

The role of a strong confining potential in a nonlinear Fokker–Planck equation

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Abstract

We show that solutions of nonlinear nonlocal Fokker–Planck equations in a bounded domain with no-flux boundary conditions can be approximated by Cauchy problems with increasingly strong confining potentials defined in the whole space. Two different approaches are analyzed, making crucial use of uniform estimates for L^2 energy functionals and free energy (or entropy) functionals respectively. In both cases, we prove that the weak formulation of the problem in a bounded domain can be obtained as the weak formulation of a limit problem in the whole space involving a suitably chosen sequence of large confining potentials. The free energy approach extends to the case degenerate diffusion.

Keywords: Fokker–Planck equations, nonlocal models, strong confinement, degenerate diffusion

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1. Introduction

In this paper, we consider a nonlinear Fokker–Planck equation of the form

$$\partial_t u = \operatorname{div} [\nabla \phi(u) + u \nabla V_0 + u \nabla (W * u)], \quad x \in \Omega, t > 0, \quad (1)$$

where $u(x, t) \geq 0$ satisfies no-flux boundary conditions on a bounded and connected domain $\Omega \subset \mathbb{R}^d$ of class C^2 , for dimension $d \geq 1$ in \mathbb{N} , and a suitable initial condition that we will specify later. The function $\phi(\cdot)$ represents nonlinear diffusion, W is a symmetric interaction potential, and V_0 is an external potential.

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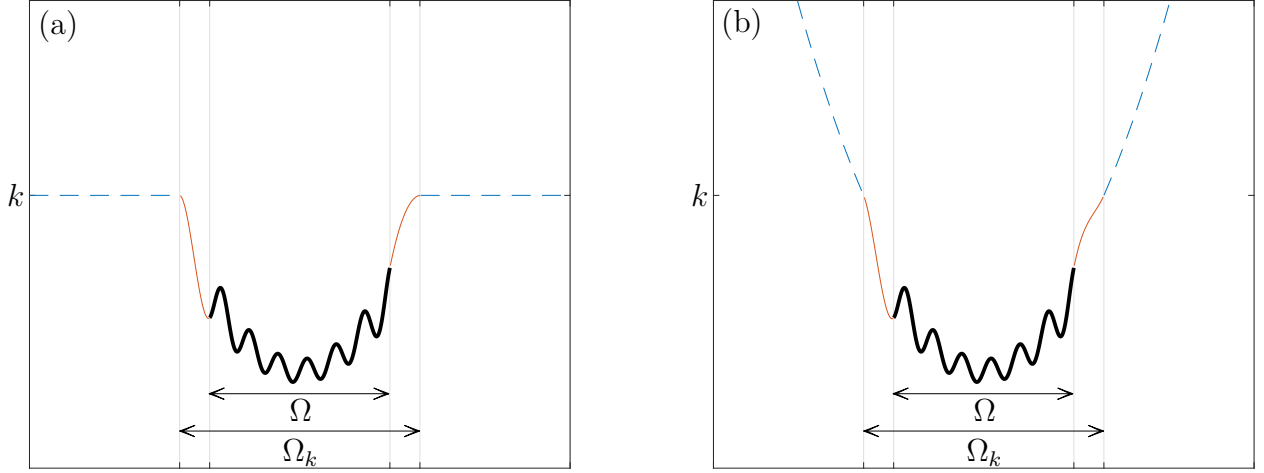


Figure 1: Sketch of the global potential $V_k(x)$ for (a) the L^2 setting (see Definition 2) and (b) the free energy setting (see Definition 4).

Equation (1) is often used to describe a system of interacting particles at the macroscopic level and explain how individual-level mechanisms give rise to population-level or collective behavior. Systems of interacting particles play a key role in many physical and biological applications, including granular materials [7, 6], self-assembly of nanoparticles [27], colloidal systems [25], ionic transport [28], cell motility [26], animal swarms [18], pedestrian dynamics [14], and social sciences [34, 38]. For example, equation (1) with $\phi = u$ and $W = 0$ can be used to describe a system of noninteracting Brownian particles under the influence of an external field V_0 , representing a chemical concentration in the case of chemotaxis. Deviations from Brownian motion, such as in the case of transport through porous media, can be modelled changing the diffusion term to $\phi = u^m$ with $m > 1$. Interactions between particles may arise in the macroscopic model in two forms: either as a modification of the diffusion term ϕ , for example $\phi = u + \beta u^2$, or as a nonlocal convolution. The former typically arises from short-range repulsive interactions between particles (such as excluded-volume interactions) [12, 15, 16], whereas the latter is used to model long-range attractive-repulsive interactions (such as electrostatic or chemoattractive interactions) [13, 16, 37].

The goal of this paper is to understand how the solutions of (1) in the bounded domain Ω relate to the solutions of the following equation in the whole space, as $k \rightarrow \infty$:

$$\partial_t u_k = \operatorname{div} [\nabla \phi(u_k) + u_k \nabla V_k + u_k \nabla (W * u_k)], \quad x \in \mathbb{R}^d, t > 0, \quad (2)$$

where the confinement potential is fixed in the bounded domain, that is, $V_k(x) = V_0(x)$ for $x \in \Omega$ and it becomes stronger outside Ω as $k \rightarrow \infty$.

In particular, we consider potentials V_k of the form depicted in Figure 1 (see Definitions 2 and 4). The parameter k determines the level of confinement ($k \rightarrow \infty$ for an infinite confinement). Our aim is to control the behavior of the solution u_k to (2) when V_k becomes a strong confinement potential outside Ω . More precisely, we show that infinite confinement

is equivalent to solving equation (1) in a bounded domain with no-flux boundary conditions for certain class of initial data supported in Ω . A similar set-up was used in [2] to study the limit of a local, non-degenerate, fully nonlinear problem with a method based on super- and subsolutions.

Our approximation by strong confinement has not only a theoretical interest but also a practical added value from the numerical viewpoint. Sometimes solving (1) in a bounded domain is hindered by the geometry of Ω . The strong confinement approximation is potentially useful to approximate numerically problems of the form (1) by problems in the whole space (2) in square geometries large enough for the domain of interest. Solving in Cartesian grids is always much easier than producing good meshes for the approximated domains.

Problem (1) is sometimes referred to as *aggregation-diffusion equation* or *drift-diffusion-interaction equation* and it has been recently studied by different authors. Free energy methods have been successfully applied to study long time behaviour of the solutions of problem (3) in the setting of gradient flows and Wasserstein spaces. For example, long time asymptotics to nonlinear nonlocal Fokker–Planck equations have been studied in [21, 22]. Many results concerning convergence to equilibrium can be found in particular cases (with or without diffusive terms) in [5, 11, 20, 23]. Gradient flows in the Wasserstein sense were first described for the linear Fokker–Planck equation in [29]; an extensive theory of the subject was later given in [4]. For classical results on parabolic equations we refer to [3, 30] and more recent results on well-posedness for the aggregation-diffusion equations are presented in [8]. A comparison between free energy and classical L^2 methods can be found in [33]. Some of the techniques we use in Section 3 involving the weight $\exp(V)$ were also employed in [1, 31]. In this paper we consider the problem of strong confinement using both an L^2 approach and a free energy approach. The L^2 approach is restricted to non-degenerate diffusion, while the free energy approach allows us to consider also degenerate diffusion but provides weaker estimates.

The paper is organized as follows. In Section 2 we introduce the key definitions and give an outline of the main results. Section 3 is devoted to the special cases in which $\phi(u) = u$ or $W = 0$, and the problem is treated with classical energy methods in the L^2 setting. We consider the fully nonlinear nonlocal problem in the free energy setting by free energy methods in Section 4. Finally, we conclude in Section 5 with some numerical examples illustrating the main results of this paper together with some numerical exploration for initial data supported outside the confined domain Ω .

2. Outline of the results

Let us consider the nonlinear nonlocal Fokker–Planck equation with confinement potential V and interaction potential W given by

$$\begin{aligned} \partial_t u &= \operatorname{div} [\nabla \phi(u) + u \nabla V + u \nabla (W * u)], \quad x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x). \end{aligned} \tag{3}$$

We denote by Q_T the space $\mathbb{R}^d \times [0, T]$ and by Ω_T the space $\Omega \times [0, T]$. The main results of this work are obtained under two different set of assumptions on the potentials V , W , the

nonlinearity of the diffusion ϕ , and the initial data. We will label them as the L^2 setting and the free energy setting.

2.1. L^2 setting, main results

Assumption 1 (L^2 setting). We consider the following set of assumptions:

1. $0 \leq u_0(x) \leq M_0$, $\text{supp}(u_0) \subseteq \Omega$.
2. $W \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ is symmetric, $\nabla W \in L^1(\mathbb{R}^d)$ and, without loss of generality, $W \geq 0$.
3. $V \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ and, without loss of generality, we also assume $V \geq 0$.
4. $\phi \in C^1([0, \infty))$ has the form $\phi(s) = s + \sigma(s)$, $\phi(0) = 0$, and is increasing. Moreover, there exist constants $\mu > 0$ and $b \geq a > 0$ such that

$$\mu s^a \leq \sigma'(s) \leq \frac{1}{\mu} s^b. \quad (4)$$

In the L^2 setting, weak solutions to (3) are defined as follows.

Definition 1 (L^2 solution). Suppose that Assumption 1 holds. We say that u is an L^2 weak solution of (3) if

$$u \in L^2(0, T; H^1(\mathbb{R}^d)) \cap C^0(0, T; L^2(\mathbb{R}^d)), \quad \nabla \phi(u) \in L^2(Q_T), \quad \partial_t u \in L^2(0, T; H^{-1}(\mathbb{R}^d)),$$

and, for all test functions $\eta \in C^\infty(Q_T)$,

$$\int_{\mathbb{R}^d} u(t) \eta(t) dx \Big|_{t=0}^{t=T} - \int_0^T \int_{\mathbb{R}^d} [(\nabla \phi(u) + u \nabla V + u \nabla W * u) \cdot \nabla \eta - u \partial_t \eta] dx dt = 0. \quad (5)$$

The initial datum is satisfied in the L^2 sense.

We now define the sequences of confinement potentials associated to a fixed confinement potential $V_0 \in \mathcal{W}^{1,\infty}(\Omega)$, $V_0 \geq 0$, that we will use in Section 3 (see Figure 1(a)).

Definition 2 (Sequence of potentials, L^2 setting). We define the following sequence of potentials $V_k \in \mathcal{W}^{1,\infty}(\mathbb{R}^d)$ for $k \geq 1$ as

$$V_k(x) = \begin{cases} V_0(x) & x \in \Omega, \\ \psi_k(x) & x \in \Omega_k \setminus \Omega, \\ k & x \in \mathbb{R}^d \setminus \Omega_k, \end{cases} \quad (6)$$

where Ω_k is an extended domain around Ω ,

$$\Omega_k = \left\{ x + \frac{1}{k} e \mid x \in \Omega, e \in S^d \right\},$$

so that $\Omega_k \searrow \Omega$ as $k \rightarrow \infty$, and ψ_k is a suitable C^1 extension of V .

Our main result in the L^2 setting concerns the convergence of the sequence of solutions u_k , which are defined in \mathbb{R}^d , to a limit function u that solves a problem in the bounded domain Ω .

Theorem 1 (Main result - L^2 setting). *Assume that the conditions in Assumption 1 are satisfied and, additionally, suppose that one of the following conditions holds: either $\phi(s) = s$ or $W = 0$.*

Consider a solution u_k of the Cauchy problem (3) in the sense of Definition 1 with $V = V_k$ satisfying the conditions in Definition 2. Then u_k converges for $k \rightarrow \infty$ to a function u such that

$$u \in L^2(0, T; H^1(\Omega)) \cap C^0(0, T; L^2(\Omega)), \nabla \phi(u) \in L^2(\Omega_T), \partial_t u \in L^2(0, T; H^1(\Omega)'),$$

and, for all $\eta \in C^\infty(\Omega_T)$, we have:

$$\int_{\Omega} u(t)\eta(t)dx \Big|_{t=0}^{t=T} - \int_0^T \int_{\Omega} [(\nabla \phi(u) + u \nabla V_0 + u \nabla W * u) \cdot \nabla \eta - u \partial_t \eta] dx dt = 0. \quad (7)$$

In other words, u is a weak solution of the Cauchy problem to (1) with no-flux boundary conditions and initial datum u_0 .

Remark 1 (Convolution on a bounded domain). Given $f : \Omega \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$, we use the following convention:

$$(W * f)(x) = \int_{\Omega} W(x - y) f(y) dy.$$

2.2. Free energy setting, main results

We now introduce a different set of assumptions and we prove a convergence result corresponding to Theorem 1 in a new setting. This allows us to consider confinement and interaction potentials diverging as $|x| \rightarrow \infty$, as well as degenerate diffusion terms.

Assumption 2 (Free energy setting). We consider the following set of assumptions:

1. $u_0(x) \geq 0$, $\int_{\mathbb{R}^d} u_0 dx = 1$ and $\text{supp}(u_0) \subseteq \Omega$.
2. $W \in \mathcal{W}_{loc}^{1,\infty}(\mathbb{R}^d)$ is symmetric and, without loss of generality, $W \geq 0$.
3. $V \in \mathcal{W}_{loc}^{1,\infty}(\mathbb{R}^d)$ and we assume that $V \geq c|x|^2$ for $|x| \rightarrow \infty$ and for some $c > 0$.
4. $\phi \in C^1([0, \infty))$ has the form $\phi(s) = s + \sigma(s)$, $\phi(0) = 0$, and is increasing. We suppose that there exist constants $\mu > 0$ and $b \geq a > 0$ such that relation (4) holds.

Notice that we do not assume boundedness or decay of V and W at infinity, but only local regularity. The initial datum can also be unbounded as we only assume non-negativity and integrability. The key ingredient is the gradient flow structure of equations (1) and (2). We denote the 2-Wasserstein space of probability measures by $\mathcal{P}_2(\mathbb{R}^d)$ endowed by the 2-Wasserstein distance d_2 .

Definition 3 (Free energy solution). Suppose that Assumption 2 is satisfied. We say that $u \in C([0, T], \mathcal{P}_2(\mathbb{R}^d)) \cap L^\infty(0, T; L^1(\mathbb{R}^d))$ is an entropy solution of (3) if u is a distributional solution in $\mathcal{D}'(\mathbb{R}^d)$ and it satisfies

$$u(t, \cdot) \rightarrow u_0 \text{ in } \mathcal{P}_2(\mathbb{R}^d) \text{ as } t \rightarrow 0,$$

for all $T > 0$,

$$\int_{Q_T} u \left| \frac{1}{u} \nabla \phi(u) + \nabla V + \nabla W * u \right|^2 dx dt < \infty,$$

and, for $\Theta(u) = \int_0^u \int_1^s \frac{\phi'(r)}{r} dr ds$, it is a gradient flow in $\mathcal{P}_2(\mathbb{R}^d)$ for the entropy functional:

$$E[u(t)] = \int_{\mathbb{R}^d} \left[\Theta(u) + uV + \frac{1}{2}u(W * u) \right] dx.$$

In Section 4 we will consider the following sequence of potentials (see Figure 1(b)).

