JOURNAL OF COMBINATORIAL THEORY, Series B **70**, 118–133 (1997)
ARTICLE NO. TB971755

Lê Numbers of Arrangements and Matroid Identities

David B. Massey*

Department of Mathematics, Northeastern University, Boston, Massachusetts 02115

Rodica Simion†

Department of Mathematics, George Washington University, Washington, DC 20052

Richard P. Stanley‡

Department of Mathematics, MIT, Cambridge, Massachusetts 02139

Dirk Vertigan$

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

Dominic J. A. Welsh¶

Mathematical Institute, Oxford University, 24–29 St. Giles, Oxford OX1 3LB, England

and

Günter M. Ziegler#

Fachbereich Mathematik 6–1, Technische Universität Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

Received August 16, 1996

DEDICATED TO PROFESSOR W. T. TUTTE ON THE OCCASION OF HIS EIGHTIETH BIRTHDAY

We present several new polynomial identities associated with matroids and geometric lattices and relate them to formulas for the characteristic polynomial and the Tutte polynomial. The identities imply a formula for the Lê numbers of complex hyperplane arrangements. © 1997 Academic Press

* E-mail: dmassey@neu.edu.
† E-mail: simion@math.gwu.edu.
‡ Corresponding author. Research supported in part by ESPRIT RAND. E-mail: rstan@math.mit.edu.
$ E-mail: vertigan@math.lsu.edu.
¶ E-mail: dwelsh@maths.ox.ac.uk.
# E-mail: ziegler@math.tu-berlin.de.

0095-8956/97 $25.00

1. FROM LÊ NUMBERS TO MATROID IDENTITIES

In July 1994 the CIRM hosted a conference on hyperplane arrangements in Luminy, France. At that conference the first author gave a talk in which he described an application to hyperplane arrangements of the more general theory of Lê cycles and Lê numbers of an arbitrary complex analytic hypersurface singularity (see Massey [4, 5]).

The Lê numbers of a hypersurface singularity are important because they provide a nice generalization of the Milnor number of an isolated hypersurface singularity: the Lê numbers are effectively calculable, the Milnor fibre has a handle decomposition in which the number of handles of each index is specified by the appropriate Lê number, and the constancy of the Lê numbers in a family implies both that Thom's a_f holds and that the Milnor fibrations are constant in the family.

In the case of hyperplane arrangements, the so-called Lê-Iomdine formulas lead to a recursive way to compute the Lê numbers. Namely, let $\mathscr{A}$ denote a central essential complex hyperplane arrangement in $\mathbf{C}^n$. We use h to denote hyperplanes in $\mathscr{A}$, and the letters v and w to denote proper flats of arbitrary dimension, that is, intersections of one or more hyperplanes in $\mathscr{A}$. Let $e(v)$ be the number of hyperplanes of $\mathscr{A}$ which contain the flat v. Then the Lê numbers $\lambda^k_{\mathscr{A}}$ are obtained by summing $\eta(y)$ over all flats y of dimension k, where the function η is defined inductively on the flats by the following rule. For all $h \in \mathscr{A}$, $\eta(h) = 0$, and for all proper flats w,

$$\sum_{w \subseteq v \subset \mathbf{C}^n} (e(w)-1)^{\dim v}\, \eta(v) = (e(w)-1)^n.$$

Now let L denote the lattice of all flats of $\mathscr{A}$, ordered by reverse inclusion (as is customary). This includes the flat $\mathbf{C}^n$, which arises as the intersection of the empty set of flats. If we set $\eta(\mathbf{C}^n) := -1$, then we obtain a sum

$$\sum_{w \subseteq v} (e(w)-1)^{\dim v}\, \eta(v) = 0. \tag{1}$$

A first curiosity is that the function η is everywhere positive except on $\mathbf{C}^n$ and on the hyperplanes $h \in \mathscr{A}$ (where it is -1 resp. 0 by definition). From its recursive definition it is surprising that this should be the case, but it is a consequence of the geometrical counting interpretation described above.

At this point, we made two observations. The first one was that the Lê numbers seemed to be closely related to the Möbius function μ on the lattice of flats. The unusual appearance of the resulting formula, (2) in the next section, initiated our investigations. In fact, in view of the recursion formula (1), the identity (2) is just the statement that the η-function on a hyperplane arrangement is given by

$$\eta(v) = (e(v) - 1)\,|\mu(\mathbf{C}^n, v)|.$$

It was this observation which led to this paper.

The purpose of this paper is to try to understand the combinatorics underlying this curious formula, which started off as a result in geometry, and to explore its generalizations and extensions. (Chapter 5 of Massey [5] contains a full account of the algebraic geometry background and arguments.) In particular, Theorem 5.6 of [5], about central hyperplane arrangements in $\mathbf{C}^n$, is a special case of a result about geometric lattices (the lattice of flats of a central hyperplane arrangement is geometric). This in turn is a special case of a result about the Tutte polynomial of a matroid which we describe in Section 5.

