Topics in Ricci flow with symmetry

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Abstract

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In this thesis, we study the Ricci flow and Ricci soliton equations on Riemannian manifolds which admit a certain degree of symmetry. More precisely, we investigate the Ricci soliton equation on connected Riemannian manifolds, which carry a cohomogeneity one action by a compact Lie group of isometries, and the Ricci flow equation for invariant metrics on a certain class of compact and connected homogeneous spaces. In the first case, we prove that the initial value problem for a cohomogeneity one gradient Ricci soliton around a singular orbit of the group action always has a solution, under a technical assumption. However, this solution is in general not unique. This is a generalisation of the analogous result for the Einstein equation, which was proved by Eschenburg and Wang in their paper "Initial value problem for cohomogeneity one Einstein metrics". In the second case, by studying the corresponding system of nonlinear ODEs, we identify a class of singular behaviours for the homogeneous Ricci flow on these spaces. The singular behaviours that we find all correspond to type I singularities, which are modelled on rigid shrinking solitons. In the case where the isotropy representation decomposes into two invariant irreducible inequivalent summands, we also investigate the existence of ancient solutions and relate this to the existence and non existence of invariant Einstein metrics. Furthermore, in this special case, we also allow the initial metric to be pseudo-Riemannian and we investigate the existence of immortal solutions. Finally, we study the behaviour of the scalar curvature for this more general situation and show that in the Riemannian case it always has to turn positive in finite time, if it was negative initially. By contrast, in the pseudo-Riemannian case, there are certain initial conditions which preserve negativity of the scalar curvature.

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Statement of Originality

This thesis contains no material that has already been accepted, or is concurrently being submitted, for any degree or diploma or certificate or other qualification in this University or elsewhere. To the best of my knowledge and belief this thesis contains no material previously published or written by another person, except where due reference is made in the text.

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Contents

Introduction

Let (M, g) be a Riemannian manifold of dimension n. A one-parameter family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$ on M is said to be a *Ricci flow* with initial metric g if it satisfies the equation

$$\frac{\partial}{\partial t}(g(t)) = -2\,\mathrm{Ric}(g(t)), \tag{1}$$

with $g(0) = g$.

A motivation for (1) is that the steady case corresponds to Einstein metrics. Hence, one hopes that in good cases the Ricci flow would converge to an Einstein metric [33].

Another reason to consider this PDE is as follows. If we choose normal coordinates around a point $p \in M$, we have that at that point

$$R_{ij} = -\frac{3}{2}\Delta(g_{ij}),$$

where Δ is the Laplace-Beltrami operator. So we can see the Ricci flow as a variation of the usual heat equation.

Note that the Ricci flow is only weakly parabolic. This is due to the fact that it is invariant under the action of the whole diffeomorphism group, which is infinite-dimensional. Despite this fact, on compact manifolds we have short time existence and uniqueness of solutions to (1). This was first proved by Hamilton [33] using the Nash-Moser theorem. Subsequently, De Turck [24] gave a simpler proof, by considering an equivalent equation, which is strictly parabolic, so that one can then apply the inverse function theorem.

The Ricci flow was first introduced by Hamilton in [33], who showed that compact 3-manifolds with strictly positive Ricci curvature are spherical space forms. It is possible to normalise the Ricci flow in such a way that the volume remains constant. One can do this by considering, instead of (1), the equation

$$\frac{\partial}{\partial t}(g(t)) = -2\,\mathrm{Ric}(g(t)) + \frac{2}{n}\overline{R}(t), \tag{2}$$

where $\overline{R}(t)$ is the average of the scalar curvature $R_{g(t)}$ of $g(t)$ and is defined as

$$\overline{R}(t) = \frac{\int_M R_{g(t)} d\mu}{\int_M d\mu},$$

where $d\mu$ is the volume form. Equation (2) is called *normalised Ricci flow.* Hamilton showed that if the initial metric has strictly positive Ricci curvature, this condition is preserved under the Ricci flow in dimension three. Then, he proved that on compact 3-manifolds with strictly positive Ricci curvature the solution to the normalised Ricci flow exists for every $t > 0$ and for $t \to \infty$ it converges to a metric with constant positive curvature. His proof is based on three a priori estimates. The first shows that the Ricci curvature remains positive, the second shows that the eigenvalues of the Ricci tensor at each point approach each other, and the third shows that it is possible to compare the curvature at distant points, because the gradient of the scalar curvature tends to zero. Moreover, all three of these estimates follow from the maximum principle for parabolic equations.

In 1986, Hamilton [34] extended this result by proving that a compact 4-manifold with positive curvature operator is diffeomorphic to either the sphere S^4 or the projective space $\mathbb{RP}^4$. The Ricci flow was then used by Perelman [46] in 2002 to prove the Poincare conjecture. In these cases, the Ricci flow has been used to get information about the topology of the underlying manifold. The strategy is to stop the flow when a singularity has formed in finite time and then perform surgery on the evolved manifold, by eliminating the singular regions, and continue the flow. One hopes that this process is finite and that after a finite number of surgeries, we obtain a flow which exists for all times and converges to a limiting flow. Then, provided that we understand the structure of the singularities which have formed, we can reconstruct the topology of the original manifold from the limiting flow and the singular regions which have been removed.

We would also like to mention that a particular normalisation of the Ricci flow in dimension 2 has also been used to transform metrics conformally into metrics of constant curvature, giving a proof of the 2-dimensional uniformization theorem, see[21].

More recently, the Ricci flow was used by Brendle and Schoen to prove the differentiable sphere theorem [12, 13, 14, 15]. They proved that every compact Riemannian manifold of dimension $n \geq 4$ which is weakly $\frac{1}{4}$-pinched in the pointwise sense has to be diffeomorphic to either a sphere or a locally symmetric space. They also proved that if the manifold is strictly $\frac{1}{4}$-pinched in the pointwise sense, then it has to be diffeomorphic to a sphere. These results follow from a more general result due to Brendle [12], in which he proves that any compact Riemannian manifold (M, g) of dimension $n \geq 4$ and such that $M \times \mathbb{R}$ has positive isotropic curvature is diffeomorphic to a spherical space form.

Some special solutions to (1) are as follows. Suppose that the initial metric $g(0)$ is Einstein, i.e. there exists $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric}(g(0)) = \lambda g(0).$$

Then, a solution to the Ricci flow with initial metric $g(0)$ is given by

$$g(t) = (1 - 2\lambda t)g(0).$$

Hence, if $\lambda > 0$, $g(t)$ will exist up to a finite time $T = \frac{1}{2\lambda}$ and, as $t \to T$, the space will shrink to a round point. By this we mean, as we approach the final time T, the space looks asymptotically like a round sphere. On the other hand, if $\lambda < 0$, $g(t)$ will exist for $t \in [0, +\infty)$ and, as $t \to +\infty$, the space expands homothetically for all time. Finally, if $\lambda = 0$, the metric $g(0)$ is a fixed point of the flow.

The natural symmetries of the Ricci flow are given by diffeomorphisms and homotheties. In fact, given any diffeomorphism φ of (M, g), we have that

$$\mathrm{Ric}(\varphi^* g) = \varphi^* \,\mathrm{Ric}(g).$$

Moreover, given any real constant σ, we have that

$$\mathrm{Ric}(\sigma g) = \mathrm{Ric}(g).$$

We can then define another class of self-similar solutions to the Ricci flow, which generalises the ones defined by Einstein metrics. These are called *Ricci soliton flows* and they are solutions which evolve by the natural symmetries of the Ricci flow. More precisely, a solution $g(t)$ to the Ricci flow is a Ricci soliton flow if there exist a one-parameter family of diffeomorphisms $\varphi(t)$ and a positive function $\sigma(t)$ such that

$$g(t) = \sigma(t)\varphi(t)^* g(0), \tag{3}$$

for all t such that a solution exists. For such a solution, by differentiating the above equation with respect to t, it is possible to show that, at each time t, the initial metric $g(0)$ has to satisfy the following PDE:

$$\mathrm{Ric}(g(0)) + \frac{1}{2}\mathcal{L}_{\widetilde{X}} g(0) + \frac{\dot{\sigma}(t)}{2} g(0) = 0,$$

where $\widetilde{X}$ is a time dependent vector field given by $\sigma(t)X$, where X is such that $X_{\varphi(t)} = \dot{\varphi}(t)$. The Ricci soliton flow is said to be *shrinking*, *expanding* or *steady* at time t if $\dot{\sigma}(t) < 0$, $\dot{\sigma}(t) > 0$ or $\dot{\sigma}(t) = 0$, respectively. In general, we can prove that $\sigma(t)$ is linear in t every time the Ricci flow $g(t)$ defined by (3) is unique among Ricci soliton flows with initial condition $g(0)$, cf. [22]. We could also consider the above equation on its own for t fixed and look for a triple (M, g, X), where (M, g) is a complete Riemannian manifold and X is a vector field on M, such that

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g + \frac{\epsilon}{2} g = 0, \tag{4}$$

for some $\epsilon \in \mathbb{R}$. To avoid confusion, we point out that for us a *Ricci soliton* will be a solution to (4). The Ricci soliton is called shrinking, expanding or steady, if $\epsilon < 0$, $\epsilon > 0$ or $\epsilon = 0$, respectively. This equation may be written as

$$\mathrm{Ric}(g) + \delta^*\omega + \frac{\epsilon}{2} g = 0,$$

where $\omega = X^\flat$ and δ^* is the symmetrized covariant derivative. If the dual form of X is exact, i.e. if there exists a smooth function u on M such that $X^\flat = du$, the Ricci soliton is called *gradient Ricci soliton* and equation (4) takes the following form:

$$\mathrm{Ric}(g) + \mathrm{Hess}(u) + \frac{\epsilon}{2} g = 0, \tag{5}$$

where $\mathrm{Hess}(u)$ is the Hessian of u. Note that if (M, g, X) satisfies (4), then it generates a solution to (1), which is of the type described by equation (3). In fact, considering the one-parameter family of vector fields on M given by

$$Y(t) = \frac{X}{1 + \epsilon t},$$

and integrating it to a one-parameter family $\varphi(t)$ of diffeomorphisms of M, we have that the one-parameter family of Riemannian metrics

$$g(t) = (1 + \epsilon t)\varphi(t)^* g$$

evolves under the Ricci flow equation, with initial metric $g(0) = g$.

Note that the Ricci soliton equation is a generalisation of the Einstein condition. In fact, if we take X to be the zero vector field in (4), we recover the Einstein equation for the metric g. So, in general, we say that a Ricci soliton is trivial if $\mathcal{L}_X g = 0$. In particular, trivial Ricci solitons are Einstein metrics.

Ricci solitons are also important, because they have motivated the discovery of monotonicity formulas for the Ricci flow [46], which have many geometric applications. Moreover, they often appear as limits of dilations of singularities in the Ricci flow [36]. In fact, suppose that we have a Ricci flow on a closed manifold which develops a singularity in finite time. Then, a sequence of suitably rescaled Ricci flows converges in the smooth Cheeger-Gromov sense to a Ricci flow which is defined on a time interval $(-\infty, a)$, with $a < \infty$. These solutions are called *ancient*. In some cases, it has been shown that this limit is a gradient Ricci soliton [27]. For example, in [1], Angenent and Knopf proved that on a sphere with a rotationally symmetric metric, there is a class of initial conditions such that the Ricci flow develops a neckpinch singularity in finite time. They showed that this singularity is modelled by a shrinking cylinder. Later, Gu and Zhu [32] showed that on the same manifold there is an initial metric such that the Ricci flow develops a degenerate neckpinch singularity. Isenberg and Garfinkle [31] proved numerically that this singularity is modelled by the Bryant soliton, which is a rotationally symmetric steady soliton on $\mathbb{R}^{n+1}$.

As we have uniqueness of the solution to the Ricci flow equation on closed manifolds, the invariance by diffeomorphisms of the Ricci tensor implies that the Ricci flow preserves symmetries of the initial metric. In particular, this is always true for homogeneous spaces, as the Ricci flow reduces to an ODE. It is then natural to investigate equations (1) and (4) on spaces which admit a certain degree of symmetry. These might include homogeneous and cohomogeneity one Riemannian manifolds. On this kind of spaces, we can restrict our attention to those Riemannian metrics which are invariant under the action of the Lie group and, as we have already mentioned above, this property will be preserved under the Ricci flow. In some cases, this allows us to reduce equations (1) and (4) to systems of nonlinear ODEs. So, in general, imposing some symmetries on the underlying manifold is a way of reducing the complexity of the PDEs corresponding to the Ricci flow and Ricci soliton equations, which are difficult to analyse.

In particular, in the case of Ricci flow of invariant metrics on compact homogeneous spaces, this topic turns out to be closely related to the existence and non-existence of invariant Einstein metrics on these spaces. A rough classification of homogeneous Einstein metrics is as follows. Let (M, g) be a homogeneous Einstein metric with scalar curvature R_g. Then,

- If $R_g > 0$, then M is compact with finite fundamental group;
- If $R_g = 0$, then M is flat;
- If $R_g < 0$, then M is non compact.

From this classification, we immediately have that a homogeneous space G/K, with G compact and infinite fundamental group, admits an invariant Einstein metric if and only if the space is flat. A characterisation of these spaces can be found in [7, Proposition 7.5].

However, a compact and simply connected homogeneous space always carries a homogeneous metric with positive Ricci curvature, cf. [49]. Therefore, it was natural to ask whether every compact and simply connected homogeneous space carries a homogeneous Einstein metric. Wang and Ziller [49] produced a counter example to this conjecture. They showed that the 12-dimensional manifold $SU(4)/SU(2)$ does not admit any homogeneous Einstein metric. In their paper, Wang and Ziller also proved a general existence theorem for a certain class of compact homogeneous Riemannian manifolds. They use a variational method, which characterises G-invariant Einstein metrics of volume one as the critical points of the total scalar curvature functional on G-invariant Riemannian metrics of unit volume:

$$T : g \longmapsto T(g) := \int_M R_g \, d\mu_g.$$

Let G be a compact and connected Lie group and K a closed connected subgroup such that G/K is effective. Wang and Ziller proved that K is a maximal connected subgroup of G if and only if the functional T restricted to the space of homogeneous Riemannian metrics of volume one is bounded from above and proper. In this case, this functional attains a maximum, which corresponds to a G-invariant Einstein metric. This variational approach has been used extensively by Böhm, Wang and Ziller [7, 10].

This thesis is divided into three chapters. In the first chapter, we are going to investigate the gradient Ricci soliton equation (5) in the cohomogeneity one setting. Note that by the work of Petersen and Wylie [47], we have that the maximal amount of symmetry on a nontrivial gradient Ricci soliton is given by a cohomogeneity one action. Also observe that a motivation to study the gradient case is that gradient shrinking Ricci solitons arise as blowing up limits of Ricci flows which develop a type I singularity in finite time [27]. In particular, we consider the initial value problem for cohomogeneity one gradient Ricci solitons around a singular orbit, generalising a result of Eschenburg and Wang [28] for the Einstein case. Our main theorem in this chapter is the following.

Theorem 0.0.1. *Let (M, g) be a connected Riemannian manifold endowed with a cohomogeneity one action by a compact Lie group of isometries G. Let $Q = G/H$ be a singular orbit of codimension $k+1$, $k \geq 1$. Suppose that H is the stabiliser of $q \in Q$ under the action of G. Then, H acts linearly with cohomogeneity one on $V = \mathbb{R}^{k+1}$, which is the normal space at $q \in Q$, and the Lie algebra of G splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_-$, where $\mathfrak{h}$ is the Lie algebra of H. Let $v_0 \in S^k$ have isotropy group K with respect to the H-action. Then, G/K is a principal orbit for the action. Assume that V and $\mathfrak{p}_-$ have no irreducible common factors as K- representations. Then, given any G-invariant metric g_Q on Q and shape operator $L_1 : NQ \to \mathrm{Sym}^2(T^*Q)$, where $NQ = G \times_H V$ is the normal bundle over Q, there exists a G-invariant gradient Ricci soliton on some open disk bundle of NQ.*

This result has already been published in the Journal of Geometry and Physics: [17].

The second chapter is dedicated to the study of the Ricci flow equation for invariant metrics on a certain class of compact and connected homogeneous Riemannian manifolds such that the isotropy representation decomposes into invariant irreducible inequivalent summands. One of the main goals of this chapter is to relate the behaviour of the Ricci flow of invariant metrics to the existence and non-existence of invariant Einstein metrics on compact homogeneous spaces. As Ricci flows can only converge to Einstein metrics, it is interesting to see what singular behaviours can occur when we consider the Ricci flow of homogeneous metrics on the non-existence examples. Furthermore, as the equation becomes more tractable, we hope that the study of the Ricci flow of homogeneous metrics might be useful to understand the Ricci flow in higher dimensions. In the case of two summands, we were able to prove the following theorems.

Theorem 0.0.2. *Let G/K be a compact and connected homogeneous space such that the isotropy representation decomposes into two inequivalent $\mathrm{Ad}_{|_K}$-invariant summands. Then, the Ricci flow starting at any invariant Riemannian metric develops a type I singularity in finite time. If K is maximal in G, then the singular time is always characterised by the shrinking of the whole space to a point. If K is not maximal in G, then another singular behaviour can occur. More precisely, suppose that there exists an intermediate Lie group H, with $K < H < G$. Then, the singular time is characterised either by the shrinking of G/K or by the shrinking of H/K and the convergence of G/K to G/H in the Hausdorff-Gromov sense. Moreover, the shrinking of G/K to a point implies that the homogeneous space carries a G-invariant Einstein metric.*

Before stating the next theorem, we would like to mention that the phase space in this case is given by $\{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 > 0\}$. Then, invariant Einstein metrics are lines through the origin in $\mathbb{R}^2$. When we talk about initial conditions, we mean a specific point in the phase space.

Theorem 0.0.3. *Let G/K be as in theorem* 0.0.2. *If K is maximal in G, then ancient solutions exist if and only if the homogeneous space carries at least two G-invariant Einstein metrics. More precisely, we have an ancient solution every time the initial condition lies between two G-invariant Einstein metrics. If K is not maximal in G, then ancient solutions exist if and only if the homogeneous space carries at least one G-invariant Einstein metric. In particular, ancient solutions occur every time the initial condition lies between two G-invariant Einstein metrics, or when the singular behaviour is characterised by the shrinking of H/K to a point.*

We were also able to generalise theorem 0.0.2 to the three summands case, when there exists an intermediate Lie group H such that H/K is isotropy irreducible and every G-invariant Riemannian metric on G/K is given by a submersion metric

$$H/K \to G/K \to G/H.$$

In this case, as we approach the final time, more singular behaviours can occur and these are all determined by the Lie algebra structure of G. Finally, we conjecture that this theorem might hold for a general number of summands and provide some evidence on why this conjecture should be true.

The third chapter expands the section about compact and connected homogeneous spaces whose isotropy representation decomposes into two invariant irreducible inequivalent summands. This chapter is divided into two main parts. In the first part, in order to complete the study carried out in the previous chapter, we allow the initial metric to have indefinite or negative signature. So we consider initial profiles which are given by pseudo-Riemannian metrics. We investigate the formation of singularities and the existence of *immortal solutions.* These are solutions defined on $[-T, +\infty)$, with $T > 0$. In particular, we prove the following theorem.

Theorem 0.0.4. *Every time the initial condition lies between two invariant Einstein* *metrics, the Ricci flow has an immortal solution, which converges to an invariant Einstein metric, as the time tends to* $+\infty$.

In the second part, we study the behaviour of the scalar curvature. We show that it always has to turn positive, when a singularity forms, so that the only solutions which preserve negativity of the scalar curvature are the immortal ones. In particular, we prove that given any invariant Riemannian metric, the Ricci flow always forces the scalar curvature to turn positive in finite time, if it was negative initially, which then implies that a singularity will develop in finite time.

Chapter 1

Initial value problem for cohomogeneity one gradient Ricci solitons

1.1 Introduction

In this chapter, we are going to investigate the gradient Ricci soliton equation in the following context. Let (M, g) be a connected Riemannian manifold endowed with a cohomogeneity one action by a compact Lie group G of isometries. We then have that the orbits with maximal dimension are hypersurfaces in M and they are called *principal orbits*. We can also have orbits with higher codimension which are called *singular*. With this kind of action, the orbit space is one-dimensional and can be an interval (open, closed or semi-open), $\mathbb{R}$ itself or the circle S^1, depending on the number of singular orbits, cf. [4]. Many examples of cohomogeneity one gradient Ricci solitons have been constructed; e.g., the Bryant soliton [39, 22], the cigar soliton [35], the $U(n)$-symmetric soliton on $\mathbb{C}^n$ discovered by Cao [18, 19] and the Kähler generalisations [40, 51, 23, 26, 29]. In [28], Eschenburg and Wang consider local existence and uniqueness of a smooth G-invariant

Einstein metric around a singular orbit in the cohomogeneity one setting. They prove, under a technical assumption, that, given any G-invariant Riemannian metric and any shape operator in a neighbourhood of a singular orbit Q, there always exists an invariant Einstein metric around Q with any prescribed sign of the Einstein constant. Here, we generalise the Einstein case to the case of gradient Ricci solitons. In particular, we are going to prove the following theorem.

Theorem 1.1.1. *Let (M, g) and G be as above. Let $Q = G/H$ be a singular orbit of codimension $k+1$, $k \geq 1$. Suppose that H is the stabiliser of $q \in Q$ under the action of G. Then, H acts linearly with cohomogeneity one on $V = \mathbb{R}^{k+1}$, which is the normal space at $q \in Q$, and the Lie algebra of G splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_-$, where $\mathfrak{h}$ is the Lie algebra of H. Let $v_0 \in S^k$ have isotropy group K with respect to the H-action. Then, G/K is a principal orbit for the action. Assume that V and $\mathfrak{p}_-$ have no irreducible common factors as K- representations. Then, given any G-invariant metric g_Q on Q and shape operator $L_1 : NQ \to \mathrm{Sym}^2(T^*Q)$, where $NQ = G \times_H V$ is the normal bundle over Q, there exists a G-invariant gradient Ricci soliton on some open disk bundle of NQ.*

We write the gradient Ricci soliton condition around Q as an initial value problem with initial data given by a G-invariant metric and shape operator L_1 around Q. We note that the smoothness condition of the metric implies that L_1 is a smooth H-equivariant linear map from V to $S^2(\mathfrak{p}_-)$, which then implies that $\mathrm{tr}\,(L_1) = 0$, so Q must be a minimal submanifold in M. As we are working around a singular orbit, we only need to solve the gradient Ricci soliton equation in the directions tangent and orthogonal to the orbits. Moreover, we can write the initial value problem as a system of ordinary non-linear differential equations with a singular point at the origin. We solve this system using the same technique as in [28], which consists of applying the method of asymptotic series to find a solution and then showing that this solution is in fact a smooth G-invariant gradient Ricci soliton. We can always show existence, but the initial data given are not sufficient to ensure uniqueness and the indeterminacy of the problem, which is the same as in the Einstein case, is always finite and can be computed using representation theory. Finally, the technical assumption about the irreducible summands of V and $\mathfrak{p}_-$ is motivated by [28] and, as it is explained in [28, Remark 2.7], it appears quite natural in the context of the Kaluza-Klein construction.

1.2 Homogeneous Riemannian manifolds

In this section, we will recall the definition of a homogeneous Riemannian manifold.

Definition 1.2.1. A Riemannian manifold (M, g) is said to be *homogeneous* if its group of isometries $I(M, g)$ acts transitively on it. (M, g) is said to be G*-homogeneous* if there exists a closed Lie group G of isometries which acts transitively on it.

Remark 1.2.2. The same Riemannian manifold can be homogeneous under different Lie groups.

Let (M, g) be a G-homogeneous Riemannian manifold. Recall that the isotropy group of a point $p \in M$ is defined as

$$K = \{f \in G \,|\, f(p) = p\}.$$

Then, we can define the *isotropy representation* of K to be

$$\chi : K \longrightarrow GL(T_pM),$$
$$f \longmapsto T_pf.$$

Since every isometry is determined by giving the image of a point p and the tangent map at that point, the isotropy representation defined above is injective. Moreover, as G is a closed subgroup of $I(M, g)$, K is a compact subgroup of $G \cap I_p(M, g)$, where $I_p(M, g) = \{f \in I(M, g) \,|\, f(p) = p\}$, and M is diffeomorphic to the quotient G/K. In particular, M is compact if and only if G is compact.

Let now $\mathfrak{g}$ be the Lie algebra of G and exp be the exponential map of $\mathfrak{g}$. Every $X \in \mathfrak{g}$ generates a one-parameter subgroup of G, which is given by $\exp(tX)$. We will now identify X with the vector field on M generated by $\exp(tX)$. Through this identification, we can identify $\mathfrak{g}$ with the set of those Killing vector fields which generate one-parameter subgroups of G. In this way, the Lie subalgebra $\mathfrak{k}$ of K is identified with those Killing vector fields which vanish at p.

Let $\mathrm{Ad}_{|_K}$ be the adjoint representation of G on its Lie algebra restricted to K. Then, there exists an $\mathrm{Ad}_{|_K}$-invariant complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

We can then identify $\mathfrak{p}$ with T_pM, by assigning to a Killing vector field its value at p. In this way, the isotropy representation χ is identified with the adjoint representation $\mathrm{Ad}_{|_K}$ of K on $\mathfrak{p}$. We then have that every G-invariant Riemannian metric on G/K is uniquely determined by an $\mathrm{Ad}_{|_K}$-invariant scalar product on $\mathfrak{p}$.

1.3 The cohomogeneity one Ricci soliton equation

Following the approach and notation of [28] and [23], we will now recall the Ricci soliton equation in the cohomogeneity one setting.

Let $(M, \widehat{g})$ be a connected Riemannian manifold of dimension $n+1$ and let G be a compact Lie group which acts on M by isometries and with cohomogeneity one.

Now, let us choose a unit speed geodesic

$$\gamma : I \longrightarrow M,$$

where $I \subset \mathbb{R}$ is an open interval, and take γ such that it intersects all the principal orbits orthogonally. Then, it is possible to define an equivariant diffeomorphism

$$\begin{aligned}\Phi : I \times G/K &\longrightarrow M_0 \subset M,\\ (t, g\cdot K) &\longmapsto g\cdot\gamma(t),\end{aligned}$$

where K is the isotropy group of $\gamma(t)$ with respect to the G-action. Hence, $\Phi(t, G/K)$ is the principal orbit P_t passing through $\gamma(t)$ and M_0 is an open dense subset in M which is the union of all the principal orbits.

Every orbit P_t is naturally equipped with a G-invariant Riemannian metric, which depends on t. Hence, through Φ we obtain a family $g(t)$ of G-invariant metrics on the homogeneous space P, where P denotes an abstract copy of the principal orbit G/K. Moreover, the map Φ sends $I \times \{p\}$, with $p \in P$, to the geodesic through p orthogonal to the principal orbits, and we have that the canonical parametrisation of I corresponds to the arclength parametrisation of the geodesic. Then, if we pullback the metric $\widehat{g}$ on M through Φ we get

$$\Phi^*(\widehat{g}) = dt^2 + g_t,$$

where g_t is a family of G-invariant Riemannian metrics on P.

Let $\widehat{\nabla}$ and $\widehat{\mathrm{Ric}}$ denote the Levi-Civita connection and the Ricci tensor of the manifold $(M, \widehat{g})$, respectively. Let ∇_t and Ric_t denote the Levi-Civita connection and the Ricci tensor of (P_t, g_t), respectively. Let L_t be the shape operator on P_t defined by

$$L_t(X) = \widehat{\nabla}_X N,$$

where X is a vector field on P_t and $N = \Phi_*(\frac{\partial}{\partial t})$ is a unit normal G-invariant vector field along P_t such that $\widehat{\nabla}_N N = 0$. We then have a family $(L_t)_{t\in I}$ of G-invariant, g_t-symmetric endomorphisms of the tangent space of P. In particular, the trace of L_t is constant along P_t. Moreover, the following equality holds

$$\dot{g}_t(X, Y) = 2g_t(L_t(X), Y), \tag{1.1}$$

for every pair of vector fields X, Y on P_t and for every $t \in I$.

Consider now the Ricci soliton equation for $(M, \widehat{g}, \widehat{\omega})$

$$\widehat{\mathrm{Ric}}(\widehat{g}) + \widehat{\delta}^*\widehat{\omega} + \frac{\epsilon}{2}\widehat{g} = 0.$$

First of all, we can take the form $\widehat{\omega}$ to be G-invariant, or, if we are dealing with gradient Ricci solitons, we can take the potential function to be G-invariant (cf. [23], p. 4). Hence, if we consider the pull-back of $\widehat{\omega}$ through Φ, we obtain

$$\Phi^*\widehat{\omega} = \xi(t)dt + \omega_t, \tag{1.2}$$

where $\xi = \xi(t)$ is a function on I and ω_t is a one-parameter family of G-invariant one-forms on P.

Let $\mathfrak{X}(P_t)$ denote all the vector fields on P_t. We then have the following proposition.

Proposition 1.3.1 ([23]). *Let $(M^{n+1}, \widehat{g})$ be a connected Riemannian manifold which admits a cohomogeneity one action by a compact Lie group G of isometries of $\widehat{g}$. Let $\widehat{\omega}$ be a G-invariant one-form on M. Under the parametrisation induced by a unit speed geodesic orthogonal to the principal orbits, the Ricci soliton equation for $\widehat{g}$ and the vector* *field dual to $\widehat{\omega}$ is given by*

$$
\begin{aligned}
&-(\delta^{\nabla_t} L_t)^\flat - d(\operatorname{tr}(L_t)) + \frac{1}{2}\dot{\omega}_t - \omega_t \circ L_t = 0,\\
&-\operatorname{tr}(\dot{L}_t) - \operatorname{tr}(L_t^2) + \dot{\xi}(t) + \frac{\epsilon}{2} = 0,\\
&\operatorname{Ric}_t(X,Y) - \operatorname{tr}(L_t) g_t(L_t(X),Y) - g_t(\dot{L}_t(X),Y)\\
&+\xi(t) g_t(L_t(X),Y) + \delta_t^* \omega_t(X,Y) + \frac{\epsilon}{2} g_t(X,Y) = 0,
\end{aligned}
$$

for all $X, Y \in \mathfrak{X}(P_t)$ and $t \in I$, where, viewing L_t as an endomorphism of $\mathfrak{X}(P_t)$, the operator $\delta^{\nabla_t} : \mathfrak{X}^*(P_t) \otimes \mathfrak{X}(P_t) \to \mathfrak{X}(P_t)$ is the codifferential.

Conversely, if g_t and ω_t are one-parameter families of metrics and one-forms on P_t, respectively, and $\xi = \xi(t)$ is a smooth function on I such that the above system is satisfied with L_t defined by $\dot{g}_t(X,Y) = 2g_t(L_t(X),Y)$ for all $X, Y \in \mathfrak{X}(P_t)$, then $\widehat{g} = dt^2 + g_t$ and $\widehat{\omega} = \xi(t)dt + \omega_t$ give a local Ricci soliton on M_0.

If we are looking for gradient Ricci solitons, that is when there exists a G-invariant smooth function u such that $\widehat{\omega} = du$, equation (1.2) becomes

$$\Phi^* \widehat{\omega} = \dot{u}(t)dt,$$

and the Ricci soliton equation in the cohomogeneity one setting is equivalent to the following system

$$-(\delta^{\nabla_t} L_t)^\flat - d(\operatorname{tr}(L_t)) = 0, \tag{1.3}$$

$$-\operatorname{tr}(\dot{L}_t) - \operatorname{tr}(L_t^2) + \ddot{u}(t) + \frac{\epsilon}{2} = 0, \tag{1.4}$$

$$
\begin{aligned}
&\operatorname{Ric}_t(X,Y) - \operatorname{tr}(L_t) g_t(L_t(X),Y) - g_t(\dot{L}_t(X),Y) + \dot{u}(t) g_t(L_t(X),Y)\\
&+\frac{\epsilon}{2} g_t(X,Y) = 0,
\end{aligned} \tag{1.5}
$$

for all $X, Y \in \mathfrak{X}(P_t)$ and $t \in I$, where $u(t)(p) = u \circ \Phi(t,p)$, for all $p \in P_t$.

1.4 Smoothness of tensors around a singular orbit

In this section, following [28], we will discuss briefly the smoothness criterion for the metric $\widehat{g}$ and the one-form $\widehat{\omega}$, in the case when there is a special orbit.

Let $(M, \widehat{g})$ be a connected $(n+1)$-dimensional Riemannian manifold and G a compact Lie group which acts on M by isometries of $\widehat{g}$ and with cohomogeneity one. Let $Q = G \cdot q$ be a singular orbit of codimension $k+1$, with $k \geq 1$, with isotropy group $H = G_q$. As $Q = G/H$ is a homogeneous space, the Lie algebra $\mathfrak{g}$ of G decomposes in the following way:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_-, \tag{1.6}$$

where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{p}_-$ is the $\mathrm{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$, which can be identified with the tangent space T_qQ.

Let $V = T_qM/T_qQ \backsimeq \mathbb{R}^{k+1}$ be the normal space at q of Q, on which H acts linearly with cohomogeneity one, i.e. it acts transitively on the sphere $S^k = H/K$, where $K \subset H$. We can identify a tubular neighbourhood of Q with the total space of the normal bundle $NQ = G \times_H V$ of Q. We have that

$$T(NQ)_{|V} = V \times (V \oplus \mathfrak{p}_-).$$

In fact, using the reductive complement $\mathfrak{p}_-$ of $\mathfrak{h}$ in $\mathfrak{g}$, we can define a G-invariant connection on NQ, which induces a splitting of the tangent space to NQ into horizontal and vertical parts:

$$T(NQ) = \pi^*NQ \oplus \pi^*TQ,$$

where $\pi : NQ \to Q$ is the bundle map, and the two pull-back bundles are trivial on V, which can be viewed as the fibre of NQ over $q \in Q$. Hence, a smooth G-invariant symmetric bilinear form a is determined by an H-equivariant smooth map

$$a : V \longrightarrow \mathrm{Sym}^2(V \oplus \mathfrak{p}_-).$$

Let W be the vector space of all smooth H-equivariant maps $L : S^k \to \mathrm{Sym}^2(V \oplus \mathfrak{p}_-)$. We then have that, for $v_0 \in S^k$, the evaluation map

$$\mathrm{ev} : W \longrightarrow \mathrm{Sym}^2(V \oplus \mathfrak{p}_-)^K,$$
$$L \longmapsto \mathrm{ev}(L) = L(v_0),$$

is a linear isomorphism. Here, $\mathrm{Sym}^2(V \oplus \mathfrak{p}_-)^K$ denotes the elements of $\mathrm{Sym}^2(V \oplus \mathfrak{p}_-)$ which are K-invariant, where $K = H_{v_0}$. Let W_m be the subspace of W consisting of all maps which are restrictions to S^k of H-equivariant homogeneous polynomials of degree m. We then have a necessary and sufficient condition for a to be smooth.

Lemma 1.4.1 ([28])**.** *Let $t \mapsto a_t$, where $a_t : S^k \to \mathrm{Sym}^2(V \oplus \mathfrak{p}_-)^K$ for all $t \in [0, \infty)$, be a smooth curve, i.e. at zero the right-hand derivatives of all orders exist and are continuous from the right. Let $\sum_p a_p t^p$ be its Taylor expansion at zero. Then the map a defined by*

$$a : V \backslash \{0\} \longrightarrow \mathrm{Sym}^2(V \oplus \mathfrak{p}_-),$$
$$v \longmapsto a(v) = a_{|v|}\left(\frac{v}{|v|}\right)$$

can be extended smoothly at zero if and only if $a_p \in ev(W_p)$ for all $p \geq 0$.

Motivated by [28], we now assume that the representations of K on $\mathfrak{p}_-$ and V have no irreducible common factors. As a consequence of this, we have that

$$\mathrm{Sym}^2(V \oplus \mathfrak{p}_-)^K = \mathrm{Sym}^2(V)^K \oplus \mathrm{Sym}^2(\mathfrak{p}_-)^K, \tag{1.7}$$

and each W_m splits as $W_m^+ \oplus W_m^-$, where the polynomials in W_m^+ take values in $\mathrm{Sym}^2(V)$

and the ones in W_m^- take values in $\mathrm{Sym}^2(\mathfrak{p}_-)$.

The smoothness criterion for $\widehat{\omega}$ is obtained essentially in the same way as for the metric $\widehat{g}$. Under the above assumption, we have that on a tubular neighbourhood around Q, $\widehat{\omega}$ is determined by an H-equivariant map

$$\widehat{\omega} : V \longrightarrow V^* \oplus \mathfrak{p}_-^*.$$

In this case, W_m is defined as the space of H-equivariant maps $L : V \to V^* \oplus \mathfrak{p}_-^*$ which are restrictions to the unit sphere S^k of homogeneous polynomials of degree m. The necessary and sufficient condition for $\widehat{\omega}$ of the form (1.2) to be smooth is that its pth Taylor coefficient, viewing $\widehat{\omega}$ as function of t, lives in $\mathrm{ev}(W_p)$, for all $p \geq 0$.

Finally, if we consider gradient Ricci solitons with potential function u, in the case of a special orbit the smoothness criterion for $\widehat{\omega} = du$ implies that $u(t)$ must be even in t. In fact, around zero, u is given by

$$u(t) = \sum_{t=0}^{\infty} \frac{u_p}{p!} t^p, \tag{1.8}$$

where

$$u_p = \frac{d^p}{dt^p} u(t) \Big|_{t=0}.$$

The smoothness condition implies that u_p must be a homogeneous polynomial of degree p on the sphere S^k, on which H acts transitively. Moreover, we can take u to be H-invariant. We now show that $u_p = 0$ if p is odd. Given $x \in S^k$, there exists $h \in H$ such that $h \cdot x = -x \in S^k$. If p is odd we have that

$$-u_p(x) = u_p(-x) = u_p(h \cdot x) = u_p(x) \Longrightarrow u_p(x) = 0.$$

Hence, if p is odd, $u_p(x) = 0$, for all $x \in S^k$. This implies that $u(t)$ given by (1.8) is even in t.

1.5 Initial value problem for gradient Ricci solitons around a singular orbit

First of all note that by [23, Proposition 2.17], if we are looking for gradient Ricci solitons in the case when there is a singular orbit, instead of considering the system (1.3)-(1.5), we can consider the following system

$$\frac{d^3}{dt^3}u(t) + \operatorname{tr}(L_t)\ddot{u}(t) + \operatorname{tr}(\dot{L}_t)\dot{u}(t) - 2\ddot{u}(t)\dot{u}(t) - \epsilon\dot{u}(t) = 0 \tag{1.9}$$

$$\begin{aligned} &\operatorname{Ric}_t(X,Y) - \operatorname{tr}(L_t)g_t(L_t(X),Y) - g_t(\dot{L}_t(X),Y) + \dot{u}(t)g_t(L_t(X),Y) \\ &+ \frac{\epsilon}{2}g_t(X,Y) = 0, \end{aligned} \tag{1.10}$$

together with (1.1). Note that equation (1.9), which can be viewed as an equation in $\dot{u}(t)$, is the first integral which arises from the contracted second Bianchi identity and was observed in [39, p. 242] and more generally in [20, p. 123] and [36, pp. 84-85]. Moreover, as we saw in the previous page, the smoothness condition on the function u implies that

$$\dot{u}(0) = 0.$$

Using the Ricci endomorphisms r_t on P_t, defined by

$$\operatorname{Ric}_t(X,Y) = g_t(r_t(X),Y),$$

for all $X, Y \in \mathfrak{X}(P_t)$, equation (1.10) becomes

$$r_t - \operatorname{tr}(L_t)L_t - \dot{L}_t + \dot{u}(t)L_t + \frac{\epsilon}{2}\,\mathrm{I} = 0, \tag{1.11}$$

where I is the identity matrix.

