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**A THEORY OF RATIONAL DEMAND FOR INDEX  
INSURANCE**

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# A THEORY OF RATIONAL DEMAND FOR INDEX INSURANCE

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## Abstract

Rational demand for hedging products, where there is a risk of contractual nonperformance, is fundamentally different to that for indemnity insurance. In particular, optimal demand is zero for infinitely risk averse individuals, and is nonmonotonic in risk aversion, wealth and price. For commonly used families of utility functions, demand is hump-shaped in the degree of risk aversion when the price is actuarially unfair, first increasing then decreasing, and either decreasing or decreasing-increasing-decreasing in risk aversion when the price is actuarially favourable. For a given belief, upper bounds are derived for the optimal demand from risk averse and decreasing absolute risk averse decision makers. The apparently low level of demand for consumer hedging instruments, particularly from the most risk averse, is explained as a rational response to deadweight costs and the risk of contractual nonperformance. A numerical example is presented for maize in a developing county which suggests that some unsubsidised weather derivatives, currently being designed for and marketed to poor farmers, may in fact be poor products, in that objective financial advice would recommend low or zero purchase from all risk averse expected utility maximisers.

Keywords: Index insurance; Derivative; Basis risk; Hedge; Microinsurance.

JEL codes: D14, D81, G20, O16.

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### 1 INTRODUCTION

When should consumers use financial contracts to hedge against a potentially material loss, and when should they not? The question is not trivial to answer. Weighed against any benefit from hedging is both the deadweight cost of such a contract, typically passed on to the purchaser through an increased premium, and any risk that the net income will not accurately reflect an incurred loss. This risk of contractual nonperformance, or *basis risk* as it is referred to in derivative markets, renders the decision problem fundamentally different to an indemnity insurance purchase decision problem.

One particularly instructive hedging product is that of the weather derivative, which over the last ten years has begun to be sold by a variety of well-meaning institutions to poor farmers as *weather indexed insurance*. The rationale typically given is quite convincing: agriculture is an uncertain business, leaving households vulnerable to serious hardship (Dercon 2004, Collins et al. 2009), and traditional indemnity-based approaches to crop insurance were unsustainably expensive, plagued by moral hazard, adverse selection and high loss adjustment costs (Hazell 1992, Skees et al. 1999). By comparison, contracts conditional only on weather indices can be fairly cheap whilst still offering much needed protection against extreme weather events such as droughts (Hess et al. 2005).

The sale of weather derivatives to farmers in developing countries has led to careful empirical studies by academic economists who have in turn noted two puzzles. First, demand for such products has been lower than expected, prompting attention from empirical economists hoping to disentangle various candidate causes of low demand such as financial illiteracy, lack of trust, poor marketing, credit constraints, basis risk and price (Giné et al. 2008, Giné and Yang 2009, Cole et al. 2009, Cai et al. 2009). For example, Cole et al. (2009) report on a series of rainfall derivative trials in two Indian states in which only 5 – 10% of households in study areas purchased cover, despite rainfall being overwhelmingly cited as the most important risk faced. Moreover, the vast

majority of purchases were for one policy only, which the authors estimate would hedge only 2 – 5% of household agricultural income.

The second empirical puzzle is that demand seems to be particularly low from the most risk averse (Giné et al. 2008, Cole et al. 2009). Guided by intuitions from indemnity insurance, which carry over to the mean variance model of Giné et al. (2008), these and other authors attribute this result to ‘behavioural’ decision making or external constraints:

‘The most likely explanation [for demand falling with risk aversion] is that it is uncertainty about the product itself (Is it reliable? How fast are pay-outs? How great is basis risk?) that drives down demand.’  
(Karlan and Morduch 2009)

‘Poor farmers on the other hand are not sufficiently well insured and would benefit from purchase of insurance, but they are severely cash and credit constrained, so that they cannot translate potential demand into purchases.’ (Binswanger-Mkhize 2011)

This paper takes a different approach, arguing that the key to solving the puzzles is to note that the weather indexed insurance products currently being sold to farmers are derivatives, not indemnity insurance products.<sup>1</sup> Our model is one of rational demand, where the consumer is assumed to be a price taking risk averse expected utility maximiser with, for some results, decreasing absolute risk aversion (DARA).

One critical aspect of the model is the nature of the joint probability structure of the index insurance product and the consumer’s loss. The net transfer from index

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<sup>1</sup> Accountants have recently revisited the specific question of how to classify weather derivatives as part of the process of developing the International Financial Reporting Standards (International Accounting Standards Board 2007, pp.450-451). IFRS contains a principles-based distinction between an insurance contract ‘in which an adverse effect on the policyholder is a contractual precondition for payment’, and a derivative contract in which it is not. Under this definition, a weather derivative is a derivative, not any kind of insurance. A weather derivative is also classed as a derivative under US Generally Accepted Accounting Principles.

insurance is assumed be imperfectly correlated with the consumer's net loss, and so index insurance purchase both worsens the worst possible outcome and improves the best possible outcome; a consumer might incur a loss but receive no net income from the index insurance product or incur no loss but receive a positive net income. This model of basis risk is fundamentally different to the independent, additive, uninsurable background risk often considered by insurance theorists, under which purchase of indemnity insurance results in a contraction of net wealth in the sense of Rothschild and Stiglitz (1970).

The model is structurally similar to that of Doherty and Schlesinger's (1990) model of indemnity insurance with contractual exclusion clauses or a risk of insurer default; indeed theirs is a mathematical special case of ours in which the insurance premium is actuarially unfair and there is no risk of upside contractual nonperformance, where a claim payment is made even though no loss has been incurred. Whilst these authors interpreted the risk of contractual nonperformance as the perceived probability that an insured individual would not be indemnified against their loss due to insurer default or contractual exclusion clauses, we allow basis risk to act as another candidate cause.

Many of the results of Doherty and Schlesinger (1990) follow through to our model, namely that the risk of contractual nonperformance leads rational demand to be nonmonotonic in risk aversion, wealth and price. Demand for indexed insurance from infinitely risk averse, maximin decision makers, is shown to be zero and so demand cannot be everywhere increasing in risk aversion. For the classes of constant absolute and constant relative risk aversion we find demand for actuarially unfair indexed cover to be hump-shaped in the degree of risk aversion, first increasing then decreasing, and demand for actuarially favourable indexed cover to be either decreasing or decreasing-increasing-decreasing in risk aversion. Given these results it is perhaps not surprising that empirical economists have been finding a negative relationship between demand for weather derivatives and risk aversion for poor farmers in developing countries.

In the key theoretical contribution of this paper, we derive upper bounds for rational purchase of hedging instruments. For the case of indemnity insurance, that is insurance without basis risk, risk aversion and DARA alone cannot bound the purchase of indemnity insurance below full insurance; an infinitely risk averse individual would rationally purchase full insurance. However, tighter bounds may be derived for actuarially fair or unfair hedging products with basis risk, both under the restriction of risk aversion alone, and that of risk aversion and DARA. Loosely speaking the bound for DARA arises because an individual who cares enough about the risk to want to purchase a sizeable hedge must, under the assumption of DARA, care enough about the downside risk of contractual nonperformance and, for the case of an actuarially unfair price, the deadweight cost of hedging to limit the size of the hedge.

We close the paper by applying the framework to the numerical example of maize in a developing country. With a belief constructed from the empirical joint distribution function of yields and weather indexed claim payments we are able to characterise the level of optimal demand for the classes of risk averse, constant relative risk averse and decreasing absolute risk averse decision makers. In general we find low rational demand, and show that optimal demand from any risk averse expected utility maximiser is zero if the price for index insurance is more than 1.75 times the expected claim income. All pricing multiples calculated by Cole et al. (2009) for unsubsidised weather indexed insurance policies sold in India are greater or equal to 1.75, suggesting that the low observed demand for weather derivatives sold to poor farmers may be consistent with objective financial advice, rather than being a result of poor understanding, an unwillingness to experiment, or credit constraints.

The rest of the paper is organised as follows: Section 2 presents the model and characterises rational demand; Section 3 reports on a numerical analysis of rational demand for weather indexed insurance for a developing country; and Section 4 concludes.

**Table 1.** *Joint probability structure*

	<i>Index = 0</i>	<i>Index = I</i>	
<i>Loss = 0</i>	$1 - q - r$	$q + r - p$	$1 - p$
<i>Loss = L</i>	$r$	$p - r$	$p$
	$1 - q$	$q$	

## 2 THE MODEL

To capture the essence of rational purchase of index insurance consider the following model. A decision maker holds strictly risk averse preferences over wealth, with a von Neumann-Morgenstern utility function  $u$  satisfying  $u' > 0$  and  $u'' < 0$ .<sup>2</sup> The decision maker is endowed with constant wealth  $w$ , is exposed to uninsurable zero mean background risk  $\tilde{z}$  with bounded support, and suffers a loss which can take the value  $L$  with probability  $p$  or zero with probability  $1 - p$ . There is also an index which can take the value  $I$  with probability  $q$  or zero with probability  $1 - q$ . The index is not necessarily perfectly correlated with the loss and so there are four possible joint realisations of the index and loss.

The index and loss are jointly statistically independent of background risk  $\tilde{z}$  and so, as usual, we may define the indirect utility function  $v$  by

$$v(x) = \mathbb{E}u(x + \tilde{z}) \text{ for all } x \in \mathbb{R} \quad (1)$$

and reduce the problem to one with four states  $s \in S = \{00, 0I, L0, LI\}$ . Note that both risk aversion and decreasing absolute risk aversion of the direct utility function  $u$  are inherited by the indirect utility function  $v$ .<sup>3</sup>

For the purpose of talking about an increase or decrease in the risk of contractual nonperformance, or basis risk as it is more commonly known in the context of

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<sup>2</sup> This restriction on the utility function is equivalent to assuming that, endowed with certain wealth, such an individual would never accept a non-degenerate lottery if the expected net gain from the lottery was nonpositive (Pratt 1964, Arrow 1965).

<sup>3</sup> The optimal decision of agent  $u$  with background risk of  $\tilde{z}$  is the same as for agent  $v$  who faces no background risk.  $v^n(z) = \mathbb{E}u^n(x + \tilde{z})$  where  $u^n$  is the  $n$ th derivative of  $u$ , and so  $v$  inherits the properties  $v' > 0$  and  $v'' < 0$ . Moreover  $v$  inherits decreasing absolute risk aversion from  $u$  (Gollier 2001, p116).

derivatives, it is perhaps natural to consider changes in the joint distribution of loss and index whilst holding the marginal distributions fixed. In our four state model  $p$  and  $q$  fully specify the marginal distributions and with  $p$  and  $q$  fixed there is only degree of freedom in the joint probability distribution. We may therefore without loss of generality define basis risk parameter  $r$  as the joint probability that the index is 0 and the loss is  $L$ , and interpret an increase in  $r$ , without any change in  $p$  or  $q$ , as an increase in basis risk. Note that whilst this parameterisation effectively restricts a change in basis risk to be suitably symmetric, it does not restrict the level of basis risk; any four state joint probability distribution can be constructed by suitable choice of  $p$ ,  $q$  and  $r$ .

The parametrisation of basis risk will have a natural interpretation in our model:  $r$  is the probability that an individual will incur a loss but the index will be good. For example, a farmer could lose her entire crop due to pestilence or localised weather conditions, but receive no claim payment from a weather derivative due to good weather having been observed at the contractual weather station. Regardless of how clever the design, weather derivatives are not able to accurately capture perils such as insects or disease, or localised weather events that can occur on the farmer's land without being observed at the contractual weather station.

