

SOME PROBLEMS IN BANACH SPACE THEORY.

by

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Thesis submitted for the Degree of Doctor of Philosophy,  
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Balliol College,  
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ABSTRACT.

Types were introduced by Krivine and Maurey, in a refinement of a result by Aldous showing that infinite dimensional subspaces of  $L_r$  contain  $\mathcal{Q}_p$  for some  $1 \leq p < \infty$ . A synthesis of these ideas was provided by Garling whose representation of types as random measures was the motivation for much of this work. This thesis aims to investigate the structure of the representation, and to provide concrete representations for differing Banach spaces.

Chapter one contains the necessary preliminaries for the later chapters, and finishes by introducing the representation due to Garling of types on  $L_\phi(X)$  as random measures on  $\mathcal{T}(X)$ .

The second chapter consists of two parts. In the first part we examine the structure of the map between types on  $L_p(X)$  and random measures on  $\mathcal{T}(X)$ . We show that convolution is preserved by the mapping, and give an explicit representation of the space of types on  $L_1(\mathcal{Q}_p)$ . The second part is concerned with representations of  $\mathcal{T}(X)$ . We give conditions for the decomposition of  $\mathcal{T}(X)$  into  $X * \mathcal{G}(X)$ , and derive representations for the space of types on  $L_1(L_2k)$ .

The third chapter studies differentiability of types. We extend differentiability from  $X$  to  $\mathcal{T}(X)$ , and develop ideas that will be used in the study of uniqueness.

In chapter four we consider questions concerning the uniqueness of measures and random measures on  $X$  and  $\mathcal{T}(X)$ . We construct spaces where the representation of types as random measures is not uniquely determined. We prove that if a certain uniqueness property for measures on  $X$  fails then  $\mathcal{Q}_1^n$  embeds in  $X$ .

It was because I was involved in the bleak world of strong contrasts, between fear and exultation, danger and security, between life and death, that the finer balances of hopes and fears of people living hard-working lives began to take on new meaning. The grim struggle of the miners in northern France against appalling working and social conditions involved me deeply, and threw a question mark over our adventure.

Peter Boardman,  
The Shining Mountain.

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## INTRODUCTION.

The simplest infinite dimensional Banach spaces are the  $\mathcal{Q}_p$  ( $1 \leq p < \infty$ ) and the  $c_0$ -spaces. It is natural to ask whether every subspace of an  $L_r(\nu)$ -space ( $1 \leq r \leq \infty$ ) or  $C(S)$ -space contains some of them. The final step in this analysis was made by Aldous (1981) [A1] who showed that every infinite dimensional subspace of  $L_r$ ,  $1 \leq r < 2$ , contains  $\mathcal{Q}_p$  for some  $p \in [1, 2]$ . Krivine and Maurey, [Kr&Ma] 1981, gave an alternative proof of his result by introducing the concept of "stable spaces" and "types", the terminology coming from model theory and the stable theories of S. Shelah. They showed that every infinite dimensional stable Banach space contains  $\mathcal{Q}_p$  for some  $1 \leq p < \infty$ .

The concept of a type has proved very fruitful in the isomorphic theory of Banach spaces since its introduction. Using the techniques of stable spaces, S. Guerre and M. Levy [Gu&Le], improved the result of Aldous, and showed that if  $X$  is an infinite dimensional subspace of  $L_1$  then, for every  $\epsilon > 0$ ,  $X$  contains a subspace  $Y$  with  $d(Y, \mathcal{Q}_{p(X)}) < 1 + \epsilon$  where  $p(X)$  is the type index of  $X$ .

Maurey proved in [Ma2] that a separable Banach space  $X$  contains  $\mathcal{Q}_1$  if and only if there exists  $g \in X^{**}$  such that  $\|x+g\| = \|x-g\| \quad \forall x \in X$ . In the language of types, this means that  $X$

admits a symmetric second dual type as defined by Haydon and Maurey [Hay&Ma]. Rosenthal later refined Maurey's results in [Ros2], and introduced the class of  $\mathcal{Q}_{1+}$ -types, which coincide with the class of second dual types. In [Fa], Farmaki introduced the notion of a  $c_{0+}$ -type which are precisely the symmetric  $\mathcal{Q}_{\infty}$ -types, and proved that the existence of a non-trivial  $c_{0+}$ -type is equivalent to the presence of  $c_0$  in  $X$ . The idea of stability was weakened in [A&N&Z] where it is shown that if  $X$  is infinite dimensional and weakly stable then either  $\mathcal{Q}_p$  for some  $p \geq 1$  or  $c_0$  embeds isomorphically in  $X$ .

The original approach of Aldous used mainly probabilistic methods of random measures and exchangeable sequences of random variables, this argument was later simplified by Maurey [Ma1]. Garling then produced a synthesis of the two approaches in [Ga], where he proved that an Orlicz function space  $L_{\phi}(X)$  was stable if  $X$  was stable, and exhibited a direct correspondence between the space of types on  $L_{\phi}(X)$  and the space of random measures on  $\mathcal{T}(X)$ . It is this representation in the setting of  $L_p(X)$  spaces that is central to this thesis.

It was an objective of this work that the use of such a representation would yield results about the structure of symmetric subspaces of  $L_1(L_4)$ , by considering appropriate random measures. The programme was interrupted by the question of uniqueness arising, that is the need to consider whether there exists two distinct random measures which represent the same type. This later question then became the focal point of much of this thesis, and this and related questions occupy chapter 4.

## PRELIMINARIES.

We will use the letters  $X, Y, \dots$  for Banach spaces, by which we mean Banach spaces over the real or complex field. An operator or map means a bounded linear operator, for an operator  $u$  from  $X$  to  $Y$  we write  $u: X \rightarrow Y$ .

If  $X$  is a space and  $Y$  a subset of  $X^*$  we write  $\sigma(X, Y)$  for the weakest linear topology on  $X$  for which each  $x \in X$  is continuous. Two common topologies used are:

$\sigma(X, X^*)$ , the weak topology or  $w$ -topology on  $X$

$\sigma(X^*, X)$ , the weak  $*$ -topology or  $w*$ -topology on  $X$

We shall abbreviate the notation by adding a  $w$  or  $w*$  as appropriate; for example " $x_n \xrightarrow{w} x$ " means " $x$  is the limit in the  $w$ -topology of the sequence  $\{x_n\}$ " that is  $(w)\lim_n x_n = x$ .

A Polish space is a complete separable metric space.

A vector lattice is a partially ordered vector space  $(X, \leq)$ :  $\forall x, y \in X$   $x \vee y = \sup\{x, y\}$  exists. Let  $x^+ = x \vee 0$ ,  $x^- = (-x) \vee 0$  and  $|x| = x^+ + x^-$ . A Banach lattice is a Banach space which is also a vector lattice and is such that  $\|x\| \leq \|y\|$  whenever  $|x| \leq |y|$ .

We denote measure spaces as  $(\Omega, \Sigma, \mu)$ ,  $(\Omega', \mathcal{F}, \nu), \dots$  where for example  $\Omega$  is a set,  $\Sigma$  a  $\sigma$ -algebra of subsets of  $\Omega$  and  $\mu$  a countably additive  $\overline{\mathbb{R}}_+$ -valued function on  $\Sigma$ . A probability space  $(\Omega, \Sigma, P)$  is a measure space such that  $P(\Omega) = 1$ .

If  $(\Omega, \Sigma, \mu)$  is a  $\sigma$ -finite measure space and  $p \geq 1$ ,  $L_p(\Omega, \Sigma, \mu)$

or more briefly  $L_p$ , is the space of all measurable  $f: \Omega \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ) such that  $\int_{\Omega} |f|^p d\mu < \infty$  (or for  $p = \infty$   $\text{ess sup } |f| < \infty$ ), equipped with the norm  $\|f\|_p = \begin{cases} (\int |f|^p d\mu)^{1/p} & 1 \leq p < \infty \\ \text{ess sup } |f| & p = \infty \end{cases}$

By  $L_p(\Omega, \Sigma, \mu; X)$  or  $L_p(X)$  we mean the space of all measurable  $f: \Omega \rightarrow X$  such that  $\int \|f\|^p d\mu < \infty$  where  $(X, \|\cdot\|)$  is a normed space,  $L_p(X)$  has the obvious norm on it.

An Orlicz function  $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a convex function which vanishes only at the origin. Let  $\Phi(f) = \int \phi(\|f\|_X) d\mu$  and  $\|f\| = \inf\{\theta \in \mathbb{R}_+ : \Phi(f/\theta) \leq 1\}$  when  $f \in L_0(X)$  and  $X$  is a Banach space. Then define the Orlicz function space  $L_{\phi}(\Omega, \Sigma, \mu) = \{f \in L_0(X) : \Phi(f) < \infty\}$  where  $L_0(X)$  is the space of all measurable functions on  $X$ .

We let  $C_b(T)$  be the Banach space of bounded  $\mathbb{R}$ -valued continuous functions on  $T$ , with the supremum norm. For  $p \geq 1$ ,  $\mathcal{L}_p$  will be the space of all sequences  $\{x_m\}$  in  $\mathbb{R}^{\mathbb{N}}$  (or  $\mathbb{C}^{\mathbb{N}}$ ) such that  $\sum |x_m| < \infty$ , or for  $p = \infty$  such that  $\sup_m |x_m| < \infty$  equipped with the norm  $\|(x_m)\| = \begin{cases} (\sum |x_m|^p)^{1/p} & 1 \leq p < \infty \\ \sup\{|x_m| : m \in \mathbb{N}\} & p = \infty \end{cases}$   $c_0$  is the subspace of  $\mathcal{L}_{\infty}$  consisting of all sequences such that  $x_m \rightarrow 0$  as  $m \rightarrow \infty$ , [D&S].

If  $\{X_i\}_{i \in I}$  is a family of Banach spaces and  $p \geq 1$  then we will write  $(\oplus X_i)_p$  for the space of all families  $(x_i)$  such that  $x_i \in X_i$  and  $\sum \|x_i\|_{X_i}^p < \infty$  or for  $p = \infty$  such that  $\sup\{\|x_i\| : i \in I\} < \infty$  with the norm  $\|\cdot\|_p$

$$\|(x_i)\|_p = \begin{cases} (\sum \|x_i\|^p)^{1/p} & 1 \leq p < \infty \\ \sup\{\|x_i\|\} & p = \infty \end{cases}$$

Let  $T$  be a complete  $\sigma$ -compact locally compact metric

space. Put  $M(T)$  to be the space of bounded signed Borel measures on  $T$ , we let  $M_+(T)$  be the subset of non-negative Borel measures on  $T$  and  $\mathcal{P}(T)$  be the subset of Borel probability measures on  $T$ . Give  $M(T)$  the narrow topology ie.  $\sigma(M(T), C_b(T))$  and although  $M(T)$  is not metrizable in this topology,  $M_+(T)$  is a Polish space.

A random variable  $X$  on a finite measure space  $(\Omega, \Sigma, \mu)$  is a Borel measurable function from  $\Omega$  to  $\mathbb{R}$ . The distribution function of  $X$  is the function  $F_X: \mathbb{R} \rightarrow [0, 1]$  given by  $F_X(x) = \mu(\{\omega: X(\omega) \leq x\})$  for  $x \in \mathbb{R}$ . Then  $F_X$  is increasing and right continuous. We then have  $\mu(X \in B) = \int_B dF_X$  for  $B \in \mathcal{B}(\mathbb{R})$ . We let  $E[X] = \int_{\Omega} X d\mu$  and call  $E(\cdot)$  the expectation and also call  $F_X$  the law of the random variable  $X$ .

We write  $(a.e.) \lim_n X_n = X$  in the case that  $\{X_n\}$  converges to  $X$  almost everywhere convergence and  $(\mu) \lim_n X_n = X$  in the case that  $\lim_m \{\omega: |X_m(\omega) - X(\omega)| \geq \delta\} = 0 \forall \delta \geq 0$ . We say  $X_n$  converges weakly to  $X$  if the distribution functions converge for every continuity point  $x$  of  $F_X$  and we write both  $(w) \lim_n X_n = X$  and  $(w) \lim_n F_{X_n} = F_X$ . See [As] or [Bu].

## CHAPTER 1: TYPES AND RANDOM MEASURES.

### 1.1 INTRODUCTION.

This chapter is mainly introductory, and is concerned with discussing the preliminary results needed later in the thesis. Beginning by defining the space of types on  $X$ ,  $\mathcal{T}(X)$ , we introduce the concept of stable spaces and give several examples. Later in this thesis we shall be concerned with making explicit representations of  $\mathcal{T}(X)$ , in particular for  $L_p$ -spaces, and we introduce some simple representations such as for example  $\mathcal{T}(\mathcal{Q}_p)$  and  $\mathcal{T}(c_0)$ . The operations of convolution and scalar multiplication in  $\mathcal{T}(X)$  are defined, and the classes of symmetric and  $\mathcal{Q}_p$ -types introduced.

The second part of the chapter is devoted to the introduction of random measures. The final part develops a representation which is central to this thesis, Garling's representation of types on an Orlicz function space  $L_\phi(X)$  as random measures on  $\mathcal{T}(X)$ , [Ga]. Much work here is concerned with the correspondence between  $\mathcal{T}(L_p(X))$  and  $\pi_p(\mathcal{T}(X))$ , in particular chapter 4 takes up the theme in the final paragraph of [Ga] and asks the question of whether there exists distinct random measures representing the same type.

## 1.2 TYPES AND STABILITY.

### Definition.

Let  $X$  be a separable Banach space. For  $x \in X$ , the type on  $X$  realized by  $x$  is a function  $\tau_x: X \rightarrow \mathbb{R}_+$  defined by  $\tau_x(z) = \|z+x\|$ . Let  $\tilde{X} = \{\tau_x: x \in X\} \subseteq \mathbb{R}_+^X$  with the pointwise convergence topology inherited from  $\mathbb{R}_+^X$ . We call the closure of  $\tilde{X}$  in  $\mathbb{R}_+^X$  the space of types on  $X$  written  $\mathcal{T}$  or  $\mathcal{T}(X)$ . Thus  $\tau \in \mathcal{T}$  is a function  $\tau: X \rightarrow \mathbb{R}_+$  such that there exists a sequence  $a_n \in X$  and an ultrafilter  $\mathcal{U}$  over  $\mathbb{N}$  with  $\tau(x) = \lim_{\mathcal{U}} \|a_n + x\|$ . Then in the space of types  $\mathcal{T}$  we have  $\tau = \lim_{\mathcal{U}} \tau_{a_n}$ , so we identify  $X$  with  $\tilde{X}$  and write  $\tau = \lim_{\mathcal{U}} a_n$ . See [Kr&Ma].

### Definition

Let  $Y$  be a closed infinite dimensional subspace of  $X$ . A function  $\tau: X \rightarrow \mathbb{R}$  is called a type on  $X$  generated by  $Y$  if there exists a sequence  $(y_n)$  in  $Y$  such that  $\tau(x) = \lim_n \|x + y_n\| \forall x \in X$ . Denote by  $\mathcal{T}(Y, X)$  the set of all types generated by  $Y$ . See [Ros1].

We can put various topologies on  $\mathcal{T}(X)$ , for example

(a) the topology inherited from  $\mathbb{R}_+^X$ .

(b) "the very strong topology", given by

$$d_\omega(\sigma, \tau) = \sup_{z \in X} |\sigma(z) - \tau(z)| = \sup_z \lim_m |\|x_m + z\| - \|y_m + z\||$$

(c) "the strong topology" with pseudo-metrics

$$\rho_M(\sigma, \tau) = \sup_{\|z\| \leq M} |\sigma(z) - \tau(z)|.$$

With topology (a) [Kr&Ma], put  $\mathcal{T}_r = \{\tau \in \mathcal{T} : \tau(0) \leq r\}$  for  $r \in \mathbb{R}_+$ , then  $\mathcal{T}_r$  is compact and  $\tau(x) \leq \tau(0) + \|x\|$ .  $\mathcal{T}(X)$  is then locally compact, metrizable and separable.

Definition.

The Banach space  $X$  is stable if whenever  $(x_m)$  and  $(y_n)$  are bounded sequences in  $X$  and ultrafilters  $\mathcal{U}, \mathcal{V}$  on  $\mathbb{N}$ , we have  $\lim_{\mathcal{U}, m} \lim_{\mathcal{V}, n} \|x_m + y_n\| = \lim_{\mathcal{V}, n} \lim_{\mathcal{U}, m} \|x_m + y_n\|$ . If  $X$  is a stable space and if  $\tau$  and  $\sigma$  are defined by  $a_m$  and  $b_n$ , let  $[\sigma, \tau] = \lim_{\mathcal{V}, n} \lim_{\mathcal{U}, m} \|a_m + b_n\|$ .

A separable Banach space  $X$  is weakly stable if for every weakly compact subset  $K$  of  $X$ , all sequences  $(x_m)$  and  $(y_n)$  in  $K$  and ultrafilters  $\mathcal{U}, \mathcal{V}$ , we have  $\lim_{\mathcal{V}, n} \lim_{\mathcal{U}, m} \|x_m + y_n\| = \lim_{\mathcal{U}, m} \lim_{\mathcal{V}, n} \|x_m + y_n\|$ . Every stable space is weakly stable. Every weakly stable, reflexive space is stable.

Theorem 1.2.1

A separable metric space is stable iff whenever  $(x_m), (y_n)$  are bounded sequences then  $\sup_{m > n} \|x_m + y_n\| \geq \inf_{m < n} \|x_m + y_n\|$ , see [Kr&Ma].

Examples of stable spaces.

Every finite dimensional space is stable and  $\mathcal{Q}_p$  is stable for  $1 \leq p < \infty$ , [Ga]. Moreover the property of stability is inherited by all subspaces. In [Kr&Ma], Krivine and Maurey show that the  $\mathcal{Q}_p$ -sum of stable spaces is again stable for  $1 \leq p < \infty$ , and that for a stable space  $E$ ,  $L_p(E)$  is also stable for  $1 \leq p < \infty$ . The spaces  $c_0, \mathcal{Q}_\infty, L_\infty, C(K)$ , James space and Tsirelson's space are not stable. Further duals of stable spaces, for example  $L_\infty = (L_1)^*$ , double duals of stable spaces,

for example  $[\mathcal{L}_1(\mathcal{L}_\infty^n)]^{**}$  and quotients of stable spaces, eg)  $L_1/H_1$ , aren't necessarily stable.

The Lorentz spaces  $\mathcal{L}_{p,q}$  and  $L_{p,q}$  for  $1 \leq p, q < \infty$  are stable [R]. If  $E$  is stable and the Orlicz function  $\phi$  satisfies the  $\Delta_2$  condition, then  $L_\phi(E)$  is stable, [Ga]. In [Gu2], S. Guerre shows that quotients of stable reflexive spaces, quotients of  $L_1$  by reflexive subspaces and duals of stable reflexive spaces aren't necessarily stable.

Theorem 1.2.2

A Banach space is stable iff <sup>given p</sup> there exists a reflexive space  $\mathcal{E}$  and two bounded maps  $U: B \rightarrow \mathcal{E}$ ,  $V: B \rightarrow \mathcal{E}^*$  where  $B$  is dense in the unit ball of  $X$  such that  $\forall x, y \in B \forall p > 0$   
 $\|x+y\|^p = \langle U(x), U(y) \rangle$ , [Kr&Ma].

As an example, we show how this construction works in

$L_1(\mu)$ : Recall  $|\alpha| = C \int_0^\infty \frac{\sin^2 \alpha t}{t^2} dt$  so that we have

$$\begin{aligned} & \int_{\Omega} |f(\omega) + g(\omega)| - |f(\omega)| - |g(\omega)| d\mu \\ &= C \int_{\Omega} \int_0^\infty \frac{-1 - \cos(f(\omega) + g(\omega))t + \cos f(\omega)t + \cos g(\omega)t}{t^2} dt d\mu \\ &= C \iint \left\{ \frac{\sin f(\omega)t \cdot \sin g(\omega)t}{t^2} - \frac{(1 - \cos f(\omega)t)(1 - \cos g(\omega)t)}{t^2} \right\} dt d\mu \\ &= \left\langle \left( \frac{\sin f(\omega)t}{t}, \left[ \frac{1 - \cos f(\omega)t}{t} \right] \right), \left( \frac{\sin g(\omega)t}{t}, - \left[ \frac{1 - \cos g(\omega)t}{t} \right] \right) \right\rangle_{L_2 \oplus L_2} \end{aligned}$$

Thus  $\|f+g\| = \langle \phi(f), \psi(g) \rangle_{L_2(\Omega \times \mathbb{R}_+)^2} + \langle \|f\|, 1 \rangle_{\mathbb{R}} + \langle 1, \|g\| \rangle_{\mathbb{R}}$   
 where  $\phi(f) = \left( \frac{\sin f(\omega)t}{t}, \left[ \frac{1 - \cos f(\omega)t}{t} \right] \right)$  and similarly for  $\psi$ .

$$\begin{aligned} \text{So } \phi: L_1 &\longrightarrow L_2(\Omega \times \mathbb{R}_+) \oplus L_2(\Omega \times \mathbb{R}_+) \oplus \mathbb{R} \oplus \mathbb{R} \\ f &\longmapsto \left( \frac{\sin f(\omega)t}{t}, \left[ \frac{1 - \cos f(\omega)t}{t} \right], \|f\|, 1 \right) \\ \psi: g &\longmapsto \left( \frac{\sin g(\omega)t}{t}, - \left[ \frac{1 - \cos g(\omega)t}{t} \right], 1, \|g\| \right) \end{aligned}$$

are the required functions and  $L_2(\Omega \times \mathbb{R}_+)^2 \oplus \mathbb{R}^2$  the required space.

Examples of types.

1.  $X=L_1(\mu)$ . Then for  $(x_n)$  a "spikey sequence" we have a type given by  $\tau(z)=\lim_n \|x_n+z\|=\|z\|+\lim_n \|x_n\|$ .

2.  $X=L_1(\Omega, \mathcal{F}, P; \mathbb{R})$ . For  $f, g \in L_1(\Omega)$ ,  $\tau_g(f)=\|f+g\|_{L_1}=E[|f(\omega)+g(\omega)|]$   
 $= E\left(\int_X |f(\omega)+g(\omega)| d\mu_\omega(x)\right) = E\left(\int_X |f(\omega)+x| d\delta_{g(\omega)}(x)\right)$ .

3.  $X=c_0$ . Let  $y_n=(y_n(r))_{r=0}^\infty \in X$  such that  $y_n \rightarrow y$  as  $n \rightarrow \infty$  then  $y \in \mathcal{Q}_\infty$ .  $\tau(x)=\lim_n \|x+y_n\|=\lim_n \sup_{r \in \mathbb{N}} \{|x(r)+y_n(r)|\}$   
 $=\max\{\sup_r |x(r)+y(r)|, \beta\}=\max\{\|x+y\|, \beta\}$ , where  $\beta=\lim_n \|y_n-y\|$ .

4.  $X=\mathcal{Q}_p$ . Let  $\tau$  be a type defined by the sequence  $(y^{(n)})$ , wlog suppose  $y^{(n)}$  converges coordinatewise to  $\bar{y} \in \mathcal{Q}_p$ . Let  $z^{(n)}=y^{(n)}-\bar{y}$  so  $z^{(n)} \rightarrow 0$  coordinatewise, and  $\|z^{(n)}\| \rightarrow \alpha$   
 $\tau(x)=\lim_n \|x+y^{(n)}\|=\lim_n \|x+\bar{y}+z^{(n)}\|=\lim_n (\|x+\bar{y}\|^p + \|z^{(n)}\|^p)^{1/p}$   
 $=(\|x+\bar{y}\|^p + \alpha^p)^{1/p}$ . Hence  $\bar{y}$  and  $\alpha$  are determined uniquely by  $\tau$ , so we have  $y^{(n)} \rightarrow \bar{y}$  and  $\|y^{(n)}-\bar{y}\| \rightarrow \alpha$ .

5. Let  $\mathcal{D}$  be an ultrafilter on  $\mathbb{N}$ , put  $F=E^{\mathbb{N}}/\mathcal{D}$ . Let  $\theta \in \mathcal{T}(E)$ ,  $\theta=\lim_{\mathcal{D}} a_n$ . The natural extension of  $\theta$  to  $F$  is defined by  $\theta(\xi)=\lim_{\mathcal{D}} \theta(x_n)$  where  $\xi=(x_n) \in F$ . By stability of  $E$   
 $\theta(\xi)=\lim_{\mathcal{D}} \lim_{\mathcal{D}} \|x_n+a_n\|=\lim_{\mathcal{D}} \lim_{\mathcal{D}} \|x_n+a_n\|=\lim_{\mathcal{D}} \|x_n+a_n\|$ ,  
so that  $\theta \in \mathcal{T}(F)$ . We can identify  $\mathcal{T}(E)$  with a conic class of  $\mathcal{T}(F)$ .

If  $\tau \in \mathcal{T}(F)$ ,  $\tau=\lim_{\mathcal{D}} \xi_m$ , the restriction  $\sigma$  of  $\tau$  to  $E$  is a type on  $E$ . Since for all  $m$ ,  $\xi_m$  is defined by  $(\xi_m^n)$  a sequence

of elements of  $E$ , if  $x \in E$   $\sigma(x) = \tau(x) = \lim_{m, \mathcal{U}} \|x + \xi_m\|$   
 $= \lim_{m, \mathcal{U}} \lim_{n, \mathcal{G}} \|x + \xi_m^n\|$ .

If  $\tau \in \mathcal{T}(X)$  and  $\tilde{X}$  is the ultrapower of  $X$ , wrt the ultrafilter  $\mathcal{U}$  then we have  $\tau(x) = \lim_{i, \mathcal{U}} \|x + y_i\| = \|x + (y_i)_{\mathcal{U}}\|_{\tilde{X}}$ .

6.  $X = L_1(\mu)$ . Then for all types there exists an extension  $L_1(\mu) \subseteq L_1(\hat{\mu})$  which is an isometric lattice homomorphism, and  $\hat{f} \in L_1(\hat{\mu})$  such that  $\forall g \in L_1(\mu)$   $\tau(g) = \|\hat{f} + g\|$ . Conversely, for any extension  $L_1(\mu) \subseteq L_1(\nu)$  and  $h \in L_1(\nu)$  the function  $f \mapsto \|f + h\|_{L_1(\nu)}$  defines a type on  $L_1(\mu)$ .

So for  $X = L_1[0,1]$ , for each  $h \in L_1([0,1] \times [0,1])$  the function  $\tau(g) = \|g + h\| \quad \forall g \in L_1[0,1]$ , defines a type on  $L_1[0,1]$ .

7.  $X = c_0$ . Then  $\forall y \in \mathcal{Q}_\infty$  the function  $\tau: X \longrightarrow \mathbb{R}$  defines a type  
 $x \mapsto \|x + y\|$   
on  $X$ . Define  $y_n(m) = \begin{cases} y(m) & m \leq n \\ 0 & m > n \end{cases}$  then clearly  $y_n \in c_0$  and  
 $\|x + y\| = \lim_n \|x + y_n\| = \tau(x)$ , hence  $\tau \in \mathcal{T}(c_0)$ .

### 1.3 Operations on types.

#### Proposition 1.3.1

Let  $X$  be a stable space and  $\tau, \sigma \in \mathcal{T}(X)$  and  $(x_m)$  be an approximating sequence for  $\sigma$ . Then  $\lim_m \tau(x_m)$  exists and is independent of the choice of  $(x_m)$ . Then  $d(\tau, \sigma) = d(\sigma, \tau)$  and  $\forall \lambda \in \mathcal{T} \quad d(\tau, \sigma) \leq d(\sigma, \lambda) + d(\lambda, \tau)$ , where  $d(\tau, \sigma) = [\tau, \sigma]$ .

Let  $X$  be a stable space. Then for all  $\sigma \in \mathcal{T}(X)$ , the function  $\tau \mapsto [\sigma, \tau]$  is continuous on  $\mathcal{T}(X)$ , see [Kr&Ma].

On stable Banach spaces, we can define scalar multiplication of types, convolution products, and typical norms, viz:

Definition.

For  $0 \neq \alpha \in \mathbb{R}$  define  $D_\alpha \tau(x) = \begin{cases} \|x\| & \alpha=0. \\ |\alpha| \tau(x/\alpha) & \text{o.w.} \end{cases}$ . Then  $D_\alpha$ , for  $\alpha \neq 0$ , is a homeomorphism of  $\mathcal{T}$  which extends scalar multiplication on  $X$ , i.e.  $\forall x \in X, D_\alpha j(x) = j(\alpha x)$ . We can define translation, by setting, for  $\tau \in \mathcal{T}(X), x \in X$   $(\tau * x)(y) = \tau(x+y)$ , thus  $\tau * x \in \mathcal{T}$  and  $*$  is a homeomorphism of  $\mathcal{T}(X)$ .

To extend this further, we define the convolution of types  $\tau, \sigma$  defined by  $(a_n), (b_m)$  resp., as  $(\sigma * \tau)(x) = \lim_{m, q} \lim_{n, \gamma} \|a_m + b_n + x\| = [\sigma * x, \tau]$ . Then  $\sigma * \tau \in \mathcal{T}$  and  $*$  is commutative and associative, and for scalar  $\alpha$ ,  $[D_\alpha \sigma, D_\alpha \tau] = D_\alpha [\sigma, \tau], (D_\alpha \sigma) * (D_\alpha \tau) = D_\alpha (\sigma * \tau)$ .

Definition.

A type  $\tau \in \mathcal{T}(X)$  is symmetric if  $\forall x \in X \tau(x) = \tau(-x)$ . The space  $\mathcal{S}(X)$  of symmetric types on  $X$  is closed in  $\mathcal{T}(X)$ . If  $X$  is infinite dimensional then there exists a bounded sequence with no convergent subsequence hence there exists an approximating sequence  $(x_m)$  with no convergent subsequence.

Let  $\tau = \lim_m x_m$  then  $\tau * D_{-1} \tau$  is a non-trivial symmetric type, so  $\mathcal{S}(X) \neq \{0\}$ . An  $\mathcal{L}_p$ -type is a type  $\tau$  such that  $D_\alpha \tau * D_\beta \tau = D_{\gamma(\alpha, \beta)} \tau \quad \forall \alpha, \beta$  where  $\gamma(\alpha, \beta) = (|\alpha|^p + |\beta|^p)^{1/p}$ . Similarly  $\tau$  is a  $c_0$ -type when  $\gamma(\alpha, \beta) = \sup(|\alpha|, |\beta|)$ .

$\tau \in \mathcal{T}(X)$  is an  $\mathcal{L}_{1+}$ -type if there exists a sequence  $(x_n)$  in  $X$  such that  $\tau(x) = \lim_n \lim_m \|x + ax_n + bx_m\|$  for  $x \in X$  and  $a, b, \geq 0$  with  $a+b=1$ . [Ros2].

$\tau \in \mathcal{T}(X)$  is a  $c_{0+}$ -type if there exists a sequence  $(x_n)$  in  $X$  such that  $\tau(x) = \lim_n \lim_m \|x + ax_n + bx_m\|$  for  $x \in X$  and  $a, b, \geq 0$  with  $\max(a, b) = 1$ . See [Fa].

#### 1.4 RANDOM MEASURES.

In this section we are particularly concerned with random measures on  $\mathcal{T}(X)$ , that is with measurable maps from  $\Omega$  to  $\mathcal{P}(\mathcal{T}(X))$ , since, when  $X$  is separable, we can represent a type on an Orlicz function space  $L_\phi(X)$  as a random measure on  $\mathcal{T}(X)$ , see [Ga].

Random measures began life as point processes on the line. Originally introduced for a particular statistical model they were studied for their own sake in the 1960's, where they were placed on an abstract footing. For a brief history and further references see [Ka].

Here we outline the first stages of the abstract theory including the topologies and convergence of random measures. We finish with some operations definable on  $\pi(T)$ .