Let $\mathfrak{h}$ in (1.6) decompose in the following way

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_+,$$

and let $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, so that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Hence, $\mathfrak{p}$ is the tangent space at a point to the principal orbit G/K, while $\mathfrak{p}_+$ and $\mathfrak{p}_-$ can be identified with the tangent spaces to H/K and $G/H = Q$, respectively.

As we are working around the singular orbit Q, by assumption (1.7), we have that g_t and L_t split in $+$ and $-$ parts. Hence, following [28], let us choose $x(t), \eta(t) \in \mathrm{End}(\mathfrak{p})^K$ preserving the splitting of $\mathfrak{p}$ and such that

$$\begin{aligned} g_t &= t^2 x_+(t) \oplus x_-(t), \\ L_t &= \left(\frac{1}{t}\,\mathrm{I}_+ + \eta_+(t)\right) \oplus \eta_-(t), \end{aligned}$$

with initial conditions given by

$$\begin{aligned} & x(0) = \mathrm{I}, \\ & \eta_+(0) = 0 \quad \text{and} \quad \eta_-(0) = L_1(v_0), \end{aligned}$$

where L_1 is the shape operator of the singular orbit Q, which is an H-equivariant linear map from $V \longrightarrow \mathrm{Sym}^2(\mathfrak{p}_-)$, which in particular implies that the trace of L_1 vanishes, i.e. Q is minimal in M. We will now drop the t-dependence in order to simplify the notation. In these new variables, equations (1.1), (1.9) and (1.11) become

$$\dot{x} = 2x\eta,$$
$$\frac{d^3}{dt^3}u + \frac{k}{t}\ddot{u} + \mathrm{tr}\,(\eta)\ddot{u} + \mathrm{tr}(\dot{\eta})\dot{u} - \frac{k}{t^2}\dot{u} - 2\ddot{u}\dot{u} - \epsilon\dot{u} = 0,$$
$$\dot{\eta} = -\frac{k}{t^2}\,\mathrm{I}_+ + \frac{1}{t^2}\,\mathrm{I}_+ - \frac{k}{t}\eta - \frac{1}{t}\,\mathrm{tr}\,(\eta)\,\mathrm{I}_+ + \frac{1}{t}\dot{u}\,\mathrm{I}_+ + r - \mathrm{tr}\,(\eta)\eta + \dot{u}\eta + \frac{\epsilon}{2}\,\mathrm{I}\,.$$

It is convenient to change variables again using

$$y = x\eta,$$

so that we do not have to deal with the quadratic term $x\eta$. We then obtain

$$\dot{x} = 2y, \tag{1.12}$$
$$\frac{d^3}{dt^3}u + \frac{k}{t}\ddot{u} + \mathrm{tr}\,(x^{-1}y)\ddot{u} - \frac{k}{t^2}\dot{u} - 2\,\mathrm{tr}\,(x^{-1}yx^{-1}y)\dot{u} + \mathrm{tr}\,(x^{-1}\dot{y})\dot{u}$$
$$- 2\ddot{u}\dot{u} - \epsilon\dot{u} = 0, \tag{1.13}$$
$$\dot{y} = (1-k)\frac{1}{t^2}x_+ - \frac{k}{t}y - \frac{1}{t}\,\mathrm{tr}\,(x^{-1}y)x_+ + \frac{1}{t}\dot{u}x_+ + 2yx^{-1}y + xr$$
$$- \mathrm{tr}\,(x^{-1}y)y + \dot{u}y + \frac{\epsilon}{2}x, \tag{1.14}$$

with initial conditions on y given by

$$y_+(0) = 0 \quad \text{and} \quad y_-(0) = L_1(v_0).$$

At this point, we need the formula for the Ricci tensor of the homogeneous metric g on the homogeneous space $P = G/K$, where G is a compact Lie group. It has the following expression (see [5, p. 185] for a derivation of this formula):

$$\begin{aligned}\operatorname{Ric}(X,Y) = &-\frac{1}{2}\operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(X)\operatorname{ad}(Y)) - \frac{1}{2}\sum_{ij} g([X,X_i]_{\mathfrak{p}},[Y,X_j]_{\mathfrak{p}})g^{ij} \\ &+ \frac{1}{4}\sum_{ijpq} g(X,[X_i,X_p])g(Y,[X_j,X_q])g^{ij}g^{pq},\end{aligned}$$

for any basis $\{X_i\}_{i=1}^n$ of $\mathfrak{p}$ and for all $X, Y \in \mathfrak{p}$. Note that our expression for the Ricci tensor is simpler than (7.38) in [5]. This is due to the fact that G is compact and hence unimodular.

The metric $\widehat{g}$ induces a G-invariant background metric $\widehat{g}_0$ on NQ. In fact, $\widehat{g}$ induces inner products on $\mathfrak{p}_-$, which can be identified with T_qQ, and on V, which can be identified with N_qQ. Considering bases $\{U_\alpha\}_{\alpha=1}^k$ of $\mathfrak{p}_+$ and $\{Z_i\}_{i=k+1}^n$ of $\mathfrak{p}_-$, which are orthonormal with respect to the background metric $\widehat{g}_0$, the inverse of g splits as follows

$$g^{\alpha\beta} = \frac{1}{t^2}x_+^{\alpha\beta}, \quad g^{ij} = x_-^{ij}.$$

Consequently, the Ricci endomorphism splits into a regular part and a singular part:

$$r = \frac{1}{t^2}r_{\text{sing}} + r_{\text{reg}},$$

which are given in Lemma 3.1 of [28].

We also have that

$$\begin{aligned}x_+r_+ &= \frac{1}{t^2}g_+r_+ = \frac{1}{t^2}\operatorname{Ric}_+, \\ x_-r_- &= g_-r_- = \operatorname{Ric}_-.\end{aligned}$$

Hence, equation (1.14) becomes

$$\dot{y} = \frac{1}{t^2}A(x) + \frac{1}{t}B(x,y) + C(x,y,t),$$

where

$$
\begin{aligned}
A(x) &= (1-k)x_+ + x r_{\text{sing}},\\
B(x,y) &= -ky - \operatorname{tr}(x^{-1}y)x_+ + \dot{u}x_+, \qquad (1.15)\\
C(x,y,t) &= 2yx^{-1}y + x r_{\text{reg}} - \operatorname{tr}(x^{-1}y)y + \dot{u}y + \frac{\epsilon}{2}x.
\end{aligned}
$$

We can now substitute $\dot{y}$ in (1.13) with the expression given by (1.14). Then, equation (1.13) becomes

$$\frac{d^3}{dt^3}u = \frac{1}{t^2}\widetilde{A}(\dot{u}) + \frac{1}{t}\widetilde{B}(\dot{u},\ddot{u}) + \widetilde{C}(\dot{u},\ddot{u},t),$$

where

$$\widetilde{A}(\dot{u}) = k\dot{u} + (k-1)\operatorname{tr}(x^{-1}x_+)\dot{u} - \operatorname{tr}(r_{\text{sing}})\dot{u}, \qquad (1.16)$$

$$\widetilde{B}(\dot{u},\ddot{u}) = -k\ddot{u} + k\operatorname{tr}(x^{-1}y)\dot{u} + \operatorname{tr}(x^{-1}y)\operatorname{tr}(x^{-1}x_+)\dot{u} - \operatorname{tr}(x^{-1}x_+)\dot{u}^2 \qquad (1.17)$$

$$
\begin{aligned}
\widetilde{C}(\dot{u},\ddot{u},t) = &-\operatorname{tr}(x^{-1}y)\ddot{u} - \operatorname{tr}(r_{\text{reg}})\dot{u} + \operatorname{tr}(x^{-1}y)\operatorname{tr}(x^{-1}y)\dot{u} - \operatorname{tr}(x^{-1}y)\dot{u}^2\\
&-(n+1)\frac{\epsilon}{2}\dot{u} + 2\ddot{u}\dot{u} + \epsilon\dot{u},
\end{aligned} \qquad (1.18)
$$

are analytic functions.

We then obtain that the system (1.12)-(1.14), with the above initial conditions, becomes the initial value problem given by

$$
\begin{aligned}
&\dot{x} = 2y,\\
&\frac{d^3}{dt^3}u = \frac{1}{t^2}\widetilde{A}(\dot{u}) + \frac{1}{t}\widetilde{B}(\dot{u},\ddot{u}) + \widetilde{C}(\dot{u},\ddot{u},t),\\
&\dot{y} = \frac{1}{t^2}A(x) + \frac{1}{t}B(x,y) + C(x,y,t),\\
&x(0) = \mathrm{I},\\
&y_+(0) = 0 \quad \text{and} \quad y_-(0) = L_1(v_0),\\
&\dot{u}(0) = 0.
\end{aligned}
$$

Therefore, the initial value problem for cohomogeneity one Ricci solitons has been reduced to an initial value problem for a system of nonlinear ordinary differential equations of order one in x, y and of order two in $\dot{u}$ with a singular point at the origin.

By [23, Lemma 2.2], Ricci solitons are real analytic. Therefore, we can solve the system by applying the method of asymptotic power series, which is described in [50, Chapter 9]. This method consists, first of all, of showing that there always exists a formal power series solution of an appropriate type. Then, after having a formal power series solution, one can apply [43, Theorem 7.1] to get a genuine solution.

We will see that the reason why there always exists a formal power series solution, is due to the geometric nature of the equations.

1.6 Solution to the initial value problem

The initial value problem considered in Section 1.5 has the following general form

$$\dot{x} = 2y, \tag{1.19}$$
$$\frac{d^3}{dt^3}u = \frac{1}{t^2}\widetilde{A}(\dot{u}) + \frac{1}{t}\widetilde{B}(\dot{u}, \ddot{u}) + \widetilde{C}(\dot{u}, \ddot{u}, t) \tag{1.20}$$
$$\dot{y} = \frac{1}{t^2}A(x) + \frac{1}{t}B(x, y) + C(x, y, t), \tag{1.21}$$
$$x(0) = a, \tag{1.22}$$
$$y(0) = b, \tag{1.23}$$
$$\dot{u}(0) = 0, \tag{1.24}$$

where $x(t), y(t), a, b \in \mathrm{Sym}^2(V \oplus \mathfrak{p}_-)^K$, $u(t)$ is smooth function and A, B, C, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are analytic functions.

As the left-hand sides of (1.20) and (1.21) do not have $\frac{1}{t}$ or $\frac{1}{t^2}$ terms, A and $\widetilde{A}$ must satisfy the following initial conditions

$$A(x(0)) = A(a) = 0 \quad \text{and} \quad 2(dA)_a \cdot b + B(a,b) = 0, \tag{1.25}$$

$$\widetilde{A}(\dot{u}(0)) = \widetilde{A}(0) = 0 \quad \text{and} \quad (d\widetilde{A})|_{t=0}\ddot{u}(0) + \widetilde{B}(0, \ddot{u}(0)) = 0. \tag{1.26}$$

We want to show that there always exists a formal power series solution. So let

$$x(t) = \sum_{m=0}^{\infty} \frac{x_m}{m!} t^m, \quad y(t) = \sum_{m=0}^{\infty} \frac{y_m}{m!} t^m,$$

with

$$x_{m+1} = 2y_m, \quad \forall m \geq 0,$$

and

$$u(t) = \sum_{m=0}^{\infty} \frac{u_m}{m!} t^m.$$

Then, let

$$A(x(t)) = \sum_{m=0}^{\infty} \frac{A_m}{m!} t^m, \quad B(x(t), y(t)) = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m, \quad C(x(t), y(t), t) = \sum_{m=0}^{\infty} \frac{C_m}{m!} t^m,$$

and

$$\widetilde{A}(\dot{u}(t)) = \sum_{m=0}^{\infty} \frac{\widetilde{A}_m}{m!} t^m, \quad \widetilde{B}(\dot{u}(t), \ddot{u}(t)) = \sum_{m=0}^{\infty} \frac{\widetilde{B}_m}{m!} t^m, \quad \widetilde{C}(\dot{u}(t), \ddot{u}(t), t) = \sum_{m=0}^{\infty} \frac{\widetilde{C}_m}{m!} t^m.$$

Substituting the above expressions in (1.20) and (1.21), respectively, we get

$$\frac{1}{2} x_{m+2} = \frac{A_{m+2}}{(m+2)(m+1)} + \frac{B_{m+1}}{m+1} + C_m, \tag{1.27}$$

$$u_{m+3} = \frac{\widetilde{A}_{m+2}}{(m+2)(m+1)} + \frac{\widetilde{B}_{m+1}}{m+1} + \widetilde{C}_m. \tag{1.28}$$

By definition, we have that

$$\begin{aligned}
A_{m+2} &= \frac{d^{m+2}}{dt^{m+2}}(A(x(t))\Big|_{t=0} = \frac{d^{m+1}}{dt^{m+1}}\left(\frac{d}{dt}(A(x(t)))\right)\Big|_{t=0} \\
&= \frac{d^{m+1}}{dt^{m+1}}(d_{x(t)}A\cdot \dot{x}(t))\Big|_{t=0} \\
&\equiv (d_{x(t)}A)_a \cdot x_{m+2} \pmod{x_1,\dots,x_{m+1}}, \\
B_{m+1} &= \frac{d^{m+1}}{dt^{m+1}}(B(x(t),y(t)))\Big|_{t=0} = \frac{d^{m}}{dt^{m}}\left(\frac{d}{dt}(B(x(t),y(t)))\right)\Big|_{t=0} \\
&= \frac{d^{m}}{dt^{m}}(\partial_{x(t)}B\cdot \dot{x}(t) + \partial_{y(t)}B\cdot \dot{y}(t) + x_{+}\ddot{u})\Big|_{t=0} \\
&\equiv \frac{1}{2}(\partial_{y(t)}B)_{(a,b)}\cdot x_{m+2} \pmod{x_1,\dots,x_{m+1}}, \\
C_m &\equiv 0 \pmod{x_1,\dots,x_{m+1}}.
\end{aligned}$$

Using the same strategy, we also have that

$$\begin{aligned}
\widetilde{A}_{m+2} &\equiv (d_{\dot{u}(t)}\widetilde{A})|_{t=0}u_{m+3} \pmod{u_1,\dots,u_{m+2}}, \\
\widetilde{B}_{m+1} &\equiv (\partial_{\ddot{u}(t)}\widetilde{B})|_{t=0}u_{m+3} \pmod{u_1,\dots,u_{m+2}}, \\
\widetilde{C}_m &\equiv 0 \pmod{u_1,\dots,u_{m+2}}.
\end{aligned}$$

Hence, equations (1.27) and (1.28) become

$$x_{m+2} = 2\frac{(d_{x(t)}A)_a\cdot x_{m+2}}{(m+2)(m+1)} + \frac{(\partial_{y(t)}B)_{(a,b)}\cdot x_{m+2}}{m+1} + \frac{D_m}{m+1}, \tag{1.29}$$

$$u_{m+3} = \frac{(d_{\dot{u}(t)}\widetilde{A})|_{t=0}u_{m+3}}{(m+2)(m+1)} + \frac{(\partial_{\ddot{u}(t)}\widetilde{B})|_{t=0}u_{m+3}}{m+1} + \frac{\widetilde{D}_m}{m+1}, \tag{1.30}$$

for some functions D_m of $x_1,\dots,x_{m+1}$ and $\widetilde{D}_m$ of $u_1,\dots,u_{m+2}$. If we now define the following two operators

$$\mathcal{L}_m = (m+1)\,\mathrm{I} - \frac{2}{m+2}(d_{x(t)}A)_a - (\partial_{y(t)}B)_{(a,b)}, \tag{1.31}$$
$$\widetilde{\mathcal{L}}_m = (m+1) - \frac{1}{m+2}(d_{\dot{u}(t)}\widetilde{A})|_{t=0} - (\partial_{\ddot{u}(t)}\widetilde{B})|_{t=0},$$

we need to have that

$$\mathcal{L}_m \cdot x_{m+2} = D_m \quad \text{and} \quad \widetilde{\mathcal{L}}_m u_{m+3} = \widetilde{D}_m,$$

which give necessary and sufficient conditions to the existence of a formal power series solution:

$$D_m \in \mathrm{Im}(\mathcal{L}_m) \quad \text{and} \quad \widetilde{D}_m \in \mathrm{Im}(\widetilde{\mathcal{L}}_m), \tag{1.32}$$

for all $m \geq 0$. We have that $\mathcal{L}_m$ and $\widetilde{\mathcal{L}}_m$ are invertible for all $m \geq m_0$, for some m_0. In fact, $\widetilde{\mathcal{L}}_m$ is bounded and dA and dB are bounded as well. For this reason, if m is large, $\mathcal{L}_m$ is close to a multiple of the identity and hence invertible. This implies that, if (1.32) is satisfied for $m < m_0$, we can fix further initial conditions, namely $x_1, \dots, x_{m_0}$ and $u_1, \dots, u_{m_0}$ satisfying equations (1.29) and (1.30) respectively, such that the formal power series solution is uniquely determined.

As we said before, a and b are two K-invariant endomorphisms which preserve the splitting of $\mathfrak{p}$ in a $+$ part and in a $-$ part. Moreover, as we saw in section 1.5, they are given by

$$a = \mathrm{I}, \quad b_+ = 0 \quad \text{and} \quad b_- = L_1(v_0),$$

where we have that $L_1 \in W_1^-$.

Now, using expressions given in (1.15) and Lemmas 4.2 and 4.4 of [28], we can write the operator $\mathcal{L}_m$ as follows. First of all, we have that

$$
\begin{aligned}
&(d_{x(t)}A)_a \cdot \xi = (d_{x(t)} r_{\text{sing}})_{\mathrm{I}} \cdot \xi, \\
&B(a,b) = -kb, \\
&(\partial_y B)_{(a,b)} \cdot \xi = -k\xi - \operatorname{tr}(\xi)\, \mathrm{I}_+, \\
&C(x,y,t) = 2yx^{-1}y + x r_{\text{reg}} - \operatorname{tr}(x^{-1}y)y + \dot{u}y + \frac{\epsilon}{2}x,
\end{aligned}
$$

where $\xi \in \operatorname{Sym}^2(V \oplus \mathfrak{p}_-)^K$ and

$$
\begin{aligned}
(d_{x(t)} r_{\text{sing}})_{\mathrm{I}} \cdot \xi_+ &= (k+1)\xi_+ - 2\operatorname{tr}(\xi_+)\, \mathrm{I}_+, \\
(d_{x(t)} r_{\text{sing}})_{\mathrm{I}} \cdot \xi_- &= \frac{1}{2}\mathcal{C} \cdot \xi_-,
\end{aligned}
$$

where $\mathcal{C}$ is an operator defined by $\mathcal{C} = -\sum_{\alpha=1}^{k} \operatorname{ad}(U_\alpha)^2$. Note that, if the H-homogeneous standard metric on S^k is normal, we have a bi-invariant metric on $\mathfrak{h}$ and we can extend the basis $\{U_\alpha\}$ of $\mathfrak{p}_+ \subset \mathfrak{h}$ to an orthonormal basis $\{V_\alpha\}$ of $\mathfrak{h}$, equipped with this bi-invariant metric, and we have that $\mathcal{C} = -\sum_\alpha \operatorname{ad}(V_\alpha)^2$ is the Casimir operator for the adjoint representation on $\mathfrak{p}_-$ and $\operatorname{End}(\mathfrak{p}_-)$. Note that we obtained, apart from C, the same expressions as [28, p. 129].

Substituting these expression in (1.31), we have that

$$
\mathcal{L}_m \cdot \xi = (m+1)\xi - \frac{2}{m+2}(dr_{\text{sing}})_{\mathrm{I}} \cdot \xi + k\xi + \operatorname{tr}(\xi)\, \mathrm{I}_+ .
$$

Furthermore, by [28, Lemma 4.6], we have that

$$
\begin{aligned}
(\mathcal{L}_m \cdot \xi)_+ &= m\left(1 + \frac{k+1}{m+2}\right)\xi_+ + \left(\frac{4}{m+2}\operatorname{tr}(\xi_+) + \operatorname{tr}(\xi)\right)\mathrm{I}_+, \\
(\mathcal{L}_m \cdot \xi)_- &= (m+1+k)\xi_- - \frac{1}{m+2}\mathcal{C} \cdot \xi_-.
\end{aligned}
$$

Again by Lemmas 4.2 and 4.4 of [28] and by (1.16)-(1.17), we also have that

$$\begin{aligned}(d_{\dot{u}(t)}\widetilde{A})|_{t=0}f &= kf + (k-1)\,\mathrm{tr}\,(x^{-1}(0)x_+(0))f - \mathrm{tr}\,(r_{\mathrm{sing}}(0))f \\ &= kf + (k-1)kf - \mathrm{tr}\,((k-1)\,\mathrm{I}_+)f \\ &= kf + (k-1)kf - (k-1)kf \\ &= kf, \\ (\partial_{\ddot{u}(t)}\widetilde{B})|_{t=0}f &= -kf,\end{aligned}$$

so that the operator $\widetilde{\mathcal{L}}_m$ becomes

$$\widetilde{\mathcal{L}}_m = (m+1) - \frac{k}{m+2} + k. \tag{1.33}$$

We now have to verify that the initial conditions (1.25) and (1.26) hold and that (1.32) is satisfied. Note that (1.25) is satisfied as explained in [28, p. 130]. Then, we have that $\widetilde{A}(\dot{u}(0)) = 0$, because $\dot{u}(0) = 0$. Moreover,

$$\begin{aligned}(d_{\dot{u}(t)}\widetilde{A})|_{t=0}\ddot{u}(0) =& k\ddot{u}(0) + (k-1)\,\mathrm{tr}(x^{-1}(0)x_+(0))\ddot{u}(0) - \mathrm{tr}\,(r_{\mathrm{sing}}(0))\ddot{u}(0) \\ =& k\ddot{u}(0)\end{aligned}$$

by Lemmas 4.2 and 4.4 of [28] and $\widetilde{B}(0, \ddot{u}(0)) = -k\ddot{u}(0)$ by definition. Hence, (1.26) holds as well. As the initial conditions hold, we need to verify equation (1.32) and to show that

$$x^l(t) := \sum_{m=0}^{l} \frac{x_m}{m!} t^m$$

defines a smooth G-invariant metric around Q for all l. By Lemma 1.4.1 this means that we need to prove that $x_m \in \mathrm{ev}(W_m)$ for all m.

In [28], the authors show that $\mathcal{L}_m(\mathrm{ev}(W_{m+2})) \subset \mathrm{ev}(W_m)$, by decomposing $\mathrm{ev}(W_{m+2})$ into eigenspaces of $\mathcal{L}_m$ and showing that the only eigenspaces corresponding to nonzero eigenvalues lie in $\mathrm{ev}(W_m) \subset \mathrm{ev}(W_{m+2})$, (we can always modify the degree of a homogeneous polynomial by an even factor without changing its value on the sphere). Moreover, $\mathcal{L}_m$ maps $\mathrm{ev}(W_m)$ bijectively onto itself if $m > 0$. Hence, the equation $\mathcal{L}_m \cdot x_{m+2} = D_m$ has a solution if and only if $D_m \in \mathrm{ev}(W_m)$. Moreover, we also have that the kernel of $\mathcal{L}_m$ is isomorphic to W^-_{m+2}/W^-_m. We need now to show that $x_p \in \mathrm{ev}(W_p)$ for all p and that $D_m \in \mathrm{ev}(W_m)$ for all m. In [28], the authors show this by induction over m. They show that there exists a solution x_{m+2} of $\mathcal{L}_m \cdot x_{m+2} = D_m$, but we can add an arbitrary element of the kernel of the operator considered, as it is not trivial.

Considering the multiplicative operator defined by (1.33), we can see that it maps $\mathrm{ev}(W_{m+3})$ to itself. So we need to show that $\widetilde{D}_m \in \mathrm{ev}(W_{m+3})$ and that

$$u^m(t) := \sum_{p=0}^{m} \frac{u_p}{p!} t^p$$

defines a smooth G-invariant function around Q for all m. This means that we have to show that $u_p \in \mathrm{ev}(W_p)$ for all p. We can prove that $\widetilde{D}_m \in \mathrm{ev}(W_{m+3})$ by induction over m. We have that $u_1 = 0 \in \mathrm{ev}(W_1)$. Then, suppose that $u_p \in \mathrm{ev}(W_p)$ for $p = 2, \ldots, m+2$, which implies that $u^{m+2}(t)$ is even in t, and consider

$$\widehat{u}(t) = u^{m+2}(t) = \sum_{p=0}^{m+2} \frac{u_p}{p!} t^p,$$

which defines a smooth G-invariant function around Q, because of the discussion in Section 1.4. By definition, we have that $\widehat{u}$ satisfies equation (1.20). Let $\widehat{A}, \widehat{B}$ and $\widehat{C}$ be the analogues of $\widetilde{A}, \widetilde{B}, \widetilde{C}$ for $\widehat{u}$. Moreover, let $\widehat{D}_m$ be some function of $\widehat{u}_1, \ldots, \widehat{u}_{m+2}$ which satisfies an analogue of equation (1.30) for $\widehat{u}$. Now, as $\widehat{u}_{m+3} = 0 \in \mathrm{ev}(W_{m+3})$ and $\widetilde{\mathcal{L}}_m \widehat{u}_{m+3} = \widehat{D}_m$, we have that $\widehat{D}_m = 0$. Then, by equations (1.28) and (1.30) and by recalling the expression of $\widetilde{C}$ given by (1.18), we have that

$$\frac{\widetilde{D}_m}{m+1} = \frac{\widetilde{D}_m - \widehat{D}_m}{m+1} = \widetilde{C}_m - \widehat{C}_m = 0 \in \mathrm{ev}(W_{m+3}).$$

Hence, $\widetilde{D}_m \in \mathrm{ev}(W_{m+3})$ and a solution u_{m+3} to $\widetilde{\mathcal{L}}_m u_{m+3} = \widetilde{D}_m$ exists in $\mathrm{ev}(W_{m+3})$.

The indeterminacy is just in the operator $\mathcal{L}_m$ and it is the same as in the Einstein case. In [28], the authors describe this indeterminacy in the formal power series solution. If $m > 0$, after solving the $-$ part of $\mathcal{L}_m \cdot x_{m+2} = D_m$, the $+$ part is uniquely determined. On the contrary, if $m = 0$, the trace free part of $(x_2)_+$ is arbitrary. This is to be expected, as the values of $x(0)$ and $y_+(0)$ are fixed by the geometry of the problem and this implies that the usual freedom in the initial value problem lies in the trace free part of $(x_2)_+$. Furthermore, by [28, Section 1], we see that the spaces W_m^- eventually stabilise. So, suppose that

$$W_{2m}^- = W_{2m_0}^- \quad \text{and} \quad W_{2m+1}^- = W_{2m_1+1}^-,$$

for all $m > m_0$ and for all $m > m_1$, respectively. Hence, as $\ker(\mathcal{L}_m) \backsimeq W_{m+2}^-/W_m^-$, we have that the indeterminacy of the initial value problem considered is given by

$$(W_{2m_0}^-/W_0^-) \oplus (W_{2m_1+1}^-/W_1^-),$$

where, in particular,

$$W_0 = \mathrm{Sym}^2(V)^H \oplus \mathrm{Sym}^2(\mathfrak{p}_-)^H.$$

The formal solution to the initial value problem (1.19)-(1.24) has the property that if truncated at any order it gives a smooth G-invariant metric and a smooth G-invariant function on NQ. Now, by in [43, Theorem 7.1], we obtain a genuine solution to the problem considered which is defined on a small interval $[0, T]$. The genuine solution may also be obtained by carrying out the Picard iteration directly, as shown in [28]. From this, one can see that $x_m(t)$ defines a smooth G-invariant metric on a tubular neighbourhood of radius T around Q. Moreover, by [28, Lemma 1.2], we can choose m to be at least 3. Hence, the solution x gives a C^3 G-invariant metric which satisfies the appropriate equation. Finally, by [23, Lemma 2.2], the pair (x, u) gives a smooth G-invariant Ricci soliton on the tubular neighbourhood of radius T around the singular orbit Q.

To conclude this section, as the reader may not be familiar with [28], we describe some examples, which show how to compute the indeterminacy explicitly.

Example 1.6.1 ([28]). Let

$$G = SO(p+n), \quad H = SO(p) \times SO(n), \quad \text{and} \quad K = SO(p) \times SO(n-1).$$

Then, the manifold we are dealing with has dimension $np + n + 1$. H acts effectively and transitively on the unit sphere in V. By [28, Lemma 1.2], we have that $W_m^+ \backsimeq \mathrm{Hom}(\mathrm{Sym}^m(V), \mathrm{Sym}^2(V))^H$ is isomorphic to zero if m is odd and that all these spaces are isomorphic if m is even. Hence, we can compute the indeterminacy in $(x_2)_+$, which is given by the dimension of $\mathrm{Hom}(\mathrm{Sym}^2(V), \mathrm{Sym}^2(V))^H$, which is 1, by [28, Lemma 1.2]. Now, we need to compute the dimension of $W_m^- \backsimeq \mathrm{Hom}(\mathrm{Sym}^m(V), \mathrm{Sym}^2(\mathfrak{p}_-))^H$. As an H-representation $\mathfrak{p}_-$ is the tensor product of the standard representation ρ_p of $SO(p)$ and the standard representation ρ_n of $SO(n)$. On the other hand, V is the tensor product of the trivial representation $\mathbf{1}$ of $SO(p)$ and ρ_n. It is well known (see [30, p. 296]) that

$$\mathrm{Sym}^m(V) = \sigma_m \oplus \sigma_{m-2} \oplus \cdots \oplus \sigma_{m-2[\frac{m}{2}]},$$

where σ_k is the irreducible representation of $SO(n)$ with dominant weight k times that of ρ_n. We also have that

$$\mathrm{Sym}^2(\mathfrak{p}_-) = (\mathrm{Sym}^2(\rho_p) \otimes \mathrm{Sym}^2(\rho_n)) \oplus (\Lambda^2(\rho_p) \otimes \Lambda^2(\rho_n)),$$

which decomposes as

$$(\sigma_2 \otimes \sigma_2) \oplus (\mathbf{1} \otimes \sigma_2) \oplus (\sigma_2 \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1}) \oplus (\mathrm{ad}_p \otimes \mathrm{ad}_n),$$

where we used the fact that the adjoint representation of $SO(n)$ is equivalent to $\Lambda^2 V$.

Applying Schur's Lemma, we have that the dimension of W_m^- is 2 when $m \geq 2$ is even, it is zero if m is odd. Finally, $\dim(W_0^-) = \dim(\mathrm{Sym}^2(\mathfrak{p}_-)^H) = 1$. Thus, the indeterminacy which occurs with $(x_2)_-$ has dimension 1. We have that the choice of the initial metric is unique up to homothety and the only choice for the initial shape operator is zero, because V is not a summand in $\mathrm{Sym}^2(\mathfrak{p}_-)$. This means that the singular orbit must be totally geodesic. Anyway, there is a one-dimensional freedom in choosing $(x_2)_+$.

Example 1.6.2. Let

$$G = SO(n+2), \quad K = SO(n), \quad H = SO(n+1).$$

The manifold M on which G acts has dimension $2n+2$, the principal orbits for this action are given by the Stiefel manifold, which is the homogeneous manifold $SO(n+2)/SO(n)$ for $n \geq 2$, and the singular orbit is given by the sphere S^{n+1}, which has dimension $n+1$ in M.

We want to compute the indeterminacy in the initial value problem for cohomogeneity one gradient Ricci solitons. We have that $V \backsimeq \mathbb{R}^{n+1}$, as H-representation, is given by the standard orthogonal representation ρ_{n+1}. Similarly, $\mathfrak{p}_- \backsimeq \mathbb{R}^{n+1}$, as H-representation, is given by ρ_{n+1}. Hence,

$$\mathrm{Sym}^2(\mathfrak{p}_-) = \mathrm{Sym}^2(\rho_{n+1}) = \sigma_2 \oplus \mathbf{1}.$$

Note that the assumption (1.7) is satisfied, because we assume that the metric g_t is diagonal with respect to the following decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \underbrace{\mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3}_{\mathfrak{p}},$$

where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are n-dimensional $SO(n)$-representations and $\mathfrak{p}_3$ is the trivial $SO(n)$-representation. With this decomposition we have that $\mathfrak{p}_- = \mathfrak{p}_2 \oplus \mathfrak{p}_3$. By Schur's Lemma we then have that the dimension of W_m^- is zero if m is odd and it is one if $m \geq 2$ is even. Moreover, W_0^- has dimension one. So, we have that the indeterminacy in $(x_2)_-$ is zero. We can now compute the dimension of W_m^+, which is zero if m is odd and 1 if either m is zero or $m \geq 2$ is even. So the indeterminacy in $(x_2)_+$, which is given by the dimension of W_2^+, is one. So we just have a one-dimensional freedom in choosing $(x_2)_+$.

Instead of considering $H = SO(n+1)$, we could consider $H = SO(2) \times SO(n)$, so that the singular orbit $Q = G/H$ has codimension 2. We have that $V \backsimeq \mathbb{R}^2$ and, as an H-representation, it is given by $\rho_2 \otimes \mathbf{1}$. On the other hand, $\mathfrak{p}_- \backsimeq \mathbb{R}^{2n}$, as an H-representation, is given by $\mathbf{1} \otimes \rho_n \otimes \rho_n$. We then have that

$$\begin{aligned}\mathrm{Sym}^2(\mathfrak{p}_-) &= \mathrm{Sym}^2(\rho_n \otimes \rho_n) = (\mathrm{Sym}^2(\rho_n) \otimes \mathrm{Sym}^2(\rho_n)) \oplus (\Lambda^2(\rho_n) \otimes \Lambda^2(\rho_n)) \\ &= (\sigma_2 \otimes \sigma_2) \oplus (\mathbf{1} \otimes \sigma_2) \oplus (\sigma_2 \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathbf{1}) \oplus (\mathrm{ad}_n \otimes \mathrm{ad}_n).\end{aligned}$$

Then we have that the dimension of W_m^- is two when $m \geq 2$ is even and it is zero when m is odd. Moreover, the dimension of W_0^- is one. So we have that the indeterminacy in $(x_2)_-$ is one. As the dimension of W_2^+ is one, the indeterminacy in $(x_2)_+$ is one.

Chapter 2

Homogeneous Ricci flow

2.1 Introduction

In this chapter, we are going to consider a certain class of compact and connected homogeneous spaces such that the Ricci flow of invariant metrics, which we will call *homogeneous Ricci flow* (HRF), always develops a singularity in finite time. We will also show that this singularity is always of type I and we will describe different singular behaviours that can occur, as we approach the singular time. Moreover, in some cases, we will investigate the existence of ancient solutions to the HRF.

Note that the study of the Ricci flow of invariant metrics on homogeneous spaces is closely related to the existence of invariant Einstein metrics on these spaces. In particular, some of the homogeneous spaces that we consider are interesting because they include many examples of compact homogeneous spaces which do not admit any invariant Einstein metric. The existence and non-existence of invariant Einstein metrics on compact homogeneous spaces have been studied extensively by Wang, Ziller, Böhm, Kerr and Dickinson [49, 6, 7, 8, 10, 9, 25]. The lowest dimensional non-existence example is the 12-dimensional manifold $SU(4)/SU(2)$ and it was found by Wang and Ziller [49]. Later, Böhm and Kerr [9] showed that this is the least dimensional example of a compact homogeneous space which does not carry any invariant Einstein metric. New non-existence examples were produced by Böhm in [8]. When the isotropy representation decomposes into pairwise inequivalent irreducible summands, these non-existence examples are of the type that we consider in this paper.

We would also like to mention that HRF on other spaces has been studied before by many authors [37, 38, 41, 42, 45]. Here we focus more on the relation between the Ricci flow of invariant metrics and the existence and non-existence of homogeneous Einstein metrics, without any restriction on the dimension.

2.2 The Ricci tensor of a homogeneous Riemannian manifold

In this section, we are going to explore in more detail the Ricci tensor of a homogeneous Riemannian manifold. Let G/K be a compact connected homogeneous space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{p}$ be an orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Let Q be an $\mathrm{Ad}_{|_K}$-invariant scalar product on $\mathfrak{p}$. It is well known that for every G-invariant Riemannian metric g, $\mathfrak{p}$ decomposes into $\mathrm{Ad}_{|_K}$-invariant irreducible summands:

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_l, \tag{2.1}$$

such that g is diagonal with respect to Q:

$$g = x_1 Q_{|_{\mathfrak{p}_1}} \oplus x_2 Q_{|_{\mathfrak{p}_2}} \oplus \cdots \oplus x_l Q_{|_{\mathfrak{p}_l}}, \tag{2.2}$$

where $x_i > 0$, for all $i = 1, \ldots, l$. In general, the decomposition (2.1) is not uniquely determined, but we do have uniqueness for the decomposition of $\mathfrak{p}$ into *isotypical summands.* Each isotypical summand is given by the direct sum of irreducible summands of $\mathfrak{p}$ which are equivalent to a fixed summand. By Schur's lemma, every invariant metric g and its Ricci tensor $\mathrm{Ric}(g)$ respect the splitting of $\mathfrak{p}$ into isotypical summands. We have the following definitions.

Definition 2.2.1. If the isotropy representation $\mathfrak{p}$ is irreducible as a K-representation, then the homogeneous space is called *isotropy irreducible.*

Definition 2.2.2. If the irreducible summands in (2.1) are pairwise inequivalent, then the isotropy representation is called *monotypic.*

Remark 2.2.3. In the monotypic case, by Schur's Lemma the Ricci tensor and the metric respect the splitting of $\mathfrak{p}$.

Let Q be an $Ad_{|_K}$-invariant scalar product on $\mathfrak{p}$ such that its restriction to every irreducible summand is a negative multiple of the Killing form B of G. Consider now the decomposition (2.1) of $\mathfrak{p}$ into $\mathrm{Ad}_{|_K}$-invariant irreducible summands such that the G-invariant Riemannian metric g diagonalises with respect to Q, as in (2.2). Let r_g be the Ricci endomorphism, which is defined as

$$\mathrm{Ric}(g)(X, Y) = g(r_g(X), Y),$$

for all $X, Y \in \mathfrak{p}$. By [49] and [44], for every $i \in \{1, \ldots, l\}$, the Ricci endomorphism on an isotypical summand $\mathfrak{p}_i$ is given by

$$(r_g)_{|\mathfrak{p}_i} = \left(\frac{b_i}{2x_i} - \frac{1}{2d_i} \sum_{j,k=1}^{l} [ijk] \frac{x_k}{x_i x_j} + \frac{1}{4d_i} \sum_{j,k=1}^{l} [ijk] \frac{x_i}{x_j x_k} \right) \mathrm{id}_{|\mathfrak{p}_i},$$

where $d_i = \dim(\mathfrak{p}_i)$ and $b_i \geq 0$ is defined by

$$-B_{|\mathfrak{p}_i} = b_i Q_{|\mathfrak{p}_i},$$

for all $i = 1, \ldots, l$. Moreover, the *structure constants* $[ijk]$, which appear in the above formula, are defined by

$$[ijk] = \sum Q([e_\alpha, e_\beta], e_\gamma)^2,$$

where the sum is taken over the Q-orthonormal bases $\{e_\alpha\}_\alpha$, $\{e_\beta\}_\beta$ and $\{e_\gamma\}_\gamma$ of $\mathfrak{p}_i$, $\mathfrak{p}_j$ and $\mathfrak{p}_k$, respectively. Note that $[ijk]$ is symmetric in i, j and k.

