *or the background state* $(\bar{U}^i_+, \bar{U}^i_-, \bar{f}^i)$ *and if there is more than one superscript then we will write the superscript* i *last with a comma before the* i*. In particular, we use the following obvious notation.*

$$\begin{aligned}
(u^\psi)^i &:= (u^i)^{\psi^i} \\
n^i &:= (-\nabla' f^i, 1) \\
\hat{n}^i &:= \frac{n^i}{|n^i|}
\end{aligned}$$

Definition 14.1.1. For notational convenience we define

$$\begin{aligned}
\mathcal{E}_\Delta(t) &:= \left|\left|(U^1_+, U^1_-, f^1, u^1 \cdot n^1) - (U^2_+, U^2_-, f^2, u^2 \cdot n^2)\right|\right|^2_{Z(t)} \\
\bar{\mathcal{E}}_\Delta(t) &:= \left|\left|(\bar{U}^1_+, \bar{U}^1_-, \bar{f}^1, \bar{u}^1 \cdot \bar{n}^1) - (\bar{U}^2_+, \bar{U}^2_-, \bar{f}^2, \bar{u}^2 \cdot \bar{n}^2)\right|\right|^2_{Z(t)}
\end{aligned}$$

We also define

$$\mathcal{E}_\Delta(\omega)(t) := \sum_{\pm} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\omega^1-\omega^2\right|\right|^2_{H^2(\Omega_\pm)} + \sum_{\pm} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial_t(\omega^1-\omega^2)\right|\right|^2_{L^2(\Omega_\pm)}$$
$$+ \sum_{\pm} \int_0^t \left|\left|\partial_t(\omega^1-\omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau$$

We aim to show that

$$\mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t) \leq \frac{1}{\epsilon}\mathcal{K}_L(t)(\int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau)$$
$$+ \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))$$

for all $t \in (0,T)$ where we are free to choose $\epsilon > 0$. In this section we modify the definition of $\mathcal{K}_L(t)$ slightly to include the energies $\mathcal{E}^i_L(t)$ of both (U^1_+, U^1_-, f^1) and (U^2_+, U^2_-, f^2) and the energies $\bar{\mathcal{E}}^i_L(t)$ of their background states and the energies $\mathcal{E}_L(\omega^i)(t)$ of their associated curls.

14.2 A Mean Value Lemma

The following simple lemma will be very useful.

Lemma 14.2.1. *Let $|\alpha| \geq 0$. Suppose that for $i = 1, 2$ and $0 \leq |\beta| \leq |\alpha|$, we have $\partial^\beta v^i \in L^\infty((0,T)\times\Omega_\pm) \cap L^\infty((0,T); L^2(\Omega_\pm))$. Let G be a smooth function. Then*

$$\left|\left|\partial^\alpha(G(v^1) - G(v^2))\right|\right|_{L^2(\Omega_\pm)} \leq F(\sum_{i=1}^2 \sum_{0\leq\gamma\leq\alpha} \left|\left|\partial^\gamma v^i\right|\right|_{L^\infty(\Omega_\pm)}) \sum_{0\leq\beta\leq\alpha} \left|\left|\partial^\beta(v^1-v^2)\right|\right|_{L^2(\Omega_\pm)}$$

for some smooth increasing function F.

Proof. Note that using the Fundamental Theorem of Calculus, we have

$$G(v^1) - G(v^2) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} G(v^2 + s(v^1-v^2)\mathrm{d}s$$
$$\int_0^1 DG(v^2 + s(v^1-v^2)\mathrm{d}s(v^1-v^2)$$

where DG denotes the derivative of G with respect to its arguments. Now we apply ∂^α and use the chain rule, then square both sides and integrate over $\Omega_\pm$ to obtain the result. □

14.3 Contraction in the Front

Proposition 14.3.1. *Define*

$$\mathcal{E}_\Delta(f)(t) := \operatorname*{ess\,sup}_{\tau\in(0,t)} \sum_{0\leq j\leq 1} \left|\left|\partial^j_t(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)} + \int_0^t \left|\left|\partial^2_t(f^1-f^2)\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau$$

The front satisfies the following estimates for $t \in (0, T)$.

$$\mathcal{E}_\Delta(f)(t) \leq \frac{C}{\epsilon}\int_0^t \mathcal{E}_\Delta(\tau)\mathrm{d}\tau + \epsilon\bar{\mathcal{E}}_\Delta(t) \tag{14.3.1}$$

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} \sum_{0\leq j\leq 1} \left|\left|\partial_t^j(f^1 - f^2)\right|\right|^2_{H^{2-j}(\Gamma)} + \sum_{0\leq j\leq 1}\int_0^t \left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)} \leq C\int_0^t \bar{\mathcal{E}}_\Delta(\tau)\mathrm{d}\tau \tag{14.3.2}$$

$$\sum_{0\leq j\leq 1}\int_0^t \left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau + \sum_{0\leq j\leq 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)} \leq C\bar{\mathcal{E}}_\Delta(t) \tag{14.3.3}$$

Proof. We note that

$$\begin{aligned}
&\sum_{0\leq j\leq 1}\int_0^t \left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau + \sum_{0\leq j\leq 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)}\\
&= \sum_{0\leq j\leq 1}\int_0^t \left|\left|\partial_t^{j}(\bar{u}^1\cdot\bar{n}^1 - \bar{u}^2\cdot\bar{n}^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau + \sum_{0\leq j\leq 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^{j}(\bar{u}^1\cdot\bar{n}^1 - \bar{u}^2\cdot\bar{n}^2)\right|\right|^2_{H^{2-j}(\Gamma)}\\
&\leq C\bar{\mathcal{E}}_\Delta(t)
\end{aligned}$$

by definition of $\bar{\mathcal{E}}_\Delta(t)$. This proves the third estimate.

Now note that, for $0 \leq j \leq 1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\partial_t^j(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)} &= 2(\partial_t^{j+1}(f^1-f^2), \partial_t^j(f^1-f^2))_{H^{3.5-j}(\Gamma)}\\
&\leq \epsilon\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)} + \frac{1}{\epsilon}\left|\left|\partial_t^j(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}
\end{aligned}$$

Integrating in time we obtain

$$\begin{aligned}
\left|\left|\partial_t^j(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)} &\leq \epsilon\int_0^t\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau + \frac{1}{\epsilon}\int_0^t\left|\left|\partial_t^j(f^1-f^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau\\
&\leq C\epsilon\bar{\mathcal{E}}_\Delta(t) + \frac{1}{\epsilon}\int_0^t \mathcal{E}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the third estimate for the second term. Together with a direct integrating in time of the third estimate, this proves the first estimate.

To prove the second estimate we note that for $0 \leq j \leq 1$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)} + \int_0^t\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)}\mathrm{d}\tau\\
&\leq C\int_0^t\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)}\mathrm{d}\tau + \int_0^t\left|\left|\partial_t^{j+1}(f^1-f^2)\right|\right|^2_{H^{2-j}(\Gamma)}\mathrm{d}\tau\\
&\leq C\int_0^t \bar{\mathcal{E}}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the third estimate. □

14.4 Contraction in the Entropy, Pressure and Curl

Lemma 14.4.1. *Let $0 \leq |\alpha| \leq 1$. We have that $\overline{u^\psi}$ satisfies the following estimate.*

$$\left|\left|\partial^\alpha((\overline{u^\psi})^1 - (\overline{u^\psi})^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t)) \tag{14.4.1}$$

Proof. Let us note that

$$\overline{u^\psi} = G(\partial_t\psi, \nabla\psi, \nabla'\bar{\psi}, \bar{u})$$

for some smooth function G (at least in the region where $J^\psi > 0$). We apply lemma 14.2.1 to obtain

$$\left|\left|\partial^\alpha(\overline{u^\psi})^1 - (\overline{u^\psi})^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \mathcal{K}_L(t) \sum_{0\leq\gamma\leq\alpha} \left|\left|\partial^\gamma(v^1 - v^2)\right|\right|^2_{H^1(\Omega_\pm)}$$

where

$$v = (\partial_t\psi, \nabla\psi, \nabla\bar{\psi}, \bar{u})$$

Hence

$$\begin{aligned}\left|\left|\partial^\alpha(\overline{u^\psi})^1 - (\overline{u^\psi})^2)\right|\right|^2_{H^1(\Omega_\pm)} &\leq \mathcal{K}_L(t) \sum_{0\leq\gamma\leq\alpha} (\left|\left|\partial^\gamma(\bar{u}^1 - \bar{u}^2)\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|\nabla'\partial^\gamma(\bar{\psi}^1 - \bar{\psi}^2)\right|\right|^2_{H^1(\Omega_\pm)} \\ &\quad + \left|\left|\partial^\gamma\partial_t(\psi^1 - \psi^2)\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|\partial^\gamma\nabla(\psi^1 - \psi^2)\right|\right|^2_{H^1(\Omega_\pm)}) \\ &\leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))\end{aligned}$$

where we have used the estimate (14.3.3) for the front. □

Proposition 14.4.2. *Define*

$$\mathcal{E}_\Delta(s)(t) := \sum_\pm \sum_{0\leq|\alpha|\leq 1} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\alpha(s^1 - s^2)\right|\right|^2_{H^1(\Omega_\pm)} + \sum_\pm \int_0^t \left|\left|\partial_t^2(s^1 - s^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau$$

The entropy s satisfies the following estimate.

$$\mathcal{E}_\Delta(s)(t) \leq \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau)\mathrm{d}\tau \tag{14.4.2}$$

for $t \in (0, T)$.

Proof. Let $0 \leq |\alpha| \leq 1$ and apply ∂^α to the linearised entropy equation (9.1.3) to obtain

$$\partial_t\partial^\alpha s + \overline{u^\psi} \cdot \nabla\partial^\alpha s = R^\alpha_L(s)$$

where

$$R^\alpha_L(s) = -\sum_{\gamma+\delta=\alpha,\delta\neq\alpha} \partial^\gamma\overline{u^\psi} \cdot \partial^\delta\nabla s$$

We claim that

$$\left|\left|R^\alpha_L(s^1) - R^\alpha_L(s^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$

Indeed, we have

$$\begin{aligned}&\left|\left|R^\alpha_L(s^1) - R^\alpha_L(s^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \sum_{\gamma+\delta=\alpha,\delta\neq\alpha} \left|\left|\partial^\gamma(\overline{u^\psi})^1 \cdot \partial^\delta\nabla(s^1 - s^2)\right|\right|^2_{H^1(\Omega_\pm)} \\ &+ \sum_{\gamma+\delta=\alpha,\delta\neq\alpha} \left|\left|\partial^\gamma((\overline{u^\psi})^1 - (\overline{u^\psi})^2) \cdot \partial^\delta\nabla s^2\right|\right|^2_{H^1(\Omega_\pm)} \\ &\leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))\end{aligned}$$

where we have used (14.4.1).

Taking the difference of the differentiated equations for s^1 and s^2, we obtain

$$\partial_t\partial^\alpha(s^1-s^2)+(\overline{u^\psi})^1\cdot\nabla\partial^\alpha(s^1-s^2)=-((\overline{u^\psi})^1-(\overline{u^\psi})^2)\cdot\nabla\partial^\alpha s^2+R_L^\alpha(s^1)-R_L^\alpha(s^2)$$

Applying the estimate (12.5.3) with $w=\partial^\alpha(s^1-s^2)$, $b=(\overline{u^\psi})^1$ and h equal to the right hand side of the above equation, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial^\alpha(s^1-s^2)\right|\right|^2_{H^1(\Omega_\pm)}\\
&\le(1+\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|(\overline{u^\psi})^1\right|\right|^2_{W^{1,\infty}(\Omega_\pm)})\int_0^t\left|\left|\partial^\alpha(s^1-s^2)\right|\right|^2_{H^1(\mathbb{R}^d)}\,\mathrm{d}\tau\\
&+\int_0^t\left|\left|R_L^\alpha(s^1)-R_L^\alpha(s^2)\right|\right|^2_{H^1(\Omega_\pm)}\,\mathrm{d}\tau+\int_0^t\left|\left|((\overline{u^\psi})^1-(\overline{u^\psi})^2)\cdot\nabla\partial^\alpha s^2\right|\right|^2_{H^1(\Omega_\pm)}\,\mathrm{d}\tau\\
&\le\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the above estimate for $R_L^\alpha(s^1)-R_L^\alpha(s^2)$ and the estimate (14.4.1).

We now apply the estimate (12.5.2) to the differentiated equation to obtain

$$\begin{aligned}
&\int_0^t\left|\left|\partial_t\partial^\alpha(s^1-s^2)\right|\right|^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&\le C\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|(\overline{u^\psi})^1\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}\int_0^t\left|\left|\nabla\partial^\alpha(s^1-s^2)\right|\right|^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&+C\int_0^t\left|\left|R_L^\alpha(s^1)-R_L^\alpha(s^2)\right|\right|^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau+C\int_0^t\left|\left|((\overline{u^\psi})^1-(\overline{u^\psi})^2)\cdot\nabla\partial^\alpha s^2\right|\right|^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau\\
&\le\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the above estimate for $R_L^\alpha(s^1)-R_L^\alpha(s^2)$.

This completes the proof. □

Proposition 14.4.3. *Define*

$$\mathcal{E}_\Delta(p)(t):=\sum_\pm\sum_{0\le|\alpha|\le1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial^\alpha(p^1-p^2)\right|\right|^2_{H^1(\Omega_\pm)}+\sum_\pm\int_0^t\left|\left|\partial_t^2(p^1-p^2)\right|\right|^2_{L^2(\Omega_\pm)}\,\mathrm{d}\tau$$

The pressure p satisfies the following estimate.

$$\mathcal{E}_\Delta(p)(t)\le\frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t))\tag{14.4.3}$$

for $t\in(0,T)$.

Proof. Let $0\le|\alpha|\le1$ and apply ∂^α to the linearised pressure equation (9.1.4) to obtain

$$\partial_t\partial^\alpha p+\overline{u^\psi}\cdot\nabla\partial^\alpha p=R_L^\alpha(p)+\partial^\alpha G_p$$

where

$$R_L^\alpha(p)=-\sum_{\gamma+\delta=\alpha,\delta\neq\alpha}\partial^\gamma\overline{u^\psi}\cdot\partial^\delta\nabla p$$

and

$$G_p := -\bar{\rho}\bar{c}^2\nabla^\psi \cdot \bar{u}$$

Note that we have

$$\left|\left|R_L^\alpha(p^1) - R_L^\alpha(p^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$

exactly as in the estimate of the entropy. We claim that

$$\int_0^t \left|\left|\partial^\alpha(G_p^1 - G_p^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$

and

$$\int_0^t \left|\left|\partial^\alpha(G_p^1 - G_p^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t) \int_0^t (\bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau))\mathrm{d}\tau$$

We note that $G_p = G(v)$ where $v = (\bar{p}, \bar{s}, \nabla\psi, \nabla\bar{u})$ and G is a smooth function. From the definition of $\mathcal{E}_\Delta(t)$,

$$\left|\left|\partial^\gamma(v^1 - v^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq C(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t)) \text{ for } 0 \leq |\gamma| \leq 1$$

$$\int_0^t \left|\left|\partial^\gamma(v^1 - v^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq C(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t)) \text{ for } |\gamma| = 1$$

Applying lemma 14.2.1 proves the claim.

Taking the difference of the differentiated equations for p^1 and p^2, we obtain

$$\begin{aligned}\partial_t\partial^\alpha(p^1 - p^2) + (\overline{u^\psi})^1 \cdot \nabla\partial^\alpha(p^1 - p^2) = -((\overline{u^\psi})^1 - (\overline{u^\psi})^2) \cdot \nabla\partial^\alpha p^2 + R_L^\alpha(p^1) - R_L^\alpha(p^2)\\ + \partial^\alpha(G_p^1 - G_p^2)\end{aligned}$$

Applying the estimate (12.5.3) with $w = \partial^\alpha(p^1 - p^2)$, $b = (\overline{u^\psi})^1$ and h equal to the right hand side of the above equation, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\alpha(p^1 - p^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq (\frac{1}{\epsilon} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|(\overline{u^\psi})^1\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}) \int_0^t \left|\left|\partial^\alpha(p^1 - p^2)\right|\right|^2_{H^1(\mathbb{R}^d)} \mathrm{d}\tau\\
&+ \int_0^t \left|\left|R_L^\alpha(p^1) - R_L^\alpha(p^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \int_0^t \left|\left|((\overline{u^\psi})^1 - (\overline{u^\psi})^2) \cdot \nabla\partial^\alpha p^2\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau\\
&+ \epsilon \int_0^t \left|\left|\partial^\alpha(G_p^1 - G_p^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau\\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))
\end{aligned}$$

where we have used the above estimates for $R_L^\alpha(p^1) - R_L^\alpha(p^2)$ and $\partial^\alpha(G_p^1 - G_p^2)$, and the estimate (14.4.1).

We now apply the estimate (12.5.2) to the differentiated equation to obtain

$$\begin{aligned}
&\int_0^t \left|\left|\partial_t\partial^\alpha(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq C \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|(\overline{u^\psi})^1\right|\right|^2_{W^{1,\infty}(\Omega_\pm)} \int_0^t \left|\left|\nabla\partial^\alpha(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau\\
&+ C \int_0^t \left|\left|R_L^\alpha(p^1) - R_L^\alpha(p^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + C \int_0^t \left|\left|((\overline{u^\psi})^1 - (\overline{u^\psi})^2) \cdot \nabla\partial^\alpha p^2\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau\\
&+ \int_0^t \left|\left|\partial^\alpha(G_p^1 - G_p^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau\\
&\leq \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the above estimates for $R^\alpha_L(p^1) - R^\alpha_L(p^2)$ and $\partial^\alpha(G^1_p - G^2_p)$.

This completes the proof. $\square$

Proposition 14.4.4. *The curl ω satisfies the following estimates.*

$$\mathcal{E}_\Delta(\omega)(t) \leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t)) \tag{14.4.4}$$

$$\int_0^t \left|\left|\partial_t(\omega^1 - \omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau \tag{14.4.5}$$

$$\int_0^t \sum_\pm \left|\left|\partial_t^2(\omega^1 - \omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t)) \tag{14.4.6}$$

for $t \in (0, T)$.

Proof. Let $0 \leq |\alpha| \leq 1$ and apply ∂^α to the linearised curl equation (9.1.5) to obtain

$$(\partial_t + \overline{u^\psi}\cdot\nabla)\partial^\alpha\omega = \partial^\alpha G_\omega + R^\alpha_L(\omega)$$

where

$$R^\alpha_L(\omega) = -\sum_{\gamma+\delta=\alpha,\delta\neq\alpha} \partial^\gamma\overline{u^\psi}\cdot\partial^\delta\nabla\omega$$

and

$$\begin{aligned} G_\omega &= G(\rho_\mu, \nabla\rho_\mu, \nabla\psi, \overline{u^\psi}, \nabla\bar{u}, \partial_t\bar{u}, \nabla g_1) \\ &:= \frac{1}{\rho_\mu}(-\nabla^\psi\rho_\mu \times (\partial_t + \overline{u^\psi}\cdot\nabla)\bar{u} - \rho_\mu(\epsilon_{ijk}\partial^\psi_j\bar{u}_l\partial^\psi_l\bar{u}_k) + \nabla^\psi \times g_1) \end{aligned}$$

so that G is a smooth function for $\rho_\mu > 0$.

Taking the difference of the differentiated equations for ω^1 and ω^2, we obtain

$$\begin{aligned} \partial_t\partial^\alpha(\omega^1 - \omega^2) + (\overline{u^\psi})^1\cdot\nabla\partial^\alpha(\omega^1 - \omega^2) &= -((\overline{u^\psi})^1 - (\overline{u^\psi})^2)\cdot\nabla\partial^\alpha\omega^2 + R^\alpha_L(\omega^1) - R^\alpha_L(\omega^2) \\ &\quad + \partial^\alpha(G^1_\omega - G^2_\omega) \end{aligned}$$

We claim that

$$\int_0^t \left|\left|R^\alpha_L(\omega^1) - R^\alpha_L(\omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))$$

Indeed, we have

$$\begin{aligned} &\left|\left|R^\alpha_L(\omega^1) - R^\alpha_L(\omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \sum_{\gamma+\delta=\alpha,\delta\neq\alpha}\left|\left|\partial^\gamma(\overline{u^\psi})^1\cdot\partial^\delta\nabla(\omega^1 - \omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \\ &+ \sum_{\gamma+\delta=\alpha,\delta\neq\alpha}\left|\left|\partial^\gamma((\overline{u^\psi})^1 - (\overline{u^\psi})^2)\cdot\partial^\delta\nabla\omega^2\right|\right|^2_{H^1(\Omega_\pm)} \\ &\leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t)) + \mathcal{K}_L(t)\sum_{\delta\leq\alpha,\delta\neq\alpha}\left|\left|\partial^\delta\nabla(\omega^1 - \omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \end{aligned}$$

where we have used (14.4.1). Using the definition of $\mathcal{E}_\Delta(\omega)(t)$ applied to the last term proves the claim.

Now we claim that

$$\int_0^t \left|\left|\partial^\alpha(G_\omega^1 - G_\omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$

and if $\alpha_0 = 0$, then

$$\int_0^t \left|\left|\partial^\alpha(G_\omega^1 - G_\omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$
$$\left|\left|\partial^\alpha(G_\omega^1 - G_\omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$

Indeed, we write $G_\omega = G(v)$ where $v = (\rho_\mu, \nabla\rho_\mu, \nabla\psi, \overline{u^\psi}, \nabla\bar{u}, \partial_t\bar{u}, \nabla g_1)$ and G is a smooth function. From the definition of $\mathcal{E}_\Delta(t)$,

$$\left|\left|v^1 - v^2\right|\right|^2_{H^1(\Omega_\pm)} \leq C(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$
$$\int_0^t \left|\left|\partial^\gamma(v^1 - v^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq C(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t)) \text{ for } |\gamma| = 1$$
$$\int_0^t \left|\left|v^1 - v^2\right|\right|^2_{H^2(\Omega_\pm)} \mathrm{d}\tau \leq C(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))$$

Combining these estimates and using 14.2.1 proves the claim.

Let $\alpha_0 = 0$. Applying the estimate (12.5.3) with $w = \partial^\alpha(\omega^1 - \omega^2)$, $b = (\overline{u^\psi})^1$ and h equal to the right hand side of the above equation, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\partial^\alpha(\omega^1 - \omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \leq \left(\frac{1}{\epsilon} + \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|(\overline{u^\psi})^1\right|\right|^2_{W^{1,\infty}(\Omega_\pm)}\right) \int_0^t \left|\left|\partial^\alpha(\omega^1 - \omega^2)\right|\right|^2_{H^1(\mathbb{R}^d)} \mathrm{d}\tau \\
&+ \int_0^t \left|\left|R_L^\alpha(\omega^1) - R_L^\alpha(\omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \int_0^t \left|\left|((\overline{u^\psi})^1 - (\overline{u^\psi})^2)\cdot\nabla\partial^\alpha\omega^2\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\
&+ \epsilon \int_0^t \left|\left|\partial^\alpha(G_\omega^1 - G_\omega^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t))
\end{aligned}$$

where we have used the above estimates for $R_L^\alpha(\omega^1) - R_L^\alpha(\omega^2)$ and $\partial^\alpha(G_\omega^1 - G_\omega^2)$ with $\alpha_0 = 0$, and the estimate (14.4.1).

We now apply the estimate (12.5.2) to the differentiated equation to obtain

$$\begin{aligned}
&\int_0^t \left|\left|\partial_t\partial^\alpha(\omega^1 - \omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq C \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|(\overline{u^\psi})^1\right|\right|^2_{W^{1,\infty}(\Omega_\pm)} \int_0^t \left|\left|\nabla\partial^\alpha(\omega^1 - \omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ C\int_0^t \left|\left|R_L^\alpha(\omega^1) - R_L^\alpha(\omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + C\int_0^t \left|\left|((\overline{u^\psi})^1 - (\overline{u^\psi})^2)\cdot\nabla\partial^\alpha\omega^2\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ C\int_0^t \left|\left|\partial^\alpha(G_\omega^1 - G_\omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq \mathcal{K}_L(t) \int_0^t \left|\left|\nabla\partial^\alpha(\omega^1 - \omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau + C\int_0^t \left|\left|\partial^\alpha(G_\omega^1 - G_\omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&+ \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the above estimate for $R_L^\alpha(\omega^1) - R_L^\alpha(\omega^2)$. In the case $\alpha_0 = 0$, we obtain

$$\int_0^t \left|\left|\partial_t\partial^\alpha(\omega^1 - \omega^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau$$

where we have used the above estimate for $\partial^\alpha G_\omega$. This completes the estimate of $\mathcal{E}_\Delta(\omega)(t)$ and also proves the second estimate. Without the restriction on α_0, we obtain

$$\begin{aligned}&\int_0^t \left|\left|\partial_t\partial^\alpha(\omega^1-\omega^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \le \mathcal{K}_L(t)\int_0^t \left|\left|\nabla\partial^\alpha(\omega^1-\omega^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\ &+\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))\\ &\le \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))\end{aligned}$$

where we have used the above estimate for $\partial^\alpha G_\omega$ and the estimate just obtained for $\nabla\partial^\alpha\omega$.

This completes the proof. □

14.5 Contraction in the Velocity

Definition 14.5.1. We define

$$\begin{aligned}&\mathcal{E}_\Delta(u)(t) := \sum_\pm\sum_{0\le j\le 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j(u^1-u^2)\right|\right|^2_{H^{2-j}(\Omega_\pm)}\\ &+\sum_\pm\sum_{0\le j\le 1}\int_0^t\left|\left|\partial_t^j(u^1-u^2)\right|\right|^2_{H^{3-j}(\Omega_\pm)}\mathrm{d}\tau+\sum_\pm\int_0^t\left|\left|\partial_t^2(u^1-u^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\ &+\sum_{0\le j\le 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)\right|\right|^2_{H^{2-j}(\mathbb{R}^2)}+\sum_{0\le j\le 1}\int_0^t\left|\left|\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)\right|\right|^2_{H^{3-j}(\mathbb{R}^2)}\mathrm{d}\tau\end{aligned}$$

for $t\in(0,T)$.

Proposition 14.5.2. *The velocity u satisfies the following estimate.*

$$\mathcal{E}_\Delta(u)(t)\le\frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

for $t\in(0,T)$, where we are free to choose $\epsilon>0$.

Proof. See the rest of this section for the proof □

We use the weak form of the equations. As in 11.3, we set

$$\widehat{u}^i=(N^i)^{-1}u^i$$

where the matrix N^i is defined in 11.3.1 with $\nabla' f$ replaced by $\nabla' f^i$. We define

$$\begin{aligned}&\mathcal{E}_\Delta(\widehat{u})(t) := \sum_\pm\sum_{0\le j\le 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j(\widehat{u}^1-\widehat{u}^2)\right|\right|^2_{H^{2-j}(\Omega_\pm)}\\ &+\sum_\pm\sum_{0\le j\le 1}\int_0^t\left|\left|\partial_t^j(\widehat{u}^1-\widehat{u}^2)\right|\right|^2_{H^{3-j}(\Omega_\pm)}\mathrm{d}\tau+\sum_\pm\int_0^t\left|\left|\partial_t^2(\widehat{u}^1-\widehat{u}^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\ &+\sum_{0\le j\le 1}\operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial_t^j(\widehat{u}^1_3-\widehat{u}^2_3)\right|\right|^2_{H^{2-j}(\mathbb{R}^2)}+\sum_{0\le j\le 1}\int_0^t\left|\left|\partial_t^j(\widehat{u}^1_3-\widehat{u}^2_3)\right|\right|^2_{H^{3-j}(\mathbb{R}^2)}\mathrm{d}\tau\end{aligned}$$

Lemma 14.5.3. *We claim that*

$$\mathcal{E}_\Delta(u)(t)\le\mathcal{K}_L(t)(\mathcal{E}_\Delta(\widehat{u})(t)+\mathcal{E}_\Delta(f)(t)) \tag{14.5.1}$$

and conversely

$$\mathcal{E}_\Delta(\widehat{u})(t)\le\mathcal{K}_L(t)(\mathcal{E}_\Delta(u)(t)+\mathcal{E}_\Delta(f)(t)) \tag{14.5.2}$$

Proof. Recall that $\widehat{u}_3 = (N^{-1}u)_3 = u \cdot n$, so the estimate of the boundary terms is immediate.

Now we note that for $1 \leq |\alpha| \leq k$, where $k = 2$ or $k = 3$, and $\alpha_0 \leq 1$, we have

$$||\partial^\alpha(u^1 - u^2)||^2_{L^2(\Omega_\pm)} = ||\partial^\alpha(G(v^1) - G(v^2))||^2_{L^2(\Omega_\pm)}$$

where G is a smooth function and $v = (u, N)$. Hence, applying 14.2.1, we have

$$||\partial^\alpha(u^1 - u^2)||^2_{L^2(\Omega_\pm)} \leq \mathcal{K}_L(t)(\sum_{0\leq\gamma\leq\alpha}(||\partial^\gamma(u^1 - u^2)||^2_{L^2(\Omega_\pm)} + ||\partial^\gamma(f^1 - f^2)||^2_{H^{0.5}(\Omega_\pm)}))$$

where we have used the lifting property for N. Thus we obtain in the case $k = 2$,

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha(u^1 - u^2)||^2_{L^2(\Omega_\pm)} \leq \mathcal{K}_L(t)(\mathcal{E}_\Delta(\widehat{u})(t) + \mathcal{E}_\Delta(f)(t))$$

In the case $k = 3$ we obtain

$$\int_0^t ||\partial^\alpha u||^2_{L^2(\Omega_\pm)}\, \mathrm{d}\tau \leq \mathcal{K}_L(t)(\mathcal{E}_\Delta(\widehat{u})(t) + \mathcal{E}_\Delta(f)(t))$$

where we have again used the lifting property for N.

