

Endomorphism rings of permutation modules over maximal Young subgroups [☆]

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Abstract

Let K be a field of characteristic two, and let λ be a two-part partition of some natural number r . Denote the permutation module corresponding to the (maximal) Young subgroup Σ_λ in Σ_r by M^λ . We construct a full set of orthogonal primitive idempotents of the centraliser subalgebra $S_K(\lambda) = 1_\lambda S_K(2, r) 1_\lambda = \text{End}_{K \Sigma_r}(M^\lambda)$ of the Schur algebra $S_K(2, r)$. These idempotents are naturally in one-to-one correspondence with the 2-Kostka numbers.

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1. Introduction

Objects of central interest in the representation theory of symmetric groups are permutation modules coming from actions on set partitions. They provide a natural link with the representation theory of general linear groups, via Schur algebras. Assume K is a field of positive characteristic p . Fix natural numbers n and r and fix partitions λ and μ of r of not more than

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n -parts. The permutation module M^λ over Σ_r is the module obtained by inducing the trivial representation from the Young subgroup Σ_λ to the symmetric group Σ_r . The indecomposable direct summands of M^λ are known as Young modules. By James' submodule theorem [9, 7.1.7] there is a unique indecomposable summand of M^λ containing the Specht module S^λ . This summand is by definition the Young module Y^λ . The module M^λ is in general a direct sum of Young modules Y^μ , and if Y^μ occurs as a summand then $\mu \geq \lambda$, in the usual dominance order on partitions. The p -Kostka number $[M^\lambda : Y^\mu]$ is the number of indecomposable summands of M^λ isomorphic with Y^μ . Thus we have:

$$M^\lambda \simeq \bigoplus_{\mu \geq \lambda} [M^\lambda : Y^\mu] Y^\mu.$$

Note that if λ is an n -part composition of r then we still have the permutation module M^λ defined as above. In this situation if λ_0 is the partition obtained from λ by ordering its parts then $M^{\lambda_0} \simeq M^\lambda$.

Let $S_K(n, r)$ be the Schur algebra of degree r , then $S_K(n, r)$ is given by

$$S_K(n, r) = \text{End}_{K\Sigma_r}(E^{\otimes r}) \simeq \text{End}_{K\Sigma_r}\left(\bigoplus_{\lambda} M^\lambda\right),$$

where E is a given n -dimensional K -vector space, and λ varies over the set $\Lambda(n, r)$ of n -part compositions of r . For the connection of Schur algebras with general linear groups, see Green [6]. The idempotent $1_\lambda \in S_K(n, r)$ corresponds to the projection onto M^λ with kernel $\bigoplus_{\mu \neq \lambda} M^\mu$. We define the centraliser subalgebra $S_K(\lambda)$ of $S_K(n, r)$ by

$$S_K(\lambda) = 1_\lambda S_K(n, r) 1_\lambda \simeq \text{End}_{K\Sigma_r}(M^\lambda).$$

In this paper we study these algebras when λ is a partition of at most two parts; that is, the associated Young subgroup is maximal. Then it is known (see, for example, [10, Example 14.4]) that the ordinary character of M^λ is multiplicity-free. It follows that the algebra $S_K(\lambda)$ is commutative (see [13], and see also Remark 3.5 below), and that any given Young module Y^μ occurs at most once as a direct summand of M^λ . All idempotents of $S_K(\lambda)$ are central, and there are finitely many primitive idempotents, in one-to-one correspondence with the indecomposable summands of M^λ . The blocks of $S_K(\lambda)$ are therefore precisely the endomorphism rings of the Young modules Y^μ which occur as a direct summand of M^λ .

Our main result is an explicit construction of a full set of orthogonal primitive idempotents of the algebra $S_K(\lambda)$ where λ is a partition of at most two parts and $\text{char}(K) = 2$. These idempotents are naturally in one-to-one correspondence with the 2-Kostka numbers. The philosophy is to consider an infinite family of algebras at the same time, as was done in [2]. This is possible, by exploiting the presentation obtained in [4] of the Schur algebra as quotient of the universal enveloping algebra. This approach allows one to keep $m = \lambda_1 - \lambda_2$ fixed and let $r = \lambda_1 + \lambda_2$ vary arbitrarily. Our motivation is to describe idempotents explicitly; this is a notoriously hard problem, in general, especially in the modular setting. We solve this problem completely, for our situation, when $p = 2$; the case of odd primes seems to be more complicated. The results on idempotents in this paper can be thought of as an algebraic realisation of the combinatorial description of the quarter-infinite Kostka matrix in Section 2.1 below.

2. Main results

2.1. The p -Kostka matrix

We fix some notation. Let K be a field of positive characteristic p . For any natural number r , we let $\lambda = (r - k, k)$ and $\mu = (r - s, s)$ vary over the two-part partitions of r . The p -Kostka numbers $[M^{(r-k,k)} : Y^{(r-s,s)}]$ do not depend on r but only on $m := \lambda_1 - \lambda_2 = r - 2k$ and $g := \lambda_2 - \mu_2 = k - s$. So they can be described by a quarter-infinite matrix with (m, g) th-entry the above p -Kostka number. Set

$$B(m, g) = \binom{m+2g}{g}.$$

By [7,8] it is known that $Y^{(r-s,s)}$ is a direct summand of $M^{(r-k,k)}$ if and only if $B(m, g) = \binom{r-2s}{k-s} \neq 0$ modulo p . Since the multiplicity $[M^{(r-k,k)} : Y^{(r-s,s)}]$ is at most one, the (m, g) th entry of the p -Kostka matrix is one if $B(m, g) \neq 0$ modulo p and zero otherwise. This latter result is based on a general formula by Klyachko [11], Corollary 9.2, reformulated by Donkin [5] in (3.6). However, neither reference gives an explicit answer.

2.2. Notation

We need the p -adic expansion of integers. If $a = \sum_{j=0}^s a_j p^j$ with $0 \leq a_j \leq p-1$ for all j then we will write $a = [a_0, a_1, \dots, a_s]$. It is a well-known property of binomial coefficients that

$$B(m, g) \equiv \prod_i \binom{(m+2g)_i}{g_i} \pmod{p}.$$

We call $\prod_i \binom{(m+2g)_i}{g_i}$ modulo p the binomial expansion of $B(m, g)$ and we write $B(m, g)_i$ for the i th factor of this product. Sometimes we will also write the binomial coefficient modulo p as a matrix with two rows where the i th column is the i th factor of the product, for $i \geq 0$:

$$B(m, g) = \begin{pmatrix} (m+2g)_0 & (m+2g)_1 & \dots & (m+2g)_i & \dots \\ g_0 & g_1 & \dots & g_i & \dots \end{pmatrix}.$$

2.3. The canonical basis of $S_K(\lambda)$

Let $m \geq 0$. We consider the infinite family of algebras $S_K(\lambda)$ where λ runs through all partitions $\lambda = (r - k, k)$ such that $r - 2k = m$. The presentation from [4] provides $S_K(\lambda)$ with a basis $\{b(a) : 0 \leq a \leq k\}$ with good properties. In fact, it is not difficult to see that this basis is inherited from the canonical basis of Lusztig's modified form $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ of the enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$, which is worked out as an example in [12, §29.4.3]. In particular, the product $b(i)b(j)$ depends only on m (and not on the degree r), where terms $b(s)$ appearing with $s > k$ are set zero. See Section 3 below for more details on this basis, upon which our computation of primitive idempotents is based.

