Hybrid Fragments of Halpern-Shoham Logic and their Expressive Power

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Abstract

Halpern and Shoham modal logic of time intervals (HS in short) is an elegant and highly influential propositional interval-based modal logic. Its sub-propositional fragments, that is fragments obtained by restricting use of propositional connectives, and their hybrid extensions with nominals and satisfaction operators have been recently studied and successfully applied in real-world use cases. Detailed investigation of their decidability and computational complexity has been conducted, however, there has been significantly less research on their expressive power. In this paper we make a step towards filling this gap. In particular, we (1) compare classes of frames definable in full HS and in its hybrid extension, and (2) determine in which sub-propositional HS-fragments we can express the difference operator, nominals, and satisfaction operators. The obtained results enable us to classify HS, its sub-propositional fragments, and their hybrid extensions according to their expressive power.

Keywords: Temporal logic; Expressive power; Hybrid logic; Halpern-Shoham logic; Interval logic.

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1. Introduction

The aim of this paper is to investigate the expressive power of the temporal logic of Halpern and Shoham (HS in short) [1] and its sub-propositional fragments in which the use of propositional connectives is restricted [2, 3, 4]. These fragments are especially interesting due to a relatively low computational complexity [2, 5, 6] and a range of potential applications [7]. Although decidability and computational complexity of these fragments have been intensively studied [2, 5, 6], their expressive power is yet to be studied in any significant depth. Our research aims at filling this gap.

Halpern-Shoham logic is a propositional multimodal logic which enables reasoning about relations between time-intervals in a one dimensional timeline. The HS language contains 12 modal operators, each corresponding to one of the Allen's binary relations between intervals [8], namely *adjacent to*, *begins*, *during*, *ends*, *later than*, *overlaps*, and their inverses (Allen's algebra contains also *identity* as the 13th relation). In the paper we will consider HS-models in which the time-line consists of an unbounded linear ordering of time-points. Propositional variables are interpreted by sets of intervals over this temporal frame. The HS-language is very expressive and the satisfiability problem of its formulas is undecidable over a range of interesting linear orders including $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ [1]. As a result, restrictions on HS have been intensively investigated in order to identify fragments of relatively low computational complexity, whose expressive power is high enough for a variety of applications. A number of methods to specify HS-fragments have been proposed, for example, restricting the set of modal operators occurring in the language [9, 10, 11], softening semantics of modal operators [12], and restricting the nesting-depth of modal operators [13].

Another method of restricting HS-formulas is based on their clausal representation [2]. The method is two folded: first, clauses are restricted to *Horn* form in which consequent of each clause consists of exactly one literal or *core* form in which both consequent and antecedent of each clause consists of exactly one literal. Additionally, the diamond modal operators are disallowed, that is only box operators occur in the modal language or only diamond modal operators are allowed. This new approach has led to the identification of tractable fragments (precisely P-complete) [6], which were already applied in real-world use cases within temporal OBDA [7]. The success of this approach motivated applying the same technique to other logics in order to establish their low complexity fragments. In particular, it was applied in modal logics K, T, K4, S4 [14], and in Metric Temporal Logic [15]. Moreover, hybrid versions of HS and its fragments have been introduced by adding to their languages *nominals*, that is atoms satisfied in exactly one interval and *satisfaction operators* indexed by nominals which allow to refer to the interval in which the particular nominals holds [5].

In this paper we investigate the expressiveness of HS, its sub-fragments, and hybrid extensions over various classes of frames. Following the approach by [3] we will distinguish between (i) irreflexive frames which were originally introduced by Halpern and Shoham in [1] and reflexive frames obtained by weakening definitions of interval relations [3, 12] (ii) non-strict frames in which punctual-intervals are allowed and strict frames in which punctual-intervals are forbidden, and (iii) discrete frames and dense frames. Combinations of the above 3 lines of distinction give us 8 basic classes of frames.

It has been showed already in [1] that discrete and dense frames are definable in full HS under the assumption that frames are irreflexive and non-strict. A fragment of HS containing only modal operators corresponding to the *adjacent to* relation and its converse (this fragment is known as the *propositional neighborhood logic* PNL) and its expressive power were studied in [16]. In particular, it was shown that discrete and dense frames are definable in PNL-language if we restrict attention to irreflexive strict frames. Moreover, PNL allows us to express the difference operator (stating that a formula holds in some interval different than the current one) and nominals [16]. There is a number of papers focused on determining which modal operators from the HS-language can be expressed in HS-fragments obtained by dropping specific operators from the language. Such classifications under the assumption that frames are irreflexive and strict have been performed with respect to all linear frames [17], dense linear frames [18], and discrete linear frames [19].

In this paper we analyse expressiveness of HS, its sub-propositional fragments, and their hybrid version in various combinations of reflexive/irreflexive, non-strict/strict, and discrete/dense frames. In particular, in Section 4 we determine which classes of frames are definable in HS and its extension with nominals HS^i. Then, in Sections 5, 6, and 7 we study expressibility of the difference operator, nominals, and satisfaction operators, respectively, in HS and its sub-propositional fragments. Finally, in Section 8 we present cumulative results of our analysis.

2. Halpern-Shoham Logic

The language of Halpern-Shoham logic consists of a countably infinite set PROP of propositional variables, classical propositional connectives: negation ($\neg$) and conjunction ($\wedge$), and 12 *diamond* modal operators: $\langle \mathsf{B}\rangle$, $\langle \overline{\mathsf{B}}\rangle$, $\langle \mathsf{D}\rangle$, $\langle \overline{\mathsf{D}}\rangle$, $\langle \mathsf{E}\rangle$, $\langle \overline{\mathsf{E}}\rangle$, $\langle \mathsf{O}\rangle$, $\langle \overline{\mathsf{O}}\rangle$, $\langle \mathsf{A}\rangle$, $\langle \overline{\mathsf{A}}\rangle$, $\langle \mathsf{L}\rangle$, and $\langle \overline{\mathsf{L}}\rangle$.

Definition 1 (HS-formula)**.** Well-formed HS-formulas are defined by the following grammar, where $p \in \mathsf{PROP}$ and $\mathsf{R} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{D}, \overline{\mathsf{D}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{O}, \overline{\mathsf{O}}, \mathsf{A}, \overline{\mathsf{A}}, \mathsf{L}, \overline{\mathsf{L}}\}$ (in what follows we will denote this set of 12 symbols by Rel):

$$\varphi \overset{\text{df}}{=} p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle \mathsf{R}\rangle\varphi.$$

Disjunction ($\vee$), implication ($\rightarrow$), equivalence ($\leftrightarrow$), and propositional constants 'true' ($\top$) and 'false' ($\bot$) are defined in the standard manner. The *box* modal operators $[\mathsf{B}]$, $[\overline{\mathsf{B}}]$, $[\mathsf{D}]$, $[\overline{\mathsf{D}}]$, $[\mathsf{E}]$, $[\overline{\mathsf{E}}]$, $[\mathsf{O}]$, $[\overline{\mathsf{O}}]$, $[\mathsf{A}]$, $[\overline{\mathsf{A}}]$, $[\mathsf{L}]$, $[\overline{\mathsf{L}}]$, are dual to $\langle \mathsf{B}\rangle$, $\langle \overline{\mathsf{B}}\rangle$, $\langle \mathsf{D}\rangle$, $\langle \overline{\mathsf{D}}\rangle$, $\langle \mathsf{E}\rangle$, $\langle \overline{\mathsf{E}}\rangle$, $\langle \mathsf{O}\rangle$, $\langle \overline{\mathsf{O}}\rangle$, $\langle \mathsf{A}\rangle$, $\langle \overline{\mathsf{A}}\rangle$, $\langle \mathsf{L}\rangle$, and $\langle \overline{\mathsf{L}}\rangle$ respectively, that is for any HS-formula φ and $\mathsf{R} \in \mathsf{Rel}$ we have $[\mathsf{R}]\varphi \overset{\text{df}}{=} \neg\langle \mathsf{R}\rangle\neg\varphi$.

Next, we define a schema of an HS-frame and discuss its properties. As usual, we will use $x < y$ as an abbreviation for $x \leq y \wedge x \neq y$.

Definition 2 (HS-frame)**.** An HS-frame is a tuple $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ such that:

- $\mathbb{D} = (D, \leq)$ is a non-strict linear ordering which is unbounded (that is $\leq$ is a reflexive, antisymmetric, transitive, and connected relation such that each element of $\mathbb{D}$ has a $<$-successor and a $<$-predecessor);
- $I(\mathbb{D})$ is the set of intervals over $\mathbb{D}$ which is either equal to:

  $$I^{+}(\mathbb{D}) \overset{\text{df}}{=} \{\langle x, y\rangle \mid x, y \in D \text{ and } x \leq y\},$$

  or to:

  $$I^{-}(\mathbb{D}) \overset{\text{df}}{=} \{\langle x, y\rangle \mid x, y \in D \text{ and } x < y\},$$

- $\mathcal{R}$ is a set of 12 binary relations between intervals which is either equal to:

  $$\mathcal{R}_{\leq} \overset{\text{df}}{=} \{\mathsf{B}_{\leq}, \overline{\mathsf{B}}_{\leq}, \mathsf{D}_{\leq}, \overline{\mathsf{D}}_{\leq}, \mathsf{E}_{\leq}, \overline{\mathsf{E}}_{\leq}, \mathsf{O}_{\leq}, \overline{\mathsf{O}}_{\leq}, \mathsf{A}_{\leq}, \overline{\mathsf{A}}_{\leq}, \mathsf{L}_{\leq}, \overline{\mathsf{L}}_{\leq}\},$$

  or to:

  $$\mathcal{R}_{<} \overset{\text{df}}{=} \{\mathsf{B}_{<}, \overline{\mathsf{B}}_{<}, \mathsf{D}_{<}, \overline{\mathsf{D}}_{<}, \mathsf{E}_{<}, \overline{\mathsf{E}}_{<}, \mathsf{O}_{<}, \overline{\mathsf{O}}_{<}, \mathsf{A}_{<}, \overline{\mathsf{A}}_{<}, \mathsf{L}_{<}, \overline{\mathsf{L}}_{<}\},$$

  where the relations in $\mathcal{R}_{\leq}$ and $\mathcal{R}_{<}$ are defined in Table 1[1].

The definition gives rise to four basic and pairwise disjoint classes of frames, namely: a) frames of the form $(\mathbb{D}, I^{+}(\mathbb{D}), \mathcal{R}_{\leq})$, b) of the form $(\mathbb{D}, I^{+}(\mathbb{D}), \mathcal{R}_{<})$, c) of the form $(\mathbb{D}, I^{-}(\mathbb{D}), \mathcal{R}_{\leq})$, and d) of the form $(\mathbb{D}, I^{-}(\mathbb{D}), \mathcal{R}_{<})$. In frames from the classes a) and b) punctual intervals are allowed. Frames from these classes are referred to as *non-strict*, in short $\mathsf{Non\text{-}S}$-frames. Whereas, in frames from the classes c) and d) punctual intervals are disallowed, and we will call such frames *strict*, in short S-frames.

It can be easily checked that in frames from the classes b) and d), that is in frames with $\mathcal{R} = \mathcal{R}_{<}$, all the relations between intervals are irreflexive. For this reason, frames from these classes are referred to as *irreflexive*, in short $<$-frames, while frames from the classes a) and c), in which $\mathcal{R} = \mathcal{R}_{\leq}$, are called *reflexive*, in short $\leq$-frames. This terminology can be a bit confusing, since in irreflexive frames – regardless of whether frames are strict or non-strict – all the relations between intervals are irreflexive, so by analogy one could expect that in reflexive frames all the relations between intervals are reflexive, but it is not the case. Nevertheless, we adopt this terminology as it is widely used in literature (see [3] and references therein). In particular, in strict HS-frames the following relations are reflexive:

$$\mathsf{B}_{\leq}, \overline{\mathsf{B}}_{\leq}, \mathsf{D}_{\leq}, \overline{\mathsf{D}}_{\leq}, \mathsf{E}_{\leq}, \overline{\mathsf{E}}_{\leq}, \mathsf{O}_{\leq}, \overline{\mathsf{O}}_{\leq},$$

whereas the following relations are irreflexive:

$$\mathsf{A}_{\leq}, \overline{\mathsf{A}}_{\leq}, \mathsf{L}_{\leq}, \overline{\mathsf{L}}_{\leq}.$$

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[1] We use the definitions from [3] but it should be noticed that in some papers the definitions of $\langle \mathsf{A} \rangle_{<}$ and $\langle \overline{\mathsf{A}} \rangle_{<}$ do not contain the condition $x' < y'$.

Table 1: Definitions of the relations between intervals in reflexive and irreflexive frames.

Reflexive frames:	Irreflexive frames:
$\langle x,y\rangle \overline{\mathsf{L}}_{\leq} \langle x',y'\rangle$ iff $y' \leq x$	$\langle x,y\rangle \overline{\mathsf{L}}_{<} \langle x',y'\rangle$ iff $y' < x$
$\langle x,y\rangle \overline{\mathsf{A}}_{\leq} \langle x',y'\rangle$ iff $x' \leq y'$ and $y' = x$	$\langle x,y\rangle \overline{\mathsf{A}}_{<} \langle x',y'\rangle$ iff $x' < y'$ and $y' = x$
$\langle x,y\rangle \overline{\mathsf{O}}_{\leq} \langle x',y'\rangle$ iff $x' \leq x \leq y' \leq y$	$\langle x,y\rangle \overline{\mathsf{O}}_{<} \langle x',y'\rangle$ iff $x' < x < y' < y$
$\langle x,y\rangle \mathsf{B}_{\leq} \langle x',y'\rangle$ iff $x = x'$ and $y' \leq y$	$\langle x,y\rangle \mathsf{B}_{<} \langle x',y'\rangle$ iff $x = x'$ and $y' < y$
$\langle x,y\rangle \mathsf{D}_{\leq} \langle x',y'\rangle$ iff $x \leq x'$ and $y' \leq y$	$\langle x,y\rangle \mathsf{D}_{<} \langle x',y'\rangle$ iff $x < x'$ and $y' < y$
$\langle x,y\rangle \mathsf{E}_{\leq} \langle x',y'\rangle$ iff $x \leq x'$ and $y = y'$	$\langle x,y\rangle \mathsf{E}_{<} \langle x',y'\rangle$ iff $x < x'$ and $y = y'$
$\langle x,y\rangle \mathsf{O}_{\leq} \langle x',y'\rangle$ iff $x \leq x' \leq y \leq y'$	$\langle x,y\rangle \mathsf{O}_{<} \langle x',y'\rangle$ iff $x < x' < y < y'$
$\langle x,y\rangle \mathsf{A}_{\leq} \langle x',y'\rangle$ iff $y = x'$ and $x' \leq y'$	$\langle x,y\rangle \mathsf{A}_{<} \langle x',y'\rangle$ iff $y = x'$ and $x' < y'$
$\langle x,y\rangle \mathsf{L}_{\leq} \langle x',y'\rangle$ iff $y \leq x'$	$\langle x,y\rangle \mathsf{L}_{<} \langle x',y'\rangle$ iff $y < x'$
$\langle x,y\rangle \overline{\mathsf{E}}_{\leq} \langle x',y'\rangle$ iff $x' \leq x$ and $y = y'$	$\langle x,y\rangle \overline{\mathsf{E}}_{<} \langle x',y'\rangle$ iff $x' < x$ and $y = y'$
$\langle x,y\rangle \overline{\mathsf{D}}_{\leq} \langle x',y'\rangle$ iff $x' \leq x$ and $y \leq y'$	$\langle x,y\rangle \overline{\mathsf{D}}_{<} \langle x',y'\rangle$ iff $x' < x$ and $y < y'$
$\langle x,y\rangle \overline{\mathsf{B}}_{\leq} \langle x',y'\rangle$ iff $x = x'$ and $y \leq y'$	$\langle x,y\rangle \overline{\mathsf{B}}_{<} \langle x',y'\rangle$ iff $x = x'$ and $y < y'$

In non-strict HS-frames $\mathsf{A}_{\leq}$ and $\overline{\mathsf{A}}_{\leq}$ become not irreflexive (which, however, does not make them reflexive). Indeed, in non-strict HS-frames punctual intervals are allowed, and for any punctual interval $\langle x,x\rangle$ it holds that $\langle x,x\rangle \mathsf{A}_{\leq} \langle x,x\rangle$ and $\langle x,x\rangle \overline{\mathsf{A}}_{\leq} \langle x,x\rangle$.

In strict HS-frames the relations from $\mathcal{R}_{<}$ are jointly exhaustive and pairwise disjoint among distinct intervals, that is between any two distinct intervals holds exactly one of the relations from $\mathcal{R}_<$ [8]. On the other hand, in non-strict frames the relations from $\mathcal{R}_<$ are not pairwise disjoint, for example if $x < y$, then $\langle x, x\rangle \mathsf{A}_< \langle x, y\rangle$ and $\langle x, x\rangle \overline{\mathsf{B}}_< \langle x, y\rangle$. The relations from $\mathcal{R}_\leq$ are pairwise disjoint neither in non-strict nor strict frames for the following inclusions hold in both classes:

$$\overline{\mathsf{A}}_\leq \subseteq \overline{\mathsf{L}}_\leq; \quad \overline{\mathsf{A}}_\leq \subseteq \overline{\mathsf{O}}_\leq; \quad \mathsf{B}_\leq \subseteq \overline{\mathsf{O}}_\leq; \quad \mathsf{B}_\leq \subseteq \mathsf{D}_\leq; \quad \mathsf{E}_\leq \subseteq \mathsf{D}_\leq; \quad \mathsf{E}_\leq \subseteq \mathsf{O}_\leq;$$

$$\mathsf{A}_\leq \subseteq \mathsf{L}_\leq; \quad \mathsf{A}_\leq \subseteq \mathsf{O}_\leq; \quad \overline{\mathsf{B}}_\leq \subseteq \mathsf{O}_\leq; \quad \overline{\mathsf{B}}_\leq \subseteq \overline{\mathsf{D}}_\leq; \quad \overline{\mathsf{E}}_\leq \subseteq \overline{\mathsf{D}}_\leq; \quad \overline{\mathsf{E}}_\leq \subseteq \overline{\mathsf{O}}_\leq.$$

We will consider additional constraints imposed on the relation $\leq$ in the above-mentioned classes of frames, namely: discreteness and density. Frames in which $\mathbb{D}$ is discrete (dense) will be referred to as *discrete* (respectively *dense*) frames, in short Dis-frames, (respectively Den-frames). Combinations of the 4 main classes of frames with these 2 further constraints put on the order of time points give rise to 8 classes of frames. Each combination will be denoted by a sequence of symbols abbreviating the chosen type of a frame, for example irreflexive, non-strict, and discrete frames will be denoted by $(<, \mathsf{Non\text{-}S}, \mathsf{Dis})$. If one of the elements in the tuple is missing, it means that it is not specified, for example $(<, \mathsf{Dis})$ denotes irreflexive and discrete frames which can be non-strict or strict.

Definition 3 (HS-model)**.** An HS-model is a tuple $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ such that $(\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ is an HS-frame and $V : \mathsf{PROP} \longrightarrow \mathcal{P}(I(\mathbb{D}))$.

In what follows, we will say that an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ is *based on* the HS-frame $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$. The type of a model is determined by the type of the frame it is based on, hence we will say that a model is reflexive/irreflexive, non-strict/strict, discrete/dense whenever the frame it is based on is.

Definition 4 (Satisfaction in HS)**.** *Satisfaction* of a formula φ in an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ and an interval $\langle x, y\rangle \in I(\mathbb{D})$ is defined inductively by the following conditions, where ψ and χ are any HS-formulas and $\mathsf{R} \in \mathsf{Rel}$[2]:

$$
\begin{array}{lll}
\mathcal{M}, \langle x, y\rangle \models p & \text{iff} & \langle x, y\rangle \in V(p), \text{ for any } p \in \mathsf{PROP}; \\
\mathcal{M}, \langle x, y\rangle \models \neg\psi & \text{iff} & \mathcal{M}, \langle x, y\rangle \not\models \psi; \\
\mathcal{M}, \langle x, y\rangle \models \psi \wedge \chi & \text{iff} & \mathcal{M}, \langle x, y\rangle \models \psi \text{ and } \mathcal{M}, \langle x, y\rangle \models \chi; \\
\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{R}\rangle\psi & \text{iff} & \text{there is } \langle x', y'\rangle \in I(\mathbb{D}) \text{ such that } \mathcal{M}, \langle x', y'\rangle \models \psi \text{ and:} \\
 & & \langle x, y\rangle \mathsf{R}_{\leq} \langle x', y'\rangle \text{ if } \mathcal{R} = \mathcal{R}_{\leq}, \text{ or} \\
 & & \langle x, y\rangle \mathsf{R}_{<} \langle x', y'\rangle \text{ if } \mathcal{R} = \mathcal{R}_{<}.
\end{array}
$$

An HS-formula φ is *true in an* HS-*model* $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ (denoted by $\mathcal{M} \models \varphi$) whenever for all $\langle x, y\rangle \in I(\mathbb{D})$ it holds that $\mathcal{M}, \langle x, y\rangle \models \varphi$. Then, an HS-formula φ is *true in an* HS-*frame* $\mathcal{F}$ (formally: $\mathcal{F} \models \varphi$) whenever for any HS-model $\mathcal{M}$ based on $\mathcal{F}$ it holds that $\mathcal{M} \models \varphi$. By $\langle x, y\rangle \neq \langle x', y'\rangle$ we will denote that $\langle x, y\rangle$ is distinct from $\langle x', y'\rangle$, that is $x \neq x'$ or $y \neq y'$.

Observe that for any class of HS-frames the following equivalences hold for any HS-formula φ:

$$
\begin{array}{lll}
\langle \mathsf{D}\rangle\varphi \leftrightarrow \langle \mathsf{B}\rangle\langle \mathsf{E}\rangle\varphi; & \langle \mathsf{O}\rangle\varphi \leftrightarrow \langle \mathsf{E}\rangle\langle \overline{\mathsf{B}}\rangle\varphi; & \langle \mathsf{L}\rangle\varphi \leftrightarrow \langle \mathsf{A}\rangle\langle \mathsf{A}\rangle\varphi; \\
\langle \overline{\mathsf{D}}\rangle\varphi \leftrightarrow \langle \overline{\mathsf{B}}\rangle\langle \overline{\mathsf{E}}\rangle\varphi; & \langle \overline{\mathsf{O}}\rangle\varphi \leftrightarrow \langle \mathsf{B}\rangle\langle \overline{\mathsf{E}}\rangle\varphi; & \langle \overline{\mathsf{L}}\rangle\varphi \leftrightarrow \langle \overline{\mathsf{A}}\rangle\langle \overline{\mathsf{A}}\rangle\varphi.
\end{array}
$$

It follows that removing the operators $\langle \mathsf{D}\rangle$, $\langle \overline{\mathsf{D}}\rangle$, $\langle \mathsf{O}\rangle$, $\langle \overline{\mathsf{O}}\rangle$, $\langle \mathsf{L}\rangle$, and $\langle \overline{\mathsf{L}}\rangle$ from the language does not decrease expressiveness of the logic.