2. MATROID IDENTITIES

Let M be a matroid of rank $r \geqslant 1$ without loops on a finite ground set E, and let $L = L(M)$ be its geometric lattice of flats. For any subset $A \subseteq E$, we denote its closure by $\bar{A} \in L$, and its rank by $r(A) = r(\bar{A})$. In particular, we have $r(E) = r$, and $r(F, G) = r(G) - r(F)$ is the rank of the interval $[F, G]$ in L. As will be apparent from the context, inclusions $A \subseteq B$ refer to subsets of E, while inequalities $A \leqslant B$ refer to the order relation in L for flats A, B. The minimum and maximum elements of L are $\hat{0} = \varnothing$ and $\hat{1} = E$, and it will be convenient to use the notation $\mathscr{A}_U = \{a \in L : r(a) = 1, a \leqslant U\}$, for $U \in L$. The cardinality $|U|$ pertains to subsets of E. Thus, for a flat U, $|\mathscr{A}_U| \leqslant |U|$ with equality when U contains no parallel elements. We will use the Möbius function on L, denoted $\mu(F, G)$, which for flats $F \leqslant G$ in L satisfies $(-1)^{r(G) - r(F)}\,\mu(F, G) > 0$. The Möbius invariant of the matroid M is denoted by $\mu(M) := \mu(\varnothing, E)$. (See [7, Chap. 15] or [6, Chap. 3] for more about the basic concepts.)

Our first main result consists of the following, apparently new, identities.

THEOREM 2.1. *For a matroid M without loops, with lattice L of flats, the following two identities are true for every flat $G \neq \varnothing$:*

$$\sum_{F \in L:\, F \leqslant G} (|G| - 1)^{r(G) - r(F)}\,(|F| - 1)\,|\mu(\varnothing, F)| = 0 \tag{2}$$

and

$$\sum_{A:\, A \subseteq G} (|G| - 1)^{r(G) - r(A)}\,(|\bar{A}| - 1)(-1)^{|A| - r(A)} = 0. \tag{3}$$

The two formulas in Theorem 2.1 are equivalent via the well-known "boolean expansion formula"

$$\mu(\varnothing, F) = \sum_{A:\, \bar{A} = F} (-1)^{|A|}, \tag{4}$$

using $|\mu(\varnothing, F)| = (-1)^{r(F)}\, \mu(\varnothing, F)$. (Matroids with loops have $\mu(\bar{\varnothing}, F) = 0$ for all flats F, so for these matroids we get only trivial extensions of the formulas of Theorem 2.1.)

To generalize these identities, one can associate a variable x_e with every element $e \in E$, and identify sets of elements $A \subseteq E$, flats in particular, with the corresponding sums of variables, $X_A := \sum_{e \in A} x_e$.

THEOREM 2.2. *Let $H \leqslant G$ be flats of the matroid M, and let $\chi([H, G]; \lambda)$ be the characteristic polynomial of the interval $[H, G]$ in $L = L(M)$, that is,*

$$\chi([H, G]; \lambda) := \sum_{F:\, H \leqslant F \leqslant G} \mu(H, F)\, \lambda^{r(F, G)},$$

where λ is an indeterminate. Then the following polynomial identities hold (in a polynomial ring $R[\lambda, t][x_e\colon e \in E]$ over an arbitrary ring R, for example $R = \mathbf{Z}$):

$$\sum_{\substack{F:\, F \in L \\ H \leqslant F \leqslant G}} \lambda^{r(F, G)}(X_F + t)\, \mu(H, F) = \frac{\chi([H, G]; \lambda)}{\lambda - 1}\, [\lambda(X_H + t) - (X_G + t)] \tag{5}$$

and

$$\sum_{A:\, H \subseteq A \subseteq G} \lambda^{r(G) - r(A)}(X_{\bar{A}} + t)(-1)^{|A \setminus H|} = \frac{\chi([H, G]; \lambda)}{\lambda - 1}\, [\lambda(X_H + t) - (X_G + t)]. \tag{6}$$

As in the case of Theorem 2.1, the two identities (5) and (6) are equivalent. This time, the left-hand sides are equal via the application of

$$\mu(H, F) = \sum_{\substack{A:\, \bar{A} = F \\ H \subseteq A \subseteq G}} (-1)^{|A \setminus H|}.$$

Setting $x_e = 1$ for all $e \in E$ and $t = -1$, we recover Theorem 2.1 from Theorem 2.2. However, Theorem 2.2 can also be deduced from Theorem 2.1. Namely, if M is a simple matroid (without loops or parallel elements) and if w_e is a positive integer for each $e \in E$, then we can construct from M a new matroid $M(w)$, by replacing every element e of M by w_e parallel elements. Now Theorem 2.1, applied to $M(w)$, yields that Theorem 2.2 is valid whenever the variables x_e have positive integer values. However, polynomial identities that hold for positive integers must be valid in any commutative ring with identity. Although the two theorems are equivalent, independent proofs, of formula (2) of Theorem 2.1 and of formula (5) of Theorem 2.2, are given in the next section.

