

Variations on the theme of zeta functions of groups and rings



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To my beloved family...

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Abstract

In recent years the study of zeta functions of groups and rings has been expanded rapidly. Many new zeta functions enumerating different sub-objects have been introduced and studied. However, we still encounter difficulties when one wants to explicitly compute them.

This thesis comprises three parts. The first part contains the computation of ideal zeta functions for low dimensional, class-2 nilpotent Lie rings. In addition, we discuss connections to Higman's PORC conjecture. The second part, which is the main part of the thesis, consists of the various approximations of zeta functions of groups and rings, namely: graded ideal zeta functions (Chapter 3 and 4), finite zeta functions enumerating subalgebras (or ideals) of finite Lie $\mathbb{F}_p$ -algebras over $\mathbb{F}_p$ (Chapter 5), and the i -th partial zeta functions of groups and rings (Chapter 6). We provide explicit calculations in many cases, where the classic subgroup/normal subgroup zeta functions are not available. In addition we study properties of the zeta functions obtained, such as uniformity properties or the existence of functional equations. In the final part of the thesis we are concerned with the connection between the study of zeta functions of groups and the study of finite p -groups. In particular we show how our methods from zeta functions of groups and rings can be adapted to study finite p -groups. With these new approaches we end our thesis with several interesting conjectures.

Most of the explicit calculations in this thesis have been checked and verified with computer algebra system MAGMA [3].

Statement of Originality

The results in this thesis are my own work and include nothing which is the outcome of work done in collaboration, except Chapter 3, Subsection 4.3.1, and Section 7.2 which was a joint work with Christopher Voll (e.g. [22]).

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Chapter 1

Introduction

Let G be a finitely generate group. It is well-known that a finitely generated group has only a finite number of subgroups of each finite index. So for any family $\mathfrak{F}(G)$ of subgroups of G , we can define $a_n^{\mathfrak{F}}(G)$ to be the number of subgroups $H \in \mathfrak{F}(G)$ of index n . The question is: how do these numbers vary with the index n ?

1.1 Zeta functions of groups and rings

1.1.1 Zeta functions of groups

Let G be a finitely generated group. In their seminal paper [15] Grunewald, Segal and Smith defined a *zeta function of a group* G by associating a Dirichlet series

$$\zeta_G^*(s) = \sum_{H * G} |G : H|^{-s} = \sum_{n=1}^{\infty} a_n^*(G) n^{-s},$$

where $* \in \{\leq, \triangleleft\}$ and

$$a_n^*(G) = |\{H \mid H * G, |G : H| = n\}|.$$

We call $\zeta_G^{\leq}(s)$ the subgroup zeta function of G and $\zeta_G^{\triangleleft}(s)$ the normal subgroup zeta function of G .

In particular, they were interested in the zeta function of a torsion-free finitely generated nilpotent group (a $\mathfrak{T}$ -group for short). In this case the nilpotency of G lends itself to a natural Euler product

$$\zeta_G^*(s) = \prod_{p \text{ prime}} \zeta_{G,p}^*(s),$$

where

$$\zeta_{G,p}^*(s) = \sum_{i=0}^{\infty} a_{p^i}^*(G) p^{-is}.$$

We call $\zeta_{G,p}^{\leq}(s)$ the *local subgroup zeta function* of G and $\zeta_{G,p}^{\triangleleft}(s)$ the *local normal subgroup zeta function* of G .

Originally introduced as a new tool to study the subgroup growth of $\mathfrak{T}$ -groups, the theory of zeta functions of groups turned out to be a very interesting topic in its own right, encompassing a surprising broad sweep of mathematics. As the theory developed, deep and unexpected connections between group theory, algebra, algebraic geometry, number theory, model theory, and combinatorics have been revealed. To name a few: the rationality of local zeta functions ([15]) depends on results from model theory and p -adic integrations, the analytic properties of local zeta functions and their uniformity, ([9], [8]) involves deep results from number theory and the resolutions of singularities, the functional equation of local zeta functions and their invariants ([40], [41]) generalises work of Denef and Igusa's local zeta function.

1.1.2 Zeta functions of rings

In an analogous way we can also define zeta functions of rings. Let L be a not-necessarily-associative ring additively isomorphic to $\mathbb{Z}^d$ for some d . In the same paper [15], Grunewald, Segal and Smith also defined *zeta functions of a ring* L :

$$\zeta_L^*(s) := \sum_{H * L} |L : H|^{-s} = \sum_{n=1}^{\infty} a_n^*(L) n^{-s},$$

where $*$ $\in \{\leq, \triangleleft\}$ and

$$a_n^*(L) = |\{H \mid H * L, |L : H| = n\}|.$$

As before, we call $\zeta_L^{\leq}(s)$ the *subring zeta function* of L and $\zeta_L^{\triangleleft}(s)$ the *ideal zeta function* of L . The good thing about working with rings is that they always satisfy the Euler product

$$\zeta_L^*(s) = \prod_{p \text{ prime}} \zeta_{L,p}^*(s),$$

where the Euler factor

$$\zeta_{L,p}^*(s) = \sum_{i=0}^{\infty} a_{p^i}^*(L) p^{-is}$$

is the zeta function of the $\mathbb{Z}_p$ -algebra $L_p := L \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p -adic integers. In this thesis we are interested in the zeta functions of Lie rings.

Definition 1.1.1. A *Lie ring* is a $\mathbb{Z}$ -module L together with a bi-additive operation

$$[\cdot, \cdot] : L \times L \rightarrow L$$

which satisfies

$$\begin{aligned} [x, x] &= 0 \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0 \end{aligned}$$

for all $x, y, z \in L$. If R is a commutative ring and L has the additional structure of a R -module and if $[\cdot, \cdot]$ is bilinear with respect to scalar multiplication by elements of R , then we call L a *Lie R -algebra* over R .

The motivation to introduce zeta functions of rings, in particular Lie rings, was to provide a easier way of calculating zeta functions of groups. The Mal'cev correspondence tells us that to each $\mathfrak{T}$ -group G there is an associate Lie $\mathbb{Q}$ -algebra $\mathcal{L}(G)$ over $\mathbb{Q}$, of dimension equal to $h(G)$, the Hirsch length of G . There is an injective map

$$\log : G \rightarrow \mathcal{L}(G),$$

such that $\log G$ spans $\mathcal{L}(G)$ (for details see [30, Chapter 6], [14] or [31]). Call G an *LR group* if $\log G$ is actually a Lie subring of $\mathcal{L}(G)$.

Theorem 1.1.2 ([15], Theorem 4.1). *Let G be a $\mathfrak{T}$ -group of Hirsch length n . Then there exists $f \in \mathbb{N}$, depending only on n , such that G^f is an LR group, and such that if $L = \log G^f$ then*

$$\zeta_{G,p}^*(s) = \zeta_{L,p}^*(s)$$

for all primes p not dividing f .

It gets even better if G is nilpotent of class 2 (a $\mathfrak{T}_2$ -group for short). In this case we can work with the associated graded Lie ring

$$L := G/Z(G) \oplus Z(G),$$

where $Z(G)$ is the centre of G , with the Lie bracket on L induced by the commutator on G . It is easy to see that

$$\zeta_{G,p}^*(s) = \zeta_{L,p}^*(s)$$

for all primes p . Hence one can work with $\zeta_{L,p}^*(s)$ instead of $\zeta_{G,p}^*(s)$. Many of the known results on the study of zeta functions of groups were actually achieved from the study of the zeta functions of corresponding Lie rings. This thesis is also mainly concerned with the zeta functions of Lie rings and their variations.

1.2 What do we want to know?

So, what do we want to know about zeta functions of groups and rings?

1.2.1 Uniformity

As we said, it is now known that the local zeta functions $\zeta_{G,p}^*(s)$ (and $\zeta_{L,p}^*(s)$) are rational functions in p and p^{-s} . But how do these functions behave for different primes?

Definition 1.2.1. The zeta function $\zeta_G^*(s)$ is said to be *finitely uniform* if there exist rational functions $W_1(X, Y), \dots, W_k(X, Y) \in \mathbb{Q}(X, Y)$ for $k \in \mathbb{N}$ such that, for every prime p ,

$$\zeta_{G,p}^*(s) = W_i(p, p^{-s})$$

for some $i \in [k]$. It is said to be *uniform* if $k = 1$.

Initially people expected $\zeta_G^*(s)$ to be finitely uniform for any G . But it turned out to be a much deeper question. In [9] du Sautoy and Grunewald discovered that the behaviour of $\zeta_{G,p}^*$ is related to the problem of counting the number of points mod p on varieties.

Theorem 1.2.2 ([9], Theorem 1.6). *Let G be a $\mathfrak{T}$ -group, and $*$ $\in \{\leq, \triangleleft\}$. There exists an algebraic variety V^* defined over $\mathbb{Q}$, with irreducible components E_i^* , $i \in T^* := \{1, \dots, t^*\}$, all of which are smooth and intersect normally, and rational functions $P_I^*(X, Y) \in \mathbb{Q}(X, Y)$, $I \subseteq T^*$ such that, for almost all primes p ,*

$$\zeta_{G,p}^*(s) = \sum_{I \subseteq T^*} c_{p,I}^* P_I^*(p, p^{-s}),$$

where

$$c_{p,I}^* = |\{a \in \overline{V^*}(\mathbb{F}_p) : a \in \overline{E_i^*} \text{ if and only if } i \in I\}|$$

and $\overline{V}$ denotes the reduction mod p of a $\mathbb{Q}$ -variety V .

Du Sautoy then showed in [8] that there exists a Hirsch length 9, class-2 nilpotent group G_E whose zeta functions are not finitely uniform. Determining the uniformity of zeta functions of a given group G or a family of groups is now one of the most fundamental problems in the study of zeta functions of groups.

1.2.2 Local Functional Equations

We say $\zeta_{G,p}^*(s)$ satisfies a *local functional equation* if it satisfies the local functional equation of the form

$$\zeta_{G,p}^*(s) \Big|_{p \rightarrow p^{-1}} = (-1)^a p^{(b-cs)} \zeta_{G,p}^*(s),$$

where a, b, c are integers which are independent of the prime p , and $p \rightarrow p^{-1}$ denotes the formal inversion of parameter p .

In [40] Voll proved that

1. $\zeta_{G,p}^{\leq}(s)$ always satisfies the local functional equation for any $\mathfrak{T}$ -groups G .
2. $\zeta_{G,p}^{\triangleleft}(s)$ always satisfies the local functional equation for any $\mathfrak{T}$ -groups G of nilpotency class 2 (a $\mathfrak{T}_2$ -group G for short).

This was the best result in the sense that there exist $\mathfrak{T}$ -groups G of nilpotency class $c = 3$ where $\zeta_{G,p}^{\triangleleft}(s)$ fails to have a local functional equation ([44, Theorem 3.4]). Voll recently improved this in [41] by proving that $\zeta_{G,p}^{\triangleleft}(s)$ satisfies the local functional equation for any $\mathfrak{T}$ -group G satisfying a certain condition ([41, Condition 1.1]). His theorems in [40, Theorem A] and [41, Theorem 1.2] even give an explicit formula for a, b and c in

$$\zeta_{G,p}^*(s) \Big|_{p \rightarrow p^{-1}} = (-1)^a p^{(b-cs)} \zeta_{G,p}^*(s).$$

Note that one can define the *uniformity* and the *local functional equation* of $\zeta_L^*(s)$ in an analogous way.

1.2.3 p -Adic integrals

Let $\mathbb{Q}_p$ denote the p -adic numbers, and $\mathbb{Z}_p$ denotes the p -adic integers. Write $v(\cdot)$ for the p -adic valuation, and $|\cdot|_p$ for the p -adic norm. Let μ be the additive Haar measure on $\mathbb{Z}_p$ with the following properties:

- $\mu(\mathbb{Z}_p) = 1$,
- If $S \subseteq \mathbb{Z}_p$ is measurable and $a \in \mathbb{Z}_p$, then $\mu(S) = \mu(a + S)$. In other words, μ is translation invariant.
- If S_1 and S_2 are disjoint measurable sets, then $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$, so μ is additive.

Once we have our Haar measure, we can define a p -adic integral, such as

$$\int_{\mathbb{Z}_p^n} |f(\mathbf{x})|^s d\mu,$$

where $f(\mathbf{x})$ is some function in $\mathbf{x} = (x_1, \dots, x_n)$. One particular type of p -adic integrals, called *cone integrals*, are especially important for us. Let $f_i(\mathbf{x}), g_i(\mathbf{x}) \in$

$\mathbb{Z}[\mathbf{x}]$ be polynomials in $\mathbf{x}$ for $0 \leq i \leq l$. The cone integral with *cone data* $\mathcal{D} = \{f_0(\mathbf{x}), g_0(\mathbf{x}), \dots, f_l(\mathbf{x}), g_l(\mathbf{x})\}$ is defined to be

$$Z_{\mathcal{D}}(s, p) = \int_{\psi(\mathcal{D})} |f_0(\mathbf{x})|_p^s |g_0(\mathbf{x})|_p d\mu,$$

where

$$\psi(\mathcal{D}) = \{\mathbf{x} \in \mathbb{Z}_p^n \mid v(f_i(\mathbf{x})) \leq v(g_i(\mathbf{x})) \text{ for } i = 1, \dots, l\}.$$

The crucial idea is the following. Let L be a Lie ring, additively isomorphic to $\mathbb{Z}^n$, and $L_p = L \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^n$. Let $(e_1, \dots, e_n)$ be the basis of L , and let M be an upper-triangular $n \times n$ matrix $M = (m_{i,j})$ with entries in $\mathbb{Z}_p$. We may consider the rows $\mathbf{m}_1, \dots, \mathbf{m}_n$ of this matrix to be additive generators of a submodule of L_p . This submodule will be a subalgebra if

$$[\mathbf{m}_i, \mathbf{m}_j] \in \langle \mathbf{m}_1, \dots, \mathbf{m}_n \rangle_{\mathbb{Z}_p} \text{ for all } 1 \leq i < j \leq n,$$

and an ideal if

$$[e_i, \mathbf{m}_j] \in \langle \mathbf{m}_1, \dots, \mathbf{m}_n \rangle_{\mathbb{Z}_p} \text{ for all } 1 \leq i, j \leq n.$$

For $i, j \in [n]$, let C_j denote an $n \times n$ matrix whose rows are $\mathbf{c}_i = [e_i, e_j]$. Let M' denote the adjoint matrix of M and

$$M^\natural = M' \text{diag}(m_{2,2}^{-1} \cdots m_{n,n}^{-1}, m_{3,3}^{-1} \cdots m_{n,n}^{-1}, \dots, m_{n,n}^{-1}, 1).$$

Since M is an upper-triangular matrix, the (i, k) -th entry of $M^\natural$ is a homogeneous polynomial of degree $k - 1$ in the variables $m_{r,s}$ with $1 \leq r \leq s \leq k - 1$.

The rows of M generate an ideal if we can solve, for each $1 \leq i, j \leq n$, the system of equations

$$\mathbf{m}_i C_j M^\natural = (m_{1,1} Y_{i,j}^{(1)}, \dots, m_{1,1} \cdots m_{n,n} Y_{i,j}^{(n)})$$

for $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) \in \mathbb{Z}_p^n$. Set $g_{i,j,k}^\triangleleft(\mathbf{x})$ to be the k -th entry of the n -tuple $\mathbf{m}_i C_j M^\natural$. Then $g_{i,j,k}^\triangleleft(\mathbf{x})$ is a homogeneous polynomial of degree k in the $m_{r,s}$, and if we set $f_k(\mathbf{x}) = m_{1,1} \cdots m_{k,k}$, we obtain the conditions $v(f_k(\mathbf{x})) \leq v(g_{i,j,k}^\triangleleft(\mathbf{x}))$ with $\deg(f_k(\mathbf{x})) = \deg(g_{i,j,k}^\triangleleft(\mathbf{x}))$.

Similarly, the rows of M generate a subalgebra if we can solve, for each $1 \leq i < j \leq n$, the system of equations

$$\mathbf{m}_i \left(\sum_{r=j}^n m_{j,r} C_r \right) M^\natural = (m_{1,1} Y_{i,j}^{(1)}, \dots, m_{1,1} \cdots m_{n,n} Y_{i,j}^{(n)})$$

for $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) \in \mathbb{Z}_p^n$. Again set $g_{i,j,k}^{\leq}(\mathbf{x})$ to be the k -th entry of the n -tuple $\mathbf{m}_i \left(\sum_{r=j}^n m_{j,r} C_r \right) M^\natural$. This time $g_{i,j,k}^{\leq}(\mathbf{x})$ is a homogeneous polynomial of degree $k+1$ in the $m_{r,s}$, and we obtain the conditions $v(f_k(\mathbf{x})) \leq v(g_{i,j,k}^{\leq}(\mathbf{x}))$ with $\deg(f_k(\mathbf{x})) + 1 = \deg(g_{i,j,k}^{\leq}(\mathbf{x}))$.

Finally, define

$$V_p^{\triangleleft} = \{M \in \text{Tr}_n(\mathbb{Z}_p) \mid v(f_k(\mathbf{x})) \leq v(g_{i,j,k}^{\triangleleft}(\mathbf{x}))\}$$

and

$$V_p^{\leq} = \{M \in \text{Tr}_n(\mathbb{Z}_p) \mid v(f_k(\mathbf{x})) \leq v(g_{i,j,k}^{\leq}(\mathbf{x}))\}.$$

We have

$$\zeta_{L_p}^*(s) = (1 - p^{-1})^{-n} \int_{V_p^*} |m_{1,1}|^{s-1} \cdots |m_{n,n}|^{s-n} d\mu$$

where μ is the normalized Haar measure on $\mathbb{Z}_p^{n(n-1)/2}$. By the Mal'cev correspondence we mentioned earlier, this also gives $\zeta_{G,p}^*(s)$. One can refer to [15], [9] and [11] for more details.

The first application of p -adic integrals were made by Grunewald, Segal and Smith in [15] when they used this to prove that the local zeta functions are rational functions in p^{-s} for all primes p . Du Sautoy and Grunewald then obtained Theorem 1.6 in [9] by studying the cone integrals. Many explicit calculations of $\zeta_{G,p}^*(s)$ and $\zeta_{L,p}^*(s)$ for various groups and rings ([33], [44], [11] and [28]) were computed by this cone integral method.

1.2.4 Zeta functions of groups and rings and finite p -groups

Zeta functions of groups and rings also have interesting applications to the theory of finite p -group. The most well-known, and perhaps also the most important connection may be their application on Higman's PORC conjecture.

Let $f_n(p)$ denote the number of p -groups of order p^n . In 1960, Higman [16] conjectured that for a fixed n , there exists a fixed integer N and finitely many polynomials $g_i(x)$ ($i = 1, 2, \dots, N$) such that if $p \equiv i \pmod{N}$ then

$$f_n(p) = g_i(p).$$

This is now known as Higman's PORC (Polynomial On Residue Classes) Conjecture. In the same paper Higman proved that for each n the function enumerating the number of groups of order p^n which have Frattini subgroups which are central and

elementary abelian is PORC. Evseev [12] has extended Higman's result to groups where the Frattini subgroup is central. In the more general case, Higman's PORC Conjecture is known to hold true for $n \leq 7$ (see [24] and [26]). For $n = 8$, Vaughan-Lee [34] proved that the number of groups of order p^8 with exponent p is PORC. However, it is still not proved for all groups of order p^8 . For $n \geq 9$, the conjecture is open.

In [7] du Sautoy exposed the link between the normal subgroup zeta functions and Higman's PORC conjecture. Let $F_{c,d}$ be a finitely generated free nilpotent group of class c on d generators. He explained ([7], Conjecture 5.10 and 5.16) that proving the uniformity of $\zeta_{F_{c,d}}^{\triangleleft}(s)$ might help proving Higman's PORC conjecture. Later du Sautoy and Vaughan-Lee ([10]) further exploited this connection. They proved that there exists a parametrised family of groups, one for each prime $p > 3$, of order p^9 whose number of immediate descendants of order p^{10} is not PORC.

1.3 Overview of results and methods employed

This thesis has three main objectives.

1. Enhance our existing knowledge on zeta functions of groups and rings of nilpotency class-2. (Chapter 2)
2. Study various ‘‘approximations’’ of zeta functions of groups and rings. (Chapter 3, 4, 5 and 6)
3. Further investigate the applications to zeta functions of groups and rings on the theory of finite Lie algebras over $\mathbb{F}_p$ and finite p -groups. (Chapter 2 and 5)

Chapter 2 contains two parts. In the first part we study the normal subgroup zeta function $\zeta_G^{\triangleleft}(s)$ for $\mathfrak{T}_2$ -groups G of Hirsch length $h(G) \leq 8$. We prove that for G of $h(G) < 8$ all these zeta functions are all finitely uniform and ‘Rational Functions On Residue Classes’ (Definition 2.2.1). We then construct two $\mathfrak{T}_2$ -groups of Hirsch length 8, G_{np8} and $G_{np8'}$, whose normal subgroup zeta functions are non-RFORC. In the second part of Chapter 2 we show how G_{np8} can be used to construct a parametrised family, G_p , of groups of order p^8 , one for each prime $p > 3$, whose number of immediate descendants of order p^9 is not PORC. This is an improvement of [10]. This part of Chapter 2 is based on the paper [21].

In Chapter 3, let R be a commutative ring and L be a nilpotent Lie R -algebra of nilpotency class c , free of finite rank over R , with lower central series $(\gamma_i(L))_{i=1}^c$.

We define a *graded ideal zeta function* $\zeta_L^{\triangleleft \text{gr}}(s)$, counting only a graded ideals of finite index in L . This can be seen as an approximation of $\zeta_L^{\triangleleft}(s)$. We calculate the explicit formula for various concrete examples using combinatoric methods. In particular we give the exact formula of $\zeta_L^{\triangleleft \text{gr}}(s)$ for $\mathfrak{f}_{3,3}$ and $\mathfrak{f}_{4,2}$, where $\mathfrak{f}_{c,d}$ denotes the free nilpotent Lie ring of nilpotency class c on d generators. Our results show that $\zeta_{\mathfrak{f}_{3,3}}^{\triangleleft \text{gr}}(s)$ and $\zeta_{\mathfrak{f}_{4,2}}^{\triangleleft \text{gr}}(s)$ are both uniform and satisfy local functional equations. Note that we still do not have explicit formulas for $\zeta_L^{\triangleleft}(s)$ for neither of them. This chapter is based on the paper [22], joint work with Voll.

In Chapter 4 we carry on the combinatoric methods used in Chapter 3 and calculate the graded ideal zeta functions of another important family of nilpotent Lie rings called $\mathcal{L}_{c(d)}$, which is basically a direct sum of maximal class Lie lattices with amalgamation. This family includes interesting examples such as maximal class Lie rings and Grenham Lie rings. We prove a general formula for different values of c and d , and provide various explicit calculations.

In Chapter 5 we switch our attention from infinite world to finite world. Let L be a Lie ring and $L(\mathbb{F}_p) = L \otimes \mathbb{F}_p$ to be a Lie $\mathbb{F}_p$ -algebra over $\mathbb{F}_p$. We calculate $\zeta_{L \otimes \mathbb{F}_p}^*(s)$, the zeta functions of $L(\mathbb{F}_p)$ enumerating subalgebras/ideals of $L(\mathbb{F}_p)$. Although the definition of zeta functions of groups and rings clearly embrace these finite Lie algebras, not many observations have been made on this approach. We show how by modifying the p -adic integral method one can calculate this (finite) zeta functions with various explicit examples.

In Chapter 6 recall that if L is a Lie ring, additively isomorphic to $\mathbb{Z}^n$, $B = (e_1, \dots, e_n)$ is the basis of L and M is an upper-triangular $n \times n$ matrix $M = (m_{i,j})$ with entries in $\mathbb{Z}_p$ then with a suitable cone data V_p^* we have

$$\zeta_{L_p}^*(s) = (1 - p^{-1})^{-n} \int_{V_p^*} |m_{1,1}|^{s-1} \cdots |m_{n,n}|^{s-n} d\mu.$$

Let

$$\zeta_{L_p, B, i}^*(s) = (1 - p^{-1})^{-n} \int_{V_p^*} |m_{1,1}|^{s-1} \cdots |m_{n,n}|^{s-n} d\mu,$$

$v(m_{j,j}) < i$

where $1 \leq j \leq n$, be the i -th partial zeta function of L . This is another approximation of $\zeta_{L_p}^*(s)$. We want to see whether we can apply Hensel-like approach to study $\zeta_{L_p}^*(s)$ from $\zeta_{L_p, B, i}^*(s)$. In other words, we would like to investigate what information on $\zeta_{L_p}^{\triangleleft}(s)$ can be captured just by looking at their i -th partial zeta functions.

In Chapter 7 we give further questions and conjectures, based on our results and observations.

Summary of Notation

$\mathbb{N} = \{1, 2, \dots\}$	the natural numbers
$\mathbb{N}_0 = \{0, 1, 2, \dots\}$	the natural numbers including zero
p	a prime number
$\mathbb{Z}_p$	the p -adic integers
$\mathbb{Q}_p$	the p -adic numbers
$\mathbb{F}_p$	the field with p elements
$v(\cdot)$	the p -adic valuation
$ \cdot _p$	the p -adic norm
G	a $\mathfrak{T}$ -group
$H \leq_f G$	H is a subgroup of finite index in G
$H \triangleleft_f G$	H is a normal subgroup of finite index in G
$h(G)$	the Hirsch length of G
L	a Lie ring
$L_p = L \otimes \mathbb{Z}_p$	a Lie $\mathbb{Z}_p$ -algebra
$H \leq_f L$	H is a subring of finite index in L
$H \triangleleft_f L$	H is an ideal of finite index in L
$\zeta(s)$	the Riemann zeta function
$\zeta_p(s)$	the local Riemann zeta function $\frac{1}{1-p^{-s}}$
$\binom{m}{r}_q$	$= \begin{cases} \frac{(1-q^m)(1-q^{m-1})\dots(1-q^{m-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} & r \leq m, \\ 0 & r > m, \end{cases}$
	the Gaussian binomial coefficients
$F_{c,d}$	the free nilpotent group of class c on d generators
$\mathfrak{f}_{c,d}$	the free nilpotent Lie ring of class c on d generators

Given $n \in \mathbb{N}$, we write $[n]$ for $\{1, 2, \dots, n\}$. We write $\mathbb{N}_{\geq 2}$ for $\{2, 3, \dots\} \subseteq \mathbb{N}$. Given a subset $I \subseteq \mathbb{N}$, we write I_0 for $I \cup \{0\}$. If $I \subseteq [n-1]$, we denote $n - I = \{n - i \mid i \in I\}$. Given $a, b \in \mathbb{N}_0$, we denote the interval $\{a+1, \dots, b\}$ by $]a, b]$ and the interval $\{a+1, \dots, b-1\}$ by $]a, b[$.

Chapter 2

The local normal subgroup zeta functions of $\mathfrak{T}_2$ -group G and Higman's PORC conjecture

2.1 Higman's PORC conjecture

Let $f_n(p)$ denote the number of p -groups of order p^n . Higman's PORC conjecture predicts that for a fixed n , there exists a fixed integer N and finitely many polynomials $g_i(x)$ ($i = 1, 2, \dots, N$) such that if $p \equiv i \pmod{N}$ then

$$f_n(p) = g_i(p).$$

In each of [24], [26] and [34] the following approach is used to classify the relevant groups: certain parametrised families of groups of prime-power order are described, and for each family the theory behind the p -group generation algorithm [25] is used to compute their immediate descendants.

Definition 2.1.1. If G is a group then the *lower exponent- p -central series* of G ,

$$G = G_1 \geq G_2 \geq \dots \geq G_i \geq \dots$$

is defined by setting $G_1 = G$, $G_2 = [G, G]G^p$, and in general setting $G_{i+1} = [G_i, G]G_i^p$. If G is a finite p -group then $G_{c+1} = 1$ for some c , and G has *p -class c* if $G_c \neq \{1\}$, $G_{c+1} = \{1\}$. If G is a finite p -group of p -class $c > 1$ then G is an *immediate descendant* of G/G_c .

The idea underpinning these classifications is the following: every p -group of order p^n , apart from the elementary abelian group, is an immediate descendant of a group of order p^k for some $k < n$. So to compute $f_n(p)$, we first list the groups of order p^k for all $k < n$, for each such group G we use the p -group generation algorithm to

compute the number of immediate descendants of G of order p^n , and add together each of these individual contributions. Finally, we add 1 to this total to account for the elementary abelian group of order p^n . For example, the formula

$$f_6(p) = 3p^2 + 39p + 344 + 24 \gcd(p-1, 3) + 11 \gcd(p-1, 4) + 2 \gcd(p-1, 5)$$

given in [24] for the number of p -groups of order p^6 can be obtained by first listing the groups of order less than p^6 which have immediate descendants of order p^6 (42 groups in this case), summing the number of immediate descendants of order p^6 for each group, and adding 1. For each of the relevant 42 parametrised families of groups of order dividing p^5 , the number of immediate descendants of order p^6 is PORC, hence the total is PORC.

So how is this PORC conjecture related to our study of zeta functions? In [7], du Sautoy discovered that the behaviour of local normal subgroup zeta functions of $\mathfrak{T}$ -groups, the uniformity problem, is closely related to Higman's PORC conjecture. With this idea du Sautoy and Vaughan-Lee [10] proved that there exists a parametrised family of groups, one for each prime $p > 3$, of order p^9 whose number of immediate descendants of order p^{10} is not PORC.

To be precise, they constructed a finite p -group G_p which has the following presentation for all $p > 3$:

$$G_p = \left\langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3 \mid [x_1, x_4] = [x_2, x_5] = y_3, \right. \\ \left. [x_1, x_5] = [x_2, x_4] = [x_3, x_6] = y_1, [x_1, x_6] = [x_3, x_4] = y_2 \right\rangle$$

where all other commutators (commutators in the generators such that neither they nor their inverses are explicitly written) are defined to be 1 and $g^p = 1$ for all $g \in G_p$.

This group G_p is a class 2 group of exponent p and of order p^9 with the following properties:

1. The number of conjugacy classes of G_p is not PORC,
2. The order of the automorphism group of G_p is not PORC,
3. The number of immediate descendants of G_p with exponent p and order p^{10} is not PORC.

Furthermore, this family of groups was carefully constructed from du Sautoy's "Elliptic Curve-group" G_E in [8], the first example of a $\mathfrak{T}$ -group whose zeta function is non-uniform.

This was an unexpected result, since it showed that there exists a “natural” partition of the groups of order p^{10} into subsets, at least one of whose cardinalities is not PORC, and seems like the uniformity problem of zeta functions of groups and rings can be exploited to find these partitions. Of course, this does not disprove the PORC conjecture, since the total number of groups of order p^{10} might still be PORC. Clearly, it may be useful to construct a family of groups of order smaller than p^{10} exhibiting non-PORC behaviour.

Motivated by these observations and results, we study the uniformity of $\zeta_G^\triangleleft(s)$ for small $\mathfrak{T}_2$ -groups G of Hirsch length $h(G) \leq 8$. Then we construct another parametrised family G_p with the following presentation for $p > 3$:

$$G_p = \left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \mid [x_1, x_4] = [x_2, x_5] = x_6, \right. \\ \left. [x_1, x_5]^2 = [x_3, x_4] = x_7^2, [x_2, x_4] = [x_3, x_5] = x_8 \right\rangle$$

where all other commutators are defined to be 1 and $g^p = 1$ for all $g \in G_p$. This gives groups of order p^8 and exponent p , one for each prime $p > 3$. We prove that the number of immediate descendants of G_p of order p^9 is not PORC.

Remark 2.1.2. Throughout the rest of this thesis, whenever we have a presentation like this for a group or a ring, we always assume that all other unlisted commutators are trivial (defined to be 1 for groups and 0 for rings).

2.2 The local normal zeta functions of groups with small Hirsch length

As one can see, Higman’s PORC conjecture refers specifically to the dependence of $f_n(p)$ on residue classes. Given the connection between $f_n(p)$ and $\zeta_{G,p}^\triangleleft(s)$ we observed in [7] and [10] it therefore seemed natural to investigate how $\zeta_{G,p}^\triangleleft(s)$ depends on residue classes. Recall the definition of *uniformity*.

Definition 2.2.1. Let G be a $\mathfrak{T}$ -group. We say that the zeta function $\zeta_G^\triangleleft(s)$ is *rational functions on residue classes* (RFORC) if there exists a fixed integer N and finitely many rational functions $W_i(X, Y) \in \mathbb{Q}(X, Y)$ for $i = 1, \dots, N$ such that if $p \equiv i \pmod N$ then

$$\zeta_{G,p}^*(s) = W_i(p, p^{-s}).$$

We say $\zeta_G^\triangleleft(s)$ is non-RFORC if there is no such N . One can analogously define the RFORC-ness for $\zeta_L^\triangleleft(s)$.

Note that being RFORC is stronger than being finitely uniform. We will see examples of zeta functions that are finitely uniform but non-RFORC.

As we discussed in the introduction, du Sautoy and Grunewald proved in [9] that the behaviour of $\zeta_{G,p}^{\triangleleft}$ is related to the problems of counting the number of points mod p on varieties. Recall

Theorem 1.2.2 ([9], Theorem 1.6). *Let G be a $\mathfrak{T}$ -group, and $*$ $\in \{\leq, \triangleleft\}$. There exists an algebraic variety V^* defined over $\mathbb{Q}$, with irreducible components E_i^* , $i \in T^* := \{1, \dots, t^*\}$, all of which are smooth and intersect normally, and rational functions $P_I^*(X, Y) \in \mathbb{Q}(X, Y)$, $I \subseteq T^*$ such that, for almost all primes p ,*

$$\zeta_{G,p}^*(s) = \sum_{I \subseteq T^*} c_{p,I}^* P_I^*(p, p^{-s}),$$

where

$$c_{p,I}^* = \left| \{a \in \overline{V^*}(\mathbb{F}_p) : a \in \overline{E_i^*} \text{ if and only if } i \in I\} \right|,$$

and $\overline{V}$ means the reduction mod p of the variety V .

This implies that the behaviour of the number of points mod p on a variety can vary wildly with the prime p . For example, the normal subgroup zeta function of the elliptic curve group $\zeta_{G_E}^{\triangleleft}$ introduced in [8] is not finitely uniform because the behaviour of the local factors of the group G_E depend on the behaviour of the number of points mod p on the elliptic curve $E = Y^2 - X^3 + X$.

So the next question is, what varieties V^* can arise from $\mathfrak{T}$ -groups G ? For a general $\mathfrak{T}$ -group G , this is still an open problem. Fortunately, if we restrict our G to be a finitely generated torsion-free class-2 nilpotent group (a $\mathfrak{T}_2$ -group for short), then this question becomes much simpler.

In [37] and [38], Voll introduced a new method for computing normal subgroup zeta functions of $\mathfrak{T}_2$ -groups based on an enumeration of vertices in the affine Bruhat-Tits building associated to $\mathrm{SL}_n(\mathbb{Q}_p)$.

Definition 2.2.2. Let G be a $\mathfrak{T}_2$ -group with $Z(G) = G'$ and suppose G has a presentation

$$G = \langle x_1, \dots, x_d, y_1, \dots, y_{d'} \mid [x_i, x_j] = M(\mathbf{y})_{ij} \rangle, \quad (2.2.1)$$

where $M(\mathbf{y})$ is an anti-symmetric $d \times d$ matrix of $\mathbb{Z}$ -linear forms in $\mathbf{y} = (y_1, \dots, y_{d'})$, and all other commutators not listed are defined to be 1. If the polynomial $\mathrm{Pf}(M(\mathbf{y})) := \sqrt{\det(M(\mathbf{y}))}$ is not identically zero we call the hypersurface

$$\mathcal{P}_G := (\mathrm{Pf}(M(\mathbf{y})) = 0)$$

in $\mathbb{P}^{d'-1}$ the Pfaffian hypersurface associated to G . Conversely, every such matrix $M(\mathbf{y})$ can define a $\mathfrak{T}_2$ -group G via (2.1).

Theorem 2.2.3 ([38], Theorem 3). *Assume that $Pf(M(\mathbf{y})) \in \mathbb{Z}[\mathbf{y}]$ is non-zero and irreducible. Assume that the Pfaffian hypersurface $\mathcal{P}_G$ is smooth and contains no lines. For a prime p let*

$$n_{\mathcal{P}_G}(p) = |\mathcal{P}_G(\mathbb{F}_p)|$$

denote the number of $\mathbb{F}_p$ -rational points of $\mathcal{P}_G$. Then there are (explicitly determined) rational functions $W_0(X, Y), W_1(X, Y) \in \mathbb{Q}(X, Y)$ such that if $\mathcal{P}_G$ has good reduction mod p ,

$$\zeta_{G,p}^{\triangleleft}(s) = W_1(p, p^{-s}) + n_{\mathcal{P}_G}(p)W_2(p, p^{-s}). \quad (2.2.2)$$

With his method, Voll managed to explicitly calculate $\zeta_{G_E,p}^{\triangleleft}$, the local normal subgroup zeta function of the elliptic curve group G_E ([37]). By adapting this method, we are going to prove Theorem 2.2.4 and 2.2.5.

Theorem 2.2.4. *Let G be a $\mathfrak{T}_2$ -group where the Pfaffian hypersurface associated to G is smooth and contains no lines. For G of $h(G) \leq 7$, its normal zeta functions are always RFORC.*

Theorem 2.2.5. *Let G_{np8} be a $\mathfrak{T}_2$ -group with the presentation*

$$G_{np8} = \left\langle \begin{array}{l} x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \mid [x_1, x_4] = [x_2, x_5] = y_3, \\ [x_1, x_5]^2 = [x_3, x_4] = y_1^2, [x_2, x_4] = [x_3, x_5] = y_2 \end{array} \right\rangle.$$

Then $\zeta_{G_{np8},p}^{\triangleleft}(s)$ is Non-RFORC.

2.2.1 Preliminary results from Number Theory

As customary in number theory we represent a quadratic form over a ring R by specifying a homogeneous polynomial of degree 2. In [23] Lidl and Niederreiter proved the following theorem.

Theorem 2.2.6 ([23], Theorem 6.21). *Let w be a quadratic form over $\mathbb{F}_p$, where p is an odd prime. Then w is equivalent to a diagonal quadratic form $a_1x_1^2 + \cdots + a_my_m^2$.*

If the quadratic form $w \in \mathbb{F}_p[y_1, \dots, y_n]$ is equivalent to a diagonal quadratic form $a_1x_1^2 + \cdots + a_ny_n^2$, some of a_i 's might be 0. If none of them are 0, in other words the rank of w is n , then we say w is nondegenerate.

Now let us prove the following result.

Theorem 2.2.7 ([23]). *Let $w = w(y_1, \dots, y_n)$ be a quadratic form in n variables $y_1, \dots, y_n$ over $\mathbb{Z}$. Let*

$$n_w(p) = \left| \left\{ \mathbf{y} \in \mathbb{P}^{n-1}(\mathbb{F}_p) \mid \overline{w(\mathbf{y})} = 0 \right\} \right|,$$

then $n_w(p)$ is PORC.

Proof. Instead of $n_w(p)$, we start with

$$F_w(p) = \left| \left\{ \mathbf{y} \in \mathbb{F}_p^n \mid \overline{w(\mathbf{y})} = 0 \right\} \right|.$$

The trick is that for any given w there are only finitely many primes p such that p is even or $p \mid D = \det(w)$. Therefore, to prove PORC one only needs to consider odd primes which do not divide D . For these primes, since they are odd, Theorem 2.2.6 showed that any nonzero quadratic form $w \in \mathbb{F}_p[y_1, \dots, y_n]$ would be equivalent to a diagonal quadratic form $a_1x_1^2 + \dots + a_my_m^2$ where $1 \leq m \leq n$ and all $a_i \neq 0$. Since the number of solutions of $a_1x_1^2 + \dots + a_my_m^2 = 0$ in $\mathbb{F}_p^n$ is p^{n-m} times the number of solutions of $a_1x_1^2 + \dots + a_my_m^2 = 0$ in $\mathbb{F}_p^m$, it suffices to consider the case where $n = m$, that is, where w is nondegenerate. In this case Theorem 6.26 and 6.27 in [23] then tells us that

$$F_w(p) = \begin{cases} p^{n-1} & \text{if } n \text{ is odd,} \\ p^{n-1} + \left(\frac{(-1)^{n/2}D}{p} \right) (p-1)p^{\frac{n}{2}-1} & \text{if } n \text{ is even} \end{cases}$$

where $\left(\frac{a}{p} \right)$ is the Legendre symbol. Finally, we have

$$n_w(p) = \frac{F_w(p) - 1}{p - 1} = \begin{cases} p + p^2 + \dots + p^{n-2} & \text{if } n \text{ is odd,} \\ p + p^2 + \dots + p^{n-2} + \left(\frac{(-1)^{n/2}D}{p} \right) p^{\frac{n}{2}-1} & \text{if } n \text{ is even} \end{cases}$$

as required. □

The other important result is the following:

Theorem 2.2.8 ([5], [42]). *Let $f(x) = x^3 - 2$. The splitting behaviour of $f(x)$ modulo p is not determined by congruence conditions on p . Furthermore, $f(x)$ splits modulo p if and only if $p \equiv 1 \pmod{3}$ and $p = a^2 + 27b^2$ for integers a and b .*

Proof. This follows from Theorem 4.15, Theorem 9.8 in [5] and Theorem 2.2.1, Theorem 2.2.3 in [42]. □

Later on, in Section 2.5, we use this result to prove that $D(p)$ in Theorem 2.5.1 is not PORC.

Corollary 2.2.9. *Let $n(p)$ denote the number of roots of $x^3 - 2$ modulo p . Then*

$$n(p) = \begin{cases} 0 & \text{if } p \equiv 1 \pmod{3} \text{ and } p \neq a^2 + 27b^2 \text{ for integers } a \text{ and } b, \\ 3 & \text{if } p \equiv 1 \pmod{3} \text{ and } p = a^2 + 27b^2 \text{ for integers } a \text{ and } b, \\ 1 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Furthermore, $n(p)$ is not PORC.

2.3 Proof of Theorem 2.2.4

In this section we prove Theorem 2.2.4:

Theorem 2.2.4. *Let G be a $\mathfrak{T}_2$ -group where $Z(G) = G'$ and the Pfaffian hypersurface associated to G is smooth and contains no lines. For G of $h(G) \leq 7$, its normal zeta functions are always RFORC.*

In general both $Z(G)/G'$ and G/G' are finitely generate abelian groups. However we can restrict ourselves to primes p not dividing the orders of the respective torsion parts as we are interested in proving results for all but finitely many primes. Also we need this condition otherwise the associated Pfaffian hypersurface $\mathcal{P}_G$ would be identically zero.

As we saw from Definition 2.2.2, such $\mathfrak{T}_2$ -group G can be written as

$$G = \langle x_1, \dots, x_d, y_1, \dots, y_{d'} \mid [x_i, x_j] = M(\mathbf{y})_{ij} \rangle, \quad (2.3.1)$$

where $M(\mathbf{y})$ is an anti-symmetric $d \times d$ matrix of $\mathbb{Z}$ -linear forms in $\mathbf{y} = (y_1, \dots, y_{d'})$, and all other commutators are trivial. The strategy for proving Theorem 2.2.4 is to look at the behaviour of $\zeta_{G,p}^{\triangleleft}(s)$ for different values of d and d' . Let us call G a (d, d') -group if $h(G/G') = d$ and $h(G') = d'$. As we are only looking at the $\mathfrak{T}_2$ -groups G of $h(G) \leq 7$, the possible cases are (2,1), (3,1), (4,1), (5,1), (6,1), (3,2), (4,2), (5,2), (3,3), (4,3).

2.3.1 $d' = 1$

When $d' = 1$, (2,1), (4,1) and (6,1) are the central products of 1, 2 and 3 copies of the Heisenberg group H respectively, and their local normal zeta functions are given in [11, Theorem 2.22] and [15]. They are RFORC.

2.3.2 $d' = 2$

For $d' = 2$ recall Voll's method in [37].

Theorem 2.3.1 ([37], Theorem 2). *Let G be a $\mathfrak{T}$ -group of class 2 with derived group G' of Hirsch length 2. Then there are irreducible polynomials $f_1(t), \dots, f_m(t) \in \mathbb{Q}[t]$ and rational functions $W_I(X, Y)$, $I \subseteq \{1, \dots, m\}$ such that for almost all primes p*

$$\zeta_{G,p}^{\triangleleft}(s) = \sum_{I \subseteq \{1, \dots, m\}} c_{p,I} W_I(p, p^{-s}),$$

where

$$c_{p,I} = |\{x \in \mathbb{P}^1(\mathbb{F}_p) : f_i(x) \equiv 0 \pmod{p} \text{ if and only if } i \in I\}|.$$

In particular, $\zeta_{G,p}^{\triangleleft}(s)$ is finitely uniform.

Although it proves finite uniformity for $d' = 2$, it does not prove whether $\zeta_{G,p}^{\triangleleft}(s)$ is RFORC or not. The key observation here is that the polynomials f_i come from either the anti-symmetric $d \times d$ matrix $M(\mathbf{y})$ or its decomposed parts, and as long as $d \leq 5$ (which is our case) the behaviour of $c_{p,I}$ satisfies the RFORC condition as we shall now show.

To make this paper self-contained we recall the definition of indecomposable groups and Theorem 6.2 and 6.3 in [14].

Definition 2.3.2. Let G be a radicable $\mathfrak{T}_2$ -group of finite Hirsch length with centre of Hirsch length 2, a so-called $\mathfrak{D}^*$ -groups. A *central decomposition* of G is a family $\{H_1, \dots, H_m\}$ of subgroups of G such that:

1. $Z(H_i) = Z(G)$ for each i ;
2. $G/Z(G)$ is the direct product of the subgroups $H_i/Z(G)$; and
3. $[H_i, H_j] = 1$ whenever $i \neq j$.

The group G is (*centrally*) *indecomposable* if the only such decomposition is $\{G\}$.

Theorem 2.3.3 ([14], Theorem 6.2). *Every $\mathfrak{D}^*$ -group G has a central decomposition into indecomposable constituents, and the decomposition is unique up to an automorphism of G . In particular, the constituents are unique up to isomorphism.*

Theorem 2.3.4 ([14], Theorem 6.3). (i) Let G be an indecomposable $\mathfrak{D}^*$ -group of Hirsch length $n + 2$. Then, with respect to a suitable basis of $G/Z(G)$ and a suitable basis (y_1, y_2) of $Z(G)$, the alternating bilinear map

$$\begin{aligned}\varphi_G : G/Z(G) \times G/Z(G) &\rightarrow Z(G) \\ (aZ(G), bZ(G)) &\mapsto [a, b]\end{aligned}$$

is represented by a matrix $M(\mathbf{y})$ as follows:

- $n = 2r + 1$.

$$M(\mathbf{y}) = M_0^r(\mathbf{y}) = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix},$$

where

$$B = B(\mathbf{y}) = \begin{pmatrix} y_2 & 0 & 0 & \dots & 0 \\ y_1 & y_2 & 0 & \dots & 0 \\ 0 & y_1 & y_2 & \dots & 0 \\ 0 & 0 & y_1 & \dots & 0 \\ 0 & & & \ddots & \\ 0 & 0 & 0 & \dots & y_2 \\ 0 & 0 & 0 & \dots & y_1 \end{pmatrix}_{(r+1) \times r}$$

- $n = 2r$.

$$M(\mathbf{y}) = M_{(f,e)}(\mathbf{y}) = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix},$$

where

$$B = B(\mathbf{y}) = \begin{pmatrix} y_1 + a_1 y_2 & y_2 & 0 & 0 & \dots & 0 \\ -a_2 y_2 & y_1 & y_2 & 0 & \dots & 0 \\ a_3 y_2 & 0 & y_1 & y_2 & \dots & 0 \\ -a_4 y_2 & 0 & 0 & y_1 & \dots & 0 \\ \vdots & & & & \ddots & \\ (-1)^r a_{r-1} y_2 & 0 & 0 & 0 & \dots & y_2 \\ (-1)^{r+1} a_r y_2 & 0 & 0 & 0 & \dots & y_1 \end{pmatrix}_{r \times r}$$

and $\det(B(\mathbf{y})) = g(y_1, y_2) = y_1^r + a_1 y_1^{r-1} y_2 + \dots + a_r y_2^r \in \mathbb{Q}[y_1, y_2]$ is such that $g(y_1, 1)$ is primary, say $g = f^e$ for f irreducible over $\mathbb{Q}$, $e \in \mathbb{N}$.

(ii) If G is any $\mathfrak{D}^*$ -group, then with respect to a suitable basis as above, φ_G is represented by the diagonal sum of matrices like $M(\mathbf{y})$ above.

Since we assumed $G' = Z(G)$, these are precisely the cases where $d' = 2$. We start from $(d, d') = (3, 2)$. For $(3, 2)$, Theorem 2.3.4 implies that G has to be an indecomposable $\mathfrak{D}^*$ -group. Hence [37, Proposition 2] gives

$$\zeta_{G,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^3}(s) \zeta_p(5s - 6) \zeta_p(3s - 4) W(p, p^{-s})$$

where $W(X, Y) = 1 + X^3Y^3$.

For $(d, d') = (4, 2)$, if G is an indecomposable $\mathfrak{D}^*$ -group then with respect to a suitable basis of G/G' and a suitable basis (y_1, y_2) of G' , we have

$$M(\mathbf{y}) = \begin{pmatrix} 0 & 0 & y_1 + a_1y_2 & y_2 \\ 0 & 0 & -a_2y_2 & y_1 \\ -y_1 - a_1y_2 & a_2y_2 & 0 & 0 \\ -y_2 & -y_1 & 0 & 0 \end{pmatrix}$$

where $\sqrt{\det(M(\mathbf{y}))} = g(y_1, y_2) = y_1^2 + a_1y_1y_2 + a_2y_2^2 \in \mathbb{Q}[y_1, y_2]$ is such that $g(y_1, 1)$ is primary, say $g = f^e$ for f irreducible over $\mathbb{Q}$, $e \in \mathbb{N}$. Then if we let p be a prime unramified in $\mathbb{Q}[t]/f(t)$, [37, Proposition 3] gives

$$\zeta_{G,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^4}(s)\zeta_p(6s-8)\zeta_p(3s-5)\zeta_p(5s-5)\zeta_p(3es-(5e-1))(P_1(p, p^{-s})-n_f(p)P_2(p, p^{-s}))$$

where $n_f(p)$ is the number of distinct linear factors in $\overline{f(t)}$ and

$$\begin{aligned} P_1(X, Y) &= (1 - X^5Y^3)(1 + X^4Y^5)(1 - X^{5e-1}Y^{3e}) \\ P_2(X, Y) &= (1 - Y)(1 + Y)X^4Y^3(1 - X^{5e}Y^{3e}). \end{aligned}$$

Now, as f is only either quadratic or linear, e is either 1 or 2, and $n_{p,f}$ is always RFORC by Theorem 2.2.7. Hence $\zeta_{G,p}^{\triangleleft}(s)$ is RFORC.

If it is decomposable, then the only way to decompose such G is to decompose it into two groups H_1 and H_2 . With respect to a suitable basis we can have

$$M(\mathbf{y}) = \begin{pmatrix} 0 & y_1 + a_1y_2 & 0 & 0 \\ -y_1 - a_1y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 + a_2y_2 \\ 0 & 0 & -y_1 - a_2y_2 & 0 \end{pmatrix}$$

where

$$M_{H_1}(\mathbf{y}) = \begin{pmatrix} 0 & y_1 + a_1y_2 \\ -y_1 - a_1y_2 & 0 \end{pmatrix}$$

and

$$M_{H_2}(\mathbf{y}) = \begin{pmatrix} 0 & y_1 + a_1y_2 \\ -y_1 - a_1y_2 & 0 \end{pmatrix}.$$

Thus there are irreducible polynomial $f_1(t) = t + a_1$, $f_2(t) = t + a_2 \in \mathbb{Q}[t]$ and rational function $W_I(X, Y)$, $I \subseteq \{1, 2\}$ such that for almost all primes p

$$\zeta_{G,p}^{\triangleleft}(s) = \sum_{I \subseteq \{1, 2\}} c_{p,I} W_I(p, p^{-s}),$$

where

$$c_{p,I} = \left| \left\{ x \in \mathbb{P}^1(\mathbb{F}_p) \mid f_i(x) \equiv 0 \pmod{p} \text{ if and only if } i \in I \right\} \right|.$$

Since both f_1 and f_2 are linear, we can see that $\zeta_{G,p}^{\triangleleft}(s)$ is RFORC.

For $(d, d') = (5, 2)$, again if it is indecomposable [37, Proposition 2] gives

$$\zeta_{G,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^5}(s) \zeta_p(7s - 10) \zeta_p(5s - 6) W(p, p^{-s})$$

where $W(X, Y) = 1 + X^5 Y^5$.