An important representation of an HS-frame $(\mathbb{D}, I(\mathbb{D}), \mathcal{R})$, called *compass representation* is obtained by treating an interval $\langle x, y\rangle$ as a point in a two-dimensional Cartesian space $\mathbb{D} \times \mathbb{D}$, such that the abscissa of this point has a value x and its ordinate has a value y [20]. In the compass representation non-punctual intervals correspond to points lying in the north-western half-plane of

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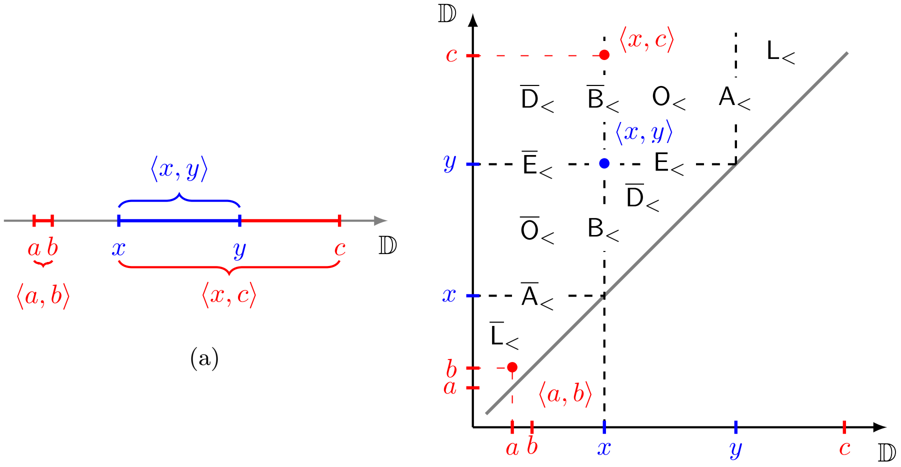
[2] For simplicity of the presentation, a symbol R is used both for relational symbols that occur in modal operators in the language and for relations in the semantics that are used to interpret corresponding modal operators. Thus, on the left-hand side of the condition, the symbol R denotes an element of Rel, whereas on the right-hand side, it denotes a relation from $\mathcal{R}_{\leq}$ or $\mathcal{R}_{<}$. It should be clear from the context in which of the above meanings a symbol is used.

$\mathbb{D} \times \mathbb{D}$ (the points whose abscissa is strictly smaller than the ordinate) and the points lying on the diagonal correspond to punctual intervals. In such a setting interval relations gain spatial interpretations. That is, given an interval $\langle x, y\rangle$ intervals accessible from $\langle x, y\rangle$ with relations from Rel can be determined on the basis of a relative position of the corresponding points in the two-dimensional Cartesian space as depicted in Figure 1. Compass representation was used, for example, to construct an axiomatic system for the Halpern-Shoham logic [20] and to determine the decidability as well as computational complexity of some fragments of the logic [3].



(a)

(b)

Figure 1: (a) one-dimensional and (b) compass representations of a frame of the Halpern-Shoham logic, in which $\langle x, y\rangle \mathsf{L}_< \langle a, b\rangle$, and $\langle x, y\rangle \overline{\mathsf{B}}_< \langle x, c\rangle$.

The satisfiability problem for the Halpern-Shoham logic formulas – that is the problem of checking whether a given formula of the Halpern-Shoham logic has a model – was shown to be undecidable over most interesting frame classes, for instance for any class of frames in which the ordering of time points contains an infinite ascending chain such as natural numbers, integers, and rational numbers [1]. The task of restricting HS to obtain fragments with a good trade-off between expressive power and complexity is studied up to the present days [10].

One of the main methods of restricting the Halpern-Shoham logic – which was already proposed by the authors of the logic – is to limit the number of modal operators in the language [10, 9]. Since the set $\mathcal{R}$ consists of 12 interval relations, there are $2^{12} = 4096$ possible combinations of modal operators that may occur in the language. A systematic study of their expressive power and computational complexity resulted in a nearly complete classification. It was proved that the easiest fragments are NP-complete, whereas the other are PSpace-complete, NExpTime-complete, ExpSpace-complete, or undecidable [11, 21]. Other ways of restricting the Halpern-Shoham logic consist for example in weakening the definitions of relations interpreting the modal operators, and imposing additional constraints on the flow of time (for example, discreteness or density) [3, 14, 7]. In the following section we will introduce another way of restricting HS which is especially important for this paper, namely imposing restriction on the use of propositional connectives.

3. Language Modifications

We will describe the recently introduced sub-propositional fragments of HS [3, 14, 7, 2]. First, we present the clausal form of HS-formulas.

Definition 5 (Positive temporal literal)**.** *Positive propositional literals* (literals in short) are defined by the following grammar, for a propositional variable $p \in \mathsf{PROP}$ and $\mathsf{R} \in \mathsf{Rel}$:

$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid \langle \mathsf{R} \rangle \lambda \mid [\mathsf{R}]\lambda.$$

Let $[\mathsf{U}]$ be the universal modality, that is $[\mathsf{U}]\varphi$ is true in a model, whenever φ is satisfied in every interval in this model. It is easy to see that $[\mathsf{U}]$ can be expressed in HS-language, for example in unbounded frames $[\mathsf{U}]\varphi$ can be defined as $[\mathsf{L}][\overline{\mathsf{L}}]\varphi$. The clausal form of HS-formulas is as follows:

Definition 6 (HS-formula in a clausal form)**.** HS-formulas in clausal form are generated by the following grammar, where λ is a positive temporal literal:

$$\varphi \overset{\text{df}}{=} \lambda \mid \neg\lambda \mid [\mathsf{U}](\lambda \wedge \ldots \wedge \lambda \rightarrow \lambda \vee \ldots \vee \lambda) \mid \varphi \wedge \varphi.$$

We say that formulas φ and ψ are *equisatisfiable* whenever φ is satisfiable if and only if ψ is. It is relatively easy to show that each HS-formula defined in Definition 1 can be transformed into an equisatisfiable formula in the clausal form and vice versa. The clausal form of HS-formulas corresponds to the Fisher's representation of formulas of linear temporal logic in *separated normal form* (SNF in short) [22]. SNF enables us to treat formulas as sets of initial conditions of the form λ and $\neg\lambda$, together with universal rules which are of the form $[\mathsf{U}](\lambda \wedge \ldots \wedge \lambda \rightarrow \lambda \vee \ldots \vee \lambda)$. We adopt this point of view and for a formula φ in clausal form we distinguish (similarly as in [3]):

- *Initial conditions* of φ, which are conjuncts in φ that are not within the range of any universal modal operator $[\mathsf{U}]$;
- *Clauses* of φ, which are subformulas of φ of the form $[\mathsf{U}]\psi$, where ψ is a subformula of φ.

The class of clausal forms of HS-formulas can be further restricted by imposing additional constraints on its grammar which give raise to the so-called *Horn* and *core* fragments of HS [2, 3].

Horn HS-formulas (HS_{horn}-formulas in short) are defined by the following modification of the grammar from Definition 6:

$$\varphi \overset{\mathrm{df}}{=} \lambda \mid [\mathsf{U}](\lambda \wedge \ldots \wedge \lambda \rightarrow \lambda) \mid \varphi \wedge \varphi.$$

Core HS-formulas (HS_{core}-formulas in short) are defined by the following grammar:

$$\varphi \overset{\mathrm{df}}{=} \lambda \mid [\mathsf{U}](\lambda \rightarrow \lambda) \mid [\mathsf{U}](\lambda \wedge \lambda \rightarrow \bot) \mid \varphi \wedge \varphi.$$

Modifications of the grammar of positive temporal literals from Definition 5 give rise to further HS-fragments, namely we distinguish fragments whose modal language is restricted to diamond modalities, that is operators of the form $\langle \mathsf{R} \rangle$, where $\mathsf{R} \in \mathsf{Rel}$, and fragments in which only box modalities are allowed, that is operators of the form $[\mathsf{R}]$, where $\mathsf{R} \in \mathsf{Rel}$. Fragments in which modal operators are limited to diamonds will be denoted by $\Diamond$ in the superscript. For instance,

$\mathsf{HS}^{\Diamond}_{horn}$ denotes the Horn fragment of HS-formulas in which modalities are restricted to diamond modalities. Fragments with modal operators restricted to diamonds are obtained by the following modification of the grammar given in Definition 5:

$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid \langle \mathsf{R} \rangle \lambda.$$

In a similar way, fragments in which modalities are restricted to box modalities will be denoted by $\Box$ in the superscript. Such fragments are obtained by the following modification of the grammar from Definition 5:

$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid [\mathsf{R}] \lambda.$$

One of the crucial constructs in temporal knowledge representation is *referentiality*, that is possibility to label time intervals (or time points in the case of point-based temporal logics) and then to refer to a chosen interval (or a time point) with a concrete label [23, 24]. The most straightforward way to provide referentiality in a modal logic is to *hybridize* this logic, that is to extend the language with, for example, the following symbols [24]:

- NOM – a countable set of nominals different than PROP;
- $\{@_i \mid i \in \mathsf{NOM}\}$ – a set of satisfaction operators indexed with nominals.

In what follows, we consider HS-languages with nominals or with both nominals and satisfaction operators. In the former case nominals are added to the grammar of positive temporal literals, that is the grammar of positive temporal is defined as follows for $p \in \mathsf{PROP}$, $\mathsf{R} \in \mathsf{Rel}$, and $i \in \mathsf{NOM}$:

$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid \langle \mathsf{R} \rangle \lambda \mid [\mathsf{R}] \lambda \mid i.$$

The class of positive temporal literals with nominals can be further extended with satisfaction operators, that is expressions of the form $@_i \lambda$, where i is a nominal. Such a class is defined by the following grammar, for $p \in \mathsf{PROP}$, $\mathsf{R} \in \mathsf{Rel}$, and $i \in \mathsf{NOM}$:

$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid \langle \mathsf{R} \rangle \lambda \mid [\mathsf{R}] \lambda \mid i \mid @_i \lambda.$$

The languages with nominals are denoted by i in the upper index, whereas the languages with nominals and satisfaction operators by the sequence $i, @$ in the upper index. For instance, the language $\mathsf{HS}^{\Box,i,@}_{horn}$ is the extension of $\mathsf{HS}^{\Box}_{horn}$ obtained by augmenting the set of positive temporal literals with the expressions of the forms i and $@_i\lambda$. That is for $p \in \mathsf{PROP}$, $\mathsf{R} \in \mathsf{Rel}$, and $i \in \mathsf{NOM}$, the $\mathsf{HS}^{\Box,i,@}_{horn}$-formulas are defined by:

$$\varphi \overset{\text{df}}{=} \lambda \mid [\mathsf{U}](\lambda \wedge \ldots \wedge \lambda \to \lambda) \mid \varphi \wedge \varphi;$$
$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid [\mathsf{R}]\lambda \mid i \mid @_i\lambda.$$

A Hasse diagram of sub-propositional HS-fragments and their hybrid versions is depicted in Figure 2.

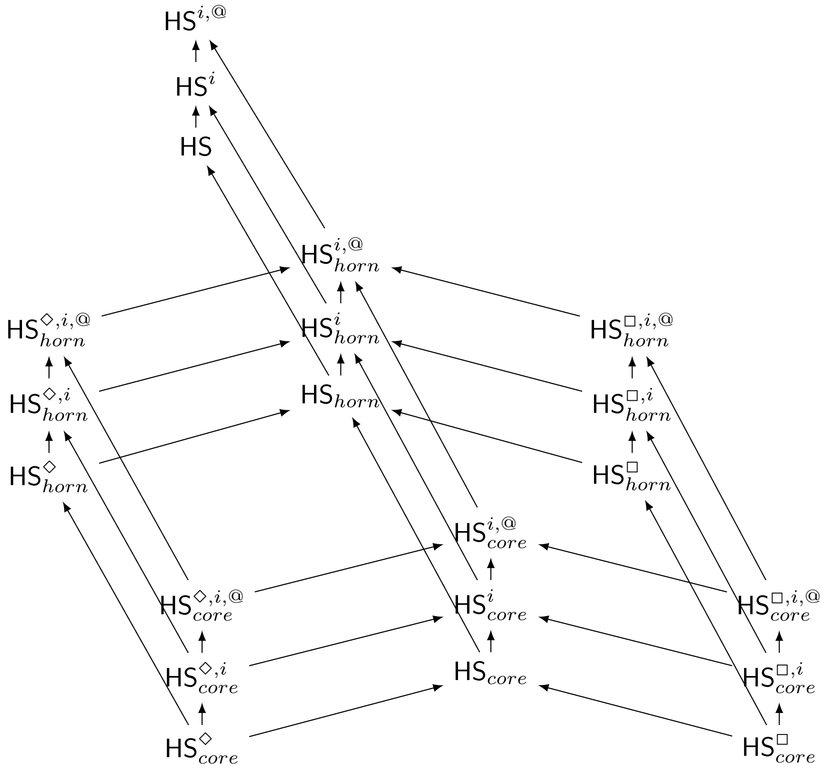


Figure 2: Hasse diagram of Horn and core HS-fragments and their hybrid versions, where an arrow indicates a syntactical extension.

A hybrid HS-model $\mathcal{M}$ is a pair $(\mathcal{F}, V)$ such that $\mathcal{F}$ is an HS-frame and a valuation $V : \mathsf{PROP} \cup \mathsf{NOM} \longrightarrow \mathcal{P}(I(\mathbb{D}))$ assigns a set of intervals to each nominal and propositional variable with an additional assumption that $V(i)$ is a singleton for any $i \in \mathsf{NOM}$. The satisfaction relation that holds between hybrid HS-models, intervals, and hybrid HS-formulas is defined as in Definition 4 with the following additional clauses, for all formulas φ and nominals $i \in \mathsf{NOM}$:

$$\begin{array}{lll} \mathcal{M}, \langle x, y\rangle \models i & \text{iff} & V(i) = \{\langle x, y\rangle\}; \\ \mathcal{M}, \langle x, y\rangle \models @_i\varphi & \text{iff} & \mathcal{M}, \langle x', y'\rangle \models \varphi, \text{ where } \langle x', y'\rangle \text{ is such that} \\ & & V(i) = \{\langle x', y'\rangle\}. \end{array}$$

Hybrid machinery usually extends the expressive power of a modal language and enables to overcome the local nature of standard modal logic [23, 25]. Although the language of HS does not contain hybrid machinery, it is expressive enough to define nominals and satisfaction operators in irreflexive frames [23]. Despite the fact that hybridization of interval temporal logics was already recognized as a promising line of research [24], it has received only a limited attention in the research community and our research is the first step towards a classification of HS-fragments according to their capability of expressing the hybrid machinery.

In the following sections we will present new results on expressive power of HS and its fragments. In particular, we will consider the following questions in various HS-fragments:

- Which classes of frames are definable?
- Which hybrid operators are expressible?

4. Frames Definability

Recall that we consider HS-frames of the form $(\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ such that $\mathbb{D} = (D, \leq)$ is a non-strict unbounded linear order, $I(\mathbb{D})$ is either $I^+(\mathbb{D})$ or $I^-(\mathbb{D})$, and $\mathcal{R}$ is either $\mathcal{R}_\leq$ or $\mathcal{R}_<$ (see Definition 2). In what follows, we will analyse definability of various classes of frames. Formally, we say that a formula φ *defines* a class

$\mathcal{K}$ of HS-frames *with respect to* a class $\mathcal{K}' \supseteq \mathcal{K}$ of HS-frames if and only if the following condition holds for any $\mathcal{F} \in \mathcal{K}'$:

$$\mathcal{F} \models \varphi \text{ if and only if } \mathcal{F} \in \mathcal{K}.$$

We say that a class $\mathcal{K}$ of HS-frames is *L-definable* with respect to a class $\mathcal{K}'$ of HS-frames if there is a formula φ from the language L such that φ defines $\mathcal{K}$ with respect to $\mathcal{K}'$.

As the choice between reflexive and irreflexive frames makes a significant difference for the meaning of modal operators we will consider definability with respect to reflexive and irreflexive frames separately. We will determine definability of the following classes:

- Non-S-frames, that is frames in which punctual intervals are allowed;
- S-frames, that is frames in which punctual intervals are disallowed;
- Dis-frames, that is discrete frames;
- Den-frames, that is dense frames.

We will consider HS-definability and HS^i-definability (where HS^i is an extension of HS with nominals – see Section 3). The obtained classification will allow us to capture differences in expressive power between HS and HS^i. Moreover, we will be able to compare expressibility in reflexive and irreflexive frames.

We start by showing results on definability with respect to irreflexive HS-frames. We will show that non-strict frames are defined with respect to irreflexive frames by the following HS-formula:

$$\mathsf{non\text{-}strict}_{(<)} \overset{\text{df}}{=} [\mathsf{A}]\neg[\mathsf{B}]\bot \wedge \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot.$$

Theorem 7. *The class of non-strict frames is HS-definable with respect to irreflexive frames.*

Proof. We claim that the formula $\mathsf{non\text{-}strict}_{(<)}$ defines non-strict frames with respect to the class of irreflexive frames. We will show that the following statements are equivalent for each irreflexive HS-frame $\mathcal{F}$:

1. $\mathcal{F} \models \mathsf{non\text{-}strict}_{(<)}$.

2. $\mathcal{F}$ is a non-strict frame.

($1 \Rightarrow 2$) Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<)$ be an irreflexive frame with $\mathcal{F} \models \mathsf{non\text{-}strict}_{(<)}$. Fix a model $\mathcal{M}$ based on $\mathcal{F}$ and an arbitrary $y \in \mathbb{D}$. To prove that $\mathcal{F}$ is non-strict we need to show that $\langle y, y\rangle \in I(\mathbb{D})$.

Let $x \in \mathbb{D}$ be such that $\langle x, y\rangle \in I(\mathbb{D})$. Since $\mathsf{non\text{-}strict}_{(<)}$ is valid in $\mathcal{F}$, we have $\mathcal{M}, \langle x, y\rangle \models \mathsf{non\text{-}strict}_{(<)}$, so $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg[\mathsf{B}]\bot$ and $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$. By $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$ there is z such that $y < z$ and $\mathcal{M}, \langle y, z\rangle \models \langle\mathsf{B}\rangle[\mathsf{B}]\bot$. Which means that there is $w < z$ such that $\langle y, w\rangle \in I(\mathbb{D})$ and $\mathcal{M}, \langle y, w\rangle \models [\mathsf{B}]\bot$.

Since $\langle y, w\rangle \in I(\mathbb{D})$, we have $y \leq w$. We claim that $y = w$. Suppose that it is not the case, that is $y < w$. Then, $\langle x, y\rangle \mathsf{A}_< \langle y, w\rangle$. Since $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg[\mathsf{B}]\bot$, by $\langle x, y\rangle \mathsf{A}_< \langle y, w\rangle$ we get $\mathcal{M}, \langle y, w\rangle \models \neg[\mathsf{B}]\bot$. On the other hand, we have showed that $\mathcal{M}, \langle y, w\rangle \models [\mathsf{B}]\bot$, so a contradiction arises. It follows that $y = w$. Since $\langle y, w\rangle \in I(\mathbb{D})$, we obtain $\langle y, y\rangle \in I(\mathbb{D})$.

($2 \Rightarrow 1$) Let $\mathcal{F} = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_<)$ be an irreflexive non-strict HS-frame. We need to show that $\mathcal{F} \models \mathsf{non\text{-}strict}_{(<)}$. Fix an interval $\langle x, y\rangle \in I^+(\mathbb{D})$ and an arbitrary model $\mathcal{M}$ based on $\mathcal{F}$. Now, our aim is to show $\mathcal{M}, \langle x, y\rangle \models \mathsf{non\text{-}strict}_{(<)}$, that is $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg[\mathsf{B}]\bot \wedge \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$.

First, we will show that $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg[\mathsf{B}]\bot$. Let $\langle y, z\rangle \in I^+(\mathbb{D})$ be such that $\langle x, y\rangle \mathsf{A}_< \langle y, z\rangle$, that is $y < z$. We need to show that $\mathcal{M}, \langle y, z\rangle \models \neg[\mathsf{B}]\bot$, that is $\mathcal{M}, \langle y, z\rangle \models \langle\mathsf{B}\rangle\top$. Since $y < z$ and $\langle y, y\rangle \in I^+(\mathbb{D})$ we have $\langle y, z\rangle \mathsf{B}_< \langle y, y\rangle$. Clearly, $\mathcal{M}, \langle y, y\rangle \models \top$, hence $\mathcal{M}, \langle y, z\rangle \models \langle\mathsf{B}\rangle\top$. Thus, $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg[\mathsf{B}]\bot$.

It remains to show that $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$. By unboundedness of $\mathbb{D}$ there exists $w \in \mathbb{D}$ such that $y < w$. Hence, $\langle x, y\rangle \mathsf{A}_< \langle y, w\rangle$. In what follows, we show $\mathcal{M}, \langle y, w\rangle \models \langle\mathsf{B}\rangle[\mathsf{B}]\bot$. By $y < w$ and $\langle y, y\rangle \in I^+(\mathbb{D})$ we obtain $\langle y, w\rangle \mathsf{B}_< \langle y, y\rangle$. Then, it suffices to show that $\mathcal{M}, \langle y, y\rangle \models [\mathsf{B}]\bot$, that is $\mathcal{M}, \langle y, y\rangle \models \neg\langle\mathsf{B}\rangle\top$. Clearly, there is no v such that $y \leq v < y$, so there is no $\langle y, v\rangle$ such that $\langle y, y\rangle \mathsf{B}_< \langle y, v\rangle$. Hence, $\mathcal{M}, \langle y, y\rangle \models \neg\langle\mathsf{B}\rangle\top$ and consequently $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$.

Therefore, $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg[\mathsf{B}]\bot \wedge \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$, so $\mathcal{M}, \langle x, y\rangle \models \mathsf{non\text{-}strict}_{(<)}$

which concludes the proof. ☐

Moreover, the negation of $\mathsf{non\text{-}strict}_{(<)}$ defines strict frames with respect to irreflexive frames.

Theorem 8. *The class of strict frames is* HS*-definable with respect to irreflexive frames.*

In the following proofs we will use two formulas which were originally introduced in [1], namely:

$$\mathsf{length0} \overset{\text{df}}{=} [\mathsf{B}]\bot;$$
$$\mathsf{length1} \overset{\text{df}}{=} \langle \mathsf{B} \rangle \top \wedge [\mathsf{B}][\mathsf{B}]\bot.$$

In irreflexive frames the formula $\mathsf{length0}$ is satisfied exactly in intervals $\langle x, y \rangle$ for which there is no $\langle x, z \rangle \in I(\mathbb{D})$ such that $z < y$. In particular, in non-strict irreflexive frames $\mathsf{length0}$ is satisfied precisely in punctual-intervals, that is in intervals 'of length 0.' The formula $\mathsf{length1}$ is satisfied in non-strict irreflexive frames precisely in intervals $\langle x, y \rangle$ such that y is the immediate $<$-successor of x, that is in intervals 'of length 1.'