For random variables  $X_1, X_2, \dots$ , we write  $X_n \xrightarrow{p} X$  for convergence in probability and  $X_n \xrightarrow{s} X$  (resp.  $X_n \xrightarrow{w} X$ ) for strong (resp. weak) convergence in  $L_1$ . Let  $\mathcal{P}(T)$  denote the set of probability measures on  $T$ . We give  $\mathcal{P}(T)$  the usual (narrow) topology, namely  $\sigma(\mathcal{P}(T), C_b(T))$ , so  $\mu_n \longrightarrow \mu$  iff  $\forall f \in C_b(T), \int f \cdot d\mu_n \longrightarrow \int f \cdot d\mu$ . We sometimes write  $\langle f, \mu \rangle$  for  $\int f \cdot d\mu$ .  $\mathcal{P}(T)$  is a Polish space.

For  $\mu \in \mathcal{P}$ , let  $\phi_\mu(t) = \int e^{itx} d\mu(x)$ , the Fourier transform, and  $|\mu| = \int |x| \cdot d\mu(x)$ .  $\delta_\alpha \in \mathcal{P}$  is the degenerate measure at  $\alpha$ .

Definition.

A random measure is a random probability measure i.e. a measurable function  $\xi: \Omega \longrightarrow \mathcal{P}(T)$ , in other words a  $\mathcal{P}(T)$ -valued random variable.

Let  $\pi(T)$  be the space of all  $T$ -valued random measures on  $\Omega$ . If  $\xi \in \pi(T)$  and  $f$  a bounded measurable function then  $\langle f, \xi \rangle: \omega \mapsto \langle f, \xi(\omega) \rangle$  defines a random variable. Call  $\xi$  constant if  $\xi \equiv \xi_0$  for some  $\xi_0 \in \mathcal{P}(T)$ , and degenerate when  $\xi = \delta_\alpha$  for some random variable  $\alpha$ , [Ka].

Proposition 1.4.1

$\pi(T)$  is closed under addition and multiplication by  $\mathbb{R}_+$ -valued random variables. A series  $\sum_j \xi_j$  of random measures is itself a random measure iff  $\forall B \in \mathcal{B} \quad \sum_j \xi_j(B) < \infty$ .

Examples.

We now give some simple examples of random measures,

(i) Let  $X \in L_0(\Omega; T) = L_0(T)$ , the space of all  $T$  measurable functions on  $\Omega$ , let  $x$  be a representative of  $X$ . Then  $\mu(\omega) = \delta_{x(\omega)}$  defines a random measure. Up to equality a.e. it is independent of the choice of  $x$ .

To show this is a random measure we have to show  $\xi$  is a measurable function, that is to say  $\omega \mapsto \mu_\omega \in \mathcal{P}(T)$  is measurable in the sense that  $\forall B \in \mathcal{B} \quad \omega \mapsto \mu_\omega(B)$  is  $\mathcal{F}$ -measurable. This is equivalent to  $\omega \mapsto \mu_\omega$  being measurable for the Borel  $\sigma$ -algebra on  $\mathcal{P}(T)$  and  $\mathcal{F}$ .

Define  $T_B: (\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$  by  $\forall B \in \mathcal{B}(T) \quad T_B(\omega) = \mu_\omega(B) = \delta_{x(\omega)}(B)$ . Then we have for  $A \in \mathcal{B}(T)$

$$T_B^{-1}(A) = \{\omega: \delta_{X(\omega)}(B) \in A\} = \left\{ \begin{array}{ll} \Omega & \text{if } 0, 1 \in A \\ \emptyset & \text{if } 0, 1 \notin A \\ X^{-1}(B) & \text{if } 1 \in A, 0 \notin A \\ (X^{-1}(B))^c & \text{if } 1 \notin A, 0 \in A \end{array} \right\} \in \mathcal{F}$$

as  $X$  is measurable.

Hence  $\xi$  is measurable so it is a random measure. Thus we have a mapping

$$\begin{array}{ccc} i: L_0(T) & \longrightarrow & \pi(T) \\ X & \longmapsto & \delta_{X(\omega)} \end{array}$$

(ii) Let  $\xi(\omega) = \mu \in \mathcal{P}(T) \forall \omega \in \Omega$ . This is called a constant random measure, and is measurable since

$$T_B^{-1}(A) = \{\omega: \mu(B) \in A\} = \begin{cases} \Omega & \text{if } \mu(B) \in A \\ \emptyset & \text{o.w.} \end{cases}$$

(iii) Let  $0 < q \leq 2$  and  $a \in \mathbb{R}$ . Then there exists a measure  $\sigma(q, a)$  on  $\mathbb{R}$  with characteristic function  $\exp(-|at|^q)$ , we call  $\sigma(q, a)$  a symmetric  $q$ -stable measure with index  $q$  and scale factor  $|a|$ .

Then  $\sigma(q, a) * \sigma(q, b) = \sigma(q, (|a|^q + |b|^q)^{1/q})$  since  $\phi_a(t) \cdot \phi_b(t) = \exp\{-|t(|a|^q + |b|^q)^{1/q}|^q\}$ .

Let  $\alpha \geq 0$  be a  $\mathbb{R}$ -valued random variable on  $\Omega$  and  $0 < q \leq 2$ . We define the  $q$ -stable random measure  $\sigma(q, \alpha)(\omega) = \sigma(q, \alpha(\omega))$ . This is measurable as the characteristic function is a measurable map.

We are now in a position to define topologies on  $\pi(T)$ , here we define two, the sm and wm topologies which Aldous uses in [A1].

Definition.

$$\begin{array}{l} \text{Define } \xi_n \xrightarrow{\text{sm}} \xi \quad \text{iff} \quad \langle f, \xi_n \rangle \xrightarrow{\text{S}} \langle f, \xi \rangle \quad f \in C_b(T) \\ \xi_n \xrightarrow{\text{wm}} \xi \quad \text{iff} \quad \langle f, \xi_n \rangle \xrightarrow{\text{W}} \langle f, \xi \rangle \quad f \in C_b(T). \end{array}$$

Let  $\phi_{\xi}(t, \omega) = \phi_{\xi(\omega)}(t) = \langle \exp(it \cdot), \xi(\omega, \cdot) \rangle$  be the random Fourier transform for  $\xi \in \pi(T)$ , then we also have

$$\begin{aligned} \xi_n \xrightarrow{sm} \xi & \text{ iff } \phi_{\xi_n}(t) \xrightarrow{s} \phi_{\xi}(t) \quad \forall t \\ \xi_n \xrightarrow{wm} \xi & \text{ iff } \phi_{\xi_n}(t) \xrightarrow{w} \phi_{\xi}(t) \quad \forall t \end{aligned}$$

Garling, in [Ga], also defines the wm-topology by considering  $\langle \cdot, \xi(\cdot) \rangle$  as a map from  $C_b(T)$  into  $L_{\infty}(\Omega)$ . Let  $T_{\xi}(f)(\omega) = \int_T f(t) \xi(\omega, dt)$ , so  $T_{\xi}$  is a norm-decreasing linear mapping of  $C_b(T)$  into  $L_{\infty}(\Omega)$ .

Giving  $L_{\infty}(\Omega)$  the weak topology  $\sigma(L_{\infty}, L_1)$ , the resulting topology defines the wm-topology on  $\pi(T)$ . Thus a subbase of neighbourhoods of  $\xi$  is given by

$$N_{(f, g)}(\xi) = \left\{ \eta: \left| \int_{\Omega} g(\omega) [(T_{\eta} f)(\omega) - (T_{\xi} f)(\omega)] \cdot P(d\omega) \right| \leq 1 \right\}$$

for  $f \in C_b(T)$ ,  $g \in L_1(\Omega)$ . It is sufficient to let  $g$  run through a separating subset of  $L_1$ , eg) we can take

$$N_{(f, E)}(\xi) = \left\{ \eta: \left| \int_E [(T_{\eta} f)(\omega) - (T_{\xi} f)(\omega)] \cdot P(d\omega) \right| \leq 1 \right\}$$

for  $f \in C_b(T)$ ,  $E \in \mathcal{F}$ .

The map  $i: L_0(T) \longrightarrow i(L_0(T))$  is a homeomorphism of  $L_0(T)$  into  $\pi(T)$  and in fact  $i(L_0)$  is wm-dense in  $\pi(T)$ , see [Ga]. By considering  $\mathcal{P}(T)$  as a subset of  $M(\beta T)$ , we can identify  $\pi(T)$  with a subset of  $(L_1(C_b(T)))^*$ . Then the wm-topology is that induced by the weak  $\ast$ -topology  $\sigma((L_1(C_b(T)))^*, L_1(C_b(T)))$ .

### 1.5 Operations on measures and random measures.

Let  $X$  be a separable Banach space and  $T=\mathcal{T}(X)$ . We are interested here in operations on  $\pi(T)$ .

#### Definition.

Let  $\mu \in \mathcal{P}(T)$  and  $0 \neq \alpha \in \mathbb{R}$ . Define  $\forall E \in \mathcal{B}(T) D_\alpha(\mu)(E) = \mu(D_{\alpha^{-1}}(E))$  so that  $D_\alpha(\mu) \in \mathcal{P}(T)$ . If  $f \in C_b(T)$  then 
$$\int f(t)(D_\alpha(\mu))(dt) = \int f(D_\alpha t)\mu(dt)$$
 so  $D_\alpha$  is a homeomorphism of  $\pi(T)$ . For  $\alpha=0$  set  $D_0(\mu) = \delta_0$ .

Similarly for  $\xi \in \pi(T)$ , set  $D_\alpha(\xi)(\omega) = D_\alpha(\xi(\omega))$ ,  $\omega \in \Omega$ . So for  $\alpha \neq 0$ ,  $D_\alpha$  is a homeomorphism of  $\pi(T)$  with the  $wm$ -topology.

If  $D_1(\xi) = \xi$  we call  $\xi$  a symmetric measure. We then define the translation of measures and random measures. Let  $f \in C_b(T)$  and that  $\mu \in \mathcal{P}(T)$  and  $x \in X$ . Define  $T_x(f)(t) = f(T_x(t))$  and  $T_x(\mu)(E) = \mu(T_{-x}(E))$  where  $T_x(t)(y) = (t * x)(y) = t(x+y)$ .

Then  $T_x: C_b(T) \longrightarrow C_b(T)$  is an isometry,  $T_x(\mu) \in \mathcal{P}(T)$ , 
$$\int f(t)(T_x(\mu))(dt) = \int (T_x(f))(t)\mu(dt)$$
 and  $T_x: \mathcal{P}(T) \longrightarrow \mathcal{P}(T)$  is a homeomorphism.

The map  $X \times \mathcal{P}(T) \longrightarrow \mathcal{P}(T)$  is jointly continuous.

$$(x, \mu) \longmapsto T_x(\mu)$$

#### Definition.

To extend the translation of measures to random measures we define, for  $\xi \in \pi(T)$  and  $x \in L_0(X)$ ,  $T_x(\xi)(\omega) = T_{x(\omega)}(\xi(\omega))$  so that  $T_x(\xi) \in \pi(T)$ .

The map  $L_0(X) \times \pi(T) \longrightarrow \pi(T)$  is jointly continuous.

$$(x, \xi) \longmapsto T_x(\xi)$$

We are now in a position to define convolution in  $\mathcal{P}(T)$  and  $\pi(T)$ . To do so we must suppose  $X$  is stable thus convolution is well defined in  $T$ . Let  $\mu, \nu \in \mathcal{P}(T)$ ,  $f \in C_b(T)$  then the function  $(s, t) \mapsto f(s * t)$  is measurable on  $T \times T$ . Let  $\phi(f) = \int f(s * t) \mu(ds) \nu(dt)$ . Thus  $\phi$  is a positive linear functional on  $C_b(T)$  and  $\phi(1) = 1$ , hence there exists  $\pi \in M(\beta T)$  such that  $\int_{\beta T} \tilde{f}(u) \pi(du) = \phi(f)$  where  $\tilde{f}$  is the continuous extension of  $f$  to  $\beta T$ .

As  $\mu$  and  $\nu$  are regular,  $\forall 0 < \epsilon < 1$  there exists  $n$  such that  $\mu(K_n), \nu(K_n) \geq 1 - \epsilon$  where  $K_n = \{t : t(0) \leq n\}$ ; and since  $(s * t)(0) \leq s(0) + t(0)$  we have  $K_n * K_n \subseteq K_{2n}$  whence  $\pi(K_{2n}) \geq (1 - \epsilon)^2$  which implies that  $\pi(T) = 1$ . So we regard  $\pi$  as a member of  $\mathcal{P}(T)$ . We denote  $\pi$  by  $\mu * \nu$ , the convolution of  $\mu$  and  $\nu$ . Let  $\xi, \eta \in \pi(T)$ , define  $\xi * \eta$  by  $(\xi * \eta)(\omega) = \xi(\omega) * \eta(\omega)$ .

Proposition 1.5.1

1. The map  $\mathcal{P}(T) \times \mathcal{P}(T) \longrightarrow \mathcal{P}(T)$  is separately continuous, and  $(\mu, \nu) \longmapsto \mu * \nu$  jointly continuous iff  $(X, \|\cdot\|)$  is finite-dimensional.
2. The map  $\pi(T) \times \pi(T) \longrightarrow \pi(T)$  is separately continuous.  $(\xi, \eta) \longmapsto \xi * \eta$

Definition.

For a random measures  $\xi$ , we can now define an  $\mathcal{Q}_p$ -random measures analogous to  $\mathcal{Q}_p$ -types. We say  $\xi$  is an  $\mathcal{Q}_p$ -random measure if for all scalars  $\alpha$  and  $\beta$ ,  $D_\alpha \xi * D_\beta \xi = D_{\gamma(\alpha, \beta)} \xi$ , where  $\gamma(\alpha, \beta) = (|\alpha|^p + |\beta|^p)^{1/p}$ . Similarly  $\xi$  is an  $c_0$ -random measure if  $\forall \alpha, \beta$ ,  $D_\alpha \xi * D_\beta \xi = D_{\gamma(\alpha, \beta)} \xi$  with  $\gamma(\alpha, \beta) = \sup(|\alpha|, |\beta|)$

Theorem 1.5.2

The following are equivalent:

(a)  $\tau$  is a type on  $L_1[0,1]$ .

(b) there exists an isometric lattice homomorphism  $L_1[0,1] \longrightarrow L_1(\Omega, \mathcal{F}, \mu)$  and  $\exists h \in L_1(\Omega)$  such that  $\forall g \in L_1[0,1]$   
 $\tau(g) = \|g+h\|_{L_1(\Omega)}$ .

(c)  $\exists \alpha > 0 \exists k \in L_1([0,1] \times [0,1])$  such that  $\forall g \in L_1[0,1]$

$$\tau(g) = \alpha + \|g+k\| = \alpha + \int_0^1 \int_0^1 |g(s) + k(s,t)| dt ds.$$

(d)  $\exists \alpha > 0 \exists$  a random measure  $\xi: [0,1] \longrightarrow \pi(\mathbb{R})$  such that

$$\forall g \in L_1[0,1] \quad \tau(g) = \alpha + \int_0^1 \left[ \int_{\mathbb{R}} |g(s) + u| d\mu_s(u) \right] ds.$$

(e)  $\exists \alpha > 0 \exists$  a probability space  $(\Omega', \mathcal{F}', \mu')$ ,

$\exists h \in L_1(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mu \otimes \mu')$  such that  $\forall g \in L_1(\Omega, \mathcal{F}, \mu)$

$$\tau(g) = \alpha + \int_{\Omega} \int_{\Omega'} |g(\omega) + h(\omega, \omega')| d\mu'(\omega') d\mu(\omega).$$

Proof.

(a)  $\Rightarrow$  (b) Let  $X = L_1[0,1]$ . Then for some ultrafilter  $\mathcal{U}$  we have  $\tau(g) = \lim_{i \rightarrow \mathcal{U}} \|g+h_i\| = \|g+(h_i)_{\mathcal{U}}\|_{\tilde{X}}$  where  $(h_i)_{i \in I}$  is an approximating sequence for  $\tau$ . Then  $\tilde{X} = L_1(\mu)$  for some measure  $\mu$ , see [Dc&Kr1] and [He], and  $L_1[0,1] = X \longrightarrow (X)_{\mathcal{U}} = L_1(\mu)$  is an isometric lattice homomorphism. Lastly  $(h_i)_{\mathcal{U}} = h \in L_1(\mu)$  so (b) holds.

(b)  $\Rightarrow$  (e) We have an isometric lattice homomorphism  $L_1[0,1] \longrightarrow L_1(\Omega, \mathcal{F}, \mu)$  and  $h \in L_1(\Omega)$  such that  $\forall g \in L_1[0,1]$   
 $\tau(g) = \|g+h\|_{L_1(\Omega)}$ . Now identify  $L_1[0,1] \cong L_1(\Omega_1, \mathcal{G}_1, \mu|_{\mathcal{G}_1})$  where  $\mathcal{G}_1$  is  $\sigma$ -subalgebra of  $\mathcal{G}|_{\Omega_1}$  which is possible since we have an isometric lattice homomorphism. Let  $\alpha = \|h\chi_{\Omega \setminus \Omega_1}\|_{L_1(\Omega)}$ . So that

$\|g+h\|_{L_1(\Omega)} = \|g+h\chi_{\Omega_1}\|_{L_1(\Omega)} + \alpha$ , by Maharam's product theorem [Mah] there exists an isometric lattice embedding  $U$  such that  $Uh\chi_{\Omega_1} \in L_1(\Omega \times \Omega')$  preserving  $L_1(\Omega_1, \mathcal{G}_1, \mu)$ .

Hence  $\|g+h\chi_{\Omega_1}\|_{L_1(\Omega)} = \|g+Uh\chi_{\Omega_1}\|_{L_1(\Omega \times \Omega')}$ . Then we set  $k=Uh\chi_{\Omega_1} \in L_1(\Omega \times \Omega')$  whence  $\tau(g) = \alpha + \|g+k\|_{L_1(\Omega \times \Omega')} = \alpha + \int_{\Omega} \int_{\Omega'} |g(\omega) + k(\omega, \omega')| d\mu'(\omega') d\mu(\omega)$ .

(e) $\Rightarrow$ (d) We have an element  $h \in L_1(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mu \otimes \mu')$ , hence  $h(s, \cdot) \in L_1(\Omega', \mathcal{F}', \mu')$  a.e. for every  $s \in \Omega$ . By Fubini's theorem, almost every section of  $h(s, t)$  is measurable, that is  $h(s, \cdot)$  is measurable for almost every  $s \in \Omega$ . For each  $s \in \Omega$  we can define  $\mu_s = \text{law of } h(s, \cdot) \text{ a.e.} = \mathcal{L}(h(s, \cdot)) \text{ a.e.}$ . That is  $\mu_s(B) = \mathcal{L}(h(s, \cdot))(B) = \mu'(h(s, \cdot)^{-1}(B))$  for  $B \in \mathcal{F}'$ . Letting  $\xi: s \longmapsto \mu_s$  we wish  $\xi$  to be a random measure, i.e. the map  $s \longmapsto \mu_s$  to be measurable. So we are required to prove that  $s \longmapsto \mu_s(-\infty, x]$  is measurable.

Let  $H_x = \{(\omega, \omega') : h(\omega, \omega') \leq x\}$  which is measurable in  $[0, 1] \times [0, 1]$ . Then by Fubini's theorem almost every section of  $H_x$  is measurable, in particular the section determined by  $\omega$  is measurable, i.e.  $\{\omega' : h(\omega, \omega') \leq x\}$  is measurable, that is  $h(s, \cdot)^{-1}(-\infty, x]$  is measurable. Let  $K(s, \cdot) = \chi_{h(s, \cdot)^{-1}(-\infty, x]}$ . Then as  $h(s, \cdot)^{-1}(-\infty, x]$  is measurable,  $K(s, \cdot)$  is a measurable function. By Fubini's theorem  $H(s) = \int_{\Omega'} K(s, t) d\mu'(t)$  is measurable, but  $H(s) = \int_{h(s, t)^{-1}(-\infty, x]} d\mu'(t) = \mu'(h(s, \cdot)^{-1}(-\infty, x]) = \mu_s(-\infty, x]$ . Hence the map  $s \longmapsto \mu_s(-\infty, x]$  is measurable.

Thus  $\xi$  is a random measure. Now the distribution function of  $h(s, \cdot)$  is  $F_{h(s, \cdot)}(x) = \mu'(\{t : h(s, t) \leq x\}) = \mu_s(-\infty, x]$ .

By Theorem 11-2A in [Bu], we have  $E[g(X)] = \int_{\mathbb{R}} g dF_X$  so in

particular  $E[|X|] = \int_{\mathbb{R}} |x| dF_X(x)$ . Thus

$$\int_{\Omega'} |h(\omega, \omega')| d\mu'(\omega') = E[|h(\omega, \omega')|] = \int_{\mathbb{R}} |u| d\mu_s(u), \text{ whence}$$

$$\int_{\Omega'} |g(s) + h(\omega, \omega')| d\mu'(\omega') = \int_{\mathbb{R}} |g(s) + u| d\mu_s(u).$$

Then as  $g \in L_1[0, 1] \equiv L_1(\Omega_1, \mathcal{G}_1, \mu|_{\mathcal{G}_1}) \subseteq L_1(\Omega, \mathcal{G}, \mu)$  and  $h = U h' \chi_{\Omega_1} \in L_1(\Omega \times \Omega')$  so that  $h(\omega, \cdot) = 0$  a.e. on  $\omega \notin \Omega_1$ . Thus

$$\int_{\Omega} \int_{\Omega'} |g(\omega) + h(\omega, \omega')| d\mu'(\omega') d\mu(\omega) = \int_0^1 \int_{\Omega'} |g(\omega) + h(\omega, \omega')| d\mu' d\mu.$$

So that we have as required  $\tau(g) = \alpha + \int_0^1 \int_{\mathbb{R}} |g(s) + u| d\mu_s(u) ds$ .

### Lemma 1.5.3

Suppose  $F: \mathbb{R} \rightarrow [0, 1]$  is increasing, right continuous and that  $F(-\infty) = 0$ ,  $F(\infty) = 1$ . Let  $X: (0, 1) \rightarrow \mathbb{R}$  be defined by  $X(t) = \sup\{x \in \mathbb{R}: F(x) \leq t\}$ . Then  $\{t: X(t) \leq \xi\} = \{t: t \leq F(\xi)\}$ .

### Proof.

Let  $A = \{t: X(t) \leq \xi\}$ ,  $B = \{t: t \leq F(\xi)\}$ .  $t \in A$  iff  $\xi$  is an upper bound for  $\{x: F(x) \leq t\}$  iff  $\nexists x > \xi$  such that  $F(x) > t \Rightarrow F(\xi) \geq t \Rightarrow t \in B$ . Now  $t \in B \Rightarrow t \leq F(\xi) \Rightarrow t \in A$  as  $F(x) \leq t \leq F(\xi)$ .

(d)  $\Rightarrow$  (c) Given the random measure  $\xi: [0, 1] \rightarrow \mathcal{P}(\mathbb{R})$  we define

$$s \longmapsto \mu_s$$

a function  $h(s, t) \in L_1([0, 1] \times [0, 1])$  by

$h(s, t) = \sup\{x \in \mathbb{R}: \mu_s(-\infty, x] \leq t\}$ . Now the map  $s \longmapsto \mu_s$  is

measurable, let  $I = \{y_1, y_2, \dots\}$  be an enumeration of  $\mathbb{Q}$ . Let

$I_1 = \{y_1\}$ ,  $I_{n+1} = I_n \cup \{y_{n+1}\}$ . Then define  $f_n^{(s)}(t)$  to be the

largest  $x \in I_n$  such that  $\mu_s(-\infty, x] \leq t$ . So  $\sup f_n^{(s)}(t) = h(s, t)$  and

as  $f_n^{(s)}$  is measurable,  $h(s, t)$  is measurable as we require. In

the lemma above, let  $F_X$  be the distribution function of  $X$  and

$\mu$  be the Lebesgue measure on  $[0,1]$ . So

$F_X(\xi) = \mu(\{t: X(t) \leq \xi\}) = \mu(\{t: t \leq F(\xi)\}) = F(\xi)$ . Let  $F(x) = \mu_s(-\infty, x]$ ,

$X(t) = h(s, t)$ . Thus  $F_{h(s, \cdot)}(x) = \mu_s(-\infty, x]$  which finally gives us

$$\int_{\mathbb{R}} |g(s) + u| d\mu_s(u) = \int_{\mathbb{R}} |g(s) + u| dF_{h(s, \cdot)}(u) = \int_0^1 |g(s) + h(s, t)| dt.$$

#### Lemma 1.5.4

Let  $h \in L_1(\{0,1\}^{\mathbb{N}})$ . If  $g$  depends only on the first  $N$  coordinates and  $\pi$  is a permutation of  $\mathbb{N}$  which fixes the first  $N$  coordinates, then  $\int |g(\underline{x}) + h(\underline{x}_\pi)| = \int |g(\underline{x}) + h(\underline{x})|$ .

(c)  $\Rightarrow$  (a) Given  $h \in L_1[0,1]^2$  identify  $[0,1] \cong \{0,1\}^{\mathbb{N}}$ , then

$h(x_1, x_2, \dots; y_1, y_2, \dots) \in L_1(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}})$ , define

$f_N \in L_1(\{0,1\}^{\mathbb{N}}) \cong L_1[0,1]$  by

$$f_N(x_1, x_2, \dots) = h(x_1, x_2, \dots, x_N, x_{N+2}, x_{N+4}, \dots; x_{N+1}, x_{N+3}, x_{N+5}, \dots)$$

Suppose  $g$  depends only on the first  $N$  coordinates of  $\mathbb{N}$ , then by the lemma above  $\|g+h\| = \|g+f_N\|$ . The set of functions which depend only on the first  $N$  coordinates is dense (as  $N \rightarrow \infty$ ) in  $L_1(\{0,1\}^{\mathbb{N}})$ , so  $\tau(g) = \alpha + \|g+h\| = \alpha + \lim_n \|g+f_n\|_{L_1[0,1]}$  for  $g \in L_1[0,1]$ . Define  $f'_n$  by  $\alpha + \|g+f_n\| = \|g+f'_n\|$ , thus we have  $\tau(g) = \alpha + \lim_n \|g+f_n\| = \lim_n \|g+f'_n\|$ . So that  $\tau \in \mathcal{T}(L_1[0,1])$ .

### 1.6 Representing a type as a random measure.

Fundamental to this thesis is the representation theorem of Garling [Ga], where he represents a type on  $L_\phi(X)$  in terms of a suitable random measure in  $\pi_\phi(T)$ .

Definition

Let  $\psi$  be a continuous non-negative function on  $[0, \infty)$ . Define  $\pi_\psi(\mathcal{T}(X)) = \left\{ \xi \in \pi(\mathcal{T}(X)) : \int_{\Omega} \int_{\mathcal{T}(X)} \psi(\|t\|) d\xi_\omega(t) dP(\omega) < \infty \right\}$ . When  $\psi(\alpha) = \alpha^p$ ,  $p \geq 1$ , we write  $\pi_p(\mathcal{T}(X))$  for  $\pi_\psi(\mathcal{T}(X))$ .

Definition

If  $\xi \in \pi(\mathcal{T}(X))$  and  $x \in L_0(X)$  define  $I_p^x(\xi) = \int_{\Omega} \int_{\mathcal{T}(X)} (t[x(\omega)])^p d\xi_\omega(t) dP(\omega)$ , and we write  $I_p(\xi)$  for  $I_p^0(\xi)$ .

There is a homeomorphic mapping  $\lambda: L_0(X) \longrightarrow L_0(\mathcal{T}(X))$ , given by  $x \longmapsto \tau_x$ , and a map  $i: L_0(\mathcal{T}(X)) \longrightarrow \pi(\mathcal{T}(X))$  given by  $t \longmapsto \delta_t$ . Let  $h = i\lambda: L_0(X) \longrightarrow \pi(\mathcal{T}(X))$ .

We say that a subset  $A$  of  $\pi_p(\mathcal{T}(X))$  is  $p$ -uniformly integrable if  $I_{\phi_n}(\xi) \longrightarrow I_p(\xi)$  uniformly on  $A$ . Where  $\phi_n$  is

$$\text{defined by } \phi_n(\gamma) = \begin{cases} \gamma^p & \text{if } 0 \leq \gamma \leq n \\ (n+1-\gamma)\gamma^p & \text{if } n \leq \gamma \leq n+1 \\ 0 & \text{if } n+1 \leq \gamma \end{cases}$$

So  $C$  is a uniformly integrable subset of  $L_p$  iff  $h(C)$  is 1-uniformly integrable.

Definition

Let  $\Omega_p = \{f: \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : f(0) = 0, f \text{ is convex and } f(2\lambda) \leq 2^p f(\lambda) \forall \lambda \geq 0\}$ . So if  $f \in \Omega_p$  then  $f$  has the form  $f(\beta) = \alpha_0 \beta^p$  for some constant  $\alpha_0$ . Let  $j: L_p(X) \longrightarrow \pi_p(\mathcal{T}(X)) \times [0, \infty)$  be the homeomorphic map  $j(x) = (h(x), 0)$ . Define  $J_p(X)$  to be the closure of  $j(L_p(X))$  in  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$ .

$J_p(X)$  can be identified with a subspace of  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$ . If  $(\Omega, \Sigma, P)$  is atom-free, then  $J_p(X)$  is homeomorphic to  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$ .

We define a topology on  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$  to give the convergence  $(\xi_n, \alpha_n) \longrightarrow (\xi, \alpha)$  iff  $\xi_n \longrightarrow \xi$  in the wm-topology, and  $I_p(\xi_n) + \alpha_n \longrightarrow I_p(\xi) + \alpha$ .

We can now state Theorem 20 of [Ga], which gives the representation between  $\mathcal{T}(L_p(X))$  and  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$ .

Theorem 1.6.1

Let  $X$  be a separable Banach space. Let  $1 \leq p < \infty$ . Then there is a map  $Q$  from  $J_p(X)$  continuously onto the space of types on  $L_p(X)$  such that for all  $f \in L_p(X)$  and all  $(\xi, \alpha) \in \pi_p(\mathcal{T}(X)) \times [0, \infty)$ ,

$$Q((\xi, \alpha))(f) = \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} t[f(\omega)]^p d\xi_{\omega}(t) dP(\omega) + \alpha^p \right\}^{1/p}.$$

If  $X$  is finite-dimensional, we can identify  $X$  with  $\mathcal{T}(X)$ .

Types on  $L_p(X)$  can be split into two parts, a uniformly integrable type and a "spikey" part. Let  $\tau$  be a type on  $L_p(X)$  with approximating sequence  $\{x_n\}$ . Then there exists a subsequence  $(z_j) = (x_{n_j})$  such that when  $y_j = z_j \cdot 1_{\{\|z_j\| < j\}}$ ,  $\{y_j\}$  is a uniformly integrable subset of  $L_p(X)$ , and  $\|w_j\|_{L_p(X)} \longrightarrow \beta$ , where  $w_j = z_j \cdot 1_{\{\|z_j\| \geq j\}}$ .

Suppose  $j(\{y_n\}) \longrightarrow (\xi, \alpha)$  in  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$ . Since  $h(\{y_n\})$  is  $p$ -uniformly integrable, by Proposition 9 of [Ga],  $I_p(h(y_n)) \longrightarrow I_p(\xi)$ . Thus  $I_p(h(y_n)) \longrightarrow I_p(\xi)$ , and hence  $\alpha = 0$ . So a uniformly integrable type  $\tau$  has the representation

$$\tau(f) = \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} t[f(\omega)]^p d\xi_{\omega}(t) dP(\omega) \right\}^{1/p}.$$

To see how the remainder of the type behaves, suppose  $j(\{w_n\}) \longrightarrow (\eta, \beta)$  in  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$ . Then  $I_p(h(w_n)) = I_p(\delta(\tau_{w_n})) = \int_{\Omega} \|w_n(\omega)\|_X^p \cdot dP(\omega) \longrightarrow \beta$ . So that  $\eta \equiv 0$ , and thus a general type on  $L_p(X)$  is composed of a uniformly integrable part and a "spikey" part, and has the form

$$\tau(f) = \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} t [f(\omega)]^p d\xi_{\omega}(t) dP(\omega) + \alpha^p \right\}^{1/p} \quad (*)$$

We shall often have cause to speak of a random measure of a type  $\tau$ , by this we mean a random measure  $\xi$  representing  $\tau$  as in (\*); we do not assume uniqueness when using this notation.