The identities in Theorem 2.2 lend themselves to a variety of other specializations of the x_e's, λ, H, and G, which produce particular identities. For example, setting all variables x_e to 0, the identity (5) reduces to the definition of the characteristic polynomial. Thus for $\lambda = 1$ and $H < G$ we recover the recursion of the Möbius function:

$$\sum_{F \in [H, G]} \mu(H, F) = 0. \tag{7}$$

By setting $\lambda = 2$, $H = \hat{0} < G$ and $t = X_G$ in (5) we obtain the identity

$$\sum_{\hat{0} \leqslant F \leqslant G} 2^{r(F, G)} (X_F + X_G)\, \mu(\hat{0}, F) = 0.$$

When we put $\lambda = (X_G + t)/t$ in (5) we obtain

$$\sum_{F:\, H \leqslant F \leqslant G} t^{r(H, F)} (X_G + t)^{r(F, G)}\, (X_F + t)\, \mu(H, F)$$
$$= t^{r(H, G)} \frac{\chi([H, G]; (X_G + t)/t)}{X_G}\, X_H (X_G + t).$$

Since the characteristic polynomial (for any nontrivial finite graded poset) has $\lambda = 1$ as a root, $\lambda - 1$ is a factor of $\chi([H, G]; \lambda)$ if $H < G$. Hence, the right-hand side is divisible by X_H (as well as by $X_G + t$, but this is obvious since every term on the left has a factor of $X_G + t$). In particular, when $H = \hat{0} < G$, the factor $X_H = 0$ annihilates the right-hand side and we obtain

$$\sum_{F:\, \hat{0} \leqslant F \leqslant G} t^{r(F)} (X_G + t)^{r(F, G)}\, (X_F + t)\, \mu(\hat{0}, F) = 0. \tag{8}$$

Setting $x_e = 1$ for all e, $t = 1$, $\lambda = |E| + 1$, $H = \hat{0}$, and $G = \hat{1}$, one obtains from formula (5) the identity

$$\sum_{F \in L} (|E| + 1)^{r - r(F)}\, (|F| + 1)\, \mu(\hat{0}, F) = 0.$$

There are many other interesting evaluations. Not even the binomial identities that one gets for the special case $L = B_n$ are entirely trivial. Their q-analogues are obtained by setting $L = \mathrm{PG}(n, q)$.

We end this section with one more specialization of Theorem 2.2. Namely, putting $t=0$, every $x_e=1$, $H=\hat{0}$ and $G=\hat{1}$ in the identity (6) gives an expression for the characteristic polynomial $\chi(M;\lambda)$ which seems new.

COROLLARY 2.3. *For any loop free matroid M on E, the characteristic polynomial χ is given by*

$$\chi(M;\lambda)=|E|^{-1}(1-\lambda)\sum_{A\subseteq E}(-1)^{|A|}\,|\bar{A}|\,\lambda^{r(E)-r(A)}.$$

3. PROOFS

In this section we give (independent) proofs of Theorems 2.1 and 2.2.

The first proof is by induction over the rank. It verifies (8), which specializes to formula (2) of Theorem 2.1 by setting $x_e=1$ and $t=-1$.

To simplify things for our proof of (8), we first note that it suffices to deal with the case $G=E$, since in the general case we can replace M by the restriction $M\mid G$. Also every summand has X_E+t as a factor, so we can divide this out, still retaining a polynomial identity. Furthermore, it is sufficient to verify the formula for $t=1$, since the general case arises from this by homogenization (that is, by substituting x_e/t for x_e, and then multiplying by t^{r+1}). We may also assume that there are no parallel elements in the matroid: only the sums of parallel classes appear in the formula, and each of them can be replaced by a single variable for the corresponding atom of L. Thus we only need to prove

$$\sum_{F\in L}(X_E+1)^{r-r(F)-1}(X_F+1)\,\mu(\varnothing,F)=0 \tag{9}$$

for a matroid M of rank $r\geqslant 1$ without loops or parallel elements.

The case $r=1$ is trivial to verify. Thus assume $r\geqslant 2$. Let $\mathscr{K}$ denote the set of coatoms of L. By L' we denote the geometric lattice of rank $r-1$ obtained by (upper) truncation of L. Correspondingly, let μ' denote the Möbius function on the lattice L'. Note that L and L' have the same set of atoms. By comparing the recursion (7) for the Möbius function for the lattice L and the truncation LM', we get that the Möbius function of the top flat of the truncation is

$$\mu'(\varnothing,E)=\mu(\varnothing,E)+\sum_{K\in\mathscr{K}}\mu_L(\varnothing,K). \tag{10}$$

A further ingredient we need is Weisner's formula [7, p. 259; 6, p. 125]: for every atom e of L one has

$$\mu(\varnothing, E)+\sum_{K:\, e\notin K}\mu(\varnothing, K)=0.$$

Summing this identity over all atoms e, weighted by the corresponding variable x_e, we get

$$X_E\mu(\varnothing, E)+\sum_{K\in\mathcal{K}}(X_E-X_K)\,\mu(\varnothing, K)=0. \tag{11}$$