Halpern and Shoham showed in [1] that discrete frames are defined with respect to irreflexive frames by the formula:

$$\mathsf{dis}_{(<)} \overset{\text{df}}{=} \mathsf{length0} \vee \mathsf{length1} \vee \big(\langle \mathsf{B} \rangle \mathsf{length1} \wedge \langle \mathsf{E} \rangle \mathsf{length1}\big),$$

whose intuitive meaning is that each interval is either punctual or 'of length 1', or contains a beginning and an ending intervals 'of length 1.' Moreover, Halpern and Shoham showed that in irreflexive and non-strict frames we can define dense frames by means of the formula $\neg\mathsf{length1}$, which states that there is no interval 'of length 1' [1]. However, this formula does not define dense frames in strict frames. We claim that dense frames are defined with respect to the class of all (that is non-strict or strict) irreflexive frames by the formula:

$$\mathsf{den}_{(<)} \overset{\text{df}}{=} (\neg \mathsf{non\text{-}strict}_{(<)} \wedge \langle \mathsf{B} \rangle \top) \vee (\mathsf{non\text{-}strict}_{(<)} \wedge \neg \mathsf{length1}).$$

This formula states that either the frame is strict and there is no interval of a minimal length or the frame is non-strict and there is no interval 'of length 1.'

Theorem 9. *The class of dense frames is* HS*-definable with respect to irreflexive frames.*

The theorem is proved in a similar manner as Theorem 7, so we move it to the Appendix (Appendix contains all proofs that we skip in the main body of the paper).

Next we will consider frame definability with respect to reflexive HS-frames. First, the class of non-strict frames is definable with respect to reflexive frames by the following formula:

$$\mathsf{non\text{-}strict}_{(\leq)} \overset{\mathrm{df}}{=} [\mathsf{E}]p \to \langle \mathsf{A} \rangle p,$$

where p is any propositional variable. This formula states that if p is satisfied in all intervals ending the current interval, then p is satisfied in some interval adjacent to the current interval. This is the case exactly in non-strict reflexive frames. Indeed, in such frames the punctual interval containing the endpoint of the current interval is in both $\mathsf{E}_{\leq}$ and $\mathsf{A}_{\leq}$ relations with the current interval.

Theorem 10. *The class of non-strict frames is* HS*-definable with respect to reflexive frames.*

Next, we will prove a negative result showing that a particular class of frames is not definable. In the proof we will make use of *surjective bounded morphisms* between HS-frames, which are defined analogously as in the standard modal logic [25].

Definition 11 (Surjective bounded morphism between HS-frames)**.** A *surjective bounded morphism* from a frame $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ to a frame $\mathcal{F}' = (\mathbb{D}', I'(\mathbb{D}'), \mathcal{R})$ is a surjection $f : I(\mathbb{D}) \longrightarrow I'(\mathbb{D}')$ satisfying the following conditions for all $\mathsf{R} \in \mathcal{R}$, $\langle x, y \rangle \in I(\mathbb{D})$, $\langle u, w \rangle \in I(\mathbb{D})$ and $\langle u', w' \rangle \in I'(\mathbb{D}')$:

(ZIG) If $\langle x, y \rangle \mathsf{R} \langle u, w \rangle$, then $f(\langle x, y \rangle) \mathsf{R} f(\langle u, w \rangle)$;

(ZAG) If $f(\langle x,y\rangle)\mathsf{R}\langle u',w'\rangle$, then there is $\langle u,w\rangle \in I(\mathbb{D})$ such that $\langle x,y\rangle\mathsf{R}\langle u,w\rangle$ and $f(\langle u,w\rangle) = \langle u',w'\rangle$.

If there exists a surjective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$, then we say that $\mathcal{F}'$ is a *bounded morphic image* of $\mathcal{F}$.

Similarly as in the case of standard modal logic a surjective bounded morphism between HS-frames preserves truth of HS-formulas in these frames.

Theorem 12. *Let $\mathcal{F}$ and $\mathcal{F}'$ be* HS*-frames such that $\mathcal{F}'$ is a bounded morphic image of $\mathcal{F}$. Then, for any* HS*-formula φ:*

$$\mathcal{F} \models \varphi \textit{ implies } \mathcal{F}' \models \varphi.$$

Proof of this theorem is analogous to the proof for standard modal logic [25]. We will use this result to show that dense frames are not HS-definable with respect to reflexive frame. Note that by Theorem 9 dense frames are HS-definable with respect to irreflexive frames, so the next result will show a difference between expressiveness of HS in irreflexive and reflexive frames.

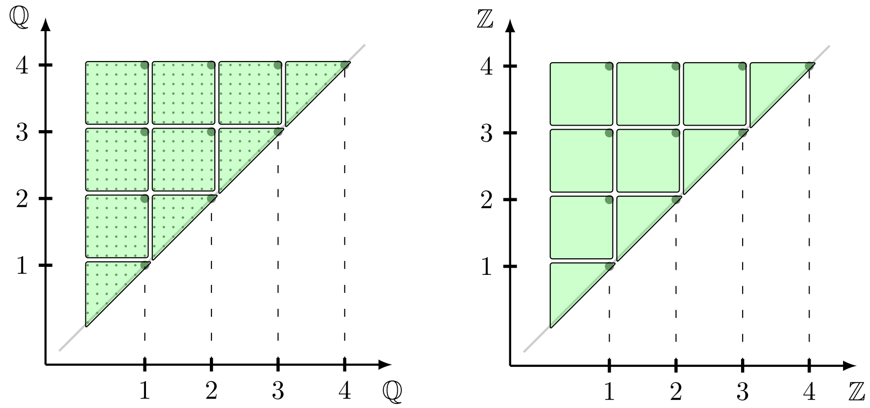
Theorem 13. *The class of dense frames is not* HS*-definable with respect to reflexive frames.*

Proof. Suppose that there is an HS-formula φ which defines dense frames with respect to reflexive frames. Let $\mathcal{F} = (\mathbb{Q}, I^+(\mathbb{Q}), \mathcal{R}_{\leq})$ and $\mathcal{F}' = (\mathbb{Z}, I^+(\mathbb{Z}), \mathcal{R}_{\leq})$ be reflexive HS-frames such that $\mathbb{Q}$ is the standard order of rational numbers and $\mathbb{Z}$ the standard order of integers. Clearly, $\mathbb{Q}$ is dense and $\mathbb{Z}$ is not, so we have $\mathcal{F} \models \varphi$ and $\mathcal{F}' \not\models \varphi$.

We will construct a surjective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$. Then, $\mathcal{F} \models \varphi$ will imply $\mathcal{F}' \models \varphi$. Hence, a contradiction with $\mathcal{F}' \not\models \varphi$ will be raised, which shows that there is no HS-formula defining dense frames with respect to reflexive frames. The intended surjective bounded morphism $f : I^+(\mathbb{Q}) \longrightarrow I^+(\mathbb{Z})$ is defined as follows:

$$f(\langle x,y\rangle) \overset{\mathrm{df}}{=} \langle \lceil x \rceil, \lceil y \rceil\rangle,$$

where $\lceil\cdot\rceil$ is the ceiling function, that is for any $q \in \mathbb{Q}$, $\lceil q\rceil$ is the least integer that is greater than or equal to q. For a pictorial representation of $\mathcal{F}$, $\mathcal{F}'$, and f see Figure 3.



($\mathcal{F}$) ($\mathcal{F}'$)

Figure 3: Surjective bounded morphism f from the frame $\mathcal{F}$ to the frame $\mathcal{F}'$.

Since ceiling is a surjective function from rational numbers to integers, we obtain that f is a surjective function from $I^+(\mathbb{Q})$ to $I^+(\mathbb{Z})$. To show that (ZIG) and (ZAG) hold a routine inspection of all $\mathsf{R} \in \mathcal{R}_{\leq}$ needs to be performed. In what follows, we prove these conditions for $\mathsf{A}_{\leq}$ as the other cases are similar.

Let $\langle x, y\rangle \in I^+(\mathbb{Q})$ and $\langle u, w\rangle \in I^+(\mathbb{Q})$ be such that $\langle x, y\rangle \mathsf{A}_{\leq} \langle u, w\rangle$. We will show that $f(\langle x, y\rangle)\mathsf{A}_{\leq} f(\langle u, w\rangle)$. Since $\langle x, y\rangle \in I^+(\mathbb{Q})$, we have $x \leq y$, so $\lceil x\rceil \leq \lceil y\rceil$. By $\langle x, y\rangle \mathsf{A}_{\leq} \langle u, w\rangle$ we have $y = u$, so $\lceil y\rceil = \lceil u\rceil$. Finally, $\langle u, w\rangle \in I^+(\mathbb{Q})$, hence $u \leq w$ and consequently $\lceil u\rceil \leq \lceil w\rceil$. Therefore, $\lceil x\rceil \leq \lceil y\rceil$, $\lceil y\rceil = \lceil u\rceil$, and $\lceil u\rceil \leq \lceil w\rceil$, which implies $\langle\lceil x\rceil, \lceil y\rceil\rangle \mathsf{A}_{\leq} \langle\lceil u\rceil, \lceil w\rceil\rangle$, and so $\langle x, y\rangle \mathsf{A}_{\leq} \langle u, w\rangle$. Hence, (ZIG) holds for $\mathsf{A}_{\leq}$.

On the other hand, let $f(\langle x, y\rangle) \in I^+(\mathbb{Z})$ and $\langle u', w'\rangle \in I^+(\mathbb{Z})$ be such that $f(\langle x, y\rangle)\mathsf{A}_{\leq}\langle u', w'\rangle$. By $f(\langle x, y\rangle)\mathsf{A}_{\leq}\langle u', w'\rangle$ we have $\langle\lceil x\rceil, \lceil y\rceil\rangle \mathsf{A}_{\leq} \langle u', w'\rangle$, so $\lceil y\rceil = u'$ and $u' \leq w'$. Hence, $y \leq w'$ and so $\langle y, w\rangle \in I^+(\mathbb{Q})$. We have $\langle x, y\rangle \mathsf{A}_{\leq} \langle y, w\rangle$, so (ZAG) holds for $\mathsf{A}_{\leq}$.

Since f is a surjective bounded morphism from $\mathcal{F}$ to $\mathcal{F}'$, dense frames are not

HS-definable with respect to reflexive frames. $\square$

It is worth noticing that the proof does not apply to the case of irreflexive frames. In particular, the function f from the proof does not satisfy (ZIG) for $\mathsf{A}_<$ and $\overline{\mathsf{A}}_<$. Indeed, if $\mathcal{F}$ and $\mathcal{F}'$ were irreflexive, we would have $\langle 1.1, 1.2\rangle \mathsf{A}_< \langle 1.2, 1.3\rangle$ but not $f(\langle 1.1, 1.2\rangle)\mathsf{A}_< f(\langle 1.2, 1.3\rangle)$, since $f(\langle 1.1, 1.2\rangle) = \langle 2, 2\rangle$, $f(\langle 1.2, 1.3\rangle) = \langle 2, 2\rangle$, and it is not the case that $\langle 2, 2\rangle \mathsf{A}_< \langle 2, 2\rangle$. Furthermore, observe that in the proof we have constructed models over non-strict frames, so the proof implies the following:

Corollary 14. *The class of dense frames is not* HS*-definable with respect to non-strict reflexive frames.*

However, Theorem 13 does not imply that dense frames are not definable with respect to strict reflexive frames, which we will leave as a future work.

If we extend HS-language with nominals (which results in HS^i-language), then dense frames are definable relatively to reflexive frames by the formula:

$$\mathsf{den}^i_{(\leq)} \overset{\text{df}}{=} \big(i \wedge \langle \overline{\mathsf{B}}\rangle(j \wedge \neg i)\big) \rightarrow \langle \overline{\mathsf{B}}\rangle(\neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j),$$

where i, j are any distinct nominals. Intuitively, this formula states that if i is satisfied in an interval $\langle x, y\rangle$ and j is satisfied in $\langle x, z\rangle$ such that $y < z$, then there exists an interval $\langle x, w\rangle$ such that $y < w < z$. Hence, the formula states that between any two time points there exists a third one, that is the frame is dense.

Theorem 15. *The class of dense frames is* HS^i*-definable with respect to reflexive frames.*

It is worth pointing out that using nominals (and not propositional variables) in $\mathsf{den}^i_{(\leq)}$ is essential. Indeed, by Theorem 13 it is not possible to define dense frames with respect to reflexive frames in pure HS-language. Hence, we obtain that HS^i-formulas allow us to define strictly more classes of frames than HS-formulas.

Furthermore, strict frames are HS^i-definable with respect to reflexive frames by the following formula:

$$\mathsf{strict}^i_{(\leq)} \overset{\mathrm{df}}{=} i \to \neg\langle \mathsf{A}\rangle i,$$

where i is a nominal. This formula states that no interval is in relation $\mathsf{A}_{\leq}$ with itself, that is $\mathsf{A}_{\leq}$ is irreflexive. With respect to reflexive frames this condition holds exactly in strict frames.

Theorem 16. *The class of strict frames is HS^i-definable with respect to reflexive frames.*

If the nominal in $\mathsf{strict}^i_{(\leq)}$ were replaced with a propositional variable, then $\mathsf{strict}^i_{(\leq)}$ would not define strict frames with respect to reflexive frames. However, it remains an open problem whether there is a way to HS-define strict frames with respect to reflexive frames.

5. Difference Operator

In this section we will study the possibility to express the *difference operator* D, for which the satisfaction condition in an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ and interval $\langle x, y\rangle \in I(\mathbb{D})$ is defined for any HS-formula φ as follows:

$$\begin{array}{lll} \mathcal{M}, \langle x, y\rangle \models \mathsf{D}\varphi & \text{iff} & \text{there exists } \langle x', y'\rangle \in I(\mathbb{D}) \text{ such that} \\ & & \langle x', y'\rangle \neq \langle x, y\rangle \text{ and } \mathcal{M}, \langle x', y'\rangle \models \varphi. \end{array}$$

We say that an operator O is *L-expressible* if for any formula from the language L extended with the operator O there exists an equisatisfiable formula in L. In full HS-language a formula $\mathsf{D}\varphi$ for arbitrary HS-formula φ is expressible in irreflexive frames by the formula $\bigvee_{\mathsf{R}\in\mathsf{Rel}}\langle \mathsf{R}\rangle\varphi$ [23]. Indeed, the relations in $\mathcal{R}_{<}$ are irreflexive and jointly exhaustive among non-identical intervals. Hence, $\bigvee_{\mathsf{R}\in\mathsf{Rel}}\langle \mathsf{R}\rangle\varphi$ is satisfied in an interval $\langle x, y\rangle$ if and only if φ is satisfied in some interval distinct from $\langle x, y\rangle$. In reflexive frames the above formula does not allow us to express $\mathsf{D}\varphi$, since not all relations in $\mathcal{R}_{\leq}$ are irreflexive.

Nevertheless, we will show that in reflexive and strict frames we can express $\mathsf{D}\varphi$ in HS-language with the formula:

$$\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi) \overset{\mathrm{df}}{=} \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi \vee \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi.$$

Let $\langle x, y\rangle$ be the current interval. Then:

- $\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi$ states that φ holds in an interval which begins in some time point x' such that $x < x'$;
- $\langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi$ states that φ holds in an interval which begins in some time point x' such that $x' < x$;
- $\langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi$ states that φ holds in an interval which ends in some time point y' such that $y' < y$;
- $\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi$ states that φ holds in an interval which ends in some time point y' such that $y < y'$.

It follows that $\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$ is satisfied in $\langle x, y\rangle$ if and only if φ holds in some interval distinct from $\langle x, y\rangle$. Observe that the first two disjuncts of $\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$ are not sufficient to express $\mathsf{D}\varphi$, since they do not cover the case in which φ holds in an interval which begins in x and ends in a time point different than y. Similarly, the latter two disjuncts of $\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$ are not sufficient since they do not cover the case in which φ holds in an interval which begins in some time point different than x and ends in y.

Theorem 17. *The difference operator is* HS*-expressible in* ($\leq$,S)*-frames.*

The formula $\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$ makes a significant use of irreflexivity of $\mathsf{A}_{\leq}$ and $\mathsf{A}_{\leq}$ in strict frames. In non-strict frames these relations are not irreflexive since each punctual interval is in both $\mathsf{A}_{\leq}$ and $\mathsf{A}_{\leq}$ relations with itself. As a result, $\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$ does not express $\mathsf{D}\varphi$ in reflexive non-strict frames. In Section 6 we will show that the difference operator cannot be expressed in HS-language in ($\leq$,$\mathsf{Non\text{-}S}$)-frames.

In the next theorem we show that D is neither $\mathsf{HS}^{\Diamond}_{horn}$-expressible in discrete nor in dense frames. The proof is based on the following property which holds in each HS-frame $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$. If $\langle x', y' \rangle$ and $\langle x'', y'' \rangle$ are reachable from $\langle x, y \rangle$ with a composition $\mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n$ of (possibly repeating) relations from $\mathcal{R}$, then each interval 'lying between' $\langle x', y' \rangle$ and $\langle x'', y'' \rangle$ is also reachable from $\langle x, y \rangle$ with $\mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n$.

To state precisely what 'lying between' two intervals means, let $\min(x, y)$ for all $x, y \in \mathbb{D}$ be a $\leq$-minimal among x and y. Similarly, let $\max(x, y)$ be a $\leq$-maximal among x and y. Moreover, for $x \leq y$ let $y - x$ be the number of distinct time points $z \in \mathbb{D}$ such that $x < z \leq y$. Note that if $\mathbb{D}$ is dense, then for any $x, y \in \mathbb{D}$ such that $x < y$ we have $x - y = \infty$. Then, we say that $\langle s, t \rangle$ 'lies between' $\langle x', y' \rangle$ and $\langle x'', y'' \rangle$ if the following conditions hold simultaneously:

- $\min(x', x'') \leq s \leq \max(x', x'')$, that is s lies between x' and x'';
- $\min(y', y'') \leq t \leq \max(y', y'')$, that is t lies between y' and y'';
- $\min(y' - x', y'' - x'') \leq t - s$, that is in compass representation $\langle s, t \rangle$ lies not closer to a diagonal than both $\langle x', y' \rangle$ and $\langle x'', y'' \rangle$.

A pictorial representation of 'lying between' is given in Figure 4.

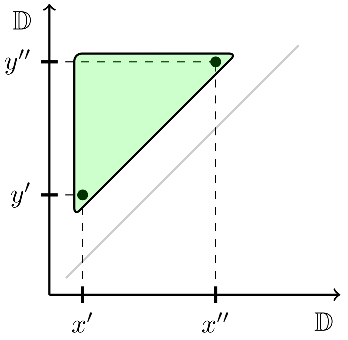


Figure 4: The triangle area corresponds to intervals which 'lie between' $\langle x', y' \rangle$ and $\langle x'', y'' \rangle$.

Now, the precise formulation of the property we have mentioned is as follows:

Lemma 18. *Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an* HS*-frame. Let $\langle x, y\rangle \mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n \langle x', y'\rangle$ and $\langle x, y\rangle \mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n \langle x'', y''\rangle$ for arbitrary intervals $\langle x, y\rangle, \langle x', y'\rangle, \langle x'', y''\rangle \in I(\mathbb{D})$ and $\mathsf{R}_1, \ldots, \mathsf{R}_n \in \mathcal{R}$. Then, for each $\langle s, t\rangle \in I(\mathbb{D})$ which satisfies all of the following conditions:*

- $\min(x', x'') \leq s \leq \max(x', x'')$;
- $\min(y', y'') \leq t \leq \max(y', y'')$;
- $\min(y' - x', y'' - x'') \leq t - s$,

we have $\langle x, y\rangle \mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n \langle s, t\rangle$.

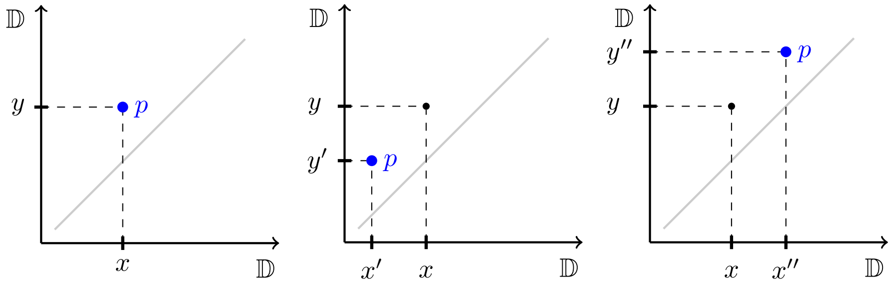
The proof of this lemma is inductive on the length n of the sequence $\mathsf{R}_1, \ldots, \mathsf{R}_n$. Since the proof is technical, we move it to the Appendix. Now, we will use the above lemma to show that the difference operator is not $\mathsf{HS}^{\Diamond}_{horn}$-expressible in Dis-frames and Den-frames.

Theorem 19. *The difference operator is not $\mathsf{HS}^{\Diamond}_{horn}$-expressible in Dis-frames nor in Den-frames.*

Proof. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be a discrete or dense HS-frame. Let $\langle x, y\rangle \in I(\mathbb{D})$, $\langle x', y'\rangle \in I(\mathbb{D})$, and $\langle x'', y''\rangle \in I(\mathbb{D})$ be isomorphic intervals in $\mathcal{F}$ such that $x' < x < x''$ and $y' < y < y''$. Observe that such $\langle x, y\rangle$, $\langle x', y'\rangle$ and $\langle x'', y''\rangle$ exist by unboundedness of $\mathbb{D}$ and by the fact that $\mathbb{D}$ is (everywhere) dense or (everywhere) discrete. We define models $\mathcal{M} = (\mathcal{F}, V)$, $\mathcal{M}' = (\mathcal{F}, V')$, and $\mathcal{M}'' = (\mathcal{F}, V'')$ such that:

$$V(p) = \{\langle x, y\rangle\}; \qquad V'(p) = \{\langle x', y'\rangle\}; \qquad V''(p) = \{\langle x'', y''\rangle\},$$

and $V(q) = V'(q) = V''(q) = \emptyset$ for each propositional variable q distinct from p. Hence, the isomorphism from $\mathcal{F}$ to $\mathcal{F}$ mapping $\langle x, y\rangle$ to $\langle x', y'\rangle$ is an isomorphism from $\mathcal{M}$ to $\mathcal{M}'$ because it preserves satisfaction of propositional variables. Similarly, the isomorphism from $\mathcal{F}$ to $\mathcal{F}$ mapping $\langle x, y\rangle$ to $\langle x'', y''\rangle$ is an isomorphism from $\mathcal{M}$ to $\mathcal{M}''$. Hence, $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$ are pairwise isomorphic. A pictorial presentation of models $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$ is given in Figure 5.



($\mathcal{M}$) ($\mathcal{M}'$) ($\mathcal{M}''$)

Figure 5: Isomorphic HS-models $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$.