## CHAPTER 2: REPRESENTATIONS OF TYPES.

### 2.1 INTRODUCTION.

This chapter is concerned with structure results about the space of types, and in particular the structure of  $\mathcal{T}(L_p(X))$ . The study of the representation of a type on  $L_1(X)$  as a random measure on  $\mathcal{T}(X)$  forms the first half of this chapter. Here we show that convolution and symmetry are preserved by the map between  $\mathcal{T}(L_1(X))$  and  $\pi_1(\mathcal{T}(X))$ . We produce a specific representation of types on  $L_1(\mathcal{Q}_p)$ , using a disintegration of probabilities on  $\mathcal{Q}_p \times \mathbb{R}_+$  into random measures on  $\mathcal{Q}_p$  and  $[0, \infty)$ . We look at the form convolution takes within  $L_1(\mathcal{Q}_p)$  in terms of this representation.

The second half of the chapter is concerned with decompositions of types in terms of their weak and symmetric parts. We look at conditions for decomposing  $\mathcal{T}(X)$  as  $X * \mathcal{P}(X)$  or  $X * \mathcal{T}_{wn}(X)$ , where  $\mathcal{T}_{wn}(X)$  is the space of "weakly null" types. A sufficient condition being that  $X$  possesses an unconditional basis, but does not contain any subspace isomorphic to  $c_0$ .

Decompositions of  $\mathcal{T}(L_{2k})$  are obtained in terms of the weak limits (in various subspaces) of the sequence defining the type. We thus decompose  $\mathcal{T}(L_{2k})$ ,  $k \geq 1$ , as a subset of  $L_{2k} \times L_k \times L_{2k, 3} \times \dots \times L_{2k, 2k-1} \times [0, \infty)$ . This is then used in representations of  $\mathcal{T}(L_1(L_2))$  and  $\mathcal{T}(L_1(L_4))$  as random measures, where we look at the form of symmetric types and convolution.

Part 1 : Representations of types as random measures.

2.2 Convolutions of types and random measures.

We saw in chapter one that if  $(\Omega, \Sigma, P)$  was an atom-free probability space, then there exists a map from  $\pi_p(\mathcal{T}(X)) \times [0, \infty)$  continuously onto the space of types on  $L_p(X)$ . There are several natural questions that one can ask concerning this map. Here we ask whether the map preserves the operation of convolution for types on  $L_1(X)$ , where  $X$  is stable. That is if  $\tau, \sigma$  are uniformly integrable types on  $L_1(X)$ , where  $X$  is stable, having corresponding random measures  $\xi$  and  $\eta$  respectively in  $\pi_1(\mathcal{T}(X))$ , then  $\tau * \sigma$  has corresponding random measure  $\xi * \eta$ . So that if

$$\tau(f) = \int_{\Omega} \int_{\mathcal{T}(X)} t[f(\omega)] d\xi_{\omega}(t) dP(\omega) \text{ and}$$

$$\sigma(f) = \int_{\Omega} \int_{\mathcal{T}(X)} t[f(\omega)] d\eta_{\omega}(t) dP(\omega), \text{ then}$$

$$(\tau * \sigma)(f) = \int_{\Omega} \int_{\mathcal{T}(X)} t[f(\omega)] d(\xi * \eta)_{\omega}(t) dP(\omega). \text{ We know that}$$

$\pi(T) = \overline{i(L_0(X))}^{wm}$ , see [Al], [Ga]. We will show that for random measures in  $\overline{i(L_1(X))}^{wm}$  convolution is preserved.

Firstly we consider the scalar case.

Following the paper "Tout sous espace de  $L_1$  contient un  $\mathcal{Q}_p$ " [Ma1], the convolution of two random measures  $\mu, \nu \in \pi(\mathbb{R})$  is defined by  $(\mu * \nu)_{\omega} = \mu_{\omega} * \nu_{\omega}$ . The convolution of two measures  $m, n$  is given by  $\int_{\mathbb{R}} f(u) d(m * n)(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) d(m \otimes n)(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) dm(x) dn(y)$ , where  $m \otimes n$  is the product measure on  $\mathbb{R}^2$ .

Maurey is concerned with a proof of the result that every infinite dimensional subspace of  $L_1$  contains a subspace isomorphic to  $\mathcal{Q}_p$ . We can easily conclude from his paper that in certain cases convolution in  $\mathcal{T}(L_1)$  corresponds to convolution in  $\pi(\mathbb{R})$ . Since by Proposition II.2 in [Ma1], we have :

If  $(x_n)$  is a sequence in a reflexive subspace of  $L_1(\Omega, P)$  such that  $\mu = \lim_n \delta_{x_n}$  we have

$\|x + \sum_{i=1}^k \alpha_i e_i\|_{\mu} = \lim_{n_1} \lim_{n_2} \dots \lim_{n_k} \|x + \sum_{i=1}^k \alpha_i x_{n_i}\|$ . Where  $(e_i)$  is the standard basis in  $\mathbb{R}^{(\mathbb{N})}$ , and for a symmetric measure  $\mu$

$\|\cdot\|_{\mu}$  is a norm on  $X \oplus \mathbb{R}^{(\mathbb{N})}$  given by

$$\|x + \sum_{i=1}^k \alpha_i e_i\|_{\mu} = E \int |x + \sum_{i=1}^k \alpha_i u_i| d\mu(u_1) \dots d\mu(u_k) = \|\delta_x * D_{\alpha_1} \mu * \dots * D_{\alpha_k} \mu\|.$$

If  $\sigma$  has an associated random measure  $\xi = \lim_n \delta_{x_n}$ , then

$$\begin{aligned} (\alpha_1 \sigma) * \dots * (\alpha_k \sigma)(x) &= E \int |x + \sum_{i=1}^k \alpha_i u_i| d\xi(u_1) \dots d\xi(u_k) \\ &= E \int |x+t| d(\alpha_1 \xi * \dots * \alpha_k \xi)(t). \end{aligned}$$

Here Maurey makes some restrictions. Firstly that  $L_1(\Omega, \Sigma, P)$  is separable, secondly that he considers types on a reflexive subspace  $X$  of  $L_1(P)$  and finally that the associated random measure  $\xi$  is given by  $\xi = \lim_n \delta_{x_n}$  for  $x_n \in X$ . We can now remove the first two conditions.

Given  $\sigma \in \mathcal{T}(L_1(P))$  define  $\|\sigma\| = \sigma(0) = \int_{\Omega} \int_{\mathbb{R}} |t| d\xi_{\omega}(t) dP(\omega)$ , and  $\|\xi\| = E \int_{\mathbb{R}} |u| d\xi(u)$ . Then  $\|\xi\| = \|\sigma\|$  if  $\xi$  is the random measure of  $\sigma$ . Let  $H$  be a subspace of  $L_1(\Omega)$  whose unit ball is uniformly integrable (then  $H$  is reflexive). Set  $D = \overline{i(H)^{wm}}$ . Then the map  $\xi \longmapsto \|\xi\|$  is finite and continuous on  $D$ , see [Ma1] or [A1]. We shall see later, Theorem 4.2.6, that for each  $\sigma \in \mathcal{T}(L_1)$  there

is a unique  $\xi \in \pi_1(\mathbb{R})$  given by Garlings integral representation, [Ga].

Let  $\phi$  be the continuous map from  $\pi_1(\mathcal{T})$  to  $\mathcal{T}(L_1(X))$ , that is  $\phi\xi = Q((\xi, 0))$  in the notation of [Ga]. So that

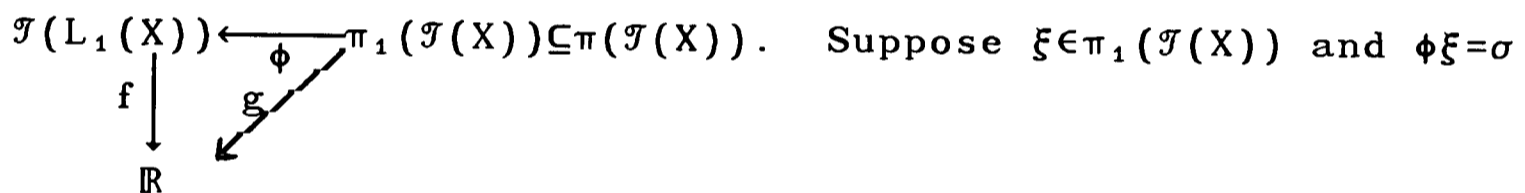
$$\phi(\xi) = \int_{\Omega} \int_{\mathcal{T}(X)} \tau(\cdot(\omega)) d\xi_{\omega}(\tau) dP(\omega).$$

Proposition 2.2.1

The function  $\xi \mapsto \|\xi\|$  is finite and continuous on  $\pi_1(\mathcal{T}(X))$ .

Proof

Let  $f: \mathcal{T} \rightarrow \mathbb{R}$  and  $g: \pi \rightarrow \mathbb{R}$ . Then  
 $\sigma \mapsto \sigma(0)$  and  $\xi \mapsto \|\xi\|$



say. Then  $\|\phi\xi\| = \|\sigma\|$  that is  $f(\phi\xi) = g(\xi)$ . Let  $\tau = 0$ , then  $[\sigma, \tau] = \sigma(0)$  so the map  $f$  is continuous. Thus  $g$  is continuous, so  $\|\cdot\|$  is continuous on  $\pi_1(\mathcal{T}(X))$ , it is also finite since  $\|\xi\| = (\phi\xi)(0) < \infty$ .

Let  $Y$  be a uniformly integrable subset of  $L_1(\Omega, P)$ . Let  $W = \overline{i(Y)^{wm}}$ . Suppose  $\xi \in W$  is given by  $\xi = \lim_n \delta_{x_n}$  where  $x_n \in Y$ . Then  $\{x_n\}$  is uniformly integrable, so  $E|x_n| \wedge R \rightarrow E|x_n|$  uniformly as  $R \rightarrow \infty$  for each  $n$ . Since  $E[|x_n| \wedge R] = E \int |u| \wedge R d\delta_{x_n}(u) \rightarrow E \int |u| \wedge R d\xi(u)$  as  $n \rightarrow \infty$ , we have  $E \int |u| d\delta_{x_n}(u) \rightarrow E \int |u| d\xi(u)$  as  $n \rightarrow \infty$ . That is  $\|\xi\| = \lim_n \|\delta_{x_n}\| = \lim_n \|x_n\|$ . Define  $\|\cdot\|_{\xi}$  as before but now on  $L_1 \oplus \mathbb{R}^{\mathbb{N}}$ . Since  $*$  is

separately continuous

$$\begin{aligned} \delta_x * D_{\alpha_1} \xi * \dots * D_{\alpha_k} \xi &= \|\lim_{n_1} \dots \lim_{n_k} (\delta_x * \delta_{\alpha_1 x_{n_1}} * \dots * \delta_{\alpha_k x_{n_k}})\| \\ &= \lim_{n_1} \dots \lim_{n_k} \|\delta_x * \delta_{\alpha_1 x_{n_1}} * \dots * \delta_{\alpha_k x_{n_k}}\| = \lim_{n_1} \dots \lim_{n_k} \|x + \sum_{i=1}^k \alpha_i x_i\|_1 \end{aligned}$$

Proposition 2.2.2

Let  $\xi = \lim_n \delta_{x_n}$  and  $\eta = \lim_n \delta_{y_n}$  for some  $x_n, y_n \in W$ , a uniformly integrable subset of  $L_1(\Omega)$ . Then

$$\lim_n \lim_m \|x + \alpha x_n + \beta y_m\| = E \int |x + \alpha u + \beta v| d\xi(u) d\eta(v) = \|\delta_x * D_\alpha \xi * D_\beta \eta\|.$$

Let  $\sigma$  and  $\tau$  be types with corresponding random measures  $\xi$  and  $\eta$  respectively, then  $(\alpha\sigma) * (\beta\tau)(x) = E \int |x + u| d(\alpha\xi * \beta\eta)(u)$ .

Thus for the class of types with random measures in

$\overline{i(L_1(\Omega; \mathbb{R}))}^{wm}$ , convolution in  $\mathcal{T}(L_1)$  corresponds to convolution in  $\pi(\mathbb{R})$ .

We now consider the non-scalar case, that is convolution on  $\mathcal{T}(L_1(X))$ .

1. Let  $X$  be a separable stable Banach space.  $T = \mathcal{T}(X)$  is a locally compact commutative semi-group under the weak topology and  $*$ . Then  $\pi(T)$  is separable and metrizable in the  $w_m$ -topology on  $\pi(T)$ . The map  $*$  is separately continuous and associative. Let  $\mu, \nu \in \mathcal{P}(T)$  and  $f \in \mathcal{C}_b(T)$ . Then the function  $(s, t) \rightarrow f(s * t)$  is measurable on  $T \times T$ , see [Ga] proposition 2. Let  $\phi(f) = \int_T \int_T f(s * t) d\mu(s) d\nu(t)$ . Then  $\phi$  is a positive linear functional on  $\mathcal{C}_b(T)$  and  $\phi(1) = 1$ . So there exists a regular Borel probability measure  $m$  on the Stone-Cech

compactification  $\beta T$  of  $T$  such that  $\phi(f) = \int_{\beta T} \tilde{f}(u) dm(u)$  where  $\tilde{f}$  is the continuous extension of  $f$  to  $\beta T$ .

Now  $\mu$  and  $\nu$  are regular, and so given  $0 < \epsilon < 1$  there exists  $n$  such that if  $K_n = \{t : t(0) \leq n\}$ ,  $\mu(K_n), \nu(K_n) \geq 1 - \epsilon$ . Now  $(s * t)(0) \leq s(0) + t(0)$ , so that  $K_n * K_n \subseteq K_{2n}$  and hence  $m(K_{2n}) \geq (1 - \epsilon)^2$ . This implies that  $m(T) = 1$ , and so we can consider  $m$  as an element of  $\mathcal{P}(T)$ . We denote  $m$  by  $\mu * \nu$ , that is the convolution of  $\mu$  and  $\nu$ . Thus we have  $\forall f \in \mathcal{C}_b(T)$

$$\int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} f(s * t) d\mu(s) d\nu(t) = \int_{\mathcal{T}(X)} f(u) d(\mu * \nu)(u).$$

2. Define  $\|\sigma\| = \sigma(0)$  and  $\|\xi\| = \int_{\Omega} \int_T \rho(0) d\xi_{\omega}(\rho) dP(\omega)$ . Then for  $g \in L_1(X)$

$$\|\delta_g * \xi\| = E \int_T \rho(0) d(\delta_{g, \omega} * \xi_{\omega})(\rho) =$$

$$E \int_T \int_T (s * t)(0) d\delta_{g, \omega}(t) d\xi_{\omega}(s) = E \int_T (s * g(\omega))(0) d\xi_{\omega}(s)$$

$$= E \int_T s(g(\omega)) d\xi_{\omega}(s) = \sigma(g).$$

### Proposition 2.2.3

Let  $\tau, \sigma \in \mathcal{T}(L_1(X))$  have associated random measures  $\xi, \eta \in \pi_1(\mathcal{T}(X))$  respectively. Suppose  $\xi, \eta \in \overline{i(L_1(X))}^{wm}$ . Then  $\tau * \sigma$  has associated random measure  $\xi * \eta$ .

### Proof

There exists sequences  $\{x_n\}, \{y_m\}$  in a uniformly integrable subset of  $L_1(X)$  with  $\xi = \lim_n \delta_{x_n}$  and  $\eta = \lim_m \delta_{y_m}$ . Then  $\{x_n\}$  is uniformly integrable, so  $E|x_n| \wedge R \rightarrow E|x_n|$  uniformly as  $R \rightarrow \infty$  for each  $n$ . Since  $E[|x_n| \wedge R] = E \int_X |u| \wedge R d\delta_{x_n}(u)$

$$= E \int_{\mathcal{T}(X)} |u| \wedge R d\delta_{t_{x_n}}(u) \longrightarrow E \int_{\mathcal{T}(X)} |u| \wedge R d\xi(u) \text{ as } n \longrightarrow \infty, \text{ we have}$$

$$E \int_{\mathcal{T}(X)} |u| d\delta_{t_{x_n}}(u) \longrightarrow E \int_{\mathcal{T}(X)} |u| d\xi(u) \text{ as } n \longrightarrow \infty. \text{ That is}$$

$$\|\xi\| = \lim_n \|\delta_{x_n}\| = \lim_n \|x_n\|.$$

$$\text{For all } x \in L_1(X), E \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (D_{t * D_{\beta} s})(x) \wedge R d\xi(t) d\eta(s) =$$

$$E \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (r * D_{\alpha} t * D_{\beta} s)(0) \wedge R d\delta_x(r) d\xi(t) d\eta(s) =$$

$$E \int_{\mathcal{T}(X)} u[0] \wedge R d(\delta_x * \alpha \xi * \beta \eta)(u) \longrightarrow E \int_{\mathcal{T}(X)} u[0] d(\delta_x * \alpha \xi * \beta \eta)(u) \text{ as}$$

$$R \longrightarrow \infty. \text{ Then } E \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (D_{t * D_{\beta} s})(x) \wedge R d\xi(t) d\eta(s) \longrightarrow$$

$$E \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (D_{t * D_{\beta} s})(x) d\xi(t) d\eta(s). \text{ So that}$$

$$\tau(f) = E \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (D_{t * D_{\beta} s})(x) d\xi(t) d\eta(s)$$

$$= \int_{\Omega} \int_{\mathcal{T}(X)} t[x(\omega)] d(D_{\alpha} \xi * D_{\beta} \eta)_{\omega}(t) dP(\omega)$$

### 2.3 Types on $L_1(\Omega; \mathcal{Q}_p)$

In the following sections we aim to do two things. Firstly obtain a specific representation of the uniformly integrable types on  $L_1(\mathcal{Q}_p)$ , via Garling's integral representation, in terms of a family of random measures  $(\xi_y)$ ,  $y \in \mathcal{Q}_p$ , on  $[0, \infty)$  and a random measure  $\xi_2$  on  $\mathcal{Q}_p$ . This will be achieved by using results of Torrat on disintegration of measures [T], and we will use this disintegration repeatedly in the following chapters. Having obtained our representation in section 2.5, we consider what form the convolution operation in  $\mathcal{T}(L_1(\mathcal{Q}_p))$  takes in terms of our random measures on  $\mathcal{Q}_p$  and  $[0, \infty)$ .

Let  $\sigma$  be a uniformly integrable type on  $L_1(\Omega; \mathcal{L}_p)$ . Then for some  $\xi \in \pi_1(\mathcal{T}(\mathcal{L}_p))$   $\sigma$  is given by the representation 
$$\sigma(g) = \int_{\Omega} \int_{\mathcal{T}(\mathcal{L}_p)} t[g(\omega)] d\xi_{\omega}(t) dP(\omega) \quad \text{for all } g \in L_1(\Omega; \mathcal{L}_p), \quad [\text{Ga}].$$
 We will use the following result : Let  $T: (\Sigma, \mathcal{B}) \longrightarrow (\Sigma_0, \mathcal{A}_0)$  be a measurable mapping, and  $\xi$  be a measure on  $\mathcal{B}$ . Define a measure  $\xi_0$  by  $\xi_0 = \xi T^{-1}$  on  $\mathcal{A}_0$ . If  $A \in \mathcal{A}_0$  and  $f: (\Sigma_0, \mathcal{A}_0) \longrightarrow (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$  is measurable, then 
$$\int_{T^{-1}A} f(T(s)) d\xi(s) = \int_A f(s) d\xi_0(s).$$
 See [As] or [Bu].

Given a type  $\tau$  on  $\mathcal{L}_p$  then  $\tau(x) = (\|x+y\|^p + \alpha^p)^{1/p}$  for some  $(y, \alpha) \in \mathcal{L}_p \times [0, \infty)$ . Now let  $\Sigma = \mathcal{T}(\mathcal{L}_p)$ ,  $\mathcal{B}$  be the Borel  $\sigma$ -field of  $\mathcal{T}(\mathcal{L}_p)$  and  $\xi$  a (random) measure on  $\mathcal{T}(\mathcal{L}_p)$ . Define  $T$  and  $f$  by 
$$T: \mathcal{T}(\mathcal{L}_p) \longrightarrow \mathcal{L}_p \times [0, \infty) \quad \text{and} \quad f: \mathcal{L}_p \times [0, \infty) \longrightarrow \mathbb{R}$$
 
$$\tau \longmapsto (y, \alpha) \quad (y, \alpha) \longmapsto (\|x+y\|^p + \alpha^p)^{1/p}$$
 so that  $\Sigma_0 = \mathcal{L}_p \times [0, \infty)$  and  $\mathcal{A}_0$  is the product  $\sigma$ -field of the two Borel  $\sigma$ -fields on  $\mathcal{L}_p$  and  $[0, \infty)$ .

Given  $(y, \alpha) \in \mathcal{L}_p \times [0, \infty)$ , does  $T^{-1}(y, \alpha)$  define a type on  $\mathcal{L}_p$ ? Let  $y = (y_i)$ , and let 
$$y^{(n)} = (y_1 - \alpha/n^{1/p}, \dots, y_n - \alpha/n^{1/p}, y_{n+1}, y_{n+2}, \dots).$$
 Then  $y^{(n)} \longrightarrow y$  pointwise and  $\|y - y^{(n)}\| \longrightarrow \alpha$ . Thus  $T$  is onto.

### Lemma 2.3.1

$T$  is measurable from  $(\mathcal{T}(\mathcal{L}_p), \mathcal{B})$  to  $(\mathcal{L}_p \times [0, \infty), \mathcal{A}_0)$ . The map  $f: (\mathcal{L}_p \times [0, \infty), \mathcal{A}_0) \longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}))$  is measurable.

### Proof

On  $\mathcal{T}(\mathcal{L}_p)$  we put the weak topology, which is Hausdorff. On  $\mathcal{L}_p \times [0, \infty)$  we put the usual (Borel) topology, which is Polish. The Borel  $\sigma$ -algebra of a Polish space coincides with that of any coarser Hausdorff topology. So  $(y, \alpha) \longmapsto (\|\cdot + y\|^p + \alpha^p)^{1/p}$  is a Borel isomorphism, hence  $T$  is measurable as required.

$f$  is the composition of maps  
 $(y, \alpha) \longmapsto (\|x+y\|_p, \alpha) \longmapsto (\|x+y\|^p + \alpha^p)^{1/p}$  which are measurable,  
so  $f$  itself is measurable.

Proposition 2.3.2

Let  $\xi$  be a probability on  $\mathcal{T}(\mathcal{Q}_p)$ . Let  $x \in \mathcal{Q}_p$ . Then  

$$\int_{\mathcal{T}(\mathcal{Q}_p)} \tau[x] d\xi(\tau) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} \{\|x+y\|^p + \alpha^p\}^{1/p} d\xi_0(y, \alpha),$$
where  $\xi_0 = \xi_0 \circ T^{-1}$  and  $T$  is as above.

Lemma 2.3.3

The map  $\xi_0: \Omega \longrightarrow \mathcal{P}(\mathcal{Q}_p \times \mathbb{R}_+)$  is measurable. Thus  $\xi_0$  is a random measure.

Proof

Define a map  $\theta_B: (\Omega, \mathcal{A}, P) \longrightarrow \mathbb{R}$  for  $B \in \mathcal{B}(\mathcal{Q}_p \times \mathbb{R}_+)$ . It is  

$$\omega \longmapsto \xi_{0, \omega}(B)$$

then sufficient to prove that  $\theta_B$  is  $\mathcal{A}$ -measurable for the Borel  $\sigma$ -field of  $\mathbb{R}$ . Let  $A \in \mathcal{B}(\mathbb{R})$ .

$\theta_B^{-1}(A) = \{\omega: \xi_{0, \omega}(B) \in A\} = \{\omega: \xi_{\omega}(T^{-1}B) \in A\}$  which is a member of  $\mathcal{A}$  as  $T^{-1}B \in \mathcal{B}(\mathcal{T}(\mathcal{Q}_p))$  and  $\xi$  is a random measure. Thus  $\xi_0 \in \pi(\mathcal{Q}_p \times \mathbb{R}_+)$  as required.

We will suppress the subscript 0 on  $\xi_0$ , and henceforth write  

$$\int_{\mathcal{T}(\mathcal{Q}_p)} \tau[g(\omega)] d\xi(\tau) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} \{\|g(\omega)+y\|^p + \alpha^p\}^{1/p} d\xi(y, \alpha)$$
without ambiguity.

## 2.4 The Disintegration of Measures.

We have represented a uniformly integrable type  $\tau$  on  $L_1(\mathcal{Q}_p)$  as a random measure  $\xi$  on  $\mathcal{Q}_p \times \mathbb{R}_+$ . We will now use a disintegration of measures on  $\mathcal{Q}_p \times \mathbb{R}_+$  to enable us to write

$\tau(f) = \int_{\Omega} \int_{\mathcal{Q}_p} \int_0^{\infty} \{ \|f(\omega) + y\|^p + \alpha^p \}^{1/p} d\xi_y^\omega(\alpha) d\xi_2^\omega(y) dP(\omega)$ , where for each  $\omega \in \Omega$ ,  $\xi_2^\omega$  is a probability on  $\mathcal{Q}_p$  and  $\{\xi_y^\omega\}$  an  $\mathcal{Q}_p$ -measurable family of probabilities on  $\mathbb{R}_+$ .

The theory of disintegrations that we use is as found in [T]. Let  $P$  be a probability on  $\mathcal{X} \times \mathcal{Y}$ , with the  $\sigma$ -algebra  $\mathcal{C} \times \mathcal{B}$ . Let  $\mathcal{Y}$  be a Polish space. Let  $T: \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$  and  $Y: \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$  be the projections. The aim is to disintegrate  $P$  by

$$P(dt \times dy) = P_T(dt) P_t(dy), \text{ so that}$$

$$\int_{\mathcal{X} \times \mathcal{Y}} f \cdot P(dt \times dy) = \int_{\mathcal{X}} \int_{\mathcal{Y}} f \cdot P_t(dy) P_T(dt).$$

### Theorem 2.4.1

Let  $P$  be a probability on  $\mathcal{X} \times \mathcal{Y}$ , with the  $\sigma$ -algebra  $\mathcal{C} \times \mathcal{B}$ . Let  $\mu$  and  $\nu$  be the marginals  $P_T$  and  $P_Y$  respectively. Let  $\mathcal{B}$  be the Borel  $\sigma$ -field of  $\mathcal{Y}$ . Let  $\nu$  be Radon and  $\mathcal{C}$  complete for  $\mu$ .

Then there is a disintegration of  $\nu$  into a family  $\nu_t = P_t$ , such that  $P(dt \times dy) = P_T(dt) P_t(dy)$ .

When  $Y$  is real and integrable we need not impose any conditions, and  $P(A \times B) = \int_{\mathcal{X}} \int_{\mathcal{Y}} 1_{A \times B}(y, t) dP_t(y) dP_T(t)$ , thus  $\int_{\mathcal{X} \times \mathcal{Y}} f dP = \int_{\mathcal{X}} \int_{\mathcal{Y}} f(y, t) dP_t(y) dP_T(t)$ . The probability  $P_T(dt)$  is unique, and  $P_t$  is unique a.e.  $[P_T]$  due to the uniqueness of conditional probabilities.

We can apply this to the case when  $\mathcal{X}=\mathcal{Q}_p$  and  $\mathcal{Y}=\mathbb{R}_+$ . Let  $\mu$  be a probability on  $\mathcal{Q}_p \times \mathbb{R}_+$ . Then via the disintegration we obtain a probability  $\mu_2$  on  $\mathcal{Q}_p$ , and a family of probabilities  $\{\mu_y\}$  on  $\mathbb{R}_+$  such that  $y \mapsto \mu_y$  is measurable, and for all measurable, integrable  $f$

$$\int_{\mathcal{Q}_p \times \mathbb{R}_+} f(y, t) d\mu(y, t) = \int_{\mathcal{Q}_p} \int_{\mathbb{R}_+} f(y, t) d\mu_y(t) d\mu_2(y).$$

Example.

Now consider the situation in which we have

$$\int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d\xi_{\omega}(t) dP(\omega) = \int_{\Omega} \int_{X \times \mathbb{R}_+} f[g(\omega), y, \alpha] d\xi_{\omega}(y, \alpha) dP(\omega).$$
 Then

for each  $\omega \in \Omega$ ,  $\xi_{\omega} \in \mathcal{P}(X \times \mathbb{R}_+)$ , where  $X$  is separable. Thus there exists a disintegration of  $\xi_{\omega}$  into a family of probabilities  $\xi_y^{\omega}$  on  $[0, \infty)$  ( $y \in X$ ), and a probability  $\xi_2$  on  $X$ , such that for each  $\omega \in \Omega$ , the map  $y \mapsto \xi_y^{\omega}$  is measurable, and thus  $\xi_{(\cdot, \cdot)}^{\omega}$  is a random measure. Further for all  $\xi_{\omega}$ -measurable, integrable  $f$

$$\text{we have } \int_{X \times \mathbb{R}_+} f(y, \alpha) \cdot d\xi_{\omega}(y, \alpha) = \int_X \int_{\mathbb{R}_+} f(y, \alpha) \cdot d\xi_y^{\omega}(\alpha) \cdot d\xi_2^{\omega}(y).$$
 So in

particular in the situation we are interested in on  $\mathcal{Q}_p \times \mathbb{R}_+$ :

$$\int_{\mathcal{Q}_p \times \mathbb{R}_+} (\|x+y\|^p + \alpha^p)^{1/p} d\xi_{\omega}(y, \alpha) = \int_{\mathcal{Q}_p} \int_0^{\infty} (\|x+y\|^p + \alpha^p)^{1/p} d\xi_y^{\omega}(\alpha) d\xi_2^{\omega}(y).$$

Lemma 2.4.2

$\xi_2$  is a random measure.

Proof

For each  $B \in \mathcal{B}(X)$  define a map  $\theta_B : (\Omega, \mathcal{A}, P) \xrightarrow{\omega \mapsto \xi_2^{\omega}(B)} \mathbb{R}$ . To show that  $\omega \mapsto \xi_2^{\omega}$  is measurable it suffices to show that  $\theta_B$  is  $\mathcal{A}$ -measurable for the Borel  $\sigma$ -field of  $\mathbb{R}$ . That is for each Borel subset  $A$  of  $\mathbb{R}$   $\theta_B^{-1}(A) = \{\omega : \xi_2^{\omega}(B) \in A\}$  is a member of  $\mathcal{A}$ .

$$\int_{X \times \mathbb{R}_+} 1_{B \times \mathbb{R}_+} d\xi_\omega = \int_X \int_0^\infty 1_B(y) \cdot 1_{\mathbb{R}_+}(\alpha) \cdot d\xi_y^\omega(\alpha) d\xi_2^\omega(y) = \xi_2^\omega(B) \xi_y^\omega(\mathbb{R}_+) \\ = \xi_2^\omega(B) \text{ since } \xi_y^\omega \text{ is a probability.}$$

If  $f=1_F$  then  $\int 1_F d\xi_\omega = \xi_\omega(F)$  and  $\omega \mapsto \xi_\omega(F)$  is measurable. Thus by considering simple functions  $\omega \mapsto \int f d\xi_\omega$  is measurable for measurable, integrable  $f$ . Thus  $\omega \mapsto \xi_2^\omega(B)$  is measurable and  $\xi_2$  is a random measure, that is  $\xi_2 \in \pi(\mathbb{R}_+)$ .

### 2.5 Convolution of types on $L_1(\Omega: \mathcal{Q}_p)$ .

Following on from the last two sections we define convolution in  $\mathcal{Q}_p \times [0, \infty)$ , and use this to construct a representation of  $\sigma * \tau$  if  $\sigma$  and  $\tau$  are uniformly integrable types on  $L_1(\mathcal{Q}_p)$ .

