Supplementary Information

Testing quantum mechanics in non Minkowski space-time with high power lasers and 4th generation light sources

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Derivation of Equation (7)

The matrix elements of the perturbation operator, $\hat{\mathcal{H}}' \equiv \hat{\mathcal{H}} - \hat{\mathcal{H}}_0 = \mathcal{H}'_{\mathrm{int}}(0)$, defined by equations (4) and (5) in the main paper, between the states (β,i), denoting the initial state of the system, and (α,f), denoting a possible final state, in which the labels i,f denote the states of the scattered photon and α,β denote the states of the scatterer system, are straightforwardly given by

$$\begin{aligned}\langle \alpha\,\mathrm{f}|\hat{\mathcal{H}}'|\beta\,\mathrm{i}\rangle &= \frac{e^2}{2mV\epsilon_0}\frac{\mathbf{e}_\mathrm{f}\cdot\mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}}\int_V \langle\alpha|\hat{\rho}(\mathbf{r})\mathrm{e}^{-\mathrm{i}(\mathbf{k}_\mathrm{f}-\mathbf{k}_\mathrm{i}-\mathrm{i}\boldsymbol{\gamma})\cdot\mathbf{r}}|\beta\rangle \mathrm{d}^3\mathbf{r} \\ &= \frac{e^2}{2mV\epsilon_0}\frac{\mathbf{e}_\mathrm{f}\cdot\mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}}\langle\alpha|\hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}|\beta\rangle\end{aligned} \tag{S.1}$$

where $\mathbf{q} = \mathbf{k}_\mathrm{i} - \mathbf{k}_\mathrm{f}$, $\boldsymbol{\gamma} = 2\mathbf{a}/c^2$ and

$$\hat{\rho}_\mathbf{k} \equiv \int_V \hat{\rho}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3\mathbf{r} \tag{S.2}$$

Expressing the operators in the interaction picture gives the time evolution of this matrix element according to

$$
\begin{aligned}
\langle \alpha \mathrm{f} | \hat{\mathcal{H}}'_{\mathrm{int}}(t) | \beta \mathrm{i} \rangle &= \frac{e^2}{2mV\epsilon_0} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} \langle \alpha | \hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}(0) | \beta \rangle \exp\left(-\mathrm{i}\left(E_{\beta\mathrm{i}} - E_{\alpha\mathrm{f}}\right)t\right) \\
&= \frac{e^2}{2mV\epsilon_0} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} \exp\left(\mathrm{i}\left(\omega_\mathrm{f} - \omega_\mathrm{i}\right)t\right) \langle \alpha | \hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}(0) | \beta \rangle \exp\left(-\mathrm{i}\left(\varepsilon_i - \varepsilon_\mathrm{f}\right)t\right) \\
&= \frac{e^2}{2mV\epsilon_0} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} \exp\left(\mathrm{i}\left(\omega_\mathrm{f} - \omega_\mathrm{i}\right)t\right) \langle \alpha | \mathrm{e}^{\mathrm{i}\hat{\mathcal{H}}_0 t} \hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}(0) \mathrm{e}^{-\mathrm{i}\hat{\mathcal{H}}_0 t} | \beta \rangle \\
&= \frac{e^2}{2mV\epsilon_0} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} \exp\left(\mathrm{i}\left(\omega_\mathrm{f} - \omega_\mathrm{i}\right)t\right) \langle \alpha | \hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}(t) | \beta \rangle
\end{aligned}
\tag{S.3}
$$

where $\hat{\mathcal{H}}'_{\mathrm{int}}$ is the perturbation Hamiltonian in the interaction picture, ie as defined by $\hat{\mathcal{H}}'_{\mathrm{int}}(t) = \exp\left(\mathrm{i}\hat{\mathcal{H}}_0 t\right) \hat{\mathcal{H}}' \exp\left(-\mathrm{i}\hat{\mathcal{H}}_0 t\right)$, and $E_{\beta\mathrm{i}} = \varepsilon_\beta + \omega_\mathrm{i}$, $E_{\alpha\mathrm{f}} = \varepsilon_\alpha + \omega_\mathrm{f}$. Substituting according to (S.3) into equation (6) in the paper, yields

