Trinity Term Project: Modeling Commodity Futures Contracts

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Chapter 1

Wheat Futures Basics

1.1 Futures vs. Forwards

If a market participant wishes to lock in a price at time t for delivery of an asset at time T, he can either engage in a *forward contract* or a *futures contract*. While similar in many respects, forwards and futures have a number of differences, which ultimately lead to futures being more attractive and traded on liquid markets all over the world.

A forward contract is essentially a binding agreement between two parties, the buyer (long) and the seller (short), that the seller will deliver an asset with current price at time t of S_t to the buyer at time T. The forward price at time t for delivery at time T will be denoted as $For(t,T)$. The contract, while binding, costs nothing initially to enter either for the buyer or seller, and pays $S_T - For(t,T)$ at expiry to the buyer (or the negative to the seller). Thus, in order that no arbitrage exist, the forward price is set to:

$$For(t,T) = S_t B(t,T) \tag{1.1.1}$$

where $B(t,T)$ is the value at time T of investing 1 unit of the numeraire (which for the subsequent discussion will be dollars) in a bank account at time t. Thus, the forward price is set to be the equivalent of putting S_t dollars in the bank at time t and removing them at time T. This relationship is easily

proved by the assumption that there is no arbitrage.

While forwards allow market participants to lock in a price for the future at no initial cost, in practice most forward agreements are privately negotiated, and therefore non-standardized and difficult to unwind (cancel) should either party choose to do so. Futures contracts, which also allow participants to enter into an agreement at time t for a price to be paid at time T, mainly differ from forward contracts in that they are standardized and traded on highly liquid markets. Thus, exiting a futures position can easily be accomplished by *offset*, or taking the exact opposite position, and hence negating the contract. Specific futures markets and contracts have various features which will be subsequently discussed in further detail. In the simplest mathematical terms, the futures price $F(t,T)$ is defined to be:

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] \tag{1.1.2}$$

where $\mathbb{Q}$ denotes the risk neutral world, under which discounted asset price are martingales. The spot price S_t follows the SDE:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t \tag{1.1.3}$$

Assuming that the bank account with interest rate r has the dynamics and solution under continuous compounding:

$$dB(t,T) = rB(t,T)dt \tag{1.1.4}$$

$$B(t,T) = e^{r(T-t)} \tag{1.1.5}$$

Therefore, as discounted asset prices are martingales, the futures price is given as:

$$F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] = S_t e^{r(T-t)} \tag{1.1.6}$$

Similarly to forwards, futures contracts cost nothing to enter. If the above relationship did not hold, it can be easily shown that there would be an arbitrage. Note that if the bank account is defined as in

equation (1.1.5) then the forward price and futures price are equal in this simple, risk neutral world with continuous compounding and constant interest rate r. However, even in the real world this relationship should also hold true if forwards and futures are written on the same contract with same expiry, as if this were not the case, the buyer or seller of the privately agreed forward could trade futures contracts on the liquid market to make a risk free profit. Under the assumption that no such simple arbitrage exists in the market, forwards and futures prices should be equal. Despite this equivalence, the remainder of this discussion will focus on futures contracts and futures markets.

1.2 Futures Contract Basics [4]

In order to accurately model futures contracts, it is first necessary to fully understand how these contracts work in a real-world trading environment. Few authors actually consider these real trading environments, which severely limits their analysis and ability to create applicable models.

This discussion will use the wheat futures contract listed on the Chicago Board of Trade (CBOT) in Chicago as the primary example of a futures contract. For a summary of contract specifications, refer to table (1.1). Wheat contracts on several varieties and grades of wheat, deliverable at various locations in the American mid-west during various months, are available in multiples of 5,000 bushels. However, not all, and in fact relatively few futures contracts are held until delivery, but are used as hedging instruments by producers, such as farmers, or consumers such as cereal makers. Speculators are also very active in commodity futures markets, though they have no intention of physical delivery. The trading experience of all three is relatively similar, but for the sake of demonstation consider that of a wheat-producing farmer.

Suppose a farmer in Iowa has an expected yield of 100,000 bushels of wheat during the spring, which he will want to deliver the following September. The CBOT futures contract is trading at \$3.50 per bushel. This reflects wheat prices in one of the delivery locations and may be different from his home market in Iowa, but it is reasonable to assume the wheat spot prices will be reasonably to highly correlated. The farmer knows that if he receives \$3.50 per bushel then he will make a good profit, and thus wants to lock in this price by entering in a futures contract.

The farmer first establishes a short futures contract at the current \$3.50 per bushel price through the

clearing house of the CBOT. The clearing house technically acts as the buyer to all sellers and the sellers to all buyers, but really exists to match buyers and sellers, and to increase liquidity. The farmer is then required to put up a margin (typically between 3-10% of the entire value of the contract). Margins are essentially performance bonds, and designed to ensure traders meet their obligations. Margin accounts are marked-to-market everyday, and if the margin account falls below a minimum level, cash infusion is necessary to keep the position open. Thus, the payoff of a futures contract is really the sum of daily payments. While this telescoping sum simplifies, the risk for a participant in a futures contract is that margin calls (requiring cash infusion) force a premature closure of the position, as the very purpose of marking to market everyday is to prevent large defaults. For the case of the short farmer, his payoff at delivery time T, with t_n, $n = 0, 1, 2, ...N$ the days from $t_0 = t$ until $T_N = T$:

$$(F(t_0,T)-F(t_1,T))+(F(t_1,T)-F(t_2,T))+.....(F(t_N,T)-F(t_{N-1},T)) = F(t_0,T)-F(t_N,T) = F(t,T)-S_T \tag{1.2.1}$$

In this example, assume the margin maintenance level (minimum margin requirement) is \$650 (about 3.7%) per contract, and thus the farmer must maintain a minimum of \$13,000 in his margin account at all times. If the futures price declines on a certain day, the farmer's margin account is credited, and if the futures price increases on a certain day, his account is debited (hence marked-to-market). It is very important to note the difference between how the academic literature treats futures and how they are traded in practice. The necessity of the margin account and the practice of marking-to-market every day means that futures contracts do, in fact, require capital to open, which often does not earn interest. Futures contracts are not simply the "binding agreements" that are discussed in academic work that have zero initial value.

Suppose that its been a bumper year for wheat in the midwest, and by September, the farmer has his 100,000 bushels and the price of wheat has fallen to \$3.00. The farmer now wants to unwind his position, which he can do by physically delivering the wheat to one of the CBOT delivery locations (incurring transportation costs) and collecting both his margin account and \$3.00 per bushel, or he can liquidate his futures position by buying 20 futures contracts, and thus collecting the profit in his margin

account (.50$*100,000=50,000$). He would then sell on his local Iowa market,where the price will likely be highly correlated with wheat prices at the CBOT delivery locations. Assuming it is also $3.00, then he has successfully hedged and locked in his initial $3.50 price. A similar argument (with the margin account having suffered a loss but with a higher sale price) shows that he has also locked in the price if the spot price at time of delivery has risen.

1.3 Basis Risk

In the previous example, it was assumed that the spot price in the Iowa farmer's local market matches the CBOT futures price, which converges to the spot prices at the delivery locations, and that price differentials between cash and futures (**basis**) remain constant in the different markets. This is not always the case, and can affect, for example, the farmer's ability to hedge. In this particular case, Iowa spot prices may go down 55 cents per bushel, while CBOT futures prices, and spot prices at delivery locations may only go down 45 cents. This situation, referred to as **basis risk**, will affect the farmer's ability to hedge. The profit on his margin account will be $45,000, but now he can only sell his 100,000 bushels for $2.95 per bushel, and thus his effective price is $3.40 per bushel. Thus, the change in basis can have a substantial impact on his ability to hedge and make a profit, particularly if his profit level is between $3.40 and $3.50 per bushel.

A price differential of 10 cents between two highly correlated markets (3%) is perhaps on the high side, but certainly not impossible. This trivial example demonstrates the importance of basis risk not only for this hypothetical farmer but also other players in the market, given the requirement of margin accounts and the tendency of many players to unwind contracts rather than take physical delivery. The basis is determined by multiple factors, including the fact that traders can collect interest on the difference between margin deposits and the total denomination of the contract. Other factors include savings on storage costs, since the farmer (or shorter) is not actually holding the commodity. For these reasons, futures prices tend to be in **contango**, or higher than risk-neutral expected future spot prices (Se^{rt}). However, futures prices can also be lower than spot prices for various reasons, such as expectations of future supply or demand, which is known as **backwardation**.

1.4 Embedded Options and Basis Risk

Many academic papers have been written on the **embedded options** in a futures contract. These are generally divided in the **timing option**, giving the seller flexibility of the time of delivery (usually just in the spot month); the **location option**, the ability to decide where to deliver; and the **quality option**, the ability to decide what grade to deliver. Consider the exact contract specifications as given by the CBOT on the wheat futures that the aforementioned farmer would be shorting in table (1.1):

Table 1.1: CBOT Wheat Futures Contract Specifications

Specification	Description
Contract Size	5,000 Bushels
Deliverable Grades	No.1/2: Hard/Soft Red Winter, Dark Northern Spring, Northern Spring; No. 2 delivered at par, No.1 at 3 cents per bushel over contract price
Tick Size	\$12.50/ contract
Contract Month	Jul, Sep, Dec, Mar, May
Last Trading Day	The business day prior to the 15th calendar day of the contract month
Last Delivery Day	Seventh business day following the last trading day of the delivery month
Delivery Point	Chicago, Illinois; Alton, Illinois; Burns Harbor, Indiana; Toledo, Ohio; St. Louis, Missouri

Clearly, his quality option is to deliver grades 1 or 2 of 3 kinds of wheat, receiving a fixed premium per bushel if he delivers no.1. However, as he will always receive a fixed premium for delivery of the higher quality, this is a rather simple option as the theoretical price of premium grade should always be 3 cents higher than the standard grade. His location option is delivering to one of the CBOT delivery locations, or as shown above, to a local market that maximizes the trade-off between spot price and transportation cost (the latter is not necessarily a direct location option, but one could argue that the multiple delivery locations specified in the CBOT contract increase the correlation between local markets, hence increasing the liquidity and optionality of delivering to these markets). His timing option is the ability to deliver his desired grade to his desired location over a set time interval during the spot month.