If it is decomposable, then again it can be decomposed into H_1 and H_2 where with respect to a suitable bases we have

$$M(\mathbf{y}) = \begin{pmatrix} 0 & y_1 + a_1 y_2 & 0 & 0 & 0 \\ -y_1 - a_1 y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & y_1 \\ 0 & 0 & -y_2 & -y_1 & 0 \end{pmatrix}$$

where

$$M_{H_1}(\mathbf{y}) = \begin{pmatrix} 0 & y_1 + a_1 y_2 \\ -y_1 - a_1 y_2 & 0 \end{pmatrix}$$

and

$$M_{H_2}(\mathbf{y}) = \begin{pmatrix} 0 & 0 & y_2 \\ 0 & 0 & y_1 \\ -y_2 & -y_1 & 0 \end{pmatrix}.$$

Now we get nothing from H_2 and just one irreducible linear polynomial $f_1(t) = t + a_1 \in \mathbb{Q}[t]$. Hence we have

$$\zeta_{G,p}^{\triangleleft}(s) = W_1(p, p^{-s}) + c_p W_2(p, p^{-s})$$

where $W_1(X, Y)$, $W_2(X, Y)$ are rational functions and

$$c_p = \{x \in \mathbb{P}^1(\mathbb{F}_p) \mid f_1(x) \equiv x + a_1 \equiv 0 \pmod{p}\} = 1$$

for all p , which is clearly RFORC.

2.3.3 $d' = 3$

For $(d, d') = (3, 3)$, any $(3, 3)$ -group G would have a presentation

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid [x_1, x_2] = f_1(y), [x_1, x_3] = f_2(y), [x_2, x_3] = f_3(y) \rangle$$

where f_1 , f_2 and f_3 are linear polynomials in $\mathbb{Z}[y_1, y_2, y_3]$. For now, consider a corresponding Lie ring $L = L(G) = G/G' \oplus G'$. Now, as $Z(L) = L'$, it implies

$$\langle y_1, y_2, y_3 \rangle = \langle f_1(y), f_2(y), f_3(y) \rangle.$$

Since y_1, y_2 and y_3 are linearly independent, the f_i 's are linearly independent too. Hence with a suitable base exchange we can see that there exists a ring $R \cong L$ with a presentation

$$\langle x'_1, x'_2, x'_3, y_1, y_2, y_3 \mid [x'_1, x'_2] = y_1, [x'_1, x'_3] = y_2, [x'_2, x'_3] = y_3 \rangle.$$

In fact R is a free class-2 nilpotent Lie ring on 3 generators, and we know its local normal zeta function are RFORC (in fact they are uniform, [11], [15]). As we have

$$\zeta_{R,p}^{\triangleleft}(s) = \zeta_{L,p}^{\triangleleft}(s) = \zeta_{G,p}^{\triangleleft}(s),$$

this covers the case $(d, d') = (3, 3)$.

Finally, suppose G is a $(4,3)$ -group. The Pfaffian hypersurface $\mathcal{P}_G$ of a $(4,3)$ -group G would be a quadric form w in $\mathbb{P}^2$ (i.e. conic). As long as w is not identically zero, smooth and contains no line, we will have

$$\zeta_{G,p}^{\triangleleft}(s) = W_1(p, p^{-s}) + n_w(p)W_2(p, p^{-s})$$

where $n_w(p)$ is RFORC by Theorem 2.2.7. This concludes the proof of Theorem 2.2.4.

Remark 2.3.5. Note that the condition on Pfaffian hypersurface associated to G being smooth and containing no lines is in fact only required for $(d, d') = (4, 3)$ case. We know by [1] and [38] that this condition holds *generically* for $d' \leq 6$, which includes our case.

2.4 $\mathfrak{T}_2$ -groups of Hirsch length 8

In this section we present two different $\mathfrak{T}_2$ -groups G_{np8} and $G_{np8'}$ of Hirsch length 8 whose local normal zeta functions are not RFORC. The name G_{np8} and $G_{np8'}$ come from "Non-PORC of Hirsch length 8".

Theorem 2.4.1. *Let G_{np8} be a $\mathfrak{T}_2$ -group with the associated Lie ring*

$$L(G_{np8}) = \left\langle x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 \mid \begin{array}{l} [x_1, x_4] = [x_2, x_5] = y_3, \\ 2[x_1, x_5] = [x_3, x_4] = 2y_1, [x_2, x_4] = [x_3, x_5] = y_2 \end{array} \right\rangle$$

Then $\zeta_{G_{np8},p}^{\triangleleft}(s)$ is not RFORC.

Proof. We will calculate $\zeta_{G_{np8},p}^{\triangleleft}(s)$ by computing the local ideal zeta functions of the corresponding Lie ring

$$L(G_{np8}) = \langle x_1, \dots, x_5, y_1, y_2, y_3 \mid [x_i, x_j] = M(\mathbf{y})_{ij} \rangle$$

where

$$M(\mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 & y_3 & y_1 \\ 0 & 0 & 0 & y_2 & y_3 \\ 0 & 0 & 0 & 2y_1 & y_2 \\ -y_3 & -y_2 & 2y_1 & 0 & 0 \\ -y_1 & -y_3 & -y_2 & 0 & 0 \end{pmatrix}.$$

Now this is exactly the case of [37, Theorem.1] where $d' = 3$, except that our $d = 5$ is odd. Hence we need a crucial modification.

Let L be a Lie ring additively isomorphic to $\mathbb{Z}^{d+d'}$. We denote by $L' = [L, L]$ the derived ring of L . A lattice $\Lambda \subseteq L$ gives rise to a pair (Λ_{ab}, Λ') of a lattice

$$\begin{aligned} \Lambda' &:= \Lambda \cap L' \leq L', \\ \Lambda_{ab} &:= (\Lambda + L')/L' \leq L/L'. \end{aligned}$$

One can see that this map is $|L' : \Lambda'|^d$ to 1. A lattice Λ defines an ideal if and only if its associated pair (Λ_{ab}, Λ') is admissible, that is, $[\Lambda_{ab}, L] \leq \Lambda'$. Let $X(\Lambda')/\Lambda' := Z(L/\Lambda')$. Then we get

$$\zeta_{L,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^d}(s) \sum_{\Lambda' \subseteq L'} |L' : \Lambda'|^{d-s} |L : X(\Lambda')|^{-s}.$$

This is essentially the Lie ring version of [15, Lemma 6.1].

Let us call a lattice $\Lambda' \subseteq \mathbb{Z}_p^{d'}$ *maximal* if $p^{-1}\Lambda' \not\subseteq \mathbb{Z}_p^{d'}$. It is not difficult to see that we only need to consider maximal lattices of p -power index, since in general $\Lambda' = p^r \Lambda'_{\max}$ where $\Lambda'_{\max}$ is maximal in its class. Thus we have

$$\begin{aligned} \zeta_{L,p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^d}(s) \sum_{\Lambda' \subseteq L'} |L' : \Lambda'|^{d-s} |L : X(\Lambda')|^{-s} \\ &= \zeta_{\mathbb{Z}_p^d}(s) \zeta_p((d + d')s - dd') A(p, p^{-s}) \end{aligned}$$

where

$$A(p, p^{-s}) = \sum_{\Lambda' \subseteq L', \Lambda' \text{ maximal}} |L' : \Lambda'|^{d-s} |L : X(\Lambda')|^{-s}.$$

As mentioned in Section 1, Voll realised this generating function can be computed using the enumeration of vertices in the affine Bruhat-Tits buildings Δ_n associated to $\mathrm{SL}_n(\mathbb{Q}_p)$. Recall from [37] the definition of the weight functions

$$\begin{aligned} w(\Lambda') &:= \log_p(|L' : \Lambda'|) \\ w'(\Lambda') &:= w(\Lambda') + \log_p(|L : X(\Lambda')|). \end{aligned}$$

Put $T := p^{-s}$ we get

$$A(p, T) = \sum_{\Lambda' \subseteq L', \text{ maximal}} p^{dw(\Lambda')} T^{w'(\Lambda')}.$$

Let us compute this generating function associated to the vertices of the building Δ_3 as Voll did in [37]. First we put to use the decomposition of Δ_3 into sector-families S_F , $F \in \mathcal{F}(p, 3)$. We have

$$A(p, T) = \sum_{F \in \mathcal{F}(p, 3)} A(p, T, F)$$

where

$$\begin{aligned} A(p, T, F) := & \frac{1}{\binom{3}{2}_p \binom{2}{1}_p} + \frac{1}{\binom{2}{1}_p} \sum_{[\mathbb{Z}_p^3] \neq [\Lambda'] \in \partial S_F} p^{dw[\Lambda']} T^{w'[\Lambda']} \\ & + \sum_{[\Lambda'] \in S_F^\circ} p^{dw[\Lambda']} T^{w'[\Lambda']} \end{aligned}$$

(we use the notation S_F , ∂S_F , S_F° to denote the sector family indexed by the flag F , its boundary and interior, respectively).

Now, for a given lattice Λ' of type $(p^{s+t}, p^t, 1)$ where $s, t \geq 0$ there is a unique coset

$$\alpha G_v \in \text{SL}_3(\mathbb{Z}_p)/G_v,$$

where $G_v := \text{Stab}_{\text{SL}_3(\mathbb{Z}_p)}(\mathbb{Z}_p^3 \cdot \text{diag}(p^{s+t}, p^t, 1))$, such that the admissibility condition becomes

$$\Lambda_{ab} M(\alpha^1) \equiv 0 \pmod{p^{s+t}} \tag{2.4.1}$$

$$\Lambda_{ab} M(\alpha^2) \equiv 0 \pmod{p^t}, \tag{2.4.2}$$

where we denote by α^j the j th column of the matrix α . We now have to analyse the elementary divisors of the system of linear equations (5) and (6). This only depends on how the flag F meets the degeneracy locus of the matrix $M(\mathbf{y})$. Here comes the tricky part. In case $d = 2r$, this degeneracy locus of $M(\mathbf{y})$ is simply the Pfaffian hypersurface. This is why we have

$$\zeta_{G,p}^{\triangleleft}(s) = W_1(p, p^{-s}) + n_{\mathcal{P}_G}(p) W_2(p, p^{-s})$$

in [37, Theorem. 1]. However, when $d = 2r + 1$ (like our case), $\text{Pf}(M(\mathbf{y})) := \sqrt{\det(M(\mathbf{y}))}$ is identically zero so there is no Pfaffian hypersurface. So we have to

manually identify and analyse the degeneracy locus of $M(\mathbf{y})$. I carefully designed the group G_{np8} so that the degeneracy locus of

$$M(\mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 & y_3 & y_1 \\ 0 & 0 & 0 & y_2 & y_3 \\ 0 & 0 & 0 & 2y_1 & y_2 \\ -y_3 & -y_2 & 2y_1 & 0 & 0 \\ -y_1 & -y_3 & -y_2 & 0 & 0 \end{pmatrix}$$

is

$$V = \{\mathbf{y} \in \mathbb{P}^2(\mathbb{Q}) \mid 2y_1^2 - y_2y_3 = y_2^2 - 2y_1y_3 = y_3^2 - y_1y_2 = 0\}.$$

It is not difficult to see that this variety V is a zero-dimensional smooth subvariety in $\mathbb{P}^2$, namely

$$V = \{(1, k^2, k) \mid k^3 = 2\},$$

Since we have identified our degeneracy locus, we can compute $A(p, T, F)$ for the following two cases:

1. $M(\overline{\alpha^1})$ maximal, $A(p, T, F) =: A_{\text{off/off}}(p, T)$,
2. $M(\overline{\alpha^1})$ not maximal, $A(p, T, F) =: A_{\text{sm.pt/off}}(p, T)$.

We start with Case (1). We know $M(\alpha^1)$ always has a $(d-1)$ -minor which is a p -adic unit. So we have

$$w'([\Lambda']) = s + 3t + (d-1)(s+t) = ds + (d+2)t.$$

The boundary of a sector-family in Δ_3 falls into three components: the root vertex, (maximal) lattices of type $(p^s, 1, 1)$, and lattices of type $(p^t, p^t, 1)$. For $s, t \geq 1$ there are $p^{2(s-1)}$, $p^{2(t-1)}$ and $p^{s-1}p^{t-1}p^{s+t-1} = p^{2s+2t-3}$ lattices in S_F of type $(p^s, 1, 1)$, $(p^t, p^t, 1)$ and $(p^{s+t}, p^t, 1)$ respectively. Therefore we get

$$\begin{aligned} A_{\text{off/off}}(p, T) &:= \frac{1}{\binom{3}{2}_p \binom{2}{1}_p} + \frac{1}{\binom{2}{1}_p} \left(\sum_{s \geq 1} p^{(d+2)s-2} T^{ds} + \sum_{t \geq 1} p^{(2d+2)t-2} T^{(d+2)t} \right) \\ &\quad + \sum_{s, t \geq 1} p^{((d+2)s + (2d+2)t - 3)} T^{ds + (d+2)t} \\ &= \frac{1 + p^d T^d + p^{d+1} T^d + p^{2d} T^{d+2} + p^{2d+1} T^{d+2} + p^{3d+1} T^{2d+2}}{\binom{3}{2}_p \binom{2}{1}_p (1 - p^{2d+2} T^{d+2}) (1 - p^{d+2} T^d)}. \end{aligned}$$

Now we deal with Case (2). In this case, the matrix $M(\alpha^1)$ will always have a $(d-3)$ -minor which is a p -adic unit. So by the same logic, we can choose affine local coordinates $(x, y, 1)$ around $\alpha^1 \in \mathbb{P}^2(\mathbb{Z}_p)$ such that

$$w'([\Lambda']) = ds + (d+2)t - 2\min(s, v_p(x), v_p(y))$$

where $x, y \in p\mathbb{Z}/(p^s)$. Start with

$$\sum_{[\Lambda'] \neq [\Lambda] \in \partial S_F} p^{dw[\Lambda']} T^{w'[\Lambda']}$$

consisting of lattices of type $(p^s, 1, 1)$, $s \geq 1$. The map

$$[\Lambda'] \mapsto p^{dw[\Lambda']} T^{w'[\Lambda']}$$

factorises over the set

$$N := \{(a, b, c) \in \mathbb{N}_{>0}^3 \mid a \geq b, a \geq c\},$$

which we view as the intersection of $\mathbb{N}_{>0}^3$ with some closed polyhedral cone C in $\mathbb{R}_{>0}^3$ as $\psi\varphi$ where

$$\varphi : [\Lambda'] \mapsto (s, v_p(x), v_p(y)) \quad (2.4.3)$$

$$\psi : (a, b, c) \mapsto p^{da} T^{da-2\min(a,b)}. \quad (2.4.4)$$

Note that

$$|\varphi^{-1}(a, b, c)| = \begin{cases} 1 & \text{if } a = b = c, \\ (1 - p^{-1})p^{a-b} & \text{if } a > b, a = c, \\ (1 - p^{-1})p^{a-c} & \text{if } a > c, a = b, \\ (1 - p^{-1})^2 p^{2a-b-c} & \text{if } a > b, a > c. \end{cases} \quad (2.4.5)$$

We decompose the cone N into sub-cones N_j on which the values $|\varphi^{-1}(a, b, c)|\psi(a, b, c)$ are easier to sum over. We choose the decomposition

$$\begin{aligned} N &= N_0 + N_1 + N_2 + N_3 + N_4 + N_5 \\ N_0 &:= \{(a, b, c) \in N \mid a = b = c \geq 1\}, \\ N_1 &:= \{(a, b, c) \in N \mid a = c > b \geq 1\}, \\ N_2 &:= \{(a, b, c) \in N \mid a = b > c \geq 1\}, \\ N_3 &:= \{(a, b, c) \in N \mid a > c > b \geq 1\}, \\ N_4 &:= \{(a, b, c) \in N \mid a > b > c \geq 1\}, \\ N_5 &:= \{(a, b, c) \in N \mid a > b = c \geq 1\}. \end{aligned}$$

Let $n_j := \dim(N_j)$. Table 1 below records the generating functions $F_j(X, Y, Z)$ together with the integers n_j and Laurent monomials $m_{jX}(p, T)$, $m_{jY}(p, T)$, $m_{jZ}(p, T)$.

Table 2.4.1:

j	n_j	$F_j(X, Y, Z)$	$m_{jX}(p, T)$	$m_{jY}(p, T)$	$m_{jZ}(p, T)$
0	1	$\frac{XYZ}{1-XYZ}$	$p^d T^d$	T^{-2}	1
1	2	$\frac{X^2 Y Z^2}{(1-XYZ)(1-XZ)}$	$p^{d+1} T^d$	$p^{-1} T^{-2}$	1
2	2	$\frac{X^2 Y^2 Z}{(1-XYZ)(1-XY)}$	$p^{d+1} T^d$	1	$p^{-1} T^{-2}$
3	3	$\frac{X^3 Y Z^2}{(1-XYZ)(1-XZ)(1-X)}$	$p^{d+2} T^d$	$p^{-1} T^{-2}$	p^{-1}
4	3	$\frac{X^3 Y^2 Z}{(1-XYZ)(1-XY)(1-X)}$	$p^{d+2} T^d$	p^{-1}	$p^{-1} T^{-2}$
5	3	$\frac{X^2 Y Z}{(1-XYZ)(1-X)}$	$p^{d+2} T^d$	$p^{-1} T^{-2}$	p^{-1}

Thus

$$\begin{aligned}
\sum_{[\mathbb{Z}_p^3] \neq [\Lambda'] \in \partial S_F} p^{dw[\Lambda']} T^{w'[\Lambda']} &= \sum_{j=0}^5 \sum_{(a,b,c) \in N_j} |\varphi^{-1}(a, b, c)| \psi(a, b, c) \\
&= \sum_{j=0}^5 (1 - p^{-1})^{n_j-1} F_j(X, Y, Z) \Bigg|_{\substack{X=m_{jX}(p,T) \\ Y=m_{jY}(p,T) \\ Z=m_{jZ}(p,T)}} \\
&= \frac{p^d T^{d-2} (1 - p^d T^d)}{(1 - p^d T^{d-2})(1 - p^{d+2} T^d)}.
\end{aligned}$$

The generating function counting over lattices of type $(p^t, p^t, 1)$, $T \geq 1$, is clearly the same as Case (1), i.e. is given by

$$\sum_{t \geq 1} p^{2t-2} \cdot p^{2dt} T^{(d+2)t} = \frac{p^{2d} T^{d+2}}{1 - p^{2d+2} T^{d+2}}.$$

Finally, we count over $[\Lambda'] \in S_F^\circ$, i.e. over lattices of type $(p^{s+t}, p^t, 1)$, $s, t \geq 1$. This is easy, since the generating function equals p times the product of the previous generating functions counting over the boundary. Thus we have

$$\sum_{[\Lambda'] \in S_F^\circ} p^{dw[\Lambda']} T^{w'[\Lambda']} = p \cdot \frac{p^d T^{d-2} (1 - p^d T^d)}{(1 - p^d T^{d-2})(1 - p^{d+2} T^d)} \cdot \frac{p^{2d} T^{d+2}}{1 - p^{2d+2} T^{d+2}}$$

Hence we get

$$\begin{aligned}
A_{\text{sm.pt/off}}(p, T) &= \frac{1}{\binom{3}{2}_p \binom{2}{1}_p} + \frac{1}{\binom{2}{1}_p} \left(\frac{p^d T^{d-2} (1 - p^d T^d)}{(1 - p^d T^{d-2})(1 - p^{d+2} T^d)} + \frac{p^{2d} T^{d+2}}{1 - p^{2d+2} T^{d+2}} \right) \\
&\quad + p \cdot \frac{p^d T^{d-2} (1 - p^d T^d)}{(1 - p^d T^{d-2})(1 - p^{d+2} T^d)} \cdot \frac{p^{2d} T^{d+2}}{1 - p^{2d+2} T^{d+2}}
\end{aligned}$$

and

$$\begin{aligned}
A(p, T) &= \left(\binom{3}{2}_p - |V(\mathbb{F}_p)| \right) \binom{2}{1}_p A_{\text{off/off}}(p, T) \\
&\quad + |V(\mathbb{F}_p)| \binom{2}{1}_p A_{\text{sm.pt/off}}(p, T) \\
&= A_1(p, T) + |V(\mathbb{F}_p)| A_2(p, T),
\end{aligned}$$

where

$$\begin{aligned}
A_1(p, T) &= \frac{1 + p^d T^d + p^{d+1} T^d + p^{2d} T^{d+2} + p^{2d+1} T^{d+2} + p^{3d+1} T^{2d+2}}{(1 - p^{2d+2} T^{d+2})(1 - p^{d+2} T^d)} \\
A_2(p, T) &= \frac{(1 - T)(1 + T)p^d T^{d-2}(1 + p^{2d+1} T^{d+2})}{(1 - p^d T^{d-2})(1 - p^{d+2} T^d)(1 - p^{2d+2} T^{d+2})}.
\end{aligned}$$

As $d = 5$ we have

$$\begin{aligned}
\zeta_{G_{np8}, p}^{\triangleleft}(s) &= \zeta_{L(G)_p}^{\triangleleft}(s) \\
&= \zeta_{\mathbb{Z}_p^5}(s) \zeta_p(8s - 15) A(p, p^{-s}) \\
&= \zeta_{\mathbb{Z}_p^5}(s) \zeta_p(8s - 15) \zeta_p(7s - 12) \zeta_p(5s - 7) W_1(p, p^{-s}) \\
&\quad + |V(\mathbb{F}_p)| \zeta_{\mathbb{Z}_p^5}(s) \zeta_p(8s - 15) \zeta_p(3s - 5) \zeta_p(5s - 7) \zeta_p(7s - 12) W_2(p, p^{-s})
\end{aligned}$$

where

$$\begin{aligned}
W_1(X, Y) &= 1 + X^5 Y^5 + X^6 Y^5 + X^{10} Y^7 + X^{11} Y^7 + X^{16} Y^{12} \\
W_2(X, Y) &= (1 - Y)(1 + Y) X^5 Y^3 (1 + X^{11} Y^7).
\end{aligned}$$

Finally, from Corollary 2.2.9 we know

$$|V(\mathbb{F}_p)| = |\{(1, k^2, k)\}| = \begin{cases} 3 & p \equiv 1 \pmod{3} \text{ and } p = a^2 + 27b^2, \\ 0 & p \equiv 1 \pmod{3} \text{ and } p \neq a^2 + 27b^2, \\ 1 & p \equiv 2 \pmod{3}, \end{cases}$$

for integers a and b , is not PORC. Thus we conclude $\zeta_{G_{np8}, p}^{\triangleleft}(s)$ is Non-RFORC as required. $\square$

Note that this is the first example of finitely-uniform but Non-RFORC ideal zeta function. In fact, we can encode the same variety V (in a different form) into another $\mathfrak{T}_2$ -group $G_{np8'}$ of Hirsch length 8 such that the behaviour of its normal local factor is also governed by $|V(\mathbb{F}_p)|$.

Theorem 2.4.2. *Let $G_{np8'}$ be a $\mathfrak{T}_2$ -group whose associated Lie ring $L(G_{np8'})$ has a presentation*

$$L(G_{np8'}) = \langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2 : [x_i, x_j] = M(\mathbf{y}) \rangle$$

where

$$M(\mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 & y_1 & y_2 & 0 \\ 0 & 0 & 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & 2y_2 & 0 & y_1 \\ -y_1 & 0 & -2y_2 & 0 & 0 & 0 \\ -y_2 & -y_1 & 0 & 0 & 0 & 0 \\ 0 & -y_2 & -y_1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\zeta_{G_{np8'}, p}^{\triangleleft}(s)$ is Non-RFORC.

Proof. This group $G_{np8'}$ is in fact an indecomposable $\mathfrak{D}^*$ -group. Look at the shape of $M(\mathbf{y})$. The matrix $M(\mathbf{y})$ is in the form

$$\begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}$$

in Theorem 3.4 with $a_1 = 0$, $a_2 = 0$, $a_3 = -2$.

Now $\det(B(\mathbf{y})) = g(y_1, y_2) = y_1^3 + 2y_2^3$ and $g(y_1, 1) = y_1^3 + 2 = f$ where f is irreducible over $\mathbb{Q}$. Hence [37, Proposition 3] gives

$$\zeta_{G, p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^6}(s) \zeta_p(8s - 12) \zeta_p(5s - 7) \zeta_p(7s - 7) \zeta_p(5s - 6) (P_1(p, p^{-s}) + n_f(p)(P_2(p, p^{-s})))$$

where

$$\begin{aligned} P_1(X, Y) &= (1 - X^7 Y^5)(1 + X^6 Y^7)(1 - X^6 Y^5) \\ P_2(X, Y) &= (1 - Y)(1 + Y)X^6 Y^5(1 - X^7 Y^5) \end{aligned}$$

and $n_f(p)$ is the number of distinct linear factors in $\overline{f(t)} = \overline{(y_1^3 + 2)}$. Again,

$$n_f(p) = \begin{cases} 3 & p \equiv 1 \pmod{3} \text{ and } p = a^2 + 27b^2, \\ 0 & p \equiv 1 \pmod{3} \text{ and } p \neq a^2 + 27b^2, \\ 1 & p \equiv 2 \pmod{3}, \end{cases}$$

for integers a and b . Hence $\zeta_{G_{np8'}, p}^{\triangleleft}(s)$ is Non-RFORC. □

2.5 G_{np8} and Higman's PORC conjecture

Let G_p be a class-2 group of order p^8 and exponent p given by the presentation:

$$G_p = \left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \mid \begin{array}{l} [x_1, x_4] = [x_2, x_5] = x_6, \\ [x_1, x_5]^2 = [x_3, x_4] = x_7^2, [x_2, x_4] = [x_3, x_5] = x_8 \end{array} \right\rangle$$

where $g^p = 1$ for all $g \in G_p$. One can see this group as a finite p -group version of G_{np8} in Theorem 2.2.5. Throughout the remaining chapter we prove the following theorems:

Theorem 2.5.1. *Let $D(p)$ denote the function enumerating the number of descendants of G_p of order p^9 and exponent p . For $p > 3$,*

$$D(p) = \begin{cases} (p^2 + p + 1)/3 + 5 & \text{if } p \equiv 1 \pmod{3} \text{ and } p \neq a^2 + 27b^2, \\ (p^2 + 10p + 1)/6 + 41 & \text{if } p \equiv 1 \pmod{3} \text{ and } p = a^2 + 27b^2, \\ (p^2 + 4p + 1)/2 + 34 & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

where a and b are integers. Furthermore, $D(p)$ is not PORC.

Theorem 2.5.2. *The number of immediate descendants of G_p of order p^9 is not PORC.*

Vaughan-Lee [35] recently listed all class-2 groups of exponent p with order p^k for $k \leq 8$. He then showed that for every group G in his list, the number of conjugacy classes of the group is a polynomial in p and the order of the automorphism group of G is also a polynomial in p . Using this result he proved that the number of groups of class-3, order p^9 and exponent p is indeed PORC.

At face value this is a counter-example to Vaughan-Lee's result in [35]. However, it turns out that this group G_p switches back and forth between three of Vaughan-Lee's groups in a non-PORC manner. This is a first example that clearly shows the existence of "local" non-PORC behaviour does not necessarily reflect "global" non-PORC outcomes.

In Vaughan-Lee's [35] list of all class-2 groups of exponent p with order p^k for $k \leq 8$, there are 22 class-2, 5-generator groups (namely 8.5.1, 8.5.2, ..., 8.5.22) of order p^8 and exponent p . For each group in the list the number of conjugacy classes is given by a polynomial in p , and the order of the automorphism group is also given by a polynomial in p . The polynomials giving the orders of the automorphism groups of these 22 groups are all different. Thus we can identify which group among those 22 is isomorphic to G_p by computing the order of the automorphism group of G_p .

In Section 2.7, we will show that the order of the automorphism group of G_p depends on $n(p)$. Comparing these polynomials with the polynomials given in [35] we obtain:

Lemma 2.5.3. *The group G_p is isomorphic to*

- *group 8.5.7 in [35] when $p = 1 \bmod 3$ and $p \neq a^2 + 27b^2$ for integers a and b*
- *group 8.5.18 in [35] when $p = 1 \bmod 3$ and $p = a^2 + 27b^2$ for integers a and b ,*
- *group 8.5.13 in [35] when $p = 2 \bmod 3$.*

This confirms that our G_p straddles three of Vaughan-Lee's groups in a non-PORC way. The number of descendants of order p^9 with exponent p of each of the three groups 8.5.7, 8.5.18 and 8.5.13 is PORC.

2.6 Associated Lie ring $L(G_p)$ and its covering algebra

Similar to the zeta functions of groups and rings, when counting the number of descendants of a finite p -group G it is often easier to work not directly with the group G , but with its associated Lie ring L . In our case G_p has exponent p , and its associated Lie ring is a Lie algebra over $\mathbb{F}_p$. The Lazard correspondence gives an isomorphism between the category of nilpotent Lie algebras of dimension n over the field $\mathbb{F}_p$ and class at most $p - 1$ and the category of groups of exponent p of order p^n and class at most $p - 1$. (Refer to [19, Chapter 10] or [24] for more details.) This implies that we can count the descendants of G_p of exponent p by counting the descendants of its associated Lie algebra $L(G_p)$ which we are about to define.

The Lie algebra $L(G_p)$ over $\mathbb{F}_p$ corresponding to the group G_p is

$$L(G_p) = \left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \mid [x_1, x_4] = [x_2, x_5] = x_6, \right. \\ \left. 2[x_1, x_5] = [x_3, x_4] = 2x_7, [x_2, x_4] = [x_3, x_5] = x_8 \right\rangle,$$

with all other Lie commutators trivial. For $p > 3$ the immediate descendants of $L(G_p)$ of dimension 9 correspond to the immediate descendants of G_p of order p^9 and exponent p .

This Lie algebra $L(G_p)$ is nilpotent of class 2 and of dimension 8, with $[L(G_p), L(G_p)]$ having dimension 3 and vector space basis $[x_1, x_4]$, $[x_1, x_5]$ and $[x_2, x_4]$. A Lie algebra L over $\mathbb{F}_p$ is an immediate descendant $L(G_p)$ if L is nilpotent of class 3 and

if $L/[[L, L], L] \cong L(G_p)$. So, how do we count such L ? We start with finding the covering algebra for $L(G_p)$.

The covering algebra for $L(G_p)$ is the largest Lie algebra M which is nilpotent of class 3 and contains an ideal I satisfying the following properties:

1. $M/I \cong L(G_p)$,
2. $I \leq [M, M]$,
3. I is contained in the centre of M .

Since very similar calculations are available in the literature, including [10] and [17, Section 9.4], we will not give full step-by-step calculations. Call this covering algebra M . Then M has dimension 21, with the ideal I of dimension 13. The nucleus of M is $[[M, M], M]$ which has dimension 6, and is contained in I . The immediate descendants of $L(G_p)$ are algebras M/J where J is a proper subspace of I with

$$J + [[M, M], M] = I.$$

There is an action of the automorphism group of $L(G_p)$ on M , and two descendants M/J_1 and M/J_2 are isomorphic if and only if J_1 is in the same orbit as J_2 under the action of the automorphism group. We obtained Theorem 2.5.1 by computing the number of orbits of subspaces J of codimension 1 in I under the action of the automorphism group.

2.7 The automorphism group of $L(G_p)$

Let V be the vector subspace of $L(G_p)$ spanned by x_1, x_2, x_3, x_4, x_5 . It is sufficient to compute H , the subgroup of the automorphism group of $L(G_p)$, which maps V onto V .

If $\vartheta \in H$, then there exist

$$S = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ B_1 & B_2 & B_3 & B_4 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 \\ D_1 & D_2 & D_3 & D_4 & D_5 \\ E_1 & E_2 & E_3 & E_4 & E_5 \end{pmatrix} \in \text{GL}(5, p)$$

such that

$$\begin{aligned} \vartheta(x_1) &= A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_5x_5, \\ \vartheta(x_2) &= B_1x_1 + B_2x_2 + B_3x_3 + B_4x_4 + B_5x_5, \\ \vartheta(x_3) &= C_1x_1 + C_2x_2 + C_3x_3 + C_4x_4 + C_5x_5, \\ \vartheta(x_4) &= D_1x_1 + D_2x_2 + D_3x_3 + D_4x_4 + D_5x_5, \\ \vartheta(x_5) &= E_1x_1 + E_2x_2 + E_3x_3 + E_4x_4 + E_5x_5. \end{aligned}$$

We now count the number of automorphisms by counting S .

Lemma 2.7.1. *The only 3 dimensional subspace of V which generates an abelian subalgebra of $L(G_p)$ is $\langle x_1, x_2, x_3 \rangle$.*

Proof. Let W be such a subspace of V and $U = \langle x_1, x_2, x_3 \rangle$.

First assume $U \cap W = \{0\}$. Then $\dim(W \oplus U) = 6$. However $W \oplus U \leq V$, hence this is impossible.

So $U \cap W \neq \{0\}$. Let $u = U \cap W \setminus \{0\}$. Then W must be a subspace of the centraliser of u in V , namely $C_V(u)$. We consider the possibilities for $C_V(u)$.

First note that

$$\begin{aligned} C_v(x_1) &= \langle x_1, x_2, x_3 \rangle, \\ C_v(x_2) &= \langle x_1, x_2, x_3 \rangle, \\ C_v(x_3) &= \langle x_1, x_2, x_3 \rangle, \end{aligned}$$

and that if $\lambda \neq 0$ then

$$\begin{aligned} C_v(x_1 + \lambda x_2) &= \langle x_1, x_2, x_3 \rangle, \\ C_v(x_2 + \lambda x_3) &= \langle x_1, x_2, x_3 \rangle, \\ C_v(x_3 + \lambda x_1) &= \langle x_1, x_2, x_3 \rangle. \end{aligned}$$

Also we have

$$\begin{aligned} [x_4, x_1 + ax_2 + bx_3] &= -[x_5, x_2] + a[x_5, x_3] - 2b[x_5, x_1], \\ [x_5, x_1 + ax_2 + bx_3] &= [x_5, x_1] + a[x_5, x_2] + b[x_5, x_3]. \end{aligned}$$

It follows that $C_V(x_1 + ax_2 + bx_3) = U$ unless

$$\begin{aligned} a &\equiv -2b^2 \pmod{p}, \\ b &\equiv a^2 \pmod{p}, \end{aligned}$$

in which case $C_V(x_1 + ax_2 + bx_3) = \langle x_1, x_2, x_3, ax_5 - bx_4 \rangle$. Since $W \leq C_V(u)$ this implies $W = U$ or $W \leq \langle x_1, x_2, x_3, ax_5 - bx_4 \rangle$. However, the latter implies that W has non-trivial intersection with $\langle x_1, x_2 \rangle$. We know

$$\begin{aligned} C_v(x_1) &= \langle x_1, x_2, x_3 \rangle, \\ C_v(x_2) &= \langle x_1, x_2, x_3 \rangle, \\ C_v(x_1 + \lambda x_2) &= \langle x_1, x_2, x_3 \rangle. \end{aligned}$$

Hence $W = U$ as required. □

Corollary 2.7.2. *Let $\vartheta \in H$ and M be a corresponding matrix. Then $A_4 = A_5 = B_4 = B_5 = C_4 = C_5 = 0$ and*

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \in GL(3, p).$$

Proof. Since ϑ is an automorphism from V onto V ,

$$\vartheta(\langle x_1, x_2, x_3 \rangle) = \langle \vartheta(x_1), \vartheta(x_2), \vartheta(x_3) \rangle.$$

It is clear that $\langle \vartheta(x_1), \vartheta(x_2), \vartheta(x_3) \rangle$ is a 3 dimensional subspace which generates an abelian subalgebra of $L(G_p)$. From Lemma 2.7.1 we know $\langle \vartheta(x_1), \vartheta(x_2), \vartheta(x_3) \rangle = \langle x_1, x_2, x_3 \rangle$. $\square$

So at this point, we know that our S looks like

$$S = \begin{pmatrix} A_1 & A_2 & A_3 & 0 & 0 \\ B_1 & B_2 & B_3 & 0 & 0 \\ C_1 & C_2 & C_3 & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & D_5 \\ E_1 & E_2 & E_3 & E_4 & E_5 \end{pmatrix} \in GL(5, p),$$

where

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \in GL(3, p).$$

This obviously satisfies our relations $[\vartheta(x_1), \vartheta(x_2)] = [\vartheta(x_1), \vartheta(x_3)] = [\vartheta(x_2), \vartheta(x_3)] = 0$. Now we need to get extra conditions from the remaining relations:

$$\begin{aligned} [\vartheta(x_4), \vartheta(x_5)] &= 0, \\ [\vartheta(x_1), \vartheta(x_4)] &= [\vartheta(x_2), \vartheta(x_5)], \\ 2[\vartheta(x_1), \vartheta(x_5)] &= [\vartheta(x_3), \vartheta(x_4)], \\ [\vartheta(x_2), \vartheta(x_4)] &= [\vartheta(x_3), \vartheta(x_5)]. \end{aligned}$$

If we solve all these simultaneous equations, we get the following results:

Lemma 2.7.3. (1) *If $p = 2 \bmod 3$*

$$S = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 2a_3 & a_1 & a_2 & 0 & 0 \\ 2a_2 & 2a_3 & a_1 & 0 & 0 \\ 2a_4 & a_5 & a_6 & \lambda & 0 \\ a_5 & a_6 & a_4 & 0 & \lambda \end{pmatrix} \text{ or } S = \begin{pmatrix} b_1 & b_2 & b_3 & 0 & 0 \\ k^2b_2 & k^2b_3 & \frac{k^2}{2}b_1 & 0 & 0 \\ 2kb_3 & kb_1 & kb_2 & 0 & 0 \\ b_4 & b_5 & b_6 & 0 & k^2\lambda \\ kb_6 & \frac{k}{2}b_4 & \frac{k}{2}b_5 & \lambda & 0 \end{pmatrix}$$

where $\det(S) \neq 0$ and $k^3 = 2 \pmod{p}$.

(2) If $p = 1 \pmod{3}$ then

$$S = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 2\omega_i a_3 & \omega_i a_1 & \omega_i a_2 & 0 & 0 \\ 2\omega_i^2 a_2 & 2\omega_i^2 a_3 & \omega_i^2 a_1 & 0 & 0 \\ 2a_4 & a_5 & a_6 & \lambda & 0 \\ \omega_i^2 a_5 & \omega_i^2 a_6 & \omega_i^2 a_4 & 0 & \omega_i^2 \lambda \end{pmatrix} \text{ or } S = \begin{pmatrix} b_1 & b_2 & b_3 & 0 & 0 \\ k_j^2 b_2 & k_j^2 b_3 & \frac{k_j^2}{2} b_1 & 0 & 0 \\ 2k_j b_3 & k_j b_1 & k_j b_2 & 0 & 0 \\ b_4 & b_5 & b_6 & 0 & k_j^2 \lambda \\ k_j b_6 & \frac{k_j}{2} b_4 & \frac{k_j}{2} b_5 & \lambda & 0 \end{pmatrix}$$

where $\det(S) \neq 0$, $\omega_i^3 = 1 \pmod{p}$ and $k_j^3 = 2 \pmod{p}$ for $1 \leq i, j \leq 3$.

Now we calculate the order of the automorphism group for different cases. Let $h(p)$ denote the order of automorphism group of $L(G_p)$. One can check manually (as in [10] and [17, Section 9.4]) that

$$h(p) = \begin{cases} N \cdot 3(p-1)p^{18} & \text{if } p = 1 \pmod{3} \text{ and } p \neq a^2 + 27b^2, \\ N \cdot 6(p-1)p^{18} & \text{if } p = 1 \pmod{3} \text{ and } p = a^2 + 27b^2, \\ N \cdot 2(p-1)p^{18} & \text{if } p = 2 \pmod{3}, \end{cases}$$

for integers a and b where N is the order of the group of matrices in $\text{GL}(3, p)$ of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 2a_3 & a_1 & a_2 \\ 2a_2 & 2a_3 & a_1 \end{pmatrix}.$$

Since the determinant of this matrix is $a_1^3 + 2a_2^3 + 4a_3^3 - 6a_1a_2a_3$, we can compute

$$N = |\{(a_1, a_2, a_3) \in \mathbb{F}_p^3 \mid a_1^3 + 2a_2^3 + 4a_3^3 - 6a_1a_2a_3 \not\equiv 0 \pmod{p}\}|.$$

When $x^3 - 2$ has no roots over the splitting field, then $a_1^3 + 2a_2^3 + 4a_3^3 - 6a_1a_2a_3$ is irreducible and nonzero unless $a_1 = a_2 = a_3 = 0$. Therefore $N = p^3 - 1$ in this case.

When $x^3 - 2$ has one unique root k , then

$$a_1^3 + 2a_2^3 + 4a_3^3 - 6a_1a_2a_3 = (a_1 + ka_2 + k^2a_3)(a_1^2 + k^2a_2^2 + k^4a_3^2 - ka_1a_2 - k^2a_1a_3 - k^3a_2a_3),$$

and $N = (p^2 - 1)(p - 1)$. Finally, when $x^3 - 2$ has three distinct roots k_1, k_2 and k_3 , then

$$a_1^3 + 2a_2^3 + 4a_3^3 - 6a_1a_2a_3 = (a_1 + k_1a_2 + k_1^2a_3)(a_1 + k_2a_2 + k_2^2a_3)(a_1 + k_3a_2 + k_3^2a_3),$$

and $N = (p - 1)^3$. Corollary 2.2.9 then gives us:

Corollary 2.7.4. *Let $h(p)$ denote the order of automorphism group of $L(G_p)$. Then*

$$h(p) = \begin{cases} 3(p-1)(p^3-1)p^{18} & \text{if } p = 1 \pmod{3} \text{ and } p \neq a^2 + 27b^2, \\ 6(p-1)^4p^{18} & \text{if } p = 1 \pmod{3} \text{ and } p = a^2 + 27b^2, \\ 2(p-1)^2(p^2-1)p^{18} & \text{if } p = 2 \pmod{3}, \end{cases}$$

for integers a and b . Thus $h(p)$ is not PORC.

2.8 Immediate descendants of order p^9

We proved Theorem 2.5.1 by counting the immediate descendants of $L(G_p)$ of dimension 9, as described in Section 2.5. As we explained in that section, the counting problem for these descendants is equivalent to the problem of enumerating the number of orbits of subspaces J of codimension 1 in I under the action of the automorphism group. The fact that the number of descendants is not PORC is directly related to the fact that the order of the automorphism group is not PORC.

Finally, let $E(p)$ be the number of descendants of G_p of order p^9 which do not have exponent p . Then the total number of descendants of G_p is $D(p) + E(p)$. From Theorem 2.5.1 we can check that if $p \equiv 1 \pmod{3}$, then $D(p)$ is smaller when $p = a^2 + 27b^2$ than it is when $p \neq a^2 + 27b^2$. Since we know that the order of the automorphism group is bigger when $p = a^2 + 27b^2$ than it is when $p \neq a^2 + 27b^2$, we can deduce that $E(p)$ will either be the same or smaller when $p = a^2 + 27b^2$ than it is when $p \neq a^2 + 27b^2$. This proves Theorem 2.5.2.

Chapter 3

Enumerating graded ideals in graded Lie rings

3.1 Graded ideals in graded Lie rings

As we discussed in the introduction, one of the main obstacles for studying zeta functions of groups and rings is that they are simply too difficult to compute. In this chapter, we define *graded ideal zeta functions* which enumerate graded ideals in graded Lie rings. This can be regarded as an “approximation” of ideal zeta functions of Lie rings.

Let R be a commutative ring and L be a nilpotent R -Lie algebra of nilpotency class c , free of finite rank over R , with lower central series $(\gamma_i(L))_{i=1}^c$. For $i = 1, \dots, c$ set $L^{(i)} := \gamma_i(L)/\gamma_{i+1}(L)$. The *associated graded R -Lie algebra* is $\text{gr}L = \bigoplus_{i=1}^c L^{(i)}$. An R -ideal I of $\text{gr}L$ (of finite index in $\text{gr}L$) is *graded* or *homogeneous* if it is generated by homogeneous elements or, equivalently, if $I = \bigoplus_{i=1}^c (I \cap L^{(i)})$. In this case we write $I \triangleleft_{\text{gr}} \text{gr}L$. Assume now that L is free and finitely generated as an R -module. We define the *graded ideal zeta function of L* as the Dirichlet generating series

$$\zeta_L^{\triangleleft_{\text{gr}}}(s) = \sum_{I \triangleleft_{\text{gr}} \text{gr}L} |\text{gr}L : I|^{-s}, \quad (3.1.1)$$

enumerating graded ideals in $\text{gr}L$ of finite index in $\text{gr}L$. Here s is a complex variable; our assumptions on L guarantee that $\zeta_L^{\triangleleft_{\text{gr}}}(s)$ converges on a complex half-plane. (For a similar definition in a slightly more general setting, see [28, Section 3.1]).

Assume now that $R = \mathcal{O}$ is the ring of integers of a number field. For a (non-zero) prime ideal $\mathfrak{p} \in \text{Spec}(\mathcal{O})$ we write $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ for the completion of $\mathcal{O}$ at $\mathfrak{p}$, a complete discrete valuation ring of characteristic zero and residue field $\mathcal{O}/\mathfrak{p}$ of cardinality q and

characteristic p , say. Considering $L(\mathcal{O}_{\mathfrak{p}}) := L \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}$ as an $\mathcal{O}_{\mathfrak{p}}$ -algebra, (3.1.1) becomes

$$\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft \text{gr}}(s) = \sum_{I \triangleleft_{\text{gr}} L(\mathcal{O}_{\mathfrak{p}})} |\text{gr} L(\mathcal{O}_{\mathfrak{p}}) : I|^{-s}.$$

A simple application of the Chinese Remainder Theorem yields the Euler product

$$\zeta_L^{\triangleleft \text{gr}}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft \text{gr}}(s),$$

expressing the “global” zeta function $\zeta_L^{\triangleleft \text{gr}}(s)$ as an infinite product of “local”, or $\mathfrak{p}$ -adic ones. By abuse of notation we denote here by $\text{Spec}(\mathcal{O})$ the set of non-zero prime ideals of $\mathcal{O}$. Each individual Euler factor is a rational function in the parameter q^{-s} . In fact, the graded zeta function in (3.1.1) is an Euler product of *cone integrals* in the sense of [9] (cf. [28, Theorem 3.3]), and the far-reaching results of this paper – regarding both the factors and the product’s analytic properties – apply. The fact that the analysis of [9] is restricted to the case $R = \mathbb{Z}$ is insubstantial for this conclusion.

3.1.1 Main results

In this chapter we are mainly concerned with graded ideal zeta functions of free nilpotent Lie rings of finite rank. Given $c \in \mathbb{N}$ and $d \in \mathbb{N}_{\geq 2}$, let $\mathfrak{f}_{c,d}$ be the free nilpotent Lie ring (scheme over $\mathbb{Z}$) of nilpotency class c on d Lie generators. One may think of $\mathfrak{f}_{c,d}$ as the quotient of the free $\mathbb{Z}$ -Lie algebra $\mathfrak{f}_d$ on d generators by the $c + 1$ -th term $\gamma_{c+1}(\mathfrak{f}_d)$ of its lower central series. For $i \in [c]$, the $\mathbb{Z}$ -rank of the i -th lower-central-series quotient $\gamma_i(\mathfrak{f}_{c,d})/\gamma_{i+1}(\mathfrak{f}_{c,d})$ is given by the Witt function

$$W_d(i) := \frac{1}{i} \sum_{j|i} \mu(j) d^{i/j}, \quad (3.1.2)$$

where μ denotes the Möbius function; cf. [43, Satz 3]. Hence $\text{rk}_{\mathbb{Z}}(\mathfrak{f}_{c,d}) = \sum_{i=1}^c W_d(i)$. Given a commutative ring R – in this paper always a number ring $\mathcal{O}$ or the completion $\mathcal{O}_{\mathfrak{p}}$ of such a ring at a non-zero prime ideal $\mathfrak{p}$ – we write $\mathfrak{f}_{c,d}(R) = \mathfrak{f}_{c,d} \otimes_{\mathbb{Z}} R$, considered as an R -Lie algebra.

For $c = 1$, the (graded ideal) zeta function of the free abelian $\mathcal{O}$ -Lie ring $\mathfrak{f}_{1,d}(\mathcal{O}) \cong \mathcal{O}^d$, enumerating all finite index $\mathcal{O}$ -sublattices of $\mathcal{O}^d$, is well known to be equal to

$$\zeta_{\mathfrak{f}_{1,d}(\mathcal{O})}^{\triangleleft \text{gr}}(s) = \prod_{i=1}^d \zeta_K(s - i + 1), \quad (3.1.3)$$

where ζ_K denotes the Dedekind zeta function of K ; cf. [15, Proposition 1.1]. The Euler product (3.1.4) reflects the well-known Euler product $\zeta_K(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} (1 - (\mathcal{O}/\mathfrak{p})^{-s})^{-1}$.

$$\zeta_{\mathfrak{f}_{1,d}(\mathcal{O})}^{\triangleleft \text{gr}}(s) = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \prod_{i=1}^d \frac{1}{1 - (\mathcal{O}/\mathfrak{p})^{i-1-s}} = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{\mathcal{O}_{\mathfrak{p}}^d}(s). \quad (3.1.4)$$

For $c = 2$ Voll computed in [39] the *ideal* zeta functions enumerating $(\mathcal{O})$ -ideals of finite index in the rings $\mathfrak{f}_{2,d}(\mathcal{O})$. (The paper only discusses the case $R = \mathbb{Z}$, but its computations carry over – *mutatis mutandis* – to the case of general number rings.) We compute the *graded ideal* zeta functions $\zeta_{\mathfrak{f}_{2,d}(\mathcal{O})}^{\triangleleft \text{gr}}(s)$ for all $d \geq 2$ in Section 3.4 and those for $(c, d) \in \{(3, 2), (4, 2)\}$ in Section 3.5 of the current paper.

This chapter's most involved result is the computation, in Section 3.3, of the graded ideal zeta function of $\mathfrak{f}_{3,3}(\mathcal{O})$. To this end we compute an explicit formula for $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$, valid for all finite extension $\mathfrak{o}$ of the p -adic integers $\mathbb{Z}_p$, where p is a prime, viz. a local ring of the form $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ for a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$ lying above p .

Theorem 3.1.1. *There exists an explicitly determined rational function $W_{3,3}^{\triangleleft \text{gr}} \in \mathbb{Q}(X, Y)$ such that, for all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$, with residue cardinality q , say,*

$$\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{3,3}^{\triangleleft \text{gr}}(q, q^{-s}).$$

It may be written as $W_{3,3}^{\triangleleft \text{gr}} = N_{3,3}/D_{3,3}$ where

$$\begin{aligned} D_{3,3}(X, Y) = & (1 - Y)(1 - XY)(1 - X^2Y)(1 - X^3Y^4)(1 - XY^5)(1 - X^2Y^5)(1 - Y^6) \\ & (1 - X^7Y^7)(1 - X^8Y^7)(1 - X^8Y^8)(1 - X^{12}Y^8)(1 - X^6Y^9)(1 - X^{15}Y^9) \\ & (1 - X^{16}Y^{10})(1 - X^{15}Y^{11})(1 - X^{12}Y^{12})(1 - X^7Y^{13})(1 - Y^{14}), \end{aligned}$$

is a polynomial of degree 115 in X and 131 in Y , and $N_{3,3} \in \mathbb{Q}[X, Y]$ is a polynomial of degree 81 in X and 108 in Y .

The rational function $W_{3,3}^{\triangleleft \text{gr}}$ satisfies the functional equation

$$W_{3,3}^{\triangleleft \text{gr}}(X^{-1}, Y^{-1}) = X^{34}Y^{23}W_{3,3}^{\triangleleft \text{gr}}(X, Y). \quad (3.1.5)$$

We note that the Witt function W_3 (cf. (3.1.2)) takes the values $(W_3(1), W_3(2), W_3(3)) = (3, 3, 8)$ and that $115 - 81 = 34 = \binom{3}{2} + \binom{3}{2} + \binom{8}{2}$ and $131 - 108 = 23 = 3 \cdot 3 + 2 \cdot 3 + 1 \cdot 8$; cf. Conjecture 7.2.2.

Our proof of Theorem 3.1.1 yields $W_{3,3}^{\triangleleft \text{gr}}$ as a sum of 15 explicitly given summands, listed essentially in Section 3.3.2. We do not reproduce the “final” outcome of this

summation here, as the numerator $N_{3,3}$ of $W_{3,3}^{\triangleleft \text{gr}}$ fills several pages. We do record, however, several corollaries of the explicit formula for $W_{3,3}^{\triangleleft \text{gr}}$.

The first corollary concerns analytic properties of the global zeta function $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}(s)$.

Corollary 3.1.2. *The global graded ideal zeta function $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}(s)$ converges on $\{s \in \mathbb{C} \mid \text{Re}(s) > 3\}$ and may be continued meromorphically to $\{s \in \mathbb{C} \mid \text{Re}(s) > 14/9\}$.*

This follows from the observation that the Euler product $\prod_{\mathfrak{p} \in \text{Spec}(\mathfrak{o})} D_{3,3}(q, q^{-s})^{-1}$ – a product of finitely many translates of the Dedekind zeta function $\zeta_K(s)$ – has abscissa of convergence 3 and the Euler product $\prod_{\mathfrak{p} \in \text{Spec}(\mathfrak{o})} N_{3,3}(q, q^{-s})$ may be continued meromorphically to $\{s \in \mathbb{C} \mid \text{Re}(s) > 14/9\}$. The latter reflects the facts that, if $N_{3,3}(X, Y) = 1 + \sum_{i \in I} \alpha_i X^{a_i} Y^{b_i}$ for a finite index set I and $\alpha_i \in \mathbb{Z} \setminus \{0\}$, $a_i \in \mathbb{N}_0$, $b_i \in \mathbb{N}$ then $\beta := \max\{\frac{a_i}{b_i} \mid i \in I\} = 14/9$; cf. [11, Lemma 5.5].