Suppose that there exists an $\mathsf{HS}^{\Diamond}_{horn}$-formula $\varphi_{\mathsf{D}p}$ expressing $\mathsf{D}p$. Therefore, $\mathcal{M}, \langle x, y\rangle \not\models \varphi_{\mathsf{D}p}$, $\mathcal{M}', \langle x, y\rangle \models \varphi_{\mathsf{D}p}$, and $\mathcal{M}'', \langle x, y\rangle \models \varphi_{\mathsf{D}p}$. To reach a contradiction we will show that:

($\star$) For all $\mathsf{HS}^{\Diamond}_{horn}$-formulas φ if $\mathcal{M}', \langle x, y\rangle \models \varphi$ and $\mathcal{M}'', \langle x, y\rangle \models \varphi$, then $\mathcal{M}, \langle x, y\rangle \models \varphi$.

Indeed, we have $\mathcal{M}', \langle x, y\rangle \models \varphi_{\mathsf{D}p}$ and $\mathcal{M}'', \langle x, y\rangle \models \varphi_{\mathsf{D}p}$, so by ($\star$) we will obtain $\mathcal{M}, \langle x, y\rangle \models \varphi_{\mathsf{D}p}$. Then, a contradiction arises by $\mathcal{M}, \langle x, y\rangle \not\models \varphi_{\mathsf{D}p}$. Thus, we will obtain that $\varphi_{\mathsf{D}p}$ does not express $\mathsf{D}p$.

To prove ($\star$) let φ be an $\mathsf{HS}^{\Diamond}_{horn}$-formula. By the definition of $\mathsf{HS}^{\Diamond}_{horn}$-formulas φ is a conjunction, so it suffices to show that each conjunct ψ of φ satisfies ($\star$). Assume that $\mathcal{M}', \langle x, y\rangle \models \psi$ and $\mathcal{M}'', \langle x, y\rangle \models \psi$. Clearly, ψ is either of the form $[\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1})$ or λ, where λ_i's and λ are generated by the grammar $\lambda = \top \mid \bot \mid r \mid \langle \mathsf{R}\rangle\lambda$, where r is a propositional variable.

Assume that ψ is of the form $[\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1})$. By $\mathcal{M}', \langle x, y\rangle \models \psi$, we have $\mathcal{M}', \langle x, y\rangle \models [\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1})$. Since $[\mathsf{U}]$ is the universal modal operator stating that a formula is satisfied in all intervals in a model, we obtain $\mathcal{M}' \models \lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1}$. Since $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ and isomorphisms preserve truth of formulas, we get $\mathcal{M} \models \lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1}$. As a result, we obtain $\mathcal{M}, \langle x, y\rangle \models [\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1})$.

Now, let ψ be generated by the grammar $\lambda = \top \mid \bot \mid r \mid \langle \mathsf{R}\rangle\lambda$, where r is a propositional variable. Then, ψ has one of the following forms: $\top$, $\bot$, p, q, $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$, $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \bot$, $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$, or $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle q$, where $q \in \mathsf{PROP}$ is distinct from p and $\mathsf{R}_i \in \mathsf{Rel}$ for all $i \in \{1, \dots, n\}$. Clearly, $\top$ is satisfied in all three models in $\langle x, y\rangle$. On the other hand, $\bot$, q, and $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle q$ are false in each interval in $\mathcal{M}'$ and $\mathcal{M}''$. Moreover, p is not satisfied in $\langle x, y\rangle$ neither in $\mathcal{M}'$ nor in $\mathcal{M}''$, so $(\star)$ holds for ψ in any of these forms. It remains to consider the cases when ψ is of the form $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$ or $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$.

Assume that $\mathcal{M}', \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$. Since $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$ does not contain propositional variables, its satisfiability in $\langle x, y\rangle$ does not depend on the valuation. Hence, $\mathcal{F}, \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$. Since $\mathcal{M}$ is based on $\mathcal{F}$, we obtain that $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$. Thus, $(\star)$ holds for ψ of the form $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle \top$.

Finally, consider ψ of the form $\langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$. Let $\mathcal{M}', \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$ and $\mathcal{M}'', \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$. We need to show $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$. Since $V'(p) = \{\langle x', y'\rangle\}$ and $V''(p) = \{\langle x'', y''\rangle\}$, we have $\langle x, y\rangle \mathsf{R}_1 \circ \dots \circ \mathsf{R}_n \langle x', y'\rangle$ and $\langle x, y\rangle \mathsf{R}_1 \circ \dots \circ \mathsf{R}_n \langle x'', y''\rangle$. To use Lemma 18 we need to show that:

- $\min(x', x'') \leq x \leq \max(x', x'')$;
- $\min(y', y'') \leq y \leq \max(y', y'')$;
- $y - x \geq \min(y' - x', y'' - x'')$.

First two conditions follow from $x' < x < x''$ and $y' < y < y''$, respectively. By the fact that $\langle x, y\rangle$, $\langle x', y'\rangle$, and $\langle x'', y''\rangle$ are isomorphic images, we have $y - x = y' - x' = y'' - x''$, so the third condition holds as well. Then, by Lemma 18 we obtain that $\langle x, y\rangle \mathsf{R}_1 \circ \dots \circ \mathsf{R}_n \langle x, y\rangle$. Since $\mathcal{M}, \langle x, y\rangle \models p$, we get $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{R}_1\rangle \dots \langle \mathsf{R}_n\rangle p$.

We have showed that the statement $(\star)$ holds for all $\mathsf{HS}^{\Diamond}_{horn}$-formulas, hence the proof is done. $\square$

Next, we will show that the difference operator is not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of frames. To prove this result we will use properties of the *canonical model* for an $\mathsf{HS}^{\Box}_{horn}$-formula φ, frame $\mathcal{F}$, and interval $\langle a, b\rangle$, which was introduced in

[3]. Intuitively, the canonical model is a minimal model based on $\mathcal{F}$ in which φ is satisfied in $\langle a, b\rangle$.

Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an HS-frame, $\langle a, b\rangle \in I(\mathbb{D})$, and φ an $\mathsf{HS}^{\Box}_{horn}$-formula. We say that φ is $\langle a, b\rangle$-*satisfiable* in $\mathcal{F}$ if there exists an HS-model $\mathcal{M}$ based on $\mathcal{F}$ such that $\mathcal{M}, \langle a, b\rangle \models \varphi$. For fixed frame $\mathcal{F}$, interval $\langle a, b\rangle$, and $\mathsf{HS}^{\Box}_{horn}$-formula φ we will define a set of triples of the form (ψ, x, y), whose intended meaning is that in the canonical model ψ holds in $\langle x, y\rangle$. We start the construction by defining:

$$\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}} \overset{\text{df}}{=} \big\{(\lambda, a, b) \mid \lambda \text{ is an initial condition of } \varphi\big\} \cup \big\{(\top, x, y) \mid \langle x, y\rangle \in I(\mathbb{D}\big\},$$

where an initial condition is defined as in Section 3.

We denote by $\mathsf{cl}_{\varphi,\mathcal{F}}(A)$ the result of applying exhaustively and non-recursively the following rules to the set A:

($\mathsf{cl1}_{\varphi,\mathcal{F}}$) If $([\mathsf{R}]\lambda, x, y) \in A$, then add to A all (λ, x', y') such that $\langle x', y'\rangle \in I(\mathbb{D})$ and $\langle x, y\rangle \mathsf{R}_* \langle x', y'\rangle$;

($\mathsf{cl2}_{\varphi,\mathcal{F}}$) If $(\lambda, x', y') \in A$ for all $\langle x', y'\rangle \in I(\mathbb{D})$ such that $\langle x, y\rangle \mathsf{R}_* \langle x', y'\rangle$ and $[\mathsf{R}]\lambda$ occurs in φ, then add $([\mathsf{R}]\lambda, x, y)$ to A;

($\mathsf{cl3}_{\varphi,\mathcal{F}}$) If $[\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_k \rightarrow \lambda)$ occurs in φ and $(\lambda_j, x, y) \in A$ for all $j \in \{1, \ldots, k\}$, then add (λ, x, y) to A,

where $\mathsf{R} \in \mathsf{Rel}$, $\mathsf{R}_* = \mathsf{R}_{\leq}$ if $\mathcal{R} = \mathcal{R}_{\leq}$, and $\mathsf{R}_* = \mathsf{R}_{<}$ if $\mathcal{R} = \mathcal{R}_{<}$. We define the following sets, obtained by subsequent applications of $\mathsf{cl}_{\varphi,\mathcal{F}}$ to $\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}$:

$$\mathsf{cl}^{0}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right) \overset{\text{df}}{=} \mathfrak{V}^{a,b}_{\varphi,\mathcal{F}};$$

$$\mathsf{cl}^{\alpha+1}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right) \overset{\text{df}}{=} \mathsf{cl}_{\varphi,\mathcal{F}}\left(\mathsf{cl}^{\alpha}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right)\right), \text{ for } \alpha+1 \text{ a successor ordinal;}$$

$$\mathsf{cl}^{\beta}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right) \overset{\text{df}}{=} \bigcup_{\gamma<\beta} \mathsf{cl}^{\gamma}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right), \text{ for } \gamma \text{ an ordinal, and } \beta \text{ a limit ordinal;}$$

$$\mathsf{cl}^{*}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right) \overset{\text{df}}{=} \bigcup_{\gamma \text{ an ordinal}} \mathsf{cl}^{\gamma}_{\varphi,\mathcal{F}}\left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}}\right).$$

Then, the canonical model $\mathcal{K}^{a,b}_{\varphi,\mathcal{F}} \stackrel{\text{df}}{=} (\mathcal{F}, V)$ is such that for any $p \in \mathsf{PROP}(\varphi)$ we have:

$$V(p) \stackrel{\text{df}}{=} \left\{ \langle x, y \rangle \mid (p, x, y) \in \mathsf{cl}^*_{\varphi,\mathcal{F}} \left(\mathfrak{V}^{a,b}_{\varphi,\mathcal{F}} \right) \right\}.$$

As proved in [3], the canonical model enjoys the following property.

Lemma 20 ([3])**.** *Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an HS-frame, $\langle a, b \rangle$ an interval in $I(\mathbb{D})$, φ an $\mathsf{HS}^{\Box}_{horn}$-formula, and $\mathcal{M}$ an HS-model based on $\mathcal{F}$. If $\mathcal{M}, \langle a, b \rangle \models \varphi$, then the following conditions hold:*

1. *$\mathcal{K}^{a,b}_{\varphi,\mathcal{F}}, \langle a, b \rangle \models \varphi$;*
2. *For all $\langle x, y \rangle \in I(\mathbb{D})$ and for all $p \in \mathsf{PROP}$, if $\mathcal{K}^{a,b}_{\varphi,\mathcal{F}}, \langle x, y \rangle \models p$, then $\mathcal{M}, \langle x, y \rangle \models p$.*

We use this lemma to show that D is not $\mathsf{HS}^{\Box}_{horn}$-expressible.

Theorem 21. *The difference operator is not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of frames.*

Proof. Suppose that the difference operator is $\mathsf{HS}^{\Box}_{horn}$-expressible. Then, there is an $\mathsf{HS}^{\Box}_{horn}$-formula φ expressing $\mathsf{D}p$ for some $p \in \mathsf{PROP}$. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an HS-frame and $\langle x, y \rangle, \langle x', y' \rangle$ distinct intervals in $I(\mathbb{D})$ (by unboundedness of $\mathbb{D}$ there are always two distinct intervals in $I(\mathbb{D})$). We define models $\mathcal{M} = (\mathcal{F}, V)$ and $\mathcal{M}' = (\mathcal{F}, V')$ as follows:

$$V(p) = \{\langle x, y \rangle\}; \qquad V'(p) = \{\langle x', y' \rangle\}.$$

Let $\langle a, b \rangle \in I(\mathbb{D})$ be an interval distinct from $\langle x, y \rangle$ and $\langle x', y' \rangle$ (such $\langle a, b \rangle$ exists by unboundedness of $\mathbb{D}$). Hence, $\mathcal{M}, \langle a, b \rangle \models \varphi$ and $\mathcal{M}', \langle a, b \rangle \models \varphi$.

Since $\mathcal{M}, \langle a, b \rangle \models \varphi$, by Lemma 20 we obtain $\mathcal{K}^{a,b}_{\varphi,\mathcal{F}}, \langle a, b \rangle \models \varphi$. Since φ simulates $\mathsf{D}p$ and $\mathcal{K}^{a,b}_{\varphi,\mathcal{F}}, \langle a, b \rangle \models \varphi$, there exists an interval $\langle u, w \rangle$ such that $\mathcal{K}^{a,b}_{\varphi,\mathcal{F}}, \langle u, w \rangle \models p$. Then, by Lemma 20 $\mathcal{M}, \langle u, w \rangle \models p$ and $\mathcal{M}', \langle u, w \rangle \models p$. Since $\langle x, y \rangle \neq \langle x', y' \rangle$, we have either $\langle u, w \rangle \neq \langle x, y \rangle$ or $\langle u, w \rangle \neq \langle x', y' \rangle$.

(Case 1): $\langle u, w \rangle \neq \langle x, y \rangle$. We have $V(p) = \{\langle x, y \rangle\}$ and $\langle u, w \rangle \neq \langle x, y \rangle$, therefore $\mathcal{M}, \langle u, w \rangle \not\models p$. Then, by $\mathcal{M}, \langle u, w \rangle \models p$ we obtain a contradiction.

(Case 2): $\langle u, w\rangle \neq \langle x', y'\rangle$. By $V'(p) = \{\langle x', y'\rangle\}$ and $\langle u, w\rangle \neq \langle x', y'\rangle$ we have $\mathcal{M}', \langle u, w\rangle \not\models p$. Then, $\mathcal{M}', \langle u, w\rangle \models p$ raises a contradiction.

It follows that the difference operator is not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of HS-frames. $\Box$

6. Nominals

It was proved in [23] that nominals can be expressed in any modal language in which the difference operator is expressible and the use of propositional connectives is not restricted. Since in full HS-language there is no restriction on the use of propositional connectives, the possibility to express the difference operator implies expressibility of nominals. Therefore, Theorem 17, stating that the difference operator is HS-expressible in ($\leq$,S)-frames, implies that nominals are HS-expressible in ($\leq$,S)-frames. As we have stated in Section 5 the difference operator is HS-expressible in irreflexive frames, so we obtain the following corollary:

Corollary 22. *Nominals are* HS*-expressible in* $<$*-frames and* ($\leq$,S)*-frames.*

Next, we will show that even in the full HS-language nominals are not expressible in ($\leq$,Non-S,Dis)-frames and ($\leq$,Non-S,Den)-frames. We will construct a bisimulation between HS-models which is defined analogously as in the standard modal logic [25], namely:

Definition 23 (Bisimulation in HS)**.** A *bisimulation* between HS-models $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ and $\mathcal{M}' = (\mathbb{D}', I'(\mathbb{D}'), \mathcal{R}, V')$ is a relation $Z \subseteq I(\mathbb{D}) \times I'(\mathbb{D}')$ which is nonempty and satisfies the below conditions for all $\mathsf{R} \in \mathcal{R}$, $\langle x, y\rangle \in I(\mathbb{D})$, $\langle u, w\rangle \in I(\mathbb{D})$, and $\langle u', w'\rangle \in I'(\mathbb{D}')$:

(atom) If $\langle x, y\rangle Z \langle x', y'\rangle$, then the same propositional variables are satisfied in $\langle x, y\rangle$ and $\langle x', y'\rangle$;

(zig) If $\langle x, y\rangle Z \langle x', y'\rangle$ and $\langle x, y\rangle \mathsf{R} \langle u, w\rangle$, then there exists $\langle u', w'\rangle \in I'(\mathbb{D}')$ such that $\langle u, w\rangle Z \langle u', w'\rangle$ and $\langle x', y'\rangle \mathsf{R} \langle u', w'\rangle$;

(zag) If $\langle x, y\rangle Z\langle x', y'\rangle$ and $\langle x', y'\rangle \mathsf{R}\langle u', w'\rangle$, then there exists $\langle u, w\rangle \in I(\mathbb{D})$ such that $\langle u, w\rangle Z\langle u', w'\rangle$ and $\langle x, y\rangle \mathsf{R}\langle u, w\rangle$.

Similarly as in the case of standard modal logic, bisimilation preserves satisfaction of formulas.

Lemma 24 (Bisimulation Invariance Lemma in HS)**.** *Let $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ and $\mathcal{M}' = (\mathbb{D}', I'(\mathbb{D}'), \mathcal{R}, V')$ be* HS *models and let Z be a bisimulation between them. Then, for all $\langle x, y\rangle \in I(\mathbb{D})$ and $\langle x', y'\rangle \in I'(\mathbb{D}')$ such that $\langle x, y\rangle Z\langle x', y'\rangle$, and for every* HS*-formula φ:*

$$\mathcal{M}, \langle x, y\rangle \models \varphi \text{ if and only if } \mathcal{M}', \langle x', y'\rangle \models \varphi.$$

Proof of this lemma is analogous to the proof for standard modal logic [25]. We will use this result to show that in some frames even the full language of HS does not allow to express nominals.

Theorem 25. *Nominals are neither* HS*-expressible in ($\leq$,Non-S,Dis)-frames nor in ($\leq$,Non-S,Den)-frames.*

Proof. Let $\mathcal{M} = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_\leq, V)$ be a reflexive non-strict HS-model. To show that the result holds for discrete frames assume that $\mathbb{D}$ is $(\mathbb{Z}, \leq)$ (that is the standard ordering of integers), whereas to show that it holds for dense frames assume that $\mathbb{D}$ is $(\mathbb{Q}, \leq)$ (that is the standard ordering of rational numbers). Suppose that there is an HS-formula φ which simulates a nominal. Hence, φ is satisfied in exactly one interval in $\mathcal{M}$, say $\langle a, b\rangle$.

In what follows, we will construct an HS-model $\mathcal{M}' = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_\leq, V')$ which is bisimilar to $\mathcal{M}$ and such that $\langle a, b\rangle$ is bisimilar to more than one interval. Then, by Bisimulation Invariance Lemma 24, the formula φ will be satisfied in more than one interval in $\mathcal{M}'$. Hence, we will obtain a contradiction which implies that nominals are not HS-expressible in $(\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_\leq)$.

We start by dividing $I^+(\mathbb{D})$ into areas $A1$–$A6$ as follows:

$$\begin{array}{ll}
\langle x, y\rangle \in A1 \text{ iff } (x < a \text{ and } y < a); & \langle x, y\rangle \in A4 \text{ iff } (x = a \text{ and } y = a); \\
\langle x, y\rangle \in A2 \text{ iff } (x < a \text{ and } y = a); & \langle x, y\rangle \in A5 \text{ iff } (x = a \text{ and } y > a); \\
\langle x, y\rangle \in A3 \text{ iff } (x < a \text{ and } y > a); & \langle x, y\rangle \in A6 \text{ iff } (x > a \text{ and } y > a).
\end{array}$$

We exploit areas $A1$–$A6$ to define the intended bisimulation $Z \subseteq I^+(\mathbb{D}) \times I^+(\mathbb{D})$ between intervals in $\mathcal{M}$ and $\mathcal{M}'$ (for a pictorial representation of this bisimulation see Figure 6) as follows (we use the standard functional notation for Z, that is we identify $Z(\langle x, y\rangle)$ with the set $\{\langle x', y'\rangle \mid \langle x, y\rangle Z \langle x', y'\rangle\}$):

$$\begin{array}{ll}
\forall\langle x, y\rangle \in A1 & Z(\langle x, y\rangle) \overset{\text{df}}{=} \{\langle x-1, y-1\rangle\}; \\
\forall\langle x, y\rangle \in A2 & Z(\langle x, y\rangle) \overset{\text{df}}{=} \{\langle x-1, y'\rangle \mid a-1 \leq y' \leq a\}; \\
\forall\langle x, y\rangle \in A3 & Z(\langle x, y\rangle) \overset{\text{df}}{=} \{\langle x-1, y\rangle\}; \\
\forall\langle x, y\rangle \in A4 & Z(\langle x, y\rangle) \overset{\text{df}}{=} \{\langle x', y'\rangle \mid a-1 \leq x' \leq a, \text{ and } a-1 \leq y' \leq a\}; \\
\forall\langle x, y\rangle \in A5 & Z(\langle x, y\rangle) \overset{\text{df}}{=} \{\langle x', y\rangle \mid a-1 \leq x' \leq a\}; \\
\forall\langle x, y\rangle \in A6 & Z(\langle x, y\rangle) \overset{\text{df}}{=} \{\langle x, y\rangle\}.
\end{array}$$

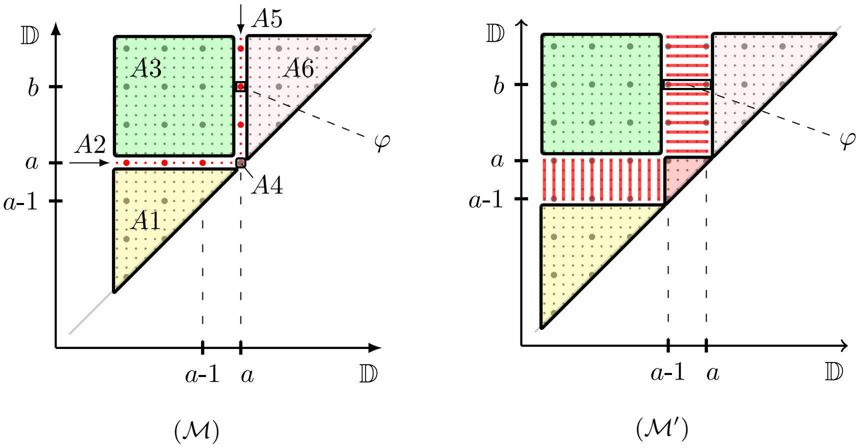


Figure 6: Bisimulation Z between models $\mathcal{M}$ and $\mathcal{M}'$.

We define V' such that for each $\langle x, y\rangle \in I^+(\mathbb{D})$ and each $p \in \mathsf{PROP}$:

$$\langle x, y\rangle \in V'(p) \quad \text{if and only if} \quad Z^{-1}\langle x, y\rangle \in V(p),$$

where Z^{-1} is the converse of Z, that is we have $\langle x', y'\rangle Z^{-1}\langle x, y\rangle$ if and only if $\langle x, y\rangle Z\langle x', y'\rangle$.

The condition (atom) follows directly from the definition of V'. To show that (zig) and (zag) hold a routine inspection of all $\mathsf{R} \in \mathcal{R}_{\leq}$ needs to be performed. In what follows, we will consider only $\mathsf{A}_{\leq}$ as the other cases are similar.

To show (zig) for $\mathsf{A}_{\leq}$ assume that $\langle x, y\rangle Z\langle x', y'\rangle$ and $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$. We need to show that there exists $\langle u', w'\rangle \in I^{+}(\mathbb{D})$ such that $\langle u, w\rangle Z\langle u', w'\rangle$ and $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$. Since $A1$–$A6$ divide $I^{+}(\mathbb{D})$, clearly $\langle x, y\rangle$ belongs to one of the areas $A1$–$A6$.