$$
\begin{aligned}
\frac{\mathrm{d}\sigma_\mathrm{i}}{\mathrm{d}\Omega_\mathrm{f}\,\mathrm{d}\omega_\mathrm{f}} &= r_\mathrm{e}^{\,2} \frac{4\pi^2 c^3}{V} \mathcal{D}_\mathrm{f} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} \sum_\alpha \int_{-\infty}^{+\infty} \left\langle \langle \beta | \hat{\rho}_{\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}\left(\tfrac{1}{2}t\right) | \alpha \rangle \langle \alpha | \hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}\left(-\tfrac{1}{2}t\right) | \beta \rangle \right\rangle \exp\left(\mathrm{i}\left(\omega_\mathrm{i} - \omega_\mathrm{f}\right)t\right) \mathrm{d}t \\
&= r_\mathrm{e}^{\,2} \frac{4\pi^2 c^3}{V} \mathcal{D}_\mathrm{f} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} \int_{-\infty}^{+\infty} \left\langle \hat{\rho}_{\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}\left(-\tfrac{1}{2}t\right) \hat{\rho}_{-\mathbf{q}-\mathrm{i}\boldsymbol{\gamma}}\left(\tfrac{1}{2}t\right) \right\rangle \exp\left(\mathrm{i}\left(\omega_\mathrm{i} - \omega_\mathrm{f}\right)t\right) \mathrm{d}t \\
&= \frac{N_\mathrm{e}}{V} r_\mathrm{e}^{\,2} \left(2\pi c\right)^3 \mathcal{D}_\mathrm{f} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} S\left(\mathbf{q} + \mathrm{i}\boldsymbol{\gamma}, \omega\right) \\
&= \frac{N_\mathrm{e}}{V} r_\mathrm{e}^{\,2} \left(2\pi c\right)^3 \mathcal{D}_\mathrm{f} \frac{\mathbf{e}_\mathrm{f} \cdot \mathbf{e}_\mathrm{i}}{\sqrt{\omega_\mathrm{f}\omega_\mathrm{i}}} S\left(\mathbf{k}_\mathrm{i} - \mathbf{k}_\mathrm{f} + \mathrm{i}\boldsymbol{\gamma}, \omega_\mathrm{i} - \omega_\mathrm{f}\right)
\end{aligned}
\tag{S.4}
$$

where $\omega = \omega_\mathrm{i} - \omega_\mathrm{f}$ and $r_\mathrm{e} = e^2 / 4\pi\epsilon_0 mc^2$ is the classical electron radius and where the Fourier-transformed Van Hove density-density correlation function, aka the dynamic structure factor, is defined by[1]

$$S(\mathbf{k},\omega)=\frac{1}{2\pi N_{\mathrm{e}}}\int_{-\infty}^{+\infty}\left\langle\hat{\rho}_{\mathbf{k}^*}\left(\tfrac{1}{2}t\right)\hat{\rho}_{-\mathbf{k}}\left(-\tfrac{1}{2}t\right)\right\rangle\exp(\mathrm{i}\,\omega t)\,\mathrm{d}t$$
$$=\frac{1}{2\pi N_{\mathrm{e}}}\int_{-\infty}^{+\infty}\left\langle\hat{\rho}_{-\mathbf{k}}^{\dagger}\left(\tfrac{1}{2}t\right)\hat{\rho}_{-\mathbf{k}}\left(-\tfrac{1}{2}t\right)\right\rangle\exp(\mathrm{i}\,\omega t)\,\mathrm{d}t \tag{S.5}$$

which incorporates the generalisation to *complex* $\mathbf{k}$. It is readily verified that the function $S(\mathbf{k},\omega)$ defined by (S.5) is always real for real ω.

Inserting the density of final states, $\mathcal{D}_{\mathrm{f}}=\mathcal{D}(\omega_{\mathrm{f}})=\dfrac{V\omega_{\mathrm{f}}^{2}}{(2\pi c)^{3}}$, into (S.4) leads to equation (7). Apart from the presence of an imaginary part, γ, this is the known result[2,3] for the double-differential cross-section for the Thomson scattering of photons by a many-electron system.