These embedded options enhance the attractiveness of entering into a futures contract for the farmer, and thus have value. For instance, futures prices can have very large swings in just a week, so the American put feature of the timing option clearly has value. Any claims that there is no value in the timing option is akin to claiming that short-dated (perhaps three weeks or less) European and American put options have the same value, which can easily be disproved by looking at actual option prices. The

location option also has value to the farmer, as transportation to different delivery locations will be different, and the farmer can choose that which minimizes his costs. The quality option can also have value given that the premiums for delivery of no.1 wheat are preset. If it costs the farmer less than the premium per bushel to produce the no.1 wheat, then he will profit by delivering the higher grade.

Previous academic work has dismissed embedded options as worthless, for example by proving that it is theoretically optimal to deliver on the first or last possible day. Consider, for example, Biagini and Bjork's [1] treatment of the timing option. The authors derive this result utilizing the fundamental pricing equation, which the futures price process F must satisfy:

$$\sup_{t\leq\tau\leq T} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^\tau r_s ds}(F_\tau - S_\tau) - \int_t^\tau e^{-\int_t^u r_s ds} dF_u | \mathcal{F}_t\right] = 0 \tag{1.4.1}$$

where τ varies over the class of optional stopping times. The first term on the LHS represents the cash flow for actual delivery and the second term represents the cash flow of the continuous resettlement. Essentially, Biagini and Bjork state that if the underlying asset pays no dividends, it is optimal to exercise as early as possible. If there are dividends, it is optimal to hold until the last day. However, these optimal strategies only hold in a mathematical martingale world which largely does not reflect the real world trading environment.

Some authors, such as Boyle [3] admit that timing options and quality options, when considered together, may actually have value, but make no attempt to quantify it. Boyle's arguments will be discussed in greater detail in the next chapter. Others authors have even gone so far as to admit that much of the theoretical treatment of the embedded options poorly reflects the real world, and have attempted to build models to value these options using data. Hranaiova, Jarrow, Tomek [5] make a solid attempt at estimating the value of the timing option and the joint value of the timing and location options using real-world spot prices as seed values for their model. They use a binomial tree in the one-dimensional case and a trinomial tree in the two dimensional case to estimate the value of the timing option by itself and the joint option.

Nonetheless, though Hranaiova et al. do come up with estimates for values of the timing option and joint options, they conclude their analysis but stating that these embedded options values represent only a small fraction of the average futures price. However, they go on to state that these option values

perhaps account for 5%-28% of the average basis! Considering the requirement of margin accounts, which implicitly means shorting a futures contract is inherently highly *leveraged*, this is not at all an insignificant finding as it could represent large profits or losses and necessitates the development of models that attempt to value embedded options in a more realistic setting than is currently done.

Chapter 2

Models

2.1 Buying Futures vs. Taking Spot Price

The first problem the hypothetical wheat farmer must consider when deciding to buy futures is simply *how much?* As buying futures contracts, which locks in a future price, is less risky then selling wheat on the open market once it is harvested, the farmer's allocation to futures depends on his risk preferences. The more risk averse he is, the more he will allocate to futures.

The problem of determining allocations in risky and risk-less assets has been around for decades, since Merton's work on the bequest problem [7]. Essentially, the problem of allocating crop in futures or in the spot market is a bequest problem, which can either be solved using a martingale approach, or a dynamic programming approach similar to that of Merton.

2.1.1 Martingale Approach

Assume the farmer lives in a Black-Scholes world , in which case there is a bank account and one risky asset with the dynamics (under the physical measure $\mathbb{P}$):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad dB(t,T) = rB(t,T)dt \tag{2.1.1}$$

The farmer has an initial crop wealth H_0, (the units of which could be dollars, acres of fields, bushels

of harvest, etc) which he can determine at an initial time $t = 0$ should be allocated to futures with expiry T or left to sell on the spot market at time T. Hence, he can choose between allocating his wealth (crop) to a risk-less asset (buy futures contracts) and invest in the risky asset (sell on the spot market). His planning "horizon" is T, and the strategy he initially chooses remains constant $\forall \quad 0 \leq t \leq T$. Buying futures is equivalent to investing in a risk-less asset in a risk-neutral world with no transaction costs. Under the measure $\mathbb{Q}$ the futures price satisfies $F(t,T) = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] = S_t e^{r(T-t)}$, as discussed in the previous chapter. If there were an excess return to be made from buying futures contracts, it is reasonable to assume this would be priced out of the market as it would represent an arbitrage. Assuming the proportion of wealth the farmer allocates to the risky asset (selling on the spot market) is denoted as π, the farmer's wealth equation will satisfy:

$$dH_t = H_t \left(r(1-\pi) + \mu\pi\right) dt + H_t \pi\sigma dW_t \tag{2.1.2}$$

This is a Geometric Brownian Motion, which will have the solution:

$$H_T = H_0 \exp\left\{\left(r(1-\pi) + \mu\pi - \frac{1}{2}\pi^2\sigma^2\right)T + \pi\sigma W_T\right\} \tag{2.1.3}$$

Incorporating utility theory and given the above wealth equation, the farmer's choice of π will be to maximize his expected utility:

$$\pi = \arg\sup_{\pi} \mathbb{E}U[H_T] \tag{2.1.4}$$

This equation depends on the choice of utility function U. For this discussion, assume U is the common power utility function $U(H) = H^\gamma/\gamma$, $0 < \gamma < 1$, which has the property of Constant Relative Risk Aversion (CRRA). In economic terms, CRRA means that the farmer's utility preferences with regards to gambling on a percentage return do not change as his total wealth changes, which is a reasonable assumption in the case of the farmer. Equation (2.1.4) can therefore be written as:

$$\pi = \arg\sup_{\pi} \mathbb{E}\left[\frac{H_0}{\gamma}\exp\left\{\left(r(1-\pi) + \mu\pi - \frac{1}{2}\pi^2\sigma^2\right)T\gamma + \pi\sigma\gamma W_T\right\}\right] \tag{2.1.5}$$

$$= \arg\sup_{\pi} \frac{H_0}{\gamma} \exp\left\{\gamma T\left[r + \pi(\mu - r) - \frac{(1-\gamma)\pi^2\sigma^2}{2}\right]\right\} \tag{2.1.6}$$

Equation (2.1.5) is obtained by use of the moment generating function for normal random variables appearing in equation (2.1.5). Since π only appears in the exponential, the value of π that maximizes the exponential will also maximize the entire function. Taking the derivative of the exponential with respect to π and setting to zero gives:

$$(\mu - r) - (1-\gamma)\pi\sigma^2 = 0$$

$$\pi^* = \frac{\mu - r}{(1-\gamma)\sigma^2} \tag{2.1.7}$$

While this is a theoretical result which is far from a policy recommendation as it depends on a number of assumptions and parameters, this analytic formula shows that as the value of γ, the risk aversion parameter for the power utility function, decreases and hence risk aversion increases, the value of π decreases. This validates the argument that the farmer should buy more futures the higher his level of risk aversion.

2.1.2 Dynamic Programming Approach

The approach taken by Merton was a dynamic programming approach, which utilizes the Hamilton-Jacobi-Bellman (HJB) equation for stochastic optimal control problems as well as Bellman's principle of optimality. [9] First, consider again the objective of maximizing expected utility and denote this as a value function $v(t, H)$. Since maximization is equivalent to minimization of the negative, this leads to:

$$v(t, H) = \sup_{\pi} \mathbb{E}[U(H_T)] = -\inf \mathbb{E}[-U(H_T)] \tag{2.1.8}$$

This equation is necessary in order to use Bellman's Principle of Optimality, which is written in terms of an infimum. Next, write down the HJB equation in terms of the generalized Hamiltonian $G(t, H, \pi, -v_H(t, H), -v_{HH}(t, H))$, where v_H and v_{HH} denote first and second derivatives w.r.t H:

$$v_t + \sup_{\pi} G(t, H, \pi, -v_H(t, H), -v_{HH}(t, H)) = 0$$

$$v(T, H) = U(H) \tag{2.1.9}$$

Given the wealth equation (2.1.2), the Hamiltonian G will be defined as:

$$G(t, H, \pi, -v_H(t, H), -v_{HH}(t, H)) = \frac{1}{2}\sigma^2\pi^2H^2v_{HH} + (\mu - r)\pi H + rHv_H \tag{2.1.10}$$

Thus, equation (2.1.9) can be rewritten as:

$$v_t + \sup_{\pi}\left(\frac{1}{2}\sigma^2\pi^2H^2v_{HH} + (\mu - r)\pi H\right) + rHv_H = 0$$

$$v(T, H) = U(H) \tag{2.1.11}$$

Therefore, the optimal value of π can be determined by setting the derivative of the expression inside the supremum w.r.t π to zero:

$$\pi^* = \arg\sup_{\pi}\left(\frac{1}{2}\sigma^2\pi^2H^2v_{HH} + (\mu - r)\pi H\right) = -\frac{\mu - r}{H\sigma^2}\frac{v_H}{v_{HH}} \tag{2.1.12}$$

To show that this leads to an equivalent result as with the martingale approach, consider the same power utility function $U(H) = H^\gamma/\gamma$, $0 < \gamma < 1$ and make the conjecture: $v(t, H) = g(t)H^\gamma$, for some arbitrary function g. Calculating derivatives, and substituting into equation (2.1.11), this will lead to an ODE in g (H does not appear):

$$\frac{\partial g}{\partial t} + g(t)\left[r\gamma - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\frac{\gamma}{\gamma - 1}\right] = 0$$

$$g(T) = \frac{1}{\gamma} \tag{2.1.13}$$

The equation for π^* is written as:

$$\pi^* = -\frac{\mu - r}{H\sigma^2}\frac{\gamma g(t)H^{\gamma-1}}{\gamma(\gamma - 1)g(t)H^{\gamma-2}} = \frac{H}{1-\gamma}\frac{\mu - r}{H\sigma^2} = \frac{\mu - r}{(1-\gamma)\sigma^2} \tag{2.1.14}$$

This is an equivalent expression as equation (2.1.7), demonstrating that the martingale approach and dynamic programming approach yield the same result.