The relations between these quantities have been described in [49]:

$$d_i b_i = 2 d_i c_i + \sum_{j,k=1}^{l} [ijk], \quad 1 \leq i \leq l, \tag{2.3}$$

where c_i are the non negative constants defined by:

$$\mathcal{C}_{\mathfrak{p}_i, Q} = c_i \cdot \mathrm{id}_{|\mathfrak{p}_i}, \quad 1 \leq i \leq l,$$

where $\mathcal{C}_{\mathfrak{p}_i, Q} = -\sum_\alpha \mathrm{ad}(e_\alpha) \circ \mathrm{ad}(e_\alpha)$ is the Casimir operator for the adjoint representation of $\mathfrak{p}_i$, for all $i = 1, \ldots, l$.

2.3 Singularities in the Ricci flow

Now let us consider the Ricci flow equation in the homogeneous case. We note that, if M is G-homogeneous, the Ricci flow equation becomes a system of nonlinear ordinary differential equations. More precisely, choose a background metric Q. Then consider the following decomposition of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_l,$$

where the $\mathfrak{p}_i$'s are pairwise inequivalent $\mathrm{Ad}_{|K}$-invariant irreducible summands, so that we are in the monotypic case. As every G-invariant Riemannian metric diagonalises as in (2.2), if the initial metric is G-homogeneous, every solution to the Ricci flow equation will take the form

$$g(t) = x_1(t)Q_{|\mathfrak{p}_1} \oplus x_2(t)Q_{|\mathfrak{p}_2} \oplus \cdots \oplus x_l(t)Q_{|\mathfrak{p}_l},$$

where $x_1(t), \ldots, x_l(t)$ are smooth functions of t, which are strictly positive for all t for which $g(t)$ is defined. We then have that $g(t)$ defined above is a Ricci flow if and only if $x_1(t), \ldots, x_l(t)$ satisfy the system

$$\dot{x}_i(t) = -b_i + \frac{1}{d_i}\sum_{j,k=1}^{l}[ijk]\frac{x_k(t)}{x_j(t)} - \frac{1}{2d_i}\sum_{j,k=1}^{l}[ijk]\frac{x_i^2(t)}{x_j(t)x_k(t)}, \tag{2.4}$$

for all $i = 1, \ldots, l$, and where $\dot{}$ indicates the derivative with respect to t, together with the condition $x_i(t) > 0$, for all $i = 1, \ldots, l$.

We will now recall the definition of singular solution to the Ricci flow.

Definition 2.3.1. Let (M, g) be a closed Riemannian manifold. A solution $g(t)$ to the Ricci flow on $M \times [0, T)$, with $T \leq +\infty$ is a *maximal solution* if either $T = +\infty$ or $T < +\infty$ and the norm of the curvature tensor $|\operatorname{Rm}(g(t))(x,t)|$ is unbounded, as $t \to T$. In the latter case, the maximal solution is called *singular.*

According to Hamilton [36], we can classify singular solutions to the Ricci flow into type I and type II singularities. We say that a solution $g(t)$, with $t \in [0, T)$, to the Ricci flow develops a *type I singularity* at $t = T$ if

i) $T < +\infty$,

ii) $\sup_{t\in[0,T)}\left(\sup_{p\in M}|\operatorname{Rm}(g(t))|_{g(t)}(p,t)\right) = +\infty$,

iii) $\sup_{t\in[0,T)}\big((T-t)\sup_{p\in M}|\operatorname{Rm}(g(t))|_{g(t)}(p,t)\big) < +\infty$.

On the other hand, a solution $g(t)$ to the Ricci flow, with $t \in [0,T)$, develops a *type II singularity* at $t = T$ if i) and ii) above are satisfied and if

$$\sup_{t\in[0,T)}\left((T-t)\sup_{p\in M}|\operatorname{Rm}(g(t))|_{g(t)}(p,t)\right) = +\infty.$$

2.4 Ancient solutions to the Ricci flow

Definition 2.4.1. Let (M, g) be a closed Riemannian manifold. A solution $g(t)$ to the Ricci flow on M which is defined on the time interval $(-\infty, T)$, with $T \leq \infty$, is called an *ancient solution.*

According to Hamilton [36], we can classify ancient solutions to the Ricci flow in type I and type II. We say that an ancient solution $g(t)$ to the Ricci flow is of *type I* if

$$\lim_{t\to-\infty}\left(|t|\sup_{p\in M}|\operatorname{Rm}(g(t))|_{g(t)}(p,t)\right) < \infty.$$

Otherwise, we say that the ancient solution is of *type II.*

These kind of solutions are important, because they arise as limits of blow ups of singular solutions to the Ricci flow near finite time singularities. Suppose $g(t)$ is Ricci flow on M which is defined on the maximal time interval $[0,T)$, with $T < \infty$. Consider a sequence $\{(p_j,t_j)\}_{j=1}^{\infty}$, with $p_j \in M$ and $t_j \to T$, such that

$$|\operatorname{Rm}|(p_j,t_j) = \sup_{p\in G/K,\, t\in[0,t_j]}|\operatorname{Rm}(g(t))|_{g(t)}(p,t) \to +\infty.$$

Let

$$g_j(t) = |\operatorname{Rm}|(p_j,t_j)\, g\left(t_j + \frac{t}{|\operatorname{Rm}|(p_j,t_j)}\right).$$

Then $(M, g_j(t), p_j)$ converges in the smooth pointed Cheeger-Gromov sense to

$$(N, g_\infty(t), p_\infty) = \lim_{j\to\infty} (M, g_j(t), p_j),$$

which is an ancient solution to the Ricci flow on N, see for example [48, Theorem 8.5.1].

2.5 A notion of convergence for metric spaces

The *Hausdorff-Gromov* distance is a way of measuring distances between metric spaces. Let (A, d_A) and (B, d_B) be two metric spaces. An *ϵ-approximation* between A and B is a subset $S \subseteq A \times B$ with the following properties:

i) both the projections of S to A and B are onto,

ii) for all $(p_1, p_2), (q_1, q_2) \in S$, $|d_A(p_1, q_1) - d_B(p_2, q_2)| < \epsilon$.

If there exists an ϵ-approximation between A and B, we write $A \sim_\epsilon B$. The *Hausdorff-Gromov distance* between A and B is defined as

$$d_{\text{H-G}}(A, B) = \inf\{\epsilon | A \sim_\epsilon B\}.$$

If such an ϵ does not exist, we write $d_{\text{H-G}}(A, B) = \infty$. We say that a sequence of metric spaces $\{(A_n, d_{A_n})\}_n$ converges to (A, d_A) in the Hausdorff-Gromov topology if $d_{\text{H-G}}(A_n, A) \to 0$, as $n \to \infty$. An example of this kind of convergence is given by the following proposition.

Proposition 2.5.1. *Let G/K be a compact and connected homogeneous space. Suppose that there exists an intermediate Lie group H, with $G > H > K$. Let $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of G, H and K, respectively. Suppose that $\mathfrak{h}$ is $\text{Ad}_{|_K}$-invariant and that* *every G-invariant Riemannian metric on G/K is a submersion metric*

$$H/K \to G/K \to G/H.$$

If we have a sequence of G-invariant Riemannian metrics g_i on G/K such that the fibre H/K shrinks to a point, as $i \to +\infty$, then G/K converges in the Hausdorff-Gromov sense to G/H.

Proof. Let $S \subseteq G/K \times G/H$ to be

$$\{(gK, gH), g \in G\}.$$

Clearly, S projects onto both G/K and G/H. Now, we can decompose $\mathfrak{g}$ into two different ways:

$$\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{q}$ and $\mathfrak{p}$ are orthogonal complements of $\mathfrak{h}$ and $\mathfrak{k}$ in $\mathfrak{g}$, respectively. Because of our assumption that every G-invariant Riemannian metric on G/K is obtained from a Riemannian submersion, the tangent space at each point of G/K splits into vertical and horizontal subspaces:

$$T_{gK}(G/K) = \mathcal{V}_{gK} \oplus \mathcal{H}_{gK},$$

where the vertical space $\mathcal{V}_{gK} \simeq \mathfrak{q}$ is the tangent subspace to the fibre H/K and the horizontal space $\mathcal{H}_{gK} \simeq \mathfrak{q}^{\perp}$ is its orthogonal complement in $T_{gK}(G/K) \simeq \mathfrak{p}$. Moreover, the submersion map $gK \mapsto gH$ defines an isometry between $\mathcal{H}_{gK}$ and $T_{gH}(G/H)$ and distances in the $\mathfrak{q}$-direction shrink. Hence, given $(gK, gH), (hK, hH) \in S$,

$$|d^i_{G/K}(gK, hK) - d^i_{G/H}(gH, hH)| \to 0,$$

as $i \to +\infty$, which proves convergence of G/K to G/H in the Hausdorff-Gromov sense.

□

For further details on these matters we refer the reader to [16].

2.6 Type I singularities in HRF

Let $g(t)$ be a Ricci flow on $G/K \times [0,T)$, with $T < \infty$ and $g(0)$ G-invariant, and suppose that T is a type I singularity for $g(t)$. Then, as $t \to T$, the scalar curvature blows up:

$$|R(g(t))| \to +\infty,$$

at the type I rate. Moreover, if $\mathrm{Vol}_{g(0)}(G/K) < \infty$, then

$$\mathrm{Vol}_{g(t)}(G/K) \to 0.$$

This follows from [27] and the homogeneity. For, given a complete Ricci flow $(M, g(t))$ on $[0,T)$, which develops a type I singularity in the finite time T, in [27], Enders, Müller and Topping define the following *singular sets* for $g(t)$.

Definition 2.6.1. Let Σ, Σ_I and Σ_R be as follows. Σ is the set of points $p \in M$ such that there does not exist a neighbourhood $U_p \ni p$ on which $|\operatorname{Rm}(g(t))|_{g(t)}$ stays bounded, as $t \to T$. Σ_I is defined to be the set of points $p \in M$ such that there exists a sequence $(p_i, t_i) \in M \times [0,T)$, with $p_i \to p$ and $t_i \to T$, such that there exists a constant $C > 0$ such that

$$|\operatorname{Rm}(g(t_i))|_{g(t_i)}(p_i, t_i) \geq \frac{C}{T - t_i},$$

for all i. Finally, Σ_R is the set of points $p \in m$ such that the scalar curvature $R(g(t))(p)$ blows up at the type I rate, as $t \to T$.

In their paper, Enders, Müller and Topping proved that

$$\Sigma = \Sigma_I = \Sigma_R,$$

for a type I Ricci flow. They also proved that, if $\mathrm{Vol}_{g(0)}(\Sigma) < \infty$,

$$\mathrm{Vol}_{g(t)}(\Sigma) \to 0,$$

as we approach the singular time.

If $M = G/K$, then the homogeneity implies that

$$\Sigma_I = G/K,$$

from which the above follows.

2.7 Ricci flow on isotropy irreducible spaces

Let G/K be a compact and connected homogeneous space such that it is effective. Suppose that G/K is isotropy irreducible. By Schur's lemma, isotropy irreducible spaces carry a unique (up to rescaling) invariant Riemannian metric, which must also be Einstein. Furthermore, as G/K is compact, this unique G-invariant Einstein metric has positive scalar curvature. Hence, the HRF on G/K is just a rescaling of the initial metric by a function which becomes zero in finite time. We are now going to deduce the behaviour of the HRF directly from the flow equations as this will be instructive for more complicated examples. In the case where G/K is isotropy irreducible, the system (2.4) becomes as follows. Let $d = \dim(\mathfrak{p})$ and choose the $\mathrm{Ad}_{|_K}$-invariant background metric Q to be $-\frac{1}{b}B$, where B is the Killing form of $\mathfrak{g}$ and $b > 0$. We, then, have that the only nonzero structure constant is given by

$$[111] = \sum_{\alpha,\beta,\gamma=1}^{d} Q([e_\alpha, e_\beta], e_\gamma)^2,$$

where $\{e_\alpha\}_\alpha$ is a Q-orthonormal basis of $\mathfrak{p}$. Then, $g(t) = x(t)Q_{|\mathfrak{p}}$ is a solution to the Ricci flow equation if $x(t)$ satisfies

$$\dot{x}(t) = -\left(b - \frac{[111]}{2d}\right) = -C,$$

where $C > 0$, by (2.3). Hence,

$$x(t) = C(T - t),$$

where $T = \frac{x(0)}{C}$ is the singular time of the HRF. As $t \to T$, G/K shrinks to a point, because $x(t) \to 0$, and the curvature tensor blows up like $(T-t)^{-1}$, which means that T is a type I singularity. Moreover, if we blow up the metric $g(t) = x(t)Q_{|\mathfrak{p}}$ by $(T-t)^{-1}$, we get an invariant Einstein metric which is homothetic to the initial metric $g(0)$.

If we want to look for the existence of ancient solutions, it is useful to change time parameter from t to $\tau = -t$. Clearly, in this case, the solution exists for all $\tau > 0$ and, as $\tau \to +\infty$, G/K expands homothetically. As $x(\tau)$ increases linearly in τ, this ancient solution is of type I. If we rescale the metric $g(\tau)$ by $\frac{1}{\tau}$, we get an invariant Einstein metric homothetic to the initial one.

Example 2.7.1. Consider the n-dimensional sphere (S^n, g_{S^n}) with constant sectional curvature $+1$. The metric g_{S^n} is Einstein with Einstein constant given by $n-1$. Hence, the solution to the Ricci flow equation with initial metric g_{S^n} is given by

$$g(t) = (1 - 2(n-1)t)g_{S^n}.$$

2.8 The two isotropy summands case

Let G/K be a compact and connected homogeneous space which is also effective. Let $\mathfrak{p}$ be the isotropy representation of K and suppose that it splits into two inequivalent irreducible $\mathrm{Ad}_{|_K}$-invariant summands.

This section will be divided into two parts. In the first part, we will suppose that there exists an intermediate Lie group H, with $K < H < G$, such that its Lie algebra $\mathfrak{h}$ is $\mathrm{Ad}_{|_K}$-invariant. Note that if both G and K are connected, every intermediate Lie algebra is automatically invariant under the adjoint action. In particular, H/K is isotropy irreducible and every G-invariant Riemannian metric on G/K is given by a fixed Riemannian submersion

$$H/K \to G/K \to G/H, \tag{2.5}$$

by rescaling the metric on the fibre and on the base. We will prove that the HRF starting at any G-invariant Riemannian metric always develops a type I singularity in finite time. We will also describe the different singular behaviours which may occur. Then, we will associate to the different singular behaviours the corresponding subsets of initial conditions. Finally, we will investigate the existence of ancient solutions to the HRF. In particular, we will show that, in some cases, it is possible to generalise to G/K described above a result of Bakas, Kong and Ni (see [2, Section 7]).

In the second part of this section, we will consider the case in which K is maximal in G. We will show that the HRF always develops a type I singularity in finite time, no matter what initial condition we pick. As in the previous case, some classes of initial conditions give ancient solutions to the HRF. These solutions always converge to an invariant Einstein metric. More precisely, the flow in this case is converging to the shrinking flow for Einstein metrics, which is characterised by having the ratio between the two functions which define the metric fixed.

2.8.1 When the isotropy group is not maximal

Let G/K be a compact and connected homogeneous space. Suppose that the isotropy representation of K decomposes into two inequivalent irreducible $\mathrm{Ad}_{|K}$-invariant summands:

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2.$$

Suppose further that K is not maximal in G, so that there exists an intermediate Lie group H, with $G > H > K$. Let

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1$$

be the Lie algebra of H and suppose that it is $\mathrm{Ad}_{|K}$-invariant. Note that the $\mathrm{Ad}_{|K}$-invariance of $\mathfrak{h}$ is automatically true if both G and K are connected. Choose a background metric Q such that it is a negative multiple of the Killing form on both $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Then,

$$[112] = 0$$

and the Ricci flow equation for the one-parameter family of homogeneous Riemannian metrics

$$g_t = x_1(t)Q_{|\mathfrak{p}_1} \oplus x_2(t)Q_{|\mathfrak{p}_2}$$

is given by the following system of nonlinear ODEs:

$$\dot{x}_1(t) = -\left(b_1 - \frac{[111]}{2d_1} - \frac{[122]}{d_1}\right) - \frac{[122]}{2d_1}\frac{x_1(t)^2}{x_2(t)^2}, \tag{2.6}$$

$$\dot{x}_2(t) = -\left(b_2 - \frac{[222]}{2d_2}\right) + \frac{[122]}{d_2}\frac{x_1(t)}{x_2(t)}, \tag{2.7}$$

together with the condition that $x_1(t), x_2(t) > 0$.

Remarks 2.8.1.

1. We note that, if $b_1 = b_2$, $d_1 = d_2$ and $[111] = [222]$, the above system of equations

coincides with the system (7.2)-(7.3) of [2]. This is, for example, the case of G simple and both G/H and H/K symmetric spaces such that $d_1 = d_2$.

2. Observe that, if $[122] = 0$, the system (2.6)-(2.7) reduces to

$$\dot{x}_1(t) = -\left(b_1 - \frac{[111]}{2d_1}\right),$$
$$\dot{x}_2(t) = -\left(b_2 - \frac{[222]}{2d_2}\right).$$

This is the situation in which the universal cover of G/K is given by a product of isotropy irreducible homogeneous spaces (cf. [49, Theorem 2.1]). In this case, the analysis is just the same as the one performed in section 2.7 on each factor.

Motivated by the above remark, we will assume that $[122] > 0$. Now, let

$$A = \frac{[122]}{2d_1}, \tag{2.8}$$
$$B = \frac{[122]}{d_2}, \tag{2.9}$$
$$C = b_1 - \frac{[111]}{2d_1} - \frac{[122]}{d_1} = 2c_1 + \frac{[111]}{2d_1}, \tag{2.10}$$
$$D = b_2 - \frac{[222]}{2d_2} = 2c_2 + \frac{[222]}{2d_2} + \frac{2}{d_2}[122]. \tag{2.11}$$

A, B and D are all strictly positive, because $[122], d_1, d_2 > 0$ and by (2.3). Moreover, (2.3) also implies that $C \geq 0$. Note that $C = 0$ if and only if $c_1 = 0$ and $[111] = 0$. This is the case of $\mathfrak{p}_1$ being a trivial 1-dimensional summand in the decomposition of $\mathfrak{p}$. Using these quantities, the system (2.6)-(2.7) can be written as

$$\dot{x}_1(t) = -C - A\frac{x_1(t)^2}{x_2(t)^2}, \tag{2.12}$$
$$\dot{x}_2(t) = -D + B\frac{x_1(t)}{x_2(t)}. \tag{2.13}$$

It is useful to consider the following quantity. Let

$$y(t) = \frac{x_1(t)}{x_2(t)}.$$

Note that (2.12)-(2.13) can be written as

$$\dot{x}_1(t) = -C - Ay(t)^2, \tag{2.14}$$

$$\dot{x}_2(t) = -D + By(t). \tag{2.15}$$

Under the Ricci flow (2.12)-(2.13), $y(t)$ evolves as follows:

$$\dot{y}(t) = \frac{1}{x_2(t)}\big(- C + Dy(t) - (A+B)y(t)^2\big). \tag{2.16}$$

Note that the equation

$$C - Dy + (A+B)y^2 = 0 \tag{2.17}$$

has solutions if and only if the following inequality is satisfied:

$$\left(b_2 - \frac{[222]}{2d_2}\right)^2 - 4[122]\left(\frac{1}{2d_1} + \frac{1}{d_2}\right)\left(b_1 - \frac{[111]}{2d_1} - \frac{[122]}{d_1}\right) \geq 0.$$

By [49], we know that the above inequality is satisfied if and only if G/K admits a G-invariant Einstein metric. More precisely, $(x_1(t), x_2(t))$ defines a homogeneous Einstein metric on G/K if and only if $\frac{x_1(t)}{x_2(t)}$ is a (positive) root of (2.17). Observe that the roots of (2.17) are always positive because the positivity of A, B, C and D implies that the sum and the product of the roots are positive. In particular, G/K carries at most two distinct G-invariant Einstein metrics, up to scaling. We will denote by y_1 and y_2 the solutions to equation (2.17).

Remark 2.8.2. If $y(0) = y_i$, then we will get the trivial solution corresponding to $y(t) = y_i$, for all $i = 1, 2$. We then have that y_i, with $i = 1, 2$, are fixed by the HRF. Moreover, as we already mentioned in the introduction, $x_i(t)$, with $i = 1, 2$, just scale homothetically for all t such that a solution exists.

The following lemma holds.

Lemma 2.8.3. *$y(t)$ is monotonically increasing or decreasing under the HRF.*

Proof. According to the number of roots of (2.17), we have that only three possibilities can occur:

- Equation (2.17) has no solutions and

$$-C + Dy(t) - (A + B)y(t)^2$$

  is always negative;

- Equation (2.17) has a unique solution $\bar{y}$ such that

$$-C + Dy(t) - (A + B)y(t)^2 = -(A + B)(y(t) - \bar{y})^2;$$

- Equation (2.17) has two distinct solutions y_1 and y_2 such that

$$-C + Dy(t) - (A + B)y(t)^2 = -(A + B)(y(t) - y_1)(y(t) - y_2).$$

By (2.16), we have that, when (2.17) has no roots or only one root, $y(t)$ is monotonically decreasing for all t such that a solution to the HRF exists. Now suppose that (2.17) has two distinct roots y_1 and y_2. Suppose without loss of generality that $y_1 < y_2$. Then, when $y(t) < y_1$ and $y(t) > y_2$, $y(t)$ is monotonically decreasing in t. On the other hand, when $y_1 < y(t) < y_2$, $y(t)$ is monotonically increasing in t. Because of the uniqueness of the solution and of remark 2.8.2, if $y(0) \neq y_i$, with $i = 1, 2$, then $y(t) \neq y_i$ for all t such that a solution to the HRF exists. Hence, $y(t)$ is monotonic along every solution to the HRF. □

We now have to distinguish between two cases: $C > 0$ and $C = 0$. We will begin by considering the situation in which $C > 0$ and then, at the end of the section, we will study the case $C = 0$. Let G/K be as above and such that $C > 0$. Then the following theorem holds.

Theorem 2.8.4. *There exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$. Moreover, T is a type I singularity and, as $t \to T$, one of the following singular behaviours occurs:*

i) The whole space shrinks to a point in finite time.

ii) The fibre H/K in (2.5) shrinks to a point in finite time and the total space G/K converges in the Hausdorff-Gromov topology to G/H.

Moreover, a necessary condition for i) to happen is that G/K carries G-invariant Einstein metrics.

Proof. Let

$$f_1(x_1, x_2) = -C - A\frac{x_1^2}{x_2^2},$$
$$f_2(x_1, x_2) = -D + B\frac{x_1}{x_2},$$

be the functions defined by the right-hand side of (2.12)-(2.13). Since A and B are strictly positive, f_1, f_2 and their derivatives with respect to x_1 and x_2 are continuous if and only if (x_1, x_2) belongs to

$$\boldsymbol{D} = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \neq 0\}.$$

We can then apply a standard theorem of ODEs, see for example [11, Theorem 1.1], which says that, given any initial condition in $((x_1)_0, (x_2)_0) \in \boldsymbol{D}$, there exists a unique solution $(x_1(t), x_2(t))$ to (2.12)-(2.13) such that $x_1(t_0) = (x_1)_0, x_2(t_0) = (x_2)_0$ and which depends continuously on t and the initial data. Moreover, the solution $(x_1(t), x_2(t))$ exists on any interval I containing t_0 and such that $(x_1(t), x_2(t)) \in \boldsymbol{D}$, for every $t \in I$.

From (2.12), we have that $x_1(t)$ is decreasing in t and

$$\dot{x}_1(t) \leq -C,$$

which integrated gives

$$x_1(t) \leq -Ct + x_1(0).$$

Hence, there exists $T \leq \frac{x_1(0)}{C} < \infty$ such that the unique solution to the Ricci flow equation will be defined on the maximal time interval $[0, T)$, otherwise $x_1(t)$ becomes negative. In particular, $x_1(t)$ will approach a finite limit, as $t \to T$.

Now, lemma 2.8.3 enables us to conclude that $x_2(t)$ approaches a limit, as $t \to T$. In fact, as $\dot{x}_2(t)$ is given by

$$\dot{x}_2(t) = -D + By(t)$$

and $y(t)$ is monotone, so it approaches a limit in $[0, \infty)$, there exists $\bar{t} \leq T$ such that $x_2(t)$ is monotonically decreasing or increasing for all $t > \bar{t}$. Hence, it approaches a limit, as $t \to T$. We also have that this limit cannot be $+\infty$. In fact, from (2.13) this would imply that $\dot{x}_2(t) \to -D$, which is a negative value. By the mean value theorem, this is a contradiction. Hence, $x_2(t)$ has a finite limit, as $t \to T$. Similarly, if $\dot{x}_2(t) \to +\infty$, $x_2(t)$ cannot tend to zero through positive values. From (2.13), we see that, if $x_2(t) \to 0$ and the limit of $x_1(t)$ is nonzero, then $\dot{x}_2(t) \to +\infty$, which is a contradiction. We can then conclude that, as $t \to T$, $y(t)$ approaches a finite limit and there are only two possible singular behaviours:

i) Both $x_1(t)$ and $x_2(t)$ tend to zero.

ii) $x_1(t)$ tends to zero and $x_2(t)$ has a finite limit, which is strictly positive.

We will now analyse these two singular behaviours separately.

Let us begin with case i). The singular time T is characterised by the shrinking of the whole space to a point, as both $x_1(t)$ and $x_2(t)$ tend to zero, as $t \to T$. Let

$$x_i(t) = k_i(T-t)^{n_i} + o((T-t)^{n_i}), \quad i = 1, 2, \tag{2.18}$$

where k_1 and k_2 are positive coefficients. First of all, we observe that $x_1(t)/x_2(t)$ being bounded for all $t \in [0, T)$ implies

$$n_1 \geq n_2 > 0.$$

Then, by substituting (2.18) into (2.12)-(2.13), we obtain that $n_1 = n_2 = 1$, which means that $x_1(t)$ and $x_2(t)$ tend to zero linearly in t. Moreover, we also have that k_1 and k_2 have to satisfy the following system of nonlinear equations:

$$\frac{C}{k_1} + A\frac{k_1}{k_2^2} = 1,$$
$$\frac{D}{k_2} - B\frac{k_1}{k_2^2} = 1.$$

The above system is satisfied if and only if

$$\tilde{g} = k_1 Q_{|_{\mathfrak{p}_1}} \oplus k_2 Q_{|_{\mathfrak{p}_2}}$$

defines a G-invariant Einstein metric on G/K. Finally, T is a type I singularity, because $|\operatorname{Rm}(g(t))|_{g(t)}$ is asymptotically given by a rescaling of $|\operatorname{Rm}(g(0))|_{g(0)}$ by $(T-t)^{-1}$ times a positive constant.

We can now consider case ii). Here, the singular time T is characterised by the fact that $x_1(t)$ becomes zero, as $t \to T$. The Ricci flow then has to stop because the metric has collapsed on $\mathfrak{p}_1$. Let

$$x_1(t) = k_1(T-t)^n + o((T-t)^n),$$
$$x_2(t) = k_2 + o(1),$$

where k_1 and k_2 are two positive constants and n is a positive integer. Substituting these expression into (2.12)-(2.13), we get that $n = 1$, which means that $x_1(t)$ tends to zero linearly in t.

Claim 2.8.5. *T is a type I singularity.*

Proof of the claim. We note that, asymptotically, on $\mathfrak{p}_1$, the Ricci flow is simply given by a rescaling of the initial metric by a constant times $(T-t)$. We, then, observe that

$$|\operatorname{Rm}(g(t))|^2_{g(t)} = \sum_{i=1}^{2} |\operatorname{Rm}(g(t)_{|\mathfrak{p}_i})|^2_{g(t)_{|\mathfrak{p}_i}} = \sum_{i=1}^{2} \frac{1}{x_i^2(t)} |\operatorname{Rm}(Q_{|\mathfrak{p}_i})|^2_{Q_{|\mathfrak{p}_i}},$$

because the squared norm of the curvature tensor respects the splitting of the metric $g(t)$. Hence, asymptotically, $|\operatorname{Rm}(g(t))|^2_{g(t)}$ behaves like $(T-t)^{-2}$.

Hence, $|\operatorname{Rm}(g(t))|_{g(t)}$ blows up to $+\infty$ like $(T-t)^{-1}$, when $t \to T$. This implies that T is a type I singularity for the Ricci flow. □

Finally, by proposition 2.5.1, we have that, as $t \to T$, G/K converges in the Hausdorff-Gromov sense to G/H. This concludes the proof of the theorem. □

We now want to investigate the existence of ancient solutions on G/K as above and with $C > 0$ and associate to each singular behaviour the corresponding subset of initial conditions. In order to do this, we have to distinguish between three different cases:

(a) G/K carries two G-invariant Einstein metrics, up to scaling;

(b) G/K carries one G-invariant Einstein metric, up to scaling;

(c) G/K does not carry any G-invariant Einstein metric.

Remark 2.8.6. Note that in the case where G/K does not admit G-invariant Einstein metrics, C is always strictly positive.

Case (a)

Let us first consider the case in which G/K admits two non isometric G-invariant Einstein metrics, up to scaling. Let $y_1 > 0$ and $y_2 > 0$ correspond to the these two G-invariant Einstein metrics. In particular, y_1 and y_2 are solutions to the following equation:

$$C - Dy + (A+B)y^2 = 0. \tag{2.19}$$

Suppose without loss of generality that $y_2 > y_1$. We then have that

$$\begin{aligned}\frac{\dot{x}_2(t)}{x_2(t)} &= \dot{y}(t)\frac{D - By(t)}{(A+B)(y(t)-y_1)(y(t)-y_2)} \\ &= \frac{\dot{y}(t)}{A+B}\left(-\frac{1}{y(t)-y_1}\left(B + \frac{D-By_2}{y_2-y_1}\right) + \frac{1}{y(t)-y_2}\frac{D-By_2}{y_2-y_1}\right).\end{aligned}$$

We can now integrate this expression and obtain:

$$\Lambda x_2(t) = |y(t)-y_1|^{-\frac{1}{A+B}\left(B+\frac{D-By_2}{y_2-y_1}\right)}|y_2 - y(t)|^{\frac{1}{A+B}\frac{D-By_2}{y_2-y_1}},$$

where Λ is a non negative constant. We then have a first integral for the Ricci flow (2.12)-(2.13). This first integral is given by

$$\Lambda = \frac{1}{x_2(t)}\left|\frac{x_1(t)}{x_2(t)} - y_1\right|^{-\frac{1}{A+B}\left(B+\frac{D-By_2}{y_2-y_1}\right)}\left|y_2 - \frac{x_1(t)}{x_2(t)}\right|^{\frac{1}{A+B}\frac{D-By_2}{y_2-y_1}}. \tag{2.20}$$

We would like to point out that we could use this first integral to solve the equations (2.14)-(2.15) by integration. In fact, we could express y as a function of x_2 using equation (2.20) and then we could write t as a function of x_2 using equation (2.15). Then x_1 would be defined using $y = \frac{x_1}{x_2}$.

We will now consider three different possible initial conditions:

(a)(1) $y(0) < y_1$,

(a)(2) $y_1 < y(0) < y_2$,

(a)(3) $y(0) > y_2$,

where $y(0) = \frac{x_1(0)}{x_2(0)}$. We have the following proposition:

Proposition 2.8.7. *The initial conditions* (a)(1), (a)(2) *and* (a)(3) *above are preserved under the HRF.*

Proof. This is essentially the same argument as in the proof of lemma 2.8.3. In fact, because of the uniqueness of the solution and remark 2.8.2, if $y(0) \neq y_i$, then $y(t) \neq y_i$ for all $i = 1, 2$ and for all t such that a solution to the HRF exists. □

Note that this proposition also follows from the conserved quantity Λ.

We will now associate to each initial condition (a)(1), (a)(2) and (a)(3) the corresponding behaviour of the Ricci flow and, then, we will investigate the existence of ancient Ricci flows on G/K.

If the initial condition $y(0)$ lies between y_1 and y_2, then we can prove the following theorem, which is a generalisation of [2, Theorem 7.1].

Theorem 2.8.8. *If $y_1 < y(0) < y_2$, there exists a positive constant $T < \infty$ such that there exists a unique type I ancient solution to the Ricci flow* (2.12)-(2.13) *defined on $(-\infty, T)$. Moreover, T is a type I singularity and, as $t \to T$, G/K shrinks to a point. This solution flows the invariant Einstein metric corresponding to y_2 to the invariant Einstein metric corresponding to y_1, as t goes from T to $-\infty$.*

Proof. We begin by noticing that theorem 2.8.4 implies that there exists a positive constant $T < \infty$ such that there exists a unique solution to the Ricci flow with initial condition $y(0)$ and defined on the maximal time interval $[0, T)$. Moreover, $x_1(t) \to 0$, as $t \to T$. By proposition 2.8.7, we have that $y_1 < y(t) < y_2$, for all $t \in [0, T)$. Hence, we also have that $x_2(t) \to 0$, as $t \to T$. This means that G/K shrinks to a point, as t approaches the singular time T. The evolution equation of $y(t)$ is given by

$$\dot{y}(t) = -\frac{(A+B)}{x_2(t)}(y(t) - y_1)(y(t) - y_2). \tag{2.21}$$

Hence, $y(t)$ is increasing in t. By the proof of theorem (2.8.4), we have that both $x_1(t)$ and $x_2(t)$ tend to zero linearly in t. Recall that, as we approach the singular time T, $y(t)$ is increasing in t and it approaches a limit. Let

$$\begin{aligned} x_1(t) &= k_1(T-t) + o((T-t)), \\ x_2(t) &= k_2(T-t) + o(T-t), \end{aligned}$$

where k_1 and k_2 are two positive constants. Substituting these expressions in (2.12)-(2.13) and taking the limit as $t \to T$, we obtain

$$\begin{aligned} -k_1 &= -C - A\frac{k_1^2}{k_2^2}, \\ -k_2 &= -D + B\frac{k_1}{k_2}, \end{aligned}$$

which means that $k_1 Q_{|_{\mathfrak{p}_1}} \oplus k_2 Q_{|_{\mathfrak{p}_2}}$ corresponds to a G-invariant Einstein metric on G/K. This implies that

$$\lim_{t \to T} y(t) = y_2.$$

We now have to show the existence of ancient solutions to the Ricci flow. In order to do this, it is convenient to change the time parameter from t to $\tau = -t$. Let $'$ denote the derivative with respect to τ. Then the system (2.12)-(2.13) becomes

$$x_1'(\tau) = C + Ay(\tau)^2, \tag{2.22}$$

$$x_2'(\tau) = D - By(\tau), \tag{2.23}$$

together with the condition $x_1(\tau) > 0$ and $x_2(\tau) > 0$. The evolution equation of $y(\tau)$ becomes:

$$y'(\tau) = \frac{A+B}{x_2(\tau)}(y(\tau) - y_1)(y(\tau) - y_2). \tag{2.24}$$

By proposition 2.8.7, $y_1 < y(\tau) < y_2$ along any solution to the Ricci flow. Hence, $y(\tau)$ is decreasing in τ. Moreover, the right-hand side of (2.22)-(2.23) are bounded for all τ such that a solution to the Ricci flow exists and the bounds depend on y_2. From (2.22), we see that $x_1'(\tau) > 0$ for all τ. We also have that

$$x_2'(\tau) > D - By_2 > D - \frac{BD}{A+B} > 0$$

where we have used the fact that

$$y_2 = \frac{D + \sqrt{D^2 - 4C(A+B)}}{2(A+B)},$$

as it is the biggest solution to (2.17). Hence, both $x_1(\tau)$ and $x_2(\tau)$ are increasing in τ with bounded derivatives. Moreover, from (2.24), we have that $y(\tau)$ is decreasing in τ. We can then apply standard ODE theory and conclude that a solution to the Ricci flow exists for all $\tau > 0$, as long as $y(\tau) > y_1$. By the uniqueness of the solution to the Ricci flow equation, $y(\tau)$ cannot become y_1, as long as $x_1(\tau)$ and $x_2(\tau)$ are positive. Hence, we can conclude that for every initial condition $y_1 < y(0) < y_2$, there exists a unique ancient solution to the Ricci flow defined for $\tau \in [0, +\infty)$, i.e. for $t \in (-\infty, 0]$. To finish the proof we notice that, by the asymptotic analysis and the fact that $y(\tau)$ remains bounded, we have that both $x_1(\tau)$ and $x_2(\tau)$ increase linearly in τ, as $\tau \to +\infty$. This implies that the ancient solution is of type I and that, as $\tau \to +\infty$, $y(\tau) \to y_1$. □

To conclude the study of the Ricci flow in this case, we will consider the other two possible initial conditions (a)(1) and (a)(3). The following two theorems hold.

Theorem 2.8.9. *If $y(0) < y_1$, there exists a positive constant $T < \infty$ such that there exists a unique type I ancient solution to the Ricci flow equation defined on $(-\infty, T)$. As $t \to T$, the fibre H/K in (2.5) shrinks to a point and G/K collapses in the Hausdorff-Gromov sense to G/H. Furthermore, as $t \to -\infty$, $y(t) \to y_1$.*

Proof. As the initial condition is preserved under the Ricci flow, equation (2.21) implies that $y(t)$ is decreasing in t. By theorem 2.8.4, there exists $T < \infty$ such that there exists a unique solution to the HRF which is defined on the maximal time interval $[0, T)$. Moreover, as we approach the singular time $y(t) \to 0$. In fact, suppose that $y(t) \to y_0$, as $t \to T$, where $y_0 > 0$. This would be the case of both $x_1(t)$ and $x_2(t)$ going to zero, as $t \to T$. From the proof of theorem 2.8.4, we know that $x_1(t)$ and $x_2(t)$ tend to zero linearly in t. Using this fact and (2.14)-(2.15), we can compute that $y_0 < y_1$ has to be a G-invariant Einstein metric on G/K. However, this cannot happen, because G/K carries exactly two homogeneous Einstein metrics, which correspond to y_1 and y_2. We can then conclude that, as $t \to T$, $y(t) \to 0$, which means that $x_1(t)$ tends to zero, while $x_2(t)$ remains strictly positive. This tells us that the singular behaviour which characterises the HRF in this case is the shrinking of the fibre H/K in (2.5) and the collapsing of G/K to G/H in the Hausdorff-Gromov sense.