One can also consider the infinite-dimensional generic algebra $\dot{\mathcal{U}}(\lambda) = 1_\lambda \dot{\mathcal{U}}(\mathfrak{gl}_2) 1_\lambda$, as in [2], which embeds naturally in the inverse limit of the following sequence of surjections

$$\dots S_{\mathbb{Q}}(\lambda + \delta) \rightarrow S_{\mathbb{Q}}(\lambda) \rightarrow \dots \rightarrow S_{\mathbb{Q}}(\nu + \delta) \rightarrow S_{\mathbb{Q}}(\nu),$$

where $\delta = (1, 1)$ and $v = (r - 2k, 0) = (m, 0)$. Here the maps are induced from corresponding maps on the Schur algebra level, corresponding to tensoring by the determinant representation. (More precisely, one tensors a corresponding coalgebra by the determinant to get an embedding of coalgebras, and then dualises to get a surjection between Schur algebras.)

The algebra $S_{\mathbb{Q}}(\lambda)$ is a homomorphic image of $\hat{\mathcal{U}}(\lambda)$ and our multiplication formula for the canonical basis elements of $S_K(\lambda)$ is coming from a corresponding multiplication formula in the generic algebra. (The setup is compatible with change of base ring.) The generic point of view is closely related to the approach of [1]; see [2] for further details.

2.4. Construction of primitive idempotents

We now take $\text{char}(K) = 2$ and keep $m \geq 0$ fixed. We work in an algebra $S_K(\lambda)$ of large enough degree r (of the right parity). For any $m \geq 0$ and $g \geq 0$ such that $B(m, g)$ is non-zero modulo two and the degree r is large enough (that is $r \geq m + 2g$) we will now define elements in the algebra $S_K(\lambda)$. First we introduce two index sets: let

$$I_{m,g} := \{u : g_u = 0 \text{ and } (m + 2g)_u = 1\},$$

$$J_{m,g} := \{u : g_u = 1 \text{ and } (m + 2g)_u = 1\}.$$

Then for a natural number t define elements in the algebra $S_K(\lambda)$ by

$$e_{m,g} := \prod_{u \in J_{m,g}} b(2^u) \prod_{u \in I_{m,g}} (1 - b(2^u)),$$

$$(e_{m,g})_{\leq t} := \prod_{u \in J_{m,g}, u \leq t} b(2^u) \prod_{u \in I_{m,g}, u \leq t} (1 - b(2^u)). \quad (1)$$

Remark. We can associate to each factor of the binary expansion of $B(m, g)$ a factor of an element $e_{m,g}$ by the following rule:

$B(m, g)_u$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Factor of $e_{m,g}$	$b(2^u)$	$(1 - b(2^u))$	1	0

In particular, an element $e_{m,g}$ defined in this way would be zero if and only if $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ occurs in the binary expansion of $B(m, g)$, that is if and only if $B(m, g) = 0$ modulo two.

The main result of the paper is the following:

Idempotent Theorem. *For any fixed $m \geq 0$, the set of elements $e_{m,g}$ with $B(m, g) \neq 0$ modulo two and $m + 2g \leq r$ is a complete set of primitive orthogonal idempotents for the algebra $S_K(\lambda)$.*

This theorem will be proved at the end of Section 6. In fact parts of the proof of this result are not so difficult to see. Observe the following:

- (i) The element $e_{m,g}$ is non-zero. By Proposition 3.6 or Lemma 3.7 one can even express it explicitly as a linear combination of the basis elements.

(ii) If $g \neq d$ and $B(m, g)$ and $B(m, d)$ are both non-zero modulo two then

$$e_{m,g}^2 \cdot e_{m,d}^2 = 0.$$

Proof of (ii). Let i be minimal such that $B(m, g)_i \neq B(m, d)_i$. Since columns $< i$ are the same, and both binomial coefficients are non-zero, the i th columns cannot be zero: Suppose that one is zero, the other not. Then $(m + 2d)_i \not\equiv (m + 2g)_i$. However, in column i the carry overs from the previous columns are the same, say x , and $d_{i-1} = g_{i-1}$. This implies a contradiction:

$$(m + 2d)_i = m_i + d_{i-1} + x = m_i + g_{i-1} + x = (m + 2g)_i \pmod{2}.$$

Hence one of them is $\binom{1}{1}$ and the other is $\binom{1}{0}$. So the squares of the elements in the algebra have factors $b(2^i)^2$ and $(1 - b(2^i)^2)$, respectively. In Section 4 we will show that the elements $b(2^i)^2$ are idempotents (see Proposition 4.3), and this implies that the product is zero. \square

Furthermore the number of primitive idempotents of $S_K(\lambda)$ is equal to the number of non-zero binomial coefficients, by [8]. So, once we have established that the elements in question are idempotents then the theorem is proved. The fact that the elements in question are idempotents will follow from an orthogonality result:

Orthogonality Lemma. Suppose $B(m, g)_s$ is zero, then $e_{m,g}^2 \cdot b(2^s)^2 = 0$.

The proof of the Orthogonality Lemma is given in Section 6.1.

2.5. Blocks of the algebra $S_K(\lambda)$

Recall that a block of a finite-dimensional algebra A is given by eA where e is a central idempotent which is primitive, viewed as an element of the centre of A . By the Idempotent Theorem, a block of the algebra $S_K(\lambda)$ has the form $e_{m,g}S_K(\lambda)$. The block has basis

$$\{e_{m,g}b(a): a = [a_0, a_1, \dots] \text{ where } a_s = 1 \text{ only for } (m + 2g)_s = 0\}.$$

To see that this set is linearly independent one uses Lemma 3.7; and to show that it spans, one observes using Lemmas 3.8 and 4.2 that if $(m + 2g)_s \neq 0$ then $e_{m,g}b(2^s)$ can be expressed in terms of the given set, by elements of ‘lower degree.’ By Lemma 3.7, the block is (minimally) generated as an algebra by all

$$\{e_{m,g}b(2^s): s \geq 0 \text{ and } (m + 2g)_s = 0\}.$$

By the Orthogonality Lemma, the block has a set of generators with square zero. Hence for a general degree r , this block is isomorphic to a quotient of an algebra of the form

$$\bigotimes K[x_i]/\langle x_i^2 \rangle.$$

a tensor product of finitely many local two-dimensional algebras.

3. Basis and multiplication structure in $S_K(\lambda)$

In this section we take $\lambda = (\lambda_1, \lambda_2)$ to be a two-part partition and we study the multiplicative structure of $S_K(\lambda)$ over a field K of characteristic $p \geq 0$. The results from this section will then be used to obtain in characteristic two a reduction formula for $b(2^s)^2$ (see Section 4).

We describe briefly some results from [3,4]. Over \mathbb{Q} , the Schur algebra $S_{\mathbb{Q}}(2, r)$ is isomorphic to the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_2)$ modulo the ideal generated by

$$H_1(H_1 - 1) \cdots (H_1 - r);$$

alternatively, $S_{\mathbb{Q}}(2, r)$ can be described as the quotient of $\mathcal{U}(\mathfrak{sl}_2)$ modulo the ideal generated by

$$(h + r)(h + r - 2) \cdots (h - r + 2)(h - r).$$

Here, as a basis for the Lie algebra \mathfrak{gl}_2 one takes $e = e_{12}$, $f = e_{21}$ as usual and H_1, H_2 respectively the diagonal matrices e_{11} and e_{22} , where e_{ij} is the usual matrix unit, and as basis for \mathfrak{sl}_2 the usual e, f, h where $h = H_1 - H_2$ is the commutator of e and f .