We have represented a uniformly integrable type  $\sigma \in \mathcal{T}(L_1(\Omega: \mathcal{Q}_p))$  as

$$\sigma(g) = \int_\Omega \int_{\mathcal{Q}_p} \int_0^\infty [\|y+g(\omega)\|_p^p + \alpha^p]^{1/p} d\mu_{y,\omega}(\alpha) d\lambda_\omega(y) dP(\omega) \quad \text{for all}$$

$g \in L_1(\Omega: \mathcal{Q}_p)$ , where  $(\Omega, \mathcal{A}, P)$  is a probability space and for each  $\omega \in \Omega$ ,  $\{\mu_{y,\omega}\}$  is a family of measures on  $\mathbb{R}_+$  and  $\lambda_\omega$  a measure on  $\mathcal{Q}_p$ . To study convolution of types we need to consider the convolution of measures on  $\mathcal{Q}_p \times [0, \infty)$ , and hence convolution in  $\mathcal{Q}_p \times [0, \infty)$ .

Let  $(y, \alpha), (z, \beta) \in \mathcal{Q}_p \times [0, \infty)$ . Define  $*$  by  $(y, \alpha) * (z, \beta) = (y+z, (\alpha^p + \beta^p)^{1/p})$ . Then  $(y, \alpha) * (z, \beta) \in \mathcal{Q}_p \times [0, \infty)$ .

Lemma 2.5.1

Let  $\tau$  and  $\sigma$  be types on  $\mathcal{Q}_p$ , so that there exists  $y, z \in \mathcal{Q}_p$  and  $\alpha, \beta \in \mathbb{R}_+$  such that  $\sigma(x) = [\|x+y\|^p + \alpha^p]^{1/p}$ ,  $\tau(x) = [\|x+z\|^p + \beta^p]^{1/p}$ . Let  $T: \mathcal{T}(\mathcal{Q}_p) \longrightarrow \mathcal{Q}_p \times \mathbb{R}_+$  be the map  $\sigma \longmapsto (y, \alpha)$ . Then  $T(\tau * \sigma) = T\tau * T\sigma$ .

Let  $f \in \mathcal{C}_b(\mathcal{Q}_p \times \mathbb{R}_+)$ . Then the map  $(s, t) \longmapsto f(s * t)$  is measurable and  $fT \in \mathcal{C}_b(\mathcal{T}(\mathcal{Q}_p))$ .

Proof

Suppose  $\sigma(x) = \lim_n \|x+y_n\|$  and  $\tau(x) = \lim_m \|x+z_m\|$ , say. Then  $(\sigma * \tau)(x) = \lim_m \lim_n \|x+y_n+z_m\| = \lim_m [\|x+y+z_m\|^p + \alpha^p]^{1/p} = [\|x+y+z\|^p + \alpha^p + \beta^p]^{1/p}$ . Thus  $T(\tau * \sigma) = (y+z, (\alpha^p + \beta^p)^{1/p}) = T\tau * T\sigma$  as required.

The maps  $+: \mathcal{Q}_p \times \mathcal{Q}_p \longrightarrow \mathcal{Q}_p$  and  $+: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$  are measurable, so the map  $*$  on  $(\mathcal{Q}_p \times \mathbb{R}_+) \times (\mathcal{Q}_p \times \mathbb{R}_+)$  is measurable.  $f$  is Borel measurable, thus  $(s, t) \longmapsto f(s * t)$  is measurable, and since  $T$  is measurable  $fT \in \mathcal{C}_b(\mathcal{T}(\mathcal{Q}_p))$ .

Suppose  $\mu$  and  $\nu$  are measures on  $\mathcal{T}(\mathcal{Q}_p)$ , then  $\mu T^{-1}$  and  $\nu T^{-1}$  are measures on  $\mathcal{Q}_p \times \mathbb{R}_+$ . We can now define  $\mu T^{-1} * \nu T^{-1}$  as before so that  $(\mu * \nu) T^{-1} = \mu T^{-1} * \nu T^{-1}$ . Then we have  $\forall f \in \mathcal{C}_b(\mathcal{Q}_p \times [0, \infty))$

$$\int_{\mathcal{Q}_p \times \mathbb{R}_+} \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(s * t) d\mu T^{-1}(s) d\nu T^{-1}(t) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(u) d(\mu T^{-1} * \nu T^{-1})(u).$$

Lemma 2.5.2

Let  $f \in \mathcal{C}_b(\mathcal{Q}_p \times [0, \infty))$  and  $T$  be as above. Let  $g(t) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(s * t) d\mu T^{-1}(s)$ . Then  $g$  is a Borel measurable map from  $\mathcal{Q}_p \times \mathbb{R}_+$  to  $\mathbb{R}$ .

### Proof

The map  $r:(s,t) \mapsto f(s * t)$  is measurable. If  $r = 1_{E \times F}$  with  $E \times F \in \mathcal{A}_0$ , then  $g(t) = 1_F(t) \mu T^{-1}(E)$  thus  $g$  is measurable. Next suppose that  $r = \sum_{i=1}^n a_i 1_{E_i \times F_i}$ , where the  $a_i$ 's are distinct and the  $E_i$ 's and  $F_i$ 's both disjoint. Then  $g(t) = \sum a_i 1_{F_i}(t) \mu T^{-1}(E_i)$ , which again is measurable. Finally suppose  $r = \lim_i r_i$ , with  $r_i$  are finite-valued simple functions with  $|r_i| \leq |r| \forall i$ .  $r$  is Borel measurable thus  $|r|$  is integrable and we can apply the Dominated Convergence Theorem. Then  $g(t) = \lim_i \int r_i$ , so that  $g$  is measurable as a limit of a sequence of measurable functions.

### Proposition 2.5.3

Let  $f \in \mathcal{C}_b(\mathcal{Q}_p \times [0, \infty))$  and  $T$  be as above. Then

$$\int_{\mathcal{Q}_p \times \mathbb{R}_+} \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(s * t) d\mu T^{-1}(s) dv T^{-1}(t) = \int_{\mathcal{T}(\mathcal{Q}_p)} \int_{\mathcal{T}(\mathcal{Q}_p)} f(Ts * Tt) d\mu(s) dv(t)$$

### Proof

Since  $f$  is Borel measurable,  $\int_{\mathcal{Q}_p \times \mathbb{R}_+} f(s * t) d\mu T^{-1}(s) = \int_{\mathcal{T}(\mathcal{Q}_p)} f(Ts * t) d\mu(s) = g(t)$ . Then from the above lemma,  $g$  is a

Borel measurable function, so that

$$\int_{\mathcal{Q}_p \times \mathbb{R}_+} g(t) dv T^{-1}(t) = \int_{\mathcal{T}(\mathcal{Q}_p)} g(Tt) dv(t) = \int_{\mathcal{T}(\mathcal{Q}_p)} \int_{\mathcal{T}(\mathcal{Q}_p)} f(Ts * Tt) d\mu(s) dv(t).$$

### Theorem 2.5.4

Let  $\mu$  and  $\nu$  be probabilities on  $\mathcal{T}(\mathcal{Q}_p)$ . Let  $T$  be as above. Then  $(\mu * \nu) T^{-1} = \mu T^{-1} * \nu T^{-1}$ .

### Proof

Since  $Ts * Tt = T(s * t)$  and  $fT \in \mathcal{C}_b(\mathcal{T}(\mathcal{Q}_p))$  whenever  $f \in \mathcal{C}_b(\mathcal{Q}_p \times [0, \infty))$  we have

$$\begin{aligned}
& \int_{\mathcal{T}(\mathcal{Q}_p)} \int_{\mathcal{T}(\mathcal{Q}_p)} f(Ts * Tt) d\mu(s) dv(t) = \\
& \int_{\mathcal{T}(\mathcal{Q}_p)} \int_{\mathcal{T}(\mathcal{Q}_p)} f(T(s * t)) d\mu(s) dv(t) \\
& = \int_{\mathcal{T}(\mathcal{Q}_p)} f(Tu) d(\mu * \nu)(u) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(u) d(\mu T^{-1} * \nu T^{-1})(u). \text{ Therefore} \\
& \forall f \in \mathcal{C}_b(\mathcal{Q}_p \times [0, \infty)) \quad \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(u) d(\mu T^{-1} * \nu T^{-1})(u) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(u) d(\mu * \nu) T^{-1}(u), \\
& \text{and hence } (\mu * \nu) T^{-1} = \mu T^{-1} * \nu T^{-1}.
\end{aligned}$$

We now find an explicit representation of  $\tau * \sigma$ , where  $\tau$  and  $\sigma$  are types on  $L_1(\mathcal{Q}_p)$ .

Let  $\tau$  and  $\sigma$  be uniformly integrable types on  $L_1(\Omega; \mathcal{Q}_p)$  represented by random measures  $\xi, \eta$  respectively. Then  $\sigma$  and  $\tau$  can be represented as

$$\begin{aligned}
\sigma(g) &= \int_{\Omega} \int_{\mathcal{Q}_p} \int_0^{\infty} [\|y + g(\omega)\|^p + \alpha^p]^{1/p} d\mu_{y, \omega}(\alpha) d\lambda_{\omega}(y) dP(\omega) \text{ and} \\
\tau(g) &= \int_{\Omega} \int_{\mathcal{Q}_p} \int_0^{\infty} [\|z + g(\omega)\|^p + \beta^p]^{1/p} dv_{z, \omega}(\beta) d\varepsilon_{\omega}(z) dP(\omega), \text{ where for} \\
& \text{each } \omega \in \Omega, \{\mu_{y, \omega}\} \text{ and } \{v_{z, \omega}\} \text{ are families of measures on } \mathbb{R}_+, \\
& \text{and } \lambda_{\omega}, \varepsilon_{\omega} \text{ are measures on } \mathcal{Q}_p. \text{ Then if } \sigma * \tau \text{ is represented by} \\
& \xi * \eta, \text{ we have } (\sigma * \tau)(g) = \int_{\Omega} \int_{\mathcal{T}(\mathcal{Q}_p)} \rho[g(\omega)] d(\xi * \eta)_{\omega}(\rho) dP(\omega) \\
& = \int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} \{\|u + g(\omega)\|^p + \gamma^p\}^{1/p} d(\xi * \eta)_{\omega} T^{-1}(u, \gamma) dP(\omega) = \\
& \int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} \{\|u + g(\omega)\|^p + \gamma^p\}^{1/p} d(\xi_{\omega}^0 * \eta_{\omega}^0)(u, \gamma) dP(\omega) = \\
& \int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} \int_{\mathcal{Q}_p \times \mathbb{R}_+} \{\|y + z + g(\omega)\|^p + \alpha^p + \beta^p\}^{1/p} d\xi_{\omega}^0(y, \alpha) d\eta_{\omega}^0(z, \beta) dP(\omega).
\end{aligned}$$

From the above lemma, the map

$$g: (z, \beta) \longmapsto \int_{\mathcal{Q}_p \times \mathbb{R}_+} \{\|y + z + g(\omega)\|^p + \alpha^p + \beta^p\}^{1/p} d\xi_{\omega}^0(y, \alpha) \text{ is measurable}$$

(and non-negative). Thus  $(\sigma * \tau)(g) =$

$$\int_{\Omega} \int_{\mathcal{Q}_p} \int_0^{\infty} \int_{\mathcal{Q}_p} \int_0^{\infty} \{\|y + z + g(\omega)\|^p + \alpha^p + \beta^p\}^{1/p} d\mu_{y, \omega}^{\omega}(\alpha) d\lambda_{\omega}(y) dv_{z, \omega}^{\omega}(\beta) d\varepsilon_{\omega}(z) dP.$$

If  $p=1$ , then let

$$I_1 = \int_{\Omega} \int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} \int_0^{\infty} \int_0^{\infty} \|y+z+g(\omega)\|_1 d\mu_y^{\omega}(\alpha) dv_z^{\omega}(\beta) d\lambda_{\omega}(y) d\varepsilon_{\omega}(z) dP(\omega).$$

$$\begin{aligned} \text{Let } I_2 &= \int_{\Omega} \int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} \int_0^{\infty} \int_0^{\infty} (\alpha+\beta) d\mu_y^{\omega}(\alpha) d\lambda_{\omega}(y) dv_z^{\omega}(\beta) d\varepsilon_{\omega}(z) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} \int_0^{\infty} \gamma d(\mu_y^{\omega} * v_z^{\omega})(\gamma) d\lambda_{\omega}(y) d\varepsilon_{\omega}(z) dP(\omega) \end{aligned}$$

If we define convolution on  $\mathcal{Q}_1$  by  $y*z=y+z$ , then we can

define convolution in  $\mathcal{P}(\mathcal{Q}_1)$  so that  $\forall f \in \mathcal{C}_b(\mathcal{Q}_1)$

$$\int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} f(y*z) d\mu(y) dv(z) = \int_{\mathcal{Q}_1} f(u) d(\mu*v)(u). \text{ Then}$$

$$I_1 = \int_{\Omega} \int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} \int_0^{\infty} \|y+z+g(\omega)\| d(\mu_y * v_z)_{\omega}(\gamma) d\lambda_{\omega}(y) d\varepsilon_{\omega}(z) dP(\omega). \text{ So that}$$

$$\begin{aligned} (\sigma*\tau)(g) &= \int_{\Omega} \int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} \int_0^{\infty} \|y+z+g(\omega)\| d(\mu_y * v_z)_{\omega}(\gamma) d\lambda_{\omega}(y) d\varepsilon_{\omega}(z) dP(\omega) \\ &\quad \int_{\Omega} \int_{\mathcal{Q}_1} \int_{\mathcal{Q}_1} \int_0^{\infty} \gamma d(\mu_y^{\omega} * v_z^{\omega})(\gamma) d\lambda_{\omega}(y) d\varepsilon_{\omega}(z) dP(\omega) \end{aligned}$$

## 2.6 The map Q between types and random measures.

Here we shall look at the structure of the map Q between  $\pi_1(\mathcal{T}(X))$  and  $\mathcal{T}(L_1(X))$ . In particular we show that the concept of symmetry is preserved by the mapping, as are the classes of  $\mathcal{Q}_p$ -types and  $\mathcal{Q}_p$ -random measures; while we show by example that  $\tau \in \mathcal{T}(L_1(\mathcal{Q}_p))$  will not necessarily be represented by a random measure  $\xi \in \pi_1(\mathcal{T}(\mathcal{Q}_p))$ . Unless otherwise stated X is a separable Banach space.

Proposition 2.6.1

Let  $\tau \in \mathcal{T}(L_1(\Omega: X))$  be uniformly integrable and  $\xi \in \pi_1(\mathcal{T}(X))$ . Suppose that random measures representing uniformly integrable types are uniquely determined. Then  $\tau$  is represented by  $\xi$  iff  $\alpha \cdot \tau$  is represented by  $D_\alpha \xi$ , and  $\tau$  represents  $\xi$  iff  $\alpha \cdot \tau$  represents  $D_\alpha \xi$ . In particular  $\tau$  is symmetric iff  $\xi$  is symmetric.

Proof

Suppose that  $\forall g \in L_1(\Omega: X)$   $\tau(g) = \int_{\Omega} \int_{\mathcal{T}(X)} t(g(\omega)) d\xi_\omega(t) dP(\omega)$ , then  $\alpha \cdot \tau(g) = |\alpha| \tau(g/\alpha) = \int_{\Omega} \int_{\mathcal{T}(X)} \alpha \cdot t[g(\omega)] d\xi_\omega(t) dP(\omega)$ .

The map  $D_\alpha : \mathcal{T}(X) \longrightarrow \mathcal{T}(X)$  is a homeomorphism with inverse  $\tau \longmapsto \alpha \cdot \tau$

$D_{1/\alpha}$ . It is measurable since if  $\tau_n \longrightarrow \tau$  then for  $\alpha \neq 0$ ,

$|\alpha| \tau_n(x/\alpha) \longrightarrow |\alpha| \tau(x/\alpha)$  so that  $A$  closed implies  $D_\alpha A$  closed.

Define  $D_\alpha \xi$  by  $(D_\alpha \xi)_\omega = D_\alpha \xi_\omega$  with  $(D_\alpha \mu)(E) = \mu(D_\alpha^{-1} E)$  for  $\mu \in \mathcal{P}(\mathcal{T}(X))$  and  $E \subset \mathcal{T}(X)$ . So that

$$\int_{\mathcal{T}(X)} \alpha \cdot t[g(\omega)] d\xi_\omega(t) = \int_{\mathcal{T}(X)} t[g(\omega)] d(D_\alpha \xi)_\omega(t)$$

thus  $\alpha \cdot \tau(g) = \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d(D_\alpha \xi)_\omega(t) dP(\omega)$  so that uniqueness

implies that  $\alpha \cdot \tau$  is represented by  $D_\alpha \xi$ . If  $\tau(g) = \tau(-g)$

$\forall g \in L_1(\Omega: X)$  then  $\tau$  is represented by  $\xi = D_1 \xi$ , thus  $\xi$  is

symmetric.

Conversely if  $\xi$  has associated type  $\tau$  then  $D_\alpha \xi$  has associated type  $|\alpha| \tau(\bullet/\alpha) = D_\alpha \tau(\bullet)$ . Thus if  $\xi$  is symmetric then  $\tau$  is symmetric.

Example

The uniqueness condition is necessary. For consider  $L_1(\mathcal{Q}_1)$ . Let  $\xi_\omega = \xi$  be given by  $\xi = 1/2[\delta(e_1, 0) + \delta(-e_1, 1)]$ . Then  $\tau(g) = \int_{\Omega} \int_{\mathcal{Q}_1 \times \mathbb{R}_+} \{\|g(\omega) + y\| + \alpha\} d\xi_\omega(y, \alpha) dP(\omega) = E_\omega \{\|g(\omega) + e_1\| + \|g(\omega) - e_1\| + 1\} / 2 = \tau(-g)$ . Hence  $\tau \in \mathcal{S}(L_1(\mathcal{Q}_1))$ . Now  $\xi$  is symmetric iff

$\forall E \subseteq \mathcal{Q}_1 \times [0, \infty) \quad \xi(E) = \xi(D_1 E)$ , but

$$\xi(E) = \begin{cases} 1 & \text{if } (e_1, 0), (-e_1, 1) \in E \\ 1/2 & \text{if either } (e_1, 0) \text{ or } (-e_1, 1) \in E \text{ but not both,} \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\xi(D_1 E) = \begin{cases} 1 & \text{if } (-e_1, 0), (e_1, 1) \in E \\ 1/2 & \text{if either } (-e_1, 0) \text{ or } (e_1, 1) \in E \text{ but not both.} \\ 0 & \text{otherwise} \end{cases}$$

Hence  $\xi \neq D_1 \xi$ , by taking for example  $E = \{(e_1, 0)\}$ .

Although we can find a symmetric random measure representing  $\tau$ , for example use  $\xi = 1/2[\delta(e_1, 1/2) + \delta(-e_1, 1/2)]$ .

Lemma 2.6.2

Suppose  $\xi$  is a random measure on the symmetric types, that is  $\xi_\omega(\mathcal{S}(X)^c) = 0$  a.e.  $[P]$  where  $\mathcal{S}(X)$  is the set of symmetric types. If  $\xi$  represents a uniformly integrable type  $\tau$ , then  $\tau$  is symmetric.

Proof

$$\begin{aligned} \text{By hypothesis } \forall g \in L_1(\Omega; X) \quad \tau(g) &= \int_{\Omega} \int_{\mathcal{S}(X)} t(g(\omega)) \cdot d\xi_\omega(t) dP(\omega) = \\ \int_{\Omega} \int_{\mathcal{S}(X)} t(g(\omega)) \cdot d\xi_\omega(t) dP(\omega) &= \int_{\Omega} \int_{\mathcal{S}(X)} t(-g(\omega)) d\xi_\omega(t) dP(\omega) = \\ \int_{\Omega} \int_{\mathcal{S}(X)} t(-g(\omega)) d\xi_\omega(t) dP(\omega) &= \tau(-g). \end{aligned}$$

Lemma 2.6.3

Let  $X$  be stable. Suppose that random measures representing uniformly integrable types are uniquely determined, then uniformly integrable  $\mathcal{Q}_p$ -types on  $\mathcal{S}(L_1(\Omega; X))$  correspond to symmetric  $\mathcal{Q}_p$ -random measures.

Proof

Let  $\tau \in \mathcal{T}(L_1(\Omega: X))$  be a uniformly integrable  $\mathcal{Q}_p$ -type. Then  $\forall \alpha, \beta \in \mathbb{R}_+$ ,  $D_\alpha \tau * D_\beta \tau = D_{\gamma(\alpha, \beta)} \tau$  where  $\gamma(\alpha, \beta) = (\alpha^p + \beta^p)^{1/p}$ . We also know that  $\forall g \in L_1(\Omega: X)$

$$(D_\alpha \tau * D_\beta \tau)(g) = \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d(D_\alpha \xi * D_\beta \xi)_\omega(t) dP(\omega)$$

and  $D_\gamma \tau(g) = \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d(D_\gamma \xi)_\omega(t) dP(\omega)$ .

So by uniqueness,  $D_\alpha \xi * D_\beta \xi = D_{\gamma(\alpha, \beta)} \xi$  as required.

Proposition 2.6.4

Let  $\tau$  be a uniformly integrable type on  $L_1(\Omega: \mathcal{Q}_p)$  represented by the random measure  $\xi' \in \pi_1(\mathcal{T}(\mathcal{Q}_p))$ , which has a corresponding  $\xi \in \pi_1(\mathcal{Q}_p \times \mathbb{R}_+)$ . Suppose that random measures representing uniformly integrable types are uniquely determined. Let  $\xi^y, \xi^z$  be the disintegration of  $\xi$  into probabilities on  $[0, \infty)$  and  $\mathcal{Q}_p$  respectively. Then  $\tau$  is symmetric iff  $\xi^z$  is symmetric and  $\xi^y = \xi^{-y}$  a.e.  $[\xi^z]$ .

Proof

$$\begin{aligned} \forall g \in L_1(\Omega: \mathcal{Q}_p), \quad \tau(g) &= \int_{\Omega} \int_{\mathcal{T}(\mathcal{Q}_p)} t(g(\omega)) d\xi'_\omega(t) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} (\|g(\omega) + y\|^p + \beta^p)^{1/p} d\xi_\omega(y, \beta) dP(\omega) \end{aligned}$$

and  $\tau$  is symmetric iff  $\xi'$  is symmetric, i.e.  $D_{-1} \xi' = \xi'$ .

Then  $D_{-1} t(g(\omega)) = (\|g(\omega) - y\|^p + \beta^p)^{1/p}$ . Let  $T$  be the map decomposing  $t$  into  $(y, \beta) \in \mathcal{Q}_p \times [0, \infty)$ . So that  $\xi = \xi' \circ T^{-1}$ . Then  $\xi'$  is symmetric iff  $\forall F \subseteq \mathcal{T}(\mathcal{Q}_p)$ ,  $\xi'(F) = D_{-1} \xi'(F) = \xi'(D_{-1} F)$ . There exists  $E \subseteq \mathcal{Q}_p \times [0, \infty)$  with  $T^{-1} E = F$ , so that  $\xi'$  is symmetric iff  $\xi' T^{-1}(E) = D_{-1} \xi' T^{-1}(E) = \xi'(T^{-1} D_{-1} E)$ , where  $D_{-1} E = \{(-y, \beta) : (y, \beta) \in E\}$ .

Thus  $\xi'$  is symmetric iff  $\xi$  is symmetric in  $\mathcal{Q}_p$ .

Disintegrate  $\xi$  into  $\xi_y$  and  $\xi_2$ . Then for  $\xi$ -measurable,

$$\text{integrable } f \int_{\mathcal{Q}_p \times \mathbb{R}_+} f(y, \beta) \cdot d\xi_\omega(y, \beta) = \int_{\mathcal{Q}_p} \int_{\mathbb{R}_+} f(y, \beta) \cdot d\xi_y^\omega(\beta) \cdot d\xi_2^\omega(y)$$

a.e. [P].

Suppose  $\xi'$  is symmetric, then  $\forall g$

$$\int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} (\|g(\omega) + y\|^p + \beta^p)^{1/p} \cdot d\xi_\omega(y, \beta) \cdot dP(\omega) \\ = \int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} (\|g(\omega) - y\|^p + \beta^p)^{1/p} \cdot d\xi_\omega(y, \beta) \cdot dP(\omega)$$

$$\text{so that } \int_{\Omega} \int_{\mathcal{Q}_p} \int_0^\infty (\|g(\omega) + y\|^p + \beta^p)^{1/p} \cdot d\xi_y^\omega(\beta) d\xi_2^\omega(y) \cdot dP(\omega) \\ = \int_{\Omega} \int_{\mathcal{Q}_p} \int_0^\infty (\|g(\omega) - y\|^p + \beta^p)^{1/p} \cdot d\xi_y^\omega(\beta) d\xi_2^\omega(y) \cdot dP(\omega) \\ = \int_{\Omega} \int_{\mathcal{Q}_p} \int_0^\infty (\|g(\omega) + y\|^p + \beta^p)^{1/p} \cdot d\xi_{-y}^\omega(\beta) d\eta_2^\omega(y) \cdot dP(\omega)$$

where  $\eta_2(G) = \xi_2(-G)$ .

Uniqueness of random measures implies that  $\eta_2 = \xi_2$  and that  $\xi_{-y} = \xi_y$ . That is  $\xi_2$  is symmetric and  $\xi_y = \xi_{-y}$  a.e. [ $\xi_2$ ].

### Example

There are symmetric types on  $L_1(\mathcal{Q}_p)$  which have representing random measures that are non-zero on the non-symmetric types.

Let  $\xi_\omega = (\delta_{(e_1, 0)} + \delta_{(-e_1, 0)})/2$  for every  $\omega \in \Omega$ . The symmetric types on  $\mathcal{Q}_p$  are of the form  $t(x) = (\|x\|^p + \alpha^p)^{1/p}$ , so that  $\xi_\omega$  has non-zero measure on the non-symmetric types. But  $\tau$  is symmetric since  $\forall g \in L_1(\Omega; \mathcal{Q}_p)$

$$\tau(g) = E_\omega [\|g(\omega) + e_1\| + \|g(\omega) - e_1\|] / 2 = \tau(-g).$$

## Part 2 : Representations of $\mathcal{T}(X)$ .

### 2.7 Representation of $\mathcal{T}(X)$ as $X*\mathcal{Y}(X)$ .

In many situations the symmetric types have a convenient representation, for example a symmetric type on  $\mathcal{L}_p$  is of the form  $\sigma(x) = \{\|x\|^p + \alpha^p\}^{1/p}$  for some  $\alpha \in [0, \infty)$ , and we consider now possible decompositions of  $\mathcal{T}(X)$  in terms of  $\mathcal{Y}(X)$ . In particular we find conditions on a stable Banach space  $X$  to enable us to write  $\mathcal{T}(X) = X*\mathcal{Y}(X)$ , that is if  $\tau \in \mathcal{T}(X)$ , then there exists  $y \in X$ ,  $\sigma \in \mathcal{Y}(X)$  with  $\tau = \tau_y * \sigma$ .

This of course is the case in  $\mathcal{T}(\mathcal{L}_p)$ , where we have for a general type  $\tau$ ,  $\tau(x) = (\|x+y\|^p + \alpha^p)^{1/p} = (\tau_y * \sigma)(x)$  where  $\tau_y(x) = \|x+y\|$  and  $\sigma$  is as above. We shall find conditions for the decomposition of  $\mathcal{T}(X)$  as  $X*\mathcal{Y}(X)$  and  $X*\mathcal{T}_{wn}(X)$  where  $\mathcal{T}_{wn}(X)$  is the set of weakly null types on  $X$  (to be defined later).

#### Definition

1. A basis  $\{x_n\}_1^\infty$  of  $X$  is a sequence such that if  $x \in X$ , there exists unique scalars  $a_n$  with  $x = \sum a_n x_n$  (norm convergence).
2. A basic sequence is a sequence which is a basis of its closed linear span.
3. An unconditional basis is a basis  $\{x_n\}$  such that for every  $x \in X$ , its expansion in terms of the basis  $\sum a_n x_n$  converges unconditionally, that is convergence of  $\sum_1^\infty a_n x_n$  implies convergence of  $\sum_{n \in \sigma} a_n x_n$  for all subsets  $\sigma$  of  $\mathbb{N}$ .
4. Let  $\{x_n\}$  be a basic sequence in  $X$ . Let  $\{u_j\}$  be non-zero of the form  $u_j = \sum_{n \in \sigma_j} a_n x_n$ , with scalars  $a_n$  and integers  $p_1 < p_2 < \dots$ , and  $\sigma_j = \{p_j+1, \dots, p_{j+1}\}$ . Then  $\{u_j\}$  is a block basic sequence of the  $\{x_n\}$ .

### Lemma 2.7.1

A Banach space  $X$  with unconditional basis  $\{x_n\}$  is weakly sequentially complete iff  $\{x_n\}$  is boundedly complete. Since stable spaces are weakly sequentially complete [Gu&La1], a stable Banach space cannot contain a copy of  $c_0$ .

The basis  $\{x_n\}$  is shrinking iff every block basis  $\{a_n\}$  is weakly null, which happens iff  $\mathcal{Q}_1$  does not imbed in  $X$ .  $X$  is reflexive iff  $c_0 \not\hookrightarrow X$  and  $\mathcal{Q}_1 \not\hookrightarrow X$ , hence  $X$  is reflexive iff  $\{x_n\}$  is shrinking and boundedly complete.

### Lemma 2.7.2

Let  $X$  be a Banach space with an unconditional basis  $(e_i)_i$ . Suppose that  $X$  does not contain any subspace isomorphic to  $c_0$ . Let  $(x_n)$  be a bounded sequence in  $X$ , then there exists a subsequence  $(x_{n_k})_k$  of  $(x_n)$  such that  $x_{n_k} = y_k + w_k$  with  $y_k \wedge w_k = 0$ ,  $(y_k)$  converging in  $X$  and  $(w_k)$  being a block basic sequence.

See J. Bastero [Ba].

### Definition

A type is weakly-null if it can be defined by a weakly convergent to zero sequence. Let  $\mathcal{T}_{wn}(X)$  be the space of weakly-null types.

A type,  $\tau$ , is a block type wrt a basic sequence  $\underline{x}$  if there is a block subsequence  $\underline{y}$  of  $\underline{x}$ , such that  $\tau$  can be defined by  $\underline{y}$ , i.e.  $\tau(z) = \lim_n \|z + y_n\| \quad \forall z \in X$ .

null type is a block type wrt  $(x_n)$ , that is  $\mathcal{T}_{wn}(X) \subseteq \mathcal{T}_b(X)$ .

Proof

If  $\tau \in \mathcal{T}_{wn}(X)$  is defined by  $(y_k)$ , then  $y_k \rightarrow 0$  weakly but  $\|y_k\| \not\rightarrow 0$ . Thus there exists a subsequence of  $\{y_k\}$  which is equivalent to a small perturbation of the block basis of  $\{x_n\}$ , hence  $\tau \in \mathcal{T}_b(X)$ .

Let  $X$  be a Banach space with an unconditional basis. We may give an asymptotic representation of the types on  $X$ . If  $\sigma$  is a type, given by a norm-bounded sequence  $(x_n)_n$ , then by 2.7.2 there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $x_{n_k} = y_k + a_k$  with  $y_k$  convergent to  $a$  in  $X$ , and  $a_n$  is a block basic sequence with  $\sigma(x) = \lim_U \|x + a + a_n\|$ .

Lemma 2.7.4

Suppose  $c_0 \hookrightarrow X$  and that  $X$  has an unconditional basis. If the constant of unconditionality is one and the basis is shrinking, then  $\mathcal{T}_b(X) = \mathcal{S}(X) = \mathcal{T}_{wn}(X)$ .

Proof

By 2.7.3, the weakly null types form a subset of the block basis types. By [Ba]  $\mathcal{T}_{wn}(X) = \mathcal{S}(X)$ , and so under this hypothesis  $\mathcal{T}_b(X) \subseteq \mathcal{S}(X) \cap \mathcal{T}_{wn}(X)$ . Hence symmetric types are precisely the weakly-null types which are exactly the block types wrt the basis  $(x_n)$  of  $X$ .

It is a theorem of H.P. Rosenthal that if  $Y \subseteq X$  is infinite dimensional and  $X$  is separable with  $\mathcal{Q}_1 \hookrightarrow X$ , then  $\mathcal{T}_{wn}(Y, X) \subseteq \mathcal{S}(Y, X)$  iff every normalized weakly null sequence in  $Y$  has a subsequence almost 1-unconditional over  $X$ , see [Ros1]. If  $\mathcal{T}_{wn}(X) \subseteq \mathcal{S}(X)$  then  $X$  contains an unconditional basic sequence.