We can further generalise the dynamic structure factor (S.5) to time-dependent non-equilibrium situations by means of

$$S(\mathbf{k},\omega;t)=\frac{1}{2\pi N_{\mathrm{e}}}\int_{-\infty}^{+\infty}\left\langle\hat{\rho}_{\mathbf{k}^*}\left(t+\tfrac{1}{2}\tau\right)\hat{\rho}_{-\mathbf{k}}\left(t-\tfrac{1}{2}\tau\right)\right\rangle\exp(\mathrm{i}\,\omega\tau)\,\mathrm{d}\tau$$
$$=\frac{1}{2\pi N_{\mathrm{e}}}\int_{-\infty}^{+\infty}\left\langle\hat{\rho}_{-\mathbf{k}}^{\dagger}\left(t+\tfrac{1}{2}\tau\right)\hat{\rho}_{-\mathbf{k}}\left(t-\tfrac{1}{2}\tau\right)\right\rangle\exp(\mathrm{i}\,\omega\tau)\,\mathrm{d}\tau \tag{S.6}$$

in which we have found it convenient to change the integration variable to τ.

Derivation of Equations (9) – (11)

We now consider how the dynamic structure factor (S.6) is modified by the collective acceleration and by $\mathbf{k}$ having an imaginary part. First of all there is the spatial integral involved in Fourier transforming the densities. This leads to a change in the normalisation of the integral that seems to depend on the large-scale spatial inhomogeneity and is thus regarded as indeterminate. We shall not be concerned with this factor.

The time integral is more interesting. To evaluate this, we write the position of the *j*th particle as

$$\mathbf{r}_j(t)=\mathbf{r}_{0j}(t)+\mathbf{R}(t) \tag{S.7}$$

in which $\mathbf{r}_{0j}(t)$ represents the background thermal motion of each particle and the component $\mathbf{R}(t)$ represents a superimposed collective motion that is independent of the background motion and of the individual particle labels, and where $\mathbf{R}(t)$ is an analytic function of time which is real on $\operatorname{Im} t = 0$. For a system subject to an initial drift velocity $\mathbf{v}_0$ and a constant acceleration $\mathbf{a}$,

$$\mathbf{R}(t) = \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2 \tag{S.8}$$

Ignoring the effect of large-scale inhomogeneities, we then have

$$\left\langle \hat{\rho}(\mathbf{r},t)\,\hat{\rho}(\mathbf{r}',t')\right\rangle \simeq \left\langle \hat{\rho}^0\left(\mathbf{r}-\mathbf{R}(t),t\right)\hat{\rho}^0\left(\mathbf{r}'-\mathbf{R}(t'),t'\right)\right\rangle \tag{S.9}$$

where $\hat{\rho}^0$ is the ambient equilibrium density, in the absence of any induced collective motion.

By the application of some straightforward algebra, a combination of (S.6), (S.9), (S.2) and (S.8) yields the frequency dependent part of $S(\mathbf{k},\omega;t)$ according to

$$S(\mathbf{k},\omega;t) = A(\boldsymbol{\gamma})\int_{-\infty}^{+\infty} S^0(\mathbf{q},\omega')\,F(\mathbf{k},\omega-\omega';t)\,\mathrm{d}\omega' \tag{S.10}$$

where $A(\boldsymbol{\gamma})$ is the undetermined normalisation, which is constant (independent of time) for times $t \ll c/a$ provided that $\mathrm{v}_0 \ll c$; $S^0(\mathbf{q},\omega)$ is the equilibrium dynamic structure factor defined by

$$\begin{aligned} S^0(\mathbf{q},\omega) &= \frac{1}{2\pi N_\mathrm{e}}\int_{-\infty}^{+\infty}\left\langle \hat{\rho}^{0\dagger}_{-\mathbf{q}}\left(\tfrac{1}{2}t\right)\hat{\rho}^0_{-\mathbf{q}}\left(-\tfrac{1}{2}t\right)\right\rangle \exp(\mathrm{i}\,\omega t)\,\mathrm{d}t \\ &= \frac{1}{2\pi N_\mathrm{e}}\int_{-\infty}^{+\infty}\left\langle \hat{\rho}^0_{\mathbf{q}}(t)\,\hat{\rho}^0_{-\mathbf{q}}(0)\right\rangle \exp(\mathrm{i}\,\omega t)\,\mathrm{d}t \end{aligned} \tag{S.11}$$