The second corollary concerns the *reduced graded ideal zeta function*

$$\zeta_{\mathfrak{f}_{3,3}, \text{red}}^{\triangleleft \text{gr}}(Y) := W_{3,3}^{\triangleleft \text{gr}}(1, Y) \in \mathbb{Q}(Y);$$

cf. Section 3.2.4. This concept was introduced in [13], albeit not in the context of graded ideal zeta functions. We expect that our ad hoc definition will fit into a general definition of reduced graded ideal zeta functions along the lines of [13].

Corollary 3.1.3. *The reduced graded ideal zeta function $\zeta_{\mathfrak{f}_{3,3}, \text{red}}^{\triangleleft \text{gr}}(Y)$ satisfies*

$$\zeta_{\mathfrak{f}_{3,3}, \text{red}}^{\triangleleft \text{gr}}(Y) = \frac{N_{3,3, \text{red}}(Y)}{(1 - Y)^3(1 - Y^3)(1 - Y^4)^2(1 - Y^5) \prod_{i=8}^{14} (1 - Y^i)}, \quad (3.1.6)$$

where $N_{3,3, \text{red}}(Y) \in \mathbb{Z}[Y]$ is equal to

$$\begin{aligned} &1 + 2Y^3 + 4Y^4 + 5Y^5 + 16Y^6 + 34Y^7 + 53Y^8 + 77Y^9 + 98Y^{10} + 121Y^{11} + 182Y^{12} + \\ &302Y^{13} + 483Y^{14} + 712Y^{15} + 953Y^{16} + 1187Y^{17} + 1425Y^{18} + 1689Y^{19} + 2046Y^{20} + \\ &2579Y^{21} + 3298Y^{22} + 4162Y^{23} + 5059Y^{24} + 5826Y^{25} + 6398Y^{26} + 6894Y^{27} + \\ &7475Y^{28} + 8270Y^{29} + 9265Y^{30} + 10260Y^{31} + 11041Y^{32} + 11529Y^{33} + 11745Y^{34} + \\ &11798Y^{35} + 11811Y^{36} + 11811Y^{37} + 11798Y^{38} + 11745Y^{39} + 11529Y^{40} + 11041Y^{41} + \\ &10260Y^{42} + 9265Y^{43} + 8270Y^{44} + 7475Y^{45} + 6894Y^{46} + 6398Y^{47} + 5826Y^{48} + \\ &5059Y^{49} + 4162Y^{50} + 3298Y^{51} + 2579Y^{52} + 2046Y^{53} + 1689Y^{54} + 1425Y^{55} + \\ &1187Y^{56} + 953Y^{57} + 712Y^{58} + 483Y^{59} + 302Y^{60} + 182Y^{61} + 121Y^{62} + 98Y^{63} + \\ &77Y^{64} + 53Y^{65} + 34Y^{66} + 16Y^{67} + 5Y^{68} + 4Y^{69} + 2Y^{70} + Y^{73}. \end{aligned}$$

It seems remarkable that $W_{3,3}^{\triangleleft \text{gr}}(1, Y)$ has a simple pole at $Y = 1$ of order $14 = 3+3+8$, the $\mathbb{Z}$ -rank of $\mathfrak{f}_{3,3}$, and that $N_{3,3,\text{red}}(Y)$ has nonnegative, unimodal coefficients. This is consistent with the speculation that $W_{3,3}^{\triangleleft \text{gr}}(1, Y)$ is in fact the Hilbert-Poincaré series of a 14-dimensional graded algebra, associated to $\mathfrak{f}_{3,3}$ in a natural way; cf. Remark 7.2.5. Note that the right hand side of (3.1.6) is not in lowest terms. The palindromic symmetry of the coefficients of $N_{3,3,\text{red}}(Y)$ is implied by the functional equation (3.1.5).

The third corollary concerns another limiting object of the local graded ideal zeta functions $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$, viz. the *topological graded ideal zeta function* $\zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s)$; cf. Section 3.2.5. In [27], Rossmann introduced and studied so-called topological zeta functions associated to a range of Dirichlet generating series. Informally speaking, topological zeta functions may be viewed as suitably defined limits of local zeta functions. Whilst topological versions of zeta functions such as $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ have not yet been studied specifically, one may define $\zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s)$ *ad hoc* as the leading term of the expansion of $\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ in $q - 1$; cf. [27, Definition 5.13].

Corollary 3.1.4. *The topological graded ideal zeta function $\zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s)$ satisfies*

$$\zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s) = \frac{(33250s^4 - 81537s^3 + 66573s^2 - 20800s + 1920)/56}{D_{3,3,\text{top}}(s)},$$

where $D_{3,3,\text{top}}(s) \in \mathbb{Z}[s]$ is defined to be

$$s^3(s-1)^4(s-2)(3s-2)(4s-3)(5s-1)(5s-2)(7s-8)(2s-3)(3s-5)(5s-8)(11s-15)(13s-7)$$

A number of features of the rational functions considered above seem remarkable. By the *degree* of a rational function $f = P/Q \in \mathbb{Q}(Y)$ we mean $\deg_Y f = \deg_Y P - \deg_Y Q$.

Corollary 3.1.5. 1.

$$\deg_s \left(\zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s) \right) = -14 = -(3 + 3 + 8) = -\text{rk}_{\mathbb{Z}}(\mathfrak{f}_{3,3}),$$

2.

$$s^{-14} \zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s^{-1})|_{s=0} = \frac{19}{288288} = (1 - Y)^{14} \zeta_{\mathfrak{f}_{3,3},\text{red}}^{\triangleleft \text{gr}}(Y)|_{Y=1} \in \mathbb{Q}_{>0},$$

3.

$$s^3 \zeta_{\mathfrak{f}_{3,3},\text{top}}^{\triangleleft \text{gr}}(s)|_{s=0} = \frac{-1}{70560} = \frac{(-1)^{2+2+7} 3 \cdot 3 \cdot 8}{3 \cdot 6 \cdot 14 \cdot 2! \cdot 2! \cdot 7!},$$

4.

$$\left. \frac{\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^3}(s) \zeta_{\mathfrak{o}^3}(s) \zeta_{\mathfrak{o}^8}(s)} \right|_{s=0} = \frac{2}{7} = \frac{3 \cdot 3 \cdot 8}{3 \cdot 6 \cdot 14}.$$

If the above-mentioned connection between the reduced graded ideal zeta function and Hilbert-Poincaré series of graded algebras were to hold, the rational number in part (2) – for which we currently do not have an interpretation – were to be interpreted in terms of the multiplicity of the associated graded algebra.

Based on the explicit computations in this chapter we present, in Chapter 6, a number of general conjectures pertaining to graded ideal zeta functions of free nilpotent Lie rings generalizing these features.

3.1.2 Background, motivation, and methodology

Recall that L is a nilpotent Lie algebra over the ring of integers $\mathcal{O}$ of a number field. The *ideal zeta function* of L is the Dirichlet generating series enumerating the $\mathcal{O}$ -ideals of L of finite index, viz.

$$\zeta_L^{\triangleleft}(s) = \sum_{I \triangleleft L} |L : I|^{-s} = \prod_{\mathfrak{p} \in \text{Spec}(\mathcal{O})} \zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s),$$

where s is a complex variable and, for a (non-zero) prime ideal $\mathfrak{p}$ of $\mathcal{O}$,

$$\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s) = \sum_{I \triangleleft L(\mathcal{O}_{\mathfrak{p}})} |L(\mathcal{O}_{\mathfrak{p}}) : I|^{-s}.$$

Ideal zeta functions of nilpotent Lie rings are well-studied predecessors of the graded ideal zeta functions studied in the present paper. One of the main results of [15], which introduced these zeta functions, establishes the rationality of each of the Euler factors $\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s)$ in q^{-s} , where $q = |\mathcal{O} : \mathfrak{p}|$; cf. [15, Theorem 3.5]. For numerous examples of ideal zeta functions of nilpotent Lie rings, see [11].

By the Mal'cev correspondence, ideal zeta functions of nilpotent Lie rings are closely related to the *normal subgroup zeta functions* enumerating finite index normal subgroups of finitely generated nilpotent groups. In particular, given $c, d \in \mathbb{N}$, almost all (i.e. all but finitely many) of the Euler factors of $\zeta_{f_{c,d}}^{\triangleleft}(s)$ coincide with those of the normal subgroup zeta function of the free class- c -nilpotent d -generator group $F_{c,d}$; cf. [15, Section 4]. The study of the normal subgroup growth of free nilpotent groups is connected with the enumeration of finite p -groups up to isomorphism; cf. [7].

Informally speaking, graded ideal zeta functions of nilpotent Lie rings may be seen as “approximations” of their ideal zeta functions. Indeed, almost all Euler factors $\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft, \text{gr}}(s)$ actually enumerate a sublattice of the lattice of ideals enumerated by $\zeta_{L(\mathcal{O}_{\mathfrak{p}})}^{\triangleleft}(s)$. In general, this approximation is quite coarse. In nilpotency class $c \leq 2$, however, the problems of computing ideal zeta functions and *graded* ideal zeta functions are all but equivalent, as the following example shows.

Example 3.1.6. Assume that L is nilpotent of class 2, with isolated commutator ideal L' such that $\text{rk}_{\mathcal{O}}(L/L') = d$. Then, essentially by [15, Lemma 6.1],

$$\zeta_L^{\triangleleft}(s) = \sum_{\Lambda_1 \leq L/L'} |L/L' : \Lambda_1|^{-s} \sum_{[\Lambda_1, L] \leq \Lambda_2 \leq L'} |L' : \Lambda_2|^{d-s}, \quad (3.1.7)$$

whereas

$$\zeta_{\mathcal{L}}^{\triangleleft_{\text{gr}}}(s) = \sum_{\Lambda_1 \leq L/L'} |L/L' : \Lambda_1|^{-s} \sum_{[\Lambda_1, \text{gr} L] \leq \Lambda_2 \leq L'} |L' : \Lambda_2|^{-s}. \quad (3.1.8)$$

In the special cases $L = \mathfrak{f}_{2,d}$, the proximity between (3.1.7) and (3.1.8) explains the proximity between the explicit formulae recorded in Theorems 3.4.1 and 3.4.2 of the current paper.

In higher nilpotency classes, we are not aware of any such simple parallels. In the realm of free nilpotent Lie rings of class greater than two, explicit computations of ideal zeta functions seem all but unfeasible. In particular, we do not know of formulae for the ideal zeta functions of the Lie rings $\mathfrak{f}_{3,3}(\mathcal{O})$ and $\mathfrak{f}_{4,2}(\mathcal{O})$.

Numerous questions regarding ideal zeta functions have analogues regarding the “approximating” *graded* ideal zeta functions, and one may speculate that the latter are easier to answer than the former. On p. 188 of [15], Grunewald, Segal, and Smith formulate, for example, a conjecture which would imply that, for any $c, d \in \mathbb{N}$ there exists a rational function $W_{c,d}(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$, $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft}(s) = W_{c,d}(q, q^{-s})$. This consequence is known to hold for $c \leq 2$ and $(c, d) = (3, 2)$ but wide open in general, including the cases $(c, d) \in \{(3, 3), (4, 2)\}$. Our Conjecture 7.2.1 – which is verified by the explicit computations in the current paper in particular for $c = 2$ and $(c, d) \in \{(3, 3), (3, 2), (4, 2)\}$ – may be viewed as a “graded” version of this conjecture.

In [27], Rossmann formulates a number of conjectures on certain special values of $\mathfrak{p}$ -adic, topological, and reduced zeta functions, pertaining in particular to ideal zeta functions of nilpotent Lie rings. Our Conjectures 7.2.7, 7.2.9 and 7.2.11 are “graded” counterparts.

In [41, Theorem 4.4], Voll proved a local functional equation for the generic Euler factors of the ideal zeta functions $\zeta_{\mathfrak{f}_{c,d}(\mathcal{O})}^{\triangleleft}(s)$ upon inversion of the prime for all $c \in \mathbb{N}$ and $d \in \mathbb{N}_{\geq 2}$. Our results suggest that this phenomenon also appears for graded ideal zeta functions of free nilpotent Lie rings; cf. Conjecture 7.2.2.

This chapter’s methodology is combinatorial. All our computations owe their feasibility to the fact that, for the parameter values considered, viz. $c = 2$ and $(c, d) \in \{(3, 3), (3, 2), (4, 2)\}$, the enumeration of graded ideals is equivalent to the enumeration

of various flags of lattices in free abelian $\mathfrak{o}$ -modules which depend only, and in a linear fashion, on the lattices' *elementary divisor types*. Our computations rely on a simple polynomial formula, due to Birkhoff, for the numbers of $\mathfrak{o}$ -submodules of given type in a finite $\mathfrak{o}$ -module of given type; cf. Proposition 3.2.2. To obtain closed formulae for the relevant graded ideal zeta functions we need to organize the enumeration of infinitely many values of Birkhoff's formula in a manageable way. We meet this challenge by organizing pairs of partitions, encoding two lattices' elementary divisor types, by their *overlap type*, formally one of finitely many multiset permutations, viz. words in two letters (each with multiplicity); cf. Section 3.3.1. For $(c, d) = (3, 3)$, for instance, we are led to consider 15 specific words in the alphabet $\{\mathbf{1}, \mathbf{2}\}$; cf. Table 3.3.1. Restricting the relevant enumerations to a fixed word yields formulae which are products of q -multinomial coefficients and so-called *Igusa functions*; cf. Section 3.2.2. In Theorem 3.3.6 we compute such formulae for each of the 15 words relevant for $(c, d) = (3, 3)$. Similar methodology has been applied before to compute ideal zeta functions of Heisenberg Lie rings over number rings ([29]) and free nilpotent Lie rings of class 2 ([39]).

That we are able to reduce our computations to combinatorial considerations with partitions is essentially owed to the fact that, for the parameters (c, d) considered, the automorphism groups of the free Lie rings $\mathfrak{f}_{c,d}$ act transitively on the lattices of given elementary divisor types in relevant sections of $\mathfrak{f}_{c,d}$. This allows us to assume that the relevant lattices are generated by multiples of elements of *Hall bases*; cf. Section 3.2.3. That a lattice may always be generated by multiples of *some* linear basis follows, of course, from the elementary divisor theorem; that (subsets of) Hall bases may be used is a lucky consequence of the existence of “many” automorphisms, as we now explain. The automorphism group of $\mathfrak{f}_{c,d}(\mathfrak{o})$ contains a copy of $\mathrm{GL}_d(\mathfrak{o})$ which allows us to perform arbitrary invertible $\mathfrak{o}$ -linear transformations of Lie generators $x_1, \dots, x_d$ of $\mathfrak{f}_{c,d}(\mathfrak{o})$. For $c = 2$, this observation – which was already exploited in the proof of [15, Theorem 2] – in conjunction with Birkhoff's formula is sufficient to compute the (graded) ideal zeta function of $\mathfrak{f}_{c,d}(\mathfrak{o})$; cf. Section 3.4. On the Lie commutators $[x_i, x_j]$, $1 \leq i < j \leq n$, these linear transformations act via the exterior square representation. In general, the image of the natural map $\bigwedge^2 \mathrm{GL}_d(\mathfrak{o}) \rightarrow \mathrm{GL}_{\binom{d}{2}}(\mathfrak{o})$ is rather small. The map is surjective, however, if $d \leq 3$, and an isomorphism if $d = 3$. This facilitates our computations for $(c, d) \in \{(3, 3), (3, 2)\}$. If $d = 2$, then the above-mentioned copy of $\mathrm{GL}_2(\mathfrak{o})$ even induces the full automorphism group of the $\mathfrak{o}$ -module generated by the weight-3-commutators $[[x_1, x_2], x_1]$ and $[[x_1, x_2], x_2]$, which we exploit for $(c, d) = (4, 2)$.

3.1.3 Settings

We write 2^S for the power set of a set S . By a partition we mean a tuple $(\lambda_1, \dots, \lambda_n)_\geq \in \mathbb{N}_0^n$ of nonincreasing nonnegative integers. The notation $I = \{i_1, \dots, i_h\}_<$ for a subset of $I \subset \mathbb{N}_0$ indicates that $i_1 < i_2 < \dots < i_h$. Similarly, $(\lambda_1, \dots, \lambda_n)_\geq \in \mathbb{N}_0^n$ denotes a partition with non-ascending parts $\lambda_1 \geq \dots \geq \lambda_n \geq 0$.

We write $\mathfrak{o}$ for a compact discrete valuation ring of characteristic zero, i.e. a finite extension of the ring $\mathbb{Z}_p$ of p -adic integers or, equivalently, a ring of the form $\mathcal{O}_{\mathfrak{p}}$, the completion of the $\mathcal{O}$, the ring of integers of a number field K , at a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}$. We write q for the cardinality of the residue field of $\mathfrak{o}$ and p for its residue characteristic. We write $t = q^{-s}$, where s is a complex variable. We write ζ_K for the Dedekind zeta function of K .

3.2 General preliminaries

Let $c \in \mathbb{N}$, $d \in \mathbb{N}_{\geq 2}$, and set $\mathfrak{f} = \mathfrak{f}_{c,d}$ be as above. The following is analogous to [15, Lemma 6.1].

Lemma 3.2.1. *Let p be a prime and $\mathfrak{o}$ be a finite extension of $\mathbb{Z}_p$. For each $\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}$ let*

$$(X(\Lambda_2) \oplus \mathfrak{f}(\mathfrak{o})^{(2)})/\Lambda_2 = Z((\mathfrak{f}(\mathfrak{o})^{(1)} \oplus \mathfrak{f}(\mathfrak{o})^{(2)})/\Lambda_2). \quad (3.2.1)$$

Then

$$\begin{aligned} \zeta_{\mathfrak{f}(\mathfrak{o})}^{\triangleleft_{\text{gr}}} (s) &= \sum_{\substack{\Lambda_1 \leq \mathfrak{f}(\mathfrak{o})^{(1)}, \dots, \Lambda_c \leq \mathfrak{f}(\mathfrak{o})^{(c)} \\ \forall i \in]1, c]: [\Lambda_{i-1}, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_i}} \prod_{i=1}^c |\mathfrak{f}(\mathfrak{o})^{(i)} : \Lambda_i|^{-s} \\ &= \zeta_{\mathfrak{o}^d}(s) \sum_{\substack{\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}, \dots, \Lambda_c \leq \mathfrak{f}(\mathfrak{o})^{(c)} \\ \forall i \in]2, c]: [\Lambda_{i-1}, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_i}} |\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s} \prod_{i=2}^c |\mathfrak{f}(\mathfrak{o})^{(i)} : \Lambda_i|^{-s}. \end{aligned} \quad (3.2.2)$$

Proof. A graded additive sublattice $\Lambda \leq \text{gr } \mathfrak{f}(\mathfrak{o})$ determines and is determined by the sequence $(\Lambda_1, \dots, \Lambda_c)$, where $\Lambda_i := \Lambda \cap \mathfrak{f}(\mathfrak{o})^{(i)}$ for all $i \in [c]$. We have $\Lambda \triangleleft_{\text{gr}} \mathfrak{f}(\mathfrak{o})$ if and only if $[\Lambda_{i-1}, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_i$ for all $i \in]1, c]$. This proves the first equality. The second follows from the definition of $X(\Lambda_2)$ given in (3.2.1), noting that $[\Lambda_1, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_2$ if and only if $\Lambda_1 \leq X(\Lambda_2)$ and that, for each $\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}$,

$$\sum_{\Lambda_1 \leq X(\Lambda_2)} |\mathfrak{f}(\mathfrak{o})^{(1)} : \Lambda_1|^{-s} = \zeta_{\mathfrak{o}^d}(s) |\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s}. \quad \square$$

3.2.1 Birkhoff's formula

Given a pair of partitions (σ, τ) , let $\alpha(\sigma, \tau; q)$ denote the number of torsion $\mathfrak{o}$ -modules of type τ contained in a fixed torsion $\mathfrak{o}$ -module of type σ . Clearly $\alpha(\sigma, \tau; q) = 0$ unless $\tau \leq \sigma$. Notice that, for $\tau = (\tau_1, \dots, \tau_n)_{\geq}$,

$$\alpha(\tau_1^{(n)}, \tau; q) = \#\{\Lambda \leq \mathfrak{o}^n \mid \mathfrak{o}^n/\Lambda \cong \bigoplus_{j=1}^n \mathfrak{o}/\mathfrak{p}^{\tau_j}\}.$$

The following explicit general formula for $\alpha(\sigma, \tau; q)$ is attributed to Birkhoff in [4].

Proposition 3.2.2 (Birkhoff). *Let $\tau \leq \sigma$ be partitions, with dual partitions $\tau' \leq \sigma'$. Then*

$$\alpha(\sigma, \tau; q) = \prod_{k \geq 1} q^{\tau'_k(\sigma'_k - \tau'_k)} \binom{\sigma'_k - \tau'_{k+1}}{\sigma'_k - \tau'_k}_{q^{-1}} \in \mathbb{Z}[q].$$

3.2.2 Igusa functions and their functional equations

Definition 3.2.3. ([29, Definition 2.5]) Let $h \in \mathbb{N}$. Given variables $\mathbf{X} = (X_1, \dots, X_h)$ and Y , we set

$$I_h(Y; \mathbf{X}) = \frac{1}{1 - X_h} \sum_{I \subseteq [h-1]} \binom{h}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_h),$$

$$I_h^\circ(Y; \mathbf{X}) = \frac{X_h}{1 - X_h} \sum_{I \subseteq [h-1]} \binom{h}{I}_Y \prod_{i \in I} \frac{X_i}{1 - X_i} \in \mathbb{Q}(Y, X_1, \dots, X_h).$$

We set $I_0(Y) = I_0^\circ(Y) = 1 \in \mathbb{Q}(Y)$.

Note that $I_1(Y; X) = \frac{1}{1-X}$ and $I_1^\circ(Y; X) = \frac{X}{1-X}$. We will make repeated use of the functional equations ([29, Proposition 4.2])

$$I_h(Y^{-1}; \mathbf{X}^{-1}) = (-1)^h X_h Y^{-\binom{h}{2}} I_h(Y; \mathbf{X}),$$

$$I_h^\circ(Y^{-1}; \mathbf{X}^{-1}) = (-1)^h X_h^{-1} Y^{-\binom{h}{2}} I_h^\circ(Y; \mathbf{X}). \quad (3.2.3)$$

Remark 3.2.4. The “symmetry centres” $(-1)^h X_h Y^{-\binom{h}{2}}$ resp. $(-1)^h X_h^{-1} Y^{-\binom{h}{2}}$ in the functional equations (3.2.3) depend on Y and X_h , but not on $X_1, \dots, X_{h-1}$.

Throughout this paper, we always substitute q^{-1} for Y and hence write $I_h(\mathbf{X})$ instead of $I_h(q^{-1}; \mathbf{X})$ and $I_h^\circ(\mathbf{X})$ instead of $I_h^\circ(q^{-1}; \mathbf{X})$.

Igusa functions are ubiquitous in the theory of zeta functions of groups and rings. Note, for instance, that

$$\zeta_{\mathfrak{o}^d}(s) = I_d((q^{(d-j)j} t^j)_{j \in [d]}) = \frac{1}{\prod_{j=0}^{d-1} (1 - q^j t)};$$

cf. (3.1.4).

3.2.3 Hall bases for free nilpotent Lie rings

Let $x_1, \dots, x_d$ be Lie generators for $\mathfrak{f} = \mathfrak{f}_{c,d}$. A *Hall basis* for $\mathfrak{f}_{c,d}$ (on $x_1, \dots, x_d$) is a $\mathbb{Z}$ -basis $\mathcal{H}_{c,d}$ for $\mathfrak{f}_{c,d}$ which may be constructed by selecting inductively certain *basic commutators* in the x_i ; see [18] for details. Note that $|\mathcal{H}_{c,d}| = \sum_{i=1}^c W_d(i)$, where W_d denotes the Witt function (3.1.2).

In Table 3.2.1 we record some Hall bases which are relevant for the current paper. Here we adopt the standard abbreviation $x_{i_1}x_{i_2}x_{i_3} \dots x_{i_r}$ for left-normed commutators $[\dots [[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_r}]$. In the case that $d = 2$ (resp. $d = 3$), we write x, y (and z) instead of x_1, x_2 (and x_3).

(c, d)	Hall basis $\mathcal{H}_{c,d}$	$(W_d(i))_{i \in [c]}$
$(1, d)$	$\{x_1, \dots, x_d\}$	(d)
$(2, d)$	$\{x_1, \dots, x_d, x_i x_j \mid 1 \leq i < j \leq d\}$	$(d, \binom{d}{2})$
$(3, 3)$	$\{x, y, z, xy, xz, yz, xyx, xzx, yzx, xyy, xzy, yzy, xzz, yzz\}$	$(3, 3, 8)$
$(3, 2)$	$\{x, y, xy, xyx, xyy\}$	$(2, 1, 2)$
$(4, 2)$	$\{x, y, xy, xyx, xyy, xyxx, xyxy, xyyy\}$	$(2, 1, 2, 3)$

Table 3.2.1: Some Hall bases

3.2.4 Reduced zeta functions

In [13], Evseev introduced a certain “limit as $\mathfrak{p} \rightarrow 1$ ” of various local zeta functions. Informally speaking, the idea is to exploit the fact that the coefficients of these generating functions enumerate $\mathbb{F}_q$ -rational points of constructible sets. The desired limit is obtained by replacing these constructible sets by their Euler-Poincaré characteristics. The resulting rational functions are called *reduced zeta functions*; cf. [13, Section 3]. This paper does not define reduced *graded* ideal zeta functions explicitly, but provides a theoretical framework that allows this quite readily. If c, d are such that $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s})$ for a rational function $W_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$, for almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$, then $\zeta_{\mathfrak{f}_{c,d,\text{red}}}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(1, Y) \in \mathbb{Q}(Y)$. Evseev proved in [13, Section 4] that reduced ideal zeta functions of Lie rings with so-called *nice* and *simple* bases are Hilbert-Poincaré series enumerating integral points in rational polyhedral cones. The Hall bases $\mathcal{H}_{2,d}$ given in Table 3.2.1, for instance, have this property. Comparing (3.1.7) and (3.1.8) one sees that, for nilpotent Lie rings L of nilpotency class 2, $\zeta_{L,\text{red}}^{\triangleleft \text{gr}}(Y) = \zeta_{L,\text{red}}^{\triangleleft}(Y)$; cf. also Remark 3.5.2.

3.2.5 Topological zeta functions

Another means of defining limits of $\mathfrak{p}$ -adic zeta functions are topological zeta functions. If c, d are such that $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s})$ for a rational function $W_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$, for almost all primes p and all finite extensions of $\mathbb{Z}_p$, then $\zeta_{\mathfrak{f}_{c,d}, \text{top}}^{\triangleleft \text{gr}}(s)$ is simply the leading term of the expansion of $W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s})$ in $q - 1$.

Example 3.2.5. For $a \in \mathbb{N}_0$ and $b \in \mathbb{N}$, $\frac{1}{1-q^{a-bs}} = \frac{1}{bs-a}(q-1)^{-1} + O(1)$. Thus

$$\zeta_{\mathfrak{f}_{1,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^d}(s) = \frac{1}{s(s-1)\dots(s-d+1)}(q-1)^{-d} + O((q-1)^{-d+1}),$$

whence

$$\zeta_{\mathfrak{f}_{1,d}, \text{top}}^{\triangleleft \text{gr}}(s) = \frac{1}{s(s-1)\dots(s-d+1)}.$$

More generally, Definition [27, Definition 5.13] applies to any system of local zeta functions of Denef type over a number field, in the sense of [27, Definition 5.7]. Examples of such systems are families of $\mathfrak{p}$ -adic zeta functions of the form $(W(q, q^{-s}))_{\mathfrak{p} \in \text{Spec}(\mathcal{O})}$, where $W(X, Y) \in \mathbb{Q}(X, Y)$ (and, as usual, $q = |\mathcal{O}/\mathfrak{p}|$ for $\mathfrak{p} \in \text{Spec}(\mathcal{O})$). All the local graded ideal zeta functions considered in this paper fit into such “uniform” families. We expect this phenomenon to be universal in the context of free nilpotent Lie rings; cf. Conjecture 7.2.1.

For a formal and far more general definition of topological zeta functions we refer to [27, Section 5]. In [27, Section 8], Rossmann collects a number of intriguing conjectures about analytic properties of topological zeta functions associated to various group- and ring-theoretic counting problems. We expect most of these conjectures to have analogues in the realm of topological graded ideal zeta functions. Motivated by the computation of various topological graded ideal zeta functions of free nilpotent Lie rings made throughout the current paper, we make a number of such conjectures in Section 7.2.

3.3 Proof of Theorem 3.1.1 $((c, d) = (3, 3))$

Let $\mathfrak{f} = \mathfrak{f}_{3,3}$ be the free nilpotent Lie ring on 3 generators of nilpotency class 3. Let p be a prime and $\mathfrak{o}$ be a finite extension of $\mathbb{Z}_p$, with uniformizer π . Note that $\mathfrak{f}(\mathfrak{o})^{(1)} \cong \mathfrak{o}^3 \cong \mathfrak{f}(\mathfrak{o})^{(2)}$ and $\mathfrak{f}(\mathfrak{o})^{(3)} \cong \mathfrak{o}^8$. In order to parameterize lattices in $\mathfrak{f}(\mathfrak{o})^{(2)}$ and $\mathfrak{f}(\mathfrak{o})^{(3)}$ we denote by

$$\mu = (\mu_1, \mu_2, \mu_3)_{\geq}, \quad \nu = (\nu_1, \dots, \nu_8)_{\geq}$$

integer partitions of at most 3 and 8 parts respectively, and set

$$\bar{\mu} = (\mu_1^{(2)}, \mu_2^{(3)}, \mu_3^{(3)}) = (\mu_1, \mu_1, \mu_2, \mu_2, \mu_2, \mu_3, \mu_3, \mu_3).$$

Proposition 3.3.1.

$$\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^3}(s) \sum_{\mu} \alpha(\mu_1^{(3)}, \mu; q) q^{-s(3\mu_1+2\mu_2+\mu_3)} \sum_{\nu \leq \bar{\mu}} \alpha(\bar{\mu}, \nu; q) q^{-s \sum_{k=1}^8 \nu_k}. \quad (3.3.1)$$

Proof. Our starting point is (3.2.2) in Lemma 3.2.1, which in this case reads

$$\zeta_{\mathfrak{f}_{3,3}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^3}(s) \sum_{\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}} |\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s} |\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2|^{-s} \sum_{\substack{\Lambda_3 \leq \mathfrak{f}(\mathfrak{o})^{(3)}, \\ [\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_3}} |\mathfrak{f}(\mathfrak{o})^{(3)} : \Lambda_3|^{-s}.$$

Let $\mu = (\mu_1, \mu_2, \mu_3)_{\geq}$ be a partition. There are $\alpha(\mu_1^{(3)}, \mu; q)$ lattices $\Lambda_2 \leq \mathfrak{f}(\mathfrak{o})^{(2)}$ whose elementary divisor type with respect to $\mathfrak{f}(\mathfrak{o})^{(2)}$ is given by μ . Clearly $|\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2| = q^{\sum_{j=1}^3 \mu_j}$. We claim that $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu_1+\mu_2}$. Indeed, lattices such as Λ_2 may be parametrized by their elementary divisor type and a coset $\alpha\Gamma_{\mu} \in \text{SL}_3(\mathfrak{o})/\Gamma_{\mu}$ of a certain stabilizer subgroup $\Gamma_{\mu} \leq \text{SL}_3(\mathfrak{o})$, where $\alpha = (\alpha_{ij}) \in \text{SL}_3(\mathfrak{o})$. By solving the simultaneous congruences

$$(x_1, x_2, x_3) \begin{pmatrix} 0 & \alpha_{1j} & \alpha_{2j} \\ -\alpha_{1j} & 0 & \alpha_{3j} \\ -\alpha_{2j} & -\alpha_{3j} & 0 \end{pmatrix} \equiv 0 \pmod{\mathfrak{p}^{\mu_j}}$$

for $j = 1, 2, 3$, we compute that $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu_1+\mu_2}$; cf. [37, Section 2.2] for details.

In fact, after a change of generators for $\mathfrak{f}(\mathfrak{o})$ if necessary, we may assume that

$$\Lambda_2 = \mathfrak{p}^{\mu_1}xy \oplus \mathfrak{p}^{\mu_2}xz \oplus \mathfrak{p}^{\mu_3}yz.$$

At this point we crucially use the fact that $d = 3$, as $\text{GL}_d(\mathfrak{o}) \cong \bigwedge^2 \text{GL}_d(\mathfrak{o})$ if and only if $d = 3$. We obtain

$$[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] = \langle \pi^{\mu_1}xyz, \pi^{\mu_1}xyy, \underline{\pi^{\mu_1}xyz}, \pi^{\mu_2}xxz, \underline{\pi^{\mu_2}xzy}, \pi^{\mu_2}xzz, \underline{\pi^{\mu_3}yzx}, \pi^{\mu_3}yzy, \pi^{\mu_3}yzz \rangle_{\mathfrak{o}}. \quad (3.3.2)$$

The Jacobi identity involving the three underlined terms is the only nontrivial relation between these terms. Indeed, the relation $xyz + zxy + yzx = 0$ implies that

$$\langle \pi^{\mu_1}xyz, \pi^{\mu_2}xzy, \pi^{\mu_3}yzx \rangle_{\mathfrak{o}} = \langle \pi^{\mu_1}(xzy - yzx), \pi^{\mu_2}xzy, \pi^{\mu_3}yzx \rangle_{\mathfrak{o}} = \mathfrak{p}^{\mu_2}xzy \oplus \mathfrak{p}^{\mu_3}yzx.$$

Hence (3.3.2) implies that

$$[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] = \mathfrak{p}^{\mu_1}xyx \oplus \mathfrak{p}^{\mu_1}xyy \oplus \mathfrak{p}^{\mu_2}xzx \oplus \mathfrak{p}^{\mu_2}xzy \oplus \mathfrak{p}^{\mu_2}xzz \oplus \mathfrak{p}^{\mu_3}yzx \oplus \mathfrak{p}^{\mu_3}yzy \oplus \mathfrak{p}^{\mu_3}yzz,$$

whence $[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}]$ has type $\bar{\mu}$ with respect to $\mathfrak{f}(\mathfrak{o})^{(3)}$. This shows that the number of lattices $\Lambda_3 \leq \mathfrak{f}(\mathfrak{o})^{(3)}$ whose elementary divisor type with respect to $\mathfrak{f}(\mathfrak{o})^{(3)}$ is given by a partition $\nu = (\nu_1, \dots, \nu_8)_{\geq}$ and which satisfy $[\Lambda_2, \mathfrak{f}(\mathfrak{o})^{(1)}] \leq \Lambda_3$ is equal to $\alpha(\bar{\mu}, \nu; q)$. Each such lattice satisfies $|\mathfrak{f}(\mathfrak{o})^{(3)} : \Lambda_3| = q^{-s \sum_{k=1}^8 \nu_k}$. $\square$

3.3.1 Overlap types and 2-dimensional words

Our approach to computing the right hand side of (3.3.1) is similar to the one taken in [29] to compute local factors of the $(\mathbb{Z})$ -ideal zeta functions of Lie rings of the form $\mathfrak{f}_{2,2}(\mathcal{O})$.

To compute the right hand side of (3.3.1) we carry out a case distinction with respect to the finitely many ways in which the partitions $\bar{\mu}$ and ν may “overlap”. To be precise, let μ and ν be partitions of 3 and 8 parts, respectively, satisfying $\nu \leq \bar{\mu}$. There are uniquely determined numbers $r \in \mathbb{N}_0$ and $M_i, N_i \in \mathbb{N}$ ($i \in [r-1]$), such that

$$\begin{aligned} \mu_1 &\geq \dots \geq \mu_{M_1} \geq \nu_1 \geq \dots \geq \nu_{N_1} > \\ \mu_{M_1+1} &\geq \dots \geq \mu_{M_2} \geq \nu_{N_1+1} \geq \dots \geq \nu_{N_2} > \dots \\ \mu_{M_{r-1}+1} &\geq \dots \geq \mu_3 \geq \nu_{N_{r-1}+1} \geq \dots \geq \nu_8. \end{aligned}$$

Define $M_r = 3$, $N_r = 8$, and $M_0 = N_0 = 0$. We call the integer sequence $(M_i, N_i)_{i \in [r-1]}$ arising the *overlap type* of the pair (μ, ν) . Set

$$\hat{\cdot} : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad n \mapsto n \cdot (2, 3, 3) := 2\delta_{n \geq 1} + 3\delta_{n \geq 2} + 3\delta_{n \geq 3},$$

where

$$\delta_{n \geq i} = \begin{cases} 1 & n \geq i, \\ 0 & \text{otherwise.} \end{cases}$$

An overlap type determines and is determined by the *2D-word*

$$v = \mathbf{1}^{M_1} \mathbf{2}^{N_1} \mathbf{1}^{M_2-M_1} \mathbf{2}^{N_2-N_1} \dots \mathbf{1}^{3-M_{r-1}} \mathbf{2}^{8-N_{r-1}},$$

i.e. a word on the alphabet $\{\mathbf{1}, \mathbf{2}\}$ of length eleven such that $\mathbf{1}$ occurs three times and $\mathbf{2}$ eight times, and $\widehat{M}_i \geq N_i$ for $i \in [r-1]$.

Observe that the condition $\nu \leq \bar{\mu}$ is equivalent to the condition $\widehat{M}_i \geq N_i$ for $i \in [r-1]$. In this case we write $v(\mu, \nu) = v$. We write $\mathcal{D}^{(2)}$ for the set of 15 2D-words arising in this way.

Remark 3.3.2. Our notion of 2D-word is related to the classical one of (2-dimensional) *Dyck word of length $2n$* . The latter are words in $\{\mathbf{1}, \mathbf{2}\}$, featuring n occurrences of each letter. They may be used to model the overlap types of two partitions, each of at most n parts; cf. [29, Section 2.4]. 2D-words, in contrast, model the overlaps of two partitions ν and $\bar{\mu}$; whilst ν has genuinely at most 8 parts, $\bar{\mu}$ has at most 3 distinct parts, with ties in blocks of respective sizes 2, 3, and 3.

We chose to phrase our results and the supporting notation as similar to the relevant material in [29] as possible, leaving the reader free to concentrate on the crucial technical differences. Our Lemmata 3.3.4 and 3.3.5, for instance, are similar to, but subtly different from Lemmata 2.16 and 2.17 in [29]. Theorem 3.3.6 is analogous to [29, Theorem 3.1].

For $v \in \mathcal{D}^{(2)}$ we set

$$D_v(q, t) := \sum_{\substack{(\mu, \nu) \\ v(\mu, \nu) = v}} \alpha(\mu_1^{(3)}, \mu; q) \alpha(\bar{\mu}, \nu; q) q^{-(3\mu_1 + 2\mu_2 + \mu_1 + \sum_{k=1}^8 \nu_k)s}. \quad (3.3.3)$$

Proposition 3.3.1 allows us to write

$$\zeta_{\mathbf{f}_{3,3}(\mathbf{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathbf{o}^3}(s) \sum_{v \in \mathcal{D}^{(2)}} D_v(q, t). \quad (3.3.4)$$

This section's main result is Theorem 3.3.6, giving a general formula for the functions $D_v(q, t)$. Table 3.3.1 lists the 15 words in $\mathcal{D}^{(2)}$ together with their overlap types and an indication where in Section 3.3.2 an explicit formula for $D_v(q, t)$ may be found.

In order to obtain such formulae, it is useful for us to have notation for the successive differences of the parts of μ and ν . For $j \in [8]$ we set

$$r_j = \begin{cases} \nu_j - \nu_{j+1} & \text{if } j \notin \{N_1, \dots, N_r\}, \\ \nu_{N_i} - \mu_{M_i+1} & \text{if } j = N_i, \end{cases} \quad (3.3.5)$$

where we define $\mu_4 = 0$. Similarly, for $j \in [3]$ we set

$$s_j = \begin{cases} \mu_j - \mu_{j+1} & \text{if } j \notin \{M_1, \dots, M_r\}, \\ \mu_{M_i} - \nu_{N_{i-1}+1} & \text{if } j = M_i. \end{cases} \quad (3.3.6)$$

By our definition, $r_j, s_j \geq 0$ for all j . Note also that $r_j > 0$ if $j \in \{N_1, \dots, N_{r-1}\}$ and observe that $\mu_{M_i} > \mu_{M_i+1}$ and $\nu_{N_i} > \nu_{N_i+1}$ for each $i \in [r-1]$. Set

$$\sim : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad n \mapsto n \cdot (3, 2, 1) := 3\delta_{n \geq 1} + 2\delta_{n \geq 2} + 1\delta_{n \geq 3}.$$

Table 3.3.1: Overlap types and 2D-words

r	$(M_i, N_i)_{i \in [r-1]}$	$v \in \mathcal{D}^{(2)}$	Section
1	–	$\mathbf{1^3 2^8}$	3.3.2
2	(2, 1)	$\mathbf{1^2 2^1 2^7}$	3.3.2
2	(2, 2)	$\mathbf{1^2 2^2 1^2 2^6}$	
2	(2, 3)	$\mathbf{1^2 2^3 1^2 2^5}$	
2	(2, 4)	$\mathbf{1^2 2^4 1^2 2^4}$	
2	(2, 5)	$\mathbf{1^2 2^5 1^2 2^3}$	
2	(1, 1)	$\mathbf{1 2^1 2^7}$	3.3.2
3	(1, 1; 2, 2)	$\mathbf{1 2^1 2^1 2^6}$	
3	(1, 1; 2, 3)	$\mathbf{1 2^1 2^2 1^2 2^5}$	
3	(1, 1; 2, 4)	$\mathbf{1 2^1 2^3 1^2 2^4}$	
3	(1, 1; 2, 5)	$\mathbf{1 2^1 2^4 1^2 2^3}$	
2	(1, 2)	$\mathbf{1 2^2 1^2 2^6}$	3.3.2
3	(1, 2; 2, 3)	$\mathbf{1 2^2 1^2 2^1 2^5}$	
3	(1, 2; 2, 4)	$\mathbf{1 2^2 1^2 2^2 1^2 2^4}$	
3	(1, 2; 2, 5)	$\mathbf{1 2^2 1^2 2^3 1^2 2^3}$	

Then we have

$$\begin{aligned}
 3\mu_1 + 2\mu_2 + \mu_3 &= \sum_{j=1}^3 \check{j} s_j + \sum_{i=1}^r \check{M}_i (r_{N_{i-1}+1} + \cdots + r_{N_i}), \\
 \nu_1 + \cdots + \nu_8 &= \sum_{j=1}^8 j r_j + \sum_{i=1}^{r-1} N_i (s_{M_i+1} + \cdots + s_{M_{i+1}}).
 \end{aligned}$$

Finally, for each $i \in [r]$ we define

$$J_i^\mu = \{j \in [M_i - M_{i-1} - 1] \mid \mu_{M_i-j} > \mu_{M_i-j+1}\}, \quad (3.3.7)$$

$$J_i^\nu = \{j \in [N_i - N_{i-1} - 1] \mid \nu_{N_i-j} > \nu_{N_i-j+1}\}. \quad (3.3.8)$$

We start with an immediate consequence of Proposition 3.2.2.

Lemma 3.3.3. *Let μ and ν be partitions of 3 and 8 parts, respectively, satisfying $\nu \leq \bar{\mu}$. Then*

$$\sum_{\mu} \alpha(\mu_1^{(3)}, \mu; q) \alpha(\bar{\mu}, \nu; q) = \prod_{k \geq 1} q^{\mu'_k(3-\mu'_k) + \nu'_k(\widehat{\mu'_k} - \nu'_k)} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}}.$$

Let μ and ν be partitions as in Lemma 3.3.3. Let $(M_i, N_i)_{i \in [r-1]}$ be their overlap type and μ' resp. ν' their dual partitions. For $k \in \mathbb{N}$, set

$$d_k := \mu'_k(3 - \mu'_k) + \nu'_k(\widehat{\mu'_k} - \nu'_k).$$

Also, set

$$\begin{aligned} a_j &:= M_i(3 - M_i) + j \left(\widehat{M}_i - j \right), & j \in]N_{i-1}, N_i], \\ b_j &:= j(3 - j) + N_{i-1} \left(\widehat{j} - N_{i-1} \right), & j \in]M_{i-1}, M_i]. \end{aligned}$$

Note that j determines a unique i .

Lemma 3.3.4. *For $i \in [r]$,*

$$\begin{aligned} \prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} q^{d_k} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}}. \end{aligned}$$

Proof. Since all the indices k appearing in the product on the left hand side satisfy $\nu_{N_{i-1}} > \mu_{M_{i-1}+1} \geq k \geq \mu_{M_i} \geq \nu_{N_{i-1}+1}$, we have $\nu'_k = \nu'_{k+1} = N_{i-1}$ for all k in the interval $[\mu_{M_i}, \mu_{M_{i-1}+1}]$. Thus

$$\prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} = 1.$$

For $j \in [M_i - M_{i-1}]$ we have $\mu'_k = M_{i-1} + j$ when $\mu_{M_{i-1}+j+1} < k \leq \mu_{M_{i-1}+j}$; observe that it may be the case for some j that no index k satisfies this condition. As a result, we see that for each $j \in [M_i - M_{i-1}]$, there are exactly $s_{M_{i-1}+j}$ elements k of the segment $[\mu_{M_i}, \mu_{M_{i-1}+1}]$ for which $\mu'_k = M_{i-1} + j$. Now, note that

$$\binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \neq 1$$

if and only if $\mu'_{k+1} \neq \mu'_k$. This is the case exactly when there exists an i such that $\mu_i = k$. It follows that if $J_i^\mu = \{j_{i,1}, \dots, j_{i,\gamma_i}\}$, with $j_{i,1} < \dots < j_{i,\gamma_i}$, and write $j_{i,0} = 0, j_{i,\gamma_i+1} = M_i - M_{i-1}$, then

$$\prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}+1}} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} = \prod_{m=0}^{\gamma_i} \binom{3 - M_i + j_{i,m+1}}{3 - M_i + j_{i,m}}_{q^{-1}}. \quad (3.3.9)$$

We make use of the well-known identity

$$\binom{\alpha}{\beta}_X = \frac{1 - X^\alpha}{1 - X^{a-\beta}} \binom{\alpha - 1}{\beta}_X$$

for Gaussian binomial coefficients. Applying it inductively, we see that for all $m \in [\gamma_i - 1]$,

$$\binom{3 - M_i + j_{i,m+1}}{3 - M_i + j_{i,m}}_{q^{-1}} = \binom{j_{i,m+1}}{j_{i,m}}_{q^{-1}} \frac{\binom{3 - M_i + j_{i,m+1}}{3 - M_i}_{q^{-1}}}{\binom{3 - M_i + j_{i,m}}{3 - M_i}_{q^{-1}}}.$$

Hence the right-hand side of (3.3.9) is equal to

$$\binom{M_i - M_{i-1}}{J_i^\lambda}_{q^{-1}} \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}}.$$

Thus

$$\begin{aligned} \prod_{k=\mu_{M_i}}^{\mu_{M_{i-1}}+1} q^{d_k} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}} \end{aligned}$$

as required. $\square$

Lemma 3.3.5. *For $i \in [r]$,*

$$\begin{aligned} \prod_{k=\mu_{M_i+1}+1}^{\mu_{M_i}-1} q^{d_k} \binom{3 - \mu'_{k+1}}{3 - \mu'_k}_{q^{-1}} \binom{\widehat{\mu'_k} - \nu'_{k+1}}{\widehat{\mu'_k} - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{N_i - N_{i-1}} q^{a_{N_{i-1}+j} r_{N_{i-1}+j}} \binom{N_i - N_{i-1}}{J_i^\nu}_{q^{-1}} \binom{\widehat{M_i} - N_{i-1}}{\widehat{M_i} - N_i}_{q^{-1}}. \end{aligned}$$

Proof. Note that the product on the left hand side may be empty; this happens in the case $\mu_{M_i} = \nu_{N_{i-1}+1} = \dots = \nu_{N_i} = \mu_{M_{i+1}} + 1$. Since $\mu_{M_i} > k > \mu_{M_{i+1}}$, we have $\mu'_k = \mu'_{k+1} = M_i$ for all k in the interval $]\mu_{M_{i+1}}, \mu_{M_{i-1}}]$. Finally, observe that for $j \in [N_i - N_{i-1}]$ we have $\nu'_k = N_{i-1} + j$ when $\nu_{N_{i-1}+j+1} < k \leq \nu_{N_{i-1}+j}$. The claim follows as in the proof of the previous lemma. $\square$

With these preparations in place we now give a formula for the functions $D_v(q, t)$.

Theorem 3.3.6. *Let $v = \prod_{i=1}^r (\mathbf{1}^{M_i - M_{i-1}} \mathbf{2}^{N_i - N_{i-1}}) \in \mathcal{D}^{(2)}$. Then*

$$\begin{aligned} D_v(q, t) = \prod_{i=1}^r \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}} \binom{\widehat{M_i} - N_{i-1}}{\widehat{M_i} - N_i}_{q^{-1}} \prod_{i=1}^r I_{M_i - M_{i-1}}(Y_{M_{i-1}+1}, \dots, Y_{M_i}) \\ \cdot \left(\prod_{i=1}^{r-1} I_{N_i - N_{i-1}}^\circ(X_{N_{i-1}+1}, \dots, X_{N_i}) \right) \cdot I_{8 - N_{r-1}}(X_{N_{r-1}+1}, \dots, X_8), \end{aligned}$$

with numerical data

$$X_j = q^{a_j} t^{\check{M}_i + j} \quad j \in]N_{i-1}, N_i], \quad (3.3.10)$$

$$Y_j = q^{b_j} t^{\check{j} + N_{i-1}} \quad j \in]M_{i-1}, M_i]. \quad (3.3.11)$$

Proof. Given $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{N}_0^3$ and $\mathbf{v}' = (v'_1, \dots, v'_8) \in \mathbb{N}_0^8$, we set, for each $i \in [r]$,

$$\begin{aligned} \text{supp}_i^M(\mathbf{v}) &= \{j \in [M_i - M_{i-1} - 1] \mid v_{M_{i-1}+j} > 0\}, \\ \text{supp}_i^N(\mathbf{v}') &= \{j \in [N_i - N_{i-1} - 1] \mid v'_{N_{i-1}+j} > 0\}. \end{aligned}$$

In practice, $\mathbf{v}$ will be one of the vectors of successive differences $\mathbf{s} = (s_1, s_2, s_3)$ (cf. (3.3.6)) and $\mathbf{v}'$ will be one of the vectors of successive differences $\mathbf{r} = (r_1, \dots, r_8)$ (cf. (3.3.5)). Given a set of partitions satisfying $\nu \leq \bar{\mu}$, recall our definitions (3.3.7) and (3.3.8). It is easy to see that, for every $i \in [r]$,

$$\text{supp}_i^M(\mathbf{v}) = M_i - M_{i-1} - J_i^\mu \quad \text{and} \quad \text{supp}_i^N(\mathbf{v}') = N_i - N_{i-1} - J_i^\nu.$$

Thus it follows that

$$\binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} = \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}} \quad \text{and} \quad \binom{N_i - N_{i-1}}{J_i^\nu}_{q^{-1}} = \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v}')}_{q^{-1}}.$$

Let δ_{ij} be the usual Kronecker delta function. Substituting Lemma 3.3.4 and 3.3.5, rewriting the expressions in terms of r_j and s_j and using (3.3.1), we find that formula (3.3.3) for $D_v(q, t)$ splits into a product as follows:

$$D_v(q, t) = \prod_{i=1}^r \binom{3 - M_{i-1}}{3 - M_i}_{q^{-1}} \binom{\widehat{M}_i - N_{i-1}}{\widehat{M}_i - N_i}_{q^{-1}} \prod_{i=1}^r A_i B_i,$$

where, for $i \in [r]$,

$$\begin{aligned} A_i &= \sum_{r_{N_{i-1}+1}=0}^{\infty} \cdots \sum_{r_{N_{i-1}=0}}^{\infty} \sum_{r_{N_i=1-\delta_{ir}}}^{\infty} \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v}')}_{q^{-1}} \prod_{j=N_{i-1}+1}^{N_i} \left(q^{a_j} t^{\check{M}_i + j} \right)^{r_j}, \\ B_i &= \sum_{s_{M_{i-1}+1}=0}^{\infty} \cdots \sum_{s_{M_i-1}=0}^{\infty} \sum_{s_{M_i}=0}^{\infty} \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}} \prod_{j=M_{i-1}+1}^{M_i} \left(q^{b_j} t^{N_{i-1} + \check{j}} \right)^{s_j}. \end{aligned}$$

We now show that all of the factors A_i and B_i are products of Igusa functions and Gaussian binomial coefficients. Given $i \in [r]$ and $I \subseteq [M_i - M_{i-1} - 1]$, we define $\mathbf{S}^i(I)$ to be the set of vectors $\mathbf{s}^i = (s_{M_{i-1}+1}, \dots, s_{M_i}) \in \mathbb{N}_0^{M_i - M_{i-1}}$ such that $s_j = 0$

unless $j \in \{M_{i-1} + k \mid k \in I\} \cup \{M_i\}$. With the numerical data defined in (3.3.11), we have

$$\begin{aligned}
B_i &= \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I}_{q^{-1}} \sum_{\mathbf{s}^i \in \mathbf{S}^i(I)} \prod_{j \in (I + M_{i-1}) \cup \{M_i\}} \left(q^{b_j} t^{N_{i-1} + j} \right)^{s_j} \\
&= \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I}_{q^{-1}} \left(\prod_{\iota \in I} \left(\sum_{s_{M_{i-1} + \iota} = 1}^{\infty} (Y_{M_{i-1} + \iota})^{s_{M_{i-1} + \iota}} \right) \right) \sum_{s_{M_i} = 0}^{\infty} (Y_{M_i})^{s_{M_i}} \\
&= \frac{1}{1 - Y_{M_i}} \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I}_{q^{-1}} \prod_{\iota \in I} \frac{Y_{M_{i-1} + \iota}}{1 - Y_{M_{i-1} + \iota}} \\
&= I_{M_i - M_{i-1}}(Y_{M_{i-1} + 1}, \dots, Y_{M_i}).
\end{aligned}$$

Analogously one shows that, with the numerical data defined in (3.3.10),

$$A_i = \begin{cases} I_{N_i - N_{i-1}}^{\circ}(X_{N_{i-1} + 1}, \dots, X_{N_i}) & \text{for } i < r, \\ I_{8 - N_{r-1}}(X_{N_{r-1} + 1}, \dots, X_8) & \text{for } i = r. \end{cases}$$

This completes the proof of Theorem 3.3.6. $\square$

3.3.2 Explicit formulae for the functions D_v

$v = \mathbf{1}^3 \mathbf{2}^8$:

$$D_{\mathbf{1}^3 \mathbf{2}^8}(q, t) = I_3(q^2 t^3, q^2 t^5, t^6) I_8((q^{i(8-i)} t^{6+i})_{i=1, \dots, 8}).$$

Remark 3.3.7. Note that $\zeta_{\mathbf{f}_{2,3}(\mathbf{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathbf{o}^3}(s) I_3(q^2 t^3, q^2 t^5, t^6)$; cf. Proposition 3.4.10.