Proposition 2.7.5

If  $X$  is a separable Banach space not containing  $\mathcal{Q}_1$ , then  $\mathcal{T}_{wn}(X)$  is a closed subspace of  $\mathcal{T}(X)$  (with the pointwise topology). Let  $\mathcal{S}_{wn}(X)$  be the set of weakly-null and symmetric types. Then  $\mathcal{T}_{wn}(X)$  and  $\mathcal{S}_{wn}(X)$  are locally compact,  $\sigma$ -compact spaces. See [A&N&Z].

Proposition 2.7.6

Let  $X$  be a stable Banach space with unconditional basis  $(e_i)$ . Then  $\mathcal{T}(X) = X * \mathcal{T}_b(X)$ , where  $\mathcal{T}_b(X) = \mathcal{T}_b(X; \underline{e})$  is the space of block types with respect to  $\underline{e} = (e_i)$ .

Proof

Since  $X$  is stable, it contains no copy of  $c_0$ . We can represent any type  $\sigma$  as  $\sigma(x) = \lim \|x + a + a_n\| = (\tau_a * \tau)(x)$ , where  $\tau_a(x) = \|x + a\|$  and  $\tau(x) = \lim \|x + a_n\|$ . Then  $\tau_a \in \mathcal{T}_X(X) \subseteq X$ , and  $\tau \in \mathcal{T}_b(X)$ .

Corollary 2.7.7

Let  $X$  be a reflexive stable space with unconditional basis (and hence the basis is shrinking). Then  $\mathcal{T}(X) = X * \mathcal{T}_{wn}(X)$ . If the unconditional constant of the basis is one, then  $\mathcal{S}(X) = \mathcal{T}_b(X)$ , and so  $\mathcal{T}(X) = X * \mathcal{S}(X) = X * \mathcal{T}_{wn}(X) = X * \mathcal{T}_b(X)$ .

Let  $1 < p < \infty$ . Then  $\mathcal{Q}_p$  and  $L_p$  have unconditional bases, and  $\mathcal{C}[0,1]$  has a basis but not an unconditional basis. If  $L_p([0,1], X)$  has an unconditional basis, then  $1 < p < \infty$  and  $X$  is superreflexive.  $L_p(\mu, X)$  contains  $c_0, \mathcal{Q}_1$  iff  $X$  contains  $c_0, \mathcal{Q}_1$  respectively. In the case of  $c_0$  this works for  $p=1$  as well.

See Diestel & Uhl, [Di&U2].

## 2.8 Types on $L_1(L_p)$ .

The representation of  $\mathcal{T}(X)$  as  $X \times \mathcal{P}(X)$  in the section 2.7 depended upon an unconditional basis with basis constant one. While  $L_p$  has an unconditional basis (for  $1 < p$ ), the constant is not one. Here we aim to produce similar representations for  $\mathcal{T}(L_{2k})$   $k \geq 1$ , and hence for  $\mathcal{T}(L_1(L_p))$  via random measures. These representations are found by considering weak limits of the defining sequence of the type in  $L_p$  and its subspaces. This is possible since in a reflexive Banach space every bounded sequence has a weakly convergent subsequence, [D&S].

By this method we gain representations of  $\mathcal{T}(L_2)$  as a subset of  $L_2 \times [0, \infty)$ , and  $\mathcal{T}(L_4)$  as a subset of  $L_4 \times L_2 \times L_{4/3} \times [0, \infty)$ , which can be reduced to  $L_4 \times [0, \infty)$  if the weak limits sit in  $L_\infty$ . We finish by applying these representations to the study of  $\mathcal{T}(L_1(L_{2k}))$ , looking at the form of symmetric types and convolution.

### Representing $\mathcal{T}(L_2)$ as $L_2 \times [0, \infty)$ .

We can represent  $\mathcal{T}(L_2)$  as a subset of  $L_2 \times [0, \infty)$ . For let  $\tau \in \mathcal{T}(L_2)$ , then  $\forall f \in L_2$   $\tau(f)^2 = \lim \|f + g_n\|^2 = \|f\|^2 + \lim \|g_n\|^2 + 2 \lim \langle f, g_n \rangle = \|f\|^2 + \tau(0)^2 + 2 \langle f, g \rangle$ , if  $g = (w)\text{-}\lim_n g_n$ . Hence  $\tau(f) = [\|f\|^2 + 2 \langle f, g \rangle + \tau(0)^2]^{1/2}$ , so that  $\tau$  can be represented as  $(g, \alpha) \in L_2 \times [0, \infty)$ , where  $\alpha = \tau(0)$ .

Suppose  $\tau, \sigma \in \mathcal{T}(L_2)$  have representations

$$\tau(f) = \lim \|f + g_n\| = [\|f\|^2 + 2\langle f, g \rangle + \tau(0)^2]^{1/2} \text{ and}$$

$$\sigma(f) = \lim \|f + h_m\| = [\|f\|^2 + 2\langle f, h \rangle + \sigma(0)^2]^{1/2}. \text{ Then}$$

$$(\tau * \sigma)(f)^2 = \lim_n \lim_m \|f + g_n + h_m\|^2 =$$

$$\lim_{n, m} [\|f\|^2 + \|g_n\|^2 + \|h_m\|^2 + 2\langle h_m, g_n \rangle + 2\langle f, g_n \rangle + 2\langle f, h_m \rangle]$$

$$= [\|f\|^2 + 2\langle f, g+h \rangle + \tau(0)^2 + 2\langle h, g \rangle + \sigma(0)^2], \text{ and}$$

$$(\tau * \sigma)(0)^2 = \lim_n \lim_m \|g_n + h_m\|^2 =$$

$$\lim_{n, m} [\|g_n\|^2 + \|h_m\|^2 + 2\langle h_m, g_n \rangle] = [\tau(0)^2 + 2\langle h, g \rangle + \sigma(0)^2], \text{ so that}$$

$$(\tau * \sigma)(f)^2 = [\|f\|^2 + 2\langle f, g+h \rangle + (\tau * \sigma)(0)^2]. \text{ Thus convolution in}$$

$$L_2 \times [0, \infty) \text{ becomes } (g, \alpha) * (h, \beta) = (g+h, [\alpha^2 + \beta^2 + 2\langle g, h \rangle]^{1/2}).$$

There is a further representation of  $\mathcal{T}(L_2)$  as  $L_2 \times [0, \infty)$  similar to the representation of  $\mathcal{T}(\mathfrak{L}_2)$  as  $\mathfrak{L}_2 \times [0, \infty)$ . Let

$\tau(f) = \lim \|f + g_n\|$ , and  $g$  be the weak limit of  $\{g_n\}$ . Then

$$\tau(f) = \lim_n \|f + g + g_n - g\| = [\|f + g\|^2 + \lim \|g_n - g\|^2 + 2\lim \langle f + g, g_n - g \rangle]^{1/2}$$

$$= [\|f + g\|^2 + \gamma^2]^{1/2}. \text{ Then } \gamma^2 = \|f\|^2 - \|f + g\|^2 + 2\langle f, g \rangle + \tau(0)^2$$

$= \tau(0)^2 - \|g\|^2 = \alpha^2 - \|g\|^2 = \lim \|g_n - g\|^2 = \lim \|g_n\|^2 - \|g\|^2$ . Thus there is a map between the representations given by

$$T: (g, \alpha) \longrightarrow (g, \alpha^2 - \|g\|^2).$$

Convolution in this new representation is given as

before, viz:  $(\tau * \sigma)(f) = \lim_n \lim_m \|f + g_n + h_m\| = \lim_n [\|f + g_n + h\|^2 + \delta^2]^{1/2}$

$= [\|f + g + h\|^2 + \gamma^2 + \delta^2]^{1/2}$ . That is in  $L_2 \times [0, \infty)$ ,

$$(g, \gamma) * (h, \delta) = (g+h, [\gamma^2 + \delta^2]^{1/2})$$

Representations of  $\mathcal{T}(L_4)$ .

For a uniformly integrable type  $\tau$  on  $L_1(L_4)$  we can represent it by  $\xi \in \pi_1(\mathcal{T}(L_4))$ , so that for  $g \in L_1(L_4)$ ,  
 $\tau(g) = \int_{\Omega} \int_{\mathcal{T}(L_4)} t[g(\omega)] d\xi_{\omega}(t) dP(\omega)$ . For  $t \in \mathcal{T}(L_4)$ ,  $\exists \eta \in \pi_4(\mathbb{R})$  and  $\beta \geq 0$   
with  $\forall h \in L_4(\Sigma, m)$   $t(h) = \left\{ \int_{\Sigma} \int_{\mathbb{R}} |s+h(u)|^4 d\eta_u(s) dm(u) + \beta^4 \right\}^{1/4}$ .

Instead of the representation of  $\mathcal{T}(L_4)$  as  $\pi_4(\mathbb{R}) \times [0, \infty)$ , we derive a representation of  $\mathcal{T}(L_4)$  as a subset of  $L_4 \times L_2 \times L_4, 3 \times [0, \infty)$ .

Proposition 2.8.1

There are representations identifying  $\mathcal{S}(L_4)$  with a subset of  $L_2^+ \times [0, \infty)$ , and  $\mathcal{T}(L_4)$  with a subset of  $L_4 \times L_2 \times L_4, 3 \times [0, \infty)$ .

Proof

Let  $\tau \in \mathcal{S}(L_4)$  with  $\tau(f) = \lim \|f + g_n\|_4 = \lim \|f - g_n\|_4$ . So that  
 $\forall f \in L_4$   $\tau(f)^4 = \lim 1/2 [\|f + g_n\|_4^4 + \|f - g_n\|_4^4] = \frac{1}{2} \lim \int \{ |f + g_n|^4 + |f - g_n|^4 \} =$   
 $\lim \int \{ f^4 + 6f^2 g_n^2 + g_n^4 \} = \|f\|_4^4 + \lim \|g_n\|_4^4 + 6 \lim \langle f^2, g_n^2 \rangle$ . Now if we let  
 $u = (w) \lim g_n^2$ , then  $\langle f^2, g_n^2 \rangle \longrightarrow \langle f^2, u \rangle$  considered as a limit in  
 $L_2 \supseteq L_4$ . So  $\forall h \in L_2$ ,  $\langle h, g_n^2 \rangle \longrightarrow \langle h, u \rangle$ . Then as before  $\exists \alpha \in \mathbb{R}_+$  with  
 $\|g_n^2\|_2 \longrightarrow (\|u\|_2^2 + \alpha^2)^{1/2}$ , thus  $\|g_n\|_4^4 = \int |g_n|^4 = \int |(g_n)^2|^2 = \|g_n^2\|_2^2$   
 $\longrightarrow \|u\|_2^2 + \alpha^2$ . So that  $\tau(f) = [\|f\|_4^4 + \|u\|_2^2 + \alpha^2 + 6\langle f^2, u \rangle]^{1/4}$ . That is  
every symmetric type  $\tau$  can be represented by a pair  $(u, \alpha)$  in  
 $L_2^+ \times [0, \infty)$ .

Suppose now that  $\tau \in \mathcal{T}(L_4)$ . Then

$$\begin{aligned} \tau(f)^4 &= \lim_n \|f + g_n\|_4^4 = \lim \| (f + g_n)^2 \|_2^2 \\ &= \|f\|_4^4 + \lim_n \|g_n\|_4^4 + 6 \lim_n \langle f^2, g_n^2 \rangle + 4 \lim \langle f^3, g_n \rangle + 4 \lim \langle f, g_n^3 \rangle \\ &= \|f\|_4^4 + \|u\|_2^2 + \alpha^2 + 6 \langle f^2, u \rangle + 4 \langle f^3, u_1 \rangle + 4 \langle f, u_3 \rangle, \quad \text{where } u_2 \text{ is the} \\ &\text{weak limit in } L_2 \text{ of } g_n^2 \in L_2, u_1 \text{ is the weak limit in } L_4 \text{ of} \end{aligned}$$

$g_n \in L_4$ , and  $u_3$  is the weak limit in  $L_{4,3}$  of  $g_n^3 \in L_{4,3}$ . Thus  $\mathcal{T}(L_4) \subseteq L_4 \times L_2 \times L_{4,3} \times [0, \infty)$ .

The following lemma places a restriction on the subset that  $\mathcal{T}(L_4)$  is identified with, namely

$$\mathcal{T}(L_4) \subseteq \{(u_1, u_2, u_3, \alpha) \in L_4 \times L_2 \times L_{4,3} \times [0, \infty) : u_1^2 \leq u_2, u_2^2 \leq u_1 u_3\}.$$

Lemma 2.8.2

If  $(g_n)$  is a bounded sequence in  $L_4$  such that  $g_n \rightarrow g$  weakly in  $L_4$ ,  $g_n^2 \rightarrow h$  weakly in  $L_2$  and  $g_n^3 \rightarrow k$  weakly in  $L_{4,3}$ , then (a)  $g^2 \leq h$  and (b)  $h^2 \leq gk$ .

Proof

(a) For any measurable set  $A$ ,  $\int_A |g| = \int_A 1_A \cdot \text{sgn}(g) \cdot g =$

$$\lim_n \int_A 1_A \cdot \text{sgn}(g) \cdot g_n \leq \lim \left( \int_A 1_A \cdot g_n^2 \right)^{1/2} (P(A))^{1/2} =$$

$$\left( \int_A h \right)^{1/2} (P(A))^{1/2}. \text{ Thus } g^2 \leq h.$$

(b) For any  $A$ ,  $\int_A h^2 = \lim_m \lim_n \int_A g_m^2 g_n^2 \leq$

$$\lim_m \lim_n \left( \int_A g_m^3 g_n \right)^{1/2} \left( \int_A g_m g_n^3 \right)^{1/2} \longrightarrow \int_A gk.$$

We can now work out the structure that convolution takes in  $L_4 \times L_2 \times L_{4,3} \times [0, \infty)$ . For a symmetric type  $\sigma \in \mathcal{G}(L_4)$  we have the following.

Let  $\sigma(f) = \lim_n \|f + h_n\| = [\|f\|_4^4 + \|v\|_2^2 + \beta^2 + 6\langle f^2, v \rangle]^{1/4}$ , where  $v = (w) - \lim_n h_n^2$ . Then  $\tau * \sigma \in \mathcal{G}(L_4)$  and so can be represented as

$$(\tau * \sigma)(f) = [\|f\|_4^4 + \|w\|_2^2 + \tau^2 + 6\langle f^2, w \rangle]^{1/4}, \text{ since}$$

$$(\tau * \sigma)^4(f) = \lim_n \lim_m \|f + g_n + h_m\|^4 =$$

$$\|f\|^4 + \lim_{n,m} \|g_n + h_m\|^4 + 6 \lim_{n,m} \langle f^2, (g_n + h_m)^2 \rangle =$$

$$\|f\|^4 + \lim_{n,m} \|g_n^2 + 2g_n h_m + h_m^2\|_2^2 + 6 \lim_{n,m} \langle f^2, g_n^2 + 2g_n h_m + h_m^2 \rangle \xrightarrow{n,m} \\ \|f\|^4 + \|w\|_2^2 + \gamma^2 + 6 \langle f^2, w \rangle. \text{ Thus } w = (w) - \lim_n \lim_m (g_n + h_m)^2 \text{ and} \\ \gamma = \lim_n \lim_m \| (g_n + h_m)^2 \|_2^2 - \|w\|_2^2 = \lim_{n,m} \|g_n + h_m\|_4^4 - \|w\|_2^2.$$

Proposition 2.8.3

Let  $\tau$  and  $\sigma$  be types on  $L_4$  with representations  $\tau = (u_1, u_2, u_3, \alpha)$  and  $\sigma = (v_1, v_2, v_3, \beta)$  in  $L_4 \times L_2 \times L_{4,3} \times [0, \infty)$ . Then  $\tau * \sigma$  has the representation

$$(u_1 + v_1, u_2 + 2u_1 v_1 + v_2, u_3 + 3u_2 v_1 + 3u_1 v_2 + v_3, (\alpha^2 + \beta^2)^{1/2}) \text{ in } L_4 \times L_2 \times L_{4,3} \times [0, \infty).$$

Proof

Let  $\tau = (u_1, u_2, u_3, \alpha)$  and  $\sigma = (v_1, v_2, v_3, \beta)$ . Then

$$\begin{aligned} (\tau * \sigma)(f)^4 &= \lim_n \sigma(f + g_n)^4 = \\ \lim [ &\|f + g_n\|^4 + \|v_2\|_2^2 + \beta^2 + 6 \langle (f + g_n)^2, v_2 \rangle + 4 \langle (f + g_n)^3, v_1 \rangle + 4 \langle f + g_n, v_3 \rangle] \\ &= \tau(f)^4 + 4 \langle u_1, v_3 \rangle + 6 \langle u_2, v_2 \rangle + 12 \langle u_1, f v_2 \rangle + \|v_2\|_2^2 + \beta^2 + 6 \langle f^2, v_2 \rangle + \\ &\quad 4 \langle f^3, v_1 \rangle + 4 \langle f, v_3 \rangle + 4 \langle u_3, v_1 \rangle + 12 \langle u_1, f^2 v_1 \rangle + 12 \langle u_2, f v_1 \rangle \\ &= \|f\|^4 + \|v_2\|_2^2 + \beta^2 + \|u_2\|_2^2 + \alpha^2 + 6 \langle f^2, u_2 + 2u_1 v_1 + v_2 \rangle + 4 \langle f^3, u_1 + v_1 \rangle + \\ &\quad 4 \langle f, u_3 + 3u_2 v_1 + 3u_1 v_2 + v_3 \rangle + 4 \langle u_1, v_3 \rangle + 4 \langle u_3, v_1 \rangle + 6 \langle u_2, v_2 \rangle \\ &= \|f\|^4 + \|w_2\|_2^2 + \gamma^2 + 6 \langle f^2, w_2 \rangle + 4 \langle f^3, w_1 \rangle + 4 \langle f, w_3 \rangle, \text{ say. So that} \\ w_1 &= u_1 + v_1, w_2 = u_2 + 2u_1 v_1 + v_2, \text{ and } w_3 = u_3 + 3u_2 v_1 + 3u_1 v_2 + v_3. \end{aligned}$$

$$\begin{aligned} \text{Indeed } \gamma^2 + \|w_2\|_2^2 &= \lim_n \lim_m \|h_m^2 + 2g_n h_m + g_n^2\|_2^2 \\ &= \lim_n \lim_m [\|g_n^2\|_2^2 + \|h_m^2\|_2^2 + 6 \langle h_m^2, g_n^2 \rangle + 4 \langle g_n^3, h_m \rangle + 4 \langle g_n, h_m^3 \rangle] \\ &= \|v_2\|_2^2 + \beta^2 + \|u_2\|_2^2 + \alpha^2 + 6 \langle u_2, v_2 \rangle + 4 \langle v_3, u_1 \rangle + 4 \langle v_1, u_3 \rangle. \text{ So that} \\ (u_1, u_2, u_3, \alpha) * (v_1, v_2, v_3, \beta) &= (u_1 + v_1, u_2 + 2u_1 v_1 + v_2, \\ &\quad u_3 + 3u_2 v_1 + 3u_1 v_2 + v_3, (\alpha^2 + \beta^2)^{1/2}) \text{ in } L_4 \times L_2 \times L_{4,3} \times [0, \infty). \end{aligned}$$

We now show that a uniformly bounded type can be represented by an element of  $L_\infty \times [0, \infty)$ .

Consider a type  $\tau \in \mathcal{T}(L_4)$ , defined by  $\{g_n\}$ . Then  $g_n \rightarrow u_1$  weakly. Let  $u_2$  be the weak limit of  $g_n^2$  in  $L_2$ . Suppose that there exists a subsequence of  $g_n$  in  $L_\infty$ . Then

$$\left| \int_E g_n^2 - u_1^2 d\mu \right| = \left| \int_E [g_n(t) - u_1(t)][g_n(t) + u_1(t)] d\mu(t) \right| \leq (\|g_n\|_\infty + \|u_1\|_\infty) \left| \int_E g_n(t) - u_1(t) d\mu(t) \right| \rightarrow 0 \text{ as } n \rightarrow \infty. \text{ Thus } u_2 = u_1^2,$$

and similarly  $u_3 = u_1^3$ . Hence  $\|f + u_1\|^4 = \|f^2 + 2fu_1 + u_1^2\|_2^2$

$$= \|f\|^4 + \|u_1^2\|_2^2 + 6\langle f^2, u_1^2 \rangle + 4\langle f^3, u_1 \rangle + 4\langle f, u_1^3 \rangle. \text{ So that}$$

$$\tau(f) = [\|f + u_1\|^4 + \alpha^2]^{1/4}, \text{ and}$$

$$\|u_1 - g_n\|^4 = \|g_n^2\|_2^2 + \|u_1^2\|_2^2 + 6\langle g_n^2, u_1^2 \rangle - 4\langle g_n^3, u_1 \rangle - 4\langle g_n, u_1^3 \rangle \rightarrow \alpha \text{ as } n \rightarrow \infty.$$

### Representations of $\mathcal{T}(L_{2k})$ .

S. Guerre gives, in [Gu1] and [Gu3], a representation for symmetric types on  $L_{2k}$ , for  $k \in \mathbb{N}$   $k > 2$ . As in the case of  $L_2$  and  $L_4$ , we can produce a representation of  $\mathcal{T}(L_{2k})$  as a subset of  $L_{2k} \times L_k \times \dots \times L_{2k, 2k-1} \times [0, \infty)$ .

If  $\tau(f) = \lim_n \|f + g_n\|$ , where  $\tau \in \mathcal{T}(L_{2k})$ , then

$$\|f + g_n\|^{2k} = \sum_{i=0}^{2k} \binom{2k}{i} \int f^i g_n^{2k-i} \rightarrow \sum_{i=0}^{2k} \binom{2k}{i} \langle f^i, u_{2k-i} \rangle, \text{ where } u_{2k-i}$$

is the weak limit of  $g_n^{2k-i}$ , so  $u_{2k-i} \in L_{2k, 2k-i}$ . Then

$$(\tau * \sigma)(f)^{2k} = \lim_n \sigma(f + g_n)^{2k} = \lim_n \sum_{i=0}^{2k} \binom{2k}{i} \langle (f + g_n)^i, v_{2k-i} \rangle =$$

$$\lim_n \sum_{i=0}^{2k} \binom{2k}{i} \sum_{j=0}^i \binom{i}{j} \langle f^j g_n^{i-j}, v_{2k-i} \rangle =$$

$$\sum_{i=0}^{2k} \binom{2k}{i} \sum_{j=0}^i \binom{i}{j} \langle f^j u_{i-j}, v_{2k-i} \rangle, \text{ then } w_r = \sum_{t=0}^r \binom{r}{t} u_t v_{r-t}.$$

Having indentified  $\mathcal{T}(L_2)$  as a subset of  $L_2^+ \times [0, \infty)$ , and  $\mathcal{T}(L_4)$  as a subset of  $L_4 \times L_2 \times L_{4/3} \times [0, \infty)$ , we use these to study the representation of  $\mathcal{T}(L_1(L_p))$  as  $\pi_1(\mathcal{T}(L_p)) \times [0, \infty)$ .

#### Theorem 2.8.4

Assume that the representation of a symmetric uniformly integrable types on  $L_1(L_2)$  as a random measure is uniquely determined. Let  $\tau$  be a uniformly integrable type on  $L_1(L_2)$ , represented by  $\xi \in \pi_1(\mathcal{T}(L_2))$ . Suppose that the representation of types as random measures preserves convolution.

Then  $\tau$  is symmetric iff  $\xi_2$  is symmetric and  $\xi_g = \xi_{-g}$  a.e.  $[\xi_2]$ , where  $\{\xi_g\}$ ,  $\xi_2$  is the disintegration of  $\xi$ .

#### Proof

The general form of a type on  $L_2$  is  $t(h) = [\|h+g\|^2 + \alpha^2]^{1/2}$ , where  $g = (w)\text{-lim} g_n$  and  $\alpha = \lim \|g_n - g\|$ , and  $\{g_n\}$  is an approximating sequence for  $t$ .

Let  $\tau$  be symmetric. Let  $\eta$  represent  $(-\tau)$ . Since there are Fréchet differentiable points in  $L_2$ , by Theorem 4.2.1 we have  $\int_{\mathcal{T}(L_2)} t(h) d\xi_\omega(t) = \int_{\mathcal{T}(L_2)} t(h) d\eta_\omega(t)$  a.e.  $[P]$  for all  $h \in L_2$ . Let

$T: L_2 \times [0, \infty) \longrightarrow L_2 \times [0, \infty)$ . Then

$$(g, \alpha) \longmapsto (-g, \alpha)$$

$$\int_{\Omega} \int_{L_2 \times [0, \infty)} [\|f(\omega) + g\|^2 + \alpha^2]^{1/2} d\xi_\omega(g, \alpha) dP(\omega) =$$

$$\int_{\Omega} \int_{L_2 \times [0, \infty)} [\|f(\omega) - g\|^2 + \alpha^2]^{1/2} d\xi_\omega(g, \alpha) dP(\omega) =$$

$$\int_{\Omega} \int_{L_2 \times [0, \infty)} [\|f(\omega) + g\|^2 + \alpha^2]^{1/2} d\eta_\omega(g, \alpha) dP(\omega), \text{ where}$$

$\eta_\omega(E) = \xi_\omega \circ T^{-1}(E) = \xi_\omega\{(-y, \alpha) : (y, \alpha) \in E\}$ . Uniqueness implies that  $\xi_\omega = \eta_\omega$  a.e.  $[P]$ . Hence  $\tau$  is symmetric iff  $\xi$  is symmetric in  $L_2$ .

Further if we disintegrate  $\xi$  into  $\xi_g$  and  $\xi_2$ , then  $\tau(f) =$

$$\int_{\Omega} \int_{L_2} \int_0^\infty [\|f(\omega) + g\|^2 + \alpha^2]^{1/2} d\xi_g^\omega(\alpha) d\xi_2^\omega(g) dP(\omega) \text{ and so}$$

$$\begin{aligned} & \int_{L_2} \int_0^\infty [\|f(\omega) - g\|^2 + \alpha^2]^{1/2} d\xi_g(\alpha) d\xi_2(g) \\ &= \int_{L_2} \int_0^\infty [\|f(\omega) + g\|^2 + \alpha^2]^{1/2} d\xi_{-g}(\alpha) d\xi_2 \circ T^{-1}(g). \end{aligned}$$

Then  $\xi_2 = \xi_2 \circ T^{-1}$  and  $\xi_g = \xi_{-g}$  a.e.  $[\xi_2]$ . So that  $\tau$  is symmetric iff  $\xi_2$  is symmetric and  $\xi_g = \xi_{-g}$  a.e.  $[\xi_2]$ .

### Example

A random measure representing a symmetric type doesn't necessarily have to be concentrated on the symmetric types. For example fix a non-zero  $e \in L_2(\mathbb{R})$  and let  $\xi_\omega = 1/2[\delta(e, 0) + \delta(-e, 0)] \forall \omega$ . Then the corresponding type on  $L_1(L_2)$  is given by  $\tau(f) = E_\omega[\|f(\omega) + e\| + \|f(\omega) - e\|]/2 = \tau(-f)$ , so  $\tau$  is indeed symmetric. Symmetric types in  $L_2$  are given by  $(0, \alpha) \in L_2 \times [0, \infty)$ , so we must have  $\xi_\omega(\mathcal{Y}(L_2)) = 0$ .

### Convolution.

The calculation of convolution runs as follows :

$$\begin{aligned} (\sigma * \tau)(f) &= E \int_{L_2 \times [0, \infty)} \int_{L_2 \times [0, \infty)} [\|f(\omega) + g + h\|^2 + \alpha^2 + \beta^2]^{1/2} d\xi_\omega(g, \alpha) d\eta_\omega(h, \beta) \\ &= E \int_{L_2} \int_0^\infty \int_{L_2} \int_0^\infty [\|f(\omega) + g + h\|^2 + \alpha^2 + \beta^2]^{1/2} d\xi_g^\omega(\alpha) d\xi_2^\omega(g) d\eta_h^\omega(\beta) d\eta_2^\omega(h) \\ &= \int_\Omega \int_{L_2 \times [0, \infty)} [\|f(\omega) + k\|^2 + \gamma^2]^{1/2} d(\xi * \eta)_\omega(k, \gamma) dP(\omega) \\ &= \int_\Omega \int_{L_2} \int_0^\infty [\|f(\omega) + k\|^2 + \gamma^2]^{1/2} d(\xi * \eta)_k^\omega(\gamma) d(\xi * \eta)_2^\omega(k) dP(\omega) \end{aligned}$$

Then if  $\sigma$  and  $\tau$  are symmetric, so is  $\sigma * \tau$ . Then by above  $(\xi * \eta)_2$  is symmetric and  $(\xi * \eta)_k = (\xi * \eta)_{-k}$  a.e.  $[(\xi * \eta)_2]$ , and these conditions are equivalent to the similar conditions on  $\xi$  and  $\eta$ .

In the examples below we show how this works for simple cases.

Examples

1. Let  $g \in L_2$ ,  $\xi_2 = [\delta_g + \delta_{-g}]/2$  and  $\xi_h = \begin{cases} \delta_3 & \text{if } h = \pm g \\ \mu_h & \text{o.w.} \end{cases}$  for any  $\mu$ .

Then  $\tau(f) = E_\omega \left\{ \int_{L_2} \int_0^\infty [\|f(\omega) + h\|^2 + \alpha^2]^{1/2} d\xi_h(\alpha) d\xi_2(h) \right\} =$   
 $E_\omega \{ [\|f(\omega) + g\|^2 + 9]^{1/2} + [\|f(\omega) - g\|^2 + 9]^{1/2} \} / 2$ , so  $\tau$  is symmetric.

Also  $\xi_2$  is symmetric and  $\xi_h = \xi_{-h}$  a.e.  $[\xi_2]$  as required.

2. Let  $h, k \in L_2$ . Let  $\eta_2 = [\delta_h + \delta_{-h}]/3 + [\delta_k + \delta_{-k}]/6$  and

$\eta_1 = \begin{cases} \delta_1 & \text{if } l = \pm h \\ \delta_2 & \text{if } l = \pm k \\ 0 & \text{o.w.} \end{cases}$ . Then

$\sigma(f) = E_\omega \{ [\|f(\omega) + h\|^2 + 1]^{1/2} + [\|f(\omega) - h\|^2 + 1]^{1/2} +$   
 $1/2 [\|f(\omega) + k\|^2 + 4]^{1/2} + 1/2 [\|f(\omega) - k\|^2 + 4]^{1/2} \} / 3$ .

Then with  $\tau$  and  $\xi$  as above, we calculate  $(\sigma * \tau)(f)$  as

$E_\omega \{ ([\|f(\omega) + h + g\|^2 + 10]^{1/2} + [\|f(\omega) + h - g\|^2 + 10]^{1/2}$   
 $+ [\|f(\omega) - h + g\|^2 + 10]^{1/2} + [\|f(\omega) - h - g\|^2 + 10]^{1/2}) / 6$   
 $+ ([\|f(\omega) + k + g\|^2 + 13]^{1/2} + [\|f(\omega) - g + k\|^2 + 13]^{1/2}$   
 $+ [\|f(\omega) - k + g\|^2 + 13]^{1/2} + [\|f(\omega) - k - g\|^2 + 13]^{1/2}) / 2 \}.$

Then

$(\xi * \eta)_2 = [\delta_{h+g} + \delta_{h-g} + \delta_{-h+g} + \delta_{-h-g}] / 6 + [\delta_{k+g} + \delta_{k-g} + \delta_{g-k} + \delta_{-k-g}] / 12$

$(\xi * \eta)_1 = \begin{cases} \delta(\sqrt{10}) & \text{if } l = \pm(h \pm g) \\ \delta(\sqrt{13}) & \text{if } l = \pm(k \pm g) \\ \mu_1 & \text{o.w.} \end{cases}$  so that  $(\xi * \eta)_1 = \xi_1 * \eta_1$  and

$(\xi * \eta)_2 = \xi_2 * \eta_2$ .