2.2 Understanding Simple Models for Embedded Options

After looking at the farmer's decision of how many futures to buy, the next aspect of futures contracts to study is the value of the various embedded options. In order to develop more useful models for valuing embedded options, it is first necessary to understand the basic models. Consider the argument proposed by Boyle [3] for assigning zero value to the timing option:

- assumes frictionless market
- no dividends and no associated transaction costs
- given three times $T_1 \leq T_2 \leq T_3$, with the contract expiring at T_3, T_1 corresponding to the first day of the month of expiry, and T_2 the day that the contract is delivered
- F_i is the futures price at time t of a futures contract where delivery takes place at T_i, and S is the spot price
- considering a long position in the contract with no timing option and a short position in a contract with the timing position, the initial cost is zero, and the payoff will be: $S(t+T_1) - F_1 - (S(t+T_1) - F_2) = F_2 - F_1$. Thus $F_1 = F_2$ if there is to be no arbitrage, and therefore the timing option has no value.

Boyle then admits that for the joint option of timing and quality (choosing among n deliverable assets), the above argument leading to zero value of the timing option no longer holds. He argues:

- $F_1(t, [S_i, S_j], (t+T_1)) = F_1$ denotes the current futures price of a contract which involves the delivery of the asset with the lowest price at time $(t+T_1)$
- $F_2(t, [S_i, S_j], (t+T_1)) = F_2$ denotes the futures price of the contract where delivery of either asset i or j can take place at any time during the interval $[t+T_1, t+T_3]$

- since it is not possible to guarantee which of S_i, S_j will be cheaper, it is not possible to guarantee a risk-less profit as with the case of the timing option on one asset

Furthermore, Boyle notes that the value of a contract at time $(t + T_2)$ where the short can deliver either asset i or j is written as: (for the short position, such as the farmer)

$$F_2 - \min(S_i(t + T_2), S_j(t + T_2))$$

Boyle then attempts to estimate the value of the timing option in the presence of two deliverable assets using a binomial lattice and assuming both assets have the same initial price and same constant volatility. His findings indicate that even in this theoretical model, even under high correlation of asset prices S_i and S_j, the joint timing and quality option can significantly affect the futures price and basis. The useful question, of course, is by how much? This will be explored in further detail in the next chapter.

2.3 Argument for Interchanging Quality and Location Options

Though physically different ideas, from a modeling and profit-maximization approach, the quality and location options are very similar. In Boyle's example above, he is considering delivering different assets (such as no.1 and 2 grade wheat) on the same market with prices S_i and S_j. Similarly, these prices could be for the same asset in two different markets i and j. This would lead to the same basic pricing formula that Boyle gives. It would also be possible to easily include fixed transportation costs K_i and K_j leading to the payoff:

$$F_2 - \min(S_i(t + T_2) - K_i, S_j(t + T_2) - K_j)$$

It should also be noted that given the contract specifications for the wheat futures contracts on the CBOT, there is no real quality option as the option to deliver a premium grade always yields a fixed premium. If the farmer is able to produce the higher quality wheat more cheaply than the premium, than he can choose to deliver the higher grade and will know exactly how much excess profit he will make

over the standard quality. However, since the premium is specified in the futures contract, the price of the higher and standard grade are separated by a fixed premium, which removes much of the optionality from any quality option. Rather, the quality option as discussed by Boyle is analogous to the location option on the real CBOT contract, but the option to deliver a premium grade on the CBOT contract is certainly not equivalent to a location option.

2.4 Quantities and Parameters that Should be Included in the Model

Consider the example of the farmer shorting wheat futures contracts to hedge against price movements for his expected future harvest. In order to determine how valuable the embedded options in this futures contract are to his position, it is reasonable to at least consider the following quantities and parameters:

- expected future harvest (H_t)
- futures price (F_t)
- spot prices in delivery and local markets ($S_{t,i}$)
- time until delivery (T)
- delivery window (w)
- futures-cash basis (b_t)
- spot price correlations between delivery locations and local markets (ρ_{ij})
- transportation and storage costs (K)
- spot price volatility (σ)
- interest rates (r_t)

Clearly, a model incorporating all of the above will be significantly more complicated than those proposed by Boyle and other authors, such as more recent papers by Hranaiova, Jarrow, Tomek [5]. To further add to the complexity, all of quantities could be made to be deterministic or even stochastic

functions of time. If all are taken to be stochastic, modeling becomes inordinately complex, especially since parameter estimation for each of the stochastic quantities can be quite difficult. Thus, while Boyle-type models are perhaps too simple, assuming all of the above to be stochastic is too complicated. Naturally, this suggests a model lying somewhere between the two.

2.5 Example Parameter Values

Consider a farmer that knows he will produce at least a certain quantity of wheat that he will want to bring to market in approximately six months (180 days). He knows he can deliver the contract at any location for 20 days before it expires. He lives in Iowa, and thus can deliver to his home market, which has negligible delivery cost, or one of the delivery locations for the CBOT wheat futures. The parameters of the problem are formulated as follows in table (2.1):

Table 2.1: Example Model Parameters

Type	Variable	Example Value	Deterministic	Stochastic
Expected Future Harvest	H	100,000	$\surd$	$\surd$
Time until Delivery	T	180	$\surd$	
Delivery Window	w	20	$\surd$	
Futures Price	F_t	\$3.50		$\surd$
Spot Prices	S_i, $i = 1, 2$	Chicago, Indiana		$\surd$
Futures- Cash Basis	b_t	+5 cents in all markets		$\surd$
Transportation Costs	K_i, $i = 1, 2$	5,10, cents/bushel	$\surd$	
Interest Rates	r	%5 per annum	$\surd$	
Spot Price Correlations	ρ_{ij}	$1 - \varepsilon$	$\surd$	
Spot Price Volatility	σ_i, $i = 1, 2$	?	$\surd$	$\surd$

As discussed in the previous example of the farmer (many of the values listed above are similar), as expiration approaches and once he is in the delivery period, there are various ways in which he can deliver. Assume that he will buy to cover on the futures market, and then deliver to one of the delivery locations or his local market. Theoretically, the futures price and the spot price at the delivery locations should converge, and thus this may be a redundant treatment. However, futures-cash basis often does not converge to zero either on the first or last possible delivery, and various papers exist to explain this basis variability (see Hraniova and Tomek [6]). Thus, upon covering his short and (simultaneously delivering on the spot market) the farmer's payoff will be:

$$H\left[\max_i(S_i - K_i) + (F_0 - F_t)\right]$$

The first term represents the amount he obtains that maximizes profit over the spot prices and fixed costs (like strike prices) at locations $i = 1, 2$, while the second term represents the approximate contribution of his margin account. Since he can deliver anywhere over the window $[T - w, T]$, he essentially has a forward-start American option.

Chapter 3

Valuation

3.1 Finite Difference Scheme for the Timing Option

Given the hypothetical example of the farmer discussed in previous chapters, it is possible to devise a finite difference scheme as an attempt to quantify the value of the embedded options. Initially, only one underlying spot price S was considered, with the primary goal of exploring the value of the short-dated American-style timing option as a percentage of futures-spot basis of the first possible delivery day. In order to discuss the finite difference scheme, it is first necessary to introduce the Black-Scholes Partial Differential Equation (BSPDE). [2] Denote the price of an option that depends on underlying price S and time t as $V(S,t)$, where S follows the SDE given in equation (2.1.1). Using Itô's formula, the dynamics of V are:

$$dV = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial S}dS \tag{3.1.1}$$

Consider a portfolio of cash and shares in the asset S in order to replicate the option. In terms of cash, this can be written as $\Pi = V - \Delta S$, where Δ is the number of shares in the stock. The portfolio is to have no risk, and thus the dynamics should be $d\Pi = r\Pi dt$, which is one equation for $d\Pi$. The other equation, given by using Itô's formula as in equation (3.1.1), is given as:

$$d\Pi = dV - \Delta dS = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial S}dS - \Delta dS \tag{3.1.2}$$

Thus, in order to remove the stochastic terms from equation (3.1.2), Δ is set to $\frac{\partial V}{\partial S}$, and equation (3.1.2) set equal to $r\Pi dt = r(V - \Delta S) = r(V - \frac{\partial V}{\partial S}S)dt$, which gives:

$$\frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dt = rVdt - rS\frac{\partial V}{\partial S}dt \tag{3.1.3}$$

After dropping the dt and rearranging, this gives the BSPDE in its most common form, with t defined as calendar time:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.1.4}$$

The finite difference scheme implemented for estimating the option value was the θ-scheme (specifically Crank-Nicolson). While finite-difference methods are not the main purpose of this discussion and therefore some details may be omitted, the θ scheme arises from discretization of the partial derivatives in the the BSPDE, written in terms of time until expiry:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV = 0 \tag{3.1.5}$$

To implement the θ scheme, and omitting how the following is exactly derived, let:

$$a_n^m = -\frac{1}{2}\theta\tau(\sigma^2 n^2 - rn)$$

$$b_n^m = 1 + \theta\tau(\sigma^2 n^2 + r)$$

$$c_n^m = -\frac{1}{2}\theta\tau(\sigma^2 n^2 + rn)$$

and

$$A_n^m = -\frac{1}{2}(1-\theta)\tau(\sigma^2 n^2 - rn)$$

$$B_n^m = 1 - (1-\theta)\tau(\sigma^2 n^2 + r)$$

$$C_n^m = -\frac{1}{2}(1-\theta)\tau(\sigma^2 n^2 + rn)$$

The values V_n^m are obtained by solving, for each set of grid points:

$$a_n^m V_{n-1}^m + b_n^m V_n^m + c_n^m V_{n+1}^m = A_n^m V_{n-1}^{m-1} + B_n^m V_n^{m-1} + C_n^m V_{n+1}^{m-1} \tag{3.1.6}$$