Now we consider the case $\partial^\alpha = \partial_t^2$. Note that if $|\beta| \leq 2$ then

$$\int_0^t \left|\left|\partial^\beta(N^1 - N^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq C\left|\left|\partial^\beta\nabla'(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} \leq C\mathcal{E}_\Delta(f)(t)$$

Combining this with the estimates above we see that

$$\int_0^t ||\partial^\alpha(u^1 - u^2)||^2_{L^2(\Omega_\pm)}\, \mathrm{d}\tau \leq \mathcal{K}_L(t)(\mathcal{E}_\Delta(\widehat{u})(t) + \mathcal{E}_\Delta(f)(t))$$

This completes the proof of the first inequality. The second inequality follows simply by replacing N by N^{-1} and reversing u and $\widehat{u}$ in the above estimates. □

Let $0 \leq |\alpha| \leq 1$ with $\alpha_3 = 0$ be a multi-index. Recall from the proof of the linearised energy estimate that $\partial^\alpha\, \widehat{u}^i$ for $i = 1, 2$ satisfy the weak equation (12.6.4).

Notation. *We will write $A^{j,1}$ and $A^{j,2}$ for $0 \leq j \leq 2$ to denote the coefficients A^j defined in 11.6.1 evaluated at the states $p^1_\pm, s^1_\pm, \bar{u}^1_\pm$ etc and $p^2_\pm, s^2_\pm, \bar{u}^2_\pm$ etc respectively. We define $G^{j,\alpha,1}$ and $G^{j,\alpha,2}$ for $1 \leq j \leq 4$ in a similar way, where $G^{j,\alpha}$ are defined in* (12.6.5)-(12.6.8).

We will also write $(\cdot,\cdot)_{B^{0,1}(t)}$ and $(\cdot,\cdot)_{B^{0,2}(t)}$ to denote the inner product $(\cdot,\cdot)_{B^0(t)}$ as defined in 11 associated with the states $p^1_\pm, s^1_\pm, \bar{u}^1_\pm$ etc and $p^2_\pm, s^2_\pm, \bar{u}^2_\pm$ etc respectively, which we recall can be written as

$$(v, w)_{B^{0,1}(t)} = \sum_\pm (A^{0,1}v, w)_{L^2(\Omega_\pm)}$$
$$(v, w)_{B^{0,2}(t)} = \sum_\pm (A^{0,2}v, w)_{L^2(\Omega_\pm)}$$

Similarly we define

$$(v, w)_{B^{1,1}(t)} = \sum_\pm (A^{1,1}(v, \nabla v), (w, \nabla w))_{L^2(\Omega_\pm)} + (v_3, w_3)_{H^1(\Gamma)}$$
$$(v, w)_{B^{1,2}(t)} = \sum_\pm (A^{1,2}(v, \nabla v), (w, \nabla w))_{L^2(\Omega_\pm)} + (v_3, w_3)_{H^1(\Gamma)}$$

Now let us subtract the equation (12.6.4) for $\partial^\alpha \widehat{u}^2$ from that for $\partial^\alpha \widehat{u}^1$ where $0 \le |\alpha| \le 1$ to obtain

$$
\begin{aligned}
&(\partial_t \partial^\alpha(\widehat{u}^1 - \widehat{u}^2), w)_{B^{0,1}(t)} + \mu\{\partial^\alpha(\widehat{u}^1 - \widehat{u}^2), w\}_{B^{1,1}(t)} \\
&+ \sum_{\pm} (A^{2,1}(\partial^\alpha(\widehat{u}^1 - \widehat{u}^2), \nabla\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)), w)_{L^2(\Omega_\pm)} \\
&= \sum_{\pm} (G^{1,\alpha}_\Delta, w)_{L^2(\Omega_\pm)} + \sum_{\pm} (G^{2,\alpha}_\Delta, \nabla w)_{L^2(\Omega_\pm)} + (G^{3,\alpha}_\Delta, w_3)_{L^2(\Gamma)} + (G^{4,\alpha}_\Delta, \nabla' w_3)_{L^2(\Gamma)}
\end{aligned}
$$

for all $w \in B^1$ and almost every $t \in (0, T)$, where

$$
\begin{aligned}
G^{1,\alpha}_\Delta &:= G^{1,\alpha,1} - G^{1,\alpha,2} - (A^{0,1} - A^{0,2})\partial_t\partial^\alpha \widehat{u}^2 - \mu(A^{1,1} - A^{1,2})(\partial^\alpha \widehat{u}^2, \nabla\partial^\alpha \widehat{u}^2) \\
&\quad - (A^{2,1} - A^{2,2})(\partial^\alpha \widehat{u}^2, \nabla\partial^\alpha \widehat{u}^2) \\
G^{2,\alpha}_\Delta &:= G^{2,\alpha,1} - G^{2,\alpha,2} - \mu(A^{1,1} - A^{1,2})(\partial^\alpha \widehat{u}^2, \nabla\partial^\alpha \widehat{u}^2) \\
G^{3,\alpha}_\Delta &:= G^{3,\alpha,1} - G^{3,\alpha,2} = 0 \\
G^{4,\alpha}_\Delta &:= G^{4,\alpha,1} - G^{4,\alpha,2}
\end{aligned}
$$

Lemma 14.5.4. *We claim that*

$$
\begin{aligned}
\int_0^t \left\|G^{1,\alpha}_\Delta\right\|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\le \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau \\
\int_0^t \left\|\partial_t G^{2,\alpha}_\Delta\right\|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau &\le \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t)) \\
\int_0^t \left\|G^{2,\alpha}_\Delta\right\|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau &\le \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau \\
\sum_{k=0}^1 \int_0^t \left\|\partial^k G^{3,\alpha}_\Delta\right\|^2_{L^2(\Gamma)} \mathrm{d}\tau &= 0 \\
\sum_{k=0}^1 \int_0^t \left\|\partial^k G^{4,\alpha}_\Delta\right\|^2_{L^2(\Gamma)} \mathrm{d}\tau &\le \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}
$$

Furthermore, for $2 \le i \le 4$,

$$
\left\|G^{i,\alpha}_\Delta\right\|^2_{L^2(\Omega_\pm)} \le \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
$$

Proof. Firstly, we recall that $G^{i,\alpha}$ have the form

$$
\begin{aligned}
G^{1,\alpha} &= G(\{\partial^\beta(N, \nabla N, \nabla\psi, p, s, \omega, g_1, \bar{u}, \nabla\bar{\psi}, \partial_t\psi, \partial_t N) : 0 \le \beta \le \alpha\}, u, \partial_t u, \nabla u) \\
G^2 &= G(\{\partial^\beta(N, \nabla N, \nabla\psi, p, s, \omega) : 0 \le \beta \le \alpha\}, u, \nabla u) \\
G^{3,\alpha} &= \partial^\alpha g_2 \\
G^{4,\alpha} &= G(\{\partial^\beta \nabla' f : 0 \le \beta \le \alpha\})
\end{aligned}
$$

for G a generic smooth function which may be scalar, vector, matrix valued etc and A^i have the form

$$
\begin{aligned}
A^0 &= A(p, s, N) \\
A^1 &= A(p, s, \nabla\psi, N, \nabla N) \\
A^2 &= A(p, s, \bar{u}, \nabla\psi, \nabla\bar{\psi}, \partial_t\psi, N, \nabla N, \partial_t N)
\end{aligned}
$$

We start by estimating $\left|\left|G^{2,\alpha,1} - G^{2,\alpha,2}\right|\right|_{L^2(\Omega_\pm)}$ and $\left|\left|\partial(G^{2,\alpha,1} - G^{2,\alpha,2})\right|\right|_{L^2(\Omega_\pm)}$. Applying the lemma 14.2.1, we see that

$$\left|\left|\partial^\beta(G^{2,\alpha,1} - G^{2,\alpha,2})\right|\right|^2_{L^2(\Omega_\pm)} \leq \mathcal{K}_L(t) \sum_{0 \leq \partial^\gamma \leq \partial^\beta} \left|\left|\partial^\gamma(v^1 - v^2)\right|\right|^2_{L^2(\Omega_\pm)}$$

where

$$v = (\{\partial^\beta(N, \nabla N, \nabla\psi, p, s, \omega) : 0 \leq \beta \leq \alpha\}, u, \nabla u)$$

Note that

$$\begin{aligned}\int_0^t \left|\left|v^1 - v^2\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq C \sum_{0\leq l\leq 1} \int_0^t & \left|\left|\partial^l((p^1, s^1, \nabla\psi^1) - (p^2, s^2, \nabla\psi^2))\right|\right|^2_{H^1(\Omega_\pm)} \\ &+ \left|\left|\partial^l(f^1 - f^2)\right|\right|^2_{H^{2.5}(\Omega_\pm)} + \left|\left|\partial^l(\omega^1 - \omega^2)\right|\right|^2_{H^1(\Omega_\pm)} + \left|\left|u^1 - u^2\right|\right|^2_{H^2(\Omega_\pm)} \mathrm{d}\tau \\ &\leq \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau\end{aligned}$$

where we have used the curl estimate (14.4.5) for the term involving ω in the case of 1 time derivative, and we have also used the lifting property for N. We also have

$$\begin{aligned}&\int_0^t \left|\left|\partial_t(v^1 - v^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \sum_{0\leq l\leq 2} \int_0^t \left|\left|\partial^l((p^1, s^1, \nabla\psi^1) - (p^2, s^2, \nabla\psi^2))\right|\right|^2_{L^2(\Omega_\pm)} \\ &+ \left|\left|\partial^l(f^1 - f^2)\right|\right|^2_{H^{1.5}(\Omega_\pm)} + \left|\left|\partial^l(\omega^1 - \omega^2)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial_t(u^1 - u^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \\ &\leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))\end{aligned}$$

where we have used the estimate (14.3.3) for the front in the first and second terms and the estimate (14.4.6) for the curl in the fourth term in the case of the highest order time derivative. Combining these estimates, we see that

$$\int_0^t \left|\left|\partial_t(G^{2,\alpha,1} - G^{2,\alpha,2})\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))$$

and

$$\int_0^t \left|\left|(G^{2,\alpha,1} - G^{2,\alpha,2})\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau \leq \mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau$$

Now let us estimate $\left|\left|G^{4,\alpha,1} - G^{4,\alpha,2}\right|\right|_{L^2(\Gamma)}$ and $\left|\left|\partial G^{4,\alpha,1} - G^{4,\alpha,2}\right|\right|_{L^2(\Gamma)}$. Applying the lemma 14.2.1, we see that

$$\left|\left|\partial^\beta(G^{4,\alpha,1} - G^{4,\alpha,2})\right|\right|^2_{L^2(\Gamma)} \leq \mathcal{K}_L(t) \sum_{0 \leq \partial^\gamma \leq \partial^\beta} \left|\left|\partial^\gamma(v^1 - v^2)\right|\right|^2_{L^2(\Omega_\pm)}$$

where

$$v = (\{\partial^\beta(\nabla' f) : 0 \leq \beta \leq \alpha\})$$

Note that if $0 \leq l \leq 1$ then

$$\int_0^t \left|\left|\partial^l(v^1 - v^2)\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \leq C \int_0^t \left|\left|\partial^l \nabla'(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} \mathrm{d}\tau \leq C \int_0^t \bar{\mathcal{E}}_\Delta(\tau)\mathrm{d}\tau$$

where we have used the estimate (14.3.2) for the front in the case of the highest order time derivative. Thus we see that

$$\int_0^t \left|\left|G^{4,\alpha,1}-G^{4,\alpha,2}\right|\right|^2_{L^2(\Gamma)}+\left|\left|\partial(G^{4,\alpha,1}-G^{4,\alpha,2})\right|\right|^2_{L^2(\Gamma)}\,\mathrm{d}\tau\le\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)\mathrm{d}\tau$$

Clearly we have

$$\int_0^t \left|\left|G^{3,\alpha,1}-G^{3,\alpha,2}\right|\right|^2_{L^2(\Gamma)}+\left|\left|\partial(G^{3,\alpha,1}-G^{3,\alpha,2})\right|\right|^2_{L^2(\Gamma)}\,\mathrm{d}\tau=0$$

Now we estimate $\left|\left|G^{1,\alpha,1}-G^{1,\alpha,2}\right|\right|_{L^2(\Omega_\pm)}$. Applying the lemma 14.2.1, we see that

$$\left|\left|G^{1,\alpha,1}-G^{1,\alpha,2}\right|\right|^2_{L^2(\Omega_\pm)}\le\mathcal{K}_L(t)\left|\left|v^1-v^2\right|\right|^2_{L^2(\Omega_\pm)}$$

where

$$v=(\{\partial^\beta(N,\nabla N,\nabla\psi,p,s,\omega,g_1,\bar{u},\nabla\bar{\psi},\partial_t\psi,\partial_tN):0\le\beta\le\alpha\},u,\partial_tu,\nabla u)$$

Note that

$$\begin{aligned}&\int_0^t\left|\left|v^1-v^2\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\le C\sum_{0\le l\le 1}\int_0^t\left|\left|\partial^l((p^1,s^1,u^1,\nabla\psi^1)-(p^2,s^2,u^2,\nabla\psi^2))\right|\right|^2_{L^2(\Omega_\pm)}\\&+\left|\left|\partial^l(f^1-f^2)\right|\right|^2_{H^2(\Omega_\pm)}+\left|\left|\partial^l\partial_t(f^1-f^2)\right|\right|^2_{H^1(\Omega_\pm)}+\left|\left|\partial^l(\omega^1-\omega^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\&\le\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau\end{aligned}$$

where we have used the estimate (14.3.2) for the front in the third term. Thus we see that

$$\int_0^t\left|\left|(G^{1,\alpha,1}-G^{1,\alpha,2})\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\le\mathcal{K}_L(t)\int_0^t\bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau$$

Now we estimate the terms involving $A^{i,1}-A^{i,2}$, which are in fact lower order. Firstly, we estimate $\partial^l(A^{1,1}-A^{1,2})$ where $0\le l\le 1$. Applying the lemma 14.2.1, we see that

$$\left|\left|\partial^l(A^{1,1}-A^{1,2})\right|\right|^2_{L^2(\Omega_\pm)}\le\mathcal{K}_L(t)\sum_{0\le l\le 1}\left|\left|\partial^l(v^1-v^2)\right|\right|^2_{L^2(\Omega_\pm)}$$

where $v=(p,s,\nabla\psi,N,\nabla N)$. We then note that

$$\begin{aligned}&\int_0^t\left|\left|\partial^l(v^1-v^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\&\le C\int_0^t\left|\left|\partial^l((p^1,s^1)-(p^2,s^2))\right|\right|^2_{L^2(\Omega_\pm)}+\left|\left|\partial^l\nabla'(f^1-f^2)\right|\right|^2_{H^{0.5}(\Omega_\pm)}\mathrm{d}\tau\\&\le C\int_0^t\mathcal{E}_\Delta(\tau)\mathrm{d}\tau\end{aligned}$$

Now, we estimate $A^{0,1}-A^{0,2}$. Applying the lemma 14.2.1, we see that

$$\left|\left|A^{0,1}-A^{0,2}\right|\right|^2_{L^2(\Omega_\pm)}\le\mathcal{K}_L(t)\left|\left|v^1-v^2\right|\right|^2_{L^2(\Omega_\pm)}$$

where $v=(p,s,N)$. We then note that

$$\begin{aligned}\int_0^t\left|\left|v^1-v^2\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau&\le C\int_0^t\left|\left|(p^1,s^1)-(p^2,s^2)\right|\right|^2_{L^2(\Omega_\pm)}+\left|\left|\nabla'(f^1-f^2)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\&\le C\int_0^t\mathcal{E}_\Delta(\tau)\mathrm{d}\tau\end{aligned}$$

Now, we estimate $A^{2,1} - A^{2,2}$. Applying the lemma 14.2.1, we see that

$$||A^{2,1} - A^{2,2}||^2_{L^2(\Omega_\pm)} \leq \mathcal{K}_L(t)\, ||v^1 - v^2||^2_{L^2(\Omega_\pm)}$$

where $v = (p, s, \bar{u}, \nabla\psi, \nabla\bar{\psi}, \partial_t\psi, N, \nabla N, \partial_t N)$. We then note that

$$\begin{aligned}
&\int_0^t ||v^1 - v^2||^2_{L^2(\Omega_\pm)}\, \mathrm{d}\tau \\
&\leq C\int_0^t ||(p^1, s^1, \bar{u}^1) - (p^2, s^2, \bar{u}^2)||^2_{L^2(\Omega_\pm)}\, \mathrm{d}\tau \\
&+ C\int_0^t ||(\nabla' f^1, \nabla' \bar{f}^1, \partial_t f^1, \nabla'\partial f^1) - (\nabla' f^2, \nabla' \bar{f}^2, \partial_t f^2, \nabla'\partial f^2)||^2_{L^2(\Gamma)}\, \mathrm{d}\tau \\
&\leq C\int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

Combining all of the above, and estimating factors not involving differences in terms of $\mathcal{K}_L(t)$, we have proved the first set of inequalities. Now note that for $2 \leq i \leq 4$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|G^{i,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} &= 2(G^{i,\alpha}_\Delta, \partial_t G^{i,\alpha}_\Delta)^2_{L^2(\Omega_\pm)} \\
&\leq \frac{1}{\epsilon}\left|\left|G^{i,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} + \epsilon\left|\left|\partial_t G^{i,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)}
\end{aligned}$$

Thus, integrating in time,

$$\begin{aligned}
\left|\left|G^{i,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} &\leq \frac{1}{\epsilon}\int_0^t \left|\left|G^{i,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau + \epsilon\int_0^t \left|\left|\partial_t G^{i,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
\end{aligned}$$

This completes the proof. □

Now we apply the energy estimate (11.6.3) to the equation for $\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)$, where $0 \leq |\alpha| \leq 1$ and $\alpha_3 = 0$ to obtain the following estimate.

$$\begin{aligned}
&\int_0^t \left|\left|\partial_t\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{0,1}(\tau)}\mathrm{d}\tau + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{1,1}(t)} \\
&\leq \mathcal{K}_L(t)\frac{1}{\epsilon}\int_0^t \sum_\pm \left|\left|G^{1,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm \left|\left|G^{2,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|G^{3,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)} + \left|\left|G^{4,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau \\
&+ \mathcal{K}_L(t)(\sum_\pm \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|G^{2,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|G^{3,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)} + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|G^{4,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)}) \\
&+ \epsilon\int_0^t \sum_\pm \left|\left|\partial_t G^{2,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\partial_t G^{3,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)} + \left|\left|\partial_t G^{4,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)}\mathrm{d}\tau
\end{aligned}$$

To estimate the right hand side we use the estimate 14.5.4 for the $G^{i,\alpha}_\Delta$ and we obtain

$$\begin{aligned}
&\int_0^t \left|\left|\partial_t\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{0,1}(\tau)}\mathrm{d}\tau + \operatorname*{ess\,sup}_{\tau\in(0,t)}\left|\left|\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{1,1}(t)} \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
\end{aligned}$$

Now to estimate the highest order space derivative, we apply the energy estimate 11.6.7 to the equation for $\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)$, where $0 \leq |\alpha| \leq 1$ and $\alpha_3 = 0$. We obtain

$$\begin{aligned}
&\left|\left|\nabla'\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{0,1}(t)} + \int_0^t \left|\left|\nabla'\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{1,1}(\tau)} \mathrm{d}\tau \\
&\leq \mathcal{K}_L(t) \int_0^t \sum_\pm \left|\left|G^{1,\alpha}_\Delta\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm \left|\left|G^{2,\alpha}_\Delta\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \left|\left|G^{3,\alpha}_\Delta\right|\right|^2_{L^2(\Gamma)} + \sum_\pm \left|\left|G^{4,\alpha}_\Delta\right|\right|^2_{H^1(\Gamma)} \mathrm{d}\tau \\
&+ \mathcal{K}_L(t)\frac{1}{\epsilon} \int_0^t \left|\left|\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{H^1(\Omega_\pm)} \mathrm{d}\tau + \epsilon \int_0^t \left|\left|\partial_t\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau
\end{aligned}$$

Applying the estimate 14.5.4 to the right hand side, we obtain

$$\begin{aligned}
&\left|\left|\nabla'\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{0,1}(t)} + \int_0^t \left|\left|\nabla'\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{B^{1,1}(\tau)} \mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
\end{aligned}$$

It remains to estimate the highest order normal derivatives. Let $0 \leq |\alpha| \leq 1$. We claim that

$$\begin{aligned}
&\int_0^t \left|\left|\partial^\alpha(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{H^2(\Omega_\pm)} \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
\end{aligned}$$

Indeed, we already know from the above estimates that this formula holds in the case $\alpha_3 = 0$, so it remains to prove the case $\alpha_3 = 1$. Let $\partial^\beta = \partial_{x_3}$. We recall from the proof of the linearised energy estimate that $\partial^2_{x_3}\partial^\beta\ \widehat{u}^i$ for $i = 1, 2$ satisfy the equation (12.6.11). Subtracting these equations for $\widehat{u}^1$ and $\widehat{u}^2$, we obtain

$$\begin{aligned}
\partial^2_3\partial^\beta(\widehat{u}^1 - \widehat{u}^2) &= \partial^\beta(G(\partial_{x_3}\psi^1, \partial^2_{x_3}\psi^1, v^1)w^1 - G(\partial_{x_3}\psi^2, \partial^2_{x_3}\psi^2, v^2)w^2) \\
&=: \partial^\beta(\tilde{G}(\tilde{v}^1) - \tilde{G}(\tilde{v}^2))
\end{aligned}$$

where the function G is defined in (12.6.11), the vectors v and w are defined in (12.6.12)-(12.6.13), $\tilde{v} := (\partial_{x_3}\psi, \partial^2_{x_3}\psi, v, w)$ and we have defined the function $\tilde{G}$ such that $G(\partial_{x_3}\psi, \partial^2_{x_3}\psi, v)w = \tilde{G}(\tilde{v})$.

Now we note that, using the lemma 14.2.1, we have

$$\left|\left|\partial^\beta(\tilde{G}(\tilde{v}^1) - \tilde{G}(\tilde{v}^2))\right|\right|^2_{L^2(\Omega_\pm)} \leq \mathcal{K}_L(t) \sum_{0\leq\gamma\leq\beta} ||\partial^\gamma(\tilde{v}^1 - \tilde{v}^2)||^2_{L^2(\Omega_\pm)}$$

Using the existing estimate (14.3.1) for f, the existing time and tangential derivative estimates and the lifting property of N we have, for $0 \leq |\gamma| \leq 1$,

$$\begin{aligned}
&\int_0^t ||\partial^\gamma(\tilde{v}^1 - \tilde{v}^2)||^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
\end{aligned}$$

Combining these estimates we obtain

$$\begin{aligned}
&\int_0^t \left|\left|\partial^2_{x_3}\partial^\beta(\widehat{u}^1 - \widehat{u}^2)\right|\right|^2_{L^2(\Omega_\pm)} \mathrm{d}\tau \\
&\leq \frac{1}{\epsilon}\mathcal{K}_L(t) \int_0^t \bar{\mathcal{E}}_\Delta(\tau) + \mathcal{E}_\Delta(\tau) + \mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau + \epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t) + \mathcal{E}_\Delta(t) + \mathcal{E}_\Delta(\omega)(t))
\end{aligned}$$

Combining this with the existing tangential estimates, we obtain

$$\sum_{\pm}\int_0^t \left|\left|\partial^\beta(\widehat{u}^1-\widehat{u}^2)\right|\right|^2_{H^2(\Omega_\pm)}\mathrm{d}\tau$$
$$\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

This completes the estimate of the normal derivatives.

Combining all the above estimates, we have shown that

$$\mathcal{E}_\Delta(\widehat{u})(t)\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

Now using the estimate (14.5.1) and the estimate (14.3.1) for the front, we obtain

$$\mathcal{E}_\Delta(u)(t)\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

This completes the proof of the estimate of the velocity 14.5.2.

Proposition 14.5.5. *The normal component of the velocity, $u\cdot n$, satisfies the following additional estimate.*

$$\sum_{j=0}^{1}\int_0^t \left|\left|\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau$$
$$\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

for $t\in(0,T)$.

Proof. Let $0\leq j\leq 1$. Applying ∂_t^j to the pressure interface condition (9.1.9) we have

$$(1-\Delta')\partial_t^j(u\cdot n)=\partial_t^j([\rho c^2\nabla^\psi\cdot u-\frac{1}{\mu}p]-\frac{1}{\mu}\sigma\nabla'\cdot\hat{n}+\frac{1}{\mu}g_2)=:\partial_t^j G_{u\cdot n}$$

We subtract the equations for $u^1\cdot n^1$ and $u^2\cdot n^2$ to obtain

$$(1-\Delta')\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)=\partial_t^j(G^1_{u\cdot n}-G^2_{u\cdot n})$$

We apply lemma 8.16.1 with $z=\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)$ and $h=\partial_t^j(G^1_{u\cdot n}-G^2_{u\cdot n})$ to conclude that

$$\left|\left|\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\leq C\left|\left|\partial_t^j(G^1_{u\cdot n}-G^2_{u\cdot n})\right|\right|^2_{H^{1.5-j}(\Gamma)}$$

Now we use lemma 14.2.1 and the Sobolev trace lemma to conclude that

$$\left|\left|\partial_t^j(G^1_{u\cdot n}-G^2_{u\cdot n})\right|\right|^2_{H^{1.5-j}(\Gamma)}\leq \mathcal{K}_L(t)\sum_{k=0}^{1}(\left|\left|\partial_t^k(f^1-f^2)\right|\right|^2_{H^{3.5-k}(\Gamma)}+\sum_{\pm}\left|\left|\partial_t^k(v^1-v^2)\right|\right|^2_{H^{2-k}(\Omega_\pm)})$$

where $v=(p,s,\nabla u,\nabla\psi)$. From the definition of $\mathcal{E}_\Delta(t)$ and the estimate of the velocity 14.5.2, for $0\leq k\leq 1$, we have

$$\int_0^t \left|\left|\partial^k(v^1-v^2)\right|\right|^2_{H^{2-k}(\Omega_\pm)}\mathrm{d}\tau$$
$$\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

Thus, after integrating in time, we obtain

$$\int_0^t \left|\left|\partial_t^j(u^1\cdot n^1-u^2\cdot n^2)\right|\right|^2_{H^{3.5-j}(\Gamma)}\mathrm{d}\tau$$
$$\leq \frac{1}{\epsilon}\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))$$

as required. □

14.6 Conclusion of the Contraction Estimate

Putting together what we have so far we have shown that

$$\begin{aligned}&\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t)\\&\leq \frac{1}{\epsilon}\mathcal{K}_L(t)(\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau)+\epsilon\mathcal{K}_L(t)(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))\end{aligned}$$

for $t\in(0,T)$.

Choosing $\epsilon=\frac{1}{5\mathcal{K}_L(t)}$, we obtain

$$\begin{aligned}&\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t)\\&\leq \mathcal{K}_L(t)(\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau)+\frac{1}{5}(\bar{\mathcal{E}}_\Delta(t)+\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t))\end{aligned}$$

for $t\in(0,T)$ where we have redefined $\mathcal{K}_L(t)$. Rearranging we obtain

$$\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t)\leq \mathcal{K}_L(t)(\int_0^t \bar{\mathcal{E}}_\Delta(\tau)+\mathcal{E}_\Delta(\tau)+\mathcal{E}_\Delta(\omega)(\tau)\mathrm{d}\tau)+\frac{1}{4}\bar{\mathcal{E}}_\Delta(t)$$

for $t\in(0,T)$.

Now we apply Gronwall's lemma to conclude

$$\mathcal{E}_\Delta(t)+\mathcal{E}_\Delta(\omega)(t)\leq (\mathcal{K}_L(t)\int_0^t \bar{\mathcal{E}}_\Delta(\tau)\mathrm{d}\tau+\frac{1}{4}\bar{\mathcal{E}}_\Delta(t))\exp(t\mathcal{K}_L(t))$$

for $t\in(0,T)$.

Now, let us note that

$$\mathcal{K}_L(T)\leq F_\mu(R^0,M^0_\mu)$$

for some smooth increasing function F_μ which may depend on μ. Hence if $T>0$ is chosen such that

$$\begin{aligned}T&\leq \frac{1}{8F_\mu(R^0,M^0_\mu)}\\T&\leq \frac{\log(2)}{F_\mu(R^0,M^0_\mu)}\end{aligned}$$

then

$$\mathcal{E}_\Delta(T)+\mathcal{E}_\Delta(\omega)(T)\leq \frac{3}{4}\bar{\mathcal{E}}_\Delta(T)$$

and in particular

$$d_Y((U^1_+,U^1_-,f^1),(U^2_+,U^2_-,f^2))\leq \lambda d_Y((\bar{U}^1_+,\bar{U}^1_-,\bar{f}^1),(\bar{U}^2_+,\bar{U}^2_-,\bar{f}^2))$$

where $\lambda=\sqrt{\frac{3}{4}}<1$.

This completes the proof of 13.2.2.

15. Proof of Uniqueness and Stability

Here we aim to prove the statement of stability in the main theorem in the fixed domains, 4.6.1, from which the statement of uniqueness immediately follows. We will use the fact that we have already proved the rest of theorem 4.6.1. For $i = 1, 2$, we let $(U_+^{0,i}, U_-^{0,i}, f^{0,i})$ be initial data satisfying the hypotheses of theorem 4.6.1, and we let $T^{0,i} > 0$ be the associated existence times depending on this initial data as given by 4.6.1. We set $T^0 = \min\{T^{0,1}, T^{0,2}\}$. We let (U_+^1, U_-^1, f^1) and (U_+^2, U_-^2, f^2) be two solutions of the equations in the fixed domains as defined in 4.3.1 on the time interval $(0, T^0)$ with initial data $(U_+^{0,i}, U_-^{0,i}, f^{0,i})$ respectively, with the properties

$$E^i(T^0) =: C^i < \infty \tag{15.0.1}$$

$$\inf_{t\in(0,T^0)} \inf_{x\in\Omega_\pm} \rho_\pm^i =: \delta^i > 0 \tag{15.0.2}$$

$$\inf_{t\in(0,T^0)} \inf_{x\in\mathbb{R}^3} J^{\psi^i} =: \kappa^i > 0 \tag{15.0.3}$$

for $i = 1, 2$, where the superscript ^{i} for $i = 1, 2$ is used to denote quantities associated with the solutions (U_+^i, U_-^i, f^i), and the energy $E(t)$ is defined in 4.5.1. We assume that $U^{0,1} - U^{0,2} \in L^2(\Omega_\pm)$ and $f^{0,1} - f^{0,2} \in L^2(\Gamma)$ so that $E_\Delta^0 < \infty$, where the initial difference energy E_Δ^0 is defined in 4.5.2.