Denoting the probability of each state  $s$  by  $\pi_s$ , we therefore have  $\{\pi_{00}, \pi_{0I}, \pi_{L0}, \pi_{LI}\} = \{1 - q - r, q + r - p, r, p - r\}$  (see Table 1).<sup>4</sup> For an index realisation of  $I$  to be a signal that the loss has been  $L$ , we require that  $\frac{\pi_{LI}}{\pi_{0I}} > \frac{\pi_{L0}}{\pi_{00}}$ , that is  $r < p(1 - q)$ . We will also assume that basis risk  $r$  is strictly positive and that all states have nonnegative probability of occurrence, and so

$$0 < r < p(1 - q) \text{ and } p - q \leq r. \quad (2)$$

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<sup>4</sup> Other specifications of basis risk are possible. One alternative specification would be that the probability of a loss was  $p$  and that the probability that the index was 'wrong', that is  $\frac{\pi_{L0}}{\pi_{L0} + \pi_{LI}} = \frac{\pi_{0I}}{\pi_{0I} + \pi_{00}}$ , was some  $\rho$ . However, under this specification  $\mathbb{P}[\text{Index} = I] = p + (1 - 2p)\rho$  and so for  $p \neq 1/2$  a change in basis risk parameter  $\rho$  would change the marginal distribution of the index in addition to increasing basis risk.

**Table 2.** *Four state framework*

State $s$	$L0$	$LI$	$00$	$0I$
Probability $\pi_s$	$r$	$p - r$	$1 - q - r$	$q + r - p$
Wealth, no indexed cover	$w - L$	$w - L$	$w$	$w$
Wealth, indexed cover of $\alpha L$	$w - P - L$	$w - P - L + \alpha L$	$w - P$	$w - P + \alpha L$

The loss is observable to the individual but not to the insurer, and so there is no market for indemnity insurance, with claim payment conditional only on the loss. However, the individual can purchase an indexed security that pays a proportion  $\alpha \geq 0$  of the potential loss  $L$  when the index realisation is  $I$ . Loss and index are imperfectly correlated and so the indexed security only offers a hedge against the potential loss; indexed insurance does not pay a claim in the state  $L0$ . The indexed product is priced with a *multiple* of  $m > 0$ ; that is the individual pays a premium of  $P = qm\alpha L$  for cover of  $\alpha L$ , receiving a claim payment of  $\alpha L$  if the index realisation is  $I$  or zero if the index is 0.<sup>5</sup> For  $m > 1$ ,  $m = 1$ ,  $m \geq 1$  and  $m < 1$  the premium will be said to be actuarially unfair, actuarially fair and actuarially favourable, respectively. We will ignore the case in which  $qm \geq 1$  for which zero indexed coverage is trivially optimal.

The individual's objective is therefore to choose a level of indexed coverage  $\alpha \geq 0$  to maximise expected indirect utility:

$$\begin{aligned} \mathbb{E}V = & (p - r)v(w - \alpha qmL - (1 - \alpha)L) + (q + r - p)v(w - \alpha qmL + \alpha L) \\ & + (1 - q - r)v(w - \alpha qmL) + rv(w - \alpha qmL - L) \end{aligned} \quad (3)$$

The first-order condition for an interior solution to this program, after cancelling  $Lq(1 - qm) > 0$ , is:

$$Av'_{LI} + (1 - A)v'_{0I} - B \times [Cv'_{00} + (1 - C)v'_{L0}] = 0, \quad (4)$$

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<sup>5</sup> Using the standard insurance terminology, the corresponding *loading* would be  $m - 1$ .

where for the sake of notational convenience,

$$A = \frac{p-r}{q}, \quad B = \frac{m-qm}{1-qm}, \quad C = \frac{1-q-r}{1-q}, \quad (5)$$

and  $v'_s$  denotes marginal indirect utility in state  $s$ .  $\mathbb{E}V$  is concave in  $\alpha$  and so the second-order condition is trivially satisfied. The parameter restrictions of equation (2) ensure that  $p < A < 1$  and  $1-p < C < 1$  and therefore that  $A + C > 1$ .  $B > 0$ , with  $B \gtrless 1$  corresponding to  $m \gtrless 1$ .

We begin by stating a basic result, for which we claim no novelty.

**Proposition 1.** *If the premium is actuarially unfair then full indexed coverage is never optimal,  $\alpha^* < 1$ . If the premium is actuarially fair then positive, partial indexed coverage is always optimal,  $0 < \alpha^* < 1$ . If the premium is actuarially favourable then positive indexed coverage is always optimal,  $\alpha^* > 0$ .*

*Proof.* If  $\alpha \geq 1$  and  $m \geq 1$  we have  $B \geq 1$ ,  $v'_{00} \geq \max(v'_{LI}, v'_{0I})$  and  $v'_{L0} > \max(v'_{LI}, v'_{0I})$  and so the LHS of (4) is strictly negative, violating first-order condition (4). If  $\alpha \leq 0$  and  $m \leq 1$  the LHS of (4) is strictly positive since  $B \leq 1$  and  $A + C > 1$ , violating first-order condition (4).  $\square$

In the model with  $r > 0$  it is impossible to eliminate all uncertainty, due to the existence of an uninsurable basis risk. This may be compared to the well known results that for actuarially fair indemnity insurance, that is when  $m = 1$ ,  $p = q$  and  $r = 0$ , full insurance is optimal (Mossin 1968, Smith 1968), and remains optimal on addition of an independent background risk.

For  $0 \leq \alpha < 1$ , objective function (3) has strictly decreasing differences in  $(\alpha; r)$ , that is  $\frac{\partial^2 \mathbb{E}V}{\partial \alpha \partial r} < 0$ , yielding the following proposition:

**Proposition 2.** *The optimal level of coverage  $\alpha^*$  is decreasing in basis risk  $r$ , and strictly decreasing if  $\alpha^* > 0$ .*

Whilst it is somewhat intuitive that risk aversion is sufficient to ensure that an increase in basis risk reduces demand, it bears mentioning that such a result doesn't hold for Doherty and Schlesinger's (1990) model of one-sided basis risk. In their model demand is nonmonotonic in the probability of insurer default, except for the special cases of constant absolute risk aversion and quadratic utility. Whilst in both models an increase in basis risk reduces the correlation between the available cover and the loss, in our model it does so in a symmetric fashion, resulting in a mean preserving spread of wealth for any fixed  $\alpha$  (Rothschild and Stiglitz 1970); the movements of probability mass from inner state  $00$  to outer state  $L0$  and from inner state  $LI$  to outer state  $0I$  each reduce the incentive to purchase indexed cover without changing the insurance premium. By comparison, in Doherty and Schlesinger's (1990) model, an increase in basis risk moves probability mass from inner state  $LI$  to outer state  $L0$  and decreases the insurance premium, the combined effect of which is ambiguous.

### 2.1 Risk aversion

To appreciate the effect of basis risk on optimal demand for different levels of risk aversion, first consider extreme risk aversion. Ignoring background risk  $\tilde{z}$ , an infinitely risk averse, maximin, individual's objective is to maximise the lowest possible wealth realisation, and therefore full purchase of indemnity insurance ( $r = 0, p = q$ ) is optimal so long as the net transfer to the individual is positive in the loss states ( $qm < 1$ ); full insurance purchase increases the lowest possible wealth from  $w - L$  to  $w - qmL$ . In stark contrast, an infinitely risk averse individual would optimally purchase zero indexed cover ( $r > 0$ ) at any positive premium since purchase of cover of  $\alpha$  would decrease the minimum wealth from  $w - L$  to  $w - L - \alpha qmL$ .

Recall that if  $V'$  is a strongly more risk averse preference ordering than  $V$  in the sense of Ross (1981) then there exists  $G : \mathbb{R} \rightarrow \mathbb{R}, G' < 0, G'' < 0$  and  $\lambda > 0$  such that  $V'(W) = \lambda V(W) + G(W)$ . The following theorem characterises

demand for indexed cover for extreme risk aversion.

**Theorem 1.** *Let  $\{u_n\}_{n=1}^{\infty}$  be any sequence of utility functions satisfying:*

- (i) *each  $u_n$  is everywhere twice differentiable with  $u'_n > 0$ ;*
- (ii)  *$u_{n+1}$  is more risk averse than  $u_n$  in the sense of Ross; and*
- (iii)  *$\lim_n (\inf_{x \in \mathbb{R}} -u''_n(x)/u'_n(x)) = +\infty$ .*

*Then optimal demand  $\alpha_n^*$  from a decision maker with utility function  $u_n$  satisfies  $\lim_n \alpha_n^* = 0$ .*

The purchase of indexed insurance reduces the difference in wealth between states  $LI$  and  $00$ , where the hedge performs, but increases the difference in wealth between states  $L0$  and  $LI$ , where the hedge does not perform. Even though the value to an infinitely risk averse individual of hedging states  $LI$  and  $00$  is high, the disutility arising from allowing a decrease in realised wealth in state  $L0$  is so high as to justify zero cover.

One immediate implication of Theorem 1 is that an increase in risk aversion, either in the sense of Arrow-Pratt or even in the stronger sense of Ross (1981), does not necessarily lead to an increase in demand for indexed cover, echoing the earlier results of Schlesinger and Schulenburg (1987) and Doherty and Schlesinger (1990) for the case of a one-sided risk of contractual nonperformance. This differs from both classical models of demand for indemnity insurance with or without background risk and predictions from the mean variance model of demand for index insurance (Giné et al. 2008) in which optimal demand is increasing in risk aversion.

If the indirect utility function satisfies constant absolute risk aversion, that is  $v'(x) \propto e^{-\gamma x}$  for some coefficient of absolute risk aversion  $\gamma$ , first order condition (4) may be rearranged to give the optimal cover as an explicit function of  $p, q, r, m, L$  and  $\gamma$ .

**Lemma 1.** *For any individual with indirect constant absolute risk aversion of  $\gamma > 0$  the optimal level of indexed cover is:*

$$\alpha_{CARA}^*(\gamma) = \max \left[ 0, \frac{1}{\gamma L} \ln \left( \frac{A + (1 - A)e^{-\gamma L}}{BC'e^{-\gamma L} + B(1 - C)} \right) \right] \quad (6)$$

For both the classes of constant absolute risk averse (CARA) and constant relative risk averse (CRRA) expected utility maximisers, the shape of optimal demand is characterised in the following theorem.