The family of algebras $\{S_K(2, r)\}_K$ (K a field) is defined over \mathbb{Z} using the usual divided powers. In this presentation, the idempotent 1_λ which we earlier defined as projection corresponding to λ , is equal to the image (in the Schur algebra) of

$$1_\lambda = \begin{pmatrix} H_1 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} H_2 \\ \lambda_2 \end{pmatrix};$$

see [2, Lemma 5.3]. Now $S_{\mathbb{Q}}(\lambda) = 1_\lambda S_{\mathbb{Q}}(2, r) 1_\lambda$. There is a natural \mathbb{Z} -form $S_{\mathbb{Z}}(2, r)$ of $S_{\mathbb{Q}}(2, r)$, namely the image of the Kostant \mathbb{Z} -form of $\mathcal{U}(\mathfrak{gl}_2)$ (or of $\mathcal{U}(\mathfrak{sl}_2)$) under the quotient map $\mathcal{U}(\mathfrak{gl}_2) \rightarrow S_{\mathbb{Q}}(2, r)$. Set $S_{\mathbb{Z}}(\lambda) = S_{\mathbb{Q}}(\lambda) \cap S_{\mathbb{Z}}(2, r)$. If we set

$$b(i) := 1_\lambda f^{(i)} e^{(i)} 1_\lambda \in S_{\mathbb{Q}}(2, r)$$

then $S_{\mathbb{Z}}(\lambda)$ is the subalgebra of $S_{\mathbb{Q}}(\lambda)$ with basis $\{b(0), b(1), \dots, b(\lambda_2)\}$. Here

$$e^{(i)} := \frac{e^i}{i!} \quad \text{and} \quad f^{(i)} := \frac{f^i}{i!}$$

are the usual divided powers in the enveloping algebra.

We next describe the multiplicative structure of $S_{\mathbb{Z}}(\lambda)$. Most important for us is a multiplication formula for the basis elements $b(i)$, given in Proposition 3.6, which also proves again that the $b(i)$ generate a \mathbb{Z} -form of $S_{\mathbb{Q}}(\lambda)$. In what follows, we identify generators e, f, h, H_1, H_2 with their images in the quotient $S_{\mathbb{Q}}(\lambda)$.

The following formula, valid in $\mathcal{U}(\mathfrak{sl}_2)$, is easily derived by induction on a :

$$ef^a = f^a e + af^{a-1}(h - a + 1). \quad (2)$$

Since this formula holds in the enveloping algebra (over \mathbb{Q}), it is valid in its homomorphic image $S_{\mathbb{Q}}(2, r)$. The first part of the next lemma is contained in [4].

Lemma 3.1. In $S_{\mathbb{Q}}(2, r)$ we have the equality $h1_{\lambda} = m1_{\lambda}$, where $m = \lambda_1 - \lambda_2$. Moreover, for any k we have

$$b(1) \cdot b(k) = (k+1)^2 b(k+1) + k(m+k+1)b(k).$$

Proof. To see this, first calculate using formula (2):

$$\begin{aligned} (k!)^2 \cdot b(1) \cdot b(k) &= f e f^k e^k 1_{\lambda} \\ &= f(f^k e + k f^{k-1}(h-k+1))e^k 1_{\lambda} \\ &= f^{k+1} e^{k+1} 1_{\lambda} + k f^k (h-k+1) e^k 1_{\lambda} \\ &= f^{k+1} e^{k+1} 1_{\lambda} + k f^k e^k (h+k+1) 1_{\lambda}, \end{aligned}$$

where we have used the fact that $h e^k = e^k (h+2k)$. This holds in the enveloping algebra of \mathfrak{gl}_2 , and hence is valid in $S_{\mathbb{Q}}(2, r)$. Now apply the first statement of this lemma to obtain the desired formula. \square

Lemma 3.2. Set $x = b(1)$. Then we have in $S_{\mathbb{Q}}(2, r)$ for any $k \geq 1$ the equality

$$b(k+1) = \frac{1}{(k+1)!^2} x(x-(m+2))(x-2(m+3)) \cdots (x-k(m+k+1)).$$

Proof. Proceed by induction on k . Define

$$F_{k+1}(x) = x(x-(m+2))(x-2(m+3)) \cdots (x-k(m+k+1)).$$

The case $k = 1$ in the preceding lemma gives the equality

$$b(2) = \frac{1}{2^2} (x^2 - (m+2)x) = \frac{F_2(x)}{(2!)^2}.$$

Thus the formula of the lemma is valid in case $k = 1$. Assume that $b(k) = \frac{F_k(x)}{(k!)^2}$. By the preceding lemma and the inductive hypothesis we then have

$$\begin{aligned} b(k+1) &= \frac{1}{(k+1)^2} \cdot (b(1)b(k) - k(m+k+1)b(k)) \\ &= \frac{1}{(k+1)!^2} \cdot (x - k(m+k+1)) F_k(x) = \frac{1}{(k+1)!^2} F_{k+1}(x). \quad \square \end{aligned}$$

Proposition 3.3. The algebra $S_{\mathbb{Q}}(\lambda)$ is semisimple and generated by $b(1)$.

Proof. The semisimplicity statement is clear, since M^{λ} is completely reducible as a Σ_r -module in characteristic zero. Thus only the claim about generation needs to be proved. It follows from Lemma 3.2 that we have the equality

$$b(k) = 1_{\lambda} f^{(k)} e^{(k)} 1_{\lambda} = \frac{F_k(x)}{(k!)^2}$$

for all $k \geq 2$. This formula holds in $S_{\mathbb{Q}}(2, r)$ and hence any element in $S_{\mathbb{Q}}(\lambda)$ is generated by $x = b(1)$. \square

Proposition 3.4. *The algebra $S_{\mathbb{Q}}(\lambda)$ is isomorphic with $\mathbb{Q}[T]/(F_{\lambda_2+1}(T))$.*

Proof. By commutation formulas appearing in [4] we have

$$b(\lambda_2 + 1) = 1_{\lambda} f^{(\lambda_2+1)} e^{(\lambda_2+1)} 1_{\lambda} = 0 = F_{\lambda_2+1}(x)$$

since $\lambda + (\lambda_2 + 1)(1, -1) = (\lambda_1 + \lambda_2 + 1, -1)$ is not a polynomial weight belonging to $\Lambda(2, r)$, for any λ . The proposition now follows from Lemma 3.2. \square

Remark 3.5. It follows immediately from the preceding proposition that the algebra $S_{\mathbb{Q}}(\lambda)$ is a commutative algebra. In fact, the commutativity of $S_{\mathbb{Q}}(\lambda)$ is a consequence of the fact that the permutation module M^{λ} is multiplicity-free (see [13]). This Σ_n -module is semisimple and its composition factors are absolutely irreducible, so by Schur's Lemma $\text{End}_{\mathbb{Q}\Sigma_n}(M^{\lambda})$ is a direct sum of copies of the field \mathbb{Q} .

Proposition 3.6. *A multiplication formula for the basis elements is given by:*

$$b(i) \cdot b(j) = \sum_{k=0}^i \binom{j+k}{i} \binom{j+k}{k} \binom{m+j+i}{i-k} b(j+k). \quad (3)$$

When $a > \lambda_2$ then $b(a)$ is zero in this formula.