Notation. *In analogy with the proof of the energy estimate proved in 8, we will write*

$$K(t) = F(E^1(t), E^2(t), \frac{1}{\delta^1}, \frac{1}{\delta^2}, \frac{1}{\kappa^1}, \frac{1}{\kappa^2})$$

for $t \in (0, T^0]$, where F is a smooth increasing function, and note that

$$K(t) \le K(T^0) = F(C^1, C^2, \frac{1}{\delta^1}, \frac{1}{\delta^2}, \frac{1}{\kappa^1}, \frac{1}{\kappa^2}) < \infty$$

We also write, for $i = 1, 2$,

$$\begin{aligned}
(u^\psi)^i &:= (u^i)^{\psi^i} \\
n^i &:= (-\nabla' f^i, 1) \\
\hat{n}^i &:= \frac{n^i}{|n^i|}
\end{aligned}$$

We aim to show that

$$E_\Delta(t) \le K(t) E_\Delta^0 + \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t)$$

where we are free to choose $\epsilon \in (0, 1]$ and $E_\Delta(t)$ is defined in 4.5.2.

Note that the proof is almost identical to the proof of the μ-independent energy estimate in 8, hence we will omit some details. See 3.9 for a sketch of how the proof of the μ-independent energy estimate in 8 is modified.

15.1 Remainder Estimates

Notation. *If we write $R^k_\infty(U,\psi)$, $R^k(U,\psi)$ etc. without placing a superscript ^{1} or ^{2} on U or ψ of f, we mean that the remainders could include terms involving both U^1 and U^2 etc, but are still of the same form as defined in 8.18. Also, we set $g_1 = g_2 = 0$ when we refer to any remainder such as $R^k_\infty(U,\psi)$, $R^k(U,\psi)$ defined in 8.18.*

Definition 15.1.1. We write $R_{\Delta,\infty}(U,\psi)$ for terms of the form

$$R^0_\infty(U,\psi)(R^0_\infty(U^1,\psi^1) - R^0_\infty(U^2,\psi^2))$$

Similarly we write $R_{\Delta,\infty,\Gamma}(U,f)$ for terms of the form

$$R^0_{\infty,\Gamma}(U,f)(R^0_{\Gamma,\infty}(U^1,f^1) - R^0_{\Gamma,\infty}(U^2,f^2))$$

We also write $R_{\Delta,\infty,\Gamma}(f)$ for terms of the form

$$R_{\infty,\Gamma}(f)(R_{\infty,\Gamma}(f^1) - R_{\infty,\Gamma}(f^2))$$

Note that $R^0_\infty(U,\psi)$, $R^0_{\infty,\Gamma}(U,f)$ and $R_{\infty,\Gamma}(f)$ are defined in section 8.18.

Note that although the remainder terms are generic, when we take a difference such as $R^0_\infty(U^1,\psi^1) - R^0_\infty(U^2,\psi^2)$, the two terms are the same functions of the unknowns, in this case (U,ψ).

Lemma 15.1.2. *We have the estimates*

$$||R_{\Delta,\infty}(U,\psi)||^2_{L^\infty(\Omega_\pm)} \le K(t)E_\Delta(t) \tag{15.1.1}$$

$$||R_{\Delta,\infty,\Gamma}(U,f)||^2_{L^\infty(\Gamma)} \le K(t)E_\Delta(t) \tag{15.1.2}$$

$$||R_{\Delta,\infty,\Gamma}(f)||^2_{L^\infty(\Gamma)} \le K(t)E_\Delta(t) \tag{15.1.3}$$

Proof. Using the mean value inequality, we have that

$$\begin{aligned}
&\big|\big|R^0_\infty(U^1,\psi^1) - R^0_\infty(U^2,\psi^2)\big|\big|^2_{L^\infty(\Omega_\pm)} \\
&\le \big|\big|R^0_\infty(U,\psi)\big|\big|^2_{L^\infty(\Omega_\pm)}(\big|\big|U^1-U^2\big|\big|^2_{L^\infty(\Omega_\pm)} + \big|\big|\psi^1-\psi^2\big|\big|^2_{L^\infty(\Omega_\pm)} + \big|\big|\partial(\psi^1-\psi^2)\big|\big|^2_{L^\infty(\Omega_\pm)}) \\
&\le K(t)(\big|\big|U^1-U^2\big|\big|^2_{H^2(\Omega_\pm)} + \big|\big|\psi^1-\psi^2\big|\big|^2_{H^2(\Omega_\pm)} + \big|\big|\partial(\psi^1-\psi^2)\big|\big|^2_{H^2(\Omega_\pm)}) \\
&\le K(t)E_\Delta(t)
\end{aligned}$$

where we have used (8.18.8), which we apply again to obtain

$$||R_{\Delta,\infty}(U,\psi)||^2_{L^\infty(\Omega_\pm)} \le K(t)E_\Delta(t)$$

as required. Note that $R_{\Delta,\infty,\Gamma}(U,f)$ and $R_{\Delta,\infty,\Gamma}(f)$ both have the form $R_{\Delta,\infty}(U,\psi)|_\Gamma$ so we may reuse the above estimate together with uniform continuity to conclude the estimates (15.1.2)-(15.1.3). □

Definition 15.1.3. For $0 \le k \le 2$, we write $R^k_\Delta(U,\psi)$ for terms of the form

$$R^j(U,\psi)(R^l(U^1,\psi^1) - R^l(U^2,\psi^2)) \text{ or } R^{k+1}(U,\psi)R_{\Delta,\infty}(U,\psi)$$

where $1 \le l, j \le k$ and $j + l \le k + 1$.

Similarly for $0 \le k \le 1$ we write $R^k_{\Delta,\Gamma}(U,f)$ for terms of the form

$$R^j_\Gamma(U,f)(R^l_\Gamma(U^1,f^1) - R^l_\Gamma(U^2,f^2)) \text{ or } R^{k+1}_\Gamma(U,f)R_{\Delta,\Gamma,\infty}(U,f)$$

where $1 \le l, j \le k$ and $j + l \le k + 1$.

For $0 \le k \le 2$, we also write $R^k_{\Delta,\Gamma}(f)$ for terms of the form

$$R^j_\Gamma(f)(R^l_\Gamma(f^1) - R^l_\Gamma(f^2)) \text{ or } R^{k+1}_\Gamma(f)R_{\Delta,\Gamma,\infty}(f)$$

where $1 \le l, j \le k$ and $j + l \le k + 1$.

Lemma 15.1.4. *Let $0 \le k \le 2$. We have the estimates*

$$\left|\left|R^k_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \le K(t)E_\Delta(t) \tag{15.1.4}$$

$$\left|\left|R^{k-1}_{\Delta,\Gamma}(U,f)\right|\right|^2_{L^2(\Gamma)} \le K(t)E_\Delta(t) \text{ (for } k \ge 1\text{)} \tag{15.1.5}$$

$$\left|\left|R^k_{\Delta,\Gamma}(f)\right|\right|^2_{L^2(\Gamma)} \le K(t)E_\Delta(t) \tag{15.1.6}$$

Proof. First of all, let us note that if the remainder terms are vector-valued then we only need to prove the estimate for the components. Applying the mean value theorem and using the form of the product in the remainder term $R^2(U,\psi)$ we obtain

$$\begin{aligned}
&R^2(U^1,\psi^1) - R^2(U^2,\psi^2) = R^2(U,\psi)\cdot((U^1,\psi^1,\partial\psi^1) - (U^2,\psi^2,\partial\psi^2)) \\
&+ R^2(U,\psi)\cdot\partial((U^1,\psi^1,\partial\psi^1) - (U^2,\psi^2,\partial\psi^2)) \\
&+ R^1(U,\psi)\cdot\partial^2((U^1,\psi^1,\nabla'\psi^1) - (U^2,\psi^2,\nabla'\psi^2))
\end{aligned}$$

Now, using the Sobolev product lemma A.3.1, we obtain

$$\begin{aligned}
&||R^2(U^1,\psi^1) - R^2(U^2,\psi^2)||_{L^2(\Omega_\pm)} \le ||R^2(U,\psi)||_{H^1(\Omega_\pm)}\,||(U^1,\psi^1,\partial\psi^1) - (U^2,\psi^2,\partial\psi^2)||_{H^1(\Omega_\pm)} \\
&+ ||R^2(U,\psi)||_{H^1(\Omega_\pm)}\,||\partial((U^1,\psi^1,\partial\psi^1) - (U^2,\psi^2,\partial\psi^2))||_{H^1(\Omega_\pm)} \\
&+ ||R^1(U,\psi)||_{H^2(\Omega_\pm)}\,||\partial^2((U^1,\psi^1,\nabla'\psi^1) - (U^2,\psi^2,\nabla'\psi^2))||_{L^2(\Omega_\pm)}
\end{aligned}$$

Thus, squaring and using the remainder estimate (8.18.3), we obtain

$$||R^2(U^1,\psi^1) - R^2(U^2,\psi^2)||^2_{L^2(\Omega_\pm)} \le K(t)E_\Delta(t)$$

Let $1 \le l, j \le k$ and $j + l \le k + 1$. Setting $m_1 = 3 - j$ and $m_2 = 2 - l$, we have $0 \le m_i \le 2$ and $m_1 + m_2 \ge 2$, hence we may apply the Sobolev product lemma A.3.1 to obtain

$$\begin{aligned}
&\left|\left|R^j(U,\psi)(R^l(U^1,\psi^1) - R^l(U^2,\psi^2))\right|\right|^2_{L^2(\Omega_\pm)} \\
&\le C\,||R^j(U,\psi)||^2_{H^{3-j}(\Omega_\pm)}\left|\left|R^l(U^1,\psi^1) - R^l(U^2,\psi^2)\right|\right|^2_{H^{2-l}(\Omega_\pm)} \\
&\le C\,||R^3(U,\psi)||^2_{L^2(\Omega_\pm)}\,||R^2(U^1,\psi^1) - R^2(U^2,\psi^2)||^2_{L^2(\Omega_\pm)} \\
&\le K(t)E_\Delta(t)
\end{aligned}$$

where we have again used the estimate (8.18.3). We also note that

$$\begin{aligned}\left|\left|R^{k+1}(U,\psi)R_{\Delta,\infty}(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} &\leq \left|\left|R^3(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)}\left|\left|R_{\Delta,\infty}(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)}\\ &\leq K(t)E_\Delta(t)\end{aligned}$$

where we have again used the estimate (8.18.3) and also the estimate (15.1.1).

The estimate (15.1.6) is similar. To prove (15.1.5), we merely have to note that $R^{k-1}_{\Delta,\Gamma}(U,f)$ has the form $R^{k-1}_\Delta(U,\psi)|_\Gamma$ (using the fact that we can replace any high-order terms in $\partial_t f$ with $u{\cdot}n$ as mentioned before), then apply the Sobolev trace estimate together with the fact that $\nabla R^{k-1}_\Delta(U,\psi) = R^k_\Delta(U,\psi)$. $\square$

Lemma 15.1.5. *Let* $0 \leq k \leq 1$. *We have the estimates*

$$\left|\left|R^k_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau \tag{15.1.7}$$

$$\left|\left|R^0_{\Delta,\Gamma}(U,f)\right|\right|^2_{L^2(\Gamma)} \leq K(t)E^0_\Delta + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau \tag{15.1.8}$$

$$\left|\left|R^k_{\Delta,\Gamma}(f)\right|\right|^2_{L^2(\Gamma)} \leq K(t)E^0_\Delta + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau \tag{15.1.9}$$

Proof. We simply note that $\partial_t R^k_\Delta(U,\psi) = R^{k+1}_\Delta(U,\psi)$ then we use the estimate (15.1.4) to obtain

$$\begin{aligned}\left|\left|R^k_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} &\leq C\left|\left|R^k_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)}\Big|_{t=0} + C\int_0^t \left|\left|\partial_t R^k_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)}\mathrm{d}\tau\\ &\leq K(t)E^0_\Delta + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau\end{aligned}$$

The proof of the estimates (15.1.8)-(15.1.9) is exactly similar. $\square$

Lemma 15.1.6. *Let* α *be a multi-index with* $\alpha_3 \leq |\alpha|-1$ *and* $1 \leq |\alpha| \leq 2$. *Thus we have either* $\partial^\alpha = \partial_{x_j}\partial^\beta$ *for some* $1 \leq j \leq 2$ *and* $|\beta| = |\alpha|-1$, *or* $\partial^\alpha = \partial_t\partial^\beta$ *for some* $|\beta| = |\alpha|-1$. *Then*

$$\left|\int_0^t\int_\Gamma R^{|\alpha|-1}_{\Delta,\Gamma}(U,f)\partial^\alpha(U^1-U^2)\mathrm{d}x'\mathrm{d}\tau\right| \leq K(t)E^0_\Delta + \frac{1}{\epsilon}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon K(t)E_\Delta(t) \tag{15.1.10}$$

where we are free to choose ϵ *in the range* $0 < \epsilon \leq 1$.

Proof. This is proved exactly as the estimate (8.18.11) but without the term $\mu\dot{p}$ and using the estimate (15.1.4) instead of (8.18.3). $\square$

Definition 15.1.7. For $1 \leq |\alpha| \leq 2$, we define a slight variant $\tilde{R}^{|\alpha|}_\Delta(U,\psi)$ of the remainder $R^{|\alpha|}_\Delta(U,\psi)$ as a sum of terms of the form

$$R^{|\alpha|}_\Delta(U,\psi) \text{ or } \partial\partial^j\psi R^l_\Delta(U,\psi) \text{ or } R^{j+1}(U,\psi)\partial^l(\psi^1-\psi^2)$$

where $1 \leq j,l \leq |\alpha|$ and $j+l \leq |\alpha|+1$.

Lemma 15.1.8. *For* $1 \leq |\alpha| \leq 2$, *the remainder* $\tilde{R}^{|\alpha|}_\Delta(U,\psi)$ *satisfies the estimate.*

$$\left|\left|\tilde{R}^{|\alpha|}_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E_\Delta(t) \tag{15.1.11}$$

Proof. Let $1 \leq j, l \leq |\alpha|$ and $j + l \leq |\alpha| + 1$. Let $m_1 = 2 - j$ and $m_2 = 2 - l$. Using the Sobolev product lemma A.3.1 and the remainder estimate (15.1.4), we have

$$\begin{aligned}\left|\left|\partial\partial^j\psi R^l_\Delta(U,\psi)\right|\right|^2_{L^2(\Omega_\pm)} &\leq \left|\left|\partial\partial^j\psi\right|\right|^2_{H^{m_1+1}(\Omega_\pm)}\left|\left|R^l_\Delta(U,\psi)\right|\right|^2_{H^{m_2}(\Omega_\pm)}\\ &\leq K(t)E_\Delta(t)\end{aligned}$$

Similarly, using the remainder estimate (8.18.3), we have

$$\begin{aligned}\left|\left|R^{j+1}(U,\psi)\partial^l(\psi^1-\psi^2)\right|\right|^2_{L^2(\Omega_\pm)} &\leq \left|\left|R^{j+1}(U,\psi)\right|\right|^2_{H^{m_1}(\Omega_\pm)}\left|\left|\partial^l(\psi^1-\psi^2)\right|\right|^2_{H^{m_2+1}(\Omega_\pm)}\\ &\leq K(t)E_\Delta(t)\end{aligned}$$

Combining this with the estimate (15.1.4) completes the proof. □

15.2 Difference Estimate of the Lower Order Terms in the Energy

Let $0 \leq |\alpha| \leq 1$. Using the fact that the norms $\left|\left|\partial_t\partial^\alpha(U^1-U^2)\right|\right|^2_{L^2(\Omega_\pm)}$, $\left|\left|\partial_t\partial^\alpha(f^1-f^2)\right|\right|^2_{H^1(\Gamma)}$ and $\left|\left|\partial_t(f^1-f^2)\right|\right|^2_{H^{2.5}(\Gamma)}$ are all bounded by $E_\Delta(t)$ together with the fundamental theorem of calculus we obtain

$$\left|\left|\partial^\alpha(U^1-U^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq CE^0_\Delta + C\int_0^t E_\Delta(t)\mathrm{d}\tau \tag{15.2.1}$$

$$\left|\left|\partial^\alpha(f^1-f^2)\right|\right|^2_{H^1(\Gamma)} + \left|\left|f^1-f^2\right|\right|^2_{H^{2.5}(\Gamma)} \leq CE^0_\Delta + C\int_0^t E_\Delta(t)\mathrm{d}\tau \tag{15.2.2}$$

15.3 The Equations for the Difference of the Derivatives of the Solutions

15.3.1 The Difference of the Differentiated Interior Equations

We let $1 \leq |\alpha| \leq 2$, and we subtract the equations (8.6.1)-(8.6.3) for (U^2, ψ^2) from those for (U^1, ψ^1) to obtain

$$\begin{aligned}&\frac{1}{\rho^1(c^1)^2}(\partial_t + (u^\psi)^1\cdot\nabla)\partial^\alpha(p^1-p^2) -\\ &\frac{1}{\rho^1(c^1)^2J^{\psi^1}}(\partial_t\partial^\alpha(\psi^1-\psi^2) + u^1\cdot\nabla\partial^\alpha(\psi^1-\psi^2) - \frac{1}{J^{\psi^1}}\partial_{x_3}\partial^\alpha(\psi^1-\psi^2)(\partial_t\psi^1 + u^1\cdot\nabla\psi^1))\partial_{x_3}p^1\\ &+ \nabla^{\psi^1}\cdot\partial^\alpha(u^1-u^2) - \frac{\nabla\partial^\alpha(\psi^1-\psi^2)}{J^{\psi^1}}\cdot\partial_{x_3}u^1 + \frac{\nabla\psi^1}{(J^{\psi^1})^2}\partial_{x_3}\partial^\alpha(\psi^1-\psi^2)\cdot\partial_{x_3}u^1\\ &= \tilde{R}^{|\alpha|}_\Delta(U,\psi)\end{aligned} \tag{15.3.1}$$

$$\begin{aligned}&\rho^1(\partial_t + (u^\psi)^1\cdot\nabla)\partial^\alpha(u^1-u^2)\\ &- \frac{\rho^1}{J^{\psi^1}}(\partial_t\partial^\alpha(\psi^1-\psi^2) + u^1\cdot\nabla\partial^\alpha(\psi^1-\psi^2) - \frac{1}{J^{\psi^1}}(\partial_{x_3}\partial^\alpha(\psi^1-\psi^2))(\partial_t\psi^1 + u^1\cdot\nabla\psi^1))\partial_{x_3}u^1\\ &+ \nabla^{\psi^1}\partial^\alpha(p^1-p^2) - \frac{\nabla\partial^\alpha(\psi^1-\psi^2)}{J^{\psi^1}}\partial_{x_3}p^1 + \frac{\nabla\psi^1}{(J^{\psi^1})^2}\partial_{x_3}\partial^\alpha(\psi^1-\psi^2)\partial_{x_3}p^1\\ &= \tilde{R}^{|\alpha|}_\Delta(U,\psi)\end{aligned} \tag{15.3.2}$$

$$\begin{aligned}&(\partial_t + (u^\psi)^1\cdot\nabla)\partial^\alpha(s^1-s^2)\\ &- \frac{1}{J^{\psi^1}}(\partial_t\partial^\alpha(\psi^1-\psi^2) + u^1\cdot\nabla\partial^\alpha(\psi^1-\psi^2) - \frac{1}{J^{\psi^1}}(\partial_{x_3}\partial^\alpha(\psi^1-\psi^2))(\partial_t\psi^1 + u^1\cdot\nabla\psi^1))\partial_{x_3}s^1\\ &= \tilde{R}^{|\alpha|}_\Delta(U,\psi)\end{aligned} \tag{15.3.3}$$

in $(0, T^0) \times \Omega_\pm$.

Now assume that $\alpha_0 \leq 1$. Because ∂^α contains a maximum of 1 time derivative, we may absorb the terms involving high-order derivatives of ψ in the equations (15.3.1)-(15.3.3) into the remainder terms and we may use $R^{|\alpha|}_\Delta(U,\psi)$ for the remainder terms instead of $\tilde{R}^{|\alpha|}_\Delta(U,\psi)$. We obtain the following.

$$\frac{1}{\rho^1(c^1)^2}(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\alpha(p^1 - p^2) + \nabla^{\psi^1} \cdot \partial^\alpha(u^1 - u^2) = R^{|\alpha|}_\Delta(U,\psi) \tag{15.3.4}$$

$$\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\alpha(u^1 - u^2) + \nabla^{\psi^1}\partial^\alpha(p^1 - p^2) = R^{|\alpha|}_\Delta(U,\psi) \tag{15.3.5}$$

$$(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\alpha(s^1 - s^2) = R^{|\alpha|}_\Delta(U,\psi) \tag{15.3.6}$$

in $(0, T^0) \times \Omega_\pm$.

15.3.2 The Corrected Pressure, Velocity and Entropy

Definition 15.3.1. For $1 \leq |\alpha| \leq 2$ and $i = 1, 2$, we define

$$U^{\alpha,i} := \partial^\alpha U^i - \frac{\partial^\alpha \psi^i}{J^{\psi^1}}\partial_{x_3}U^1$$

Note that here we have used J^{ψ^1} and $\partial_{x_3}U^1$ in the second term, even for $i = 2$.

Lemma 15.3.2. *For $1 \leq |\alpha| \leq 2$ and $|\beta| = 1$, we have*

$$||\partial^\alpha(U^1 - U^2)||^2_{L^2(\Omega_\pm)} \leq C\,||U^{\alpha,1} - U^{\alpha,2}||^2_{L^2(\Omega_\pm)} + K(t)\,||\partial^\alpha(f^1 - f^2)||^2_{L^2(\Gamma)} \tag{15.3.7}$$

$$\begin{aligned}\left|\left|D^{u^1_-}_t \partial^\beta(U^1 - U^2)\right|\right|^2_{L^2(\Omega_\pm)} &\leq C\left|\left|D^{u^1_-}_t (U^{\beta,1} - U^{\beta,2})\right|\right|^2_{L^2(\Omega_\pm)} \\ &\quad + K(t)\left|\left|(\partial^\beta(f^1 - f^2), \partial'\partial^\beta(f^1 - f^2))\right|\right|^2_{L^2(\Gamma)}\end{aligned} \tag{15.3.8}$$

$$||U^{\alpha,1} - U^{\alpha,2}||^2_{L^2(\Omega_\pm)} \leq C\,||\partial^\alpha(U^1 - U^2)||^2_{L^2(\Omega_\pm)} + K(t)\,||\partial^\alpha(f^1 - f^2)||^2_{L^2(\Gamma)} \tag{15.3.9}$$

Proof. Let $1 \leq |\alpha| \leq 2$ and $0 \leq |\gamma| \leq 2 - |\alpha|$. Using the above definition, we have

$$\begin{aligned}
&\frac{1}{C}\,||\partial^\gamma\partial^\alpha(U^1 - U^2)||^2_{L^2(\Omega_\pm)} \leq ||\partial^\gamma(U^{\alpha,1} - U^{\alpha,2})||^2_{L^2(\Omega_\pm)} + \left|\left|\partial^\gamma(\frac{\partial^\alpha(\psi^1 - \psi^2)}{J^{\psi^1}}\partial_{x_3}U^1)\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq ||\partial^\gamma(U^{\alpha,1} - U^{\alpha,2})||^2_{L^2(\Omega_\pm)} + \left|\left|\frac{1}{J^{\psi^1}}\partial_{x_3}U^1\right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial^\gamma\partial^\alpha(\psi^1 - \psi^2)||^2_{L^2(\Omega_\pm)} \\
&+ \left|\left|\frac{1}{J^\psi}\right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial^\gamma\partial_{x_3}U||^2_{H^1(\Omega_\pm)}\,||\partial^\alpha(\psi^1 - \psi^2)||^2_{H^1(\Omega_\pm)} \\
&+ \left|\left|\frac{\partial^\gamma\partial_{x_3}\psi^1}{(J^{\psi^1})^2}\right|\right|^2_{L^\infty(\Omega_\pm)} ||\partial_{x_3}U^1||^2_{H^2(\Omega_\pm)}\,||\partial^\alpha(\psi^1 - \psi^2)||^2_{L^2(\Omega_\pm)} \\
&\leq ||\partial^\gamma(U^{\alpha,1} - U^{\alpha,2})||^2_{L^2(\Omega_\pm)} + K(t)(||\partial^\gamma\partial^\alpha(f^1 - f^2)||^2_{L^2(\Gamma)} + ||\partial^\alpha(f^1 - f^2)||^2_{H^{|\gamma|}(\Gamma)})
\end{aligned}$$

where we have used the the Sobolev embedding theorem to estimate the L^2 norm of a product in terms of the product of H^1 norms, and the last two terms in the second to last line do not exist in the case $|\alpha| = 2$. Along with the lower order estimate (8.5.1) for $(u^\psi_-)^1$, this proves (15.3.7)-(15.3.8). Similarly, we have

$$\begin{aligned}
\frac{1}{C}\,||(U^{\alpha,1} - U^{\alpha,2})||^2_{L^2(\Omega_\pm)} &\leq ||\partial^\alpha(U^1 - U^2)||^2_{L^2(\Omega_\pm)} + \left|\left|\frac{\partial^\alpha(\psi^1 - \psi^2)}{J^{\psi^1}}\partial_{x_3}U^1\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq ||\partial^\alpha(U^1 - U^2)||^2_{L^2(\Omega_\pm)} + K(t)\,||\partial^\alpha(f^1 - f^2)||^2_{L^2(\Gamma)}
\end{aligned}$$

□

Lemma 15.3.3. *For* $1 \le |\alpha| \le 2$*, we have*

$$\begin{aligned}(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\alpha(U^1 - U^2) &= (\partial_t + (u^\psi)^1 \cdot \nabla)(U^{\alpha,1} - U^{\alpha,2}) \\ &+ \frac{(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\alpha(\psi^1 - \psi^2)}{J\psi^1}\partial_{x_3}U^1 + \tilde{R}_\Delta^{|\alpha|}(U,\psi) \\ &= (\partial_t + (u^\psi)^1 \cdot \nabla)(U^{\alpha,1} - U^{\alpha,2}) + \frac{1}{J\psi^1}((\partial_t + u^1 \cdot \nabla)\partial^\alpha(\psi^1 - \psi^2) \\ &- \frac{1}{J\psi^1}(\partial_t\psi^1 + u^1 \cdot \nabla\psi^1)\partial_{x_3}\partial^\alpha(\psi^1 - \psi^2))\partial_{x_3}U^1 + \tilde{R}_\Delta^{|\alpha|}(U,\psi)\end{aligned}$$

and

$$\begin{aligned}\nabla^{\psi^1}\partial^\alpha(U^1 - U^2) &= \nabla^{\psi^1}(U^{\alpha,1} - U^{\alpha,2}) + \frac{\nabla^{\psi^1}\partial^\alpha(\psi^1 - \psi^2)}{J\psi^1}\partial_{x_3}U^1 + \tilde{R}_\Delta^{|\alpha|}(U,\psi) \\ &= \nabla^{\psi^1}(U^{\alpha,1} - U^{\alpha,2}) + \frac{1}{J\psi^1}(\nabla\partial^\alpha(\psi^1 - \psi^2) - \frac{\nabla\psi^1}{J\psi^1}\partial_{x_3}\partial^\alpha(\psi^1 - \psi^2))\partial_{x_3}U^1 + \tilde{R}_\Delta^{|\alpha|}(U,\psi)\end{aligned}$$

Proof. We may simply differentiate the formula for $U^{\alpha,1} - U^{\alpha,2}$ and rearrange to obtain

$$\begin{aligned}\partial\partial^\alpha(U^1 - U^2) &= \partial(U^{\alpha,1} - U^{\alpha,2}) + \frac{\partial\partial^\alpha(\psi^1 - \psi^2)}{J\psi^1}\partial_{x_3}U^1 + \frac{\partial^\alpha(\psi^1 - \psi^2)}{J\psi^1}\partial\partial_{x_3}U^1 \\ &- \frac{\partial^\alpha(\psi^1 - \psi^2)}{(J\psi^1)^2}\partial\partial_{x_3}\psi^1\partial_{x_3}U^1\end{aligned}$$

and we note that the last two terms on the right have the form as the remainder $\tilde{R}_\Delta^{|\alpha|}(U,\psi)$. ☐

Using the above lemma in the preceding equations we obtain

$$\frac{1}{\rho^1(c^1)^2}(\partial_t + (u^\psi)^1 \cdot \nabla)(p^{\alpha,1} - p^{\alpha,2}) + \nabla^{\psi^1} \cdot (u^{\alpha,1} - u^{\alpha,2}) = \tilde{R}_\Delta^{|\alpha|}(U,\psi) \tag{15.3.10}$$

$$\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla)(u^{\alpha,1} - u^{\alpha,2}) + \nabla^{\psi^1}(p^{\alpha,1} - p^{\alpha,2}) = \tilde{R}_\Delta^{|\alpha|}(U,\psi) \tag{15.3.11}$$

$$(\partial_t + (u^\psi)^1 \cdot \nabla)(s^{\alpha,1} - s^{\alpha,2}) = \tilde{R}_\Delta^{|\alpha|}(U,\psi) \tag{15.3.12}$$

in $(0,T^0) \times \Omega_\pm$ for $1 \le |\alpha| \le 2$.

15.3.3 The Difference of the Tangential and Time Derivatives of the Jump Conditions

We let $1 \le |\alpha| \le 2$ with $\alpha_3 = 0$ and we subtract the equations (8.6.14)-(8.6.15) for (U^2, f^2) from those for (U^1, f^1) to obtain

$$\begin{aligned}\partial_t\partial^\alpha(f^1 - f^2) &= \partial^\alpha(u_\pm^1 - u_\pm^2) \cdot n^1 - (u_\pm^1)' \cdot \nabla'\partial^\alpha(f^1 - f^2) + R_\Gamma^{|\alpha|-1}(U^1, f^1) - R_\Gamma^{|\alpha|-1}(U^2, f^2) \\ &+ \partial^\alpha u_\pm^2 \cdot (n^1 - n^2) - (u_\pm^1 - u_\pm^2)' \cdot \nabla'\partial^\alpha f^2 \\ &= \partial^\alpha(u_\pm^1 - u_\pm^2) \cdot n^1 - (u_\pm^1)' \cdot \nabla'\partial^\alpha(f^1 - f^2) + R_{\Delta,\Gamma}^{|\alpha|-1}(U,f)\end{aligned} \tag{15.3.13}$$

$$[\partial^\alpha(p^1 - p^2)] = -\sigma\nabla' \cdot \partial^\alpha(\hat{n}^1 - \hat{n}^2) \tag{15.3.14}$$

on $(0,T^0) \times \Gamma$.