Now we compute

$$\begin{aligned}
&\sum_{F\in L}(X_F+1)^{r-r(F)-1}(X_F+1)\,\mu(\varnothing, F)\\
&\quad=\sum_{F:\, r(F)\leqslant r-2}(X_E+1)^{r-r(F)-1}(X_F+1)\,\mu(\varnothing, F)\\
&\qquad+\sum_{K\in\mathcal{K}}(1+X_K)\,\mu(\varnothing, K)+\mu(\varnothing, E)\\
&\qquad-(X_E+1)\,\mu'(\varnothing, E)+\sum_{K\in\mathcal{K}}(1+X_K)\,\mu(\varnothing, K)+\mu(\varnothing, E)\\
&\quad=-(X_E+1)(\mu(\varnothing, E)+\sum_{K\in\mathcal{K}}\mu(\varnothing, K))\\
&\qquad+\sum_{K\in\mathcal{K}}(1+X_K)\,\mu(\varnothing, K)+\mu(\varnothing, E)\\
&\qquad-X_E\mu(\varnothing, E)+(X_K-X_E)\sum_{K\in\mathcal{K}}\mu(\varnothing, K))=0,
\end{aligned}$$

where the first equality is just a split according to rank, the second one uses the identity (9) for L' (which is true by induction, with $r(L')=r-1\geqslant 1$), the third one substitutes the Möbius function of L' given by (10), the fourth equality is a simple rearrangement of terms, and the last one is the Weisner sum (11). ∎

We now give a proof of Theorem 2.2, specifically of the identity (5), through an order-theoretic approach. It uses a theorem of Stanley [1, p. 177]: if P is a finite geometric lattice and α is a modular element (that is, $r(\alpha\wedge\beta)+r(\alpha\vee\beta)=r(\alpha)+r(\beta)$ for every $\beta\in P$), then

$$\chi(P;\lambda)=\chi([\hat{0},\alpha];\lambda)\cdot\sum_{z:\, z\wedge\alpha=\hat{0}}\mu(\hat{0}, z)\,\lambda^{r(\hat{1})-r(\alpha)-r(z)}.$$

Since $X_F + t = X_H + t + X_F - X_H$, we may write

$$\sum_{F\colon H \leqslant F \leqslant G} \lambda^{r(F,\,G)} (X_F + t)\, \mu(H, F)$$

$$= (X_H + t)\, \chi([H, G]; \lambda) + \sum_{F\colon H \leqslant F \leqslant G}\ \sum_{a \in \mathscr{A}_F \setminus \mathscr{A}_H} X_a \lambda^{r(F,\,G)} \mu(H, F),$$

where, for atoms $a \in L$, we have $X_a = \sum x_e$ with e ranging over the parallel elements whose class is a. Thus, for $U \in L$ we have $X_U = \sum_{a \in \mathscr{A}_U} X_a$, and the double sum can be rewritten as

$$\sum_{a \in \mathscr{A}_G \setminus \mathscr{A}_H} X_a \left[\sum_{F\colon H \leqslant F \leqslant G} \lambda^{r(F,\,G)} \mu(H, F) - \sum_{F\colon F \wedge (H \vee a) = H} \lambda^{r(F,\,G)} \mu(H, F) \right].$$

Since the interval $[H, G]$ is itself a geometric lattice and atoms are modular elements, the last sum can be simplified using Stanley's theorem. We obtain

$$(X_H + t)\, \chi([H, G]; \lambda) + \sum_{a \in \mathscr{A}_G \setminus \mathscr{A}_H} X_a \left[\chi([H, G]; \lambda) - \frac{\lambda \chi([H, G]; \lambda)}{\lambda - 1} \right]$$

$$= \frac{\chi([H, G]; \lambda)}{\lambda - 1} \, [(\lambda - 1)(X_H + t) - (X_G - X_H)]$$

which completes the proof. ∎

4. A BIJECTIVE PROOF

Note that the formula (2) has only one negative term. It can be rewritten as

$$\sum_{F\colon r(F) 0} (|E| - 1)^{r - r(F)} \, (|F| - 1) \, |\mu(\varnothing, F)| = (|E| - 1)^r \qquad (12)$$

In this form the identity has nonnegative integer terms on both sides and requires a bijective proof. In this section we present a bijective proof of the identity (12) for the lattice of flats of a matroid without loops.

First we establish some helpful terminology and notation. If $e \in E$ is an element in the parallel class of the atom a and $u \in L$, we will write, abusing notation in the interest of simplicity, $u \vee e$ for $u \vee a = u \cup \{e\}$. We will write $u \lessdot v$ if v *covers* u, that is, if there exists some element $e \in E \setminus u$ with $u \vee e = v$. Similarly, we write $u \lessdot\!\!= v$ if either v covers u or $u = v$, that is, if there exists some element $e \in E$ with $u \vee e = v$.

Now replace the Hasse diagram of the geometric lattice L by its *Cayley–Hasse diagram*, that is, if $u \leqslant v$ in L and $e \in E$ is such that $v = u \vee e$, then put a (directed) edge labeled e from u to v, including loops when $u = v$ and $e \leqslant u$. Thus, paths in the Cayley–Hasse diagram correspond to labeled saturated multichains in L. Paths starting at $\hat{0}$ correspond to finite sequences of elements from the ground set of the matroid.