Symmetric types on  $L_1(L_4)$ .

Let  $\tau \in \mathcal{T}(L_1(L_4))$  be given by,

$$\tau(f) = \int_{\Omega} \int_{L_4 \times L_2 \times L_{4,3} \times \mathbb{R}_+} [\|f(\omega)\|^4 + \|u_2\|_2^2 + 6\langle f(\omega)^2, u_2 \rangle + 4\langle f(\omega)^3, u_1 \rangle + 4\langle f(\omega), u_3 \rangle]^{1/4} d\xi_{\omega}(u_1, u_2, u_3, \alpha) dP(\omega).$$

Then  $\tau$  is symmetric iff  $\xi$  is symmetric on  $\mathcal{T}(L_4)$  which happens iff  $\xi$  is symmetric in  $L_4$  on  $L_4 \times L_2 \times L_{4,3} \times \mathbb{R}_+$ . Let

$$F[f(\omega), u_1, u_2, u_3, \alpha] = [\|f(\omega)\|^4 + \|u_2\|_2^2 + 6\langle f(\omega)^2, u_2 \rangle + 4\langle f(\omega)^3, u_1 \rangle + 4\langle f(\omega), u_3 \rangle]^{1/4}.$$

We can however find symmetric types with associated random measure concentrated wholly on the non-symmetric types. For example let  $\xi = 1/2[\delta(e, 0, 0, 0) + \delta(-e, 0, 0, 0)]$  for a fixed non-zero  $e \in L_4$ . Let  $\xi_{\omega} = \xi$ , then  $\xi$  is a random measure concentrated on the non-symmetric types of  $L_4$ , where we are representing  $\mathcal{T}(L_4)$  as  $L_4 \times L_2 \times L_{4,3} \times \mathbb{R}_+$ . Then

$$2\tau(f) = E_{\omega} [\|f(\omega)\|^4 + 4\langle f(\omega)^3, e \rangle]^{1/4} + E_{\omega} [\|f(\omega)\|^4 + 4\langle f(\omega)^3, -e \rangle]^{1/4} = 2\tau(-f),$$

so that  $\tau$  is indeed symmetric.

## CHAPTER 3 : DERIVATIVES OF TYPES.

### 3.1 INTRODUCTION

The aim of this chapter is to study various differential properties in  $\mathcal{T}(X)$ , in particular with reference to the concept of smoothness in  $X$ . The ideas evolved out of work on the uniqueness of representing types as random measures, where we were led to consider cases when  $\tau(-nx) + \tau(nx) - 2n\|x\| \rightarrow 0$  as  $n \rightarrow \infty$ . By looking at the functional  $f_x : \mathcal{T}(X) \rightarrow \mathbb{R}$  defined by  $f_x(\tau) = \lim_n [\tau(nx) - n\|x\|]$ , we develop a tool that reduces the question of uniqueness for random measures on  $\mathcal{T}(X)$ , to one of measures on  $\mathcal{T}(X)$ .

We develop the properties of  $f_x$  and compute some simple examples, where it can be seen as extracting some information about the weak limit of the defining sequence of the type  $\tau$ .

Gateaux and Fréchet differentiability can be defined for arbitrary real-valued functions on  $X$ , and we consider differentiability of types on  $X$ , which leads, after looking at some examples, to the extension of the norm-duality map from  $X$  to  $\mathcal{T}(X)$ .

### 3.2 The functional $f_x$ .

#### Definition

The Banach space  $X$  is said to be smooth at  $x \in U(X)$  whenever there exists a unique  $f \in U(X^*)$  such that  $f(x) = 1$ . If  $X$  is smooth at each point of  $U(X)$  then we say that  $X$  is smooth.

$X$  is said to have a Gateaux differentiable norm at  $x$  whenever given  $y \in X$ ,  $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t} \equiv \rho'(x;y)$  exists. If the norm of  $X$  is Gateaux differentiable at each point of  $X$  then we say that  $X$  has a Gateaux differentiable norm.

Then  $X$  is smooth at  $x$  iff the norm of  $X$  is Gateaux differentiable at  $x$ . See [Di ].

### Definition

$X$  is said to have a Fréchet differentiable norm at  $x$  whenever  $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$  exists uniformly for  $y \in X$ . If the norm of  $X$  is Fréchet differentiable at each point of  $X$  then we say that  $X$  has a Fréchet differentiable norm.

Thus smoothness is equivalent to the existence of a unique norm to norm weak-star continuous support map, and Fréchet differentiability is equivalent to the norm to norm continuity of this support mapping.

For a general discussion of smoothness and related topics, the reader is referred to Diestel [Di ], Cudia [Cu] or Day [Day1]. In [Day2] it is shown that the  $L_p$ -spaces are smooth, . . . . The differential of the norm,  $\rho'$ , is computed for  $\mathbb{R}^n$ ,  $L_p$ , and  $C(\Omega, \mathbb{R})$  in [Ho]; where it is also shown that the set of smooth points of a solid closed convex subset of a separable Banach space is a residual (i.e. dense and of second category) subset of  $\partial(A)$ . For  $1 < p < \infty$ ,  $L_p(X)$  is smooth iff  $X$  is smooth.

Although  $\mathcal{T}(X)$  is not in general a Banach space, in certain situations we are able to extend functionals to  $\mathcal{T}(X)$ . Here we use a uniform convergence argument to enable us to define  $f_x(\tau)$ , a linear functional on  $\mathcal{T}(X)$  in the sense that  $f_x(\alpha\tau + \beta\sigma) = \alpha f_x(\tau) + \beta f_x(\sigma)$ . One can view  $f_x(\tau)$  as the derivative of the weak part of  $\tau$ , that is in many cases  $f_x(\tau) = \rho'(x; u)$  where  $u$  is the weak limit of the approximating sequence defining  $\tau$ . We now define  $f_x$  and explore some of its basic properties. Assume  $X$  is stable.

Lemma 3.2.1

Let  $(a_{nk})$  be a sequence of real numbers. Suppose  $\lim_k a_{nk}$  exists for each  $n$ , and  $\lim_n a_{nk}$  exists uniformly in  $k$ . Then

$$\lim_n \lim_k a_{nk} = \lim_k \lim_n a_{nk}.$$

Suppose a type  $\tau$  has approximating sequence  $\{y_k\}$ . Let  $a_{nk} = \frac{\|x + y_k/n\| - \|x\| - \rho'(x, y_k/n)}{\|y_k/n\|}$ . If  $\|\cdot\|_X$  is Fréchet differentiable at  $x$ , then  $\forall \epsilon > 0 \exists \delta > 0$  such that  $\|\Delta y\| < \delta$  implies that  $\left| \frac{\|x + \Delta y\| - \|x\| - \rho'(x; \Delta y)}{\|\Delta y\|} \right| < \epsilon$ .

Assume that  $\tau$  isn't the trivial type, so that  $(y_k)$  has no subsequence convergent to 0. Thus there exists a greatest positive integer  $R$  such that  $\|y_k\| = 0$ . Let  $P_\delta = \inf\{\|y_j\|/\delta : j > R\}$ . Since the  $y_k$  are bounded,  $0 < P < \infty$ . Let  $\epsilon > 0$ , then  $\exists \delta > 0$  as above. Then there exists  $P_\delta$  such that for  $n > P_\delta$  and  $j > R$   $\|y_j/n\| < \delta$  thus  $|a_{nk}| < \epsilon$ . That is  $\lim_n a_{nk}$  converges uniformly in  $k$ . We know already that  $\lim_n a_{nk} = 0$  and that  $\lim_k a_{nk}$  exists, so  $\lim_n \lim_k a_{nk} = \lim_k \lim_n a_{nk} = 0$ .

That is  $\lim_n \lim_k a_{nk} = \lim_n [\tau(nx) - n\|x\| - \lim_k \rho'(x; y_k)] = 0$ . So that  $\lim_n [\tau(nx) - n\|x\|] = \lim_k \rho'(x; y_k)$ . We are thus motivated to make the following definitions.

Definition

Let  $x \in X$ . Let  $\tau \in \mathcal{T}(X)$ .

- (A) Define  $f_x(\tau) = \lim_n [\tau(nx) - n\|x\|]$  whenever this exists.  
 (B) We say that  $\tau$  is a continuity point of  $f_x$  if  $f_x(\tau)$  exists and equals  $\lim_m \rho'(x; y_m)$  whenever  $(y_m)$  is an approximating sequence for  $\tau$ .

So if  $x$  is a point of Fréchet differentiability, then for all types  $\tau$  on  $X$ ,  $f_x(\tau)$  exists and  $\tau$  is a continuity point of  $f_x$ .

Lemma 3.2.2

Suppose that  $\tau$  is a continuity point of  $f_x$ , that is both (A) and (B) hold, then

- (i)  $f_x$  is well defined.  
 (ii)  $f_{-x} = -f_x$   
 (iii)  $f_x(-\tau) = -f_x(\tau)$

Proof

(i) Suppose  $\tau(x) = \lim_m \|x + y_m\| = \lim_m \|x + z_m\|$ . Extract subsequences so that  $\rho'(x; y_m)$  and  $\rho'(x; z_m)$  converge. Then  
 $0 = \lim_n [\tau(nx) - \tau(nx) + n\|x\| - n\|x\|]$   
 $= \lim_n [\tau(nx) - n\|x\|] - \lim_n [\tau(nx) - n\|x\|] = \lim_m \rho'(x; y_m) - \lim_m \rho'(x; z_m).$

- (ii)  $f_{-x}(\tau) = \lim_m \rho'(-x; y_m) = -\lim_m \rho'(x; y_m) = -f_x(\tau).$   
 (iii)  $f_x(-\tau) = \lim_m \rho'(x; -y_m) = -\lim_m \rho'(x; y_m) = -f_x(\tau).$

Proposition 3.2.3

Let  $X$  be a Banach space and  $x \in X$ . Suppose that for all  $\tau \in \mathcal{T}(X)$ ,  $f_x(\tau) = \lim_n [\tau(nx) - n\|x\|]$  exists and that  $\tau$  is a continuity point of  $f_x$ . Then  $\|\cdot\|_X$  is Fréchet differentiable at  $x$ .

Proof

The second assumption tells us that  $f_x(\tau) = \lim_m f_x(\tau_{y_m})$  whenever  $\tau_{y_m}$  is a sequence of degenerate types converging to  $\tau$ . Using the density of these types in  $\mathcal{T}(X)$  and an easy lemma about metric spaces we see that  $f_x$  is continuous on  $\mathcal{T}(X)$ .

Now  $\tau(nx) - n\|x\|$  is a decreasing function of  $n$ . So Dini's theorem tells us that  $\tau(nx) - n\|x\|$  converges to  $f_x(\tau)$  uniformly on the compact set  $\{\tau \in \mathcal{T}(X) : \tau(0) \leq 1\}$ . In particular  $[\|x+hy\| - \|x\|]/h \rightarrow \rho'(x;y)$  uniformly over  $y \in \text{ball}(X)$ , which is the criterion of Fréchet differentiability.

Proposition 3.2.4

Suppose  $\tau, \sigma \in \mathcal{T}(X)$  and that  $f_x(\tau), f_x(\sigma)$  exist for some  $0 \neq x \in X$ . Suppose that  $\tau$  and  $\sigma$  are continuity points of  $f_x$  then  $f_x(\tau * \sigma)$  exists,  $f_x(\tau * \sigma) = f_x(\tau) + f_x(\sigma)$  and  $\tau * \sigma$  is a continuity point of  $f_x$ .

Proof

If  $(y_m)$  and  $(z_k)$  are approximation sequences for  $\tau$  and  $\sigma$  respectively, then  $f_x(\tau) = \lim_m \rho'(x; y_m)$  and  $f_x(\sigma) = \lim_k \rho'(x; z_k)$ .

Then  $\frac{\|x+y_m/n\| - \|x\|}{1/n} \xrightarrow{n} \rho'(x; y_m)$  and

$$\begin{aligned} \rho'(x; z_k) &= \lim_m \rho'(x; z_k) = \lim_m \rho'(x; y_m + z_k) - \lim_m \rho'(x; y_m) \\ &= \lim_n \lim_m \left\{ \frac{\|x + (y_m + z_k)/n\| - \|x\|}{1/n} - \frac{\|x + y_m/n\| - \|x\|}{1/n} \right\} \\ &= \lim_n \lim_m \frac{\|x + (y_m + z_k)/n\| - \|x + y_m/n\|}{1/n}. \end{aligned}$$

Thus  $0 = \rho'(x; z_k) - \rho'(x; z_k)$

$$\begin{aligned} &= \lim_n \lim_m \frac{\|x + (y_m + z_k)/n\| - \|x + y_m/n\|}{1/n} - \lim_n \frac{\|x + z_k/n\| - \|x\|}{1/n} \\ &= \lim_n [\lim_m (\|nx + y_m + z_k\| - \|nx + y_m\|) + n\|x\| - \|nx + z_k\|] \\ &= \lim_n [\lim_m \|nx + y_m + z_k\| - \|nx + z_k\| - \lim_m \|nx + y_m\| + n\|x\|] \\ &= \lim_n [\tau(nx + z_k) - \|nx + z_k\| - \tau(nx) + n\|x\|] \end{aligned}$$

Therefore  $\lim_n [\tau(nx + z_k) - \|nx + z_k\|] = \lim_n [\tau(nx) - n\|x\|]$

$$= \lim_m \rho'(x; y_m) = f_x(\tau)$$

$$f_x(\tau) + f_x(\sigma) = \lim_m \rho'(x; y_m) + \lim_k \rho'(x; z_k)$$

$$\begin{aligned} &= \lim_n [\tau(nx + z_k) - \|nx + z_k\|] + \lim_n [\lim_k \|nx + z_k\| - n\|x\|] \\ &= \lim_k \lim_n [\tau(nx + z_k) - \|nx + z_k\|] + \lim_n \lim_k [\|nx + z_k\| - n\|x\|] \end{aligned}$$

Let  $a_{nk} = \tau(nx + z_k) - \|nx + z_k\|$ . Then  $\lim_n a_{nk}$  exists uniformly in  $k$ , and  $\lim_k a_{nk} = (\tau * \sigma)(nx) - \sigma(nx)$  which exists for each  $n$ . Thus  $\lim_k \lim_n a_{nk} = \lim_n \lim_k a_{nk}$ .

$$\begin{aligned} f_x(\tau) + f_x(\sigma) &= \lim_n \lim_k [\tau(nx + z_k) - \|nx + z_k\|] + \lim_n \lim_k [\|nx + z_k\| - n\|x\|] \\ &= \lim_n \lim_k [\tau(nx + z_k) - n\|x\|] = \lim_n [(\tau * \sigma)(nx) - n\|x\|] = f_x(\tau * \sigma) \end{aligned}$$

$$\text{and } f_x(\tau) + f_x(\sigma) = \lim_m \rho'(x; y_m) + \lim_k \rho'(x; z_k)$$

$$= \lim_k \lim_m \rho'(x; y_m + z_k) = \lim_m \lim_k \rho'(x; y_m + z_k).$$

So  $\tau * \sigma$  is a continuity point of  $f_x$ .

Lemma 3.2.5

Suppose  $\tau$  is a continuity point of  $f_x$  and  $f_y$ . Then

(i) If  $\alpha \neq 0$  then  $\alpha \cdot \tau$  is a continuity point of  $f_x$ ,

$$f_x(\alpha \cdot \tau) = \alpha f_x(\tau) \text{ and } f_{\alpha X}(\tau) = [\text{sign}(\alpha)] \cdot f_x(\tau).$$

(ii)  $f_x(\tau_y) = \rho'(x; y)$  so that  $f_x(\alpha \cdot \tau_x) = \alpha \|x\|$ .

Proof

$$(i) f_{\alpha X}(\tau) = \lim_n [\tau(n\alpha x) - n\|\alpha x\|]$$

$$= |\alpha| \lim_n \lim_m [\|nx + y_m / \alpha\| - n\|x\|] = |\alpha| \lim_m \rho'(x; y_m / \alpha) = |\alpha| / \alpha \cdot f_x(\tau).$$

$$\text{Let } \sigma = \alpha \cdot \tau. \text{ Then } \lim_n [\sigma(nx) - n\|x\|] = \lim_n [\lim_m \|nx + \alpha y_m\| - n\|x\|]$$

$$= |\alpha| \lim_n [\tau(nx/\alpha) - n\|x/\alpha\|] = |\alpha| f_{x/\alpha}(\tau) = \alpha f_x(\tau) = \alpha \cdot \lim_m \rho'(x; \alpha y_m)$$

$$= \lim_m \rho'(x; \alpha y_m). \text{ So } f_x(\sigma) \text{ exists and equals } \lim_m \rho'(x; \alpha y_m).$$

(ii) is trivial.

Lemma 3.2.6

Suppose  $\tau$  is a symmetric type. Suppose that  $\tau$  is a continuity point of  $f_x$ . Then  $f_x(\tau) = 0$ . If  $f_x(\tau) = 0$  for some

$(\tau, x) \in \mathcal{T}(X) \times X$ , then  $\forall y \in X$

$$\lim_n [\tau(nx) - \tau(-nx)] = \lim_n [\tau(nx+y) - \tau(-nx-y)] = 0.$$

Proof

$$\forall z \in X, \tau(z) = \tau(-z). \text{ Thus } f_x(\tau) = \lim_n [\tau(nx) - n\|x\|] \\ = \lim_n [\tau(-nx) - n\|x\|] = f_{-x}(\tau) = -f_x(\tau). \text{ Hence } f_x(\tau) = 0.$$

$$\text{If } f_x(\tau) = 0 \text{ then } 0 = f_x(\tau) - f_{-x}(\tau) = \lim_n [\tau(nx) - \tau(-nx)].$$

$$\lim_n [\tau(nx+y) - \|nx+y\|] = \lim_n [\tau(nx) - n\|x\|] = 0, \text{ thus}$$

$$\lim_n [\tau(nx+y) - \tau(-nx-y)] = 0.$$

Proposition 3.2.7

Let  $\sigma \in \mathcal{T}(L_1(\Omega: X))$  be uniformly integrable and represented by the random measure  $\xi \in \pi_1(\mathcal{T}(X))$ . Let  $g \in L_1(\Omega: X)$ . If for each  $\tau \in \mathcal{T}(X)$   $f_{g(\omega)}(\tau)$  exists for almost all  $\omega \in \Omega$ , then  $f_g(\sigma)$  exists and

$$f_g(\sigma) = \int_{\Omega} \int_{\mathcal{T}(X)} f_{g(\omega)}(\tau) \cdot d\xi_{\omega}(\tau) \cdot dP(\omega)$$

Proof

$$\begin{aligned} f_g(\sigma) &= \lim_n [\sigma(n g) - n \|g\|_{L_1}] \\ &= \lim_n \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} \tau(n g(\omega)) d\xi_{\omega}(\tau) dP(\omega) - n \int_{\Omega} \|g(\omega)\|_X dP(\omega) \right\} \\ &= \lim_n \int_{\Omega} \int_{\mathcal{T}(X)} \{ \tau(n g(\omega)) - n \|g(\omega)\|_X \} d\xi_{\omega}(\tau) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} \lim_n [ \tau(n g(\omega)) - n \|g(\omega)\|_X ] d\xi_{\omega}(\tau) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} f_{g(\omega)}(\tau) d\xi_{\omega}(\tau) dP(\omega) \end{aligned}$$

by applying the dominated convergence theorem twice. Thus if for almost all  $\omega \in \Omega$  [P],  $f_{g(\omega)}$  exists, then  $f_g$  exists.

Proposition 3.2.8

Let  $\tau, \sigma \in \mathcal{T}(L_1(\Omega: X))$  be uniformly integrable and represented by the random measures  $\xi, \eta$  respectively. Let  $g \in L_1(\Omega: X)$ .

Suppose that  $\tau$  and  $\sigma$  are continuity points of  $f_g$ . Suppose

further that every type  $t \in \mathcal{T}(X)$  is a continuity point of  $f_{g(\omega)}$

a.e. [P]. Then

$$\begin{aligned} &\int_{\Omega} \int_{\mathcal{T}(X)} f_{g(\omega)}(r) d(\xi * \eta)_{\omega}(r) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} f_{g(\omega)}(t * s) d\xi_{\omega}(t) d\eta_{\omega}(s) dP(\omega). \end{aligned}$$

Proof

$$\begin{aligned} f_g(\tau) + f_g(\sigma) &= \int_{\Omega} \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} f_{g(\omega)}(t) + f_{g(\omega)}(s) d\xi_{\omega}(t) d\eta_{\omega}(s) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} f_{g(\omega)}(t * s) d\xi_{\omega}(t) d\eta_{\omega}(s) dP(\omega) \end{aligned}$$

$$f_g(\tau) + f_g(\sigma) = f_g(\tau * \sigma) = \int_{\Omega} \int_{\mathcal{T}(X)} f_{g(\omega)}(r) d(\xi * \eta)_{\omega}(r) dP(\omega)$$

Thus the two sides are equal whenever the set  $\{\omega : g(\omega) = 0\}$  is  $[P]$ -null. If  $g \equiv 0$ , then

$$f_g(\tau) + f_g(\sigma) = \int_{\Omega} \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (t * s)(0) d\xi_{\omega}(t) d\eta_{\omega}(s) dP(\omega) \text{ and}$$

$$f_g(\tau * \sigma) = \int_{\Omega} \int_{\mathcal{T}(X)} r(0) d(\xi * \eta)_{\omega}(r) dP(\omega). \text{ If } g \neq 0 \text{ but } P\{\omega : g(\omega) = 0\} \neq 0$$

then let  $F = \{\omega : g(\omega) = 0\}$ . Thus

$$f_g(\tau) + f_g(\sigma) = \int_{F^c} \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} f_{g(\omega)}(t * s) d\xi_{\omega}(t) d\eta_{\omega}(s) dP(\omega) + \int_F \int_{\mathcal{T}(X)} \int_{\mathcal{T}(X)} (t * s)(0) d\xi_{\omega}(t) d\eta_{\omega}(s) dP(\omega) = f_g(\tau * \sigma)$$

### Proposition 3.2.9

Suppose that for all  $i$ ,  $f_x(\tau_i)$  and  $f_x(\tau)$  exist. Then

(i) Let  $\tau_i \longrightarrow \tau$  in the weak topology. Then  $f_o(\tau_i) \longrightarrow f_o(\tau)$ .

(ii) If  $\tau_i \longrightarrow \tau$  in the uniform topology then  $f_x(\tau_i) \longrightarrow f_x(\tau)$ .

(iii)  $f_x(\tau_i) \longrightarrow f_x(\tau)$  for all  $x$  does not imply that  $\tau_i \longrightarrow \tau$

weakly, (iv)  $\lim_i f_x(\tau_{y_i}) = f_x(\lim_i \tau_{y_i})$ , if  $(y_i)$  is a bounded sequence.

### Proof

The uniform topology on  $\mathcal{T}(X)$  is given by the metric  $d_{\infty}(\sigma, \tau) = \sup\{|\tau(x) - \sigma(x)| : x \in X\}$ . Suppose  $d_{\infty}(\tau_i, \tau) \longrightarrow 0$ , then  $d_{\infty}(\tau_i, \tau_j) \leq d_{\infty}(\tau_i, \tau) + d_{\infty}(\tau_j, \tau) \longrightarrow 0$ , thus  $|\tau_i(nx) - \tau_j(nx)| \leq \sup\{|\tau_i(x) - \tau_j(x)| : x \in X\} = d_{\infty}(\tau_i, \tau) \longrightarrow 0$  as  $i, j \longrightarrow \infty$ .

Thus  $|f_x(\tau_i) - f_x(\tau_j)| = \lim_n |\tau_i(nx) - \tau_j(nx)| \longrightarrow 0$  as  $i, j \longrightarrow \infty$ .

So that  $\lim_i [\tau_i(nx) - n\|x\|]$  converges uniformly in  $n$ , and

$\lim_n [\tau_i(nx) - n\|x\|]$  exists. Thus  $\lim_i \lim_n [\tau_i(nx) - n\|x\|] =$

$\lim_n \lim_i [\tau_i(nx) - n\|x\|] = \lim_n [\tau(nx) - n\|x\|] = f_x(\tau)$ .

In  $c_0$ ,  $\rho'(x; y) = (\text{sgn} \cdot x_i) \cdot y_i$ . Let  $\tau_i(x) = \max\{\|x + y_i\|, \beta\}$  and  $\tau(x) = \max\{\|x + y\|, \alpha\}$  with  $y = (\text{ptwise}) \lim_i y_i$  and  $\alpha = \|y - y_i\|$ . Then

$\lim_i \tau_i(x) = \max\{\|x+y\|, \alpha, \beta\} \neq \tau(x)$  if  $\beta > \alpha$ . But  $f_x(\tau_i) = (\operatorname{sgn} x_j) \cdot y_j$  and  $f_x(\tau) = (\operatorname{sgn} x_j) \cdot y_j$ , so that  $f_x(\tau_i) \longrightarrow f_x(\tau)$  while  $\tau_i(x) \not\longrightarrow \tau(x)$ .

We now look at some illustrative examples, calculating  $f_x$  on  $\mathcal{T}(X)$  for some simple Banach spaces  $X$ . We also consider the question of whether  $f_x(\tau) = 0$  only on the symmetric types, as we will be interested in such cases in chapter 4.

### Examples

1.  $X = \mathcal{Q}_1$ . In this case  $f_x(\tau)$  exists precisely when  $\tau$  is a symmetric type. Let  $\tau \in \mathcal{T}(\mathcal{Q}_1)$  with  $\tau(x) = \|x\| + \alpha$  for some  $\alpha \geq 0$ . Then  $f_x(\tau) = \alpha$ . But no types are continuity points.

2.  $X = \mathcal{Q}_p$ . Every non-zero point of  $\mathcal{Q}_p$  is Fréchet differentiable. Types on  $\mathcal{Q}_p$  are given by  $\sigma(x) = (\|x+y\|^p + \alpha^p)^{1/p}$ , and symmetric types by  $\tau(x) = (\|x\|^p + \alpha^p)^{1/p}$ . Then if  $\tau$  is a symmetric continuity point, we have

$$(\|nx\|^p + \alpha^p)^{1/p} - n\|x\| \longrightarrow f_x(\tau) = 0 \quad \text{as } n \longrightarrow \infty.$$

$$\begin{aligned} 0 &= \lim_n [(\|nx\|^p + \alpha^p)^{1/p} - n\|x\|] = \lim_n [(\|nx-y+y\|^p + \alpha^p)^{1/p} - n\|x\|] \\ &= \lim_n [\sigma(nx-y) - n\|x\|] = \lim_n \lim_m \frac{\|x + (y_m - y)/n\| - \|x\|}{1/n} \\ &= \lim_m \lim_n \frac{\|x + (y_m - y)/n\| - \|x\|}{1/n} = \lim_m \rho'(x; y_m - y) \\ &= \lim_m \rho'(x; y_m) - \rho'(x; y). \end{aligned}$$

Thus  $f_x(\sigma) = \lim_m \rho'(x; y_m) = \rho'(x; y)$ . But since every symmetric type is a continuity point we have that all types are continuity points, and that  $f_x(\sigma) = \rho'(x; y)$ .

3. Let  $\tau$  be a uniformly integrable type on  $L_1(\Omega; X)$ . Let  $g \in L_1(X)$ . If for each  $t \in \mathcal{T}(X)$ ,  $f_{g(\omega), \omega}(t)$  exists a.e.  $[P]$ , then  $f_g(\tau)$  exists, and  $f_g(\tau) = \int_{\Omega} \int_{\mathcal{T}(X)} f_{g(\omega), \omega}(t) d\xi_{\omega}(t) dP(\omega)$ .

For  $X = \mathcal{L}_p$ , every type is a continuity point of  $f_{g(\omega)}$  and  $f_{g(\omega), \omega}(t) = \rho'(g(\omega); y)$ . So that for each  $\xi \in \mathcal{P}(\mathcal{T}(\mathcal{L}_p))$

$$\int_{\mathcal{T}(\mathcal{L}_p)} f_x(t) d\xi(t) = \int_{\mathcal{L}_p \times \mathbb{R}_+} \rho'(x; y) d\xi(y, \beta).$$

Disintegrating  $\xi$  into a family of probabilities  $\{\xi_y\}$ ,  $y \in \mathcal{L}_p$ , on  $[0, \infty)$  and a probability  $\xi_2$  on  $\mathcal{L}_p$ , we have

$$f_g(\tau) = \int_{\Omega} \int_{\mathcal{L}_p} \rho'(g(\omega); y) d\xi_2^{\omega}(y) dP(\omega)$$

In the next set of examples we consider cases when  $f_x(\tau) = 0$  implies that  $\tau$  is symmetric.

### Examples

1. For  $X = \mathcal{L}_p$ .  $f_x(\tau) = 0$  for some  $x$  does not imply that  $\tau$  is symmetric. Let  $y = (1, 0, \dots)$ ,  $x = (0, 1, 0, \dots)$ . Then  $\forall \alpha$   
 $\tau(z) = (\|z + y\|^p + \alpha^p)^{1/p}$  isn't symmetric.  $f_x(\tau) = 0$  iff  $\rho'(x; y) = 0$  and the Gateaux derivative at  $x$  defines a linear functional on  $\mathcal{L}_p$ , which is given by  $\rho'(x; \bullet) = \frac{x |x|^{p-2}}{\|x\|^{p-1}} \in \mathcal{L}_q$  where  $q^1 + p^1 = 1$ .

So that  $\rho'(x; y) = \sum \frac{x_n |x_n|^{p-2}}{\|x\|^{p-1}} y_n$ , then  $\rho'(x; y) = 0$  but  $\tau$  isn't symmetric.

2. For  $X = \mathcal{L}_p$ . If  $\forall x \in X$ ,  $f_x(\tau) = 0$ , then  $\tau$  is symmetric.

Letting  $x$  run through  $\{e_n\}_{\mathbb{N}}$ , the unit basis of  $\mathcal{L}_p$ , we see that for each  $n$ ,  $y_n = 0$ , so that  $y = 0$  and hence  $\tau$  is symmetric.

3. For  $\tau \in \mathcal{T}(L_1(\Omega; \mathcal{L}_p))$ ,  $f_g(\tau) = 0 \forall g$  does not imply that  $\tau$  is symmetric. Let  $e_1 = (1, 0, \dots) \in \mathcal{L}_p$ . Put,  $\forall \omega$ ,  $\xi_\omega^2 = (\delta_{e_1} + \delta_{-e_1})/2$ ,

$$\xi_\omega^y = \begin{cases} \delta_1 & y = e_1 \\ \delta_2 & y = -e_1 \\ 0 & \text{o.w.} \end{cases}. \text{ Then } \forall g \in L_1(\Omega; \mathcal{L}_p) \quad \tau(g) \text{ equals}$$

$$\frac{E}{2} \left\{ \int_0^\infty (\|g(\omega) + e_1\|^p + \beta^p)^{1/p} \cdot d\xi_{e_1}(\beta) + \int_0^\infty (\|g(\omega) - e_1\|^p + \beta^p)^{1/p} \cdot d\xi_{-e_1}(\beta) \right\} \\ = \frac{E}{2} \left\{ [\|g(\omega) + e_1\|^p + 1^p]^{1/p} + [\|g(\omega) - e_1\|^p + 2^p]^{1/p} \right\}.$$

Let  $g(\omega) = e_1$ . Then  $\tau(g) = \{(2^p + 1)^{1/p} + 2\}/2$ ,  $\tau(-g) = \{1 + 2 \cdot 2^{1/p}\}/2$ . So that  $\tau(g) = \tau(-g)$  iff  $p=1$ , giving a contradiction. So that  $\tau \notin \mathcal{S}(L_1(\Omega; \mathcal{L}_p))$ , but  $\forall g \quad f_g(\tau) = \int_\Omega \int_{\mathcal{L}_p} \rho'(g(\omega); y) d\xi_\omega^2(y) dP(\omega) = E\{\rho'(g(\omega); e_1) + \rho'(g(\omega); -e_1)\}/2 = 0$ .