$v = \mathbf{1}^2 \mathbf{2}^j \mathbf{12}^{8-j}$, $j \in \{1, \dots, 5\}$:

$$\begin{aligned}
D_{\mathbf{1}^2 \mathbf{2}^j \mathbf{12}^{8-j}}(q, t) &= \binom{5}{j}_{q^{-1}} \binom{3}{1}_{q^{-1}} I_2(q^2 t^3, q^2 t^5) \\
&\quad \cdot I_j^{\circ}((q^{2+i(5-i)} t^{5+i})_{i=1, \dots, j}) I_1(q^{j(8-j)} t^{6+j}) I_{8-j}((q^{i(8-i)} t^{6+i})_{i=j+1, \dots, 8}).
\end{aligned}$$

$v = \mathbf{1212}^{j-1} \mathbf{12}^{8-j}$, $j \in \{1, \dots, 5\}$:

For $j = 1$,

$$D_{\mathbf{1212}^7}(q, t) = \binom{3}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(q^2 t^3) I_1^{\circ}(q^3 t^4) I_2(q^6 t^6, q^7 t^7) I_7((q^{i(8-i)} t^{6+i})_{i=2, \dots, 8}).$$

For $j \in \{2, \dots, 5\}$,

$$\begin{aligned}
D_{\mathbf{1212}^{j-1} \mathbf{12}^{8-j}}(q, t) &= \binom{4}{j-1}_{q^{-1}} \binom{3}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}}^2 I_1(q^2 t^3) I_1^{\circ}(q^3 t^4) I_1(q^6 t^6) \\
&\quad \cdot I_{j-1}^{\circ}((q^{2+i(5-i)} t^{5+i})_{i=2, \dots, j}) I_1(q^{j(8-j)} t^{6+j}) I_{8-j}((q^{i(8-i)} t^{6+i})_{i=j+1, \dots, 8}).
\end{aligned}$$

$v = \mathbf{12}^2\mathbf{12}^{j-1}\mathbf{12}^{7-j}$, $j \in \{1, \dots, 4\}$:

For $j = 1$,

$$D_{\mathbf{12}^2\mathbf{12}^6}(q, t) = \binom{3}{1}_{q^{-1}} I_1(q^2t^3)I_2^\circ(q^3t^4, q^2t^5)I_2(q^8t^7, q^{12}t^8)I_6((q^{i(8-i)}t^{6+i})_{i=3,\dots,8}).$$

For $j \in \{2, 3, 4\}$,

$$D_{\mathbf{12}^2\mathbf{12}^{j-1}\mathbf{12}^{7-j}}(q, t) = \binom{3}{j-1}_{q^{-1}} \binom{3}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(q^2t^3)I_2^\circ(q^3t^4, q^2t^5)I_1(q^8t^7) \\ \cdot I_{j-1}^\circ((q^{2+i(5-i)}t^{5+i})_{i=3,\dots,j+1})I_1(q^{(j+1)(8-(j+1))}t^{7+j})I_{8-(j+1)}((q^{i(8-i)}t^{6+i})_{i=j+2,\dots,8}).$$

3.3.3 Completion of the proof

The first two claims of Theorem 3.1.1 follow from (3.3.4) and the explicit formulae for D_v , $v \in \mathcal{D}^{(2)}$ given in Section 3.3.2. To deduce the local functional equation (3.1.5), one checks, repeatedly using the functional equations (3.2.3) for the Igusa functions I_h resp. I_h° and the well-known fact that, for $a, b \in \mathbb{N}_0$, $a \geq b$, $\binom{a}{b}_q = q^{b(a-b)}\binom{a}{b}_{q^{-1}}$, that each of the functions D_v satisfies

$$D_v(q^{-1}t^{-1}) = -q^{31}t^{20}D_v(q, t).$$

The functional equation (3.1.5) follows from (3.3.4) as

$$\zeta_{\mathfrak{o}^3}(s)|_{q \rightarrow q^{-1}} = -q^3t^3\zeta_{\mathfrak{o}^3}(s).$$

3.4 Nilpotency class two ($c = 2$)

Let $d \in \mathbb{N}_{\geq 2}$. In this section we compute the local graded ideal zeta functions $\zeta_{\mathfrak{f}_{d,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$. We use them to determine the abscissae of convergence of the global graded ideal zeta functions $\zeta_{\mathfrak{f}_{d,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ and some properties of the associated topological and reduced graded ideal zeta functions. Throughout we write d' for $\binom{d}{2} = W_2(d)$; cf. (3.1.2).

3.4.1 p-Adic formulae

The paper [39] gives the *normal subgroup zeta functions* of the free class-2-nilpotent d -generator groups $F_{2,d}$, enumerating these groups' normal subgroups of finite index. By the Mal'cev correspondence, these are the ideal zeta functions of the free nilpotent

$\mathbb{Z}$ -Lie rings $\mathfrak{f}_{2,d}$. The computations generalize to the case of general number rings in a straightforward manner. To recall the paper's main result define the function

$$\varphi : [d-1]_0 \rightarrow [d'], \quad i \mapsto id - \binom{i+1}{2}.$$

Given $(I, J) \in 2^{[d-1]_0} \times 2^{[d']}$, the paper [39] defines a total order $<_{\varphi(I),J}$ on the disjoint union $I \cup J$. Without loss of generality, we may assume that the sets I and J have the same cardinality h . For $i \in I = \{i_1, \dots, i_h\}_{<}$ and $j \in J = \{j_1, \dots, j_h\}_{<}$, set

$$\begin{aligned} j(i) &:= \min \{j \in J \cup \{d'\} \mid \varphi(i) <_{\varphi(I),J} j\}, \\ i(j) &:= \max \{i \in I \cup \{0\} \mid \varphi(i) <_{\varphi(I),J} j\}, \end{aligned}$$

and $j_0 := i_0 := 0 <_{\varphi(I),J} \varphi([d-2]) \uplus [d'-1] <_{\varphi(I),J} j_{h+1} := d'$. The following is essentially the main result of [39], in notation compatible with the current paper. (The underlining of terms of the form $d+$ in (3.4.3) is meant to facilitate comparison with the “graded” numerical data (3.4.5) in Theorem 3.4.2, and may be ignored.)

Theorem 3.4.1. [39, Theorem 4] *For all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s) = \zeta_{\mathfrak{o}^d}(s) \sum_{\substack{I \subseteq [d-2], J \subseteq [d'-1] \\ |I|=|J|, \varphi(I) \leq J}} \mathcal{A}_{I,J}^{\triangleleft}(q, t), \quad (3.4.1)$$

where

$$\begin{aligned} \mathcal{A}_{I,J}^{\triangleleft}(q, t) &= \zeta_{\mathfrak{o}^{i_h}}(s)^{-1} I_{j_1}((X_\alpha)_{\alpha \in [j_1-1]}, X_0) \cdot \prod_{r=1}^h \binom{j_{r+1} - \varphi(i_r)}{j_r - \varphi(i_r)}_{q^{-1}} \binom{d - i_{r-1}}{d - i_r}_{q^{-1}} \\ &\quad \cdot I_{i_r - i_{r-1}}^{\circ}((Y_\beta)_{\beta \in [i_r-1, i_r]}, Y'_{i_r}) I_{j_{r+1} - j_r}((X_\alpha)_{\alpha \in [j_r, j_{r+1}-1]}, X_{j_r}), \end{aligned} \quad (3.4.2)$$

with numerical data

$$\begin{aligned} X_j &= q^{i(j)(d-i(j))+(d'-j)(\underline{d+j}-\varphi(i(j)))} t^{d-i(j)+d'-j}, & j \in [d'-1]_0, \\ Y_i &= q^{i(d-i)+(d'-j(i))(\underline{d+j(i)}-\varphi(i))} t^{d-i+d'-j(i)}, & i \in [d-2], \\ Y'_{i_r} &= q^{i_{r-1}(d-i_{r-1})+(d'-j(i_r))(\underline{d+j(i_r)}-\varphi(i_{r-1}))} t^{d-i_{r-1}+d'-j(i_r)}, & r \in [h]. \end{aligned} \quad (3.4.3)$$

The following is a graded analogue of Theorem 3.4.1.

Theorem 3.4.2. *For all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^d}(s) \sum_{\substack{I \subseteq [d-2], J \subseteq [d'-1] \\ |I|=|J|, \varphi(I) \leq J}} \mathcal{A}_{I,J}^{\triangleleft_{\text{gr}}}(q, t), \quad (3.4.4)$$

where $\mathcal{A}_{I,J}^{\triangleleft \text{gr}}(q, t)$ is defined as $\mathcal{A}_{I,J}^{\triangleleft}(q, t)$ in (3.4.2), but with numerical data

$$\begin{aligned} X_j &= q^{i(j)(d-i(j))+(d'-j)(j-\varphi(i(j)))} t^{d-i(j)+d'-j}, & j \in [d'-1]_0, \\ Y_i &= q^{i(d-i)+(d'-j(i))(j(i)-\varphi(i))} t^{d-i+d'-j(i)}, & i \in [d-2], \\ Y'_{i_r} &= q^{i_{r-1}(d-i_{r-1})+(d'-j(i_r))(j(i_r)-\varphi(i_{r-1}))} t^{d-i_{r-1}+d'-j(i_r)}, & r \in [h]. \end{aligned} \quad (3.4.5)$$

Proof. The difference between (3.4.3) and (3.4.5) reflects the difference between (3.1.7) and (3.1.8). $\square$

Corollary 3.4.3.

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{d+d'} q^{\binom{d+d'}{2}} t^{2d+d'} \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s), \quad (3.4.6)$$

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{d+d'} q^{\binom{d}{2} + \binom{d'}{2}} t^{2d+d'} \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s). \quad (3.4.7)$$

Proof. Eq. (3.4.6) is [39, Theorem 3(a)]. The proof proceeds by establishing the symmetry in question for each of the summands in (3.4.1), using (3.2.3). The formulae (3.4.1) and (3.4.4) only differ in their numerical data. By Remark 3.2.4, it suffices to control the value of $X_0 \cdot \prod_{r=1}^h \frac{X_{j_r}}{Y'_{i_r}}$ in both cases. Inspection of (3.4.3) and (3.4.5), respectively, reveals that in the ideal case this term is by a factor $q^{dd'}$ larger than in the graded ideal case. Thus

$$\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{d+d'} q^{\binom{d+d'}{2} - dd'} t^{2d+d'} \zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

As $\binom{d+d'}{2} - dd' = \binom{d}{2} + \binom{d'}{2}$, this establishes (3.4.7). $\square$

3.4.2 p-Adic behaviour at zero

We describe the behaviours of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$ and $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ at $s = 0$.

Theorem 3.4.4. *For all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)}{\zeta_{\mathfrak{o}^{d+d'}}(s)} \Big|_{s=0} = 1, \quad (3.4.8)$$

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^d}(s) \zeta_{\mathfrak{o}^{d'}}(s)} \Big|_{s=0} = \frac{d \cdot d'}{d \cdot (d + d')} = \frac{d-1}{d+1}. \quad (3.4.9)$$

Proof. To prove (3.4.8) we note that, by (3.4.1),

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)}{\zeta_{\mathfrak{o}^{d+d'}}(s)} = \prod_{j=d}^{d+d'-1} (1 - q^j t) \left(\mathcal{A}_{\emptyset, \emptyset}^{\triangleleft}(q, t) + \sum_{(I,J) \neq (\emptyset, \emptyset)} \mathcal{A}_{I,J}^{\triangleleft}(q, t) \right).$$

By inspection of (3.4.3) we see that $\mathcal{A}_{\emptyset, \emptyset}^{\triangleleft}(q, t) = I_{d'}((q^{(d+(d'-j))j}t^{d+j})_{j \in [d']})$ and that

$$\mathcal{A}_{I, J}^{\triangleleft}(q, 1) = \begin{cases} 0 & \text{if } (I, J) \neq (\emptyset, \emptyset), \\ I_{d'}((q^{(d+(d'-j))j})_{j \in [d']}) = \prod_{j=d}^{d+d'-1} (1 - q^j) & \text{if } (I, J) = (\emptyset, \emptyset). \end{cases}$$

To prove (3.4.9) we note that

$$\frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^d}(s)\zeta_{\mathfrak{o}^{d'}}(s)} = \prod_{j=0}^{d'-1} (1 - q^j t) \left(\mathcal{A}_{\emptyset, \emptyset}^{\triangleleft \text{gr}}(q, t) + \sum_{(I, J) \neq (\emptyset, \emptyset)} \mathcal{A}_{I, J}^{\triangleleft \text{gr}}(q, t) \right).$$

By inspection of (3.4.5) we see that $\mathcal{A}_{\emptyset, \emptyset}^{\triangleleft \text{gr}}(q, t) = I_{d'}((q^{(d'-j)j}t^{d+j})_{j \in [d']})$, which has a simple pole at $t = 1$. In contrast, $\mathcal{A}_{I, J}^{\triangleleft \text{gr}}(q, t)$ has no pole at $t = 1$ when $(I, J) \neq (\emptyset, \emptyset)$. Hence

$$\left. \frac{\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^d}(s)\zeta_{\mathfrak{o}^{d'}}(s)} \right|_{s=0} = \left. \frac{I_{d'}((q^{(d'-j)j}t^{d+j})_{j \in [d']})}{I_{d'}((q^{(d'-j)j}t^j)_{j \in [d']})} \right|_{t=1} = \left. \frac{1 - t^{d'}}{1 - t^{d+d'}} \right|_{t=1} = \frac{d'}{d + d'}. \quad \square$$

Remark 3.4.5. In the pertinent special cases, (3.4.8) confirms [27, Conjecture IV ($\mathfrak{P}$ -adic form)].

3.4.3 Global analytic properties

The following result compares some of the known analytic properties of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$ (cf. [39, Theorem 3]) with those of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$.

Theorem 3.4.6. *The abscissae of convergence of $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$ resp. $\zeta_{\mathfrak{f}_{2,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ are*

$$\alpha^{\triangleleft}(\mathfrak{f}_{2,d}) = \max \left\{ d, \frac{((\binom{d}{2}) - j)(d + j) + 1}{\binom{d+1}{2} - j} \mid j \in [(\binom{d}{2}) - 1] \right\} \text{ resp. ,} \quad (3.4.10)$$

$$\alpha^{\triangleleft \text{gr}}(\mathfrak{f}_{2,d}) = \max \left\{ d, \frac{((\binom{d}{2}) - j)j + 1}{\binom{d+1}{2} - j} \mid j \in [(\binom{d}{2}) - 1] \right\}. \quad (3.4.11)$$

Both zeta functions allow for some meromorphic continuation beyond their respective abscissa of convergence and have a simple pole at $s = \alpha^{\triangleleft}(\mathfrak{f}_{2,d})$ resp. $s = \alpha^{\triangleleft \text{gr}}(\mathfrak{f}_{2,d})$.

Proof. Eq. (3.4.10) and the ensuing claim about the meromorphic continuation is essentially [39, Theorem 3]. Eq. (3.4.11) and the analogous claim are proved analogously. $\square$

3.4.4 Topological zeta functions – degree and behaviour at zero

Theorem 3.4.7.

$$\deg_s \left(\zeta_{f_{2,d,\text{top}}}^{\triangleleft}(s) \right) = -(d + d') = \deg_s \left(\zeta_{f_{2,d,\text{top}}}^{\triangleleft_{\text{gr}}}(s) \right).$$

Moreover,

$$s \zeta_{f_{2,d,\text{top}}}^{\triangleleft}(s) \Big|_{s=0} = \frac{(-1)^{d+d'-1}}{(d + d' - 1)!}, \quad (3.4.12)$$

$$s^2 \zeta_{f_{2,d,\text{top}}}^{\triangleleft_{\text{gr}}}(s) \Big|_{s=0} = \frac{(-1)^{(d-1)+(d'-1)} d'}{(d + d') (d - 1)! (d' - 1)!}. \quad (3.4.13)$$

Proof. Note that, given $h \in \mathbb{N}$ and, for $i \in [h]$, $X_i = q^{a_i} t^{b_i}$ for $a_i \in \mathbb{N}_0$, $b_i \in \mathbb{N}$,

$$I_h(X_1, \dots, X_h) = \frac{h!}{\prod_{i=1}^h (b_i s - a_i)} (q - 1)^{-h} + O((q - 1)^{-h+1}).$$

The summands in the formula (3.4.1) for $\zeta_{f_{2,d}(\mathfrak{o})}^{\triangleleft}(s)$ are all products of Igusa functions and Gaussian binomial coefficients. Hence there exist $a_{I,J,i}^{\triangleleft} \in \mathbb{N}_0$ and $b_{I,J,i}^{\triangleleft}, c_{I,J}^{\triangleleft} \in \mathbb{N}$ such that

$$\zeta_{f_{2,d}(\mathfrak{o})}^{\triangleleft}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft} s - a_{I,J,i}^{\triangleleft})} (q - 1)^{-(d+d')} + O((q - 1)^{-(d+d'+1)}).$$

(Here and in the sequel, the sums are over pairs (I, J) as in (3.4.1).) Hence

$$\zeta_{f_{2,d,\text{top}}}^{\triangleleft}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft} s - a_{I,J,i}^{\triangleleft})} \quad (3.4.14)$$

is a rational function in s of degree $-(d + d')$, confirming the first claim.

The second claim, on the degree of $\zeta_{f_{2,d,\text{top}}}^{\triangleleft_{\text{gr}}}(s)$, follows from analogous considerations based on (3.4.4). Indeed, there exist $a_{I,J,i}^{\triangleleft_{\text{gr}}} \in \mathbb{N}_0$ and $b_{I,J,i}^{\triangleleft_{\text{gr}}}, c_{I,J}^{\triangleleft_{\text{gr}}} \in \mathbb{N}$ such that

$$\zeta_{f_{2,d}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft_{\text{gr}}}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft_{\text{gr}}} s - a_{I,J,i}^{\triangleleft_{\text{gr}}})} (q - 1)^{-(d+d')} + O((q - 1)^{-(d+d'+1)}).$$

Hence

$$\zeta_{f_{2,d,\text{top}}}^{\triangleleft_{\text{gr}}}(s) = \sum_{I,J} \frac{c_{I,J}^{\triangleleft_{\text{gr}}}}{\prod_{i=1}^{d+d'} (b_{I,J,i}^{\triangleleft_{\text{gr}}} s - a_{I,J,i}^{\triangleleft_{\text{gr}}})} \quad (3.4.15)$$

is a rational function in s of degree $-(d + d')$, too.

Turning to the local behaviour around $s = 0$, we start with the observation that $\zeta_{\mathfrak{o}^a}(s) \mathcal{A}_{I,J}^{\triangleleft}(q, t)$ has no pole at $t = 1$ unless $(I, J) = (\emptyset, \emptyset)$, in which case it is simple.

Thus the summand $\frac{c_{I,J}^{\triangleleft}}{\prod_{i=1}^{d+d'}(b_{I,J,i}^{\triangleleft}s - a_{I,J,i}^{\triangleleft})}$ in (3.4.14) has no pole at $s = 0$ unless $(I, J) = (\emptyset, \emptyset)$, in which case it is simple. Specifically,

$$\zeta_{\mathfrak{o}^d}(s)\mathcal{A}_{\emptyset,\emptyset}^{\triangleleft}(q,t) = I_d((q^{i(d-i)}t^i)_{i \in [d]})I_{d'}((q^{(d+d'-j)j}t^{d+j})_{j \in [d']}),$$

whence

$$\frac{c_{\emptyset,\emptyset}^{\triangleleft}}{\prod_{i=1}^{d+d'}(b_{\emptyset,\emptyset,i}^{\triangleleft}s - a_{\emptyset,\emptyset,i}^{\triangleleft})} = \frac{1}{\prod_{i=0}^{d-1}(s-i)} \cdot \frac{(d')!}{\prod_{j=1}^{d'}((d+j)s - (d+d'-j)j)}$$

and thus

$$s\zeta_{\mathfrak{f}_{2,d},\text{top}}^{\triangleleft}(s)\Big|_{s=0} = \frac{(d')!}{\prod_{i=1}^{d-1}(-i) \prod_{j=1}^{d'}(-(d+d'-j)j)} = \frac{(-1)^{d+d'-1}}{(d+d'-1)!},$$

which establishes (3.4.12).

The proof of (3.4.13) goes along similar lines. We observe that $\zeta_{\mathfrak{o}^d}(s)\mathcal{A}_{I,J}^{\triangleleft_{\text{gr}}}(q,t)$ has a simple pole at $t = 1$ unless $(I, J) = (\emptyset, \emptyset)$, in which case it is double. Thus the summand $\frac{c_{I,J}^{\triangleleft_{\text{gr}}}}{\prod_{i=1}^{d+d'}(b_{I,J,i}^{\triangleleft_{\text{gr}}}s - a_{I,J,i}^{\triangleleft_{\text{gr}}})}$ in (3.4.15) has a simple pole at $s = 0$ unless $(I, J) = (\emptyset, \emptyset)$, in which case it is double. Specifically,

$$\zeta_{\mathfrak{o}^d}(s)\mathcal{A}_{\emptyset,\emptyset}^{\triangleleft_{\text{gr}}}(q,t) = I_d((q^{i(d-i)}t^i)_{i \in [d]})I_{d'}((q^{(d'-j)j}t^{d+j})_{j \in [d']}),$$

whence

$$\frac{c_{\emptyset,\emptyset}^{\triangleleft_{\text{gr}}}}{\prod_{i=1}^{d+d'}(b_{\emptyset,\emptyset,i}^{\triangleleft_{\text{gr}}}s - a_{\emptyset,\emptyset,i}^{\triangleleft_{\text{gr}}})} = \frac{1}{\prod_{i=0}^{d-1}(s-i)} \cdot \frac{(d')!}{\prod_{j=1}^{d'}((d+j)s - (d'-j)j)}$$

and thus

$$s^2\zeta_{\mathfrak{f}_{2,d},\text{top}}^{\triangleleft_{\text{gr}}}(s)\Big|_{s=0} = \frac{(d')!}{(d+d') \prod_{i=1}^{d-1}(-i) \prod_{j=1}^{d'-1}(-(d'-j)j)} = \frac{(-1)^{(d-1)+(d'-1)}d'}{(d+d')(d-1)!(d'-1)!},$$

which proves (3.4.13). $\square$

Remark 3.4.8. In the pertinent special cases, the theorem's first statement confirms [27, Conjecture 1], whereas (3.4.12) confirms [27, Conjecture IV (topological form)].

3.4.5 Explicit examples

We record the following consequences of Theorem 3.4.2.

Proposition 3.4.9 $((c, d) = (2, 2) - \text{Heisenberg})$.

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s)I_1(t^3) =: W_{2,2}^{\triangleleft \text{gr}}(q, t), \\ \zeta_{\mathfrak{f}_{2,2}, \text{red}}^{\triangleleft \text{gr}}(Y) &= W_{2,2}^{\triangleleft \text{gr}}(1, Y) = \frac{1}{(1-Y)^2(1-Y^3)}, \\ \zeta_{\mathfrak{f}_{2,2}, \text{top}}^{\triangleleft \text{gr}}(s) &= \frac{1}{3s^2(s-1)}.\end{aligned}$$

The global graded ideal zeta function $\zeta_{\mathfrak{f}_{2,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_K(s)\zeta_K(s-1)\zeta_K(3s)$ has abscissa of convergence 2 and meromorphic continuation to the whole complex plane.

Proposition 3.4.10 $((c, d) = (2, 3))$.

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^3}(s)I_3(q^2t^3, q^2t^5, t^6) =: W_{2,3}^{\triangleleft \text{gr}}(q, t), \\ \zeta_{\mathfrak{f}_{2,3}, \text{red}}^{\triangleleft \text{gr}}(Y) &= W_{2,3}^{\triangleleft \text{gr}}(1, Y) = \frac{1 + 2Y^3 + 2Y^5 + Y^8}{(1-Y)^3(1-Y^3)(1-Y^4)(1-Y^5)(1-Y^6)}, \\ \zeta_{\mathfrak{f}_{2,3}, \text{top}}^{\triangleleft \text{gr}}(s) &= \frac{1}{s^2(s-1)(s-2)(3s-2)(5s-2)}.\end{aligned}$$

The global graded ideal zeta function $\zeta_{\mathfrak{f}_{2,3}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ has abscissa of convergence 3 and may be continued meromorphically to $\{s \in \mathbb{C} \mid \text{Re}(s) > 1/3\}$.

We omit the (largish) formula for $\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$, but do note the following consequences.

Proposition 3.4.11 $((c, d) = (2, 4))$. *There exists a rational function $W_{2,4}(X, Y) \in \mathbb{Q}[X, Y]$ such that*

$$\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{2,4}^{\triangleleft \text{gr}}(q, q^{-s}).$$

Setting

$$\begin{aligned}N_{2,4}(Y) &= Y^{33} + 5Y^{33} + 12Y^{28} + 6Y^{27} + 15Y^{26} + 26Y^{25} + 11Y^{24} + 39Y^{23} + 40Y^{22} + 43Y^{21} + \\ &\quad 62Y^{20} + 45Y^{19} + 66Y^{18} + 61(Y^{17} + Y^{16}) + 66Y^{15} + 45Y^{14} + 62Y^{13} + \\ &\quad 43Y^{12} + 40Y^{11} + 39Y^{10} + 11Y^9 + 26Y^8 + 15Y^7 + 6Y^6 + 12Y^5 + 5Y^3 + 1 \in \mathbb{Q}[Y]\end{aligned}$$

we have

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,4}, \text{red}}^{\triangleleft \text{gr}}(Y) &= W_{2,4}^{\triangleleft \text{gr}}(1, Y) = \frac{N_{2,4}(Y)}{(1-Y)^4(1-Y^3) \prod_{i=6}^{10} (1-Y^i)}, \\ \zeta_{\mathfrak{f}_{2,4}, \text{top}}^{\triangleleft \text{gr}}(s) &= \frac{3}{10} \frac{36s^2 - 43s + 12}{s^2(s-1)^3(s-2)(s-3)(3s-4)^2(2s-1)(7s-9)(9s-5)}.\end{aligned}$$

The global graded ideal zeta function $\zeta_{\mathfrak{f}_{2,4}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ has abscissa of convergence 4 and may be continued meromorphically to $\{s \in \mathbb{C} \mid \text{Re}(s) > 7/6\}$.

3.5 Two generators ($d = 2$)

We compute the graded ideal zeta functions $\zeta_{\mathfrak{f}_{c,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ for $c \in \{3, 4\}$ as well as their reduced and topological counterparts. (For $c = 1$ see (3.1.3), for $c = 2$ see Proposition 3.4.9.)

Proposition 3.5.1. $((c, d) = (3, 2))$

$$\begin{aligned}\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s) \frac{1 + t^4}{(1 - t^3)(1 - qt^4)(1 - t^5)} =: W_{3,2}^{\triangleleft \text{gr}}(q, t), \\ \zeta_{\mathfrak{f}_{3,2,\text{red}}}^{\triangleleft \text{gr}}(Y) &= W_{3,2}^{\triangleleft \text{gr}}(1, Y) = \frac{1 + Y^4}{(1 - Y)^2(1 - Y^3)(1 - Y^4)(1 - Y^5)}, \\ \zeta_{\mathfrak{f}_{3,2,\text{top}}}^{\triangleleft \text{gr}}(s) &= \frac{2}{15s^3(s - 1)(4s - 1)}.\end{aligned}$$

The global graded ideal zeta function

$$\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \frac{\zeta_K(s)\zeta_K(s - 1)\zeta_K(3s)\zeta_K(4s)\zeta_K(4s - 1)\zeta_K(5s)}{\zeta_K(8s)}$$

has abscissa of convergence 2 and meromorphic continuation to the whole complex plane.

Proof. Let $\mathfrak{f} = \mathfrak{f}_{3,2}$ and recall the Hall basis $\mathcal{H}_{3,2}$ for $\mathfrak{f}$ from Table 3.2.1. Recall formula (3.2.2) for the zeta function of $\mathfrak{f}(\mathfrak{o})$ in terms of pairs (Λ_2, Λ_3) of sublattices of $\mathfrak{f}(\mathfrak{o})^{(2)} \cong \mathfrak{o}$ and $\mathfrak{f}(\mathfrak{o})^{(3)} \cong \mathfrak{o}^2$, respectively. Note that $|\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2| = q^\mu$ for some $\mu \in \mathbb{N}_0$, $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu}$, and (Λ_2, Λ_3) satisfies the condition in (3.2.2) if and only if

$$\mathfrak{p}^\mu \mathfrak{f}(\mathfrak{o})^{(3)} \leq \Lambda_3. \quad (3.5.1)$$

This proves the first claim; the others are trivial consequences. $\square$

Remark 3.5.2. Comparison with $\zeta_{\mathfrak{f}_{3,2}(\mathfrak{o})}^{\triangleleft}(s)$ (cf. [11, Theorem 2.35]) yields $\zeta_{\mathfrak{f}_{3,2,\text{red}}}^{\triangleleft \text{gr}}(Y) = \zeta_{\mathfrak{f}_{3,2,\text{red}}}^{\triangleleft}(Y)$. Note that the Hall basis $\mathcal{H}_{3,2}$ is nice and simple in the sense of [13].

Proposition 3.5.3. $((c, d) = (4, 2))$ *Set*

$$\begin{aligned}N_{4,2}(X, Y) &= -X^2(Y^{22} + Y^{18} + Y^{17} + Y^{16} + Y^{15} + Y^{11}) \\ &\quad - X(Y^{16} + Y^{15} - Y^{13} + Y^9 - Y^7 - Y^6) + Y^{11} + Y^7 + Y^6 + Y^5 + Y^4 + 1 \in \mathbb{Q}[X, Y].\end{aligned}$$

Then

$$\begin{aligned}\zeta_{\mathfrak{f}_{4,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s) \frac{N_{4,2}(q, t)}{(1-t^3)(1-qt^4)(1-t^5)(1-qt^5)(1-q^2t^6)(1-q^2t^7)(1-t^8)} \\ \zeta_{\mathfrak{f}_{4,2,\text{red}}(\mathfrak{o})}^{\triangleleft \text{gr}}(Y) &= \frac{Y^{17} + Y^{13} + 2(Y^{12} + Y^{11} + Y^{10} + Y^7 + Y^6 + Y^5) + Y^4 + 1}{(1-Y)^2 \prod_{i=3}^8 (1-Y^i)}, \\ \zeta_{\mathfrak{f}_{4,2,\text{top}}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \frac{1}{60} \frac{(20s-3)}{s^4(s-1)(5s-1)(4s-1)(7s-2)(3s-1)}.\end{aligned}$$

The global graded ideal zeta function $\zeta_{\mathfrak{f}_{4,2}(\mathfrak{o})}^{\triangleleft}(s)$ has abscissa of convergence 2 and may be continued meromorphically to $\{s \in \mathbb{C} \mid \text{Re}(s) > 2/11\}$.

Proof. Let $\mathfrak{f} = \mathfrak{f}_{4,2}$ and recall the Hall basis $\mathcal{H}_{4,2}$ for $\mathfrak{f}$ from Table 3.2.1. Recall formula (3.2.2) for the zeta function of $\mathfrak{f}(\mathfrak{o})$ in terms of triples $(\Lambda_2, \Lambda_3, \Lambda_4)$ of sublattices of $\mathfrak{f}(\mathfrak{o})^{(2)} \cong \mathfrak{o}$, $\mathfrak{f}(\mathfrak{o})^{(3)} \cong \mathfrak{o}^2$, and $\mathfrak{f}(\mathfrak{o})^{(4)} \cong \mathfrak{o}^3$, respectively. Note that $|\mathfrak{f}(\mathfrak{o})^{(2)} : \Lambda_2| = q^\mu$ for some $\mu \in \mathbb{N}_0$, $|\mathfrak{f}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu}$, and $(\Lambda_2, \Lambda_3, \Lambda_4)$ satisfies the condition in (3.2.2) if and only if (Λ_2, Λ_3) satisfies (3.5.1) and $[\Lambda_3, \text{gr } \mathfrak{f}(\mathfrak{o})] \leq \Lambda_4$. To investigate the latter condition, we may assume that $\Lambda_3 = \langle \pi^{\nu_1}xyx, \pi^{\nu_2}xyy \rangle_{\mathfrak{o}}$ for a partition $\nu = (\nu_1, \nu_2)_{\geq}$. Then

$$[\Lambda_3, \text{gr } \mathfrak{f}(\mathfrak{o})] = \langle \pi^{\nu_1}xyxx, \pi^{\nu_1}xyxy, \pi^{\nu_2}xyyx, \pi^{\nu_2}xyyy \rangle_{\mathfrak{o}}$$

As $xyxy = xyxx$ by the Jacobi identity, the underlined term may be omitted, whence

$$[\Lambda_3, \text{gr } \mathfrak{f}(\mathfrak{o})] = \pi^{\nu_1}xyxx \oplus \pi^{\nu_2}xyyx \oplus \pi^{\nu_2}xyyy.$$

To enumerate the lattices $\Lambda_4 \leq \mathfrak{f}(\mathfrak{o})^{(4)}$ satisfying $[\Lambda_3, \text{gr } \mathfrak{f}(\mathfrak{o})] \leq \Lambda_4$, we distinguish according to the “overlap type” such a lattice may have with $[\Lambda_3, \text{gr } \mathfrak{f}(\mathfrak{o})]$. Indeed, if Λ_4 has elementary divisor type given by a partition $(\xi_1, \xi_2, \xi_3)_{\geq}$, then either

$$\nu_1 \geq \nu_2 \geq \xi_1 \geq \xi_2 \geq \xi_3 \quad \text{or} \quad \nu_1 \geq \xi_1 > \nu_2 \geq \xi_2 \geq \xi_3.$$

Each of the two cases is covered by a formula similar to the one established in Theorem 3.3.6. Adding them yields a formula for the zeta function, viz.

$$\begin{aligned}\zeta_{\mathfrak{f}_{4,2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) &= \zeta_{\mathfrak{o}^2}(s) I_1(t^3) \\ &\cdot \left(I_2(qt^4, t^5) I_3(q^2t^6, q^2t^7, t^8) + \binom{2}{1}_{q^{-1}} I_1(qt^4) I_1^{\circ}(qt^5) I_1(q^2t^6) I_2(q^2t^7, t^8) \right).\end{aligned}$$

This proves the first claim; the others are trivial consequences. The claim about the meromorphic continuation follows from [11, Lemma 5.5]. $\square$

Chapter 4

Enumerating graded ideals of $\mathcal{L}_c(d)$

In this chapter we introduce another interesting family of nilpotent Lie lattices. Let $c, d \in \mathbb{N}$ and let $\mathcal{L}_{c(d)}$ be the class- c -nilpotent $\mathbb{Z}$ -Lie lattice defined as follows:

$$\mathcal{L}_{c(d)} = \langle x_0, \{x_{ij}\}_{i=1, \dots, d, j=1, \dots, c} \mid \forall i = 1, \dots, d, j = 1, \dots, c-1 : [x_0, x_{ij}] = x_{i,j+1} \rangle_{\mathbb{Z}};$$

cf. [41, Section 4.3].

Note that this is essentially a direct sum of maximal class Lie lattice with amalgamation. This is a generalisation of many important Lie lattices. For $c = 1$ we get $\mathcal{L}_{1(d)}(\mathfrak{o}) \cong \mathfrak{o}^{d+1}$. For $d = 1$ this yields the maximal class Lie lattice $\mathcal{L}_{c(1)}(\mathfrak{o}) \cong M_c(\mathfrak{o})$. For $c = 2$, $\mathcal{L}_{2(d)}(\mathfrak{o})$ is isomorphic to the Grenham Lie lattice $\mathcal{G}_{d+1}(\mathfrak{o})$. We give explicit formulas for $\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ for general $c, d \in \mathbb{N}$.

4.1 General setting

For $c = 1$, $\mathcal{L}_{1(d)}(\mathfrak{o}) \cong \mathfrak{o}^{d+1}$, and $\zeta_{\mathcal{L}_{1(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s)$. So we assume $c \geq 2$. In order to parametrize lattices in $\mathcal{L}_{c(d)}(\mathfrak{o})$ we write

$$\mu_k = (\mu_{1,k}, \dots, \mu_{d,k})_{\geq}$$

for partitions of at most d parts, where $2 \leq k \leq c$. Recall the definition of $\alpha(\sigma, \tau; q)$ in Section 3, denoting the number of torsion $\mathfrak{o}$ -modules of type τ contained in a fixed torsion $\mathfrak{o}$ -module of type σ . Our stating point is the following result:

Proposition 4.1.1. *Let $d \in \mathbb{N}$.*

$$\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s) \left(\sum_{\mu_2} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) q^{-s(3\mu_{1,2} + 2 \sum_{i=2}^d \mu_{i,2})} \prod_{k=3}^c \sum_{\mu_k \leq \mu_{k-1}} \alpha(\mu_{k-1}, \mu_k; q) q^{-s(\sum_{j=1}^d \mu_{j,k})} \right)$$

where the sum runs over partitions μ_k of at most d parts.

Proof. Like the proof of Proposition 3.3.1, our starting point is (3.2.2) in Lemma 3.2.1, which in this case reads

$$\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^{d+1}}(s) \sum_{\substack{\Lambda_2 \leq \mathcal{L}_{c(d)}(\mathfrak{o})^{(2)}, \dots, \Lambda_c \leq \mathcal{L}_{c(d)}(\mathfrak{o})^{(c)} \\ \forall i \in]2, c]: [\Lambda_{i-1}, \mathcal{L}_{c(d)}(\mathfrak{o})^{(1)}] \leq \Lambda_i}} |\mathcal{L}_{c(d)}(\mathfrak{o})^{(1)} : X(\Lambda_2)|^{-s} \prod_{i=2}^c |\mathcal{L}_{c(d)}(\mathfrak{o})^{(i)} : \Lambda_i|^{-s}. \quad (4.1.1)$$

Let $\mu_2 = (\mu_{1,2}, \dots, \mu_{d,2})_{\geq}$ be a partition. There are $\alpha(\mu_{1,2}^{(d)}, \mu_2; q)$ lattices $\Lambda_2 \leq \mathcal{L}_{c(d)}(\mathfrak{o})^{(2)}$ whose elementary divisor type with respect to $\mathcal{L}_{c(d)}(\mathfrak{o})^{(2)}$ is given by μ_2 . Clearly $|\mathcal{L}_{c(d)}(\mathfrak{o})^{(2)} : \Lambda_2| = q^{\sum_{j=1}^d \mu_{j,2}}$. We claim that $|\mathcal{L}_{c(d)}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu_{1,2} + \sum_{j=2}^d \mu_{j,2}}$. Indeed, lattices such as Λ_2 may be parametrized by their elementary divisor type and a coset $\alpha\Gamma_{\mu_2} \in \text{SL}_d(\mathfrak{o})/\Gamma_{\mu_2}$ of a certain stabilizer subgroup $\Gamma_{\mu_2} \leq \text{SL}_d(\mathfrak{o})$, where $\alpha = (\alpha_{ij}) \in \text{SL}_d(\mathfrak{o})$. This time we solve the simultaneous congruences

$$(x_1, x_2, \dots, x_{d+1}) \begin{pmatrix} 0 & \alpha_{1j} & \alpha_{2j} & \cdots & \alpha_{dj} \\ -\alpha_{1j} & 0 & 0 & \cdots & 0 \\ -\alpha_{2j} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{dj} & 0 & 0 & \cdots & 0 \end{pmatrix} \equiv 0 \pmod{\mathfrak{p}^{\mu_{j,2}}}$$

for $j \in [d]$ to get $|\mathcal{L}_{c(d)}(\mathfrak{o})^{(1)} : X(\Lambda_2)| = q^{2\mu_{1,2} + \sum_{j=2}^d \mu_{j,2}}$; cf. [36, Chapter 5] for details.

In fact, after a change of generators for $\mathfrak{f}(\mathfrak{o})$ if necessary, we may assume that

$$\Lambda_2 = \mathfrak{p}^{\mu_{1,2}} x_{12} \oplus \cdots \oplus \mathfrak{p}^{\mu_{d,2}} x_{d2}.$$

We obtain

$$[\Lambda_2, \mathcal{L}_{c(d)}(\mathfrak{o})^{(1)}] = \mathfrak{p}^{\mu_{1,2}} x_{13} \oplus \cdots \oplus \mathfrak{p}^{\mu_{d,2}} x_{d3}, \quad (4.1.2)$$

hence $[\Lambda_2, \mathcal{L}_{c(d)}(\mathfrak{o})^{(1)}]$ has type μ_2 with respect to $\mathcal{L}_{c(d)}(\mathfrak{o})^{(3)}$.

This shows that the number of lattices $\Lambda_3 \leq \mathcal{L}_{c(d)}(\mathfrak{o})^{(3)}$ whose elementary divisor type with respect to $\mathcal{L}_{c(d)}(\mathfrak{o})^{(3)}$ is given by a partition $\mu_3 = (\nu_{1,3}, \dots, \nu_{d,3})_{\geq}$ and which satisfy $[\Lambda_2, \mathcal{L}_{c(d)}(\mathfrak{o})^{(1)}] \leq \Lambda_3$ is equal to $\alpha(\mu_2, \mu_2; q)$. Each such lattice satisfies $|\mathcal{L}_{c(d)}(\mathfrak{o})^{(3)} : \Lambda_3| = q^{-s \sum_{j=1}^d \mu_{j,3}}$. Applying the same argument inductively, one can see that for $3 < k \leq c$, the number of lattices $\Lambda_k \leq \mathcal{L}_{c(d)}(\mathfrak{o})^{(k)}$ whose elementary divisor type with respect to $\mathcal{L}_{c(d)}(\mathfrak{o})^{(k)}$ is given by a partition $\mu_k = (\nu_{1,k}, \dots, \nu_{d,k})_{\geq}$ and which satisfy $[\Lambda_{k-1}, \mathcal{L}_{c(d)}(\mathfrak{o})^{(1)}] \leq \Lambda_k$ is equal to $\alpha(\mu_{k-1}, \mu_k; q)$. Each such lattice satisfies $|\mathcal{L}_{c(d)}(\mathfrak{o})^{(k)} : \Lambda_k| = q^{-s \sum_{j=1}^d \mu_{j,k}}$. $\square$

4.2 Explicit formulas for $d = 1$ and for $c \leq 3$

In this section we provide explicit formulas of $\zeta_{\mathcal{L}_{c^d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ for $d = 1$ and for $c \leq 3$.

For $d = 1$, the graded ideal zeta function of the maximal-class $\mathfrak{o}$ -Lie lattices $\mathcal{L}_{c^1}$ have been computed in [28, Proposition 3.5].

Proposition 4.2.1 ([28, Proposition 3.5]). *Let $c \in \mathbb{N}$.*

$$\zeta_{\mathcal{L}_{c^1}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \frac{1}{(1-t)(1-qt)(1-t^3)(1-t^4)\dots(1-t^{c+1})}.$$

Remark 4.2.2. Note that we still do not have an explicit formula for ungraded ideal zeta function of maximal class c Lie lattice, $\zeta_{\mathcal{L}_{c^1}(\mathfrak{o})}^{\triangleleft}(s)$ (or $\zeta_{M_c(\mathfrak{o})}^{\triangleleft}(s)$) for $c \geq 5$. We do not even know whether they are (finitely) uniform or not.

For $c = 1$, $\mathcal{L}_{1(d)}(\mathfrak{o}) \cong \mathfrak{o}^{d+1}$, and $\zeta_{\mathcal{L}_{1(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s)$.

Theorem 4.2.3.

$$\zeta_{\mathcal{L}_{2(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s) I_d \left(\left(q^{(d-j)j} t^{1+2j} \right)_{j \in [d]} \right). \quad (4.2.1)$$

Proof. This follows from Proposition 4.1.1, since we have

$$\zeta_{\mathcal{L}_{2(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s) \left(\sum_{\mu_2} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) q^{-s(3\mu_{1,2} + 2\sum_{i=2}^d \mu_{i,2})} \right). \quad (4.2.2)$$

□

Remark 4.2.4. Compare this with [38, Theorem 5]

$$\zeta_{\mathcal{L}_{2(d)}(\mathfrak{o})}^{\triangleleft}(s) = \zeta_{\mathfrak{o}^{1+d}}(s) I_d \left(\left(q^{(d+d-j)j} t^{1+2j} \right)_{j \in [d]} \right). \quad (4.2.3)$$

Corollary 4.2.5 (Uniformity). *There exists $V_{2,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathcal{L}_{2(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = V_{c,d}^{\triangleleft \text{gr}}(q, q^{-s}).$$

Corollary 4.2.6 (Local functional equations).

$$\zeta_{\mathcal{L}_{2(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{1+2d} q^{\binom{d+1}{2} + \binom{d}{2}} t^{2+3d} \zeta_{\mathcal{L}_{2d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s). \quad (4.2.4)$$

We now turn to $c = 3$.

4.2.1 Graded ideal zeta functions of $\mathcal{L}_{3(d)}$

For $c = 3$, Proposition 4.1.1 gives

$$\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s) \left(\sum_{\mu_2} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) q^{-s(3\mu_{1,2} + 2\sum_{i=2}^d \mu_{i,2})} \sum_{\mu_3 \leq \mu_2} \alpha(\mu_2, \mu_3; q) q^{-s(\sum_{j=1}^d \mu_{j,3})} \right) \quad (4.2.5)$$

To compute the right hand side of (4.2.5), let μ_2 and μ_3 be partitions of d parts satisfying $\mu_3 \leq \mu_2$. There are uniquely determined numbers $r \in \mathbb{N}_0$ and $M_i, N_i \in \mathbb{N}$ ($i \in [r-1]$), such that

$$\begin{aligned} \mu_{1,2} &\geq \cdots \geq \mu_{M_1,2} \geq \mu_{1,3} \geq \cdots \geq \mu_{N_1,3} > \\ \mu_{M_1+1,2} &\geq \cdots \geq \mu_{M_2,2} \geq \mu_{N_1+1,3} \geq \cdots \geq \mu_{N_2,3} > \cdots \\ \mu_{M_{r-1}+1,2} &\geq \cdots \geq \mu_{d,2} \geq \mu_{N_{r-1}+1,3} \geq \cdots \geq \mu_{d,3}. \end{aligned}$$

Define $M_r = N_r = d$ and $M_0 = N_0 = 0$, and observe that the condition $\mu_3 \leq \mu_2$ is equivalent to the condition $M_i \geq N_i$ for $i \in [r-1]$. In this case we write $v(\mu_2, \mu_3) = v$. Note that v determines and is determined by M_i 's and N_i 's.

Recall that a *Dyck word of length $2d$* is a word w in the letters **2** and **3**, such that **2** and **3** each occur d times and no initial segment of w contains more threes than twos. We denote the set of Dyck words of length $2d$ by $\mathcal{D}_d^2$ and note that the cardinality of $\mathcal{D}_d^2$ is the n -th Catalan number $\text{Cat}_d = \frac{1}{d+1} \binom{2d}{d}$. The good thing about $\mathcal{L}_{3(d)}$ is that unlike $\mathfrak{f}_{3,3}$, we do not have “artificial” Dyck words in the sense that both **2** and **3** actually happens d times each. It is easy to see that our $v(\mu_2, \mu_3) = v$ is indeed a Dyck word of length $2d$.

For $v \in \mathcal{D}_d^2$ we set

$$D_v(q, t) := \sum_{\substack{(\mu_2, \mu_3) \\ v(\mu_2, \mu_3) = v}} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) \alpha(\mu_2, \mu_3; q) q^{-(3\mu_{1,2} + 2\sum_{i=2}^d \mu_{i,2} + \sum_{j=1}^d \mu_{j,3})s}. \quad (4.2.6)$$

Proposition 4.1.1 and (4.2.5) allows us to write

$$\zeta_{\mathcal{L}_{3d}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}_p^{d+1}}(s) \sum_{v \in \mathcal{D}_d^2} D_v(q, t). \quad (4.2.7)$$

This section's main result is Theorem 4.2.10, giving a general formula for the functions D_v . The proof of Theorem 4.2.10 is very similar to the proof of Theorem 3.3.6, replacing 3 and 8 with d and μ, ν with μ_2, μ_3 respectively.

For $j \in [d]$ we set the successive differences of the parts of μ_2 and μ_3

$$r_j = \begin{cases} \mu_{j,3} - \mu_{j+1,3} & \text{if } j \notin \{N_1, \dots, N_r\}, \\ \mu_{N_i,3} - \mu_{M_i+1,2} & \text{if } j = N_i, \end{cases} \quad (4.2.8)$$

where we define $\mu_{d+1,2} = 0$. Similarly, we set

$$s_j = \begin{cases} \mu_{j,2} - \mu_{j+1,2} & \text{if } j \notin \{M_1, \dots, M_r\}, \\ \mu_{M_i,2} - \mu_{N_{i-1}+1,3} & \text{if } j = M_i. \end{cases} \quad (4.2.9)$$

By our definition, $r_j, s_j \geq 0$ for all j . Note also that $r_j > 0$ if $j \in \{N_1, \dots, N_{r-1}\}$ and observe that $\mu_{M_i,2} > \mu_{M_i+1,2}$ and $\mu_{N_i,3} > \mu_{N_i+1,3}$ for each $i \in [r-1]$.

The difference between $\mathcal{L}_{3(d)}$ and $\mathfrak{f}_{3,3}$ is that, as we mentioned above, both μ_2 and μ_3 are partitions of d parts, so one can naturally define $\mu_3 \leq \mu_2$ without any extra effort.

Set

$$\smile : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad n \mapsto n \cdot (3, 2, \dots, 2) := 3\delta_{n \geq 1} + 2\delta_{n \geq 2} + \dots + 2\delta_{n \geq d}.$$

Then we have

$$\begin{aligned} 3\mu_{1,2} + 2 \sum_{k=2}^d \mu_k &= \sum_{j=1}^d \smile s_j + \sum_{i=1}^r \widetilde{M}_i (r_{N_{i-1}+1} + \dots + r_{N_i}), \\ \mu_{1,3} + \dots + \mu_{d,3} &= \sum_{j=1}^d j r_j + \sum_{i=1}^{r-1} N_i (s_{M_i+1} + \dots + s_{M_{i+1}}). \end{aligned}$$

Finally, for each $i \in [r]$ we define

$$\begin{aligned} J_i^{\mu_2} &= \{j \in [M_i - M_{i-1} - 1] \mid \mu_{M_i-j,2} > \mu_{M_i-j+1,2}\}, \\ J_i^{\mu_3} &= \{j \in [N_i - N_{i-1} - 1] \mid \mu_{N_i-j,3} > \mu_{N_i-j+1,3}\}. \end{aligned} \quad (4.2.10)$$

We start with an immediate consequence of Proposition 3.2.2.