Definition 4 (Sequence of potentials, free energy setting). We define the following sequence of potentials $V_k \in \mathcal{W}_{loc}^{1,\infty}(\mathbb{R}^d)$:

$$V_k(x) = \begin{cases} V_0(x) & x \in \Omega, \\ \psi_k(x) & x \in \Omega_k \setminus \Omega, \\ \zeta_k(x) & x \in \mathbb{R}^d \setminus \Omega_k, \end{cases} \quad (8)$$

where Ω_k is an extended domain around Ω ,

$$\Omega_k = \left\{ x + \frac{1}{k}e \mid x \in \Omega, e \in S^d \right\},$$

so that $\Omega_k \searrow \Omega$ as $k \rightarrow \infty$. Here $\zeta_k(x) \geq k$ is such that $V_k(x) \geq c|x|^2$ for $|x|$ sufficiently large (in particular $\int_{\mathbb{R}^d} e^{-\zeta_k(x)} dx < \infty$), and $\psi_k(x)$ is a C^1 interpolant between the values of V_0 on $\partial\Omega$ and ζ_k outside Ω_k .

Theorem 2 (Main result - Free energy setting). *Assume that the conditions in Assumption 2 are met. Consider a solution u_k of problem (3) in \mathbb{R}^d in the sense of Definition 3 with $V = V_k$ satisfying the conditions in Definition 4. Then u_k converges for $k \rightarrow \infty$ to a function u satisfying the following weak formulation in Ω :*

$$\int_{\Omega} u(t)\eta(t) dx \Big|_{t=0}^{t=T} - \int_0^T \int_{\Omega} [(\nabla \phi(u) + u \nabla V_0 + u \nabla W * u) \cdot \nabla \eta - u \partial_t \eta] dx dt = 0,$$

for any $\eta \in C_0^\infty(\Omega_T)$. The initial datum is also satisfied in $\mathcal{P}_2(\Omega)$.

The same result holds for $\phi(\cdot) = \sigma(\cdot)$, that is, in the case of degenerate diffusion.

3. Analysis via L^2 estimates

The well-posedness of problem (3) in the L^2 setting can be deduced from the existence and uniqueness results in [8]. Let us discuss some of the properties of the solutions. Without loss of generality we assume that ϕ is the restriction of a C^1 function on the whole line to the positive semi-axis.

Lemma 3 (Non-negativity and conservation of mass). *Let Assumption 1 hold. Any weak solution u of problem (3) in the sense of Definition 1 is non-negative and furthermore it preserves mass, that is,*

$$\int_{\mathbb{R}^d} u \, dx = \int_{\mathbb{R}^d} u_0 \, dx.$$

Proof. To obtain conservation of mass, it is sufficient to test with the function $\eta = 1$ and integrate by parts to obtain $0 = \langle \partial_t u, 1 \rangle = \partial_t \int_{\mathbb{R}^d} u \, dx$.

In order to obtain non-negativity, we consider a solution u of problem (3) (in the sense of Definition 1) and we test the equation against $\theta = (u)_-$ (which is non-negative and supported in the set $\{u \leq 0\}$). In particular, for a.e. t we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \frac{1}{2} \theta^2 \, dx + \int_{\mathbb{R}^d} [\phi'(u) |\nabla \theta|^2 + \theta \nabla (V + W * u) \cdot \nabla \theta] \, dx = 0,$$

which implies, since $\phi'(s) \geq 1$,

$$\int_{\mathbb{R}^d} \frac{1}{2} \theta^2 \, dx + \int_{Q_T} |\nabla \theta|^2 \, dx \leq \|\nabla (V + W * u)\|_{L^\infty(Q_T)} \int_{Q_T} \left[\frac{\varepsilon}{2} |\nabla \theta|^2 + \frac{1}{2\varepsilon} |\theta|^2 \right] \, dx,$$

for $\varepsilon = \|\nabla (V + W * u)\|_{L^\infty(Q_T)}^{-1}$. Notice that we have already established that $\int_{\mathbb{R}^d} u \, dx = \int_{\mathbb{R}^d} u_0 \, dx$ and that the quantity $\|\nabla (V + W * u)\|_{L^\infty(Q_T)}$ is finite thanks to Assumption 1. Thus we have

$$\int_{\mathbb{R}^d} \theta^2 \, dx + \int_{Q_T} |\nabla \theta|^2 \, dx \leq \|\nabla (V + W * u)\|_{L^\infty(Q_T)}^2 \int_{Q_T} |\theta|^2 \, dx.$$

Using Gronwall's inequality we obtain $\theta = (u)_- = 0$ a.e. $(x, t) \in Q_T$. \square

3.1. Linear Fokker–Planck equation

We begin with the simplest case with non-interacting particles ($W = 0$, $\phi(s) = s$); in this case it is possible to work in an L^2 setting.

Lemma 4 (Energy identity and boundedness). *Consider the scalar equation*

$$\begin{aligned} \partial_t u &= \Delta u + \operatorname{div}(u \nabla V), \quad x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x). \end{aligned} \tag{9}$$

For every weak solution of problem (9) (in the sense of Definition 1), the following identity holds

$$\int_{\mathbb{R}^d} u(T)^2 e^V dx + 2 \int_0^T \int_{\mathbb{R}^d} e^V |\nabla u + u \nabla V|^2 dx dt = \int_{\mathbb{R}^d} u_0^2 e^V dx, \quad (10)$$

for a.e. $T > 0$. In addition, u has exponential tails in $L^\infty(Q_T)$ in the following sense

$$u_0 \leq m e^{-V} \Rightarrow u \leq m e^{-V} \text{ a.e. } (t, x) \in Q_T, \quad (11)$$

for a fixed $m \geq 0$.

Proof. We test the equation against $u e^V$. We obtain

$$\int_{\mathbb{R}^d} \partial_t u (u e^V) dx = \int_{\mathbb{R}^d} [\Delta u + \operatorname{div}(u \nabla V)] u e^V dx. \quad (12)$$

Integrating the left-hand side of (12) in time, we deduce

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u (u e^V) dx dt = \int_0^T \int_{\mathbb{R}^d} \partial_t \left(\frac{1}{2} u^2 e^V \right) dx dt = \int_{\mathbb{R}^d} \frac{1}{2} u(T)^2 e^V dx - \int_{\mathbb{R}^d} \frac{1}{2} u_0^2 e^V dx.$$

Using integration by parts in the right-hand side of (12) we get

$$\begin{aligned} \int_{\mathbb{R}^d} [\Delta u - \operatorname{div}(u \nabla V)] \cdot (u e^V) dx &= - \int_{\mathbb{R}^d} (\nabla u + u \nabla V) \cdot \nabla (u e^V) dx \\ &= - \int_{\mathbb{R}^d} e^V |\nabla u + u \nabla V|^2 dx \leq 0. \end{aligned}$$

Collecting terms yields

$$\int_{\mathbb{R}^d} \frac{1}{2} e^V u(T)^2 dx = \int_{\mathbb{R}^d} \frac{1}{2} u_0^2 e^V dx - \int_0^T \int_{\mathbb{R}^d} e^V |\nabla u + u \nabla V|^2 dx dt,$$

as required. In order to prove boundedness, we consider the function $\tilde{u} = u e^V$. Let us rewrite equation (9) in terms of \tilde{u} :

$$e^{-V} \partial_t \tilde{u} = \operatorname{div} (e^{-V} \nabla \tilde{u}).$$

We integrate the equation above against the test function $(\tilde{u} - m)_+$:

$$\int_{\mathbb{R}^d} \frac{1}{2} e^{-V} (\tilde{u}(t) - m)_+^2 dx + \int_0^t \int_{\mathbb{R}^d} e^{-V} |\nabla (\tilde{u} - m)_+|^2 dx d\tau = \int_{\mathbb{R}^d} \frac{1}{2} e^{-V} (\tilde{u}(0) - m)_+^2 dx.$$

Notice that $\tilde{u}(0) = u_0 e^V$, thus the right-hand side in the equality above vanishes. This means that $\tilde{u}(t) \leq m$ for a.e. $(t, x) \in Q_T$ and the proof is complete. \square

Corollary 5. Consider the assumptions of Lemma 4 and recall that $\operatorname{supp}(u_0) \subseteq \Omega$. Let $V = V_k$ as in Definition 2 and consider $u = u_k$ as in Lemma 4. Then $u_k(x, t) \rightarrow 0$ in $L^\infty([0, T] \times \Omega_k^c)$ for $k \rightarrow \infty$.

Proof. It is a direct consequence of (11) (recalling that $V_k \rightarrow \infty$ on Ω_k^c). \square

3.2. Nonlocal Fokker–Planck equation

Here we consider the extension of the previous linear case to include a nonlinear interaction potential W .

Lemma 6 (L^2 energy estimate, case $\phi(s) = s$). *Let u be a weak solution of problem (3) with $\phi(s) = s$ in the sense of Definition 1. Let \bar{u}_0 be the (constant) mass of u and recall that $\text{supp}(u_0) \subset \Omega$. The following inequality holds:*

$$\int_{\mathbb{R}^d} u(T)^2 e^V dx + \int_0^T \int_{\mathbb{R}^d} e^{-V} |\nabla(e^V u)|^2 dx dt \leq C(\nabla W, \bar{u}_0, T) \int_{\Omega} u_0^2 e^V dx. \quad (13)$$

Proof. We are going to use $u \exp(V)$ as our test function.

$$\int_{\mathbb{R}^d} u(T)^2 e^V dx + 2 \int_0^T \int_{\mathbb{R}^d} [e^V |\nabla u + u \nabla V|^2 + u(\nabla W * u) \cdot \nabla(e^V u)] dx dt = \int_{\Omega} u_0^2 e^V dx,$$

and

$$\int_0^T \int_{\mathbb{R}^d} u e^{V/2} (\nabla W * u) \cdot e^{-V/2} \nabla(e^V u) dx dt \leq \|u e^{V/2}\|_{L^2} \|e^{-V/2} \nabla(e^V u)\|_{L^2} \|\nabla W\|_{L^\infty} \|u\|_{L^1},$$

resulting into

$$\begin{aligned} & \int_{\mathbb{R}^d} u(T)^2 e^V dx + \int_0^T \int_{\mathbb{R}^d} e^{-V} |\nabla(e^V u)|^2 dx dt \\ & \leq \int_{\Omega} u_0^2 e^V dx + \|\nabla W\|_{L^\infty}^2 \|u_0\|_{L^1}^2 \int_0^T \int_{\mathbb{R}^d} u^2 e^V dx dt. \end{aligned}$$

Using Gronwall's lemma we obtain

$$\int_{\mathbb{R}^d} u(T)^2 e^V dx + \int_0^T \int_{\mathbb{R}^d} e^{-V} |\nabla(e^V u)|^2 dx dt \leq \exp(T \|\nabla W\|_{L^\infty}^2 \|u_0\|_{L^1}^2) \int_{\mathbb{R}^d} u_0^2 e^V dx,$$

as announced. \square

Remark 2. Unlike in the linear case, boundedness of (weak) solutions is not so straightforward. It can be shown arguing as in [30], Chapter 3, Theorem 7.1 or [35]. Since we do not need boundedness in our analysis, we do not discuss it further.

Corollary 7. *Under the same assumptions of Lemma 6, we have the following decay estimate for $V = V_k$:*

$$\int_{\Omega_k^c} u(T)^2 dx + \int_0^T \int_{\Omega_k^c} |\nabla u + u \nabla V|^2 dx dt \leq \exp(-k) C(T, \nabla W, u_0) \int_{\Omega} u_0^2 e^{V_0} dx. \quad (14)$$

Proof. This is a direct consequence of Lemma 6, recalling that fact that $\text{supp}(u_0) \subseteq \Omega$ and that $V_k \geq k$ on Ω_k^c . \square

3.3. Nonlinear local Fokker–Planck equation

In the case of nonlinear diffusion, we generalize the familiar procedure often used to obtain energy estates and we introduce a new quantity, indicated by $P(u)$, which coincides with u in the linear case.

Lemma 8 (Energy inequality, $W = 0$). *Let u be a weak solution the following equation*

$$\begin{aligned} \partial_t u &= \Delta \phi(u) + \operatorname{div}(u \nabla V), \quad x \in \mathbb{R}^d, t > 0, \\ u(x, 0) &= u_0(x), \end{aligned} \tag{15}$$

where $\phi(s) = s + \sigma(s)$ satisfies Assumptions 1. Then:

$$\int_{\mathbb{R}^d} e^V Q(u(T)) dx + \int_{\mathbb{R}^d} e^V \frac{P(u)}{u} |\nabla \phi(u) + u \nabla V(x)|^2 dx \leq \int_{\Omega} e^V Q(u_0) dx, \tag{16}$$

for a.e. $T > 0$, with

$$P(u) = \exp \left(\int_1^u \frac{\phi'(s)}{s} ds \right),$$

and

$$Q(u) = \int_0^u P(s) ds.$$

Remark 3. Notice that, thanks to Assumption 1 (point 4), we have

$$\frac{\phi'(s)}{s} = \frac{1 + \sigma'(s)}{s} \geq \frac{1}{s} + \mu s^{a-1},$$

for any $s \geq 0$ and some given $\mu > 0$, $a > 0$. From the definition of P , for $u \geq 0$, we obtain

$$P(u) = \exp \left(\int_1^u \frac{1 + \sigma'(s)}{s} ds \right) \geq \exp \left(\log(u) + \mu \int_1^u s^{a-1} ds \right) = u \exp \left(\frac{\mu}{a} (u^a - 1) \right).$$

Consequently, $P(u)$ is well defined and, using the lower bound just obtained, we observe that it satisfies

$$P(u) \geq c(\mu, a) u \exp(u^a), \tag{17}$$

for $c(\mu, a) = \exp(-\frac{\mu}{a})$. It follows that, for any $u \geq 0$,

$$\frac{P(u)}{u} \geq c(\mu, a) \quad \text{and} \quad \lim_{s \rightarrow 0^+} \frac{P(s)}{s} = c(\mu, a).$$

In addition, notice that since $P(u) \geq c(\mu, a)u$, we also have

$$Q(u) \geq \frac{1}{2} c(\mu, a) u^2.$$

These facts will be useful in the proof of Theorem 1, providing useful lower bounds for the left-hand side of inequality (16).

Proof of Lemma 8. First of all, we notice that

$$\begin{aligned}\nabla(e^V P(u)) &= e^V P(u) \nabla \left(V(x) + \int^u \frac{\phi'(s)}{s} ds \right) = e^V P(u) \left(\nabla V(x) + \frac{\phi'(u)}{u} \nabla u \right) \\ &= e^V \frac{P(u)}{u} (\nabla \phi(u) + u \nabla V(x)).\end{aligned}$$

We test equation (15) against $e^V P(u)$ and obtain

$$\int_{\mathbb{R}^d} \partial_t u e^V P(u) dx = \int_{\mathbb{R}^d} \operatorname{div} [\nabla \phi(u) + u \nabla V] e^V P(u) dx. \quad (18)$$

Considering the left-hand side of (18), we deduce

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u [e^V P(u)] dx dt = \int_0^T \int_{\mathbb{R}^d} \partial_t [Q(u) e^V] dx dt = \int_{\mathbb{R}^d} Q(u(T)) e^V dx - \int_{\mathbb{R}^d} Q(u_0) e^V dx,$$

where Q is a primitive of P .