We will now show the existence of ancient solutions to the Ricci flow. As we did in the proof of theorem 2.8.8, let us change time parameter from t to $\tau = -t$. Then, from (2.24), we have that $y(\tau)$ is increasing in τ. As $y(\tau) < y_1$ for all τ such that a solution to the above system exists, the derivatives $x_1'(\tau)$ and $x_2'(\tau)$ remain bounded. We then have that, as τ increases, the solution to the Ricci flow exists as long as $x_1(\tau)$ and $x_2(\tau)$

remain positive and $y(\tau) < y_1$. As $x_1(\tau)$ and $x_2(\tau)$ are increasing in τ, the solution to the Ricci flow exists as long as $y(\tau) < y_1$. By the uniqueness of the solution, $y(\tau)$ cannot reach y_1 as long as $x_1(\tau)$ and $x_2(\tau)$ remain positive. Hence, the solution exists for all $\tau > 0$. Moreover, using the asymptotic analysis and the fact that $y(\tau)$ remains bounded, it is possible to compute that $x_1(\tau)$ and $x_2(\tau)$ both increase linearly in τ, as $\tau \to +\infty$. This fact implies that the ancient solution is of type I and, using (2.22) and (2.23), it also implies that $y(\tau) \to y_1$, as $\tau \to +\infty$. This concludes the proof of the theorem. □

Theorem 2.8.10. *If $y(0) > y_2$, there exists a positive constant $T < \infty$ such that there exists a unique solution to the Ricci flow equation defined on $[0, T)$. As $t \to T$, G/K shrinks to a point and $y(t) \to y_2$. In particular, there are no ancient solutions to the HRF in this case.*

Proof. By (2.21) and the fact that the initial condition is preserved under the Ricci flow, we have that $y(t)$ is decreasing in t. By theorem 2.8.4, there exists a positive constant $T < \infty$ such that there exists a unique solution to the Ricci flow with initial condition given by $y(0)$ and defined on the maximal time interval $[0, T)$. Theorem 2.8.4 also tells us that, as $t \to T$, $x_1(t) \to 0$. Moreover, as $y(t) > y_2$ for all $t \in [0, T)$, we have that $x_1(t) \to 0$ implies that $x_2(t) \to 0$, as $t \to T$. Hence, the behaviour of the Ricci flow, as t approaches the singular time T, is given by the shrinking of the whole space to a point in finite time. Moreover, by the proof of theorem 2.8.4, we know that $x_1(t)$ and $x_2(t)$ tend to zero linearly in t. Using this fact, from (2.12)-(2.13), we can compute that $y(t) \to y_2$, as $t \to T$.

It remains to show that, with this initial condition, there are no ancient solutions to the HRF. Let us change time parameter from t to $\tau = -t$. We are then considering the system given by (2.22)-(2.23). By (2.24), $y(\tau)$ is increasing in τ. The evolution equation (2.22) implies that $x_1(\tau)$ is increasing in τ. Moreover, $x_1'(\tau) > C > 0$. We then need to understand the behaviour of $x_2(\tau)$, for all τ such that a solution exists. If $x_2'(0) < 0$, by (2.23) $x_2'(\tau) < 0$ for all τ such that a solution exists. If $x_2'(0)$ is non negative, then $x_2'(\tau)$ will be positive until $\tau = \tau_0$ such that $x_2'(\tau_0) = 0$. In fact, $x_2'(\tau) = 0$ if and only if $y(\tau) = \frac{D}{B} > y_2$. Moreover, if $y(0) < \frac{D}{B}$, then $y(\tau)$ will become $\frac{D}{B}$ in finite time. In fact, if $y(\tau)$ approaches a limit, then this limit has to correspond to a homogeneous Einstein metric on G/K, but this is not possible because the only two invariant Einstein metrics are y_1 and y_2 and $y(\tau) > y_2$, for all τ such that a solution exists. As

$$x_2''(\tau) = -By'(\tau) < 0,$$

for all τ, τ_0 is the maximum point of $x_2(\tau)$. Then, $x_2'(\tau) < 0$ for all $\tau > \tau_0$. If a solution to (2.22)-(2.23) existed for all $\tau > 0$, then $x_1(\tau)$ would diverge to $+\infty$ and $x_2(\tau)$ would remain bounded and positive. From (2.23), this implies that $x_2'(\tau) \to -\infty$, as $\tau \to +\infty$, which is not possible. Hence, the solution will have to stop at a finite $\bar{\tau}$, which is characterised by $x_2(\tau)$ becoming zero. This concludes the proof of the theorem. □

Case (b)

We will now consider the case in which G/K admits exactly one G-invariant Einstein metric, up to scaling. Let $\bar{y}$ be the unique solution to (2.19). We then have that

$$\frac{\dot{x}_2(t)}{x_2(t)} = \frac{\dot{y}(t)}{A+B}\frac{D - y(t)}{(y(t) - \bar{y})^2},$$

which integrated gives the first integral

$$\widetilde{\Lambda} = \frac{1}{x_2(t)} \exp\left(- \frac{1}{A+B}\frac{D - y(t)}{y(t) - \bar{y}}\right) |y(t) - \bar{y}|^{-\frac{1}{A+B}}, \tag{2.25}$$

where $\widetilde{\Lambda}$ is a positive constant.

We will now describe the behaviour of the Ricci flow, according to the initial condition. We can have two possible initial conditions:

(b)(1) $y(0) < \bar{y}$,

(b)(2) $y(0) > \bar{y}$.

Remark 2.8.11. We note that the evolution equation of $y(t)$ is given by

$$\dot{y}(t) = -\frac{A+B}{x_2(t)}(y(t) - \bar{y})^2.$$

Hence, $y(t)$ is always decreasing in t.

Using the first integral (2.25) and the fact that the solution to the HRF is unique we can prove the following proposition, which is an analogue of proposition 2.8.7.

Proposition 2.8.12. *The initial conditions* (b)(1) *and* (b)(2) *above are preserved under the Ricci flow.*

We then have that the following theorem holds:

Theorem 2.8.13. *If $y(0) < \bar{y}$, there exists a positive constant $T < \infty$ such that there exists a unique type I ancient solution to the Ricci flow* (2.12)-(2.13) *with initial condition $y(0)$ and defined on the maximal time interval $(-\infty, T)$. As $t \to T$, the fibre H/K in* (2.5) *shrinks to a point and G/K collapses to G/H in the Hausdorff-Gromov sense. Moreover, as $t \to -\infty$, $y(t) \to \bar{y}$.*

On the other hand, if $y(0) > \bar{y}$, there exists a positive constant $T < \infty$ such that there exists a unique solution to the Ricci flow with initial metric given by $y(0)$ and defined on the maximal time interval $[0, T)$. As $t \to T$, the whole space G/K shrinks to a point and $y(t) \to \bar{y}$. In particular, there are no ancient solutions to the Ricci flow with initial condition $y(0)$.

To prove the first part of the theorem, we proceed as in the proof of theorem 2.8.9 and the proof of the second part of the theorem works in the same way as the one of theorem 2.8.10. As these proofs are very similar, we will omit it.

Case (c)

In this case, by theorem 2.8.4, the behaviour of the Ricci flow starting at any G-invariant Riemannian metric will be characterised by the shrinking of the fibre H/K in (2.5) and the collapsing in the Hausdorff-Gromov sense of G/K to G/H. Moreover, the following proposition is true.

Proposition 2.8.14. *If G/K described above does not carry any G-invariant Einstein metric, then the Ricci flow on G/K starting at any homogeneous Riemannian metric does not have ancient solutions.*

Proof. Let us change time parameter from t to $\tau = -t$. In particular, we have that

$$y'(\tau) = \frac{1}{x_2(t)}\big(C - Dy(\tau) + (A+B)y(\tau)^2\big),$$

where, as before, $'$ refers to the derivative with respect to τ. From this, we can see that $y(\tau)$ is increasing in τ. Now, consider the system (2.22)-(2.23). From the evolution equation of $x_1(\tau)$, we have that it is always increasing in τ. Moreover, $x_1'(\tau) > C > 0$, for all τ such that a solution exists. We will now determine the behaviour of $x_2(\tau)$. We need to distinguish between two possible situation, depending on the sign of $x_2'(0)$. If $x_2'(0) < 0$, then $x_2'(\tau) < 0$ for all τ such that a solution to the HRF exists. On the other hand, if $x_2'(0)$ is non negative, $x_2(\tau)$ will be increasing in τ until $\tau = \tau_0$ such that $x_2'(\tau_0) = 0$. As $x_2''(\tau) = -By'(\tau)$, τ_0 is the maximum point for $x_2(\tau)$. Then, $x_2(\tau)$ will be decreasing in τ for all $\tau > \tau_0$. We note that τ_0 has to exist, because $y(\tau)$ cannot approach a finite limit, as this has to correspond to a homogeneous Einstein metric on G/K. If a solution existed for all $\tau > +\infty$, then $x_1(\tau)$ would diverge to $+\infty$, as $\tau \to +\infty$. Furthermore, $x_2(\tau)$ would remain bounded and positive. This implies that the first derivative of $x_2(\tau)$ would diverge to $-\infty$, as $\tau \to +\infty$, which is not possible. We can then conclude that the solution has to stop at a finite $\bar{\tau}$, which is characterised by $x_2(\tau)$ becoming zero. □

To conclude this section, we will consider G/K as described above and such that $C = 0$. In this case, equation (2.16) becomes

$$\dot{y}(t) = \frac{1}{x_2(t)}(Dy(t) - (A+B)y(t)^2).$$

Hence, G/K always admits a G-invariant Einstein metric which is unique up to scaling. Let

$$\bar{y} = \frac{D}{A+B}$$

denote this homogeneous Einstein metric. Then the evolution equation of $y(t)$ becomes

$$\dot{y}(t) = -(A+B)\frac{y(t)}{x_2(t)}(y(t) - \bar{y}).$$

We have two possible initial conditions: $y(0) < \bar{y}$ or $y(0) > \bar{y}$. These are preserved under the HRF, because we have uniqueness of the solution. Then, if $y(0) > \bar{y}$, $y(t)$ will be decreasing in t and it will be increasing in t if $y(0) < \bar{y}$. Moreover, $\bar{y}$ is fixed point of the HRF, so the regions $\{y > \bar{y}\}$ and $\{y < \bar{y}\}$ are preserved. We can prove the following theorem.

Theorem 2.8.15. *If $y(0) > \bar{y}$, there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$. T is a type I singularity and, as $t \to T$, G/K shrinks to a point and $y(t)$ approaches the unique invariant Einstein metric $\bar{y}$. Moreover, there are no ancient solutions.*

Whereas, if $y(0) < \bar{y}$, there exists $T < \infty$ such that there exists a unique type II ancient solution to the HRF on G/K which is defined on $(-\infty, T)$. T is a type I singularity and, as $t \to T$, G/K shrinks to a point and $y(t)$ approaches $\bar{y}$. Finally, as $t \to -\infty$, $y(t) \to 0$ and G/K collapses in the Hausdorff-Gromov sense to G/H.

Proof. The HRF in this case corresponds to the following system of nonlinear ODEs:

$$\dot{x}_1(t) = -Ay^2(t),$$
$$\dot{x}_2(t) = -D + By(t).$$

We will first show that the HRF always develops a singularity in finite time. In fact, if $y(0) < \bar{y}$, then $y(t) < \bar{y}$, which implies that

$$\dot{x}_2(t) < -D + B\bar{y} < 0.$$

Hence the HRF has to stop before $x_2(t)$ becomes zero. Moreover, $x_2(t) \to 0$ implies that $x_1(t) \to 0$, because $y(t) = \frac{x_1(t)}{x_2(t)} < \bar{y}$, for all t such that a solution exists. On the other hand, if $y(0) > \bar{y}$, we have that

$$\dot{x}_1(t) < -A\bar{y}^2 < 0,$$

so the HRF will have to stop before $x_1(t)$ becomes zero. This implies that also $x_2(t)$ tends to zero, because $y(t) > \bar{y}$, for all t such that a solution exists. Hence, every solution to the HRF will develop a singularity in a finite time T and, as $t \to T$, $y(t) \to \bar{y}$. Moreover, we can classify this singularity to be of type I, because both $x_1(t)$ and $x_2(t)$ tend to zero linearly in t.

We will now investigate the existence of ancient solutions. As in the proof of theorem 2.8.10, we can show that there are no ancient solutions when $y(0) > \bar{y}$, because $x_2(t)$ becomes zero in finite time. Whereas, if $y(0) < \bar{y}$, both $x_1(\tau)$ and $x_2(\tau)$ are increasing with bounded derivatives for all $\tau \geq 0$. Hence, there always exists a unique ancient solution to the HRF with initial condition $y(0)$. We can then classify this ancient solution as of type II. This is due to the fact that, as $t \to -\infty$, $y(t) \to 0$, which implies that

$x_1(t)$ increases logarithmically in t, while $x_2(t)$ increases linearly in t. Hence,

$$|\operatorname{Rm}(g(t))|^2_{g(t)} \geq \frac{1}{|t|},$$

as $t \to -\infty$. This implies that the ancient solution is of type II. □

2.8.2 When the isotropy group is maximal

Let G/K be a compact and connected homogeneous space such that it also effective. Suppose that the isotropy representation of K decomposes into two inequivalent irreducible $\mathrm{Ad}_{|K}$-invariant summands:

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2.$$

Suppose further that K is maximal in G. This implies that the structure constants [112] and [122] are both nonzero. Then,

$$g(t) = x_1(t)Q_{|\mathfrak{p}_1} \oplus x_2(t)Q_{|\mathfrak{p}_2}$$

is a solution to the Ricci flow on G/K if and only if $x_1(t)$ and $x_2(t)$ satisfy the following system of nonlinear ODEs:

$$\dot{x}_1(t) = -\left(b_1 - \frac{[111]}{2d_1} - \frac{[122]}{d_1}\right) + \frac{[112]}{d_1}\frac{x_2(t)}{x_1(t)} - \frac{[122]}{2d_1}\frac{x_1(t)^2}{x_2(t)^2}, \tag{2.26}$$

$$\dot{x}_2(t) = -\left(b_2 - \frac{[222]}{2d_2} - \frac{[112]}{d_2}\right) + \frac{[122]}{d_2}\frac{x_1(t)}{x_2(t)} - \frac{[112]}{2d_2}\frac{x_2(t)^2}{x_1(t)^2}, \tag{2.27}$$

together with the condition $x_1(t), x_2(t) > 0$.

Let

$$A_1 = b_1 - \frac{[111]}{2d_1} - \frac{[122]}{d_1},\ B_1 = \frac{[112]}{d_1},\ C_1 = \frac{[122]}{2d_1}, \tag{2.28}$$

$$A_2 = b_2 - \frac{[222]}{2d_2} - \frac{[112]}{d_2},\ B_2 = \frac{[122]}{d_2},\ C_2 = \frac{[112]}{2d_2}. \tag{2.29}$$

Remark 2.8.16. Because of the relations (2.3) and the fact that K is maximal in G, the quantities A_i, B_i and C_i, with $i = 1, 2$, defined above are strictly positive.

Consider

$$y(t) = \frac{x_1(t)}{x_2(t)}.$$

Then, the system (2.26)-(2.27) becomes

$$\dot{x}_1(t) = -A_1 + \frac{B_1}{y(t)} - C_1 y^2(t), \tag{2.30}$$

$$\dot{x}_2(t) = -A_2 + B_2\, y(t) - \frac{C_2}{y(t)^2}. \tag{2.31}$$

The evolution equation for $y(t)$ is given by

$$\begin{aligned}\dot{y}(t) &= \frac{1}{y(t)x_2(t)}\left(y(t)\dot{x}_1(t) - y(t)^2\dot{x}_2(t)\right)\\ &= \frac{1}{y(t)x_2(t)}\Big(- \big(B_2 + C_1\big)\, y(t)^3 + A_2\, y(t)^2 - A_1 y(t) + B_1 + C_2\Big).\end{aligned}$$

Let

$$g_1(y(t)) = y(t)\dot{x}_1(t) = -C_1 y(t)^3 - A_1\, y(t) + B_1,$$

$$g_2(y(t)) = y(t)^2\dot{x}_2(t) = B_2\, y(t)^3 - A_2\, y(t)^2 - C_2,$$

which are two cubics in $y(t)$ with the following properties. The cubic $g_1(y(t))$ tends to $-\infty$ when $y(t) \to +\infty$ and it equals $B_1 > 0$ when $y(t) = 0$. Moreover, $g_1(y(t))$ is monotonically decreasing in $y(t)$. Hence, it only has one root, which is strictly positive.

On the other hand, the cubic $g_2(y(t))$ tends to $+\infty$ when $y(t) \to +\infty$ and it becomes $-C_2 < 0$ when $y(t) = 0$. We also have that $g_2(y(t))$ has two critical points: one at $y(t) = 0$, which is a local maximum, and one at a positive $y(t)$, which is a local minimum. Hence, as $y(t)$ increases, $g_2(y(t))$ decreases until it reaches its minimum and then it increases monotonically in $y(t)$. We can then conclude that also $g_2(y(t))$ has only one root, which is strictly positive. As $y(t) > 0$ for all t, the zeroes of $g_1(y(t))$ and $g_2(y(t))$ correspond exactly to the critical points of $\dot{x}_1(t)$ and $\dot{x}_2(t)$, respectively. We denote these points by $\tilde{y}_1$ and $\tilde{y}_2$, respectively. Then, note that $y(t)$ is a root of the equation

$$-\big(B_2 + C_1\big)\, y(t)^3 + A_2\, y(t)^2 - A_1 y(t) + B_1 + C_2 = 0 \tag{2.32}$$

if and only if $g_1(y(t)) = g_2(y(t))$. We have that the roots of (2.32) are always strictly positive. In fact, if $y(t)$ is negative, by remark 2.8.16, the above expression is strictly positive. Moreover, these roots are located between $\tilde{y}_1$ and $\tilde{y}_2$ defined above. In particular, this tells us that $\tilde{y}_1 < \tilde{y}_2$. In fact, if $\tilde{y}_2 < \tilde{y}_1$, $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are positive for all t such that $y(t) \in (\tilde{y}_2, \tilde{y}_1)$. Now, the fact that the roots of (2.32) are all strictly positive implies that equation (2.32) has solutions if and only if G/K carries homogeneous Einstein metrics. As the space is compact, every invariant Einstein metric has positive scalar curvature. If we start the Ricci flow at a homogeneous Einstein metric, the whole space shrinks to a point in finite time. So, $x_1(t)$ and $x_2(t)$ cannot be increasing when $y(t)$ equals to an invariant Einstein metric. Hence $\tilde{y}_1 < \tilde{y}_2$ and $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are negative for all t such that $y(t) \in (\tilde{y}_1, \tilde{y}_2)$.

By the above discussion, G/K carries at most three G-invariant Einstein metrics, up to scaling, which correspond to the roots of (2.32). In particular, G/K carries at least one G-invariant Einstein metric (cf. [49, Theorem 2.2]). In fact,

$$-\big(B_2 + C_1\big)\, y^3 + A_2\, y^2 - A_1 y + B_1 + C_2$$

tends to $-\infty$ when $y \to +\infty$ and it becomes $B_1 + C_2 > 0$ when $y = 0$, which implies that it has to have at least one positive root. Hence we need to distinguish the following three cases:

(d) G/K carries three homogeneous Einstein metrics, up to scaling;

(e) G/K carries two homogeneous Einstein metrics, up to scaling;

(f) G/K carries one homogeneous Einstein metric, up to scaling.

Remark 2.8.17. Note that $y(t)$ is monotone along any solution to the HRF. This is due to the uniqueness of the solution and the fact that the critical points of $y(t)$ correspond to homogeneous Einstein metrics on G/K.

Before considering the three cases listed above separately, some remarks about the ODE are as follows. By standard ODE theory, there exists $T \leq \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$. Moreover, the solution exists as long as both functions are positive and the norm of the solution is bounded. Consider the phase space

$$X = \{(x_1, x_2) \in \mathbb{R}^2, x_1, x_2 > 0\},$$

to which the solution $(x_1(t), x_2(t))$ to the HRF belongs. We have that $x_1 = \tilde{y}_1 x_2$ and $x_1 = \tilde{y}_2 x_2$ define two lines in X, which separate the regions in which $x_1(t)$ and $x_2(t)$ are monotonically increasing or decreasing. As $\tilde{y}_1 < \tilde{y}_2$, X can be divided into three connected regions:

i) $X_1 = \{(x_1, x_2) \in X, x_1 < \tilde{y}_1 x_2\}$,

ii) $X_2 = \{(x_1, x_2) \in X, \tilde{y}_1 x_2 < x_1 < \tilde{y}_2 x_2\}$,

iii) $X_3 = \{(x_1, x_2) \in X, x_1 > \tilde{y}_2 x_2\}$.

We note that the only region which is invariant under the HRF is the one given by X_2 above, as the tangent vector $(\dot{x}_1(t), \dot{x}_2(t))$ on boundary

$$\{(x_1, x_2) \in X, x_1 = \tilde{y}_1 x_2\} \cup \{(x_1, x_2) \in X, x_1 = \tilde{y}_2 x_2\}$$

of X_2 always points inside it. Moreover, in this region both $x_1(t)$ and $x_2(t)$ are monotonically decreasing. On the other hand, X_1 is characterised by $x_1(t)$ being monotonically increasing and bounded and $x_2(t)$ being monotonically decreasing. Whereas, in X_3, $x_1(t)$ is monotonically decreasing and $x_2(t)$ is monotonically increasing and bounded. We then have that if $(x_1(0), x_2(0))$ belongs to X_1 or X_3 above, the HRF $(x_1(t), x_2(t))$ with this initial condition will always enter the region X_2 in finite time and will stay there. Once in X_2, as $x_1(t)$ and $x_2(t)$ are monotonically decreasing in t, the solution to the HRF exists and it is unique as long as both $x_1(t)$ and $x_2(t)$ are strictly positive. By the mean value theorem and (2.26)-(2.27) $x_1(t)$ and $x_2(t)$ can only go to zero simultaneously, as $t \to T$, and in such a way that $\frac{x_1(t)}{x_2(t)}$ remains bounded and strictly positive. Hence, the flow will stop at a finite time T, which is characterised by both $x_1(t)$ and $x_2(t)$ becoming zero.

Case (d)

In this case, there are three different solutions to (2.32). We will denote them by y_1, y_2 and y_3. Suppose without loss of generality that $y_1 < y_2 < y_3$. The evolution equation of $y(t)$ is given by

$$\dot{y}(t) = -\frac{1}{x_2(t)y(t)}\big(B_2 + C_1\big)\big(y(t) - y_1\big)\big(y(t) - y_2\big)\big(y(t) - y_3\big).$$

We have four possible initial conditions:

(d)(1) $y(0) < y_1$,

(d)(2) $y_1 < y(0) < y_2$,

(d)(3) $y_2 < y(0) < y_3$,

(d)(4) $y(0) > y_3$.

Remark 2.8.18. We observe that if we start the Ricci flow at $y(0) = y_i$, then the solution will be given by $y(t) = y_i$, for all $i = 1, \ldots, 4$.

Because of the above remark and the uniqueness of the solution, we have the following proposition.

Proposition 2.8.19. *The initial conditions* (d)(1)*,* (d)(2)*,* (d)(3) *and* (d)(4) *are preserved under the HRF.*

We will now analyse these different initial conditions separately. We begin by noticing the following. If $y(0)$ satisfies (d)(1) or (d)(3), then $y(t)$ will be increasing in t. On the contrary, if $y(0)$ satisfies (d)(2) or (d)(4), then $y(t)$ will be decreasing in t. Moreover, as

$$\begin{aligned}
\ddot{x}_1(t) &= -\frac{B_1}{y(t)^2}\dot{y}(t) - 2C_1 y(t)\dot{y}(t) = -\left(\frac{B_1}{y(t)^2} + 2C_1 y(t)\right)\dot{y}(t), \\
\ddot{x}_2(t) &= B_2\dot{y}(t) + 2\frac{C_2}{y(t)^3}\dot{y}(t) = \left(B_2 + 2\frac{C_2}{y(t)^3}\right)\dot{y}(t),
\end{aligned}$$

we also have that $\tilde{y}_1$ and $\tilde{y}_2$ are maximum points of $x_1(t)$ and $x_2(t)$, respectively. By performing an ODE analysis of (2.30)-(2.31) very similar to the one previously done, we can prove the following theorem.

Theorem 2.8.20. *If* (d)(1) *is satisfied, then there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K defined on the maximal time interval* $[0, T)$*. T is a type I singularity and, as $t \to T$, the whole space shrinks to a point and $y(t)$ approaches the invariant Einstein metric y_1. In this case, there are no ancient solutions.*

If (d)(2) *is satisfied, then there exists $T < \infty$ such that there exists a unique type I ancient solution to the HRF on G/K which defined on the maximal time interval*

$(-\infty, T)$. *T is a type I singularity and, as $t \to T$, G/K shrinks to a point and $y(t) \to y_1$. Moreover, as $t \to -\infty$, $y(t) \to y_2$.*

If (d)(3) *is satisfied, then there exists $T < \infty$ such that there exists a unique type I ancient solution to the HRF on G/K which is defined on the maximal time interval* $(-\infty, T)$. *T is a type I singularity and, as $t \to T$, G/K shrinks to a point and $y(t) \to y_3$. Furthermore, as $t \to -\infty$, $y(t) \to y_2$.*

Finally, if (d)(4) *is satisfied, then there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K defined on the maximal time interval* $[0, T)$. *T is a type I singularity and, as $t \to T$, G/K shrinks to a point and $y(t) \to y_3$. There are no ancient solutions in this case.*

Case (e)

We suppose now that (2.32) has 2 distinct roots. Let us denote them by y_1 and y_2. In this case, we can write the evolution equation of $y(t)$ in the following way:

$$\dot{y}(t) = -\frac{1}{x_2(t)y(t)}\big(B_2 + C_1\big)\big(y(t) - y_1\big)\big(y(t) - y_2\big)^2.$$

We then need to distinguish two possible situations, i.e. $y_1 < y_2$ or $y_1 > y_2$. Moreover, for each one of them, we have three possible initial conditions. If $y_1 < y_2$, we can have

(e)(1) $y(0) < y_1$,

(e)(2) $y_1 < y(0) < y_2$,

(e)(3) $y(0) > y_2$.

In this case, $y(t)$ will be increasing in t for $y(0)$ satisfying (e)(1) and it will be decreasing in t otherwise. If $y_1 > y_2$, we can have

(e)(4) $y(0) < y_2$,

(e)(5) $y_2 < y(0) < y_1$,

(e)(6) $y(0) > y_1$.

Then, $y(t)$ will be increasing in t when $y(0)$ satisfies (e)(4) or (e)(5) and it will be decreasing in t otherwise.

Note that, in both situations, because of the uniqueness of the solution to the HRF, these initial conditions are preserved forward and backwards in time. By studying the ODEs (2.30)-(2.31) in the various possible situations, we can prove the following theorem.

Theorem 2.8.21. *Suppose that $y_1 < y_2$ (resp. $y_1 > y_2$). Then if $y(0) < y_1$ (resp. $y(0) < y_2$), there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$. T is a type I singularity and, as t $\to T$, G/K shrinks to a point and $y(t) \to y_1$ (resp. $y(t) \to y_2$). There are no ancient solutions in this case.*

If $y_1 < y(0) < y_2$ (resp. $y_2 < y(0) < y_1$), there exist a positive constant $T < \infty$ such that there exists a unique type I ancient solution on G/K which is defined on the maximal time interval $(-\infty, T)$. T is a type I singularity and, as t $\to T$, G/K shrinks to a point. Moreover, the HRF flows $y(t)$ from y_2 (resp. y_1) to y_1 (resp. y_2), as t goes from $-\infty$ to T.

Finally, if $y(0) > y_2$ (resp. $y(0) > y_1$), there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$. T is a type singularity and, as t $\to T$, G/K shrinks to a point and $y(t) \to y_2$ (resp. $y(t) \to y_1$). In particular, there are no ancient solutions.

Case (f)

In this case, equation (2.32) has exactly one root, which will be denoted by $\bar{y}$. We have two possible situations. Either $\bar{y}$ has order three or it has order one. In the first case, the evolution equation for $y(t)$ is given by

$$\dot{y} = -\frac{1}{x_2(t)y(t)}\big(B_2 + C_1\big)\big(y(t) - \bar{y}\big)^3.$$

In the second case, the evolution equation for $y(t)$ becomes

$$\dot{y} = -\frac{1}{x_2(t)y(t)}\big(B_2 + C_1\big)\big(y(t) - \bar{y}\big)P(y(t)),$$

where $P(y(t))$ is a polynomial of degree two in $y(t)$ which is strictly positive for all t. In both cases, if $y(0) < \bar{y}$, $y(t)$ will be increasing in t, while, if $y(0) > \bar{y}$, $y(t)$ will be decreasing in t. Note that because of the uniqueness of the solution to the HRF, these initial conditions are preserved. Then, the analysis of the ODE system (2.30)-(2.31) leads to the following theorem.

Theorem 2.8.22. *There exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$. T is a type singularity and, as $t \to T$, G/K shrinks to a point and $y(t) \to \bar{y}$. In particular, there are no ancient solutions.*

2.9 A more general case

In this section, we are going to generalise theorem 2.8.4 to a more general class of compact homogeneous spaces. We will also analyse the Ricci soliton that we obtain by blowing up the solution to the HRF near the singularity and, to conclude, we will give a few examples of homogeneous spaces to which the theorem can be applied. In particular, some of these are also examples of compact homogeneous spaces which do not carry invariant Einstein metrics and can be found in [8].

2.9.1 Singular behaviours

Let G/K be a compact and connected homogeneous space and such that G acts effectively on G/K. Suppose that there exists an intermediate Lie group H, with $K < H < G$, such that H/K is isotropy irreducible. Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of H and K, respectively. Suppose that $\mathfrak{h}$ is $\mathrm{Ad}_{|_K}$-invariant. Note that this is automatically true if both G and H are connected. Let $\mathfrak{p}$ be the isotropy representation of K. We assume that $\mathfrak{p}$ decomposes into pairwise inequivalent irreducible $\mathrm{Ad}_{|_K}$-invariant summands:

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_l.$$

Suppose, without loss of generality, that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1$. We will also assume that $\mathfrak{p}_i$ is an H-module, for all $i = 2, \ldots, l$. Hence, every G-invariant Riemannian metric is given by a submersion metric

$$H/K \to G/K \to G/H. \tag{2.33}$$

We will first of all derive the system of nonlinear ODEs which is equivalent to the Ricci flow in this case. Choose a background metric Q such that, when restricted to each of the $\mathfrak{p}_i$'s, it is a negative multiple of the Killing form B. Then, the following structure constants on G/K vanish:

$$\begin{aligned} &[11i] = 0,\ 2 \le i \le l, \\ &[1ij] = 0,\ 2 \le i < j \le l. \end{aligned} \tag{2.34}$$

By (2.34), the Ricci endomorphism on $\mathfrak{p}_1$ is given by

$$\begin{aligned}
r_{|\mathfrak{p}_1} &= \left(\frac{b_1}{2x_1} - \frac{1}{2d_1} \sum_{j,k=1}^{l} [1jk] \frac{x_k}{x_1 x_j} + \frac{1}{4d_1} \sum_{j,k=1}^{l} [1jk] \frac{x_1}{x_k x_j} \right) \mathrm{id}_{|\mathfrak{p}_1} \\
&= \left(\frac{b_1}{2x_1} - \frac{1}{2d_1} \frac{[111]}{x_1} - \frac{1}{2d_1} \sum_{j=2}^{l} \frac{[1jj]}{x_1} + \frac{1}{4d_1} \frac{[111]}{x_1} + \frac{1}{4d_1} \sum_{j=2}^{l} [1jj] \frac{x_1}{x_j^2} \right) \mathrm{id}_{|\mathfrak{p}_1} \\
&= \left(\left(\frac{b_1}{2} - \frac{[111]}{4d_1} - \frac{1}{2d_1} \sum_{j=2}^{l} [1jj] \right) \frac{1}{x_1} + \frac{1}{4d_1} \sum_{j=2}^{l} [1jj] \frac{x_1}{x_j^2} \right) \mathrm{id}_{|\mathfrak{p}_1},
\end{aligned}$$

and the Ricci endomorphism on $\mathfrak{p}_i$ is given by

$$\begin{aligned}
r_{|\mathfrak{p}_i} =& \left(\frac{b_i}{2x_i} - \frac{1}{2d_i} \sum_{j,k=1}^{l} [ijk] \frac{x_k}{x_i x_j} + \frac{1}{4d_i} \sum_{j,k=1}^{l} [ijk] \frac{x_i}{x_k x_j} \right) \mathrm{id}_{|\mathfrak{p}_i} \\
=& \left(\frac{b_i}{2x_i} - \frac{1}{2d_i} \sum_{j,k \notin \{1,i\}, j \neq k} [ijk] \frac{x_k}{x_i x_j} - \frac{1}{2d_i} \sum_{j \notin \{1,i\}} \frac{[ijj]}{x_i} - \frac{1}{2d_i} \sum_{k \notin \{i\}} [iik] \frac{x_k}{x_i^2} \right. \\
& - \frac{1}{2d_i} \frac{[iii]}{x_i} - \frac{1}{2d_i} \sum_{j \notin \{i\}} \frac{[jii]}{x_j} + \frac{1}{4d_i} \sum_{k \notin \{i\}} \frac{[iik]}{x_k} + \frac{1}{4d_i} \frac{[iii]}{x_i} + \frac{1}{4d_i} \sum_{j \notin \{i\}} \frac{[iij]}{x_j} \\
& \left. + \frac{1}{4d_i} \sum_{j,k \notin \{1,i\}, j \neq k} [ijk] \frac{x_i}{x_j x_k} + \frac{1}{4d_i} \sum_{j \notin \{1,i\}} [ijj] \frac{x_i}{x_j^2} \right) \mathrm{id}_{|\mathfrak{p}_i} \\
=& \left(\left(\frac{b_i}{2} - \frac{[iii]}{4d_i} - \frac{1}{2d_i} \sum_{j \notin \{1,i\}} [ijj] \right) \frac{1}{x_i} - \frac{1}{2d_i} \sum_{j,k \notin \{1,i\}, j \neq k} [ijk] \frac{x_k}{x_i x_j} \right. \\
& \left. + \frac{1}{4d_i} \sum_{j,k \notin \{1,i\}, j \neq k} [ijk] \frac{x_i}{x_j x_k} + \frac{1}{4d_i} \sum_{j \notin \{1,i\}} [ijj] \frac{x_i}{x_j^2} - \frac{1}{2d_i} \sum_{k \notin \{i\}} [iik] \frac{x_k}{x_i^2} \right) \mathrm{id}_{|\mathfrak{p}_i},
\end{aligned}$$

for all $i = 2, \ldots, l$. We use the convention that the sum over an empty set of indices is zero. Then,

$$g(t) = x_1(t) Q_{|\mathfrak{p}_1} \oplus \cdots \oplus x_l(t) Q_{|\mathfrak{p}_l}$$

is a solution to the Ricci flow equation on G/K if and only if $x_1(t), \ldots, x_l(t)$ satisfy the following system of nonlinear ODEs:

$$\dot{x}_1(t) = -\left(b_1 - \frac{[111]}{2d_1} - \frac{1}{d_1}\sum_{j=2}^{l}[1jj]\right) - \frac{1}{2d_1}\sum_{j=2}^{l}[1jj]\frac{x_1(t)^2}{x_j(t)^2}, \tag{2.35}$$

$$\begin{aligned} \dot{x}_i(t) = &-\left(b_i - \frac{[iii]}{2d_i} - \frac{1}{d_i}\sum_{j\notin\{1,i\}}[ijj]\right) - \frac{1}{2d_i}\sum_{j\notin\{1,i\}}[ijj]\frac{x_i(t)^2}{x_j(t)^2} \\ &- \frac{1}{2d_i}\sum_{j,k\notin\{1,i\},j\neq k}[ijk]\frac{x_i(t)^2}{x_j(t)x_k(t)} + \frac{1}{d_i}\sum_{j,k\notin\{1,i\},j\neq k}[ijk]\frac{x_k(t)}{x_j(t)} \\ &+ \frac{1}{d_i}\sum_{j\notin\{i\}}[iij]\frac{x_j(t)}{x_i(t)}, \end{aligned} \tag{2.36}$$

for all $i = 2, \ldots, l$, together with the condition that $x_1(t), \ldots, x_l(t) > 0$.

Definition 2.9.1. Let $J \subseteq \{1, \ldots, l\}$ be the set of indices such that, if $i \in J$, then $[iij] = 0$ and $[ijk] = 0$, for all $j, k \neq i$, with $j \neq k$.

Remark 2.9.2. More precisely, $i \in J$ means that $\mathfrak{k} \oplus \mathfrak{p}_i$ is an intermediate Lie algebra such that $\mathfrak{p}_j$, with $j \neq i$, is an H_i-module, where H_i is the intermediate Lie group with Lie algebra $\mathfrak{k}\oplus\mathfrak{p}_i$. Hence, the only structure constants involving i which could be nonzero are given by $[iii]$ and $[ijj]$, with $j \notin J$. In particular, we have that

$$\mathfrak{k} \oplus \bigoplus_{i\in J} \mathfrak{p}_i$$

is an intermediate Lie algebra. Note that J is nonempty, as $1 \in J$.

Moreover, observe that if $J = \{1 \ldots l\}$, then G/K is such that its universal cover is a product of isotropy irreducible homogeneous spaces and the analysis reduces to the one performed in section 2.7 on each summand.

Finally, if $i \in J$, equation (2.35) simplifies and $x_i(t)$ is decreasing in t, because all positive terms have vanishing coefficients. In particular, there are no terms which contain $\frac{1}{x_i(t)}$.

Now, let

$$\boldsymbol{D} = \{(x_1, \ldots, x_l) \in \mathbb{R}^l | x_i \neq 0, \text{ for all } i \in \{1, \ldots, l\} \backslash J\}.$$

Suppose that the initial time is given by $t_0 = 0$. We can then apply a standard theorem of ODEs, see for example [11, Theorem 1.1], which says that, given any initial condition $((x_1)_0, \ldots, (x_l)_0) \in \boldsymbol{D}$, there exists a unique solution $(x_1(t), \ldots, x_l(t))$ to the above system such that

$$(x_1(0), \ldots, x_l(0)) = ((x_1)_0, \ldots, (x_l)_0)$$

and which depends continuously on t and the initial data. Moreover, the solution $(x_1(t), \ldots, x_l(t))$ exists on any interval I containing 0 and such that $(x_1(t), \ldots, x_l(t)) \in \boldsymbol{D}$, for every $t \in I$.

We observe that some of the coefficients of the system may vanish, depending on which intermediate invariant Lie algebras there are between $\mathfrak{g}$ and $\mathfrak{k}$. In particular, using (2.3), we have that

$$\begin{aligned} A_1 :=& b_1 - \frac{[111]}{2d_1} - \frac{1}{d_1}\sum_{j=2}^{l}[1jj] = 2c_1 + \frac{[111]}{2d_1} \geq 0, \\ A_i :=& b_i - \frac{[iii]}{2d_i} - \frac{1}{d_i}\sum_{j\notin\{1,i\}}[ijj] \\ =& 2c_i + 2\sum_{j\notin\{i\}}\frac{[iij]}{d_i} + \frac{[iii]}{2d_i} + \sum_{j,k\notin\{1,i\}, j\neq k}\frac{[ijk]}{d_i} \geq 0, \end{aligned}$$

for every $i = 2, \ldots, l$. More precisely, $A_i = 0$, with $i \in \{1, \ldots, l\}$, if and only if $\mathfrak{p}_i$ is a trivial summand.