Lemma 4.2.7. *Let μ_2 and μ_3 be partitions of d parts, respectively, satisfying $\mu_3 \leq \mu_2$. Then*

$$\sum_{\mu_2} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) \alpha(\mu_2, \mu_3; q) = \prod_{k \geq 1} q^{\mu'_{k,2}(d - \mu'_{k,2}) + \mu'_{k,3}(\mu'_{k,2} - \mu'_{k,3})} \binom{d - \mu'_{k+1,2}}{d - \mu'_{k,2}}_{q^{-1}} \binom{\mu'_{k,2} - \mu'_{k+1,3}}{\mu'_{k,2} - \mu'_{k,3}}_{q^{-1}}. \quad (4.2.11)$$

Let μ_2 and μ_3 be partitions as in Lemma 4.2.7. Let $(M_i, N_i)_{i \in [r-1]}$ be the parameters associated to them and μ'_2 and ν'_3 their dual partitions. For $k \in \mathbb{N}$, set

$$d_k := \mu'_{k,2}(d - \mu'_{k,2}) + \mu'_{k,3}(\mu'_{k,2} - \mu'_{k,3}).$$

Also, set

$$\begin{aligned} a_j &:= M_i (d - M_i) + j (M_i - j), & j &\in]N_{i-1}, N_i], \\ b_j &:= j (d - j) + N_{i-1} (j - N_{i-1}), & j &\in]M_{i-1}, M_i]. \end{aligned}$$

Lemma 4.2.8. *For $i \in [r - 1]$,*

$$\begin{aligned} \prod_{k=\mu_{M_i,2}}^{\mu_{M_{i-1}+1,2}} q^{d_k} \binom{d - \mu'_{k+1,2}}{d - \mu'_{k,2}}_{q^{-1}} \binom{\mu'_{k,2} - \mu'_{k+1,3}}{\mu'_{k,2} - \mu'_{k,3}}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^{\mu_2}}_{q^{-1}} \binom{d - M_{i-1}}{d - M_i}_{q^{-1}}. \end{aligned}$$

Proof. Since $\mu_{N_{i-1},3} > \mu_{M_{i-1}+1,2} \geq k \geq \mu_{M_i,2} \geq \mu_{N_{i-1}+1,3}$, we have $\mu'_{k,3} = \mu'_{k+1,3} = N_{i-1}$ for all k in the interval $[\mu_{M_i,2}, \mu_{M_{i-1}+1,2}]$. Thus we always have

$$\prod_{k=\mu_{M_i,2}}^{\mu_{M_{i-1}+1,2}} \binom{\mu'_{k,2} - \mu'_{k+1,3}}{\mu'_{k,2} - \mu'_{k,3}}_{q^{-1}} = 1.$$

For $j \in [M_i - M_{i-1}]$ we have $\mu'_{k,2} = M_{i-1} + j$ when $\mu_{M_{i-1}+j+1,2} < k \leq \mu_{M_{i-1}+j,2}$; observe that it may be the case for some j that no index k satisfies this condition. As a result, we see that for each $j \in [M_i - M_{i-1}]$, there are exactly $s_{M_{i-1}+j}$ elements of k of the segment $[\mu_{M_i,2}, \mu_{M_{i-1}+1,2}]$ for which $\mu'_{k,2} = M_{i-1} + j$. Now, note that

$$\binom{d - \mu'_{k+1,2}}{d - \mu'_{k,2}}_{q^{-1}} \neq 1$$

if and only if $\mu'_{k+1,2} \neq \mu'_{k,2}$. This is the case exactly when there exists an i such that $\mu_{i,2} = k$. It follows that if $J_i^\mu = \{j_{i,1}, \dots, j_{i,\gamma_i}\}$, with $j_{i,1} < \dots < j_{i,\gamma_i}$, and write $j_{i,0} = 0, j_{i,\gamma_i+1} = M_i - M_{i-1}$, then

$$\prod_{k=\mu_{M_i,2}}^{\mu_{M_{i-1}+1,2}} \binom{d - \mu'_{k+1,2}}{d - \mu'_{k,2}}_{q^{-1}} = \prod_{m=0}^{\gamma_i} \binom{d - M_i + j_{i,m+1}}{d - M_i + j_{i,m}}_{q^{-1}}. \quad (4.2.12)$$

We make use of the well-known identity

$$\binom{\alpha}{\beta}_X = \frac{1 - X^\alpha}{1 - X^{a-\beta}} \binom{\alpha - 1}{\beta}_X$$

for Gaussian binomial coefficients. Applying it inductively, we see that for all $m \in [\gamma_i - 1]$,

$$\binom{d - M_i + j_{i,m+1}}{d - M_i + j_{i,m}}_{q^{-1}} = \binom{j_{i,m+1}}{j_{i,m}}_{q^{-1}} \frac{\binom{d - M_i + j_{i,m+1}}{d - M_i}_{q^{-1}}}{\binom{d - M_i + j_{i,m}}{d - M_i}_{q^{-1}}}.$$

Hence the right-hand side of (4.2.12) is equal to

$$\binom{M_i - M_{i-1}}{J_i^{\mu_2}}_{q^{-1}} \binom{d - M_{i-1}}{d - M_i}_{q^{-1}}.$$

Thus

$$\begin{aligned} \prod_{k=\mu_{M_i,2}}^{\mu_{M_{i-1}+1,2}} q^{d_k} \binom{d - \mu'_{k+1,2}}{d - \mu'_{k,2}}_{q^{-1}} \binom{\mu'_{k,2} - \mu'_{k+1,3}}{\mu'_{k,2} - \mu'_{k,3}}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^{\mu_2}}_{q^{-1}} \binom{d - M_{i-1}}{d - M_i}_{q^{-1}} \end{aligned}$$

as required. $\square$

Lemma 4.2.9. *For $i \in [r]$,*

$$\begin{aligned} \prod_{k=\mu_{M_i+1,2}+1}^{\mu_{M_i,2}-1} q^{d_k} \binom{d - \mu'_{k+1,2}}{d - \mu'_{k,2}}_{q^{-1}} \binom{\mu'_{k,2} - \mu'_{k+1,3}}{\mu'_{k,2} - \mu'_{k,3}}_{q^{-1}} \\ = \prod_{j=1}^{N_i - N_{i-1}} q^{a_{N_{i-1}+j} r_{N_{i-1}+j}} \binom{N_i - N_{i-1}}{J_i^{\mu_3}}_{q^{-1}} \binom{M_i - N_{i-1}}{M_i - N_i}_{q^{-1}}. \end{aligned}$$

Proof. Note that the product on the left hand side may be empty; this happens in the case $\mu_{M_i,2} = \mu_{N_{i-1}+1,3} = \dots = \mu_{N_i,3} = \mu_{M_i+1,2} + 1$. Since $\mu_{M_i,2} > k > \mu_{M_i+1,2}$, we have $\mu'_{k,2} = \mu'_{k+1,2} = M_i$ for all k in the interval $]\mu_{M_i+1,2}, \mu_{M_i-1,2}]$. Finally, observe that for $j \in [N_i - N_{i-1}]$ we have $\mu'_{k,3} = N_{i-1} + j$ when $\mu_{N_{i-1}+j+1,3} < k \leq \mu_{N_{i-1}+j,3}$. The claim follows as in the proof of the previous lemma. $\square$

With these preparations in place we now give a formula for the functions D_v .

Theorem 4.2.10. *Let $v = \prod_{i=1}^r (\mathbf{2}^{M_i - M_{i-1}} \mathbf{3}^{N_i - N_{i-1}}) \in \mathcal{D}_d^2$. Then*

$$\begin{aligned} D_v(q, t) = \prod_{i=1}^r \binom{d - M_{i-1}}{d - M_i}_{q^{-1}} \binom{M_i - N_{i-1}}{M_i - N_i}_{q^{-1}} \\ \left(\prod_{i=1}^r I_{M_i - M_{i-1}}(Y_{M_{i-1}+1}, \dots, Y_{M_i}) \right) \cdot \prod_{i=1}^{r-1} I_{N_i - N_{i-1}}^\circ(X_{N_{i-1}+1}, \dots, X_{N_i}) \\ \cdot I_{d - N_{r-1}}(X_{N_{r-1}+1}, \dots, X_d), \end{aligned}$$

with numerical data

$$X_j = q^{M_i(d-M_i)+j(M_i-j)} t^{2M_i+1+j} \quad j \in]N_{i-1}, N_i], \quad (4.2.13)$$

$$Y_j = q^{j(d-j)+N_{i-1}(j-N_{i-1})} t^{N_{i-1}+2j+1} \quad j \in]M_{i-1}, M_i]. \quad (4.2.14)$$

Proof. Given vectors $\mathbf{v} = (v_1, v_2, \dots, v_d) \in \mathbb{N}_0^d$ and $\mathbf{v}' = (v'_1, v'_2, \dots, v'_d) \in \mathbb{N}_0^d$, we set, for each $i \in [r]$,

$$\begin{aligned}\text{supp}_i^M(\mathbf{v}) &= \{j \in [M_i - M_{i-1} - 1] \mid v_{M_{i-1}+j} > 0\}, \\ \text{supp}_i^N(\mathbf{v}) &= \{j \in [N_i - N_{i-1} - 1] \mid v'_{N_{i-1}+j} > 0\}.\end{aligned}$$

In practice, $\mathbf{v}$ will be one of the vectors of successive differences $\mathbf{r} = (r_1, \dots, r_d)$ and $\mathbf{v}'$ will be one of the vectors of successive differences $\mathbf{s} = (s_1, \dots, s_d)$. Given a set of partitions satisfying $\mu_3 \leq \mu_2$, recall our definition (4.2.10). It is easy to see that, for every $i \in [r]$,

$$\begin{aligned}\text{supp}_i^M(\mathbf{v}) &= M_i - M_{i-1} - J_i^{\mu_2}, \\ \text{supp}_i^N(\mathbf{v}) &= N_i - N_{i-1} - J_i^{\mu_3}.\end{aligned}\tag{4.2.15}$$

Thus it follows that

$$\begin{aligned}\binom{M_i - M_{i-1}}{J_i^{\mu_2}}_{q^{-1}} &= \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}}, \\ \binom{N_i - N_{i-1}}{J_i^{\mu_3}}_{q^{-1}} &= \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v}')}_{q^{-1}}.\end{aligned}\tag{4.2.16}$$

Let δ_{ij} be the usual Kronecker delta function. Substituting Lemma 4.2.8 and 4.2.9, rewriting the expressions in terms of r_j and s_j and using (4.2.16), we find that formula (4.2.6) for $D_v(q, t)$ splits into a product as follows:

$$D_v(q, t) = \prod_{i=1}^r \binom{d - M_{i-1}}{d - M_i}_{q^{-1}} \binom{M_i - N_{i-1}}{M_i - N_i}_{q^{-1}} \prod_{i=1}^r A_i B_i,$$

where, for $i \in [r]$,

$$\begin{aligned}A_i &= \sum_{r_{N_{i-1}+1}=0}^{\infty} \cdots \sum_{r_{N_i-1}=0}^{\infty} \sum_{r_{N_i}=1-\delta_{ir}}^{\infty} \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v}')}_{q^{-1}} \prod_{j=N_{i-1}+1}^{N_i} \left(q^{a_j} t^{\tilde{M}_i+j}\right)^{r_j}, \\ B_i &= \sum_{s_{M_{i-1}+1}=0}^{\infty} \cdots \sum_{s_{M_i-1}=0}^{\infty} \sum_{s_{M_i}=0}^{\infty} \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}} \prod_{j=M_{i-1}+1}^{M_i} \left(q^{b_j} t^{N_{i-1}+j}\right)^{s_j}.\end{aligned}$$

We now show that all of the factors A_i and B_i are products of Igusa functions and Gaussian binomial coefficients. Given $i \in [r]$ and $I \subseteq [M_i - M_{i-1} - 1]$, we define $\mathbf{S}^i(I)$ to be the set of vectors $\mathbf{S}^i = (s_{M_{i-1}+1}, \dots, s_{M_i}) \in \mathbb{N}_0^{M_i - M_{i-1}}$ such that $s_j = 0$ unless $j \in \{M_{i-1} + k \mid k \in I\} \cup \{M_i\}$. With the numerical data defined in Theorem

4.2.10, we have

$$\begin{aligned}
B_i &= \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I} \sum_{q^{-1} \mathbf{s}^i \in \mathbf{S}^i(\mathbf{I})} \prod_{j \in (I + M_{i-1}) \cup \{M_i\}} \left(q^{b_j} t^{N_{i-1} + \check{j}} \right)^{s_j} \\
&= \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I} \sum_{q^{-1}} \left(\prod_{\iota \in I} \left(\sum_{s_{M_{i-1} + \iota} = 1}^{\infty} (Y_{M_{i-1} + \iota})^{s_{M_{i-1} + \iota}} \right) \right) \sum_{s_{M_i} = 0}^{\infty} (Y_{M_i})^{s_{M_i}} \\
&= \frac{1}{1 - Y_{M_i}} \sum_{I \subseteq [M_i - M_{i-1} - 1]} \binom{M_i - M_{i-1}}{I} \prod_{q^{-1} \iota \in I} \frac{Y_{M_{i-1} + \iota}}{1 - Y_{M_{i-1} + \iota}} \\
&= I_{M_i - M_{i-1}} (Y_{M_{i-1} + 1}, \dots, Y_{M_i}).
\end{aligned}$$

Analogously one can show that, with the numerical data defined in Theorem 4.2.10,

$$A_i = \begin{cases} I_{N_i - N_{i-1}}^{\circ} (X_{N_{i-1} + 1}, \dots, X_{N_i}) & \text{for } i < r, \\ I_{d - N_{r-1}} (X_{N_{r-1} + 1}, \dots, X_d) & \text{for } i = r. \end{cases}$$

This completes the proof of Theorem 4.2.10. $\square$

Now with (4.2.7) Theorem 4.2.10 implies:

Theorem 4.2.11 (Uniformity). *There exists $V_{3,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathcal{L}_{3(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = V_{c,d}^{\triangleleft \text{gr}}(q, q^{-s}).$$

Theorem 4.2.12 (Local functional equations).

$$\zeta_{\mathcal{L}_{3(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = (-1)^{1+3d} q^{\binom{d+1}{2} + 2\binom{d}{2}} t^{3+6d} \zeta_{\mathcal{L}_{3d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s). \quad (4.2.17)$$

4.2.2 Explicit examples

We record the following consequences of Theorem 4.2.10.

Proposition 4.2.13. $((c, d) = (3, 2))$

$$\zeta_{\mathcal{L}_{3(2)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}^3}(s) \frac{1 + t^3 + t^4 + t^6 - qt^7 - qt^9 - qt^{10} - qt^{13}}{(1 - qt^3)(1 - qt^4)(1 - t^5)(1 - qt^6)(1 - t^7)} =: V_{3,2}^{\triangleleft \text{gr}}(q, t),$$

and

$$\zeta_{\mathcal{L}_{3(2)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = -q^5 t^{15} \zeta_{\mathcal{L}_{32}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

Setting

$$\begin{aligned}
M_{3,2}(Y) &= 11 + Y + Y^2 + 2Y^3 + 3Y^4 + 3Y^5 + 4Y^6 + 3Y^7 + 3Y^8 + 2Y^9 + Y^{10} \\
&\quad + Y^{11} + Y^{12} \in \mathbb{Q}[Y],
\end{aligned}$$

we have

$$\begin{aligned}\zeta_{\mathcal{L}_{3(2)},\text{red}}^{\triangleleft\text{gr}}(s) &= V_{3,2}^{\triangleleft\text{gr}}(1, Y) \\ &= \frac{M_{3,2}(Y)}{(1-Y)^2(1-Y^3)(1-Y^4)(1-Y^5)(1-Y^6)(1-Y^7)}, \\ \zeta_{\mathcal{L}_{3(2)},\text{top}}^{\triangleleft\text{gr}}(s) &= \frac{2}{35} \frac{13s-2}{s^3(s-1)(s-2)(3s-1)(4s-1)(6s-1)}.\end{aligned}$$

Proof. For $(c, d) = (3, 3)$, (4.2.7) becomes

$$\zeta_{\mathcal{L}_{3^2}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) = \zeta_{\mathfrak{o}_{\mathfrak{p}}^3}(s) \sum_{v \in \mathcal{D}_2^2} D_v(q, t)$$

We have

$$D_{2^2 3^2}(q, t) = I_2(qt^3, t^5) I_2(qt^6, t^7),$$

and

$$D_{2^2 3^2}(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^{\circ}(qt^4) I_1(qt^6) I_1(t^7).$$

□

Proposition 4.2.14. *((c,d)=3,3) There exists a rational function $V_{3,3}^{\triangleleft\text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ such that*

$$\zeta_{\mathcal{L}_{3(3)}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) = V_{3,3}^{\triangleleft\text{gr}}(X, Y),$$

and

$$\zeta_{\mathcal{L}_{3(3)}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = q^{12} t^{21} \zeta_{\mathcal{L}_{3^3}(\mathfrak{o})}^{\triangleleft\text{gr}}(s).$$

Setting

$$\begin{aligned}M_{3,3}(Y) &= 1 + 2Y^3 + 3Y^4 + 3Y^5 + 5Y^6 + 3Y^7 + 8Y^8 + 6Y^9 + 9Y^{10} + 10Y^{11} + 10Y^{12} \\ &\quad + 9Y^{13} + 10Y^{14} + 10Y^{15} + 9Y^{16} + 6Y^{17} + 8Y^{18} + 3Y^{19} + 5Y^{20} \\ &\quad + 3Y^{21} + 3Y^{22} + 2Y^{23} + Y^{26} \in \mathbb{Q}[Y]\end{aligned}$$

we have

$$\begin{aligned}\zeta_{\mathcal{L}_{3(3)},\text{red}}^{\triangleleft\text{gr}}(s) &= V_{3,3}^{\triangleleft\text{gr}}(1, Y) = \frac{M_{3,3}(Y)}{(1-Y)^4(1-Y^3) \prod_{i=6}^{10} (1-Y^i)}, \\ \zeta_{\mathcal{L}_{3(3)},\text{top}}^{\triangleleft\text{gr}}(s) &= \frac{1}{70} \frac{-36 + 396s - 1383s^2 + 1505s^3}{s^3(s-3)(s-2)(s-1)(2s-1)^2(3s-2)(4s-1)(5s-2)(7s-2)(9s-2)}.\end{aligned}$$

Proof. For $(c, d) = (3, 3)$, (4.2.7) becomes

$$\zeta_{\mathcal{L}_{33}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}_{\mathfrak{p}}^4}(s) \sum_{v \in \mathcal{D}_3^2} D_v(q, t).$$

We have

$$\begin{aligned} D_{\mathbf{2}^3\mathbf{3}^3}(q, t) &= I_3(q^2t^3, q^2t^5, t^7) I_2(q^2t^8, q^2t^9, t^{10}), \\ D_{\mathbf{2}^2\mathbf{3}^2\mathbf{3}^2}(q, t) &= \binom{3}{2}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_2(q^2t^3, q^2t^5) I_1^\circ(q^3t^6) I_1(q^2t^8) I_2(q^2t^9, t^{10}), \\ D_{\mathbf{2}^2\mathbf{3}^2\mathbf{2}^3}(q, t) &= \binom{3}{2}_{q^{-1}} I_2(q^2t^3, q^2t^5) I_2^\circ(q^3t^6, q^2t^7) I_1(q^2t^9) I_1(t^{10}), \\ D_{\mathbf{2}^3\mathbf{2}^2\mathbf{3}^2}(q, t) &= \binom{3}{1}_{q^{-1}} I_1(q^2t^3) I_1^\circ(q^2t^4) I_2(q^3t^6, q^2t^8) I_2(q^2t^9, t^{10}), \\ D_{\mathbf{2}^3\mathbf{2}^3\mathbf{2}^3}(q, t) &= \binom{3}{2}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(q^2t^3) I_1^\circ(q^2t^4) I_1(q^3t^6) I_1^\circ(q^2t^7) I_1(q^2t^9) I_1(t^{10}). \end{aligned}$$

□

Proposition 4.2.15. *((c,d)=3,4) There exists a rational function $V_{3,4}^{\triangleleft_{\text{gr}}}(X, Y) \in \mathbb{Q}(X, Y)$ such that*

$$\zeta_{\mathcal{L}_{3(4)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = V_{3,4}^{\triangleleft_{\text{gr}}}(X, Y),$$

and

$$\zeta_{\mathcal{L}_{3(4)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) \Big|_{q \rightarrow q^{-1}} = -q^{22} t^{27} \zeta_{\mathcal{L}_{3^4}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s).$$

Setting

$$\begin{aligned} M_{3,4}(Y) &= 1 + 2Y + 2Y^2 + 4Y^3 + 10Y^4 + 20Y^5 + 35Y^6 + 49Y^7 + 69Y^8 + 96Y^9 \\ &\quad + 135Y^{10} + 187Y^{11} + 251Y^{12} + 327Y^{13} + 417Y^{14} + 519Y^{15} \\ &\quad + 629Y^{16} + 740Y^{17} + 861Y^{18} + 980Y^{19} + 1107Y^{20} + 1232Y^{21} \\ &\quad + 1349Y^{22} + 1448Y^{23} + 1532Y^{24} + 1591Y^{25} + 1630Y^{26} + 1642Y^{27} \\ &\quad + 1630Y^{28} + 1591Y^{29} + 1532Y^{30} + 1448Y^{31} + 1349Y^{32} + 1232Y^{33} \\ &\quad + 1107Y^{34} + 980Y^{35} + 861Y^{36} + 740Y^{37} + 629Y^{38} + 519Y^{39} \\ &\quad + 417Y^{40} + 327Y^{41} + 251Y^{42} + 187Y^{43} + 135Y^{44} + 96Y^{45} + 69Y^{46} \\ &\quad + 49Y^{47} + 35Y^{48} + 20Y^{49} + 10Y^{50} + 4Y^{51} + 2Y^{52} + 2Y^{53} + Y^{54} \in \mathbb{Q}[Y] \end{aligned}$$

and

$$\begin{aligned} T_{3,4}(s) &= s^3(s-1)^2(s-2)(s-3)(s-4)(4s-1)(4s-3)(5s-4)(6s-5)(7s-3) \\ &\quad \cdot (7s-4)(8s-5)(9s-5)(10s-3)(11s-4). \end{aligned}$$

we have

$$\zeta_{\mathcal{L}_{3(4),\text{red}}}^{\triangleleft\text{gr}}(s) = V_{3,4}^{\triangleleft\text{gr}}(1, Y) = \frac{M_{3,4}(Y)}{(1-Y)^3(1-Y^2)(1-Y^3)^2 \prod_{i=7}^{13}(1-Y^i)},$$

$$\zeta_{\mathcal{L}_{3(4),\text{top}}}^{\triangleleft\text{gr}}(s) = \frac{8}{117} \frac{561540s^5 - 1409277s^4 + 1365673s^3 - 636170s^2 + 141700s - 12000}{T_{3,4}(s)}.$$

Proof. For $(c, d) = (3, 4)$, (4.2.7) becomes

$$\zeta_{\mathcal{L}_{34}(\mathfrak{o})}^{\triangleleft\text{gr}}(s) = \zeta_{\mathfrak{o}_{\mathfrak{p}}^5}(s) \sum_{v \in \mathcal{D}_4^2} D_v(q, t).$$

$v = \mathbf{2^4 3^4}$:

$$D_{\mathbf{2^4 3^4}}(q, t) = I_4(q^3 t^3, q^4 t^5, q^3 t^7, t^9) I_4((q^{i(4-i)} t^{9+i})_{i=1, \dots, 4}).$$

$v = \mathbf{2^3 3^j 2^{3^{4-j}}}$, $j \in \{1, \dots, 3\}$:

$$D_{\mathbf{2^3 3^j 2^{3^{4-j}}}}(q, t) = \binom{4}{1}_{q^{-1}} \binom{3}{j}_{q^{-1}} I_3(q^3 t^3, q^4 t^5, q^3 t^7) \\ \cdot I_j^\circ((q^{3+i(3-i)} t^{7+i})_{i=1, \dots, j}) I_1(q^{j(4-j)} t^{9+j}) I_{4-j}((q^{i(4-i)} t^{9+i})_{i=j+1, \dots, 4}).$$

$v = \mathbf{2^2 3^{2j} 2^{3^{4-j}}}$, $j \in \{1, \dots, 3\}$:

For $j = 1$,

$$D_{\mathbf{2^2 3^{2j} 2^{3^{4-j}}}}(q, t) = \binom{4}{2}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_2(q^3 t^3, q^4 t^5) I_1^\circ(q^5 t^6) I_2(q^5 t^8, q^3 t^{10}) I_3((q^{i(4-i)} t^{9+i})_{i=2, \dots, 4}).$$

For $j \in \{2, 3\}$,

$$D_{\mathbf{2^2 3^{2j} 2^{3^{4-j}}}}(q, t) = \binom{4}{2}_{q^{-1}} \binom{2}{j}_{q^{-1}} \binom{2}{1}_{q^{-1}}^2 I_2(q^3 t^3, q^4 t^5) I_1^\circ(q^5 t^6) I_1(q^5 t^8) \\ \cdot I_{j-1}^\circ((q^{3+i(3-i)} t^{7+i})_{i=2, \dots, j}) I_1(q^{j(4-j)} t^{9+j}) I_{4-j}((q^{i(4-i)} t^{9+i})_{i=j+1, \dots, 4}).$$

$v = \mathbf{2^2 3^2 2^2 3^2}$:

$$D_{\mathbf{2^2 3^2 2^2 3^2}}(q, t) = \binom{4}{2}_{q^{-1}} I_2(q^3 t^3, q^4 t^4) I_2^\circ(q^5 t^6, q^4 t^7) I_2(q^5 t^9, q^4 t^{11}) I_2(q^3 t^{12}, t^{13}).$$

$v = \mathbf{2^2 3^2 2^{323}}$:

$$D_{\mathbf{2^2 3^2 2^{323}}}(q, t) = \binom{4}{2}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_2(q^3 t^3, q^4 t^5) I_2^\circ(q^5 t^6, q^4 t^7) I_1(q^5 t^9) \\ \cdot I_1^\circ(q^3 t^{10}) I_1(q^3 t^{12}) I_1(t^{13}).$$

$$v = \mathbf{232}^2\mathbf{3}^{j-1}\mathbf{23}^{4-j} :$$

For $j = 1$,

$$D_{\mathbf{232}^3\mathbf{3}^3}(q, t) = \binom{4}{3}_{q^{-1}} I_1(q^3 t^3) I_1^\circ(q^3 t^4) I_3(q^5 t^6, q^5 t^8, q^3 t^{10}) I_3((q^{i(4-i)} t^{9+i})_{i=2, \dots, 4}).$$

For $j \in \{2, 3\}$,

$$D_{\mathbf{232}^2\mathbf{3}^{j-1}\mathbf{23}^{4-j}}(q, t) = \binom{4}{3}_{q^{-1}} \binom{3}{2}_{q^{-1}} \binom{2}{j-1}_{q^{-1}} I_1(q^3 t^3) I_1^\circ(q^3 t^4) I_2(q^5 t^6, q^5 t^8) \\ \cdot I_{j-1}^\circ((q^{3+i(3-i)} t^{7+i})_{i=2, \dots, j}) I_1(q^{j(4-j)} t^{9+j}) I_{4-j}((q^{i(4-i)} t^{9+i})_{i=j+1, \dots, 4}).$$

$$v = \mathbf{23232}^2\mathbf{3}^2 :$$

$$D_{\mathbf{23232}^2\mathbf{3}^2}(q, t) = \binom{4}{3}_{q^{-1}} \binom{3}{1}_{q^{-1}} I_1(q^3 t^3) I_1^\circ(q^3 t^4) I_1(q^5 t^6) I_1^\circ(q^4 t^7) \\ \cdot I_2(q^5 t^9, q^4 t^{11}) I_2(q^3 t^{12}, t^{13}).$$

$$v = \mathbf{23232323} :$$

$$D_{\mathbf{23232323}}(q, t) = \binom{4}{3}_{q^{-1}} \binom{3}{2}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(q^3 t^3) I_1^\circ(q^3 t^4) I_1(q^5 t^6) I_1^\circ(q^4 t^7) \\ \cdot I_1(q^5 t^9) I_1^\circ(q^3 t^{10}) I_1(q^3 t^{12}) I_1^\circ(t^{13}).$$

□

4.3 $\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft}(s)$ for $c = 4$ and higher

In this section we prove and give the formula for $\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft}(s)$ for $c = 4$ in detail. Then we would suggest how one can extend this method for higher c .

Recall that for $c > 3$, Proposition 4.1.1 gives

$$\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^{1+d}}(s) \left(\sum_{\mu_2} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) q^{-s(3\mu_{1,2} + 2 \sum_{i=2}^d \mu_{i,2})} \right. \\ \left. \prod_{k=3}^c \sum_{\mu_k \leq \mu_{k-1}} \alpha(\mu_{k-1}, \mu_k; q) q^{-s(\sum_{j=1}^d \mu_{j,k})} \right) \quad (4.3.1)$$

To compute the right hand side of (4.3.1), let μ_k be partitions of d parts satisfying $\mu_k \leq \mu_{k-1}$ for $k \in [3, c]$. There are uniquely determined numbers $r \in \mathbb{N}_0$ and

$M_i^{(2)}, \dots, M_i^{(c)} \in \mathbb{N}$ ($i \in [r-1]$), such that

$$\begin{aligned} \mu_{1,2} \geq \dots \geq \mu_{M_1^{(2)},2} \geq \mu_{1,3} \geq \dots \geq \mu_{M_1^{(3)},3} \geq \dots \geq \mu_{1,c} \geq \dots \geq \mu_{M_1^{(c)},c} > \\ \mu_{M_1^{(2)}+1,2} \geq \dots \geq \mu_{M_2^{(2)},2} \geq \mu_{M_1^{(3)}+1,3} \geq \dots \geq \mu_{M_2^{(3)},3} \geq \dots \geq \mu_{M_1^{(c)}+1,c} \geq \dots \geq \mu_{M_2^{(c)},c} > \\ \dots \mu_{M_{r-1}^{(2)}+1,2} \geq \dots \geq \mu_{d,2} \geq \mu_{M_{r-1}^{(3)}+1,3} \geq \dots \geq \mu_{d,3} \geq \dots \geq \mu_{M_{r-1}^{(c)}+1,c} \geq \dots \geq \mu_{d,c}. \end{aligned}$$

Define $M_r^{(k)} = d$ and $M_0^{(k)} = 0$ for all $k \in [2, c]$. Observe that the condition $\mu_k \leq \mu_{k-1}$ is equivalent to the condition $M_i^{(k-1)} \geq M_i^{(k)}$ for $i \in [r-1]$. In this case we write $v(\mu_2, \dots, \mu_c) = v$. Note that v determines and is determined by $M_i^{(k)}$'s.

This time the *Dyck word of length $(c-1)d$* is a word w in the letters $\mathbf{2}, \mathbf{3}, \dots, \mathbf{c}$ such that each occur d times and no initial segment of w contains more k 's than $k-1$'s. We denote the set of $(c-1)$ -dimensional Dyck words of length $(c-1)d$ by $\mathcal{D}_d^{c-1}$ and note that the cardinality of $\mathcal{D}_d^{c-1}$ is the multidimensional Catalan numbers

$$\text{Cat}_d^{c-1} = \left[\prod_{i=0}^{c-2} \frac{i!}{(d+i)!} \right] ((c-1)d)!.$$

Note that $c = 3$ indeed gives $\text{Cat}_d^2 = \frac{1}{d+1} \binom{2d}{d}$. It is easy to see that our $v(\mu_2, \dots, \mu_c) = v$ is indeed a $(c-1)$ -dimensional Dyck word.

For $v \in \mathcal{D}_d^{c-1}$ we set

$$\begin{aligned} D_v^{c-1}(q, t) := \\ \sum_{\substack{(\mu_2, \dots, \mu_c) \\ v(\mu_2, \dots, \mu_c) = v}} \alpha(\mu_{1,2}^{(d)}, \mu_2; q) \alpha(\mu_2, \mu_3; q) \dots \alpha(\mu_c, \mu_{c-1}; q) q^{-(3\mu_{1,2} + 2 \sum_{i=2}^d \mu_{i,2} + \sum_{k=3}^c \sum_{j=1}^d \mu_{j,k})s}. \end{aligned} \quad (4.3.2)$$

Proposition 4.1.1 and (4.2.5) allows us to write

$$\zeta_{\mathcal{L}_{c^d}^{\triangleleft \text{gr}}(\mathfrak{o})}(s) = \zeta_{\mathfrak{o}_{\mathfrak{p}}^{d+1}}(s) \sum_{v \in \mathcal{D}_d^{c-1}} D_v^{c-1}(q, t). \quad (4.3.3)$$

4.3.1 For $c = 4$

Since we fixed our c to be 4, to improve the readability we would like to introduce different notations. Let

$$\lambda = (\lambda_1, \dots, \lambda_d), \mu = (\mu_1, \dots, \mu_d), \nu = (\nu_1, \dots, \nu_d)$$

be partitions of d parts satisfying $\mu \leq \lambda$ and $\nu \leq \mu$. There are uniquely determined numbers $r \in \mathbb{N}_0$ and $L_i, M_i, N_i \in \mathbb{N}$ ($i \in [r-1]$), such that

$$\begin{aligned} \lambda_1 &\geq \cdots \geq \lambda_{L_1} \geq \mu_1 \geq \cdots \geq \mu_{M_1} \geq \nu_1 \geq \cdots \geq \nu_{N_1} > \\ \lambda_{L_1+1} &\geq \cdots \geq \lambda_{L_2} \geq \mu_{M_1+1} \geq \cdots \geq \mu_{M_2} \geq \nu_{N_1+1} \geq \cdots \geq \nu_{N_2} > \cdots \\ \lambda_{L_{r-1}+1} &\geq \cdots \geq \lambda_d \geq \mu_{M_{r-1}+1} \geq \cdots \geq \mu_d \geq \nu_{N_{r-1}+1} \geq \cdots \geq \nu_d. \end{aligned} \quad (4.3.4)$$

We call the integer sequence $(L_i, M_i, N_i)_{i \in [r-1]}$ arising the *overlap type* of the triple (λ, μ, ν) . This determines and is determined uniquely by the 3-dimensional Dyck word of length $3d$. In this case we write $v(\lambda, \mu, \nu) = v$. For $v \in \mathcal{D}^{(3)}$ we set

$$D_v^3(q, t) := \sum_{\substack{(\lambda, \mu, \nu) \\ v(\lambda, \mu, \nu) = v}} \alpha(\lambda_1^{(d)}, \lambda; q) \alpha(\lambda, \mu; q) \alpha(\mu, \nu; q) q^{-(3\lambda_1 + \sum_{i=2}^d 2\lambda_i + \sum_{j=1}^d \mu_j + \sum_{k=1}^d \nu_k)s}. \quad (4.3.5)$$

We know

$$\zeta_{\mathcal{L}_{cd}^{gr}(o)}^{\triangleleft}(s) = \zeta_{\mathcal{O}_{\mathfrak{p}}^{d+1}}(s) \sum_{v \in \mathcal{D}_d^{(3)}} D_v^3(q, t). \quad (4.3.6)$$

With this new notation, we have the following result.

Theorem 4.3.1. *Let $v = \prod_{i=1}^r (2^{L_i-L_{i-1}} 3^{M_i-M_{i-1}} 4^{N_i-N_{i-1}}) \in \mathcal{D}_d^{(3)}$. Then*

$$\begin{aligned} D_v^3(q, t) &= \prod_{i=1}^r \binom{d-L_{i-1}}{d-L_i}_{q^{-1}} \binom{L_i-M_{i-1}}{L_i-M_i}_{q^{-1}} \binom{M_i-N_{i-1}}{M_i-N_i}_{q^{-1}} \\ &\quad \cdot \left(\prod_{i=1}^r I_{L_i-L_{i-1}}(Z_{L_{i-1}+1}, \dots, Z_{L_i}) \right) \left(\prod_{i=1}^{r-1} I_{M_i-M_{i-1}}^*(Y_{M_{i-1}+1}, \dots, Y_{M_i}) \right) \\ &\quad \cdot \prod_{i=1}^{r-1} I_{N_i-N_{i-1}}^\circ(X_{N_{i-1}+1}, \dots, X_{N_i}) \\ &\quad \cdot I_{d-M_{r-1}}(Y_{M_{r-1}+1}, \dots, Y_d) I_{d-N_{r-1}}(X_{N_{r-1}+1}, \dots, X_d), \end{aligned}$$

with numerical data

$$X_j = q^{L_i(3-L_i)+M_i(L_i-M_i)+j(M_i-j)} t^{\check{L}_i+M_i+j} \quad j \in]N_{i-1}, N_i], \quad (4.3.7)$$

$$Y_j = q^{L_i(3-L_i)+j(L_i-j)+N_{i-1}(j-N_{i-1})} t^{\check{L}_i+N_{i-1}+j} \quad j \in]M_{i-1}, M_i], \quad (4.3.8)$$

$$Z_j = q^{j(3-j)+M_{i-1}(j-M_{i-1})+N_{i-1}(M_{i-1}-N_{i-1})} t^{M_{i-1}+N_{i-1}+j} \quad j \in]L_{i-1}, L_i], \quad (4.3.9)$$

and

$$I_{M_i-M_{i-1}}^* = \begin{cases} I_{M_i-M_{i-1}} & \text{if } N_i - N_{i-1} > 0, \\ I_{M_i-M_{i-1}}^\circ & \text{if } N_i - N_{i-1} = 0. \end{cases}$$

To prove Theorem 4.3.1 we start with definitions. First we set, for $i \in [r]$,

$$\begin{aligned} a_j &:= L_i(d - L_i) + M_i(L_i - M_i) + j(M_i - j), & j \in]N_{i-1}, N_i], \\ b_j &:= L_i(d - L_i) + j(L_i - j) + N_{i-1}(j - N_{i-1}), & j \in]M_{i-1}, M_i], \\ c_j &:= j(d - j) + M_{i-1}(j - M_{i-1}) + N_{i-1}(M_{i-1} - N_{i-1}), & j \in]L_{i-1}, L_i]. \end{aligned}$$

It is also useful for us to have notation for the successive differences of the parts of λ , μ , and ν . For $j \in [d]$ we set

$$r_j = \begin{cases} \lambda_j - \lambda_{j+1} & \text{if } j \notin \{L_1, \dots, L_r\}, \\ \lambda_{L_i} - \mu_{M_{i-1}+1} & \text{if } j = L_i > L_{i-1}, M_i > M_{i-1}, \\ \lambda_{L_i} - \nu_{N_{i-1}+1} & \text{if } j = L_i > L_{i-1}, M_i = M_{i-1}, \end{cases} \quad (4.3.10)$$

and

$$s_j = \begin{cases} \mu_j - \mu_{j+1} & \text{if } j \notin \{M_1, \dots, M_r\}, \\ \mu_{M_i} - \nu_{N_{i-1}+1} & \text{if } j = M_i > M_{i-1}, N_i > N_{i-1}, \\ \mu_{M_i} - \lambda_{L_{i+1}} & \text{if } j = M_i > M_{i-1}, N_i = N_{i-1}, \end{cases} \quad (4.3.11)$$

where we define $\lambda_{d+1} = \mu_{d+1} = 0$. Similarly, for $j \in [d]$ we set

$$t_j = \begin{cases} \nu_j - \nu_{j+1} & \text{if } j \notin \{N_1, \dots, N_r\}, \\ \nu_{N_i} - \lambda_{L_{i+1}} & \text{if } j = N_i > N_{i-1}, L_{i+1} > L_i, \\ \nu_{N_i} - \mu_{M_{i+1}} & \text{if } j = N_i > N_{i-1}, L_{i+1} = L_i. \end{cases} \quad (4.3.12)$$

Again set

$$\checkmark : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad n \mapsto n \cdot (3, 2, \dots, 2) := 3\delta_{n \geq 1} + 2\delta_{n \geq 2} + \dots + 2\delta_{n \geq d}.$$

We have

$$\begin{aligned} 3\lambda_1 + 2 \sum_{k=2}^d \lambda_k &= \sum_{j=1}^d \checkmark r_j + \sum_{i=1}^r \checkmark L_i (s_{M_{i-1}+1} + \dots + s_{M_i} + t_{N_{i-1}+1} + \dots + t_{N_i}), \\ \mu_1 + \dots + \mu_d &= \sum_{j=1}^d j s_j + \sum_{i=1}^{r-1} M_i (r_{L_{i+1}} + \dots + r_{L_{i+1}}) + \sum_{i=1}^r M_i (t_{N_{i-1}+1} + \dots + t_{N_i}), \\ \nu_1 + \dots + \nu_d &= \sum_{j=1}^d j t_j + \sum_{i=1}^{r-1} N_i (r_{L_{i+1}} + \dots + r_{L_{i+1}} + s_{M_{i+1}} + \dots + s_{M_{i+1}}). \end{aligned}$$

By our definition, r_j, s_j and $t_j \geq 0$ for all j . Note also that $s_j > 0$ if $N_{i-1} = N_i$, $t_j > 0$ if $j \in \{N_1, \dots, N_{r-1}\}$ and observe that $\lambda_{L_i} > \lambda_{L_{i+1}}, \mu_{M_i} > \mu_{M_{i+1}}$, and $\nu_{N_i} > \nu_{N_{i+1}}$ for each $i \in [r-1]$. Finally, for each $i \in [r]$ we define

$$\begin{aligned} J_i^\lambda &= \{j \in [L_i - L_{i-1} - 1] \mid \lambda_{L_i-j} > \lambda_{L_{i-j}+1}\}, \\ J_i^\mu &= \{j \in [M_i - M_{i-1} - 1] \mid \mu_{M_i-j} > \mu_{M_{i-j}+1}\}, \\ J_i^\nu &= \{j \in [N_i - N_{i-1} - 1] \mid \nu_{N_i-j} > \nu_{N_{i-j}+1}\}. \end{aligned} \quad (4.3.13)$$

Lemma 4.3.2. *Let λ , μ , and ν be partitions of d parts satisfying $\mu \leq \lambda$ and $\nu \leq \mu$. Then*

$$\begin{aligned} & \sum_v \alpha(\lambda_1^{(d)}, \lambda; q) \alpha(\lambda, \mu; q) \alpha(\mu, \nu; q) \\ &= \prod_{k \geq 1} q^{\lambda'_k(d-\lambda'_k) + \mu'_k(\lambda'_k - \mu'_k) + \nu'_k(\mu'_k - \nu'_k)} \binom{d - \lambda'_{k+1}}{3 - \lambda'_k}_{q^{-1}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{q^{-1}} \binom{\mu'_k - \nu'_{k+1}}{\mu'_k - \nu'_k}_{q^{-1}}. \end{aligned}$$

Let λ , μ , and ν be partitions as in Lemma 4.3.2. Let $(L_i, M_i, N_i)_{i \in [r-1]}$ be the parameters associated to them and λ' , μ' , and ν' their dual partitions. For $k \in \mathbb{N}$, set

$$d_k := \lambda'_k(3 - \lambda'_k) + \mu'_k(\lambda'_k - \mu'_k) + \nu'_k(\mu'_k - \nu'_k).$$

Lemma 4.3.3. *For $i \in [r-1]$,*

$$\begin{aligned} & \prod_{k=\mu_{M_{i-1}+1}+1}^{\lambda_{L_{i-1}+1}} q^{d_k} \binom{d - \lambda'_{k+1}}{d - \lambda'_k}_{q^{-1}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{q^{-1}} \binom{\mu'_k - \nu'_{k+1}}{\mu'_k - \nu'_k}_{q^{-1}} \\ &= \prod_{j=1}^{L_i - L_{i-1}} q^{a_{L_{i-1}+j} r_{L_{i-1}+j}} \binom{L_i - L_{i-1}}{J_i^\lambda}_{q^{-1}} \binom{d - L_{i-1}}{d - L_i}_{q^{-1}}. \end{aligned}$$

Proof. Note that the product on the left-hand side may be empty; this happens in the case $\lambda_{L_i} = \lambda_{L_{i-1}}$. Also since $\mu_{M_{i-1}} \geq \nu_{N_{i-1}} > \lambda_{L_{i-1}+1}$, we have $\nu'_k = \nu'_{k+1} = N_{i-1}$ and $\mu'_k = \mu'_{k+1} = M_{i-1}$, for all k in the interval $]\mu_{M_{i-1}+1}, \lambda_{L_{i-1}+1}]$. Thus we always have

$$\prod_{k=\mu_{M_{i-1}+1}+1}^{\lambda_{L_{i-1}+1}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{q^{-1}} \binom{\mu'_k - \nu'_{k+1}}{\mu'_k - \nu'_k}_{q^{-1}} = 1.$$

For $j \in [L_i - L_{i-1}]$ we have $\lambda'_k = L_{i-1} + j$ when $\lambda_{L_{i-1}+j+1} < k \leq \lambda_{L_{i-1}+j}$; observe that it may be the case for some j that no index k satisfies this condition. As a result, we see that for each $j \in [L_i - L_{i-1}]$, there are exactly $r_{L_{i-1}+j}$ elements of k of the segment $]\mu_{M_{i-1}+1}, \lambda_{L_{i-1}+1}]$ for which $\lambda'_k = L_{i-1} + j$. Now, note that

$$\binom{d - \lambda'_{k+1}}{d - \lambda'_k}_{q^{-1}} \neq 1$$

if and only if $\lambda'_{k+1} \neq \lambda'_k$. This is exactly when there exists an i such that $\lambda_i = k$. It follows that if $J_i^\lambda = \{j_{i,1}, \dots, j_{i,\gamma_i}\}$, with $j_{i,1} < \dots < j_{i,\gamma_i}$, then

$$\begin{aligned} & \prod_{k=\mu_{M_{i-1}+1}+1}^{\lambda_{L_{i-1}+1}} \binom{d - \lambda'_{k+1}}{d - \lambda'_k}_{q^{-1}} \\ &= \binom{d - L_i + j_{i,\gamma_i}}{d - L_i}_{q^{-1}} \cdot \prod_{m=1}^{\gamma_i-1} \binom{d - L_i + j_{i,m+1}}{d - L_i + j_{i,m}}_{q^{-1}} \cdot \binom{d - L_{i-1}}{d - L_i + j_{i,\gamma_i}}_{q^{-1}}. \end{aligned} \tag{4.3.14}$$

We make use of the well-known identity

$$\binom{\alpha}{\beta}_X = \frac{1 - X^\alpha}{1 - X^{a-\beta}} \binom{\alpha - 1}{\beta}_X$$

for Gaussian binomial coefficients. Applying it inductively, we see that for all $m \in [\gamma_i - 1]$,

$$\binom{d - L_i + j_{i,m+1_i}}{d - L_i + j_{i,m}}_{q^{-1}} = \binom{j_{i,m+1}}{j_{i,m}}_{q^{-1}} \frac{\binom{d - L_i + j_{i,m+1}}{d - L_i}_{q^{-1}}}{\binom{d - L_i + j_{i,m}}{d - L_i}_{q^{-1}}}.$$

Hence the right-hand side of 4.3.14 is equal to

$$\binom{L_i - L_{i-1}}{J_i^\lambda}_{q^{-1}} \binom{d - L_{i-1}}{d - L_i}_{q^{-1}}.$$

Thus

$$\begin{aligned} \prod_{k=\mu_{M_{i-1}+1}+1}^{\lambda_{L_{i-1}+1}} q^{d_k} \binom{d - \lambda'_{k+1}}{d - \lambda'_k}_{q^{-1}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{q^{-1}} \binom{\mu'_k - \nu'_{k+1}}{\mu'_k - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{L_i - L_{i-1}} q^{a_{L_{i-1}+j} r_{L_{i-1}+j}} \binom{L_i - L_{i-1}}{J_i^\lambda}_{q^{-1}} \binom{d - L_{i-1}}{d - L_i}_{q^{-1}} \end{aligned}$$

as required. $\square$

Lemma 4.3.4. For $i \in [r - 1]$,

$$\begin{aligned} \prod_{k=\nu_{N_{i-1}+1}+1}^{\mu_{M_{i-1}+1}} q^{d_k} \binom{d - \lambda'_{k+1}}{d - \lambda'_k}_{q^{-1}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{q^{-1}} \binom{\mu'_k - \nu'_{k+1}}{\mu'_k - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{M_i - M_{i-1}} q^{b_{M_{i-1}+j} s_{M_{i-1}+j}} \binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} \binom{L_i - M_{i-1}}{L_i - M_i}_{q^{-1}}. \end{aligned}$$

Proof. Analogous to the proof of Lemma 4.3.3. $\square$

Lemma 4.3.5. For $i \in [r - 1]$,

$$\begin{aligned} \prod_{k=\lambda_{L_i+1}+1}^{\nu_{N_{i-1}+1}} q^{d_k} \binom{d - \lambda'_{k+1}}{d - \lambda'_k}_{q^{-1}} \binom{\lambda'_k - \mu'_{k+1}}{\lambda'_k - \mu'_k}_{q^{-1}} \binom{\mu'_k - \nu'_{k+1}}{\mu'_k - \nu'_k}_{q^{-1}} \\ = \prod_{j=1}^{N_i - N_{i-1}} q^{c_{N_{i-1}+j} t_{N_{i-1}+j}} \binom{N_i - N_{i-1}}{J_i^\nu}_{q^{-1}} \binom{M_i - N_{i-1}}{M_i - N_i}_{q^{-1}}. \end{aligned}$$

Proof. Analogous to the proof of Lemma 4.3.3. $\square$

Proof of Theorem 4.3.1. Given vectors $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{N}_0^d$, we set, for each $i \in [r]$,

$$\begin{aligned}\text{supp}_i^L(\mathbf{v}) &= \{j \in [L_i - L_{i-1} - 1] \mid v_{L_{i-1}+j} > 0\}, \\ \text{supp}_i^M(\mathbf{v}) &= \{j \in [M_i - M_{i-1} - 1] \mid v_{M_{i-1}+j} > 0\}, \\ \text{supp}_i^N(\mathbf{v}) &= \{j \in [N_i - N_{i-1} - 1] \mid v_{N_{i-1}+j} > 0\}.\end{aligned}$$

In practice, $\mathbf{v}$ will be one of the vectors of successive differences $\mathbf{r} = (r_1, \dots, r_d)$ or $\mathbf{s} = (s_1, \dots, s_d)$ or $\mathbf{t} = (t_1, \dots, t_d)$. Given a set of partitions satisfying $\mu \leq \lambda$ and $\nu \leq \mu$, recall our definition 4.3.13. It is easy to see that, for every $i \in [r]$,

$$\begin{aligned}\text{supp}_i^L(\mathbf{v}) &= L_i - L_{i-1} - J_i^\lambda, \\ \text{supp}_i^M(\mathbf{v}) &= M_i - M_{i-1} - J_i^\mu, \\ \text{supp}_i^N(\mathbf{v}) &= N_i - N_{i-1} - J_i^\nu.\end{aligned}\tag{4.3.15}$$

Thus it follows that

$$\begin{aligned}\binom{L_i - L_{i-1}}{J_i^\lambda}_{q^{-1}} &= \binom{L_i - L_{i-1}}{\text{supp}_i^L(\mathbf{v})}_{q^{-1}}, \\ \binom{M_i - M_{i-1}}{J_i^\mu}_{q^{-1}} &= \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}}, \\ \binom{N_i - N_{i-1}}{J_i^\nu}_{q^{-1}} &= \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v})}_{q^{-1}}.\end{aligned}\tag{4.3.16}$$

Let δ_{ij} be the usual Kronecker delta function. Substituting Lemma 4.3.3, 4.3.4, and 4.3.5, rewriting the expressions in terms of r_j , s_j and t_j and using (4.3.16), we find that formula (4.3.5) for $D_w(q, t)$ splits into a product as follows:

$$D_v^3(q, t) = \prod_{i=1}^r \binom{d - L_{i-1}}{d - L_i}_{q^{-1}} \binom{L_i - M_{i-1}}{L_i - M_i}_{q^{-1}} \binom{M_i - N_{i-1}}{M_i - N_i}_{q^{-1}} \prod_{i=1}^r A_i B_i C_i,$$

where, for $i \in [r]$,

$$\begin{aligned}A_i &= \sum_{t_{N_{i-1}+1}=0}^{\infty} \cdots \sum_{t_{N_i-1}=0}^{\infty} \sum_{t_{N_i}=1-\delta_{ir}}^{\infty} \binom{N_i - N_{i-1}}{\text{supp}_i^N(\mathbf{v})}_{q^{-1}} \prod_{j=N_{i-1}+1}^{N_i} \left(q^{a_j} t^{\tilde{L}_i + M_i + j}\right)^{t_j}, \\ B_i &= \sum_{s_{M_{i-1}+1}=0}^{\infty} \cdots \sum_{s_{M_i-1}=0}^{\infty} \sum_{s_{M_i}=\delta_{N_{i-1}N_i}}^{\infty} \binom{M_i - M_{i-1}}{\text{supp}_i^M(\mathbf{v})}_{q^{-1}} \prod_{j=M_{i-1}+1}^{M_i} \left(q^{b_j} t^{\tilde{L}_i + N_{i-1} + j}\right) s_j, \\ C_i &= \sum_{r_{L_{i-1}+1}=0}^{\infty} \cdots \sum_{r_{L_i}=0}^{\infty} \binom{L_i - L_{i-1}}{\text{supp}_i^L(\mathbf{v})}_{q^{-1}} \prod_{j=L_{i-1}+1}^{L_i} \left(q^{c_j} t^{M_{i-1} + N_{i-1} + j}\right)^{r_j}.\end{aligned}$$

We now show that all of the factors A_i , B_i , and C_i are products of Igusa functions and Gaussian binomial coefficients. Given $i \in [r]$ and $I \subseteq [L_i - L_{i-1} - 1]$, we define

$\mathbf{R}^i(I)$ to be the set of vectors $\mathbf{r}^i = (r_{L_{i-1}+1}, \dots, r_{L_i}) \in \mathbb{N}_0^{L_i-L_{i-1}}$ such that $r_j = 0$ unless $j \in \{L_{i-1} + k \mid k \in I\} \cup \{L_i\}$. With the numerical data defined in (4.3.7), we have

$$\begin{aligned}
C_i &= \sum_{I \subseteq [L_i-L_{i-1}-1]} \binom{L_i-L_{i-1}}{I} \sum_{q^{-1} \mathbf{r}^i \in \mathbf{R}^i(I)} \prod_{j \in (I+L_{i-1}) \cup \{L_i\}} \left(q^{c_j} t^{M_{i-1}+N_{i-1}+j} \right)^{r_j} \\
&= \sum_{I \subseteq [L_i-L_{i-1}-1]} \binom{L_i-L_{i-1}}{I} \left(\prod_{\iota \in I} \left(\sum_{r_{L_{i-1}+\iota}=1}^{\infty} (Z_{L_{i-1}+\iota})^{r_{L_{i-1}+\iota}} \right) \right) \sum_{r_{L_i}=0}^{\infty} (Z_{L_i})^{r_{L_i}} \\
&= \frac{1}{1-Z_{L_i}} \sum_{I \subseteq [L_i-L_{i-1}-1]} \binom{L_i-L_{i-1}}{I} \prod_{\iota \in I} \frac{Z_{L_{i-1}+\iota}}{1-Z_{L_{i-1}+\iota}} \\
&= I_{L_i-L_{i-1}}(Z_{L_{i-1}+1}, \dots, Z_{L_i}).
\end{aligned}$$

Analogously one can show that, with the numerical data defined in (4.3.7),

$$B_i = \begin{cases} I_{M_i-M_{i-1}}^{\circ}(Y_{M_{i-1}+1}, \dots, Y_{M_i}) & \text{for } N_{i-1} = N_i, \\ I_{M_i-M_{i-1}}(Y_{M_{i-1}+1}, \dots, Y_{M_i}) & \text{for } N_{i-1} < N_i, \end{cases}$$

and

$$A_i = \begin{cases} I_{N_i-N_{i-1}}^{\circ}(X_{N_{i-1}+1}, \dots, X_{N_i}) & \text{for } i \in [r-1], \\ I_{N_i-N_{i-1}}(X_{N_{i-1}+1}, \dots, X_{N_i}) & \text{for } i = r. \end{cases}$$

This completes the proof of Theorem 4.3.1. $\square$

4.3.2 Computation of the functions D_v^{c-1} for higher c – theory

The main difference between $c = 3$ and $c \geq 4$ is that there are “middle values”. What do we mean by this? When $c = 3$, the 2-dimensional Dyck words only contain **2** and **3**, and it is guaranteed that there are always **2**’s after **3**’s, except the very last part where it ends. However, when $c > 3$, there are “jumps” between numbers with difference more than one. For example, **232434** $\in \mathcal{D}_2^3$ and we have **4** after **2**. That gives the alternating behaviour of the Igusa function $I_{M_i^{(k)}-M_{i-1}^{(k)}}^*$ in Theorem 4.3.1. Once we deal with this different jumps using alternating Igusa functions, then higher c only means we have higher dimensional Dyck words with more case distinctions. However, the principle logic behind the proofs of these theorems are the same.