Proof. The proof is by induction on i . The induction beginning for $i = 1$ is given by Lemma 3.1. Let now $i > 1$. Then using Lemma 3.1, the product $P := b(i+1) \cdot b(j)$ equals:

$$\begin{aligned} P &= \frac{b(i) \cdot (b(1) - i(m+i+1))}{(i+1)^2} b(j) \\ &= \frac{b(1) - (j+k)(m+j+k+1) + (j+k)(m+j+k+1) - i(m+i+1)}{(i+1)^2} b(i)b(j) \\ &= \frac{b(1) - (j+k)(m+j+k+1) + (j+k-i)(m+j+k+i+1)}{(i+1)^2} b(i)b(j) \\ &= \sum_{k=0}^i \frac{b(1) - (j+k)(m+j+k+1)}{(i+1)^2} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} b(j+k) \\ &\quad + \sum_{k=0}^i \frac{(j+k-i)(m+j+k+i+1)}{(i+1)^2} \binom{j+k}{i} \binom{j+k}{k} \binom{m+i+j}{i-k} b(j+k) \\ &= \sum_{k=0}^i \frac{k+1}{i+1} \binom{j+k+1}{i+1} \binom{j+k+1}{k+1} \binom{m+i+j}{i-k} b(j+k+1) \\ &\quad + \sum_{k=0}^i \frac{(m+j+k+i+1)}{(i+1)} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i-k} b(j+k) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{i+1} \frac{k}{i+1} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i+1-k} b(j+k) \\
&\quad + \sum_{k=0}^i \frac{(m+j+k+i+1)}{(i+1)} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+i+j}{i-k} b(j+k) \\
&= \sum_{k=0}^{i+1} \binom{j+k}{i+1} \binom{j+k}{k} \binom{m+j+i+1}{i+1-k} \cdot b(j+k)
\end{aligned}$$

and the induction is complete. Note that the last equality above is justified as follows:

$$\begin{aligned}
&\frac{k}{i+1} \binom{m+i+j}{i+1-k} + \frac{m+j+k+i+1}{i+1} \binom{m+i+j}{i-k} \\
&= \frac{k}{i+1} \binom{m+i+j}{i+1-k} + \frac{i+1-k}{i+1} \binom{m+i+j}{i+1-k} + \binom{m+i+j}{i-k} \\
&= \binom{m+i+j}{i+1-k} + \binom{m+i+j}{i-k} \\
&= \binom{m+i+1+j}{i+1-k}
\end{aligned}$$

for any k satisfying $1 \leq k \leq i$. \square

The previous proposition verifies again that $S_{\mathbb{Z}}(\lambda)$ is a \mathbb{Z} -form. We have $S_K(\lambda) \simeq S_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} K$ for any field K , and by abuse of notation we still write $b(i)$ for $1_{\lambda} f^{(i)} e^{(i)} 1_{\lambda} \otimes 1_K$. Then the multiplication formula (3) is also valid in the K -algebra $S_K(\lambda)$; and again $b(a) = 0$ whenever $a > \lambda_2$.

For the remainder of this section, we work over the field K of positive characteristic p .

Lemma 3.7. Write $i = [i_0, i_1, \dots]$ p -adically. Then $b(i) = \prod_{t \geq 0} b(i_t \cdot p^t)$ in $S_K(\lambda)$.

Proof. This is shown by induction on the length of the p -adic decomposition of i . Assume that $j = [i_0, i_1, \dots, i_{t-1}]$; then by Eq. (3):

$$b(j) \cdot b(i_t p^t) = \sum_{k=0}^j \binom{i_t p^t + k}{j} \binom{i_t p^t + k}{k} \binom{m + i_t p^t + j}{j-k} b(i_t p^t + k).$$

In the above sum, the binomial coefficient $\binom{i_t p^t + k}{j} = \binom{k}{j}$ is non-zero if and only if j is p -contained in k , that is $i_s \leq k_s$ for all $s \leq t-1$. Hence $j \leq k$ and by assumption also $k \leq j$. So $\binom{i_t p^t + k}{j} \neq 0$ precisely if $k = j$. In that case, $\binom{i_t p^t + k}{j} = \binom{i_t p^t + j}{j} = \binom{j}{j} = 1$, $\binom{i_t p^t + k}{k} = \binom{k}{k} = 1$, and $\binom{m + i_t p^t + j}{j-k} = 1$. Hence $b(j) \cdot b(i_t p^t) = b(i_t p^t + j)$. \square

Lemma 3.8. We define the degree of the basis element $b(i)$ to be i . Let $1 < n \leq p-1$, then in $S_K(\lambda)$ we have

$$(b(p^t))^n = (n!)^2 b(n \cdot p^t) + \text{terms of lower degree}.$$

Proof. This follows by induction on n , using the multiplication formula given in Proposition 3.6. More precisely, let $2 \leq c \leq p-1$, then

$$b(p^t) \cdot b((c-1)p^t) = \sum_{k=0}^{p^t} \binom{(c-1)p^t + k}{p^t} \binom{(c-1)p^t + k}{k} \binom{m + cp^t}{p^t - k} b((c-1)p^t + k),$$

where for $k = p^t$ we obtain $\binom{(c-1)p^t + k}{p^t} = \binom{cp^t}{p^t} = \binom{c}{1}$ and $\binom{(c-1)p^t + k}{k} = \binom{cp^t}{p^t} = \binom{c}{1}$ and $\binom{m + cp^t}{p^t - k} = \binom{m + cp^t}{0} = 1$. Thus the above formula takes the form

$$b(p^t) \cdot b((c-1)p^t) = c^2 b(c \cdot p^t) + \text{terms of lower degree}. \quad (4)$$

Taking $c = 2$ in this formula gives

$$b(p^t)^2 = 2^2 b(2 \cdot p^t) + \text{terms of lower degree}$$

and multiplying this through by $b(p^t)$ and using Eq. (4) again yields

$$b(p^t)^3 = 3^2 \cdot 2^2 b(3 \cdot p^t) + \text{terms of lower degree}$$

and so forth. \square

Corollary 3.9. Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of r and assume t is such that $p^t \leq \lambda_2 < p^{t+1}$. Then the algebra $S_K(\lambda)$ is generated by the elements $b(p^0), b(p^1), \dots, b(p^t)$.

Proof. We know already from Lemma 3.7 a factorisation of a basis element $b(i)$. Write $i = [i_0, i_1, \dots]$ p -adically. Then

$$b(i) = \prod_{t \geq 0} b(i_t \cdot p^t).$$

Hence we need to show that the elements $b(c \cdot p^t)$ for $1 \leq c \leq p-1$ are generated by the elements $b(p^t)$. This follows by induction on t using Lemma 3.8. \square

Remark. For odd primes it seems hard to find explicit expressions for $b(p^t)^n$ in terms of generators, and arithmetic conditions seem to be quite complicated. For $p = 2$ this is done in the next section.

Example 3.10. Let $m = 0$ and $p = 2$. Then r is even and $\lambda = (r/2, r/2)$; in this case the algebra $S_K(\lambda)$ has dimension $r/2 + 1$. It is generated by $b(0), \dots, b(2^k)$ where $2^k \leq r/2 + 1 < 2^{k+1}$ subject to the relations

$$b(2^i)^2 = 0, \quad 0 \leq i \leq k;$$

$$\prod_{i \in I} b(2^i) = 0, \quad \text{whenever } I \subseteq \{0, 1, \dots, k\} \text{ and } \sum_{i \in I} 2^i \geq r/2 + 1.$$

It follows that there are no non-zero primitive idempotents except 1, and hence $S_K(\lambda)$ is indecomposable; that is, the algebra is a block.

4. The elements $b(i)^2$ are idempotents

From now we assume that the characteristic of the underlying field K is $p = 2$. Then Lemma 3.7 shows that the basis element $b(i)$ in $S_K(\lambda)$ is equal to the product of the $b(2^t)$ for which $i_t = 1$. So to understand the multiplication completely we need to understand the squares of the basis elements $b(2^t)$.

Example 4.1. Let m be fixed with 2-adic expansion $m = [m_0, \dots, m_t, \dots]$. Suppose $t = 0, 1$ then we see directly from multiplication formula (3) that

$$b(2^0)^2 = m_0 \cdot b(2^0),$$

$$b(2^1)^2 = b(2^1)[m_1 \cdot 1 + m_0 \cdot b(2^0)].$$

So we can write $b(2^1)^2 = b(2^1)(m_1 + b(2^0)^2)$. This has the following generalisation.