Assume that $\langle x, y\rangle \in A1$, then $x < a$ and $y < a$. Since $\langle x, y\rangle Z\langle x', y'\rangle$, we have $x' = x - 1$ and $y' = y - 1$. By $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$ we obtain $y = u$ and $u \leq w$. We have either $w < a$, $w = a$, or $a < w$.

(Case 1): $w < a$. Let $u' = u - 1$ and $w' = w - 1$. Since $u \leq w$, by $u' = u - 1$ and $w' = w - 1$ we obtain $u' \leq w'$, so $\langle u', w'\rangle \in I^{+}(\mathbb{D})$. We will show that $\langle u, w\rangle Z\langle u', w'\rangle$ and $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$.

Since $u = y$ and $y < a$, we have $u < a$. Moreover, $w < a$, so $\langle u, w\rangle \in A1$. By the definition of Z we obtain $\langle u, w\rangle Z\langle u-1, w-1\rangle$, so we have $\langle u, w\rangle Z\langle u', w'\rangle$.

To show that $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$ observe that $y' = y - 1$ and $u' = u - 1$. Then, by $y = u$ we have $y' = u'$, hence $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$.

(Case 2): $w = a$. Let $u' = u - 1$ and $w' = w$. Since $u \leq w$, by $u' = u - 1$ and $w' = w$ we obtain $u' < w'$, so $\langle u', w'\rangle \in I^{+}(\mathbb{D})$.

Since $w = a$, we obtain that $\langle u, w\rangle \in A2$. By the definition of Z we have $\langle u, w\rangle Z\langle u-1, w\rangle$, so $\langle u, w\rangle Z\langle u', w'\rangle$.

We have $y' = y - 1$, $u' = u - 1$, and $y = u$, so $y' = u'$. It follows that $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$.

(Case 3): $a < w$. Let $u' = u - 1$ and $w' = w$. By the same argument as in the Case 2 we have $u' < w'$, so $\langle u', w'\rangle \in I^{+}(\mathbb{D})$.

By $a < w$ we have $\langle u, w\rangle \in A3$. Hence, by the definition of Z we have $\langle u, w\rangle Z\langle u-1, w\rangle$, so $\langle u, w\rangle Z\langle u', w'\rangle$.

We have $y' = y-1$, $u' = u-1$, and $y = u$, so $y' = u'$. As a result, $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$.

The cases when $\langle x, y\rangle$ belongs to areas $A2$–$A6$ are similar.

Next, we will consider (zag) condition for $\mathsf{A}_{\leq}$. Assume that $\langle x, y\rangle Z\langle x', y'\rangle$ and $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$. We need to show that there exists $\langle u, w\rangle \in I^{+}(\mathbb{D})$ such that $\langle u, w\rangle Z\langle u', w'\rangle$ and $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$. Clearly, $\langle x, y\rangle$ belongs to one of the areas $A1$–$A6$. Assume that $\langle x, y\rangle \in A1$, that is $x < a$ and $y < a$. Since $\langle x, y\rangle Z\langle x', y'\rangle$, we have $x' = x-1$ and $y' = y-1$. Hence, $x' < a-1$ and $y' < a-1$. By $\langle x', y'\rangle \mathsf{A}_{\leq}\langle u', w'\rangle$ we obtain $y' = u'$ and $u' \leq w'$. We have either $w' < a-1$, $a-1 \leq w' \leq a$, or $a < w'$.

(Case 1): $w' < a-1$. Let $u = u'+1$ and $w = w'+1$. Since $u' \leq w'$, we have $u \leq w$, so $\langle u, w\rangle \in I^{+}(\mathbb{D})$. We need to show that $\langle u, w\rangle Z\langle u', w'\rangle$ and $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$.

Since $w' < a-1$ and $w = w'+1$, we obtain $w < a$. Then, by $u \leq w$ and $w < a$ we obtain $u < a$ and consequently $\langle u, w\rangle \in A1$. Hence, $\langle u, w\rangle Z\langle u-1, w-1\rangle$, so $\langle u, w\rangle Z\langle u', w'\rangle$.

Next, we will show $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$. We have $y' = y-1$, $u = u'+1$, and $y' = u'$, so $y = u$. It follows that $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$.

(Case 2): $a-1 \leq w' \leq a$. Let $u = u'+1$ and $w = a$. We have $u = u'+1$ and $u' = y'$, so $u = y'+1$. Moreover, $y' < a-1$, therefore $u < a$. Since $u < a$ and $w = a$, we obtain $u < w$, so $\langle u, w\rangle \in I^{+}(\mathbb{D})$.

By $w = a$ we have $\langle u, w\rangle \in A2$. Then, by $a-1 \leq w' \leq a$ we obtain $\langle u, w\rangle Z\langle u-1, w'\rangle$, that is $\langle u, w\rangle Z\langle u', w'\rangle$.

Moreover, $y' = y-1$, $u = u'+1$, and $y' = u'$, so $y = u$. It follows that $\langle x, y\rangle \mathsf{A}_{\leq}\langle u, w\rangle$.

(Case 3): $a < w'$. Let $u = u'+1$ and $w = w'$. We have $y' < a-1$ and $y' = u'$, so $u' < a-1$. Then, by $u = u'+1$ we obtain $u < a$. On the other hand, $a < w'$

and $w = w'$, so $a < w$. Then, by $u < a$ and $a < w$ we have $\langle u, w\rangle \in I^{+}(\mathbb{D})$ and $\langle u, w\rangle \in A3$. Hence, $\langle u, w\rangle Z \langle u-1, w\rangle$, that is $\langle u, w\rangle Z \langle u', w'\rangle$.

It remains to show $\langle x, y\rangle \mathsf{A}_{\le} \langle u, w\rangle$. We have $y' = y - 1$, $u = u' + 1$, and $y' = u'$, so $y = u$. Hence, $\langle x, y\rangle \mathsf{A}_{\le} \langle u, w\rangle$.

The cases when $\langle x, y\rangle$ belongs to areas $A2$–$A6$ are similar.

It follows that Z is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. By the definition of Z, for each x such that $a-1 \le x \le a$ we have $\langle a, b\rangle Z \langle x, b\rangle$. Hence, there are at least two intervals bisimilar to $\langle a, b\rangle$, namely $\langle a-1, b\rangle$ and $\langle a, b\rangle$. By $\mathcal{M}', \langle a, b\rangle \models \varphi$ and the Bisimulation Invariance Lemma in HS we obtain $\mathcal{M}', \langle a-1, b\rangle \models \varphi$ and $\mathcal{M}', \langle a, b\rangle \models \varphi$. Since φ is satisfied in more than one interval in $\mathcal{M}'$, it follows that φ does not simulate a nominal, contradiction. Thus, nominals are not HS-expressible in ($\le$,Non-S,Dis)-frames and ($\le$,Non-S,Den)-frames. ☐

As a consequence of the above theorem we obtain that the difference operator is not HS-expressible in ($\le$,Non-S,Dis)-frames and ($\le$,Non-S,Den)-frames. Indeed, suppose that D is HS-expressible in the mentioned classes of HS-frames. Then, by the discussed result from [23] we obtain that nominals are HS-expressible in these frames. However, by Theorem 25 we know that this is not the case, so we obtain a contradiction.

Corollary 26. *The difference operator is not* HS*-expressible in ($\le$,Non-S,Dis)-frames and ($\le$,Non-S,Den)-frames.*

By Theorem 19 the difference operator is not $\mathsf{HS}^{\Diamond}_{horn}$-expressible in Dis-frames and in Den-frames. Interestingly, we will show that nominals are expressible in an even more restricted fragment, namely in $\mathsf{HS}^{\Diamond}_{core}$.

Theorem 27. *Nominals are $\mathsf{HS}^{\Diamond}_{core}$-expressible in $<$-frames and ($\le$,S)-frames.*

Proof. Let φ be an $\mathsf{HS}^{\Diamond,i}_{core}$-formula. We will show how to construct an equisatisfiable $\mathsf{HS}^{\Diamond}_{core}$-formula φ'. We will show separate (but similar) constructions for $<$-frames and ($\le$,S)-frames. In both cases we simulate each nominal i occurring in φ with a fresh propositional variable p_i.

First, we introduce an $\mathsf{HS}^{\Diamond}_{core}$-formula $\psi^i_{(<)}$ expressing in $<$-frames that p_i simulates a nominal, that is $\psi^i_{(<)}$ states that p_i is satisfied in exactly one interval. Define:

$$\psi^i_{(<)} \overset{\mathrm{df}}{=} \langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_i \wedge \tag{1}$$

$$[\mathsf{U}](p_i \wedge \langle \overline{\mathsf{B}} \rangle \langle \overline{\mathsf{E}} \rangle \langle \mathsf{E} \rangle p_i \rightarrow \bot) \wedge \tag{2}$$

$$[\mathsf{U}](p_i \wedge \langle \overline{\mathsf{E}} \rangle \langle \overline{\mathsf{B}} \rangle \langle \mathsf{B} \rangle p_i \rightarrow \bot). \tag{3}$$

Let $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<, V)$ be an HS-model based on an irreflexive frame. We need to show that $\mathcal{M} \models \psi^i_{(<)}$ if and only if p_i is satisfied in exactly one interval in $I(\mathbb{D})$. By (1) we have $\mathcal{M} \models \langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_i$, so there exists $\langle x', y' \rangle \in I(\mathbb{D})$ such that $\mathcal{M}, \langle x', y' \rangle \models p_i$. Next, we need to show that there is no $\langle x'', y'' \rangle \neq \langle x', y' \rangle$ such that $\mathcal{M}, \langle x'', y'' \rangle \models p_i$. Suppose that there is such $\langle x'', y'' \rangle$, then we have $y' \neq y''$ or $x' \neq x''$.

(Case 1): $y' \neq y''$. Hence, either $y' < y''$ or $y'' < y'$. If $y' < y''$, then it is easy to show that $\langle x', y' \rangle \overline{\mathsf{B}}_< \circ \overline{\mathsf{E}}_< \circ \mathsf{E}_< \langle x'', y'' \rangle$. Hence, $\mathcal{M}, \langle x', y' \rangle \models p_i \wedge \langle \overline{\mathsf{B}} \rangle \langle \overline{\mathsf{E}} \rangle \langle \mathsf{E} \rangle p_i$, and so by (2) we obtain a contradiction.

On the other hand, if $y'' < y'$, then $\langle x'', y'' \rangle \overline{\mathsf{B}}_< \circ \overline{\mathsf{E}}_< \circ \mathsf{E}_< \langle x', y' \rangle$. As a result, $\mathcal{M}, \langle x'', y'' \rangle \models p_i \wedge \langle \overline{\mathsf{B}} \rangle \langle \overline{\mathsf{E}} \rangle \langle \mathsf{E} \rangle p_i$, so by (2) we obtain a contradiction.

(Case 2): $x' \neq x''$. Then, $x' < x''$ or $x'' < x'$. We will consider only the case $x' < x''$ since the argument for $x'' < x'$ is analogous (compare with Case 1). If $x' < x''$, then $\langle x', y' \rangle \overline{\mathsf{E}}_< \circ \overline{\mathsf{B}}_< \circ \mathsf{B}_< \langle x'', y'' \rangle$. Hence, $\mathcal{M}, \langle x', y' \rangle \models p_i \wedge \langle \overline{\mathsf{E}} \rangle \langle \overline{\mathsf{B}} \rangle \langle \mathsf{B} \rangle p_i$, so by (3) we obtain a contradiction.

It follows that $\mathcal{M} \models \psi^i_{(<)}$ if and only if p_i is satisfied in exactly one interval in $I(\mathbb{D})$. In ($\leq$,S)-frames a formula stating that p_i is satisfied in exactly one interval is as follows:

$$\psi^i_{(\leq,\mathsf{S})} \overset{\mathrm{df}}{=} \langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_i \wedge \tag{4}$$

$$[\mathsf{U}](p_i \wedge \langle \mathsf{A} \rangle \langle \overline{\mathsf{E}} \rangle \langle \mathsf{E} \rangle p_i \rightarrow \bot) \wedge \tag{5}$$

$$[\mathsf{U}](p_i \wedge \langle \overline{\mathsf{A}} \rangle \langle \overline{\mathsf{B}} \rangle \langle \mathsf{B} \rangle p_i \rightarrow \bot). \tag{6}$$

Clearly, (4) states that p_i is satisfied in some interval. Then, by a similar argument as in the case of $\psi^i_{(<)}$, we can show that by (5) the propositional variable p_i cannot be satisfied in two intervals $\langle x, y\rangle$, $\langle x', y'\rangle$ such that $y < y'$, whereas (6) disallows p_i to be satisfied in two intervals $\langle x, y\rangle$, $\langle x', y'\rangle$ such that $x' < x$. As a result, $\psi^i_{(\leq,\mathsf{S})}$ forces p_i to be satisfied in exactly one interval.

In the case of an irreflexive frame, φ' is a conjunction of a formula obtained by replacing each nominal i in φ with a fresh propositional variable p_i and $\bigwedge_{i\in\mathsf{NOM}(\varphi)} \psi^i_{(<)}$, where $\mathsf{NOM}(\varphi)$ is the set of nominals occurring in φ. For a frame from the class ($\leq$,S), the construction of φ' is analogous but $\psi^i_{(<)}$ is replaced with $\psi^i_{(\leq,\mathsf{S})}$. It follows that φ' is equisatisfiable with φ and since we have eliminated all nominals from φ, the formula φ' belongs to the $\mathsf{HS}^{\Diamond}_{core}$-language. $\square$

Observe that by Theorem 25 nominals are not HS-expressible in ($\leq$,$\mathsf{Non\text{-}S}$,Dis)-frames and ($\leq$,$\mathsf{Non\text{-}S}$,Den)-frames. Hence, we cannot use the construction from the proof of Theorem 27 to express nominals in ($\leq$,$\mathsf{Non\text{-}S}$,Dis)-frames and in ($\leq$,$\mathsf{Non\text{-}S}$,Den)-frames. In particular, in such frames we cannot construct an HS-formula stating that a propositional variable p_i holds in exactly one interval. In ($\leq$,$\mathsf{Non\text{-}S}$,Dis)-frames and ($\leq$,$\mathsf{Non\text{-}S}$,Den)-frames the formula $\psi^i_{(<)}$ from the above proof does not express that p_i holds in exactly one interval because in reflexive frames $\mathsf{E}_{\leq}$, $\overline{\mathsf{E}}_{\leq}$, $\mathsf{B}_{\leq}$, and $\overline{\mathsf{B}}_{\leq}$ are not irreflexive relations (they are reflexive in such frames). On the other hand, also $\psi^i_{(\leq,\mathsf{S})}$ cannot be used because in ($\leq$,$\mathsf{Non\text{-}S}$,Dis) and ($\leq$,$\mathsf{Non\text{-}S}$,Den)-frames the relations $\mathsf{A}_{\leq}$ and $\overline{\mathsf{A}}_{\leq}$ are not irreflexive (they are reflexive in punctual-intervals).

Furthermore, we have proved that nominals are not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of frames.

Theorem 28. *Nominals are not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of frames.*

The proof is similar to the proof of Theorem 21, so we move it to the Appendix.

7. Satisfaction Operators

Satisfaction operators (similarly as nominals) can be expressed by means of the difference operator whenever there is no restriction on the use of propositional connectives in a language (see [23]). Hence, by the fact that the difference operator is HS-expressible in $<$-frames and in $(\leq,\mathsf{S})$-frames (see Section 5), we obtain that satisfaction operators are also expressible in these classes of frames.

Corollary 29. *Satisfaction operators are* HS*-expressible in* $(\leq,S)$*-frames and* $<$*-frames.*

Next, we will consider some modification of HS-language. First, we show that satisfaction operators are $\mathsf{HS}^{\Diamond,i}_{horn}$-expressible in any class of frames.

Theorem 30. *Satisfaction operators are* $\mathsf{HS}^{\Diamond,i}_{horn}$*-expressible in any class of frames.*

Proof. Let φ be an $\mathsf{HS}^{\Diamond,i,@}_{horn}$-formula. We will construct an $\mathsf{HS}^{\Diamond,i}_{horn}$-formula φ' which in any class of frames is equisatisfiable with φ.

We start the construction by taking φ and in each iteration we modify it. In each iteration of the construction we choose the right-most occurrence of a satisfaction operator, say $@_i$, in the so far constructed formula. Let $@_i\eta$ be the literal containing this occurrence of $@_i$. Since we have chosen the right-most occurrence of a satisfaction operator, η does not contain satisfaction operators. Then, we perform two steps:

Step 1: We replace $@_i\eta$ with:

$$\langle \mathsf{L}\rangle\langle\overline{\mathsf{L}}\rangle p_{@i\eta}, \tag{7}$$

where $p_{@i\eta}$ is a fresh propositional variable which did not occur in the so far constructed formula.

Step 2: We concatenate the obtained formula with:

$$\wedge\, [\mathsf{U}](i \wedge \eta \to p_{@i\eta}) \wedge [\mathsf{U}](p_{@i\eta} \to i) \wedge [\mathsf{U}](p_{@i\eta} \to \eta). \tag{8}$$

Intuitively, we force $p_{@i\eta}$ to be satisfied in the single interval in which $i \wedge \eta$ holds. We continue construction with the obtained formula until all occurrences of satisfaction operators are deleted. We denote the resulting formula by φ'.

We will prove by induction on the number of iterations of the construction that φ and φ' are equisatisfiable. Assume that in the beginning of some iteration of the construction we had a formula φ_n and in the end of this iteration we have obtained a formula φ_{n+1}. We claim that in any class of frames φ_n is equisatisfiable with φ_{n+1}. Formula φ_{n+1} was obtained by replacing in φ_n a subformula $@_i\eta$ with $\langle \mathsf{L}\rangle\langle \overline{\mathsf{L}}\rangle p_{@i\eta}$ and by concatenating the obtained formula with (8).

Assume that φ_n is satisfiable. Hence, there exists an HS-model $\mathcal{M} = (\mathcal{F}, V)$ and an interval $\langle x, y\rangle$ such that $\mathcal{M}, \langle x, y\rangle \models \varphi_n$. We will to show that φ_{n+1} is also satisfiable in $\mathcal{F}$. Let $\langle x', y'\rangle$ be such that $V(i) = \{\langle x', y'\rangle\}$. Define $\mathcal{M}' = (\mathcal{F}, V')$ such that V' extends V with:

$$V'(p_{@i\eta}) \overset{\mathrm{df}}{=} \begin{cases} \{\langle x', y'\rangle\} & \text{if } \mathcal{M}, \langle x', y'\rangle \models \eta; \\ \emptyset & \text{if } \mathcal{M}, \langle x', y'\rangle \not\models \eta. \end{cases}$$

It is easy to see that by the definition of V' we obtain that (8) is true in $\mathcal{M}'$. Moreover, by the definition of V', for each interval $\langle u, w\rangle$ we have $\mathcal{M}', \langle u, w\rangle \models p_{@i\eta}$ if and only if $\mathcal{M}, \langle u, w\rangle \models i \wedge \eta$. Hence, $\mathcal{M}' \models \langle \mathsf{L}\rangle\langle \overline{\mathsf{L}}\rangle p_{@i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. It follows that replacing $@_i\eta$ with $\langle \mathsf{L}\rangle\langle \overline{\mathsf{L}}\rangle p_{@i\eta}$ in φ_n does not change the truth value of this formula, so $\mathcal{M}', \langle x, y\rangle \models \varphi_{n+1}$.

On the other hand, assume that φ_{n+1} is satisfiable, that is $\mathcal{M}, \langle x, y\rangle \models \varphi_{n+1}$ for some HS-model $\mathcal{M}$ and interval $\langle x, y\rangle$. Then, (8) is satisfied in $\mathcal{M}$ in $\langle x, y\rangle$, so for each interval $\langle u, w\rangle$ we have $\mathcal{M}, \langle u, w\rangle \models p_{@i\eta}$ if and only if $\mathcal{M}, \langle u, w\rangle \models i \wedge \eta$. It follows that $\mathcal{M} \models \langle \mathsf{L}\rangle\langle \overline{\mathsf{L}}\rangle p_{@i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. Thus, $\mathcal{M}, \langle x, y\rangle \models \varphi_n$.

In each iteration of the construction we delete one occurrence of a satisfaction operator and add a formula of a constant length which does not contain satisfaction operators. It follows that the construction terminates and number of iterations is linear in the length of φ. □

We can use a similar translation to express satisfaction operators in $\mathsf{HS}^{\Box,i}_{horn}$ as described in the proof of the following theorem.

Theorem 31. *Satisfaction operators are* $\mathsf{HS}^{\Box,i}_{horn}$*-expressible in any class of frames.*

Proof Sketch. Let φ be an $\mathsf{HS}^{\Box,i,@}_{horn}$-formula. The construction of an $\mathsf{HS}^{\Box,i}_{horn}$-formula φ' which is equisatisfiable with φ is analogous to the construction presented in the proof of Theorem 30, except that:

- (7) is replaced by $[\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$, and
- (8) is replaced by:

$$\wedge\, [\mathsf{U}](i \wedge \eta \to [\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}) \wedge [\mathsf{U}]([\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta} \wedge i \to \eta). \tag{9}$$

Intuitively, we force $p_{@i\eta}$ to be satisfied in all intervals if and only if η holds in the interval in which i holds. The construction is clearly linear since it is analogous to the one from the proof of Theorem 30 and it is relatively easy to show that in each HS-frame φ and φ' are equisatisfiable. $\Box$

Our last result on expressive power concerns HS^{i}_{core}-language which, as we will show, enables to express satisfaction operators in ($\leq$,S)-frames and $<$-frames.

Theorem 32. *Satisfaction operators are* HS^{i}_{core}*-expressible in (*$\leq$*,S)-frames and* $<$*-frames.*

Proof. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an HS-frame belonging to one of the classes ($\leq$,S) or $<$. Let φ be an HS^{i}_{core}-formula. The construction of an equisatisfiable HS^{i}_{core}-formula φ' is as follows. Initially take φ and in each iteration of the construction modify the so far constructed formula. First, find the right-most occurrence of a satisfaction operator, say $@_i$, in the so far constructed formula. Let $@_i\eta$ be the literal containing this occurrence of $@_i$, hence η does not contain satisfaction operators. Then, depending on the location of $@_i$ we perform different modification of the formula.

If the chosen occurrence of $@_i$ is in an antecedent of some clause, then perform the following operations:

Step 1a: Replace $@_i\eta$ with $[\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$, where $p_{@_i\eta}$ is a fresh propositional variable which did not occur in the so far constructed formula.