Theorem 4.3.6. *Let $v = \prod_{i=1}^r \left(\mathbf{2}^{M_i^{(2)}-M_{i-1}^{(2)}} \mathbf{3}^{M_i^{(3)}-M_{i-1}^{(3)}} \dots \mathbf{c}^{M_i^{(c)}-M_{i-1}^{(c)}} \right) \in \mathcal{D}_d^{(c-1)}$. Then*

$$\begin{aligned}
D_v^{c-1}(q, t) = & \prod_{i=1}^r \binom{d - M_{i-1}^{(2)}}{d - M_i^{(2)}}_{q^{-1}} \left(\prod_{k=3}^c \binom{M_i^{(k-1)} - M_{i-1}^{(k)}}{M_i^{(k-1)} - M_i^{(k)}}_{q^{-1}} \right) \\
& \cdot \left(\prod_{i=1}^r I_{M_i^{(2)} - M_{i-1}^{(2)}} \left(X_{M_{i-1}^{(2)}+1}^{(2)}, \dots, X_{M_i^{(2)}}^{(2)} \right) \right) \\
& \cdot \left(\prod_{j=3}^{c-1} \prod_{i=1}^{r-1} I_{M_i^{(j)} - M_{i-1}^{(j)}}^* \left(X_{M_{i-1}^{(j)}+1}^{(j)}, \dots, X_{M_i^{(j)}}^{(j)} \right) \right) \\
& \cdot \prod_{i=1}^{r-1} I_{M_i^{(c)} - M_{i-1}^{(c)}}^o \left(X_{M_{i-1}^{(c)}+1}^{(c)}, \dots, X_{M_i^{(c)}}^{(c)} \right) \prod_{k=3}^c I_{d - M_{r-1}^{(k)}} \left(X_{M_{r-1}^{(k)}+1}^{(k)}, \dots, X_d^{(k)} \right),
\end{aligned}$$

with numerical data

$$X_j^{(c)} = q^{a_j^{(c)}} t^{\sum_{i=2}^{c-1} M_i^{(i)} + j} \quad j \in \left[M_{i-1}^{(c)}, M_i^{(c)} \right], \quad (4.3.17)$$

$$X_j^{(k)} = q^{a_j^{(k)}} t^{\sum_{i=2}^{k-1} M_i^{(i)} + \sum_{i=k+1}^c M_{i-1}^{(i)} + j} \quad j \in \left[M_{i-1}^{(k)}, M_i^{(k)} \right], \quad (4.3.18)$$

$$X_j^{(2)} = q^{a_j^{(2)}} t^{\sum_{i=3}^c M_{i-1}^{(i)} + j} \quad j \in \left[M_{i-1}^{(2)}, M_i^{(2)} \right], \quad (4.3.19)$$

where $k \in [3, c-1]$,

$$\begin{aligned}
a_j^{(c)} = & M_i^{(2)} \left(d - M_i^{(2)} \right) + \sum_{\iota=3}^{c-1} \left(M_i^{(\iota)} \left(M_i^{(\iota-1)} - M_i^{(\iota)} \right) \right) + j \left(M_i^{(c-1)} - j \right), \\
a_j^{(k)} = & M_i^{(2)} \left(d - M_i^{(2)} \right) + \sum_{\iota=3}^{k-1} \left(M_i^{(\iota)} \left(M_i^{(\iota-1)} - M_i^{(\iota)} \right) \right) + j \left(M_i^{(k-1)} - j \right) \\
& + M_{i-1}^{(k+1)} \left(j - M_{i-1}^{(k+1)} \right) + \sum_{\iota=k+2}^c \left(M_{i-1}^{(\iota)} \left(M_{i-1}^{(\iota-1)} - M_{i-1}^{(\iota)} \right) \right), \\
a_j^{(2)} = & j(d - j) + M_{i-1}^{(3)} \left(j - M_{i-1}^{(3)} \right) + \sum_{\iota=4}^c \left(M_{i-1}^{(\iota)} \left(M_{i-1}^{(\iota-1)} - M_{i-1}^{(\iota)} \right) \right),
\end{aligned}$$

and

$$I_{M_i^{(k)} - M_{i-1}^{(k)}}^* = \begin{cases} I_{M_i^{(k)} - M_{i-1}^{(k)}} & \text{if } \exists \iota \in [k+1, c] \ M_i^{(\iota)} - M_{i-1}^{(\iota)} > 0, \\ I_{M_i^{(k)} - M_{i-1}^{(k)}}^o & \text{if } \forall \iota \in [k+1, c] \ M_i^{(\iota)} - M_{i-1}^{(\iota)} = 0. \end{cases}$$

Theorem 4.3.7 (Uniformity). *There exists $V_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = V_{c,d}^{\triangleleft \text{gr}}(q, q^{-s}).$$

4.3.3 Explicit examples for $c > 3$

We record the following consequences of Theorem 4.3.1 and 4.3.6

Proposition 4.3.8. *$((c,d)=4,2)$ Let*

$$\begin{aligned} U_{4,2}(q, t) = & 1 + t^3 + t^4 + t^5 + t^6 + t^7 - qt^7 + t^8 - qt^8 - 2qt^9 - 2qt^{10} \\ & + t^{11} - 3qt^{11} - q^2t^{11} - 2qt^{12} + q^2t^{12} - 3qt^{13} \\ & - 2qt^{14} + q^2t^{14} - 3qt^{15} + q^2t^{15} - qt^{16} + 2q^2t^{16} \\ & - 2qt^{17} + q^2t^{17} - qt^{18} + 3q^2t^{18} - qt^{19} + 2q^2t^{19} \\ & + 3q^2t^{20} - qt^{21} + 2q^2t^{21} + qt^{22} + 3q^2t^{22} - q^3t^{22} \\ & + 2q^2t^{23} + 2q^2t^{24} + q^2t^{25} - q^3t^{25} + q^2t^{26} \\ & - q^3t^{26} - q^3t^{27} - q^3t^{28} - q^3t^{29} - q^3t^{30} - q^3t^{33} \in \mathbb{Q}(q, t). \end{aligned}$$

We have

$$\zeta_{\mathcal{L}_{4(2)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = \zeta_{\mathfrak{o}^3}(s) \cdot \frac{U_{4,2}(q, t)}{(1 - qt^3)(1 - qt^4)(1 - t^5)(1 - qt^5)(1 - qt^6)(1 - t^7)(1 - qt^7)(1 - qt^8)(1 - t^9)},$$

and

$$\zeta_{\mathcal{L}_{4(2)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) \Big|_{q \rightarrow q^{-1}} = -q^6 t^{24} \zeta_{\mathcal{L}_{42}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s).$$

Let

$$\zeta_{\mathcal{L}_{4(2)}(\mathfrak{o})}^{\triangleleft_{\text{gr}}}(s) = V_{4,2}^{\triangleleft_{\text{gr}}}(X, Y).$$

Setting

$$\begin{aligned} M_{4,2}(Y) = & 1 + 2Y^3 + Y^4 + 2Y^5 + 3Y^6 + 2Y^7 + 3Y^8 + 2Y^9 + 3Y^{10} \\ & + 2Y^{11} + 3Y^{12} + 2Y^{13} + Y^{14} + 2Y^{15} + Y^{18} \in \mathbb{Q}[Y], \end{aligned}$$

we have

$$\begin{aligned} \zeta_{\mathcal{L}_{4(2)}, \text{red}}^{\triangleleft_{\text{gr}}}(s) &= V_{3,2}^{\triangleleft_{\text{gr}}}(1, Y) \\ &= \frac{M_{4,2}(Y)}{(1 - Y)^3 \prod_{i=4}^9 (1 - Y^i)}, \\ \zeta_{\mathcal{L}_{3(2)}, \text{top}}^{\triangleleft_{\text{gr}}}(s) &= \frac{2}{315} \frac{1575s^3 - 648s^2 + 88s - 4}{s^4(s-1)(s-2)(3s-1)(4s-1)(5s-1)(6s-1)(7s-1)(8s-1)}. \end{aligned}$$

Proof. For $(c, d) = (4, 2)$, (4.3.6) becomes

$$\zeta_{\mathcal{L}_{4^2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \zeta_{\mathfrak{o}_{\mathfrak{p}}^3}(s) \sum_{v \in \mathcal{D}_2^3} D_v^3(q, t).$$

We have

$$\begin{aligned} D_{\mathbf{2}^3 \mathbf{3}^2 \mathbf{4}^2}^3(q, t) &= I_2(qt^3, t^5) I_2(qt^6, t^7) I_2(qt^8, t^9), \\ D_{\mathbf{2}^3 \mathbf{2}^3 \mathbf{3}^4}^3(q, t) &= \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1(t^7) I_2(qt^8, t^9), \\ D_{\mathbf{2}^3 \mathbf{3}^4 \mathbf{3}^4}^3(q, t) &= \binom{2}{1}_{q^{-1}} I_2(qt^3, t^5) I_1(qt^6) I_1^\circ(qt^7) I_1(qt^8) I_1(t^9), \\ D_{\mathbf{2}^3 \mathbf{3}^4 \mathbf{2}^3 \mathbf{4}}^3(q, t) &= \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1(qt^4) I_1^\circ(qt^5) I_1(qt^7) I_1(qt^8) I_1(t^9), \\ D_{\mathbf{2}^3 \mathbf{2}^3 \mathbf{4}^3 \mathbf{4}}^3(q, t) &= \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1^\circ(qt^7) I_1(qt^8) I_1(t^9). \end{aligned}$$

□

Proposition 4.3.9. *((c,d)=5,2) There exists a rational function $V_{5,2}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ such that*

$$\zeta_{\mathcal{L}_{5(2)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = V_{5,2}^{\triangleleft \text{gr}}(X, Y),$$

and

$$\zeta_{\mathcal{L}_{5(2)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) \Big|_{q \rightarrow q^{-1}} = -q^7 t^{35} \zeta_{\mathcal{L}_{5^2}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

Setting

$$\begin{aligned} M_{5,2}(Y) &= 1 + 2Y + 2Y^2 + 3Y^3 + 5Y^4 + 8Y^5 + 13Y^6 + 17Y^7 + 21Y^8 + 25Y^9 \\ &\quad + 28Y^{10} + 31Y^{11} + 35Y^{12} + 40Y^{13} + 44Y^{14} + 47Y^{15} \\ &\quad + 47Y^{16} + 44Y^{17} + 40Y^{18} + 35Y^{19} + 31Y^{20} + 28Y^{21} \\ &\quad + 25Y^{22} + 21Y^{23} + 17Y^{24} + 13Y^{25} + 8Y^{26} + 5Y^{27} \\ &\quad + 3Y^{28} + 2Y^{29} + 2Y^{30} + Y^{31} \in \mathbb{Q}[Y] \end{aligned}$$

we have

$$\begin{aligned} \zeta_{\mathcal{L}_{5(2)}, \text{red}}^{\triangleleft \text{gr}}(s) &= V_{5,2}^{\triangleleft \text{gr}}(1, Y) = \frac{M_{5,2}(Y)}{\prod_{i=1}^{11} (1 - Y^i)}, \\ \zeta_{\mathcal{L}_{5(2)}, \text{top}}^{\triangleleft \text{gr}}(s) &= \frac{2}{3465} \frac{50175s^4 - 22725s^3 + 3798s^2 - 284s + 8}{s^5(s-1)(s-2) \prod_{k=3}^{11} (ks-1)}. \end{aligned}$$

Proof. For $(c, d) = (5, 2)$, (4.3.6) becomes

$$\zeta_{\mathcal{L}_{5^2}^{\triangleleft \text{gr}}(\mathfrak{o})}(s) = \zeta_{\mathbb{O}_{\mathfrak{p}}^3}(s) \sum_{v \in \mathcal{D}_2^4} D_v^4(q, t).$$

We have

$$D_{\mathbf{2}^2\mathbf{3}^2\mathbf{4}^2\mathbf{5}^2}^4(q, t) = I_2(qt^3, t^5) I_2(qt^6, t^7) I_2(qt^8, t^9) I_2(qt^{10}, t^{11}),$$

$$D_{\mathbf{2}^3\mathbf{2}^3\mathbf{4}^2\mathbf{5}^2}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1(t^7) I_2(qt^8, t^9) I_2(qt^{10}, t^{11}),$$

$$D_{\mathbf{2}^2\mathbf{3}^4\mathbf{3}^4\mathbf{5}^2}^4(q, t) = \binom{2}{1}_{q^{-1}} I_2(qt^3, t^5) I_1(qt^6) I_1^\circ(qt^7) I_1(qt^8) I_1(t^9) I_2(qt^{10}, t^{11}),$$

$$D_{\mathbf{2}^3\mathbf{4}^2\mathbf{3}^4\mathbf{5}^2}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1(qt^4) I_1^\circ(qt^5) I_1(qt^7) I_1(qt^8) I_1(t^9) I_2(qt^{10}, t^{11}),$$

$$D_{\mathbf{2}^3\mathbf{2}^4\mathbf{3}^4\mathbf{5}^2}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1^\circ(qt^7) I_1(qt^8) I_1(t^9) I_2(qt^{10}, t^{11}),$$

$$D_{\mathbf{2}^2\mathbf{3}^2\mathbf{4}^5\mathbf{4}^5}^4(q, t) = \binom{2}{1}_{q^{-1}} I_2(qt^3, t^5) I_2(qt^6, t^7) I_1(qt^8) I_1^\circ(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}^3\mathbf{2}^3\mathbf{4}^5\mathbf{4}^5}^4(q, t) = \binom{2}{1}_{q^{-1}} \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1(t^7) I_1(qt^8) I_1^\circ(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}^2\mathbf{3}^4\mathbf{3}^5\mathbf{4}^5}^4(q, t) = \binom{2}{1}_{q^{-1}} I_2(qt^3, t^5) I_1(qt^6) I_1^\circ(qt^7) I_1(qt^8) I_1^\circ(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}^3\mathbf{4}^2\mathbf{3}^5\mathbf{4}^5}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1(qt^4) I_1^\circ(qt^5) I_1(qt^7) I_1(qt^8) I_1^\circ(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}^3\mathbf{2}^4\mathbf{3}^5\mathbf{4}^5}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1^\circ(qt^7) I_1(qt^8) I_1^\circ(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}^2\mathbf{3}\mathbf{4}\mathbf{5}\mathbf{3}\mathbf{4}\mathbf{5}}^4(q, t) = \binom{2}{1}_{q^{-1}} I_2(qt^3, t^5) I_1(qt^6) I_1(qt^7) I_1^\circ(qt^8) I_1(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}\mathbf{3}\mathbf{4}\mathbf{2}\mathbf{5}\mathbf{3}\mathbf{4}\mathbf{5}}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1(qt^4) I_1^\circ(qt^5) I_1(qt^7) I_1^\circ(qt^8) I_1(qt^9) I_1(qt^{10}) I_1(t^{11}),$$

$$D_{\mathbf{2}\mathbf{3}\mathbf{2}\mathbf{4}\mathbf{5}\mathbf{3}\mathbf{4}\mathbf{5}}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1^\circ(qt^4) I_1(qt^6) I_1(qt^7) I_1^\circ(qt^8) I_1(qt^9) I_1(qt^{10}) I_1(t^{11}).,$$

$$D_{\mathbf{2}\mathbf{3}\mathbf{4}\mathbf{5}\mathbf{2}\mathbf{3}\mathbf{4}\mathbf{5}}^4(q, t) = \binom{2}{1}_{q^{-1}} I_1(qt^3) I_1(qt^4) I_1(qt^5) I_1^\circ(qt^6) I_1(qt^8) I_1(qt^9) I_1(qt^{10}) I_1(t^{11}).$$

□

4.3.4 p-Adic behaviour at zero

We describe the behaviours of $\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ at $s = 0$.

Theorem 4.3.10. *For all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\left. \frac{\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^{d+1}}(s) (\zeta_{\mathfrak{o}^d}(s))^{c-1}} \right|_{s=0} = \frac{(d+1) \cdot d^{c-1}}{(d+1)(2d+1) \cdots (cd+1)} = \frac{d^{c-1}}{\prod_{k=2}^c (kd+1)}. \quad (4.3.20)$$

Proof. To prove (4.3.20), let

$$v(\mu_2, \dots, \mu_c)_{\emptyset} = v_{\emptyset} \in \mathcal{D}_d^{(c-1)}$$

such that $v = \mathbf{2}^d \mathbf{3}^d \cdots \mathbf{c}^d$, so there is no overlap between μ_i and μ_j . In other words this is the case when $r = 1$.

Now,

$$\frac{\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^{d+1}}(s) (\zeta_{\mathfrak{o}^d}(s))^{c-1}} = \left(\prod_{j=0}^{d-1} (1 - q^j t) \right)^{c-1} \left(D_{v_{\emptyset}}^{c-1}(q, t) + \sum_{v \neq v_{\emptyset}} D_v^{c-1}(q, t) \right).$$

By inspection of Theorem 4.3.6 we see that

$$D_{v_{\emptyset}}^{c-1}(q, t) = I_d((q^{(d-j)j} t^{1+2j})_{j \in [d]}) \prod_{k=3}^c I_d((q^{(d-j)j} t^{1+(k-1)d+j})_{j \in [d]}),$$

which has a pole at $t = 1$ with multiplicity $c - 1$. In contrast, $D_v^{c-1}(q, t)$ has a pole at $t = 1$ with multiplicity $c - 2$ when $v \neq v_\emptyset$. Hence

$$\begin{aligned} \left. \frac{\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)}{\zeta_{\mathfrak{o}^{d+1}}(s) (\zeta_{\mathfrak{o}^d}(s))^{c-1}} \right|_{s=0} &= \left. \frac{I_d((q^{(d-j)j} t^{1+2j})_{j \in [d]}) \prod_{k=3}^c I_d((q^{(d-j)j} t^{1+(k-1)d+j})_{j \in [d]})}{I_d((q^{(d-j)j} t^j)_{j \in [d]})^{c-1}} \right|_{t=1} \\ &= \left. \frac{(1-t^d)^{c-1}}{\prod_{k=2}^c (1-t^{1+kd})} \right|_{t=1} \\ &= \frac{d^{c-1}}{\prod_{k=2}^c (kd+1)} \end{aligned}$$

□

4.3.5 Topological zeta functions – degree and behaviour at zero

Theorem 4.3.11.

$$\deg_s \left(\zeta_{\mathcal{L}_{c(d), \text{top}}}^{\triangleleft \text{gr}}(s) \right) = -(1 + cd).$$

Moreover,

$$s^c \zeta_{\mathcal{L}_{c(d), \text{top}}}^{\triangleleft \text{gr}}(s) \Big|_{s=0} = \frac{(-1)^{(d)+(d-1)^{c-1}} (d+1) d^{c-1}}{(d!) ((d-1)!)^{c-1} \prod_{j=1}^c (1+jd)}. \quad (4.3.21)$$

Proof. Note that, given $h \in \mathbb{N}$ and, for $i \in [h]$, $X_i = q^{a_i} t^{b_i}$ for $a_i \in \mathbb{N}_0$, $b_i \in \mathbb{N}$,

$$I_h(X_1, \dots, X_h) = \frac{h!}{\prod_{i=1}^h (b_i s - a_i)} (q-1)^{-h} + O((q-1)^{-h+1}).$$

The summands in the Theorem 4.3.6 for $\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ are all products of Igusa functions and Gaussian binomial coefficients. Hence there exist $a_{v,i}^{\triangleleft \text{gr}} \in \mathbb{N}_0$ and $b_{v,i}^{\triangleleft \text{gr}}, c_v^{\triangleleft \text{gr}} \in \mathbb{N}$ such that

$$\zeta_{\mathcal{L}_{c(d)}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = \sum_v \frac{c_v^{\triangleleft \text{gr}}}{\prod_{i=1}^{1+cd} (b_{v,i}^{\triangleleft \text{gr}} s - a_{v,i}^{\triangleleft \text{gr}})} (q-1)^{-(1+cd)} + O((q-1)^{-(1+cd)+1}).$$

Hence

$$\zeta_{\mathcal{L}_{c(d), \text{top}}}^{\triangleleft \text{gr}}(s) = \sum_v \frac{c_v^{\triangleleft \text{gr}}}{\prod_{i=1}^{1+cd} (b_{v,i}^{\triangleleft \text{gr}} s - a_{v,i}^{\triangleleft \text{gr}})} \quad (4.3.22)$$

is a rational function in s of degree $-(1 + cd)$, confirming the first claim.

Turning to the local behaviour around $s = 0$, we observe that $\zeta_{\mathfrak{o}^{d+1}}(s) \mathbb{D}_v^{c-1}(q, t)$ has a pole at $t = 1$ with multiplicity $c - 1$ unless $v = v_\emptyset$, in which case it is c . Thus

the summand $\frac{c_v^{\triangleleft \text{gr}}}{\prod_{i=1}^{1+cd}(b_{v,i}^{\triangleleft \text{gr}} s - a_{v,i}^{\triangleleft \text{gr}})}$ in (4.3.22) has a pole with multiplicity $c - 1$ at $s = 0$ unless $v = v_\emptyset$, in which case it is c . Specifically,

$$\zeta_{\mathfrak{o}^{d+1}}(s) D_{v_\emptyset}^{c-1}(q, t) = I_{d+1}((q^{i(d+1-i)} t^i)_{i \in [d+1]}) I_d((q^{(d-j)j} t^{1+2j})_{j \in [d]}) \cdot \prod_{k=3}^c I_d((q^{(d-j)j} t^{1+(k-1)d+j})_{j \in [d]}),$$

whence

$$\begin{aligned} & \frac{c_{v_\emptyset}^{\triangleleft \text{gr}}}{\prod_{i=1}^{1+cd}(b_{v_\emptyset,i}^{\triangleleft \text{gr}} s - a_{v_\emptyset,i}^{\triangleleft \text{gr}})} \\ &= \frac{1}{\prod_{i=0}^d (s - i)} \cdot \frac{(d!)^{c-1}}{\prod_{j=1}^d ((1+2j)s - (d-j)j) \prod_{k=3}^c \prod_{l=1}^d ((1+(k-1)d+l)s - (d-l)l)} \end{aligned}$$

and thus

$$\begin{aligned} s^c \zeta_{\mathcal{L}_{c(d), \text{top}}}^{\triangleleft \text{gr}}(s) \Big|_{s=0} &= \frac{(d!)^{c-1}}{\prod_{k=2}^c (1+kd) \prod_{i=1}^d (-i) \left(\prod_{j=1}^{d-1} (-(d-j)j) \right)^{c-1}} \\ &= \frac{(-1)^{(d)+(d-1)^{c-1}} (d+1) d^{c-1}}{(d!) ((d-1)!)^{c-1} \prod_{j=1}^c (1+jd)}. \end{aligned}$$

which proves (4.3.21). □

In fact, one can manually check Theorem 4.3.10 and 4.3.11 with explicit formulas given here.

Chapter 5

Zeta functions of Lie $\mathbb{F}_p$ -algebras and finite p -groups

Let p be a prime. Most of the research on zeta functions of groups and rings, including this thesis, studied infinite objects. However, our definition of zeta functions of groups and rings is clearly applicable to finite p -groups and Lie algebras as well. For example, given a Lie ring L and write $L(\mathbb{F}_p) = L \otimes \mathbb{F}_p$, we can compute the zeta functions of Lie $\mathbb{F}_p$ -algebra $L(\mathbb{F}_p)$, which would enumerate subalgebras or ideals of $L(\mathbb{F}_p)$. One can think of this as an approximation of the zeta function of $L_p = L \otimes \mathbb{Z}_p$. In this direction we study how this approximation can be used to study the more complicated zeta functions $\zeta_{L_p}^*(s)$.

One can also treat this as an interesting object on its own right. As we already exploited in Section 2.6, when we used the associated Lie ring $L(G_p)$ instead of the original group G_p to count the number of immediate descendants of G_p of order p^9 and of exponent p , the Lazard correspondence gives an isomorphism between the category of nilpotent Lie algebras of dimension n over the field $\mathbb{F}_p$ and class at most $p - 1$ and the category of groups of exponent p of order p^n and class at most $p - 1$. (refer to [19, Chapter 10] or [24] for more details). One can therefore use this to study the zeta functions of finite p -groups as well. In this chapter we study and calculate the zeta functions of finite Lie $\mathbb{F}_p$ -algebra $L(\mathbb{F}_p)$, and investigate how these results can be applied in various ways.

5.1 Zeta functions of Lie $\mathbb{F}_p$ -algebras

Let L be a Lie ring, additively isomorphic to $\mathbb{Z}^n$, and let $L(\mathbb{F}_p) = L \otimes \mathbb{F}_p$ be a Lie $\mathbb{F}_p$ -algebra over $\mathbb{F}_p$ of dimension n , additively isomorphic to $\mathbb{F}_p^n$. One can define the

zeta functions of the Lie $\mathbb{F}_p$ -algebras $L(\mathbb{F}_p)$ as follows:

$$\zeta_{L(\mathbb{F}_p)}^*(s) := \sum_{H * L(\mathbb{F}_p)} |L(\mathbb{F}_p) : H|^{-s} = \sum_{i=0}^n a_{p^i}^*(L(\mathbb{F}_p)) p^{-is},$$

where $*$ $\in \{\leq, <\}$ and

$$a_m^*(L(\mathbb{F}_p)) = |\{H \mid H * L(\mathbb{F}_p), |L(\mathbb{F}_p) : H| = m\}|.$$

Note that they are also (finite) Dirichlet polynomials.

One can regard these “polynomials” as approximations of zeta functions in the sense that this is the starting points of zeta functions of $L_p = L \otimes \mathbb{Z}_p$. They do not in general have functional equations, or palindromic behaviour, but they do have their own uniformity property.

Definition 5.1.1. The finite zeta functions over finite fields $\zeta_{L(\mathbb{F}_p)}^*(s)$ is said to be *finitely uniform* if there exist polynomials $W_1[x, y], \dots, W_k[x, y] \in \mathbb{Z}[x, y]$ for suitable $k \in \mathbb{N}$ such that, for every prime p ,

$$\zeta_{L(\mathbb{F}_p)}^*(s) = W_i[p, p^{-s}]$$

for some $i \in [k]$. It is said to be *uniform* if $k = 1$.

So how do we compute them? Let L be a Lie ring additively isomorphic to $\mathbb{Z}^d$ with basis $\mathcal{B} = (e_1, e_2, \dots, e_n)$, and $L_p = L \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^n$. There is one-to-one correspondence between submodules of L_p of index p^k and integer upper triangular matrices

$$M = (m_{i,j}) = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ 0 & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n,n} \end{pmatrix}$$

with entries in $\mathbb{Z}$ satisfying

1. $m_{i,i} = p^{a_i}$, where $a_i \geq 0$
2. $0 \leq m_{i,j} < m_{j,j}$,
3. $m_{1,1} \cdots m_{n,n} = p^k$,

for $1 \leq i < j \leq n$. We give a new unique matrix representation for $L(\mathbb{F}_p)$.

Lemma 5.1.2. *There is one-to-one correspondence between additive subgroups of $L(\mathbb{F}_p)$ of index p^k and the upper triangular matrix*

$$M = (m_{i,j}) = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ 0 & m_{2,2} & \cdots & m_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n,n} \end{pmatrix}$$

with entries in $\mathbb{F}_p$ satisfying

1. $m_{i,i} = 0$ or 1 ,
2. $m_{i,j} = 0$ if $m_{j,j} = 1$, and $0 \leq m_{i,j} \leq p-1$ if $m_{j,j} = 0$
3. $k = n - \text{rank}(M)$,

for $1 \leq i < j \leq n$.

Proof. Let M be an additive subgroup of $L(\mathbb{F}_p)$. Then M has a ‘triangular’ basis

$$\begin{aligned} \mathbf{m}_1 &= m_{1,1}e_1 + m_{1,2}e_2 + \cdots + m_{1,n}e_n \\ \mathbf{m}_2 &= m_{2,2}e_2 + \cdots + m_{2,n}e_n \\ &\vdots \\ \mathbf{m}_n &= m_{n,n}e_n \end{aligned}$$

where $m_{i,j} \in \mathbb{F}_p$. First, note that since $L(\mathbb{F}_p)$ is itself a finite Lie ring of order p^n , any additive subgroup M of $L(\mathbb{F}_p)$ has finite index. In particular, this implies that we can have $\mathbf{m}_i = 0$, and

$$|L(\mathbb{F}_p) : M| = n - \text{rank}(M).$$

As $\langle \mathbf{m}_j \rangle = \langle u \cdot \mathbf{m}_j \rangle$ for any non-zero unit u in $\mathbb{F}_p$, each $m_{i,i}$ can be either 0 or 1, and we can deduce that each M can be uniquely represented by a matrix in reduced row echelon forms. These reduced row echelon forms can be re-written into our required forms once we exchange the i -th row with j -th row where the leading coefficient of i -th row is $m_{i,j} = 1$. $\square$

As a direct consequence of Lemma 5.1.2 we obtain the following corollary.

Corollary 5.1.3. *Let 0_i denote an $i \times i$ zero matrix, I_j an $j \times j$ identity matrix, and $D_{i,j}$ an arbitrary $i \times j$ matrix over $\mathbb{F}_p$. Then there is one-to-one correspondence between additive subgroups of $L(\mathbb{F}_p)$ of index p^k and the upper triangular matrix*

$$M = \begin{pmatrix} 0_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & D_{b_1, a_2} & 0 & \cdots & D_{b_1, a_r} & 0 \\ 0 & 0 & 0_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & D_{b_2, a_r} & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix} \in \text{Tr}(\mathbb{F}_p)$$

where $\sum_{i=1}^r (a_i) = k$, $\sum_{i=1}^r (a_i + b_i) = n$, $a_1, b_r \geq 0$ and $a_2, \dots, a_r, b_1, \dots, b_{r-1} > 0$.

Corollary 5.1.3 implies that for any given additive subgroup M of index p^k in $L(\mathbb{F}_p)$, there are uniquely determined numbers $r \in \mathbb{N}$ and $\underline{a} = (a_1, \dots, a_r), \underline{b} = (b_1, \dots, b_r)$ such that

$$(m_{1,1}, m_{2,2}, \dots, m_{n,n}) = (0^{(a_1)}, 1^{(b_1)}, \dots, 0^{(a_r)}, 1^{(b_r)}).$$

Let us call $(\underline{a}, \underline{b})$ the *diagonal type* of M , and write $d(M) = (\underline{a}, \underline{b})$. Let $M_{\underline{a}, \underline{b}}$ denote a matrix of diagonal type $(\underline{a}, \underline{b})$, and $\mathcal{M}_{\underline{a}, \underline{b}}$ denote the set of $n \times n$ upper triangular matrices over $\mathbb{F}_p$ with diagonal type $(\underline{a}, \underline{b})$. Since each a_i, b_i can be either 0 or 1, there are 2^n possible $\mathcal{M}_{\underline{a}, \underline{b}}$'s, and each of them satisfies $|\mathcal{M}_{\underline{a}, \underline{b}}| = \sum_{i=1}^r \sum_{j=1}^{i-1} a_i b_j$.

For example, let

$$L = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3 \rangle$$

be the Heisenberg Lie ring. Then $L(\mathbb{F}_p)$ becomes a 3-dimensional Lie $\mathbb{F}_p$ -algebra of order p^3 . One can check that the additive subgroups of $L(\mathbb{F}_p)$ with index p^k are precisely the following, represented by suitable generator matrices:

- $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ if $k = 0$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{1,2} & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & m_{1,3} \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ if $k = 1$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 0 \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{1,2} & m_{1,3} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ if $k = 2$,

$$\bullet \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \text{ if } k = 3.$$

Here we have $8 = 2^3$ distinct partitions. Note that each $M_{\underline{a}, \underline{b}} \in \mathcal{M}_{\underline{a}, \underline{b}}$ corresponds to an additive subgroup of index $p^{\sum_{i=1}^r a_i}$.

5.1.1 Uniformity and the behaviour of $\zeta_{L(\mathbb{F}_p)}^*(s)$

Not every member of $\mathcal{M}_{\underline{a}, \underline{b}}$ can give arise to subalgebra/ideal of $L(\mathbb{F}_p)$. Hence we need to find out the conditions which make those matrices become subalgebras/ideals.

Like the p -adic integrations for original zeta functions of groups and rings, let

$$F_p^{\leq} := \{M \in \text{Tr}(\mathbb{F}_p) \mid \text{rows of } M \text{ defines a subalgebra in } L(\mathbb{F}_p)\}$$

and

$$F_p^{\triangleleft} := \{M \in \text{Tr}(\mathbb{F}_p) \mid \text{rows of } M \text{ defines an ideal in } L(\mathbb{F}_p)\}$$

Also let $F_{\underline{a}, \underline{b}}^{\leq} = F_p^{\leq} \cap \mathcal{M}_{\underline{a}, \underline{b}}$ and $F_{\underline{a}, \underline{b}}^{\triangleleft} = F_p^{\triangleleft} \cap \mathcal{M}_{\underline{a}, \underline{b}}$. Then we have

$$\zeta_{L(\mathbb{F}_p)}^{\leq}(s) = \sum_{\underline{a}, \underline{b}} |F_{\underline{a}, \underline{b}}^{\leq}| p^{-(\sum_{i=1}^r a_i)s}$$

and

$$\zeta_{L(\mathbb{F}_p)}^{\triangleleft}(s) = \sum_{\underline{a}, \underline{b}} |F_{\underline{a}, \underline{b}}^{\triangleleft}| p^{-(\sum_{i=1}^r a_i)s}.$$

One now needs to find formulas for $|F_{\underline{a}, \underline{b}}^{\leq}|$ and $|F_{\underline{a}, \underline{b}}^{\triangleleft}|$. We start with the following lemma:

Lemma 5.1.4. *For given $r \in \mathbb{N}$, $\underline{a}$ and $\underline{b}$, we have*

$$\begin{pmatrix} 0_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & D_{b_1, a_2} & 0 & \cdots & D_{b_1, a_r} & 0 \\ 0 & 0 & 0_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & D_{b_2, a_r} & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix} \begin{pmatrix} I_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & -D_{b_1, a_2} & 0 & \cdots & -D_{b_1, a_r} & 0 \\ 0 & 0 & I_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & -D_{b_2, a_r} & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix} \\ = \begin{pmatrix} 0_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix}$$

Proof. Let

$$M_{\underline{a},\underline{b}} = \begin{pmatrix} 0_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & D_{b_1,a_2} & 0 & \cdots & D_{b_1,a_r} & 0 \\ 0 & 0 & 0_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & D_{b_2,a_r} & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix},$$

$$M_{\underline{a},\underline{b}}^* = \begin{pmatrix} I_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & -D_{b_1,a_2} & 0 & \cdots & -D_{b_1,a_r} & 0 \\ 0 & 0 & I_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & -D_{b_2,a_r} & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix},$$

and

$$M_{\underline{a},\underline{b}}^\flat = \begin{pmatrix} 0_{a_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{b_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0_{a_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{b_2} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{a_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{b_r} \end{pmatrix}.$$

We prove our lemma by induction on r . Note that $M_{\underline{a},\underline{b}}$ defines $M_{\underline{a},\underline{b}}^*$ and $M_{\underline{a},\underline{b}}^\flat$.

First, we have

$$\begin{pmatrix} 0_{a_1} & 0 \\ 0 & I_{b_1} \end{pmatrix} \begin{pmatrix} I_{a_1} & 0 \\ 0 & I_{b_1} \end{pmatrix} = \begin{pmatrix} 0_{a_1} & 0 \\ 0 & I_{b_1} \end{pmatrix}$$

and

$$\begin{pmatrix} 0_{a_1} & 0 & 0 & 0 \\ 0 & I_{b_1} & D_{b_1,a_2} & 0 \\ 0 & 0 & 0_{a_2} & 0 \\ 0 & 0 & 0 & I_{b_2} \end{pmatrix} \begin{pmatrix} I_{a_1} & 0 & 0 & 0 \\ 0 & I_{b_1} & -D_{b_1,a_2} & 0 \\ 0 & 0 & I_{a_2} & 0 \\ 0 & 0 & 0 & I_{b_2} \end{pmatrix} = \begin{pmatrix} 0_{a_1} & 0 & 0 & 0 \\ 0 & I_{b_1} & 0 & 0 \\ 0 & 0 & 0_{a_2} & 0 \\ 0 & 0 & 0 & I_{b_2} \end{pmatrix}.$$

So it holds for $r = 1, 2$.

Now, suppose it holds for $r = n$. Then for $r = n + 1$,

$$\begin{aligned}
M_{\underline{a}, \underline{b}} M_{\underline{a}, \underline{b}}^* &= \left(\begin{array}{c|cc} & 0 & 0 \\ & D_{b_1, a_{n+1}} & 0 \\ M_{\underline{a}', \underline{b}'} & 0 & 0 \\ & \vdots & \vdots \\ & D_{b_n, a_{n+1}} & 0 \\ \hline 0 & 0_{a_{n+1}} & 0 \\ & 0 & I_{b_{n+1}} \end{array} \right) \left(\begin{array}{c|cc} & 0 & 0 \\ & -D_{b_1, a_{n+1}} & 0 \\ M_{\underline{a}', \underline{b}'}^* & 0 & 0 \\ & \vdots & \vdots \\ & -D_{b_n, a_{n+1}} & 0 \\ \hline 0 & I_{a_{n+1}} & 0 \\ & 0 & I_{b_{n+1}} \end{array} \right) \\
&= \left(\begin{array}{c|cc} & 0 & 0 \\ & D_{b_1, a_{n+1}} & -D_{b_1, a_{n+1}} & 0 \\ M_{\underline{a}', \underline{b}'} M_{\underline{a}', \underline{b}', n}^* & 0 & 0 \\ & \vdots & \vdots \\ & D_{b_n, a_{n+1}} & -D_{b_n, a_{n+1}} & 0 \\ \hline 0 & 0_{a_{n+1}} & 0 \\ & 0 & I_{b_{n+1}} \end{array} \right) \\
&= \left(\begin{array}{c|cc} & 0 & 0 \\ & 0 & 0 \\ M_{\underline{a}', \underline{b}'}^b & 0 & 0 \\ & \vdots & \vdots \\ & 0 & 0 \\ \hline 0 & 0_{a_{n+1}} & 0 \\ & 0 & I_{b_{n+1}} \end{array} \right) \\
&= M_{\underline{a}, \underline{b}}^b
\end{aligned}$$

where $\underline{a}' = (a_1, \dots, a_n)$, $\underline{b}' = (b_1, \dots, b_n)$. Hence it holds for all $r \in \mathbb{N}$. $\square$

With this we are ready to prove the following theorem, analogous to [9, Theorem 5.7]

Theorem 5.1.5. *Let L be a Lie ring, additively isomorphic to $\mathbb{Z}^n$, and $L(\mathbb{F}_p) = L \otimes \mathbb{F}_p$. There exists an explicit system of subvarieties E_i ($i \in \mathcal{T}$, $\mathcal{T}$ finite) of a variety V defined over $\mathbb{Z}$ and, for each subset I of $\mathcal{T}$, a polynomial $W_I[X, Y] \in \mathbb{Z}[X, Y]$ such that for almost all primes p*

$$\zeta_{L(\mathbb{F}_p)}^*(s) = \sum_{I \in \mathcal{T}} c_I(p) W_I[p, p^{-s}]$$

where $c_I(p) = |\{a \in V(\mathbb{F}_p) : a \in E_i(\mathbb{F}_p) \text{ if and only if } i \in I\}|$

Proof. As before let C_j denote the matrix whose rows are $\mathbf{c}_i = \beta(\mathbf{e}_i, \mathbf{e}_j)$. Then $M_{\underline{a}, \underline{b}} \in F_{\underline{a}, \underline{b}}^{\leq}$ if and only if we can solve for each $1 \leq i, j \leq n$ the equation

$$\mathbf{m}_i \left(\sum_{l=j}^n m_{j,l} C_l \right) = \left(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)} \right) M_{\underline{a}, \underline{b}}$$

with $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) \in \mathbb{Z}^n$. Similarly, $M_{\underline{a}, \underline{b}} \in F_{\underline{a}, \underline{b}}^{\triangleleft}$ if and only if we can solve for each $1 \leq i, j \leq n$ the equation

$$\mathbf{m}_i C_j = (Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) M_{\underline{a}, \underline{b}}$$

with $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) \in \mathbb{Z}^n$.

Let us consider the ideals first. We can rewrite this equation as

$$\mathbf{m}_i C_j M_{\underline{a}, \underline{b}}^* = (Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) M_{\underline{a}, \underline{b}}^b.$$

Now, let $g_{ijk}^{\triangleleft}(m_{r,s})$ denote the k -th entry of the n -tuple $\mathbf{m}_i C_j M_{\underline{a}, \underline{b}}^*$. Then again the way we defined our M implies that $g_{ijk}^{\triangleleft}(m_{r,s})$ is at most degree-2 polynomial in $m_{r,s}$. Since $M_{\underline{a}, \underline{b}}^b = \text{diag}(0^{(a_1)}, 1^{(b_1)}, \dots, 0^{(a_r)}, 1^{(b_r)})$, it is clear that the k -th entry of the n -tuple $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) M_{\underline{a}, \underline{b}}^b$ is either 0 or $Y_{i,j}^{(k)}$. Since the condition $g_{ijk}^{\triangleleft}(m_{r,s}) = Y_{i,j}^{(k)}$ is redundant, we can conclude that $M_{\underline{a}, \underline{b}}$ gives an ideal if and only if we can solve a system of equations $g_{ijk}^{\triangleleft}(m_{r,s}) = 0$ for $1 \leq i, j, k \leq n$ where $m_{k,k} = 0$.

Finally, let $\mathbf{Y}_{\underline{a}, \underline{b}}^{\triangleleft}$ denote the variety defined by the corresponding system of equations $g_{ijk}^{\triangleleft}(m_{r,s}) = m_{k,k} Y_{i,j}^{(k)}$, and let $c_{\underline{a}, \underline{b}}^{\triangleleft} = |\mathbf{Y}_{\underline{a}, \underline{b}}^{\triangleleft}(\mathbb{F}_p)|$. Then we get $|F_{\underline{a}, \underline{b}}^{\triangleleft}| = c_{\underline{a}, \underline{b}}^{\triangleleft}$.

Hence

$$\zeta_{L(\mathbb{F}_p)}^{\triangleleft}(s) = \sum_{\underline{a}, \underline{b}} c_{\underline{a}, \underline{b}}^{\triangleleft} \cdot p^{-(\sum_{i=1}^r a_i)s}.$$

For subalgebras, one can similarly define $g_{ijk}^{\leq}(m_{r,s})$, $\mathbf{Y}_{\underline{a}, \underline{b}}^{\leq}$ and $c_{\underline{a}, \underline{b}}^{\leq}$ from our initial equation

$$\mathbf{m}_i \left(\sum_{l=j}^n m_{j,l} C_l \right) = (Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) M_{\underline{a}, \underline{b}}$$

and check that

$$\zeta_{L(\mathbb{F}_p)}^{\leq}(s) = \sum_{\underline{a}, \underline{b}} c_{\underline{a}, \underline{b}}^{\leq} \cdot p^{-(\sum_{i=1}^r a_i)s}.$$

This proves Theorem 5.1.5. □

This proof is important for two reasons. First, as we said this is an $\mathbb{F}_p$ -analogy of [9, Theorem 5.7]. As we have seen in Chapter 2, the original result was significant as it showed the local zeta functions of groups and rings can behave very wildly. Our proof of Theorem 5.1.5 suggests that this much-simpler-looking approximation over $\mathbb{F}_p$ can also behave wildly. Second, this proof gives an explicit algorithm for $\zeta_{L(\mathbb{F}_p)}^{\triangleleft}(s)$.

5.1.2 Zeta functions of finite p -groups

Let G_p be a finite p -group of order p^n . One can define the zeta functions of finite p -groups G_p as follows:

$$\zeta_{G_p}^*(s) := \sum_{H * G_p} |G_p : H|^{-s} = \sum_{i=0}^n a_{p^i}^*(G_p) p^{-is},$$

where $*$ $\in \{\leq, <\}$ and

$$a_m^*(G_p) = |\{H \mid H * G_p, |G_p : H| = m\}|.$$

Note that they are also (finite) Dirichlet polynomials.

Let L be an n -dimensional class- c nilpotent Lie ring and $L(\mathbb{F}_p) = L \otimes \mathbb{F}_p$ be a finite n -dimensional class- c Lie $\mathbb{F}_p$ -algebra, additively isomorphic to $\mathbb{F}_p^n$. If $c < p$, the Lazard correspondence gives a corresponding finite p -group $G(L(\mathbb{F}_p))$ of class c , order p^n and exponent p using the Baker-Campbell-Hausdorff formula ([19, Chapter 10]). In particular, the Lazard correspondence also gives isomorphism between the subalgebras (and ideals) of $L(\mathbb{F}_p)$ and the subgroups (and normal subgroups) of its corresponding $G(L(\mathbb{F}_p))$ ([19, Example 10.24]). This implies that for primes $p > c$ we have

$$\zeta_{G(L(\mathbb{F}_p))}^*(s) = \zeta_{L(\mathbb{F}_p)}^*(s).$$

Therefore, similar to what we normally do for the zeta functions of groups and rings, one can study the zeta functions of finite Lie $\mathbb{F}_p$ -algebra to prove analogous results on finite p -groups.

5.2 Algorithm in theory: the Heisenberg Lie algebra H over $\mathbb{F}_p$

In theory our algorithm is very simple.

(1) For a given Lie $\mathbb{F}_p$ -algebra $L(\mathbb{F}_p)$ with $\dim(L(\mathbb{F}_p)) = n$, we write down 2^n distinct partitions $\mathcal{M}_{\underline{a}, \underline{b}}$.

(2) For each $\mathcal{M}_{\underline{a}, \underline{b}}$ we compute

$$c_{\underline{a}, \underline{b}}^* := |F_{\underline{a}, \underline{b}}^*|.$$

(3) We have

$$\zeta_{L(\mathbb{F}_p)}^*(s) = \sum_{\underline{a}, \underline{b}} c_{\underline{a}, \underline{b}}^* \cdot p^{-(\sum_{i=1}^r a_i)s}.$$

We see how this can be applied in practice.

Theorem 5.2.1.

$$\begin{aligned}\zeta_{H(\mathbb{F}_p)}^{\triangleleft}(s) &= 1 + (1+p)p^{-s} + p^{-2s} + p^{-3s}, \\ \zeta_{H(\mathbb{F}_p)}^{\leq}(s) &= 1 + (1+p)p^{-s} + (1+p+p^2)p^{-2s} + p^{-3s}.\end{aligned}$$

Proof. As we saw earlier, there are 8 possible partitions corresponding to the additive subgroups of H of index p^k .

- $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ for $k = 0$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{1,2} & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & m_{1,3} \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ for $k = 1$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 0 \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{1,2} & m_{1,3} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ for $k = 2$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ for $k = 3$.

We now show how we can explicitly calculate $\zeta_{L(\mathbb{F}_p)}^{\triangleleft}(s)$ using these partitions.

Start with $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$. They trivially give 1 ideal of index 1 and p^3 respectively.

Consider $M_{\underline{a}, \underline{b}} = \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$. Then our equation

$$\mathbf{m}_i C_j M_{\underline{a}, \underline{b}}^* = \left(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)} \right) M_{\underline{a}, \underline{b}}^b$$

gives rise to the system of equations

$$\begin{aligned}
i = 1, j = 1 : (0, 0, 0) &= (0, Y_{1,1}^{(2)}, Y_{1,1}^{(3)}), \\
i = 1, j = 2 : (0, 0, 0) &= (0, Y_{1,2}^{(2)}, Y_{1,2}^{(3)}), \\
i = 1, j = 3 : (0, 0, 0) &= (0, Y_{1,3}^{(2)}, Y_{1,3}^{(3)}), \\
i = 2, j = 1 : (0, 0, -1) &= (0, Y_{2,1}^{(2)}, Y_{2,1}^{(3)}), \\
i = 2, j = 2 : (0, 0, 0) &= (0, Y_{2,2}^{(2)}, Y_{2,2}^{(3)}), \\
i = 2, j = 3 : (0, 0, 0) &= (0, Y_{2,3}^{(2)}, Y_{2,3}^{(3)}), \\
i = 3, j = 1 : (0, 0, 0) &= (0, Y_{3,1}^{(2)}, Y_{3,1}^{(3)}), \\
i = 3, j = 2 : (0, 0, 0) &= (0, Y_{3,2}^{(2)}, Y_{3,2}^{(3)}), \\
i = 3, j = 3 : (0, 0, 0) &= (0, Y_{3,3}^{(2)}, Y_{3,3}^{(3)}).
\end{aligned}$$

Since we can solve this for all $i, j \in [3]$, we get $c_{\underline{a}, \underline{b}}^{\triangleleft} = 1$.

Consider the next case $M_{\underline{a}, \underline{b}} = \begin{pmatrix} 1 & m_{1,2} & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$. This gives

$$\begin{aligned}
i = 1, j = 1 : (0, 0, -m_{1,2}) &= (Y_{1,1}^{(1)}, 0, Y_{1,1}^{(3)}), \\
i = 1, j = 2 : (0, 0, 1) &= (Y_{1,2}^{(1)}, 0, Y_{1,2}^{(3)}), \\
i = 1, j = 3 : (0, 0, 0) &= (Y_{1,3}^{(1)}, 0, Y_{1,3}^{(3)}), \\
i = 2, j = 1 : (0, 0, -1) &= (Y_{2,1}^{(1)}, 0, Y_{2,1}^{(3)}), \\
i = 2, j = 2 : (0, 0, 0) &= (Y_{2,2}^{(1)}, 0, Y_{2,2}^{(3)}), \\
i = 2, j = 3 : (0, 0, 0) &= (Y_{2,3}^{(1)}, 0, Y_{2,3}^{(3)}), \\
i = 3, j = 1 : (0, 0, 0) &= (Y_{3,1}^{(1)}, 0, Y_{3,1}^{(3)}), \\
i = 3, j = 2 : (0, 0, 0) &= (Y_{3,2}^{(1)}, 0, Y_{3,2}^{(3)}), \\
i = 3, j = 3 : (0, 0, 0) &= (Y_{3,3}^{(1)}, 0, Y_{3,3}^{(3)}).
\end{aligned}$$

Since we can solve this for all $i, j \in [3]$ and $m_{1,2} \in \{0, \dots, p-1\}$, we get $c_{\underline{a}, \underline{b}}^{\triangleleft} = p$.