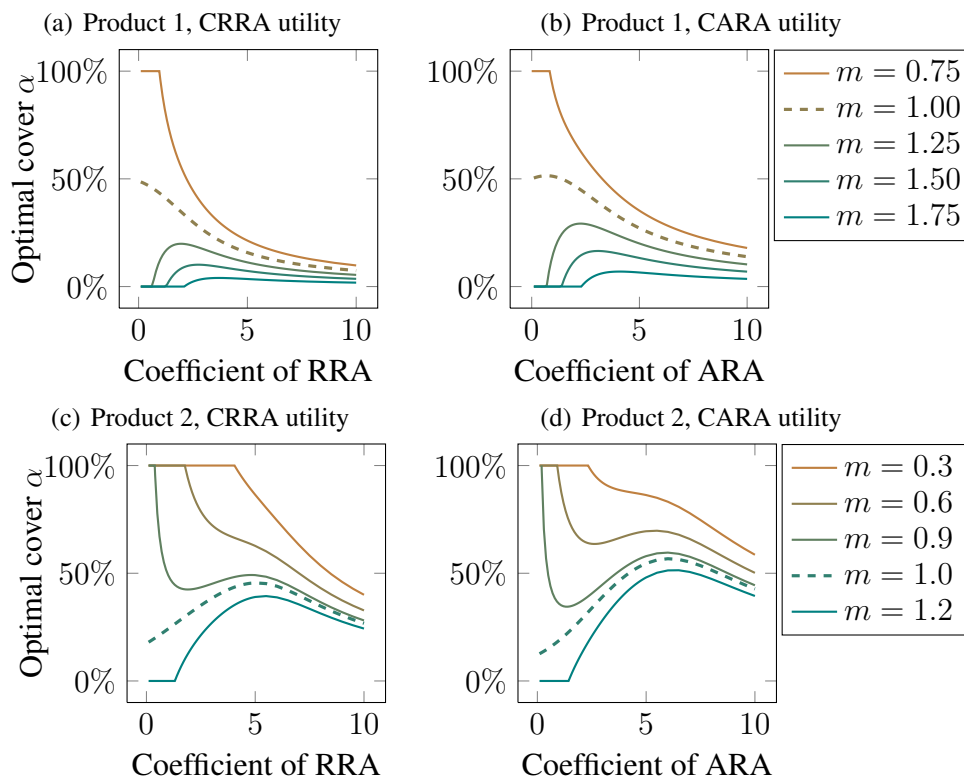
**Theorem 2.** *The following conditions on the optimal level of indexed cover hold both for the class of risk averse indirect utility functions that satisfy constant relative risk aversion and for the class of risk averse indirect utility functions that satisfy constant absolute risk aversion, where  $\gamma$  denotes the coefficient of relative or absolute risk aversion, respectively:*

- (i) **Actuarially unfair products ( $m > 1$ ):**  $\alpha^*(\gamma) = 0$  for all  $\gamma \in (0, \infty)$  if  $r \geq p(1 - qm)$ . Otherwise  $\alpha^*(\gamma)$  is zero for  $\gamma \leq \gamma_1$ , strictly increasing for  $\gamma_1 < \gamma < \gamma_2$  and strictly decreasing for  $\gamma_2 < \gamma < \infty$  for some  $0 < \gamma_1 < \gamma_2 < \infty$ ;
- (ii) **Actuarially fair products ( $m = 1$ ):**  $\alpha^*(\gamma) > 0$  is either strictly decreasing for all  $\gamma \in (0, \infty)$  or strictly increasing for  $0 < \gamma < \gamma_1$  and strictly decreasing for  $\gamma_1 < \gamma < \infty$  for some  $0 < \gamma_1 < \infty$ ; and
- (iii) **Actuarially favourable products ( $m < 1$ ):**  $\alpha^*(\gamma) > 0$  is either strictly decreasing for all  $\gamma \in (0, \infty)$  or strictly decreasing for  $0 < \gamma < \gamma_1$  and  $\gamma_2 < \gamma < \infty$ , and strictly increasing for  $\gamma_1 < \gamma < \gamma_2$ , for some  $0 < \gamma_1 \leq \gamma_2 < \infty$ .

These relationships can be seen in Figure 1, which plots the optimal purchase of indexed cover for constant absolute and relative risk averse expected indirect

utility maximisers against the respective coefficient of absolute and relative risk aversion for two different products.

**Figure 1.** Rational hedging and risk aversion for CRRA and CARA



*Note:* Assumptions underlying figures (a) to (d) are as follows.  $p = q = 1/3$ ,  $r = 1/9$  for product 1 (figures (a) and (b)) and  $p = 1/20$ ,  $q = 4/20$  and  $r = 1/1000$  for product 2 (figures (c) and (d)).  $w = 3L/2$  for CRRA (figures (a) and (c)) and  $L = 1$  for CARA (figures (b) and (d)).

The result that the optimal level of actuarially unfair indexed cover is hump-shaped in the coefficient of relative or absolute risk aversion, first increasing then decreasing, may be understood as follows. For actuarially unfair premiums,  $m > 1$ , indexed cover decreases mean wealth and so risk neutral individuals would optimally purchase zero cover. Indexed cover also decreases the minimum possible wealth and so infinitely risk averse individuals would optimally purchase zero cover (Theorem 1). It is therefore clear that only individuals with intermediate levels of risk aversion might wish to purchase material amounts of actuarially unfair cover. For the cases of CARA and CRRA, this hump-shaped relationship is smooth.

For actuarially fair cover,  $m = 1$ , there are two factors affecting demand; an increase in cover both reduces the risk for states in which the hedge performs (00 and  $LI$ ) and increases the risk for states in which the hedge does not perform ( $L0$  and  $0I$ ). For high risk aversion, the utility cost from basis risk will dominate decisions and demand will be decreasing in risk aversion. However, the benefit from the hedge when it performs may dominate for low risk aversion and so the optimal level of actuarially fair cover will either be decreasing or increasing then decreasing in risk aversion for CARA or CRRA utility.

For actuarially favourable premiums,  $m < 1$ , indexed cover increases mean wealth and so risk neutral individuals would optimally purchase as much cover as possible, but it decreases minimum wealth and so infinitely risk averse individuals would optimally purchase zero cover (Theorem 1 and Corollary ??). Between these two extremes, the decision maker must trade off the benefits from the increase in expected wealth and from hedging when the contract performs (reduction in risk between states 00 and  $LI$ ) against the cost of basis risk when the contract does not perform (particularly state  $L0$ ). For CARA and CRRA, demand is either monotonically decreasing in risk aversion or first decreasing then increasing then decreasing in risk aversion. Panes (c) and (d) of Figure 1 provide a numerical example of demand ‘decreasing with a hump’, where the premium is only slightly lower than the actuarially fair premium and basis risk is low.

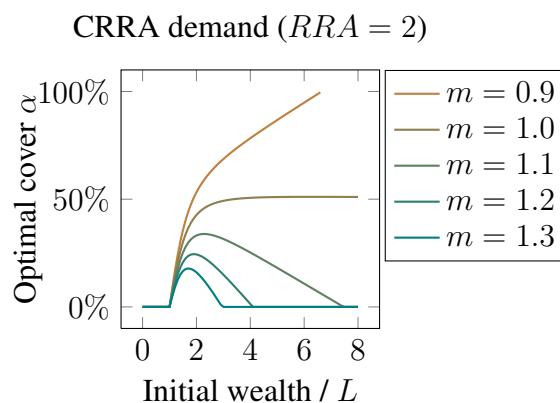
### 2.2 *Wealth and price*

Other comparative statics results do follow through from Doherty and Schlesinger (1990), as would be expected since their model is a reparameterised, special case of ours. To begin with, there is no monotonic relationship between demand and wealth  $w$ , pricing multiple  $m$  or loss  $L$ , even if one restricts preferences to satisfy increasing absolute risk aversion (IARA) or decreasing absolute risk aversion (DARA).

The first result stands in direct contrast to Mossin's (1968) observation that indemnity insurance is an inferior good under DARA. In models without basis risk, an increase in insurance purchase transfers wealth from high to low wealth states, subject to some deadweight cost. However, in the presence of basis risk an increase in indexed cover transfers wealth from the lowest and intermediate wealth states to the highest and other intermediate wealth states, and the restriction of DARA is no longer relevant for determining whether indexed cover is an inferior good.

The second result, that indexed cover may be a Giffen good, stands in direct contrast to the result of Mossin (1968) and Smith (1968) who show that that, in the absence of basis risk, demand for indemnity insurance is lower for  $m > 1$  than for  $m = 1$ . This result also arises from indexed cover not being an inferior good, even under DARA. Although an increase in premium increases  $B$ , acting to decrease demand, it also uniformly decreases wealth, which has an ambiguous effect on optimal demand for reasons described in the previous paragraph. If one restricts preference to satisfy constant absolute risk aversion (CARA) this wealth effect disappears, ruling out the possibility that indexed cover is a Giffen good. However, without the restriction to CARA the effect on demand of an increase in  $m$  is ambiguous. Figure 2 plots the optimal purchase of indexed cover for constant relative risk averse expected utility maximisers with respect to initial wealth.

**Figure 2.** *Rational hedging and wealth* ( $p = q = 1/3$ ,  $r = 1/9$ )



The third result follows trivially from the observation that demand is nonmonotone in loss  $L$  for CARA (see equation (6)), and therefore by extension for DARA or IARA. Although this result was not mentioned in Doherty and Schlesinger (1990), it also holds in the case of one-sided basis risk. By way of comparison, demand for indemnity insurance under DARA is monotonically increasing in loss  $L$ .

### 2.3 Bounds for rational demand

It is also possible to derive some strong results about the welfare implications of hedging. A financial advisor advising on the purchase of indemnity insurance cannot reasonably rule out any level of purchase without understanding a client's preferences; zero purchase may be advisable for a risk neutral client, and nearly full cover advisable for a risk averse client. As we will show in this section, this intuition does not follow through to the case of indexed cover with basis risk.

First, there are indexed products for which zero coverage is optimal for any risk averse expected utility maximiser.<sup>6</sup>

**Theorem 3.** *For any risk averse individual the optimal level of indexed cover is zero if  $\mathbb{E}[i|l = L] \leq m\mathbb{E}[i]$ .*

*Proof.* First note that the condition is equivalent to  $r \geq p(1 - qm)$  and, combined with the assumption of affiliation (2), can only hold if  $m > 1$ . Indirect utility function  $v$  inherits strict risk aversion from  $u$  and so  $v'_{L0} \geq v'_{LI}$  and  $v'_{00} \geq v'_{0I}$  for  $\alpha \geq 0$ .<sup>3</sup> Since  $C > 1 - A$  and  $B \geq 1$  for  $m > 1$ ,  $(1 - A)v'_{0I} - BCv'_{00} < 0$  and so first order condition (4) cannot hold unless  $Av'_{LI} > B(1 - C)v'_{L0}$ . This in turn implies that  $A > B(1 - C)$  which can be rearranged to give the restriction  $\frac{p-r}{p} > qm$  or  $\mathbb{E}[i|l = L] > m\mathbb{E}[i]$ .  $\square$

---

<sup>6</sup> Theorem 3 holds in a more general setting with index  $i \in [0, I]$  affiliated with loss  $l \in [0, L]$  in the sense of Milgrom and Weber (1982).

Theorem 3 may be understood as follows. When  $m \leq 1$ , the condition of Theorem 3 can never hold due to our assumption that the index and loss are strictly positively correlated (equation (2)); with positive correlation and an actuarially fair or favourable price there will be a risk averse individual for whom positive purchase is optimal.

However, for the case of actuarially fair or unfair cover, purchase does not increase average wealth and so for positive purchase to be optimal for a risk averse individual it must at least result in an increase in average wealth in low wealth states, that is the states in which a loss has occurred. Purchase of cover of  $\alpha$  increases the premium by  $\alpha qmL$  and results in average claim income, conditional on a loss  $L$  having occurred, of  $\alpha L \frac{p-r}{p}$ . The condition ensuring that the average net gain in these low wealth states is positive on purchase of indexed cover is therefore  $\alpha L \left( \frac{p-r}{p} - qm \right) > 0$ , or equivalently  $\mathbb{E}[i|l = L] > m\mathbb{E}[i]$ .

When basis risk is low, for example when the only source of basis risk is a 1% chance of insurer default, the restriction of Theorem 3 is not much tighter than the restriction that the maximum possible claim payment  $\alpha L$  is larger than the premium  $\alpha qmL$ . However, when there is both a sizeable basis risk  $r$  and premium multiple  $m$ , the restriction of Theorem 3 is tighter.