Notice that, since $P(u) > 0$, $Q(u)$ is increasing and we can choose $Q(0) = 0$. Using integration by parts in the right-hand side of (18) we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \operatorname{div} [\nabla \phi(u) + u \nabla V] e^V P(u) dx &= - \int_0^T \int_{\mathbb{R}^d} [\nabla \phi(u) + u \nabla V] \cdot \nabla [e^V P(u)] dx dt \\ &= - \int_0^T \int_{\mathbb{R}^d} e^V \frac{P(u)}{u} |\nabla \phi(u) + u \nabla V(x)|^2 dx dt.\end{aligned}$$

Collecting terms we finally conclude that

$$\int_{\mathbb{R}^d} e^V Q(u(T)) dx + \int_0^T \int_{\mathbb{R}^d} e^V \frac{P(u)}{u} |\nabla \phi(u) + u \nabla V(x)|^2 dx \leq \int_{\Omega} e^V Q(u_0) dx. \quad (19)$$

□

The following Corollary will help us pass to the limit in the proof of Theorem 1, case $W = 0$.

Corollary 9. *Under the same assumptions of Lemma 8, we have the estimate*

$$\int_{\mathbb{R}^d} \frac{1}{2} e^V u(T)^2 dx + \int_0^T \int_{\mathbb{R}^d} e^V |\nabla \phi(u) + u \nabla V(x)|^2 dx dt \leq \int_{\Omega} \frac{1}{c(\mu, a)} e^V Q(u_0) dx. \quad (20)$$

Additionally, for $V = V_k$, the following decay estimate holds:

$$\int_{\Omega_k^c} u_k(T)^2 dx + \int_0^T \int_{\Omega_k^c} |\nabla \phi(u_k) + u_k \nabla V_k(x)|^2 dx dt \leq \exp(-k) \int_{\Omega} \frac{2e^{V_0}}{c(\mu, a)} Q(u_0) dx. \quad (21)$$

Proof. Combining estimate (16) and the lower bounds for P and Q from Remark 3 we have

$$\int_{\mathbb{R}^d} e^V \frac{1}{2} c(\mu, a) u(T)^2 dx + \int_0^T \int_{\mathbb{R}^d} e^V c(\mu, a) \exp(u^a) |\nabla \phi(u) + u \nabla V(x)|^2 dx dt \leq \int_{\Omega} e^V Q(u_0) dx,$$

which directly implies (20). This implies the decay recalling that $V_k \geq k$ on Ω_k^c . \square

We are now going to exploit the properties of the function Q in order to obtain a bound in L^p . First, we need the following simple result.

Lemma 10. *Let $p > 1$, $a > 0$. The following inequality holds for any $s > 0$:*

$$s^p \leq \theta_{p,a} s \exp(s^a), \quad \text{where} \quad \theta_{p,a} = \left(\frac{p-1}{ea} \right)^{\frac{p-1}{a}}. \quad (22)$$

Proof. Maximizing the function $s^{p-1} \exp(-s)$ we obtain the optimal value for $\theta_{p,a}$, which is attained at $s = p-1$ and hence $\theta_{p,a} = \left(\frac{p-1}{ea} \right)^{\frac{p-1}{a}}$. \square

Corollary 11 (L^p estimate). *Under the hypotheses of Lemma 8, for any $1 \leq p < \infty$ and $\theta_{p,a}$ as in Lemma 10, it holds*

$$\int_{\mathbb{R}^d} e^V u(t)^p dx \leq \theta_{p,a} \int_{\Omega} e^V Q(u_0) dx, \quad \text{for a.e. } t \in [0, T]. \quad (23)$$

Proof. The result follows combining the inequalities (16), (17) and (22). \square

Remark 4. Notice that the term $\operatorname{div}(u \nabla V)$ can be generalized to the form $\operatorname{div}(\alpha(u) \nabla V)$, where α is such that $\alpha(s) = s(1 + \rho(s))$ and there exists a constant $\mu \in [0, 1)$ such that

$$\mu \sigma'(s) \leq \rho(s) \leq \sigma'(s). \quad (24)$$

In this case we can define P as follows:

$$P(u) = \exp \left(\int_1^u \frac{\phi'(s)}{\alpha(s)} ds \right),$$

and we can obtain a result analogous to Lemma 8.

3.4. Proof of Theorem 1

We are going to present the two cases, $W = 0$ or $\phi(s) = s$, separately.

Remark 5. Suppose that the function $ue^{V/2}$ belongs to $L^2(0, T; H^1(\Omega))$, then u belongs to the same space. Indeed, u belongs to $L^2(0, T; L^2(\Omega))$ since $V \in \mathcal{W}^{1,\infty}(\Omega)$ and we have $\nabla(ue^{V/2}) = e^{V/2}(\nabla u + u \nabla V/2)$, so that $\nabla u = e^{-V/2} \nabla(ue^{V/2}) - u \nabla V/2$, where both terms on the right-hand side are square-integrable.

Proof of Theorem 1, case $\phi(s) = s$. The weak formulation (5) gives us

$$\int_{\mathbb{R}^d} u_k(t)\eta(t)dx \Big|_{t=0}^{t=T} - \int_0^T \int_{\mathbb{R}^d} [(\nabla u_k + u_k \nabla(V_k + W * u_k)) \cdot \nabla \eta - u_k \partial_t \eta] dx dt = 0. \quad (25)$$

for all test functions $\eta \in H^1(\mathbb{R}^d)$. We notice that, from the energy inequality (13), the function $u_k e^{\frac{V_k}{2}}$ is bounded in $L^2(0, T; H^1(\mathbb{R}^d)) \cap L^\infty(0, T; L^2(\mathbb{R}^d))$. Since the term $e^{\frac{V_k}{2}}$ is bounded from below, we also have that u_k belongs to the same space (see Remark 5).

We divide \mathbb{R}^d into three parts, namely Ω , Ω_k^c and $\Omega_k \setminus \Omega$. Consequently, we split (25) as $I_\Omega + I_{\Omega_k^c} + I_{\Omega_k \setminus \Omega} + J_\Omega + J_{\Omega_k^c} + J_{\Omega_k \setminus \Omega}$, where

$$\begin{aligned} I_A &= \int_0^T \int_A (\nabla u_k + u_k \nabla V_k + u_k \nabla W * u_k) \cdot \nabla \eta dx dt, \\ J_A &= \int_A [u_k(T)\eta(T) - u_0\eta(0)] dx - \int_0^T \int_A u_k \partial_t \eta dx dt. \end{aligned}$$

We want to show that all terms but I_Ω and J_Ω vanish in the limit $k \rightarrow \infty$. Then $I_\Omega + J_\Omega$ will characterize the limit problem defined in Ω .

- **I_Ω .** Restricting our attention to I_Ω , from (13) we obtain

$$\int_\Omega u_k(T)^2 e^{V_k} dx + \int_0^T \int_\Omega e^{-V_k} |\nabla(e^{V_k} u_k)|^2 dx dt \leq C \int_\Omega u_0^2 e^V dx. \quad (26)$$

This implies that

$$u_k \in L^\infty(0, T; L^2(\Omega)), \quad \nabla(e^{V_0} u_k) \in L^2(0, T; L^2(\Omega)),$$

uniformly in k . Since V_0 is bounded and sufficiently smooth, thanks to Remark 5 we obtain $u_k \in L^2(0, T; H^1(\Omega))$ uniformly in k , hence we can extract a subsequence u_{k_n} that converges weakly in $L^2(0, T; H^1(\Omega))$ to a limit denoted by u .

We now estimate the time derivative $\partial_t u_k$ in order to prove a strong convergence result for u_k . Notice that, rewriting (5), we obtain the bound

$$\begin{aligned} \langle \partial_t u_k, \eta \rangle_{H^1(\Omega)', H^1(\Omega)} &= \int_\Omega [u_k(T)\eta(T) - u_k(0)\eta(0)] dx - \int_0^T \int_{\mathbb{R}^d} u_k \partial_t \eta dx dt, \\ &= \int_{\Omega_T} [(\nabla u_k + u_k \nabla V_0 + u_k \nabla W * u_k) \cdot \nabla \eta] dx dt \\ &\leq C \left(\nabla V, \nabla W, \int_\Omega u_0 dx \right) \|u_k\|_{L^2(0, T; H^1(\Omega))} \|\eta\|_{L^2(0, T; H^1(\Omega))}, \end{aligned}$$

where C is a constant independent of k . We also know that $\|u_k\|_{L^2(0, T; H^1(\Omega))}$ is bounded uniformly with respect to k , we deduce that $\partial_t u_k \in L^2(0, T; H^1(\mathbb{R}^d)')$. We can now apply Aubin–Lions Lemma and deduce that, by compactness, u_{k_n} converges strongly

(up to a subsequence) in $L^2(\Omega_T)$ as well. To simplify the notation, in what follows we write u_k instead of u_{k_n} . In particular, thanks to the strong convergence of u_k in L^2 , we have

$$u_k \nabla V_k \rightarrow u \nabla V_0 \text{ in } L^2(\Omega_T),$$

and

$$u_k \nabla(W * u_k) \rightarrow u \nabla(W * u) \text{ in } L^1(\Omega_T).$$

In fact, for $k \rightarrow \infty$,

$$\|\nabla W * (u_k - u)\|_{L^2(Q_T)} \leq \|\nabla W\|_{L^1(\Omega)} \|(u_k - u)\|_{L^2(Q_T)} \rightarrow 0.$$

Thus we have obtained

$$I_\Omega \rightarrow \int_\Omega [(\nabla u + u \nabla V + u \nabla W * u) \cdot \nabla \eta] dx,$$

as $k \rightarrow \infty$.

- $I_{\Omega_k^c}$. Considering $I_{\Omega_k^c}$ and using again (13), we have

$$\int_{\Omega_k^c} u_k(T)^2 e^k dx + \int_0^T \int_{\Omega_k^c} e^k |\nabla u_k|^2 dx dt \leq C \int_\Omega u_0^2 e^{V_0} dx,$$

and therefore, for $k \rightarrow \infty$, we have that $I_{\Omega_k^c} \rightarrow 0$ since, by the decay estimate (14),

$$\|u_k\|_{L^2(0,T;H^1(\Omega_k^c)) \cap L^\infty(0,T;L^2(\Omega_k^c))} \rightarrow 0. \quad (27)$$

- $I_{\Omega_k \setminus \Omega}$. It remains to be checked that $I_{\Omega_k \setminus \Omega}$ also vanishes. Once more, from the energy identity (13), we obtain

$$\int_0^T \int_{\Omega_k \setminus \Omega} e^{\psi_k} |\nabla u_k + u_k \nabla(\psi_k + W * u_k)|^2 dx dt \leq C \int_\Omega u_0^2 e^{V_0} dx.$$

Since $\exp(V_k) \geq 1$, we deduce that

$$\begin{aligned} & \int_0^T \int_{\Omega_k \setminus \Omega} (\nabla u_k + u_k \nabla(\psi_k + W * u_k)) \cdot \nabla \eta dx dt \\ & \leq \left(\int_0^T \int_{\Omega_k \setminus \Omega} |\nabla u_k + u_k \nabla(\psi_k + W * u_k)|^2 dx dt \right)^{1/2} \left(\int_0^T \int_{\Omega_k \setminus \Omega} |\nabla \eta|^2 dx dt \right)^{1/2} \\ & \leq \left(C \int_\Omega u_0^2 e^{V_0} dx \right)^{1/2} \left(\int_0^T \int_{\Omega_k \setminus \Omega} |\nabla \eta|^2 dx dt \right)^{1/2} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Indeed, the first term in the last line is bounded and the second one vanishes since $\nabla \eta \in L^2(Q_T)$ and $|\Omega_k \setminus \Omega| \rightarrow 0$ as $k \rightarrow \infty$.

- J_Ω . The sequence u_k converges weakly in $H^1(\Omega)$ to a limit u , hence

$$J_\Omega \rightarrow \int_\Omega [u(T)\eta(T) - u_0\eta(0)]dx - \int_0^T \int_\Omega u \partial_t \eta \, dx dt \quad \text{as } k \rightarrow \infty, \quad (28)$$

- $J_{\Omega_k^c}$. Notice that $u_0 = 0$ in Ω_k^c . The remaining terms in $J_{\Omega_k^c}$ vanish thanks to (27).
- $J_{\Omega_k \setminus \Omega}$. The integral $J_{\Omega_k \setminus \Omega}$ goes to zero because the integrand is uniformly bounded in L^1 (thanks to the conservation of mass) and $|\Omega_k \setminus \Omega| \rightarrow 0$.

The weak formulation we obtain in the limit is the following:

$$\int_\Omega [u(T)\eta(T) - u(0)\eta(0)]dx - \int_0^T \int_\Omega [(\nabla \phi(u) + u \nabla V_0 + u \nabla W * u) \cdot \nabla \eta - u \partial_t \eta] \, dx dt = 0.$$

Notice that the initial datum is still satisfied in the L^2 sense and that since the test function can be any element of $H^1(\Omega)$ this implies that no-flux conditions on $\partial\Omega$ are implicitly enforced. It is easy to see that the initial datum is satisfied in the L^2 sense. \square

Proof of Theorem 1, case $W = 0$. We now consider the case $W = 0$ and ϕ generic. We can repeat most of the steps above using the energy inequality (16) instead of (13), therefore we will focus only on the main differences. First of all, we observe that the quotient $\frac{P(u)}{u} \geq c(\mu, a)$ for any $u \geq 0$, see Remark 3. In particular, just like in the previous proof, we can split the weak formulation into terms denoted by I_A and J_A . The terms denoted by J_A above are unchanged and are treated in the same way. The terms $I_{\Omega_k^c}$ and $I_{\Omega_k \setminus \Omega}$ vanish in an analogous way since, also in this case we have exponential decay with respect to k , as shown in (21).

Considering I_{Ω_k} , from (16), Remark 3 and recalling that $Q(0) = 0$, it follows that

$$\int_{\mathbb{R}^d} e^{V_k} \frac{1}{2} u_k(T)^2 dx + \int_{\mathbb{R}^d} e^{V_k} |\nabla \phi(u_k) + u_k \nabla V_k(x)|^2 dx \leq \int_{\Omega} e^{V_0} Q(u_0) dx. \quad (29)$$

We now proceed as in the proof of the case $\phi(s) = s$, in the sense that, thanks to inequalities (21) and (29), we have that, up to a subsequence, u_k converges to 0 outside of Ω and it converges strongly in $L^2(Q_T)$ (hence almost everywhere) and weakly in $L^2(0, T; H^1(\Omega))$ to a function u satisfying the limit weak formulation (7). In particular, let us focus on the nonlinear diffusion term. We know that $u_k \rightarrow u$ a.e. in Ω , ϕ' is continuous and it has polynomial growth by assumption (4). Furthermore, thanks to Corollary 11 (with $V = V_k$) we know that $u_k \in L^\infty(0, T; L^p(\Omega))$, for any $p \geq 1$. Combining such bounds and a.e. convergence, we deduce that $\phi'(u_k) \rightarrow \phi'(u)$ strongly in $L^2(\Omega_T)$. Additionally, we have $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^2(\Omega_T)$ as $k \rightarrow \infty$, thus the product $\phi'(u_k) \nabla u_k$ converges to $\phi'(u) \nabla u$ weakly in $L^1(\Omega_T)$ and we have

$$\int_\Omega \nabla \phi(u_k) \cdot \nabla \eta dx = \int_\Omega \phi'(u_k) \nabla u_k \cdot \nabla \eta dx \rightarrow \int_\Omega \phi'(u) \nabla u \cdot \nabla \eta dx \quad \text{as } k \rightarrow \infty,$$

The remaining term I_Ω is treated just like in the previous case. \square

Remark 6. We do not treat the more general case with nonlinear ϕ as well as $W \neq 0$ in the present section since it is not clear how to obtain a suitable energy estimate that is uniform with respect to k . However, the general equation can be studied using entropy techniques as shown in the next section.