Fix a linear ordering $<_E$ of E such that if $e_1 <_E e_2$ and e_1, e_2 are not parallel, and if $e_1' \parallel e_1$ and $e_2' \parallel e_2$, then $e_1' <_E e_2'$. Let p be the first (smallest) element with respect to $<_E$. For $A \subseteq E$ we write m_A for the smallest element in A with respect to $<_E$. Thus, $p = m_E$. A saturated chain is *decreasing* if the labels along its coverings (starting from the minimum of the chain) form a decreasing sequence of elements with respect to the chosen linear ordering $<_E$. A chain $u = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_m = v$ is a *minimally labeled u-v-chain* if the label of each covering $x_i \lessdot x_{i+1}$ is the least element e such that $e \vee x_i = x_{i+1}$. Since L is geometric, hence semimodular, the number of decreasing minimally labeled u-v-chains is equal to $|\mu(u, v)|$ (see, e.g., [3, Corollary 2.3]). It is an easy observation that the top covering of a decreasing minimally labeled $\hat{0}$-v-chain is labeled by m_v. Given an element e, a (multi)chain is called *e-free* if none of its labels is e.

The left-hand side of (12) can be interpreted as the cardinality of the set $\mathscr{T}$ of triples (D, e, C), where D is a decreasing minimally labeled $\hat{0}$-F-chain, $e \in F \backslash \{m_F\}$, and C is a p-free saturated multichain of length $r - r(F)$ starting at F.

The right-hand side of (12) has a completely transparent interpretation: it is the cardinality of the set $\mathscr{M}$ of p-free saturated multichains $\hat{0} = y_0 \lessdot \cdots \lessdot y_r$ in the Cayley–Hasse diagram of the geometric lattice. Due to the labeling, the same underlying saturated multichain in the lattice may occur on the right hand side of (12) with multiplicity.

Observe that $M \in \mathscr{M}$ cannot be a decreasing minimally labeled chain. Otherwise, since its length is r, we would have $y_r = \hat{1}$ forcing the top covering to be labeled p, and contradicting the p-freeness of M. Let $e_1, e_2, ..., e_r$ be the labels along M and let n, $n \geqslant 0$, be the largest index for which $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_n$ is a decreasing minimally labeled chain. There are three possible reasons why $y_n \lessdot y_{n+1}$ fails to extend this chain to a longer decreasing minimally labeled chain: either (i) $y_n \lessdot y_{n+1}$ and $e_{n+1} \neq m_{y_{n+1}} <_E e_n$, or (ii) $y_n \lessdot y_{n+1}$ and $m_{y_{n+1}} \geqslant_E e_n$, or (iii) $y_n = y_{n+1}$.

We now describe a bijection $\varphi\colon \mathscr{T} \to \mathscr{M}$. Consider $(D, e, C) \in \mathscr{T}$. Let D be $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n \lessdot x_{n+1} = F$ and f_i be the label of the covering $x_{i-1} \lessdot x_i$ for each $i = 1, 2, ..., n+1$. We have $n \geqslant 0$ since $F > \hat{0}$, and $f_{n+1} = m_F$.

If e is independent of x_n (equivalently, $e \notin x_n$), then we set $\varphi((D, e, C)) = M$, where M is the saturated multichain (starting at $\hat{0}$) obtained from the sequence of lables $f_1, f_2, ..., f_n$, e concatenated with the sequence of labels on C. It is obvious that $M \in \mathcal{M}$. Moreover, M falls under case (i): its longest initial subchain with decreasing minimal labels could be extended by modifying the label on the $(n+1)$st covering (Fig. 1).

Suppose now that e is dependent on x_n and that p is not in $F = x_{n+1}$. Let k be the smallest index such that e is dependent on x_k. Since the matroid does not contain loops, we have $n \geqslant k > 0$. Let

$$y_i = \begin{cases} x_i & \text{if } 0 \leqslant i \leqslant k-1, \\ y_{k-1} \vee f_{k+1} \vee \cdots \vee f_{i+1} & \text{if } k \leqslant i \leqslant n. \end{cases}$$

We put $\varphi((D, e, C)) = M$, where M is the multichain (starting at $\hat{0}$) determined by the sequence of labels $f_1, f_2, ..., f_{k-1}, f_{k+1}, f_n, f_{n+1}, e$, followed by the labels along C. Obviously, $M \in \mathcal{M}$, and we claim that M falls in case (ii). To justify the claim we need to verify three conditions. First, $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_n$ must be a decreasing minimally labeled chain. It is obviously decreasing, and the fact that it is minimally labeled follows from

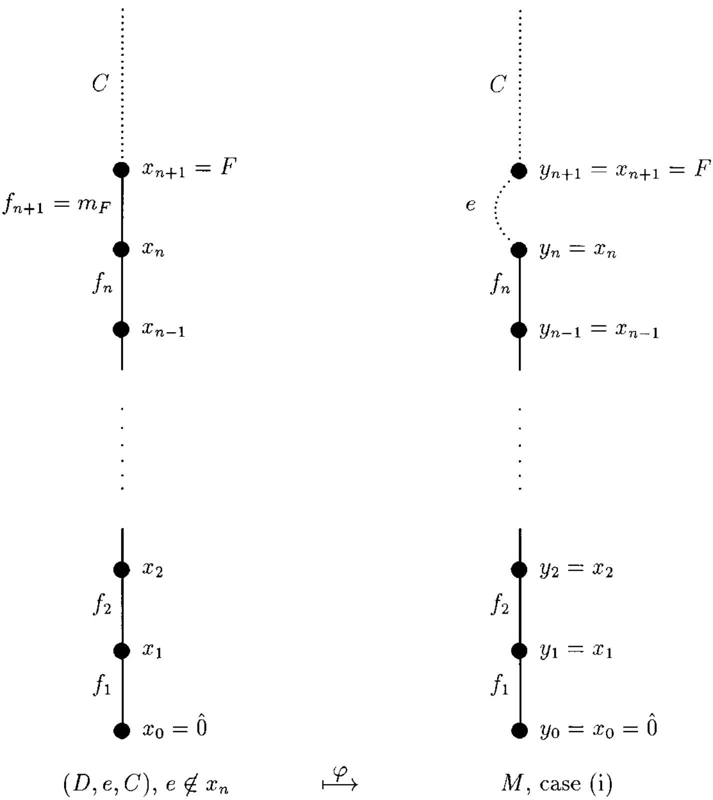

FIG. 1. The bijection φ, first restriction.