For  $\mathbb{E}[i|l = L] > m\mathbb{E}[i]$ , risk aversion alone is not sufficient to justify an upper bound for demand tighter than full cover, even for actuarially unfair cover, since we cannot rule out the possibility that the individual is approximately risk neutral except for some interval between  $w - \alpha qmL - (1 - \alpha)L$  and  $w - \alpha qmL$  in which she is very risk averse. However, we may rule out such contrived cases by assuming that absolute risk aversion decreases with wealth. This assumption of decreasing absolute risk aversion (DARA) is equivalent to assuming that if the decision maker would accept some lottery given certain endowment of  $w$  she would accept the same lottery given certain endowment of  $w' > w$ . Regardless of whether this restriction is appropriate from a positive point of view, that is whether individual's actual decisions violate this assumption, we agree with

Arrow (1965) and Pratt (1964) who argued that it is normatively sound. Moreover we consider DARA, and by extension the upper bound of the following theorem, to be an appropriate basis for generic financial advice about products designed to reduce exposure to risk.

Under DARA we may derive the following upper bound on rational demand for indexed cover, identical to the former upper bound for products with  $\mathbb{E}[i|l = L] \leq m\mathbb{E}[i]$  and tighter for products with  $\mathbb{E}[i|l = L] > m\mathbb{E}[i]$ .

**Theorem 4.** *For any strictly risk averse individual with decreasing absolute risk aversion the optimal level of actuarially fair or unfair indexed coverage is zero if  $\mathbb{E}[i|l = L] \leq m\mathbb{E}[i]$ , or otherwise bounded above by the unique  $\bar{\alpha}$  that solves*

$$A\bar{\alpha}(1 - \bar{\alpha})^{1-\bar{\alpha}} = (A + BC - 1)\bar{\alpha} \times (B(1 - C))^{1-\bar{\alpha}}$$

$$\bar{\alpha} \in \left(0, \frac{A + BC - 1}{A + B - 1}\right] \quad (7)$$

Although we relegate the full proof to the appendix, a sketch is as follows. Conditional on the marginal indirect utility of wealth in states  $LI$  and  $00$ ,  $v'_{LI}$  and  $v'_{00}$ , we first show that demand is bounded above by an indirect utility function which is risk neutral above  $v'_{00}$  and has constant absolute risk aversion of  $\gamma$  below  $v'_{00}$ . The maximisation problem is then to choose  $\gamma \geq 0$  to maximise demand  $\alpha$ , conditional on demand satisfying first order condition (4). Equation (7) defines this largest optimal  $\alpha$  for any  $\gamma \geq 0$ .

Loosely speaking, the logic of the DARA upper bound of Theorem 4 is as follows: if an individual cares enough about the risk between states  $LI$  and  $00$  to want to purchase a sizeable hedge, the individual must care enough about the downside basis risk between states  $L0$  and  $LI$  and the deadweight cost of hedging to limit the size of the hedge.

As presented in Clarke and Kalani (2011), Theorem 4 provides a one-decision problem test for DARA in experimental settings. So long as the

randomisation device used to determine net transfers in an experiment is statistically independent of participant's background wealth, demand from participants in excess of the respective upper bound may be taken as a violation of risk aversion for products with  $r \geq p(1 - qm)$  or risk aversion and DARA for other products.

Table 3 presents values of  $\bar{\alpha}$  when the probability of no claim being paid conditional on a loss having occurred is taken to be  $\frac{1}{3}$ .

**Table 3.** *Examples of upper bounds for rational purchase of indexed cover*

$\mathbb{P}[Loss] = \mathbb{P}[Claim] (p = q)$	$\mathbb{P}[No Claim Loss] (r/p)$	Premium Multiple ( $m$ )	$\bar{\alpha}$
1/10	1/3	6.0	3.9%
1/8	1/3	4.5	6.7%
1/5	1/3	3.0	4.5%
1/4	1/3	2.4	4.8%
1/3	1/3	1.8	5.4%

When premiums are actuarially fair, that is  $m = 1$ , the formula for the upper bound may be simplified.

**Corollary 1.** *For any strictly risk averse individual with decreasing absolute risk aversion the optimal level of actuarially fair indexed coverage is bounded above by  $\bar{\alpha} = \frac{p(1-q)-r}{(p-r)(1-q)}$ .*

*Proof.* For  $m = 1$ ,  $\bar{\alpha} = \frac{A+C-1}{A} = \frac{p(1-q)-r}{(p-r)(1-q)}$  is a solution to equation (7), and by Theorem 4 it is the unique solution.  $\square$

### 2.4 Discussion

#### *Bundling indexed insurance with credit*

To many proponents, the main welfare gains to be derived from indexed insurance are through the relaxation of credit constraints. For example, it is often argued that weather or area yield indexed insurance can act as collateral against loans, increasing the creditworthiness of farmers and allowing them the opportunity to

invest in appropriate inputs to increase agricultural productivity (Hazell 1992, Carter et al. 2007, Mahul and Stutley 2010).

This proposition can be analysed in the current framework. For example, suppose that a relaxation of credit constraints brought about by purchase of indexed insurance leads to a risk-free increase in wealth. The problem of how much of this bundle to purchase is then mathematically identical to the problem of how much indexed insurance to purchase, where indexed insurance is priced with a lower effective pricing multiple  $m$  than the standalone indexed insurance product. If the effective multiple is less than unity, one would expect risk neutral decision makers to purchase as much of the bundle as possible. However more risk averse decision makers would purchase less of the bundle, with demand suppressed by concerns about the increase in downside potential; for the case of bundled agricultural credit and weather indexed insurance, a farmer could take out a loan and purchase indexed insurance but have a bad year with no index insurance claim payment. The bundle would only be purchased by the infinitely risk averse if, as suggested by Carter et al. (2010), the effective pricing multiple  $m$  is nonpositive.

### *Beliefs and real insurance products*

There are many interpretations to the model presented in this section.  $p$ ,  $q$  and  $r$  could be known probabilities with  $m = \frac{P}{q\alpha L}$  as the premium multiple consistent with the individual's objective belief  $q$  and the observed price  $P$  for cover of  $L$ . Alternatively  $(p, q, r)$  could be interpreted as a subjective decision-theoretic belief, after any subjective probability weighting.

For the case of indexed insurance  $r$  has been presented as the downside basis risk but other interpretations are possible. Schlesinger and Schulenburg (1987) and Doherty and Schlesinger (1990) consider a mathematical special case of ours in which  $\pi_{0I} = 0$ , that is where there is no upside basis risk. They interpret their model as one where an insurer sells indemnity insurance but with probability  $r$  a policyholder incurs a loss but does not receive indemnification from the insurer

due to insurer insolvency or contractual exclusions. Mathematically, the cause of this risk of contractual nonperformance, whether it be downside basis risk, insurer insolvency or contractual exclusions, does not change the shape of optimal demand.

However, whilst insurer insolvency and contractual exclusions for consumer indemnity insurance products might be considered to have low probability in countries with robust capital adequacy requirements and consumer protection regulation, motivating Schlesinger and Schulenburg (1987) to end their paper with the caveat ‘we might not expect to observe [demand falling with risk aversion] very often as a matter of practice’, there may be a high risk of contractual nonperformance for indexed products, or products sold in environments without credible regulation. A negative relationship between demand and risk aversion has been well documented for the case of weather derivatives sold to poor farmers (e.g. Cole et al. 2009) and for a health microinsurance product in Kenya where consumers did not fully trust the insurer to pay valid claims (Dercon et al. 2011).

### *Gap insurance*

One immediate implication of Theorem 1 is that gap insurance, which would make a payment in state  $L0$  at least equal to the premium, would be valuable for the most risk averse if bundled with indexed cover (Doherty and Richter 2002). So long as the combined index/gap insurance premium was less than the maximum loss, an infinitely risk averse individual would optimally purchase full cover, as opposed to the zero cover of Theorem 1.

For the case of agricultural insurance it seems difficult to imagine a formal insurance company being able to offer an indexed product bundled with gap insurance at low cost. However, as suggested in Clarke (2011), local institutions could perhaps supplement formal sector indexed insurance with local informal or semiformal risk pooling that provided gap insurance using cheap local

information. Any such gap insurance would be of significant benefit to the most risk averse even if only a refund of premium was paid in state  $L0$ ; the most risk averse would then optimally purchase full cover, rather than the zero cover of Theorem 1.

### *Hedging and the poverty headcount measure*

There are, of course, sets of preferences which violate risk aversion or DARA, for which optimal purchase of indexed cover may exceed the upper bound of Theorem 4. Whilst we have argued above that such sets of preferences can reasonably be ignored for the purpose of providing generic financial advice to individuals, there is one particular class of preferences worthy of particular attention.

Suppose that policymakers choose policies to minimise the expected number of individuals below some poverty line  $\mathcal{P}$ . For individuals with wealth  $w$  and subject to loss  $L$  such that  $\mathcal{P} \in [w - L, w - qmL)$ , then purchase of cover  $\alpha L \in (\frac{P-(w-L)}{1-qm}, L]$  would reduce their expected contribution to headcount poverty from  $p$ , the probability of a loss occurring, to  $r$ , the probability of a loss occurring and no indexed claim payment being made. This result arises from a well documented feature of the headcount poverty measure, whereby it is possible to reduce expected headcount poverty by transferring wealth from states below the poverty line to states also below the poverty line but with higher wealth (Sen 1979). Providing targeted subsidies for indexed financial products may therefore meet the objectives of policymakers, even if all rational individuals would strictly prefer the subsidy to be in the form of an actuarially equivalent uncontingent cash transfer.

### 3 NUMERICAL EXAMPLE

The methodology of Section 2 can be applied to real world data to construct a numerical analysis of objectively rational demand. By objective, we refer to a decision rule under which beliefs are based on an objective probability distribution and, as above, we use the term rational to mean DARA EUT preferences over roulette lotteries.<sup>7</sup>

For example, suppose that a financial advisor is to provide advice to the following maize farmers in a developing country. In 2009, weather index insurance products designed for maize were sold in 31 nearby subdistricts in a developing country.<sup>8</sup> From 1999 to 2007, average maize yields within these subdistricts were recorded and weather readings were taken at the contractual weather station for the weather indexed insurance product. Let  $y_{ij}$  denote the yield in kilograms per hectare in year  $i$  for subdistrict  $j$  and  $x_{ij}$  denote the claim payment that would have been made, denominated in local currency, from one unit of the product for subdistrict  $j$  given the recorded weather readings for year  $i$ . 18 data items are missing, leaving a total of  $n = 261$  complete  $(x_{ij}, y_{ij})$  pairs, indexed by  $ij \in D$  (see Figure 3(a)).