Lemma 4.2. Suppose $m = [m_0, \dots, m_t, \dots]$ in 2-adic expansion. Let $0 \leq v \leq t$ be maximal such that $m_{v-1} = 0$. Then

$$b(2^t)^2 = b(2^t) \left[m_t \cdot 1 + \sum_{i=v-1}^{t-1} b(2^i)^2 \right],$$

setting $b(2^i) = 0$ and $m_i = 0$ if $i < 0$.

Proof. We make the convention that $m_i = 0$ when $i < 0$. We rewrite the product $b(2^t)^2$ using the multiplication formula given in Eq. (3). Note that $\binom{2^t+k}{2^t} = \binom{2^t+k}{k}$ is zero modulo two when $k = 2^t$, and if $k < 2^t$ it is one modulo two. Moreover for $k < 2^t$ we have

$$\binom{m+2^{t+1}}{2^t-k} \equiv \binom{m}{2^t-k} \pmod{2}. \quad (5)$$

We will change variables using the relation $2^t + k = 2^{t+1} - (2^t - k) = 2^{t+1} - l$. Hence—by Eq. (5)—we can rewrite Eq. (3) in the form

$$b(2^t)^2 = \sum_{k=0}^{2^t-1} \binom{m}{2^t-k} b(2^t+k) = \sum_{l=1}^{2^t} \binom{m}{l} b(2^{t+1}-l) = b(2^t) \left[\sum_{l=1}^{2^t} \binom{m}{l} b(2^t-l) \right]. \quad (6)$$

For the last equality note that $2^{t+1} - l = 2^t + (2^t - l)$, and so for $0 \leq 2^t - l < 2^t$ we can factor $b(2^{t+1} - l) = b(2^t)b(2^t - l)$ by Lemma 3.7. The term with $l = 2^t$ is equal to $m_t b(0) = m_t \cdot 1$. So we can write

$$b(2^t)^2 = b(2^t)[m_t \cdot 1 + \Gamma(t)] \quad \text{where } \Gamma(t) := \sum_{l=1}^{2^t-1} \binom{m}{l} b(2^t - l). \quad (7)$$

We will now prove a recursion formula for $\Gamma(t)$. We claim that

$$\begin{aligned} \Gamma(1) &= b(2^0)^2, \\ \Gamma(t) &= b(2^{t-1})^2 + m_{t-1} \Gamma(t-1) \quad \text{for } t \geq 2. \end{aligned} \quad (8)$$

First, $\Gamma(1) = \binom{m}{1} b(2^0) = m_0 b(2^0) = b(2^0)^2$, where the last equality is by Example 4.1. Suppose that $t \geq 2$, and split $\Gamma(t)$ into two sums as follows:

$$\begin{aligned} \Gamma(t) &= \sum_{l=1}^{2^t-1} \binom{m}{l} b(2^t - l) + \sum_{l=2^{t-1}+1}^{2^t-1} \binom{m}{l} b(2^t - l) \quad (\text{by definition of } \Gamma(t)) \\ &= b(2^{t-1})^2 + \sum_{l=2^{t-1}+1}^{2^t-1} \binom{m}{l} b(2^t - l) \quad (\text{by Eq. (6), Lemma 3.7}) \\ &= b(2^{t-1})^2 + m_{t-1} \sum_{r=1}^{2^{t-1}-1} \binom{m}{r} b(2^{t-1} - r) \quad (\text{the argument is given below}) \\ &= b(2^{t-1})^2 + m_{t-1} \Gamma(t-1) \quad (\text{by definition of } \Gamma(t-1)). \end{aligned}$$

For the third equality sign in the latter equation, set $l = 2^{t-1} + r$ where $1 \leq r \leq 2^{t-1} - 1$, and note that

$$\binom{m}{2^{t-1} + r} \equiv m_{t-1} \binom{m}{r} \pmod{2},$$

and $2^t - l = 2^{t-1} - r$. Hence the recursion formula for $\Gamma(t)$ claimed in Eq. (8) is shown. It in fact implies the following simpler formula for $\Gamma(t)$:

$$\Gamma(t) = \sum_{i=v-1}^{t-1} b(2^i)^2,$$

where v is as in the statement. Substituting this into (7) completes the proof. \square

Proposition 4.3. *Let $p = 2$. For $i \geq 0$, the elements $b(2^i)^2$ are idempotent in $S_K(\lambda)$. Moreover, if $m_j = 0$ for all $j \leq i$ then $b(2^i)^2 = 0$.*

Proof. This follows from Lemma 4.2 by induction. \square

5. Analysis of the binomial coefficient $B(m, g)$

Still keeping $p = 2$ fixed, we assume throughout this section that m and g are integers such that the binomial coefficient $B(m, g)$ is non-zero modulo two. We need to relate the binomial expansion of m with that of $B(m, g)$. Note that we have the following depiction of the binary addition:

$$\begin{array}{r|cccccc}
 m & m_0 & m_1 & m_2 & \dots & m_i & \dots \\
 +2g & 0 & g_0 & g_1 & \dots & g_{i-1} & \dots \\
 \hline
 m+2g & (m+2g)_0 & (m+2g)_1 & (m+2g)_2 & \dots & (m+2g)_i & \dots
 \end{array}$$

In this addition, we need to keep track over the ‘carry overs.’ So define integers $x_i \geq 0$ such that

$$m_i + g_{i-1} + x_{i-1} = (m+2g)_i + 2x_i. \quad (9)$$

Thus x_i is the carry over from column i to column $i+1$ in the binary addition of m and $2g$. Most important for the proofs later will be that $(m+2g)_i = 1$ implies that $x_i = 0$; more precisely we have the following:

Proposition 5.1. *Let $m = [m_0, m_1, \dots]$ and $g = [g_0, g_1, \dots]$ be in binary expansion. Assume that $B(m, g)$ is non-zero. Then $(m+2g)_i + 2x_i < 3$ for all i . In particular, if $(m+2g)_i = 1$ then $x_i = 0$.*

Proof. Certainly $(m+2g)_i + 2x_i \leq 3$. Assume for a contradiction that this number is equal to three for some i . Then $x_{i-1} = g_{i-1} = 1$. Since $g_{i-1} = 1$ we must have that $(m+2g)_{i-1} = 1$ as well, since otherwise the binomial coefficient $B(m, g)$ would be zero. But then it follows that $m_{i-1} + g_{i-2} + x_{i-2} = 3$, and then repeating the argument gives $m_1 + g_0 + x_0 = 3$. This implies $x_0 = 1$. On the other hand, $(m+2g)_0 = m_0$ and hence $x_0 = 0$, a contradiction. \square

We will later prove some properties by induction. The elements $e_{m,g}$ are defined as products, and it will be convenient to use factors of these which are already known to be idempotents. The basis for the induction will be the following:

Lemma 5.2 (Splitting Lemma). *Let u be a natural number and define*

$$n := [m_0, m_1, \dots, m_u] \quad \text{and} \quad d := [g_0, g_1, \dots, g_{u-1}].$$

Suppose $(m+2g)_u = 1$. Then the binary expansion of $B(n, d)$ equals the binary expansion of $B(m, g)_{<u}$ extended by one column $\binom{1}{0}$. In particular if $g_u = 0$ then $B(n, d) = B(m, g)_{\leq u}$.

Proof. By Proposition 5.1 we know that $x_u = 0$, and by Eq. (9) we hence have $m_u + g_{u-1} + x_{u-1} = 1$; the claim follows. \square

Remark. The Splitting Lemma shows that when $g_u = 0$ then the element $e_{n,d}$ is a factor of $e_{m,g}$, when written as in the definition; see Eq. (1).

We will have to use the formula from Lemma 4.2. So we need to know the digits of $B(m, g)$, given the binary expansion of m and of g . We now describe these explicitly.