the following argument (a simplification of our original argument, for which we thank the anonymous referee). Since D is decreasing minimally labeled, we have $f_i = m_{x_i}$ for $i \leqslant n+1$, hence $f_i = m_{y_i}$ for $i < k$. For $i > k$ we have $y_{i-1} \leqslant x_i$, and therefore $m_{y_{i-1}} \geqslant m_{x_i} = f_i$. On the other hand, $f_i \in y_{i-1}$ and we conclude that $f_i = m_{y_{i-1}}$. Thus the labeling is minimal.

Second, e must be independent of $f_1, f_2, \ldots, f_{k-1}, f_{k+1}, \ldots, f_n, f_{n+1}$ (this is immediate from an elementary exchange argument, since $f_1, f_2, \ldots, f_n, f_{n+1}$ are independent; in fact $x_k = x_{k-1} \vee e$). Third, we must have $m_{y_n \vee e} \geqslant f_{n+1}$ (this is obvious since $y_n \vee e = \, = x_{n+1} = F$) (Fig. 2).

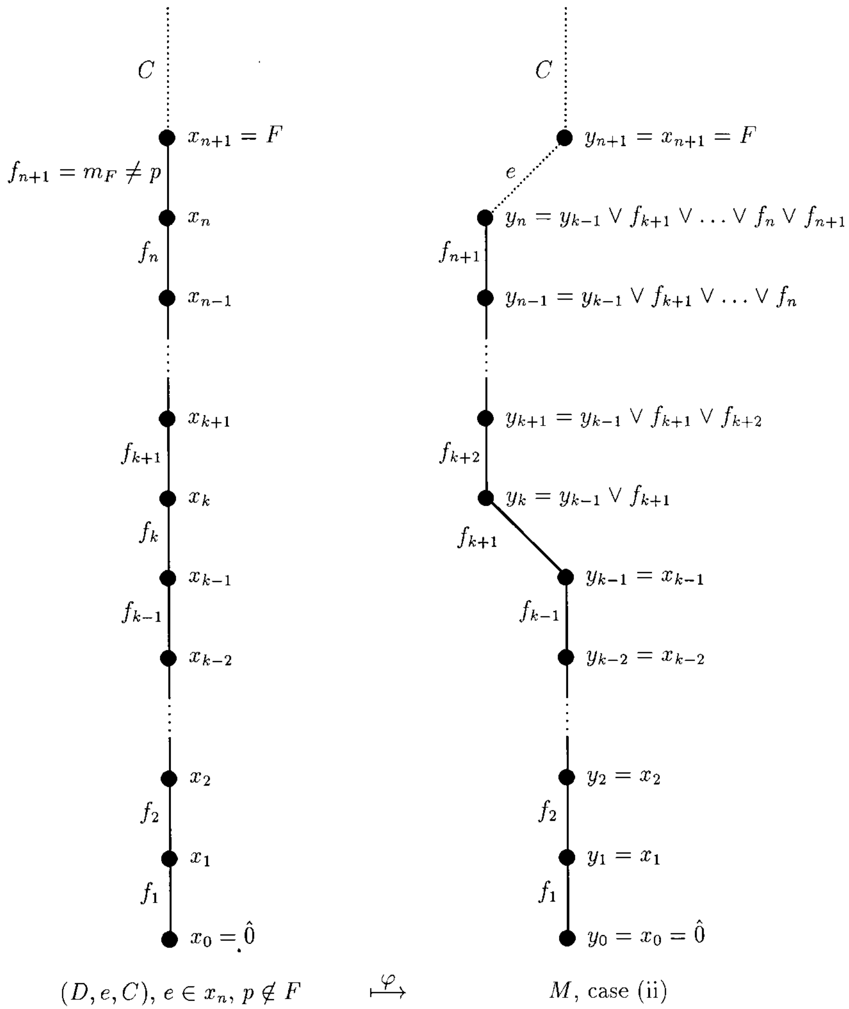

FIG. 2. The bijection φ, second restriction.