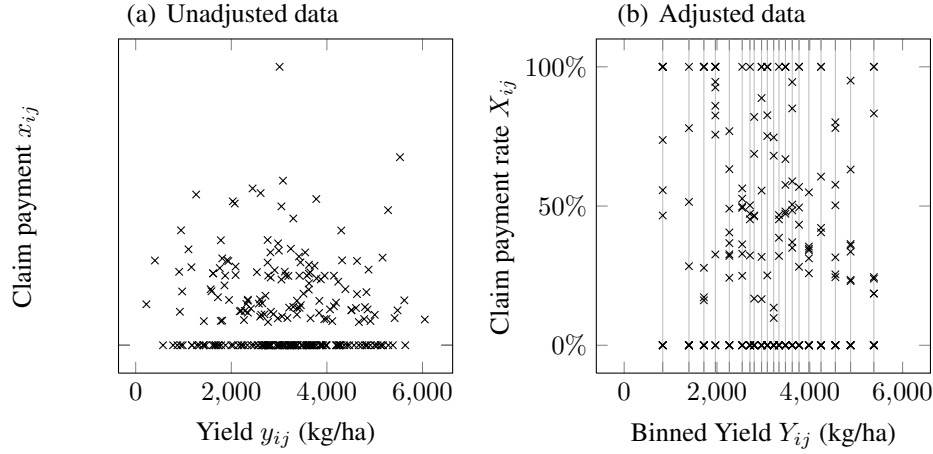
With up to nine years of matched historical data per product, the empirical joint distribution function is not a particularly useful belief for a objective EUT decision maker; optimal demand for indexed insurance is sensitive to the distribution of claim payments conditional on the yield, particularly for low yields, and the empirical conditional distribution is degenerate. With up to nine years of data, demand from the most risk averse will be driven by the year with the lowest yield for which either there was a claim payment greater than the premium in that year or there wasn't.

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<sup>7</sup> Anscombe and Aumann (1963) differentiated *roulette lotteries*, where probabilities are known, and *horse lotteries*, where they are not. An objective financial advisor might combine an objective belief, constructed as a frequentist, with an individual's preferences over roulette lotteries to arrive at a recommended decision.

<sup>8</sup> The numerical example uses data for real weather derivatives sold across a developing country in 2009. However, for reasons of confidentiality the name of the country is not disclosed.

**Figure 3.** Unadjusted and adjusted joint empirical distribution of yields and claim payments



Another approach would be to assume homogeneity of the yield and claim payment distribution between subdistricts and use the empirical distribution of the 261 pairs as the decision maker's belief. However, there are two problems with this approach. First, although products were designed using a coherent methodology, the product details and the maximum possible claim payment in local currency (the *sum insured*) differed between products. Second, out of 261 observations, only 25 yield values have been recorded more than once, and so the empirical distribution of claim payments conditional on yields is still largely degenerate.

We address these problems in the following way (see Figure 3(b)). First, historical claim payment amounts  $x_{ij}$  are divided by the maximum historical claim payment for that product  $\max_i(x_{ij})$  to give the historical claim payment rate  $X_{ij} = \frac{x_{ij}}{\max_i(x_{ij})}$ . This converts all claim payments histories to the same scale, from 0% to 100%, allowing histories from different products to be pooled.<sup>9</sup>

Second, all  $ij$  pairs are sorted in order of increasing yield  $y_{ij}$  and partitioned into

<sup>9</sup> The mean Pearson product-moment correlation coefficient for yields and claim payments over all 31 products is  $-16\%$ . Pooling all data without adjusting gives a correlation coefficient of  $-6.8\%$ . Pooling after adjusting the claim payments as described decreases this to  $-12.8\%$  and averaging yield data within 5th percentile bins decreases it further to  $-13.6\%$ . Adjusting claim payments using different rules, for example by dividing by the contractual sum insured for each product or the mean historical claim payment for each product gives similar results.

20 five-percentile bins,  $k \in \{1 \dots 20\}$ .<sup>10</sup> For each  $ij$  pair in the  $k$ th bin, the binned yield  $Y_{ij}$  is the mean yield  $y_{ij}$  over all  $ij$  pairs in the bin. For example, for the first bin, containing the thirteen  $ij$  pairs with the lowest yields,  $Y_{ij}$  is calculated to be 831 kg/ha (see Table 4).

**Table 4.** *Illustration of binning of yield data for first bin*

$y_{ij}$ rank	261	260	259	258	257	256	255	254	253	252	251	250	249
$y_{ij}$ (kg/ha)	223	402	572	783	872	909	924	943	948	972	983	1104	1166
$Y_{ij}$ (kg/ha)	831	831	831	831	831	831	831	831	831	831	831	831	831
$X_{ij}$	56%	74%	0%	0%	0%	100%	47%	0%	100%	100%	0%	100%	100%

We now interpret the adjusted joint probability distribution, displayed in Figure 3(b), as an objective belief about the joint distribution of yield and indexed claim payments for a representative maize farmer in our study area, and calculate how much index insurance such a farmer would purchase. Following the notation in the previous section, the maximum possible loss  $L$  a farmer could incur under our belief is just the difference between the maximum and minimum binned yield of  $5,381 - 831 = 4,550$  kg/ha.

A maize farmer's objective is therefore to choose a level of coverage  $\alpha \geq 0$ , providing a maximum claim payment of  $\alpha L$ , to maximise expected utility:

$$\mathbb{E}U = \frac{1}{n} \sum_{ij \in D} u(w + \tilde{z} + Y_{ij} + \alpha L(X_{ij} - m\bar{X})) \quad (8)$$

where  $\bar{X}$  denotes  $\frac{1}{n} \sum_{ij \in D} X_{ij}$ ,  $w$  is initial certain wealth,  $\tilde{z}$  is a zero mean random background risk statistically independent of the joint distribution of  $(X, Y)$ ,  $m$  is the pricing multiple and  $u$  is the farmer's utility function, assumed to satisfy  $u' > 0$  and  $u'' < 0$ .

The first-order condition for an interior solution may be written in terms of the

<sup>10</sup> All bins contain 13  $(x_{ij}, y_{ij})$  pairs except the 45th to 50th percentile bin which contains 14 products. For the three cases in which there are two pairs with equal yields, one of which must go in a higher bin than the other, and the claim payments are not equal, the allocation into bins is random.

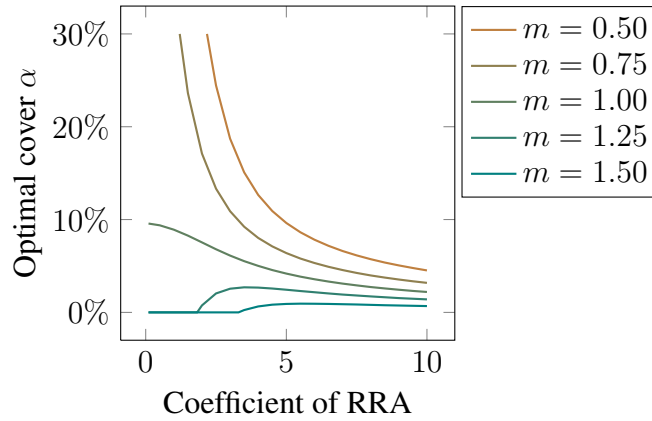
indirect utility function  $v$  (equation (1)) as

$$\sum_{ij \in D} L(w + X_{ij} - m\bar{X})v'(Y_{ij} + \alpha L(X_{ij} - m\bar{X})) = 0 \quad (9)$$

and the second-order condition is trivially satisfied.

Figure 4 presents the optimal level of demand for indexed insurance from a maize farmer with beliefs as displayed in Figure 3(b) and CRRA indirect utility function over net income from agriculture, that is with  $w = 0$ . Five different insurance multiples are considered: two actuarially favourable ( $m = 0.5, 0.75$ ); two actuarially unfair ( $m = 1.25, 1.50$ ); and actuarially fair ( $m = 1$ ). As can be seen from Figure 4, and in line with Theorem 2,  $\alpha$  is monotonically decreasing in the coefficient of relative risk aversion when  $m < 1$  and is hump-shaped for  $m \geq 1$ . Moreover, the level of demand is low when the product is actuarially fair or unfair, with a maximum level of demand of 2.7% when  $m = 1.25$  and 0.9% when  $m = 1.5$ . Ignoring the case of risk neutrality, for which the optimal level of cover is ambiguous, the maximum optimal level of demand is only 9.6% for an actuarially fair product with  $m = 1$ .

**Figure 4.** Optimal purchase of index insurance for maize from CRRA decision makers

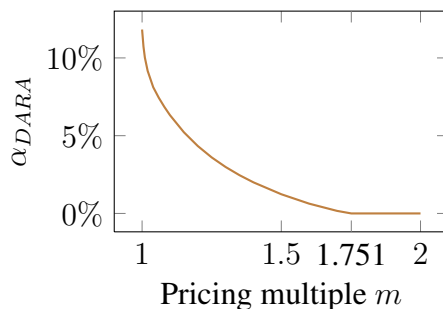


Moreover, given the belief displayed in Figure 3(b), it is possible to numerically calculate the DARA upper bound, above which no DARA EUT decision maker

would ever optimally purchase.<sup>11</sup> These rational bounds are displayed in Figure 5. As would be expected, the DARA upper bound is infinite for actuarially favourable products since a decision maker with very low levels of risk aversion could optimally purchase an infinite amount of indexed insurance. However, the DARA upper bound is decreasing in loading for a premium multiple  $m \geq 1$  and is zero for  $m \geq 1.751$ .

In fact, DARA is not necessary for this result to hold; any strictly risk averse EUT decision maker would optimally purchase zero cover for  $m \geq 1.751$  for the following reason, along the lines of Theorem 3.<sup>12</sup> The average unconditional claim payment is 29.7% and the average claim payment conditional on achieving the lowest possible binned yield is 52.0%. Moreover, the average claim payment conditional on achieving a binned yield less than or equal to any yield between 831 and 5,381 kg/ha is no more than 52.0%. If the premium multiple were above 1.751, the insurance premium would be greater than the average claim payment conditional on achieving a binned yield less than or equal to any yield between 831 and 5,381 kg/ha. The reasoning then follows that of Theorem 3.

**Figure 5.** DARA upper bound for purchase of index insurance for maize



In practice both fixed costs and variable costs seem to be high for small

<sup>11</sup> The procedure for calculating the DARA upper bound was as follows. First, the decision maker was assumed to be risk neutral for wealth above some level of wealth  $W$  and have constant absolute risk aversion of  $\gamma \geq 0$  for wealth below  $W$ .  $W$  and  $\gamma$  were chosen to maximise the optimal level of demand  $\alpha$ . Second, using this utility function as a starting point, the decision maker was free to choose levels of absolute risk aversion between each of the 261 ordered wealth realisations, consistent with DARA. The utility function was chosen to maximise the optimal level of demand  $\alpha$ . In all cases, this  $\alpha$  was virtually identical to the  $\alpha$  calculated in the first step.

<sup>12</sup> For our empirical distribution the yield and indexed claim payments are negatively correlated but not affiliated, and so the conditions of Theorem 3 are not met. However, the joint distribution is sufficiently negatively correlated for a bound to be derived.

products marketed to poor farmers. Giné et al. (2007) and Cole et al. (2009) use historical weather data to estimate objective premium multiples  $m$  for unsubsidised products sold to poor farmers in India, with the former quoting an average premium multiple of 3.4 and the latter calculating multiples individually for seven products, ranging from 1.75 to 5.26. If the joint distribution between indexed claim payments and farmer yields for these products were the same as the joint distribution between indexed claim payments and area average yields for the present example of maize, the combination of basis risk and actuarially unfair prices with multiple above 1.75 would lead to zero optimal demand from any objective, risk averse EUT decision maker. The puzzle of low demand could therefore be resolved without requiring ‘behavioural’ decision makers or credit constraints.<sup>13</sup>

## 4 CONCLUDING REMARKS

Recent work on weather derivatives for the poor by academic economists seems to have been lacking a sound theoretical basis. This paper attempts to address this by presenting a model of rational demand for indexed products. The presence of basis risk is shown to alter or reverse many of the key results arising from the theory of rational demand for indemnity insurance (see Table 5, which extends Table 1 of Doherty and Schlesinger (1990)).