and

$$F(\mathbf{k},\omega;t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\Big(\mathrm{i}\big(\omega\tau-\mathbf{k}^*\cdot\mathbf{R}\big(t+\tfrac{1}{2}\tau\big)+\mathbf{k}\cdot\mathbf{R}\big(t-\tfrac{1}{2}\tau\big)\big)\Big)\mathrm{d}\tau$$

$$=\frac{\mathrm{e}^{-2\boldsymbol{\gamma}\cdot\mathbf{R}(t)}}{2\pi}\int_{-\infty}^{+\infty}\exp\Big(\mathrm{i}\big(\omega-\mathbf{q}\cdot\mathbf{v}(t)\big)\tau-\tfrac{1}{4}\boldsymbol{\gamma}\cdot\mathbf{a}\tau^2\Big)\mathrm{d}\tau$$

$$=\frac{1}{\sqrt{\pi\boldsymbol{\gamma}\cdot\mathbf{a}}}\exp\left(-\frac{1}{\boldsymbol{\gamma}\cdot\mathbf{a}}(\omega-\mathbf{q}\cdot\mathbf{v})^2-2\boldsymbol{\gamma}\cdot\mathbf{R}(t)\right)$$

$$=\frac{1}{\sqrt{\pi\lambda}}\frac{c}{a}\exp\left(-\frac{c^2}{\lambda a^2}(\omega-\mathbf{q}\cdot\mathbf{v})^2-2\boldsymbol{\gamma}\cdot\mathbf{R}(t)\right)$$

$$\simeq\frac{1}{\sqrt{\pi\lambda}}\frac{c}{a}\exp\left(-\frac{c^2}{\lambda a^2}(\omega-\mathbf{q}\cdot\mathbf{v})^2\right) \tag{S.12}$$

where $\mathbf{k}=\mathbf{q}+\mathrm{i}\boldsymbol{\gamma}$, $\lambda=\frac{c^2}{a^2}\boldsymbol{\gamma}\cdot\mathbf{a}$ and $\mathbf{v}(t)=\mathbf{v}_0+\mathbf{a}t$. Equation (S.12) is in the form of a normalised Gaussian function of ω and tends to the delta function $\delta(\omega-\mathbf{q}\cdot\mathbf{v})$ as $a\to 0$. Note that convergence of the integration over τ requires that $\lambda\geq 0$, to which all the known metrics conform.

Finally we apply (S.10) in the case of a dilute system represented as a Boltzmann gas of particles of mass m at temperature T, for which the dynamic structure factor is[1]

$$S^0(\mathbf{q},\omega)=\sqrt{\frac{m}{2\pi q^2T}}\exp\left(-\frac{m}{2q^2T}\left(\omega-\frac{q^2}{2m}\right)^2\right) \tag{S.13}$$

Performing the convolution (S.10), using (S.12), then yields

$$S(\mathbf{k},\omega;t)=A(\boldsymbol{\gamma})\sqrt{\frac{m}{2\pi q^2T_{\mathrm{eff}}(q)}}\exp\left(-\frac{m}{2q^2T_{\mathrm{eff}}(q)}\left(\omega-\mathbf{q}\cdot\mathbf{v}-\frac{q^2}{2m}\right)^2\right) \tag{S.14}$$

where

$$T_{\mathrm{eff}}(q)=T+\frac{\lambda ma^2}{2q^2c^2} \tag{S.15}$$

If the scattering is observed in the plane perpendicular to the laser induced motion, ie the direction of polarisation of the driving laser, then we can set $\mathbf{q}\cdot\mathbf{v}=0$ in (S.14), which then reduces to the result expressed by equations (9) – (11) in the paper.

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