4. Analysis via free energy estimates

Let us consider the full problem with nonlinear diffusion and nonlocal interaction terms:

$$\begin{aligned} \partial_t u &= \operatorname{div} [\nabla \phi(u) + u \nabla V + u \nabla (W * u)], \\ u(x, 0) &= u_0(x), \end{aligned} \quad x \in \mathbb{R}^d, t > 0. \quad (30)$$

We consider a solution u_k to (30) with $V = V_k$ given in Definition 4. We have to prove that the sequence u_k converges to a limit function u solving problem (3) in Ω . The main steps involved are:

1. finding estimates for u_k independent of k ,
2. showing that $u_k \rightarrow 0$ outside Ω ,
3. passing to the limit in the weak formulation of (30).

Concerning well-posedness of problem (30), the following result has been proven in [4], Theorem 11.2.8.

Theorem 12 (Existence and uniqueness of solutions). *Let Assumption 2 hold and suppose that V and W are strictly convex. For every $u_0 \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a unique distributional solution $u \in \mathcal{P}_2(\mathbb{R}^d)$ of (30) in \mathbb{R}^d satisfying*

$$u(t, \cdot) \rightarrow u_0 \text{ in } \mathcal{P}_2(\mathbb{R}^d) \text{ as } t \rightarrow 0,$$

$\phi(u) \in L^1_{loc}(0, \infty; \mathcal{W}^{1,1}_{loc}(\mathbb{R}^d))$, and, for all $T > 0$,

$$\int_{Q_T} u \left| \frac{1}{u} \nabla \phi(u) + \nabla V + \nabla W * u \right|^2 dx dt < \infty. \quad (31)$$

Remark 7 (Existence theory). Theorem 12 holds even if $\phi = \sigma$ and the initial datum is a measure in $\mathcal{P}_2(\mathbb{R}^d)$. However, given the assumptions in this paper, we do not have to consider measure-valued solutions. For further results concerning existence, uniqueness and asymptotic properties of gradient flow/free energy solutions, we refer to [4, 21].

4.1. Uniform bounds

Lemma 13. *Let u be a weak solution of (30) and let Assumption 2 hold. Problem (30) is a gradient flow (in the Wasserstein sense) for the following associated free energy functional*

$$E[u(t)] = \int_{\mathbb{R}^d} \left[u \log u + \Xi(u) + uV + \frac{1}{2}u(W * u) \right] dx,$$

where

$$\Xi(u) = \int_0^u \xi(s) ds, \quad \xi(s) = \int_1^s \frac{\sigma'(r)}{r} dr.$$

More specifically, $\forall t \geq 0$ we have

$$E[u(t)] + \int_0^t D[u(\tau)] d\tau = E[u_0], \quad (32)$$

where

$$D[u(t)] = \int_{\mathbb{R}^d} u |\nabla(\log(u) + \xi(u) + V + W * u)|^2 dx. \quad (33)$$

Moreover, for any $t \geq 0$, we have that $u \geq 0$ and that

$$\int_{\mathbb{R}^d} u dx = \int_{\mathbb{R}^d} u_0 dx.$$

Proof. Identity (32) is obtained differentiating $E[u(t)]$ with respect to time and noticing that equation (3) can be rewritten as follows:

$$\partial_t u = \operatorname{div} [u \nabla(\log(u) + \xi(u) + V + W * u)].$$

See for example Theorem 11.2.1, Chapter 11 in [4] for further details. \square

Remark 8. Notice that, for $V = V_k$, the estimates for the L^1 norm and for the free energy in Lemma 13 are uniform with respect to k and t .

We now state a useful technical Lemma, for its proof we refer the reader to [10]. Recall that $(f(x))_- = \max\{0, -f(x)\}$.

Lemma 14 (Carleman estimate [10]). *Consider two functions $\rho \in L^1_+(\mathbb{R}^d)$ and $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}$, with $\gamma(x) \geq 0$, $e^{-\gamma} \in L^1(\mathbb{R}^d)$, such that the moment $\int_{\mathbb{R}^d} \gamma(x) \rho(x) dx$ is bounded. Then*

$$\int_{\mathbb{R}^d} \rho(x) (\log \rho(x))_- dx \leq \int_{\mathbb{R}^d} \gamma(x) \rho(x) dx + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\gamma(x)} dx. \quad (34)$$

Thanks to Lemma 14, the negative part of $u_k \log u_k$ is bounded and Lemma 15 (below) provides an estimate for u_k that involves only positive terms on the left-hand side.

Lemma 15. *Let u_k be the solution of (30) where $V = V_k$ is defined in (6). Then the following inequality holds*

$$\int_{\mathbb{R}^d} [u_k |\log u_k| + u_k V_k + u_k (W * u_k)] dx \leq C_0, \quad (35)$$

where C_0 is a constant depending on Ω , u_0 and V_0 only, given by

$$C_0 = 2E(u_0) + \frac{2}{e} \left(\int_{\Omega} e^{-\frac{1}{2} V_0} dx + \varepsilon_k \right),$$

with $\varepsilon_k = \int_{\mathbb{R}^d \setminus \Omega} e^{-\frac{1}{2}V_k} dx \rightarrow 0$ as $k \rightarrow \infty$. Additionally, we have that

$$\int_{\mathbb{R}^d \setminus \Omega_k} u_k dx \leq \frac{2C_0}{k}, \quad (36)$$

and $u_k \rightarrow 0$ in $L^1(\mathbb{R}^d \setminus \Omega_k)$ as $k \rightarrow \infty$, uniformly in $t \in [0, T]$.

Proof. Let us first observe that, from (4), we have

$$\frac{\phi'(s)}{s} = \frac{1}{s} + \frac{\sigma'(s)}{s} \geq \frac{1}{s} + \mu s^{a-1} > 0,$$

for $s > 0$. It follows that all the terms involving $\frac{\sigma'(s)}{s}$ appearing in the free energy functional E are non-negative. Before using the entropy inequality of Lemma 13, we have to ensure that the term involving $u_k \log(u_k)$ is non-negative. We have

$$\int_{\mathbb{R}^d} \left[(u_k \log u_k) + u_k V_k + \frac{1}{2} u_k (W * u_k) \right] dx \leq E(u_0).$$

In order to estimate the negative part of $u_k \log u_k$ we will apply Lemma 14 with $\rho = u_k$ and $\gamma = \frac{1}{2}V_k$ (notice that $\int_{\mathbb{R}^d} V_k u_k dx$ is bounded but we do not know that the bound is uniform in k at this stage). More specifically, we have

$$\int_{\mathbb{R}^d} u_k (\log u_k)_- dx \leq \frac{1}{2} \int_{\mathbb{R}^d} V_k u_k dx + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\frac{1}{2}V_k} dx.$$

This implies that

$$\int_{\mathbb{R}^d} \left[u_k |\log u_k| + \frac{1}{2} u_k V_k + \frac{1}{2} u_k ((W) * u_k) \right] dx \leq E(u_0) + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\frac{1}{2}V_k} dx,$$

and, in turn,

$$\frac{1}{2} \int_{\mathbb{R}^d \setminus \Omega_k} u_k V_k dx \leq E(u_0) + \frac{1}{e} \int_{\mathbb{R}^d} e^{-\frac{1}{2}V_k} dx.$$

We subsequently obtain

$$0 \leq \int_{\mathbb{R}^d \setminus \Omega_k} u_k dx \leq 2 \frac{C_0}{k} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

□

Remark 9. Notice that the assumption $W \geq 0$ is not restrictive and the same argument applies if W has any lower bound of the type $W \geq -q$, in particular

$$\int_{\mathbb{R}^d} u(W * u) dx = \int_{\mathbb{R}^d} u((W + q) * u) dx - q \left(\int_{\mathbb{R}^d} u dx \right)^2.$$

The following estimate will be used extensively in the next subsection.

Corollary 16. *Let u_k be the solution to (30) when $V = V_k$ is given by (8). Then the following estimate (uniform in k) holds*

$$\begin{aligned} & \int_{\mathbb{R}^d} [u_k |\log u_k| + \Xi(u_k) + u_k V_k + u_k ((W + q) * u_k)] dx \\ & + \int_0^T \int_{\mathbb{R}^d} u_k |\nabla(\log u_k + \xi(u_k) + V_k + W * u_k)|^2 dx dt \leq C_0, \end{aligned} \quad (37)$$

where C_0 is specified in Lemma 15.

Proof. It is a direct consequence of (32) and (35). \square

4.2. Passage to the limit and proof of Theorem 2.

Before proving Theorem 2, we state a simple interpolation lemma.

Lemma 17. *Consider a function $f : \Omega \rightarrow \mathbb{R}$ such that*

$$f \in L^{p_0}(0, T; L^{q_0}(\Omega)) \cap L^{p_1}(0, T; L^{q_1}(\Omega)),$$

where $1 \leq p_i, q_i \leq \infty$, for $i = 0, 1$. Then

$$f \in L^{p_\theta}(0, T; L^{q_\theta}(\Omega)),$$

for any $\theta \in (0, 1)$ and the following relations hold:

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof. The proof consists in applying Hölder's inequality to $f^{1-\theta} f^\theta = f$ twice, first with respect to $x \in \Omega$, then with respect to $t \in [0, T]$. Namely we have:

$$\begin{aligned} \left(\int_0^T \|f\|_{L^{q_\theta}(\Omega)}^{p_\theta} dt \right)^{\frac{1}{p_\theta}} & \leq \left(\int_0^T \|f\|_{L^{q_0}(\Omega)}^{1-\theta} \|f\|_{L^{q_1}(\Omega)}^\theta dt \right)^{\frac{1}{p_\theta}} \\ & \leq \left(\int_0^T \|f\|_{L^{q_0}(\Omega)}^{p_0} dt \right)^{\frac{1-\theta}{p_0}} \left(\int_0^T \|f\|_{L^{q_1}(\Omega)}^{p_1} dt \right)^{\frac{\theta}{p_1}}. \end{aligned}$$

\square

Similarly to the L^2 case, we consider the weak formulation (25) and we divide \mathbb{R}^d into three parts, namely Ω , Ω_k^c and $\Omega_k \setminus \Omega$, and hence we split the weak formulation as follows:

$$\begin{aligned} & \int_{\mathbb{R}^d} u_k \eta \Big|_0^T dx + \int_0^T \int_{\mathbb{R}^d} [(\nabla \phi(u_k) + u_k \nabla V_k + u_k \nabla W * u_k) \cdot \nabla \eta - u_k \partial_t \eta] dx dt \\ & = I_\Omega + I_{\Omega_k \setminus \Omega} + I_{\Omega_k^c} + J_\Omega + J_{\Omega_k \setminus \Omega} + J_{\Omega_k^c}, \end{aligned} \quad (38)$$

for any test function $\eta \in C_0^\infty(Q_T)$. Namely we have defined

$$\begin{aligned} I_A &= \int_0^T \int_A (\nabla \phi(u_k) + u_k \nabla V_k + u_k \nabla W * u_k) \cdot \nabla \eta \, dx dt, \\ J_A &= - \int_0^T \int_A u_k \partial_t \eta \, dx dt. \end{aligned}$$

Proof of Theorem 2, non-degenerate case. We are going to show that all terms except I_Ω and J_Ω vanish in the limit $k \rightarrow \infty$. It follows that the term $I_\Omega + J_\Omega$ characterizes the limit problem defined in Ω .

- **I_Ω .** First, we restrict our attention to I_Ω . We notice that restricting (37) in Ω we have

$$\int_0^T \int_\Omega u_k |\nabla(\log u_k + \xi(u_k) + V_0 + W * u_k)|^2 \, dx dt \leq C_0. \quad (39)$$

Hence, we have that $\sqrt{u_k} \nabla(\log u_k + \xi(u_k) + V_0 + W * u_k) \in L^2(\Omega_T)$. We now proceed to show that the last two terms in this expression are bounded in $L^2(\Omega_T)$. We have

$$\int_0^T \int_\Omega u_k |\nabla V_0|^2 \, dx dt \leq T \|\nabla V_0\|_{L^\infty(\Omega)}^2 \int_\Omega u_0 \, dx, \quad (40)$$

and

$$\int_0^T \int_\Omega u_k |\nabla W * u_k|^2 \, dx dt \leq T \|\nabla W\|_{L^\infty(\Omega)}^2 \left(\int_\Omega u_0 \, dx \right)^2. \quad (41)$$

We rewrite the diffusion terms as

$$\int_0^T \int_\Omega u_k |\nabla(\log u_k + \xi(u_k))|^2 \, dx dt = \int_0^T \int_\Omega (1 + \sigma'(u_k)) |\nabla(\sqrt{u_k})|^2 \, dx dt. \quad (42)$$

Recalling that $\sigma' \geq 0$ and combining (39), (40) and (41), from (42) it follows that $\nabla(\sqrt{u_k}) \in L^2(\Omega_T)$ uniformly in k . We know that the mass of u_k is constant and u_k is non-negative, therefore $\sqrt{u_k} \in L^\infty(0, T; L^2(\Omega))$. Additionally, we know that $\sqrt{u_k} \in L^2(0, T; H^1(\Omega))$, which implies $\sqrt{u_k} \in L^2(0, T; L^{2^*}(\Omega))$. By Lemma 17, we deduce that $\sqrt{u_k} \in L^r(\Omega_T)$ for any $r \in (2, 2(1 + 2/d))$, when $d \geq 2$ (if $d = 1$ we choose $r = 4$). We can now estimate $\partial_t u_k$ as follows:

$$\begin{aligned} \langle \partial_t u_k, \eta \rangle &= \int_{\Omega_T} \sqrt{u_k} \sqrt{u_k} [\nabla(\log u_k + \xi(u_k) + V_0 + W * u_k)] \cdot \nabla \eta \, dx dt \\ &\leq C_0^{\frac{1}{2}} \|\sqrt{u_k}\|_{L^r(\Omega_T)} \|\nabla \eta\|_{L^s(\Omega_T)}, \end{aligned}$$

where r and s satisfy $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$. This implies that $\partial_t u_k$ is bounded in $L^s(0, T; W^{1,s}(\Omega))'$ uniformly with respect to k . In order to obtain compactness for the sequence $\sqrt{u_k}$, we need a modified version of Aubin–Lions Lemma. In particular, if $a > 1$ we apply

Theorem 1 of [32] (with $\Phi(\cdot) = \sqrt{|\cdot|}$, so that $\sqrt{|\cdot|} \in W^{1,1}(\mathbb{R})$ and $\text{meas}(\{\sqrt{|\cdot|} > \delta\}) \rightarrow 0$ in L^1 as $\delta \rightarrow 0$), otherwise we apply Theorem 3 of [24] (with $m = \frac{1}{2}$). We deduce that the sequence $\sqrt{u_k}$ is compact in $L^2(\Omega_T)$. Therefore we can extract a subsequence (still denoted by u_k) that converges strongly in the same space, that is,

$$\sqrt{u_k} \rightarrow \sqrt{u} \quad \text{in } L^2(\Omega_T). \quad (43)$$

We now rewrite I_Ω as follows

$$I_\Omega = \int_0^T \int_\Omega \sqrt{u_k} \sqrt{u_k} \nabla(\log u_k + \xi(u_k) + V_k + W * u_k) \cdot \nabla \eta \, dx dt, \quad (44)$$

and we notice that the integrand is the product of a strongly converging sequence and a weakly converging sequence in $L^2(\Omega_T)$, in particular $\sqrt{u_k} \rightarrow \sqrt{u}$ and $F_k \rightharpoonup F \in L^2(\Omega_T)$, where $F_k = \sqrt{u_k} \nabla(\log u_k + \xi(u_k) + V_k + W * u_k)$. This means that

$$\int_0^T \int_\Omega \sqrt{u_k} F_k \cdot \nabla \eta \, dx dt \rightarrow \int_0^T \int_\Omega \sqrt{u} F \cdot \nabla \eta \, dx dt.$$