Setting $\theta = 0$ will yield a fully explicit method, $\theta = 1$ a fully implicit method, and $\theta = 0.5$ the Crank-Nicolson method, which is second-order accurate in time. In this analysis, a theta-scheme with the following parameters was used:

Table 3.1: Theta-Scheme Parameters

Specification	Value
Spatial Steps	1024
Temporal Steps	128
Theta	0.5 (Crank-Nicolson)
Interest Rate	.05
Volatility	Variable
Days until Expiry w	20
Business Days in a Year	252
Futures Price	3.5

The algorithm for solving for the initial price V for different initial values of S proceeds as follows [8]. Let:

$$\begin{pmatrix} b_0^m & c_0^m & 0 & .... & 0 \\ a_1^m & b_1^m & c_1^m & .... & 0 \\ 0 & \searrow & \searrow & \searrow & 0 \\ 0 & .... & a_{N-1}^m & b_{N-1}^m & c_{N-1}^m \\ 0 & .... & 0 & a_N^M & b_N^m \end{pmatrix} = D_1$$

$$\begin{pmatrix} B_0^m & C_0^m & 0 & .... & 0 \\ A_1^m & B_1^m & C_1^m & .... & 0 \\ 0 & \searrow & \searrow & \searrow & 0 \\ 0 & .... & A_{N-1}^m & B_{N-1}^m & C_{N-1}^m \\ 0 & .... & 0 & A_N^M & B_N^m \end{pmatrix} = D_2$$

1. Choose S_{max}, N, M, θ
2. $h = S_{max}/N; \quad \tau = T/M$
3. $S = (0, h, 2h, ...(N-1)h, Nh)$
4. $V^0 = F - S$
5. for $m = 1 : M$
   (a) $rhs = D_2 V^{m-1}$
   (b) Solve $D_1 V^m = rhs$ for V_m
   (c) if, for each $n = 0, 2, ...N$, $F - nh > V_n^m$, $V_n^m = F - nh$
6. end for

Once the above algorithm is complete, the vector V^M will represent the initial option prices for values of S (again, in the algorithm time is "time until expiry"). While many of the parameters used in this problem represent the hypothetical example of the farmer without much creativity, volatility is especially difficult to estimate. For this reason, various different volatilities were tested, and the resulting impact on price and price as an initial percentage of underlying spot were plotted in figures (3.1) and (3.2).

Immediately obvious from the graphs is the effect of volatility. While annual volatility has less of an effect on the overall option price, it can greatly effect price (or excess price in the case of in-the-money options) as a percentage of initial basis. As volatility increases, the option price makes up a larger and larger percentage of the basis, especially as the basis moves to zero from either side. As the futures contract, in this example, only has 20 days until it must be delivery, it is reasonable to expect that the basis would be relatively small. Thus, S_t should be close to F on the first delivery day, which implies

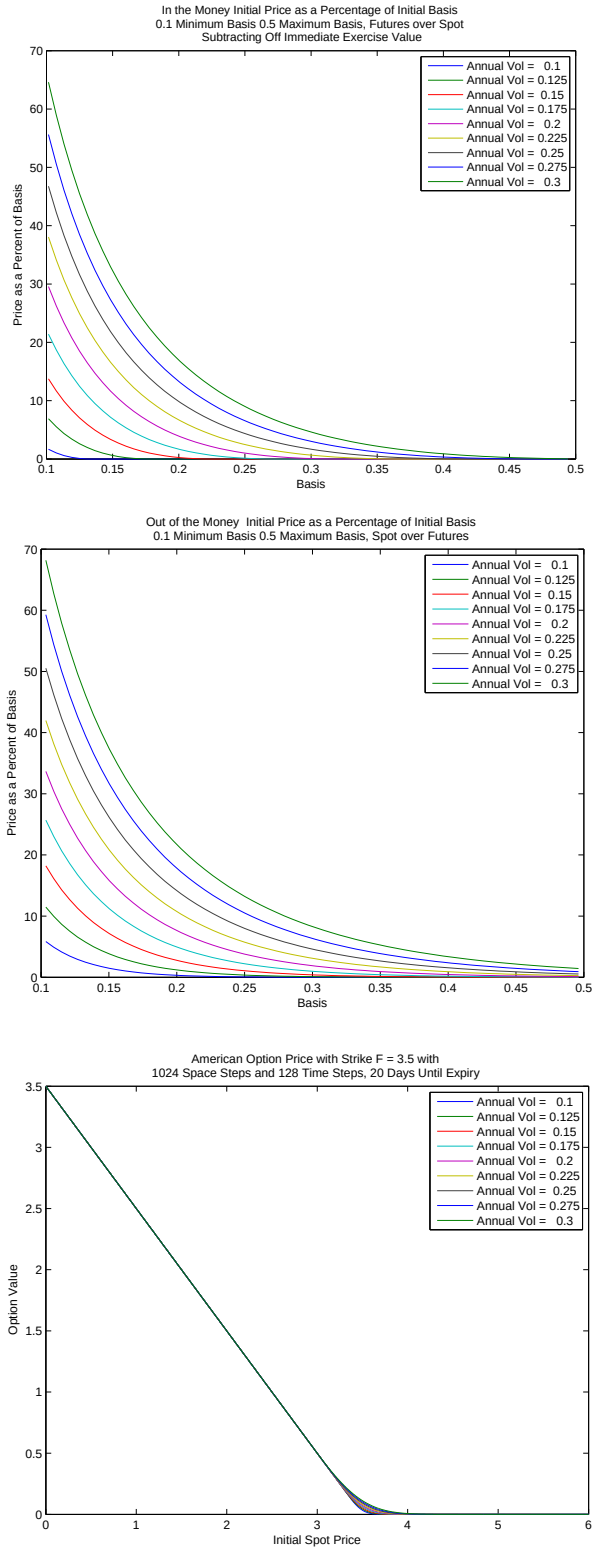
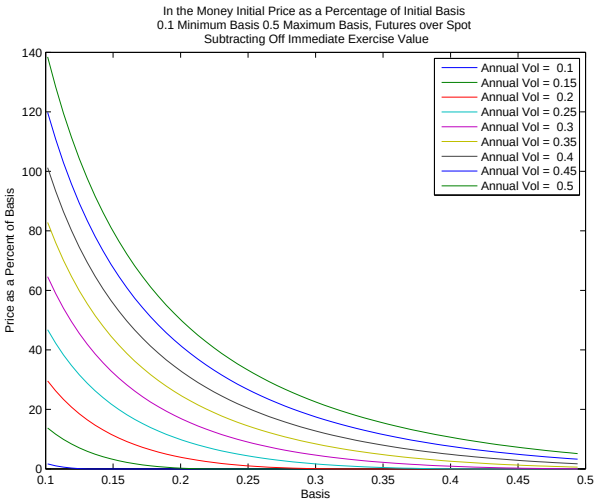
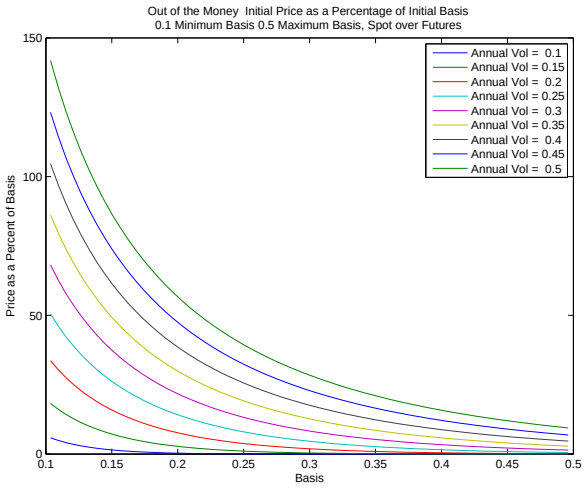


Figure 3.1: Volatility Range: 10-30%





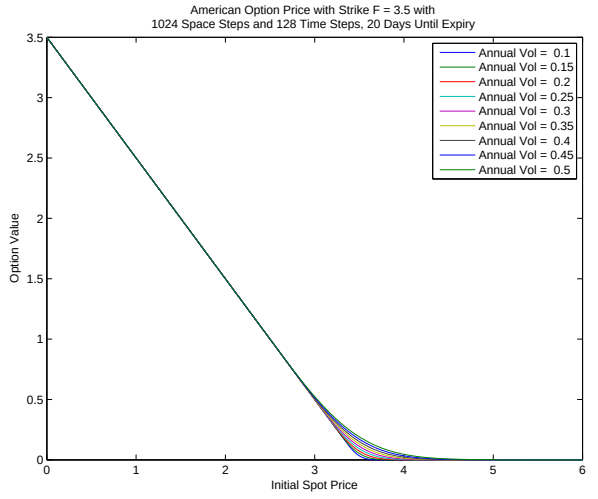


Figure 3.2: Volatility Range: 10-50%

that, especially in situations of higher volatility, the timing option can account for a significant portion of the basis. While it might not be possible to determine exactly what volatility parameter to use, the above graphs show for a wide range of volatilities, the timing option has large value as a percentage of the basis. This analysis is in keeping with and extends the results found by Hranaiova et al. who found that the timing option accounts for 5-28% of initial basis [5].