From equation (2.35), we see that $x_1(t)$ is decreasing in t. So, $x_1(t)$ always approaches a finite limit, as $t \to T$, where $T \leq +\infty$ is the maximal existence time. We would also like to show that the functions $x_2(t), \ldots, x_l(t)$ all have a limit, as $t \to T$.

Conjecture 2.9.3. *The functions $x_2(t), \dots, x_l(t)$ approach limits, which could be $+\infty$, as $t \to T$.*

Note that this conjecture is true when $l = 2$ and, as we will see in section 2.9.2, it is also true when $l = 3$.

We will now assume for the rest of this section that this conjecture is true.

We will also assume that the isotropy representation $\mathfrak{p}$ does not contain any trivial summand. In particular, this implies that $A_1 > 0$. Then,

$$\dot{x}_1(t) \leq -\left(b_1 - \frac{[111]}{2d_1} - \frac{1}{d_1} \sum_{j=2}^{l} [1jj] \right),$$

integrating the above expression, we obtain

$$x_1(t) \leq -\left(b_1 - \frac{[111]}{2d_1} - \frac{1}{d_1} \sum_{j=2}^{l} [1jj] \right) t + x_1(0) := -Ct + x_1(0).$$

Hence, there exists $T < \frac{x_1(0)}{C} < \infty$ such that the unique solution to the Ricci flow will be defined on the maximal time interval $[0, T)$, otherwise $x_1(t)$ becomes negative. We will now identify a class of singular behaviours to the HRF on G/K.

Remark 2.9.4. As $x_i(t)$, with $i \in J$, is decreasing in t, it is also bounded from above for any t such that a solution exists.

We then have the following proposition, which shows that all the functions are bounded, as $t \to T$.

Proposition 2.9.5. *The functions $x_i(t)$, with $i \notin J$, remain bounded, as $t \to T$.*

Proof. We first of all note that, if as $t \to T$, $x_i(t)$, with $i \notin J$, all go to infinity at the same rate, then the derivatives $\dot{x}_i(t)$, with $i \notin J$, have to remain bounded. In fact, we have the following derivative estimate:

$$\dot{x}_i(t) < \frac{1}{d_i} \sum_{j,k \notin J \cup \{i\}, j \neq k} [ijk] \frac{x_k(t)}{x_j(t)} + \frac{1}{d_i} \sum_{j \notin \{i\}} [iij] \frac{x_j(t)}{x_i(t)}$$

for $i \notin J$. Hence, if $\frac{x_m(t)}{x_i(t)}$, $\frac{x_j(t)}{x_i(t)}$ and $\frac{x_k(t)}{x_j(t)}$ are bounded for all $m \in J$, $i \notin J$ and $j, k \notin J \cup \{i\}$, we obtain that

$$\dot{x}_i(t) < C_i,$$

where C_i is a positive constant, for all $i \notin J$. Integrating this inequality leads to

$$x_i(t) < tC_i + x_i(0).$$

As the Ricci flow has to stop before the finite time T, this would imply that $x_i(t)$ has to remain bounded for all $i \notin J$, which is a contradiction.

We will now prove in general that the derivatives of $x_i(t)$, with $i = 2, \ldots, l$, are all bounded, as $t \to T$, which will give a contradiction as above. Suppose that there exists $I \subseteq \{2, \ldots, l\}$ such that $x_i(t)$ blows up to $+\infty$, as $t \to T$, if and only if $i \in I$. Define $I' \subseteq I$ as the set of indices $i \in I$ such that $\frac{x_i(t)}{x_j(t)}$ either blows up to $+\infty$ or remains bounded and strictly positive, as $t \to T$, for all $j \in I \backslash \{i\}$. In particular, if $i, j \in I'$, then both $\frac{x_i(t)}{x_j(t)}$ and $\frac{x_j(t)}{x_i(t)}$ either blow up to $+\infty$ or remain bounded. However, if one of these ratios blows up to $+\infty$, then its inverse would become zero, which is not possible, so both ratios have to remain bounded. This means that $x_i(t)$, with $i \in I'$, all tend to $+\infty$ at the same rate. Suppose that $i \in I'$. From (2.36), we then have that $\dot{x}_i(t)$ would diverge to $-\infty$, which is a contradiction, unless $[ijj] = 0$ and $[ijk] = 0$ for all $j, k \in I$ such that

$$\lim_{t \to T} \frac{x_i(t)}{x_j(t)} = +\infty \text{ or } \lim_{t \to T} \frac{x_i(t)}{x_k(t)} = +\infty, \tag{2.37}$$

or there exists $h \notin I$ such that $x_h(t)$ tends to zero and in such a way that there exists

$m \in \{1, \ldots, l\}$, with $[ihm] \neq 0$, such that $\frac{x_m(t)}{x_h(t)}$ blows up to $+\infty$ faster than the terms which tend to $-\infty$ in the expression of $\dot{x}_i(t)$. If such a h exists, then looking at the expression for $\dot{x}_h(t)$, as $[ihm] \neq 0$, we would have that $\frac{x_i(t)}{x_m(t)}$ diverges to $+\infty$ and the first derivative of $x_h(t)$ would as well blow up to $+\infty$, which is a contradiction. So we can conclude that such a h cannot exist. So the above statement about $[ijj]$ and $[ijk]$ vanishing is true. Moreover, $\frac{x_j(t)}{x_i(t)}$ either approaches a positive finite limit, as $t \to T$, or it tends to zero. By repeating this argument on the other functions which blow up, we have that $\dot{x}_i(t)$ is bounded, as $t \to T$, for every $i = 2, \ldots, l$, as required. □

We then have that, if conjecture 2.9.3 is assumed to be true, the singular time T is characterised by the fact that at least one between $x_1(t), \ldots, x_l(t)$ becomes zero, as $t \to T$. The following proposition explains what happens if the metric collapses in one direction.

Proposition 2.9.6. *If $x_i(t)$ tends to zero, as $t \to T$, then there exists an intermediate Lie group $\widetilde{H}$ with Lie algebra given by*

$$\mathfrak{h}_{\widetilde{J}} = \mathfrak{k} \oplus \bigoplus_{j \in \widetilde{J}} \mathfrak{p}_j, \tag{2.38}$$

for some $\widetilde{J} \subseteq \{1, \ldots, l\}$, and such that $\mathfrak{p}_i$ is a $\widetilde{H}$-module, for every $i \notin \widetilde{J}$. Moreover, $i \in \widetilde{J}$ and $x_j(t)$ tends to zero, as $t \to T$, for every $j \in \widetilde{J}$.

Proof. Suppose that $x_i(t)$ tends to zero, as $t \to T$. Then, let $\widehat{J}$ be the set of indices j such that the corresponding function $x_j(t)$ tends to zero. Now, looking at the expression for $\dot{x}_i(t)$, we also need that $x_j(t) \to 0$, for all j such that $[iij] \neq 0$, and, for every such j, also $x_k(t) \to 0$, for all k such that $[ijk] \neq 0$, with $k, j \notin \{i\}$ and $j \neq k$, and in such a way that the ratios between these functions are bounded. Otherwise, $\dot{x}_i(t)$ would blow up to $+\infty$, as $t \to T$. By the mean value theorem, this cannot happen, because $x_i(t) \to 0$, with $x_i(0) > 0$. Also note that there cannot exist k such that $[ijk] \neq 0$, for some j, and $\frac{x_i(t)}{x_k(t)}$

tends to $+\infty$ in such a way that the derivative of $x_i(t)$ tends to $-\infty$. In fact, if such a k exists, we can look at the evolution equation for $x_k(t)$ and find that its first derivative tends to a positive value, which is impossible. Hence, there has to exists $\widetilde{J} \subseteq \widehat{J}$ such that, if $i \in \widetilde{J}$, also $j, k \in \widetilde{J}$, for every j, k such that $[ijj] \neq 0$ and $[ijk] \neq 0$. The functions $x_j(t)$, with $j \in \widetilde{J}$, are those which necessarily have to tend to zero if $x_i(t)$ becomes zero. Moreover, by the calculation done in case i) below we have that the functions which tend to zero all tend to zero at the same rate. $\square$

We can then conclude that, as $t \to T$, the following singular behaviours can occur in the Ricci flow:

i) There exists an intermediate Lie group $\widetilde{H}$ with Lie algebra $\mathfrak{h}_{\widetilde{J}} = \mathfrak{k} \oplus \bigoplus_{i \in \widetilde{J}} \mathfrak{p}_i \subset \mathfrak{g}$ such that every G-invariant Riemannian metric on G/K is given by a Riemannian submersion

$$\widetilde{H}/K \to G/K \to G/\widetilde{H}. \tag{2.39}$$

Then, $x_i(t) \to 0$, for all $i \in \widetilde{J}$. Moreover, the other components of the metric remain bounded and positive.

ii) $x_i(t) \to 0$, for all $i = 1, \dots, l$.

We will now analyse these singular behaviours separately.

Case i)

The singular time T is characterised by the fact that the fibre in (2.33) shrinks to a point. As the derivatives of $x_i(t)$, with $i \in \widetilde{J}$ are all bounded, there exist positive integers n_i and real numbers $k_i > 0$ such that

$$x_i(t) = k_i(T-t)^{n_i} + o((T-t)^{n_i}), \text{ for all } i \in \widetilde{J}. \tag{2.40}$$

By substituting (2.40) into the system (2.35)-(2.36), we get that $n_i = 1$, for all $i \in \widetilde{J}$, which means that $x_i(t)$, with $i \in \widetilde{J}$, tends to zero linearly in t, as $t \to T$. We also have that k_i, with $i \in \widetilde{J}$, have to satisfy the system

$$\begin{aligned} 1 =& \frac{1}{k_i}\left(b_i - \frac{[iii]}{2d_i} - \frac{1}{d_i}\sum_{j\in\widetilde{J}\setminus\{i\}}[ijj]\right) + \frac{1}{2d_i}\sum_{j\in\widetilde{J}\setminus\{i\}}[ijj]\frac{k_i}{k_j^2} - \frac{1}{d_1}\sum_{j\in\widetilde{J}\setminus\{i\}}[iij]\frac{k_j}{k_i^2} \\ &+ \frac{1}{2d_i}\sum_{j,m\in\widetilde{J},j\neq m}[ijm]\frac{k_i}{k_jk_m} - \frac{1}{d_i}\sum_{j,m\in\widetilde{J},j\neq m}[ijm]\frac{k_j}{k_ik_m}, \end{aligned}$$

for all $i \in \widetilde{J}$. This means that k_i, with $i \in \widetilde{J}$, defines a homogeneous Einstein metric on $\widetilde{H}/K$. Hence, a necessary condition for this singular behaviour to happen is that the fibre in the Riemannian submersion (2.33) carries a homogeneous Einstein metric. The fact that, as $t \to T$, $x_i(t) \to 0$ linearly in t, for all $i \in \widetilde{J}$, implies that T is a type I singularity for the Ricci flow. In fact, on $\mathfrak{p}_i$, with $i \in \widetilde{J}$, $g(t)$ is asymptotically given by a rescaling of the initial metric by a constant times $(T-t)$. We, then, observe that

$$|\operatorname{Rm}(g(t))|^2_{g(t)} = \sum_{i=1}^{l}|\operatorname{Rm}(g(t)_{|\mathfrak{p}_i})|^2_{g(t)_{|\mathfrak{p}_i}} = \sum_{i=1}^{l}\frac{1}{x_i^2(t)}|\operatorname{Rm}(Q_{|\mathfrak{p}_i})|^2_{Q_{|\mathfrak{p}_i}},$$

because the squared norm of the curvature tensor respects the splitting of the metric in this case. Hence, as $t \to T$, $|\operatorname{Rm}(g(t))|_{g(t)}$ behaves like $(T-t)^{-1}$, which implies that the singularity is of type I. Finally, by proposition 2.5.1, as $t \to T$, the total space in the Riemannian submersion (2.33) converges in the Hausdorff-Gromov topology to the base space.

Case ii)

This time, the singular behaviour is characterised by the shrinking of G/K to a point, as $t \to T$. In fact, $x_i(t)$ tends to zero, as $t \to T$, for all $i = 1, \dots, l$. Let

$$x_i(t) = k_i(T-t)^{n_i} + o((T-t)^{n_i}), \quad i = 1, \dots, l, \tag{2.41}$$

where $n_1, \dots, n_l$ are positive integers and $k_1, \dots, k_l$ are positive coefficients. By substituting (2.41) into (2.35)-(2.36), we obtain that $n_i = 1$, for all $i = 1, \dots, l$, and that $k_1, \dots, k_l$ have to satisfy the following system:

$$
\begin{aligned}
1 =& \frac{1}{k_1}\left(b_1 - \frac{[111]}{2d_1} - \frac{1}{d_1}\sum_{j=2}^{l}[1jj]\right) + \frac{1}{2d_1}\sum_{j=2}^{l}[1jj]\frac{k_1}{k_j^2},\\
1 =& \frac{1}{k_i}\left(b_i - \frac{[iii]}{2d_i} - \frac{1}{d_i}\sum_{j\notin\{1,i\}}[ijj]\right) + \frac{1}{2d_i}\sum_{j\notin\{1,i\}}[ijj]\frac{k_i}{k_j^2}\\
&+ \frac{1}{2d_i}\sum_{j,m\notin\{1,i\},j\neq m}[ijm]\frac{k_i}{k_jk_m} - \frac{1}{d_i}\sum_{j,m\notin\{1,i\},j\neq m}[ijm]\frac{k_m}{k_jk_i}\\
&- \frac{1}{d_i}\sum_{j\notin\{i\}}[iij]\frac{k_j}{k_i^2},
\end{aligned}
$$

for all $i = 2, \dots, l$. The above system has a solution if and only if $(k_1, \dots, k_l)$ define a G-invariant Einstein metric on G/K. This tells us that a necessary condition for case ii) to happen is that G/K carries G-invariant Einstein metrics. As $x_i(t) \to 0$ linearly in t, as $t \to T$, $|\operatorname{Rm}(g(t))|_{g(t)}$ is asymptotically given by a rescaling of $|\operatorname{Rm}(g(0))|_{g(0)}$ by $(T-t)^{-1}$ times a positive constant. This implies that T is a type I singularity.

We have then proved the following theorem.

Theorem 2.9.7. *Let G/K be as above. Suppose that $\mathfrak{p}$ does not contain any trivial summand. If conjecture* 2.9.3 *is assumed to be true, there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which defined on the maximal time interval $[0, T)$. T is a type I singularity and, as $t \to T$, one of the following situations can occur:*

i) There exists an intermediate Lie group $\widetilde{H}$, with $G > \widetilde{H} > K$, such that every G-invariant Riemannian metric on G/K is a submersion metric

$$\widetilde{H}/K \to G/K \to G/\widetilde{H},$$

the fibre $\widetilde{H}/K$ shrinks to a point and the total space G/K collapses down to the *base space $G/\widetilde{H}$ in the Hausdorff-Gromov sense.*

ii) G/K shrinks to a point.

Moreover, a necessary condition for i) to happen is that $\widetilde{H}/K$ carries homogeneous Einstein metrics and a necessary condition for ii) to occur is that the whole space G/K admits G-invariant Einstein metrics.

2.9.2 The case l=3

In this section, we are going to analyse in detail the case of three inequivalent, invariant irreducible summands. The main goal is to prove that conjecture 2.9.3 is true in this case. Furthermore, we will also consider the case in which the isotropy representation contains a trivial summand.

When, $l = 3$, the system of nonlinear ODEs that we have to study is given as follows. The G-invariant metric

$$g(t) = x_1(t)Q_{|\mathfrak{p}_1} \oplus x_2(t)Q_{|\mathfrak{p}_2} \oplus x_3(t)Q_{|\mathfrak{p}_3}$$

is a solution to the HRF if and only if $x_1(t), \ldots, x_3(t)$ satisfy the following system of nonlinear ODEs:

$$\dot{x}_1(t) = -A_1 - B_1\frac{x_1(t)^2}{x_2(t)^2} - C_1\frac{x_1(t)^2}{x_3(t)^2}, \tag{2.42}$$

$$\dot{x}_2(t) = -A_2 - B_2\frac{x_2(t)^2}{x_3(t)^2} + C_2\frac{x_1(t)}{x_2(t)} + D_2\frac{x_3(t)}{x_2(t)}, \tag{2.43}$$

$$\dot{x}_3(t) = -A_3 - B_3\frac{x_3(t)^2}{x_2(t)^2} + C_3\frac{x_1(t)}{x_3(t)} + D_3\frac{x_2(t)}{x_3(t)}, \tag{2.44}$$

together with the condition that $x_1(t) > 0$, $x_2(t) > 0$ and $x_3(t) > 0$. The coefficients in the above system are given by

$$
\begin{aligned}
A_1 &= b_1 - \frac{[111]}{2d_1} - \frac{1}{d_1}([122] + [133]),\\
A_2 &= b_2 - \frac{[222]}{2d_2} - \frac{[233]}{d_2},\\
A_3 &= b_3 - \frac{[333]}{2d_3} - \frac{[322]}{d_3},\\
B_1 &= \frac{[122]}{2d_1}, \quad C_1 = \frac{[133]}{2d_1},\\
B_2 &= \frac{[233]}{2d_2}, \quad C_2 = \frac{[122]}{d_2}, \quad D_2 = \frac{[223]}{d_2},\\
B_3 &= \frac{[223]}{2d_3}, \quad C_3 = \frac{[133]}{d_3}, \quad D_3 = \frac{[233]}{d_3}.
\end{aligned}
$$

We observe that the above quantities are all non negative. A general consideration about the system (2.42)-(2.44) is as follows. We have that the functions defined by the RHSs of the above system and their derivatives with respect to x_1, x_2 and x_3 are continuous if and only if (x_1, x_2, x_3) belongs to

$$\boldsymbol{D} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_i \neq 0, \forall i \notin J\},$$

where J is defined as the set of indices i such that $[iij] = 0$ and $[ijk] = 0$, for every $j, k \notin \{i\}$ and such that $j \neq k$. In particular, $1 \in J$. Then, by a standard theorem of ODEs, we have existence and uniqueness of solutions on $\boldsymbol{D}$.

Remark 2.9.8. We observe that if put $y = \frac{x_1}{x_2}$ and $z = \frac{x_3}{x_2}$, then from the system (2.42)-(2.44) we derive a vector field in the plane (y, z) which is given by

$$P(y,z)\frac{\partial}{\partial y} + Q(y,z)\frac{\partial}{\partial z},$$

where

$$P(y,z) = -A_1 - (B_1 + C_2)y^2 - C_1\frac{y^2}{z^2} + A_2 y + B_2\frac{y}{z^2} - D_2 yz,$$
$$Q(y,z) = -A_3 - (B_3 + D_2)z^2 + C_3\frac{y}{z} + (D_3 + B_2)\frac{1}{z} + A_2 z - C_2 yz$$

are rational functions of y and z. Then, an integral curve of this vector field expresses a relation between y and z, so this is another approach which could be used to treat the flow equations.

In order to study the behaviour of the solution to (2.42)-(2.44), we need to distinguish several different cases, which depend on the intermediate Lie algebras. Let

$$\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 > 0, x_2 > 0, x_3 > 0\}$$

be the space in which the solution $(x_1(t), x_2(t), x_3(t))$ to the above system lives. We will now completely study the case in which the only intermediate Lie algebra is given by $\mathfrak{k} \oplus \mathfrak{p}_1$. The other cases can then be treated with similar techniques and we will omit them.

So, suppose that all the coefficients in the system of ODEs are positive, apart from A_1, which is non negative. Note that $A_1 = 0$ if and only if $\mathfrak{p}_1$ is a trivial summand. We define the following two hypersurfaces in Σ:

$$L_2 = \left\{(x_1, x_2, x_3) \in \Sigma | -A_2 - B_2\frac{x_2^2}{x_3^2} + C_2\frac{x_1}{x_2} + D_2\frac{x_3}{x_2} = 0\right\},$$

and

$$L_3 = \left\{(x_1, x_2, x_3) \in \Sigma | -A_3 - B_3\frac{x_3^2}{x_2^2} + C_3\frac{x_1}{x_3} + D_3\frac{x_2}{x_3} = 0\right\}.$$

We can rewrite these two hypersurfaces in the following way. L_2 is given by $(x_1, x_2, x_3) \in$

Σ such that

$$x_1 = \frac{A_2}{C_2} x_2 + \frac{B_2}{C_2} \frac{x_2^3}{x_3^2} - \frac{D_2}{C_2} x_3 =: f_2(x_2, x_3),$$

and L_3 is given by $(x_1, x_2, x_3) \in \Sigma$ such that

$$x_1 = \frac{A_3}{C_3} x_3 + \frac{B_3}{C_3} \frac{x_3^3}{x_2^2} - \frac{D_3}{C_3} x_2 =: f_3(x_2, x_3),$$

So L_2 and L_3 are two connected hypersurfaces in Σ and each of them separates Σ into two connected regions. More precisely, L_2 and L_3 respectively separate the regions in which $x_2(t)$ and $x_3(t)$ are monotonic increasing or decreasing in t. In particular,

- if $x_1(t) > f_2(x_2(t), x_3(t))$, then $x_2(t)$ will be monotonically increasing in t and
- if $x_1(t) < f_2(x_2(t), x_3(t))$, then $x_2(t)$ will be monotonically decreasing in t.

Analogously,

- if $x_1(t) > f_3(x_2(t), x_3(t))$, then $x_3(t)$ will be monotonically increasing in t and
- if $x_1(t) < f_3(x_2(t), x_3(t))$, then $x_3(t)$ will be monotonically decreasing in t.

We note that

$$\begin{aligned}
\frac{\partial f_2}{\partial x_2} &= \frac{A_2}{C_2} + 3\frac{B_2}{C_2}\frac{x_2^2}{x_3^2} > 0, \\
\frac{\partial f_2}{\partial x_3} &= -2\frac{B_2}{C_2}\frac{x_2^3}{x_3^3} - \frac{D_2}{C_2} < 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial f_3}{\partial x_2} &= -2\frac{B_3}{C_3}\frac{x_3^3}{x_2^3} - \frac{D_3}{C_3} < 0, \\
\frac{\partial f_3}{\partial x_3} &= \frac{A_3}{C_3} + 3\frac{B_3}{C_3}\frac{x_3^2}{x_2^2} > 0.
\end{aligned}$$

We now want to understand how these two hypersurfaces intersect. We have that

$$\begin{aligned} & f_2(x_2,x_3) = f_3(x_2,x_3) \\ \Leftrightarrow & \frac{B_2}{C_2}\left(\frac{x_2}{x_3}\right)^5 + \left(\frac{A_2}{C_2}+\frac{D_3}{C_3}\right)\left(\frac{x_2}{x_3}\right)^3 - \left(\frac{D_2}{C_2}+\frac{A_3}{C_3}\right)\left(\frac{x_2}{x_3}\right)^2 - \frac{B_3}{C_3} = 0 \\ \Leftrightarrow & \frac{B_2}{C_2}\left(\frac{x_2}{x_3}\right)^5 + \left(\frac{A_2}{C_2}+\frac{D_3}{C_3}\right)\left(\frac{x_2}{x_3}\right)^3 = \left(\frac{D_2}{C_2}+\frac{A_3}{C_3}\right)\left(\frac{x_2}{x_3}\right)^2 + \frac{B_3}{C_3} \end{aligned}$$

Let $y = \frac{x_2}{x_3}$. Then, the last equation above becomes

$$\frac{B_2}{C_2}y^5 + \left(\frac{A_2}{C_2}+\frac{D_3}{C_3}\right)y^3 = \left(\frac{D_2}{C_2}+\frac{A_3}{C_3}\right)y^2 + \frac{B_3}{C_3}. \tag{2.45}$$

We will now show that there is only one positive solution to the above equation. First of all we note that we can rewrite equation (2.45) as

$$\frac{B_2}{C_2}y^2 + \left(\frac{A_2}{C_2}+\frac{D_3}{C_3}\right) = \left(\frac{D_2}{C_2}+\frac{A_3}{C_3}\right)\frac{1}{y} + \frac{B_3}{C_3}\frac{1}{y^3}.$$

This shows that there are no negative roots. Then we observe that the derivative of (2.45) vanishes exactly when $y = 0$, with negative second derivative, or when y satisfies

$$5\frac{B_2}{C_2}y^3 + 3\left(\frac{A_2}{C_2}+\frac{D_3}{C_3}\right)y - 2\left(\frac{D_2}{C_2}+\frac{A_3}{C_3}\right) = 0. \tag{2.46}$$

We can easily see that equation (2.46) has at least one positive root. Moreover, the first derivative of (2.46) is positive, so this positive root is the only root of (2.46). Now, if (2.45) had two positive roots, then its first derivative would necessarily have to vanish twice for $y > 0$. As this does not happen, we conclude that there is only one positive root. Let $y = C$ be the only root to (2.45). Note that $y = C$ defines a line through the origin in $\mathbb{R}^3$. Hence, we have that the intersection $L_2 \cap L_3$ in Σ is given by a line, which we will call L. We now want to understand how these two hypersurfaces intersect the plane $\{x_1 = 0\}$ in $\Sigma \cup \{x_1 = 0\}$. We have that $L_2 \cap \{x_1 = 0\}$ is given by $(x_1, x_2, x_3) \in \Sigma \cup \{x_1 = 0\}$

such that

$$\frac{A_2}{C_2}x_2 + \frac{B_2}{C_2}\frac{x_2^3}{x_3^2} - \frac{D_2}{C_2}x_3 = 0.$$

Dividing the above equation by x_3, we obtain

$$\frac{A_2}{C_2}\frac{x_2}{x_3} + \frac{B_2}{C_2}\left(\frac{x_2}{x_3}\right)^3 - \frac{D_2}{C_2} = 0.$$

Now let $y = \frac{x_2}{x_3}$. We have that there exists a unique solution to the equation

$$\frac{A_2}{C_2}y + \frac{B_2}{C_2}y^3 - \frac{D_2}{C_2} = 0.$$

Moreover, this solution is positive. Hence, L_2 intersects $\{x_1 = 0\}$ in a line. Similarly, we have that L_3 intersects $\{x_1 = 0\}$ in a line.

We then have that the phase space is divided into four regions:

- Σ_1=$\{(x_1, x_2, x_3) \in \Sigma |\, x_1 > f_2(x_2, x_3), x_1 > f_3(x_2, x_3)\}$, where $x_2(t)$ and $x_3(t)$ are increasing in t;
- Σ_2=$\{(x_1, x_2, x_3) \in \Sigma |\, x_1 > f_2(x_2, x_3), x_1 < f_3(x_2, x_3)\}$, where $x_2(t)$ is increasing in t and $x_3(t)$ is decreasing in t;
- Σ_3=$\{(x_1, x_2, x_3) \in \Sigma |\, x_1 < f_2(x_2, x_3), x_1 > f_3(x_2, x_3)\}$, where $x_2(t)$ is decreasing in t and $x_3(t)$ is increasing in t;
- Σ_4=$\{(x_1, x_2, x_3) \in \Sigma |\, x_1 < f_2(x_2, x_3), x_1 < f_3(x_2, x_3)\}$, where $x_2(t)$ and $x_3(t)$ are decreasing in t.

We will consider firstly the case in which $A_1 > 0$. We have the following lemma.

Lemma 2.9.9. *Σ_4 is invariant under the system* (2.42)-(2.44).

Proof of the lemma. The boundary of Σ_4 is given by $(x_1, x_2, x_3) \in \Sigma$ such that one of the following holds: $x_1 = f_2(x_2, x_3)$ and $x_1 < f_3(x_2, x_3)$, $x_1 = f_3(x_2, x_3)$ and $x_1 < f_2(x_2, x_3)$

or $x_1 = f_2(x_2, x_3) = f_3(x_2, x_3)$. If we compute the vector field $(\dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t))$ on $\partial\Sigma_4$, we have that it always points towards the interior Σ_4. In fact, $\dot{x}_1(t)$ is always negative, while for the other components we need to distinguish three different possible situations. If $x_1 = f_2(x_2, x_3)$ and $x_1 < f_3(x_2, x_3)$, then $\dot{x}_2(t)$ vanishes, while $\dot{x}_3(t)$ is negative. This then implies that

$$\frac{d}{dt}(x_1(t) - f_2(x_2(t), x_3(t))) = \dot{x}_1(t) - \frac{\partial f_2}{\partial x_3}\dot{x}_3(t) < 0.$$

Similarly, if $x_1 = f_3(x_2, x_3)$ and $x_1 < f_2(x_2, x_3)$, then $\dot{x}_3(t)$ is zero, while $\dot{x}_2(t)$ is negative, which implies that

$$\frac{d}{dt}(x_1(t) - f_3(x_2(t), x_3(t))) = \dot{x}_1(t) - \frac{\partial f_3}{\partial x_2}\dot{x}_2(t) < 0.$$

Finally, if $x_1 = f_2(x_2, x_3) = f_3(x_2, x_3)$, then both $\dot{x}_2(t)$ and $\dot{x}_3(t)$ are zero. $\square$

We also observe that, if $A_1 \neq 0$, then the solution has to stop in finite time. In fact, we have that

$$\dot{x}_1(t) < -A_1,$$

which integrated gives

$$x_1(t) < -A_1 t + x_1(0).$$

Hence, the solution has to stop in finite time before $x_1(t)$ becomes zero. We have then proved the following lemma.

Lemma 2.9.10. *The solution to the system* (2.42)-(2.44) *has to stop in finite time.*

We now have to distinguish several possible initial conditions. For what concerns the behaviour of the flow in Σ_4, we have the following lemma.

Lemma 2.9.11. *If* $(x_1(0), x_2(0), x_3(0)) \in \Sigma_4$, *then there exists* $T < \infty$ *such that there exists a unique solution to* (2.42)-(2.44) *defined on the maximal time interval* $[0, T)$. *As*

$t \to T$, either all the functions become zero or $x_1(t)$ becomes zero and the other two functions approach a positive limit.

Proof of the lemma. If $(x_1(0), x_2(0), x_3(0)) \in \Sigma_4$, then due to the invariance of this region we will remain in Σ_4 for all t such that a solution to the HRF exists. Recall that in Σ_4 all the functions are decreasing in t. So they approach a non negative limit and, by lemma 2.9.10, there exists $T < \infty$ such that there exists a unique solution to the HRF which is defined on the maximal time interval $[0, T)$ and T is characterised by the fact that some of the functions which define the metric become zero. Note that we cannot have $x_2(t)$ or $x_3(t)$ becoming zero, without the trajectory going to the origin. In fact, from the evolution equation of $x_2(t)$, we can see that if $x_2(t)$ becomes zero, then also $x_3(t)$ and $x_1(t)$ have to become zero, otherwise its first derivative would diverge to $+\infty$, but this is not possible because of the mean value theorem. We have a similar result for $x_3(t)$. Hence, $x_2(t)$ or $x_3(t)$ can become zero, only if the trajectory goes to the origin. This implies that T is either characterised by all the functions becoming zero or by $x_1(t)$ becoming zero and $x_2(t)$ and $x_3(t)$ approaching a positive limit, which is finite as the functions are decreasing in t. □

If $(x_1(0), x_2(0), x_3(0)) \in \partial\Sigma_4$, then the solution to the HRF will immediately enter the region Σ_4, and one of the two singular behaviours described above will occur.

If $(x_1(0), x_2(0), x_3(0)) \in \Sigma \backslash \{\Sigma_4 \cup \partial\Sigma_4\}$, we need to distinguish three possible initial conditions, namely $(x_1(0), x_2(0), x_3(0))$ in Σ_1, Σ_2 or Σ_3. We will begin by considering $(x_1(0), x_2(0), x_3(0)) \in \Sigma_1$ and we will show that the solution has to exit this region in finite time. The region Σ_1 is characterised by the fact that both $x_2(t)$ and $x_3(t)$ are increasing in t. We have the following lemma.

Lemma 2.9.12. *In Σ_1, it is not possible that $x_2(t)$ or $x_3(t)$ diverge to $+\infty$, while $x_1(t)$ approaches a non negative limit.*

Proof of the lemma. Suppose that $x_2(t)$ tends to $+\infty$, as t approaches the final time.

Then from the evolution equation of $x_2(t)$, we see that also $x_3(t)$ has to tend to $+\infty$, otherwise the derivative of $x_2(t)$ will become negative, which is impossible because of the mean value theorem. The same is true for $x_3(t)$. So, the only possibility is that both $x_2(t)$ and $x_3(t)$ tend to $+\infty$ in such a way that the ratios $\frac{x_2(t)}{x_3(t)}$ and $\frac{x_3(t)}{x_2(t)}$ remain bounded. Hence, there exists constants C and C' such that

$$C < \frac{x_2(t)}{x_3(t)} < C',$$

for all t such that a solution exists. Hence,

$$\dot{x}_2(t) < \widetilde{C},$$
$$\dot{x}_2(t) < \overline{C},$$

where $\widetilde{C}$ and $\overline{C}$ are two positive constants. By lemma 2.9.10, we know that the existence time is finite. Integrating the above inequalities between 0 and T we have that both $x_2(t)$ and $x_3(t)$ are bounded from above by a positive constant. □

Note that this lemma shows in particular that the functions $x_2(t)$ and $x_3(t)$ are both bounded in Σ_1.

Proposition 2.9.13. *If $(x_1(0), x_2(0), x_3(0)) \in \Sigma_1$, then the solution $(x_1(t), x_2(t), x_3(t))$ has to leave the region Σ_1 in finite time.*

Proof of the proposition. Suppose that there exists $\bar{t}$ such that $(x_1(t), x_2(t), x_3(t))$ is in Σ_1 for all $t \in (\bar{t}, T)$, where T is the maximal time of existence. In Σ_1 the trajectory has a limit, because the functions are all monotonic. Hence, because of the above lemma, the only way in which the solution can stop is because one of the functions becomes zero at the final time. As $x_2(t)$ and $x_3(t)$ are both increasing in t, the only possibility is that $x_1(t)$ becomes zero.

However, $\{x_1 = 0\}$ cannot be contained in the interior of Σ_1. We will prove this by showing that $f_2(x_2, x_3)$ and $f_3(x_2, x_3)$ cannot be both negative. In fact, $f_2(x_2, x_3) = 0$ if and only if x_2 and x_3 are such that

$$-A_2 - B_2\frac{x_2^2}{x_3^2} + D_2\frac{x_3}{x_2} = 0,$$

which corresponds to a line $x_2 = \bar{y}_2 x_3$, where $\bar{y}_2$ is a positive constant, in the plane $\{x_1 = 0\}$. Similarly, $f_3(x_2, x_3) = 0$ if and only if x_2 and x_3 are such that

$$-A_3 - B_3\frac{x_3^2}{x_2^2} + D_3\frac{x_2}{x_3} = 0$$

is given by a line $x_2 = \bar{y}_3 x_3$ in $\{x_1 = 0\}$, where $\bar{y}_3$ is a positive constant. We can show that $\bar{y}_2 < \bar{y}_3$. Now, $f_2(x_2, x_3)$ is negative if $x_2 < \bar{y}_2 x_3$. However, this implies that $x_2 < \bar{y}_3 x_3$, which means that $f_3(x_2, x_3)$ is positive.

Hence, the solution has to leave Σ_1 before the final time. □

We then have that, if $(x_1(0), x_2(0), x_3(0)) \in \Sigma_1$, the solution will enter one between Σ_2, Σ_3 and Σ_4 in finite time. We already know what happens, if the trajectory enters the region Σ_4. So, we need to consider the other two possible initial conditions. Suppose that $(x_1(0), x_2(0), x_3(0)) \in \Sigma_2$. In this region, $x_2(t)$ is increasing in t, while $x_3(t)$ is decreasing in t, while $x_1(t)$ is always decreasing in t. In Σ_2 the trajectory has a limit, because all the functions are monotonic. From the evolution equation of $x_2(t)$, we have that $x_2(t)$ cannot blow up to $+\infty$, because the first derivative of $x_2(t)$ would become negative. Moreover, $x_3(t)$ cannot become zero in Σ_2, because from its evolution equation we would have that $\dot{x}_3(t)$ would diverge to $+\infty$. So, either the solution stops in finite time because it reaches the plane $\{x_1 = 0\}$, or it reaches one of the hypersurfaces L_2 and L_3, in finite time. If it reaches L_3, then we are on the boundary of Σ_4 and we will enter the region Σ_4, where we know what the behaviour of the solution is. If $(x_1(t), x_2(t), x_3(t))$ hits the hypersurface

L_2, then we will either enter the region Σ_1, in the case where $x_2(t)$ has a minimum, or, if also $\ddot{x}_2(t)$ vanishes, the solution will move along the tangent plane to the surface L_2 at that point.

We now have the following lemma.

Lemma 2.9.14. *There cannot be a sequence of times such that $x_2(t)$ or $x_3(t)$ diverge to $+\infty$ along this sequence.*

Proof of the lemma. Suppose first that there exists a sequence of times $\{t_n\}_n$ such that $x_2(t_n)$ diverges to $+\infty$, as $n \to +\infty$, while $x_3(t_n)$ remains bounded. Then,

$$f_2(x_2(t_n), x_3(t_n)) \to +\infty \text{ and } f_3(x_2(t_n), x_3(t_n)) \to -\infty,$$

as $n \to +\infty$. This implies that there exists $N > 0$ such that $(x_1(t_n), x_2(t_n), x_3(t_n))$ lies in Σ_3, for every $n > N$. However, in Σ_3, $x_2(t)$ is monotonic decreasing in t, so we get a contradiction. Similarly, if $x_2(t_n)$ is bounded and $x_3(t_n)$ diverges to $+\infty$, as $n \to +\infty$, then there exists $N > 0$ such that $(x_1(t_n), x_2(t_n), x_3(t_n))$ lies in Σ_2, for every $n > N$. As $x_3(t)$ is monotonic decreasing in Σ_2, we derive a contradiction. So such a sequence of times cannot exist.