The interesting case is when $M_{\underline{a}, \underline{b}} = \begin{pmatrix} 1 & 0 & m_{1,3} \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$. This gives

$$\begin{aligned}
i = 1, j = 1 : (0, 0, 0) &= (Y_{1,1}^{(1)}, Y_{1,1}^{(2)}, 0), \\
i = 1, j = 2 : (0, 0, 1) &= (Y_{1,2}^{(1)}, Y_{1,2}^{(2)}, 0), \\
i = 1, j = 3 : (0, 0, 0) &= (Y_{1,3}^{(1)}, Y_{1,3}^{(2)}, 0), \\
i = 2, j = 1 : (0, 0, -1) &= (Y_{2,1}^{(1)}, Y_{2,1}^{(2)}, 0), \\
i = 2, j = 2 : (0, 0, 0) &= (Y_{2,2}^{(1)}, Y_{2,2}^{(2)}, 0), \\
i = 2, j = 3 : (0, 0, 0) &= (Y_{2,3}^{(1)}, Y_{2,3}^{(2)}, 0), \\
i = 3, j = 1 : (0, 0, 0) &= (Y_{3,1}^{(1)}, Y_{3,1}^{(2)}, 0), \\
i = 3, j = 2 : (0, 0, 0) &= (Y_{3,2}^{(1)}, Y_{3,2}^{(2)}, 0), \\
i = 3, j = 3 : (0, 0, 0) &= (Y_{3,3}^{(1)}, Y_{3,3}^{(2)}, 0).
\end{aligned}$$

When $i = 1, j = 2$, one needs to solve $1 = 0$, which is impossible. Hence we get $c_{\underline{a},b}^{<} = 0$, and all together we get $1 + p$ ideals of index p .

Similarly, $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ gives

$$\begin{aligned}
i = 1, j = 1 : (0, 0, 0) &= (0, 0, Y_{1,1}^{(3)}), \\
i = 1, j = 2 : (0, 0, 0) &= (0, 0, Y_{1,2}^{(3)}), \\
i = 1, j = 3 : (0, 0, 0) &= (0, 0, Y_{1,3}^{(3)}), \\
i = 2, j = 1 : (0, 0, 0) &= (0, 0, Y_{2,1}^{(3)}), \\
i = 2, j = 2 : (0, 0, 0) &= (0, 0, Y_{2,2}^{(3)}), \\
i = 2, j = 3 : (0, 0, 0) &= (0, 0, Y_{2,3}^{(3)}), \\
i = 3, j = 1 : (0, 0, 0) &= (0, 0, Y_{3,1}^{(3)}), \\
i = 3, j = 2 : (0, 0, 0) &= (0, 0, Y_{3,2}^{(3)}), \\
i = 3, j = 3 : (0, 0, 0) &= (0, 0, Y_{3,3}^{(3)}),
\end{aligned}$$

$\begin{pmatrix} 0 & 0 & 0 \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ gives

$$\begin{aligned}
i = 1, j = 1 : (0, 0, 0) &= (0, Y_{1,1}^{(2)}, 0), \\
i = 1, j = 2 : (0, 0, 0) &= (0, Y_{1,2}^{(2)}, 0), \\
i = 1, j = 3 : (0, 0, 0) &= (0, Y_{1,3}^{(2)}, 0), \\
i = 2, j = 1 : (0, 0, -1) &= (0, Y_{2,1}^{(2)}, 0), \\
i = 2, j = 2 : (0, 0, 0) &= (0, Y_{2,2}^{(2)}, 0), \\
i = 2, j = 3 : (0, 0, 0) &= (0, Y_{2,3}^{(2)}, 0), \\
i = 3, j = 1 : (0, 0, 0) &= (0, Y_{3,1}^{(2)}, 0), \\
i = 3, j = 2 : (0, 0, 0) &= (0, Y_{3,2}^{(2)}, 0), \\
i = 3, j = 3 : (0, 0, 0) &= (0, Y_{3,3}^{(2)}, 0),
\end{aligned}$$

and $\begin{pmatrix} 1 & m_{1,2} & m_{1,3} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ gives

$$\begin{aligned}
i = 1, j = 1 : (0, 0, -m_{1,2}) &= (Y_{1,1}^{(1)}, 0, 0), \\
i = 1, j = 2 : (0, 0, 1) &= (Y_{1,2}^{(1)}, 0, 0), \\
i = 1, j = 3 : (0, 0, 0) &= (Y_{1,3}^{(1)}, 0, 0), \\
i = 2, j = 1 : (0, 0, 0) &= (Y_{2,1}^{(1)}, 0, 0), \\
i = 2, j = 2 : (0, 0, 0) &= (Y_{2,2}^{(1)}, 0, 0), \\
i = 2, j = 3 : (0, 0, 0) &= (Y_{2,3}^{(1)}, 0, 0), \\
i = 3, j = 1 : (0, 0, 0) &= (Y_{3,1}^{(1)}, 0, 0), \\
i = 3, j = 2 : (0, 0, 0) &= (Y_{3,2}^{(1)}, 0, 0), \\
i = 3, j = 3 : (0, 0, 0) &= (Y_{3,3}^{(1)}, 0, 0).
\end{aligned}$$

The first case gives 1 ideal of index p^2 , while the other two cases give no ideals.

Summing all up, we get

$$\zeta_{H(\mathbb{F}_p)}^{\triangleleft}(s) = 1 + (1 + p)p^{-s} + p^{-2s} + p^{-3s}$$

as required. Same method applies for $\zeta_{H(\mathbb{F}_p)}^{\leq}(s)$. □

Remark 5.2.2. Compare it with

$$\begin{aligned}\zeta_{H \otimes \mathbb{Z}_p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2}(s) \zeta_p(3s-2) \\ &= 1 + (1+p)p^{-s} + (1+p+p^2)p^{-2s} + (1+p+2p^2+p^3)p^{-3s} \\ &\quad + \dots\end{aligned}$$

5.3 Algorithm in practice: 1. Diagonal Conditions

Our algorithm worked well with the Heisenberg case, but it has a clear limitation: the number of partitions we need to check grows exponentially. For example, if one wants to compute $\zeta_{L(\mathbb{F}_p)}^*(s)$ for a 10-dimensional Lie $\mathbb{F}_p$ -algebra, then in theory one needs to check $2^{10} = 1024$ different cases. We still have finite number of cases to check, but this will be very impractical. Fortunately, there are some techniques one can apply to make this computation easier. The first technique is what we call a *diagonal condition*.

Forget about $\mathbb{F}_p$ for now, and let L be a Lie ring, additively isomorphic to $\mathbb{Z}^n$. Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ 0 & m_{2,2} & \cdots & m_{2,n} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & m_{n,n} \end{pmatrix}$$

be an additive subgroup of L . Let $\mathbf{m}_i$ denote the i -th row of M , M' denote the adjoint matrix of M , and

$$M^\sharp = M' \text{diag}(m_{22}^{-1} \cdots m_{nn}^{-1}, \dots, m_{nn}^{-1}, 1).$$

Then we showed that

(1) $M \triangleleft L$ if and only if

$$\mathbf{m}_i C_j M^\sharp = (m_{1,1} Y_{i,j}^{(1)}, \dots, m_{1,1} \cdots m_{n,n} Y_{i,j}^{(n)})$$

has a solution $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) \in \mathbb{Z}_p^n$ for all $i, j \in [n]$.

(2) $M \leq L$ if and only if

$$\mathbf{m}_i \left(\sum_{l=j}^n m_{j,l} C_l \right) M^\sharp = (m_{1,1} Y_{i,j}^{(1)}, \dots, m_{1,1} \cdots m_{n,n} Y_{i,j}^{(n)})$$

has a solution $(Y_{i,j}^{(1)}, \dots, Y_{i,j}^{(n)}) \in \mathbb{Z}_p^n$ for all $i, j \in [n]$.

Among these cone conditions, there are some simple conditions of the form $m_{ii} | m_{jj}$, one diagonal entry dividing the other diagonal entry for ideals, and of the form

$m_{ii}|m_{jj}m_{kk}$, one diagonal entry dividing the product of two diagonal entries for subalgebras, what we call *diagonal conditions*. One can think of these conditions as a backbone of the structure of subalgebras/ideals of L .

How does this help our algorithm?

5.3.1 The Heisenberg Lie algebra H , revisited

Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ 0 & m_{2,2} & m_{2,3} \\ 0 & 0 & m_{3,3} \end{pmatrix}$$

denote the matrix representations of subspaces of H .

Before we check each of 8 different partitions as we did before, one can easily check the diagonal conditions for H : $M \triangleleft H$ if and only if $m_{3,3}|m_{1,1}, m_{2,2}$. Hence $m_{3,3} = 0$ forces $m_{1,1} = m_{2,2} = 0$. That means we need to check only 5 cases, namely

- $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ for $n = 0$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{1,2} & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ for $n = 1$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 1 \end{pmatrix}$ for $n = 2$,
- $\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ for $n = 3$.

This also explains why we could not solve the system of equations for $\begin{pmatrix} 1 & 0 & m_{1,3} \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & m_{12} & m_{1,3} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$. For example, the first case $\begin{pmatrix} 1 & 0 & m_{1,3} \\ & 1 & m_{2,3} \\ & & 0 \end{pmatrix}$ asked us to solve

$$i = 1, j = 2 : (0, 0, 1) = (Y_{1,2}^{(1)}, Y_{1,2}^{(2)}, 0 = 0 \cdot Y_{1,2}^{(3)}),$$

which requires $Y_{1,2}^{(3)} = \frac{1}{0}$.

A reduction from eight to five cases may not look very significant, but in larger examples it could be valuable.

5.3.2 The maximal class Lie algebra M_c , counting ideals

For $c \geq 2$, let $M_c = \langle x_0, x_1, \dots, x_c \mid \forall i \in [c-1], [x_0, x_i] = x_{i+1} \rangle$ be a maximal class- c Lie ring M_c .

Theorem 5.3.1. *For $c \geq 2$,*

$$\begin{aligned} \zeta_{M_c(\mathbb{F}_p)}^\triangleleft(s) &= 1 + (1+p)p^{-s} + \sum_{k=2}^{c+1} p^{-ks} \\ &= \zeta_{\mathbb{F}_p^2}(s) + \sum_{k=3}^{c+1} p^{-ks}. \end{aligned}$$

Proof. Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,c+1} \\ 0 & m_{2,2} & \cdots & m_{2,c+1} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & m_{c+1,c+1} \end{pmatrix}$$

denote a generator matrix for a vector subspace of $M_c(\mathbb{F}_p)$. If $M \triangleleft M_c(\mathbb{F}_p)$, then for $3 \leq i \leq c+1$, our diagonal condition implies $m_{i,i} \mid m_{j,j}$ for all $1 \leq j < i$. Now, let $m_{3,3} = 1$. Then it forces $m_{4,4} = \cdots = m_{c+1,c+1} = 1$. Hence there are only 4 such available cases with $m_{3,3} = 1$,

$$\begin{aligned} M_{111\dots,1} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \\ M_{011\dots,1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, M_{101\dots,1} = \begin{pmatrix} 1 & m_{1,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \\ M_{001\dots,1} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \end{aligned}$$

This gives us one ideal of index 1, a total of $p+1$ ideals of index p and one ideal of index p^2 . For the rest, our M would look like

$$M_{0\dots 01\dots,1} = \begin{pmatrix} 0_{a_1} & 0 \\ 0 & I_{b_1} \end{pmatrix}$$

where $3 < a_1 \leq c + 1$. Each of them gives 1 ideal of index p^{a_1} . Thus we get

$$\begin{aligned}\zeta_{M_c(\mathbb{F}_p)}^\triangleleft(s) &= 1 + (1 + p)p^{-s} + \sum_{k=2}^{c+1} p^{-ks} \\ &= \zeta_{\mathbb{F}_p^2}(s) + \sum_{k=3}^{c+1} p^{-ks}.\end{aligned}$$

□

Remark 5.3.2. Note that we do get $\zeta_{H(\mathbb{F}_p)}^\triangleleft(s)$ for $c = 2$.

This result is intriguing in many ways. First, one only needs to check $c + 3$ cases instead of 2^{c+1} cases. We even have an explicit formula for any $c \geq 2$. Second, it demonstrates a well-known fact from finite p -group theory that any maximal class- c group of order p^{c+1} contains only one normal subgroup of order p^m for $1 \leq m \leq c - 1$ and $1 + p$ maximal (hence normal) subgroups of order p^c . Finally, we get the following immediate corollary.

Corollary 5.3.3. $\zeta_{M_c(\mathbb{F}_p)}^\triangleleft(s)$ is uniform for all $c \geq 2$.

It is worth noting that the original ideal zeta function $\zeta_{M_{c,p}}^\triangleleft(s)$, where $M_{c,p} = M_c \otimes \mathbb{Z}_p$, is extremely difficult to calculate. In fact, we only know the explicit formulas for $c = 3$ and $c = 4$. Interestingly, various approximations of $\zeta_{M_{c,p}}^\triangleleft(s)$ seem to be rather easy to calculate. For example, we have

$$\zeta_{M_{c,p}}^{\triangleleft gr}(s) = \zeta_{\mathbb{Z}_p^2}(s) \prod_{i=3}^{c+1} \zeta_p(is)$$

([28, Proposition 3.5]),

$$\zeta_{M_{c,p}}^{\triangleleft red}(s) = (\zeta_p(s))^2 \prod_{i=3}^{c+1} \zeta_p(is)$$

([13, Example 4.4]), and now

$$\zeta_{M_c(\mathbb{F}_p)}^\triangleleft(s) = \zeta_{\mathbb{F}_p^2}(s) + \sum_{k=3}^{c+1} p^{-ks}.$$

In all these examples the zeta functions are uniform. In contrast, it is currently not known whether the zeta functions $\zeta_{M_{c,p}}^\triangleleft(s)$ for $c > 4$ are uniform as well. Would this imply that $\zeta_{M_{c,p}}^\triangleleft(s)$ will be also uniform for all $c \geq 2$?

5.3.3 The maximal class Lie algebra M_c , counting subalgebras

Theorem 5.3.4. *Let $t = p^{-s}$. For $c \geq 2$,*

$$\begin{aligned}\zeta_{M_2(\mathbb{F}_p)}^{\leq}(s) &= 1 + (1+p)t + (1+p+p^2)t^2 + t^3, \\ \zeta_{M_3(\mathbb{F}_p)}^{\leq}(s) &= 1 + (1+p)t + (1+p+2p^2)t^2 + (1+p+p^2+p^3)t^3 + t^4, \\ \zeta_{M_4(\mathbb{F}_p)}^{\leq}(s) &= 1 + (1+p)t + (1+p+2p^2+p^3)t^2 + (1+p+2p^2+2p^3+p^4)t^3 \\ &\quad + (1+p+p^2+p^3+p^4)t^4 + t^5,\end{aligned}$$

and finally

$$\zeta_{M_{c+1}(\mathbb{F}_p)}^{\leq}(s) = \zeta_{M_c(\mathbb{F}_p)}^{\leq}(s) + \sum_{a=0}^{c-1} \binom{c-1}{a}_p p^{c-1-a} t^{a+2} + p^c t^c.$$

Proof. Start with $c = 2$, which is the Heisenberg case.

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ 0 & m_{2,2} & m_{2,3} \\ 0 & 0 & m_{3,3} \end{pmatrix}$$

gives a subalgebra of $M_2(\mathbb{F}_p)$ if and only if $m_{3,3} | m_{1,1}m_{2,2}$. Hence we check the 7 cases

- $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ for $n = 0$,
- $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{12} & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ for $n = 1$,
- $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 & 0 \\ 1 & m_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & m_{12} & m_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $n = 2$,
- $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $n = 3$.

We get

$$\zeta_{M_2(\mathbb{F}_p)}^{\leq}(s) = 1 + (1+p)t + (1+p+p^2)t^2 + t^3.$$

Now let $c = 3$ and

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} \\ 0 & 0 & m_{3,3} & m_{3,4} \\ 0 & 0 & 0 & m_{4,4} \end{pmatrix}.$$

The diagonal conditions for $M \leq M_3$ are $m_{4,4} | m_{1,1} m_{3,3}$ and $m_{3,3} | m_{1,1} m_{2,2}$. If $m_{4,4} = 1$, then

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ 0 & m_{2,2} & m_{2,3} & 0 \\ 0 & 0 & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a subalgebra of $M_3(\mathbb{F}_p)$ if and only if

$$M' = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ 0 & m_{2,2} & m_{2,3} \\ 0 & 0 & m_{3,3} \end{pmatrix}$$

is the subalgebra of $M_2(\mathbb{F}_p)$. This gives $\zeta_{M_2(\mathbb{F}_p)}^{\leq}(s)$.

If $m_{4,4} = 0$, recall that we have $m_{4,4} | m_{1,1} m_{3,3}$ and $m_{3,3} | m_{1,1} m_{2,2}$. That means at least one of $m_{1,1}, m_{2,2}$ and $m_{3,3}$ also needs to be zero. Hence we do not get any subalgebras of index p .

For ideals of index p^2 , note that by our diagonal condition the only possible case is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & m_{2,4} \\ 0 & 0 & 1 & m_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This gives p^2 subalgebras of index p^2 .

For ideals of index p^3 , we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & m_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & m_{2,3} & m_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & m_{1,2} & m_{1,3} & m_{1,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and they give $p + p^2 + p^3$ subalgebras of index p^3 . Finally we get 1 subalgebra of index p^4 . Hence we get

$$\begin{aligned} \zeta_{M_3(\mathbb{F}_p)}^{\leq}(s) &= \zeta_{M_2(\mathbb{F}_p)}^{\leq}(s) + p^2 t^2 + (p + p^2 + p^3) t^3 + t^4 \\ &= 1 + (1 + p)t + (1 + p + 2p^2)t^2 + (1 + p + p^2 + p^3)t^3 + t^4. \end{aligned}$$

Now fix $c \geq 4$. Write

$$\begin{aligned}
M &= \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,c} & m_{1,c+1} \\ 0 & m_{2,2} & \cdots & m_{2,c} & m_{2,c+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_{c,c} & m_{c,c+1} \\ 0 & 0 & \cdots & 0 & m_{c+1,c+1} \end{pmatrix} \\
&= \left(\begin{array}{c|c} M' & \begin{matrix} m_{1,c+1} \\ \vdots \\ m_{c,c+1} \end{matrix} \\ \hline 0 & m_{c+1,c+1} \end{array} \right)
\end{aligned}$$

where

$$M' = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,c} \\ 0 & m_{2,2} & \cdots & m_{2,c} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & m_{c,c} \end{pmatrix}.$$

For $m_{c+1,c+1} = 1$,

$$M = \left(\begin{array}{c|c} M' & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array} \right)$$

and we get $\zeta_{M_c(\mathbb{F}_p)}^{\leq}(s)$.

For $m_{c+1,c+1} = 0$, our diagonal condition requires

$$m_{c+1,c+1} | m_{1,1} m_{c,c}, m_{c,c} | m_{1,1} m_{c-1,c-1}, \dots, m_{3,3} | m_{1,1} m_{2,2}.$$

So again we do not get any subalgebras of index p . Suppose $m_{1,1} = 1$. Then we need to have $m_{c+1,c+1} = m_{c,c} = \dots = m_{2,2} = 0$, which means we need at least c zeroes. Thus $m_{1,1}$ has to be equal to 0 for subalgebras of index p^i where $2 \leq i \leq c-1$ or $i = c+1$.

For $m_{1,1} = 0$, note that

$$[\mathbf{m}_i, \mathbf{m}_j] = 0$$

for all $1 \leq i < j \leq n$. That means we can put whatever we want in $(c-1) \times (c-1)$ upper triangular matrices

$$M'' = \begin{pmatrix} m_{2,2} & m_{2,3} & \cdots & m_{2,c} \\ 0 & m_{3,3} & \cdots & m_{3,c} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & m_{c,c} \end{pmatrix}.$$

Such M would look like

$$M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & m_{2,2} & \cdots & m_{2,c} & m_{2,c+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m_{c,c} & m_{c,c+1} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let a denote the total number of zeroes in the diagonal entries of M'' . We know there are $\binom{c-1}{a}_p$ such M'' , and each of them gives additional p^{c-1-a} possibilities of M . This gives

$$\binom{c-1}{a}_p \cdot p^{c-1-a}$$

subalgebras of index p^{a+2} for $0 \leq a \leq c-1$. Finally we get p^c subalgebras of index p^c from

$$M = \begin{pmatrix} 1 & m_{1,2} & \cdots & m_{1,c} & m_{1,c+1} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus we get

$$\zeta_{M_{c+1}(\mathbb{F}_p)}^{\leq}(s) = \zeta_{M_c(\mathbb{F}_p)}^{\leq}(s) + \sum_{a=0}^{c-1} \binom{c-1}{a}_p p^{c-1-a} t^{a+2} + p^c t^c.$$

as required. □

Like the previous one this result is intriguing in many ways. First, we have an explicit formula for any $c \geq 2$. Second, it gives $1 + p$ maximal subalgebras of order p^c . In addition, it matches with a result by Berkovich [2] that if G is a group of order p^m and exponent p , then the number of subgroups of order p^n for $2 \leq n \leq m-2$ is congruent to $1 + p + 2p^2$ modulo p^3 . We also get

Corollary 5.3.5. $\zeta_{M_c(\mathbb{F}_p)}^{\leq}(s)$ is uniform for all $c \geq 2$.

The subring zeta function $\zeta_{M_{c,p}}^{\leq}(s)$ is even more mysterious than the ideal zeta function. Not only do we not know the explicit formulas for $c \geq 5$, we also do not have explicit formulas for

$$\zeta_{M_{c,p}}^{\leq \text{gr}}(s)$$

and

$$\zeta_{M_{c,p}}^{\leq \text{red}}(s).$$

5.4 Algorithm in practice: 2. Lower Central Series

Let c be the nilpotency class of $L(\mathbb{F}_p)$, $(\gamma_i(L))_{i=1}^c$ be the lower central series of $L(\mathbb{F}_p)$, and for $i = 1, \dots, c$ set $L(\mathbb{F}_p)^{(i)} := \gamma_i(L(\mathbb{F}_p))/\gamma_{i+1}(L(\mathbb{F}_p))$. Let $d_i = \dim(L(\mathbb{F}_p)^{(i)})$. One of the basic tricks of making p -adic integrals “simpler” is to choose the basis such that the first d_1 elements generate $\gamma_1(L(\mathbb{F}_p))/\gamma_2(L(\mathbb{F}_p))$, the next d_2 generate $\gamma_2(L(\mathbb{F}_p))/\gamma_3(L(\mathbb{F}_p))$, and so on. In fact we already used this in previous sections.

5.4.1 $\mathfrak{f}_{c,d}$, the Free nilpotent Lie algebra

Let $\mathfrak{f}_{c,d}(\mathbb{F}_p)$ be the free nilpotent Lie $\mathbb{F}_p$ -algebra of nilpotency class c on d generators.

Theorem 5.4.1.

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,2}(\mathbb{F}_p)}^{\triangleleft}(s) &= 1 + (1+p)p^{-s} + p^{-2s} + p^{-3s}, \\ \zeta_{\mathfrak{f}_{3,2}(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathfrak{f}_{2,2}(\mathbb{F}_p)}^{\triangleleft}(s) + (1+p)p^{-4s} + p^{-5s}, \\ \zeta_{\mathfrak{f}_{4,2}(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathfrak{f}_{3,2}(\mathbb{F}_p)}^{\triangleleft}(s) + (p+p^2)p^{-5s} + (1+p+p^2)p^{-6s} + (1+p+p^2)p^{-7s} + p^{-8s}.\end{aligned}$$

Proof. First, note that $\mathfrak{f}_{2,2}(\mathbb{F}_p) = H(\mathbb{F}_p)$. Now, the dimension of $\mathfrak{f}_{3,2}(\mathbb{F}_p)$ is 5, with $d_1 = 2, d_2 = 1, d_3 = 2$.

Let

$$M = \left(\begin{array}{cc|cc} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ \hline 0 & 0 & m_{3,3} & m_{3,4} & m_{3,5} \\ 0 & 0 & 0 & m_{4,4} & m_{4,5} \\ 0 & 0 & 0 & 0 & m_{5,5} \end{array} \right).$$

The good thing with choosing basis in such way is that we can easily separate the cases as we did before.

Suppose $m_{4,4} = m_{5,5} = 1$. Then this corresponds to

$$M = \left(\begin{array}{cc|cc} m_{1,1} & m_{1,2} & m_{1,3} & 0 & 0 \\ 0 & m_{2,2} & m_{2,3} & 0 & 0 \\ \hline 0 & 0 & m_{3,3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right),$$

which gives $\zeta_{\mathfrak{f}_{2,2}(\mathbb{F}_p)}^{\triangleleft}(s)$.

Since $m_{4,4} \mid m_{1,1}, m_{3,3}$ and $m_{5,5} \mid m_{2,2}, m_{3,3}$, the rest gives $(1+p)p^{-4s} + p^{-5s}$.

Similarly, for $\mathfrak{f}_{4,2}$ we have

$$M = \left(\begin{array}{cc|cc|cc|cc} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & m_{1,6} & m_{1,7} & m_{1,8} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} & m_{2,6} & m_{2,7} & m_{2,8} \\ \hline 0 & 0 & m_{3,3} & m_{3,4} & m_{3,5} & m_{3,6} & m_{3,7} & m_{3,8} \\ 0 & 0 & 0 & m_{4,4} & m_{4,5} & m_{4,6} & m_{4,7} & m_{4,8} \\ \hline 0 & 0 & 0 & 0 & m_{5,5} & m_{5,6} & m_{5,7} & m_{5,8} \\ 0 & 0 & 0 & 0 & 0 & m_{6,6} & m_{6,7} & m_{6,8} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & m_{7,7} & m_{7,8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{8,8} \end{array} \right)$$

and one can check with the aid of computer that $M \triangleleft \mathfrak{f}_{4,2}(\mathbb{F}_p)$ if $m_{6,6} | m_{1,1}, m_{4,4}, m_{7,7} | m_{1,1}, m_{2,2}, m_{4,4}, m_{5,5}, m_{8,8} | m_{2,2}, m_{5,5}$. One can check that

$$\begin{aligned} (m_{6,6}, m_{7,7}, m_{8,8}) &= (1, 1, 1) \text{ gives } \zeta_{F_{3,2}(\mathbb{F}_p)}^{\triangleleft}(s), \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (0, 1, 1) \text{ gives } pt^5 + t^6, \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (1, 0, 1) \text{ gives } pt^6, \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (1, 1, 0) \text{ gives } p^2t^5 + p^2t^6, \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (0, 0, 1) \text{ gives } t^7, \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (0, 1, 0) \text{ gives } pt^7, \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (1, 0, 0) \text{ gives } p^2t^7, \\ (m_{6,6}, m_{7,7}, m_{8,8}) &= (0, 0, 0) \text{ gives } t^8. \end{aligned}$$

□

5.4.2 L_E , the elliptic curve ring

Let L_E be a class-2 Lie ring of dimension 9 given by the following presentation:

$$L_E = \left\langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3 \mid [x_1, x_5] = [x_2, x_4] = [x_3, x_6] = y_1, \right. \\ \left. [x_1, x_6] = [x_3, x_4] = y_2, [x_1, x_4] = [x_2, x_5] = y_3 \right\rangle.$$

We will show that like $L_{E,p}$, the ideal zeta function of $L_E(\mathbb{F}_p)$ also depends on the behaviour mod p of the number of points on the elliptic curve $E : Y^2 = X^3 - X$, and hence the ideal zeta function is not finitely uniform.

Theorem 5.4.2.

$$\begin{aligned} \zeta_{L_E(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathbb{F}_p^6}(s) + p^2 |E(\mathbb{F}_p)| p^{-5s} + |E(\mathbb{F}_p)| (p + p^2) p^{-6s} \\ &\quad + \binom{3}{1}_p p^{-7s} + \binom{3}{2}_p p^{-8s} + p^{-9s}, \end{aligned}$$

where $|E(\mathbb{F}_p)| = |\{(x : y : z) \in \mathbb{P}^2(\mathbb{F}_p) : y^2z = x^3 - xz^2\}|$. In particular, $\zeta_{L_E \otimes \mathbb{F}_p}^{\triangleleft}(s)$ is not finitely uniform.

Proof. Let us first show where $|E(\mathbb{F}_p)|$ in the coefficient of p^{-5s} comes from. This is essentially the $\mathbb{F}_p$ version of [8, Theorem3.2].

Let

$$M = \begin{pmatrix} 1 & m_{1,2} & m_{1,3} & 0 & m_{1,5} & m_{1,6} & 0 & 0 & m_{1,9} \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1 & m_{4,5} & m_{4,6} & 0 & 0 & m_{4,9} \\ & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 0 & m_{7,9} \\ & & & & & & & 1 & m_{8,9} \\ & & & & & & & & 0 \end{pmatrix}$$

Then for $1 \leq i, j \leq 9$ and $k \in 2, 3, 5, 6, 9$, our system of equations $g_{ijk}^{\triangleleft}(m_{r,s}) = 0$ becomes

$$\begin{aligned} m_{1,5}m_{7,9} + m_{1,6}m_{8,9} &= 0, \\ m_{1,5} &= 0, \\ m_{1,6}m_{7,9} &= 0, \\ 1 - m_{1,2}m_{7,9} - m_{1,3}m_{8,9} &= 0, \\ m_{1,2} - m_{7,9} &= 0, \\ -m_{1,3}m_{7,9} - m_{8,9} &= 0, \\ 1 + m_{4,5}m_{7,9} + m_{4,6}m_{8,9} &= 0, \\ -m_{4,5} + m_{7,9} &= 0, \\ m_{4,6}m_{7,9} + m_{8,9} &= 0. \end{aligned}$$

Note that $m_{7,9} \neq 0$ since $m_{7,9} = 0$ implies $m_{8,9} = 0$ and $1 + m_{4,5}m_{7,9} + m_{4,6}m_{8,9} = 1 = 0$, which is impossible. After rearranging the equations and using $m_{7,9} \neq 0$, we have

$$\begin{aligned} m_{1,2} &= m_{4,5} = m_{7,9} \\ m_{1,5} &= m_{1,6} = 0 \\ m_{1,3} &= m_{4,6} = -\frac{m_{8,9}}{m_{7,9}} \\ 1 - m_{7,9}^2 + \frac{m_{8,9}^2}{m_{7,9}} &= 0. \end{aligned}$$

Hence, whenever $(m_{7,9}, m_{8,9})$ is a non-zero point on the elliptic curve $E(\mathbb{F}_p)$, there are p^2 matrices determined by $(m_{7,9}, m_{8,9})$, where p^2 comes from the free choices of $0 \leq m_{1,9}, m_{4,9} \leq p-1$. Thus

$$c_{a,b} = (|E(\mathbb{F}_p)| - 2) p^2.$$

and we get $|E(\mathbb{F}_p)|$ as required.

Now, note that this is a class-2 Lie algebra and the dimension of the derived part $d_2 = 3$. So we only need to consider eight cases and we have already done the most difficult one. One can explicitly calculate Theorem 5.4.2 by computing the rest seven cases. $\square$

5.5 Algorithm in practice: 3. Class-2 Cases

In Chapter 2 we showed that for class 2 cases, Voll's elementary divisor method is very effective for computing ideal zeta functions of Lie rings. Fortunately it also works well with Lie $\mathbb{F}_p$ -algebras.

Theorem 5.5.1. *Let L be a class-2 Lie ring with $L/L' \cong L_1$ and $L' \cong L_2$. Then we have*

$$\begin{aligned} \zeta_{L(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathbb{F}_p^{d_1}}(s) + p^{-ds} \\ &\quad + \sum_{0 < \Lambda_2 < L_2} |L_2 : \Lambda_2|^{-s} |L_1 : X(\Lambda_2)|^{-s} \sum_{\Lambda_1 \leq X(\Lambda_2)} |L_2 : \Lambda_2|^{rk(\Lambda_1)} |X(\Lambda_2) : \Lambda_1|^{-s} \end{aligned}$$

where $X(\Lambda_2)/\Lambda_2 = Z(L/\Lambda_2)$ and $\dim(L) = d$, $\dim(L_1) = d_1$ and $\dim(L_2) = d_2$.

Proof. Let Λ be the lattice in $L(\mathbb{F}_p)$. Then like over $\mathbb{Z}_p$, it gives rise to a pair (Λ_1, Λ_2) of a lattice

$$\Lambda_2 := \Lambda \cap L_2(\mathbb{F}_p) \leq L_2(\mathbb{F}_p)$$

in the derived ring and a lattice

$$\Lambda_1 := (\Lambda + L_2(\mathbb{F}_p))/L_2(\mathbb{F}_p) \leq L(\mathbb{F}_p)/L_2(\mathbb{F}_p).$$

The crucial difference between $\mathbb{Z}_p$ and $\mathbb{F}_p$ is that over $\mathbb{F}_p$, this surjective map is $|L_2(\mathbb{F}_p) : \Lambda_2|^{rk(\Lambda_1)}$ to 1. Again, $\Lambda \triangleleft L(\mathbb{F}_p)$ if and only if its associated pair (Λ_1, Λ_2) is admissible. More precisely, let

$$X(\Lambda_2)/\Lambda_2 := Z(L(\mathbb{F}_p)/\Lambda_2).$$

Then (Λ_1, Λ_2) is admissible if and only if $\Lambda_1 \leq X(\Lambda_2)$. Therefore, we have

$$\begin{aligned} \zeta_{L(\mathbb{F}_p)}^\triangleleft(s) &= \sum_{\Lambda_2 \leq L_2(\mathbb{F}_p)} |L_2(\mathbb{F}_p) : \Lambda_2|^{-s} |L(\mathbb{F}_p) : X(\Lambda_2)|^{-s} \\ &\cdot \sum_{\Lambda_1 \leq X(\Lambda_2)} |L_2(\mathbb{F}_p) : \Lambda_2|^{\text{rk}(\Lambda_1)} |X(\Lambda_2) : \Lambda_1|^{-s}. \end{aligned}$$

This is analogous to Lemma 6.1 in [15]. Note that since we are over $\mathbb{F}_p$, each Λ_2 is already maximal. Hence we first assume that Λ_2 is of elementary divisor type

$$\nu = (\underbrace{0, \dots, 0}_i, \underbrace{1, \dots, 1}_{d_2-i})$$

and say $\nu(\Lambda_2) = i$. Let

$$\begin{aligned} A_i^\triangleleft(s) &:= \sum_{\nu(\Lambda_2)=i} |L_2(\mathbb{F}_p) : \Lambda_2|^{-s} |L(\mathbb{F}_p) : X(\Lambda_2)|^{-s} \\ &\cdot \sum_{\Lambda_1 \leq X(\Lambda_2)} |L_2(\mathbb{F}_p) : \Lambda_2|^{\text{rk}(\Lambda_1)} |X(\Lambda_2) : \Lambda_1|^{-s}. \end{aligned}$$

Then one can see that

$$A_i^\triangleleft(s) = \sum_{\nu(\Lambda_2)=i} p^{-is} |L(\mathbb{F}_p) : X(\Lambda_2)|^{-s} \sum_{j=0}^{\text{rk}(X(\Lambda_2))} \binom{\text{rk}(X(\Lambda_2))}{j}_p p^{ij - (\text{rk}(X(\Lambda_2)) - j)s}$$

and

$$\zeta_{L(\mathbb{F}_p)}^\triangleleft = \zeta_{\mathbb{F}_p^{d_1}} + p^{-ds} + \sum_{i=1}^{d_2-1} A_i^\triangleleft.$$

□

Since $\text{rk}(X(\Lambda_2)) = d_1 - |L(\mathbb{F}_p) : X(\Lambda_2)|$, all we need to understand is the map

$$\Lambda_2 \mapsto |L(\mathbb{F}_p) : X(\Lambda_2)|.$$

Fortunately we already have some explicit examples that we already understand this map. The next section will show how one can compute $\zeta_{L(\mathbb{F}_p)}^\triangleleft(s)$ using this approach.

5.6 Other explicit examples

5.6.1 The Grenham Rings $\mathcal{G}_n$

Let $\mathcal{G}_n = \langle w, x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1} \mid [w, x_i] = y_i \rangle$, denote the Grenham Lie ring on n -generators. We have

Theorem 5.6.1.

$$\begin{aligned}\zeta_{\mathcal{G}_n(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathbb{F}_p^n}(s) + p^{-(2n-1)s} \\ &\quad + \sum_{i=1}^{n-2} \binom{n-1}{i}_p p^{-(2i+1)s} \sum_{k=0}^{n-(i+1)} \binom{n-(i+1)}{k}_p p^{ik-(n-(i+1)-k)s}\end{aligned}$$

Proof. Recall

$$\zeta_{L(\mathbb{F}_p)}^{\triangleleft} = \zeta_{\mathbb{F}_p^{d_1}} + p^{-ds} + \sum_{i=1}^{d_2-1} A_i^{\triangleleft}.$$

For $\mathcal{G}_n$ we have $d_1 = n, d_2 = n - 1$ and $d = 2n - 1$. So we have

$$\zeta_{\mathcal{G}_n(\mathbb{F}_p)}^{\triangleleft} = \zeta_{\mathbb{F}_p^n}(s) + p^{-(2n-1)s} + \sum_{i=1}^{n-2} A_i^{\triangleleft}.$$

Now, one needs to compute

$$A_i^{\triangleleft}(s) = \sum_{\nu(\Lambda_2)=i} p^{-is} |L(\mathbb{F}_p) : X(\Lambda_2)|^{-s} \sum_{j=0}^{\text{rk}(X(\Lambda_2))} \binom{\text{rk}(X(\Lambda_2))}{j}_p p^{ij-(\text{rk}(X(\Lambda_2))-j)s}.$$

In [36, Chapter 5], Voll already showed that for $\mathcal{G}_n$, we have $|\mathcal{G}_n(\mathbb{F}_p) : X(\Lambda_2)| = i + 1$ and $\text{rk}(X(\Lambda_2)) = n - (i + 1)$ when $\nu(\Lambda_2) = i$, as they only depends on the elementary divisor type of its maximal element. Hence we get the required results. $\square$

Example 5.6.2.

$$\begin{aligned}\zeta_{\mathcal{G}_3(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathbb{F}_p^3}(s) + p^{-5s} + (1+p)p^{-3s}(p+p^{-s}) \\ &= 1 + (1+p+p^2)p^{-s} + (1+p+p^2)p^{-2s} + (1+p+p^2)p^{-3s} \\ &\quad + (1+p)p^{-4s} + p^{-5s}.\end{aligned}$$

Note that ([15])

$$\begin{aligned}\zeta_{\mathcal{G}_{3,p}}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^3}(s) \zeta_p(3s-3) \zeta_p(3s-4) \zeta_p(5s-6) \zeta_p(6s-6)^{-1} \\ &= 1 + (1+p+p^2)p^{-s} + (1+p+2p^2+p^3+p^4)p^{-2s} \\ &\quad + (1+p+2p^2+3p^3+3p^4+p^5+p^6)p^{-3s} \\ &\quad + (1+p+2p^2+3p^3+5p^4+4p^5+3p^6+p^7+p^8)p^{-4s} \\ &\quad + (1+p+2p^2+3p^3+5p^4+6p^5+7p^6+4p^7+3p^8+p^9+p^{10})p^{-5s} \\ &\quad + \dots\end{aligned}$$

and

$$\begin{aligned}
\zeta_{\mathcal{L}_{2(2)} \otimes \mathbb{Z}_p}^{\triangleleft gr}(s) &= \zeta_{\mathbb{Z}_p^3}(s) \zeta_p(3s-1) \zeta_p(5s) (1+p^{-3s}) \\
&= 1 + (1+p+p^2)p^{-s} + (1+p+2p^2+p^3+p^4)p^{-2s} \\
&\quad + (2+2p+2p^2+2p^3+2p^4+p^5+p^6)p^{-3s} \\
&\quad + (2+3p+4p^2+3p^3+3p^4+2p^5+2p^6+p^7+p^8)p^{-4s} \\
&\quad + (3+3p+5p^2+5p^3+5p^4+4p^5+3p^6+2p^7+2p^8+p^9+p^{10})p^{-5s} \\
&\quad + \dots
\end{aligned}$$

5.6.2 Class-2 free Lie rings on d -generators

Let $\mathfrak{f}_{2,d}$ be the class-2 Free Lie ring on d generators. For $k \in \mathbb{N}$, let

$$I_k := \left\{ i \in \mathbb{N} : \binom{k-1}{2} < i \leq \binom{k}{2} \right\}.$$

Then, let

$$\pi : \mathbb{N} \rightarrow \mathbb{N}$$

where $\pi(i) = k$ for $i \in I_k$. Since I_k partitions $\mathbb{N}$ this is well-defined.

Theorem 5.6.3.

$$\begin{aligned}
\zeta_{\mathfrak{f}_{2,d}(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathbb{F}_p^d}(s) + p^{-(d+d')s} \\
&\quad + \sum_{i=1}^{d'-1} \binom{d'}{i}_p p^{-(i+\pi(i))s} \sum_{k=0}^{d-\pi(i)} \binom{d-\pi(i)}{k}_p p^{ik-(d-\pi(i)-k)s}
\end{aligned}$$

Example 5.6.4.

$$\begin{aligned}
\zeta_{\mathfrak{f}_{2,3}(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\mathbb{F}_p^3}(s) + p^{-6s} + (1+p+p^2)p^{-3s}(p+p^{-s}) + (1+p+p^2)p^{-5s} \\
&= 1 + (1+p+p^2)p^{-s} + (1+p+p^2)p^{-2s} + (1+p+p^2+p^3)p^{-3s} \\
&\quad + (1+p+p^2)p^{-4s} + (1+p+p^2)p^{-5s} + p^{-6s}.
\end{aligned}$$

5.6.3 Central product of m copies of Heisenberg H_m

Theorem 5.6.5. $\zeta_{H_m(\mathbb{F}_p)}^{\triangleleft}(s) = \zeta_{\mathbb{F}_p^{2m}}(s) + p^{-(2m+1)s}.$

Note that

$$\zeta_{H_m \otimes \mathbb{Z}_p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^{2m}}(s) \zeta_p((2m+1)s - 2m).$$

5.6.4 Other examples

Theorem 5.6.6. *Let*

$$g_{5,3} = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_4, [x_1, x_4] = [x_2, x_3] = x_5 \rangle.$$

Then

$$\zeta_{g_{5,3}(\mathbb{F}_p)}^{\triangleleft}(s) = \zeta_{\mathbb{F}_p^3}(s) + p^{(1-3s)} + p^{-4s} + p^{-5s}.$$

Theorem 5.6.7. *Let*

$$Fil_4 = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = [x_2, x_3] = x_5 \rangle.$$

$$\text{Then } \zeta_{Fil_4(\mathbb{F}_p)}^{\triangleleft}(s) = \zeta_{M_4(\mathbb{F}_p)}^{\triangleleft}(s) = 1 + (1+p)p^{-s} + p^{-2s} + p^{-3s} + p^{-4s} + p^{-5s}.$$

Theorem 5.6.8. *Let*

$$g_{6,4} = \langle x_1, x_2, x_3, x_4, y_1, y_2 \mid [x_1, x_2] = y_1, [x_1, x_3] = [x_2, x_4] = y_2 \rangle.$$

Then

$$\begin{aligned} \zeta_{g_{6,4}(\mathbb{F}_p)}^{\triangleleft}(s) = & 1 + (1 + p + p^2 + p^3)p^{-s} + (1 + p + 2p^2 + p^3 + p^4)p^{-2s} \\ & + (1 + p + 2p^2 + p^3)p^{-3s} + (1 + p + p^2)p^{-4s} \\ & + (1 + p)p^{-5s} + p^{-6s} \end{aligned}.$$

Chapter 6

The i -th partial zeta functions of groups and rings

Let L be a Lie ring, additively isomorphic to $\mathbb{Z}^n$, and $L_p = L \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^n$. Let $(e_1, \dots, e_n)$ be the basis of L , and let M be an upper-triangular $n \times n$ matrix $M = (m_{i,j})$ with entries in $\mathbb{Z}_p$. Recall Section 1.2 that one can construct a p -adic integral with a suitable cone data

$$V_p^{\triangleleft} = \{M \in \text{Tr}_n(\mathbb{Z}_p) \mid v(f_k(\mathbf{x})) \leq v(g_{i,j,k}^{\triangleleft}(\mathbf{x}))\}$$

and

$$V_p^{\leq} = \{M \in \text{Tr}_n(\mathbb{Z}_p) \mid v(f_k(\mathbf{x})) \leq v(g_{i,j,k}^{\leq}(\mathbf{x}))\}$$

such that

$$\zeta_{L_p}^*(s) = (1 - p^{-1})^{-n} \int_{V_p^*} |m_{1,1}|^{s-1} \cdots |m_{n,n}|^{s-n} d\mu.$$

Now, define

$$V_{p,i}^* = \{M \in V_p^* \mid v(m_{j,j}) < i \text{ for all } j \in [n]\}.$$

With this definition we can write $V_p^* = V_{p,\infty}^*$, and think of $V_{p,i}^*$ as layers of V_p^* where $V_{p,1}^* \subset V_{p,2}^* \subset \dots$. By counting these specific M 's for different i , integrating over $V_{p,i}^*$ instead of V_p^* , one might hope to observe the ‘Hensel-like’ behaviour of the p -adic integration. In fact we observed some interesting behaviours that motivated us to define the i -th partial zeta functions $\zeta_{L_p,i,B}^*(s)$:

$$\begin{aligned} \zeta_{L_p,i,B}^*(s) &= (1 - p^{-1})^{-n} \int_{V_{p,i}^*} |m_{1,1}|^{s-1} \cdots |m_{n,n}|^{s-n} d\mu \\ &= (1 - p^{-1})^{-n} \int_{\substack{V_p^* \\ v(m_{j,j}) < i}} |m_{1,1}|^{s-1} \cdots |m_{n,n}|^{s-n} d\mu. \end{aligned}$$

In principle, these partial zeta functions can be regarded as the approximations of layers of the original zeta functions. Unfortunately, unlike the original zeta function $\zeta_{L_p}^*(s)$, our current definition of the i -th partial zeta functions is “basis-dependent”, in the sense that different bases would give different values. We will later demonstrate explicit examples with different bases giving different calculations.

6.1 Algorithms with simple explicit examples

The main difference between $\zeta_{L_p}^*(s)$ and $\zeta_{L_p,i,B}^*(s)$ is that one is an infinite generating series, while the other one is just a finite polynomial. This means that just as in Chapter 5, we can compute $\zeta_{L_p,i,B}^*(s)$ algorithmically in the following way:

- (1) For a given L with dimension n , fix a basis B and write down the cone conditions $V_{p,i}^*$.
- (2) Since there are only finite possible cases, we run through all of them.

Throughout this chapter, let $t = p^{-s}$. We will omit B when our choice of basis is clear.

Theorem 6.1.1.

$$\begin{aligned}\zeta_{\mathbb{Z}_p^n,i}^*(s) &= (1 + t + \cdots + t^{i-1}) \cdots (1 + p^{n-1}t + \cdots + p^{(n-1)(i-1)}t^{i-1}) \\ &= \prod_{j=1}^n \sum_{k=0}^{i-1} p^{k(j-1)} t^k.\end{aligned}$$

Proof. First, recall that

$$\zeta_{\mathbb{Z}_p^n}^*(s) = \zeta_p(s) \cdots \zeta_p(s - (n-1)),$$

where

$$\zeta_p(s) = \frac{1}{1-t} = 1 + t + t^2 + \cdots.$$

Let

$$\zeta_{p,i}(s) = 1 + t + t^2 + \cdots + t^{i-1}.$$

One can check that

$$\zeta_{\mathbb{Z}_p,i}(s) = \zeta_{p,i}(s)$$

and

$$\zeta_{\mathbb{Z}_p^n,i}^*(s) = \zeta_{p,i}(s) \cdots \zeta_{p,i}(s - (n-1)).$$

□

Theorem 6.1.2. *Let*

$$H = \langle x_1, x_2, x_3 : [x_1, x_2] = x_3 \rangle$$

be a Heisenberg Lie ring. Then

$$\begin{aligned}\zeta_{H_p,2}^{\triangleleft}(s) &= 1 + (1+p)t + pt^2 + p^2t^3 \\ &= \zeta_{\mathbb{Z}_p^2,2}(s) + p^2t^3,\end{aligned}$$

and

$$\begin{aligned}\zeta_{H_p,2}^{\leq}(s) &= 1 + (1+p)t + (p+p^2+p^3)t^2 + p^3t^3 \\ &= \zeta_{\mathbb{Z}_p^2,2}(s) + (p^2+p^3)t^2 + p^3t^3.\end{aligned}$$

Proof. Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ & m_{2,2} & m_{2,3} \\ & & m_{3,3} \end{pmatrix} \in \text{Tr}_3(\mathbb{Z}_p).$$

It is a well-known fact that $M \triangleleft L_p$ if and only if $m_{3,3} | m_{1,1}, m_{1,2}, m_{2,2}$. Since we have an extra condition that $v(m_{j,j}) < 2$ for all $j \in [3]$, we get:

1. $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} p & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & m_{12} & 0 \\ & p & 0 \\ & & 1 \end{pmatrix},$ and $\begin{pmatrix} p & m_{1,2} & 0 \\ & p & 0 \\ & & 1 \end{pmatrix},$ giving $\zeta_{\mathbb{Z}_p^2,2}(s),$
2. $\begin{pmatrix} p & 0 & m_{1,3} \\ & p & m_{2,3} \\ & & p \end{pmatrix},$ giving $p^2t^3.$

Similarly, $M \leq L_p$ if and only if $m_{3,3} | m_{1,1}m_{2,2},$ and our extra condition implies that we have :

1. $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} p & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & m_{12} & 0 \\ & p & 0 \\ & & 1 \end{pmatrix},$ and $\begin{pmatrix} p & m_{1,2} & 0 \\ & p & 0 \\ & & 1 \end{pmatrix},$ giving $\zeta_{\mathbb{Z}_p^2,2}(s),$
2. $\begin{pmatrix} p & 0 & m_{1,3} \\ & 1 & m_{2,3} \\ & & p \end{pmatrix},$ giving $p^2t^2,$
3. $\begin{pmatrix} 1 & m_{1,2} & m_{1,3} \\ & p & m_{2,3} \\ & & p \end{pmatrix},$ giving $p^3t^2,$
4. $\begin{pmatrix} p & m_{1,2} & m_{1,3} \\ & p & m_{2,3} \\ & & p \end{pmatrix},$ giving $p^3t^3.$

□

The same analogous techniques that we demonstrated in Chapter 5 all work effectively here as well. The Heisenberg example shows both diagonal condition techniques reducing the cases and the lower central series technique separating $\zeta_{\mathbb{Z}_p^2,2}(s)$ and the rest.

In fact $\zeta_{L_p,2}^*(s)$ is very similar to $\zeta_{L(\mathbb{F}_p)}^*(s)$, since there are two choices for each entry on the diagonal.

Theorem 6.1.3.

$$\zeta_{H_p,3}^\triangleleft(s) = \zeta_{\mathbb{Z}_p^2,3}(s) + p^2t^3 + (1+p)p^2t^4 + p^3t^5 + p^4t^6$$

Proof. Since $i = 3$, each diagonal entry can be one of 1, p or p^2 . First, note that $m_{3,3} = 1$ gives $\zeta_{\mathbb{Z}_p^2,3}(s)$. Now $m_{3,3} = p$ gives $p \mid m_{1,1}, m_{1,2}, m_{2,2}$, so we get

1. $\begin{pmatrix} p & 0 & m_{1,3} \\ & p & m_{2,3} \\ & & p \end{pmatrix}$, giving p^2t^3 ,
2. $\begin{pmatrix} p^2 & 0 & m_{1,3} \\ & p & m_{2,3} \\ & & p \end{pmatrix}$, giving p^2t^4 ,
3. $\begin{pmatrix} p & pm'_{1,2} & m_{1,3} \\ & p & m_{2,3} \\ & & p \end{pmatrix}$, giving p^3t^4 ,
4. $\begin{pmatrix} p^2 & pm'_{1,2} & m_{1,3} \\ & p^2 & m_{2,3} \\ & & p \end{pmatrix}$, giving p^3t^5 ,

where $0 \leq m'_{1,3} < p$.

Finally we get $\begin{pmatrix} p^2 & 0 & m_{1,3} \\ & p^2 & m_{2,3} \\ & & p^2 \end{pmatrix}$, giving p^4t^6 . □

6.2 L_p of nilpotency class c and the c -th partial zeta functions

6.2.1 Rationality and variation with the prime

In [15] Grunewald, Segal and Smith proved the following rationality results for $\zeta_{L_p}^*(s)$.