3.2 Joint Option: Timing and Location

The finite difference scheme used in the previous section to value the timing option can also be expanded to value the joint option. With the added location option, the farmer can choose to exercise his American timing option by selling at different locations, each with spot price $S_{i,t}$. Instead of one underlying Geometric Brownian Motion, assume there are two correlated assets, with the SDEs:

$$dS_{1,t} = \mu_1 S_{1,t} dt + \sigma_1 S_{1,t} dW_t^{(1)}$$

$$dS_{2,t} = \mu_2 S_{2,t} dt + \sigma_2 S_{2,t} dW_t^{(2)} \tag{3.2.1}$$

The correlation between the Brownian motions is ρ, ie. $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$. Since the option price V is now a function of three variables, $V = V(t, S_{1,t}, S_{2,t})$, the Black-Scholes type PDE becomes (with time t again defined at time until expiry):

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} - \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} - \frac{1}{2}\sigma^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} - r\frac{\partial V}{\partial S_1} - r\frac{\partial V}{\partial S_2} + rV = 0 \tag{3.2.2}$$

Equation (3.2.2) can easily be generalized n-asset case. Given the SDEs:

$$dS_{i,t} = \mu_i S_{i,t} dt + \sigma_i S_{i,t} dW^{(i)}, \quad i = 1, ...n \tag{3.2.3}$$

with correlation ρ_{ij}, ie $\mathbb{E}[dW^{(i)} dW^{(j)}] = \rho_{ij} dt$. The n-dimensional Black-Scholes PDE will be:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sum_{i,j=1}^{n} \rho\sigma_1\sigma_2 S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} - \sum_{i=1}^{n} r S_i \frac{\partial V}{\partial S_i} - rV = 0 \tag{3.2.4}$$

3.3 Theta-Scheme with two Correlated Assets

Returning to equation (3.2.2), the new complexity in deriving the analogous theta scheme to the one-dimensional case arises from the mixed derivative. However, this can be approximating by taking V^m, for $m = 1.....M$ as a matrix of values, as opposed to a vector, and using the surrounding points in both spatial dimensions. The exercise function at each time step, G^m will simply change to $G^m = (F - \min(S_1^m, S_2^m)), \forall S_1, S_2$. As in the one-dimensional case, at each time step the continuation value is computed first, and then the value assigned to the node is the maximum of the continuation value and the exercise value. The theta-scheme can be implemented as follows:

First, equation (3.2.2) can be represented as:

$$V_{j,k}^{m-1} - V_{j,k}^m = \tau \left[\theta \mathcal{L} V_{j,k}^{m-1} + (1-\theta)\mathcal{L} V_{j,k}^m\right] \tag{3.3.1}$$

with τ again defined as T/M, and $S_{1,j} = jh_1$, $S_{2,k} = kh_2$, and $h1 = S_{1,max}/N_1$, $h2 = S_{2,max}/N_2$. In this formulation, $\theta = 1$ returns the fully explicit scheme, and $\theta = 0$ the fully implicit. As in the one dimensional case, setting $\theta = .5$ gives the Crank-Nicolson scheme. The operator $\mathcal{L}$ is defined as:

$$\mathcal{L} V_{j,k}^m = (S_{1,terms})^m + (S_{2,terms})^m + (S_{1,2,terms})^m \tag{3.3.2}$$

with:

$$(S_{1,terms})^m = \frac{1}{2} S_{1,j}^2 \sigma_1^2 \left[\frac{V_{j+1,k}^m - 2V_{j,k}^m + V_{j-1,k}^m}{h_1^2}\right] + r S_{1,j} \left[\frac{V_{j+1,k}^m - V_{j-1,k}^m}{2h_1}\right] - r V_{j,k}^m$$

$$(S_{2,terms})^m = \frac{1}{2} S_{2,k}^2 \sigma_2^2 \left[\frac{V_{j,k+1}^m - 2V_{j,k}^m + V_{j,k-1}^m}{h_2^2}\right] + r S_{2,k} \left[\frac{V_{j,k+1}^m - V_{j,k-1}^m}{2h_2}\right]$$

$$(S_{1,2,terms})^m = \rho \sigma_1 \sigma_2 S_{1,j} S_{2,k} \left[\frac{V_{j+1,k+1}^m - V_{j-1,k+1}^m - V_{j+1,k-1}^m - V_{j-1,k-1}}{4h_1 h_2}\right] \tag{3.3.3}$$

Note that in the above discretization equations the probability measure is changed from the physical measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$, and thus the drift rates on both assets is taken to be r, as

is done with the Black-Scholes type PDE (3.2.2). Thus, the drift rates on the individual assets do not factor into the discretization equations. For the boundary conditions, obviously at $S_1 = 0$ or $S_2 = 0$, the option will be exercised as this is the maximum payoff, and $V = F$. For $S_{1,max}$ or $S_{2,max}$, it is unlikely that the asset with the maximum spot price will ever become the minimum of the two, and thus the payoff is just a function of the other asset. As a proxy for this value, the Black-Scholes put formula is used. Although this is for European options, given the short time until expiry, it is a fairly decent proxy, especially at boundary values which have a smaller impact on the more realistic prices of interest.

The discretization equations can also be written in stencil form as:

$$[a]_{j,k}[V^{m-1}]_{j,k} = [A]_{j,k}[V^m]_{j,k} \tag{3.3.4}$$

where:

$$[a]_{j,k} = \begin{bmatrix} a^{NW}_{j,k} & a^{N}_{j,k} & a^{NE}_{j,k} \\ a^{W}_{j,k} & a^{C}_{j,k} & a^{E}_{j,k} \\ a^{SW}_{j,k} & a^{S}_{j,k} & a^{SE}_{j,k} \end{bmatrix}$$

$$[A]_{j,k} = \begin{bmatrix} A^{NW}_{j,k} & A^{N}_{j,k} & A^{NE}_{j,k} \\ A^{W}_{j,k} & A^{C}_{j,k} & A^{E}_{j,k} \\ A^{SW}_{j,k} & A^{S}_{j,k} & A^{SE}_{j,k} \end{bmatrix}$$

$$[V]_{j,k} = \begin{bmatrix} V_{j-1,k+1} & V_{j,k+1} & V_{j+1,k+1} \\ V_{j-1,k} & V_{j,k} & V_{j+1,k} \\ V_{j-1,k-1} & V_{j,k-1} & V_{j+1,k-1} \end{bmatrix}$$

Stencil multiplication will produce a scalar by multiplying element by element, and then summing. The coefficients for $[a]_{j,k}$ and $[A]_{j,k}$ are given by:

$$a^{C}_{j,k} = 1 + \theta\tau \left[S^2_{1,j}\sigma^2_1/(h^2_1) + S^2_{2,k}\sigma^2_2/(h^2_2) + r \right]$$

$$a^{N}_{j,k} = -\frac{1}{2}\theta\tau \left[S^2_{2,k}\sigma^2_2/(h^2_2) + rS_{2,k}/h_2 \right]$$

$$a^S_{j,k} = -\frac{1}{2}\theta\tau\left[S^2_{2,k}\sigma^2_2/(h^2_2) - rS_{2,k}/h_2\right]$$

$$a^E_{j,k} = -\frac{1}{2}\theta\tau\left[S^2_{1,j}\sigma^2_1/(h^2_1) + rS_{1,j}/h_1\right]$$

$$a^W_{j,k} = -\frac{1}{2}\theta\tau\left[S^2_{1,j}\sigma^2_1/(h^2_1) - rS_{1,j}/h_1\right]$$

$$a^{NE}_{j,k} = -\theta\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

$$a^{NW}_{j,k} = \theta\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

$$a^{SW}_{j,k} = -\theta\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

$$a^{SE}_{j,k} = \theta\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

and:

$$A^C_{j,k} = 1 - (1-\theta)\tau\left[S^2_{1,j}\sigma^2_1/(h^2_1) + S^2_{2,k}\sigma^2_2/(h^2_2) + r\right]$$

$$A^N_{j,k} = \frac{1}{2}(1-\theta)\tau\left[S^2_{2,k}\sigma^2_2/(h^2_2) + rS_{2,k}/h_2\right]$$

$$A^S_{j,k} = \frac{1}{2}(1-\theta)\tau\left[S^2_{2,k}\sigma^2_2/(h^2_2) - rS_{2,k}/h_2\right]$$

$$A^E_{j,k} = \frac{1}{2}(1-\theta)\tau\left[S^2_{1,j}\sigma^2_1/(h^2_1) + rS_{1,j}/h_1\right]$$

$$A^W_{j,k} = \frac{1}{2}(1-\theta)\tau\left[S^2_{1,j}\sigma^2_1/(h^2_1) - rS_{1,j}/h_1\right]$$

$$A^{NE}_{j,k} = (1-\theta)\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

$$A^{NW}_{j,k} = -(1-\theta)\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

$$A^{SW}_{j,k} = (1-\theta)\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

$$A^{SE}_{j,k} = -(1-\theta)\tau\left[\rho\sigma_1\sigma_2 S_{1,j}S_{2,k}/(4h_1h_2)\right]$$

This numerical scheme for two assets was then implemented to value the joint option of the American timing feature, and the ability to sell for the minimum of two available prices. The following parameters

were used:

Table 3.2: Tow Asset Theta-Scheme Parameters

Specification	Value
S_1 Spatial Steps	64
S_2 Spatial Steps	64
Temporal Steps	32
Theta	0.5 (Crank-Nicolson)
Interest Rate	.05
S_1 Volatility	.25
S_2 Volatility	.3
Correlation ρ	.95
Days until Expiry w	20
Business Days in a Year	252
Futures Price	3.5

The number of grid points chosen for both spatial and temporal steps is smaller than in the one-dimensional case. This is due to the higher degree of complexity of this scheme, which leads to large instabilities if the space steps $h_{1,2}$ and time step τ are chosen to be too small. The initial price V is now given in three dimensions, as V is a matrix with each point on the gird corresponding to distinct values of S_1 and S_2.

As with the one-dimensional case, the price of the option is of less concern than analyzing the price as a percentage of initial basis, as it is mostly either zero or equal to the exercise payoff for with the minimum of the two prices. In the above example and returning to the case of the hypothetical farmer mentioned in previous chapters, let S_1 be the spot price in the farmer's local market, and S_2 the spot price in one of the delivery markets. Thus, it is reasonable to assume that S_2 would be very close to F, as there are only 20 days left until expiration. Consider the plots in figure (3.4), which show the price as a percentage of basis $F - S_1$, where $S_2 = F$.