Furthermore, again combining (39), (40) and (41), it follows that $F = \nabla \phi(u) + u \nabla V_0 + u \nabla W * u$ and

$$I_\Omega \rightarrow \int_0^T \int_\Omega [(\nabla \phi(u) + u \nabla V_0 + u \nabla W * u) \cdot \nabla \eta] \, dx dt, \quad \text{as } k \rightarrow \infty. \quad (45)$$

- $I_{\Omega_k^c}$. The term $I_{\Omega_k^c}$ is dealt with in an analogous way to I_Ω since equation (44) holds replacing Ω by Ω_k^c . In particular, this term then vanishes since $u_k \rightarrow 0$ strongly in $L^1(\Omega_k^c)$ (using Lemma 15).
- $I_{\Omega_k \setminus \Omega}$. We now check that $I_{\Omega_k \setminus \Omega}$ also vanishes. Indeed we have

$$I_{\Omega_k \setminus \Omega} \leq \|u_k\|_{L^\infty(0,T;L^1(\Omega_k \setminus \Omega))}^{1/2} \left(\int_0^T \int_{\Omega_k \setminus \Omega} u_k |\nabla(\log u_k + \xi(u_k) + V_k + W * u_k)|^2 \, dx dt \right)^{\frac{1}{2}}.$$

The second factor in the right hand side is bounded by the Lemma 16. The first factor is bounded and converges to zero as $k \rightarrow \infty$ using the following argument. By Jensen's inequality we obtain, for a region R such that $|R| < 1$,

$$\int_R g(x) dx \leq (-\log |R|)^{-1} \left(\int_R g(x) \log g(x) dx + e^{-1} \right). \quad (46)$$

We use inequality (46) with $R = \Omega_k \setminus \Omega$ and $g = u_k$. Recalling that $|\Omega_k \setminus \Omega| \rightarrow 0$ as $k \rightarrow \infty$, we obtain the desired result, $I_{\Omega_k \setminus \Omega} \rightarrow 0$.

- J_Ω . Up to a subsequence, u_k converges weakly in $L^1(\Omega)$ to a limit function u , hence

$$J_\Omega \rightarrow - \int_0^T \int_A u \partial_t \eta \, dx dt, \quad \text{as } k \rightarrow \infty. \quad (47)$$

- $J_{\Omega_k^c}$. Notice that $u_0 = 0$ in Ω_k^c . The remaining terms in $J_{\Omega_k^c}$ vanish due to Lemma 15.
- $J_{\Omega_k \setminus \Omega}$. The integral $J_{\Omega_k \setminus \Omega}$ goes to zero because the integrand is uniformly bounded in L^1 (thanks to the conservation of mass, see Lemma 3) and $|\Omega_k \setminus \Omega| \rightarrow 0$.

Thanks to (45) and (47), the weak formulation we obtain in the limit is the following:

$$\int_0^T \int_{\Omega} [(\nabla \phi(u) + u \nabla V_0 + u \nabla W * u) \cdot \nabla \eta - u \partial_t \eta] dx dt = 0.$$

Notice that since the test function can take arbitrary values on $\partial\Omega$, the no-flux conditions on $\partial\Omega$ are implicitly enforced.

We now show that initial datum is satisfied in $\mathcal{P}_2(\mathbb{R}^d)$. To do so, we use the characterization of $\mathcal{P}_2(\mathbb{R}^d)$ convergence given in Proposition 7.1.5, in [4]. In particular, the second moment $\int_{\mathbb{R}^d} u_k |x|^2 dx$ is bounded uniformly ($V_k(x) \geq c|x|^2$ at infinity) and the so-called narrow convergence is implied by the L^1 bounds obtained in (36), (43) and (46). We deduce that u_k converges to u in $\mathcal{P}_2(\mathbb{R}^d)$ as well. As a consequence, we obtain that the initial datum is satisfied in the $\mathcal{P}_2(\mathbb{R}^d)$ sense, indeed, since our L^1 bounds are uniform in time, we have that

$$\lim_{t \rightarrow 0} \lim_{k \rightarrow \infty} d_2(u(t, \cdot), u_0) \leq \lim_{t \rightarrow 0} \lim_{k \rightarrow \infty} (d_2(u_k(t, \cdot), u(t, \cdot)) + d_2(u_k(t, \cdot), u_0)) = 0,$$

where d_2 denotes the 2-Wasserstein distance. Hence the result is proven. \square

Proof of Theorem 2, degenerate case. Let us now drop the linear diffusion term and consider $\phi = \sigma$. Almost all the results above remain unchanged, therefore we will only highlight the steps that differ. Notice that, as before, u_k converges to 0 in the complement of Ω . Let us discuss the passage to the limit in the term I_{Ω} in the proof of Theorem 12. More specifically, we have to obtain strong L^1 convergence in Ω finding an alternative to (42). Similarly to (39), we have

$$\int_0^T \int_{\Omega} u_k |\nabla(\xi(u_k) + V_0 + W * u_k)|^2 dx dt \leq C_0.$$

Recall that, by assumption, we have $\sigma'(s) \geq \mu s^a$ for some $a > 0$. Since u_k has constant mass (and hence it is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$), we deduce that

$$\int_0^T \int_{\Omega} \mu u_k^{2a-1} |\nabla u_k|^2 dx dt \leq \int_0^T \int_{\Omega} \frac{\sigma'(u)^2}{u} |\nabla u_k|^2 dx dt = \int_0^T \int_{\Omega} u_k |\nabla \xi(u_k)|^2 dx dt \leq C_0.$$

This implies that $\nabla \left(u_k^{a+\frac{1}{2}} \right) \in L^2(\Omega_T)$. We observe that, since $\Xi(u_k)$ and $\nabla \xi(u_k)$ are bounded in $L^2(\Omega)$ independently of k , and since u_k is non-negative, inequality (37) implies, for a constant C depending only on a and μ ,

$$\int_{\Omega} u_k(t)^{1+a} dx \leq C(a, \mu) \int_{\Omega} \Xi(u_k(t)) dx < \infty, \quad \forall t \geq 0. \quad (48)$$

It follows that $\sqrt{u_k} \in L^\infty(0, T; L^{2+2a}(\Omega))$, which implies $\sqrt{u_k} \in L^r(\Omega_T)$ for any $r \in (2, 2+2a)$. We can estimate $\partial_t u_k$ as follows:

$$\begin{aligned} \langle \partial_t u_k, \eta \rangle &= \int_{\omega_T} \sqrt{u_k} \sqrt{u_k} (\nabla(\xi(u_k) + V_0 + W * u_k)) \cdot \nabla \eta dx dt \\ &\leq C_0^{\frac{1}{2}} \|\sqrt{u_k}\|_{L^r(\Omega_T)} \|\nabla \eta\|_{L^s(\Omega_T)}, \end{aligned}$$

where r and s satisfy $\frac{1}{r} + \frac{1}{s} = \frac{1}{2}$. This implies that $\partial_t u_k$ is bounded in $L^s(0, T; W^{1,s}(\Omega))'$ uniformly with respect to k . We can now apply either the modified version of Aubin–Lions Lemma presented in Theorem 1 of [32] (with $\Phi(\cdot) = |\cdot|^{a+\frac{1}{2}}$) if $a \geq 1$, or Theorem 3 in [24] (with $m = a + \frac{1}{2}$) otherwise. Hence, by compactness, the sequence u_k converges a.e. to a limit u . Combining a.e. convergence with inequality (48) and uniqueness of weak limits, we obtain strong L^1 convergence of u_k to u for $k \rightarrow \infty$. This allows to conclude the proof. \square

Remark 10 (Moments). We have used the hypothesis of quadratic growth of V at infinity only to ensure that the second moment $\int_{\mathbb{R}^d} u_k |x|^2 dx$ is bounded and therefore that the initial datum is satisfied in $\mathcal{P}_2(\mathbb{R}^d)$. It is possible to make less restrictive assumptions, for example, if V grows linearly at infinity we get control over the first moment and the initial datum is satisfied in $\mathcal{P}_1(\mathbb{R}^d)$.

5. Numerical exploration

We will now illustrate our main results on the approximation of no-flux boundary value problems by large confinement with some numerical results. Here, we make use of a numerical scheme with excellent properties such as semidiscrete free energy decay, and positivity under a CFL condition. The numerical scheme is based on a finite volume discretization with upwinding and second order reconstruction. We refer to [9, 17] and the references therein for further details. This numerical strategy has been successfully used in many similar gradient flow type equations and systems [19], and it has recently been generalized to high order DG-approximations in [36]. All our numerical results are obtained with the original second-order version in [17]. In the next subsection we will showcase our results in one dimension and then we will explore the behavior of the solutions when the initial is not necessarily supported on the limiting domain. Finally, the last subsection explores these issues in two dimensions.

5.1. One-dimensional examples

We first run simulations of the one-dimensional problem, with $\Omega = [-1, 1]$ and a computational domain $B = [-4, 4]$ (we choose B large enough so that its size does not affect the solution in Ω).

Figure 2 shows an example with the linear Fokker–Planck equation ($\phi = u$ and $W = 0$). Figure 3 shows an example with a nonlinear local Fokker–Planck equation (with $\phi = u + \beta u^2$ and $W = 0$). Figure 4 shows an example with a nonlinear nonlocal Fokker–Planck equation (with $\phi = u + \beta u^2$ and $W = -(1 - |x|)_+$). In all three figures, the subplot (a) shows the

confining potentials V_k in thin colored lines, and the potential V_0 in Ω in a thick black line. The subplot (b) shows the solutions u_k at the final simulation time in thin colored lines and the solution of the limit problem u (only defined in Ω) as a thick black line. The subplot (c) shows the l^2 -norm between u_k and u in Ω as a function of time, for various values of k . The subplot (d) shows again the l^2 -norm between u_k and u in Ω but only at the final time (circles), as well as the norm of u_k in $B \setminus \Omega$ at the final time. As expected, we observe that u_k get closer to u in Ω and that the norms of the errors decrease as k increases.

In Figure 5 we consider an example with the same initial condition u_0 and potential V_0 for all three cases (linear, nonlinear local and nonlinear nonlocal Fokker–Planck equation) so that we can compare the effects that the different terms have in the solution. We consider a simple case with no external potential in Ω , $V_0 = 0$ (see Figure 5(a)) and initial data $u_0 = \chi_{[-1, -0.7] \cup [0.7, 1]}$. Figures 5(b-d) show the solutions u_k at $T_f = 0.2$ and for $k = 1, \dots, 10$ (colored lines) and the limit problem solution u (thick black line) in the linear, nonlinear, and nonlocal cases, respectively.

For our final one-dimensional simulation, we consider a case (not allowed in our analysis) where part of the support of the initial data u_0 lies outside $\Omega = [-1, 1]$. In particular, the initial condition for u_k is:

$$u_0(x) = \frac{C}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}, \quad x \in B = [-4, 4], \quad (49)$$

with $\sigma = 2$ and where C is a constant such that $\int_B u_0 dx = 1$. The initial condition for the limit problem is u_0 constrained in Ω and the mass that lies outside Ω placed on the $x = \pm 1$:

$$\bar{u}_0(x) = u_0(x) + M_l \delta_{-1}(x) + M_r \delta_1(x), \quad x \in \Omega = [-1, 1], \quad (50)$$

where $M_l = \int_{-4}^{-1} u_0 dx$ and $M_r = \int_1^4 u_0 dx$. We consider again a zero external potential, $V_0 = 0$ (Figure 6(a)), and linear diffusion, $\phi = u$ and $W = 0$. Figure 6(b) shows the solutions u_k at $T_f = 2$ for k up to 10, and the solution of the limit problem u . We observe nice convergence as k increases, see Figures 6(c) and (d) for the evolution of the error. Figure 7 shows the dynamics up to $t = 1$ of the limit problem, a weak confinement case ($k = 2$) and a strong confinement case ($k = 10$). To implement the initial condition (50), we placed a Dirac delta on the boundary nodes inside $\Omega = [-1, 1]$. Let us denote the grid points in $B = [-4, 4]$, regularly spaced by Δx , as $[l_1, l_2, \dots, l_M, x_1, x_2, \dots, x_N, r_1, r_2, \dots, r_M]$. Here l_i and r_i are points outside of Ω (respectively to the left or to the right), and x_i are inside Ω . The points are chosen so that $x_1 = -1 + \Delta x/2$ and $x_N = 1 - \Delta x/2$ (as standard in finite-volume schemes). The initial datum for the limit problem in Ω is set as follows:

$$\begin{aligned} M_l &= \Delta x \sum_{i=1}^M u_0(l_i), & M_r &= \Delta x \sum_{i=1}^M u_0(r_i), \\ \bar{u}_0^i &= u_0(x_i), & i &= 2, \dots, N-1, \\ \bar{u}_0^1 &= u_0(x_1) + M_l/\Delta x, & \bar{u}_0^N &= u_0(x_N) + M_r/\Delta x. \end{aligned}$$

Note that the factor $1/\Delta x$ is required as we are spreading a punctual mass over a compartment of length Δx .