A numerical example is presented which suggests that the aggregate level and shape of observed demand for weather derivatives from poor farmers may be consistent with a model of rational demand, thereby offering explanations for two outstanding empirical puzzles without the need to resort to ‘behavioural’ preferences or credit constraints.

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<sup>13</sup> This is not to say that models with behavioural preferences or credit constraints have no descriptive power for the analysis of index insurance purchase; Bryan (2010) and Cole et al. (2009) find strong relationships between purchase of indexed insurance and ambiguity aversion and credit constraints, respectively. Rather, one can explain the low demand, particularly for the risk averse, using nothing more than expected utility theory with an objective belief.

**Table 5.** *Indemnity and indexed insurance: a summary of key results*

Result without basis risk (Indemnity insurance, $r = 0$ )	Result with basis risk (Indexed cover, $r > 0$ )
<b>Shape of rational hedging:</b> <ul style="list-style-type: none"> <li>• More risk averse <math>\Rightarrow</math> buy more cover</li> <li>• Infinitely risk averse <math>\Rightarrow \alpha = 1</math></li> <li>• Fair price (<math>m = 1</math>) <math>\Rightarrow \alpha = 1</math></li> <li>• Positive loading (<math>m &gt; 1</math>) <math>\Rightarrow</math> buy less insurance</li> <li>• Insurance is inferior for DARA utility</li> <li>• Larger potential loss <math>L \Rightarrow</math> buy more insurance for DARA utility</li> </ul>	<ul style="list-style-type: none"> <li>• <math>\alpha</math> decreasing in basis risk</li> <li>• More risk averse <math>\nRightarrow</math> buy more cover</li> <li>• Unfair price (<math>m &gt; 1</math>) <math>\Rightarrow</math> CRRA/CARA cover hump-shaped (increasing then decreasing) in coefficient of RRA/ARA</li> <li>• Fair price (<math>m = 1</math>) <math>\Rightarrow</math> CRRA/CARA cover decreasing or hump-shaped in coefficient of RRA/ARA</li> <li>• Favourable price (<math>m &lt; 1</math>) <math>\Rightarrow</math> CRRA/CARA cover decreasing or decreasing-increasing-decreasing in coefficient of RRA/ARA</li> <li>• Infinitely risk averse <math>\Rightarrow \alpha = 0</math></li> <li>• Fair price (<math>m = 1</math>) <math>\Rightarrow \alpha &lt; 1</math></li> <li>• Positive loading (<math>m &gt; 1</math>) <math>\nRightarrow</math> buy less cover</li> <li>• Cover may not be inferior for DARA utility</li> <li>• Larger potential loss <math>L \nRightarrow</math> buy more cover for DARA utility</li> </ul>
<b>Level of rational hedging:</b> <ul style="list-style-type: none"> <li>• Positive loading (<math>m &gt; 1</math>) <math>\Rightarrow</math> DARA upper bound of <math>\bar{\alpha} = 1</math></li> </ul>	<ul style="list-style-type: none"> <li>• Positive loading (<math>m &gt; 1</math>) <math>\Rightarrow</math> Risk averse upper bound of <math>\bar{\alpha}_{RA}</math> and DARA upper bound of <math>\bar{\alpha}_{DARA} \leq \bar{\alpha}_{RA}</math></li> </ul>

To proponents, consumer hedging products based on cheaply observable, manipulation-free indices can offer partial cover at a lower deadweight cost than would be possible under traditional indemnity insurance contracts (Shiller 1993, Shiller 2003). However, the experience of selling weather derivatives to poor farmers suggests that the combination of deadweight costs and basis risk can render such products unattractive, particularly to the most risk averse, even if they offer a hedge against economically important risks.

Weather derivatives have changed the face of agricultural insurance for the poor by focusing the minds of academics and practitioners on a problem which, if suitably addressed, could lead to a substantial increase in welfare for many of the world's rural poor (Banerjee 2002, Collins et al. 2009, Karlan and Morduch 2009). Indices are not a silver bullet; designing a good agricultural insurance product for poor farmers will require more than just choosing the best functional

form for a weather index, and implementing products with low basis risk will require more institutional capacity building than the installation of tamper-proof weather stations. However, indices have their uses, particularly in agricultural insurance.

One approach to reducing basis risk, presented in Clarke (2011), is to combine indices that capture local aggregate shocks with local semiformal risk pooling. Groups of poor individuals, such as extended families or members of close-knit communities, are typically able to support relatively inexpensive pooling for locally idiosyncratic shocks. The role of the formal insurer should then be to ‘reinsure’ the group, paying claims when the group has suffered a large aggregate loss. The formal insurer could pay when the total group loss is large, as in Mexico’s system of agricultural insurance through Fondos (Ibarra 2004), or when a statistical sample of losses indicates a large average loss has been incurred, as in India’s system of area yield index insurance (Mahul et al. 2011). The combination of local sample-based indices and local risk pooling could lead to lower basis risk for farmers than other indexed approaches whilst offering lower cost than indemnity based approaches to formal insurance.

Regardless of how insurance for the poor develops it seems important for economists to be engaged with normative issues of welfare, generic financial advice and product design, in addition to positive issues such as the shape and level of observed demand. Paraphrasing Sen (1995), such engagement may help insurance companies and governments avoid the danger of designing products for the poor that end up being poor products.

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## A APPENDIX

*Proof of Theorem 1.* We begin with the following lemma, proved in Klibanoff et al. (2005, Lemma 8).

**Lemma 2.** *Let  $\Sigma$  be a  $\sigma$ -algebra over a set  $\Delta$ ,  $\eta$  be a finitely additive probability measure on  $\Sigma$ , and  $\psi$  be a bounded  $\sigma$ -measurable function  $\psi : \Delta \rightarrow I$  with  $I$  an interval in  $\mathbb{R}$ . Suppose  $\{\phi_n\}_n$  is a sequence of twice differentiable real-valued functions  $\phi_n : I \rightarrow \mathbb{R}$  satisfying  $\phi'_n(x) > 0$  for all  $x \in I$  and with Arrow-Pratt coefficients  $A_n : I \rightarrow \mathbb{R}$  such that  $\lim_n (\inf_{x \in I} A_n) = +\infty$  and  $A_n(x) \leq A_{n+1}(x)$  for each  $x \in I$  and each  $n$ . Then*

$$\lim_{n \rightarrow \infty} \phi_n^{-1} \left( \int_{\Delta} \phi_n(\psi) d\eta \right) = \text{ess inf } \psi, \quad (\text{A-1})$$

where  $\text{ess inf } \psi = \sup\{t \in \mathbb{R} | \eta(\{x | \psi(x) < t\}) = 0\}$

Define indirect utility functions as

$$v_n(x) = \mathbb{E}u_n(x + \tilde{z}) \text{ for all } x \in \mathbb{R}, n \in \{1, \dots, \infty\}.$$

From condition (ii), direct utility function  $u_{n+1}$  is more risk averse than  $u_n$  in the sense of Ross and so by Theorem 3 of Ross (1981) indirect utility function  $v_{n+1}$  is more risk averse than  $v_n$  in the sense of Pratt (1964).

From condition (iii) for any  $k \in \mathbb{R}$  there is some  $n_k$  such that  $-u''_n(x)/u'_n(x) \geq k$  for all  $n \geq n_k, x \in \mathbb{R}$ . So, for any  $n \geq n_k$  and  $x \in \mathbb{R}$ ,

$$\begin{aligned} -\frac{v''_n(x)}{v'_n(x)} &= \frac{\int f(z) \times [-u''_n(x+z)] dz}{\int f(z) u'_n(x+z) dz} \\ &\geq \frac{k \int f(z) u'_n(x+z) dz}{\int f(z) u'_n(x+z) dz} \\ &= k \end{aligned}$$

and so we have  $\lim_n (\inf_{x \in \mathbb{R}} -v''_n(x)/v'_n(x)) = +\infty$ .

Now, defining  $\Delta = S$ ,  $\phi_n = v_n$  and  $\psi_s(\alpha) = w - \theta_s + \alpha(\iota_s - qmL)$ , where  $\iota_s$  and  $\theta_s$  denote the realised index and loss respectively, Lemma 2 implies that for  $\alpha \geq 0$

$$\lim_{n \rightarrow \infty} v_n^{-1} \left( \sum_S \pi_s v_n(\psi_s(\alpha)) \right) = w - L - \alpha qmL \quad (\text{A-2})$$

since  $\psi(\alpha)$  is a discrete random variable with minimum of  $w - L - \alpha qmL$ .

Now consider random variables  $\psi(0)$  and  $\psi(\alpha)$ , the random wealth on purchasing 0 or  $\alpha$  units of indexed insurance. For any  $\alpha \geq 0$  by (A-2) there exists  $n_\alpha$  large enough so that, for all  $n \geq n_\alpha$ ,

$$\left| v_n^{-1} \left( \sum_S \pi_s v_n(\psi_s(\alpha)) \right) - (w - L - \alpha qmL) \right| < w - L - (w - L - \alpha qmL).$$

Hence, since  $\psi_s(0) \geq w - L$  for all  $s \in S$  and so  $v_n^{-1}(\sum_S \pi_s v_n(\psi_s(0))) \geq w - L$ ,

$$v_n^{-1} \left( \sum_S \pi_s v_n(\psi_s(0)) \right) > v_n^{-1} \left( \sum_S \pi_s v_n(\psi_s(\alpha)) \right), \quad \text{for all } n \geq n_\alpha,$$

which in turn implies that a decision maker with utility function  $u_n$  prefers index insurance purchase of  $\alpha^* = 0$  to  $\alpha^* = \alpha > 0$  for all  $n \geq n_\alpha$ .  $\square$

*Proof of Theorem 2.* For both constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) the proof will take the following form. First order condition (4) will give an equation defining the interior solution  $\alpha$  as a function of  $\gamma$ , denoted  $\alpha^*(\gamma)$ . Totally differentiating first order condition (4) with respect to  $\gamma$ , setting  $\frac{d\alpha}{d\gamma}$  to zero, and substituting in first order condition (4) will give a necessary condition for  $(\alpha, \gamma)$  pairs such that  $\frac{d\alpha^*(\gamma)}{d\gamma} = 0$ . Such pairs will define  $\alpha$  as an implicit function of  $\gamma$ , which we will denote by  $\tilde{\alpha}(\gamma)$ .  $\tilde{\alpha}(\gamma)$  will be shown to either be strictly decreasing in  $\gamma$  or be strictly increasing for  $\gamma < \Gamma$  and strictly decreasing for  $\Gamma < \gamma$  for some  $-\infty < \Gamma < \infty$ .  $\alpha^*(\gamma)$  is continuously differentiable and  $\tilde{\alpha}(\gamma)$  is continuous, and the limits as  $\gamma \rightarrow 0^+$  and  $\gamma \rightarrow \infty$  may be characterised using L'Hôpital's rule. Moreover, since  $\alpha^*(\gamma)$  is continuously differentiable it must have zero gradient when crossing  $\tilde{\alpha}(\gamma)$ , and therefore cannot cross  $\tilde{\alpha}(\gamma)$  from below when  $\tilde{\alpha}(\gamma)$  is increasing nor from above when  $\tilde{\alpha}(\gamma)$  is decreasing. These observations are combined to derive the stated conditions for each of  $m > 1$ ,  $m = 1$  and  $m < 1$ .