15.3.4 The Difference of Derivatives of the Unit Normal

Lemma 15.3.4. *Let* $1 \leq |\alpha| \leq 2$ *with* $\alpha_3 = 0$. *Then*

$$\partial^\alpha(\frac{\nabla' f^1}{|n^1|} - \frac{\nabla' f^2}{|n^2|}) = \frac{\partial^\alpha \nabla'(f^1 - f^2)}{|n^1|} - \frac{(\nabla' f^1 \cdot \partial^\alpha \nabla'(f^1 - f^2))\nabla' f^1}{|n^1|^3} + R^{|\alpha|-1}_{\Delta,\Gamma}(f)$$

where the remainder $R^{|\alpha|-1}_{\Delta,\Gamma}(f)$ *is defined in 15.1.3.*

Proof. We apply lemma 8.6.4 to obtain

$$\begin{aligned}
\partial^\alpha(\frac{\nabla' f^1}{|n^1|} - \frac{\nabla' f^2}{|n^2|}) &= \frac{\partial^\alpha \nabla' f^1}{|n^1|} - \frac{(\nabla' f^1 \cdot \partial^\alpha \nabla' f^1)\nabla' f^1}{|n^1|^3} + R^{|\alpha|-1}_{\Gamma}(f^1) \\
&- (\frac{\partial^\alpha \nabla' f^2}{|n^2|} - \frac{(\nabla' f^2 \cdot \partial^\alpha \nabla' f^2)\nabla' f^2}{|n^2|^3} + R^{|\alpha|-1}_{\Gamma}(f^2)) \\
&= \frac{\partial^\alpha \nabla'(f^1 - f^2)}{|n^1|} - \frac{(\nabla' f^1 \cdot \partial^\alpha \nabla'(f^1 - f^2))\nabla' f^1}{|n^1|^3} + R^{|\alpha|-1}_{\Delta,\Gamma}(f)
\end{aligned}$$

as required. □

15.4 A Slightly Lower Order Difference Estimate for the Front

Let $|\alpha| = 2$ with $\alpha_3 = 0$. In this section we wish to estimate $\partial^\alpha(f^1 - f^2)$ as opposed to $\partial^\alpha \nabla'(f^1 - f^2)$.

Suppose $\partial^\alpha = \partial_{x_i}\partial^\beta$ for $|\beta| = 1$. Then in fact

$$\begin{aligned}
\left|\left|\partial^\alpha(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} &\leq \left|\left|\partial^\beta \nabla'(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} \\
&\leq C E^0_\Delta + C\int_0^t E_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the lower order estimate (15.2.2).

Otherwise, $\partial^\alpha = \partial_t \partial^\beta$ for $|\beta| = 1$. Then the equation (15.3.13) implies

$$\partial_t \partial^\beta(f^1 - f^2) = \partial^\beta(u^1_\pm - u^2_\pm) \cdot n^1 - (u^1_\pm)' \cdot \nabla' \partial^\beta(f^1 - f^2) + R^0_{\Delta,\Gamma}(U, f)$$

We now proceed analogously to 8.7 to obtain eventually

$$\sum_{|\alpha|=2} \left|\left|\partial^\alpha(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon K(t)E_\Delta(t) \tag{15.4.1}$$

15.5 Difference Estimate of the Entropy

We multiply equation (15.3.12) by $s^{\alpha,1} - s^{\alpha,2}$, where $|\alpha| = 2$, and use the product rule to obtain

$$\frac{1}{2}\partial_t \left|s^{\alpha,1} - s^{\alpha,2}\right|^2 + \frac{1}{2}\nabla \cdot ((u^\psi)^1 \left|s^{\alpha,2} - s^{\alpha,2}\right|^2) = \tilde{R}^{|\alpha|}_\Delta(U, \psi)(s^{\alpha,1} - s^{\alpha,2})$$

in $(0, T^0) \times \Omega_\pm$. Note that we have absorbed the lower-order term generated by the product rule into the remainder on the right hand side. Then we integrate over $\Omega_\pm$ and apply the divergence theorem for Sobolev functions to the second term on the left. Note that since $(u^\psi)^1_{3\pm} = 0$ on Γ we are left with no boundary terms. Hence, using the remainder estimate (15.1.4), together with the estimate (15.3.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\left|s^{\alpha,1}-s^{\alpha,2}\right|^2\mathrm{d}x\leq K(t)E_\Delta(t)$$

Integrating from 0 to t, we obtain

$$\left|\left|s^{\alpha,1}-s^{\alpha,2}\right|\right|^2_{L^2(\Omega_\pm)}\leq K(t)E^0_\Delta+K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau \tag{15.5.1}$$

Applying the estimate (15.3.7) together with the estimate (15.4.1) we obtain

$$\left|\left|\partial^\alpha(s^1-s^2)\right|\right|^2_{L^2(\Omega_\pm)}\leq K(t)E^0_\Delta+\frac{1}{\epsilon}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau+\epsilon K(t)E_\Delta(t) \tag{15.5.2}$$

15.6 Difference Estimate of the Tangential Derivatives of the Pressure, Velocity and Front

Let α be a multi-index with $|\alpha|=2$ and $\alpha_3=0$, so that ∂^α contains no normal derivatives. Assume also that $\alpha_0\leq 1$, so that we may, if we wish, write $\partial^\alpha=\partial_{x_i}\partial^\beta$ for some $1\leq i\leq 2$ and $|\beta|=1$.

We multiply (15.3.4) by $\partial^\alpha(p^1-p^2)$ and (15.3.5) by $\partial^\alpha(u^1-u^2)$. We then add the resulting equations to obtain

$$\begin{aligned}&(\frac{1}{\rho^1(c^1)^2}(\partial_t+(u^\psi)^1\cdot\nabla)\partial^\alpha(p^1-p^2))\partial^\alpha(p^1-p^2)+(\rho^1(\partial_t+(u^\psi)^1\cdot\nabla)\partial^\alpha(u^1-u^2))\cdot\partial^\alpha(u^1-u^2)\\&+(\nabla^{\psi^1}\cdot\partial^\alpha(u^1-u^2))\partial^\alpha(p^1-p^2)+(\nabla^{\psi^1}\partial^\alpha(p^1-p^2))\cdot\partial^\alpha(u^1-u^2)\\&=R^{|\alpha|}_\Delta(U,\psi)\partial^\alpha(U^1-U^2)\end{aligned}$$

in $(0,T^0)\times\Omega_\pm$. Now we use the product rule to obtain

$$\begin{aligned}&\frac{1}{2}\partial_t(\frac{1}{\rho^1(c^1)^2}\left|\partial^\alpha(p^1-p^2)\right|^2)+\frac{1}{2}\partial_t(\rho^1\left|\partial^\alpha(u^1-u^2)\right|^2)+\frac{1}{2}\nabla\cdot(\frac{1}{\rho^1(c^1)^2}(u^\psi)^1\left|\partial^\alpha(p^1-p^2)\right|^2)\\&+\frac{1}{2}\nabla\cdot(\rho^1(u^\psi)^1\left|\partial^\alpha(u^1-u^2)\right|^2)\\&+\nabla'\cdot(\partial^\alpha(p^1-p^2)\partial^\alpha(u^1-u^2))+\partial_{x_3}(\frac{(1,-\nabla'\psi^1)}{J\psi^1}\partial^\alpha(p^1-p^2)\cdot\partial^\alpha(u^1-u^2))\\&=R^{|\alpha|}_\Delta(U,\psi)\partial^\alpha(U^1-U^2)\end{aligned}$$

where we have absorbed some lower order terms into the remainder on the right hand side. Now we integrate over $\Omega_\pm$ and apply the divergence theorem. The terms which are tangential divergences vanish, and $(u^\psi)^1_{3\pm}=0$ on Γ hence the third and fourth terms vanish. We obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\frac{1}{\rho^1(c^1)^2}\left|\partial^\alpha(p^1-p^2)\right|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\rho^1\left|\partial^\alpha(u^1-u^2)\right|^2\mathrm{d}x\\&\mp 2\int_\Gamma\partial^\alpha(p^1-p^2)\partial^\alpha(u^1-u^2)\cdot n^1\mathrm{d}x'\\&=\int_{\Omega_\pm}R^{|\alpha|}_\Delta(U,\psi)\partial^\alpha(U^1-U^2)\mathrm{d}x\end{aligned}$$

We integrate in time from 0 to t, apply Cauchy's inequality and the remainder estimate (15.1.4) for the remainder term on the right hand side. Hence we obtain

$$\begin{aligned}
&\int_{\Omega_\pm} \frac{1}{\rho^1 (c^1)^2} \left|\partial^\alpha (p^1 - p^2)\right|^2 \mathrm{d}x + \int_{\Omega_\pm} \rho^1 \left|\partial^\alpha (u^1 - u^2)\right|^2 \mathrm{d}x \\
&\mp 2 \int_0^t \int_\Gamma \partial^\alpha (p^1 - p^2) \partial^\alpha (u^1 - u^2) \cdot n^1 \mathrm{d}x' \mathrm{d}\tau \\
&\leq K(t) E_\Delta^0 + K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau
\end{aligned}$$

We sum over $\pm$ to obtain a jump condition. Doing this we obtain

$$\begin{aligned}
&\sum_\pm \int_{\Omega_\pm} \frac{1}{\rho^1 (c^1)^2} \left|\partial^\alpha (p^1 - p^2)\right|^2 \mathrm{d}x + \sum_\pm \int_{\Omega_\pm} \rho^1 \left|\partial^\alpha (u^1 - u^2)\right|^2 \mathrm{d}x - 2 I_\Delta^\alpha \\
&\leq K(t) E_\Delta^0 + K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau
\end{aligned}$$

where

$$I_\Delta^\alpha = \int_0^t \int_\Gamma [\partial^\alpha (p^1 - p^2) \partial^\alpha (u^1 - u^2) \cdot n^1] \mathrm{d}x' \mathrm{d}\tau$$

To estimate I_Δ^α, firstly, we use the differentiated equation for the front, (15.3.13), to replace $n^1 \cdot \partial^\alpha (u^1 - u^2)$, and we obtain the following.

$$\begin{aligned}
I_\Delta^\alpha &= \int_0^t \int_\Gamma \partial_t \partial^\alpha (f^1 - f^2) [\partial^\alpha (p^1 - p^2)] \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_\Gamma [(u^1)' \cdot \partial^\alpha \nabla' (f^1 - f^2) \partial^\alpha (p^1 - p^2)] \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma [R_{\Delta,\Gamma}^{|\alpha|-1}(U, f) \partial^\alpha (p^1 - p^2)] \mathrm{d}x' \mathrm{d}\tau \\
&= I_{\Delta,1}^\alpha + I_{\Delta,2}^\alpha + I_{\Delta,3}^\alpha
\end{aligned}$$

Firstly, we note that, using the remainder estimate (15.1.10), we have

$$\left|I_{\Delta,3}^\alpha\right| \leq K(t) E_\Delta^0 + \frac{1}{\epsilon} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon K(t) E_\Delta(t)$$

Let us consider $I_{\Delta,1}^\alpha$. Using the differentiated pressure interface condition (15.3.14) for $p^1 - p^2$, we have

$$\begin{aligned}
I_{\Delta,1}^\alpha &= -\sigma \int_0^t \int_\Gamma \partial_t \partial^\alpha (f^1 - f^2) \partial^\alpha \nabla' \cdot (\hat{n}^1 - \hat{n}^2) \mathrm{d}x' \mathrm{d}\tau \\
&= \sigma \int_0^t \int_\Gamma \partial_t \partial^\alpha \nabla' (f^1 - f^2) \cdot \partial^\alpha \left(\frac{\nabla' f^1}{|n^1|} - \frac{\nabla' f^2}{|n^2|}\right) \mathrm{d}x' \mathrm{d}\tau
\end{aligned}$$

where we have used integration by parts.

Using the lemma 15.3.4, we have

$$
\begin{aligned}
& -I^{\alpha}_{\Delta,1} \\
&= \sigma \int_0^t \int_\Gamma \partial_t \partial^\alpha \nabla'(f^1 - f^2) \cdot \frac{\partial^\alpha \nabla'(f^1 - f^2)}{|n^1|} \mathrm{d}x' \mathrm{d}\tau \\
&- \sigma \int_0^t \int_\Gamma \partial_t (\nabla' f^1 \cdot \partial^\alpha \nabla'(f^1 - f^2)) \frac{(\nabla' f^1 \cdot \partial^\alpha \nabla'(f^1 - f^2))}{|n^1|^3} \mathrm{d}x' \mathrm{d}\tau \\
&+ \sigma \int_0^t \int_\Gamma \partial_t \partial^\alpha \nabla'(f^1 - f^2) \cdot R^{|\alpha|-1}_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau + \sigma \int_0^t \int_\Gamma \partial^\alpha \nabla'(f^1 - f^2) \cdot R^{|\alpha|}_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \\
&= \frac{1}{2} \sigma \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \frac{1}{|n^1|} \left| \partial^\alpha \nabla'(f^1 - f^2) \right|^2 - \frac{1}{|n^1|^3} \left| \nabla' f^1 \cdot \partial^\alpha \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' \mathrm{d}\tau \\
&- \sigma \int_0^t \int_\Gamma \partial_t \partial^\beta \nabla'(f^1 - f^2) \cdot \partial_{x_i} R^{|\alpha|-1}_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau + \sigma \int_0^t \int_\Gamma R^{|\alpha|}_{\Delta,\Gamma}(f) \cdot R^{|\alpha|}_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \\
&\geq \frac{1}{2} \sigma \int_\Gamma \frac{1}{|n^1|} \left| \partial^\alpha \nabla'(f^1 - f^2) \right|^2 - \frac{1}{|n^1|^3} \left| \nabla' f^1 \cdot \partial^\alpha \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' \\
&- \sigma \int_0^t \left|\left| \partial_t \partial^\beta (f^1 - f^2) \right|\right|_{H^1(\Gamma)} \left|\left| R^{|\alpha|-1}_{\Delta,\Gamma}(f) \right|\right|_{H^1(\Gamma)} + \left|\left| R^{|\alpha|}_{\Delta,\Gamma}(f) \right|\right|^2_{L^2(\Gamma)} - K(t) E^0_\Delta
\end{aligned}
$$

where we have used the fact that $\partial^\alpha = \partial_{x_i} \partial^\beta$ together with integration by parts. We now apply lemma 8.6.5 to the first term on the right hand side.

Hence

$$
-I^{\alpha}_{\Delta,1} \geq \frac{1}{2} \sigma \int_\Gamma \frac{1}{|n^1|^3} \left| \partial^\alpha \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' - K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau - K(t) E^0_\Delta
$$

where we have used the remainder estimate (15.1.6).

Now we proceed to estimate $I^{\alpha}_{\Delta,2}$. We have

$$
\begin{aligned}
\left| I^{\alpha}_{\Delta,2} \right| &\leq \sum_\pm \left| \int_0^t \int_\Gamma (u^1_\pm)' \cdot \partial^\alpha \nabla'(f^1 - f^2) \partial^\alpha (p^1_\pm - p^2_\pm) \mathrm{d}x' \mathrm{d}\tau \right| \\
&= \sum_\pm \left| \int_0^t \int_\Gamma (u^1_\pm)' \cdot \partial_{x_i} \partial^\beta \nabla'(f^1 - f^2) \partial_{x_i} \partial^\beta (p^1_\pm - p^2_\pm) \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq \sum_\pm \int_0^t \left|\left| (u^1_\pm)' \cdot \partial_{x_i} \partial^\beta \nabla'(f^1 - f^2) \right|\right|_{H^{0.5}(\Gamma)} \left|\left| \partial_{x_i} \partial^\beta (p^1_\pm - p^2_\pm) \right|\right|_{H^{-0.5}(\Gamma)} \mathrm{d}\tau \\
&\leq \sum_\pm \int_0^t ||(u^1_\pm)||_{W^{1,\infty}(\Gamma)} \left|\left| \partial_{x_i} \partial^\beta \nabla'(f^1 - f^2) \right|\right|_{H^{0.5}(\Gamma)} \left|\left| \partial^\beta (p^1_\pm - p^2_\pm) \right|\right|_{H^{0.5}(\Gamma)} \mathrm{d}\tau \\
&\leq \sum_\pm \int_0^t ||(u^1_\pm)||^2_{W^{1,\infty}(\Gamma)} \left|\left| \partial_{x_i} \partial^\beta \nabla'(f^1 - f^2) \right|\right|^2_{H^{0.5}(\Gamma)} \mathrm{d}t + \sum_\pm \int_0^t \left|\left| \partial^\beta (p^1_\pm - p^2_\pm) \right|\right|^2_{H^{0.5}(\Gamma)} \mathrm{d}\tau \\
&\leq K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau
\end{aligned}
$$

where we have again used the Sobolev trace estimate.

Putting what we have so far together using the inequality (15.0.2), we have

$$
\begin{aligned}
& \sum_\pm ||\partial^\alpha (p^1 - p^2)||^2_{L^2(\Omega_\pm)} + \sum_\pm ||\partial^\alpha (u^1 - u^2)||^2_{L^2(\Omega_\pm)} + ||\partial^\alpha \nabla'(f^1 - f^2)||^2_{L^2(\Gamma)} \\
& \leq K(t) E^0_\Delta + \frac{1}{\epsilon} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon K(t) E_\Delta(t)
\end{aligned}
\tag{15.6.1}
$$

15.7 Difference Estimate of the Time Derivatives of the Corrected Pressure and Velocity and the Front

We let β be a multi-index with $|\beta| = 1$, $\beta_3 = 0$. Recall the operator $D_t^{u_-^1}$ is defined as $D_t^{u_-^1} := \partial_t + (u_-^\psi)^1 \cdot \nabla$ (with u_-^1 defined on Ω_+ by Sobolev extension).

We set $\alpha = \beta$ in the equations (15.3.10)-(15.3.11) then apply the operator $D_t^{u_-^1}$ to obtain the equations

$$\frac{1}{\rho^1(c^1)^2}(\partial_t + (u^\psi)^1 \cdot \nabla) D_t^{u_-^1}(p^{\beta,1} - p^{\alpha,2}) + \nabla^{\psi^1} \cdot D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2}) = \tilde{R}_\Delta^2(U,\psi)$$
$$\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla) D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2}) + \nabla^{\psi^1} D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2}) = \tilde{R}_\Delta^2(U,\psi)$$

in $(0,T^0) \times \Omega_\pm$. We now proceed as for the tangential derivatives. We multiply the above equations by $D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2})$ and $D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2})$ respectively, then add them together. Integrating over $\Omega_\pm$ we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \frac{1}{\rho^1(c^1)^2}\left|D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2})\right|^2 \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \rho^1 \left|D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2})\right|^2 \mathrm{d}x \\
&\mp 2\int_\Gamma D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2}) D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2}) \cdot n^1 \mathrm{d}x' \\
&= \int_{\Omega_\pm} \tilde{R}_\Delta^2(U,\psi) D_t^{u_-^1}(U^{\beta,1} - U^{\beta,2}) \mathrm{d}x
\end{aligned}$$

Integrating from 0 to t and summing over $\pm$, we obtain

$$\begin{aligned}
&\sum_\pm \int_{\Omega_\pm} \frac{1}{\rho^1(c^1)^2}\left|D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2})\right|^2 \mathrm{d}x + \sum_\pm \int_{\Omega_\pm} \rho^1 \left|D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2})\right|^2 \mathrm{d}x - 2K_\Delta^\beta \\
&\leq K(t)E_\Delta^0 + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

where we have used the remainder estimate (15.1.11) and we have defined

$$K_\Delta^\beta = \int_0^t \int_\Gamma [D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2}) D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2}) \cdot n^1] \mathrm{d}x' \mathrm{d}\tau$$

We must write out $[D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2}) D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2}) \cdot n^1]$ in terms of the quantities p and u for which we have boundary conditions.

$$K_\Delta^\beta = \int_0^t \int_\Gamma [D_t^{u_-^1}(p^{\beta,1} - p^{\beta,2}) D_t^{u_-^1}(u^{\beta,1} - u^{\beta,2}) \cdot n^1] \mathrm{d}x' \mathrm{d}\tau = K_1^\beta - K_2^\beta - K_3^\beta + K_4^\beta$$

where

$$\begin{aligned}
K_{\Delta,1}^\beta &= \int_0^t \int_\Gamma [n^1 \cdot D_t^{u_-^1} \partial^\beta (u^1 - u^2) D_t^{u_-^1} \partial^\beta (p^1 - p^2)] \mathrm{d}x' \mathrm{d}\tau \\
K_{\Delta,2}^\beta &= \int_0^t \int_\Gamma [n^1 \cdot D_t^{u_-^1} \partial^\beta (u^1 - u^2) D_t^{u_-^1} (\partial^\beta (f^1 - f^2) \partial_3 p^1)] \mathrm{d}x' \mathrm{d}\tau \\
K_{\Delta,3}^\beta &= \int_0^t \int_\Gamma [n^1 \cdot D_t^{u_-^1} \partial^\beta (p^1 - p^2) D_t^{u_-^1} (\partial^\beta (f^1 - f^2) \partial_3 u^1)] \mathrm{d}x' \mathrm{d}\tau \\
K_{\Delta,4}^\beta &= \int_0^t \int_\Gamma [n^1 \cdot D_t^{u_-^1} (\partial^\beta (f^1 - f^2) \partial_3 p^1) D_t^{u_-^1} (\partial^\beta (f^1 - f^2) \partial_3 u^1)] \mathrm{d}x' \mathrm{d}\tau
\end{aligned}$$

Applying the remainder estimates (15.1.5) and (15.1.10) we may estimate $K^{\beta}_{\Delta,2}$, $K^{\beta}_{\Delta,3}$ and $K^{\beta}_{\Delta,4}$ analogously to K^{β}_{2}, K^{β}_{3} and K^{β}_{4} as in 8.10 to obtain

$$\left|K^{\beta}_{\Delta,4}\right| + \left|K^{\beta}_{\Delta,2}\right| + \left|K^{\beta}_{\Delta,3}\right| \leq K(t)E^{0}_{\Delta} + \frac{1}{\epsilon}K(t)\int_{0}^{t} E_{\Delta}(\tau)\mathrm{d}\tau + \epsilon K(t)E_{\Delta}(t)$$

We now consider $K^{\beta}_{\Delta,1}$. We use the differentiated equation for the front, (15.3.13), to replace $n^{1} \cdot D_{t}^{u^{1}_{-}}\partial^{\beta}(u^{1}-u^{2})$, and we obtain the following.

$$\begin{aligned}
K^{\beta}_{\Delta,1} &= \int_{0}^{t}\int_{\Gamma}[n^{1}\cdot D_{t}^{u^{1}_{-}}\partial^{\beta}(u^{1}-u^{2})D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&= \int_{0}^{t}\int_{\Gamma}\partial_{t}D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})[D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{\Gamma}[u'\cdot D_{t}^{u_{-}}\partial^{\beta}\nabla'(f^{1}-f^{2})D_{t}^{u_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{\Gamma}[R^{1}_{\Delta,\Gamma}(U,f)D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&= \int_{0}^{t}\int_{\Gamma}\partial_{t}D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})[D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{\Gamma}((u^{1}_{+})'\cdot\nabla')D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})[D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{\Gamma}([(u^{1})']\cdot\nabla')D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}_{-}-p^{2}_{-})\mathrm{d}x'\mathrm{d}\tau \\
&+ \int_{0}^{t}\int_{\Gamma}[R^{1}_{\Delta,\Gamma}(U,f)D_{t}^{u^{1}_{-}}\partial^{\beta}(p^{1}-p^{2})]\mathrm{d}x'\mathrm{d}\tau \\
&= K^{\beta}_{\Delta,11} + K^{\beta}_{\Delta,12} + K^{\beta}_{\Delta,13} + K^{\beta}_{\Delta,14}
\end{aligned}$$

Applying the remainder estimate (15.1.10), we have

$$\left|K^{\beta}_{\Delta,14}\right| \leq K(t)E^{0}_{\Delta} + \frac{1}{\epsilon}K(t)\int_{0}^{t} E_{\Delta}(\tau)\mathrm{d}\tau + \epsilon K(t)E_{\Delta}(t)$$

The estimate of $K^{\beta}_{\Delta,11}$ is similar to the estimate of $I^{\alpha}_{\Delta,1}$. Using the pressure interface condition (15.3.14) and integrating by parts, we have

$$\begin{aligned}
K^{\beta}_{\Delta,11} &= -\sigma\int_{0}^{t}\int_{\Gamma}\partial_{t}D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})D_{t}^{u^{1}_{-}}\partial^{\beta}\nabla'\cdot((\hat{n})^{1}-(\hat{n})^{2})\mathrm{d}x'\mathrm{d}\tau \\
&= \sigma\int_{0}^{t}\int_{\Gamma}\partial_{t}D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})\nabla'\cdot D_{t}^{u^{1}_{-}}\partial^{\beta}(\frac{\nabla' f^{1}}{|n^{1}|}-\frac{\nabla' f^{2}}{|n^{2}|})\mathrm{d}x'\mathrm{d}\tau + K^{\beta}_{\Delta,113} \\
&= -\sigma\int_{0}^{t}\int_{\Gamma}\partial_{t}D_{t}^{u^{1}_{-}}\partial^{\beta}\nabla'(f^{1}-f^{2})\cdot D_{t}^{u^{1}_{-}}\partial^{\beta}(\frac{\nabla' f^{1}}{|n^{1}|}-\frac{\nabla' f^{2}}{|n^{2}|})\mathrm{d}x'\mathrm{d}\tau \\
&K^{\beta}_{\Delta,113} + K^{\beta}_{\Delta,115}
\end{aligned}$$

where

$$\begin{aligned}
K^{\beta}_{\Delta,113} &= -\sigma\sum_{j=1}^{2}\int_{0}^{t}\int_{\Gamma}\partial_{t}D_{t}^{u^{1}_{-}}\partial^{\beta}(f^{1}-f^{2})(\partial_{x_j}(u^{1}_{-})')\cdot\nabla'\partial^{\beta}(\frac{\partial_{x_j}f^{1}}{|n^{1}|}-\frac{\partial_{x_j}f^{2}}{|n^{2}|})\mathrm{d}x'\mathrm{d}\tau \\
K^{\beta}_{\Delta,115} &= -\sigma\sum_{j=1}^{2}\int_{0}^{t}\int_{\Gamma}\partial_{t}((\partial_{x_j}(u^{1}_{-})')\cdot\nabla'\partial^{\beta}(f^{1}-f^{2}))D_{t}^{u^{1}_{-}}\partial^{\beta}(\frac{\partial_{x_j}f^{1}}{|n^{1}|}-\frac{\partial_{x_j}f^{2}}{|n^{2}|})\mathrm{d}x'\mathrm{d}\tau
\end{aligned}$$

can be thought of as lower order terms. Let us also label the first term on the right as

$$K^{\beta}_{\Delta,111} = -\sigma \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot D_t^{u^1_-} \partial^\beta (\frac{\nabla' f^1}{|n^1|} - \frac{\nabla' f^2}{|n^2|}) \mathrm{d}x' \mathrm{d}\tau$$

The lower order terms $K^{\beta}_{\Delta,113}$ and $K^{\beta}_{\Delta,115}$ are estimated analogously to K^{β}_{113} and K^{β}_{115} in 8.10, hence we merely state the relevant estimates.

$$\left|K^{\beta}_{\Delta,113}\right| \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t)$$
$$\left|K^{\beta}_{115}\right| \leq K(t) \int_0^t E_\Delta(\tau)\mathrm{d}\tau$$

To estimate $K^{\beta}_{\Delta,111}$, we use lemma 15.3.4 to obtain

$$\begin{aligned}
- K^{\beta}_{\Delta,111} &= \sigma \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot \frac{D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2)}{|n^1|} \mathrm{d}x' \mathrm{d}\tau \\
&- \sigma \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot \frac{(\nabla'(f^1 - f^2) \cdot D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2))\nabla' f^1}{|n^1|^3} \mathrm{d}x' \mathrm{d}\tau \\
&+ \sigma \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot R^0_{\infty,\Gamma}(U, f) R^1_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \\
&= \sigma \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot \frac{D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2)}{|n^1|} \mathrm{d}x \mathrm{d}\tau \\
&- \sigma \int_0^t \int_\Gamma \partial_t (\nabla' f^1 \cdot D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2)) \frac{(\nabla' f^1 \cdot D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2))}{|n^1|^3} \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot R^0_{\infty,\Gamma}(U, f) R^1_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U, f) R^2_{\Delta,\Gamma}(f) R^2_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \\
&= \frac{1}{2} \int_0^t \sigma \frac{\mathrm{d}}{\mathrm{d}t} \int_\Gamma \frac{1}{|n^1|} \left| D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \right|^2 - \frac{1}{|n^1|^3} \left| \nabla' f^1 \cdot D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot R^0_{\infty,\Gamma}(U, f) R^1_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \\
&+ \int_0^t \int_\Gamma R^1_{\infty,\Gamma}(U, f) R^2_{\Delta,\Gamma}(f) R^2_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau
\end{aligned}$$

We now apply lemma 8.6.5 to the first term on the right hand side and we apply the remainder estimates (8.18.12) and (15.1.6) to the last term. The other term is estimated by integrating by parts in time as in 8.10 so we merely state the estimate.

$$\begin{aligned}
&\left| \int_0^t \int_\Gamma \partial_t D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \cdot R^0_{\infty,\Gamma}(U, f) R^1_{\Delta,\Gamma}(f) \mathrm{d}x' \mathrm{d}\tau \right| \\
&\leq K(t)E^0_\Delta + \frac{1}{\epsilon} K(t) \int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon K(t) E_\Delta(t)
\end{aligned}$$

Thus we have eventually shown that

$$\begin{aligned}
-K^{\beta}_{\Delta,11} \geq \frac{1}{2} \sigma \int_\Gamma \frac{1}{|n^1|^3} \left| D_t^{u^1_-} \partial^\beta \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' \\
- \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t) - K(t) E^0_\Delta
\end{aligned}$$

The estimate of $K^{\beta}_{\Delta,12}$ is similar to the estimate of $K^{\beta}_{\Delta,11}$, except this time the remainder terms are easier to deal with because we have replaced a time derivative with a tangential derivative. Using the pressure interface condition (15.3.14), we have

$$\begin{aligned}
K^{\beta}_{\Delta,12} &= -\sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta(f^1-f^2)D_t^{u^1_-}\partial^\beta\nabla'\cdot((\hat{n})^1-(\hat{n})^2)\mathrm{d}x'\mathrm{d}\tau\\
&= \sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta(f^1-f^2)\nabla'\cdot D_t^{u^1_-}\partial^\beta(\frac{\nabla' f^1}{|n^1|}-\frac{\nabla' f^2}{|n^2|})\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}(f)\cdot R^2_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\\
&= -\sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\cdot D_t^{u^1_-}\partial^\beta(\frac{\nabla' f^1}{|n^1|}-\frac{\nabla' f^2}{|n^2|})\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}(f)\cdot R^2_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma (\nabla'\nabla' u^1)R^1_{\Delta,\Gamma}(f)\cdot R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\\
&= K^{\beta}_{\Delta,121}+K^{\beta}_{\Delta,123}+K^{\beta}_{\Delta,125}
\end{aligned}$$

where we have used integration by parts.