Lemma 5.3. Given natural numbers t and a . Suppose $B(m, g)_{\leq t+a}$ in binary decomposition is of the form

$$B(m, g)_{\leq t+a} = \begin{pmatrix} \cdots & 1 & 0 & \cdots & 0 \\ \cdots & g_t & 0 & \cdots & 0 \end{pmatrix}. \quad (10)$$

Then we have:

- (a) Suppose $g_t = 0$, then $m_{t+1} = \cdots = m_{t+a} = 0$ and $x_{t+1} = \cdots = x_{t+a} = 0$.
- (b) Suppose $g_t = 1$, then $m_{t+1} = \cdots = m_{t+a} = 1$ and $x_{t+1} = \cdots = x_{t+a} = 1$.

Proof. By Proposition 5.1 we know that $x_t = 0$. By Eq. (9) we have:

$$\begin{aligned} m_{t+1} + g_t + 0 &= 0 + 2x_{t+1}, \\ m_{t+2} + 0 + x_{t+1} &= 0 + 2x_{t+2}, \\ &\vdots \\ m_{t+a} + 0 + x_{t+a-1} &= 0 + 2x_{t+a}. \end{aligned}$$

For (a), assume that $g_t = 0$. Then $x_{t+1} = 0$ and hence $m_{t+1} = 0$. Now the second equation shows that $x_{t+2} = 0$ and hence $m_{t+2} = 0$, and so on. Part (b) is similar. \square

We will have to consider sequences of digits such that $m_i = 1$ for $v \leq i \leq s$ and $m_{v-1} = 0$. For these values of i we need to know the i th columns of $B(m, g)$.

Lemma 5.4. Suppose column s of $B(m, g)$ is zero but column $s - 1$ is non-zero. Let $u \geq 0$ be minimal such that $(m + 2g)_i = 1$ for $u \leq i < s$, and let $0 \leq v \leq s$ be maximal with $m_{v-1} = 0$. Then $v \geq u$. Moreover:

- (a) If $m_s = 0$ then $g_i = 0$ for $v - 1 \leq i \leq s - 1$.
- (b) If $m_s = 1$ then $g_{s-1} = 1$ and $g_i = 0$ for $v - 1 \leq i < s - 1$.

Proof. (i) Suppose $m_u = 0$ or $u = 0$. Then by definition of v we have that $v \geq u$. So assume that $m_u = 1$ and $u > 0$. By definition of u we have that $(m + 2g)_u = 1$ and $(m + 2g)_{u-1} = 0$. Then Eq. (9) for columns u and $u - 1$ together with the assumptions and Proposition 5.1 read:

$$\begin{aligned} 1 + g_{u-1} + x_{u-1} &= 1, \\ m_{u-1} + g_{u-2} + x_{u-2} &= 0 + 2x_{u-1}. \end{aligned}$$

So $x_{u-1} = 0 = g_{u-1}$ which implies that $m_{u-1} + g_{u-2} + x_{u-2} = 0$ and hence $m_{u-1} = 0$. This shows that $u \leq v$.

(ii) For (a) and (b), use Proposition 5.1 and Eq. (9) for columns between v and $s - 1$. By assumption and (i) we have that $(m + 2g)_i = 1 = m_i$ for $v \leq i \leq s - 1$. This implies $g_{i-1} = 0 = x_{i-1}$ for $v \leq i \leq s - 1$ and $x_{s-1} = 0$. Then Eq. (9) for column s becomes

$$m_s + g_{s-1} + 0 = 0 + 2x_s.$$

If $m_s = 0$ then $x_s = 0$ and $g_{s-1} = 0$. On the other hand, if $m_s = 1$ then $x_s = 1$ and $g_{s-1} = 1$. \square

6. The proofs of the Orthogonality Lemma and the Idempotent Theorem

In this section we return to the analysis of the basis $\{b(i)\}$ of $S_K(\lambda)$, still under the assumption $\text{char}(K) = 2$. With the information obtained in the preceding section, we are now in a position to complete the proof of both the Orthogonality Lemma and the Idempotent Theorem, stated in Section 2.4.

6.1. Proof of the Orthogonality Lemma

Suppose the s th column of $B(m, g)$ is zero. The aim is to show that $e_{m,g}^2 \cdot b(2^s)^2 = 0$. Recall from Lemma 4.2 that $b(2^s)^2 = b(2^s)\psi$ with

$$\psi = \psi_{m,s} = m_s + \sum_{i=v-1}^{s-1} b(2^i)^2, \quad (11)$$

where $0 \leq v \leq s$ is maximal such that $m_{v-1} = 0$. We will prove that

$$(e_{m,g})_{<s}^2 \cdot \psi_{m,s} = 0. \quad (12)$$

Certainly this then implies the Orthogonality Lemma in Section 2. Note that if $s = 0$ then $\psi = m_0 = 0$ since $(m + 2g)_0 = m_0 = 0$. So assume $s > 0$. If all columns before column s are zero then $m_i = 0$ for $i \leq s$ and then $\psi = 0$ by Proposition 4.3. So assume now that $w < s$ is such that $(m + 2g)_w = 1$ and $(m + 2g)_i = 0$ for $w + 1 \leq i \leq s$. We use induction on the number of zero columns between w and s to prove Eq. (12).

Suppose column $s - 1$ is non-zero. Let $u \geq 0$ be minimal such that $(m + 2g)_i = 1$ for $u \leq i < s$. We apply Lemma 5.4, which shows that $v \geq u$. Moreover, suppose $m_s = 0$, then by part (a) of the lemma we know that $(e_{m,g})_{<s}$ has factors $(1 - b(2^i))$ for $v - 1 \leq i \leq s - 1$. This gives that $(e_{m,g})_{<s}^2 \cdot \psi = 0$ by Proposition 4.3.

Similarly, if $m_s = 1$ then part (b) of the lemma shows that $(e_{m,g})_{<s}$ has factors $(1 - b(2^i))$ for $v - 1 \leq i < s - 1$ and also a factor $b(2^{s-1})$. Then the claim follows again from Proposition 4.3, using that $b(2^{s-1})^2 \cdot (m_s + b(2^{s-1})^2) = 0$. This proves the base case of the induction.

For the inductive step, suppose now that column $s - 1$ is zero. The inductive hypothesis states that

$$(e_{m,g})_{<s-1}^2 \cdot \psi_{m,s-1} = 0.$$

If $g_w = 0$ then we have by Lemma 5.3 that $m_i = 0$ for $w + 1 \leq i \leq s$. Then $v = s$ and we can write

$$\psi_{m,s} = b(2^{s-1})^2 = \psi_{m,s-1} \cdot b(2^{s-1}),$$

using Lemma 4.2. By the inductive hypothesis we deduce $(e_{m,g})_{<s}^2 \cdot \psi_{m,s} = 0$. Now suppose $g_w = 1$, then by Lemma 5.3 we know that $m_i = 1$ for $w + 1 \leq i \leq s$. We rewrite and again use Lemma 4.2:

$$\psi_{m,s} = \psi_{m,s-1} + b(2^{s-1})^2 = \psi_{m,s-1} + \psi_{m,s-1} \cdot b(2^{s-1}),$$

and again using the inductive hypothesis we have $(e_{m,g})_{<s}^2 \cdot \psi_{m,s} = 0$. This completes the proof of (12), and hence also the proof of the Orthogonality Lemma.

6.2. Proof of the Idempotent Theorem

This will be done by induction on t , the largest column label of a non-zero column in the binary decomposition of $B(m, g)$, which we call the degree of $e_{m,g}$. In fact, we will prove the following:

Claim. Elements $e_{m,g}$ and $(e_{m,g})_{<t}$ are idempotents.