Clearly, the price of the option can account for a very significant portion of the basis, and in the case of out of the money initial local spot S_1, the price is many times the basis. This analysis can also be extended to include costs of transport to the delivery location with spot price S_2, which would make the farmer more inclined to deliver in his home market. The payoff for exercising, with delivery costs K per bushel is now $G^m = (F - min(S_1^m, S_2^m - K))$. The corresponding price and basis graphs change slightly with $K = \$0.10$ per bushel, as is shown in figure (3.5).

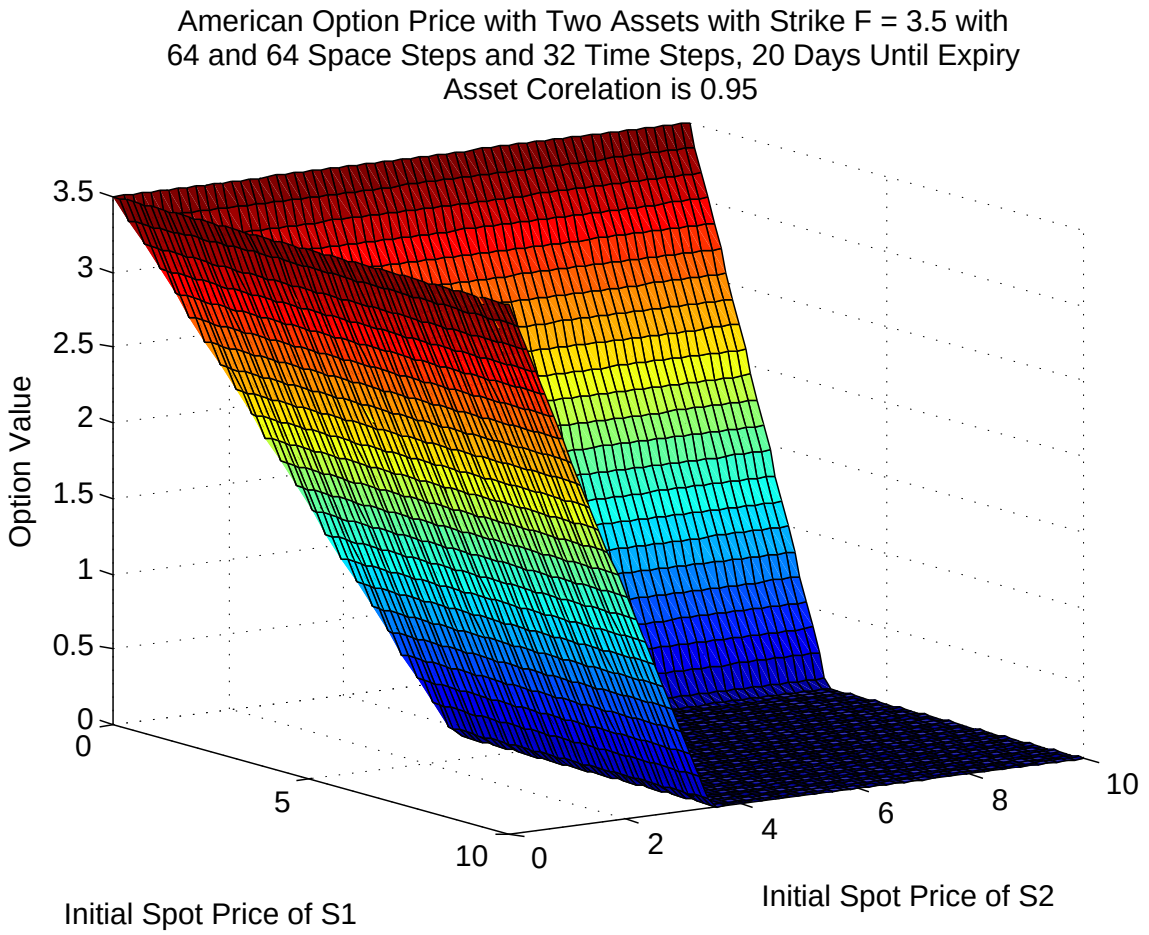


Figure 3.3: Price of Joint Option

3.4 A Model with Stochastic Volatility

The θ-scheme for the two-dimensional case is actually quite analogous to incorporating stochastic volatility for the one-dimensional case. Assume there is one-asset with stochastic variance v_t, which is of the form described in the Heston Model [8]:

$$dS_t = \mu S_t dt + \sqrt{v_t} dW_t^1$$

$$dv_t = \alpha(\beta - v_t)dt + \gamma\sqrt{v_t} dW_t^2 \tag{3.4.1}$$

where γ is the volatility of the variance process v_t and $\alpha, \beta > 0$ determine the rate of mean reversion.

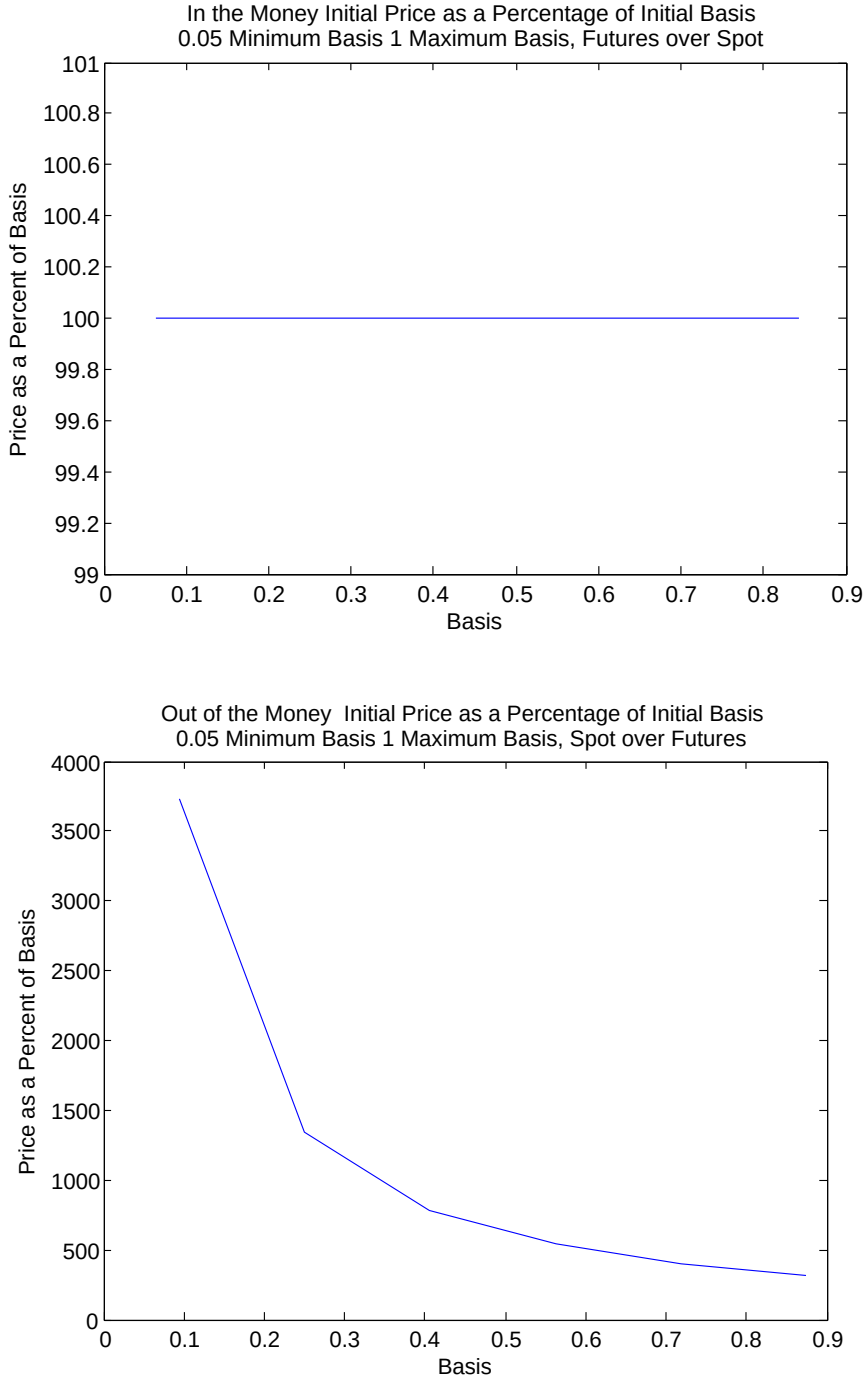


Figure 3.4: Price of Joint Option as a Percentage of Local Basis

As in the two dimensional case, the correlation between the Brownian motions is ρ. Using Itô's formula, the resulting Black-Scholes-type PDE becomes:

$$\frac{\partial V}{\partial t} - \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} - \rho\gamma Sv\frac{\partial^2 V}{\partial S\partial v} - \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2} - r\frac{\partial V}{\partial S} - \alpha(\beta - v)\frac{\partial V}{\partial v} + rV = 0 \qquad (3.4.2)$$