Step 2a: Concatenate the obtained formula with the following:

$$\wedge[\mathsf{U}](i \to [\overline{\mathsf{B}}][\mathsf{B}][\mathsf{A}]p_{@_i\eta}) \wedge [\mathsf{U}](i \to [\overline{\mathsf{A}}][\overline{\mathsf{B}}][\mathsf{B}]p_{@_i\eta})\wedge \tag{10}$$

$$[\mathsf{U}](i \to [\mathsf{A}][\overline{\mathsf{E}}][\mathsf{E}]p_{@_i\eta}) \wedge [\mathsf{U}](i \to [\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{A}}]p_{@_i\eta})\wedge \tag{11}$$

$$[\mathsf{U}](\eta \to p_{@_i\eta}). \tag{12}$$

If the chosen occurrence of $@_i$ is in a consequent of some clause or is an initial condition, then perform the following steps:

Step 1b: Replace $@_i\eta$ with $\langle\mathsf{L}\rangle\langle\overline{\mathsf{L}}\rangle r_{@_i\eta}$, where $r_{@_i\eta}$ is a fresh propositional variable which did not occur in the so far constructed formula.

Step 2b: Concatenate the obtained formula with the following:

$$\wedge\, [\mathsf{U}](r_{@_i\eta} \to i) \wedge [\mathsf{U}](r_{@_i\eta} \to \eta). \tag{13}$$

The construction is linear since it is analogous to the constructions from the proofs of Theorem 30 and Theorem 31. Hence, it remains to show that φ and φ' are equisatisfiable in $\mathcal{F}$.

Assume that in the beginning of some iteration of the construction we had a formula φ_n and in the end of this iteration we have obtained a formula φ_{n+1}. Let $@_i$ be the right-most occurrence of a satisfaction operator in φ_n and let $@_i\eta$ be the literal containing $@_i$.

(Case 1): $@_i$ is located in an antecedent of some clause. Then, φ_{n+1} is obtained by replacing in φ_n the subformula $@_i\eta$ with $[\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ and by concatenating the obtained formula with (10)–(12).

Assume that φ_n is satisfiable, hence there exists an HS-model $\mathcal{M} = (\mathcal{F}, V)$ and $\langle x, y\rangle \in I(\mathbb{D})$ such that $\mathcal{M}, \langle x, y\rangle \models \varphi_n$. We will show that φ_{n+1} is also satisfiable in $\mathcal{F}$. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ and let $\langle x', y'\rangle \in I(\mathbb{D})$ be such that $V(i) = \{\langle x', y'\rangle\}$. Define $\mathcal{M}' = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V')$ such that V' extends V with:

$$V'(p_{@_i\eta}) \overset{\text{df}}{=} \begin{cases} I(\mathbb{D}) & \text{if } \mathcal{M}, \langle x', y'\rangle \models \eta; \\ I(\mathbb{D}) \setminus \{\langle x', y'\rangle\} & \text{if } \mathcal{M}, \langle x', y'\rangle \not\models \eta. \end{cases}$$

It is relatively easy to see that (10) and (11) are true in a model based on any frame from the classes $(\leq, \mathsf{S})$ and $<$ whenever $p_{@_i\eta}$ is satisfied in all intervals in which i does not hold. Hence, by the definition of V' we obtain that (10) and (11) are true in $\mathcal{M}'$. It is also easy to see that by the definition of V' we obtain that (12) is true in $\mathcal{M}'$. Also by the definition of V' we have $\mathcal{M}' \models [\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. It follows that replacing $@_i\eta$ with $[\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ in φ_n does not change the truth value of the formula, so we have $\mathcal{M}', \langle x, y\rangle \models \varphi_{n+1}$.

On the other hand, assume that $\mathcal{M}, \langle x, y\rangle \models \varphi_{n+1}$ for some HS-model $\mathcal{M}$ which is based on a frame $\mathcal{F}$ and for some interval $\langle x, y\rangle$. Let $\langle x', y'\rangle$ be such that $\mathcal{M}, \langle x', y'\rangle \models i$. Hence, by (12) we have either $\mathcal{M}, \langle x', y'\rangle \models \eta \wedge p_{@_i\eta}$, $\mathcal{M}, \langle x', y'\rangle \models \neg\eta \wedge \neg p_{@_i\eta}$, or $\mathcal{M}, \langle x', y'\rangle \models \neg\eta \wedge p_{@_i\eta}$. If $\mathcal{M}, \langle x', y'\rangle \models \eta \wedge p_{@_i\eta}$ or $\mathcal{M}, \langle x', y'\rangle \models \neg\eta \wedge \neg p_{@_i\eta}$, then $\mathcal{M} \models [\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. Hence, replacing $[\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ with $@_i\eta$ in φ_{n+1} does not change truth value of the formula and consequently $\mathcal{M}, \langle x, y\rangle \models \varphi_n$.

If $\mathcal{M}, \langle x', y'\rangle \models \neg\eta \wedge p_{@_i\eta}$, then $\mathcal{M}, \models [\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ and $\mathcal{M} \models \neg @_i\eta$. Notice, that replacing $[\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ with $@_i\eta$ (that is a true formula with a false formula) in an antecedent of a clause in φ_{n+1} which was true cannot make this clause false. Thus, $\mathcal{M}, \langle x, y\rangle \models \varphi_n$.

(Case 2): $@_i$ occurs as a consequent of some clause or as an initial condition. Then, φ_{n+1} is obtained by replacing in φ_n the subformula $@_i\eta$ with $\langle\mathsf{L}\rangle\langle\overline{\mathsf{L}}\rangle r_{@_i\eta}$ and by concatenating the obtained formula with (13).

Assume that φ_n is satisfiable, so there exist an HS-model $\mathcal{M} = (\mathcal{F}, V)$ and $\langle x, y\rangle$ such that $\mathcal{M}, \langle x, y\rangle \models \varphi_n$. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ and $\langle x', y'\rangle \in I(\mathbb{D})$ be such that $V(i) = \{\langle x', y'\rangle\}$. Define $\mathcal{M}' = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V')$ such that V'

extends V with:

$$V'(r_{@_i\eta}) \overset{\text{df}}{=} \begin{cases} \{\langle x', y'\rangle\} & \text{if } \mathcal{M}, \langle x', y'\rangle \models \eta; \\ \emptyset & \text{if } \mathcal{M}, \langle x', y'\rangle \not\models \eta. \end{cases}$$

It is easy to see that by the definition of V' we have $\mathcal{M}' \models \langle \mathsf{L}\rangle\langle\overline{\mathsf{L}}\rangle r_{@_i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. Hence, $\mathcal{M}', \langle x, y\rangle \models \varphi_{n+1}$.

On the other hand, assume that $\mathcal{M}, \langle x, y\rangle \models \varphi_{n+1}$ for some HS-model $\mathcal{M}$ and interval $\langle x, y\rangle$. Let $\langle x', y'\rangle$ be such that $\mathcal{M}, \langle x', y'\rangle \models i$. Then, by (13) we have either $\mathcal{M}, \langle x', y'\rangle \models \eta \wedge r_{@_i\eta}$, $\mathcal{M}, \langle x', y'\rangle \models \neg\eta \wedge \neg r_{@_i\eta}$, or $\mathcal{M}, \langle x', y'\rangle \models \eta \wedge \neg r_{@_i\eta}$. In the first two cases we have $\mathcal{M} \models [\mathsf{L}][\overline{\mathsf{L}}]p_{@_i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. If $\mathcal{M}, \langle x', y'\rangle \models \eta \wedge \neg r_{@_i\eta}$, then replacing $\langle \mathsf{L}\rangle\langle\overline{\mathsf{L}}\rangle r_{@_i\eta}$ with $@_i\eta$ (that is a false formula with a true formula) in a consequent of a clause in φ_{n+1} which was true cannot make this clause false. It follows that $\mathcal{M}, \langle x, y\rangle \models \varphi_n$.

We have showed that φ and φ' are equisatisfiable which completes the proof. □

8. Conclusions

In the paper we have characterized expressive power of HS and its modifications in terms of possibility to:

- Define with respect to reflexive and irreflexive frames the classes of: non-strict, strict, discrete, and dense frames;
- Express the difference operator, nominals, and satisfaction operators.

Cumulative results on frames definability are presented in Table 2. We have recalled the result from [1] that discrete frames are HS-expressible with respect to irreflexive frames. Then, we have proved that non-strict (Theorem 7), strict (Theorem 8), and dense (Theorem 9) frames are also HS-definable with respect to irreflexive frames. Moreover, we have showed that non-strict frames are HS-definable with respect to reflexive frames (Theorem 10) and dense frames are

Table 2: Frames definable in HS and HS^i with respect to reflexive and irreflexive frames, respectively, where ✓ denotes that the given class of frames is definable, – that the class is not definable, and **?** an open problem.

		Definable classes of frames:			
	with respect to:	non-strict	strict	discrete	dense
HS	reflexive frames	✓	?	?	–
	irreflexive frames	✓	✓	✓	✓
HS^i	reflexive frames	✓	✓	?	✓
	irreflexive frames	✓	✓	✓	✓

not (Theorem 13). The question of HS-definability of strict and discrete frames with respect to reflexive frames remains open.

Furthermore, we have showed that adding nominals to HS-language allows to define strictly more classes of frames. In particular, dense frames are HS^i-definable with respect to reflexive frames (Theorem 15) but they are not HS-definable with respect to reflexive frames (Theorem 13). We have also showed that strict frames are HS^i-definable with respect to reflexive frames (Theorem 16) which in HS remains an open problem. Let us emphasize two corollaries of our analysis:

- Some classes of HS-frames are HS-definable with respect to irreflexive but not with respect to reflexive frames;
- Adding nominals to HS enables us to define strictly more classes of HS-frames.

In Section 5, Section 6, and Section 7 we have showed that the possibility to express the difference operator, nominals, and satisfaction operators in HS depends on the form of a frame. Let us distinguish two disjoint sets of HS-frames:

S: ($\leq$,S,Dis), ($\leq$,S,Den), ($<$,Non-S,Dis), ($<$,Non-S,Den), ($<$,S,Dis), ($<$,S,Den);

W: ($\leq$,Non-S,Dis) and ($\leq$,Non-S,Den).

In the frames from the first class we can express in the HS-language the difference operator (follows from Theorem 17 and [23]), nominals (Corollary 22), and satisfaction operators (Corollary 29). Surprisingly, in the frames from the second class even the full language of HS is unable to express any of these operators (Corollary 26 and Theorem 25) – we assume that if nominals are not expressible, then satisfaction operators (indexed by nominals) are neither. Hence, we will call the first class of frames *strong* (**S**) and the second *weak* (**W**). The cumulative results on operators expressible in strong and weak frames, respectively, are presented in Table 3.

Table 3: Operators expressible in HS-fragments (a) in **S**-frames and (b) in **W**-frames, where ✓ denotes that the given operator is expressible, – that the operator is no expressible, and **?** an open problem.

	D	i	@
HS	✓	✓	✓
HS_{horn}	?	✓	✓
$\mathsf{HS}^{\Diamond}_{horn}$	–	✓	✓
$\mathsf{HS}^{\Box}_{horn}$	–	–	–
HS_{core}	?	✓	✓
$\mathsf{HS}^{\Diamond}_{core}$	–	✓	?
$\mathsf{HS}^{\Box}_{core}$	–	–	–

(a)

	D	i	@
HS	–	–	–
HS_{horn}	–	–	–
$\mathsf{HS}^{\Diamond}_{horn}$	–	–	–
$\mathsf{HS}^{\Box}_{horn}$	–	–	–
HS_{core}	–	–	–
$\mathsf{HS}^{\Diamond}_{core}$	–	–	–
$\mathsf{HS}^{\Box}_{core}$	–	–	–

(b)

Furthermore, we have showed that in no class of frames the difference operator is $\mathsf{HS}^{\Box}_{horn}$-expressible (Theorem 21) nor the nominals are (Theorem 28), hence the same hold for $\mathsf{HS}^{\Box}_{core}$. Neither in discrete nor in dense frames the difference operator is $\mathsf{HS}^{\Diamond}_{horn}$-expressible (Theorem 19). Nevertheless, in **S**-frames $\mathsf{HS}^{\Diamond}_{horn}$-expressible are nominals (Theorem 27) and satisfaction operators (Theorem 30). We have showed that nominals are $\mathsf{HS}^{\Diamond}_{core}$-expressible in **S**-frames (Theorem 27) but we do not know whether the satisfaction operators are. Moreover, satisfaction operators are $\mathsf{HS}^{\Box,i}_{horn}$-expressible in any class of frames (Theorem 31) and HS^{i}_{core}-expressible in **S**-frames (Theorem 32).

Our results shed more light on the interplay between expressibility of the difference operator and hybrid machinery (that is nominals and satisfaction operators) in HS. Let us recall that whenever the use of Boolean connectives in a modal language is unrestricted, then expressibility of the difference operator implies expressibility of hybrid machinery [23]. Interestingly, we have showed that in some HS-fragments we can express hybrid machinery without using the difference operator. In particular, we proved that although the difference operator is not $\mathsf{HS}^{\Diamond}_{horn}$-expressible in **S**-frames, the hybrid machinery is.

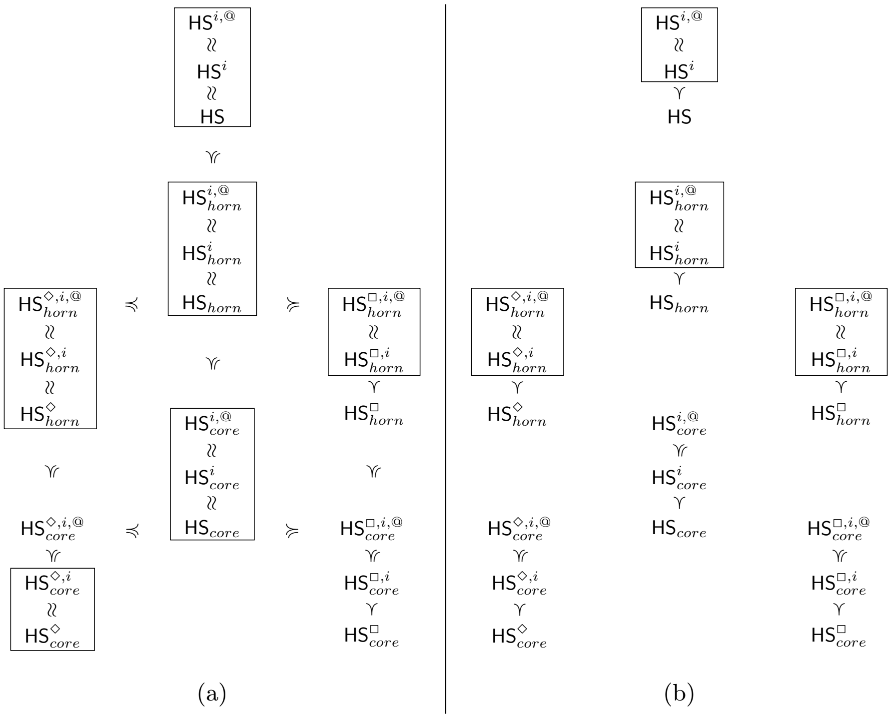


Figure 7: Relative expressive power (a) in **S**-frames and (b) in **W**-frames, where $\approx$ denotes the relation of the same expressive power, $\preccurlyeq$ smaller-or-equal, and $\prec$ strictly smaller expressive power.

We summarize obtained results by constructing diagrams showing relative expressive power of HS and its modifications. The diagrams for strong and weak frames are depicted in Figure 7, where a language L is said to have a smaller-or-equal expressive power than a language L', if for each formula from L we can construct an equisatisfiable formula from L' (possibly using additional propositional variables). Note that in weak frames the difference operator, nominals, and satisfaction operators are not expressible (see Table 3 (b)). Hence, our understanding of the relative expressiveness of HS-fragments in weak frames is much lower than in the case of strong frames.

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References

[1] J. Y. Halpern, Y. Shoham, A propositional modal logic of time intervals, Journal of the ACM (JACM) 38 (4) (1991) 935–962.

[2] D. Bresolin, E. Muñoz-Velasco, G. Sciavicco, Sub-propositional fragments of the interval temporal logic of Allen's relations, in: European Conference on Logics in Artificial Intelligence, Springer, 2014, pp. 122–136.

[3] D. Bresolin, A. Kurucz, E. Muñoz-Velasco, V. Ryzhikov, G. Sciavicco, M. Zakharyaschev, Horn fragments of the Halpern-Shoham interval temporal logic, ACM Transactions on Computational Logic (TOCL) 18 (3) (2017) 22:1–22:39.

[4] D. Bresolin, E. Muñoz-Velasco, G. Sciavicco, Fast(er) reasoning in interval temporal logic, in: LIPIcs-Leibniz International Proceedings in Informatics, Vol. 82, Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

[5] P. A. Wałęga, Computational complexity of a hybridized Horn fragment of Halpern-Shoham logic, in: Indian Conference on Logic and Its Applications, Springer, 2017, pp. 224–238.

[6] A. Artale, R. Kontchakov, V. Ryzhikov, M. Zakharyaschev, Tractable interval temporal propositional and description logics, in: Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI-15), 2015, pp. 1417–1423.

[7] R. Kontchakov, L. Pandolfo, L. Pulina, V. Ryzhikov, M. Zakharyaschev, Temporal and spatial OBDA with many-dimensional Halpern-Shoham logic., in: IJCAI, 2016, pp. 1160–1166.

[8] J. F. Allen, Maintaining knowledge about temporal intervals, Communications of the ACM 26 (11) (1983) 832–843.

[9] D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco, et al., Interval temporal logics: a journey, Bulletin of EATCS 3 (105).

[10] V. Goranko, A. Montanari, G. Sciavicco, A road map of interval temporal logics and duration calculi, Journal of Applied Non-Classical Logics 14 (1-2) (2004) 9–54.

[11] D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco, Interval temporal logics over strongly discrete linear orders: expressiveness and complexity, Theoretical Computer Science 560 (2014) 269–291.

[12] A. Montanari, I. Pratt-Hartmann, P. Sala, Decidability of the logics of the reflexive sub-interval and super-interval relations over finite linear orders, in: Temporal Representation and Reasoning (TIME), 2010 17th International Symposium on, IEEE, 2010, pp. 27–34.

[13] D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco, The light side of interval temporal logic: The Bernays-Schönfinkel's fragment of CDT, Annals of Mathematics and Artificial Intelligence 71 (2014) 11–39.

[14] D. Bresolin, E. Munoz-Velasco, G. Sciavicco, On the complexity of fragments of Horn modal logics, in: Temporal Representation and Reasoning (TIME), 2016 23rd International Symposium on, IEEE, 2016, pp. 186–195.

[15] S. Brandt, E. G. Kalayci, R. Kontchakov, V. Ryzhikov, G. Xiao, M. Zakharyaschev, Ontology-based data access with a Horn fragment of metric temporal logic., in: AAAI, 2017, pp. 1070–1076.

[16] V. Goranko, A. Montanari, G. Sciavicco, et al., Propositional interval neighborhood temporal logics, J. UCS 9 (9) (2003) 1137–1167.

[17] D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco, Expressiveness of the interval logics of Allen's relations on the class of all linear orders: complete classification, in: IJCAI Proceedings-International Joint Conference on Artificial Intelligence, Vol. 22, 2011, p. 845.

[18] L. Aceto, D. Della Monica, A. Ingólfsdóttir, A. Montanari, G. Sciavicco, A complete classification of the expressiveness of interval logics of Allen's relations over dense linear orders, in: Temporal Representation and Reasoning (TIME), 2013 20th International Symposium on, 2013, pp. 65–72.

[19] L. Aceto, D. Della Monica, A. Ingólfsdóttir, A. Montanari, G. Sciavicco, On the expressiveness of the interval logic of Allen's relations over finite and discrete linear orders, in: European Workshop on Logics in Artificial Intelligence, Springer, 2014, pp. 267–281.

[20] Y. Venema, et al., Expressiveness and completeness of an interval tense logic., Notre Dame journal of formal logic 31 (4) (1990) 529–547.

[21] D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco, On the complexity of fragments of the modal logic of Allen's relations over dense structures., in: LATA, 2015, pp. 511–523.

[22] M. Fisher, A resolution method for temporal logic., in: IJCAI, Vol. 91, 1991, pp. 99–104.

[23] C. Areces, P. Blackburn, M. Marx, The computational complexity of hybrid temporal logics, Logic Journal of IGPL 8 (5) (2000) 653–679.

[24] P. Blackburn, Representation, reasoning, and relational structures: a hybrid logic manifesto., Logic Journal of the IGPL 8 (3).

[25] P. Blackburn, M. De Rijke, Y. Venema, Modal Logic, Vol. 53, Cambridge University Press, 2002.

APPENDIX (PROOFS)

Proofs for Section 4: Frames Definability

Theorem 8. *The class of strict frames is* HS*-definable with respect to irreflexive frames.*

Proof. We will prove that the negation of the following formula:

$$\mathsf{non\text{-}strict}_{(<)} \overset{\mathrm{df}}{=} [\mathsf{A}]\neg[\mathsf{B}]\bot \wedge \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle[\mathsf{B}]\bot$$

defines the class of strict frames with respect to the class of irreflexive frames. Thus, we will show that the following statements are equivalent for each irreflexive HS-frame $\mathcal{F}$:

1. $\mathcal{F} \models \neg\mathsf{non\text{-}strict}_{(<)}$.
2. $\mathcal{F}$ is a strict frame.

($1 \Rightarrow 2$) Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<)$ be an irreflexive frame with $\mathcal{F} \models \neg\mathsf{non\text{-}strict}_{(<)}$. If $\mathcal{F}$ were a non-strict frame, then by Theorem 7 we would have $\mathcal{F} \models \mathsf{non\text{-}strict}_{(<)}$ which contradicts the assumption that $\mathcal{F} \models \neg\mathsf{non\text{-}strict}_{(<)}$. Hence, $\mathcal{F}$ is not non-strict. Each frame is either strict or non-strict, so $\mathcal{F}$ is strict.

($2 \Rightarrow 1$) Fix an irreflexive strict HS-frame $\mathcal{F} = (\mathbb{D}, I^-(\mathbb{D}), \mathcal{R}_<)$, an HS-model $\mathcal{M}$ based on $\mathcal{F}$, and $\langle x, y\rangle \in I^-(\mathbb{D})$. We will show that $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{non\text{-}strict}_{(<)}$, that is $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle[\mathsf{B}]\bot \vee [\mathsf{A}]\neg\langle\mathsf{B}\rangle[\mathsf{B}]\bot$. We will consider a case in which there exists an immediate $<$-successor of y in $\mathbb{D}$ and a case in which there is no such a time point.

(Case 1): There exists an immediate $<$-successor of y, say z, in $\mathbb{D}$. We will show that $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle[\mathsf{B}]\bot$. Since $y < z$ we obtain that $\langle y, z\rangle \in I^-(\mathbb{D})$ and $\langle x, y\rangle \mathsf{A}_< \langle y, z\rangle$. It remains to show that $\mathcal{M}, \langle y, z\rangle \models [\mathsf{B}]\bot$.