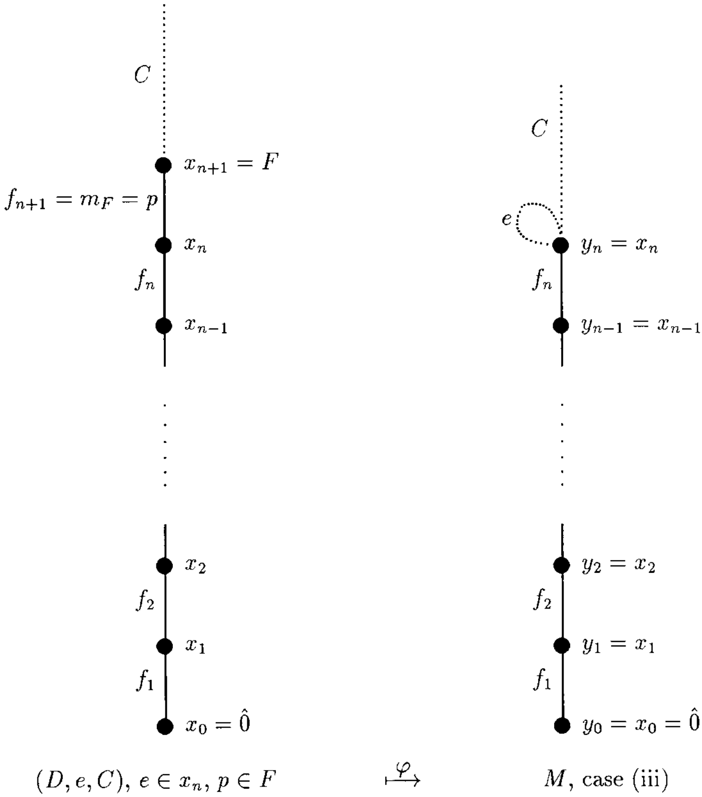

FIG. 3. The bijection φ, third restriction.

Finally, if e is dependent on x_n and p is in $F = x_{n+1}$, then $f_{n+1} = p$. Put $\varphi((D, e, C)) = M$, where M is the multichain determined by the sequence of labels $f_1, f_2, ..., f_n, e$ followed by the labels along C. Clearly, M falls under case (iii) (Fig. 3).

It is clear that φ is surjective. We omit the proof of its invertibility. Using arguments similar to those used in the construction of φ, one can show that each of the three restrictions of φ is invertible. ∎

5. THE TUTTE POLYNOMIAL

Now recall [7, 2] that for a matroid M on ground set E the *Tutte polynomial* $T(M; x, y)$ is defined by

$$T(M; x, y) := \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}.$$

In the setting of Tutte polynomials,

$$T(M \mid F; 1, 0) = (-1)^{r(F)} \mu(\varnothing, F)$$

for any flat $F \in L$, where $M \mid F$ denotes the restriction of the matroid M to the flat F.

The identity (3) is equivalent to the formula:

$$T(M;\ |E|, 0) = \sum_{A \subseteq E} |\bar{A}|\ (|E| - 1)^{r(E) - r(A)}\ (-1)^{|A| - r(A)}. \qquad (13)$$

It turns out that this is just a special case of the following much more general identity.

THEOREM 5.1. *If M is a loop free matroid on E, then*

$$|E|\ T(M; x, y)$$
$$= \sum_{A \subseteq E} \{x\ |\bar{A}| + y(1-x)\ |\bar{A} \setminus A|\}\ (x-1)^{r(E) - r(A)}\ (y-1)^{|A| - r(A)}.$$

Theorem 5.1 implies Theorem 2.1, since the substitution $x = |E|$, $y = 0$ yields (3).

Proof. Let M be a loop-free matroid of rank $r \geqslant 1$ on E. For each $A \subseteq E$, we define the polynomial $t(A)$ by

$$t(A) := (x-1)^{r(E) - r(A)}\ (y-1)^{|A| - r(A)}.$$

It would be more precise and complete to denote these polynomials by $t(A; M; x, y)$. However, for simplicity we will drop the $(M; x, y)$ from the notation, not only here, but also in the polynomials T_e^{ab}, T^{ab}, etc. that will appear below. Nevertheless it is useful to keep in mind that all these quantities are 2-variable polynomials.

With this notation, we will now prove Theorem 5.1, by verifying the formula

$$|E|\ T(M; x, y) + y(1-x) \sum_{A \subseteq E} |A|\ t(A) = (x + y - xy) \sum_{A \subseteq E} |\bar{A}|\ t(A). \qquad (14)$$

Define $f\colon 2^E \times E \to \{0, 1\}^2$ such that $f(A, e) = (a, b)$, where

$$a = 1 \qquad \text{if and only if } e \in A, \qquad \text{and}$$

$$b = 0 \qquad \text{if and only if } r(A \Delta \{e\}) = r(A).$$

For $(a, b) \in \{0, 1\}^2$, $e \in E$, and $A \subseteq E$, let

$$E^{ab} = \{(A, e) : f(A, e) = (a, b)\}$$
$$E_e^{ab} = \{A : f(A, e) = (a, b)\}$$
$$E_A^{ab} = \{e : f(A, e) = (a, b)\}.$$

Clearly, for any $A \subseteq E$, the sets E_A^{00}, E_A^{01}, E_A^{10}, E_A^{11} partition E. We note

$$A = E_A^{10} \cup E_A^{11},$$
$$\bar{A} = A \cup E_A^{00}.$$

Now, for $(a, b) \in \{0, 1\}^2$ and $e \in E$, let

$$T_e^{ab} := \sum_{A \in E_e^{ab}} t(A), \qquad T^{ab} := \sum_{e \in E} T_e^{ab}. \tag{15}$$

Then we have

$$\sum_{e \in E} T_e^{ab} = \sum_{(A, e) \in E^{ab}} t(A) = \sum_{A \subseteq E} |E_A^{ab}| \, t(A). \tag{16}$$

For $A \subseteq E$ and $e \in E$, with $e \notin A$, if $r(A \cup e) = r(A)$, then

$$t(A \cup e) = (y - 1) \, t(A).$$

If $r(A \cup e) = r(A) + 1$, then

$$(x - 1) \, t(A \cup e) = t(A).$$

(Similarly we can compare $r(A)$ and $r(A \backslash e)$ when $e \in A$.)