We now use lemma 15.3.4 to estimate $K^{\beta}_{\Delta,121}$.

$$\begin{aligned}
&-K^{\beta}_{\Delta,121} = \sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\cdot\frac{D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)}{|n^1|}\mathrm{d}x'\mathrm{d}\tau\\
&-\sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\cdot\frac{(\nabla' f^1\cdot D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2))\nabla' f^1}{|n^1|^3}\mathrm{d}x'\mathrm{d}\tau\\
&+\sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\cdot R^0_{\Gamma,\infty}(U,f))R^1_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\\
&=\sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\cdot\frac{D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)}{|n^1|}\mathrm{d}x\mathrm{d}\tau\\
&-\sigma\int_0^t\int_\Gamma (u^1_+\cdot\nabla')(\nabla' f^1\cdot D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2))\frac{(\nabla' f^1\cdot D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2))}{|n^1|^3}\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma (u^1_+\cdot\nabla')D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\cdot R^0_{\infty,\Gamma}(U,f)R^1_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma R^0_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}(f)\cdot R^2_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\\
&=\frac{1}{2}\sigma\int_0^t\int_\Gamma \nabla'\cdot(u^1_+(\frac{1}{|n^1|}\left|D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\right|^2-\frac{1}{|n^1|^3}\left|\nabla' f^1\cdot D_t^{u^1_-}\partial^\beta\nabla'(f^1-f^2)\right|^2))\mathrm{d}x'\mathrm{d}\tau\\
&+\int_0^t\int_\Gamma R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}\nabla'\cdot R^1_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau+\int_0^t\int_\Gamma R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}(f)\cdot R^2_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau
\end{aligned}$$

We now apply the divergence theorem and the remainder estimates (8.18.8) and (15.1.6) to obtain

$$\begin{aligned}
\left|K^{\beta}_{\Delta,121}\right| &\le \left|\int_0^t\int_\Gamma R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}\nabla'\cdot R^1_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\right|\\
&+\left|\int_0^t\int_\Gamma R^1_{\infty,\Gamma}(U,f)R^2_{\Delta,\Gamma}(f)\cdot R^2_{\Delta,\Gamma}(f)\mathrm{d}x'\mathrm{d}\tau\right|\\
&\le K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau
\end{aligned}$$

We estimate $K^\beta_{\Delta,123}$ and $K^\beta_{\Delta,125}$ analogously to K^β_{123} and K^β_{125} in 8.10 to obtain

$$\left|K^\beta_{\Delta,123}\right| + \left|K^\beta_{\Delta,125}\right| \le K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau$$

Thus we have shown that

$$\left|K^\beta_{\Delta,12}\right| \le K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau$$

We proceed to estimate $K^\beta_{\Delta,13}$. Note that by rearranging the differentiated equation (15.3.13) we obtain

$$D_t^{u^1_-}\partial^\alpha(f^1-f^2) = \partial^\alpha(u^1_- - u^2_-)\cdot n^1 + R^{|\alpha|-1}_{\Delta,\Gamma}(U,f) \tag{15.7.1}$$

Using the equation (15.7.1), we have

$$\begin{aligned} K^\beta_{\Delta,13} &= \int_0^t\int_\Gamma n^1\cdot([(u^1)']\cdot\nabla')\partial^\beta(u^1_- - u^2_-)D_t^{u^1_-}\partial^\beta(p^1_- - p^2_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma R^1_{\Delta,\Gamma}(U,f)\cdot\partial\partial^\beta(p^1_- - p^2_-)\mathrm{d}x'\mathrm{d}\tau \end{aligned}$$

We want to convert this into an integral over the interior Ω_-. First of all we need to go back to using the corrected pressure and velocity. We have

$$\begin{aligned} K^\beta_{\Delta,13} &= \int_0^t\int_\Gamma n^1\cdot([(u^1)']\cdot\nabla')(u^{\beta,1}_- - u^{\beta,2}_-)D_t^{u^1_-}(p^{\beta,1}_- - p^{\beta,2}_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma n^1\cdot([(u^1)']\cdot\nabla')(\partial^\beta(f^1-f^2)\partial_{x_3}u^1_-)D_t^{u^1_-}\partial^\beta(p^1_- - p^2_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma n^1\cdot([(u^1)']\cdot\nabla')\partial^\beta(u^1_- - u^2_-)D_t^{u^1_-}(\partial^\beta(f^1-f^2)\partial_{x_3}p^1_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma n^1\cdot([(u^1)']\cdot\nabla')(\partial^\beta(f^1-f^2)\partial_{x_3}u^1_-)D_t^{u^1_-}(\partial^\beta(f^1-f^2)\partial_{x_3}p^1_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma R^1_{\Delta,\Gamma}(U,f)\cdot\partial\partial^\beta(p^1_- - p^2_-)\mathrm{d}x'\mathrm{d}\tau \\ &= \int_0^t\int_\Gamma n^1\cdot([(u^1)']\cdot\nabla')(u^{\beta,1}_- - u^{\beta,2}_-)D_t^{u^1_-}(p^{\beta,1}_- - p^{\beta,2}_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma R^1_{\Delta,\Gamma}(U,f)\cdot\partial\partial^\beta(p^1_- - p^2_-)\mathrm{d}x'\mathrm{d}\tau \\ &+ \int_0^t\int_\Gamma \nabla'\partial^\beta(u^1_- - u^2)\cdot R^1_{\Delta,\Gamma}(U,f)\mathrm{d}x'\mathrm{d}\tau + \int_0^t\int_\Gamma R^1_{\Delta,\Gamma}(U,f)\cdot R^1_{\Delta,\Gamma}(U,f)\mathrm{d}x'\mathrm{d}\tau \\ &= K^\beta_{\Delta,131} + K^\beta_{\Delta,132} + K^\beta_{\Delta,133} + K^\beta_{\Delta,134} \end{aligned}$$

Using the remainder estimates (15.1.5) and (15.1.10), we have

$$\left|K^\beta_{\Delta,132} + K^\beta_{\Delta,133} + K^\beta_{\Delta,134}\right| \le K(t)E^0_\Delta + \frac{1}{\epsilon}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon K(t)E_\Delta(t)$$

To estimate $K^\beta_{\Delta,131}$ we convert the boundary integral into an integral over the interior Ω_- as for K^β_{131}

in 8.10. We obtain

$$
\begin{aligned}
K^{\beta}_{\Delta,131} &= \int_0^t \int_{\Omega_-} \nabla' \cdot (([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2})) \\
&+ \partial_{x_3}(\frac{1}{J\psi}(-\nabla'\psi, 1) \cdot ([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2}) \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \int_{\Omega_-} \nabla' \cdot (([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2})) \\
&+ \frac{(-\nabla'\psi^1, 1)}{J\psi^1} \cdot \partial_{x_3}(([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2})) \mathrm{d}x\mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} \tilde{R}^2_{\Delta}(U,\psi) \cdot \tilde{R}^2_{\Delta}(U,\psi) \mathrm{d}x\mathrm{d}\tau \\
&= \int_0^t \int_{\Omega_-} ([(u^1)'] \cdot \nabla')(\nabla' \cdot (u^{\beta,1} - u^{\beta,2}) + \frac{(-\nabla'\psi^1, 1)}{J\psi^1} \cdot \partial_{x_3}(u^{\beta,1} - u^{\beta,2})) D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2}) \mathrm{d}x\mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} ([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t (\nabla'(p^{\beta,1} - p^{\beta,2}) + \frac{(-\nabla'\psi^1, 1)}{J\psi^1} \partial_{x_3}(p^{\beta,1} - p^{\beta,2}) \mathrm{d}x\mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} \tilde{R}^2_{\Delta}(U,\psi) \cdot \tilde{R}^2_{\Delta}(U,\psi) \mathrm{d}x\mathrm{d}\tau \\
&= -\int_0^t \int_{\Omega_-} \frac{1}{\rho^1 (c^1)^2} ([(u^1)'] \cdot \nabla') D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2}) D^{u^1_-}_t (p^{\beta,1} - p^{\beta,2}) \mathrm{d}x\mathrm{d}\tau \\
&- \int_0^t \int_{\Omega_-} \rho^1 ([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2}) \mathrm{d}x\mathrm{d}\tau \\
&+ \int_0^t \int_{\Omega_-} \tilde{R}^2_{\Delta}(U,\psi) \cdot \tilde{R}^2_{\Delta}(U,\psi) \mathrm{d}x\mathrm{d}\tau \\
&= K^{\beta}_{\Delta,1311} + K^{\beta}_{\Delta,1313} + K^{\beta}_{\Delta,1314}
\end{aligned}
$$

where we have used the corrected pressure and velocity equations (15.3.10)-(15.3.11) in the second to last line.

To estimate $K^{\beta}_{\Delta,1313}$, we use integration by parts in time and space and the divergence theorem.

$$
\begin{aligned}
\left| K^{\beta}_{\Delta,1313} \right| &\leq \left| \int_0^t \int_{\Omega_-} D^{u^1_-}_t (\rho^1 ([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2})) \mathrm{d}x\mathrm{d}\tau \right| \\
&+ \left| \int_0^t \int_{\Omega_-} \rho^1 ([(u^1)'] \cdot \nabla') D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2}) \mathrm{d}x\mathrm{d}\tau \right| \\
&+ \left| \int_0^t \int_{\Omega_-} \tilde{R}^2_{\Delta}(U,\psi) \cdot \tilde{R}^2_{\Delta}(U,\psi) \mathrm{d}x\mathrm{d}\tau \right| \\
&\leq \left| \int_{\Omega_-} \rho^1 ([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2}) \mathrm{d}x \Big|_0^t \right| \\
&+ \frac{1}{2} \left| \int_0^t \int_{\Omega_-} \nabla \cdot ((u^{\psi}_-)^1 \rho^1 ([(u^1)'] \cdot \nabla')(u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2})) \mathrm{d}x\mathrm{d}\tau \right| \\
&+ \frac{1}{2} \left| \int_0^t \int_{\Omega_-} \nabla' \cdot (\rho^1 [(u^1)'] D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2}) \cdot D^{u^1_-}_t (u^{\beta,1} - u^{\beta,2})) \mathrm{d}x\mathrm{d}\tau \right| \\
&+ \left| \int_0^t \int_{\Omega_-} \tilde{R}^2_{\Delta}(U,\psi) \cdot \tilde{R}^2_{\Delta}(U,\psi) \mathrm{d}x\mathrm{d}\tau \right| \\
&\leq \left| \int_{\Omega_-} R^0_{\infty}(U,\psi) \nabla'(u^{\beta,1} - u^{\beta,2}) \cdot \partial(u^{\beta,1} - u^{\beta,2}) \mathrm{d}x \Big|_0^t \right| + K(t) \int_0^t E_{\Delta}(\tau) \mathrm{d}\tau
\end{aligned}
$$

where we have used the remainder estimate (15.1.11) and the fact that $(u_-^\psi)_3^1|_\Gamma = 0$. We now apply Cauchy's lemma to the first term on the right, followed by the estimate we already have for the tangential derivative $\nabla'(u^{\beta,1} - u^{\beta,2})$ given by (15.6.1) together with (15.3.9) to obtain

$$\begin{aligned}\left|K^\beta_{\Delta,1313}\right| &\leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{1}{2}}}\left|\left|R^0_\infty(U,\psi)\right|\right|^2_{L^\infty(\Omega_-)}\left|\left|\nabla'(u^{\beta,1} - u^{\beta,2})\right|\right|^2_{L^2(\Omega_-)} \\ &\quad + \epsilon^{\frac{1}{2}}\left|\left|\partial(u^{\beta,1} - u^{\beta,2})\right|\right|^2_{L^2(\Omega_-)} + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau \\ &\leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)\end{aligned}$$

The remainder estimate (15.1.11) implies

$$\left|K^\beta_{\Delta,1314}\right| \leq K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau$$

Using the divergence theorem and the remainder estimate (15.1.11), we have

$$\begin{aligned}\left|K^\beta_{\Delta,1311}\right| &\leq \frac{1}{2}\left|\int_0^t\int_{\Omega_-}\nabla'\cdot(\frac{1}{\rho^1(c^1)^2}[(u^1)']D_t^{u^1_-}(p^{\beta,1} - p^{\beta,2})D_t^{u^1_-}(p^{\beta,1} - p^{\beta,2}))\mathrm{d}x\mathrm{d}\tau\right| \\ &\quad + \left|\int_0^t\int_{\Omega_-}\tilde{R}^2_\Delta(U,\psi)\cdot\tilde{R}^2_\Delta(U,\psi)\mathrm{d}x\mathrm{d}\tau\right| \\ &= \left|\int_0^t\int_{\Omega_-}\tilde{R}^2_\Delta(U,\psi)\cdot\tilde{R}^2_\Delta(U,\psi)\mathrm{d}x\mathrm{d}\tau\right| \\ &\leq K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau\end{aligned}$$

Thus we have eventually shown that

$$\begin{aligned}-K^\beta_\Delta &\geq \frac{1}{2}\sigma\int_\Gamma\frac{1}{|n^1|^3}\left|D_t^{u^1_-}\partial^\beta\nabla'(f^1 - f^2)\right|^2\mathrm{d}x' \\ &\quad - \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau - \epsilon^{\frac{1}{2}}K(t)E_\Delta(t) - K(t)E^0_\Delta\end{aligned}$$

Hence, using the inequality (15.0.2), we have

$$\begin{aligned}&\sum_\pm\left|\left|D_t^{u^1_-}(p^{\beta,1} - p^{\beta,2})\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm\left|\left|D_t^{u^1_-}(u^{\beta,1} - u^{\beta,2})\right|\right|^2_{L^2(\Omega_\pm)} \\ &+ \left|\left|D_t^{u^1_-}\partial^\beta\nabla'(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} \\ &\leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)\end{aligned}\tag{15.7.2}$$

In particular, combining this with the tangential estimate for $\left|\left|\nabla'\partial^\beta\nabla'(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)}$, we obtain

$$\left|\left|\partial\partial^\beta\nabla'(f^1 - f^2)\right|\right|^2_{L^2(\Gamma)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)\tag{15.7.3}$$

Finally, we convert the estimate (15.7.2) into an estimate for the non-corrected pressure and velocity using the estimate (15.3.8) together with the estimate (15.7.3) for the front. We obtain

$$\sum_\pm\left|\left|D_t^{u^1_-}\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm\left|\left|D_t^{u^1_-}\partial^\beta(u^1 - u^2)\right|\right|^2_{L^2(\Omega_\pm)}\tag{15.7.4}$$

$$\leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)\tag{15.7.5}$$

15.8 A Weighted Normal Derivative Difference Estimate

Let w be given by either $w = \mathrm{Ext}_{\Omega_-}(u^\psi)^1_{3+}$ in $\mathbb{R}^3$ or $w = \mathrm{Ext}_{\Omega_+}(u^\psi)^1_{3-}$ in $\mathbb{R}^3$ or $w_\pm = (u^\psi)^1_{3\pm}$ in $\Omega_\pm$.

We let β be a multi-index with $|\beta| = 1$. We let $\partial^\alpha = \partial_{x_3}\partial^\beta$ in the differentiated equations (15.3.4)-(8.6.5). We then multiply by w to obtain the following.

$$\frac{1}{\rho^1(c^1)^2}(\partial_t + (u^\psi)^1 \cdot \nabla)((w\partial^\alpha(p^1 - p^2)) + \nabla^{\psi^1} \cdot ((w\partial^\alpha(u^1 - u^2)) = R^{|\alpha|}_\Delta(U,\psi)$$
$$\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla)((w\partial^\alpha(u^1 - u^2)) + \nabla^{\psi^1}((w\partial^\alpha(p^1 - p^2)) = R^{|\alpha|}_\Delta(U,\psi)$$

We multiply the first equation by $w\partial^\alpha(p^1 - p^2)$ and the second by $w\partial^\alpha(u^1 - u^2)$ then sum the equations to obtain the following.

$$\begin{aligned}
&\frac{1}{2}\partial_t(\frac{1}{\rho^1(c^1)^2}\left|w\partial^\alpha(p^1 - p^2)\right|^2) + \frac{1}{2}\partial_t(\rho^1\left|w\partial^\alpha(u^1 - u^2)\right|^2)\\
&+ \frac{1}{2}\nabla \cdot (\frac{1}{\rho^1(c^1)^2}(u^\psi)^1\left|w\partial^\alpha(p^1 - p^2)\right|^2) + \frac{1}{2}\nabla \cdot (\rho^1(u^\psi)^1\left|w\partial^\alpha(u^1 - u^2)\right|^2)\\
&+ \nabla^{\psi^1} \cdot (w\partial^\alpha(u^1 - u^2)w\partial^\alpha(p^1 - p^2)) = R^{|\alpha|}_\Delta(U,\psi) \cdot \partial^\alpha(U^1 - U^2)
\end{aligned}$$

We integrate over $\Omega_\pm$ and use the divergence theorem together with the fact that $w = 0$ on Γ. We obtain the following.

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\frac{1}{\rho^1(c^1)^2}\left|w\partial^\alpha(p^1 - p^2)\right|^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm}\rho^1\left|w\partial^\alpha(u^1 - u^2)\right|^2\mathrm{d}x\\
&= \int_{\Omega_\pm} R^{|\alpha|}_\Delta(U,\psi) \cdot \partial^\alpha(U^1 - U^2)\mathrm{d}x
\end{aligned}$$

Integrating from 0 to t, using the remainder estimate (15.1.4) and the condition (15.0.2), we obtain

$$\int_{\Omega_\pm}\left|w\partial^\alpha(p^1 - p^2)\right|^2\mathrm{d}x + \int_{\Omega_\pm}\left|w\partial^\alpha(u^1 - u^2)\right|^2\mathrm{d}x \tag{15.8.1}$$
$$\leq K(t)E^0_\Delta + \frac{1}{\epsilon}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon K(t)E_\Delta(t) \tag{15.8.2}$$

Assume now β is a multi-index with $|\beta| = 1$ and $\beta_3 = 0$. Combining the above estimate with the high-order time derivative estimate (15.7.5), we obtain the following simpler time derivative estimate.

$$\sum_\pm\left|\left|\partial_t\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} + \sum_\pm\left|\left|\partial_t\partial^\beta(u^1 - u^2)\right|\right|^2_{L^2(\Omega_\pm)} \tag{15.8.3}$$
$$\leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t) \tag{15.8.4}$$

for $|\beta| = 1$ with $\beta_3 = 0$.

15.9 Difference Estimate of the Curl

Let $|\beta| = 1$ and subtract the equation (8.12.1) for (U^2, ψ^2) from the corresponding equation for (U^1, ψ^1) to obtain

$$\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla)\nabla^{\psi^1} \times \partial^\beta(u^1 - u^2) = R^2_\Delta(U,\psi)$$

in $(0, T^0) \times \Omega_\pm$.

For β a multi-index with $|\beta| = 1$ and $1 \leq i \leq 2$, we define the corrected curl as

$$\omega^{\beta,i} = \nabla^{\psi^1} \times \partial^\beta u^i \tag{15.9.1}$$

Thus the corrected curl satisfies the equation

$$\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla)(\omega^{\beta,1} - \omega^{\beta,2}) = R^2_\Delta(U, \psi) \tag{15.9.2}$$

in $(0, T^0) \times \Omega_\pm$.

We multiply equation (15.9.2) by $\omega^{\beta,1} - \omega^{\beta,2}$ and use the product rule to obtain

$$\frac{1}{2}\partial_t(\rho^1 \left|\omega^{\beta,1} - \omega^{\beta,2}\right|^2) + \frac{1}{2}\nabla \cdot (\rho^1 (u^\psi)^1 \left|\omega^{\beta,1} - \omega^{\beta,2}\right|^2) = R^2_\Delta(U, \psi)(\omega^{\beta,1} - \omega^{\beta,2})$$

in $(0, T^0) \times \Omega_\pm$. Note that we have absorbed lower-order terms generated by the product rule into the remainder $R^2_\Delta(U, \psi)$ on the right hand side.

Then we integrate over $\Omega_\pm$ and apply the divergence theorem for Sobolev functions to the second term on the left. Note that since $(u^\psi)^1_{3\pm} = 0$ on Γ we are left with no boundary terms. Hence we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\pm} \rho^1 \left|\omega^{\beta,1} - \omega^{\beta,2}\right|^2 \mathrm{d}x = \int_{\Omega_\pm} R^2_\Delta(U, \psi)(\omega^{\beta,1} - \omega^{\beta,2})\mathrm{d}x$$

Applying Cauchy's inequality to the right hand side, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\sqrt{\rho^1}(\omega^{\beta,1} - \omega^{\beta,2})\right|\right|^2_{L^2(\Omega_\pm)} \leq \left|\left|R^2_\Delta(U, \psi)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\omega^{\beta,1} - \omega^{\beta,2}\right|\right|^2_{L^2(\Omega_\pm)}$$

Then using the estimate (15.1.4) of the remainder, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\sqrt{\rho^1}(\omega^{\beta,1} - \omega^{\beta,2})\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E_\Delta(t)$$

Integrating from 0 to t, we obtain

$$\left|\left|\sqrt{\rho^1}(\omega^{\beta,1} - \omega^{\beta,2})\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau$$

Since $\rho^1 \geq \delta^1$ by (15.0.2), we obtain

$$\left|\left|\omega^{\beta,1} - \omega^{\beta,2}\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau \tag{15.9.3}$$

15.10 Difference Estimate of the Normal Derivatives of the Pressure and Velocity

For $|\beta| = 1$ and $1 \leq i \leq 2$, we define the corrected divergence as

$$\eta^\beta = \nabla^{\psi^1} \cdot \partial^\beta u^i \tag{15.10.1}$$

Corollary 15.10.1. *Let $|\beta| = 1$. Then*

$$\left|\left|\partial_{x_3}\partial^\beta(u^1 - u^2)\right|\right|_{L^2(\Omega_\pm)}$$
$$\leq \left|\left|R^0_\infty(U, \psi)\right|\right|_{L^\infty(\Omega_\pm)}(\left|\left|\nabla'\partial^\beta(u^1 - u^2)\right|\right|_{L^2(\Omega_\pm)} + \left|\left|\eta^{\beta,1} - \eta^{\beta,2}\right|\right|_{L^2(\Omega_\pm)} + \left|\left|\omega^{\beta,1} - \omega^{\beta,2}\right|\right|_{L^2(\Omega_\pm)})$$

Proof. This is immediate from the lemma 8.13.2 applied to $v = \partial^\beta(u^1 - u^2)$, where we have replaced $G_i(\nabla\psi^1)$ by the remainder $R^0_\infty(U, \psi)$. $\square$

Now, let $|\beta| = 1$. Note that by (15.3.1) and the fact that $\eta^{\beta,i} = \nabla^{\psi^1} \cdot \partial^\beta u^i$, we have

$$\eta^{\beta,1} - \eta^{\beta,2} = -\frac{1}{\rho^1(c^1)^2}(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\beta(p^1 - p^2) + \tilde{R}^{|\beta|}_\Delta(U, \psi) + R^0_\infty(U, \psi) \cdot \nabla p^1 \partial_t \partial^\beta(\psi^1 - \psi^2)$$

where we have absorbed all the high-order terms in ψ involving a spatial derivative into the remainder $\tilde{R}^{|\beta|}_\Delta(U, \psi)$. Note also that $\left|\left|\partial_t\partial^\beta(\psi^1 - \psi^2)\right|\right|_{H^1(\Omega_\pm)}$ can be estimated by the estimates (15.4.1) and (15.7.3). Thus

$$\begin{aligned}
&\left|\left|\eta^{\beta,1} - \eta^{\beta,2}\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq K(t)(\left|\left|\partial_t\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta(p^2 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|(u^\psi)^1_3\partial_{x_3}\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)}) \\
&+ \left|\left|\tilde{R}^{|\beta|}_\Delta(U, \psi)\right|\right|^2_{L^2(\Omega_\pm)} + K(t)\left|\left|\partial_t\partial^\beta(\psi^1 - \psi^2)\right|\right|^2_{H^1(\Omega_\pm)} \\
&\leq K(t)(E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}E_\Delta(t) + \left|\left|\partial_t\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)})
\end{aligned} \tag{15.10.2}$$

where we have used the estimates (15.0.2), (15.2.1) and the lower order remainder estimate (15.1.7), together with the weighted normal derivative estimate (15.8.2).

Also, from the third component of (15.3.2), we have

$$\begin{aligned}
\partial_{x_3}\partial^\beta(p^1 - p^2) = -J^{\psi^1}\rho^1(\partial_t + (u^\psi)^1 \cdot \nabla)\partial^\beta(u^1_3 - u^2_3) + \tilde{R}^{|\beta|}_\Delta(U, \psi) \\
+ R^0_\infty(U, \psi) \cdot \nabla u^1_3 \partial_t\partial^\beta(\psi^1 - \psi^2)
\end{aligned}$$

where again we have absorbed all the high-order terms in ψ involving a spatial derivative into the remainder $\tilde{R}^{|\beta|}_\Delta(U, \psi)$ and we note also that $\left|\left|\partial_t\partial^\beta(\psi^1 - \psi^2)\right|\right|_{H^1(\Omega_\pm)}$ can be estimated by the estimates (15.4.1) and (15.7.3). Thus

$$\begin{aligned}
&\left|\left|\partial_{x_3}\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} \\
&\leq K(t)(\left|\left|\partial_t\partial^\beta(u^1_3 - u^2_3)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta(u^2_3 - u^2_3)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|(u^\psi)^1_3\partial_{x_3}\partial^\beta(u^1_3 - u^2_3)\right|\right|^2_{L^2(\Omega_\pm)}) \\
&+ \left|\left|\tilde{R}^{|\beta|}_\Delta(U, \psi)\right|\right|^2_{L^2(\Omega_\pm)} + K(t)\left|\left|\partial_t\partial^\beta(\psi^2 - \psi^2)\right|\right|^2_{H^1(\Omega_\pm)} \\
&\leq K(t)(E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}E_\Delta(t) + \left|\left|\partial_t\partial^\beta(u^1_3 - u^2_3)\right|\right|^2_{L^2(\Omega_\pm)} + \left|\left|\nabla'\partial^\beta(u^1_3 - u^2_3)\right|\right|^2_{L^2(\Omega_\pm)})
\end{aligned} \tag{15.10.3}$$

where we have used the estimates (15.0.2), (15.2.1) and the lower order remainder estimate (15.1.7), together with the weighted normal derivative estimate (15.8.2).

Now we claim that

$$\left|\left|\partial_{x_3}\partial^\beta(u^1 - u^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t) \tag{15.10.4}$$

$$\left|\left|\partial_{x_3}\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t) \tag{15.10.5}$$

We proceed by induction on β_3. In the case that $\beta_3 = 0$, we note that we may apply (15.6.1) and (15.8.4) to the right hand side of (15.10.2) and (15.10.3) to obtain

$$\left|\left|\eta^{\beta,1} - \eta^{\beta,2}\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$
$$\left|\left|\partial_{x_3}\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$

Then we apply the estimate from corollary 15.10.1 together with the lower order remainder estimate (8.18.4), the estimate (15.6.1) for the tangential derivatives, the estimate (15.9.3) for the corrected curl and the estimate just obtained for the corrected divergence to conclude that

$$\left|\left|\partial_{x_3}\partial^\beta(u^1 - u^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$

which proves the result in the case $\beta_3 = 0$.

Now we suppose the result is true for $\beta_3 \leq k$ for some $0 \leq k \leq |\beta| - 1$ and try to show it is true for $\beta_3 = k + 1$. But this is straightforward and is done in the same way. Note that the previous induction step gives us the estimate for

$$||\partial^\gamma(u^1 - u^2)||^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$
$$||\partial^\gamma(p^1 - p^2)||^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$

where $|\gamma| = 2$ and $\gamma_3 \leq k+1$. We apply these estimates to the right hand side of (15.10.2) and (15.10.3) to obtain

$$\left|\left|\eta^{\beta,1} - \eta^{\beta,2}\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$
$$\left|\left|\partial_{x_3}\partial^\beta(p^1 - p^2)\right|\right|^2_{L^2(\Omega_\pm)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t)$$

Then we apply the estimate from corollary 15.10.1 together with the lower order remainder estimate (8.18.4), the previous induction step for the term $\nabla'\partial^\beta u$, the estimate (15.9.3) for the corrected curl and the estimate just obtained for the corrected divergence to conclude the result for the case $\beta_3 = k + 1$. This proves the claim.

15.11 Elliptic-Type Difference Estimate for the Front

Lemma 15.11.1.

$$||\partial^\alpha\nabla'(f^1 - f^2)||^2_{H^{0.5}(\Gamma)} \leq K(t)E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau + \epsilon^{\frac{1}{2}}K(t)E_\Delta(t) \qquad (15.11.1)$$

for $0 \leq |\alpha| \leq 2$*,* $\alpha_0 \leq 1$ *and* $\alpha_3 = 0$*.*

Proof. Let α be a multi-index with $0 \leq |\alpha| \leq 2$, $\alpha_0 \leq 1$ and $\alpha_3 = 0$, so that we can write $\partial^\alpha = \partial^\beta\partial_{x_i}$ for some $i \in \{1, 2\}$. Let $\langle D'\rangle^{0.5}$ denote the half-order derivative operator which corresponds to multiplying by $\langle\xi'\rangle^{0.5}$ in Fourier space - see the notation in the appendix.