Assume that $t = 0$ then $B(m, g) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In particular $m_0 = 1$ and so $e_{m,g} = (1 - b(2^0))$ is idempotent. Also $(e_{m,g})_{<t} = 1$ is idempotent. We assume the statement holds for all $e_{n,d}$ of degree $< t$. Let $e_{m,g}$ be of degree t and write $e := e_{m,g} = P \cdot (1 - b(2^t))$ where $P = (e_{m,g})_{<t}$. We have

$$e^2 = P^2(1 - b(2^t))^2 = P^2(1 - \psi b(2^t)),$$

where $\psi = \psi_{m,g}$ is defined as in Eq. (11). We will show that $P^2 \cdot \psi = P^2$, and secondly that $P^2 = P$. This then implies that $e = e_{m,g}$ is idempotent.

- (a) We claim that $P^2 \cdot \psi = P^2$, that is $P^2(1 - \psi) = 0$. To see this, let $\tilde{m} := m + 2^t$, then $\tilde{m}_t = 1 + m_t$ and $\tilde{m}_i = m_i$ for $i < t$. Hence $B(\tilde{m}, g)$ differs from $B(m, g)$ in columns t and $t + 1$. Therefore

$$(e_{m,g})_{<t} = (e_{\tilde{m},g})_{<t} = P.$$

Moreover (using $p = 2$) we have $\psi_{\tilde{m},t} = 1 - \psi_{m,t}$. So we get from the Orthogonality Lemma, see Eq. (12):

$$P^2(1 - \psi_{m,t}) = (e_{\tilde{m},g})_{<t}^2 \cdot \psi_{\tilde{m},t} = 0.$$

- (b) We claim that $P^2 = P$. This is clear if $P = 1$. So suppose $P > 1$, then there is some $u < t$ maximal such that $(m + 2g)_u = 1$. If $g_u = 0$ then $P = e_{n,d}$ with d and n as in the Splitting Lemma 5.2. Hence by the inductive hypothesis P is idempotent. If $g_u = 1$, then $P = (e_{m,g})_{<u} \cdot b(2^u)$. Define n and d by

$$e_{n,d} = (e_{m,g})_{<u} \cdot (1 - b(2^u)). \quad (13)$$

By construction $e_{n,d}$ has degree $u < t$ and hence by the inductive hypothesis we get that $e_{n,d}$ and $(e_{m,g})_{<u}$ are idempotents. Since the characteristic of the underlying field is two and by Eq. (13), we have that $(e_{m,g})_{<t} = P = (e_{m,g})_{<u} \cdot b(2^u) = e_{n,d} + (e_{m,g})_{<u}$ is idempotent. \square

7. The correspondence between idempotents and Young modules

Fix an integer $g \geq 0$ such that $\binom{m+2g}{g} \neq 0$. Then we have for each $r \geq m + 2g$ of the right parity a partition λ with $\lambda_1 - \lambda_2 = m$, and a partition $\mu = (\mu_1, \mu_2)$ with $\mu_1 - \mu_2 = m + 2g$.

We also have the primitive idempotent $e_{m,g}$ defined in Eq. (1); and we know that Y^μ is a direct summand of M^λ . We will now show that in fact $e_{m,g}$ is the projection of M^λ corresponding to Y^μ .

Theorem 7.1. *Let λ, μ be two-part partitions of r such that Y^μ is a direct summand of M^λ . Let $\lambda_1 - \lambda_2 = m$, $\mu_1 - \mu_2 = m + 2g$ and $g = \lambda_2 - \mu_2$. Then the idempotent $e_{m,g}$ of $S_K(\lambda)$ is the projection onto Y^μ .*

The proof of this will take the rest of the section. We use induction on r , starting with the case $\mu_2 = 0$, that is $\mu = (r, 0)$. Then the inductive step will be to show that if the theorem is true for degree r then it is true for degree $r + 2$.

To begin the induction we make two observations:

(i) Suppose that $\mu_2 = 0$. In the special case when $\lambda = \mu$ we have $g = 0$ and $m = r$. So $\lambda_2 = 0$ and the algebra $S_K(\lambda)$ has dimension one. Furthermore, $e_{m,0} = 1$ and $M^\lambda = Y^\lambda$, so the theorem is trivially true.

(ii) Suppose next that $\mu_2 = 0$ and $\mu > \lambda$. We have then $r = \mu_1$ and $\mu_2 = 0$. By case (i), we know that $e_{r,0} \in S_K(\mu)$ is the projection corresponding to the summand Y^μ of M^μ . Both idempotents $e_{m,g}$ and $e_{r,0}$ lie in $S_K(2, r)$. To show that the summand of M^λ corresponding to the projection $e_{m,g}$ is isomorphic to Y^μ we must show that the idempotents $e_{m,g}$ and $e_{r,0}$ are associated in $S_K(2, r)$.

Proposition 7.2. *Under the assumptions in (ii), the idempotents $e_{m,g}$ and $e_{r,0}$ are associated in $S_K(2, r)$. Hence $e_{m,g}M^\lambda$ is isomorphic to Y^μ .*

Proof. (a) We first simplify the expressions for the two idempotents. Note that by definition (see Eq. (1)) we have

$$\begin{aligned} e_{m,g} &= \prod_{u \in J_{m,g}} b(2^u) \cdot \prod_{u \in I_{m,g}} (1 - b(2^u)) \quad (\text{by Eq. (1)}) \\ &= b(g) \cdot \prod_{u \in I_{m,g}} (1 - b(2^u)) \quad (\text{by Lemma 3.7}) \\ &= b(g) \cdot (1 \pm \text{sum of products of } b(i)\text{'s}) \\ &= b(g). \end{aligned}$$

To get the last equality note that $b(g) \cdot b(i) = 0$ as the algebra $S_K(\lambda)$ has basis $\{b(0), b(1), \dots, b(g)\}$ and by using Lemma 3.7. Moreover, as $M^{(r,0)} = Y^{(r,0)}$, we have $e_{r,0} = 1_{(r,0)}$.

(b) Let $\alpha = (1, -1)$ and recall from [4, Theorem 2.4], that for any composition ν we have

$$e \cdot 1_\nu = \begin{cases} 1_{\nu+\alpha} \cdot e & \text{if } \nu + \alpha \text{ is a composition,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f \cdot 1_\nu = \begin{cases} 1_{\nu-\alpha} \cdot e & \text{if } \nu - \alpha \text{ is a composition,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, by [4], Proposition 4.3 we have that $H_i \cdot 1_\lambda = \lambda_i \cdot 1_\lambda$ for $i = 1, 2$, and recall that $h = H_1 - H_2$. These formulas imply that $e \cdot 1_{(r,0)} = 0$ as $(r, 0) + \alpha$ is not a composition. Moreover, with $\lambda = (g + m, g)$ a partition of $r = m + 2g$ we have

$$e^{(g)} \cdot 1_\lambda = 1_{(r,0)} \cdot e^{(g)}, \quad 1_{(r,0)} \cdot f^{(g)} = f^{(g)} \cdot 1_\lambda, \quad \binom{h}{g} \cdot 1_{(r,0)} = \binom{r}{g} \cdot 1_{(r,0)}.$$

(c) We next give elements u and v in the Schur algebra $S_K(2, r)$ such that $e_{m,g} = uv$ and $e_{r,0} = vu$, proving that the two idempotents are associated. More precisely, let

$$u = 1_\lambda f^{(g)} 1_{(r,0)} \quad \text{and} \quad v = 1_{(r,0)} e^{(g)} 1_\lambda.$$