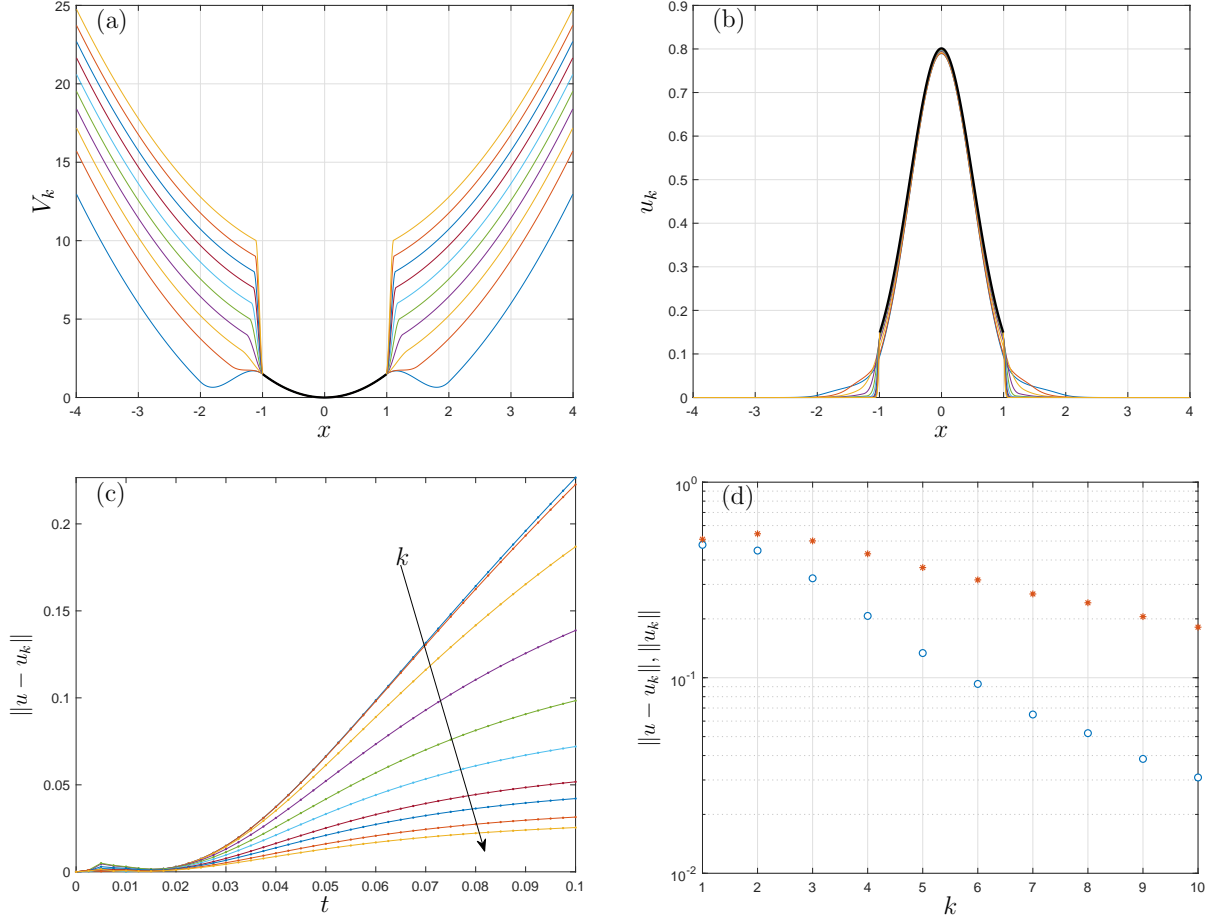


Figure 2: Example with linear diffusion, local. External potential $V(x) = 1.5x^2$, linear diffusion $\phi(u) = u$, zero nonlocal term $W \equiv 0$. Simulation with $\Omega = [-1, 1]$, computational domain for u_k is $B = [-4, 4]$ ($k \geq 1$), initial condition $u_0 = 1$, final time $T_f = 0.2$. Grid spacing $\Delta x = 0.01$ and initial time-step $\Delta t = 10^{-5}$. (a) Confining potential V_k for $k = 1, 2, \dots, 10$ (potential V shown in thick black line). (b) Solutions u (thick black line) and u_k (colored thin lines) at $t = T_f$. (c) Norm between u and u_k in Ω at times $t \in [0, 0.1]$. (d) Norm between u and u_k in Ω at $t = T_f$ (circles) and norm of u_k in $B \setminus \Omega$ at $t = T_f$ (asterisks).

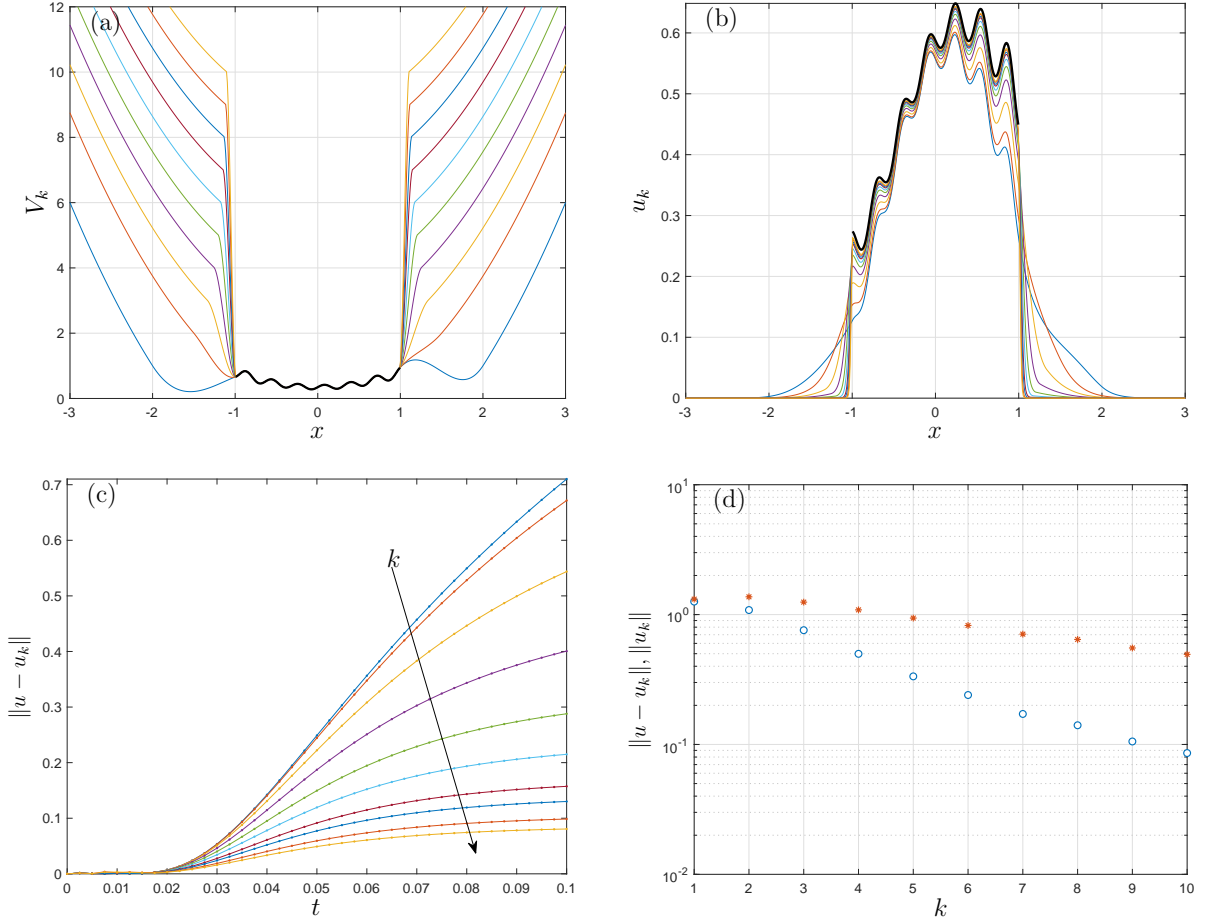


Figure 3: Example with nonlinear diffusion, local. External potential $V(x) = 0.46(1+0.2 \sin(20x))(x^2+0.75)$, nonlinear diffusion $\phi(u) = u + \beta u^2$, with $\beta = 0.49$, zero nonlocal term $W \equiv 0$. Simulation with $\Omega = [-1, 1]$, computational domain for u_k is $B = [-4, 4]$ ($k \geq 1$), initial condition $u_0 = \chi_{[0.1, 0.3]}$, final time $T_f = 0.2$. Grid spacing $\Delta x = 0.005$ and initial time-step $\Delta t = 10^{-5}$. (a) Confining potential V_k for $k = 1, 2, \dots, 10$ (potential V shown in thick black line). (b) Solutions u (thick black line) and u_k (colored thin lines) at $t = T_f$. (c) Norm between u and u_k in Ω at times $t \in [0, 0.1]$. (d) Norm between u and u_k in Ω at $t = T_f$ (circles) and norm of u_k in $B \setminus \Omega$ at $t = T_f$ (asterisks).

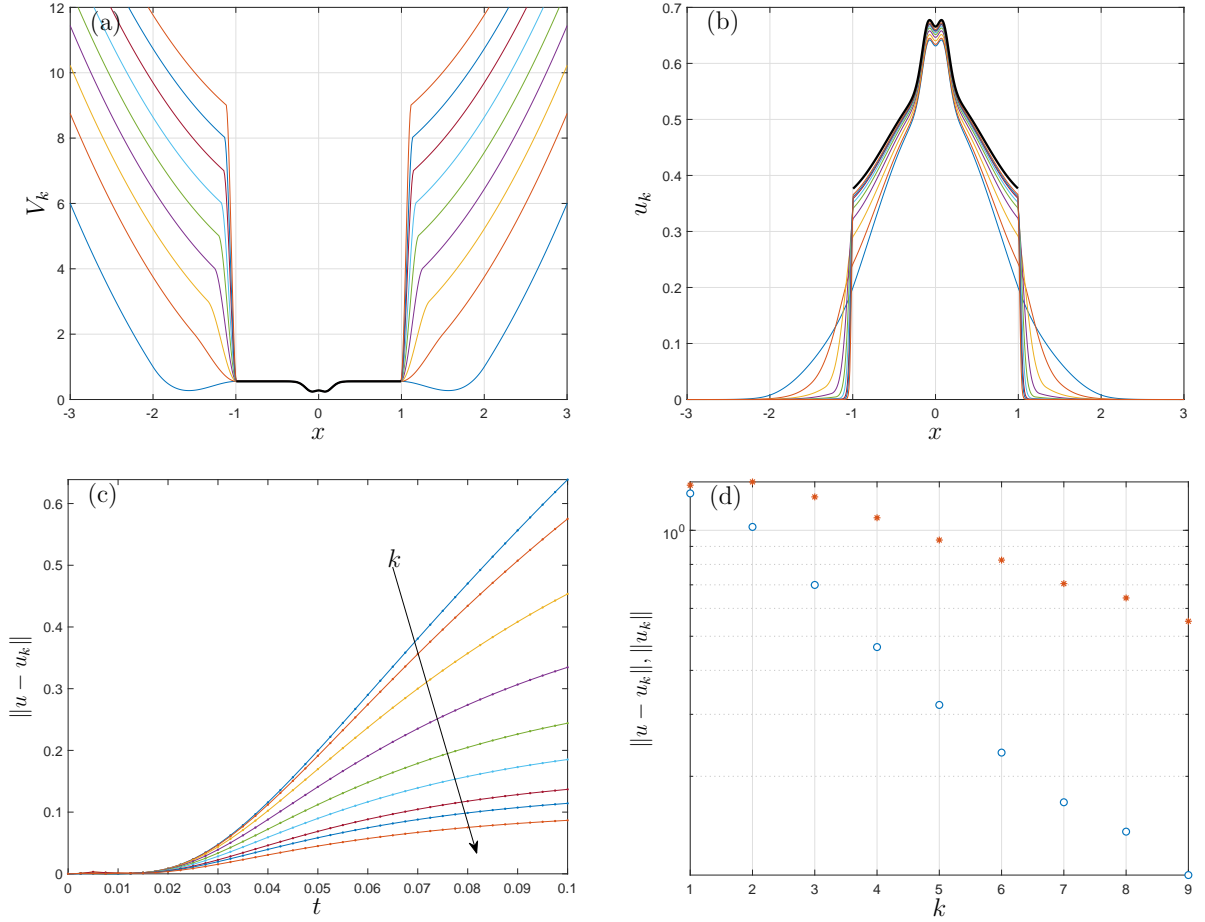


Figure 4: Example with nonlinear diffusion, nonlocal. External potential $V(x) = 0.56[\exp(-100x^2) - 1.5\exp(-50x^2) + 1]$, nonlinear diffusion $\phi(u) = u + \beta u^2$, with $\beta = 0.49$, nonlocal term $W = -(1 - |x|)_+$. Simulation with $\Omega = [-1, 1]$, computational domain for u_k is $B = [-4, 4]$ ($k \geq 1$), initial condition $u_0 = 1$, final time $T_f = 0.2$. Grid spacing $\Delta x = 0.01$ and initial time-step $\Delta t = 10^{-5}$. (a) Confining potential V_k for $k = 1, 2, \dots, 9$ (potential V shown in thick black line). (b) Solutions u (thick black line) and u_k (colored thin lines) at $t = T_f$. (c) Norm between u and u_k in Ω at times $t \in [0, 0.1]$. (d) Norm between u and u_k in Ω at $t = T_f$ (circles) and norm of u_k in $B \setminus \Omega$ at $t = T_f$ (asterisks).

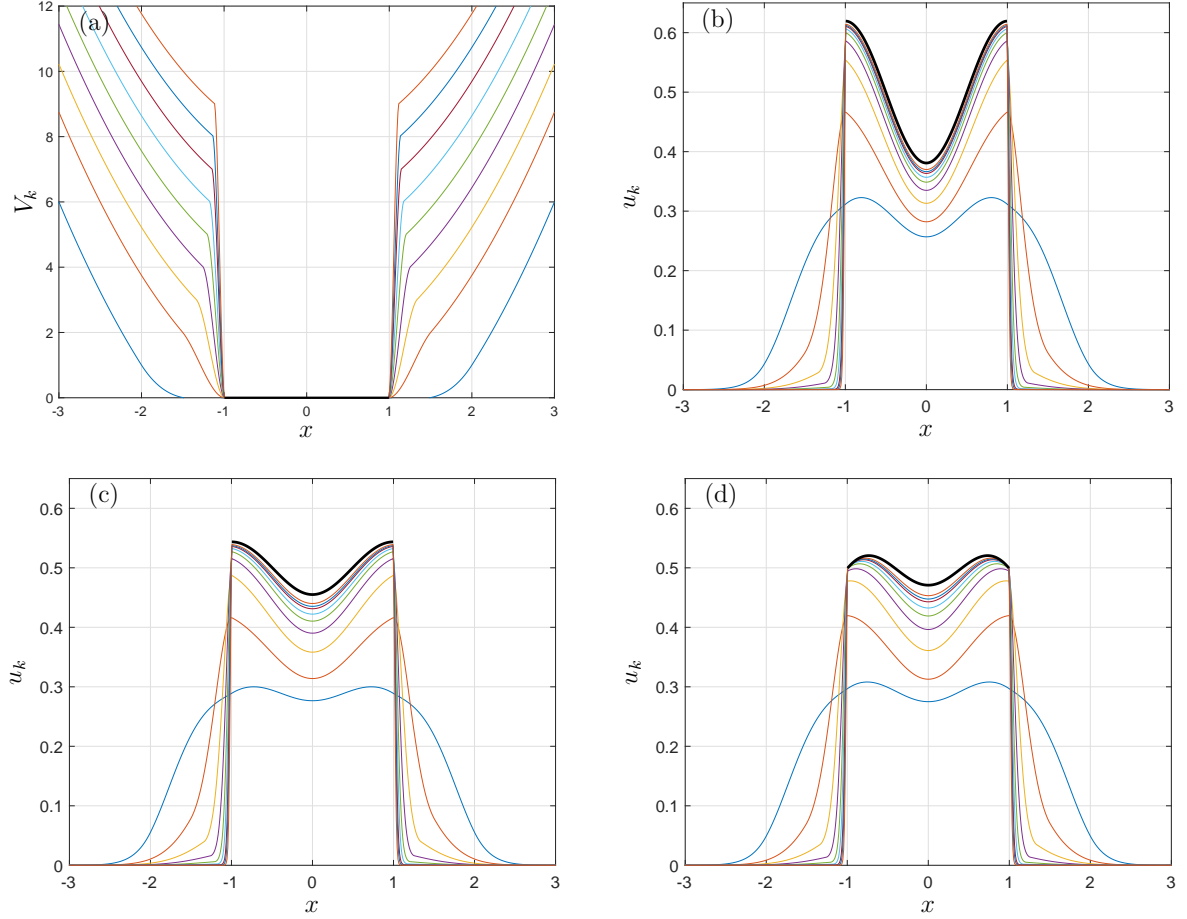


Figure 5: Comparison between models in an example with no external potential in Ω , $V_0 = 0$, and initial data $u_0 = \chi_{[-1, -0.7] \cup [0.7, 1]}$. (a) Confining potential V_k for $k = 1, \dots, 10$ (potential V shown in thick black line). (b-d) Solutions u (thick black line) and u_k (colored thin lines) at $t = 0.2$: (b) $\phi = u$ and $W = 0$, (c) $\phi(u) = u + \beta u^2$, with $\beta = 0.49$, and $W = 0$, (d) $\phi(u) = u + \beta u^2$, with $\beta = 0.49$ and $W = -(1 - |x|)_+$. Simulation with $\Omega = [-1, 1]$, computational domain for u_k is $B = [-4, 4]$ ($k \geq 1$). Grid spacing $\Delta x = 0.01$ and initial time-step $\Delta t = 10^{-5}$.