4. For  $\tau \in \mathcal{T}(L_1(\Omega; \mathcal{L}_p))$ ,  $f_g(\tau) = 0 \forall g$  does not imply that  $\xi^2$  is symmetric.

We know that  $f_g(\tau) = \int_\Omega \int_{\mathcal{L}_p} \rho'(g(\omega); y) \cdot d\xi_\omega^2(y) \cdot dP(\omega)$ . Fix  $y \neq 0$ . For each  $\omega \in \Omega$  let  $\xi_\omega^2 = (\delta_y)/3 + 2/3(\delta_{-y/2})$ . Then  $\xi^2$  obviously is not symmetric, but for each  $g$ ,

$$f_g(\tau) = E_\omega \{ 1/3 \rho'(g(\omega); y) + 2/3 \rho'(g(\omega); -y/2) \} = E_\omega(0) = 0.$$

5. Let  $X = c_0$ . When  $X = C(\Omega, \mathbb{R})$  with  $\Omega$  a compact Hausdorff space, the smooth points of the unit ball are precisely the peak functions in  $C(\Omega; \mathbb{R})$ . See [Ho]. In this case,

$\rho'(x; y) = (\text{sgn} x(p_0)) \cdot \delta_{p_0}(y)$ , where  $x$  obtains its norm at  $p_0$ , and  $\delta_{p_0}$  is the evaluation functional,  $\delta_{p_0}(y) = y(p_0)$ .

Then  $\xi = (\xi_i)$  is Gateaux differentiable iff  $\#\{j: |\xi_j| = \|\xi\|\} = 1$ . That is, considering  $c_0$  as a subspace of  $C(\mathbb{N}, \mathbb{R})$ ,  $\xi$  is a smooth point iff it is a peak function.

Let  $\tau \in \mathcal{T}(X)$ . Then  $\tau(x) = \max\{\|x+y\|, \beta\}$ , so that  $f_x(\tau)$  exists and equals  $\rho'(x;y)$  whenever  $x$  is a peak function. Since  $\rho'(x;y) = \text{sign}[x(i)] \cdot y(i)$  and if  $(y_m)$  is an approximating sequence for  $\tau$  then  $y_m \rightarrow y$ , we have  $y_m \rightarrow y$  coordinatewise. Hence  $\rho'(x;y_m) \rightarrow \rho'(x;y)$  whenever the latter exists. Thus whenever  $x$  is a peak function every type is a continuity point of  $f_x$ . Also  $f_x(\tau) = 0$  for some  $x$  does not imply that  $\tau$  is symmetric, but  $f_x(\tau) = 0 \forall x$  does mean that  $\tau$  is symmetric.

Proposition 3.2.10

Let  $X$  be stable. Suppose  $\mathcal{T}(X) = X * \mathcal{G}(X)$ , in the sense that if  $\tau \in \mathcal{T}(X)$  then there exists  $\sigma \in \mathcal{G}(X)$  and an  $x \in X$  such that  $\tau = \tau_x * \sigma$ , see chapter 2. If  $\sigma$  is a continuity point of  $f_z$ , then so is  $\tau$  and  $f_z(\tau) = \rho'(z;x)$ , and hence if  $f_z(\tau) = 0 \forall z$  then  $\tau$  is symmetric.

Proof

$$f_z(\tau) = f_z(\tau_x * \sigma) = f_z(\tau_x) + f_z(\sigma) = f_z(\tau_x) = \rho'(z;x).$$

If  $\rho'(z;x) = 0 \forall z$ , then put  $z=x$ . Hence  $\rho'(z;x) = \|x\| = 0$ , so  $x=0$  and  $\tau = \tau_0 * \sigma = \sigma \in \mathcal{G}(X)$ .

Later, we shall consider whether a random measure  $\xi$  representing a type  $\tau$ , is necessarily unique. We shall use the fact that  $f_x(\tau) + f_{-x}(\tau) = 0$  to enable us to consider the uniqueness of the integrand. We find a similar result for the functional  $f_g$  as follows:

Proposition 3.2.11

Suppose that, for some  $\tau \in \mathcal{T}(L_1(\Omega: X))$ , and all  $g \in L_1(\Omega: X)$ ,

$$\begin{aligned} f_g(\tau) &= \int_{\Omega} \int_{\mathcal{T}(X)} f_{g, \omega}(t) \cdot d\xi_{\omega}(t) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} f_{g, \omega}(t) \cdot d\eta_{\omega}(t) dP(\omega). \end{aligned}$$

If there exists  $x \neq 0$  such that every type on  $X$  is a continuity point of  $f_x$  then

$$\int_{\mathcal{T}(X)} f_{g, \omega}(t) d\xi_{\omega}(t) = \int_{\mathcal{T}(X)} f_{g, \omega}(t) d\eta_{\omega}(t) \text{ a.e. } [P].$$

Proof

Let  $h_1(\omega) = g(\omega) \cdot 1_A(\omega) + x \cdot 1_{A^c}(\omega)$ ,  $h_2(\omega) = g(\omega) \cdot 1_A(\omega) - x \cdot 1_{A^c}(\omega)$ ,

for some  $x \in X$ ,  $x \neq 0$ , such that every type is a continuity point

$$\begin{aligned} \text{of } f_x. \text{ Then } f_{h_1}(\tau) + f_{h_2}(\tau) &= \int_A \int_{\mathcal{T}(X)} 2f_{g, \omega}(t) \cdot d\xi_{\omega}(t) dP(\omega) \\ &+ \int_{A^c} \int_{\mathcal{T}(X)} f_x(t) + f_{-x}(t) \cdot d\xi_{\omega}(t) dP(\omega) \end{aligned}$$

and  $f_x(t) + f_{-x}(t) = 0 \quad \forall t \in \mathcal{T}(X)$ . Thus  $\forall A \subseteq \Omega$

$$\int_A \int_{\mathcal{T}(X)} f_{g, \omega}(t) \cdot d\xi_{\omega}(t) dP(\omega) = \int_A \int_{\mathcal{T}(X)} f_{g, \omega}(t) \cdot d\eta_{\omega}(t) dP(\omega) \text{ and}$$

hence the result.

Lemma 3.2.12

Let  $\tau$  be a uniformly integrable type on  $L_1(\Omega: X)$ . If every

type on  $X$  is a continuity point of  $f_{g(\omega)}$  and  $f_g(\tau) = 0$  for each

$$g \in L_1(\Omega: X) \text{ then } \int_{\mathcal{T}(X)} f_{g, \omega}(t) d\xi_{\omega}(t) = \int_{\mathcal{T}(X)} f_{g, \omega}(t) dD_{-1}\xi_{\omega}(t)$$

a.e. [P].

Proof

$$\begin{aligned} \int_{\Omega} \int_{\mathcal{T}(X)} f_{g, \omega}(t) \cdot d\xi_{\omega}(t) dP(\omega) &= - \int_{\Omega} \int_{\mathcal{T}(X)} f_{g, \omega}(t) \cdot d\xi_{\omega}(t) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} f_{g, \omega}(D_{-1}t) d\xi_{\omega}(t) dP(\omega) = \int_{\Omega} \int_{\mathcal{T}(X)} f_{g, \omega}(t) dD_{-1}\xi_{\omega}(t) dP(\omega). \end{aligned}$$

Hence the result by the above proposition.

Calculations of  $f_x$  on  $\mathcal{T}(E \times F)$ .

To calculate  $f_x$  on the space  $X = E \oplus F$ , we need to know  $\rho'[(x, y); (u, v)]$  for  $(x, y), (u, v) \in E \oplus F$ . Let  $(E, \|\cdot\|_E)$ ,  $(F, \|\cdot\|_F)$  be Banach spaces with dense  $G_\delta$ -sets of Fréchet differentiable points. Let  $\|(x, y)\| = \|(\|x\|_E, \|y\|_F)\|^*$ , where  $\|\cdot\|^*$  is a norm on  $\mathbb{R}^2$ .

Then one can show that there is a Fréchet differentiable point  $(x, y) \in X$ . Then the Gateaux derivative is given by  $\rho'[(x, y); (u, v)] = \rho'_U[(\|x\|, \|y\|); [\rho'_E(x; u), \rho'_F(y; v)]]$ , where  $\rho'_U(\cdot, \cdot)$  is the Fréchet derivative of  $\|\cdot\|^*$ , and  $\rho'_E, \rho'_F$  are the derivatives of  $\|\cdot\|_E$  and  $\|\cdot\|_F$  respectively.

Example

We now consider the space of types on  $X = \mathcal{L}_p \oplus \mathcal{L}_q$ , with a stable norm. We represent  $\mathcal{T}(\mathcal{L}_p \oplus \mathcal{L}_q)$  as  $(\mathcal{L}_p \times [0, \infty)) \oplus (\mathcal{L}_q \times [0, \infty))$ , and are thus able to calculate  $f_{(x, y)}(\tau)$ ,  $\tau \in \mathcal{T}(X)$ . Not all norms on  $X$  are stable, so we consider only norms given by "norms of  $\mathbb{R}^2$ " [Ra]. If  $\|\cdot\|^*$  is a norm on  $\mathbb{R}^2$ , define  $\|(x, y)\| = \|(\|x\|_p, \|y\|_q)\|^*$  for  $x \in \mathcal{L}_p, y \in \mathcal{L}_q$ . This is a stable norm giving the usual topology on  $\mathcal{L}_p \oplus \mathcal{L}_q$ .

Let  $\sigma$  be a type on  $X$ . So  $\exists z_n \in X$  with  $\sigma(x) = \lim_n \|x + z_n\|$ , then  $z_n = (x_n, y_n)$  for some  $(x_n), (y_n)$ . Then, for some  $(u, \alpha) \in \mathcal{L}_p \times [0, \infty)$  and  $(v, \beta) \in \mathcal{L}_q \times [0, \infty)$

$$\sigma(x, y) = \lim_n \|(\|x + x_n\|, \|y + y_n\|)\|^* =$$

$$\|([\|x + u\|^p + \alpha^p]^{1/p}, [\|y + v\|^q + \beta^q]^{1/q})\|^* = \|(\sigma_1(x), \sigma_2(y))\|^*,$$

where  $\sigma_1(x) = \lim_n \|x + x_n\|_p = (\|x + u\|^p + \alpha^p)^{1/p}$  and  $\sigma_2(y) = \lim_n \|y + y_n\|_q = (\|y + v\|^q + \beta^q)^{1/q}$ .

To check convolution, let  $\sigma, \tau \in \mathcal{G}(X)$ . Let  $\tau_1(z) = \lim_n \|z + r_n\|$   
 $= [\|z+r\|^p + \gamma^p]^{1/p}$ ,  $\tau_2(y) = \lim_n \|y + s_n\| = [\|y+s\|^q + \delta^q]^{1/q}$ ,  $\forall z \in \mathcal{Q}_p$ ,  
 $y \in \mathcal{Q}_q$ . Then  $(\sigma * \tau)(x, y) = \lim_n \lim_m \|(x_n, y_n) + (r_m, s_m) + (x, y)\|$   
 $= \lim_n \lim_m \|(x + x_n + r_m, y + y_n + s_m)\|$   
 $= \|([\|x+u+r\|^p + \alpha^p + \gamma^p]^{1/p}, [\|y+v+s\|^q + \beta^q + \delta^q]^{1/q})\|^*$   
 $= \|([\sigma_1 * \tau_1](x), [\sigma_2 * \tau_2](y))\|^*$ .

Then  $f_{(x,y)} = \lim_m \rho'[(x, y); (u_m, v_m)] = \rho'[(x, y); (u, v)]$  as  
expected. So  $\rho'[(x, y); (u, v)] =$

$$\rho'_{\mathbb{R}^2} \left[ (\|x\|, \|y\|); \left[ \sum_{n=1}^{\infty} \frac{x_n |x_n|^{p-2} u_n}{\|x\|^{p-1}}, \sum_{n=1}^{\infty} \frac{y_n |y_n|^{q-2} v_n}{\|y\|^{q-1}} \right] \right].$$

Let  $Y = \mathbb{R}^n$ , with the  $r$ -norm. Then for  $x = (x_n), y = (y_n) \in \mathbb{R}^n$ ,

$$\rho'(x, y) = \sum_{i=1}^n \frac{y_i |x_i|^{r-1} \text{sgn}(x_i)}{\|x\|^{r-1}}. \text{ So for } n=r=2,$$

$$\rho'_y(x, y) = \frac{x_1 y_1 + x_2 y_2}{\|x\|}.$$

The unit  $r$ -ball in  $\mathbb{R}^n$  is smooth on all  
it's boundary. For  $r=1$ , the unit 1-ball is smooth at all  
points with non-zero components, i.e.  $\forall x$  with  $x \neq 0$ . Then for

$n=2, r=1, \rho'_y(x, y) = y_1 \text{sgn}(x_1) + y_2 \text{sgn}(x_2)$ . For  $Y = (\mathbb{R}^2, \|\cdot\|_2)$  we

$$\text{get } \rho'_x[(x, y); (u, v)] = \frac{\sum_{n=1}^{\infty} \left[ \frac{x_n |x_n|^{p-2} u_n}{\|x\|^{p-2}} + \frac{y_n |y_n|^{q-2} v_n}{\|y\|^{q-2}} \right]}{(\|x\|^2 + \|y\|^2)^{1/2}}.$$

For  $Y = (\mathbb{R}^2, \|\cdot\|_1)$ ,  $\rho'_x[(x, y); (u, v)] =$

$$\sum_{n=1}^{\infty} \left[ \frac{x_n |x_n|^{p-2} u_n}{\|x\|^{p-1}} + \frac{y_n |y_n|^{q-2} v_n}{\|y\|^{q-1}} \right] = \rho'_E(x; u) + \rho'_F(y; v).$$

We now have a representation of  $\mathcal{T}(\mathcal{L}_p \oplus \mathcal{L}_q)$  as  $(\mathcal{L}_p \times [0, \infty)) \oplus (\mathcal{L}_q \times [0, \infty))$ , given by  $T: \sigma \longrightarrow ((u, \alpha), (v, \beta))$ . This is a measurable map, and so if  $\tau$  is a uniformly integrable type on  $L_1(\Omega; \mathcal{L}_p \oplus \mathcal{L}_q)$  then if  $g \in L_1$  and  $g(\omega) = (g_1(\omega), g_2(\omega)) \in \mathcal{L}_p \oplus \mathcal{L}_q$ ,

$$\tau(g) = \int_{\Omega} \int_{\Sigma} \|([\|g_1(\omega) + u\|^p + \alpha^p]^{1/p}, [\|g_2(\omega) + v\|^q + \beta^q]^{1/q})\|^* d\xi_{\omega}(u, \alpha, v, \beta) dP$$

where  $\Sigma = (\mathcal{L}_p \times [0, \infty)) \oplus (\mathcal{L}_q \times [0, \infty))$ .

### Example

Let  $X = \mathcal{L}_p \oplus c_0$ , be equipped with the norm  $\|(x, y)\| = \|(\|x\|_p, \|y\|_{\infty})\|^*$ , where  $\|\cdot\|^*$  is a norm on  $\mathbb{R}^2$ . Let  $\tau \in \mathcal{T}(X)$ . Then  $\tau(x, y) = \|(\tau_1(x), \tau_2(y))\|^*$  for some  $\tau_1 \in \mathcal{T}(\mathcal{L}_p)$ ,  $\tau_2 \in \mathcal{T}(c_0)$ .  $\tau$  is symmetric iff  $\tau_1$  and  $\tau_2$  are both symmetric iff  $u = v = 0$ .

Let  $(x, y)$  be a Fréchet differentiable point of  $X$ . Let  $\sigma$  be the symmetric restriction of  $\tau$ , i.e.  $\sigma(x, y) = \|([\|x\|_p + \alpha^p]^{1/p}, \max[\|y\|, \beta])\|^*$ . Then if  $\sigma$  is a continuity point  $0 = f_{(x, y)}(\sigma) = \lim_m \rho'[(x, y); (u_m, v_m) - (u, v)]$ . Thus  $f_{(x, y)}(\tau) = \rho'[(x, y); (u, v)]$ , whenever this exists.

### 3.3 Gateaux and Fréchet differentiability in $\mathcal{T}(X)$ .

In this section we look at Gateaux and Fréchet differentiability in  $\mathcal{T}(X)$ , and we examine how the functional  $\tau'(x; \cdot)$  behaves in different situations.

A function  $g$  is Gateaux differentiable at  $x \in X$ , in the direction of  $y$ , if  $g'(x; y) \equiv \lim_{t \rightarrow 0} \frac{g(x+ty) - g(x)}{t}$  exists.  $g$  is Fréchet differentiable if the limit exists uniformly in  $y \in \text{ball}(X)$ .  $g$  is uniformly Gateaux differentiable if for each  $y$  the limit exists uniformly in  $x \in X$ , see [Ho]. If  $\|\cdot\|_X$  is uniformly Gateaux differentiable, then  $\rho'(x+y_m; y)$  exists uniformly in  $m$ , so that

$$\tau'(x; y) \equiv \lim_{t \rightarrow 0} \frac{\tau(x+ty) - \tau(x)}{t} = \lim_m \rho'(x+y_m; y) \text{ exists.}$$

We assume throughout this section that  $X$  is a stable Banach space with a uniformly Gateaux differentiable norm. Then, of course  $\tau'(x; y)$  is independent of the sequence chosen to define  $\tau$ . That is  $\tau'$  is well defined. Obviously  $\tau'(x; \alpha y + \beta z) = \alpha \tau'(x; y) + \beta \tau'(x; z)$ .

#### Lemma 3.3.1

$$(i) (\sigma * \tau)'(x; y) = \lim_m \lim_n \rho'(x+y_m+x_n; y).$$

$$(ii) [D_a \tau]'(x; y) = a \tau'(x/a; y/a).$$

$$(iii) \tau'(ax; y)/a = [D_{1/a} \tau]'(x; y/a).$$

$$(iv) \tau'(x; y) = (\tau * \tau_x)'(0; y).$$

#### Proof

An easy exercise.

Lemma 3.3.2

- (i)  $\tau'_u(v; y) = \rho'(u+v; y)$ , thus  $\tau'(0; y) = \|y\|$ .  
(ii) If  $\sigma \in \mathcal{S}(X)$ , then  $\sigma'(-x; y) = \sigma'(x; y)$  and  $\sigma'(0; y) = 0 \forall x, y \in X$ .  
(iii) If  $\mathcal{T}(X) = X * \mathcal{S}(X)$ , then  $[\mathcal{S}(X)]' = 0$  implies that  $[\mathcal{T}(X)]' = 0$ .  
(iv) If  $\tau_i \longrightarrow \tau$  in the uniform topology, then  $\tau_i' \longrightarrow \tau'$  in the pointwise topology on  $X \times X$ .

Proof

Let  $\sigma \in \mathcal{S}(X)$ . Then  $\sigma'(x; y) = \lim_m \rho'(x+y_m; y) = \lim_m \rho'(x-y_m; y)$   
 $= -\lim_m \rho'(-x+y_m; y) = -\sigma'(-x; y)$  and  $\sigma'(x; y) = -[D_1 \sigma]'(-x; -y)$   
 $= \sigma'(-x; y)$ .

So that  $\sigma'(0; y) = -\sigma'(0; y)$  thus  $\sigma'(0; y) = 0$ .

If  $\mathcal{T}(X) = X * \mathcal{S}(X)$ , then for  $\tau \in \mathcal{T}(X)$ ,  $\tau(x) = \lim \|x+y+y_m\|$ ; then  
 $\tau'(x; z) = \lim_m \rho'(x+y+y_m; z) = \sigma'(x+y; z)$ . So that  $\sigma'(u; v) = 0 \forall u \Rightarrow$   
 $\tau'(w; v) = 0 \forall w$ .

$d_\infty(\tau_i, \tau) = \sup\{|\tau_i(x) - \tau(x)| : x \in X\} \longrightarrow 0$ , as  $i \longrightarrow \infty$ . With this topology we are able to exchange the  $\lim_i$  and  $\lim_{t \rightarrow 0}$  limits, so that  $\tau_i'(x; y) \longrightarrow \tau'(x; y) \forall x, y \in X$ .

Proposition 3.3.3

Let  $\tau \in \mathcal{T}(L_1(\Omega; X))$  be represented by the random measure  $\xi \in \pi_1(\mathcal{T}(X))$  and scalar  $\alpha \geq 0$ . Let  $f, g \in L_1(\Omega; X)$ . Then

$$\tau'(f; g) = \int_{\Omega} \int_{\mathcal{T}(X)} \sigma'[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) \cdot dP(\omega).$$

Proof

$$\tau(g) = \int_{\Omega} \int_{\mathcal{T}(X)} \sigma(g(\omega)) \cdot d\xi_{\omega}(\sigma) dP(\omega) + \alpha. \text{ Then}$$

$$\tau'(f; g) = \lim_{t \rightarrow 0} \frac{1}{t} \int_{\Omega} \int_{\mathcal{T}(X)} \sigma[f(\omega) + tg(\omega)] - \sigma[f(\omega)] \cdot d\xi_{\omega}(\sigma) dP(\omega)$$

$$= \int_{\Omega} \int_{\mathcal{T}(X)} \sigma'[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) dP(\omega), \text{ by applying the DCT}$$

twice.

Computing  $\tau'(x; z)$  is fairly straightforward, and we can now furnish ourselves with several examples of  $\tau'(x; z)$  on simple Banach spaces.

### Examples

1.  $X = \mathcal{Q}_p$ . Let  $\tau \in \mathcal{F}(X)$ . Then  $\tau(x) = \lim \|x + y_m\| = (\|x + y\|^p + \alpha^p)^{1/p}$ . Suppose that  $\tau'(x; z)$  exists. Then  $\tau'(x; z) = \lim \rho'(x + y_m; z) = \sum_{i=1}^{\infty} z_i \cdot \frac{(x + y_m)_i |(x + y_m)_i|^{p-2}}{\|x + y_m\|^{p-2}} \longrightarrow \rho'(x + y; z)$ , since  $(y_m)_i \longrightarrow y_i \quad \forall m$  as  $i \longrightarrow \infty$ .

Let  $\sigma$  be the symmetric restriction of  $\tau$ , i.e.

$$\begin{aligned} \sigma(x) &= (\|x\|^p + \alpha^p)^{1/p}. \text{ Then } \sigma'(0; z) = 0 \quad \forall z. \text{ So that } 0 = \lim_{t \rightarrow 0} \frac{\sigma(tz) - \sigma(0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(\|tz\|^p + \alpha^p)^{1/p} - \alpha}{t} = \lim_{t \rightarrow 0} \frac{[(\|tz + y - y\|^p + \alpha^p)^{1/p} - \alpha]}{t} \\ &= \lim_{t \rightarrow 0} \frac{\tau(tz - y) - \tau(-y)}{t} = \lim_m \rho'(y_m - y; z). \end{aligned}$$

By direct calculation we have that

$$\begin{aligned} \tau'(x; z) &= (\|x + y\|^p + \alpha^p)^{1/p-1} \|x + y\|^{p-1} \rho'(x + y; z) = \\ &= \tau(x)^{1-p} \|x + y\|^{p-1} \rho'(x + y; z). \end{aligned}$$

2.  $X = c_0$ . Let  $\tau(x) = \max\{\|x + z\|, \alpha\}$ . Then we have three cases: (i)  $\|x + z\| < \alpha$ . Then  $\tau(x + y) = \tau(x) = \alpha$  for sufficiently small  $\|y\|$ , so  $\tau$  is Fréchet differentiable at  $x$  and  $\tau'(x, y) = 0$ .

(ii)  $\|x + z\| > \alpha$ . Then  $\tau(x + y) = \|x + y + z\|$  for sufficiently small  $\|y\|$ , so  $\tau$  is Fréchet differentiable at  $x$  iff  $x + z$  is peak and  $\tau'(x, y) = \rho'(x + z; y)$ .

(iii)  $\|x + z\| = \alpha$ , then  $\tau$  is not Gateaux differentiable at  $x$ .

Example

1. Let  $\tau \in \mathcal{T}(L_1(\Omega; \mathcal{Q}_p))$  be represented by the pair

$(\xi, \alpha) \in \pi_1(\mathcal{T}(\mathcal{Q}_p)) \times [0, \infty)$ . Then

$$\begin{aligned} \tau'(f; g) &= \int_{\Omega} \int_{\mathcal{T}(\mathcal{Q}_p)} \sigma'[f(\omega); g(\omega)] d\xi_{\omega}(\sigma) dP(\omega) = \\ &= \int_{\Omega} \int_{\mathcal{Q}_p \times [0, \infty)} (\|f(\omega) + y\|^{-p} \alpha^{p+1})^{1/p-1} \rho'[f(\omega) + y; g(\omega)] d\xi_{\omega}(y, \alpha) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{Q}_p} (\|f(\omega) + y\|^{-p} \alpha^{p+1})^{1/p-1} \rho'[f(\omega) + y; g(\omega)] d\xi_{\omega}^2(y) dP(\omega) \end{aligned}$$

where  $\xi^2$  is the marginal (random) measure of  $\xi$  on  $\mathcal{Q}_p$ .

2.  $\tau'(0; g) = 0 \forall g$  does not imply that  $\tau$  is symmetric.

Let  $\xi_{\omega}^2 = [\delta_{(e_1, 1)} + \delta_{(-e_1, 1)/2}] / 2$ , and  $\xi_{\omega}^y = \begin{cases} \delta_1 & y = e_1 \\ \delta_{1/2} & y = -e_1/2 \\ 0 & \text{o.w.} \end{cases}$ . Then  $\tau$

is not symmetric, since  $\tau(g) = E_{\omega} \{ (\|g(\omega) + e_1\|^p + 1)^{1/p} / 2 +$

$(\|g(\omega) - e_1/2\|^p + 1/2^p)^{1/p} \}$ , but  $\forall g \tau'(0; g) =$

$E_{\omega} \{ \rho'(e_1; g(\omega)) + \rho'(-e_1; g(\omega)) \} / 2 = 0$ , and neither  $\xi^2$  or  $\xi_y$  are symmetric.

Again keeping one eye on possible applications to the uniqueness problem, we see that in the case of  $\tau'(\cdot; \cdot)$  we can again reduce the problem to that of considering just the integrand.

Proposition 3.3.4

Let  $\xi, \eta \in \pi_1(\mathcal{T}(X))$ . Suppose that for all  $f, g \in L_1(\Omega; X)$ ,

$$\begin{aligned} \int_{\Omega} \int_{\mathcal{T}(X)} t'[f(\omega); g(\omega)] d\xi_{\omega}(t) dP(\omega) &= \\ \int_{\Omega} \int_{\mathcal{T}(X)} t'[f(\omega); g(\omega)] d\eta_{\omega}(t) dP(\omega). & \text{ Then } \int_{\mathcal{T}(X)} t'[f(\omega); g(\omega)] d\xi_{\omega}(t) = \\ \int_{\mathcal{T}(X)} t'[f(\omega); g(\omega)] d\eta_{\omega}(t), & \text{ a.e. } [P]. \end{aligned}$$

Proof

For all measurable sets  $A \subseteq \Omega$ ,  $\forall f, g \in L_1(\Omega; X)$

$$\int_A \int_{\mathcal{T}(X)} t' [f(\omega); g(\omega)] d\xi_\omega(t) dP(\omega) = \int_A \int_{\mathcal{T}(X)} t' [f(\omega); g(\omega)] d\eta_\omega(t) dP(\omega). \text{ The result follows.}$$

Examples

1. The space  $X = E \oplus F$ . Let  $\|\cdot\|^*$  be a norm on  $\mathbb{R}^2$ , set  $\|(x, y)\| = \|(\|x\|_E, \|y\|_F)\|^*$ . Assume  $E, F$  are separable stable Banach spaces, and that  $E \oplus F$  is uniformly Gateaux differentiable.

Let  $(x, y)$  be a Fréchet differentiable point of  $X = E \oplus F$ . Then  $\rho'[(x, y); (u, v)] = \rho'_{\mathbb{R}^2}[(\|x\|, \|y\|); (\rho'_E(x; u), \rho'_F(y; v))]$ . For  $\tau \in \mathcal{T}(X)$ ,  $\tau(x, y) = \|(\tau_1(x), \tau_2(y))\|^*$  for some  $\tau_1 \in \mathcal{T}(E)$  and  $\tau_2 \in \mathcal{T}(F)$ .

Suppose  $\tau'[(x, y); (u, v)]$  exists. Then if  $(a_n, b_m)$  defines  $\tau$ ,  $\tau'[(x, y); (u, v)] = \lim_m \rho'_X[(x, y) + (a_m, b_m); (u, v)] = \lim_m \rho'_X[(x + a_m, y + b_m); (u, v)] = \lim_m \rho'_{\mathbb{R}^2}[(\|x + a_m\|, \|y + b_m\|); (\rho'_E(x + a_m; u), \rho'_F(y + b_m; v))]$ .

2. Let  $E = \mathcal{L}_p$ ,  $F = \mathcal{L}_q$ . Then  $\tau_1(x) = (\|x + a\|^p + \alpha^p)^{1/p}$ ,  $\tau_2(y) = (\|y + b\|^q + \beta^q)^{1/q}$  so that  $\tau'[(x, y); (u, v)] = \rho'_{\mathbb{R}^2}[(\|x + a\|^p + \alpha^p)^{1/p}, (\|y + b\|^q + \beta^q)^{1/q}; (A(x, a, u), B(y, b, v))]$ , where  $A(x, a, u) = (1 + \alpha^p \|x + a\|^{-p})^{1/p-1} \rho'_E(x + a; u)$  and  $B(y, b, v) = (1 + \beta^q \|y + b\|^{-q})^{1/q-1} \rho'_F(y + b; v)$ .

Proposition 3.3.5

Let  $\tau \in \mathcal{T}(L_p(\Omega: X))$  be uniformly integrable and represented by the random measure  $\xi \in \pi_p(\mathcal{T}(X))$ . Then  $\tau'(f; g) =$

$$\left[ \int_{\Omega} \int_{\mathcal{T}(X)} t [f(\omega)]^p d\xi_{\omega}(t) dP(\omega) \right]^{1/p-1} \int_{\Omega} \int_{\mathcal{T}(X)} t [f(\omega)]^{p-1} t' [f(\omega); g(\omega)] d\xi_{\omega}(t) dP(\omega).$$

Proof

Let  $U(\alpha) = \alpha^{1/p}$ ,  $V(f) = \tau(f)^p$ . Then  $U'(\alpha) = \alpha^{1/p-1}/p$  and  $V'(f; g) =$

$$\lim_{t \rightarrow 0} \int_{\Omega} \int_{\mathcal{T}(X)} \frac{\sigma [f(\omega) + tg(\omega)]^p - \sigma [f(\omega)]^p}{t} d\xi_{\omega}(\sigma) dP(\omega) =$$

$$\int_{\Omega} \int_{\mathcal{T}(X)} p \sigma^{p-1} [f(\omega)] \sigma' [f(\omega); g(\omega)] d\xi_{\omega}(\sigma) dP(\omega). \text{ So that } \tau'(f; g) =$$

$$\left[ \int_{\Omega} \int_{\mathcal{T}(X)} t [f(\omega)]^p d\xi_{\omega}(t) dP(\omega) \right]^{1/p-1} \int_{\Omega} \int_{\mathcal{T}(X)} p t^{p-1} [f(\omega)] t' [f(\omega); g(\omega)] d\xi_{\omega}(t) dP(\omega) / p$$

Examples

1.  $X = \mathcal{L}_q$ . Let  $\tau \in \mathcal{T}(L_p(\Omega: X))$  be uniformly integrable and represented by the random measure  $\xi \in \pi_p(\mathcal{T}(X))$ . Then  $\tau'(f; g)$  is

$$\left[ \int_{\Omega} \int_{\mathcal{L}_q \times \mathcal{L}_{0, \infty}} (\|f(\omega) + y\|^q + \beta^q)^{p/q} d\xi_{\omega}(y, \beta) dP(\omega) \right]^{1-p/p} \int_{\Omega} \int_{\mathcal{L}_q \times \mathcal{L}_{0, \infty}} C_{\omega}(y, \beta) d\xi_{\omega}(y, \beta) dP$$

where  $C_{\omega}(y, \beta) = (\|f(\omega) + y\|^q + \beta^q)^{p/q-1} \|f(\omega) + y\|^{q-1} p' [f(\omega) + y; g(\omega)]$ .