Applying $\langle D'\rangle^{0.5}\partial^\alpha$ to (15.3.1) and subtracting the equation for (U^2,f^2) from that for (U^1,f^1), we obtain

$$\left\langle D'\right\rangle^{0.5}[\partial^\alpha(p^1-p^2)]=\sigma\left\langle D'\right\rangle^{0.5}\nabla'\cdot\partial^\alpha(\frac{\nabla' f^1}{|n^1|}-\frac{\nabla' f^2}{|n^2|})$$

We multiply both sides by $\langle D'\rangle^{0.5}\partial^\alpha(f^1-f^2)$ and integrate by parts to obtain

$$\begin{aligned}&\int_\Gamma(\left\langle D'\right\rangle^{0.5}[\partial^\beta(p^1-p^2)])(\left\langle D'\right\rangle^{0.5}\partial^\alpha\partial_{x_i}(f^1-f^2))\mathrm{d}x'\\&=\sigma\int_\Gamma(\left\langle D'\right\rangle^{0.5}\partial^\alpha(\frac{\nabla' f^1}{|n^1|}-\frac{\nabla' f^2}{|n^2|}))\cdot(\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2))\mathrm{d}x'\end{aligned}$$

We now use lemma 15.3.4 to differentiate $\frac{\nabla' f^1}{|n^1|}-\frac{\nabla' f^2}{|n^2|}$, followed by the fractional product rule A.3.8 from the appendix, and using the notation of this lemma, we denote the remainder term by

$$R(G(\nabla' f^1),\partial^\alpha\nabla'(f^1-f^2))$$

where G is a smooth function. We obtain

$$\begin{aligned}&\int_\Gamma(\left\langle D'\right\rangle^{0.5}[\partial^\beta(p^1-p^2)])(\left\langle D'\right\rangle^{0.5}\partial^\alpha\partial_{x_i}(f^1-f^2))\mathrm{d}x'\\&=\sigma\int_\Gamma(\left\langle D'\right\rangle^{0.5}\partial^\alpha(\frac{\nabla' f^1}{|n^1|}-\frac{\nabla' f^2}{|n^2|}))\cdot(\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2))\mathrm{d}x'\\&=\sigma\int_\Gamma\frac{1}{|n^1|}\left|\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\right|^2-\frac{1}{|n^1|^3}\left|\nabla' f^1\cdot\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\right|^2\mathrm{d}x'\\&+\sigma\int_\Gamma\left\langle D'\right\rangle^{0.5}R^1_{\Delta,\Gamma}(f)\cdot\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\mathrm{d}x'\\&+\sigma\int_\Gamma R(G(\nabla' f^1),\partial^\alpha\nabla'(f^1-f^2))\cdot\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\mathrm{d}x'\\&\geq\sigma\int_\Gamma\frac{1}{|n^1|^3}\left|\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\right|^2\mathrm{d}x'\\&-\frac{1}{\delta}\left|\left|R(G(\nabla' f^1),\partial^\alpha\nabla'(f^1-f^2))\right|\right|^2_{L^2(\Gamma)}-\delta\int_\Gamma\left|\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\right|^2\mathrm{d}x'\\&-\left|\left|R^1_{\Delta,\Gamma}(f)\right|\right|^2_{H^1(\Gamma)}-C\left|\left|\partial^\alpha\nabla'(f^1-f^2)\right|\right|^2_{L^2(\Gamma)}\end{aligned}$$

where we have used lemma 8.6.5. We have also used Cauchy's inequality in the last step, and we are free to choose $0<\delta\leq 1$. Note that $\partial R^1_{\Delta,\Gamma}(f)$ is a sum of terms which contain factors of the form $\partial^\gamma\nabla'(f^1-f^2)$ where $|\gamma|\leq 2$. We recall that we have already estimated $\partial^\gamma\nabla'(f^1-f^2)$ in (15.7.3). Hence, using Cauchy's inequality, the remainder estimate (15.1.9), and the estimate of the remainder from A.3.8, we obtain

$$\begin{aligned}&\sigma\int_\Gamma\frac{1}{|n^1|^3}\left|\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\right|^2\mathrm{d}x'\\&\leq\frac{C}{\delta}\left|\left|\left\langle D'\right\rangle^{0.5}[\partial^\beta(p^1-p^2)]\right|\right|^2_{L^2(\Gamma)}+\delta\left|\left|\left\langle D'\right\rangle^{0.5}\partial^\alpha\nabla'(f^1-f^2)\right|\right|^2_{L^2(\Gamma)}\\&+\frac{1}{\delta}(K(t)E^0_\Delta+\frac{1}{\epsilon^{\frac{3}{2}}}K(t)\int_0^t E_\Delta(\tau)\mathrm{d}\tau+\epsilon^{\frac{1}{2}}K(t)E_\Delta(t))\end{aligned}$$

We choose $\delta \leq \frac{\sigma}{2||n^1||^3_{L^\infty(\Gamma)}}$ and rearrange and use the Sobolev trace theorem to obtain

$$\begin{aligned}
&\sigma \int_\Gamma \frac{1}{|n^1|^3} \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' \\
&\leq C \sum_\pm \frac{||n^1||^3_{L^\infty(\Gamma)}}{\sigma} \left|\left| \partial^\beta (p^1 - p^2) \right|\right|^2_{H^1(\Omega_\pm)} + K(t) E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t) \\
&\leq K(t) E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t)
\end{aligned}$$

where we have used the lower order estimate (15.2.1) and the existing energy estimates (15.6.1) and (15.10.5). Hence

$$\begin{aligned}
&\int_\Gamma \left| \langle D' \rangle^{0.5} \partial^\alpha \nabla'(f^1 - f^2) \right|^2 \mathrm{d}x' \\
&\leq K(t) E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t)
\end{aligned}$$

This completes the proof. □

15.12 Conclusion of the Proof of Uniqueness and Stability

Combining the estimates (15.2.1), (15.2.2), (15.4.1), (15.6.1), (15.8.4), (15.7.3), (15.5.2), (15.10.4), (15.10.5) and (15.11.1) we obtain

$$\begin{aligned}
E_\Delta(t) &\leq K(t) E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}} K(t) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(t) E_\Delta(t) \\
&\leq K(T^0) E^0_\Delta + \frac{1}{\epsilon^{\frac{3}{2}}} K(T^0) \int_0^t E_\Delta(\tau) \mathrm{d}\tau + \epsilon^{\frac{1}{2}} K(T^0) E_\Delta(t)
\end{aligned}$$

where we are free to choose $\epsilon \in (0, 1]$. Choosing $\epsilon^{\frac{1}{2}} = \frac{1}{2K(T^0)}$ and rearranging we obtain

$$E_\Delta(t) \leq K(T^0) E^0_\Delta + K(T^0) \int_0^t E_\Delta(\tau) \mathrm{d}\tau$$

where we have redefined the function $K(t)$, but it is still of the same form and $K(T^0) < \infty$. Applying Gronwall's lemma we immediately obtain that

$$E_\Delta(t) \leq K(T^0) E^0_\Delta \exp(K(T^0) t)$$

for $t \in (0, T^0)$. This is the statement of stability.

Note in particular this implies that if $(U^{0,1}_+, U^{0,1}_-, f^{0,1}) = (U^{0,2}_+, U^{0,2}_-, f^{0,2})$ then

$$E_\Delta(t) = 0$$

for $t \in (0, T^0) = (0, T^{0,1}) = (0, T^{0,2})$. Thus

$$\begin{aligned}
||U^1 - U^2||_{L^2(\Omega_\pm)} &= 0 \\
||f^1 - f^2||_{L^2(\Gamma)} &= 0
\end{aligned}$$

for $t \in (0, T^0)$ and hence $U^1 = U^2$ almost everywhere in $(0, T^0) \times \Omega_\pm$ and $f^1 = f^2$ almost everywhere on $(0, T^0) \times \Gamma$. This completes the proof of uniqueness.

16. Extensions

16.1 The Two-Dimensional Case

Suppose that the flow is independent of one of the horizontal directions, so that we may consider the fluid to occupy some two-dimensional regions $\Omega_\pm(t) \subset \mathbb{R}^2$ separated by a one-dimensional surface $\Gamma(t)$. Then in fact exactly the same theorem as 4.6.1 holds, where we replace the three-dimensional space variable with a two-dimensional one by, say, dropping the x_1 coordinate, then relabelling the coordinates so that x_1 becomes the horizontal direction and x_2 the normal direction. Our proof works in exactly the same way, and in fact the Sobolev embedding theorem actually gives slightly better estimates for a fixed number of derivatives, but not quite enough for us to reduce the regularity requirement on the initial data. Note that we replace the curl $\nabla \times u$ with the two-dimensional curl, which can be written as a scalar, $\partial_{x_1} u_2 - \partial_{x_2} u_1$, or, to maintain consistency with the three-dimensional notation, as a 3-vector whose first two components are zero, given by $(\partial_{x_1}, \partial_{x_2}, 0) \times (u_1, u_2, 0)$. Other changes to the proof are minimal.

16.2 Body Force

We consider here the effect of adding a body force, $h(t,x)$, to the right hand side of the velocity equation (2.1.2). This means the system (2.1.1)-(2.1.5) remains the same, except that the velocity equation (2.1.2) is replaced by the equation

$$\rho(\partial_t + u \cdot \nabla)u + \nabla p = h \tag{16.2.1}$$

First let us consider adding h to the right hand side of the velocity equation (4.3.2) in the fixed domains, where we allow $h = h_\pm$ in $\Omega_\pm$. Suppose $h = h(t,x,U,\psi)$ is a function of (t,x,U,ψ) where $U = (p,u,s)$ such that

$$E_h(t) := \sum_\pm \sum_{j=0}^{3} \int_0^t \left|\left|\partial_\tau^j (h_\pm(\tau, x, U_\pm(\tau,x), \psi(\tau,x)))\right|\right|^2_{H^{3-j}(\Omega_\pm)} \mathrm{d}\tau \le M_h + tF_h(E(t))$$

for any (U_+, U_-, f) and $\psi = L^a f$ with $E(t) < \infty$, where M_h is a constant depending on the function h and F_h is a smooth increasing function depending on the function h. Thus $h(t,x,U(t,x),\psi(t,x))$ has basically the same regularity in space as solutions with $E(t) < \infty$, with the only difference being that h is L^2 in space at zero order. Then it is straightforward to show that the main theorem in the fixed domains 4.6.1 with the addition of this body force is proved in exactly the same way. Note that when passing to the μ-approximate equations we would mollify h as we did ρ and replace it with h_μ which is smooth in space. If we allow M^0 and $K(t)$ as defined in 8 to depend on the function h, then it is clear that the energy estimate in 8 remains the same, so that we have the following bounds on T^0 and C^0.

$$T^0 \geq T_h(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) > 0$$

where $T_h(\cdot)$ is a smooth decreasing function which may depend on the function h and

$$C^0 \leq C_h(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}) < \infty$$

where $C_h(\cdot)$ is a smooth increasing function which may depend on the function h. The main other point to note is that when we come to linearise the μ-approximate equations, we will simply replace h_μ by $\bar{h}_\mu := (h(t, x, \bar{U}(t,x), \bar{\psi}(t,x)))_\mu = (\mathrm{Ext}_{\Omega_\pm}(h(t, x, \bar{U}(t,x), \bar{\psi}(t,x)))) * \eta_\mu|_{\Omega_\pm}$.

Using the above result for the fixed domains, we may add any function $h(t, x, U)$ to the right hand side of velocity equation (2.1.2) in the moving domains such that we can write $h(t, x', x_3 + \psi, U) = h(t, x, \psi, U)$ as a function of (t, x, ψ, U) for some h satisfying the conditions above. Note since the domains $\Omega_\pm(t)$ are unknown, we require that h is defined for $x \in \mathbb{R}^3$ rather than separately for $x \in \Omega_\pm$ as above. Thus we obtain the conclusions of the main theorem 2.4.1 with the addition of the body force h, where we modify the bounds on T^0 and C^0 as above (without the dependence on κ^0).

Note that, if we were working in a horizontally periodic domain with top and bottom walls or a bounded domain, the above would allow the the addition of a gravity force $h = \rho g$ where g is the constant vector acceleration due to gravity, but to fit the setup above we need $h \in L^2(\Omega_\pm)$ so we would have to multiply the gravitational force by a cut-off function or weight.

16.3 Constructing Compatible Initial Data

We know that there exist piecewise constant flat vortex sheet solutions to the vortex sheet equations (2.1.1)-(2.1.5) as described in the introduction, with $p_\pm = \bar{p}_\pm = \bar{p}$, $u_\pm = \bar{u}_\pm = (\bar{u}'_+, 0)$, $s_\pm = \bar{s}_\pm$, $f = \bar{f} = 0$ where $\bar{p}, \bar{u}'_\pm, \bar{s}_\pm$ are constants. We have also shown in theorem 2.4.1 that given initial data with $E^0 < \infty$ satisfying the compatibility conditions (2.2.5)-(2.2.6) up to order 2 we have solutions, but it is difficult to construct initial data which actually satisfy the compatibility conditions (2.2.5)-(2.2.6) up to order 2. However, one way to do so would be to use the result with a given body force described above. We take $h = h_\pm$ to be given bump functions in time so that $\partial_t^j h_\pm \in C_c((0, \tilde{T}); H^{3-j}(\Omega_\pm))$ for $0 \leq j \leq 3$ where $\tilde{T} > 0$ will be taken sufficiently small. We then solve the system (4.3.1)-(4.3.5) on the fixed domains with initial data $(\bar{U}_+, \bar{U}_-, \bar{f})$ being the flat vortex sheet solution above, and body force h. We have that the existence time T^0 is bounded below by

$$T^0 \geq T(E^0, \frac{1}{\delta^0}, \frac{1}{\kappa^0}, E_h(1)) > 0$$

where we may take, say, $\kappa^0 = \frac{2}{3}$. Now we make the energy $E_h(1)$ of h sufficiently small and the time $\tilde{T}$ smaller than the corresponding existence time T^0. Then we may take the solution at time some time $\tilde{T} < T' < T^0$ as initial data for the problem without a body force, which will be non-trivial for most non-trivial functions h. The time T' may be chosen so that this new initial data $(U_+^0, U_-^0, f^0) = (U_+(T'), U_-(T'), f(T'))$ has $E^0 < \infty$, and the compatibility conditions (4.4.6)-(4.4.7) up to order 2 are automatically satisfied since (U_+, U_-, f) is a solution of the equations (4.3.1)-(4.3.5) near time T' with energy $E(T^0) < \infty$. Defining the diffeomorphism θ from ψ as in 5.1.1, we have that $(U_+^0 \circ \theta^{-1}(T'), U_-^0 \circ \theta^{-1}(T'), f^0)$ are non-trivial initial data for the system (2.1.1)-(2.1.5) on the moving domains satisfying the compatibility conditions (2.2.5)-(2.2.6) up to order 2.

16.4 Horizontally Periodic Domain with Top and Bottom Walls

Instead of working on the unbounded domains $\Omega_\pm^T$ with interface Γ^T as defined in 2.1.1, we could work on a horizontally periodic domain with top and bottom walls, by which we mean we redefine

$$\Omega_+^T := \{(t,x) \in (0,T) \times \Omega : x_3 > f(t,x)\}$$
$$\Omega_-^T := \{(t,x) \in (0,T) \times \Omega : x_3 < f(t,x)\}$$
$$\Gamma^T := \{(t,x) \in (0,T) \times \Omega : x_3 = f(t,x)\}$$

where the fixed domain Ω is defined as

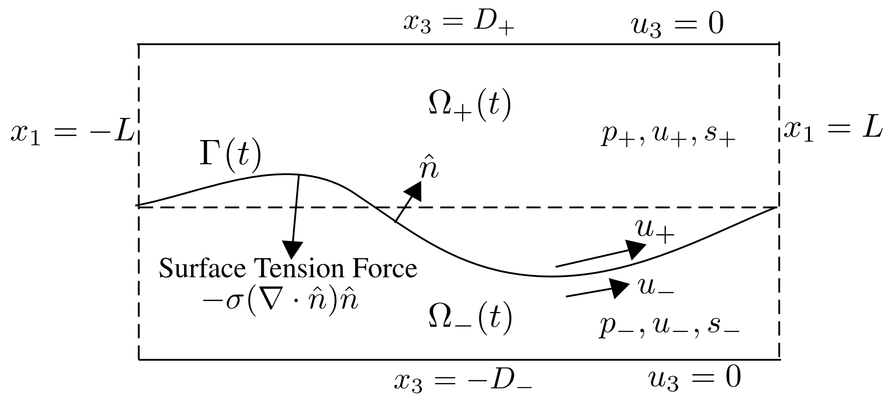
$$\Omega := \{(t,x) \in (0,T) \times \mathbb{R}^3 : x_3 < D_+, x_3 > -D_-, -L < x_1, x_2 < L\}$$

for some fixed constants $D_+ > 0$, $D_- > 0$, $L > 0$. We impose periodic boundary conditions in the horizontal directions, and we impose the solid wall boundary condition

$$u_3 = 0$$

on $(0,T) \times \{x_3 = \pm D_\pm\}$. Note that this is exactly the statement that the fluid does not flow through the walls at $x_3 = \pm D_\pm$ since u_3 is the component of the velocity normal to these walls.

Cross-Section of Ω with x_2 Constant



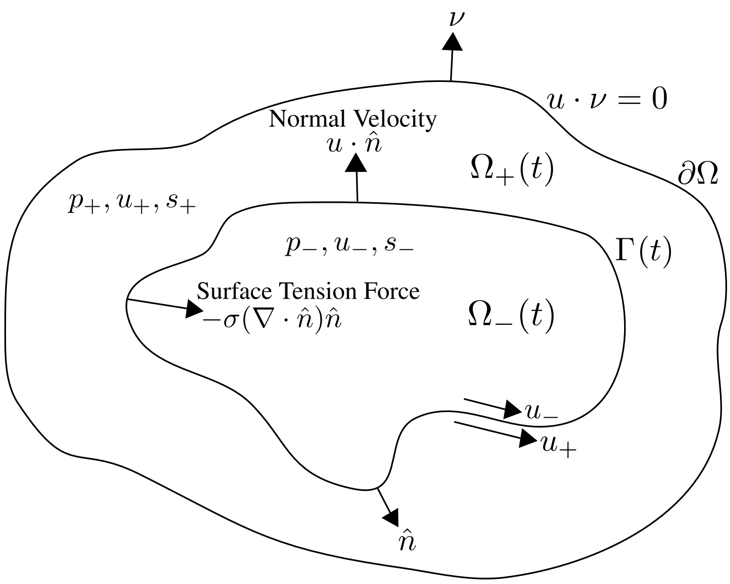
We require that initially the surface Γ^0 does not touch the walls, which means that for a short time by continuity it will still not touch the walls. We note that when constructing the lift ψ of f, we want to ensure that $\psi = 0$ in a neighbourhood of the walls so that the change of coordinates θ is the identity in a neighbourhood of the walls. Thus we want to set $\psi = \tilde{\chi}(x_3)f$ where $\tilde{\chi}$ is a smooth cut-off function with sufficiently small support depending on $D_\pm$. But then $\partial_{x_3}\psi$ might be quite large, so one way to ensure that θ remains a diffeomorphism would be to set $\theta(t,x) = (t, x', Ax_3 + \psi(t,x))$ for A a sufficiently large positive constant, to ensure that the Jacobian $J^\psi := A + \partial_{x_3}\psi$ is uniformly positive. This slight change should not cause any difficulties in adapting the proof of 2.4.1 to the case of a horizontally periodic domain with top and bottom walls. Note that the boundary condition $u_3 = 0$ on $(0, T) \times \{x_3 = \pm D_\pm\}$ ensures that the boundary terms generated in the energy estimate due to the walls are in fact zero.

Thus we conclude that it would be straightforward to adapt the proof of 2.4.1 to the case of a horizontally periodic domain with top and bottom walls, or indeed a domain which is unbounded horizontally but bounded by walls above and below.

16.5 Bounded Domain

The case of a bounded domain was considered by Shkoller et al in [6] in the incompressible setting. In this setup, $\mathbb{R}^3$ is replaced by a sufficiently smooth simply connected domain $\Omega \subset \mathbb{R}^3$ and the interface between the two fluids Γ is a closed surface which divides the interior of Ω into two pieces, Ω_+ and Ω_-, where, say, Ω_+ is the outer region.

Cross-Section of Ω with x_2 Constant



The interior equations and the pressure interface condition remain the same, and the interface Γ still moves with the fluid, which in a local chart would still be the condition $\partial_t f = u \cdot n$. We assume that the outer boundary of Ω acts as a fixed wall, so the boundary condition on $\partial\Omega$ is $u \cdot \nu = 0$ where ν is normal to $\partial\Omega$. This implies that any boundary terms generated in the energy estimate due to the outer boundary $\partial\Omega$ are lower order terms provided Ω is a sufficiently smooth domain.

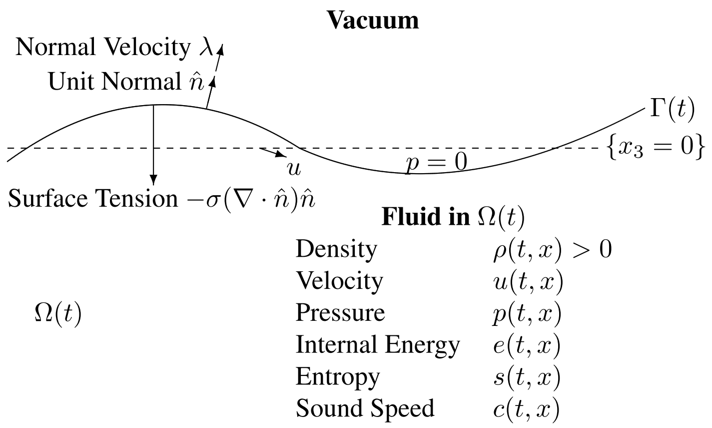
The analysis of this problem requires the introduction of coordinate charts and a partition of unity to define derivatives on Γ, which adds significant technical complications to the problem. However, since we have not imposed any smallness assumption of the graph of the from f in the above work, it seems likely that there is no fundamental obstruction in adapting the above work to the case of a bounded domain, although we will not comment any further in this direction.

17. The One-Phase Problem With and Without Surface Tension

The one-phase problem with and without surface tension has been considered by Shkoller et al in [10], so we do not devote a large section to this problem. However, we sketch the details of the results we would hope to obtain for the one-phase problem using the above methods since the exact statements are slightly different to those given by Shkoller et al in [10].

17.1 Statement of the One-Phase Problem

Perturbation of a Flat Liquid-Vacuum Boundary



When we say the one phase problem, we are referring to the problem obtained from the vortex-sheet problem when we replace one of the fluids, say the fluid in the region Ω_+^T, by vacuum. The interior equations in Ω_-^T remain the same, the interface still moves with the fluid and we replace p_+ by 0 in the pressure interface condition (2.1.5). Thus the system of equations can be written as follows, where we drop the subscript $_-$ since it is now redundant, so Ω^T refers to Ω_-^T etc.

$$\frac{1}{\rho c^2}(\partial_t + u \cdot \nabla)p + \nabla \cdot u = 0 \tag{17.1.1}$$

$$\rho(\partial_t + u \cdot \nabla)u + \nabla p = 0 \tag{17.1.2}$$

$$(\partial_t + u \cdot \nabla)s = 0 \tag{17.1.3}$$

in Ω^T,

$$\partial_t f = u \cdot n \tag{17.1.4}$$

$$p = \sigma \nabla' \cdot \hat{n} \tag{17.1.5}$$

on Γ^T.

If $\sigma > 0$ we call this the one-phase problem with surface tension, and if $\sigma = 0$ we call it the one-phase problem without surface tension.

The compatibility conditions given in 2.2.3 reduce to the following compatibility condition.

Definition 17.1.1. We say that initial data (U^0, f^0) satisfy the compatibility condition for the system (17.1.1)-(17.1.5) up to order k, for integer $k \geq 0$, if the following holds for all $x' \in \mathbb{R}^2$.

$$\partial_0^j (p^0|_{\Gamma^0})(x') = \sigma \partial_0^j (\nabla' \cdot \hat{n}^0)(x') \tag{17.1.6}$$

for $0 \leq j \leq k$, where

$$p^0|_{\Gamma^0}(x') := p^0(x', f^0(x'))$$

Note that in order to relate this to the above work, we will require that the density is bounded below, so it is non-zero at the boundary, and hence physically we would probably be modelling a compressible liquid rather than a gas. One could also replace the vacuum region with a region of constant pressure $\bar{p} > 0$, in which case the pressure interface interface condition would become $p = \sigma \nabla' \cdot \hat{n} + \bar{p}$, which would cause no difficulties in the proof since $\bar{p}$ would disappear upon differentiation.

17.2 The One-Phase Problem With Surface Tension

Here we assume $\sigma > 0$. In this case it is straightforward to adapt the proof of the main theorem 2.4.1 given above to prove an analogous result for the one-phase problem. The statement of the result for the one-phase problem with surface tension can be written as follows. Note that the energy is the same as defined in 2.3.1 but without the Ω_+ terms.

Theorem 17.2.1. *Let (U^0, f^0) be initial data with energy $E^0 < \infty$. Assume that this initial data satisfies the compatibility condition* (17.1.6) *up to order* 2*. Assume also that the initial density satisfies*

$$\inf_{x \in \Omega^0} \rho^0(x) =: \delta^0 > 0 \tag{17.2.1}$$

Then there exists a time $T^0 > 0$ and a solution (U, f) of the the system (17.1.1)-(17.1.5) *with $\sigma > 0$ on the time interval $(0, T^0)$. Moreover,*

$$E(T^0) \leq C^0$$
$$\rho \geq \frac{\delta^0}{2} \text{ in } \Omega^{T^0}$$

The time $T^0 > 0$ is bounded below as follows.

$$T^0 \geq T(E^0, \frac{1}{\delta^0}) > 0$$

where $T(\cdot)$ is a smooth decreasing function which may depend on σ. Similarly, the constant $C^0 > 0$ is bounded above as follows.

$$C^0 \leq C(E^0, \frac{1}{\delta^0}) < \infty$$

where $C(\cdot)$ is a smooth increasing function which may depend on σ.

In addition, the solution (U, f) is unique on the time interval $(0, T^0)$ in the sense that it is the only solution of the equations on the time interval $(0, T^0)$ with initial data (U^0, f^0) satisfying the following properties

$$E(T^0) < \infty$$

$$\inf_{(t,x)\in\Omega^{T^0}} \rho > 0$$

We avoid repeating all the details of the proof, but we note that there are some significant simplifications particularly in the energy estimate. Indeed in proving the energy estimate, everything is estimated in an exactly analogous way as in 8, except that a large simplification can be made when estimating the highest order time derivative. The boundary term K_{13}^{β}, which we had to estimate by going back to the equations in the interior, is no longer present at all. In particular, this means that we can use the time derivative ∂_t instead of the material time derivative D_t^{u-} in this high order time derivative estimate, and we can skip all the estimates dealing with the term K_{13}^{β}. In fact it becomes possible to estimate the tangential and time derivatives in the same way, provided that we estimate the term corresponding to I_2^{α} in the same way as K_{12}^{β}.

17.3 The One-Phase Problem Without Surface Tension

It is also possible under slightly modified assumptions to prove short-time existence and uniqueness for the one-phase problem without surface tension, that is $\sigma = 0$. Indeed, all that is required is to prove an energy estimate for the one-phase problem with surface tension that is independent of σ (which is the same as proving an energy estimate for the μ-approximate one-phase problem that is independent of σ in the case $\mu = 0$). One could then use weak-$*$ compactness and the Sobolev compact embedding theorem for solutions of the problem with surface tension to pass to the limit $\sigma = 0$ and obtain solutions to the problem without surface tension. Note that obtaining a time interval of existence independent of σ given a σ-independent energy estimate would be straightforward assuming that the time interval of existence for the case $\sigma > 0$ can be maximised in such a way that the σ-independent energy blows up t reaches the maximal time.

Note that in the case $\sigma = 0$ we no longer obtain an elliptic-type estimate for the front f from the pressure interface condition (17.1.5). In fact, to obtain a good estimate for f we need to make an additional assumption on the initial data. That is, the Taylor sign condition should be satisfied as follows.

$$-\partial_{x_3} p^0|_{\Gamma^0} \geq \lambda^0 > 0$$

for some $\lambda^0 > 0$. Note this implies by continuity that the Taylor sign condition holds for a short time $T > 0$, ie

$$-\partial_{x_3} p|_{\Gamma^T} \geq \frac{\lambda^0}{2} > 0$$

Note that in the case of an unbounded domain, the requirement $-\partial_{x_3} p^0|_{\Gamma^0} \geq \lambda^0 > 0$ conflicts with $\nabla p^0 \in L^2(\Omega^0)$. It may be possible to allow for this by adding a gravitational body force and subtracting a background pressure, but even then it seems that we would need to introduce a bottom wall to prevent the density from tending to infinity. Hence in this section for simplicity we will restrict to the case of a horizontally periodic domain with bottom wall, but we expect the same result to hold for a general sufficiently smooth bounded domain, or a horizontally unbounded domain with bottom wall with gravitational body force after subtracting an appropriate background pressure. We define

$$\Omega^T := \{(t,x) \in (0,T) \times \Omega : x_3 < f(t,x)\}$$
$$\Gamma^T := \{(t,x) \in (0,T) \times \Omega : x_3 = f(t,x)\}$$

where the fixed domain Ω is defined as

$$\Omega := \{(t,x) \in (0,T) \times \mathbb{R}^3 : x_3 > -D, -L < x_1, x_2 < L\}$$

for some fixed constants $D > 0$, $L > 0$. We impose periodic boundary conditions in the horizontal directions, and we impose the solid wall boundary condition

$$u_3 = 0 \tag{17.3.1}$$

on $(0,T) \times \{x_3 = -D\}$.