Then by repeated use of the equations in (b) we have

$$u \cdot v = 1_\lambda f^{(g)} 1_{(r,0)} e^{(g)} 1_\lambda = 1_\lambda f^{(g)} e^{(g)} 1_\lambda = b(g)$$

and

$$\begin{aligned} v \cdot u &= 1_{(r,0)} e^{(g)} 1_\lambda f^{(g)} 1_{(r,0)} \\ &= 1_{(r,0)} e^{(g)} f^{(g)} 1_{(r,0)} \\ &= 1_{(r,0)} \cdot \left[\sum_{j=0}^g f^{(g-j)} \binom{h-2g+2j}{j} e^{(g-j)} \right] \cdot 1_{(r,0)} \\ &= 1_{(r,0)} \cdot \left[f^{(0)} \binom{h}{g} e^{(0)} \right] \cdot 1_{(r,0)} \\ &= \binom{r}{g} \cdot 1_{(r,0)} = B(m, g) \cdot 1_{(r,0)} = 1_{(r,0)} \end{aligned}$$

modulo two. Hence $e_{m,g} = b(g)$ and $e_{r,0} = 1_{(r,0)}$ are associated. \square

We hence have shown that whenever $\mu = (r, 0)$ then the claim made in Theorem 7.1 is true. Now it remains to deal with the inductive step. We assume that Theorem 7.1 holds in degree r and show it holds in degree $r + 2$. Clearly any pair of partitions $\tilde{\lambda} < \tilde{\mu}$ in degree $r + 2$ with $\tilde{\mu}_2 > 0$ which satisfies the assumptions of Theorem 7.1, is obtained from partitions $\lambda > \mu$ in degree r as $\tilde{\mu} = \mu + (1, 1)$ and $\tilde{\lambda} = \lambda + (1, 1)$. We hence do the induction step by comparing M^λ and $M^{\lambda+(1^2)}$. To do so, we will first analyse more closely how the hyperalgebra actions on $E^{\otimes r}$ and $E^{\otimes r+2}$ are related. We fix a basis $\{v_1, v_2\}$ of the K -vector space E . We write briefly $v_{\underline{i}}$ for the tensor product $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$, with \underline{i} the multi-index $\underline{i} = (i_1, \dots, i_r)$. Define the linear map

$$j: E^{\otimes r} \rightarrow E^{\otimes r+2} \quad \text{by} \quad x \mapsto (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes x.$$

Recall that both tensor powers are modules for the hyperalgebra $\mathfrak{U}_K = \mathfrak{U}(\mathfrak{gl}_2)_Z \otimes K$. The map j commutes with the action of the divided powers $e^{(a)}, f^{(a)} \in \mathfrak{U}_K$: this is easy to see, noting that the map j is tensoring with $\bigwedge^2 E$, which is trivial under the action of e and f . Now we restrict

j to M^λ ; it takes M^λ to $M^{\lambda+(1^2)}$. Since the products $f^{(a)}e^{(a)}$ lie in the zero weight space of \mathfrak{U}_K , they preserve M^λ and $M^{\lambda+(1^2)}$. The idempotents 1_λ and $1_{\lambda+(1^2)}$ are the projections onto these spaces, and it follows that j intertwines the actions of elements $b(a)$ on M^λ and on $M^{\lambda+(1^2)}$. In particular this implies

$$j(e_{m,g}x) = e_{m,g}j(x), \quad \text{for all } x \in M^\lambda. \quad (14)$$

The following proposition completes the proof of Theorem 7.1.

Proposition 7.3. *Suppose $e_{m,g}$ is the projection on M^λ corresponding to Y^μ . Then $e_{m,g}$ on $M^{\lambda+(1^2)}$ is the projection corresponding to $Y^{\mu+(1^2)}$.*

Proof. We may assume $m \neq 0$; the case $m = 0$ is understood, see Example 3.10. We know that the Specht module S^μ is a submodule of Y^μ . Furthermore, $\text{Hom}_K \Sigma_r(S^\mu, M^\lambda)$ is one-dimensional (see [10, 13.13]). So M^λ has a unique submodule isomorphic to S^μ , which is contained in Y^μ . Similarly $M^{\lambda+(1^2)}$ has a unique submodule isomorphic to $S^{\mu+(1^2)}$ and it is contained in $Y^{\mu+(1^2)}$. Since the elements $e_{m,g}$ are projections onto a Young module, it suffices to show the following:

$$\text{If } e_{m,g}(S^\mu) \neq 0 \text{ in } M^\lambda \text{ then } e_{m,g}(S^{\mu+(1^2)}) \neq 0 \text{ in } M^{\lambda+(1^2)}.$$

To do so we use polytabloids, that is the standard generators for Specht modules, see James [10, Chapter 4]. We start with standard tableaux of shapes μ and $\mu + (1^2)$, respectively, the two rows of which are filled in as follows:

$$t_1 = \begin{array}{cccc} 3 & 5 & \dots & (2u-1) \\ 4 & 6 & \dots & (2u) \end{array} \quad (2u+1) \dots (r+2), \quad t_2 = \begin{array}{cccc} 1 & 3 & \dots & (2u-1) \\ 2 & 4 & \dots & (2u) \end{array} \quad (2u+1) \dots (r+2).$$

Here $u = \mu_2 + 1$. Let R_{t_i} be the row stabiliser of t_i , and C_{t_i} the column stabiliser of t_i . To write down the polytabloid generating S^μ in this setup, we must start with an appropriate element $\omega_1 \in M^\lambda$ which is fixed by all elements of R_{t_1} . Then the corresponding ‘polytabloid’ is

$$\varepsilon_{t_1} = \omega_1 \{C_{t_1}\}^-,$$

where $\{C_{t_1}\}^-$ is the alternating sum over all elements in C_{t_1} . We can take

$$\omega_1 = \sum_{\underline{i}} v_{\underline{i}}$$

summing over all \underline{i} such that $i_\rho = 2$ for ρ in the second row of t_1 , and all other $i_\rho \in \{1, 2\}$ such that the weight of \underline{i} is λ . Note that $\lambda_2 \geq \mu_2$, so such a \underline{i} exists. (When $\lambda = \mu$ then ω_1 consists of just one basis vector.) Similarly one defines the Specht module generator ε_{t_2} from t_2 . Explicitly,

$$\{C_{t_1}\}^- = (1 - (3, 4))(1 - (5, 6)) \cdots (1 - (2u - 1, 2u)).$$

This shows that $\omega_1\{C_{t_1}\}^- = \tilde{\omega}_1\{C_{t_1}\}^-$ where $\tilde{\omega}_1$ is the sum over all $v_{\underline{i}}$ such that $i_{2\rho+1} = 1$ and $i_{2\rho+2} = 2$ for $1 \leq \rho < u$ (and $i_\rho \in \{1, 2\}$ otherwise such that the weight of \underline{i} is λ). We next apply the map j to ε_{t_1} :

$$j(\varepsilon_{t_1}) = (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes \varepsilon_{t_1} = (v_1 \otimes v_2 \otimes \tilde{\omega}_1(1 - (1, 2))) \cdot \{C_{t_1}\}^-.$$

Now, $(1 - (1, 2))\{C_{t_1}\}^- = \{C_{t_2}\}^-$ and $v_1 \otimes v_2 \otimes \tilde{\omega}_1 = \tilde{\omega}_2$. This shows that j takes ε_{t_1} precisely to ε_{t_2} .

We can now complete the inductive step of the proof. Suppose $e_{m,g}(S^\mu) \neq 0$, then $e_{m,g}(\varepsilon_{t_1}) \neq 0$ since ε_{t_1} is a generator of the Specht module (and $e_{m,g}$ is a homomorphism). Then also $j \circ e_{m,g}(\varepsilon_{t_1}) \neq 0$ since j is one-to-one. Hence by Eq. (14),

$$0 \neq j(e_{m,g}(\varepsilon_{t_1})) = e_{m,g} \circ j(\varepsilon_{t_1}) = e_{m,g}(\varepsilon_{t_2}).$$

Hence $e_{m,g}(S^{\mu+(1^2)}) \neq 0$, as required. \square

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