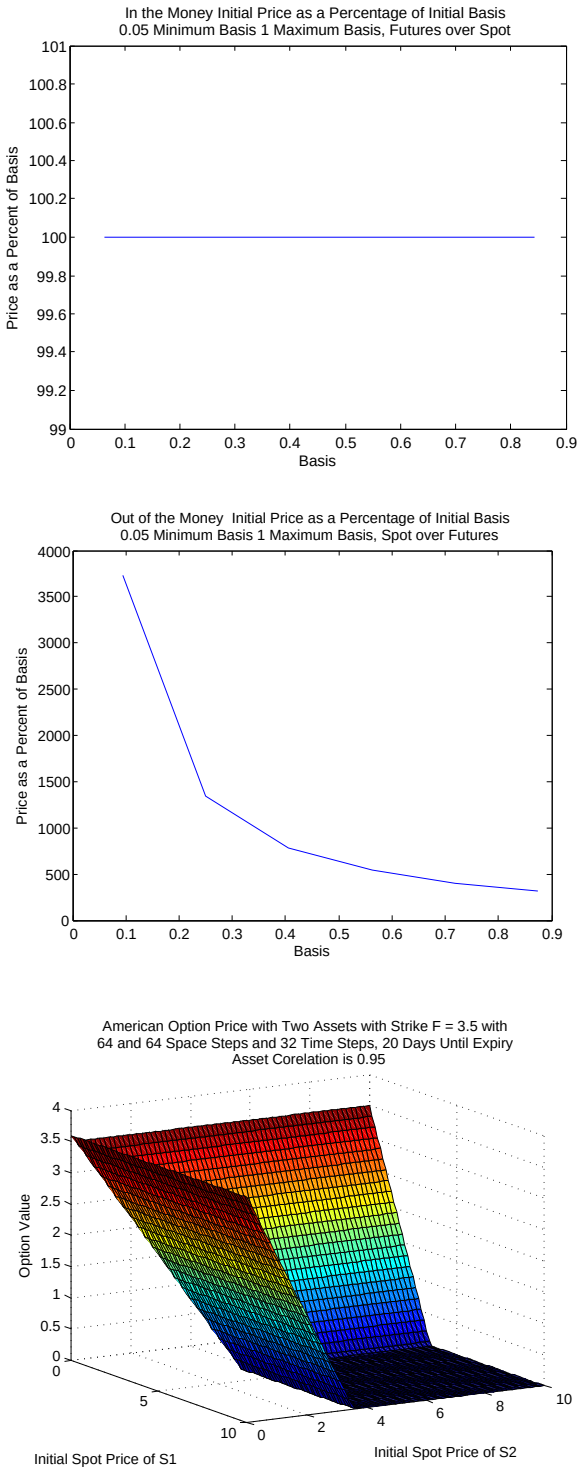


Figure 3.5: Joint Option with Transportation Costs

The payoff function is the same as in the one-dimensional case with deterministic volatility, and discretization equations are changed slightly:

$$\mathcal{L}V_{j,k}^m = (S_{terms})^m + (v_{terms})^m + (Sv_{terms})^m \tag{3.4.3}$$

with:

$$(S_{terms})^m = \frac{1}{2}S_j^2 v_k^2 \left[\frac{V_{j+1,k}^m - 2V_{j,k}^m + V_{j-1,k}^m}{h_1^2}\right] + rS_j \left[\frac{V_{j+1,k}^m - V_{j-1,k}^m}{2h_1}\right] - rV_{j,k}^m$$

$$(v_{terms})^m = \frac{1}{2}\gamma^2 v_k \left[\frac{V_{j,k+1}^m - 2V_{j,k}^m + V_{j,k-1}^m}{h_2^2}\right] + \alpha(\beta - v_k)\left[\frac{V_{j,k+1}^m - V_{j,k-1}^m}{2h_2}\right]$$

$$(Sv_{terms})^m = \rho\gamma S_j v_k \left[\frac{V_{j+1,k+1}^m - V_{j-1,k+1}^m - V_{j+1,k-1}^m - V_{j-1,k-1}}{4h_1 h_2}\right] \tag{3.4.4}$$

The equations used in the stencil notation will also change slightly:

$$a_{j,k}^C = 1 + \theta\tau\left[S_j^2 v_k/(h_1^2) + v_k\gamma^2/(h_2^2) + r\right]$$

$$a_{j,k}^N = -\frac{1}{2}\theta\tau\left[v_k\gamma^2/(h_2^2) + \alpha(\beta - v_k)/h_2\right]$$

$$a_{j,k}^S = -\frac{1}{2}\theta\tau\left[v_k\gamma^2/(h_2^2) - \alpha(\beta - v_k)/h_2\right]$$

$$a_{j,k}^E = -\frac{1}{2}\theta\tau\left[S_j^2 v_k/(h_1^2) + rS_j/h_1\right]$$

$$a_{j,k}^W = -\frac{1}{2}\theta\tau\left[S_j^2 v_k/(h_1^2) - rS_j/h_1\right]$$

$$a_{j,k}^{NE} = -\theta\tau\left[\rho v_k\gamma S_j/(4h_1 h_2)\right]$$

$$a_{j,k}^{NW} = \theta\tau\left[\rho v_k\gamma S_j/(4h_1 h_2)\right]$$

$$a_{j,k}^{SW} = -\theta\tau\left[\rho v_k\gamma S_j/(4h_1 h_2)\right]$$

$$a_{j,k}^{SE} = \theta\tau\left[\rho v_k\gamma S_j/(4h_1 h_2)\right]$$

and:

$$A^C_{j,k} = 1 - (1-\theta)\tau\left[S_j^2 v_k/(h_1^2) + v_k^2\gamma^2/(h_2^2) + r\right]$$

$$A^N_{j,k} = \frac{1}{2}(1-\theta)\tau\left[v_k\gamma^2/(h_2^2) + \alpha(\beta - v_k)/h_2\right]$$

$$A^S_{j,k} = \frac{1}{2}(1-\theta)\tau\left[v_k\gamma^2/(h_2^2) - \alpha(\beta - v_k)/h_2\right]$$

$$A^E_{j,k} = \frac{1}{2}(1-\theta)\tau\left[S_j^2 v_k/(h_1^2) + rS_j/h_1\right]$$

$$A^W_{j,k} = \frac{1}{2}(1-\theta)\tau\left[S_j^2 v_k/(h_1^2) - rS_j/h_1\right]$$

$$A^{NE}_{j,k} = (1-\theta)\tau\left[\rho v_k\gamma S_j/(4h_1h_2)\right]$$

$$A^{NW}_{j,k} = -(1-\theta)\tau\left[\rho v_k\gamma S_j/(4h_1h_2)\right]$$

$$A^{SW}_{j,k} = (1-\theta)\tau\left[\rho v_k\gamma S_j/(4h_1h_2)\right]$$

$$A^{SE}_{j,k} = -(1-\theta)\tau\left[\rho v_k\gamma S_j/(4h_1h_2)\right]$$

The difficulties in implementing this stochastic volatility of the Heston type are in estimating the parameters α, β, γ, and in determining the boundary conditions. The boundary $S = 0$ is easily determined for V as the option would cleary be exercised and thus the value set to F. Also, at $v = 0$, the PDE becomes hyperbolic:

$$-\frac{\partial V}{\partial t} = rS\frac{\partial V}{\partial S} + \alpha\beta\frac{\partial V}{\partial v} - rV \tag{3.4.5}$$

which can be approximated by one-sided finite differences. For the other two boundary conditions of the grid V, Reisinger [8] suggests that at $S = S_{max}$,

$$\frac{\partial^2 V}{\partial S^2}(S_{max}, v, t) = 0 \quad \forall v \in [0, v_{max}]$$

For $v = v_{max}$, Reisinger also suggests that the first derivative of V w.r.t v disappears as high volatility prices have limited influence on mean volatility prices:

$$\frac{\partial V}{\partial v}(S, v_{max}, t) = 0 \quad \forall S \in [0, S_{max}]$$

3.5 Exercise Value and Implicit Finite Difference Schemes

In the finite difference schemes discussed in this chapter, exercise and continuation value were compared at each node, with the maximum of the two being taken as the ultimate value for the node. In an explicit scheme, this procedure poses no problems. However, in a scheme involving implicit methods (such as the Crank-Nicolson scheme), this procedure is not actually correct. In an explicit method with exercise payoff G_n at the node corresponding to S_n, the value on that node is set to:

$$V_n^m = \begin{cases} \hat{V}_n^m & \text{if } \hat{V}_n^m \geq G_n \\ G_n & \text{if } \hat{V}_n^m < G_n \end{cases} \tag{3.5.1}$$

where $\hat{V}_n^m$ is the continuation value:

$$\hat{V}_n^m = A_n^m V_{n-1}^{m-1} + B_n^m V_n^{m-1} + C_n^m V_{n+1}^{m-1} \tag{3.5.2}$$

In order to understand why (3.5.1) is valid for the explicit method but not the implicit, consider the *linear complementarity problem* which uniquely determines the price of an American option [8]:

$$\mathcal{L}V \geq 0$$

$$V \geq G$$

$$\mathcal{L}V \cdot (V - G) = 0 \tag{3.5.3}$$

where $\mathcal{L}$ is the Black-Scholes operator with t as time until expiry:

$$\mathcal{L}V = \frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV \geq 0 \tag{3.5.4}$$

The inequalities in equation (3.5.3) arise from three main properties of American options. The first

comes from the fact that the risk free rate is an upper bound on the return of a fully hedged portfolio in a risk-neutral world. The second inequality is a result of the ability to exercise at any time, and thus the value of the option can never be less than its immediate payoff. The third inequality arises from the fact that any time t the option holder can either exercise or continue to hold, and thus his decision is either to take the immediate payoff or hedge the option with respect to time (Δ-hedging):

$$V = G \quad \vee \quad \frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \tag{3.5.5}$$

With the explicit method, it is easily verified that equation (3.5.1) satisfies the inequalities in equation (3.5.3). Clearly the value on each node is greater than or equal to the payoff, and the discretization method ensures that (at least) either the backwards Black-Scholes operator $\mathcal{L}V = 0$, or the value equals the payoff ($V = G$). These arguments cover the second two inequalities. For the first equality, it can easily be verified that the resetting the value on the node to the exercise value does not violate the requirement that $\mathcal{L}V \geq 0$, as the payoff will be greater than or equal to the continuation value:

$$V_n^m \geq \hat{V}_n^m = A_n^m V_{n-1}^{m-1} + B_n^m V_n^{m-1} + C_n^m V_{n+1}^{m-1} \tag{3.5.6}$$