The loss of the estimate for the front f from the pressure interface condition (17.1.5) means that we should modify our energy so that U and f are estimated to the same order. If we try and do this with an energy with only 3 derivatives, then the remainder terms become impossible to estimate, hence we should increase the order of the derivatives in our energy by one. If we include the terms in our energy that we would have in the case $\sigma > 0$, but this time multiplied by the appropriate factor of σ so that we can obtain a σ-independent estimate for this energy, we obtain the following definition.

Definition 17.3.1. Define the σ-independent energy $E_\sigma : (0,T] \to [0,\infty]$ by

$$E_\sigma(t) = \operatorname*{ess\,sup}_{\tau\in(0,t)} ||U||^2_{L^\infty(\Omega(\tau))} + \sum_{1\le|\alpha|\le 4} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha U||^2_{L^2(\Omega(\tau))} + \operatorname*{ess\,sup}_{\tau\in(0,t)} ||f||^2_{L^\infty(\mathbb{R}^2)}$$
$$+ \sum_{1\le|\alpha|\le 4} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha f||^2_{L^2(\mathbb{R}^2)} + \sigma \sum_{1\le|\alpha|\le 4} \operatorname*{ess\,sup}_{\tau\in(0,t)} ||\partial^\alpha f||^2_{H^1(\mathbb{R}^2)} + \sigma^2 \sum_{j=0}^{3} \operatorname*{ess\,sup}_{\tau\in(0,t)} \left|\left|\nabla' \partial_t^j f\right|\right|^2_{H^{4.5-j}(\mathbb{R}^2)}$$

We allow the case $\sigma = 0$, in which case the energy is written as $E_0(t)$.

The initial energy E^0_σ of the initial data is defined in the obvious way, replacing time derivatives in the above energy by initial time derivatives.

The statement of the result for the one-phase problem without surface tension in a horizontally periodic domain with bottom wall can be written as follows.

Theorem 17.3.2. *Let (U^0, f^0) be initial data with energy $E^0_0 < \infty$. Assume that this initial data satisfies the compatibility condition* (17.1.6) *up to order* 3 *with $\sigma = 0$, plus the compatibility condition associated with the bottom wall,*

$$\partial_0^j u_3^0|_{x_3=-D} = 0 \text{ for } 0 \le j \le 3 \tag{17.3.2}$$

Assume also that the initial density satisfies

$$\inf_{x\in\Omega^0} \rho^0(x) =: \delta^0 > 0 \tag{17.3.3}$$

and that the following Taylor sign condition is satisfied

$$-\inf_{x\in\Gamma^0} \partial_{x_3} p^0|_{\Gamma^0} =: \lambda^0 > 0 \tag{17.3.4}$$

and that the initial surface and bottom wall satisfy

$$\mathrm{dist}(\Gamma^0, \{x_3 = -D\}) =: \eta^0 > 0 \tag{17.3.5}$$

Then there exists a time $T^0 > 0$ *and a solution* (U, f) *of the the system* (17.1.1)-(17.1.5) *with* $\sigma = 0$ *plus* (17.3.1) *on the time interval* $(0, T^0)$*. Moreover,*

$$\begin{aligned} E_0(T^0) &\leq C^0 \\ \rho &\geq \frac{\delta^0}{2} \text{ in } \Omega^{T^0} \\ -\partial_{x_3} p &\geq \frac{\lambda^0}{2} \text{ on } \Gamma^{T^0} \\ \inf_{t\in(0,T^0)} \mathrm{dist}(\Gamma(t), \{x_3 = -D\}) &\geq \frac{\eta^0}{2} \end{aligned}$$

The time $T^0 > 0$ *is bounded below as follows.*

$$T^0 \geq T(E_0^0, \frac{1}{\delta^0}, \frac{1}{\lambda^0}, \frac{1}{\eta^0}) > 0$$

where $T(\cdot)$ *is a smooth decreasing function. Similarly, the constant* $C^0 > 0$ *is bounded above as follows.*

$$C^0 \leq C(E_0^0, \frac{1}{\delta^0}, \frac{1}{\lambda^0}, \frac{1}{\eta^0}) < \infty$$

where $C(\cdot)$ *is a smooth increasing function.*

In addition, the solution (U, f) *is unique on the time interval* $(0, T^0)$ *in the sense that it is the only solution of the equations on the time interval* $(0, T^0)$ *with initial data* (U^0, f^0) *satisfying the following properties*

$$\begin{aligned} E(T^0) &< \infty \\ \inf_{(t,x)\in\Omega^{T^0}} \rho &> 0 \\ -\inf_{(t,x)\in\Gamma^{T^0}} \partial_{x_3} p &> 0 \\ \inf_{t\in(0,T^0)} \mathrm{dist}(\Gamma(t), \{x_3 = -D\}) &> 0 \end{aligned}$$

We sketch some details of how to obtain the σ-independent energy estimate. Firstly, we should use the corrected unknowns as defined in (8.6.7) throughout the energy estimate in order to remove the highest order terms in ψ from the differentiated equations, as we did for the time derivative estimate in 8.10. In fact the time derivatives are now estimated in the same way as the tangential derivatives, provided we estimate the term corresponding to I_2^α in the same way as K_{12}^β. The boundary term that we obtain during the time and tangential estimates using the corrected unknowns is

$$-K^\alpha = \int_0^t \int_\Gamma n \cdot u^\alpha p^\alpha \mathrm{d}x' \mathrm{d}\tau =: K_1^\alpha - K_2^\alpha - K_3^\alpha + K_4^\alpha$$

where

$$\begin{aligned}
K_1^\alpha &= \int_0^t \int_\Gamma n \cdot \partial^\alpha u \partial^\alpha p \mathrm{d}x' \mathrm{d}\tau \\
K_2^\alpha &= \int_0^t \int_\Gamma n \cdot \partial^\alpha u \partial^\alpha f \partial_3 p \mathrm{d}x' \mathrm{d}\tau \\
K_3^\alpha &= \int_0^t \int_\Gamma n \cdot \partial^\alpha p \partial^\alpha f \partial_3 u \mathrm{d}x' \mathrm{d}\tau \\
K_4^\alpha &= \int_0^t \int_\Gamma n \cdot \partial^\alpha f \partial_3 p \partial^\alpha f \partial_3 u \mathrm{d}x' \mathrm{d}\tau
\end{aligned}$$

Note that K_4^α can be estimated directly by $tK(t)$ using notation analogous to that in 8. The term K_3^α can also be estimated by $tK(t)$ after using (17.1.5) to replace $\partial^\alpha p$ by $\sigma \partial^\alpha \nabla' \cdot \hat{n}$ and integrating by parts in space. The term K_1^α can be estimated as before, again by using (17.1.5) to replace $\partial^\alpha p$ by $\sigma \partial^\alpha \nabla' \cdot \hat{n}$.

Now let us consider K_2^α. Using the equation for the front (17.1.4), we obtain

$$\begin{aligned}
K_2^\alpha &= \int_0^t \int_\Gamma \partial^\alpha (u \cdot n) \partial^\alpha f \partial_3 p \mathrm{d}x' \mathrm{d}\tau - \int_0^t \int_\Gamma u \cdot \partial^\alpha n \partial^\alpha f \partial_3 p \mathrm{d}x' \mathrm{d}\tau + R \\
&= \int_0^t \int_\Gamma \partial^\alpha \partial_t f \partial^\alpha f \partial_3 p \mathrm{d}x' \mathrm{d}\tau + \int_0^t \int_\Gamma u \cdot \nabla' \partial^\alpha f \partial^\alpha f \partial_3 p \mathrm{d}x' \mathrm{d}\tau + R
\end{aligned}$$

where we are using R to denote a generic remainder term satisfying $|R| \leq tK(t)$ and we are not being very precise with the remainder terms. We use the product rule and apply the divergence theorem to the second term to obtain

$$\begin{aligned}
-K_2^\alpha &= -\frac{1}{2} \int_\Gamma |\partial^\alpha f|^2 \partial_3 p \mathrm{d}x' \Big|_0^t - \frac{1}{2} \int_0^t \int_\Gamma \nabla' \cdot (u |\partial^\alpha f|^2 \partial_3 p) \mathrm{d}x' \mathrm{d}\tau + R \\
&\geq -\frac{1}{2} \int_\Gamma |\partial^\alpha f|^2 \partial_3 p \mathrm{d}x' - M^0 - tK(t)
\end{aligned}$$

Hence the Taylor sign condition $-\partial_3 p|_\Gamma \geq \frac{\lambda^0}{2} > 0$ ensures we have an estimate for $||\partial^\alpha f||^2_{L^2(\Gamma)}$.

The rest of the energy estimate is much the same as in 8, except we use the corrected unknowns throughout. The estimate of the unknowns given an estimate for the corrected unknowns is simple given that we have 4 derivatives in our energy, and is obtained as follows.

$$\begin{aligned}
||\partial^\alpha U||^2_{L^2(\Omega)} &\leq C\, ||U^\alpha||^2_{L^2(\Omega)} + ||\partial^\alpha \psi||^2_{L^2(\Omega)} \left|\left| \frac{\partial_{x_3} U}{J\psi} \right|\right|^2_{L^\infty(\Omega)} \\
&\leq C\, ||U^\alpha||^2_{L^2(\Omega)} + C\, ||\partial^\alpha f||^2_{L^2(\Gamma)} (M^0 + \int_0^t \left|\left| \partial_t (\frac{\partial_{x_3} U}{J\psi}) \right|\right|^2_{H^2(\Omega)} \mathrm{d}\tau) \\
&\leq C\, ||U^\alpha||^2_{L^2(\Omega)} + ||\partial^\alpha f||^2_{L^2(\Gamma)} (M^0 + tK(t))
\end{aligned}$$

where we have used the Sobolev embedding theorem.

This completes our sketch of the proof of 17.3.2.

A. Appendix

A.1 A Hodge Decomposition Based Estimate

The following estimate based on Hodge decomposition allows us to separate an estimate for first order derivatives of a vector field into an estimate of the curl, divergence, and normal component on the boundary.

Note that in this section, $f \in C([0,T] \times W^{2,\infty}(\mathbb{R}^2))$, $\psi = L^a f$ for some $a \geq 0$ as defined in 4.2.2, the domains $\Omega_\pm$ and Γ are defined in 4.2.1, $n = (-\nabla' f, 1)$, J^ψ and the notation ∇^ψ are defined in 4.2.3, and $J^\psi \geq \kappa > 0$ for some constant $\kappa > 0$.

First we write the Hodge decomposition in our setting.

Lemma A.1.1. *Fix $t \in [0,T]$. Let $w \in C^2(\Omega_\pm; \mathbb{R}^3)$. Then*

$$\Delta^\psi w = \nabla^\psi (\nabla^\psi \cdot w) - \nabla^\psi \times (\nabla^\psi \times w) \tag{A.1.1}$$

where

$$(\Delta^\psi w)_i := \nabla^\psi \cdot (\nabla^\psi w_i)$$

Proof. Fix t. We define $z \in C^2(\theta_\pm(t)(\Omega_\pm); \mathbb{R}^3)$ by

$$z = w \circ \theta^{-1}(t)$$

where the diffeomorphisms $\theta_\pm : (0,T) \times \Omega_\pm \to \Omega_\pm^T$ are defined in terms of f in 5.1.1, and we may evaluate them at $t \in [0,T]$ using uniform continuity. We sometime write θ for just the spatial components of θ since θ maps t to itself. We recall the formula (5.1.1) for taking derivatives.

$$\nabla a = (\nabla^\psi (a \circ \theta)) \circ \theta^{-1}$$

Note that the formula also implies

$$\nabla \times a = (\nabla^\psi \times (a \circ \theta)) \circ \theta^{-1}$$
$$\nabla \cdot a = (\nabla^\psi \cdot (a \circ \theta)) \circ \theta^{-1}$$

if a is a vector. Hence

$$\nabla z_i = (\nabla^\psi w_i) \circ \theta^{-1}$$
$$\nabla \times z = (\nabla^\psi \times w) \circ \theta^{-1}$$
$$\nabla \cdot z = (\nabla^\psi \cdot w) \circ \theta^{-1}$$

Applying these formulae again to $a = \nabla z_i$ etc, we obtain

$$\nabla \cdot (\nabla z_i) = (\nabla^\psi \cdot (\nabla^\psi w_i)) \circ \theta^{-1}$$
$$\nabla(\nabla \cdot z) = (\nabla^\psi(\nabla^\psi \cdot w)) \circ \theta^{-1}$$
$$\nabla \times (\nabla \times z) = (\nabla^\psi \times (\nabla^\psi \times w)) \circ \theta^{-1}$$

Now we can apply the well-known identity

$$\Delta z = \nabla(\nabla \cdot z) - \nabla \times (\nabla \times z)$$

to obtain

$$\Delta^\psi w = \nabla^\psi(\nabla^\psi \cdot w) - \nabla^\psi \times (\nabla^\psi \times w)$$

$\square$

Proposition A.1.2. *Fix* $t \in [0, T]$*. Let* $w_\pm \in L^2(\Omega_\pm; \mathbb{R}^3)$ *with* $\nabla^\psi \cdot w \in L^2(\Omega_\pm)$*,* $\nabla^\psi \times w \in L^2(\Omega_\pm; \mathbb{R}^3)$ *and* $w_\pm \cdot n \in H^{0.5}(\Gamma)$*. Then* $w_\pm \in H^1(\Omega_\pm)$ *with*

$$\begin{aligned} ||w||_{H^1(\Omega_\pm)} &\leq C(1 + \left|\left|\nabla' f\right|\right|^3_{W^{1,\infty}(\Gamma)}) \\ &\quad \times \left(||w||_{L^2(\Omega_\pm)} + \left|\left|\nabla^\psi \cdot w\right|\right|_{L^2(\Omega_\pm)} + \left|\left|\nabla^\psi \times w\right|\right|_{L^2(\Omega_\pm)} + ||w_\pm \cdot n||_{H^{0.5}(\Gamma)} \right) \end{aligned}$$

Proof. Note we may assume $w \in C_c^\infty(\Omega_\pm)$ else we may approximate w by a sequence of $C_c^\infty(\Omega_\pm)$ functions which converges to w pointwise, and for which the norms on the right hand side are bounded by those of w up to a constant, from which we obtain that $w \in H^1(\Omega_\pm)$ with the above estimate.

We dot the formula A.1.1 with $w_\pm$, integrate over $\Omega_\pm$ and use integration by parts. We use summation convention for notational efficiency, where

$$\begin{aligned} \partial_j^\psi &:= \partial_j - \frac{1}{J^\psi}\partial_j \psi \partial_{x_3} \\ &= \partial_j' + \frac{(-\nabla'\psi, 1)_j}{J^\psi}\partial_{x_3} \end{aligned}$$

We obtain the following.

$$\begin{aligned} 0 &= -\int_{\Omega_\pm} w_i \partial_j^\psi \partial_j^\psi w_i \mathrm{d}x + \int_{\Omega_\pm} w_i \partial_i^\psi (\nabla^\psi \cdot w)\mathrm{d}x - \int_{\Omega_\pm} \epsilon_{ijk} w_i \partial_j^\psi (\nabla^\psi \times w)_k \mathrm{d}x \\ &= -\int_{\Omega_\pm} \partial_j^\psi (w_i \partial_j^\psi w_i)\mathrm{d}x + \int_{\Omega_\pm} \partial_j^\psi w_i \partial_j^\psi w_i \mathrm{d}x \\ &+ \int_{\Omega_\pm} \partial_i^\psi (w_i \nabla^\psi \cdot w)\mathrm{d}x - \int_{\Omega_\pm} \left|\nabla^\psi \cdot w\right|^2 \mathrm{d}x \\ &- \int_{\Omega_\pm} \epsilon_{ijk} \partial_j^\psi (w_i (\nabla^\psi \times w)_k)\mathrm{d}x + \int_{\Omega_\pm} \epsilon_{ijk} (\partial_j^\psi w_i)(\nabla^\psi \times w)_k \mathrm{d}x \\ &= \pm \int_\Gamma n_j w_i \partial_j^\psi w_i \mathrm{d}x' + \int_{\Omega_\pm} \partial_{x_3}(\frac{(-\nabla'\psi, 1)_j}{J^\psi}) w_i \partial_j^\psi w_i \mathrm{d}x + \sum_{j=1}^3 \int_{\Omega_\pm} \left|\partial_j^\psi w\right|^2 \mathrm{d}x \\ &\mp \int_\Gamma n_i w_i \nabla^\psi \cdot w \mathrm{d}x' - \int_{\Omega_\pm} \partial_{x_3}(\frac{(-\nabla'\psi, 1)_i}{J^\psi}) w_i \nabla^\psi \cdot w \mathrm{d}x - \int_{\Omega_\pm} \left|\nabla^\psi \cdot w\right|^2 \mathrm{d}x \\ &\pm \int_\Gamma \epsilon_{ijk} n_j w_i (\nabla^\psi \times w)_k \mathrm{d}x' + \int_{\Omega_\pm} \epsilon_{ijk} \partial_{x_3}(\frac{(-\nabla'\psi, 1)_j}{J^\psi}) w_i (\nabla^\psi \times w)_k \mathrm{d}x - \int_{\Omega_\pm} \left|\nabla^\psi \times w\right|^2 \mathrm{d}x \end{aligned}$$

Let us denote by I all the terms on the right hand side involving boundary integrals.

Note that we have

$$\epsilon_{ijk} n_j w_i (\nabla^\psi \times w)_k = \epsilon_{ijk}\epsilon_{klm} n_j w_i \partial_l^\psi w_m$$

Using the formula

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$$

we obtain

$$\epsilon_{ijk} n_j w_i (\nabla^\psi \times w)_k = n_j w_i \partial_i^\psi w_j - n_j w_i \partial_j^\psi w_i$$

Thus we have

$$\begin{aligned} I &= \pm \int_\Gamma n_j w_i \partial_j^\psi w_i \mathrm{d}x' \mp \int_\Gamma n_i w_i \nabla^\psi \cdot w \mathrm{d}x' \pm \int_\Gamma n_j w_i \partial_i^\psi w_j - n_j w_i \partial_j^\psi w_i \mathrm{d}x' \\ &= \mp \int_\Gamma n_i w_i \nabla^\psi \cdot w \mathrm{d}x' \pm \int_\Gamma n_j w_i \partial_i^\psi w_j \mathrm{d}x' \end{aligned}$$

after cancelling the first and last terms. Now we use the definition of ∂^ψ to obtain

$$\begin{aligned} I &= \mp \int_\Gamma (n\cdot w)(\nabla' \cdot w') + (n \cdot w)(n \cdot \partial_{x_3} w) \mathrm{d}x' \pm \int_\Gamma n_j w_i' \partial_i' w_j + n_j w_i n_i \partial_{x_3} w_j \mathrm{d}x' \\ &= \mp \int_\Gamma (n\cdot w)(\nabla' \cdot w')\mathrm{d}x' \pm \int_\Gamma n_j w_i' \partial_i' w_j \mathrm{d}x' \\ &= \mp \int_\Gamma (n\cdot w)(\nabla' \cdot w')\mathrm{d}x' \pm \int_\Gamma (w' \cdot \nabla')(w \cdot n) - w \cdot ((w' \cdot \nabla')n) \mathrm{d}x' \end{aligned}$$

after cancelling the second and last terms then using the product rule. Thus

$$\begin{aligned} |I| \leq{} & \left|\left|\nabla' \cdot w'\right|\right|_{H^{-0.5}(\Gamma)} \left|\left|w \cdot n\right|\right|_{H^{0.5}(\Gamma)} + \left|\left|\nabla'(w\cdot n)\right|\right|_{H^{-0.5}(\Gamma)} \left|\left|w'\right|\right|_{H^{0.5}(\Gamma)} \\ & + \left|\left|w\right|\right|_{L^2(\Gamma)}^2 \left|\left|\nabla' n\right|\right|_{L^\infty(\Gamma)} \end{aligned}$$

Note that

$$\left|\left|w\right|\right|_{L^2(\Gamma)}^2 = \mp 2 \int_{\Omega_\pm} w \cdot \partial_{x_3} w \mathrm{d}x \leq \frac{1}{\epsilon} \left|\left|w\right|\right|_{L^2(\Omega_\pm)}^2 + \epsilon \left|\left|w\right|\right|_{H^1(\Omega_\pm)}^2$$

where we are free to choose $\epsilon > 0$. Choosing $\epsilon = \frac{\delta}{||\nabla' n||_{L^\infty(\Gamma)}}$, for $\delta > 0$ to be determined, and using the Sobolev trace estimate and Cauchy's inequality, we obtain

$$|I| \leq \frac{C}{\delta} \left|\left|w \cdot n\right|\right|_{H^{0.5}(\Gamma)}^2 + \frac{C}{\delta} \left|\left|\nabla' f\right|\right|_{W^{1,\infty}(\Gamma)}^2 \left|\left|w\right|\right|_{L^2(\Omega_\pm)}^2 + C\delta \left|\left|w\right|\right|_{H^1(\Omega_\pm)}^2$$

After rearranging we obtain

$$\begin{aligned} \sum_{j=1}^3 \int_{\Omega_\pm} \left|\partial_j^\psi w\right|^2 \mathrm{d}x ={} & \int_{\Omega_\pm} \left|\nabla^\psi \cdot w\right|^2 \mathrm{d}x + \int_{\Omega_\pm} \left|\nabla^\psi \times w\right|^2 \mathrm{d}x - I \\ & - \int_{\Omega_\pm} \partial_{x_3}\left(\frac{(-\nabla'\psi, 1)_j}{J^\psi}\right) w_i \partial_j^\psi w_i \mathrm{d}x + \int_{\Omega_\pm} \partial_{x_3}\left(\frac{(-\nabla'\psi, 1)_i}{J^\psi}\right) w_i \nabla^\psi \cdot w \mathrm{d}x \\ & - \int_{\Omega_\pm} \epsilon_{ijk} \partial_{x_3}\left(\frac{(-\nabla'\psi, 1)_j}{J^\psi}\right) w_i (\nabla^\psi \times w)_k \mathrm{d}x \\ \leq{} & \left|\left|\nabla^\psi \cdot w\right|\right|_{L^2(\Omega_\pm)}^2 + \left|\left|\nabla^\psi \times w\right|\right|_{L^2(\Omega_\pm)}^2 \\ & + \frac{C}{\delta} \left|\left|w \cdot n\right|\right|_{H^{0.5}(\Gamma)}^2 + \frac{C}{\delta} \left|\left|\nabla' f\right|\right|_{W^{1,\infty}(\Gamma)}^2 \left|\left|w\right|\right|_{L^2(\Omega_\pm)}^2 + C\delta \left|\left|w\right|\right|_{H^1(\Omega_\pm)}^2 \\ & + \frac{C}{\delta} \left|\left|\nabla' f\right|\right|_{W^{1,\infty}(\Gamma)}^2 \left|\left|w\right|\right|_{L^2(\Omega_\pm)}^2 + C\delta \left|\left|w\right|\right|_{H^1(\Omega_\pm)}^2 \end{aligned}$$

Finally, we observe that

$$\partial_{x_j} w = \partial_j^\psi w + \partial_j \psi \partial_3^\psi w$$

Hence

$$\begin{aligned}
||w||_{H^1(\Omega_\pm)}^2 \le \frac{C}{\delta}(1 + \left|\left|\nabla' f\right|\right|_{W^{1,\infty}(\Gamma)}^4) \\
\times \left(\left|\left|\nabla^\psi \cdot w\right|\right|_{L^2(\Omega_\pm)}^2 + \left|\left|\nabla^\psi \times w\right|\right|_{L^2(\Omega_\pm)}^2 + ||w \cdot n||_{H^{0.5}(\Gamma)}^2 + ||w||_{L^2(\Omega_\pm)}^2\right) \\
+ C\delta(1 + \left|\left|\nabla' f\right|\right|_{W^{1,\infty}(\Gamma)}^2) ||w||_{H^1(\Omega_\pm)}^2
\end{aligned}$$

Choosing $\delta = \frac{1}{2C(1+||\nabla' f||_{W^{1,\infty}(\Gamma)}^2)}$, we obtain

$$\begin{aligned}
||w||_{H^1(\Omega_\pm)}^2 \le C(1 + \left|\left|\nabla' f\right|\right|_{W^{1,\infty}(\Gamma)}^6) \\
\times \left(\left|\left|\nabla^\psi \cdot w\right|\right|_{L^2(\Omega_\pm)}^2 + \left|\left|\nabla^\psi \times w\right|\right|_{L^2(\Omega_\pm)}^2 + ||w \cdot n||_{H^{0.5}(\Gamma)}^2 + ||w||_{L^2(\Omega_\pm)}^2\right)
\end{aligned}$$

which implies the result (up to adjusting the constant and taking square roots). □

A.2 Lifting Lemmas

Notation. *We will use the notation $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ for $\xi \in \mathbb{R}^d$.*

Lemma A.2.1. *Let $l \ge 0$ be an integer and $d \ge 2$ be an integer. Let $h^k \in H^{l-k+0.5}(\mathbb{R}^{d-1})$ for $0 \le k \le l$. Then there exists $g \in H^{l+1}(\mathbb{R}^d)$ such that*

$$\mathrm{Tr}(\partial_{x_d}^k g) = h^k \text{ for } 0 \le k \le l$$

and

$$||g||_{H^{l+1}(\mathbb{R}^d)} \le C \sum_{k=0}^{l} \left|\left|h^k\right|\right|_{H^{l-k+0.5}(\mathbb{R}^{d-1})}$$

Moreover, the map $\mathcal{L} : (h^0, \ldots, h^l) \mapsto g$ is linear.

Proof. First suppose $h^k \in C_c^\infty(\mathbb{R}^{d-1})$ for $0 \le k \le l$. Write $x = (x', x_d) \in \mathbb{R}^d$ with $x' \in \mathbb{R}^{d-1}$. Let $\chi \in C_c^\infty(\mathbb{R})$ be a smooth positive cut-off function with $\chi(s) = 1$ for $|s| \le 1$. We define $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = \sum_{k=0}^{l} \frac{1}{k!} x_d^k \chi(x_d \langle D' \rangle) h^k(x') := \sum_{k=0}^{l} \int_{\mathbb{R}^{d-1}} e^{2\pi i x' \cdot \xi'} \frac{1}{k!} x_d^k \chi(x_d \langle \xi' \rangle) \mathcal{F}(h^k)(\xi') \mathrm{d}\xi'$$

Note that, for $0 \le m \le l+1$, we have

$$\partial_{x_d}^m g(x) = \sum_{k=0}^{l} \sum_{i+j=m, i \le k} \int_{\mathbb{R}^{d-1}} e^{2\pi i x' \cdot \xi'} \frac{1}{(k-i)!} x_d^{k-i} \chi^{(j)}(x_d \langle \xi' \rangle) \langle \xi' \rangle^j \mathcal{F}(h^k)(\xi') \mathrm{d}\xi'$$

Evaluating at $x_d = 0$, we obtain, for $0 \le m \le l$,

$$\begin{aligned}
\partial_{x_d}^m g(x', 0) &= \sum_{k=0}^{l} \int_{\mathbb{R}^{d-1}} e^{2\pi i x' \cdot \xi'} \chi^{(m-k)}(x_d \langle \xi' \rangle) \langle \xi' \rangle^{m-k} \mathcal{F}(h^k)(\xi') \mathrm{d}\xi' \\
&= \int_{\mathbb{R}^{d-1}} e^{2\pi i x' \cdot \xi'} \mathcal{F}(h^m)(\xi') \mathrm{d}\xi' \\
&= h^m(x')
\end{aligned}$$

where we have used the fact that $\chi(s) = 1$ for $|s| \leq 1$ so $\chi^{(l)}(0) = 0$ for $l \geq 1$.

Now let $m, n \geq 0$ with $0 \leq m + n \leq l + 1$. Note that from the above formula we have

$$\mathcal{F}(\partial_{x_d}^m g)(\xi', x_d) = \sum_{k=0}^{l} \sum_{i+j=m, i \leq k} \frac{1}{(k-i)!} x_d^{k-i} \chi^{(j)}(x_d \langle \xi' \rangle) \langle \xi' \rangle^j \mathcal{F}(h^k)(\xi')$$

where we are taking the Fourier transform with respect to x'. Thus

$$\begin{aligned}
||\partial_{x_d}^m g||_{H^n(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \mathcal{F}(\partial_{x_d}^m g)(\xi', x_d)^2 \langle \xi' \rangle^{2n} \mathrm{d}\xi' \mathrm{d}x_d \\
&\leq C \sum_{k=0}^{l} \sum_{i+j=m, i \leq k} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left| x_d^{k-i} \chi^{(j)}(x_d \langle \xi' \rangle) \right|^2 \langle \xi' \rangle^{2j} \mathcal{F}(h^k)(\xi')^2 \langle \xi' \rangle^{2n} \mathrm{d}\xi' \mathrm{d}x_d \\
&= C \sum_{k=0}^{l} \sum_{i+j=m, i \leq k} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left| x_d^{k-i} \chi^{(j)}(x_d \langle \xi' \rangle) \right|^2 \mathrm{d}x_d \langle \xi' \rangle^{2j} \mathcal{F}(h^k)(\xi')^2 \langle \xi' \rangle^{2n} \mathrm{d}\xi' \\
&= C \sum_{k=0}^{l} \sum_{i+j=m, i \leq k} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \left| y_d^{k-i} \chi^{(j)}(y_d) \right|^2 \frac{1}{\langle \xi' \rangle^{2k-2i+1}} \mathrm{d}y_d \langle \xi' \rangle^{2j} \mathcal{F}(h^k)(\xi')^2 \langle \xi' \rangle^{2n} \mathrm{d}\xi' \\
&\leq C \sum_{k=0}^{l} \sum_{i+j=m, i \leq k} \int_{\mathbb{R}^{d-1}} \frac{1}{\langle \xi' \rangle^{2k-2i+1}} \langle \xi' \rangle^{2j} \mathcal{F}(h^k)(\xi')^2 \langle \xi' \rangle^{2n} \mathrm{d}\xi' \\
&\leq C \sum_{k=0}^{l} \int_{\mathbb{R}^{d-1}} \mathcal{F}(h^k)(\xi')^2 \langle \xi' \rangle^{2m-2k+2n-1} \mathrm{d}\xi' \\
&\leq C \sum_{k=0}^{l} \left|\left| h^k \right|\right|_{H^{l+0.5-k}}^2
\end{aligned}$$

This shows that $\mathcal{L} : (h^0, \dots, h^l) \mapsto g$ is a bounded linear map on $\prod_{k=0}^{l} (H^{l+0.5-k}(\mathbb{R}^{d-1}) \cap C_c^\infty(\mathbb{R}^{d-1}))$. Since this is a dense subset of $\prod_{k=0}^{l} (H^{l+0.5-k}(\mathbb{R}^{d-1}))$, the map extends uniquely to a bounded linear map on the whole space. This completes the proof. □

Remark A.2.2. Note that we can use the above lemma with $h^k = 0$ for $1 \leq k \leq l$ if we don't require constraints on $\partial_{x_d}^k g|_{x_d=0}$ for $k \geq 1$.