Summing over the appropriate sets A gives

$$\begin{aligned} (x - 1) \, T_e^{11} &= T_e^{01}, \\ T_e^{10} &= (y - 1) \, T_e^{00}. \end{aligned} \tag{17}$$

For $a \in \{0, 1\}$, let

$$T_e^{a*} = T_e^{a0} + T_e^{a1} \tag{18}$$

and let

$$T^{a*} = \sum_{e \in E} T_e^{a*}.$$

Then

$$T^{1*} = \sum_{e \in E} \left(\sum_{\substack{A \subseteq E \\ e \in A}} t(A) \right) = \sum_{A \subseteq E} t(A)\,|A|,$$

and, similarly,

$$T^{0*} = |E|\; T - T^{1*},$$

so that

$$T^{0*} + T^{1*} = |E|\; T. \tag{19}$$

Also, for any $e \in E$

$$T = T_e^{0*} + T_e^{1*}. \tag{20}$$

Using (17) and (18) it is now possible to express each T_e^{ab} as a linear combination of T_e^{0*} and T_e^{1*}. This leads to the following identities, with $q := (x-1)(y-1)$:

$$\begin{aligned}(1-q)\; T_e^{11} &= T_e^{1*} - (y-1)\; T_e^{0*},\\ (1-q)\; T_e^{10} &= -qT_e^{1*} + (y-1)\; T_e^{0*},\\ (1-q)\; T_e^{00} &= -(x-1)\; T_e^{1*} + T_e^{0*}.\end{aligned}$$

Summing over $e \in E$ and using (20) and (19) gives

$$\begin{aligned}(1-q)\; T^{11} &= -(y-1)\; |E|\; T + yT^{1*}\\ (1-q)\; T^{10} &= (y-1)\; |E|\; T - x(y-1)\; T^{1*}\\ (1-q)\; T^{00} &= |E|\; T - xT^{1*}.\end{aligned} \tag{21}$$

But $|\bar{A}|$ is given by

$$|\bar{A}| = |E_A^{11}| + |E_A^{10}| + |E_A^{00}|.$$

By (15) and (16),

$$\sum_{A \subseteq E} |\bar{A}|\; t(A) = T^{11} + T^{10} + T^{00}.$$

Substituting from (21) this gives the identity

$$(1-q) \sum_{A \subseteq E} |\bar{A}|\; t(A) = |E|\; T + y(1-x) \sum_{A \subseteq E} |A|\; t(A),$$

which completes the proof of (14), and thus of Theorem 5.1. ∎

A weighted version of (13) is obtained as follows. Given a variable x_e for each $e \in E$ and for each $A \subseteq E$ letting

$$X_A = \sum_{e \in A} x_e,$$

then the following identity holds—it contains all the previous ones as special cases.

THEOREM 5.2. *If M is a loop-free matroid on E, then*

$$X_E T(M; x, y) + y(1-x) \sum_{A \subseteq E} X_A t(A) = (x + y - xy) \sum_{A \subseteq E} X_{\bar{A}} t(A).$$

Proof. The proof follows exactly the proof of (13), replacing each term $\sum_{e \in E} f(e)$ by $\sum_{e \in E} x_e f(e)$ and each $|B|$ by X_B. For example, we would now define

$$T^{ab} = \sum_{e \in E} x_e T_e^{ab}.$$

Moreover, using (20) we can express any linear combination of T^{11}, T^{10}, T^{01}, T^{00} as a linear combination of T and T^{1*}. ∎

REFERENCES

1. M. Aigner, "Combinatorial Theory," Springer-Verlag, New York, 1979.
2. A. Björner, Homology and shellability of matroids and geometric lattices, *in* "Matroid Applications" (N. White, Ed.), pp. 226–283, Cambridge Univ. Press, Cambridge, 1992.
3. A. Björner, A. Garsia, and R. P. Stanley, An introduction to Cohen–Macaulay partially ordered sets, *in* "Ordered Sets" (I. Rival, Ed.), NATO Advanced Study Institutes Series, Ser. C, Vol. 3, pp. 583–615, Reidel, Dordrecht, 1982.
4. D. Massey, The Lê Varieties, I, II, *Invent. Math.* **99** (1990), 357–376; **104** (1991), 113–148.
5. D. Massey, "Lê Cycles and Hypersurface Singularities," Lecture Notes in Math., Vol. 1615, Springer-Verlag, Heidelberg, 1995.
6. R. P. Stanley, "Enumerative Combinatorics, Vol. I," Wadsworth & Brooks/Cole, Monterey, 1986.
7. D. J. A. Welsh, "Matroid Theory," Academic Press, London, 1976.