Theorem 6.2.1 ([15], Theorem 3.5). *For p primes, the local zeta function $\zeta_{L_p}^*(s)$ is a rational function in p^{-s} .*

This implies that the sequence $(a_{p^i}^*(L))$ of the numbers of subrings/ideals of L of index p^i satisfies a strong regularity property, that the numbers of finite index subalgebras/ideals of L_p are already determined by the numbers of subalgebras in some finite quotient of L_p .

In [6] du Sautoy then proved the following result:

Theorem 6.2.2 ([6], Corollary 3.3). *For each $n \in \mathbb{N}$ there exists a finite index set J_n and finitely many pairs $(a_j, b_j) \in \mathbb{N}_0^2$ for $j \in J_n$ such that, if L is a Lie ring additively isomorphic to $\mathbb{Z}^n$ then for all prime p and $* \in \{\leq, \triangleleft, \wedge\}$,*

$$\zeta_{L_p}^*(s) = \frac{\Psi_{L_p}^*(t)}{\Phi_{L_p}^*(t)},$$

where $\Psi_{L_p}^*(X), \Phi_{L_p}^*(X) \in \mathbb{Q}[X]$ and $\Phi_{L_p}^*(t)$ can be taken to divide $\prod_{j \in J_n} (1 - p^{a_j} t^{b_j})$.

In particular, this implies that for each $L \cong \mathbb{Z}^n$, prime p and $* \in \{\leq, \triangleleft, \wedge\}$, there exist $k \in \mathbb{N}$, $(a_j, b_j) \in \mathbb{N}_0^2$ where $j \in [k]$, and a polynomial $\Psi_{L_p}^*(X) \in \mathbb{Q}[X]$ such that

$$\zeta_{L_p}^*(s) = \frac{\Psi_{L_p}^*(t)}{\prod_{j \leq k} (1 - p^{a_j} t^{b_j})},$$

where $(1 - p^{a_j} t^{b_j})$'s in the denominator does not get cancelled by the numerator $\Psi_{L_p}^*(X) \in \mathbb{Q}[X]$, i.e. is in its lowest terms. For example, we have

$$\begin{aligned} \zeta_{H_p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2}(s) \zeta_p(2 - 3s) \\ &= \frac{1}{(1-t)(1-pt)(1-p^2t^3)}, \end{aligned}$$

where H is the Heisenberg Lie ring. In this case $\frac{1}{(1-t)(1-pt)(1-p^2t^3)}$ is in its lowest terms, and we have $1-t, 1-pt$ and $1-p^2t^3$ in the denominator. If we compare this with Theorem 6.1.2 we observe the following:

$$\begin{aligned} \zeta_{H_p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2}(s) \zeta_p(2 - 3s) \\ &= \frac{1}{(1-t)(1-pt)(1-p^2t^3)}, \\ \zeta_{H_p,2}^{\triangleleft}(s) &= 1 + \underline{t} + \underline{pt} + \underline{pt^2} + \underline{p^2t^3}. \end{aligned}$$

Note that we have t, pt and p^2t^3 in both formulas. Since

$$\zeta_p(a_j - b_j s) = \frac{1}{(1 - p^{a_j} t^{b_j})} = 1 + p^{a_j} t^{b_j} + (p^{a_j} t^{b_j})^2 + \dots,$$

one can say, for example, that p^2t^3 in $\zeta_{H_p,2}^{\triangleleft}(s)$ captures the factor $(1 - p^2t^3)$ of the denominator of $\zeta_{H_p}^{\triangleleft}(s)$. We make the following definition:

Definition 6.2.3. We say $\zeta_{L_p,i,B}^*(s)$ captures all the factors of $\zeta_{L_p}^*(s)$ if

$$\zeta_{L,p}^*(s) = \frac{\Psi_{L,p}^*(t)}{\prod_{j \leq k} (1 - p^{a_j} t^{b_j})}$$

is in its lowest terms, then the coefficients of $p^{a_j} t^{b_j}$ for $j \in [k]$ in $\zeta_{L_p,i,B}^*(s)$ are non-zero.

Note that $\zeta_{\mathbb{Z}_p,i}(s)$ captures all the factors $\zeta_{\mathbb{Z}_p}(s)$ for $i \geq 2$. We just showed that $\zeta_{H_p,2}^\triangleleft(s)$ captures all the factors of $\zeta_{H_p}^\triangleleft(s)$. Also we have

$$\begin{aligned} \zeta_{H_p}^{\leq}(s) &= \zeta_{\mathbb{Z}_p}^2(s) \zeta_p(2s-2) \zeta_p(2s-3) \zeta_p(3s-2)^{-1}, \\ &= \frac{1 - p^2 t^3}{(1-t)(1-pt)(1-p^2 t^2)(1-p^3 t^2)} \end{aligned}$$

and

$$\begin{aligned} \zeta_{H_p,2}^{\leq}(s) &= \zeta_{\mathbb{Z}_p,2}^2(s) + (p^2 + p^3)t^2 + p^3 t^3, \\ &= 1 + \underline{t} + \underline{pt} + \underline{pt^2} + \underline{p^2 t^2} + \underline{p^3 t^2} + p^3 t^3. \end{aligned}$$

Note that all of $t, pt, p^2 t^2$ and $p^3 t^2$ from the denominator of $\zeta_{H_p}^{\leq}(s)$ in its lowest terms are also in $\zeta_{H_p,2}^{\leq}(s)$. Hence $\zeta_{H_p,2}^{\leq}(s)$ captures all the factors of $\zeta_{H_p}^{\leq}(s)$.

The reason why we are interested in these factors is that these “survivors” are the potential candidates for the poles, and also can be seen as the building blocks of the zeta functions, controlling this strong regularity property. We now observe some interesting phenomenons between $\zeta_{L_p}^*(s)$ in its lowest terms and $\zeta_{L_p,c,B}^*(s)$, where c is the nilpotency class of L .

6.2.2 Maximal Class M_3 , M_4 and Fil_4

Theorem 6.2.4. For $c \geq 2$, let

$$M_c = \langle x_0, x_1, \dots, x_c : \forall i \in [c-1], [x_0, x_i] = x_{i+1} \rangle$$

be a maximal class- c Lie ring M_c . Then

$$\begin{aligned} \zeta_{M_3,p,2}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p,2}^2(s) + p^2 t^3 + p^2 t^4 \\ \zeta_{M_4,p,2}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p,2}^2(s) + p^2 t^3 + p^2 t^4 + p^2 t^5 \end{aligned}$$

Proof. Note that $M_2 = H$.

For M_3 , recall Theorem 5.3.4. We have

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ 0 & m_{2,2} & m_{2,3} & 0 \\ 0 & 0 & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

giving $\zeta_{M_2,p,2}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^2,2}(s) + p^2 t^3$.

For

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} \\ 0 & 0 & m_{3,3} & m_{3,4} \\ 0 & 0 & 0 & p \end{pmatrix},$$

one requires $p \mid m_{1,1}, m_{1,2}, m_{2,2}, m_{1,3}, m_{2,3}, m_{3,3}, m_{3,4}$, and

$$M = \begin{pmatrix} p & 0 & 0 & m_{1,4} \\ 0 & p & 0 & m_{2,4} \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

gives $p^2 t^4$. Hence

$$\begin{aligned} \zeta_{M_3,p,2}^{\triangleleft}(s) &= \zeta_{M_2,p,2}^{\triangleleft}(s) + p^2 t^4 \\ &= \zeta_{\mathbb{Z}_p^2,2}(s) + p^2 t^3 + p^2 t^4. \end{aligned}$$

We do the similar calculation for M_4 to get

$$\begin{aligned} \zeta_{M_4,p,2}^{\triangleleft}(s) &= \zeta_{M_3,p,2}^{\triangleleft}(s) + p^2 t^5 \\ &= \zeta_{\mathbb{Z}_p^2,2}(s) + p^2 t^3 + p^2 t^4 + p^2 t^5 \end{aligned}$$

□

Comparing

$$\begin{aligned} \zeta_{M_3,p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2}(s) \zeta_p(3s-2) \zeta_p(4s-2) \zeta_p(5s-3) \zeta_p(5s-2)^{-1} \\ &= \frac{1 - p^2 t^5}{(1-t)(1-pt)(1-p^2 t^3)(1-p^2 t^4)(1-p^3 t^5)} \end{aligned}$$

in its reduced form with

$$\zeta_{M_3,p,2}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^2,2}(s) + p^2 t^3 + p^2 t^4,$$

one can see that $\zeta_{M_3,p,2}^{\triangleleft}(s)$ does not capture all the factors of $\zeta_{M_3,p}^{\triangleleft}(s)$. However, observe the following result:

Theorem 6.2.5.

$$\zeta_{M_3,p,3}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^2,3}(s) + \underline{p^2 t^3} + \underline{2p^2 t^4} + p^3 t^4 + \underline{3p^3 t^5} + 2p^4 t^6 + p^4 t^7 + p^4 t^8.$$

In particular, $\zeta_{M_3,p,3}^{\triangleleft}(s)$ captures all the factors of $\zeta_{M_3,p}^{\triangleleft}(s)$.

Proof. Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} \\ 0 & 0 & m_{3,3} & m_{3,4} \\ 0 & 0 & 0 & m_{4,4} \end{pmatrix},$$

denotes the matrix representations of additive subgroups of M_3 . We know ([33]) this gives an ideal if

$$\begin{aligned} m_{3,3} &| m_{1,1}, m_{1,2}, m_{2,2}, \\ m_{4,4} &| m_{1,1}, m_{3,3}, \\ m_{3,3}m_{4,4} &| m_{1,1}m_{3,4}, m_{1,3}m_{3,3} - m_{1,2}m_{3,4}, m_{2,3}m_{3,3} - m_{2,2}m_{3,4}. \end{aligned}$$

We have already shown that for $i = 2$,

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ 0 & m_{2,2} & m_{2,3} & 0 \\ 0 & 0 & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

gives $\zeta_{H_p,2}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^2,2}(s) + p^2 t^3$, and

$$M = \begin{pmatrix} p & 0 & 0 & m_{1,4} \\ 0 & p & 0 & m_{2,4} \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

gives $p^2 t^4$.

For $i = 3$, we now have more possibilities. Again,

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & 0 \\ 0 & m_{2,2} & m_{2,3} & 0 \\ 0 & 0 & m_{3,3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

gives

$$\zeta_{H_p,3}^{\triangleleft}(s) = \zeta_{\mathbb{Z}_p^2,3}(s) + p^2 t^3 + (1+p)p^2 t^4 + p^3 t^5 + p^4 t^6.$$

Then, for $m_{4,4} = p$, the cone conditions gives

- $\begin{pmatrix} p & 0 & 0 & m_{1,4} \\ 0 & p & 0 & m_{2,4} \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, giving $p^2 t^4$
- $\begin{pmatrix} p^2 & 0 & 0 & m_{1,4} \\ 0 & p & m_{3,4} & m_{2,4} \\ 0 & 0 & p & m_{3,4} \\ 0 & 0 & 0 & p \end{pmatrix}$, giving $p^3 t^5$
- $\begin{pmatrix} p & pm'_{1,2} & 0 & m_{1,4} \\ 0 & p^2 & 0 & m_{2,4} \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, giving $p^3 t^5$
- $\begin{pmatrix} p^2 & pm'_{1,2} & m'_{1,2}m_{3,4} & m_{1,4} \\ 0 & p^2 & 0 & m_{2,4} \\ 0 & 0 & p & m_{3,4} \\ 0 & 0 & 0 & p \end{pmatrix}$, giving $p^4 t^6$
- $\begin{pmatrix} p^2 & 0 & pm'_{1,3} & m_{1,4} \\ 0 & p^2 & pm'_{2,3} & m_{2,4} \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$, giving $p^4 t^7$,

Finally, $m_{4,4} = p^2$ gives

$$\begin{pmatrix} p^2 & 0 & 0 & m_{1,4} \\ 0 & p^2 & 0 & m_{2,4} \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix},$$

giving $p^4 t^8$.

Thus we have

$$\begin{aligned} \zeta_{M_3,p,3}^{\triangleleft}(s) &= \zeta_{H_p,3}(s) + p^2 t^4 + 2p^3 t^5 + p^4 t^6 + p^4 t^7 + p^4 t^8 \\ &= \zeta_{\mathbb{Z}_p^2,3}(s) + p^2 t^3 + 2p^2 t^4 + p^3 t^4 + 3p^3 t^5 + 2p^4 t^6 + p^4 t^7 + p^4 t^8, \end{aligned}$$

as required. $\square$

So for M_3 , the 2nd partial zeta function was not enough. One needs to at least get to $i = 3$ to capture the whole recurrence factors.

Theorem 6.2.6.

$$\begin{aligned} \zeta_{M_4,p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2}(s) \zeta_p(3s-2) \zeta_p(5s-2) \zeta_p(7s-4) \zeta_p(8s-5) \zeta_p(9s-6) \\ &\quad \times \zeta_p(11s-6) \zeta_p(12s-7) \zeta_p(6s-3)^{-1} W_{M_4}^{\triangleleft}(p, t) \end{aligned}$$

and

$$\begin{aligned}
\zeta_{M_4,p,2}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2,2}(s) + \underline{p^2t^3} + p^2t^4 + \underline{p^2t^5}, \\
\zeta_{M_4,p,3}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2,3}(s) + \underline{p^2t^3} + 2p^2t^4 + p^3t^4 + \underline{p^2t^5} + 3p^3t^5 + 2p^3t^6 + 2p^4t^6 + \underline{2p^4t^7} \\
&\quad + p^4t^8 + \underline{p^5t^8} + p^4t^9 + p^4t^{10}, \\
\zeta_{M_4,p,4}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2,4}(s) + \underline{p^2t^3} + 2p^2t^4 + p^3t^4 + \underline{2p^2t^5} + 3p^3t^5 + p^4t^5 + 4p^3t^6 + 4p^4t^6 + \underline{7p^4t^7} \\
&\quad + 2p^5t^7 + 2p^4t^8 + \underline{6p^5t^8} + p^4t^9 + 3p^5t^9 + \underline{4p^6t^9} + p^4t^{10} + p^5t^{10} + 3p^6t^{10} + p^7t^{10} \\
&\quad + p^5t^{11} + \underline{3p^6t^{11}} + p^7t^{11} + p^5t^{12} + p^6t^{12} + \underline{p^7t^{12}} + p^6t^{13} + p^7t^{13} + p^6t^{14} + p^6t^{15}.
\end{aligned}$$

Proof. Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ 0 & 0 & m_{3,3} & m_{3,4} & m_{3,5} \\ 0 & 0 & 0 & m_{4,4} & m_{4,5} \\ 0 & 0 & 0 & 0 & m_{5,5} \end{pmatrix},$$

denotes the matrix representations of additive subgroups of M_4 . This time we have to solve

$$\begin{aligned}
&m_{3,3}|m_{1,1}, m_{1,2}, m_{2,2}, m_{4,4}|m_{1,1}, m_{3,3}, m_{5,5}|m_{1,1}, m_{4,4} \\
&m_{3,3}m_{4,4}|m_{1,1}m_{3,4}, m_{1,3}m_{3,3} - m_{1,2}m_{3,4}, m_{2,3}m_{3,3} - m_{2,2}m_{3,4}, \\
&m_{4,4}m_{5,5}|m_{1,1}m_{4,5}, m_{3,3}m_{4,5} - m_{3,4}m_{4,4} \\
&m_{3,3}m_{4,4}m_{5,5}|m_{1,1}(m_{3,4}m_{4,5} - m_{3,5}m_{4,4}) \\
&m_{3,3}m_{4,4}m_{5,5}|m_{1,2}m_{3,4}m_{4,5} - m_{1,2}m_{3,5}m_{4,4} - m_{1,3}m_{3,3}m_{4,5} + m_{1,4}m_{3,3}m_{4,4} \\
&m_{3,3}m_{4,4}m_{5,5}|m_{2,2}m_{3,4}m_{4,5} - m_{2,2}m_{3,5}m_{4,4} - m_{2,3}m_{3,3}m_{4,5} + m_{2,4}m_{3,3}m_{4,4}
\end{aligned}$$

First, for each i , we get $\zeta_{M_3,p,i}^{\triangleleft}(s)$ for $m_{5,5} = 1$. Now for $m_{5,5} \neq 1$, we need to compute 1 matrix for $\zeta_{M_4,p,2}^{\triangleleft}(s)$, 7 matrices for $\zeta_{M_4,p,3}^{\triangleleft}(s)$, and 27 matrices to compute for $\zeta_{M_4,p,4}^{\triangleleft}(s)$. Thus we get

$$\begin{aligned}
\zeta_{M_4,p,2}^{\triangleleft}(s) &= \zeta_{M_3,p,2}(s) + p^2t^5 \\
&= \zeta_{\mathbb{Z}_p^2,2}(s) + \underline{p^2t^3} + p^2t^4 + \underline{p^2t^5},
\end{aligned}$$

$$\begin{aligned}
\zeta_{M_4,p,3}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2,3}(s) + \underline{p^2t^3} + 2p^2t^4 + p^3t^4 + \underline{p^2t^5} + 3p^3t^5 + 2p^3t^6 + 2p^4t^6 + \underline{2p^4t^7} \\
&\quad + p^4t^8 + \underline{p^5t^8} + p^4t^9 + p^4t^{10}.
\end{aligned}$$

and

$$\begin{aligned}\zeta_{M_4,p,4}^{\triangleleft}(s) = & \zeta_{\mathbb{Z}_p^2,4}(s) + \underline{p^2t^3} + 2p^2t^4 + p^3t^4 + 2\underline{p^2t^5} + 3p^3t^5 + p^4t^5 + 4p^3t^6 + 4p^4t^6 + 7\underline{p^4t^7} \\ & + 2p^5t^7 + 2p^4t^8 + 6\underline{p^5t^8} + p^4t^9 + 3p^5t^9 + 4\underline{p^6t^9} + p^4t^{10} + p^5t^{10} + 3p^6t^{10} + p^7t^{10} \\ & + p^5t^{11} + 3\underline{p^6t^{11}} + p^7t^{11} + p^5t^{12} + p^6t^{12} + \underline{p^7t^{12}} + p^6t^{13} + p^7t^{13} + p^6t^{14} + p^6t^{15}.\end{aligned}$$

□

Theorem 6.2.7. *Let the Lie ring Fil_4 have presentation*

$$Fil_4 = \langle z, x_1, x_2, x_3, x_4 : [z, x_1]x_2, [z, x_2] = x_3, [z, x_3] = x_4, [x_1, x_2] = x_4 \rangle.$$

Then

$$\begin{aligned}\zeta_{Fil_4,p}^{\triangleleft}(s) = & \zeta_{\mathbb{Z}_p^2}(s)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6) \\ & \times \zeta_p(10s-6)\zeta_p(12s-7)W_{Fil_4}^{\triangleleft}(p, t)\end{aligned}$$

and

$$\begin{aligned}\zeta_{Fil_4,p,2}^{\triangleleft}(s) &= \zeta_{M_4,p,2}^{\triangleleft}(s) \\ \zeta_{Fil_4,p,3}^{\triangleleft}(s) &= \zeta_{M_4,p,3}^{\triangleleft}(s) \\ \zeta_{Fil_4,p,4}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^2,4}(s) + \underline{p^2t^3} + 2p^2t^4 + p^3t^4 + 2\underline{p^2t^5} + 3p^3t^5 + p^4t^5 + 4p^3t^6 + 4p^4t^6 + 7\underline{p^4t^7} \\ &+ 2p^5t^7 - p^3t^8 + 4p^4t^8 + 5\underline{p^5t^8} + 5p^5t^9 + 3\underline{p^6t^9} + p^4t^{10} + p^5t^{10} + 3\underline{p^6t^{10}} + p^7t^{10} \\ &+ p^5t^{11} + 3p^6t^{11} + p^7t^{11} + p^5t^{12} + p^6t^{12} + \underline{p^7t^{12}} + p^6t^{13} + p^7t^{13} + p^6t^{14} + p^6t^{15} \\ &\neq \zeta_{M_4,p,4}^{\triangleleft}(s).\end{aligned}$$

Proof. Let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ 0 & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} \\ 0 & 0 & m_{3,3} & m_{3,4} & m_{3,5} \\ 0 & 0 & 0 & m_{4,4} & m_{4,5} \\ 0 & 0 & 0 & 0 & m_{5,5} \end{pmatrix}$$

denote the matrix representations of additive subgroups of Fil_4 . One can observe that the list of conditions we need to solve this time is almost identical to the list of conditions we had to solve for M_4 . Indeed the only differences are the ones we

underlined:

$$\begin{aligned}
& m_{3,3} | m_{1,1}, m_{1,2}, m_{2,2}, \underline{m_{4,4}} | m_{1,1}, m_{3,3}, \underline{m_{5,5}} | m_{1,1}, m_{4,4} \\
& m_{3,3} m_{4,4} | m_{1,1} m_{3,4}, m_{1,3} m_{3,3} - m_{1,2} m_{3,4}, m_{2,3} m_{3,3} - m_{2,2} m_{3,4}, \\
& m_{4,4} m_{5,5} | m_{1,2} m_{4,4} - \underline{m_{1,1} m_{4,5}}, m_{3,3} m_{4,5} - m_{3,4} m_{4,4} \\
& m_{3,3} m_{4,4} m_{5,5} | m_{1,3} m_{3,3} m_{4,4} - \underline{m_{1,1} (m_{3,4} m_{4,5} - m_{3,5} m_{4,4})} \\
& m_{3,3} m_{4,4} m_{5,5} | m_{1,2} m_{3,4} m_{4,5} - m_{1,2} m_{3,5} m_{4,4} - m_{1,3} m_{3,3} m_{4,5} + m_{1,4} m_{3,3} m_{4,4} \\
& m_{3,3} m_{4,4} m_{5,5} | m_{2,2} m_{3,4} m_{4,5} - m_{2,2} m_{3,5} m_{4,4} - m_{2,3} m_{3,3} m_{4,5} + m_{2,4} m_{3,3} m_{4,4}.
\end{aligned}$$

Interestingly, they almost give the same i -th partial zeta functions until we reach $i = 4$ and analyse

$$M = \begin{pmatrix} p^3 & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} \\ 0 & p^2 & m_{2,3} & m_{2,4} & m_{2,5} \\ 0 & 0 & p & m_{3,4} & m_{3,5} \\ 0 & 0 & 0 & p & m_{4,5} \\ 0 & 0 & 0 & 0 & p \end{pmatrix}.$$

For $\zeta_{M_4, p, 4}^{\leq}(s)$, one gets $p^5 t^8$ from

$$\begin{pmatrix} p^3 & pm'_{1,2} & m'_{1,2} m_{3,4} & m'_{1,2} m_{3,5} & m_{1,5} \\ 0 & p^2 & 0 & -m_{3,4}^2 & m_{2,5} \\ 0 & 0 & p & m_{3,4} & m_{3,5} \\ 0 & 0 & 0 & p & m_{3,4} \\ 0 & 0 & 0 & 0 & p \end{pmatrix}.$$

However, for $\zeta_{\text{Fil}_4, p, 4}^{\leq}(s)$, one gets $p^4 t^8$ from

$$\begin{pmatrix} p^3 & 0 & 0 & 0 & m_{1,5} \\ 0 & p^2 & 0 & -m_{3,4}^2 & m_{2,5} \\ 0 & 0 & p & m_{3,4} & m_{3,5} \\ 0 & 0 & 0 & p & m_{3,4} \\ 0 & 0 & 0 & 0 & p \end{pmatrix}$$

and $(p-1)p^3 t^8$ from

$$\begin{pmatrix} p^3 & pm'_{1,2} & 0 & m'_{1,2} m_{3,5} & m_{1,5} \\ 0 & p^2 & 0 & 0 & m_{2,5} \\ 0 & 0 & p & 0 & m_{3,5} \\ 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & p \end{pmatrix}$$

where $0 < m'_{1,2} < p$. □

Again we discovered that we need to have at least $i = 4$ to capture all the factors of $\zeta_{M_4,p}^\triangleleft(s)$ and $\zeta_{\text{Fil}_4,p}^\triangleleft(s)$. Furthermore, we need to have at least $i = 4$ to tell the difference between $\zeta_{M_4,p,i}^\triangleleft(s)$ and $\zeta_{\text{Fil}_4,p,i}^\triangleleft(s)$. Does this imply that in order to get some useful information on $\zeta_{L,p}^\triangleleft(s)$, we need to have at least $i = c$ where c is the nilpotency class of L ? We do not have a proof yet, but our calculations so far support this claim.

6.2.3 Other explicit examples

We now provide explicit examples that we calculated. Most of these calculations were done by hand using the algorithm we demonstrated earlier, although some of them required the aid of computer packages. They are not as difficult as the usual p -adic method (e.g [44]), but very long and tedious.

Theorem 6.2.8. *Let $\mathfrak{f}_{2,3}$ be the class-2 free Lie ring on 3 generators.*

$$\begin{aligned}\zeta_{\mathfrak{f}_{2,3},p}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p^3}(s) \zeta_p(3s-5) \zeta_p(5s-8) \zeta_p(6s-9) W_{\mathfrak{f}_{2,3}}^\triangleleft(p, t), \\ \zeta_{\mathfrak{f}_{2,3},p,2}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p^3,2}(s) + (1+p+\underline{p^2})\underline{p^3t^3} + 3p^5t^4 + (1+p+\underline{p^2})\underline{p^6t^5} + \underline{p^9t^6}.\end{aligned}$$

In particular, they are uniform and $\zeta_{\mathfrak{f}_{2,3},p,2}^\triangleleft(s)$ captures all the factors of $\zeta_{\mathfrak{f}_{2,3},p}^\triangleleft(s)$.

Theorem 6.2.9. *Let $\mathfrak{f}_{3,2}$ be the class-3 free Lie ring on 2 generators.*

$$\begin{aligned}\zeta_{\mathfrak{f}_{3,2},p}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p^2}(s) \zeta_p(3s-2) \zeta_p(4s-3) \zeta_p(5s-4)^2 \zeta_p(7s-6) W_{\mathfrak{f}_{3,2}}^\triangleleft(p, t), \\ \zeta_{\mathfrak{f}_{3,2},p,2}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p^2,2}(s) + \underline{p^2t^3} + p^2t^4 + \underline{p^3t^4} + \underline{p^4t^5}, \\ \zeta_{\mathfrak{f}_{3,2},p,3}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p^2,3}(s) + \underline{p^2t^3} + 2p^2t^4 + \underline{2p^3t^4} + 4p^3t^5 + \underline{2p^4t^5} - p^3t^6 + 5p^4t^6 + p^5t^6 \\ &\quad + 2p^5t^7 + \underline{p^6t^7} + p^4t^8 + 2p^6t^8 + p^6t^9 + p^7t^9 + p^8t^{10}.\end{aligned}$$

In particular, they are uniform and $\zeta_{\mathfrak{f}_{3,2},p,3}^\triangleleft(s)$ captures all the factors of $\zeta_{\mathfrak{f}_{3,2},p}^\triangleleft(s)$.

Theorem 6.2.10. *Let $\mathfrak{f}_{4,2}$ be the class-4 free Lie ring on 2 generators.*

$$\begin{aligned}\zeta_{\mathfrak{f}_{4,2},p,2}^\triangleleft(s) &= \zeta_{\mathbb{Z}_p^2,2}(s) + p^2t^3 + p^2t^4 + p^3t^4 + p^3t^5 + 2p^4t^5 + p^4t^6 + 2p^5t^6 \\ &\quad + p^4t^7 + p^5t^7 + p^6t^7 + p^6t^8.\end{aligned}$$

In particular, it is uniform.

Remark 6.2.11. Note that we do not know $\zeta_{\mathfrak{f}_{4,2},p}^\triangleleft(s)$ to compare.

Theorem 6.2.12. *Let*

$$\mathcal{G}_3 = \langle w, x_1, x_2, y_1, y_2 : [w, x_i] = y_i, i \in 1, 2 \rangle$$

be the Grenham Lie ring on 3 generators.

$$\begin{aligned}\zeta_{\mathcal{G}_3,p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^3}(s) \zeta_p(3s-4) \zeta_p(5s-6) (1+p^3t^3), \\ \zeta_{\mathcal{G}_3,p,2}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^3,2}(s) + p^3t^3 + \underline{p^4t^3} + 2p^5t^4 + \underline{p^6t^5}.\end{aligned}$$

In particular, they are uniform and $\zeta_{\mathcal{G}_3,p,2}^{\triangleleft}(s)$ captures all the factors of $\zeta_{\mathcal{G}_3,p}^{\triangleleft}(s)$.

Theorem 6.2.13. *Let*

$$T_3 = \langle x_1, x_2, x_3, y_1, y_2 : [x_i, x_{i+1}] = y_i, i \in 1, 2 \rangle$$

be the maximal class-two quotient of the Lie ring of unitriangular 3×3 matrices.

$$\begin{aligned}\zeta_{T_3,p}^{\triangleleft}(s) &= \zeta_{\mathcal{G}_3}^{\triangleleft}(s), \\ \zeta_{T_3,p,2}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^3,2}(s) + p^3t^3 + \underline{p^4t^3} + p^4t^4 + p^5t^4 + \underline{p^6t^5} \\ &\neq \zeta_{\mathcal{G}_3,2}^{\triangleleft}(s).\end{aligned}$$

In particular, they are uniform and $\zeta_{T_3,p,2}^{\triangleleft}(s)$ captures all the factors of $\zeta_{T_3,p}^{\triangleleft}(s)$.

Remark 6.2.14. This result is very interesting in two ways. First, note that as a Lie ring $T_3 \cong \mathcal{G}_3$. This shows that the i -th partial zeta functions are indeed basis-dependent. Second, at the same time both $\zeta_{\mathcal{G}_3,2}^{\triangleleft}(s)$ and $\zeta_{T_3,2}^{\triangleleft}(s)$ captures all the linear factors.

Theorem 6.2.15. *Let $L_{3,2}$ ($[44]$) be a class-3 Lie ring given by the presentation*

$$\langle w, x_1, x_2, y_1, y_2, z : [w, x_1] = y_1, [w, x_2] = y_2, [w, y_1] = z \rangle.$$

$$\begin{aligned}\zeta_{L_{3,2},p}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^3}(s) \zeta_p(3s-4) \zeta_p(4s-4) \zeta_p(5s-5) \zeta_p(5s-6) \zeta_p(6s-6) \\ &\quad \times \zeta_p(9s-11) W_{L_{3,2}}^{\triangleleft}(p, t), \\ \zeta_{L_{3,2},p,2}^{\triangleleft}(s) &= \zeta_{\mathbb{Z}_p^3,2}(s) + p^3t^3 + \underline{p^4t^3} + \underline{p^4t^4} + 2p^5t^4 + \underline{2p^5t^6} + \underline{p^6t^6}.\end{aligned}$$

In particular, they are uniform but $\zeta_{L_{3,2},p,2}^{\triangleleft}(s)$ does not capture all the factors of $\zeta_{L_{3,2},p}^{\triangleleft}(s)$. Since the nilpotency class of $L_{3,2}$ is 3, it supports our claim that one should look at $\zeta_{L_p,i,B}^{\triangleleft}(s)$ where $i \geq c$ to see all the recurrence factors.

6.3 p -adic integration and the resolution of singularities

Here we look at another interesting example.

Theorem 6.3.1.

$$\zeta_{sl_2(\mathbb{Z}_p),2}(s) = 1 + (1+p)t + (p^2 + 3p - 1)t^2 + (p^2 + p - 1)t^3$$

Proof. Before we calculate

$$\zeta_{sl_2(\mathbb{Z}_p),2}(s) = 1 + (1+p)t + (p^2 + 3p - 1)t^2 + (p^2 + p - 1)t^3,$$

let us re-visit Taylor's calculation of $\zeta_{sl_2(\mathbb{Z}_p)}(s)$ in [33] using p -adic integrals and the resolution of singularities. Let

$$H = \begin{pmatrix} a & c & b \\ 0 & x & y \\ 0 & 0 & z \end{pmatrix},$$

and

$$F = 16a^2c^2x^6yz^6(ax^2 + 4bxy - 4cy^2).$$

Taylor performed a resolution of singularities at

$$ax^2 + 4bxy - 4cy^2.$$

If we carry out a blow-up at $x = y = 0$, then we get four conditions, namely

1. $v(x) \leq v(y)$ and $v(y') \leq v(a)$,
2. $v(x) \leq v(y)$ and $v(y') > v(a)$,
3. $v(x) > v(y)$ and $v(x') \leq v(c)$,
4. $v(x) > v(y)$ and $v(x') > v(c)$.

Note that each case corresponds to an explicit system of irreducible varieties E_i 's, i.e.

- $a'^2c^2x^9y'^4z^6(a' + 4b - 4cy')$ for Case (1),
- $a^4c^2x^9y''z^6(1 + 4by'' - 4acy''^2)$ for Case (2),
- $a^2c'^2x'^9y^9z^6(ax' + 4b - 4c')$ for Case(3),

- $a^2 c^9 x''^6 y^9 z^6 (a''^2 + 4bx'' - 4)$ for Case (4).

We know that in the end, we get three non-trivial cone-integrals corresponding to Case (1), (2) and (3), which give

$$\zeta_{\text{sl}_2(\mathbb{Z}_p)}(s) = \zeta_p(s) \zeta_p(s-1) \zeta_p(2s-1) \zeta_p(2s-2) \zeta_p(3s-1)^{-1}.$$

Now, let

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ 0 & m_{2,2} & m_{2,3} \\ & & m_{3,3} \end{pmatrix} \in \text{Tr}_3(\mathbb{Z}_p)$$

be an additive sublattice of $\text{sl}_2(\mathbb{Z}_p)$. From the above we know that M is a subalgebra of $\text{sl}_2(\mathbb{Z}_p)$ if $m_{2,2} \mid 4m_{1,2}m_{2,3}$, $m_{2,2} \mid 4m_{1,2}m_{3,3}$, $m_{2,2}m_{3,3} \mid m_{1,1}m_{2,2}^2 + 4m_{1,3}m_{2,2}m_{2,3} - 4m_{1,2}m_{2,3}^2$. Hence we have

$$\zeta_{\text{sl}_2(\mathbb{Z}_p),2}(s) = (1-p^{-1})^{-3} \int_{\substack{m_{2,2} \mid 4m_{1,2}m_{2,3} \\ m_{2,2} \mid 4m_{1,2}m_{3,3} \\ m_{2,2}m_{3,3} \mid m_{1,1}m_{2,2}^2 + 4m_{1,3}m_{2,2}m_{2,3} - 4m_{1,2}m_{2,3}^2 \\ v(m_{1,1}), v(m_{2,2}), v(m_{3,3}) < 2}} |m_{1,1}|^{s-1} |m_{2,2}|^{s-2} |m_{3,3}|^{s-3} d\mu.$$

We again do the case distinctions:

- $\begin{pmatrix} 1 & 0 & m_{13} \\ & 1 & m_{23} \\ & & p \end{pmatrix}$ where $m_{23} \neq 0$ and $1 + 4m_{13}m_{23} = 0$ (Case (1)),
- $\begin{pmatrix} p & 0 & 0 \\ & 1 & m_{23} \\ & & p \end{pmatrix}$ where $m_{23} \neq 0$ (Case (1)),
- $\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} p & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} p & 0 & m_{13} \\ & 1 & 0 \\ & & p \end{pmatrix}, \begin{pmatrix} 1 & m_{12} & m_{13} \\ & p & 0 \\ & & p \end{pmatrix}$ (Case (2)),
- $\begin{pmatrix} p & m_{12} & m_{13} \\ & p & 0 \\ & & p \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ & p & 0 \\ & & 1 \end{pmatrix}, \begin{pmatrix} p & 0 & 0 \\ & p & 0 \\ & & 1 \end{pmatrix}$ (Case(2)),
- $\begin{pmatrix} 1 & 0 & 0 \\ & p & m_{23} \\ & & p \end{pmatrix}, m_{23} \neq 0, \begin{pmatrix} p & 0 & 0 \\ & p & m_{23} \\ & & p \end{pmatrix}, m_{23} \neq 0$ (Case(3)).

This gives

$$\zeta_{\mathrm{sl}_2(\mathbb{Z}_p),2}(s) = 1 + (1 + p)t + (p^2 + 3p - 1)t^2 + (p^2 + p - 1)t^3.$$

□

It is now well-known that we can compute $\zeta_{\mathrm{sl}_2(\mathbb{Z}_p)}(s)$ in a simpler way. Taylor showed in [33] that we can do a different blow-up $y = z = 0$, which makes the whole calculation much easier. Klopsch and Voll then showed in [20] that we do not even have to do any resolution of singularities for $\zeta_{\mathrm{sl}_2(\mathbb{Z}_p)}(s)$. However, we chose the most complicated method to demonstrate that even in this choice of blow-ups there exist non-trivial sublattices $M \in V_{p,2}^{\leq}$ for all non-trivial different blow-ups. We observed that analysing Cases (1), (2) and (3) were necessary to compute $\zeta_{\mathrm{sl}_2(\mathbb{Z}_p)}(s)$ and all these cases “survived” for $\zeta_{\mathrm{sl}_2(\mathbb{Z}_p),2}(s)$. This suggests that our approximation is, even at $i = 2$, not as simple as one might expect and might contains enough information of the original zeta function. Since the wild behaviour of $\zeta_{L_p}^*(s)$ comes from blow-ups and resolution of singularities, this suggests that studying $\zeta_{L_p,i}^*(s)$ for some finite i might be a good method to predict the behaviour (e.g. uniformity) of $\zeta_{L_p}^*(s)$.

Chapter 7

Questions and conjectures

In this chapter we propose various open conjectures based on our observations. We believe they would be interesting leads for further research.

7.1 The behaviour of ζ_L^* and its approximations

7.1.1 Uniformity of various approximations

The various approximations we introduced in this thesis were designed to overcome the difficulties of computing classical zeta functions of groups and rings. The following questions arise naturally from our work.

Question 7.1.1. Would the uniformity of these approximations imply the uniformity of the original ζ_L^* ?

For example, $\zeta_{M_c, p}^{\triangleleft \text{gr}}(s)$, $\zeta_{M_c(\mathbb{F}_p)}^{\triangleleft}(s)$, and $\zeta_{M_c}^{\triangleleft \text{red}}(s)$ all turned out to be uniform for all c . Does this imply that $\zeta_{M_c}^{\triangleleft}(s)$ is also uniform for all c ?

At the moment we do not have a single example L or G where

- (1) the original zeta functions are (finitely) uniform but some of their approximations are not uniform,
- (2) the original zeta functions are not uniform but some of their approximations are (finitely) uniform.

It would be very interesting if one could find such examples (provided that they exist).

7.1.2 The i -th partial zeta functions

Recall that we have

$$\begin{aligned}\zeta_{M_4,p}^{\triangleleft \text{gr}}(s) &= \zeta_{\text{Fil}_4,p}^{\triangleleft \text{gr}}(s), \\ \zeta_{M_4(\mathbb{F}_p)}^{\triangleleft}(s) &= \zeta_{\text{Fil}(\mathbb{F}_p)}^{\triangleleft}(s), \\ \zeta_{M_4,p}^{\triangleleft \text{red}}(s) &= \zeta_{\text{Fil},p}^{\triangleleft \text{red}}(s),\end{aligned}$$

but

$$\zeta_{M_4,p,4}^{\triangleleft}(s) \neq \zeta_{\text{Fil}_4,p,4}^{\triangleleft}(s).$$

This suggests that the c -th partial zeta function $\zeta_{L_p,c}^*(s)$, where c is the nilpotency class of L , might be the best approximation of all, since it spotted the subtle difference between M_4 and Fil_4 that other approximations could not.

Conjecture 7.1.2. $\zeta_{L_p,c,B}^*(s)$ captures all the factors of $\zeta_{L_p}^*(s)$.

If this is true it has a very powerful consequence.

Conjecture 7.1.3. Let L is a Lie ring additively isomorphic to $\mathbb{Z}^n$ and

$$\zeta_{L,p}^*(s) = \frac{\Psi_{L,p}^*(t)}{\prod_{j \leq k} (1 - p^{a_j} t^{b_j})}$$

is in its lowest terms. Then $\max\{b_1, \dots, b_k\} \leq n(c-1)$.

Proof. Note the highest t -power we can achieve in $\zeta_{L_p,c,B}^*(s)$ is indeed $n(c-1)$, corresponding to the $n \times n$ upper triangular matrices with every diagonal entry $m_{i,i} = p^{c-1}$. Assume Conjecture 7.1.2 is true. This means that we should be able to find all $p^{a_j} t^{b_j}$'s from the denominators of $\zeta_{L,p}^*(s)$ in its lowest terms in $\zeta_{L_p,c,B}^*(s)$. Therefore the maximal value of b_j 's cannot exceed $n(c-1)$. $\square$

Since these poles are the building blocks of our zeta functions, understanding them would be very important.

The main drawback of the current definition of the i -th partial zeta functions is that this is basis-dependent. So one can also ask the following questions:

Question 7.1.4. Is there an example of a Lie ring L such that for two different bases B_1 and B_2 for L , $\zeta_{L_p,i,B_1}^*(s)$ is (finitely) uniform but $\zeta_{L_p,i,B_2}^*(s)$ is not (finitely) uniform for some $i \in \mathbb{N}$? Or is the uniformity basis-independent?

Question 7.1.5. Is there an example of a Lie ring L such that for two different bases B_1 and B_2 for L , $\zeta_{L_p, i, B_1}^*(s)$ captures all the factors of $\zeta_{L_p}^*(s)$ but $\zeta_{L_p, i, B_2}^*(s)$ does not for some $i \in \mathbb{N}$? Or is this property basis-independent?

Remark 7.1.6. In an attempt to make these partial zeta functions basis-independent, I am actually considering the following alternative “ j -th partial zeta functions”

$$\zeta_{L_p, j}^*(s) := \sum_{\substack{H * L_p \\ p^j L_p \leq H}} |L_p : H|^{-s}.$$

Clearly they are basis-independent, and much more natural than the current definition of the i -th partial zeta functions. For class-2 cases, one can exploit Voll’s elementary divisor method. Unfortunately, this idea came too late for this thesis.

7.1.3 Higman’s PORC conjecture

We observed that although non-uniform ideal zeta functions might generate a family of p -groups whose number of immediate descendants are Non-PORC, the total sum can still be PORC. One might even say that looking for one single irregular local objects might not be the right method to tackle Higman’s PORC conjecture. However, proving that the normal subgroup zeta functions of $F_{c,d}$ for any $c, d \in \mathbb{N}$ are finitely uniform and PORC still seems out of reach as well.

7.2 Graded ideal zeta functions of $\mathfrak{f}_{c,d}$

Let $c \in \mathbb{N}$ and $d \in \mathbb{N}_{\geq 2}$. Recall $\mathfrak{f}_{c,d}$ denotes the free nilpotent Lie ring of class c on d generators. We record a number of general conjectures regarding the graded ideal zeta functions $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)$ as well as their topological and reduced counterparts. We write W for the Witt function W_d (cf. (3.1.2)) and set $r = \text{rk}_{\mathbb{Z}}(\mathfrak{f}_{c,d}) = \sum_{i=1}^c W(i)$. With the exception of Conjecture 7.2.7 for $c = 2$ and $d \geq 5$, all conjectures in this section are supported, in the relevant special cases, by the results in this paper.

Conjecture 7.2.1 (Uniformity). *There exists $W_{c,d}^{\triangleleft \text{gr}}(X, Y) \in \mathbb{Q}(X, Y)$ such that, for almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s) = W_{c,d}^{\triangleleft \text{gr}}(q, q^{-s}).$$

Conjecture 7.2.2 (Local functional equations). *For almost all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,*

$$\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s)|_{q \rightarrow q^{-1}} = (-1)^r q^{\sum_{i=1}^c \binom{W(i)}{2} - \sum_{i=1}^c (c+1-i)W(i)s} \zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft \text{gr}}(s).$$

Remark 7.2.3. Local functional equations are a universal feature for quite general subring zeta functions ([40, Theorem A]) and for ideal zeta functions of nilpotent Lie rings of nilpotency class 2 ([40, Theorem C]). In fact, we expect the proof of the latter result should carry over to the graded setting, making the functional equations (3.4.7) an instance of a general phenomenon in class 2; cf. Example 3.1.6. If Conjecture 7.2.1 holds, then Conjecture 7.2.2 states that

$$W_{c,d}^{\triangleleft \text{gr}}(X^{-1}, Y^{-1}) = (-1)^r X^{\sum_{i=1}^c} \binom{W(i)}{2} Y^{\sum_{i=1}^c (c+1-i)W(i)} W_{c,d}^{\triangleleft \text{gr}}(X, Y).$$

In general, the operation $q \rightarrow q^{-1}$ is defined rather more delicately in terms of inversion of Frobenius eigenvalues.

In nilpotency class greater than 2 local functional equations are not to be expected in general for ideal zeta functions of nilpotent Lie rings: see [11, Section 1.2.3] for counterexamples in the ungraded setting and [28, Section 10, Table 2] in the graded setting.

A sufficient criterion for local functional equations in an enumerative setup generalizing ideal zeta functions of nilpotent Lie rings is given in [41, Theorem 1.2]. It applies to the ideal zeta functions of the free nilpotent Lie rings $\mathfrak{f}_{c,d}$; see [41, Theorem 4.4]. Note that, in the notation of [41, Theorem 4.4], $N_0 = r$ and $\sum_{i=0}^{c-1} N_i = \sum_{i=1}^c (c+1-i)W(i)$.

For the following conjecture, assume that Conjecture 7.2.1 holds. In this case, we may define the *reduced graded ideal zeta function*

$$\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y) = W_{c,d}^{\triangleleft \text{gr}}(1, Y) \in \mathbb{Q}(Y).$$

We expect that the technology in [13] may be adapted to define the reduced graded ideal zeta function even without the assumption of Conjecture 7.2.1.

Conjecture 7.2.4 (Reduced zeta function). *The reduced graded ideal zeta function $\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y)$ has a pole of order r at 1 and there exist $e_i \in \mathbb{N}$, $i \in [r]$, as well as nonnegative integers $a_j \in \mathbb{N}_0$, $j \in [N]_0$, where $N := (\sum_{i=1}^r e_i) - \sum_{j=1}^c (c+1-j)W_d(j)$, with $a_0 = 1$ and*

$$a_j = a_{N-j} \text{ for all } j \in [N]_0 \text{ and} \tag{7.2.1}$$

$$\zeta_{\mathfrak{f}_{c,d}, \text{red}}^{\triangleleft \text{gr}}(Y) = \frac{\sum_{j=0}^N a_j Y^j}{\prod_{i=1}^r (1 - Y^{e_i})}.$$

Remark 7.2.5. The palindromic property (7.2.1) holds, of course, in particular if Conjecture 7.2.2 holds. What Conjecture 7.2.4 suggests is that the reduced graded

ideal zeta functions $\zeta_{\mathfrak{f}_{c,d},\text{red}}^{\triangleleft\text{gr}}(Y)$ share key properties with the Hilbert-Poincaré series of graded Cohen-Macaulay algebras. If there were such an algebra, its Cohen-Macaulayness would explain the nonnegativity of the coefficients a_j , whereas their palindromy would reflect its Gorensteinness (assuming that it is a domain; cf. [32]).

For $c = 2$, the validity of Conjecture 7.2.4 may be seen from the facts that $\zeta_{\mathfrak{f}_{2,d},\text{red}}^{\triangleleft\text{gr}}(Y) = \zeta_{\mathfrak{f}_{2,d},\text{red}}^{\triangleleft}(Y)$ and the Hall basis $\mathcal{H}_{2,d}$ for $\mathfrak{f}_{2,d}$ in Table 3.2.1 is nice and simple in the sense of [13]. Hence [13, Proposition 4.1] is applicable.

The following is a graded analogue of [27, Conjecture I].

Conjecture 7.2.6 (Degree of topological zeta function).

$$\deg_s \left(\zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s) \right) = -r.$$

Conjecture 7.2.7 (Behaviour of topological zeta function at ∞).

$$s^{\deg_s \left(\zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s) \right)} \zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s^{-1}) \Big|_{s=0} = (1-Y)^r \zeta_{\mathfrak{f}_{c,d},\text{red}}^{\triangleleft\text{gr}}(Y) \Big|_{Y=1} \in \mathbb{Q}_{>0}.$$

Remark 7.2.8. We have no interpretation in terms of c and d for the numbers featuring in Conjecture 7.2.7. If Conjecture 7.2.4 were to hold for the reasons we explained in Remark 7.2.5, they were related to the “multiplicity” of the associated graded algebra.

We close with two conjectures on the behaviour of $\mathfrak{p}$ -adic and topological graded ideal zeta functions at $s = 0$.

Conjecture 7.2.9 (Behaviour of topological zeta function at zero). *The topological graded ideal zeta function $\zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s)$ has a pole of order c at $s = 0$ with leading coefficient*

$$s^c \zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft\text{gr}}(s) \Big|_{s=0} = \frac{(-1)^{\sum_{i=1}^c (W(i)-1)} \prod_{i=1}^c W(i)}{\prod_{i=1}^c \left(\sum_{j=1}^i W(j) \right) (W(i)-1)!}.$$

Remark 7.2.10. [27, Conjecture IV (topological form)] implies that $\zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft}(s)$ has a simple pole at $s = 0$ with residue

$$s \zeta_{\mathfrak{f}_{c,d},\text{top}}^{\triangleleft}(s) \Big|_{s=0} = \frac{(-1)^{r-1}}{(r-1)!}.$$

Conjecture 7.2.11 (Behaviour of $\mathfrak{p}$ -adic zeta function at zero).

$$\frac{\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft\text{gr}}(s)}{\prod_{i=1}^c \zeta_{\mathfrak{o}^{W(i)}}(s)} \Big|_{s=0} = \frac{\prod_{i=1}^c W(i)}{\prod_{i=1}^c \left(\sum_{j=1}^i W(j) \right)}. \quad (7.2.2)$$

Remark 7.2.12. [27, Conjecture IV ($\mathfrak{P}$ -adic form)] implies that

$$\left. \frac{\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft}(s)}{\zeta_{\mathfrak{o}^r}(s)} \right|_{s=0} = 1. \quad (7.2.3)$$

It seems remarkable that the right-hand sides of both (7.2.2) and (7.2.3) are independent of $\mathfrak{o}$. Note that $s = 0$ is outside the domain of convergence of the series defining $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft}(s)$ resp. $\zeta_{\mathfrak{f}_{c,d}(\mathfrak{o})}^{\triangleleft, \text{gr}}(s)$.

Remark 7.2.13. If one replaces the numbers $W(i)$ by the ranks of the successive quotients of the upper central series of $\mathcal{L}_{c(d)}$ (e.g. $W(1) = d + 1$, $W(2) = 2d + 1$, and so on), our results in Chapter 4 support the conjectures we just mentioned in this section. In fact Theorem 4.3.10 and 4.3.11 show that Conjectures 7.2.6, 7.2.9 and 7.2.11 are actually true for $\mathcal{L}_{c(d)}$. The explicit examples we have in that chapter satisfy other conjectures as well. However, this is not always true for graded ideal zeta functions of general (non-free) nilpotent Lie rings. Consider, for instance, the direct product $L := \mathfrak{f}_{2,2} \times \mathfrak{f}_{2,2}$. Then, for all primes p and all finite extensions $\mathfrak{o}$ of $\mathbb{Z}_p$,

$$\zeta_{L(\mathfrak{o})}^{\triangleleft, \text{gr}}(s) = \zeta_{\mathfrak{o}^4}(s) \frac{1 - t^5}{(1 - t^3)^2(1 - qt^5)}. \quad (7.2.4)$$

Indeed, L is the “fundamental graded Lie ring” m6_2_3, whose local graded ideal zeta functions are recorded in [28, Section 10, Table 2]. Formula (7.2.4) may also easily be derived directly from formulae for the ideal zeta functions of L (cf. [11, Theorem 2.4]), using the comparison identities recorded in Example 3.1.6. In any case,

$$s^2 \zeta_{L, \text{top}}^{\triangleleft, \text{gr}}(s) \Big|_{s=0} = \frac{5}{54} \neq \frac{(-1)^{3+1} 4 \cdot 2}{4 \cdot 6 \cdot 3! \cdot 1!} = \frac{1}{18}.$$

Likewise,

$$\left. \frac{\zeta_{L(\mathfrak{o})}^{\triangleleft, \text{gr}}(s)}{\zeta_{\mathfrak{o}^4}(s) \zeta_{\mathfrak{o}^2}(s)} \right|_{s=0} = \frac{5}{9} \neq \frac{4 \cdot 2}{4 \cdot 6} = \frac{1}{3}.$$

That this value, however, is a nonnegative rational number independent of $\mathfrak{o}$, as well as the coincidence

$$s^6 \zeta_{L, \text{top}}^{\triangleleft, \text{gr}}(s^{-1}) \Big|_{s=0} = \frac{1}{9} = (1 - Y)^6 \zeta_{L, \text{red}}^{\triangleleft, \text{gr}}(Y) \Big|_{Y=1} \in \mathbb{Q}_{>0},$$

seem to indicate that some of the conjectures made in this section may have generalizations to more general graded ideal zeta functions.

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