Since z is the immediate $<$-successor of y and $\langle y, y\rangle \notin I^-(\mathbb{D})$, there is no $w < z$ such that $\langle y, w\rangle \in I^-(\mathbb{D})$. Hence, that is there is no $\langle y, w\rangle \in I^-(\mathbb{D})$ such that $\langle y, z\rangle \mathsf{B}_< \langle y, w\rangle$. Thus, $\mathcal{M}, \langle y, z\rangle \models [\mathsf{B}]\bot$. We have showed that $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{A}\rangle[\mathsf{B}]\bot$, therefore $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{A}\rangle[\mathsf{B}]\bot$. It follows that $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{non\text{-}strict}_{(<)}$.

(Case 2): There is no immediate $<$-successor of y in $\mathbb{D}$. In what follows, we will show that $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg\langle \mathsf{B}\rangle[\mathsf{B}]\bot$. Let $z \in \mathbb{D}$ be such that $\langle x, y\rangle \mathsf{A}_< \langle y, z\rangle$. Then, $y < z$. We will show $\mathcal{M}, \langle y, z\rangle \models \neg\langle \mathsf{B}\rangle[\mathsf{B}]\bot$, that is $\mathcal{M}, \langle y, z\rangle \models [\mathsf{B}]\langle \mathsf{B}\rangle\top$. Let $\langle y, w\rangle \in I^-(\mathbb{D})$ be such that $\langle y, z\rangle \mathsf{B}_< \langle y, w\rangle$. Therefore, $w < z$. Since $\langle y, w\rangle \in I^-(\mathbb{D})$, we have $y < w$. It remains to show $\mathcal{M}, \langle y, w\rangle \models \langle \mathsf{B}\rangle\top$.

By the assumption that y has no immediate $<$-successors there exists $u \in \mathbb{D}$ such that $y < u < w$. Then, $\langle y, w\rangle \mathsf{B}_< \langle y, u\rangle$, so we have $\mathcal{M}, \langle y, w\rangle \models \langle \mathsf{B}\rangle\top$. Thus, $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]\neg\langle \mathsf{B}\rangle[\mathsf{B}]\bot$, that is $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{non\text{-}strict}_{(<)}$.

We have shown that in both cases $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{non\text{-}strict}_{(<)}$. This completes the proof. □

Theorem 9. *The class of dense frames is* HS*-definable with respect to irreflexive frames.*

Proof. Let:

$$\mathsf{length1} \overset{\text{df}}{=} \langle \mathsf{B}\rangle\top \wedge [\mathsf{B}][\mathsf{B}]\bot;$$
$$\mathsf{den}_{(<)} \overset{\text{df}}{=} (\neg\mathsf{non\text{-}strict}_{(<)} \wedge \langle \mathsf{B}\rangle\top) \vee (\mathsf{non\text{-}strict}_{(<)} \wedge \neg\mathsf{length1}).$$

We claim that $\mathsf{den}_{(<)}$ defines the class of dense frames with respect to irreflexive frames. We will show that for every irreflexive HS-frame $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathsf{R}_<)$ the following statements are equivalent:

1. $\mathcal{F} \models \mathsf{den}_{(<)}$.
2. $\mathcal{F}$ is a dense frame.

$(1 \Rightarrow 2)$ Let $\mathcal{F}$ be an irreflexive HS-frame such that $\mathcal{F} \models \mathsf{den}_{(<)}$. Fix arbitrary time points $x, y \in \mathbb{D}$ such that $x < y$. We need to show that $\mathcal{F}$ is dense, that is there is $z \in \mathbb{D}$ such that $x < z < y$. Let $\mathcal{M}$ be a model based on $\mathcal{F}$. We will consider the cases in which $\mathcal{F}$ is non-strict and strict, separately.

(Case 1): $\mathcal{F}$ is a non-strict frame. By Theorem 7 we have $\mathcal{F} \models \mathsf{non\text{-}strict}_{(<)}$, therefore $\mathcal{M}, \langle x, y\rangle \not\models \neg\mathsf{non\text{-}strict}_{(<)} \wedge \langle \mathsf{B}\rangle\top$. We have $\mathcal{M}, \langle x, y\rangle \models \mathsf{den}_{(<)}$, that is $\mathcal{M}, \langle x, y\rangle \models (\neg\mathsf{non\text{-}strict}_{(<)} \wedge \langle \mathsf{B}\rangle\top) \vee (\mathsf{non\text{-}strict}_{(<)} \wedge \neg\mathsf{length1})$. As a result, we have $\mathcal{M}, \langle x, y\rangle \models \mathsf{non\text{-}strict}_{(<)} \wedge \neg\mathsf{length1}$, so in particular, $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{length1}$, that is $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{B}\rangle\top \vee \neg[\mathsf{B}][\mathsf{B}]\bot$.

By non-strictness of $\mathcal{F}$ we have $\langle x, x\rangle \in I(\mathbb{D})$. Then, by $x < y$, we obtain $\langle x, y\rangle \mathsf{B}_< \langle x, x\rangle$, so $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{B}\rangle\top$. As we have previously showed, $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{B}\rangle\top \vee \neg[\mathsf{B}][\mathsf{B}]\bot$, hence we obtain $\mathcal{M}, \langle x, y\rangle \models \neg[\mathsf{B}][\mathsf{B}]\bot$, that is $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{B}\rangle\langle \mathsf{B}\rangle\top$.

Then, there are $u, w \in \mathbb{D}$ such that $\langle x, y\rangle \mathsf{B}_< \langle x, w\rangle$ and $\langle x, w\rangle \mathsf{B}_< \langle x, u\rangle$. Hence, we have $x \leq u < w < y$. It follows that w is such that $x < w < y$, so $\mathcal{F}$ is dense.

(Case 2): $\mathcal{F}$ is a strict frame. Then, by Theorem 8, $\mathcal{F} \models \neg\mathsf{non\text{-}strict}_{(<)}$, hence $\mathcal{M}, \langle x, y\rangle \not\models \mathsf{non\text{-}strict}_{(<)}$, so $\mathcal{M}, \langle x, y\rangle \not\models \mathsf{non\text{-}strict}_{(<)} \wedge \neg\mathsf{length1}$. We have $\mathcal{F} \models \mathsf{den}_{(<)}$, so $\mathcal{M}, \langle x, y\rangle \models (\neg\mathsf{non\text{-}strict}_{(<)} \wedge \langle \mathsf{B}\rangle\top) \vee (\mathsf{non\text{-}strict}_{(<)} \wedge \neg\mathsf{length1})$. Then, we have $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{non\text{-}strict}_{(<)} \wedge \langle \mathsf{B}\rangle\top$. In particular, it holds that $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{B}\rangle\top$, so there is $\langle x, w\rangle \in I(\mathbb{D})$ such that $\langle x, y\rangle \mathsf{B}_< \langle x, w\rangle$, that is $w < y$. By strictness of $\mathcal{F}$ and the fact that $\langle x, w\rangle \in I(\mathbb{D})$ we obtain $x < w$. Hence, $x < w < y$, so $\mathcal{F}$ is dense.

($2 \Rightarrow 1$) Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<)$ be a dense irreflexive HS-frame, $\mathcal{M}$ an HS-model based on $\mathcal{F}$, and $\langle x, y\rangle \in I(\mathbb{D})$ an interval. We will show that $\mathcal{M}, \langle x, y\rangle \models \mathsf{den}_{(<)}$. In what follows, we will consider the cases in which $\mathcal{F}$ is strict and non-strict, separately:

(Case 1): $\mathcal{F}$ is strict. Then, by Theorem 8 we have $\mathcal{F} \models \neg\mathsf{non\text{-}strict}_{(<)}$. We will show that $\mathcal{F} \models \langle \mathsf{B}\rangle\top$, that is $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{B}\rangle\top$ for an arbitrary $\langle x, y\rangle \in I(\mathbb{D})$.

Fix arbitrary $\langle x, y\rangle \in I(\mathbb{D})$. Since $\mathcal{F}$ is strict, $\langle x, y\rangle \in I(\mathbb{D})$ implies $x < y$. By the assumption, $\mathcal{F}$ is dense, so there is $z \in \mathbb{D}$ such that $x < z < y$.

Then, $\langle x, y\rangle \mathsf{B}_{<} \langle x, z\rangle$, hence $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{B}\rangle\top$. As a result, we have showed that $\mathcal{M}, \langle x, y\rangle \models \mathsf{strict}_{(<)} \wedge \langle \mathsf{B}\rangle\top$, so $\mathcal{M}, \langle x, y\rangle \models \mathsf{den}_{(<)}$.

(Case 2): $\mathcal{F}$ is non-strict, so by Theorem 7 we obtain that $\mathcal{F} \models \mathsf{non\text{-}strict}_{(<)}$. Let $\langle x, y\rangle \in I(\mathbb{D})$. In what follows, we will show that $\mathcal{M}, \langle x, y\rangle \models \neg\mathsf{length1}$, that is $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{B}\rangle\top \vee \neg[\mathsf{B}][\mathsf{B}]\bot$. Clearly, either $x = y$ or $x < y$.

(Case 2.1): $x = y$. Then, there is no $z < y$ such that $\langle x, z\rangle \in I(\mathbb{D})$, so there is no interval accessible with $\mathsf{B}_{<}$ from $\langle x, y\rangle$. Hence, $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{B}\rangle\top$ and consequently $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{B}\rangle\top \vee \neg[\mathsf{B}][\mathsf{B}]\bot$.

(Case 2.2): $x < y$. By density of $\mathcal{F}$, there is z such that $x < z < y$. Then, $\langle x, y\rangle \mathsf{B}_{<} \langle x, z\rangle$. Since $\mathcal{F}$ is non-strict, we have $\langle x, x\rangle \in I(\mathbb{D})$ and consequently $\langle x, z\rangle \mathsf{B}_{<} \langle x, x\rangle$. Therefore, $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{B}\rangle\langle \mathsf{B}\rangle\top$, that is $\mathcal{M}, \langle x, y\rangle \models \neg[\mathsf{B}][\mathsf{B}]\bot$. Hence, $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{B}\rangle\top \vee \neg[\mathsf{B}][\mathsf{B}]\bot$.

We have shown that in all cases $\mathcal{F} \models \mathsf{den}_{(<)}$, so the right-to-left implication holds and the proof is done. ☐

Theorem 10. *The class of non-strict frames is* HS*-definable with respect to reflexive frames.*

Proof. We claim that the formula $\mathsf{non\text{-}strict}_{(\leq)}$ defines non-strict frames with respect to reflexive frames. Thus, we will show that for every reflexive HS-frame $\mathcal{F}$ the following conditions are equivalent:

1. $\mathcal{F} \models \mathsf{non\text{-}strict}_{(\leq)}$.
2. $\mathcal{F}$ is a non-strict frame.

(1 ⇒ 2) We will prove the contrapositive of the left-to-right implication. Recall, that if a frame is not non-strict, then it is strict. Let $\mathcal{F} = (\mathbb{D}, I^{-}(\mathbb{D}), \mathcal{R}_{\leq})$ be a reflexive strict HS-frame. We will show that there are an HS-model $\mathcal{M}$ based on $\mathcal{F}$ and $\langle x, y\rangle \in I^{-}(\mathbb{D})$ such that $\mathcal{M}, \langle x, y\rangle \not\models \mathsf{non\text{-}strict}_{(\leq)}$.

Fix an arbitrary $\langle x, y\rangle \in I^{-}(\mathbb{D})$ and let $\mathcal{M} = (\mathcal{F}, V)$ be an HS-model such that $V(p) = \{\langle z, y\rangle \in I^{-}(\mathbb{D}) \mid x \leq z\}$. Then, for any z such that $x \leq z$ and

$\langle z, y\rangle \in I^-(\mathbb{D})$ we have $\mathcal{M}, \langle z, y\rangle \models p$, hence $\mathcal{M}, \langle x, y\rangle \models [\mathsf{E}]p$. On the other hand, by $\langle x, y\rangle \in I^-(\mathbb{D})$ we have $x \neq y$, so by the definition of V there is no w such that $y < w$ and $\mathcal{M}, \langle y, w\rangle \models p$. It follows that there is no $\langle y, w\rangle \in I^-(\mathbb{D})$ such that $\langle x, y\rangle \mathsf{A}_{\leq} \langle y, w\rangle$ and $\mathcal{M}, \langle y, w\rangle \models p$. Hence, $\mathcal{M}, \langle x, y\rangle \models \neg\langle \mathsf{A}\rangle p$, so $\mathcal{M}, \langle x, y\rangle \models [\mathsf{E}]p \wedge \neg\langle \mathsf{A}\rangle p$. As a result, we obtain $\mathcal{M}, \langle x, y\rangle \not\models [\mathsf{E}]p \rightarrow \langle \mathsf{A}\rangle p$, that is $\mathcal{M}, \langle x, y\rangle \not\models \mathsf{non\text{-}strict}_{(\leq)}$ and so $\mathcal{F} \not\models \mathsf{non\text{-}strict}_{(\leq)}$.

($2 \Rightarrow 1$) Fix a non-strict reflexive HS-frame $\mathcal{F} = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_{\leq})$. We show that $\mathcal{F} \models \mathsf{non\text{-}strict}_{(\leq)}$. Let $\mathcal{M}$ be an HS-model based on $\mathcal{F}$ and let $\langle x, y\rangle \in I^+(\mathbb{D})$. We will show that $\mathcal{M}, \langle x, y\rangle \models \mathsf{non\text{-}strict}_{(\leq)}$, that is $\mathcal{M}, \langle x, y\rangle \models [\mathsf{E}]p \rightarrow \langle \mathsf{A}\rangle p$.

Assume that $\mathcal{M}, \langle x, y\rangle \models [\mathsf{E}]p$. Since $\langle x, y\rangle \in I^+(\mathbb{D})$, we have $x \leq y$, so by $\langle y, y\rangle \in I^+(\mathbb{D})$ we obtain $\langle x, y\rangle \mathsf{E}_{\leq} \langle y, y\rangle$. Then, by the assumption, $\mathcal{M}, \langle y, y\rangle \models p$. Clearly, $y \leq y$, so $\langle x, y\rangle \mathsf{A}_{\leq} \langle y, y\rangle$. By $\langle x, y\rangle \mathsf{A}_{\leq} \langle y, y\rangle$ and $\mathcal{M}, \langle y, y\rangle \models p$ we obtain $\mathcal{M}, \langle x, y\rangle \models \langle \mathsf{A}\rangle p$, hence $\mathcal{M}, \langle x, y\rangle \models [\mathsf{E}]p \rightarrow \langle \mathsf{A}\rangle p$. □

Theorem 15. *The class of dense frames is* HS^i*-definable with respect to reflexive frames.*

Proof. We claim that the following formula:

$$\mathsf{den}^i_{(\leq)} \overset{\text{df}}{=} \big(i \wedge \langle \overline{\mathsf{B}}\rangle (j \wedge \neg i)\big) \rightarrow \langle \overline{\mathsf{B}}\rangle (\neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j),$$

for any distinct nominals i, j defines the class of dense frames with respect to reflexive frames. We will show that for every reflexive frame $\mathcal{F}$ the following statements are equivalent:

1. $\mathcal{F} \models \mathsf{den}^i_{(\leq)}$.
2. $\mathcal{F}$ is a dense frame.

($1 \Rightarrow 2$) We will show that the contrapositive of the left-to-right implication holds. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_{\leq})$ be a reflexive and non-dense HS-frame. Since $\mathbb{D}$ is not dense, there are $y, z \in \mathbb{D}$ such that z is the immediate $<$-successor of y, that is $y < z$. Let $x \in \mathbb{D}$ be a time point such that $x < y$ and let $\mathcal{M} = (\mathcal{F}, V)$ be a model based on $\mathcal{F}$ such that $V(i) = \{\langle x, y\rangle\}$ and $V(j) = \{\langle x, z\rangle\}$, for nominals i and j. In what follows, we will show that $\mathcal{M}, \langle x, y\rangle \not\models \mathsf{den}^{i}_{(\leq)}$, that is $\mathcal{M}, \langle x, y\rangle \models i \wedge \langle \overline{\mathsf{B}}\rangle(j \wedge \neg i)$ and $\mathcal{M}, \langle x, y\rangle \not\models \langle \overline{\mathsf{B}}\rangle(\neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j)$.

By the definition of V, we have $\mathcal{M}, \langle x, y\rangle \models i$. Since $y < z$, we have $\langle x, y\rangle \overline{\mathsf{B}}_{\leq} \langle x, z\rangle$. By $V(j) = \{\langle x, z\rangle\}$ and $V(i) = \{\langle x, y\rangle\}$, we have $\mathcal{M}, \langle x, z\rangle \models j$ and $\mathcal{M}, \langle x, z\rangle \models \neg i$, so $\mathcal{M}, \langle x, z\rangle \models j \wedge \neg i$. Thus, $\mathcal{M}, \langle x, y\rangle \models \langle \overline{\mathsf{B}}\rangle(j \wedge \neg i)$. Hence, $\mathcal{M}, \langle x, y\rangle \models i \wedge \langle \overline{\mathsf{B}}\rangle(j \wedge \neg i)$.

Now, suppose that $\mathcal{M}, \langle x, y\rangle \models \langle \overline{\mathsf{B}}\rangle(\neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j)$. Then, there is $\langle x, w\rangle \in I(\mathbb{D})$ such that $\langle x, y\rangle \overline{\mathsf{B}}_{\leq} \langle x, w\rangle$ and $\mathcal{M}, \langle x, w\rangle \models \neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j$. By $\langle x, y\rangle \overline{\mathsf{B}}_{\leq} \langle x, w\rangle$ we have $w \leq y$. Since z is the immediate $<$-successor of y, we have $w = y$, or $w = z$, or $z < w$. In the first case $\mathcal{M}, \langle x, w\rangle \models i$, so we obtain $\mathcal{M}, \langle x, w\rangle \not\models \neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j$, a contradiction. In the second case $\mathcal{M}, \langle x, w\rangle \models j$, so $\mathcal{M}, \langle x, w\rangle \not\models \neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j$, which also leads to a contradiction. Finally, if $z < w$, then as z is the immediate $<$-successor fo y, we get $y < w$. Since $V(j) = \{\langle x, z\rangle\}$, there is no u such that $w \leq u$ and $\mathcal{M}, \langle x, u\rangle \models j$, so $\mathcal{M}, \langle x, w\rangle \models \neg\langle \overline{\mathsf{B}}\rangle j$. Hence, $\mathcal{M}, \langle x, w\rangle \not\models \neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j$, a contradiction. Therefore, we have proved that $\mathcal{M}, \langle x, y\rangle \not\models \langle \overline{\mathsf{B}}\rangle(\neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j)$. Hence, $\mathcal{M}, \langle x, y\rangle \not\models \mathsf{den}^{i}_{(\leq)}$ and so $\mathcal{F} \not\models \mathsf{den}^{i}_{(\leq)}$.

($2 \Rightarrow 1$) Assume that $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_{\leq})$ is a dense reflexive HS-frame. Let $\mathcal{M}$ be an HS-model based on $\mathcal{F}$ and $\langle x, y\rangle \in I(\mathbb{D})$. We will show that $\mathcal{M}, \langle x, y\rangle \models \mathsf{den}^{i}_{(\leq)}$, that is $\mathcal{M}, \langle x, y\rangle \models \big(i \wedge \langle \overline{\mathsf{B}}\rangle(j \wedge \neg i)\big) \rightarrow \langle \overline{\mathsf{B}}\rangle(\neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j)$.

Let us assume that $\mathcal{M}, \langle x, y\rangle \models i \wedge \langle \overline{\mathsf{B}}\rangle(j \wedge \neg i)$. Then, $\mathcal{M}, \langle x, y\rangle \models i$ and there is z such that $y \leq z$ and $\mathcal{M}, \langle x, z\rangle \models j \wedge \neg i$. If $z = y$, then $\mathcal{M}, \langle x, y\rangle \models \neg i$ which raises a contradiction. Hence, $y < z$. By $y < z$ and density of $\mathcal{F}$ there is $w \in \mathbb{D}$ such that $y < w < z$. Since $y < w$, we have $\langle x, y\rangle \overline{\mathsf{B}}_{\leq} \langle x, w\rangle$. Thus, it suffices to show that $\mathcal{M}, \langle x, w\rangle \models \neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j$. Recall that each nominal is satisfied in exactly one intervals. Thus, since $\mathcal{M}, \langle x, y\rangle \models i$ and $y \neq w$ we have $\mathcal{M}, \langle x, w\rangle \models \neg i$. Similarly, $\mathcal{M}, \langle x, z\rangle \models j$ and $z \neq w$ imply $\mathcal{M}, \langle x, w\rangle \models \neg j$. Thus, $\mathcal{M}, \langle x, w\rangle \models \neg i \wedge \neg j$. Moreover, $w < z$ implies $\langle x, w\rangle \overline{\mathsf{B}}_{\leq} \langle x, z\rangle$. Therefore, by $\mathcal{M}, \langle x, z\rangle \models j$ we obtain $\mathcal{M}, \langle x, w\rangle \models \langle \overline{\mathsf{B}}\rangle j$. Hence, we have proved that $\mathcal{M}, \langle x, w\rangle \models \neg i \wedge \neg j \wedge \langle \overline{\mathsf{B}}\rangle j$. Then, $\mathcal{M}, \langle x, y\rangle \models \mathsf{den}^{i}_{(\leq)}$ and consequently $\mathcal{F} \models \mathsf{den}^{i}_{(\leq)}$. □

Theorem 16. *The class of strict frames is* HS^i*-definable with respect to reflexive frames.*

Proof. We claim that the formula:

$$\mathsf{strict}^i_{(\leq)} \overset{\text{df}}{=} i \to \neg\langle \mathsf{A} \rangle i,$$

where i is a nominal, HS^i-defines strict HS-frames with respect to reflexive frames. Thus, we will show that for every reflexive HS-frame $\mathcal{F}$ the following statements are equivalent:

1. $\mathcal{F} \models \mathsf{strict}^i_{(\leq)}$.
2. $\mathcal{F}$ is a strict frame.

($1 \Rightarrow 2$) We will show that the contrapositive of the left-to-right implication holds. Let $\mathcal{F} = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_{\leq})$ be a reflexive and non-strict HS-frame and $\langle x, x \rangle \in I^+(\mathbb{D})$ a punctual interval. We will show that $\mathcal{F} \not\models \mathsf{strict}^i_{(\leq)}$, that is $\mathcal{F} \not\models i \to \neg\langle \mathsf{A} \rangle i$.

Let us consider an HS-model $\mathcal{M} = (\mathcal{F}, V)$ based on $\mathcal{F}$ such that $V(i) = \{\langle x, x \rangle\}$. Clearly, $\mathcal{M}, \langle x, x \rangle \models i$. Moreover, since the relation $\mathsf{A}_{\leq}$ is reflexive for punctual intervals, $\langle x, x \rangle \mathsf{A}_{\leq} \langle x, x \rangle$. Therefore, $\mathcal{M}, \langle x, x \rangle \models \langle \mathsf{A} \rangle i$ and so $\mathcal{M}, \langle x, x \rangle \not\models i \to \neg\langle \mathsf{A} \rangle i$. Hence, $\mathcal{F} \not\models \mathsf{strict}^i_{(\leq)}$.