Suppose now that there exists a sequence of times $\{t_n\}_n$ such that $x_2(t_n)$ and $x_3(t_n)$ both diverge to $+\infty$, as $n \to +\infty$. We already proved that the ratios $\frac{x_2(t_n)}{x_3(t_n)}$ and $\frac{x_3(t_n)}{x_2(t_n)}$ remain bounded. Moreover, we can assume that $x_1(t_n) \neq 0$, for every n. In fact, if there exists $n > 0$ such that $x_1(t_n) = 0$, then the solution has to stop at $T = t_n$ and we know that $x_2(t)$ and $x_3(t)$ cannot both be increasing when we are on $\{x_1 = 0\}$. We will now show that both $f_2(x_2(t_n), x_3(t_n))$ and $f_3(x_2(t_n), x_3(t_n))$ diverge to $+\infty$, as $n \to +\infty$. We observe that we can write

$$f_2(x_2(t), x_3(t)) = x_2(t) g_2(x_2(t), x_3(t)),$$
$$f_3(x_2(t), x_3(t)) = x_3(t) g_3(x_2(t), x_3(t)),$$

where

$$g_2(x_2(t), x_3(t)) = \frac{A_2}{C_2} + \frac{B_2}{C_2}\frac{x_2(t)^2}{x_3(t)^2} - \frac{D_2}{C_2}\frac{x_3(t)}{x_2(t)}, \tag{2.47}$$

$$g_3(x_2(t), x_3(t)) = \frac{A_3}{C_3} + \frac{B_3}{C_3}\frac{x_3(t)^2}{x_2(t)^2} - \frac{D_3}{C_3}\frac{x_2(t)}{x_3(t)}. \tag{2.48}$$

We have that both $g_2(x_2(t_n), x_3(t_n))$ and $g_3(x_2(t_n), x_3(t_n))$ are bounded, as $n \to +\infty$. Moreover, if there exists $n > 0$ such that $g_2(x_2(t_n), x_3(t_n)) = 0$ or $g_3(x_2(t_n), x_3(t_n)) = 0$, then one between $f_2(x_2(t_n), x_3(t_n))$ and $f_3(x_2(t_n), x_3(t_n))$ has to vanish. However, this would imply that $x_1(t_n) = 0$, which is not possible. So, $g_2((t_n), x_3(t_n)) > 0$ and $g_3(x_2(t_n), x_3(t_n)) > 0$, for every n. We then have that $f_2(x_2(t_n), x_3(t_n))$ and $f_3(x_2(t_n), x_3(t_n))$ diverge to $+\infty$, as $n \to +\infty$. This then implies that there exists $N > 0$ such that $(x_1(t_n), x_2(t_n), x_3(t_n))$ lies in Σ_4, where both $x_2(t)$ and $x_3(t)$ are decreasing in t. We then get a contradiction and conclude that such a sequence of times cannot exist. □

We have then proved that $x_i(t)$, with $i = 1, 2, 3$, are all bounded as we approach the final time. This means that the solution stops because some of the functions vanish. Hence, by proposition 2.9.13, there has to exist $\bar{t} > 0$ such that the solution does not reenter Σ_1, after $t > \bar{t}$. By the previous results, we then have that the solution either hits the hypersurface $\{x_1 = 0\}$ or it enters the invariant region Σ_4. The case in which $(x_1(0), x_2(0), x_3(0)) \in \Sigma_3$ is analogous. We can then conclude that there exists $T < \infty$ such that there exists a unique solution to the HRF on G/K which is defined on the maximal time interval $[0, T)$ and, as $t \to T$, the solution stops because either $x_1(t)$ becomes zero or the solution tends to the origin.

Using these results, we have the following theorem.

Theorem 2.9.15. *Suppose that $\mathfrak{k} \oplus \mathfrak{p}_1$ is the only intermediate Lie algebra and that $\mathfrak{p}$ does not contain any trivial summand. Then, there exists $T > 0$ and finite such that there exists a unique solution to* (2.42)-(2.44)*, which is defined on the maximal time interval*

$[0, T)$. Moreover, as $t \to T$, one of the following singular behaviours can occur:

- *All the functions become zero;*
- *$x_1(t)$ tends to zero and $x_2(t)$ and $x_3(t)$ approach a positive limit.*

Remark 2.9.16. We note that the above theorem is true in general for $\mathfrak{p}$ which does not contain any trivial summand. The other cases, which correspond to more intermediate Lie algebras, can be treated in a similar way. In particular, as t tends to the final time, the singular behaviours are characterised as follows. There exists $\widetilde{J} \subseteq \{1, 2, 3\}$ such that $x_i(t)$ becomes zero, for every $i \in \widetilde{J}$. Moreover, $\widetilde{J}$ defines the intermediate Lie algebra

$$\widetilde{\mathfrak{h}} = \mathfrak{k} \bigoplus_{i \in \widetilde{J}} \mathfrak{p}_i,$$

with $\mathfrak{k} \subset \widetilde{\mathfrak{h}} \subset \mathfrak{g}$.

We will now consider the case in which $A_1 = 0$. In this case, the main difference is that we do not know whether the existence time of the solution is finite or not.

We will first of all show that the system does not have critical points. In the case where $A_1 > 0$, this is immediate, as $\dot{x}_1(t)$ is never zero. If $A_1 = 0$, we have that $\dot{x}_1(t) = 0$ if and only if $\frac{x_1(t)}{x_2(t)} = 0$ and $\frac{x_1(t)}{x_3(t)} = 0$. By the computation we did before, we know that $\dot{x}_2(t)$ and $\dot{x}_3(t)$ vanish if and only if $(x_1(t), x_2(t), x_3(t))$ belong to the hypersurfaces L_2 and L_3, respectively. We also showed that these two hypersurfaces intersect the plane $\{x_1 = 0\}$ in two different lines through the origin. These lines are respectively given by $x_2 = \bar{y}_2 x_3$ and $x_2 = \bar{y}_3 x_3$, where $\bar{y}_2 < \bar{y}_3$. We then have that, if $x_1(t) = 0$, $\dot{x}_2(t) = 0$ is zero if and only if $\frac{x_2(t)}{x_3(t)} = \bar{y}_2$. Similarly, if $x_1(t) = 0$, $\dot{x}_3(t) = 0$ if and only if $\frac{x_2(t)}{x_3(t)} = \bar{y}_3$. As $\bar{y}_2 \neq \bar{y}_3$, we have that $\dot{x}_2(t)$ and $\dot{x}_3(t)$ can never be zero at the same time, when $x_1(t) = 0$. Hence, the system does not have critical points.

We will prove that theorem 2.9.15 holds also in this special case. In order to do this, we need to show that the results proved so far for the case $A_1 > 0$ are true also when

$A_1 = 0$. We begin by noticing that lemma 2.9.9 is still true when $A_1 = 0$ and with the same proof. We then have the following lemma.

Lemma 2.9.17. *If* $(x_1(0), x_2(0), x_3(0)) \in \Sigma_4$*, then there exists* $T < \infty$ *such that there exists a unique solution to* (2.42)-(2.44) *defined on the maximal time interval* $[0, T)$*. As* $t \to T$*, all the functions become zero.*

Proof of the lemma. The only differences with lemma 2.9.11 are that we do not have lemma 2.9.10 and we have to exclude the case in which $x_1(t)$ becomes zero and the other functions approach a positive limit, as t approaches the finite time. Apart from that the proof of this lemma is exactly the same as the proof of lemma 2.9.11.

We will begin by showing that $x_1(t)$ can become zero if and only if the trajectory goes to the origin. From the evolution equation for $x_1(t)$, we can easily see that, if the initial condition is given by $x_1(0) = 0$ and $x_2(0), x_3(0) > 0$, then the solution to the ODE will be characterised by $x_1(t) = 0$ and $x_2(t), x_3(t) \geq 0$, for every t such that a solution exists. This means that $\{x_1 = 0\}$ is invariant. We will now prove that $x_1(t)$ cannot become zero unless some of the other functions become zero as well. Suppose for a contradiction that we have a solution $(x_1(t), x_2(t), x_3(t))$ defined on $[0, T)$, with $T \leq +\infty$, and such that $x_1(t)$ tends to zero, as $t \to T$ and $x_2(t)$ and $x_3(t)$ approach a positive limit. Let α_2 and α_3 denote the limits of $x_2(t)$ and $x_3(t)$, as t tends to T. If $T < \infty$, consider another solution to the ODE with initial condition given by $(0, \alpha_1, \alpha_2)$. We will then have that this solution is characterised by $x_1(t) = 0$ both forward and backwards in time. We then derive a contradiction, because the solution is unique. The case $T = \infty$ can be excluded, because there are no critical points and $x_2(t)$ and $x_3(t)$ are bounded both from above and away from zero. We can then conclude that $x_1(t)$ can become zero if and only if the trajectory reaches the origin.

We will now show that the existence time of the solution has to be finite. Suppose for a contradiction that the solution existed for every $t > 0$. As we are in Σ_4, where all the functions are decreasing in t, we have that $x_i(t)$, with $i = 1, 2, 3$, approach a non negative limit, as $t \to +\infty$. We will now show that the limit of the trajectory has to be the origin. We proved that $x_i(t)$ can become zero if and only if the trajectory goes to the origin, for every $i = 1, 2, 3$. So, the only other possibility is that the functions all approach a positive limit, as $t \to +\infty$. However, this limit must be a critical point and we derive a contradiction. We can then conclude that the limit of the trajectory, as $t \to +\infty$, has to correspond to the origin.

We also note that if at least one of $\frac{x_1(t)}{x_2(t)}$ and $\frac{x_1(t)}{x_3(t)}$ tends to a positive limit, as $t \to +\infty$, then we can show that $x_1(t)$ has to vanish in finite time. In fact, this would imply that $\dot{x}_1(t)$ is bounded from above by a negative constant. So, suppose that the solution exists for every $t > 0$ and that both $\frac{x_1(t)}{x_2(t)}$ and $\frac{x_1(t)}{x_3(t)}$ tend to zero, as $t \to +\infty$. By the mean value theorem, $x_2(t)$ and $x_3(t)$ tend to zero in such a way that $\frac{x_2(t)}{x_3(t)}$ is bounded both from above and away from zero, for every $t > 0$. Then, there exist constants $\widehat{C}$ and $\widetilde{C}$ such that

$$\widehat{C} < \frac{x_2(t)}{x_3(t)} < \widetilde{C},$$

for every $t > 0$. Moreover, as Σ_4 is invariant, we can pick these two constants in such a way that the derivatives of $x_2(t)$ and $x_3(t)$ remain negative in the limit. This means that $\dot{x}_2(t)$ and $\dot{x}_3(t)$ are bounded from above by a negative constant, which implies that the solution has to stop in finite time. So we derive a contradiction. We can then conclude that the existence time of the solution has to be finite. □

We now have an analogue of lemma 2.9.12, but the proof is rather different, because we do not have lemma 2.9.10.

Proof of lemma 2.9.12 *when* $A_1 = 0$. We want to show that $x_2(t)$ and $x_3(t)$ are bounded in Σ_1. We have already proved that $x_2(t)$ and $x_3(t)$ cannot diverge to $+\infty$ unless they both do and in such a way that their ratio is bounded from above and away from zero.

Hence, there exists constants C and C' such that

$$C < \frac{x_2(t)}{x_3(t)} < C',$$

for all t such that a solution exists. Moreover, $x_1(t)$ will approach a non negative limit $\bar{x}_1$. We now observe that we can write

$$f_2(x_2, x_3) = x_2 g_2(x_2, x_3),$$
$$f_3(x_2, x_3) = x_3 g_3(x_2, x_3),$$

where $g_2(x_2, x_3)$ and $g_3(x_2, x_3)$ are defined by (2.47)-(2.48). As long as we are in Σ_1, both $g_2(x_2(t), x_3(t))$ and $g_3(x_2(t), x_3(t))$ are positive. This is due to the fact that $f_i(x_2(t), x_3(t)) = 0$ if and only if $g_i(x_2(t), x_3(t)) = 0$, for every $i = 2, 3$. Hence, if $x_2(t)$ and $x_3(t)$ tend to $+\infty$, then also $f_2(x_2(t), x_3(t))$ and $f_3(x_2(t), x_3(t))$ diverge to $+\infty$. As we are in Σ_1, the following two inequalities have to be satisfied:

$$x_1(t) > f_2(x_2(t), x_3(t)) \quad \text{and} \quad x_1(t) > f_3(x_2(t), x_3(t)).$$

We would then have that also $x_1(t)$ has to diverge to $+\infty$, which is not possible as it is strictly decreasing. Hence, we can exclude the fact that, in Σ_1, $x_2(t)$ and $x_3(t)$ tend to $+\infty$, while $x_1(t)$ approaches a non negative limit. □

Proposition 2.9.13 is still true and the proof is exactly the same, apart from the fact that, as we do not have lemma 2.9.10, we need to exclude the case in which the solution exists for all $t > 0$ and $x_2(t)$ and $x_3(t)$ approach a positive limit, as $t \to +\infty$. We will do this by showing that $x_1(t)$ always tends to zero. In fact, suppose for a contradiction that this was not true. Then there exists positive constants C' and C'' such that

$$\frac{x_1(t)}{x_2(t)} > C' \text{ and } \frac{x_1(t)}{x_3(t)} > C''.$$

From the evolution equation of $x_1(t)$ this would then imply that $\dot{x}_1(t)$ is less than a constant which is strictly negative, which would then imply that $x_1(t)$ would become zero in finite time and we then get a contradiction. We also note that the case in which the solution exists for every $t > 0$ and, as $t \to +\infty$, $x_1(t)$ tends to zero and $x_2(t)$ and $x_3(t)$ approach a positive limit cannot occur in Σ_1. In fact, we showed previously that if $x_1(t) = 0$, then $x_2(t)$ and $x_3(t)$ cannot both be increasing in t. Hence, $x_1(t)$ cannot tend to zero, as $t \to +\infty$, without having left Σ_1 from a certain time onwards. Now the rest of the proof is the same as in the case $A_1 > 0$ and lemma 2.9.14 is still true with the same proof.

To conclude, we also need to exclude the cases in which the solution exists for every $t > 0$ in Σ_2 or in Σ_3. We will show that there cannot be a solution which exists for every $t > 0$ in Σ_2. The case of Σ_3 is similar. Suppose for a contradiction that there was such a solution. Recall that in Σ_2, $x_2(t)$ is increasing and $x_3(t)$ is decreasing. So all the functions approach a limit as $t \to +\infty$. Moreover, by the evolution equation of $x_3(t)$, we know that $x_3(t)$ cannot become zero otherwise its first derivative would diverge to $+\infty$. As the dynamical system does not have critical points, we can exclude the cases in which $x_2(t)$ and $x_3(t)$ approach positive limits and $x_1(t)$ approaches a non negative limit, as $t \to +\infty$. So the existence time cannot be infinite. We can then conclude that theorem 2.9.15 and remark 2.9.16 are still true when $A_1 = 0$.

2.9.3 Blowing up the solution near the singularity

We are now going to analyse the rescaled Ricci flow near the singularity T, where the singular behaviours are the one described in theorem 2.9.7. Let T be a type I singularity for the HRF on G/K. Consider a sequence $\{(p_j, t_j)\}_{j=1}^{\infty}$, with $p_j \in G/K$ and $t_j \to T$, such that

$$|\operatorname{Rm}|(p_j, t_j) = \sup_{p \in G/K,\, t \in [0, t_j]} |\operatorname{Rm}(g(t))|_{g(t)}(p, t) \to +\infty.$$

Let

$$x_k^j(t) = |\operatorname{Rm}|(p_j, t_j)\, x_k\left(t_j + \frac{t}{|\operatorname{Rm}|(p_j, t_j)}\right),$$

for all $k = 1, \ldots, l$. Let $g^j(t)$ be the Riemannian metric defined by $x_1^j(t), \ldots, x_l^j(t)$. Then,

$$(N, g_\infty(t), p_\infty) = \lim_{j\to\infty} (G/K, g^j(t), p_j)$$

is an eternal Ricci flow. Note that the limit of the pointed convergence is not affected by the location of the p_j's, because we are in the homogeneous case. By [27], we know that $(N, g_\infty(t))$ is a nonflat gradient shrinking Ricci soliton.

In the case where the whole space shrinks to a point in finite time, it is easy to see that $g_\infty(t)$ is given by a homogeneous Einstein metric (with positive scalar curvature) on G/K.

Now, let us consider the other kind of singular behaviour which leads to a type I singularity in the HRF. Let $g_\infty(t)$ be given by $x_1^\infty(t), \ldots, x_l^\infty(t)$. By the proof of theorem 2.9.7, we have that

$$x_i^\infty(t) = \lim_{j\to\infty} |\operatorname{Rm}|(p_j, t_j) x_i^j\left(t_j + \frac{t}{|\operatorname{Rm}|(p_j, t_j)}\right)$$

will be given by a decreasing linear function of t, for all $i \in \widetilde{J}$, where $\widetilde{J}$ is defined by (2.38). On the other hand, we have that

$$x_k^\infty(t) = \infty,$$

for all $k \notin \widetilde{J}$, as $x_k(t)$ remains bounded, as $t \to T$.

Hence, $(N, g_\infty(t))$ will be given by the *rigid Ricci soliton*

$$\widetilde{N} \times \mathbb{R}^q,$$

where $\widetilde{N}$ is the homogeneous Einstein manifold defined by $x_i^\infty(t)$, with $i \in \widetilde{J}$, and it corresponds to the fibre $\widetilde{H}/K$ in (2.33), and the flat factor $\mathbb{R}^q$ is defined by $x_k^\infty(t)$, for $k \notin \widetilde{J}$, with $q = \dim(G/\widetilde{H})$.

2.9.4 Examples

We are now going to present two examples of homogeneous spaces to which the theory developed in this section can be applied. These are also examples of compact homogeneous spaces which do not carry any invariant Einstein metric and were found by Bohm in [8].

Example 2.9.18. Let $\widetilde{G}$ be a compact connected Lie group and consider homogeneous spaces given by

$$G/K = \widetilde{G} \times \widetilde{G}/(\Delta\widetilde{K}(\widehat{K} \times \widehat{K})), \tag{2.49}$$

where $\widetilde{K}$ is a Lie subgroup of $\widetilde{G}$ which is simple, $\widehat{K}$ is a Lie subgroup of $\widetilde{G}$ which is either simple or 1-dimensional and $\widetilde{G}/(\widetilde{K}\widehat{K})$ is a compact irreducible symmetric space, cf. [5, Table 7.102]. In the above expression, $\Delta\widetilde{K}$ denotes the diagonal embedding of $\widetilde{K}$ in $\widetilde{K}\times\widetilde{K}$ and $\Delta\widetilde{K}(\widehat{K} \times \widehat{K})$ is the group obtained by considering all the possible products between elements in $\Delta\widetilde{K}$ and $\widehat{K}\times\widehat{K}$. The Lie algebra of $\widetilde{G}\times\widetilde{G}$ is given by $\widetilde{\mathfrak{g}}_1\oplus\widetilde{\mathfrak{g}}_2$, where $\widetilde{\mathfrak{g}}_1$ and $\widetilde{\mathfrak{g}}_2$ refer to the Lie algebras of the first and the second factors in the direct product $\widetilde{G} \times \widetilde{G}$, respectively. Let $\widetilde{\mathfrak{k}}$ and $\widehat{\mathfrak{k}}$ be the Lie algebras of $\widetilde{K}$ and $\widehat{K}$, respectively. Hence, the isotropy representation decomposes into three pairwise inequivalent irreducible summands given by

$$\mathfrak{p}_1 = ((\widetilde{\mathfrak{k}})_1 \oplus (\widetilde{\mathfrak{k}})_2) \ominus \Delta\widetilde{\mathfrak{k}},$$

$$\mathfrak{p}_2 = \widetilde{\mathfrak{g}}_1 \ominus (\widetilde{\mathfrak{k}} \oplus \widehat{\mathfrak{k}})_1,$$

$$\mathfrak{p}_3 = \widetilde{\mathfrak{g}}_2 \ominus (\widetilde{\mathfrak{k}} \oplus \widehat{\mathfrak{k}})_2,$$

where the subscripts on the right-hand side refer to the factors in the direct sum $\widetilde{\mathfrak{g}} \oplus \widetilde{\mathfrak{g}}$ and where by $\ominus$ we mean the following. Let V and W be two vector spaces. Then

$V \ominus W = V \cap W^\perp$, where $W^\perp$ denotes the orthogonal complement of W. We then have the following intermediate Lie algebras:

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1,$$
$$\mathfrak{h}_1 = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$
$$\mathfrak{h}_2 = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_3.$$

These are Lie algebras of intermediate Lie groups H, H_1 and H_2, respectively. Note also that $\mathfrak{p}_3$ is irreducible also as an H_1-module and $\mathfrak{p}_2$ is irreducible also as an H_2-module. As the hypotheses of theorem 2.9.7 are satisfied, some of the singular behaviours which may occur in the HRF starting at any G-invariant Riemannian metric are given by

- H/K shrinks to a point in finite time and G/K converges to G/H in the Hausdorff-Gromov topology;
- H_1/K shrinks to a point in finite time and G/K converges to G/H_1 in the Hausdorff-Gromov topology;
- H_2/K shrinks to a point in finite time and G/K converges to G/H_2 in the Hausdorff-Gromov topology;
- G/K shrinks to a point in finite time.

Now, using the non existence criterion found by Böhm in [8], we can find sufficient conditions in order to exclude some of the above singular behaviours. Let $b = b_1 = b_2 = b_3$. The only non vanishing structure constants are given by $[122] = [133]$ and $d_2 = d_3$. Böhm proved that if

$$\left(b - \frac{2[122]}{d_1}\right)[122]\left(\frac{1}{d_2} + \frac{1}{d_1}\right) > \frac{b^2}{4} \tag{2.50}$$

is satisfied, then G/K does not carry any homogeneous Einstein metric. Moreover, looking at the Einstein equations for H_1/K and H_2/K, using Böhm's non existence criterion we have that if

$$\left(b - \frac{2[122]}{d_1}\right)[122]\left(\frac{1}{d_2} + \frac{1}{2d_1}\right) > \frac{b^2}{4} \tag{2.51}$$

is satisfied, then these two homogeneous spaces do not carry any invariant Einstein metric. Hence, if both (2.50) and (2.51) are satisfied, the only singular behaviour that we find is the one given by the shrinking of H/K to a point in finite time.

An example of such a homogeneous space is obtained by taking $\widetilde{G} = SO(n)$, $\widetilde{K} = SO(n-k)$ and $\widehat{K} = SO(k)$, with $n > k$.

Example 2.9.19. Let $\widetilde{G}$ be a compact connected Lie group. Consider two coprime integers p and q such that $(p,q) \neq \pm(1,1)$. A family of homogeneous spaces to which theorem 2.9.7 can be applied is given by

$$\widetilde{G} \times \widetilde{G}/(\Delta\widetilde{K} \cdot SO_{p,q}(2)), \tag{2.52}$$

where $\widetilde{K}$ is a Lie subgroup of $\widetilde{G}$ which is simple, $SO_{p,q}(2)$ is the diagonal embedding of $SO(2)$ in $SO(2) \times SO(2)$ with slope determined by (p,q), and $\widetilde{G}/(\widetilde{K} \cdot SO(2))$ is a symmetric space which can be found in [5, Table 7.102]. In this case, the isotropy representation $\mathfrak{p}$ decomposes into four irreducible pairwise inequivalent summands, which are given by

$$\begin{aligned}
\mathfrak{p}_1 &= (\tilde{\mathfrak{k}}_1 \oplus \tilde{\mathfrak{k}}_2) \ominus \Delta\tilde{\mathfrak{k}},\\
\mathfrak{p}_2 &= \mathfrak{g}_1 \ominus (\mathfrak{so}(2)_1 \oplus \tilde{\mathfrak{k}}_1),\\
\mathfrak{p}_3 &= \mathfrak{g}_2 \ominus (\mathfrak{so}(2)_2 \oplus \tilde{\mathfrak{k}}_2),\\
\mathfrak{p}_4 &= (\mathfrak{so}(2)_1 \oplus \mathfrak{so}(2)_2) \ominus \mathfrak{so}_{p,q}(2).
\end{aligned}$$

Hence, the only non vanishing structure constants are [221], [224], [331] and [334]. Let $G = \widetilde{G} \times \widetilde{G}$ and let $\mathfrak{k}_{p,q}$ be the Lie algebra of $K_{p,q} = \Delta\widetilde{K} \cdot SO_{p,q}(2)$. Then, $\mathfrak{k}_{p,q} \oplus \mathfrak{p}_1$ is the Lie algebra of an intermediate Lie group H_1 and $\mathfrak{p}_2, \ldots, \mathfrak{p}_4$ are also irreducible as

H_1-module. The hypotheses of theorem 2.9.7 are then satisfied. To identify some of the possible singular behaviours in the Ricci flow starting at any homogeneous Riemannian metric, we will list below all the intermediate Lie algebras.

$$\begin{aligned}\mathfrak{h}_1 &= \mathfrak{k}_{p,q} \oplus \mathfrak{p}_1,\\ \mathfrak{h}_4 &= \mathfrak{k}_{p,q} \oplus \mathfrak{p}_4,\\ \tilde{\mathfrak{h}} &= \mathfrak{k}_{p,q} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_4,\\ \bar{\mathfrak{h}} &= \mathfrak{k}_{p,q} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_3 \oplus \mathfrak{p}_4.\end{aligned}$$

These are the Lie algebras of intermediate Lie groups given by H_1, H_4, $\widetilde{H}$ and $\overline{H}$, respectively. We note that $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are irreducible also as H_4 modules, $\mathfrak{p}_3$ is irreducible also as an $\widetilde{H}$-module and $\mathfrak{p}_2$ is irreducible also as an $\overline{H}$-module. We also have that $\mathfrak{p}_4$ is a trivial summand. Hence, the Ricci flow starting at any homogeneous Riemannian metric could develop one of the singular behaviours listed below.

- $H_1/K_{p,q}$ shrinks to a point in finite time and the whole space converges in the Hausdorff-Gromov topology to G/H_1;
- $\widetilde{H}/K_{p,q}$ shrinks to a point in finite time and the whole space converges in the Hausdorff-Gromov topology to $G/\widetilde{H}$, provided that $\widetilde{H}/K_{p,q}$ carries homogeneous Einstein metrics;
- $\overline{H}/K_{p,q}$ shrinks to a point in finite time and the whole space converges in the Hausdorff-Gromov topology to $G/\overline{H}$, provided that $\overline{H}/K_{p,q}$ carries homogeneous Einstein metrics;
- The whole space shrinks to a point, provided that it carries homogeneous Einstein metrics.

We will now provide some sufficient conditions in order to exclude some of these singular behaviours. Let $b = b_1 = b_2 = b_3 = b_4$. The only nonzero structure constants are given by [122] = [133], [224] and [334]. Moreover, $d_2 = d_3$ and $d_4 = 1$. In [8], Böhm proved that if (2.50) is satisfied, then G/K does not carry any homogeneous Einstein metric. Using the same non existence criterion, we can also show that if (2.51) is satisfied then both $\widetilde{H}/K_{p,q}$ and $\overline{H}/K_{p,q}$ do not admit any invariant Einstein metric. We can then conclude that, if (2.50) and (2.51) are satisfied, the only singular behaviour that we find is the one given by the shrinking of H_1/K to a point in finite time.

Chapter 3

The pseudo-Riemannian case

3.1 Introduction

In this chapter we are going to study in more detail the dynamical systems related to the HRF on the homogeneous spaces considered in section 2.8. More precisely, we will allow the functions which define the metric to be non positive. This means that we will also consider initial conditions which do not necessarily define a Riemannian metric, but a pseudo-Riemannian one. In general, the Riemannian side and the pseudo-Riemannian one are very different. For example, for homogeneous spaces, in the pseudo-Riemannian context, the isotropy group at a point is not necessarily compact, so there might not be an invariant complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{g}$. In order to overcome this problem, one usually works with the quotient vector space $\mathfrak{g}/\mathfrak{k}$ with the action induced by the Lie bracket of $\mathfrak{g}$ [5].

In our situation, however, we are only going to study the two dynamical systems obtained in section 2.8 in full generality. As the 1-parameter family of metrics that we are considering is in general defined by

$$g(t) = x_1(t)Q_{|_{\mathfrak{p}_1}} \oplus x_2(t)Q_{|_{\mathfrak{p}_2}},$$

for $t \in [0, T)$, the signature of $g(t)$ is determined by the signs of $x_1(t)$ and $x_2(t)$. By allowing the functions to be non positive, we allow the initial metric to be pseudo-Riemannian. We will investigate the formation of singularities and the existence of immortal solutions, which are defined on $[0, +\infty)$.

Finally, the last part part of the chapter is dedicated to the study of the behaviour of the scalar curvature $R(t)$ of the metric $g(t)$ under the same dynamical systems. We prove that in the Riemannian case the scalar curvature always has to turn positive in finite time, if it was negative initially. Whereas, in the pseudo-Riemannian case, there are situations in which for a certain class of initial conditions negativity of the scalar curvature is preserved and this leads to the existence of immortal solutions.

We would like to mention that topics related to the Ricci flow in the pseudo-Riemannian case have been studied by other authors before. As an example, we refer the reader to [3].

3.2 When the isotropy group is not maximal

In this section, we are going to study a dynamical system of the following form:

$$\dot{x}_1(t) = -C - A\frac{x_1(t)^2}{x_2(t)^2}, \tag{3.1}$$

$$\dot{x}_2(t) = -D + B\frac{x_1(t)}{x_2(t)}, \tag{3.2}$$

where A, B and D and C are defined by (2.8)-(2.11). In particular, recall that A, B and D are positive and C is non negative. In the previous chapter, as we were interested in deforming Riemannian metrics under the Ricci flow, we were imposing that the functions $x_1(t)$ and $x_2(t)$ have to be positive. We will now drop this assumption, which geometrically means that we will allow the metric defined by $x_1(t)$ and $x_2(t)$ to be pseudo-Riemannian.

Let x_1, x_2 be coordinates in $\mathbb{R}^2$, with x_1 being the vertical axis and x_2 the horizontal axis. Consider

$$\boldsymbol{D}_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \neq 0\}.$$

Then, by a standard theorem of ODEs, we have existence and uniqueness of solutions on $\boldsymbol{D}_1$. So, we immediately notice that the function $x_1(t)$ is allowed to change sign along a solution of the above system, while $x_2(t)$ cannot, because the solution has to stop if $x_2(t)$ becomes zero. Suppose that the initial time is given by $t_0 = 0$. There are four different initial conditions that we have to consider:

i) $x_1(0) > 0$ and $x_2(0) > 0$,

ii) $x_1(0) \leq 0$ and $x_2(0) > 0$,

iii) $x_1(0) \leq 0$ and $x_2(0) < 0$,

iv) $x_1(0) > 0$ and $x_2(0) < 0$.

3.2.1 Case i)

This case has been considered extensively in section 2.8.1. Recall that the solutions to

$$C - Dy + (A+B)y^2 = 0 \tag{3.3}$$

correspond to stationary points of (3.1)-(3.2). We will now give an alternative proof of the fact that the solution can go to the origin in finite time only if equation (3.3) has a positive root. Consider the evolution equation of $\frac{x_1(t)}{x_2(t)}$:

$$\frac{d}{dt}\left(\frac{x_1(t)}{x_2(t)}\right) = \frac{1}{x_2(t)}\left(-C + D\frac{x_1(t)}{x_2(t)} - (A+B)\left(\frac{x_1(t)}{x_2(t)}\right)^2\right). \tag{3.4}$$

Note that if (3.3) has no roots, then $\frac{x_1(t)}{x_2(t)}$ is monotonically decreasing in t. Suppose that the solution goes to the origin in finite time, which means that both $x_1(t)$ and $x_2(t)$

become zero. We already know, by the work we did in the previous chapter that these functions go to zero linearly in t. This would then imply that $\frac{d}{dt}\left(\frac{x_1(t)}{x_2(t)}\right)$ diverges to $-\infty$ as $(T-t)^{-1}$, as t tends to the final time T. We then have that also its integral between 0 and T would diverge to $-\infty$, which would then imply that $\frac{x_1(t)}{x_2(t)}$ diverges to $-\infty$, which is a contradiction, as we know that the ratio between the functions is bounded. In this way, we have then proved the following proposition.

Proposition 3.2.1. *If $x_1(0) > 0$ and $x_2(0) > 0$, then there exists a solution to* (3.1)-(3.2) *which goes to the origin in finite time if and only if* (3.3) *has at least one root.*

Example 3.2.2. The 12-dimensional manifold $SU(4)/SU(2)$ is an example of a homogeneous space such that the above proposition can be used to show that it does not carry any $SU(4)$-invariant homogeneous Einstein metric. In fact, we proved that a solution to the HRF in this case can go to the origin if and only if the space carries a homogeneous Einstein metric. In this way, we then reprove the non existence criterion found by Wang and Ziller in [49].

3.2.2 Case ii)

If $(x_1(0), x_2(0))$ satisfies the initial condition given by ii) above, then, as long as we remain in the fourth quadrant, $\frac{x_1(t)}{x_2(t)}$ is non positive for every t such that a solution exists. Hence, both functions $x_1(t)$ and $x_2(t)$ are decreasing in t. Moreover, we have that

$$\dot{x}_2(t) < -D,$$

which integrated gives

$$x_2(t) < -Dt + x_2(0).$$

Hence, the solution will have to stop after a finite time, before $x_2(t)$ becomes zero. We will now show that $x_1(t)$ cannot diverge to $-\infty$ before $x_2(t)$ becomes zero, because this contradicts the uniqueness of the solution. In fact, suppose for a contradiction that there exists an initial condition such that $x_1(t)$ tends to $-\infty$, as $t \to T'$, with $x_2(T') = \alpha > 0$. Then, consider an initial condition with $x_1(0) << -1$ and $x_2(0) = \alpha$. If we now solve the equations backwards in time, we have that both $x_1(t)$ and $x_2(t)$ are increasing in t. So the backwards solution would cross the trajectory of the previous one and we get a contradiction, because the uniqueness still holds. We can then conclude that the final time is characterised by $x_2(t)$ becoming zero. Moreover, as t approaches the final time, $x_1(t)$ either tends to a negative limit, or it diverges to $-\infty$. We have then proved the following proposition.

Proposition 3.2.3. *If $x_1(0) \leq 0$ and $x_2(0) > 0$, then there exists $T < \infty$ such that the unique solution $(x_1(t), x_2(t))$ to (3.1)-(3.2) exists on $[0, T)$ and, as $t \to T$, $x_2(t)$ becomes zero and $x_1(t)$ tends to a negative limit, which could be $-\infty$.*

Before studying the solution backwards in time, we would like to consider the evolution equation of the quantity $x_1(t)x_2(t)$, as it is useful to understand better the behaviour of $x_1(t)$. We have that

$$\begin{aligned}\frac{d}{dt}(x_1(t)x_2(t)) &= -Cx_2(t) - A\frac{x_1(t)^2}{x_2(t)} - Dx_1(t) + B\frac{x_1(t)^2}{x_2(t)} \\ &= \frac{x_1(t)^2}{x_2(t)}\left(B - A - D\frac{x_2(t)}{x_1(t)} - C\frac{x_2(t)^2}{x_1(t)^2}\right).\end{aligned}$$

As t approaches the final time, the expression between brackets always tends to $B - A$, while $\frac{x_1(t)^2}{x_2(t)}$ tends to $+\infty$. From proposition 3.2.3, if $x_1(t)$ approaches a negative limit, which is not $-\infty$, the product $x_1(t)x_2(t)$ tends to zero. As $x_1(0)x_2(0) < 0$, it is then necessary that $B - A \geq 0$, otherwise the first derivative of $x_1(t)x_2(t)$ would tend to $-\infty$, which is not possible, because of the mean value theorem. We can then conclude that if $B - A < 0$, then necessarily $x_1(t)$ has to tend to $-\infty$, as t approaches the final time and in such a way that $x_1(t)x_2(t)$ also tends to $-\infty$. Recall that from (2.8)-(2.11), $B - A < 0$ if and only if $d_2 > 2d_1$.

We now want to understand if every solution with initial condition as in ii) comes from one in the first quadrant. In oder to do this, we investigate the existence of backwards solutions, with initial condition given by ii). Let $\tau = -t$ and consider the system of nonlinear ODEs

$$x_1(\tau)' = C + A\frac{x_1(\tau)^2}{x_2(\tau)^2}, \tag{3.5}$$

$$x_2(\tau)' = D - B\frac{x_1(\tau)}{x_2(\tau)}, \tag{3.6}$$

where $'$ indicates the derivative with respect to τ. We then have that, as long as we remain in the fourth quadrant, both $x_1(\tau)$ and $x_2(\tau)$ are increasing in τ. We now have to distinguish the cases $C = 0$ and $C > 0$. If $C > 0$, we have that

$$x_1(\tau)' > C,$$

which integrated gives

$$x_1(\tau) > C\tau + x_1(0).$$

Hence, $x_1(\tau)$ becomes zero in finite time, which means that the solution crosses the x_2-axis. Note that, as $x_2(0) > 0$ and $x_2(\tau)$ is increasing in τ, the solution cannot reach the origin from the fourth quadrant. We then have that, when $C > 0$, every solution with initial condition given by ii) comes from one starting in the first quadrant.

We will now consider briefly the case in which $C = 0$. In this case $\{x_1 = 0\}$ is invariant under the flow, so $x_1(\tau)$ cannot become zero in finite time. As $y(\tau)$ is increasing in τ, the derivatives of $x_1(\tau)$ and $x_2(\tau)$ are bounded, so the solution exists for every $\tau > 0$. This means that when $C = 0$, there exists an ancient solution and the trajectories never arise in the first quadrant.

3.2.3 Case iv)

Here, as long as we are in the second quadrant, both $x_1(t)$ and $x_2(t)$ are decreasing in t, so the solution is moving towards the x_2-axis. We need to distinguish two different situations, namely $C = 0$ or $C > 0$. Let us firstly consider the case $C = 0$. Then, the evolution equation for $x_1(t)$ becomes

$$\dot{x}_1(t) = -A\frac{x_1(t)^2}{x_2(t)^2}.$$

We know that $x_2(t) < x_2(0) < 0$, as long as $(x_1(t), x_2(t))$ belongs to the second quadrant. We then have that

$$\dot{x}_1(t) > -A\frac{x_1(t)^2}{x_2(0)^2},$$

which integrated gives

$$x_1(t) > \frac{x_1(0)}{C't+1},$$

where C' is a positive constant given by $A\frac{x_1(0)}{x_2(0)^2}$. The above inequality tells us that $x_1(t)$ cannot become zero in finite time. This means that the solution has to stay in the second quadrant. We will now show that $x_2(t)$ cannot become $-\infty$ in finite time. In fact, from its evolution equation we have that $\frac{x_1(t)}{x_2(t)}$ is increasing in t, as long as we are in the second quadrant. Hence, we have that

$$\dot{x}_2(t) > -D + B\frac{x_1(0)}{x_2(0)} = -C'',$$

which is a negative constant. If we integrate the above expression, we obtain

$$x_2(t) > -C''t + x_2(0),$$

which implies that $x_2(t)$ cannot diverge to $-\infty$ in finite time. Hence, if $C = 0$, the solution to the ODE exists for all $t > 0$. We will now prove that, as $t \to +\infty$, $x_1(t)$ tends to zero and $x_2(t)$ tends to $-\infty$. As $\frac{x_1(t)}{x_2(t)}$ is negative, we have that

$$\dot{x}_2(t) < -D,$$

which integrated gives

$$x_2(t) < -Dt + x_2(0),$$

which tends to $-\infty$ as $t \to +\infty$. We will now prove that $x_1(t)$ has to tend to zero, as $t \to +\infty$. Suppose for a contradiction that there exists a solution $(x_1(t), x_2(t))$ to (3.1)-(3.2), with $C = 0$ and with initial condition as in iv), such that $x_1(t) \to \alpha > 0$, as $t \to +\infty$. Then, consider another solution which starts at $(x_1(0) = \alpha, x_2(0) < 0)$. If we try to look for a backwards solution with this initial condition, it means that we want to solve the system (3.5)-(3.6), with $C = 0$. We then have that both $x_1(\tau)$ and $x_2(\tau)$ are increasing in τ, which means that the solution is moving towards the x_1-axis. This would then imply that this solution and the previous one intersect at a point, which contradicts the uniqueness of the solution. We have then proved the following proposition.