Lemma A.2.3. *Let $l \geq 0$ be an integer and $d \geq 2$ be an integer. Let $h : (0, T) \times \mathbb{R}^{d-1} \to \mathbb{R}$ be such that $\partial_t^j h \in H^{l+0.5-j}(\mathbb{R}^{d-1})$ for $0 \leq j \leq l$. Define $g : (0, T) \times \mathbb{R}^d \to \mathbb{R}$ by*

$$g(t) = \mathcal{L}(h(t), 0, \dots, 0)$$

where $\mathcal{L}$ is given above. Then for $0 \leq j \leq l$,

$$\partial_t^j g = \mathcal{L}(\partial_t^j h(t), 0, \dots, 0)$$

so

$$\left|\left| \partial_t^j g(t) \right|\right|_{H^{l+1-j}(\mathbb{R}^d)} \leq C \left|\left| \partial_t^j h(t) \right|\right|_{H^{l-j+0.5}(\mathbb{R}^{d-1})}$$

for some constant $C > 0$ and

$$\mathrm{Tr}(\partial_t^j g) = \partial_t^j h$$

Also, for $0 \leq j \leq l - k$ and $k \geq 1$,

$$\mathrm{Tr}(\partial_{x_d}^k \partial_t^j g) = 0$$

Proof. We apply the above result to $(h, 0 \ldots, 0)$ with t as a parameter. The result then follows using the linearity of $\mathcal{L}$ and exchanging the order of integration when calculating the weak time derivatives, to obtain that the weak derivatives of g with respect to t are given by

$$\partial_t^j g = \mathcal{L}(\partial_t^j h(t), 0 \ldots, 0)$$

□

A.3 Products and Compositions in Sobolev Spaces

The following lemma is very useful for proving chain and product rules in Sobolev spaces. This is important for obtaining high-order energy estimates.

Lemma A.3.1. *Let $p \in [1, \infty]$, $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be a domain where the standard Sobolev embedding holds and let $m > \frac{d}{p}$ be an integer. Let $0 \leq m_i \leq m$ be integers for $1 \leq i \leq n$ with $\sum_{i=1}^n m_i \geq (n-1)m$ and let $u_i \in W^{m_i,p}(\Omega)$. Then $\prod_{i=1}^n u_i \in L^p(\Omega)$ and*

$$\left|\left|\prod_{i=1}^n u_i\right|\right|_{L^p(\Omega)} \leq C \prod_{i=1}^n ||u_i||_{W^{m_i,p}(\Omega)}$$

Proof. For $p = \infty$ the result is obvious and in fact only requires $m \geq 0$, so we will assume $p < \infty$.

We will use the following Sobolev embeddings. Let $k \geq 1$ be an integer and $u \in W^{k,p}(\Omega)$. Then for $q \geq p$,

$$||u||_{L^q(\Omega)} \leq C \, ||u||_{W^{k,p}(\Omega)}$$

provided

$$\frac{1}{q} > \frac{1}{p} - \frac{k}{d}$$

and $kp \leq d$. (Note it is the case $kp = d$ that requires the inequality to be strict). If $kp > d$ then

$$||u||_{L^\infty(\Omega)} \leq C \, ||u||_{W^{k,p}(\Omega)}$$

Suppose $m_i p > d$ for some i. By renumbering if necessary, we may assume $m_n p > d$. Then

$$\begin{aligned}\left|\left|\prod_{i=1}^n u_i\right|\right|_{L^p(\Omega)} &\leq \left|\left|\prod_{i=1}^{n-1} u_i\right|\right|_{L^p(\Omega)} ||u_n||_{L^\infty(\Omega)} \\ &\leq C \left|\left|\prod_{i=1}^{n-1} u_i\right|\right|_{L^p(\Omega)} ||u_n||_{W^{m_n,p}(\Omega)}\end{aligned}$$

Also note that since $m_n \leq m$, we have $\sum_{i=1}^{n-1} m_i \geq (n-2)m$. Hence we are reduced to proving the result with n replaced by $n-1$. Thus we may assume $m_i p \leq d$ for all i.

Suppose $m_i = 0$ for some i. By renumbering if necessary, we may assume $m_n = 0$. Then $\sum_{i=1}^{n-1} m_i \geq (n-1)m$ and $0 \leq m_i \leq m$ implies $m_i = m > \frac{d}{p}$ for all $i < n$, hence

$$\begin{aligned}\left|\left|\prod_{i=1}^n u_i\right|\right|_{L^p(\Omega)} &\leq \prod_{i=1}^{n-1} ||u_i||_{L^\infty(\Omega)} \, ||u_n||_{L^p(\Omega)} \\ &\leq C \prod_{i=1}^{n-1} ||u_i||_{W^{m_i,p}(\Omega)} \, ||u_n||_{W^{m_n,p}(\Omega)}\end{aligned}$$

Thus we may assume $m_i > 0$ for all i.

Now, using Hölder's inequality,

$$\left|\left|\prod_{i=1}^{n} u_i\right|\right|_{L^p(\Omega)} \leq \prod_{i=1}^{n} ||u_i||_{L^{\frac{p}{\lambda_i}}(\Omega)}$$

where $\sum_{i=1}^{n} \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$ for all i. Hence, using Sobolev embedding, we have

$$\left|\left|\prod_{i=1}^{n} u_i\right|\right|_{L^p(\Omega)} \leq C \prod_{i=1}^{n} ||u_i||_{W^{m_i,p}(\Omega)}$$

provided

$$\frac{\lambda_i}{p} > \frac{1}{p} - \frac{m_i}{d}$$

for all i. But, summing the above inequalities, it is possible, assuming $0 < m_i \leq \frac{d}{p}$, to choose such $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^{n} \lambda_i = 1$ if and only if

$$\frac{n}{p} - \frac{\sum_{i=1}^{n} m_i}{d} < \frac{1}{p} \iff \sum_{i=1}^{n} m_i > (n-1)\frac{d}{p}$$

But this does indeed hold since $\sum_{i=1}^{n} m_i \geq (n-1)m$ and $m > \frac{d}{p}$. $\square$

Corollary A.3.2 (The Leibniz Rule or The Product Rule). *Let $p \in [1, \infty]$, $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be a domain where the standard Sobolev embedding holds and let $m > \frac{d}{p}$ be an integer. Let $0 \leq m_i \leq m$ and be integers for $1 \leq i \leq n$ and $0 \leq k \leq m$ be an integer, with $\sum_{i=1}^{n} m_i \geq (n-1)m + k$. Let $u_i \in W^{m_i,p}(\Omega)$. Then $\prod_{i=1}^{n} u_i \in W^{k,p}(\Omega)$ with weak derivatives given by the classical Leibniz rule and*

$$\left|\left|\prod_{i=1}^{n} u_i\right|\right|_{W^{k,p}(\Omega)} \leq C \prod_{i=1}^{n} ||u_i||_{W^{m_i,p}(\Omega)}$$

Proof. Let γ^i be multi-indices with $\sum_{i=1}^{n} \gamma_i = \alpha$, where $|\alpha| \leq k$. Note that $\sum_{i=1}^{n}(m_i - |\gamma_i|) \geq (n-1)m$, hence we may apply the above result to obtain

$$\left|\left|\prod_{i=1}^{n} \partial^{\gamma^i} u_i\right|\right|_{L^p(\Omega)} \leq C \prod_{i=1}^{n} ||u_i||_{W^{m_i,p}(\Omega)}$$

Assuming u_i are smooth, we immediately obtain the result, since $\partial^\alpha \prod_{i=1}^{n} u_i$ is a sum of terms of the form $\prod_{i=1}^{n} \partial^{\gamma^i} u_i$ by the classical chain rule. For non-smooth u_i we use approximation by smooth functions together with this inequality. $\square$

Corollary A.3.3 (The Chain Rule). *Let $p \in [1, \infty]$, $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be a domain where the standard Sobolev embedding holds and let $m > \frac{d}{p}$ be an integer. Let $F \in C_b^m(\mathbb{R}^{d'})$ and $u : \Omega \to \mathbb{R}^{d'}$ with $u \in W^{m,p}(\Omega)$. Let α be a multi-index with $1 \leq |\alpha| \leq m$. Let $0 < \beta \leq \alpha, 0 < \gamma^j \leq \alpha$ $(1 \leq j \leq |\beta|)$, be multi-indices with $\sum_{j=1}^{|\beta|} \left|\gamma^j\right| = |\alpha|$. Then*

$$\left|\left|(\partial^\beta F)(u) \prod_{j=1}^{|\beta|} \partial^{\gamma^j} u_{i_j}\right|\right|_{L^p(\Omega)} \leq C\, ||u||_{W^{m,p}(\Omega)}^{|\beta|}$$

where u_{i_j} denotes a component of u depending on j. Moreover, the function $F(u) \in L^\infty(\Omega)$ has a weak α-derivative in $L^p(\Omega)$ given as in the classical chain rule by sums of terms of the above form which satisfies the inequality

$$||\partial^\alpha(F(u))||_{L^p(\Omega)} \leq C\,||u||_{W^{m,p}(\Omega)}\,(1 + ||u||_{W^{m,p}(\Omega)})^{m-1}$$

In addition, if $F(0) = 0$, then $F(u) \in W^{m,p}(\Omega)$ with

$$||F(u)||_{W^{m,p}(\Omega)} \leq C\,||u||_{W^{m,p}(\Omega)}\,(1 + ||u||_{W^{m,p}(\Omega)})^{m-1}$$

Proof. Note that $\sum_{j=1}^{\beta}(m - |\gamma^j|) = |\beta|\,m - |\alpha| \geq (|\beta| - 1)m$, hence we may use the above result and the fact that $\partial^\beta F$ is bounded to obtain

$$\begin{aligned}\left|\left|(\partial^\beta F)(u)\prod_{j=1}^{|\beta|}\partial^{\gamma^j}u_{i_j}\right|\right|_{L^p(\Omega)} &\leq C\prod_{j=1}^{|\beta|}\left|\left|\partial^{\gamma^j}u_{i_j}\right|\right|_{W^{m-|\gamma^j|,p}(\Omega)}\\ &\leq C\,||u||_{W^{m,p}(\Omega)}^{|\beta|}\end{aligned}$$

Assuming u_i are smooth, we immediately obtain the required inequalities, since $\partial^\alpha(F(u))$ is a sum of terms of the form $(\partial^\beta F)(u)\prod_{j=1}^{|\beta|}\partial^{\gamma^j}u_{i_j}$ by the classical product and chain rules. For non-smooth u_i we use approximation by smooth functions together with this inequality.

Finally, if $F(0) = 0$, then we have

$$\begin{aligned}|F(u)| &= \left|\int_0^1 DF(tu)u\mathrm{d}t\right|\\ &\leq C\,|u|\end{aligned}$$

since DF is bounded.

Thus $F(u) \in L^p(\Omega)$ with $||F(u)||_{L^p(\Omega)} \leq C\,||u||_{L^p(\Omega)}$. Together with the previous part, this implies the final statement of the result. □

Corollary A.3.4. *Let $D, D' \subset \mathbb{R}^{d'}$ be open with $D' \subset\subset D$. Let $F \in C^m(D)$ and $u : \Omega \to D'$ with $u \in W^{m,p}(\Omega)$. Then the above chain rule holds with these new F and u.*

Proof. Since u takes values in D', we may modify F outside D' by multiplying by a smooth cut-off function which is identically 1 on D' and 0 outside D'' for some $D'' \subset\subset D$, so we may assume $F \in C_b^m(\mathbb{R}^{d'})$, and then we can apply the above result. □

Corollary A.3.5 (Change of Coordinates)**.** *Let $p \in [1, \infty]$ and let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ with $d \geq 1$ be domains where the standard Sobolev embedding holds. Let $\theta : \Omega_1 \to \Omega_2$ be a bijection and suppose that θ is Lipschitz. Set $J = |\det(\nabla\theta)|$ and assume $\frac{1}{J} \leq K$ almost everywhere for some constant $K > 0$. Let $v \in L^q(\Omega_2)$ for some $q \in [0, \infty]$. Then*

$$||v \circ \theta||_{L^q(\Omega_1)} \leq C_K\,||v||_{L^q(\Omega_2)}$$

where the constant $C_K < \infty$ is an increasing function of K. Let $u \in W^{1,p}(\Omega_2)$. Then $u \circ \theta \in W^{1,p}(\Omega_1)$ with weak derivatives given by the classical chain rule, and

$$||u \circ \theta||_{W^{1,p}(\Omega_1)} \leq C_{K,\mathrm{Lip}\,\theta}\,||u||_{W^{1,p}(\Omega_2)}$$

where the constant $C_{K,\mathrm{Lip}\,\theta} < \infty$ is an increasing function of K and $\mathrm{Lip}\,\theta$. Moreover, suppose that $u \in W^{m',p}(\Omega_2)$ and $\nabla^k\nabla\theta = \nabla\theta^{k,\infty} + \nabla\theta^{k,p}$ where $\nabla\theta^{k,\infty} \in L^\infty(\Omega_1)$ and $\nabla\theta^{k,p} \in L^p(\Omega_1)$ for $1 \leq k \leq m$ with $m > \frac{d}{p}$ and $0 \leq m' \leq m+1$. Then $u \circ \theta \in W^{m',p}(\Omega_1)$ with derivatives given by the classical chain rule, and

$$||u \circ \theta||_{W^{m',p}(\Omega_1)} \leq C_\theta \, ||u||_{W^{m',p}(\Omega_2)}$$

where the constant $C_\theta < \infty$ is an increasing function of K, $\mathrm{Lip}\,\theta$, $\left|\left|\nabla\theta^{k,\infty}\right|\right|_{L^\infty(\Omega_1)}$ and $\left|\left|\nabla\theta^{k,p}\right|\right|_{L^p(\Omega_1)}$ for $1 \leq k \leq m$.

Proof. Firstly, note that by the change of variables formula for integrals, we have

$$\begin{aligned}\int_{\Omega_1} |v \circ \theta|^q \,\mathrm{d}x &= \int_{\Omega_2} |v|^q \,\frac{1}{J \circ \theta^{-1}}\mathrm{d}x \\ &\leq K \, ||v||^q_{L^q(\Omega_2)}\end{aligned}$$

which proves the first part.

Now

$$\begin{aligned}||(\partial_i u) \circ \theta \partial_j \theta_i||_{L^p(\Omega_1)} &\leq ||Du \circ \theta||_{L^p(\Omega_1)} \, ||D\theta||_{L^\infty(\Omega_1)} \\ &\leq C_{K,\mathrm{Lip}\,\theta} \, ||Du||_{L^p(\Omega_2)}\end{aligned}$$

by the first part.

Thus, assuming u is smooth we may apply the classical chain rule to conclude that $u \circ \theta$ has weak derivatives of first order in $L^p(\Omega_1)$ given by the classical chain rule, satisfying the claimed inequality. If u is not smooth, we obtain the same result by considering the standard mollification u^ϵ of u and using the inequality just proved to pass to the limit $\epsilon \to 0$. Together with the first part, this proves the second part of the result.

Now we suppose that $u \in W^{m',p}(\Omega_2)$ for some integer $m' \geq 0$ and $\nabla^k\nabla\theta = \nabla\theta^{k,\infty} + \nabla\theta^{k,p}$ where $\nabla\theta^{k,\infty} \in L^\infty(\Omega_1)$ and $\nabla\theta^{k,p} \in L^p(\Omega_1)$ for $1 \leq k \leq m$ with $m > \frac{d}{p}$ and $0 \leq m' \leq m+1$. Let α be a multi-index with $|\alpha| \leq m'$. Let β, γ^j be multi-indices with $|\beta| + \sum_{j=1}^{|\beta|} \left|\gamma^j\right| = |\alpha|$. We claim that

$$\begin{aligned}&\left|\left|(\partial^\beta u) \circ \theta \prod_{j=1}^{|\beta|} \partial^{\gamma^j} \nabla\theta_{e(j)}\right|\right|_{L^p(\Omega_1)} \\ &\leq C_\theta \, ||u||_{W^{m',p}(\Omega_2)}\end{aligned}$$

where $\nabla\theta_{e(j)}$ denotes an entry of the matrix $\nabla\theta$ depending on j. Indeed, note that $(\partial^\beta u) \circ \theta \in W^{m'-|\beta|,p}(\Omega_1)$ by the first part, and that in the worst case scenario for Sobolev embedding we may assume that $\partial^{\gamma^j}\nabla\theta_{e(j)} \in W^{m-|\gamma^j|,p}(\Omega_1)$ for all j. Now $m' - |\beta| + \sum_{j=1}^{|\beta|}(m - \left|\gamma^j\right|) = m' + |\beta|\, m - |\alpha| \geq |\beta|\, m$, which is one less the number of terms in the product times m, and we note that $m' - |\beta| \leq m' - 1 \leq m$ and $m - \left|\gamma^j\right| \leq m$ for all j, hence we may apply the above result on products A.3.1 to conclude the claim.

Using the classical chain rule, this proves the result for u smooth, then we conclude by mollifying u. ∎

Lemma A.3.6 (Inverting a Change of Coordinates)**.** *Let $p \in [1, \infty]$, let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ with $d \geq 1$ be domains where the standard Sobolev embedding holds and let $m > \frac{d}{p}$ be an integer. Let $\theta : \Omega_1 \to \Omega_2$ be a bijection and suppose that $\theta \in C^1(\Omega_1)$ with $\nabla^k \nabla\theta = \nabla\theta^{k,\infty} + \nabla\theta^{k,p}$ where $\nabla\theta^{k,\infty} \in L^\infty(\Omega_1)$ and $\nabla\theta^{k,p} \in L^p(\Omega_1)$ for $1 \leq k \leq m$. Set $J = |\det(\nabla\theta)|$ and assume $\frac{1}{J} \leq K$ for some constant $K > 0$. Then $\theta^{-1} \in C^1(\Omega_2)$ and $\nabla^k\nabla(\theta^{-1}) = \nabla(\theta^{-1})^{k,\infty} + \nabla(\theta^{-1})^{k,p}$ where $\nabla(\theta^{-1})^{k,\infty} \in L^\infty(\Omega_2)$ and $\nabla(\theta^{-1})^{k,p} \in L^p(\Omega_2)$ for $1 \leq k \leq m$, and we have*

$$\begin{aligned}
\left|\left|\nabla(\theta^{-1})\right|\right|_{L^\infty(\Omega_2)} &\leq C_\theta \\
\left|\left|\nabla(\theta^{-1})^{k,\infty}\right|\right|_{L^\infty(\Omega_2)} &\leq C_\theta \\
\left|\left|\nabla(\theta^{-1})^{k,p}\right|\right|_{L^p(\Omega_2)} &\leq C_\theta
\end{aligned}$$

for $1 \leq k \leq m$, where the constant $C_\theta > 0$ is an increasing function of K, $\mathrm{Lip}\,\theta$, $\left|\left|\nabla\theta^{k,\infty}\right|\right|_{L^\infty(\Omega_1)}$ and $\left|\left|\nabla\theta^{k,p}\right|\right|_{L^p(\Omega_1)}$ for $1 \leq k \leq m$.

Proof. Since $\frac{1}{J} \leq K$, the inverse function theorem implies $\theta^{-1} \in C^1(\Omega_2)$ and

$$\nabla(\theta^{-1}) \circ \theta = (\nabla\theta)^{-1}$$

Now

$$\left|(\nabla\theta)^{-1}\right| = \frac{1}{J}\left|\mathrm{cof}\,\nabla\theta\right|$$

where the entries of the matrix $\mathrm{cof}\,\nabla\theta$ are products of entries of $\nabla\theta$. This immediately implies

$$\left|\left|\nabla(\theta^{-1})\right|\right|_{L^\infty(\Omega_2)} \leq C_\theta$$

so that θ^{-1} is Lipschitz. Now, setting $F(M) = \frac{1}{|\det(M)|}\left|\mathrm{cof}\,M\right|$ for M a $d \times d$ matrix, which is smooth on the set $|\det(M)| > 0$, we may us the fact that $\frac{1}{J} \leq K$ to apply the chain rule above to conclude that $\nabla(\theta^{-1}) \circ \theta$ satisfies the conclusions of the lemma.

Now we proceed by induction, using the change of coordinates rule above one derivative at a time. Let γ be a multi-index with $1 \leq |\gamma| \leq m$. We assume by induction that $\nabla(\theta^{-1})$ has weak α-derivatives either in $L^\infty(\Omega_2)$ or $L^p(\Omega_2)$ for $1 \leq |\alpha| \leq |\gamma| - 1$ with components satisfying one of the inequalities

$$\begin{aligned}
\left|\left|\partial^\alpha\nabla_j(\theta^{-1})_i\right|\right|_{L^\infty(\Omega_2)} &\leq C_\theta \\
\left|\left|\partial^\alpha\nabla_j(\theta^{-1})_i\right|\right|_{L^p(\Omega_2)} &\leq C_\theta
\end{aligned}$$

Note that by induction $\partial^\alpha\nabla(\theta^{-1})$ is given by the classical chain rule, so is a sum of terms of the form

$$\partial^\beta(\nabla\theta)^{-1} \circ \theta^{-1} \prod_{j=1}^{|\beta|} \partial^{\gamma^j}\nabla(\theta^{-1})_{e(j)}$$

where $\sum_{j=1}^{|\beta|}\left|\gamma^j\right| + |\beta| = |\alpha|$. But we can rewrite this as

$$(\partial^\beta(\nabla\theta)^{-1} \prod_{j=1}^{|\beta|} \partial^{\gamma^j}(\nabla\theta)^{-1}_{e(j)}) \circ \theta^{-1}$$

Now we apply the product rule to

$$\nabla(\partial^{\beta}(D\theta)^{-1}\prod_{j=1}^{|\beta|}\partial^{\gamma^j}(D\theta)^{-1}_{e(j)})$$

and use our estimates for the components of $\partial^{\gamma}(D\theta)^{-1}$ together with the fact that θ^{-1} is bijective and Lipschitz, to conclude that we can apply the change of coordinates lemma to obtain the desired result. $\square$

Proposition A.3.7. *Let $p \in [1,\infty]$, let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ with $d \geq 1$ be domains where the standard Sobolev embedding holds, let $m > \frac{d}{p}$ be an integer and let m' be an integer with $0 \leq m' \leq m+1$. Let $\theta : \Omega_1 \to \Omega_2$ be a bijection and suppose that $\theta \in C^1(\Omega_1)$ with $\nabla^k\nabla\theta = \nabla\theta^{k,\infty} + \nabla\theta^{k,p}$ where $\nabla\theta^{k,\infty} \in L^{\infty}(\Omega_1)$ and $\nabla\theta^{k,p} \in L^p(\Omega_1)$ for $1 \leq k \leq m$. Set $J = |\det(\nabla\theta)|$ and assume $\frac{1}{J} \leq K$ for some constant $K > 0$. Suppose either $u \in W^{m',p}(\Omega_2)$ or $v \in W^{m',p}(\Omega_1)$ where $v = u \circ \theta$. Then $u \in W^{m',p}(\Omega_2)$, $v \in W^{m',p}(\Omega_1)$, and we have the estimates*

$$\begin{aligned}
||v||_{W^{m',p}(\Omega_1)} &\leq C_\theta\, ||u||_{W^{m',p}(\Omega_2)} \\
||u||_{W^{m',p}(\Omega_2)} &\leq C_\theta\, ||v||_{W^{m',p}(\Omega_1)}
\end{aligned}$$

where the constant $C_\theta > 0$ is an increasing function of K, $\operatorname{Lip}\theta$, $\left|\left|\nabla\theta^{k,\infty}\right|\right|_{L^{\infty}(\Omega_1)}$ and $\left|\left|\nabla\theta^{k,p}\right|\right|_{L^p(\Omega_1)}$ for $1 \leq k \leq m$.

Proof. Note that $||u||_{W^{m',p}(\Omega_2)} \leq C_\theta\, ||v||_{W^{m',p}(\Omega_1)}$ if and only if $\left|\left|v \circ \theta^{-1}\right|\right|_{W^{m',p}(\Omega_2)} \leq C_\theta\, ||v||_{W^{m',p}(\Omega_1)}$. Now apply the above two results. $\square$

Lemma A.3.8. *Let $s \in [0,1]$ and let $r > \frac{d}{2}$ where $d \geq 1$ is an integer. Let $a \in H^{s+r}(\mathbb{R}^d)$ and $u \in H^s(\mathbb{R}^d)$. Then $au \in H^s(\mathbb{R}^d)$ and*

$$\langle D\rangle^s (au) = a\,\langle D\rangle^s u + R(a,u)$$

where the remainder $R(a,u)$ satisfies

$$||R(a,u)||_{L^2(\mathbb{R}^d)} \leq C\, ||a||_{H^{s+r}(\mathbb{R}^d)}\, ||u||_{L^2(\mathbb{R}^d)}$$

for some $C > 0$.

Proof. First of all, assume $a, u \in C_c^{\infty}(\mathbb{R}^d)$. We define $R(a,u)$ by

$$\langle D\rangle^s (au) - a\,\langle D\rangle^s u = R(a,u)$$

Taking Fourier transforms, and using the fact that products become convolutions, we have

$$\begin{aligned}
\langle\xi\rangle^s (\hat{a} * \hat{u})(\xi) - (\hat{a} * (\langle\eta\rangle^s \hat{u}))(\xi) &= \hat{R}(\xi) \\
\int_{\mathbb{R}^d} \hat{a}(\xi-\eta)\hat{u}(\eta)(\langle\xi\rangle^s - \langle\eta\rangle^s)\mathrm{d}\eta &= \hat{R}(\xi)
\end{aligned}$$

Hence

$$\left|\hat{R}(\xi)\right| \leq \int_{\mathbb{R}^d} |\hat{a}(\xi-\eta)|\, |\hat{u}(\eta)|\, |\langle\xi\rangle^s - \langle\eta\rangle^s|\, \mathrm{d}\eta$$

Now we use the inequality

$$|\langle \xi \rangle^s - \langle \eta \rangle^s| \leq C \langle \xi - \eta \rangle^s$$

which holds for some $C > 0$ provided $s \in [0, 1]$. We obtain

$$\begin{aligned}\left|\hat{R}(\xi)\right| &\leq C \int_{\mathbb{R}^d} |\hat{a}(\xi - \eta)| \, |\hat{u}(\eta)| \, \langle \xi - \eta \rangle^s \, \mathrm{d}\eta \\ &= C(\langle \eta \rangle^s |\hat{a}|) * |\hat{u}|\end{aligned}$$

Applying Young's inequality for convolutions, we have

$$\begin{aligned}\left|\left|\hat{R}(\xi)\right|\right|_{L^2(\mathbb{R}^d)} &\leq C \, ||\langle \xi \rangle^s \hat{a}||_{L^1(\mathbb{R}^d)} \, ||\hat{u}||_{L^2(\mathbb{R}^d)} \\ &= C \left|\left|\langle \xi \rangle^{s+r} \hat{a} \, \langle \xi \rangle^{-r}\right|\right|_{L^1(\mathbb{R}^d)} ||u||_{L^2(\mathbb{R}^d)} \\ &\leq C \left|\left|\langle \xi \rangle^{s+r} \hat{a}\right|\right|_{L^2(\mathbb{R}^d)} \left|\left|\langle \xi \rangle^{-r}\right|\right|_{L^2(\mathbb{R}^d)} ||u||_{L^2(\mathbb{R}^d)} \\ &\leq C \, ||a||_{H^{s+r}(\mathbb{R}^d)} \, ||u||_{L^2(\mathbb{R}^d)}\end{aligned}$$

where we have used Holder's inequality and the fact that $\langle \xi \rangle^{-r} \in L^2(\mathbb{R}^d)$ for $r > \frac{d}{2}$. Hence

$$||R(a, u)||_{L^2(\mathbb{R}^d)} \leq C \, ||a||_{H^{s+r}(\mathbb{R}^d)} \, ||u||_{L^2(\mathbb{R}^d)}$$

as claimed. If a and u are not smooth the we use approximation by mollification, noting that the result we have just proved will show that the approximating sequence $a_n u_n$ is Cauchy in H^s, which together with almost everywhere convergence shows that $au \in H^s$, and the result follows. □

Lemma A.3.9. *Let $u \in H^1(\mathbb{R}^2)$, $v \in H^{0.5}(\mathbb{R}^2)$. Then*

$$||uv||_{L^2(\mathbb{R}^2)} \leq C \, ||u||_{H^1(\mathbb{R}^2)} \, ||v||_{H^{0.5}(\mathbb{R}^2)}$$

Proof. Set $d = 2$, $p = 2$ and $s = 0.5$. Note that the standard Sobolev embedding theorem implies $||u||_{L^q(\mathbb{R}^2)} \leq C_q \, ||u||_{H^1(\mathbb{R}^2)}$ for any $q \in [2, \infty)$ since $\frac{d}{p} = \frac{2}{2} = 1$. Note also that the fractional version of the Sobolev embedding theorem implies that for $s \in (0, 1)$, $||v||_{L^r(\mathbb{R}^2)} \leq C_r \, ||v||_{H^s(\mathbb{R}^2)}$ for any $r \in [2, 2^*]$ where $2^* = \frac{4}{2-2s}$, which is 4 in the case $s = 0.5$. Now we use Holder's inequality to estimate

$$\begin{aligned}||uv||_{L^2(\mathbb{R}^2)} &\leq ||u||_{L^6(\mathbb{R}^2)} \, ||v||_{L^3(\mathbb{R}^2)} \\ &\leq C \, ||u||_{H^1(\mathbb{R}^2)} \, ||v||_{H^{0.5}(\mathbb{R}^2)}\end{aligned}$$

Note in fact we could have used an L^4-L^4 estimate and estimated both terms in the product in $H^{0.5}$ but we don't need this so we have avoided using the critical case to show that this estimate is not one of the most important ones. □

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