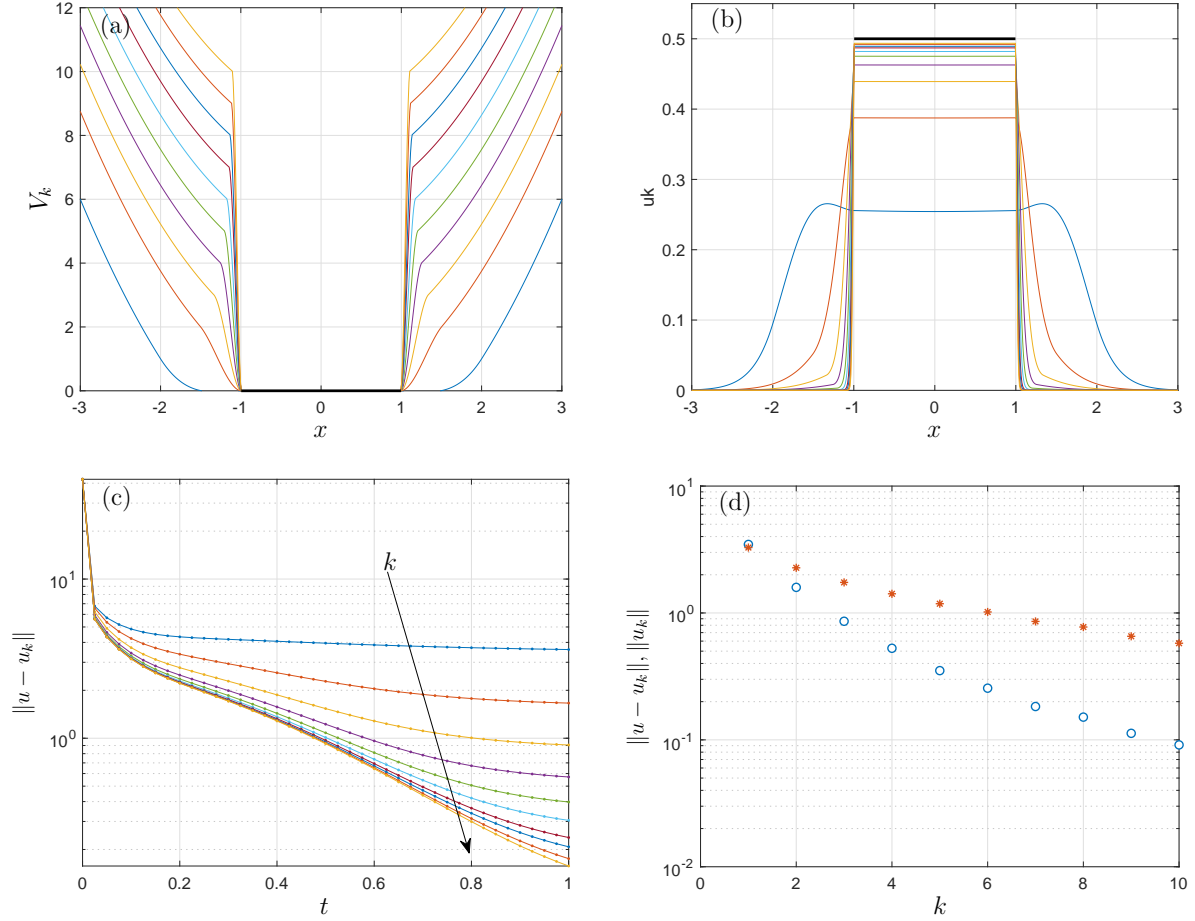


Figure 6: Example with an initial condition with support outside Ω , u_0 Gaussian with $\mu = 0$ and $\sigma = 2$. Linear diffusion, $V_0 = 0$ and $T_f = 2$. Simulation with $\Omega = [-1, 1]$, computational domain for u_k is $B = [-4, 4]$ ($k \geq 1$). Grid spacing $\Delta x = 0.01$ and initial time-step $\Delta t = 10^{-5}$. (a) Confining potential V_k for $k = 1, \dots, 10$ (potential V shown in thick black line). (b) Solutions u (thick black line) and u_k (colored thin lines) at $t = T_f$. (c) Norm between u and u_k in Ω at times $t \in [0, 0.1]$. (d) Norm between u and u_k in Ω at $t = T_f$ (circles) and norm of u_k in $B \setminus \Omega$ at $t = T_f$ (asterisks).

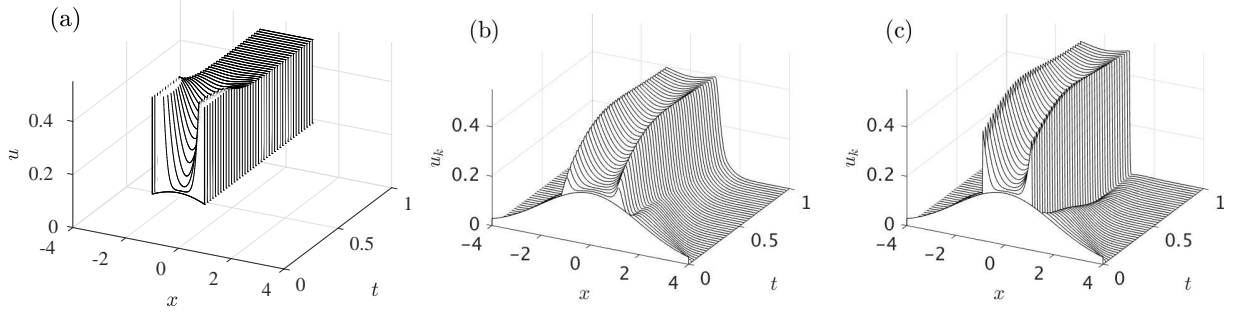


Figure 7: The dynamics of the solutions shown in Figure 6 until time $t = 1$ for (a) in Ω ($k = 0$), (b) a weak confining potential ($k = 2$), and (c) a strong confining potential ($k = 10$). In (a) the vertical axis is truncated (it'd go up to 30, mass of the delta functions at the edges).

5.2. Two-dimensional examples

In one dimension it seems reasonable to say that (50) is the only way to move the initial mass outside Ω towards $\partial\Omega$. This would imply the convergence towards a unique limit problem regardless of the confining potential V_k . However, it is not clear if the same would hold true in higher dimensions, where there are multiple ways of transporting mass from outside Ω to $\partial\Omega$. For example, suppose that $\Omega = \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Then among many options, the mass could be sent to $|x| = 1$ radially or proportionally to the strength of V_k .

To explore what happens when the initial data has support outside Ω in two dimensions, we consider the square domain $\Omega = [-1, 1]^2$ and $B = [-4, 4]^2$. Again for simplicity we work with the linear problem, setting $\phi = u$ and $W = 0$, and also $V_0 = 0$. We choose $\Omega_k = [-L_k, L_k]^2$, with $L_k = 1 + 1/k$. We consider four scenarios, combining the cases when the initial datum u_0 and/or the confinement potential V_k are radially symmetric or not. As a radially symmetric initial data we use the following volcano-shaped function (see Figure 8(a))

$$u_0 = C e^{-5(r-1)^2}, \quad (51)$$

where C is a normalization constant (so that u_0 has unit mass in B) and $r = \sqrt{x_1^2 + x_2^2}$ with $(x_1, x_2) \in B = [-4, 4]^2$. As a non-radially symmetric initial data we use (see Figure 8(b))

$$u_0 = C \left[1 + \frac{1}{2} \sin\left(\frac{x_2}{r}\right) \right] e^{-5(r-1)^2}, \quad (52)$$

where C is again the normalization constant.

We use two methods to transport the initial mass from outside Ω to its boundary. The first method consists of sending the mass perpendicular to $\partial\Omega$ and accumulate at the corners of $\partial\Omega$ all the mass in the regions $\{|x_1| > 1 \text{ and } |x_2| > 1\}$. The resulting initial conditions \bar{u}_0 for the limit problem corresponding to (51) and (52) are shown in Figures 8(c) and 8(d) respectively. This method leads to a concentration of mass at the four corners of Ω . The second method we consider consists of transporting the mass radially from the origin. Specifically, if $U_0^{ij} = u_0(x_i, x_j)$ with $(x_i, x_j) \notin \Omega$, then we compute $\theta^{ij} = \tan^{-1}(x_j, x_i)$ (we

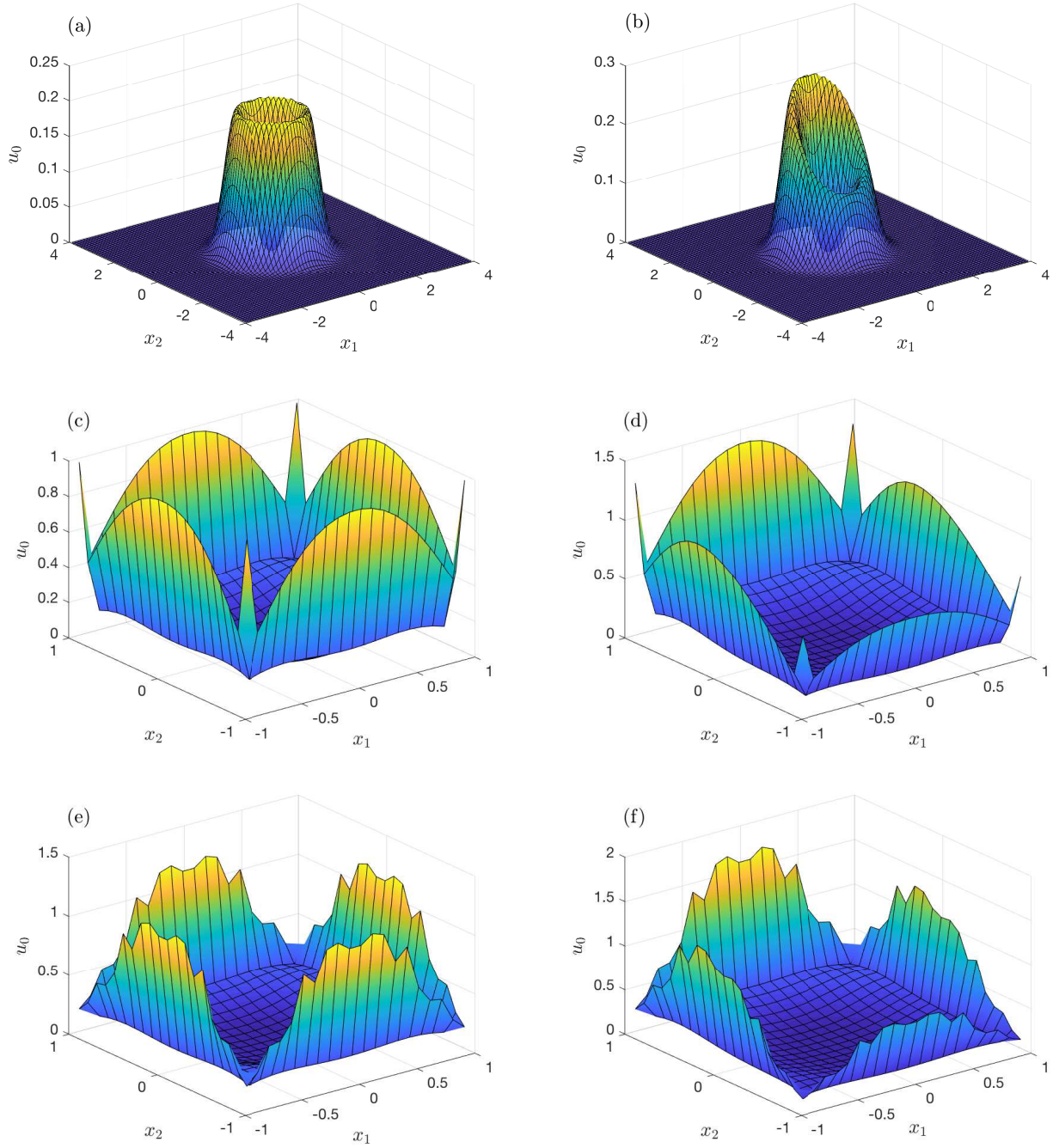


Figure 8: Initial conditions used for the two-dimensional examples in B (a-b) and in Ω (c-f). (a) Radially symmetric u_0 . (b) Non-radially symmetric u_0 . (c) Radially symmetric \bar{u}_0 obtained by the first mass transport method. (d) Non-radially symmetric \bar{u}_0 obtained by the first mass transport method. (e) Radially symmetric \bar{u}_0 obtained by the second mass transport method. (f) Non-radially symmetric \bar{u}_0 obtained by the second mass transport method.

use the four-quadrant inverse tangent, `atan2` in Matlab) and send the mass U_0^{ij} to the grid point on $\partial\Omega$ whose angle is closest to θ^{ij} . The initial conditions \bar{u}_0 for the limit problem corresponding to (51) and (52) and computed using the second method are shown in Figures 8(e) and 8(f) respectively.

For the confinement potentials, we use $V_k(x) = k + r^2 - L_k^2$ for $x \notin \Omega$ for the radially symmetric case and $V_k(x) = k + \tilde{r}^2 - L_k^2$ with $\tilde{r} = [1 + \sin(x_2/r)/2]r$ for the non-radially symmetric case. We run simulations for a short time ($T_f = 0.01$) comparing the solution u_k in B for $k = 25$ with either symmetric or asymmetric initial data (see Figures 8(a-b)) and using either a symmetric or asymmetric confinement potential V_k with the solution u to the limit problem in Ω with initial data \bar{u}_0 prescribed using one of the two methods described above (see Figures 8(c-f)).

In all possible combinations, we find that the simulation results (not shown) are not particularly sensitive to the shape of the confinement potential, and little sensitive to the method of transporting the mass of u_0 from outside Ω to $\partial\Omega$ (see Figure 8(c-d) for method 1 and (e-f) for method 2). The first transportation method (moving mass perpendicular to $\partial\Omega$) leads to a smaller difference at all times, but this could be determined by the choice of Ω and its discretization. The choice of a non-symmetric initial datum affects the shape of the solutions but it doesn't seem to affect the previous considerations concerning mass transportation particularly. Such insensitivity is somewhat counter-intuitive since, in dimension greater than one, we would expect the limit problem to vary depending on the choice of the confinement potential, and in particular on the way it diverges to infinity outside Ω .