Proposition 3.2.4. *If $C = 0$ and $x_1(0) > 0$ and $x_2(0) < 0$, then the solution $(x_1(t), x_2(t))$ to* (3.1)-(3.2) *exists for all $t > 0$ and, as $t \to +\infty$, $x_1(t)$ tends to zero and $x_2(t)$ tends to $-\infty$.*

To conclude, note that if $x_1(0) = 0$ and $x_2(0) < 0$, then $\dot{x}_1(t) = 0$ for all t such that a solution exists, so the solution will just move along the x_2-axis and $x_2(t)$ will tend to $-\infty$ linearly in t.

We now have to analyse the case in which $C > 0$. We have that

$$\dot{x}_1(t) < -C,$$

which integrated tells us that $x_1(t)$ will become zero in finite time. Moreover, using the same estimate as before, we can show that $x_2(t)$ cannot become $-\infty$ in finite time. So, if $C > 0$, the solution will cross the x_2-axis in finite time and enter the third quadrant, where both functions are negative. The following proposition is then true.

Proposition 3.2.5. *If $C > 0$, then every solution to* (3.1)-(3.2) *with initial condition in the second quadrant, crosses the x_2-axis in finite time.*

Finally, observe that if $x_1(0) = 0$ and $x_2(0) < 0$, then the solution will immediately enter the third quadrant.

3.2.4 Case iii)

Recall that in this case, $(x_1(0), x_2(0))$ belongs to the third quadrant. Here, we have to distinguish several different cases, which depend on how many roots equation (3.3) has. In particular, the following situations can occur:

a) Equation (3.3) has no roots;

b) Equation (3.3) has exactly one positive root;

c) Equation (3.3) has two distinct positive roots.

Note that in case a) and c), $C > 0$. Whereas, for case b), we have to distinguish the cases in which $C = 0$ and $C > 0$. Moreover, we will also investigate which solutions in the third quadrant come from solutions starting in the second one and which do not.

Case a)

Let L be the line in $\mathbb{R}^2$ defined by

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 | - D + B\frac{x_1}{x_2} = 0 \right\}.$$

This line separates the third quadrant into two connected regions given by

$$\Sigma_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1 < 0, x_2 < 0, x_1 < \frac{D}{B} x_2\right\},$$
$$\Sigma_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 | x_1 < 0, x_2 < 0, x_1 > \frac{D}{B} x_2\right\}.$$

In particular, if $(x_1(t), x_2(t)) \in \Sigma_1$, then $x_2(t)$ is increasing in t, while if $(x_1(t), x_2(t)) \in \Sigma_2$, then $x_2(t)$ is decreasing in t. Recall that $x_1(t)$ is always decreasing in t. Now, the second derivative of $x_2(t)$ is given by

$$\ddot{x}_2(t) = B\frac{\dot{x}_1(t)}{x_2(t)} - B\frac{x_1(t)}{x_2(t)^2}\dot{x}_2(t).$$

Then, when $\dot{x}_2(t)$ is zero, we have that $\ddot{x}_2(t)$ is positive, which means that every critical point is a minimum point for $x_2(t)$. Hence, the line L can only be crossed from Σ_2 to Σ_1. We also have that Σ_1 is an invariant region for the dynamical system. In fact, if we compute the vector field $(\dot{x}_1(t), \dot{x}_2(t))$ on L, we have that it is given by $(\dot{x}_1(t), 0)$, so it points towards the interior of Σ_1. We will begin by considering the case in which $(x_1(0), x_2(0)) \in \Sigma_1$. Here $x_1(t)$ is decreasing in t, while $x_2(t)$ is increasing. So the solution is moving towards the x_1-axis. As equation (3.3) has no roots and $x_2(t)$ is negative, from the evolution equation of $\frac{x_1(t)}{x_2(t)}$, we can see that $\frac{x_1(t)}{x_2(t)}$ is increasing in t. So, we have that

$$\dot{x}_2(t) > -D + B\frac{x_1(0)}{x_2(0)} =: C' > 0,$$

which integrated tells us that $x_2(t)$ becomes zero in finite time. This means that the solution will have to stop in finite time at or before the time in which $x_2(t)$ becomes zero. We now need to understand the behaviour of $x_1(t)$. We will now show that $x_1(t)$ cannot diverge to $-\infty$ before $x_2(t)$ becomes zero. Suppose for a contradiction that $x_1(t)$ becomes $-\infty$, while $x_2(t)$ approaches a negative limit, say α. Then consider the initial condition $(x_1(0), \alpha)$, with $x_1(0) < 0$. If we try to solve the equations backwards in time, as $x_1(t)$ is increasing and $x_2(t)$ is decreasing, the solution will cross the previous one in finite time. We then derive a contradiction, because as long as $x_2(t)$ is nonzero, we have uniqueness of the solution. So, we know that at the final time $x_2(t)$ becomes zero. For what concerns the behaviour of $x_1(t)$, we are in the same situation as in case ii) above. So, $x_1(t)$ could either become $-\infty$ or approach a negative limit, as t tends to the final time. However, if we consider the evolution equation of $x_1(t)x_2(t)$, we can show that if $x_1(t)$ tends to a negative limit, which is not $-\infty$, then necessarily $B - A \leq 0$. Moreover, if $B - A > 0$, $x_1(t)$ has to tend to $-\infty$, as t approaches the final time, and in such a way that also $x_1(t)x_2(t) \to -\infty$.

We now have to consider the case in which $(x_1(0), x_2(0)) \in \Sigma_2$. Recall that in Σ_2 both functions are decreasing in t. Moreover, their derivatives are bounded from below in Σ_2:

$$\dot{x}_1(t) > -C - A\left(\frac{D}{B}\right)^2,$$
$$\dot{x}_2(t) > -D.$$

This implies that $x_1(t)$ and $x_2(t)$ cannot diverge to $-\infty$ in finite time. We then have that two possible behaviours can occur. Either the solution exists for all $t > 0$ in Σ_2, or the solution hits the line L and enters the region Σ_1, which is invariant. We will show that it is not possible for the solution to exists for all $t > 0$ in Σ_2. Suppose that we have a solution of (3.1)-(3.2) in Σ_2 which exists for all $t > 0$. Then, as $\frac{x_1(t)}{x_2(t)}$ is increasing in t and bounded from above by $\frac{D}{B}$, we have that $\frac{x_1(t)}{x_2(t)}$ approaches a finite positive limit, as $t \to +\infty$. Let denote this limit by $\bar{y}$. In particular, $\frac{D}{B} > \bar{y} > 0$. We then have a line $x_1 = \bar{y}x_2$ in $\mathbb{R}^2$ to which the solution is asymptotic from above. Now consider another solution with initial condition given by $(x_1(0), x_2(0)) \in \Sigma_2$ such that $\frac{x_1(0)}{x_2(0)} = \bar{y}$. If we now look for solutions to (3.1)-(3.2) which go backwards in time, we find that, as both $x_1(t)$ and $x_2(t)$ are increasing as the time goes backwards, this solution crosses the previous one and we contradict the uniqueness. So, the solution has to leave Σ_2 in finite time and enter the invariant region Σ_1. We have then proved the following proposition.

Proposition 3.2.6. *If* $(x_1(0), x_2(0))$ *belongs to the third quadrant and* (3.3) *has no roots, then there exists* $T < \infty$ *such that there exists a unique solution to* (3.1)-(3.2) *which is defined on the maximal time interval* $[0, T)$. *As* $t \to T$, $x_2(t)$ *becomes zero and* $x_1(t)$ *tends to a negative limit, which could be* $-\infty$.

We conclude this subsection by showing that every solution with $C > 0$ and which starts in the third quadrant arises from one which starts in the second quadrant. First of all note that if we have a solution which starts in Σ_1, then we can solve the equation backwards and find that it comes from a solution starting in Σ_2. In fact, consider the system (3.5)-(3.6), where we changed time parameter from t to $\tau = -t$. Then, $x_1(t)$ is increasing in τ, while $x_2(\tau)$ is decreasing. This means that the trajectory is moving towards the line L, which is a line of minimum points for $x_2(\tau)$. As

$$x_1(\tau)' < C + A\frac{x_1(0)^2}{x_2(0)^2},$$
$$x_2(\tau)' > D - B\frac{x_1(0)}{x_2(0)},$$

the solution will cross the line L in finite time and enter the region Σ_2. It only remains to show that every solution starting in Σ_2 comes from a solution starting in the second quadrant. Suppose that $(x_1(0), x_2(0)) \in \Sigma_2$. Then, by considering the system (3.5)-(3.6), we have that both $x_1(\tau)$ and $x_2(\tau)$ are increasing in τ. Hence, the trajectory is moving towards the x_2-axis. Moreover,

$$x_1(\tau)' > C,$$

which integrated gives

$$x_1(\tau) > C\tau + x_1(0).$$

Then, after a finite time T', $x_1(\tau)$ becomes zero and two different behaviours can occur.

Either the solution crosses the x_2-axis or it reaches the origin and stops. However, as $\frac{x_1(\tau)}{x_2(\tau)}$ is decreasing in τ, it is not possible that $x_2(\tau)$ becomes zero, while $x_1(\tau)$ approaches a negative limit. If the solution reaches the origin, then

$$\frac{d}{d\tau}\left(\frac{x_1(\tau)}{x_2(\tau)}\right) \to -\infty,$$

because (3.3) does not have roots. By performing an asymptotic analysis, we can find that both $x_1(\tau)$ and $x_2(\tau)$ tend to zero linearly in τ. Hence, the first derivative of the ratio $\frac{x_1(\tau)}{x_2(\tau)}$ tends to $-\infty$ as $(T'-\tau)^{-1}$. This then implies that its integral between 0 and T' diverges to $+\infty$, which is not possible, as $\frac{x_1(\tau)}{x_2(\tau)}$ is decreasing in τ. We can then conclude that the backwards solution has to cross the x_2-axis in finite time, which means that every solution comes from one which starts in the second quadrant.

Case b)

Here, we have to distinguish the cases in which $C=0$ and $C>0$. We will start with the case in which $C>0$. Let $\bar{y}$ denote the unique solution to (3.3). Then, we can write the evolution equation of $\frac{x_1(t)}{x_2(t)}$ as

$$\frac{d}{dt}\left(\frac{x_1(t)}{x_2(t)}\right) = -\frac{A+B}{x_2(t)}\left(\frac{x_1(t)}{x_2(t)}-\bar{y}\right)^2,$$

which implies that $\frac{x_1(t)}{x_2(t)}$ is always increasing in t, unless we are on the line $x_1=\bar{y}x_2$, where the ratio remains constant. As in the previous case we consider the line L, which separates the regions in which $x_2(t)$ is increasing and those in which it is decreasing in t, which we denoted by Σ_1 and Σ_2, respectively. Note that $\bar{y}<\frac{D}{B}$, which means that the line $x_1=\bar{y}x_2$, with $x_1,x_2<0$, lies in the region Σ_2, in which both functions are decreasing in t. Moreover, if the initial condition $(x_1(0),x_2(0))$ is such that $x_1(0)=\bar{y}x_2(0)$, then the solution exists for every $t > 0$ and for every such t

$$\frac{x_1(t)}{x_2(t)} = \bar{y}.$$

This is due to the fact that the line $x_1 = \bar{y}x_2$ is invariant under the system (3.1)-(3.2). Clearly, if we investigate the solution backwards in time starting on this line, we have that the trajectory reaches the origin in finite time. We then have to distinguish the following remaining initial conditions:

1) $\frac{x_1(0)}{x_2(0)} < \bar{y}$;

2) $\frac{x_1(0)}{x_2(0)} > \bar{y}$;

We will now analyse these possible initial conditions separately. We will also show that every solution starting with initial condition given by 1) comes from a solution which starts in the second quadrant.

Case 1): By the uniqueness of the solution, we have that

$$\frac{x_1(t)}{x_2(t)} < \bar{y},$$

for every t such that a solution exists. Hence, the functions $x_1(t)$ and $x_2(t)$ are both decreasing in t with bounded derivatives:

$$\dot{x}_1(t) > -C - A\bar{y}^2,$$
$$\dot{x}_2(t) > -D.$$

By standard ODE theory we then have that this implies that the solution exists for every $t > 0$. As $\frac{x_1(t)}{x_2(t)}$ is increasing in t and less than $\bar{y}$, for every $t > 0$, it approaches a positive limit as $t \to +\infty$. Due to the uniqueness of the solution this limit has to correspond to $\bar{y}$. We have then proved that $\frac{x_1(t)}{x_2(t)}$ tends to $\bar{y}$, as $t \to +\infty$. We will now investigate the solution backwards in time to see from where it comes from. The equations (3.5)-(3.6) tell us that $x_1(\tau)$ and $x_2(\tau)$ are both increasing in $\tau = -t$, while their ratio is decreasing in τ. Hence, the point $(x_1(\tau), x_2(\tau))$ is moving towards the x_2-axis. Moreover, as in case a), $x_2(\tau)$ cannot become zero, which means that the point cannot go to the origin. Hence, the solution will cross the x_2-axis in finite time, which means that every solution starting with 1) comes from a solution arising in the second quadrant.

Case 2): We have to distinguish two possible initial conditions: $(x_1(0), x_2(0)) \in \Sigma_1$ and $(x_1(0), x_2(0)) \in \Sigma_2$. We will begin by considering the case in which the initial condition lies in Σ_1. As in case a), Σ_1 is an invariant region for our system of differential equations. This implies that $(x_1(t), x_2(t)) \in \Sigma_1$, for every t such that a solution exists. Then, by proceeding as in case a), we can show that the solution has to stop in finite time, because $x_2(t)$ becomes zero. Moreover, as t approaches the final time, $x_1(t)$ tends to $-\infty$. Finally, by investigating the solution backwards in time, we can show that every solution which starts with this initial condition comes from one which starts in Σ_2. We now have to consider the case in which the initial condition lies in Σ_2. Here we can proceed as in case a) and show that every solution starting with this initial condition has to cross the line L in finite time and enter the invariant region Σ_1, where we know what happens.

We have then proved the following proposition.

Proposition 3.2.7. *If* $C > 0$ *and* (3.3) *has exactly one root, then, if* $\frac{x_1(0)}{x_2(0)} < \bar{y}$, *with* $x_1(0), x_2(0) < 0$, *there exists a unique solution to the system* (3.1)-(3.2) *which is defined on* $[0, +\infty)$. *Moreover, as* $t \to +\infty$, $\frac{x_1(t)}{x_2(t)}$ *tends to* $\bar{y}$. *If* $\frac{x_1(0)}{x_2(0)} > \bar{y}$, *with* $x_1(0), x_2(0) < 0$, *then there exists* $T < \infty$ *such that there exists a unique solution to the system* (3.1)-(3.2) *which is defined on the maximal time interval* $[0, T)$. *Moreover, the final time* T *is characterised by the fact that* $x_2(t)$ *becomes zero and* $x_1(t)$ *tends to a negative limit,* *which could be $-\infty$. Finally, we have that every solution starting with $\frac{x_1(0)}{x_2(0)} < \bar{y}$ comes from one arising in the second quadrant.*

We still have to consider the case in which $C = 0$. Then, equation (3.3) has one zero root and then one positive root given by

$$\bar{y} = \frac{D}{A+B}.$$

In particular, the evolution equation for $\frac{x_1(t)}{x_2(t)}$ becomes

$$\frac{d}{dt}\left(\frac{x_1(t)}{x_2(t)}\right) = -(A+B)\frac{x_1(t)}{x_2(t)^2}\left(\frac{x_1(t)}{x_2(t)} - \frac{D}{A+B}\right). \tag{3.7}$$

As before, the line $x_1 = \bar{y}x_2$ is a particular solution to the system (3.1)-(3.2) and can never be crossed by any other solution. In contrast to the case in which $C > 0$, this time we have that also the line $x_1 = 0$ is a particular solution to the system that we are considering. So, also the x_2-axis can never be crossed by any other solution, which means that none of the solutions with this initial condition can come from solutions starting in the second quadrant. We then have to consider two possible initial conditions, which are the same as in 1) and 2) above.

Case 1): By the uniqueness of the solution, we have that

$$0 < \frac{x_1(t)}{x_2(t)} < \bar{y}, \tag{3.8}$$

for every t such that a solution exists. Equation (3.7) implies that $\frac{x_1(t)}{x_2(t)}$ is decreasing in t. We also know that both $x_1(t)$ and $x_2(t)$ are decreasing in t with bounded derivatives:

$$\dot{x}_1(t) > -A\bar{y}^2,$$

$$\dot{x}_2(t) > -D.$$

So the solution exists for every $t > 0$. Moreover, $\frac{x_1(t)}{x_2(t)}$ tends to zero, as $t \to +\infty$. We will now investigate the solution backwards in time. Let $\tau = -t$. Then, $\frac{x_1(\tau)}{x_2(\tau)}$ is increasing in τ, as well as $x_1(\tau)$ and $x_2(\tau)$. This implies that $x_1(\tau)$ cannot become zero, unless also $x_2(\tau)$ becomes zero. Moreover, we have that

$$x_2(\tau)' > D - B\bar{y} > 0,$$

which integrated tells us that $x_2(\tau)$ becomes zero in finite time. Now (3.8) implies that $x_2(\tau)$ cannot become zero, unless also $x_1(\tau)$ becomes zero. We can then conclude that the point $(x_1(\tau), x_2(\tau))$ goes to the origin in finite time, with $\frac{x_1(\tau)}{x_2(\tau)}$ which tends to $\bar{y}$ as τ tends to the final time.

Case 2): By the uniqueness of the solution we have that

$$\frac{x_1(t)}{x_2(t)} > \bar{y},$$

for every t such that a solution exists. Moreover, equation (3.7) tells us that $\frac{x_1(t)}{x_2(t)}$ is increasing in t. We now have to distinguish two possible initial conditions, namely $(x_1(0), x_2(0)) \in \Sigma_1$ or $(x_1(0), x_2(0)) \in \Sigma_2$, which were described above. We will consider firstly the case in which the initial condition lies in Σ_1. As before, this is an invariant region for the system (3.1)-(3.2). So, we will have that $(x_1(t), x_2(t)) \in \Sigma_1$, for every t such that a solution exists. Recall that in Σ_1, $x_1(t)$ is decreasing, while $x_2(t)$ is increasing in t. We have that

$$\dot{x}_2(t) > -D + B\frac{x_1(0)}{x_2(0)},$$

which is a positive constant. Hence, by integrating the above equation, we have that $x_2(t)$ has to become zero in finite time. Now, proceeding as in case a) above, we have that the final time is characterised by $x_2(t)$ becoming zero and $x_1(t)$ approaching a negative limit, which could be $-\infty$. Moreover, if $B-A>0$, then necessarily $x_1(t)\to-\infty$, as t tends to the final time, and in such a way that $x_1(t)x_2(t)\to+\infty$.

By investigating the solution backwards in time, we find that it always has to cross the line L and enter the region Σ_2, which means that every solution starting in Σ_1 comes from one starting in Σ_2.

It remains to consider the case in which the initial condition lies in Σ_2. Here both functions are decreasing in t, while their ratio is increasing. As before, we have that the solution has to cross the line L, where $x_2(t)$ attains its minimum and enter the invariant region Σ_1. If we investigate the solution backwards in time, we are then looking for solutions to the system (3.5)-(3.6), with $C=0$. We then have that both $x_1(\tau)$ and $x_2(\tau)$ are both increasing in τ, while their ratio is decreasing. Moreover,

$$
\begin{aligned}
&x_1(\tau)' > A\frac{x_1(0)^2}{x_2(0)^2},\\
&x_2(\tau)' < D.
\end{aligned}
$$

Then, the solution stops in finite time because $x_1(\tau)$ becomes zero. As $y(\tau)>\bar{y}$ for every τ such that a solution exists, if $x_1(\tau)$ vanishes, then also $x_2(\tau)$ must vanish, which means that the trajectory goes to the origin. Moreover, we can compute that $\frac{x_1(\tau)}{x_2(\tau)}$ tends to $\bar{y}$, as τ approaches the final time. We have then proved the following proposition.

Proposition 3.2.8. *Suppose that $C=0$, then, if $\frac{x_1(0)}{x_2(0)}<\bar{y}$, there exists $0<T<\infty$ such that there exists a unique solution to (3.1)-(3.2), which is defined on the maximal time interval $(-T,+\infty)$. As $t\to+\infty$, $\frac{x_1(t)}{x_2(t)}$ tends to zero and as $t\to-T$, $\frac{x_1(t)}{x_2(t)}$ tends to $\bar{y}$. Whereas, if $\frac{x_1(0)}{x_2(0)}>\bar{y}$, then there exist $0<T_1,T_2<\infty$ such that there exists a unique solution on the maximal time interval $(-T_1,T_2)$. As $t\to T_2$, $x_1(t)$ tends to a negative limit, which could be $-\infty$, and $x_2(t)$ tends to zero. Finally, as $t\to-T_1$, $\frac{x_1(t)}{x_2(t)}$ tends to $\bar{y}$.*

Case c)

First of all we note that $C > 0$, because we assume that equation (3.3) has two distinct positive roots. We denote these by y_1 and y_2. Suppose without loss of generality that $y_1 < y_2$. We then have that

$$\frac{d}{dt}\left(\frac{x_1(t)}{x_2(t)}\right) = -\frac{A+B}{x_2(t)}\left(\frac{x_1(t)}{x_2(t)} - y_1\right)\left(\frac{x_1(t)}{x_2(t)} - y_2\right). \tag{3.9}$$

As before, we can consider the line L which separates the third quadrant into two connected regions Σ_1 and Σ_2. We note that the lines $x_1 = y_1 x_2$ and $x_1 = y_2 x_2$, with $x_1, x_2 < 0$, are located in the region Σ_2, where $x_2(t)$ is decreasing in t. We also observe that if the initial condition satisfies $\frac{x_1(0)}{x_1(0)} = y_i$, with $i = 1, 2$, then equation (3.9) above implies that

$$\frac{x_1(t)}{x_2(t)} = y_i,$$

for all $t > 0$. As before, this is due to the fact that the lines $x_1 = y_i x_2$, with $i = 1, 2$ are invariant under the system (3.1)-(3.2). Whereas, if we try to solve the equations backwards in time, we have that the solution stops in finite time, because it reaches the origin. Hence, we have to consider the following initial conditions:

1) $\frac{x_1(0)}{x_2(0)} < y_1$;

2) $y_1 < \frac{x_1(0)}{x_2(0)} < y_2$;

3) $y_2 < \frac{x_1(0)}{x_2(0)}$.

We will now consider all these cases separately.

Case 1): By equation (3.9), we have that $\frac{x_1(t)}{x_2(t)}$ is increasing in t. Moreover, we know, as we are in Σ_2, that both $x_1(t)$ and $x_2(t)$ are decreasing in t. We can also show that the derivatives of $x_1(t)$ and $x_2(t)$ are bounded from below:

$$\dot{x}_1(t) > -C - Ay_1^2,$$
$$\dot{x}_2(t) > -D.$$

So, the solution exists for every $t > 0$ and, as $t \to +\infty$, $\frac{x_1(t)}{x_2(t)}$ tends to y_1. If we now investigate the solution backwards in time, as in case a, we find that it has to cross the x_2-axis in finite time, which means that every solution starting with this initial condition comes from one arising in the second quadrant.

Case 2): From equation (3.9), we have that $\frac{x_1(t)}{x_2(t)}$ is decreasing in t. Moreover, we know that

$$y_1 < \frac{x_1(t)}{x_2(t)} < y_2, \tag{3.10}$$

for every t such that a solution exists. This means that the solution $(x_1(t), x_2(t))$ never leaves Σ_2. Hence, both $x_1(t)$ and $x_2(t)$ are decreasing with derivatives bounded from below:

$$\dot{x}_1(t) > -C - Ay_2^2,$$
$$\dot{x}_2(t) > -D.$$

Then the solution exists for every $t > 0$ and, as $t \to +\infty$, $\frac{x_1(t)}{x_2(t)}$ tends to y_1. We will now investigate the solution backwards in time. We are then considering the system (3.5)-(3.6), where $\tau = -t$. We have that $x_1(\tau)$ and $x_2(\tau)$ are both increasing in τ, as well as their ratio. Moreover,

$$x_2(\tau)' > D - By_1 > 0,$$

which integrated tells us that $x_2(\tau)$ becomes zero in finite time. However, equation (3.10) tells us that $x_2(\tau)$ cannot become zero, unless also $x_1(\tau)$ becomes zero, and viceversa. Hence, the solution stops in finite time, because the solution reaches the origin. Finally, $\frac{x_1(\tau)}{x_2(\tau)}$ tends to y_2, as τ approaches the final time.

Case 3): By the uniqueness of the solution, we have that

$$\frac{x_1(t)}{x_2(t)} > y_2,$$

for every t such that a solution exists. Moreover, from equation (3.9), we have that $\frac{x_1(t)}{x_2(t)}$ is increasing in t. Here, we have to distinguish two possible initial conditions, namely $(x_1(0), x_2(0)) \in \Sigma_1$ and $(x_1(0), x_2(0)) \in \Sigma_2$. We will begin by considering the case in which the initial condition lies in Σ_1. As this region is invariant under the system (3.1)-(3.2), we have that $(x_1(t), x_2(t)) \in \Sigma_1$, for every t such that a solution exists. Here, as in case a, we have that the solution stops in finite time because $x_2(t)$ becomes zero and $x_1(t)$ tends to a negative limit, which could be $-\infty$. Moreover, if $B - A > 0$, then necessarily $x_1(t)$ tends to $-\infty$ and $x_1(t)x_2(t)$ tends to $+\infty$. Then, exactly as in the previous cases, if we investigate backwards solutions, we find that every solution starting in Σ_1 comes from one starting in Σ_2. We then have to consider the case in which the initial condition lies in Σ_2. Again as in the previous cases, the solution has to cross the line L, which corresponds to a line of minimum points for $x_2(t)$, and enter the region Σ_2. Whereas, if we investigate the solution backwards in time, we find that it exists only for a finite time, because both functions become zero. Moreover, the ratio $\frac{x_1(t)}{x_2(t)}$ tends to y_2. We have then proved the following proposition:

Proposition 3.2.9. *Suppose that* (3.3) *has two distinct roots. Then, if* $\frac{x_1(0)}{x_2(0)} < y_1$*, there a unique solution to* (3.1)-(3.2) *which is defined on* $[0, +\infty)$*. As* $t \to +\infty$*, the ratio* $\frac{x_1(t)}{x_2(t)}$ *tends to* y_1*. Moreover every solution with this initial condition comes from one arising* *in the second quadrant. If,* $y_1 < \frac{x_1(0)}{x_2(0)} < y_2$*, then there exists* $0 < T_1 < \infty$ *such that there exists a unique solution on the maximal time interval* $(-T_1, +\infty)$*. As* $t \to +\infty$*,* $\frac{x_1(t)}{x_2(t)}$ *tends to* y_1*, while, as* $t \to -T_1$*, the ratio approaches* y_2*. Finally, if* $\frac{x_1(0)}{x_2(0)} > y_2$*, there exist* $0 < T_2, T_3 < \infty$ *such that there exists a unique solution which is defined on the maximal time interval* $(-T_3, T_2)$*. As* $t \to T_2$*,* $\frac{x_1(t)}{x_2(t)}$ *diverges to* $+\infty$*, while it tends to* y_2*, as* $t \to -T_3$*.*

3.3 When the isotropy group is maximal

In this section, we are going to study the following dynamical system:

$$\dot{x}_1(t) = -A_1 + B_1 \frac{x_2(t)}{x_1(t)} - C_1 \frac{x_1(t)^2}{x_2(t)^2}, \tag{3.11}$$

$$\dot{x}_2(t) = -A_2 + B_2 \frac{x_1(t)}{x_2(t)} - C_2 \frac{x_2(t)^2}{x_1(t)^2}, \tag{3.12}$$

where the A_i's and the B_i's are defined by (2.28)-(2.29) and are all strictly positive. In the previous chapter, we were interested in the case where the functions $x_1(t)$ and $x_2(t)$ define a Riemannian metric. For this reason, we were imposing the condition that both these functions had to be strictly positive. Here, we will consider the remaining cases, in which the functions are also allowed to be negative. Let

$$\boldsymbol{D}_2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \neq 0\}.$$

Suppose that the initial time is given by $t_0 = 0$. Then by standard ODE theory, we have existence and uniqueness of solutions on $\boldsymbol{D}_2$. Hence, the functions $x_1(t)$ and $x_2(t)$ are not allowed to change sign along any solution to (3.11)-(3.12). We then have to distinguish the following cases:

I) $x_1(t), x_2(t) > 0$;

II) $x_1(t) < 0$, $x_2(t) > 0$;

III) $x_1(t) < 0, x_2(t) < 0$;

IV) $x_1(t) > 0$, $x_2(t) < 0$.

We have already discussed case I) in detail. We will now consider the other cases separately.

3.3.1 Cases II) and IV)

As the functions have opposite signs, the ratio $\frac{x_1(t)}{x_2(t)}$ is always negative. Hence, both $x_1(t)$ and $x_2(t)$ are decreasing, for every t such that a solution exists. Moreover, we have the following derivative estimates:

$$\dot{x}_1(t) < -A_1,$$
$$\dot{x}_2(t) < -A_2.$$

By integrating these inequalities, we have that in case II) the solution has to stop at or before the time in which $x_2(t)$ becomes zero, which is finite. In case IV), the solution has to stop at or before the time in which $x_1(t)$ becomes zero,which is finite. Now, as in the previous section, we have that in case II) the final time is characterised by $x_2(t)$ becoming zero and $x_1(t)$ approaching a negative limit, which could be $-\infty$. Similarly, in case IV) the final time is characterised by $x_1(t)$ becoming zero and $x_2(t)$ approaching a negative limit, which could be $-\infty$. In order to have a better understanding of the behaviour of the function which does not tend to zero, it is useful to compute the evolution equation of $x_1(t)x_2(t)$.

$$\begin{aligned}\frac{d}{dt}(x_1(t)x_2(t)) =& - A_1x_2(t) + B_1\frac{x_2(t)^2}{x_1(t)} - C_1\frac{x_1(t)^2}{x_2(t)} - A_2x_1(t)\\ &+ B_2\frac{x_1(t)^2}{x_2(t)} - C_2\frac{x_2(t)^2}{x_1(t)}\\ =& - A_1x_2(t) - A_2x_1(t) + (B_1 - C_2)\frac{x_2(t)^2}{x_1(t)}\\ &+ (B_2 - C_1)\frac{x_1(t)^2}{x_2(t)}.\end{aligned}$$

Hence, if we are in case II) and $B_2 - C_1 < 0$, necessarily $x_1(t)$ has to tend to $-\infty$ and in such a way that $x_1(t)x_2(t)$ also tends to $-\infty$, as t approaches the final time. On the other hand, if we are in case IV) and $B_1 - C_2 < 0$, then necessarily both $x_2(t)$ and $x_1(t)x_2(t)$ have to tend to $-\infty$, as t approaches the final time.

We will now investigate the solution backwards in time. Let us change time parameter from t to $\tau = -t$. Then, the system (3.11)-(3.12) becomes

$$x_1(\tau)' = A_1 - B_1\frac{x_2(\tau)}{x_1(\tau)} + C_1\frac{x_1(\tau)^2}{x_2(\tau)^2}, \tag{3.13}$$

$$x_2(\tau)' = A_2 - B_2\frac{x_1(\tau)}{x_2(\tau)} + C_2\frac{x_2(\tau)^2}{x_1(\tau)^2}, \tag{3.14}$$

where $'$ indicates the derivative with respect to τ. In this case we have that both functions are increasing in τ. With similar techniques as before, we can show that in case II) the backwards solution stops in finite time because $x_1(\tau)$ becomes zero, while $x_2(\tau)$ approaches a positive limit, which could be $+\infty$. For what concerns case IV), the backwards solution stops in finite time because $x_2(\tau)$ becomes zero, while $x_1(\tau)$ approaches a positive limit, which could be $+\infty$. We have then proved the following proposition.

Proposition 3.3.1. *In case II) (case IV)), there exist* $T_1, T_2 > 0$ *such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval* $(-T_2, T_1)$. *Moreover,* T_1 *is characterised by* $x_2(t)$ *(*$x_1(t)$*) becoming zero, while* $x_1(t)$ *(*$x_2(t)$*) ap-*

proaches a negative limit, which could be $-\infty$*. Finally,* $-T_2$ *is characterised by* $x_1(t)$ *(*$x_2(t)$*) becoming zero, while* $x_2(t)$ *(*$x_1(t)$*) approaches a positive limit, which could be* $+\infty$.

3.3.2 Case III)

We consider the following two hypersurfaces in $\mathbb{R}^2$:

$$L_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 | -A_1 + B_1\frac{x_2}{x_1} - C_1\frac{x_1^2}{x_2^2} = 0\right\},$$
$$L_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 | -A_2 + B_2\frac{x_1}{x_2} - C_2\frac{x_2^2}{x_1^2} = 0\right\}.$$

By the work we did in the previous chapter, we know that L_1 and L_2 correspond respectively to two lines in $\mathbb{R}^2$ given by $x_1 = \tilde{y}_1 x_2$ and $x_1 = \tilde{y}_2 x_2$, with $\tilde{y}_1, \tilde{y}_2 > 0$. Moreover, we also showed that $\tilde{y}_1 < \tilde{y}_2$. We note that L_1 and L_2 separate in the third quadrant the regions in which $x_1(t)$ and $x_2(t)$ are increasing or decreasing in t. In particular, we have three regions:

$$\Sigma_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 < 0,\, x_1 > \tilde{y}_1 x_2 > \tilde{y}_2 x_2\},$$
$$\Sigma_2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 < 0,\, \tilde{y}_2 x_2 < x_1 < \tilde{y}_1 x_2\},$$
$$\Sigma_3 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 < 0,\, x_1 < \tilde{y}_2 x_2 < \tilde{y}_1 x_2\}.$$

In Σ_1, $x_1(t)$ is increasing, while $x_2(t)$ is decreasing. In Σ_3, $x_1(t)$ is decreasing, while $x_2(t)$ is increasing. Finally, in Σ_2, both functions are decreasing. By computing the vector field $(\dot{x}_1(t), \dot{x}_2(t))$ on the boundary of these regions, we can easily see that the only invariant ones are Σ_1 and Σ_3. Let $y(t) = \frac{x_1(t)}{x_2(t)}$. Its evolution equation is given by

$$\dot{y}(t) = \frac{1}{y(t)x_2(t)}\big(- \big(B_2 + C_1\big)\, y(t)^3 + A_2\, y(t)^2 - A_1 y(t) + B_1 + C_2\big).$$

We can then consider the following equation in y:

$$-\left(B_2+C_1\right)y^3+A_2\,y^2-A_1y+B_1+C_2=0. \tag{3.15}$$

In the previous chapter, we showed that the roots of this equation are all positive. Moreover, they all correspond to lines through the origin in $\mathbb{R}^2$, which are located between the lines $x_1=\tilde{y}_1x_2$ and $x_1=\tilde{y}_2x_2$, i.e. in the region Σ_2. We also have that these lines correspond to fixed points of (3.11)-(3.12). Hence, by the uniqueness of the solution, they can never be crossed by any other solution. This implies that the quantity $y(t)$ is monotonic along any solution to (3.11)-(3.12).

We will begin by studying the behaviour of the solution in the invariant regions Σ_1 and Σ_3. Suppose that $(x_1(0),x_2(0))\in\Sigma_3$. Then, $(x_1(t),x_2(t))\in\Sigma_3$, for every t such that a solution exists. Recall that in Σ_3, $x_1(t)$ is decreasing, while $x_2(t)$ is increasing. So the solution is moving towards the x_1-axis. From (3.12), we have that

$$\dot{x}_2(t)>-A_2+B_2\frac{x_1(0)}{x_2(0)}-C_2\frac{x_2(0)^2}{x_1(0)^2}:=C,$$

which is a positive constant as $(x_1(0),x_2(0))\in\Sigma_3$. If we integrate the above equation we get

$$x_2(t)>Ct+x_2(0),$$

which means that the solution has to stop at or before the finite time in which $x_2(t)$ becomes zero. As before, we have that $x_1(t)$ cannot diverge to $-\infty$ before $x_2(t)$ becomes zero. So the final time is characterised by $x_2(t)$ becoming zero, while $x_1(t)$ approaches a negative limit, which could be $-\infty$. If we consider the evolution equation of $x_1(t)x_2(t)$, then we find that if $B_2-C_1>0$, necessarily $x_1(t)$ has to tend to $-\infty$ in such a way that $x_1(t)x_2(t)$ tends to $+\infty$.

Similarly, if $(x_1(0),x_2(0))\in\Sigma_1$, then $(x_1(t),x_2(t))\in\Sigma_1$, for every t such that a solution exists. In this case, we have that

$$\dot{x}_1(t) > -A_1 + B_1 \frac{x_2(0)}{x_1(0)} - C_1 \frac{x_1(0)^2}{x_2(0)^2} := C',$$

which is a positive constant. If we integrate the above expression, we obtain that the solution has to stop at or before the finite time in which $x_1(t)$ becomes zero. For what concerns the behaviour of $x_2(t)$, we have that it cannot diverge to $-\infty$ before $x_1(t)$ becomes zero. So, as t approaches the final time, $x_2(t)$ tends to a negative limit, which could be $-\infty$. Finally, from the evolution equation of $x_1(t)x_2(t)$, we have that if $B_1 - C_2 > 0$, then necessarily the limit of $x_2(t)$ is $-\infty$ and the limit of $x_1(t)x_2(t)$ is $+\infty$.

If we investigate the solution backwards in time, we find that every solution which starts in one of the two invariant regions comes from one which starts in Σ_2.

We will now study the case in which the initial condition lies in Σ_2. We have to distinguish three cases corresponding to how many roots equation (3.15) has:

A) Equation (3.15) has only one root;

B) Equation (3.15) has exactly two roots;

C) Equation (3.15) has three roots.

Case A)

Let $\bar{y}$ denote the unique solution to (3.15). Then, either

$$\dot{y}(t) = -\frac{1}{x_1(t)}(B_2 + C_1)(y(t) - \bar{t})^3$$

or

$$\dot{y} = -\frac{1}{x_1(t)}(B_2 + C_1)(y(t) - \bar{y})P(y(t)),$$

where $P(y(t))$ is a second order polynomial in $y(t)$, which is always positive. So, in both cases, if $y(t) > \bar{y}$, then $\dot{y}(t) > 0$, while if $y(t) < \bar{y}$, then $\dot{y}(t) < 0$. We then have to distinguish the following two cases:

1) $y(0) < \bar{y}$;

2) $y(0) > \bar{y}$.

Case 1): By the uniqueness of the solution, we have that $y(t) < \bar{y}$, for every t such that a solution exists. Moreover, $y(t)$ is monotonic decreasing in t. Recall that as long as we are in Σ_2, both functions are decreasing. The following lemma is true.