This inequality points to why continuation value and exercise value cannot be compared at each time step as in equation (3.5.1) when using an implicit method. For an implicit method, values of each node are calculated alongside values in the same timestep:

$$a_n^m \hat{V}_{n-1}^m + b_n^m \hat{V}_n^m + c_n^m \hat{V}_{n+1}^m = A_n^m V_{n-1}^{m-1} + B_n^m V_n^{m-1} + C_n^m V_{n+1}^{m-1} \tag{3.5.7}$$

Therefore, it is not possible to prove that resetting the options value to the exercise value if it is greater than the continuation value will satisfy the first inequality in equation (3.5.3). In other words, it is **not** possible to write a statement such as:

$$V_n^m \geq a_n^m \hat{V}_{n-1}^m + b_n^m \hat{V}_n^m + c_n^m \hat{V}_{n+1}^m = A_n^m V_{n-1}^{m-1} + B_n^m V_n^{m-1} + C_n^m V_{n+1}^{m-1}$$

Thus, with a implicit method, setting the value on a node is not consistent with the linear com-

plementarity problem, and it can be proved that if such a process is employed arbitrage will exist in the computed option price. [8] Various other resetting methods exist for American options valued using implicit methods, such as the projected Gauss-Seidel algorithm or Penalty methods. However, in practice, the numerical results obtain using one of these improved algorithms or (incorrectly) explicitly resetting are actually very similar. Nonetheless, a more complete implementation of the finite difference schemes discussed in the previous sections of this chapter would include projected Gauss-Seidel or Penalty methods. For further reference, see Reisinger. [8]

Chapter 4

Conclusions

4.1 Combining Modeling and Valuation

Chapter 3 has demonstrated that in both the simple case of the timing option with one-asset, as well as the more complicated joint option involving two assets, the-short dated options embedded in a futures contract have a quantifiable value. While it is difficult to give an exact value for these embedded options, chapter 3 shows that this value cannot be ignored when creating accurate models of futures markets. The next aspect in modeling is to reconcile this determined option value with the farmer's decision of how many futures contracts to buy as a percentage of his overall harvest discussed in chapter 2.

Assume that the farmer has observed that in his local market, the spot price is almost always less than the spot price in delivery markets, perhaps due to transportation, storage costs, demand, or other factors. How might this knowledge, combined with an estimate of value of an embedded options in a futures contract, affect his decision to buy futures? Denote b as the delivery-local basis (and hence futures-local basis under the assumption that delivery spot and futures price converge). Suppose the farmer uses the techniques described in chapter 3 to calculate the value of the embedded options in a futures contract at this level b. Denote by this value as δ, which is defined as a percentage of spot price at the basis b. Thus, as $b = F - S$, δ can be defined as: $\delta = V(F - b, t)/S$.

In the risk-neutral world discussed in chapters 1 and 2, the futures price is given as: $F = Se^{rt}$. Now, suppose that the futures price is written as:

$$F = Se^{rt} + EO = Se^{rt} + \delta S \tag{4.1.1}$$

where EO denotes the value of the embedded option. This can be rewritten as:

$$F = S(e^{rt} + \delta) = Se^{yt} \tag{4.1.2}$$

where y is a new rate for which there is an explicit solution in terms of r, t, δ:

$$y = \frac{1}{t}\ln(e^{rt} + \delta) \tag{4.1.3}$$

This new rate y is thus a function of spot price S and time until expiration t: $y = y(S, t)$. Substituting $y(S, t)$ for r into the wealth equation (2.1.2), will yield:

$$dH_t = H_t\left(y(S,t)(1-\pi) + \mu\pi\right)dt + H_t\pi\sigma dW_t \tag{4.1.4}$$

Due to the fact that y is a function of S and t, this introduces complexities in obtaining a solution. However, as was stated at the beginning of this section, the rational for the argument leading to equation (4.1.3) is that the farmer observes a "deterministic" local-delivery basis. Thus, y can be thought of as a deterministic function. Furthermore, while it is an oversimplification, if y is taken as a constant similar to r, with $y(S, t)$ replaced by y, equation (4.1.4) can be solved either via the martingale approach approach or dynamic programming approach as given in chapter 2. Obviously, y will not be constant for all t and δ, but if the farmer were to compare similar time frames or basis in the past, he might be able to determine a constant approximation for y. This would give the result for optimal crop allocation π_y^*:

$$\pi_y^* = \frac{\mu - y}{(1-\gamma)\sigma^2} \tag{4.1.5}$$

Since $y > r$ if the embedded options have positive value, it is clear from equation (4.1.5) that the risky allocation will be less when the value of embedded options is included in the futures price. Thus, a farmer with the same level of risk aversion γ would allocation more to futures when he takes considers the embedded options. Consider figure (4.1), which demonstrates how a farmer's unconstrained allocation

to futures would change with various parameter values given in equation (4.1.5), and under constant γ with small variations in time horizon.

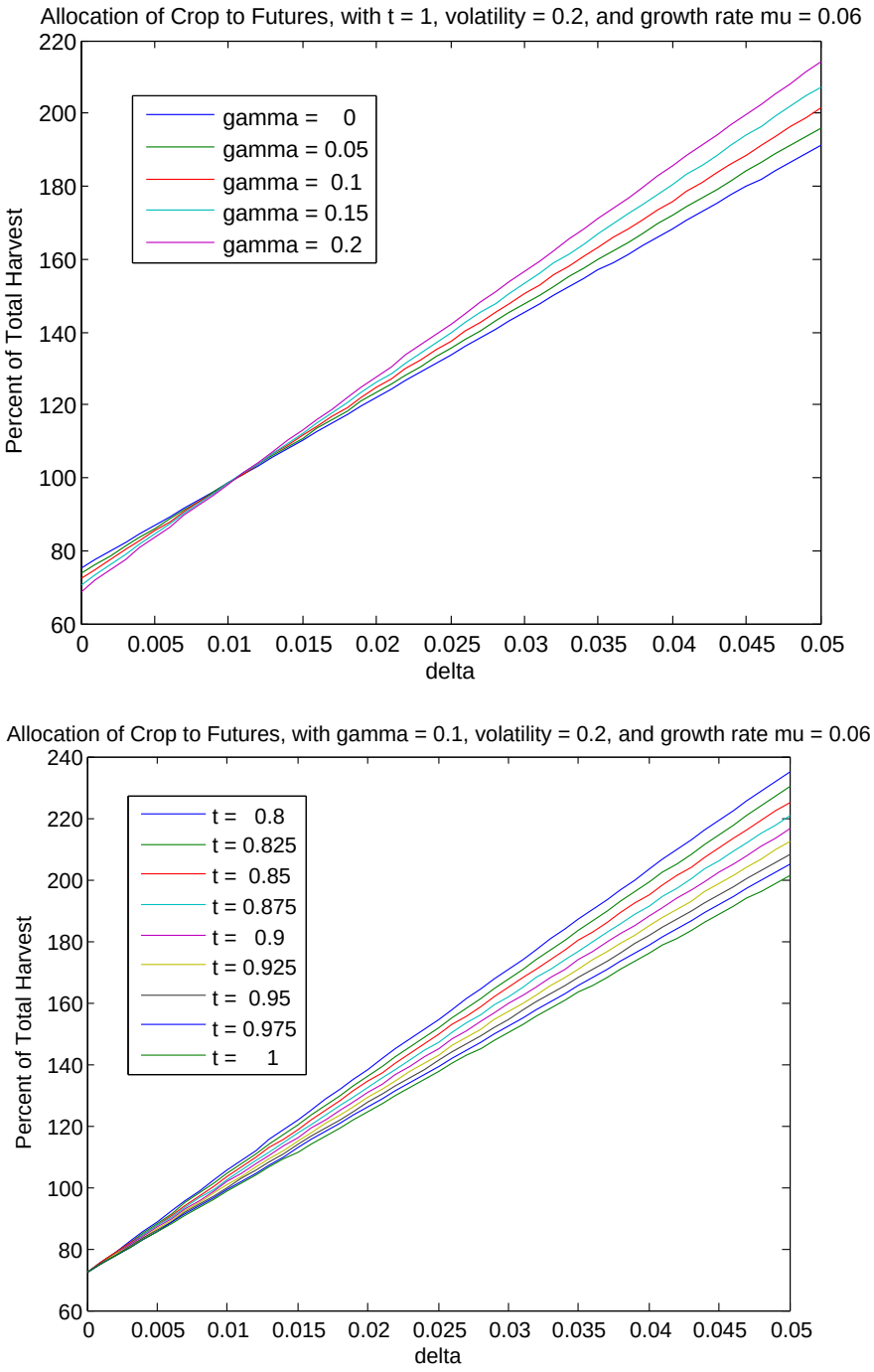


Figure 4.1: Allocation of Crop to Futures

4.2 Further Improvement

This paper provides a discussion of the commodity futures market, using primarily a hypothetical wheat farmer as an example. Suggestions are given as to what should be included in a model of a such a farmer shorting wheat futures. Specifically, the embedded options in a futures contract are investigated in further detail than is often done in academic work on this subject. Models are proposed to value the timing and joint in the one and two dimensional deterministic case, and in the one-dimensional case with stochastic volatility. Finally, suggestions are provided in the previous section on how to incorporate the value of the embedded options into the farmer's initial decision of how much of his crop to allocate to futures.

Further investigation in this area would more rigorously attempt to quantify the value of the embedded options in futures, and explore how much they impact the futures market using real world data. While this paper indicated that embedded options can make up a significant proportion of futures-cash basis, the next step in the investigation could be to compare these results from various finite difference approximations to actual data to explore observed basis in futures markets. Another area of improvement would be the the preceding section in this chapter. Rather than taking y as a constant and using existing solution techniques for the wealth equation described in chapter 2, a more thorough solution could be developed when y is taken as a function of S and t. More rigorous analytical treatment, combined with the comparison to actual data in order to accurately model the embedded options may lead to better policy rules for farmers buying futures, or other analogous market participants.

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