**CARA:** First consider the interior solution to first order condition (4) for the case of CARA

$$\alpha^*(\gamma) = \frac{1}{\gamma L} \ln \left( \frac{A + (1-A)e^{-\gamma L}}{BCe^{-\gamma L} + B(1-C)} \right)$$

(see Lemma 1).  $\alpha^*(\gamma) \leq 0$  for all  $\gamma$  if  $A \leq B(1-C)$  which, combined with the assumption that loss and index are positively correlated ( $A + C > 1$ ), also implies that  $B > 1$ . This condition may be rearranged to give  $r \geq p(1 - qm)$ . For the remainder of the proof we consider the case  $A > B(1-C)$  for which there exist  $\gamma \in (0, \infty)$  for which  $\alpha^*(\gamma) > 0$ .

Differentiating  $\alpha^*(\gamma)$  with respect to  $\gamma$ , evaluating it at some  $\gamma \in (0, \infty)$  such that  $\frac{d\alpha^*(\gamma)}{d\gamma} = 0$ , and substituting in the first order condition results in the following equation for  $\alpha$ :

$$\tilde{\alpha}(\gamma) = \frac{(A + C - 1)e^{-\gamma L}}{(A + (1-A)e^{-\gamma L})(Ce^{-\gamma L} + (1-C))} \quad (\text{A-3})$$

$\tilde{\alpha}$  is continuous with strictly positive gradient for  $\gamma \in (0, \Gamma)$  and strictly negative gradient for  $\gamma \in (\Gamma, \infty)$  where  $\Gamma := \frac{1}{2L} \ln \left( \frac{(1-A)C}{A(1-C)} \right) \in (0, \infty)$ . We also have  $\lim_{\gamma \rightarrow 0^+} \frac{\partial \tilde{\alpha}(\gamma)}{\partial \gamma} = 1$ ,  $\lim_{\gamma \rightarrow 0^+} \tilde{\alpha}(\gamma) = A + C - 1$  and  $\lim_{\gamma \rightarrow \infty} \alpha^*(\gamma) = 0$ .

For  $m > 1$ , the optimal level of cover is zero when  $\alpha^*(\gamma) \leq 0$ , that is when  $\gamma \leq \gamma_1 := \frac{1}{L} \ln \left( \frac{A+BC-1}{A+BC-B} \right)$ , and strictly positive for  $\gamma > \gamma_1$ .  $B > 1$  and so  $0 < \gamma_1 < \infty$ . Since  $\alpha^*(\gamma)$  must have zero gradient when crossing  $\tilde{\alpha}(\gamma)$ , it cannot cross  $\tilde{\alpha}(\gamma)$  from below when  $\tilde{\alpha}(\gamma)$  is increasing, that is when  $\gamma < \Gamma$ , nor from above when  $\tilde{\alpha}(\gamma)$  is decreasing, that is when  $\gamma > \Gamma$ . L'Hôpital's rule gives  $\lim_{\gamma \rightarrow 0^+} \tilde{\alpha}(\gamma) - \alpha^*(\gamma) > 0$  and so  $\alpha^*(\gamma)$  can only cross  $\tilde{\alpha}(\gamma)$ , that is have zero gradient, for at most one  $\gamma$  denoted  $\gamma_2$  and satisfying  $\gamma_2 \geq \Gamma$ . Moreover, since  $\alpha^*$  is continuously differentiable with  $\alpha^*(\gamma) \leq 0$  for  $\gamma \leq \gamma_1$ ,  $\alpha^*(\gamma) > 0$  for  $\gamma > \gamma_1$ , and  $\lim_{\gamma \rightarrow \infty} \alpha^*(\gamma) = 0$  there must be at least one  $\gamma > 0$  such that  $\frac{d\alpha^*(\gamma)}{d\gamma} = 0$ . Therefore  $\gamma_2$  exists and is unique.

For  $m = 1$ , both  $\alpha^*$  and  $\tilde{\alpha}$  tend to  $A + C - 1$  in the limit as  $\gamma \rightarrow 0^+$  and the gradient of the former may be positive or negative, calculated to be  $[A(1-A) - C(1-C)]L$  by

repeated application of L'Hôpital's Rule:

$$\begin{aligned} \lim_{\gamma \rightarrow 0^+} \frac{d\alpha^*(\gamma)}{d\gamma} &= \lim_{\gamma \rightarrow 0^+} \frac{1}{\gamma} \left[ \alpha^*(\gamma) - \frac{(1-A)}{Ae^{\gamma L} + (1-A)} + \frac{C}{C + (1-C)e^{\gamma L}} \right] \\ &= \lim_{\gamma \rightarrow 0^+} \frac{1}{\gamma} \left[ \alpha^*(\gamma) - \frac{(1-A)}{Ae^{\gamma L} + (1-A)} + \frac{C}{C + (1-C)e^{\gamma L}} \right] \\ &\quad + \frac{A(1-A)Le^{\gamma L}}{(Ae^{\gamma L} + (1-A))^2} - \frac{C(1-C)Le^{\gamma L}}{(C + (1-C)e^{\gamma L})^2} \\ &= [A(1-A)L - C(1-C)]L \end{aligned}$$

For  $\lim_{\gamma \rightarrow \infty} \alpha^*(\gamma) = 0$  to hold, if  $A(1-A) \leq C(1-C)$  then  $\alpha^*(\gamma)$  can never cross  $\tilde{\alpha}(\gamma)$  for  $\gamma > 0$  and if  $A(1-A) > C(1-C)$  then  $\alpha^*(\gamma)$  must cross  $\tilde{\alpha}(\gamma)$  exactly once for  $\gamma > 0$ .  $\alpha^*(\gamma)$  will therefore be strictly decreasing in  $\gamma$  for  $\gamma \in (0, \infty)$  if  $A(1-A) \leq C(1-C)$  or strictly increasing for  $\gamma \in (0, \gamma_1)$  and strictly decreasing for  $\gamma \in (\gamma_1, \infty)$  if  $A(1-A) > C(1-C)$ .

For  $0 < m < 1$ , L'Hôpital's rule gives  $\lim_{\gamma \rightarrow 0^+} \alpha^*(\gamma) = +\infty$  and so  $\alpha^*(\gamma)$  can only cross  $\tilde{\alpha}(\gamma)$ , with  $\frac{\partial \alpha^*(\gamma)}{\partial \gamma} = 0$  zero times, once at  $\Gamma$ , or twice, once at  $\gamma_1 \leq \Gamma$  and once at  $\gamma_2 \geq \Gamma$ . If they cross zero times,  $\alpha^*$  is strictly decreasing over  $(0, \infty)$ , if once  $\alpha^*$  is strictly decreasing over  $(0, \infty)$  except at  $\gamma_1 = \gamma_2 := \Gamma$  where it has zero gradient, and if twice  $\alpha^*$  is strictly decreasing over  $(0, \gamma_1)$  and  $(\gamma_2, \infty)$  and strictly increasing over  $(\gamma_1, \gamma_2)$ .

**CRRA:** Now consider the case of CRRA. For risk averse CRRA with coefficient of relative risk aversion  $\gamma \in (0, \infty)$ ,  $v'(x) \propto x^{-\gamma}$  and we may rewrite first order condition (4) as  $g(\alpha, \gamma) = 0$ , where

$$g(\alpha, \gamma) = Aw_{LI}^{-\gamma} + (1-A)w_{0I}^{-\gamma} - B \times [Cw_{00}^{-\gamma} + (1-C)w_{L0}^{-\gamma}]. \quad (\text{A-4})$$

and  $w_s > 0$  for all states  $s \in S$ . Since CRRA utility is undefined if wealth in any state is negative,  $\alpha^*$  is bounded by above by  $\bar{\alpha} = \frac{w-L}{qmL}$  and below by  $\underline{\alpha} = -\frac{w-L}{(1-qm)L} \cdot \frac{\partial g}{\partial \alpha} < 0$  due to the strict concavity of the objective function and  $\lim_{\alpha \rightarrow \bar{\alpha}} g(\alpha, \gamma) = -\infty$  and  $\lim_{\alpha \rightarrow \underline{\alpha}} g(\alpha, \gamma) = +\infty$  for all  $\gamma \in (0, \infty)$  and so (A-4) defines  $\alpha$  as an implicit function of  $\gamma$ , which we denote by  $\alpha^* : \mathbb{R}^+ \rightarrow (\underline{\alpha}, \bar{\alpha})$ .  $\alpha^*$  is bounded and  $g$  is continuous in both its arguments and so  $\alpha^*$  is continuous. Moreover,  $\frac{\partial g}{\partial \gamma}$  is finite for  $\alpha \in (\underline{\alpha}, \bar{\alpha})$  and  $\frac{\partial g}{\partial \alpha} < 0$  and so  $\alpha^*(\gamma)$  is continuously differentiable.

As for the case of CARA,  $\alpha^*(\gamma) \leq 0$  for all  $\gamma > 0$  if  $A \leq B(1-C)$ , that is if  $r \geq p(1-qm)$ . This can be seen by noting that  $r \geq p(1-qm)$  can only hold along with correlation equation (2) if  $m > 1$ , in which case the sum of the first two terms of  $g$  has smaller magnitude than the sum of the second two terms for  $\alpha \geq 0$ . For the remainder of the proof we consider the case  $A > B(1-C)$  for which there exist  $\gamma \in (0, \infty)$  for which  $\alpha^*(\gamma) > 0$ .

For small positive  $\gamma$  we may apply the Taylor expansion for  $e^x$  to FOC (A-4) to give:

$$\frac{1-B}{\gamma} = A \ln(w_{LI}) + (1-A) \ln(w_{0I}) - BC \ln(w_{00}) - B(1-C) \ln(w_{L0}) + O(\gamma) \quad (\text{A-5})$$

The RHS of (A-5) is strictly increasing in  $\alpha$  and so for the first order condition to hold we must have:

$$\lim_{\gamma \rightarrow 0^+} \alpha^*(\gamma) \begin{cases} = \underline{\alpha} < 0 & \text{for } m > 1, B > 1 \\ \in (\underline{\alpha}, \bar{\alpha}) & \text{for } m = 1, B = 1 \\ = \bar{\alpha} > 0 & \text{for } m < 1, B < 1 \end{cases} \quad (\text{A-6})$$

Further,

$$\lim_{\gamma \rightarrow \infty} \alpha^*(\gamma) = 0^+ \quad (\text{A-7})$$

for the following reason. (A-4) contains four terms, each with wealth in the respective state raised to the power of  $-\gamma$ . As  $\gamma \rightarrow \infty$  the two terms with lowest wealth must dominate and for (A-4) to hold they must have opposite sign with sum approximately equal to zero. For any  $\alpha$ , only the terms corresponding to states  $L0$  and  $LI$  can have this property. We must therefore have  $\lim_{\gamma \rightarrow \infty} \left( \frac{w_{L0} + \alpha L}{w_{L0}} \right) \times \left( \frac{B(1-C)}{A} \right)^{\frac{1}{\gamma}} = 1$ . Since  $A > B(1-C)$ , the second term in brackets tends to unity from below and so the first term in brackets must tend to unity from above, with  $\alpha$  also tending to zero from above.

Totally differentiating the first order condition with respect to  $\gamma$  at the optimal (unconstrained)  $\alpha$  gives

$$\frac{\partial g}{\partial \gamma} \Big|_{\alpha=\alpha^*} + \frac{\partial g}{\partial \alpha} \Big|_{\alpha=\alpha^*} \times \frac{d\alpha^*(\gamma)}{d\gamma} = 0, \quad (\text{A-8})$$

where

$$\begin{aligned} \frac{\partial g}{\partial \gamma} = & -A \ln(w_{LI}) w_{LI}^{-\gamma} - (1-A) \ln(w_{0I}) w_{0I}^{-\gamma} \\ & + B \times [C \ln(w_{00}) w_{00}^{-\gamma} + (1-C) \ln(w_{L0}) w_{L0}^{-\gamma}], \end{aligned} \quad (\text{A-9})$$

and since  $\frac{\partial g}{\partial \alpha} < 0$ , it must be that  $\frac{\partial g}{\partial \alpha} \Big|_{\alpha=\alpha^*}$  and  $\frac{d\alpha^*(\gamma)}{d\gamma}$  have the same sign.