Lemma 3.3.2. *The solution has to leave Σ_2 after a finite time.*

Proof of the lemma. Suppose for a contradiction that the solution exists for every $t > 0$ in Σ_2. Then, $y(t)$ has to approach a limit as $t \to +\infty$. Moreover, this limit has to be different from 0 and $+\infty$, because it has to be in Σ_2. We then have that the limit of $y(t)$ has to be a root of equation (3.15) different from $\bar{y}$, which is a contradiction. Hence, the solution cannot exists for all $t > 0$ in Σ_2, which means that it has to stop in finite time. We will now show that this cannot happen in Σ_2. Recall that in Σ_2, both functions are negative and decreasing, which means that they are bounded away from zero. Moreover, we have that

$$\dot{x}_1(t) > -A_1 - C_1\bar{y}^2, \tag{3.16}$$

which means that $x_1(t)$ is bounded from below by a negative constant, if the existence time is finite. Hence, in Σ_2, the solution can stop in finite time only if $x_2(t)$ diverges to $-\infty$, which would then imply that $y(t)$ tends to 0. However this is not possible in Σ_2. So the solution has to leave Σ_2 after a finite time. □

By this lemma, we have that the trajectory has to enter the invariant region Σ_1 in finite time, where we know what happens.

We can now investigate the solution backwards in time. First of all note that, as long as we are in Σ_2, $x_1(\tau)$ and $x_2(\tau)$ are increasing in $\tau = -t$. So the trajectory never leaves Σ_2 in this case. Moreover, from (3.13)-(3.14) and the fact that $y(0) < y(\tau) < \bar{y}$, for every τ such that a solution exists, we have that

$$x_1(\tau)' > A_1 - \frac{B_1}{y(0)} + C_1 y(0)^2,$$
$$x_2(\tau)' > A_2 - B_2\bar{y} + C_2\frac{1}{\bar{y}^2}.$$

So the derivatives of $x_1(\tau)$ and $x_2(\tau)$ are bounded from above by a positive constant, which means that the solution stops in finite time because one of the two functions becomes zero. However, the only possibly for the functions to become zero is that the trajectory goes to the origin, otherwise we would leave the region Σ_2, which is invariant for the backwards flow. Finally, we can compute that $\frac{x_1(\tau)}{x_2(\tau)}$ tends to $\bar{y}$, as τ approaches the final time.

Case 2): By the uniqueness of the solution, we have that $y(t) > \bar{y}$, for every t such that a solution exists. In particular, this implies that $y(t)$ is monotonic increasing in t. We then have that the following lemma is true.

Lemma 3.3.3. *The solution has to leave Σ_2 after a finite time.*

Proof of the lemma. The proof of this lemma is very similar to the proof of lemma 3.3.2. We can show in the same way that the existence time in Σ_2 has to be finite. Then, instead of (3.16), we have the following derivative estimate:

$$\dot{x}_2(t) > -A_2 - \frac{C_2}{\bar{y}^2}.$$

So in Σ_2 $x_2(t)$ is bounded from below by a negative constant, because the existence time is finite. This implies that the solution can stop in finite time only if $x_1(t)$ becomes $-\infty$, because both functions are bounded away from zero. This would then imply that $y(t)$ tends to $+\infty$, which is not possible in Σ_2. So, the solution has to leave Σ_2 after a finite time. □

We can then conclude that the solution has to enter the invariant region Σ_3 in finite time.

If we now investigate the solution backwards in time, in a similar way as for case 1) above, we find that the solution has to stop in finite time, because the trajectory reaches the origin. Moreover, the ratio $\frac{x_1(\tau)}{x_2(\tau)}$ tends to $\bar{y}$, as τ approaches the final time.

We have then proved the following proposition.

Proposition 3.3.4. *Suppose we are in case III) and equation* (3.15) *has only one root* $\bar{y}$. *Then, according to the initial condition, we have that:*

- *If* $y(0) < \bar{y}$, *there exist* $T_1, T_2 > 0$ *and finite such that there exists a unique solution to the system* (3.11)-(3.12) *which is defined on the maximal time interval* $(-T_1, T_2)$. *As* $t \to -T_1$, $y(t) \to \bar{y}$ *and, as* $t \to T_2$, $y(t) \to 0$.

- *If* $y(0) > \bar{y}$, *there exist* $T_1', T_2' > 0$ *and finite such that there exists a unique solution to the system* (3.11)-(3.12) *which is defined on the maximal time interval* $(-T_1', T_2')$. *As* $t \to -T_1'$, $y(t) \to \bar{y}$ *and, as* $t \to T_2'$, $y(t) \to +\infty$.

Case B)

Let y_1 and y_2 denote the two roots of equation (3.15). We then have that

$$\dot{y}(t) = -\frac{B_2 + C_1}{x_1(t)}(y(t) - y_1)^2(y(t) - y_2). \tag{3.17}$$

We now have to consider two possible situations, namely $y_1 < y_2$ or $y_1 > y_2$. We will only consider the case in which $y_1 < y_2$, as the other one is similar. There are three possible initial conditions:

1) $y(0) < y_1$;

2) $y_1 < y(0) < y_2$;

3) $y(0) > y_2$.

Case 1): We have that $y(t)$ is monotonic decreasing in t. Moreover,

$$y(t) < y_1,$$

for every t such that a solution exists. We now have an analogue of lemma 3.3.2, which tells us that the solution has to leave Σ_2 after a finite time and enter the invariant region Σ_1. Now as in case 1) of A), we can show that if we try to solve the equations backwards in time, it has to stop after a finite time because both functions approach zero and $y(t)$ tends to y_1.

Case 2): By the uniqueness of the solution, we have that

$$y_1 < y(t) < y_2,$$

for every t such that a solution exists. This implies that the solution remains in the region Σ_2 for all times. Moreover, $y(t)$, $x_1(t)$ and $x_2(t)$ are all monotonic decreasing in t. We also have that the derivatives of $x_1(t)$ and $x_2(t)$ are bounded from below by negative constants:

$$\begin{aligned}\dot{x}_1(t) &> -A_1 + B_1\frac{x_2(0)}{x_1(0)} - C_1\frac{x_1(0)^2}{x_2(0)^2},\\ \dot{x}_2(t) &> -A_2 + B_2 y_1 - \frac{C_2}{y_1^2}.\end{aligned}$$

Hence, the solution exists for every $t > 0$. Moreover, we can compute that $y(t)$ tends to y_1, as $t \to +\infty$. If we now investigate the solution backwards in time, we find that it has to stop after a finite time, because both functions tend to zero. This is due to the fact that $x_1(t)$ and $x_2(t)$ are negative and increasing in t with derivatives bounded from below by positive constants. Moreover, $y(t)$ tends to y_2, as we go backwards in time.

Case 3):We can treat this case in the same way as case 2) of A) above. We find that if the initial condition lies in Σ_2, then it has to leave this region in finite time and enter the invariant region Σ_3. Moreover, if we solve the equations backwards in time, we find that it has to stop in finite time because both functions become zero. Moreover, $y(t)$ tends to y_2, as we go backwards in time.

We have then proved the following proposition.

Proposition 3.3.5. *Suppose that we are in case III) and that equation* (3.15) *has exactly two roots y_1 and y_2 such that* (3.17) *holds. Then if $y_1 < y_2$ ($y_1 > y_2$), depending to the initial condition we have that:*

- *If $y(0) < y_1$ ($y(0) < y_2$), there exist $T_1, T_2 > 0$ and finite such that there exists a unique solution to the system* (3.11)-(3.12) *which is defined on the maximal time interval $(-T_2, T_1)$. As $t \to T_1$, $y(t) \to 0$ and, as $t \to -T_2$, $y(t)$ tends to y_1 ($y(t)$ tends to y_2).*

- *If $y_1 < y(0) < y_2$ ($y_2 < y(0) < y_1$), then there exists $T > 0$ and finite such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval $(-T, +\infty)$. As $t \to +\infty$, $y(t)$ approaches y_1 and, as $t \to -T$, $y(t)$ tends to y_2.*

- *If $y(0) > y_2$ ($y(0) > y_1$), there exist $T_1', T_2' > 0$ and finite such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval $(-T_2', T_1')$. As $t \to T_1'$, $y(t)$ tends to $+\infty$ and, as $t \to -T_2'$, $y(t)$ approaches y_2 ($y(t)$ approaches y_2).*

Case C)

Let y_1, y_2 and y_3 denote the three roots of equation (3.15). Suppose without loss of generality that $y_1 < y_2 < y_3$. We then have that the evolution equation of $y(t)$ can be written as

$$\dot{y}(t) = -\frac{B_2 + C_1}{x_1(t)}(y(t) - y_1)(y(t) - y_2)(y(t) - y_3).$$

We then have four possible initial conditions:

1) If $y(0) < y_1$;

2) If $y_1 < y(0) < y_2$;

3) If $y_2 < y(0) < y_3$;

4) If $y(0) > y_3$.

Note that in cases 1) and 3), $y(t)$ is monotonic decreasing in t, while in cases 2) and 4), $y(t)$ is monotonic increasing. The analysis that we need for cases 1) and 4) is exactly the same as in cases 1) and 3) of B) above. Whereas, for the other two cases, we proceed exactly in the same way as in case 2) of B) above. For this reason, we will then state the following proposition, without proving it.

Proposition 3.3.6. *Suppose that we are in case IV) and that equation* (3.15) *has exactly three roots. Denote these roots by y_1, y_2 and y_3, with $y_1 < y_2 < y_3$. Then,*

- *If $y(0) < y_1$, there exist $T_1, T_2 > 0$ and finite such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval $(-T_2, T_1)$. As $t \to T_1$, $y(t) \to 0$ and, as $t \to -T_2$, $y(t) \to y_1$.*

- *If $y_1 < y(0) < y_2$, there exists $T > 0$ and finite such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval $(-T, +\infty)$. As $t \to +\infty$, $y(t) \to y_2$ and, as $t \to -T$, $y(t) \to y_1$.*

- *If $y_2 < y(0) < y_3$, there exists $T' > 0$ and finite such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval* $(-T', +\infty)$. *As $t \to +\infty$, $y(t) \to y_2$ and, as $t \to -T'$, $y(t) \to y_3$.*

- *If $y(0) > y_3$, there exist $T'_1, T'_2 > 0$ and finite such that there exists a unique solution to* (3.11)-(3.12) *which is defined on the maximal time interval* $(-T'_2, T'_1)$. *As $t \to T'_1$, $y(t) \to +\infty$ and, as $t \to -T'_2$, $y(t) \to y_3$.*

3.4 The scalar curvature

In this section, we will study the behaviour of the scalar curvature under the evolution equations given by (3.1)-(3.2) and (3.11)-(3.12).

3.4.1 When the isotropy group is not maximal

We will firstly consider the case in which $(x_1(t), x_2(t))$ evolves under the system (3.1)-(3.2). The scalar curvature is given by

$$R(t) = \frac{1}{x_1(t)}\left(C\frac{d_1}{2} + D\frac{d_2}{2}\frac{x_1(t)}{x_2(t)} - A\frac{d_1}{2}\frac{x_1(t)^2}{x_2(t)^2}\right),$$

where d_1 and d_2 are the dimension of the two irreducible inequivalent invariant summands $\mathfrak{p}_1$ and $\mathfrak{p}_2$, as in section 2.8.1 We also have the following relation between A and B:

$$\frac{d_1}{2}A = \frac{d_2}{4}B. \tag{3.18}$$

In the Riemannian case, by the Maximum principle we know that positivity of the scalar curvature is preserved under the Ricci flow. Moreover, we also know that if the scalar curvature is positive initially, then the Ricci flow has to develop a singularity in finite time. We will now study the behaviour of the scalar curvature according to the different cases i)-iv).

First of all, we have that R vanishes when

$$C\frac{d_1}{2} + D\frac{d_2}{2}y - A\frac{d_1}{2}y^2 = 0, \tag{3.19}$$

where $y = \frac{x_1}{x_2}$ and x_1, x_2 are coordinates in $\mathbb{R}^2$. The discriminant of (3.19) is given

$$\frac{d_2^2}{4}D^2 + d_1^2 AC,$$

which is always strictly positive. So, (3.19) has always two different solutions, which are given by

$$\bar{y}_1 = \frac{1}{d_1 C}\Bigg(- \frac{d_2}{2}D + \Big(\frac{d_2^2}{4}D^2 + d_1^2 AC\Big)^{\frac{1}{2}}\Bigg),$$
$$\bar{y}_2 = \frac{1}{d_1 C}\Bigg(- \frac{d_2}{2}D - \Big(\frac{d_2^2}{4}D^2 + d_1^2 AC\Big)^{\frac{1}{2}}\Bigg),$$

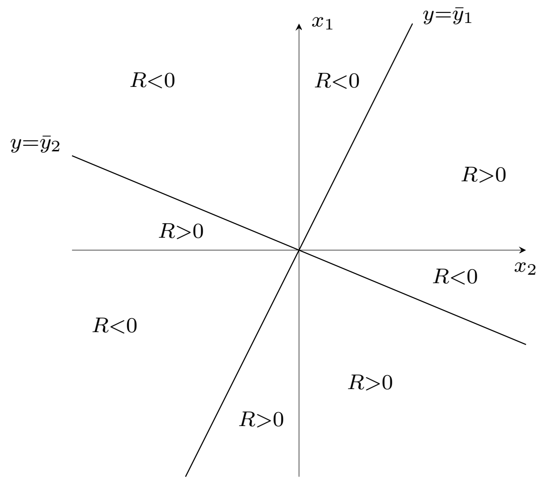
in the case where $C > 0$, and by

$$\tilde{y}_1 = 0,$$
$$\tilde{y}_2 = \frac{d_2}{d_1}\frac{D}{A},$$

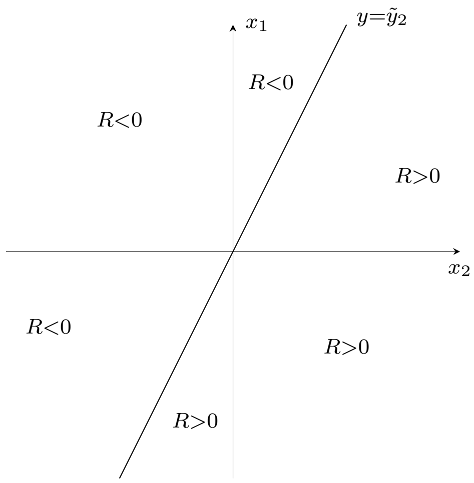
in the case where $C = 0$. Clearly, $\bar{y}_2$ is negative and $\tilde{y}_2$ is positive. We also have that $\bar{y}_1$ is positive. In fact,

$$\bar{y}_1 > \frac{1}{d_1 C}\Bigg(- \frac{d_2}{2}D + \frac{d_2}{2}D\Bigg) = 0.$$

The solution to (3.19) define lines through the origin in $\mathbb{R}^2$, which separates the regions in which R is positive and the regions in which R is negative. The following two pictures which illustrates this in the two cases in which $C > 0$ and $C = 0$:



$C > 0$



$C = 0$

We now have to distinguish the different cases i)-iv).

Case i)

We will study the behaviour of the scalar curvature in the first quadrant, where both $x_1(t)$ and $x_2(t)$ are positive. We will distinguish the cases in which $C > 0$ and $C = 0$.

Let us begin with the case in which $C > 0$. Here we have that $R > 0$ if and only if $y < \bar{y}_1$. From (3.2), we have that the region in which $x_2(t)$ is decreasing in t is given by

$$\left\{ (x_1, x_2) \in \mathbb{R}^2 | x_1 < \frac{D}{B} x_2 \right\}.$$

We will now show that in this region R is positive. If $y = \frac{D}{B}$,

$$R = \frac{1}{x_1} \left(\frac{d_1}{2} C + \frac{d_2}{2} \frac{D^2}{B} - \frac{d_1}{2} \frac{D^2}{B^2} A \right).$$

By (3.18),

$$R = \frac{1}{x_1} \left(\frac{d_1}{2} C + \frac{d_2}{2} \frac{D^2}{B} - \frac{d_2}{4} \frac{D^2}{B} \right) = \frac{1}{x_1} \left(\frac{d_1}{2} C + \frac{d_2}{4} \frac{D^2}{B} \right) > 0.$$

Hence, on the boundary of the region where $x_2(t)$ is decreasing in t, $R > 0$. This means that

$$\frac{D}{B} < \bar{y}_1,$$

which implies that R is positive for every $y < \frac{D}{B}$. We can easily see the the region $\{y < \frac{D}{B}\}$ is invariant for the system (3.1)-(3.2). In fact, the vector field $(\dot{x}_1(t), \dot{x}_2(t))$ on the boundary $\{y = \frac{D}{B}\}$ is given by $(\dot{x}_1(t), 0)$, which points towards the interior, as $\dot{x}_1(t)$ is negative. Note that this region is characterised by $x_2(t)$ being decreasing in t. So, this invariant region is located where R is positive. Moreover, if $y > \frac{D}{B}$, $x_2(t)$ is increasing and the solution is moving towards the boundary of the invariant region. In theorem 2.8.4, we proved that the solution either cross the x_2-axis or goes to the origin in finite time. This means that the trajectory has to enter the invariant region $\{y < \frac{D}{B}\}$ in finite time. We then know that if R is negative initially, it will turn positive in finite time, because the solution has to enter the invariant region. We will also show that R is strictly increasing when it is negative.

$$\begin{aligned}
\frac{d}{dt}(R(t)) =& -\frac{\dot{x}_1(t)}{x_1(t)^2}\Big(C\frac{d_1}{2} + D\frac{d_2}{2}y(t) - A\frac{d_1}{2}y(t)^2\Big) \\
&+ \frac{1}{x_1(t)}\Big(D\frac{d_2}{2}\dot{y}(t) - d_1Ay(t)\dot{y}(t)\Big) \\
>& -\frac{\dot{x}_1(t)}{x_1(t)^2}\Big(D\frac{d_2}{2} - A\frac{d_1}{2}y(t)\Big)y(t) + \frac{1}{x_1(t)}\Big(D\frac{d_2}{2} - d_1Ay(t)\Big)\dot{y}(t) \\
>& -\frac{\dot{x}_1(t)}{x_1(t)^2}\Big(D\frac{d_2}{2} - d_1Ay(t)\Big)y(t) + \frac{1}{x_1(t)}\Big(D\frac{d_2}{2} - d_1Ay(t)\Big)\dot{y}(t) \\
=&\Big(D\frac{d_2}{2} - d_1Ay(t)\Big)\Big(-\frac{\dot{x}_1(t)}{x_1(t)}y(t) + \dot{y}(t)\Big)\frac{1}{x_1(t)}.
\end{aligned}$$

We know that

$$\frac{d_2}{2}D - d_1Ay(t) < 0$$

when $R < 0$, as this inequality is equivalent to $y > \frac{D}{B}$ via (3.18). Then, in order to show that $R(t)$ is increasing in t, we only need to show that

$$-\frac{\dot{x}_1(t)}{x_1(t)}y(t) + \dot{y}(t) < 0.$$

We have that

$$\begin{aligned}
\dot{y}(t) - \frac{\dot{x}_1(t)}{x_1(t)}y(t) =& \frac{1}{x_2(t)}(-C + Dy(t) - (A+B)y(t)^2) + \frac{y(t)}{x_1(t)}(C + Ay(t)^2) \\
=& - C\frac{1}{x_2(t)} + D\frac{y(t)}{x_2(t)} - (A+B)\frac{y(t)^2}{x_2(t)} + C\frac{y(t)}{x_1(t)} + A\frac{y(t)^3}{x_1(t)} \\
=& \frac{1}{x_1(t)}(Dy(t)^2 - By(t)^3) \\
=& B\frac{1}{x_1(t)}\left(\frac{D}{B}y(t)^2 - y(t)^3\right),
\end{aligned} \tag{3.20}$$

which is negative, as $y(t) > \frac{D}{B}$.

We will now consider the case in which $C = 0$. Here, $R > 0$, when $y < \tilde{y}_2$. By (3.18),

$$\tilde{y}_2 = \frac{d_2}{d_1}\frac{D}{A} = 2\frac{d_2}{d_1}\frac{D}{d_2 B}d_1 = 2\frac{D}{B}$$

So, the invariant region in which $x_2(t)$ is decreasing is located where $R(t)$ is positive. This means that if the scalar curvature is negative initially, then it has to turn positive in finite time. Moreover, exactly as before, we can show that $R(t)$ is increasing in t when it is negative.

Case ii)

We will firstly consider the case in which $C > 0$. We are considering the case in which $x_1(t)$ is negative and $x_2(t)$ is positive. From its expression, we can see that $R(t)$ is positive if and only if $y(t) < \bar{y}_2$. We will show that $R(t)$ is increasing in t when it is negative. In fact, $R(t)$ is negative if and only if

$$C\frac{d_1}{2} + D\frac{d_2}{2}y(t) - A\frac{d_1}{2}y(t)^2 > 0,$$

because $x_1(t) < 0$. Moreover, we have that $y(t)$ is strictly decreasing in t and $y(t) < 0$. The evolution equation for $R(t)$ is given by

$$\begin{aligned}\frac{d}{dt}(R(t)) = &-\frac{\dot{x}_1(t)}{x_1(t)^2}\left(C\frac{d_1}{2} + D\frac{d_2}{2}y(t) - A\frac{d_1}{2}y(t)^2\right)\\ &+\frac{1}{x_1(t)}\left(D\frac{d_2}{2}\dot{y}(t) - d_1Ay(t)\dot{y}(t)\right).\end{aligned} \tag{3.21}$$

By the above discussion, we can easily see that this expression is positive when $R(t) < 0$. By the work that we did in section 3.2, we know that the solution stops in finite time because $x_2(t)$ becomes zero, while $x_1(t)$ approaches a negative limit. This means $R(t)$ has to turn positive in finite time, if it starts negative. To conclude the study of this case, we will show that positivity of $R(t)$ is preserved, as it is increasing in t even when it is positive. From (3.21), as we computed in the previous case,

$$\frac{d}{dt}(R(t)) > \left(D\frac{d_2}{2} - d_1Ay(t)\right)\left(\frac{\dot{y}(t)}{x_1(t)} - \frac{\dot{x}_1(t)}{x_1(t)^2}y(t)\right). \tag{3.22}$$

As $y(t) < 0$,

$$D\frac{d_2}{2} - d_1Ay(t) > 0.$$

Moreover, we showed that

$$\frac{1}{x_1(t)}\left(\dot{y}(t) - \frac{\dot{x}_1(t)}{x_1(t)}y(t)\right) = \frac{B}{x_1(t)^2}\left(\frac{D}{B} - y(t)\right)y(t)^2, \tag{3.23}$$

which is positive when $y(t) < 0$. Hence,

$$\left(D\frac{d_2}{2} - d_1Ay(t)\right)\left(\frac{\dot{y}(t)}{x_1(t)} - \frac{\dot{x}_1(t)}{x_1(t)^2}y(t)\right) > 0$$

and $R(t)$ is increasing in t and its positivity is preserved.

We will now consider the case in which $C = 0$. The scalar curvature is then given by

$$R(t) = \frac{y(t)}{x_1(t)}\left(\frac{d_2}{2} - A\frac{d_1}{2}y(t)\right),$$

which is positive in the fourth quadrant. So if $C = 0$, $R(t)$ is always positive in this case.

Case iii)

Recall that both $x_1(t)$ and $x_2(t)$ are negative in this case. We will begin with the case in which $C > 0$. From its expression we have that $R(t) > 0$ if and only if $y(t) > \bar{y}_1$. When we were considering case i), we proved that $\frac{D}{B} < \bar{y}_1$. In this case this means that the region in which $R(t)$ is positive is located where $x_2(t)$ is increasing in t. By the work we did in section 3.2, we know that the region in which $\dot{x}_2(t) > 0$ is invariant under the system (3.1)-(3.2). We will study firstly the behaviour of $R(t)$ in the invariant region, where $x_2(t)$ is increasing in t, and then we will consider the case in which $\dot{x}_2(t) < 0$. We will show that in this region the scalar curvature is always increasing in t. Using (3.18), we have that

$$D\frac{d_2}{2} - d_1Ay(t) > 0 \Leftrightarrow y(t) < \frac{d_2}{2d_1}\frac{D}{A} = \frac{D}{B}.$$

Hence, in the invariant region, $D\frac{d_2}{2} - d_1Ay(t) < 0$. So, by (3.22), in order to prove that $\dot{R}(t) > 0$, we only need to show that

$$\frac{\dot{y}(t)}{x_1(t)} - \frac{\dot{x}_1(t)}{x_1(t)^2}y(t) < 0.$$

By (3.23), we have that the above inequality is satisfied if and only if $y(t) > \frac{D}{B}$, which is true in the invariant region. So, $R(t)$ is increasing in t. This also implies that positivity of $R(t)$ is preserved under (3.1)-(3.2). In section 3.2, we proved that if the solution enters the invariant region, then it has to stop in finite time, which is characterised by $x_2(t)$

becoming zero. This means that, in this region, $R(t)$ has to turn positive in finite time, if it was negative initially.

We will now analyse the behaviour of $R(t)$ in the region where $x_2(t)$ is decreasing in t. Here we know that the scalar curvature is negative. We have to distinguish the different cases according to how many roots (3.3) has. If it has no roots, then by proposition 3.2.6 we know that the solution has to enter the invariant region in finite time. This means that $R(t)$ has to turn positive in finite time. Suppose now that (3.3) has one root, say $\bar{y}$. Then by the work we did in section 3.2, we know that if $y(0) < \bar{y}$ then the solution to (3.1)-(3.2) is defined for every $t > 0$ and it approaches $\bar{y}$, as $t \to +\infty$. We know that $\bar{y} < \bar{y}_1$, because the line $x_1 = \bar{y}x_2$ is located in the region where both functions are decreasing. This means that if $y(0) < \bar{y}$, then $R(0) < 0$ and its negativity is preserved.

The case $C = 0$ is similar. Also in this case we have that $\tilde{y}_2 > \tilde{y}$, where $\tilde{y}$ is the positive root of (3.3). If $y(0) < \tilde{y}$, then $R(0) < 0$ and its negativity is preserved for every $t > 0$. If $y(0) > \tilde{y}$, then $R(t)$ has to turn positive in finite time, if it is was negative initially. Once it turns positive, its positivity is preserved.

Case iv)

We will now consider the second quadrant, in which $x_1(t)$ is positive and $x_2(t)$ is negative. We will begin by considering the case in which $C > 0$. As $y(t) < 0$, we have that $R(t) > 0$ if and only if $y(t) > \bar{y}_2$. Recall that in this case $y(t)$ is always increasing in t. Then, from (3.22) together with (3.20), we can see that $R(t)$ is always increasing in t. This in particular means that positivity of the scalar curvature is preserved. By proposition 3.2.5, we know that every solution to (3.1)-(3.2) which starts in the second quadrant has to cross the x_2-axis in finite time. This means that $R(t)$ has to turn positive in finite time, if it starts negative. To conclude, we will consider the case in which $C = 0$. Here, we know that $R(t) < 0$ if and only if $y(t) < \tilde{y}_2$, which is always true in the second quadrant, as $y(t) < 0$ and $\tilde{y}_2 > 0$. We then have that $R(t)$ is always negative in the second quadrant. Moreover, by proposition 3.2.4, we have that the solution to (3.1)-(3.2) exists for every $t > 0$ and stays in the second quadrant. Hence, the negativity of $R(t)$ is preserved.

3.4.2 When the isotropy group is maximal

We are now considering the case in which $(x_1(t), x_2(t))$ is evolving under the system (3.11)-(3.12). When the isotropy group is maximal, the scalar curvature is given by

$$R(t) = \frac{A_1 d_1}{2}\frac{1}{x_1(t)} + \frac{d_2 A_2}{2}\frac{1}{x_2(t)} - \frac{d_1}{4}B_1\frac{x_2(t)}{x_1(t)^2} - \frac{d_2}{4}B_2\frac{x_1(t)}{x_2(t)^2},$$

where d_1 and d_2 are the dimensions of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively. Let us begin by finding the regions in which the scalar curvature has a definite sign. We have that $R(t) = 0$ if and only if

$$\frac{d_1}{2}A_1 + \frac{d_2}{2}A_2\frac{x_1(t)}{x_2(t)} - \frac{d_1}{4}B_1\frac{x_2(t)}{x_1(t)} - \frac{d_2}{4}B_2\frac{x_1(t)^2}{x_2(t)^2} = 0.$$

Let $y(t) = \frac{x_1(t)}{x_2(t)}$. Then $R(t) = 0$ if and only if $y(t)$ solves

$$-\frac{d_2}{4}B_2 y(t)^3 + \frac{d_2}{2}A_2 y(t)^2 + \frac{d_1}{2}A_1 y(t) - \frac{d_1}{4}B_1 = 0. \tag{3.24}$$

Now, as the cubic

$$\alpha(y) := -\frac{d_2}{4}B_2 y^3 + \frac{d_2}{2}A_2 y^2 + \frac{d_1}{2}A_1 y - \frac{d_1}{4}B_1$$

intersects the line $y = 0$ in a negative value and it tends to $+\infty$, as $y \to -\infty$, and to $-\infty$, as $y \to +\infty$, we have that it has to intersect the y-axis in at least one negative point. By [49, Theorem 2.2], we know that there has to exist a region in the first quadrant in which the scalar curvature is positive, because the homogeneous space G/K admits at least one invariant Einstein metric. Moreover, by [49, Theorem 2.1], we know that the scalar curvature cannot be bounded from below. Hence, there has to exists a region in the first quadrant in which the scalar curvature is negative. This implies that (3.24) has at least one positive root. So the cubic $\alpha(y)$ meets the y-axis in at least two points, one positive and one negative. We also have that $\alpha(y) = 0$ if and only if

$$-\frac{d_2}{4}B_2y^3 = -\frac{d_2}{2}A_2y^2 - \frac{d_1}{2}A_1y + \frac{d_1}{4}B_1.$$

Let

$$\begin{aligned}\alpha_1(y) &= -\frac{d_2}{4}B_2y^3\\ \alpha_2(y) &= -\frac{d_2}{2}A_2y^2 - \frac{d_1}{2}A_1y + \frac{d_1}{4}B_1.\end{aligned}$$

Observe that $\alpha_1(0) = 0$, while $\alpha_2(0) = \frac{d_1}{4}B_1 > 0$. Moreover, both these two curves tend to $-\infty$ and in such a way that $\alpha_1(y) < \alpha_2(y)$, as $y \to +\infty$. As they intersect in one positive y and $\alpha_1(y) < \alpha_2(y)$, as $y \to +\infty$, they have to intersect another time for $y > 0$. So they intersect into two positive points. Then, $\alpha(y) = 0$ has three roots. Two of these roots have to be positive and the other one has to be negative. We have then proved that $R(t) = 0$ has three roots in terms of $y(t)$, two of them being positive and one being negative. Let $\bar{y}_1$, $\bar{y}_2$ and $\bar{y}_3$ denote these three roots. Suppose without loss of generality that $\bar{y}_1$ and $\bar{y}_2$ are positive and that $\bar{y}_3$ is negative. We can then write $R(t)$ as

$$\begin{aligned}R(t) =&\frac{1}{x_1(t)y(t)}\left(\frac{d_1}{2}A_1y(t) + \frac{d_2}{2}A_2y(t)^2 - \frac{d_1}{4}B_1 - \frac{d_2}{4}B_2y(t)^3\right)\\ =&\frac{x_2(t)}{x_1(t)^2}\left(-\frac{d_2}{4}B_2\right)(y(t)-\bar{y}_1)(y(t)-\bar{y}_2)(y(t)-\bar{y}_3).\end{aligned}$$

$\bar{y}_i$, with $i = 1, 2, 3$ defines lines through the origin in $\mathbb{R}^2$, which separates the regions in which the scalar curvature is either positive or negative. We have that the following lemma is true.

Lemma 3.4.1. *If $x_1(t), x_2(t) > 0$, then $R(t)$ is positive in the region where both functions* *are decreasing in t.*

Proof of the lemma. From the evolution equation (3.11)-(3.12), we have that $x_1(t)$ and $x_2(t)$ are both decreasing in t if and only if

$$-A_1 + \frac{B_1}{y(t)} - C_1 y(t)^2 < 0,$$
$$-A_2 + B_2 y(t) - \frac{C_2}{y(t)^2} < 0.$$

We can rewrite the above inequalities as follows.

$$A_1 y(t) - B_1 + C_1 y(t)^3 > 0,$$
$$A_2 y(t)^2 - B_2 y(t)^3 + C_2 > 0.$$

Then, using (2.28)-(2.29), we have that

$$d_1 B_1 = 2 d_2 C_2,$$
$$d_2 B_2 = 2 d_1 C_1,$$

which implies that $x_1(t)$ and $x_2(t)$ are both decreasing in t if and only if $y(t)$ satisfies the system

$$\frac{d_1}{2} A_1 y(t) - \frac{d_1}{2} B_1 + \frac{d_2}{4} B_2 y(t)^3 > 0,$$
$$\frac{d_2}{2} A_2 y(t)^2 - \frac{d_2}{2} B_2 y(t)^3 + \frac{d_1}{4} B_1 > 0.$$

By putting these two inequalities together, we have that $y(t)$ satisfies the following inequality:

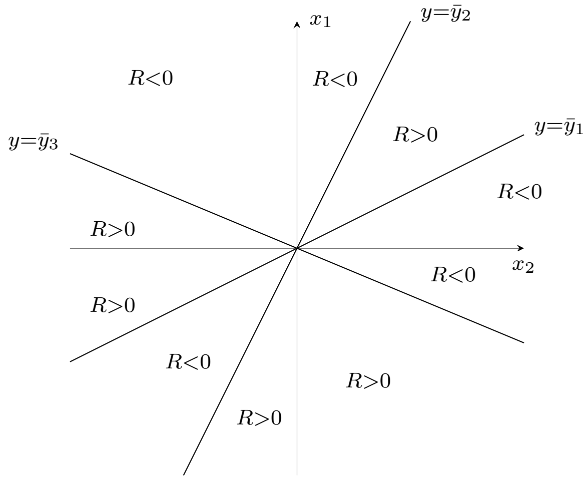
$$\frac{d_1}{2} A_1 y(t) - \frac{d_1}{4} B_1 + \frac{d_2}{2} A_2 y(t)^2 - \frac{d_2}{4} B_2 y(t)^3 > 0,$$

which implies that $R(t)$ has to be positive. □

This lemma implies that the line $x_1 = \bar{y}_1 x_2$ is located in the region where $x_1(t)$ is increasing and $x_2(t)$ is decreasing, while the line $x_1 = \bar{y}_2 x_2$ is located in the region where $x_1(t)$ is decreasing and $x_2(t)$ is increasing. We also have that the following corollary is true.

Corollary 3.4.2. *If $x_1(t), x_2(t) < 0$, then $R(t)$ is negative in the region where both functions are decreasing in t.*

The following picture illustrates the regions in which $R(t)$ is positive and those in which it is negative.



We now have to analyse the behaviour of the scalar curvature in the cases I)-IV).

Case I)

Recall that in this case both $x_1(t)$ and $x_2(t)$ are positive. Then, $R(t) > 0$ if and only if $\bar{y}_1 < y(t) < \bar{y}_2$, which includes in particular the region in which both functions are decreasing in t. As the Ricci flow preserves positivity of the scalar curvature, we know that if $R(0) > 0$ then $R(t) > 0$, for every t such that a solution exists. If the solution has negative scalar curvature initially, then it means that the initial condition does not lie in the invariant region. As the solution has to enter the invariant region in finite time, this means that the scalar curvature has to turn positive in finite time, if it was negative initially.

Cases II)-IV)

Recall that in case II), $x_1(t) < 0$ and $x_2(t) > 0$, while in case IV), $x_1(t) > 0$ and $x_2(t) < 0$. Moreover, in both cases, both functions are monotonic decreasing in t. In case II), $R(t) > 0$ if and only if $y(t) < \bar{y}_3$, while in case IV), $R(t) > 0$ if and only if $y(t) > \bar{y}_3$. So in both cases, $R(t) > 0$ if and only if $x_1(t) < \bar{y}_3 x_2(t)$. If we compute the vector field $(\dot{x}_1(t), \dot{x}_2(t))$ on $y(t) = \bar{y}_3$, we then have that the region

$$\{(x_1, x_2) \in \mathbb{R}^2 | x_1 < \bar{y}_3 x_2 \text{ and } x_1 x_2 < 0\}$$

is invariant under the system (3.11)-(3.12). Hence, the scalar curvature stays positive if it was positive initially and it has to to become positive in finite time, if it was negative initially. Moreover, in both cases, as t approaches the final time, $R(t)$ tends to $+\infty$.

Case III)

In this case, both functions are negative. Here, we have that $R(t) > 0$ if and only if $y(t) < \bar{y}_1$ and $y(t) > \bar{y}_2$. As the lines which correspond to the positive roots of (3.15) are located between $x_1 = \bar{y}_1 x_2$ and $x_1 = \bar{y}_2 x_2$, we have that positivity of the scalar curvature is preserved under the system (3.11)-(3.12). We now want to study the behaviour of $R(t)$ according to the number of roots of equation (3.15). The roots of this equation define lines through the origin, which are located in the region where all the functions are decreasing in t and where the scalar curvature is negative. We will now consider the case in which equation (3.15) has exactly two roots, the the other two cases as similar. Denote these roots by y_1 and y_2 and suppose that $y_1 < y_2$. We then have to distinguish three different initial conditions, namely

1) $y(0) < y_1$;

2) $y_1 < y(0) < y_2$;

3) $y(0) > y_2$.

As we already know what happens to the scalar curvature when we start the flow in one of the two invariant regions, we will also suppose that the initial condition always lies in the region where both $x_1(t)$ and $x_2(t)$ are decreasing. In case 1) and 3) above, we have that the solution has to enter one of the two invariant regions in finite time and stay there until one of the two functions becomes zero. This means that the scalar curvature has to turn positive in finite time in these two cases. Moreover, $R(t)$ tends to $+\infty$, as t approaches the final time. In case 2), we proved that the solution exists for every $t > 0$. Moreover, because of the uniqueness of the solution, we have that $y_1 < y(t) < y_2$, for every $t > 0$. This then implies that the scalar curvature remains negative for every $t > 0$. So, in general, we have that the following proposition is true.

Proposition 3.4.3. *Let y_1 and y_2 denote the smallest and the biggest roots of* (3.15)*, respectively. Then, depending to the initial condition we have that*

- *If $y(0) < y_1$ or $y(0) > y_2$, then the scalar curvature has to turn positive in finite time, if it was negative initially and its positivity is preserved;*

- *If $y_1 < y(0) < y_2$, then the scalar curvature remains negative for every $t > 0$.*

We conclude by mentioning briefly what happens in the case in which (3.15) has a different number of roots. If equation (3.15) has exactly one root $\bar{y}$, then, if $y(0) \neq \bar{y}$, the scalar curvature always has to turn positive in finite, if it was negative initially and its positivity is preserved. Finally, suppose that equation (3.15) has three roots y_1, y_2 and y_3, with $y_1 < y_2 < y_3$. Then if $y_i < y(0) < y_{i+1}$, with $i = 1, 2$, then the scalar curvature remains negative for every $t > 0$. Whereas, if $y(0) < y_1$ or $y(0) > y_3$, then the scalar curvature has to turn positive in finite time, if it was negative initially and its positivity is always preserved.

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