We next try a more extreme two-dimensional case, still with $\Omega = [-1, 1]^2$. In particular, we compare a simple confinement potential (without buffer zone, $\Omega_k \equiv \Omega$, and quadratic outside Ω , see Figure 9(a)) and the following potential (see Figure 9(b)):

$$V_k = \begin{cases} 0, & \text{if } x \in \Omega \text{ or } |x_1| < 1/\sqrt{k}, \\ k + x_1^2 + x_2^2, & \text{otherwise.} \end{cases} \quad (53)$$

We work with the linear Fokker–Planck equation in $B = [-4, 4]^2$, initial condition $u_0 = C \exp(-|x|/2)$ with C normalisation condition and zero potential inside Ω , $V_0 = 0$. We look at the behaviour as k increases (we solve for $k = 5, 10, 15, 20$) and very short times ($t = 0, 0.001, \dots, 0.01$).

In Figure 9 we show the two confinement potentials for $k = 20$ and their respective solutions at $t = 0.01$. In Figure 10 we plot the value of the solutions at the various times for the different k along $\partial\Omega$, parameterized with arc length s (starting from $x = (1, 1)$ and going round clockwise). The lines corresponding to the two potentials seem to slowly converge to the same profile as the parameter k increases.

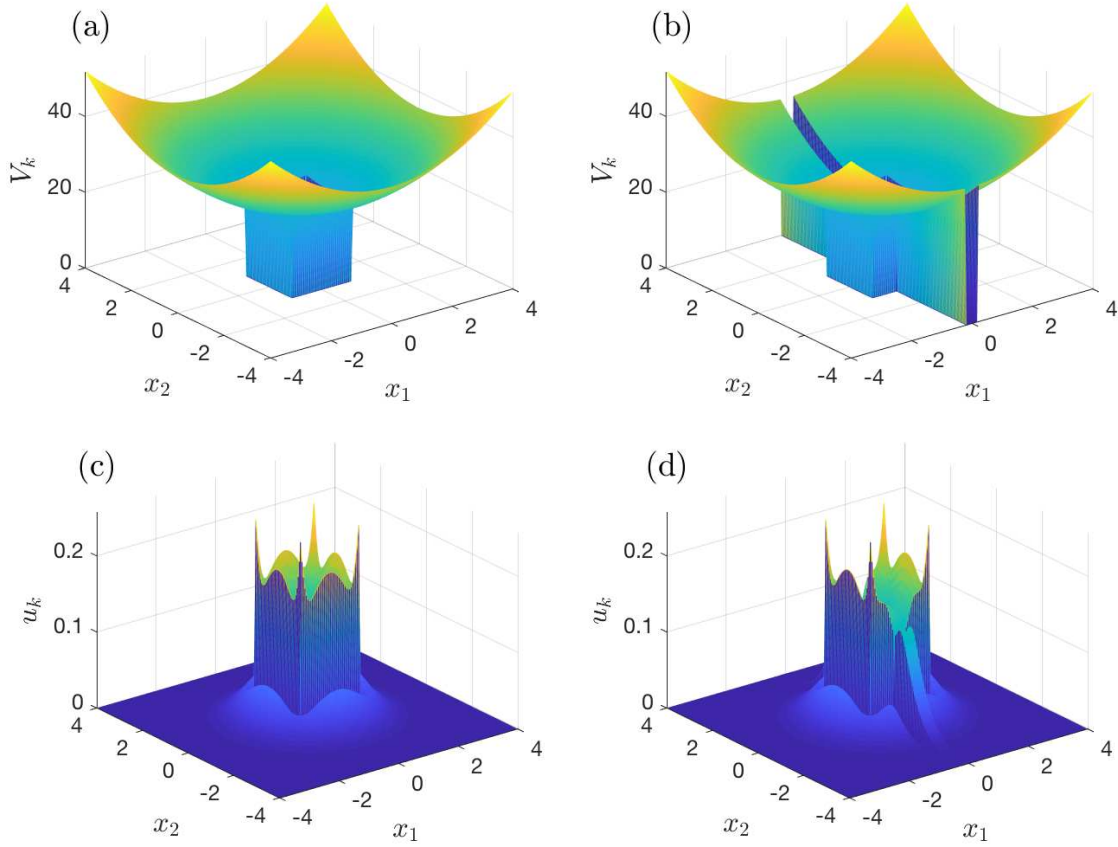


Figure 9: (a) Simple confinement potential $V_k = k + x_1^2 + x_2^2$ outside Ω and zero otherwise, with $k = 20$. (b) Moses confinement potential (53) with $k = 20$. (c) and (d) Solutions u_k of the linear Fokker–Planck equation at $t = 0.01$ corresponding to the potentials in (a) and (b), respectively. Initial data is $u_0 = C \exp(-|x|/2)$ with C normalisation condition.

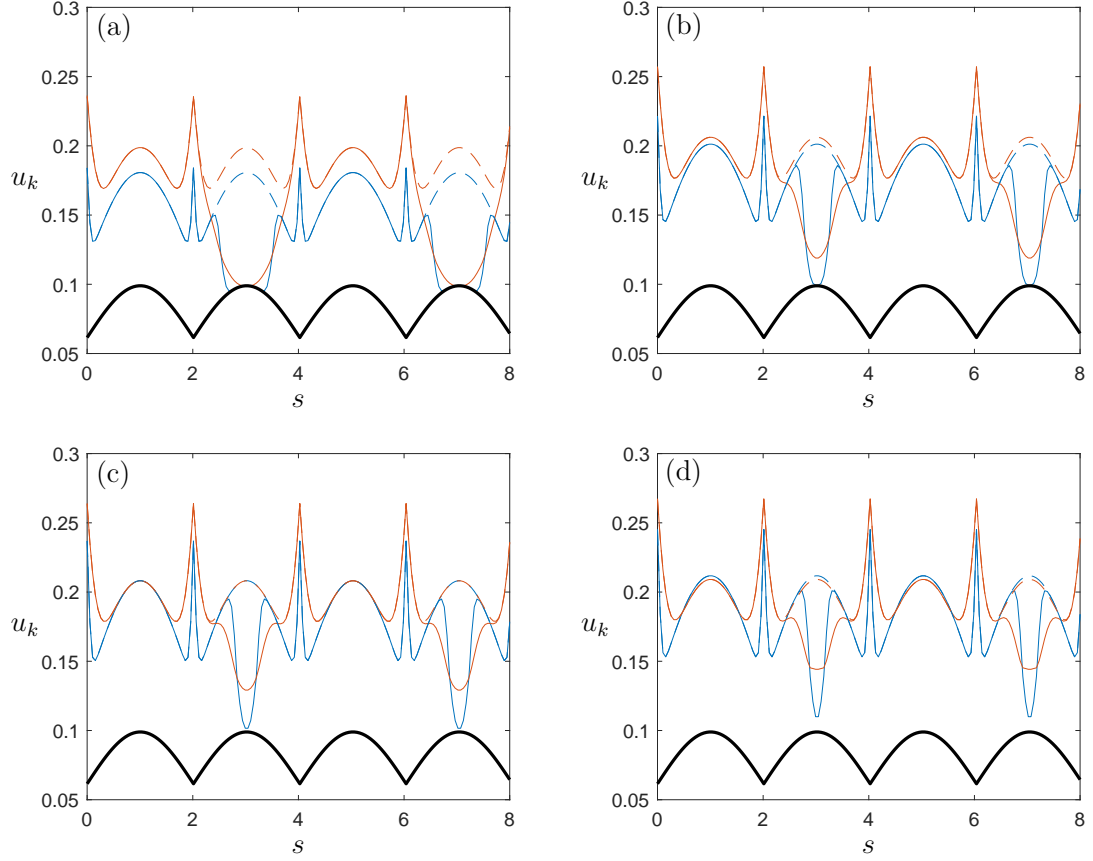


Figure 10: Solution u_k along $\partial\Omega$ at times $t = 0.001$ (shown in blue) and $t = 0.01$ (shown in red) using either the simple quadratic potential (dashed lines) or the Moses potential (solid lines) for (a) $k = 5$, (b) $k = 20$, (c) $k = 50$, and (d) $k = 100$. The initial condition is the same in all cases and shown as a black thick line. The horizontal axis represents $\partial\Omega$, parameterized with arc length s (starting from $x = (1, 1)$ and going round clockwise).

In conclusion, our numerical simulations in two dimensions did not give a clear indication in terms of (non-)uniqueness of the limit problem for $k \rightarrow \infty$. We formulate the following conjecture:

Conjecture. *Let $d \geq 2$ and suppose that the hypotheses of Theorem 1 (resp. Theorem 2) are satisfied, but assume that $\text{supp}(u_0) \not\subseteq \Omega$. Consider a solution u_k of the Cauchy problem (3) in the sense of Definition 1 (resp. Definition 3) with $V = V_k$ satisfying the conditions in Definition 2 (resp. Definition 4). Then the sequence u_k does not converge to the solution of a unique limit problem for $k \rightarrow \infty$. In fact, as $k \rightarrow \infty$, the mass outside Ω (namely $\int_{\Omega^c} u_0 dx$) accumulates on the boundary $\partial\Omega$, resulting in a singular, measure-valued initial datum of the form $u_0|_{\Omega} + \mathcal{M}|_{\partial\Omega}$, where \mathcal{M} is a non-negative measure concentrated on $\partial\Omega$. The measure \mathcal{M} is not uniquely determined and it can vary depending on the properties of Ω , u_0 and the sequence V_k .*

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References

- [1] L. Alasio, M. Bruna, and Y. Capdeboscq. Stability estimates for systems with small cross-diffusion. *ESAIM: M2AN*, 52(3), 2018.
- [2] D. Alexander and I. Kim. A Fokker-Planck type approximation of parabolic PDEs with oblique boundary data. *T. Am. Math. Soc.*, 368(8):5753–5781, 2016.
- [3] H. W. Alt and S. Luckhaus. Quasilinear elliptic-parabolic differential equations. *Math. Z.*, 183(3), 1983.
- [4] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.
- [5] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations. *Commun. Part. Diff. Eq.*, 26(1-2), 2001.
- [6] D. Benedetto, E. Caglioti, J. A. Carrillo, and M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Statist. Phys.*, 91(5-6):979–990, 1998.
- [7] D. Benedetto, E. Caglioti, and M. Pulvirenti. A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér.*, 31(5):615–641, 1997.
- [8] A. L. Bertozzi and D. Slepcev. Existence and uniqueness of solutions to an aggregation equation with degenerate diffusion. *Commun. Pur. Appl. Anal.*, 9(6), 2009.
- [9] M. Bessemoulin-Chatard and F. Filbet. A finite volume scheme for nonlinear degenerate parabolic equations. *SIAM J. Sci. Comput.*, 34(5), 2012.
- [10] A. Blanchet, E. Carlen, and J. A. Carrillo. Functional inequalities, thick tails and asymptotics for the critical mass Patlak–Keller–Segel model. *J. Funct. Anal.*, 262(5), 2012.
- [11] V. Bonnaillie-Noël, J. A. Carrillo, T. Goudon, and G. A. Pavliotis. Efficient numerical calculation of drift and diffusion coefficients in the diffusion approximation of kinetic equations. *IMA J. Numer. Anal.*, 36(4), 2016.
- [12] M. Bruna and S. J. Chapman. Excluded-volume effects in the diffusion of hard spheres. *Phys. Rev. E*, 85(1):011103, 2012.

- [13] M. Bruna, S. J. Chapman, and M. Robinson. Diffusion of Particles with Short-Range Interactions. *SIAM J. Appl. Math.*, 77(6), 2017.
- [14] M. Burger, P. A. Markowich, and J.-F. Pietschmann. Continuous limit of a crowd motion and herding model: Analysis and numerical simulations. *Kinet. Relat. Mod.*, 4(4):1025–1047, 2011.
- [15] V. Calvez and J. A. Carrillo. Volume effects in the Keller-Segel model: energy estimates preventing blow-up. *J. Math. Pures Appl. (9)*, 86(2):155–175, 2006.
- [16] V. Calvez, J. A. Carrillo, and F. Hoffmann. The geometry of diffusing and self-attracting particles in a one-dimensional fair-competition regime. In *Nonlocal and nonlinear diffusions and interactions: new methods and directions*, volume 2186 of *Lecture Notes in Math.*, pages 1–71. Springer, Cham, 2017.
- [17] J. A. Carrillo, A. Chertock, and Y. Huang. A finite-volume method for nonlinear nonlocal equations with a gradient flow structure. *Commun. Comput. Phys.*, 17(01), 2015.
- [18] J. A. Carrillo, M. R. D’Orsogna, and V. Panferov. Double milling in self-propelled swarms from kinetic theory. *Kinet. Relat. Mod.*, 2(2):363–378, 2009.
- [19] J. A. Carrillo, Y. Huang, and M. Schmidtchen. Zoology of a nonlocal cross-diffusion model for two species. *SIAM J. Appl. Math.*, 78(2), 2018.
- [20] J. A. Carrillo, P. A. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized sobolev inequalities. *Monatsh. Math.*, 133(1), 2001.
- [21] J. A. Carrillo, R. J. McCann, and C. Villani. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Rev. Mat. Iberoam.*, 19(3), 2003.
- [22] J. A. Carrillo, R. J. McCann, and C. Villani. Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Ration. Mech. An.*, 179(2), 2006.
- [23] J. A. Carrillo and G. Toscani. Exponential convergence toward equilibrium for homogeneous Fokker–Planck–type equations. *Math. Method. Appl. Sci.*, 21(13), 1998.
- [24] X. Chen, A. Jüngel, and J.-G. Liu. A note on aubin-lions-dubinskii lemmas. *Acta Appl. Math.*, 133(1):33–43, 2014.
- [25] T. Glanz, R. Wittkowski, and H. Löwen. Symmetry breaking in clogging for oppositely driven particles. *Phys. Rev. E*, 94(5):052606, 2016.
- [26] T. Hillen and K. J. Painter. A user’s guide to PDE models for chemotaxis. *J. Math. Biol.*, 58(1-2):183–217, 2008.
- [27] D. D. Holm and V. Putkaradze. Formation of clumps and patches in self-aggregation of finite-size particles. *Phys. D*, 220(2):183–196, 2006.
- [28] T.-L. Horng, T.-C. Lin, C. Liu, and B. Eisenberg. PNP Equations with Steric Effects: A Model of Ion Flow through Channels. *J. Phys. Chem. B*, 116(37):11422–11441, 2012.
- [29] R. Jordan, D. Kinderlehrer, and F. Otto. The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.*, 29(1), 1998.
- [30] O. A. Ladyzhenskaia, V. A. Solonnikov, and N. N. Ural’tseva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1988.
- [31] P. A. Markowich. *The stationary semiconductor device equations*. Springer Science & Business Media, 2013.
- [32] A. Moussa. Some variants of the classical Aubin–Lions lemma. *J. Evol. Equ.*, 16(1):65–93, 2016.
- [33] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Commun. Part. Diff. Eq.*, 26(1-2), 2001.
- [34] L. Pareschi and G. Toscani. *Interacting Multiagent Systems: Kinetic equations and Monte Carlo methods*. Oxford University Press, 2013.
- [35] G. Stampacchia. Le problème de dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. I. Fourier*, 15(1), 1965.
- [36] Z. Sun, J. A. Carrillo, and C.-W. Shu. A discontinuous galerkin method for nonlinear parabolic equations and gradient flow problems with interaction potentials. *J. Comput. Phys.*, 352, 2018.
- [37] C. M. Topaz, A. L. Bertozzi, and Mark A. Lewis. A nonlocal continuum model for biological aggregation. *Bull. Math. Biol.*, 68(7):1601–1623, 2006.
- [38] G. Toscani. Kinetic models of opinion formation. *Commun. Math. Sci.*, 4(3):481–496, 2006.