Substituting in the FOC and multiplying by  $w_{00}^\gamma$  gives

$$\begin{aligned} h(\alpha, \gamma) &:= w_{00}^\gamma \left[ \frac{\partial g}{\partial \gamma} + g \times \ln(w_{0I}) \right] \\ &= A \ln \left( \frac{w_{0I}}{w_{LI}} \right) \left( \frac{w_{00}}{w_{LI}} \right)^\gamma - BC \ln \left( \frac{w_{0I}}{w_{00}} \right) \\ &\quad - B(1-C) \ln \left( \frac{w_{0I}}{w_{L0}} \right) \left( \frac{w_{00}}{w_{L0}} \right)^\gamma \end{aligned} \quad (\text{A-10})$$

$h(\alpha, \gamma) = 0$  is a necessary condition for  $\alpha, \gamma$  pairs such that  $\frac{d\alpha^*}{d\gamma} = 0$  for  $\gamma \in (0, \infty)$ .

$h$  is strictly decreasing in  $\alpha$  for  $\alpha \in [0, \bar{\alpha})$  and

$$\begin{aligned} h(0, \gamma) &> 0 \quad \forall \gamma \in (0, \infty) \\ \lim_{\alpha \rightarrow \bar{\alpha}} h(\alpha, \gamma) &= -\infty \quad \forall \gamma \in (0, \infty), \end{aligned} \quad (\text{A-11})$$

with the first inequality from our restriction to cases with  $A > B(1-C)$ , and so equation (A-10) defines  $\alpha$  such that  $\frac{d\alpha^*}{d\gamma} = 0$  as an implicit function of  $\gamma$ , which we denote by  $\tilde{\alpha} : \mathbb{R}^+ \rightarrow (0, \bar{\alpha})$ .  $\tilde{\alpha}$  is bounded and  $h$  is continuous in both its arguments and so  $\tilde{\alpha}$  is continuous.

We may show that  $h$  is first strictly increasing then strictly decreasing in  $\gamma$  for  $\alpha \in [0, 1)$ , and strictly decreasing in  $\gamma$  for  $\alpha \geq 1$ . Denoting  $y_\alpha = \ln \left( \frac{w_{00}}{w_{LI}} \right)$ ,  $z_\alpha = \ln \left( \frac{w_{00}}{w_{L0}} \right)$ ,  $X_\alpha = BC \ln \left( \frac{w_{0I}}{w_{00}} \right)$ ,  $Y_\alpha = A \ln \left( \frac{w_{0I}}{w_{LI}} \right)$  and  $Z_\alpha = B(1-C) \ln \left( \frac{w_{0I}}{w_{L0}} \right)$ , we may rewrite  $h$  as

$$h(\alpha, \gamma) = Y_\alpha e^{y_\alpha \gamma} - X_\alpha - Z_\alpha e^{z_\alpha \gamma} \quad (\text{A-12})$$

where  $z_\alpha \geq y_\alpha > 0$  for  $\alpha \in [0, 1)$  with strict inequality when  $\alpha = 0$ , and  $z_\alpha > 0$ ,  $y_\alpha \leq 0$  for  $\alpha \geq 1$ . The gradient of (A-12) with respect to  $\gamma$  is given by

$$\frac{\partial h}{\partial \gamma} = \left( y_\alpha Y_\alpha - z_\alpha Z_\alpha e^{(z_\alpha - y_\alpha)\gamma} \right) e^{y_\alpha \gamma}. \quad (\text{A-13})$$

For the case of  $\alpha \geq 1$  the first term is nonpositive and the second is strictly negative, and so  $\frac{\partial h}{\partial \gamma}$  is strictly negative. For  $\alpha \in [0, 1)$ ,  $\frac{\partial h}{\partial \gamma}$  is strictly positive for  $\gamma < \Gamma$  and strictly negative for  $\gamma > \Gamma$  where

$$e^\Gamma = \left( \frac{z_\alpha Z_\alpha}{y_\alpha Y_\alpha} \right)^{\frac{1}{z_\alpha - y_\alpha}} \quad (\text{A-14})$$

For  $m > 1$ , the optimal level of cover is zero when  $\alpha^*(\gamma) \leq 0$ , that is when  $\gamma \leq \gamma_1$ , and strictly positive for  $\gamma > \gamma_1$  where

$$\begin{aligned} \frac{A + BC - B}{(w - L)^{\gamma_1}} - \frac{A + BC - 1}{w^{\gamma_1}} &= 0 \\ \therefore \gamma_1 &:= \ln \left( \frac{A + BC - 1}{A + BC - B} \right) / \ln \left( \frac{w}{w - L} \right). \end{aligned} \quad (\text{A-15})$$

$B > 1$  and so  $0 < \gamma_1 < \infty$ . Equations (A-6) and (A-11) imply that  $\lim_{\gamma \rightarrow 0^+} \tilde{\alpha}(\gamma) - \alpha^*(\gamma) > 0$ . The remainder of the proof for this case follows that for the case of  $0 < m < 1$  for CARA.

For  $m = 1$ ,  $\lim_{\gamma \rightarrow 0^+} \alpha^*(\gamma) - \tilde{\alpha}(\gamma) = 0$  since (A-5) and (A-10) become arbitrarily close for any  $\alpha \in [0, \bar{\alpha}]$  as  $\gamma \rightarrow 0^+$ . Following the case of  $m = 1$  for CARA, if  $\lim_{\gamma \rightarrow 0^+} \frac{d\alpha^*(\gamma)}{d\gamma} \geq 0$  then  $\alpha^*(\gamma)$  can never cross  $\tilde{\alpha}(\gamma)$  for  $\gamma > 0$  and otherwise  $\alpha^*(\gamma)$  must cross  $\tilde{\alpha}(\gamma)$  exactly once for  $\gamma > 0$ . The remainder of the proof for this case follows that for the case of  $m = 1$  for CARA.

If negative then  $\alpha^*(\gamma)$  can never cross  $\tilde{\alpha}(\gamma)$  for  $\gamma > 0$  and if positive it must cross  $\tilde{\alpha}(\gamma)$  if it is to satisfy  $\lim_{\gamma \rightarrow \infty} \alpha^*(\gamma) = 0$ .  $\alpha^*(\gamma)$  will therefore be strictly decreasing in  $\gamma$  for  $\gamma \in (0, \infty)$  if  $A(1 - A) \leq C(1 - C)$  or strictly increasing for  $\gamma \in (0, \gamma_1)$  and strictly decreasing for  $\gamma \in (\gamma_1, \infty)$  if  $A(1 - A) > C(1 - C)$ .

For  $0 < m < 1$ , equations (A-6) and (A-11) imply that  $\lim_{\gamma \rightarrow 0^+} \tilde{\alpha}(\gamma) - \alpha^*(\gamma) < 0$ . The remainder of the proof for this case follows that for the case of  $0 < m < 1$  for CARA.

□

*Proof of Theorem 4.* Indirect utility function  $v$  inherits strict risk aversion and decreasing absolute risk aversion from  $u$ .<sup>3</sup> For  $0 \leq \alpha \leq 1$  and any risk averse indirect utility function  $v$  that satisfies DARA there exist constants  $\gamma_1 \geq \gamma_2 \geq \gamma_3 > 0$  such that  $v'_{LI}(\gamma_1, \gamma_2, \gamma_3, \alpha) = v'_{L0}(\alpha)e^{-\gamma_1 \alpha L}$ ,  $v'_{00}(\gamma_1, \gamma_2, \gamma_3, \alpha) = v'_{L0}(\alpha)e^{-\gamma_1 \alpha L - \gamma_2(1-\alpha)L}$  and  $v'_{0I}(\gamma_1, \gamma_2, \gamma_3, \alpha) = v'_{L0}(\alpha)e^{-\gamma_1 \alpha L - \gamma_2(1-\alpha)L - \gamma_3 \alpha L}$ . Substituting these into first-order condition (4) and rearranging gives:

$$\begin{aligned} e^{-\gamma_1 \alpha L} v'_{L0}(\alpha) \left( A + (1 - A)e^{-\gamma_2(1-\alpha)L - \gamma_3 \alpha L} - BC v'_{L0} e^{-\gamma_2(1-\alpha)L} \right) \\ - B(1 - C)v'_{L0}(\alpha) = 0 \end{aligned} \quad (\text{A-16})$$

The LHS of (A-16) is strictly decreasing in  $\alpha$  due to the strict concavity of objective function (3). Moreover, it is strictly decreasing in both  $\gamma_3$  and  $\gamma_1$  at the optimal level of cover  $\alpha^*(\gamma_1, \gamma_2, \gamma_3)$ . Setting  $\gamma_1$  and  $\gamma_3$  to be as low as possible, that is equal to  $\gamma_2$  and 0 respectively, will therefore maximise the optimal cover  $\alpha^*$ .

Denoting  $0 \leq \Gamma = e^{-\gamma_2 L} \leq 1$ , equation (A-16) becomes

$$A\Gamma^{\bar{\alpha}} - (A + BC - 1)\Gamma - B(1 - C) = 0, \quad (\text{A-17})$$

and the requirement that  $\Gamma$  is chosen optimally becomes

$$-\bar{\alpha}A\Gamma^{\bar{\alpha}-1} + (A + BC - 1) = 0. \quad (\text{A-18})$$

Rearranging equations (A-17) and (A-18) and recalling that  $0 \leq \Gamma \leq 1$  gives:

$$0 \leq \left( \frac{B(1 - C)}{1 - \bar{\alpha}} \right)^{\frac{1}{\bar{\alpha}}} = \Gamma = \frac{\bar{\alpha}B(1 - C)}{(A + BC - 1)(1 - \bar{\alpha})} \leq 1$$

which may be rearranged to give equation (7). Now, all that remains to be proven is that there is a unique solution to equation (7). Taking the natural logarithm of both sides of equation (7) and rearranging gives:

$$H(\bar{\alpha}) = [\bar{\alpha} \ln(\bar{\alpha}) + (1 - \bar{\alpha}) \ln(1 - \bar{\alpha})] - \left[ \ln \left( \frac{B(1 - C)}{A} \right) + \bar{\alpha} \ln \left( \frac{A + BC - 1}{B(1 - C)} \right) \right] = 0 \quad (\text{A-19})$$

$H(\alpha)$  is strictly convex for  $0 < \alpha \leq \frac{A+BC-1}{A+B-1} < 1$  and therefore to prove uniqueness it is sufficient to show that  $\lim_{\alpha \rightarrow 0+} H(\alpha) > 0$  and  $H(\frac{A+BC-1}{A+B-1}) \leq 0$ . Now, using L'Hôpital's rule,  $\lim_{\alpha \rightarrow 0+} H(\alpha) = \ln \left( \frac{A}{B(1-C)} \right) > 0$  since  $r < p(1 - qm)$ . Finally,  $H(\frac{A+BC-1}{A+B-1}) = \ln \left( \frac{A}{A+B-1} \right) \leq 0$  since  $B \geq 1$ .

□