($2 \Rightarrow 1$) Let $\mathcal{F} = (\mathbb{D}, I^-(\mathbb{D}), \mathcal{R}_{\leq})$ be a strict reflexive HS-frame and let $\mathcal{M}$ be an HS-model based on $\mathcal{F}$. Take $\langle x, y \rangle \in I^-(\mathbb{D})$. We will show that $\mathcal{M}, \langle x, y \rangle \models \mathsf{strict}^i_{(\leq)}$, that is $\mathcal{M}, \langle x, y \rangle \models i \to \neg\langle \mathsf{A} \rangle i$.

Assume that $\mathcal{M}, \langle x, y \rangle \models i$. Since i is a nominal, it is satisfied in exactly one interval, namely $\langle x, y \rangle$. By $\langle x, y \rangle \in I^-(\mathbb{D})$, interval $\langle x, y \rangle$ is not punctual. Since the relation $\mathsf{A}_{\leq}$ is irreflexive for non-punctual intervals, we get $\mathcal{M}, \langle x, y \rangle \not\models \langle \mathsf{A} \rangle i$. Thus, $\mathcal{M}, \langle x, y \rangle \models i \to \neg\langle \mathsf{A} \rangle i$. Hence, $\mathcal{M}, \langle x, y \rangle \models \mathsf{strict}^i_{(\leq)}$ and so $\mathcal{F} \models \mathsf{strict}^i_{(\leq)}$. □

Proofs for Section 5: Difference Operator

Theorem 17. *The difference operator is* HS*-expressible in (≤,S)-frames.*

Proof. For any HS formula φ define:

$$\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi) \overset{\text{df}}{=} \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi \vee \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi.$$

Let $\mathcal{M} = (\mathbb{D}, I^{-}(\mathbb{D}), \mathcal{R}_{\leq})$ be an HS-model based on a reflexive strict frame. We will show that in every $\langle x, y\rangle \in I^{-}(\mathbb{D})$ the following conditions are equivalent for any HS-formula φ:

1. $\mathcal{M}, \langle x, y\rangle \models \mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$.
2. $\mathcal{M}, \langle x, y\rangle \models \mathsf{D}\varphi$.

($1 \Rightarrow 2$) Assume that $\mathcal{M}, \langle x, y\rangle \models \mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$. We will show that $\mathcal{M}, \langle x, y\rangle \models \mathsf{D}\varphi$. We have $\mathcal{M}, \langle x, y\rangle \models \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi \vee \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi$, so one of the four disjuncts in $\mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$ is satisfied in $\langle x, y\rangle$.

(Case 1): $\mathcal{M}, \langle x, y\rangle \models \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi$. Then, there exists $\langle x, y'\rangle \in I^{-}(\mathbb{D})$ such that $\langle x, y\rangle\overline{\mathsf{B}}_{\leq}\langle x, y'\rangle$ and $\mathcal{M}, \langle x, y'\rangle \models \langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi$. Hence, there is $\langle x, y''\rangle \in I^{-}(\mathbb{D})$ such that $\langle x, y'\rangle\mathsf{B}_{\leq}\langle x, y''\rangle$ and $\mathcal{M}, \langle x, y''\rangle \models \langle\mathsf{A}\rangle\varphi$. Thus, there is $\langle y'', z\rangle \in I^{-}(\mathbb{D})$ such that $\langle x, y''\rangle\mathsf{B}_{\leq}\langle y'', z\rangle$ and $\mathcal{M}, \langle y'', z\rangle \models \varphi$.

By $\langle x, y\rangle\overline{\mathsf{B}}_{\leq}\langle x, y'\rangle$ we have $y \leq y'$. Since $\langle x, y'\rangle\mathsf{B}_{\leq}\langle x, y''\rangle$ and $\langle x, y''\rangle \in I^{-}(\mathbb{D})$, we get $x < y'' \leq y'$. Hence, $x \neq y''$ and $\langle x, y\rangle \neq \langle y'', z\rangle$. Then, by $\langle x, y\rangle \neq \langle y'', z\rangle$ and $\mathcal{M}, \langle y'', z\rangle \models \varphi$ we obtain $\mathcal{M}, \langle x, y\rangle \models \mathsf{D}\varphi$.

(Case 2): $\mathcal{M}, \langle x, y\rangle \models \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi$. Then, similarly as in Case 1 there are intervals $\langle z, x\rangle, \langle z, x'\rangle, \langle z, x''\rangle \in I^{-}(\mathbb{D})$ such that $\langle x, y\rangle\overline{\mathsf{A}}_{\leq}\langle z, x\rangle$, $\langle z, x\rangle\overline{\mathsf{B}}_{\leq}\langle z, x'\rangle$, $\langle z, x'\rangle\mathsf{B}_{\leq}\langle z, x''\rangle$, and $\mathcal{M}, \langle z, x''\rangle \models \varphi$.

By $\langle z, x\rangle \in I^{-}(\mathbb{D})$ we have $z < x$. Hence, $z \neq x$ and $\langle x, y\rangle \neq \langle z, x''\rangle$. Then, by $\mathcal{M}, \langle z, x''\rangle \models \varphi$ we obtain $\mathcal{M}, \langle x, y\rangle \models \mathsf{D}\varphi$.

(Case 3): $\mathcal{M}, \langle x, y\rangle \models \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi$. By a similar argument as in Case 2 we obtain that there exists $\langle x', y'\rangle \in I^{-}(\mathbb{D})$ such that $y' < y$ and $\mathcal{M}, \langle x', y'\rangle \models \mathsf{D}\varphi$. Hence, $\langle x, y\rangle \neq \langle x', y'\rangle$. Thus, by $\mathcal{M}, \langle x', y'\rangle \models \varphi$ we get $\mathcal{M}, \langle x, y\rangle \models \mathsf{D}\varphi$.

(Case 4): $\mathcal{M}, \langle x,y\rangle \models \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi$. Similarly as in Case 1 we obtain that there exists $\langle x',y'\rangle \in I^-(\mathbb{D})$ such that $y < y'$ and $\mathcal{M}, \langle x',y'\rangle \models \mathsf{D}\varphi$. Therefore, $\langle x,y\rangle \neq \langle x',y'\rangle$. Since $\mathcal{M}, \langle x',y'\rangle \models \varphi$, we obtain $\mathcal{M}, \langle x,y\rangle \models \mathsf{D}\varphi$.

It follows that $\mathcal{M}, \langle x,y\rangle \models \mathsf{D}\varphi$.

$(2 \Rightarrow 1)$ Assume that $\mathcal{M}, \langle x,y\rangle \models \mathsf{D}\varphi$. Thus φ is satisfied in $\langle x',y'\rangle \in I^-(\mathbb{D})$ distinct from $\langle x,y\rangle$. Clearly, $x < x'$ or $x' < x$ or $y < y'$, or $y' < y$.

If $x < x'$, then it is easy to show that $\langle x,y\rangle\overline{\mathsf{B}}_{\leq} \circ \mathsf{B}_{\leq} \circ \mathsf{A}_{\leq}\langle x',y'\rangle$. It follows that $\mathcal{M}, \langle x,y\rangle \models \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi$. If $x' < x$, then we have $\mathcal{M}, \langle x,y\rangle \models \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi$. Similarly, $y < y'$ implies $\mathcal{M}, \langle x,y\rangle \models \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi$ and $y' < y$ implies $\mathcal{M}, \langle x,y\rangle \models \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi$.

Hence, $\mathcal{M}, \langle x,y\rangle \models \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi \vee \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi$ and so $\mathcal{M}, \langle x,y\rangle \models \mathsf{diff}_{(\leq,\mathsf{S})}(\varphi)$. $\square$

Lemma 18. *Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an* HS*-frame. Let $\langle x,y\rangle\mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n\langle x',y'\rangle$ and $\langle x,y\rangle\mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n\langle x'',y''\rangle$ for arbitrary intervals $\langle x,y\rangle, \langle x',y'\rangle, \langle x'',y''\rangle \in I(\mathbb{D})$ and $\mathsf{R}_1, \ldots, \mathsf{R}_n \in \mathcal{R}$. Then, for each $\langle s,t\rangle \in I(\mathbb{D})$ which satisfies all of the following conditions:*

- $\min(x',x'') \leq s \leq \max(x',x'')$;
- $\min(y',y'') \leq t \leq \max(y',y'')$;
- $\min(y'-x', y''-x'') \leq t-s$,

we have $\langle x,y\rangle\mathsf{R}_1 \circ \ldots \circ \mathsf{R}_n\langle s,t\rangle$.

Proof. We will prove the statement under the assumption that $\mathcal{F}$ is an irreflexive frame, that is $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<)$. The proof for reflexive frames in analogous.

First, we recall that in an each HS-frame the operators $\langle\mathsf{D}\rangle$, $\langle\overline{\mathsf{D}}\rangle$, $\langle\mathsf{O}\rangle$, $\langle\overline{\mathsf{O}}\rangle$, $\langle\mathsf{L}\rangle$, and $\langle\overline{\mathsf{L}}\rangle$ are HS-expressible by means of $\langle\mathsf{A}\rangle$, $\langle\overline{\mathsf{A}}\rangle$, $\langle\mathsf{B}\rangle$, $\langle\overline{\mathsf{B}}\rangle$, $\langle\mathsf{E}\rangle$, and $\langle\overline{\mathsf{E}}\rangle$ (see Section 2). In particular, in irreflexive frames $\mathsf{D}_<$, $\overline{\mathsf{D}}_<$, $\mathsf{O}_<$, $\overline{\mathsf{O}}_<$, $\mathsf{L}_<$, and $\overline{\mathsf{L}}_<$ can be defined as compositions of $\mathsf{B}_<, \overline{\mathsf{B}}_<, \mathsf{E}_<, \overline{\mathsf{E}}_<, \mathsf{A}_<$, and $\overline{\mathsf{A}}_<$. Hence, we can restrict attention to $\mathsf{R}_1, \ldots, \mathsf{R}_n \in \{\mathsf{B}_<, \overline{\mathsf{B}}_<, \mathsf{E}_<, \overline{\mathsf{E}}_<, \mathsf{A}_<, \overline{\mathsf{A}}_<\}$.

Let $\langle x, y\rangle \in I(\mathbb{D})$. We will prove the lemma inductively on the length n of the sequence $\mathsf{R}_1, \ldots, \mathsf{R}_n$. Assume that the statement holds for all sequences of length at most k for some $k \in \mathbb{N}$. We will show that the statement holds for each sequence of length $k+1$.

We fix a sequence $\mathsf{R}_1, \ldots, \mathsf{R}_k, \mathsf{R}_{k+1} \in \{\mathsf{B}_<, \overline{\mathsf{B}}_<, \mathsf{E}_<, \overline{\mathsf{E}}_<, \mathsf{A}_<, \overline{\mathsf{A}}_<\}$ and define:

$$X \overset{\mathrm{df}}{=} \{\langle a, b\rangle \in I(\mathbb{D}) \mid \langle x, y\rangle \mathsf{R}_1 \circ \ldots \circ \mathsf{R}_k \langle a, b\rangle\};$$
$$X' \overset{\mathrm{df}}{=} \{\langle a, b\rangle \in I(\mathbb{D}) \mid \langle x, y\rangle \mathsf{R}_1 \circ \ldots \circ \mathsf{R}_k \circ \mathsf{R}_{k+1} \langle a, b\rangle\}.$$

Let $\langle x', y'\rangle \in X'$, $\langle x'', y''\rangle \in X'$, and $\langle s, t\rangle \in I(\mathbb{D})$ be intervals which satisfy conditions listed in the lemma, namely: $\min(x', x'') \leq s \leq \max(x', x'')$, $\min(y', y'') \leq t \leq \max(y', y'')$, and $\min(y' - x', y'' - x'') \leq t - s$. We will show that $\langle s, t\rangle \in X'$.

(Case 1): $\mathsf{R}_{k+1} = \mathsf{B}_<$. Then, there are intervals $\langle x', w'\rangle, \langle x'', w''\rangle \in X$ such that $\langle x', w'\rangle \mathsf{B}_< \langle x', y'\rangle$ and $\langle x'', w''\rangle \mathsf{B}_< \langle x'', y''\rangle$. Let $z = \max(w', w'')$. We will show that $\langle s, z\rangle \in X$ and $\langle s, z\rangle \mathsf{B}_< \langle s, t\rangle$ which implies that $\langle s, t\rangle \in X'$.

Directly by the assumption we have $\min(x', x'') \leq s \leq \max(x', x'')$. Then, by $z = \max(w', w'')$ it follows that $\min(w', w'') \leq z \leq \max(w', w'')$. Moreover, by $z = \max(w', w'')$ and $\min(x', x'') \leq s \leq \max(x', x'')$ we obtain the following: $z - s = \max(w', w'') - s \geq \max(w', w'') - \max(x', x'')$. From the definitions of max and min we have $\max(w', w'') - \max(x', x'') \geq \min(w' - x', w'' - x'')$, so $z - s \geq \min(w' - x', w'' - x'')$. Since $\langle x', w'\rangle \in I(\mathbb{D})$, $\langle x'', w''\rangle \in I(\mathbb{D})$ and $z - s \geq \min(w' - x', w'' - x'')$, we get $\langle s, z\rangle \in I(\mathbb{D})$ (no matter whether the the frame is non-strict or strict).

We have showed that $\langle s, z\rangle \in I(\mathbb{D})$, $\min(x', x'') \leq s \leq \max(x', x'')$, $\min(w', w'') \leq z \leq \max(w', w'')$, and $z - s \geq \min(w' - x', w'' - x'')$. Hence, by the inductive hypothesis we obtain $\langle s, z\rangle \in X$. By $\langle x', w'\rangle \mathsf{B}_< \langle x', y'\rangle$ and $\langle x'', w''\rangle \mathsf{B}_< \langle x'', y''\rangle$ we have $y' < w'$ and $y'' < w''$, respectively. Hence, $\max(y', y'') < \max(w', w'')$. Then, by $z = \max(w', w'')$ and $\min(y', y'') \leq t \leq \max(y', y'')$ we obtain $t < z$. Thus, $\langle s, z\rangle \mathsf{B}_< \langle s, t\rangle$ and so $\langle s, t\rangle \in X'$.

(Case 2): $\mathsf{R}_{k+1} = \overline{\mathsf{B}}_<$. Then, there are intervals $\langle u', w'\rangle, \langle u'', w''\rangle \in X$ such that $\langle u', w'\rangle \overline{\mathsf{B}}_< \langle x', y'\rangle$ and $\langle u'', w''\rangle \overline{\mathsf{B}}_< \langle x'', y''\rangle$. Let $z = s + \min(w' - u', w'' - u'')$. Similarly as in Case 1 we can show that by inductive hypothesis $\langle s, z\rangle \in X$. Moreover, $\langle s, z\rangle \overline{\mathsf{B}}_< \langle s, t\rangle$, so $\langle s, t\rangle \in X'$.

(Case 3): $\mathsf{R}_{k+1} = \mathsf{E}_<$. Hence, there exist intervals $\langle u', w'\rangle, \langle u'', w''\rangle \in X$ such that $\langle u', w'\rangle \mathsf{E}_< \langle x', y'\rangle$ and $\langle u'', w''\rangle \mathsf{E}_< \langle x'', y''\rangle$. Let $z = \min(u', u'')$. Then, by inductive hypothesis $\langle z, t\rangle \in X$. Furthermore, $\langle z, t\rangle \mathsf{E}_< \langle s, t\rangle$, so $\langle s, t\rangle \in X'$.

(Case 4): $\mathsf{R}_{k+1} = \overline{\mathsf{E}}_<$. Then, there are intervals $\langle u', w'\rangle, \langle u'', w''\rangle \in X$ such that $\langle u', w'\rangle \overline{\mathsf{E}}_< \langle x', y'\rangle$ and $\langle u'', w''\rangle \overline{\mathsf{E}}_< \langle x'', y''\rangle$. Let $z = t - \min(w' - u', w'' - u'')$. Then, by the inductive hypothesis $\langle z, t\rangle \in X$. Moreover, $\langle z, t\rangle \overline{\mathsf{E}}_< \langle s, t\rangle$, so $\langle s, t\rangle \in X'$.

(Case 5): $\mathsf{R}_{k+1} = \mathsf{A}_<$. Then, there are intervals $\langle u', w'\rangle, \langle u'', w''\rangle \in X$ such that $\langle u', w'\rangle \mathsf{A}_< \langle x', y'\rangle$ and $\langle u'', w''\rangle \mathsf{A}_< \langle x'', y''\rangle$. Let $z = \min(u', u'')$, by the inductive hypothesis we have $\langle z, s\rangle \in X$. Furthermore, $\langle z, s\rangle \mathsf{A}_< \langle s, t\rangle$, so $\langle s, t\rangle \in X'$.

(Case 6): $\mathsf{R}_{k+1} = \overline{\mathsf{A}}_<$. It follows that there exist $\langle u', w'\rangle, \langle u'', w''\rangle \in X$ such that $\langle u', w'\rangle \overline{\mathsf{A}}_< \langle x', y'\rangle$ and $\langle u'', w''\rangle \overline{\mathsf{A}}_< \langle x'', y''\rangle$. Let $z = \max(w', w'')$. By the inductive hypothesis $\langle t, z\rangle \in X$. Moreover, $\langle t, z\rangle \overline{\mathsf{A}}_< \langle s, t\rangle$, so $\langle s, t\rangle \in X'$.

We have showed that $\langle s, t\rangle \in X'$ and so the proof completed. ☐

Proofs for Section 6: Nominals

Theorem 28. *Nominals are not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of frames.*

Proof. Suppose that nominals are $\mathsf{HS}^{\Box}_{horn}$-expressible in some class of frames. Let $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ be an HS-frame and let φ be an $\mathsf{HS}^{\Box}_{horn}$-formula expressing that $p \in \mathsf{PROP}$ simulates a nominal i. Let $\mathcal{M} = (\mathcal{F}, V)$ and $\mathcal{M}' = (\mathcal{F}, V')$ be HS-models such that for distinct intervals $\langle x, y\rangle$ and $\langle x', y'\rangle$ in $I(\mathbb{D})$ (by unboundedness of $\mathbb{D}$ there are always two distinct intervals in $I(\mathbb{D})$) we have:

$$V(p) = \{\langle x, y\rangle\}; \qquad V'(p) = \{\langle x', y'\rangle\}.$$

Then, $\mathcal{M}, \langle x, y\rangle \models \varphi$. Hence, by Lemma 20 we have $\mathcal{K}^{x,y}_{\varphi,\mathcal{F}}, \langle x, y\rangle \models \varphi$. Since φ expresses that p simulates a nominal, there exists $\langle u, w\rangle$ such that $\mathcal{K}^{x,y}_{\varphi,\mathcal{F}}, \langle u, w\rangle \models p$. Then, by Lemma 20 we have $\mathcal{M}, \langle u, w\rangle \models p$ and $\mathcal{M}', \langle u, w\rangle \models p$. Since $\langle x, y\rangle$ is distinct from $\langle x', y'\rangle$, we have either $\langle u, w\rangle \neq \langle x, y\rangle$ or $\langle u, w\rangle \neq \langle x', y'\rangle$.

(Case 1): $\langle u, w\rangle \neq \langle x, y\rangle$. We have $V(p) = \{\langle x, y\rangle\}$ and $\langle u, w\rangle \neq \langle x, y\rangle$, therefore $\mathcal{M}, \langle u, w\rangle \not\models p$. Then, by $\mathcal{M}, \langle u, w\rangle \models p$ we obtain a contradiction.

(Case 2): $\langle u, w\rangle \neq \langle x', y'\rangle$. By $V'(p) = \{\langle x', y'\rangle\}$ and $\langle u, w\rangle \neq \langle x', y'\rangle$ we have $\mathcal{M}', \langle u, w\rangle \not\models p$. Then, $\mathcal{M}', \langle u, w\rangle \models p$ raises a contradiction.

It follows that nominals are not $\mathsf{HS}^{\Box}_{horn}$-expressible in any class of frames. $\Box$

Proofs for Section 7: Satisfaction Operators

Theorem 31. *Satisfaction operators are $\mathsf{HS}^{\Box,i}_{horn}$-expressible in any class of frames.*

Proof Sketch. Let φ be an $\mathsf{HS}^{\Box,i,@}_{horn}$-formula. The construction of an $\mathsf{HS}^{\Box,i}_{horn}$-formula φ' which is equisatisfiable with φ is analogous to the construction presented in the proof of Theorem 30 except that:

- (7) is replaced by $[\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$, and
- (8) is replaced by:

$$\wedge[\mathsf{U}](i \wedge \eta \to [\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}) \wedge [\mathsf{U}]([\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta} \wedge i \to \eta).$$

The construction is linear since it is analogous to the one from the proof of Theorem 30. It remains to show that in each HS-frame φ and φ' are equisatisfiable.

Assume that in the beginning of some iteration of the construction we had a formula φ_n and in the end of this iteration we have obtained φ_{n+1}. We claim that in any frame φ_n is equisatisfiable with φ_{n+1}. Clearly, φ_{n+1} was obtained by replacing in φ_n a subformula $@_i\eta$ with $[\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$ and by concatenating the obtained formula with (9).

Assume that φ_n is satisfiable, so there is an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ and an interval $\langle x, y\rangle$ such that $\mathcal{M}, \langle x, y\rangle \models \varphi_n$. We will show that φ_{n+1} is also satisfiable in $(\mathbb{D}, I(\mathbb{D}), \mathcal{R})$. Let $\langle x', y'\rangle$ be such that $V(i) = \{\langle x', y'\rangle\}$. Let us define $\mathcal{M}' = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V')$ such that V' extends V with:

$$V'(p_{@i\eta}) \overset{\text{df}}{=} \begin{cases} I(\mathbb{D}) & \text{if } \mathcal{M}, \langle x', y'\rangle \models \eta; \\ \emptyset & \text{if } \mathcal{M}, \langle x', y'\rangle \not\models \eta. \end{cases}$$

It is easy to see that by the definition of V' the formula (9) is true in $\mathcal{M}'$. Also by the definition of V' we obtain that $\mathcal{M}' \models [\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. It follows that replacing $@_i\eta$ with $[\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$ in φ_n does not change the truth value of this formula, so $\mathcal{M}', \langle x, y\rangle \models \varphi_{n+1}$.

On the other hand, assume that φ_{n+1} is satisfiable, that is $\mathcal{M}, \langle x, y\rangle \models \varphi_{n+1}$ for some HS-model $\mathcal{M}$ and interval $\langle x, y\rangle$. Then, (9) is satisfied in $\mathcal{M}$ in $\langle x, y\rangle$, so $\mathcal{M} \models [\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$ if and only if $\mathcal{M} \models @_i\eta$. Therefore, $\mathcal{M}, \langle x, y\rangle \models \varphi_n$ and so φ_n is satisfiable. $\square$