2. Let  $\tau \in \mathcal{T}(L_p(\Omega: \mathbb{R}))$  be uniformly integrable and represented by the random measure  $\xi \in \pi_p(\mathbb{R})$ . Then  $\tau'(f; g)$  is

$$\left[ \int_{\Omega} \int_{\mathbb{R}} |f(\omega) + t|^p d\xi_{\omega}(t) dP(\omega) \right]^{1-p/p} \int_{\Omega} \int_{\mathbb{R}} C_{\omega}(t) d\xi_{\omega}(t) dP(\omega) \text{ where}$$

$$C_{\omega}(t) = |f(\omega) + t|^{p-1} \text{sign}[f(\omega) + t] \cdot g(\omega).$$

### 3.4 Higher derivatives.

Having looked at first order differentiability in  $\mathcal{T}(X)$ , we now give a cursory look at second order derivatives in  $\mathcal{T}(X)$ , and compute some simple examples.

Throughout this section let  $X$  be a separable stable Banach space with uniformly Fréchet differentiable norm. Let  $x, y, z \in X$ ,  $\tau \in \mathcal{T}(X)$ , then  $\tau''(x)(y; z) = \lim_{t \rightarrow 0} \frac{\tau'(x+ty)(z) - \tau'(x)(z)}{t} = \lim_{t \rightarrow 0} \frac{\tau'(x+ty; z) - \tau'(x; z)}{t} = \lim_{t \rightarrow 0} \lim_m \frac{\rho'(x+ty+y_m; z) - \rho'(x+y_m; z)}{t}$ . Since  $\|\cdot\|_X$  is uniformly Fréchet differentiable,  $\rho''$  will exist so that

$$\tau''(x)(y; z) = \lim_m \lim_{t \rightarrow 0} \frac{\rho'(x+ty+y_m; z) - \rho'(x+y_m; z)}{t} = \lim_m \rho''(x+y_m)(y; z).$$

#### Lemma 3.4.1

Let  $\tau \in \mathcal{T}(X)$ . Let  $x, y, z \in X$ ,  $a > 0$ . Then

1.  $[D_a \tau]''(x)(y; z) = a \tau''(x/a)(y/a; z/a)$ .
2.  $\tau''(x)(y; \cdot)$  is linear.
3.  $\tau''(ax)(y; z) = a [D_{1/a} \tau]''(x/a)(y/a; z/a)$ .
4.  $\tau''(x)(ay; z) = a [D_{1/a} \tau]''(x/a)(y; z/a)$ .

#### Proposition 3.4.2

Let  $\tau \in \mathcal{T}(L_1(\Omega; X))$  be uniformly integrable and be represented by  $\xi$  and  $\eta$ . Then  $\forall f, g, h \in L_1(\Omega; X)$

$$\tau''(f)(g; h) = \int_{\Omega} \int_{\mathcal{T}(X)} \sigma''[f(\omega)][g(\omega); h(\omega)] d\xi_{\omega}(\sigma) dP(\omega)$$

and for almost all  $\omega \in \Omega$ ,

$$\int_{\mathcal{T}(X)} \sigma''[f(\omega)][g(\omega); h(\omega)] d\xi_{\omega}(\sigma) = \int_{\mathcal{T}(X)} \sigma''[f(\omega)][g(\omega); h(\omega)] d\eta_{\omega}(\sigma).$$

Proof

$$\tau''(f)(g;h) = \lim_{t \rightarrow 0} \int_{\Omega} \int_{\mathcal{T}(X)} \frac{\sigma'[f(\omega)+tg(\omega);h(\omega)] - \sigma'[f(\omega);h(\omega)]}{t} d\xi_{\omega}(\sigma) dP$$

$$\text{Let } k=1_A h. \text{ Then } \tau''(f)(g;k) = \int_A \int_{\mathcal{T}(X)} \sigma''[f(\omega)][g(\omega);h(\omega)] d\xi_{\omega}(\sigma) dP(\omega)$$

$$= \int_A \int_{\mathcal{T}(X)} \sigma''[f(\omega)][g(\omega);h(\omega)] d\eta_{\omega}(\sigma) dP(\omega) \quad \text{for all } A \subseteq \Omega.$$

Example

Let  $\tau \in \mathcal{T}(\mathcal{L}_p)$ . Then  $\tau'(x;z) = \rho'(x+y;z)$  for some  $y \in \mathcal{L}_p$ . Thus

$$\tau''(x)(u;z) = \lim_{t \rightarrow 0} \frac{\rho'(x+tu+y;z) - \rho'(x+y;z)}{t} =$$

$$(\|x+y\|^{\beta} + \alpha^p \|x+y\|^{\gamma}) \{ (p-1)\alpha^p \rho'(x+y;u) \rho'(x+y;z) + (\|x+y\|^{p-1} + \alpha^p \|x+y\|$$

$$) \rho''(x+y)(u;z) \} = \Gamma(x,u,z,y,\alpha), \text{ where } \beta = (p^2+p)/(2p-1) \text{ and}$$

$$\gamma = (p-2)p/(1-2p). \text{ So that } \tau''(u) \text{ exists whenever } \tau'(x;z) \text{ and}$$

$$\rho''(x+y) \text{ exist } \forall z. \text{ Then for } \tau \in \mathcal{T}(L_1(\Omega;X)); f,g,h \in L_1(\Omega;X) \text{ we}$$

have  $\tau''(f)(g;h) = \int_{\Omega} \int_{\mathcal{L}_p \times \mathcal{L}_{0,\infty}} \Gamma[f(\omega),g(\omega),h(\omega),y,\alpha] d\xi_{\omega}(y,\alpha) dP(\omega).$

3.5 The extension of the norm duality map to types.

In this section we consider the extension of the norm duality map from  $X$  to  $\mathcal{T}(X)$ . Throughout this section  $X$  is a separable smoothly normed Banach space. When referring to a type  $\tau$  on an  $L_p(X)$ -space, we assume that  $\tau$  is uniformly integrable and represented by the random measure  $\xi$ , via

$$\tau(f) = \int_{\Omega} \int_{\mathcal{T}(X)} t[f(\omega)] d\xi_{\omega}(t) dP(\omega).$$

Let  $f(x) = \|x\|^2/2$ . Then  $f$  is differentiable on all of  $X$ , and  $\nabla f(x) = \begin{cases} \|x\| \rho'(x;\cdot) & x \neq 0 \\ 0 & x = 0 \end{cases}$ . The mapping  $T = \nabla f$  is called the norm-duality map. It is monotone and demicontinuous. Then  $\text{range}(T)$  is dense in  $X^*$  whenever  $X$  is complete, but equals  $X^*$  only when  $X$  is reflexive, see [Ho].  $T$  is injective exactly

when  $X$  is strictly normed.

We can extend the norm-duality map in an obvious way from  $X$  to  $\mathcal{F}(X)$ . Consider  $\tau \in \mathcal{F}(X)$ , defined by  $(y_m)$ . Then whenever  $\tau'$  is defined,  $[\tau^2/2]'(x) = \lim_m \|x+y_m\| \cdot \rho'(x+y_m; \cdot) = \tau(x) \cdot \tau'(x; \cdot)$ .

Proposition 3.5.1

1.  $T\tau(x; \cdot)$  is linear.
2.  $TD_\alpha \tau(x; y) = \alpha^2 T\tau(x/\alpha; y/\alpha)$ .
3.  $T\tau(\alpha x; y) = \alpha TD_{1/\alpha} \tau(x; y)$ .
4.  $T\tau_u(v; y) = \|u+v\| \cdot \rho'(u+v; y)$  so that  $T\tau_0(v; y) = T(v)(y)$ .
5. If  $\tau_i \longrightarrow \tau$  in the uniform topology, then  $T\tau_i \longrightarrow T\tau$  pointwise.

Proof

1.  $T\tau(x; \alpha y + \beta z) = \tau(x) \cdot \tau'(x; \alpha y + \beta z) = \alpha T\tau(x; y) + \beta T\tau(x; z)$ .
2.  $TD_\alpha \tau(x; y) = D_\alpha \tau(x) \cdot [D_\alpha \tau]'(x; y) = \alpha \tau(x/\alpha) \cdot \alpha \tau'(x/\alpha; y/\alpha) = \alpha^2 T\tau(x/\alpha; y/\alpha)$ .
3.  $T\tau(\alpha x; y) = \tau(\alpha x) \cdot \tau'(\alpha x; y) = \alpha D_{1/\alpha} \tau(x) \cdot \alpha [D_{1/\alpha} \tau]'(x; y/\alpha) = \alpha^2 TD_{1/\alpha} \tau(x; y/\alpha) = \alpha TD_{1/\alpha} \tau(x; y)$ .
4.  $T\tau_u(v; y) = \tau_u(v) \cdot \tau_u'(v; y)$ .
5.  $\tau_i \longrightarrow \tau$  in the  $d_\infty$  topology means that as  $i \longrightarrow \infty$   $\sup\{|\tau_i(x) - \tau(x)| : x \in X\} \longrightarrow 0$ . Then  $\tau_i'(x; y) \longrightarrow \tau'(x; y) \forall x, y \in X$ , as  $i \longrightarrow \infty$ .

We can easily work out  $T\tau$  for  $\tau \in \mathcal{F}(L_2(X))$  as we do below, and then see what information we can extract if the random measure representing a uniformly integrable type is not uniquely defined.

Proposition 3.5.2

Let  $\tau \in \mathcal{T}(L_2(X))$  be represented by  $(\xi, \alpha) \in \pi_2(\mathcal{T}(X)) \times [0, \infty)$ .

Then for all  $f, g \in L_2(X)$

$$T\tau(f; g) = \int_{\Omega} \int_{\mathcal{T}(X)} \sigma[f(\omega)] \sigma'[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) dP(\omega).$$

Proof

We have  $\tau(f) = \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} \sigma[f(\omega)]^2 \cdot d\xi_{\omega}(\sigma) \cdot dP(\omega) + \alpha^2 \right\}^{1/2}$ . Then

$$\begin{aligned} T\tau(f; g) &= \lim_{s \rightarrow 0} \int_{\Omega} \int_{\mathcal{T}(X)} \frac{\sigma[f(\omega) + sg(\omega)]^2 - \sigma[f(\omega)]^2}{2s} \cdot d\xi_{\omega}(\sigma) \cdot dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) \cdot dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} \sigma[f(\omega)] \sigma'[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) \cdot dP(\omega). \end{aligned}$$

Examples

1.  $X = \mathbb{R}$ . Let  $\tau \in \mathcal{T}(L_2)$ ,  $f, g \in L_2$ . Then

$$\begin{aligned} T\tau(f; g) &= \int_{\Omega} \int_{\mathcal{T}(\mathbb{R})} \sigma[f(\omega)] \sigma'[f(\omega); g(\omega)] d\xi_{\omega}(\sigma) dP(\omega) \\ &= \int_{\Omega} \int_{\mathbb{R}} |f(\omega) + t| \cdot \text{sign}[f(\omega) + t] g(\omega) d\xi_{\omega}(t) dP(\omega) \\ &= \int_{\Omega} \int_{\mathbb{R}} [f(\omega) + t] g(\omega) d\xi_{\omega}(t) dP(\omega) = E_{\omega}[f(\omega)g(\omega)] + E_{\omega} \left\{ g(\omega) \int_{\mathbb{R}} t d\xi_{\omega}(t) \right\}. \end{aligned}$$

2.  $X = \mathcal{L}_p$ . Let  $\tau \in \mathcal{T}(L_2(\mathcal{L}_p))$  be represented by

$(\xi, \beta) \in \pi_2(\mathcal{T}(\mathcal{L}_p)) \times [0, \infty)$ . Then for all  $f, g \in L_2(\mathcal{L}_p)$ .

$$\begin{aligned} T\tau(f; g) &= \int_{\Omega} \int_{\mathcal{L}_p \times [0, \infty)} (\|f(\omega) + y\|^p + \alpha^p)^{1/p} \rho'[f(\omega) + y; g(\omega)] \cdot d\xi_{\omega}(y, \alpha) dP(\omega) = \\ &= \int_{\Omega} \int_{\mathcal{L}_p \times [0, \infty)} (\|f(\omega) + y\|^p + \alpha^p)^{1/p} \sum_{i=1}^{\infty} \frac{[f(\omega) + y]_i |f(\omega)_i + y_i|^{p-2} g(\omega)_i}{\|f(\omega) + y\|^{p-1}} \cdot d\xi_{\omega}(y, \alpha) dP \end{aligned}$$

Proposition 3.5.3

Let  $\tau \in \mathcal{T}(L_2(X))$  be represented by  $\xi \in \pi_2(\mathcal{T}(X))$ ,  $\alpha \geq 0$  and suppose that for all  $f \in L_2(X)$

$$\begin{aligned} \tau(f) &= \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} \sigma[f(\omega)]^2 d\xi_{\omega}(\sigma) dP(\omega) + \alpha^2 \right\}^{1/2} \\ &= \left\{ \int_{\Omega} \int_{\mathcal{T}(X)} \sigma[f(\omega)]^2 d\eta_{\omega}(\sigma) dP(\omega) + \alpha^2 \right\}^{1/2}. \end{aligned}$$

Then for all  $f$  and  $g$  in  $L_2(X)$

$$\int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) = \int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] \cdot d\eta_{\omega}(\sigma)$$

a.e. [P].

Proof

By the above proposition  $\forall f, g$

$$\begin{aligned} T\sigma(f; g) &= \int_{\Omega} \int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] d\xi_{\omega}(\sigma) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] d\eta_{\omega}(\sigma) dP(\omega). \end{aligned}$$

Then  $\forall A \subseteq \Omega$ ,

$$T\sigma[f(\omega); g(\omega) 1_A] = \begin{cases} T\sigma[f(\omega); g(\omega)] & \omega \in A \\ 0 & \omega \notin A \end{cases}, \text{ so that } \forall A, \forall f, g \in L_2(X)$$

$$\int_A \int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] \cdot d\xi_{\omega}(\sigma) dP(\omega) = \int_A \int_{\mathcal{T}(X)} T\sigma[f(\omega); g(\omega)] \cdot d\eta_{\omega}(\sigma) dP(\omega)$$

the result follows.

CHAPTER 4 : UNIQUENESS THEOREMS FOR MEASURES  
AND RANDOM MEASURES.

4.1 INTRODUCTION.

In [Ga], Garling constructs a map between  $\pi_\phi(\mathcal{T}(X))$  and the space of uniformly integrable types on  $L_\phi(X)$ . This map is continuous and onto, thus every type  $\tau$  has an associated pair  $(\xi, \alpha)$ , where  $\xi \in \pi_\phi(\mathcal{T}(X))$  and  $\alpha \geq 0$ . In [Ga] it is remarked that even in some very simple cases this map is not one-one. That is there are spaces  $L_\phi(X)$  such that there exists a type on  $L_\phi(X)$  which has two distinct associated random measures. As his example Garling uses the space  $L_2(\mathbb{R})$ .

This chapter develops this theme, and looks at a number of different possible uniqueness questions. There are analogous uniqueness properties for  $\mathcal{T}(X)$  and for  $X$  itself. We start by looking at uniqueness results on  $\mathcal{T}(L_1(X))$ . We show that the representation of uniformly integrable types on  $L_1(\mathbb{R})$  by random measures on  $\mathbb{R}$  is unique, by that we mean the map defined in [Ga] is one-one. Later we shall show that the representation of  $\mathcal{T}(L_p(\mathbb{R}))$  by random measures is unique if and only if  $p$  is not an even integer. We consider the cases  $L_1(\mathcal{Q}_p)$  and  $L_1(c_0)$ , and while we were unable to solve these situations completely, we have found positive partial results.

The non-uniqueness of the representation of uniformly integrable types on  $L_p(\mathcal{Q}_p)$  by random measures on  $\mathcal{T}(\mathcal{Q}_p)$ ,  $1 \leq p < \infty$ , is established; and we use this to construct various spaces  $Y$  with  $L_1(Y)$  having a non-unique representation of

types.

The final section of this chapter deals with the analogous situation of uniqueness over  $X$ . Such a situation has been dealt with by Linde [Lil] amongst others. Here we consider some aspects of non-uniqueness, in particular looking at the question: given non-uniqueness, what is the structure of  $X$ ? We show that if we have non-uniqueness of a certain kind then we can embed  $\mathcal{Q}_1^n$  into  $X$ . We find that non-uniqueness will also force the behaviour of the norm on  $X$ . In particular we show that a functional norm with non-uniqueness must be essentially an  $\mathcal{Q}_1$ -norm.

Part 1: Uniqueness properties for  $X$  and  $\mathcal{T}(X)$ .

Definition.

1.  $X$  has  $(U_p)$  if the representation of types on  $L_p(X)$  by random probabilities on  $\mathcal{T}(X)$  is unique. So  $X$  has  $(U_p)$  iff for distinct random measures  $\xi, \eta: \Omega \rightarrow \mathcal{P}(\mathcal{T}(X))$  there necessarily exist  $g \in L_p(X)$  such that

$$\int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)]^p d\xi_{\omega}(t) dP(\omega) \neq \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)]^p d\eta_{\omega}(t) dP(\omega).$$

2.  $X$  has  $(u_p)$  if the analogous uniqueness property holds for probabilities on  $\mathcal{T}(X)$ . Thus  $X$  has  $(u_p)$  iff for distinct measures  $\mu, \nu \in \mathcal{P}(\mathcal{T}(X))$  there necessarily exist  $z \in X$  such that

$$\int_{\mathcal{T}(X)} t[z]^p d\mu(t) \neq \int_{\mathcal{T}(X)} t[z]^p d\nu(t).$$

3.  $X$  has  $(w_p)$  if for distinct measures  $\mu, \nu \in \mathcal{P}(X)$  there necessarily exists  $x \in X$  such that  $\int_X \|x+y\|^p d\mu(y) \neq \int_X \|x+y\|^p d\nu(y)$ .

4.  $X$  has  $(nw_p)$  if  $X$  has  $(w_p)$  for discrete measures  $\mu$  and  $\nu$ , where  $\mu = 1/n(\sum_{i=1}^n \delta_{a_i})$  and  $\nu = 1/2(\delta_b + \delta_c)$ , and all the points  $a_i, b$  and  $c$ ,  $(1 \leq i \leq n)$ , are all distinct.

5.  $X$  has (TB) if  $\mu \in \mathcal{P}(\mathcal{T}(X))$  is uniquely determined by  $\mu\{t \in \mathcal{T}(X) : t(x) \leq r\}$  for  $x \in X$ ,  $r \in \mathbb{R}_+$ .

6.  $X$  has (B) if given  $\xi, \eta \in \mathcal{P}(X)$  and  $\xi B(x, r) = \eta B(x, r)$  for all  $x \in X$  and  $r \geq 0$  then  $\xi = \eta$ .

In the first part of the chapter we show that if  $X$  admits a Fréchet differentiable point, then  $(U_1)$  reduces to  $(u_1)$ . In the case of  $X = \mathcal{Q}_p$  we show that  $(u_1)$  and  $(w_1)$  reduce to the ball properties (TB) and (B) respectively. We have a collection of positive results about the property (B) for measures on  $X$ , and a few results for measures on  $\mathcal{T}(X)$ . We do not know of any infinite dimensional space in which we can prove that  $(U_p)$  fails.

In the second part of the chapter we investigate cases where uniqueness fails. We prove that the failure of  $(nw_1)$  implies the presence of  $\mathcal{Q}_1^n$  in  $X$ . We show that  $[\mathcal{Q}_r \oplus \mathcal{Z}]_p$  doesn't have  $(U_q)$  whenever  $q/p, p/r \in \mathbb{N}$ , and that  $\mathbb{R}$  has  $(U_p)$  iff  $p \notin 2\mathbb{N}$ .

For the moment let us consider the case of uniformly integrable types on  $L_1(X)$ . A sensible first stage in proving that  $\xi$  and  $\eta$  were equal would be to show that the failure of  $(U_1)$  implies that  $\forall x \in X, \int_{\mathcal{T}(X)} \tau(x) d\xi_\omega(\tau) = \int_{\mathcal{T}(X)} \tau(x) d\eta_\omega(\tau)$  a.e. [P]. We shall see later how to prove this in certain cases when we have differentiability conditions on  $X$ . Though in general it is false both that

$$(1) \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d\xi_\omega(t) dP(\omega) = \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d\eta_\omega(t) dP(\omega) \text{ implies}$$

that  $\xi = \eta$  a.e. [P], and

$$(2) \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d\xi_\omega(t) dP(\omega) = \int_{\Omega} \int_{\mathcal{T}(X)} t[g(\omega)] d\eta_\omega(t) dP(\omega) \text{ implies}$$

that  $\int_{\mathcal{T}(X)} t[g(\omega)] d\xi_\omega(t) = \int_{\mathcal{T}(X)} t[g(\omega)] d\eta_\omega(t)$  a.e. [P].

The following two examples show that both (1) and (2) above fail in certain cases.

### Examples

1. Let  $\Omega = \mathbb{R}$  and  $P = \delta_1 + \delta_{-1}$ . Let  $X = \mathcal{Q}_1$ . Let  $\xi_\omega = \delta(a, b_\omega) + \delta(c, d_\omega)$  and  $\eta_\omega = \delta(a, f_\omega) + \delta(c, h_\omega)$  where  $b_1 = 1, b_{-1} = 2, d_1 = 2, d_{-1} = 3; f_1 = 2, f_{-1} = 1, h_1 = 3, h_{-1} = 2$ ; and  $a, c \in \mathcal{Q}_1$ . Then for all  $g \in L_1(X)$ ,

$$\int_{\Omega} \int_{\mathcal{T}(X)} \tau(g(\omega)) d\xi_\omega(\tau) dP(\omega) = \|g(1) + a\| + 1 + \|g(1) + c\| + 2 + \|g(-1) + a\| + 2 + \|g(-1) + c\| + 3 = \int_{\Omega} \int_{\mathcal{T}(X)} \tau(g(\omega)) d\eta_\omega(\tau) dP(\omega), \text{ but}$$

$$\int_{\mathcal{T}(X)} \tau(g(\omega)) d\xi_\omega(\tau) \neq \int_{\mathcal{T}(X)} \tau(g(\omega)) d\eta_\omega(\tau) \text{ on } A = [1/2, \infty) \text{ for}$$

example.

2.  $(U_1)$  fails in  $\mathcal{Q}_1$ . For any non-trivial probability space

$(\Omega, P)$  define  $\xi = 1/2[\delta(e_1, 0) + \delta(e_2, 1)]$  and

$\eta = 1/2[\delta(e_1, 1) + \delta(e_2, 0)]$ , where  $\{e_i\}$  is the standard basis of

$$\varrho_1. \text{ Then } \xi \neq \eta, \text{ but } \int_{\mathcal{T}(X)} \tau[x] d\xi(\tau) = 1/2[\|x+e_1\| + \|x+e_2\| + 1]$$

$$= \int_{\mathcal{T}(X)} \tau[x] d\eta(\tau).$$

#### 4.2 The reduction of $(U_1)$ to $(u_1)$ .

The next theorem gives us a sufficient condition for the reduction of  $(U_1)$  to  $(u_1)$ .

##### Theorem 4.2.1

Suppose that for all  $g \in L_1(\Omega; X)$

$$\int_{\Omega} \int_{\mathcal{T}(X)} \tau[g(\omega)] d\xi_{\omega}(\tau) dP(\omega) = \int_{\Omega} \int_{\mathcal{T}(X)} \tau[g(\omega)] d\eta_{\omega}(\tau) dP(\omega), \text{ and that}$$

there exists a point of Fréchet differentiability  $x \in X$ , then

$$\text{for all } y \in X, \int_{\mathcal{T}(X)} \tau(y) d\xi_{\omega}(\tau) = \int_{\mathcal{T}(X)} \tau(y) d\eta_{\omega}(\tau) \text{ a.e. } [P].$$

In fact the proof is related to the functional  $f_x$  defined in section 3.2, and the key property that  $f_x(\tau) + f_{-x}(\tau) = 0 \forall \tau$ . Though before  $f_x(\tau)$  was defined if  $\lim_n [\tau(nx) - n\|x\|]$  existed, here we do not need its existence but rather that  $\lim_n [\tau(nx) + \tau(-nx) - 2n\|x\|] = 0$ .

##### Lemma 4.2.2

Let  $y \in X$ . If  $X$  has Gateaux differentiable norm at  $x$  then

$$\tau_y(nx) + \tau_y(-nx) - 2n\|x\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

##### Proof

$$\tau_y(nx) - n\|x\| = \|nx+y\| - n\|x\| = \frac{\|x+y/n\| - \|x\|}{1/n} \longrightarrow \rho'(x;y) \text{ as } n \longrightarrow \infty$$

$$\text{Thus } \tau_y(nx) + \tau_y(-nx) - 2n\|x\|$$

$$= \frac{\|x+y/n\| - \|x\|}{1/n} - \rho'(x;y) + \frac{\|x-y/n\| - \|x\|}{1/n} - \rho'(x;-y) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Let  $\tau \in \mathcal{T}(X)$ . Suppose  $\{y_m\}$  is an approximating sequence for  $\tau$ . Let  $\|\cdot\|_X$  be Fréchet differentiable at  $x$ . Then  $\lim_m \frac{\|x+y_m/n\| - \|x\| - \rho'(x; y_m/n)}{\|y_m/n\|}$  exists whenever  $\tau(0) \neq 0$ , and  $\lim_m \lim_n \frac{\|x+y_m/n\| - \|x\| - \rho'(x; y_m/n)}{\|y_m/n\|} = 0$ , see chapter three.

Theorem 4.2.3

Let  $\|\cdot\|_X$  be Fréchet differentiable at  $x$ . Then  $\forall \tau \in \mathcal{T}(X)$ , and all  $z \in X$   $\lim_n [\tau(nx) + \tau(-nx) - 2n\|x\|] = 0$ , and

$$\lim_n [\tau(nx+z) + \tau(-nx+z) - 2n\|x\|] = 0.$$

Proof

Wlog assume  $\tau(0) \neq 0$ . Then

$$\begin{aligned} \lim_n [\tau(nx) + \tau(-nx) - 2n\|x\|] &= \lim_n \lim_m \frac{\|x+y_m/n\| + \|x-y_m/n\| - 2\|x\|}{1/n} \\ &= \lim_n \lim_m \left\{ \frac{\|x+y_m/n\| - \|x\| - \rho'(x; y_m/n)}{1/n} + \frac{\|x-y_m/n\| - \|x\| - \rho'(x; -y_m/n)}{1/n} \right\} \\ &= 0. \end{aligned}$$

Replace  $\tau$  by  $\tau * \tau_z$  in the previous result.

Theorem 4.2.4

Let  $A \in \mathcal{F}$ . Suppose  $\exists x \in X$  such that  $\|\cdot\|_X$  is Fréchet differentiable at  $x$ . Suppose  $\tau$  is a uniformly integrable type on  $L_1(\Omega; X)$ , which has two associated random measures  $\xi$  and  $\eta$ . Then for all  $g \in L_1(\Omega; X)$

$$\int_A \int_{\mathcal{T}(X)} t(g(\omega)) \cdot d\xi_\omega(t) \cdot dP(\omega) = \int_A \int_{\mathcal{T}(X)} t(g(\omega)) \cdot d\eta_\omega(t) \cdot dP(\omega)$$

Proof

By above  $\forall \tau \in \mathcal{T}(X)$   $\lim_n [\tau(nx+g(\omega)) + \tau(-nx+g(\omega)) - 2n\|x\|] = 0$   
a.e. [P]. Let  $\theta_n = 1/2 \{ \tau[nx1_{A^c} + g] + \tau[-nx1_{A^c} + g] \} - n\|x\| \cdot P(A^c)$ . Then

$$\theta_n = \int_A \int_{\mathcal{T}(X)} t(g(\omega)) \cdot d\xi_\omega(t) \cdot dP(\omega) + \int_{A^c} \int_{\mathcal{T}(X)} \left\{ \frac{t(nx+g(\omega)) + t(-nx+g(\omega))}{2} - n\|x\| \right\} d\xi_\omega(t) dP(\omega)$$

Now  $|t(nx+g(\omega)) + t(-nx+g(\omega)) - 2n\|x\|| \leq 2t(g(\omega))$ , and

$t(g(\omega))$  is  $\xi_\omega$ -integrable a.e. [P] since

$$\tau(g) = \int_\Omega \int_{\mathcal{T}(X)} t(g(\omega)) \cdot d\xi_\omega(t) \cdot dP(\omega) < \infty \text{ implies that}$$

$$\int_{\mathcal{T}(X)} t(g(\omega)) \cdot d\xi_\omega(t) < \infty \text{ a.e. [P].}$$

Thus by Dominated Convergence Theorem,

$$\int_{\mathcal{T}(X)} \frac{t(nx+g(\omega)) + t(-nx+g(\omega))}{2} - n\|x\| \cdot d\xi_\omega(t) \longrightarrow 0 \text{ a.e. as } n \longrightarrow \infty$$

Then since  $\int_{\mathcal{T}(X)} t(g(\omega)) \cdot d\xi_\omega(t)$  is P-integrable, we can apply

the DCT again to get

$$\int_{A^c} \int_{\mathcal{T}(X)} \frac{t(nx+g(\omega)) + t(-nx+g(\omega))}{2} - n\|x\| \cdot d\xi_\omega(t) \cdot dP(\omega) \longrightarrow 0$$

as  $n \longrightarrow \infty$ .

Thus  $\theta_n \longrightarrow \int_A \int_{\mathcal{T}(X)} t(g(\omega)) d\xi_\omega(t) dP(\omega)$  as  $n \longrightarrow \infty$ . Replacing  $\xi$  by

$\eta$  we obtain the result.

Corollary 4.2.5

With the same hypothesis as above we have

$$\int_{\mathcal{T}(X)} t[g(\omega)] d\xi_\omega(t) = \int_{\mathcal{T}(X)} t[g(\omega)] d\eta_\omega(t) \text{ a.e. [P].}$$

We can now prove what is sometimes known as Rudin's equimeasurability theorem fairly quickly.

Theorem 4.2.6

$\mathbb{R}$  has  $(U_1)$ .

Proof

Suppose that  $\int_{\mathbb{R}} |s+x| d\mu(s) = \int_{\mathbb{R}} |s+x| d\nu(s)$  for all  $x \in X$ . Let  $f(x) = \int_{\mathbb{R}} |s+x| d\mu(s) = \int_{-x}^{\infty} [s+x] d\mu(s) + \int_{-\infty}^{-x} -[s+x] d\mu(s)$ . We shall use

the result that if  $A$  is a positive random variable then  $E[A] = \int_{\mathbb{R}_+} 1 - P[A \leq t] dt$ , see [Bu]. Then for  $A = (Y+x)1_{[Y > -x]}$ ,

$\mu(A \leq t) = \mu[Y+x \leq t]$ , and for  $A = (Y+x)1_{[Y < -x]}$ ,  $\mu(A \leq t) = \mu[-t \leq Y+x]$ .

$$\begin{aligned} \text{Then } f(x) &= \int_{\mathbb{R}} [s+x] 1_{[s > -x]} d\mu(s) + \int_{\mathbb{R}} -[s+x] 1_{[s < -x]} d\mu(s) \\ &= \int_{\mathbb{R}_+} 1 - \mu[\omega+x \leq t] dt + \int_{\mathbb{R}_+} 1 - \mu[-t \leq \omega+x] dt = \int_{\mathbb{R}_+} \mu[\omega+x > t] dt + \int_{\mathbb{R}_+} \mu[\omega+x < -t] dt \\ &= \int_{-x}^{\infty} \mu(t, \infty) dt + \int_{-\infty}^{-x} \mu(-\infty, t) dt \end{aligned}$$

Lemma 4.2.7

Let  $g: [a, b] \rightarrow \mathbb{R}$  be Borel measurable and integrable wrt Lebesgue measure. If  $f(x) - f(a) = \int_a^x g(t) dt$ ,  $a \leq x \leq b$ , and  $g$  is continuous at  $x$ , then  $f$  is differentiable at  $x$  and  $f'(x) = g(x)$ . If the continuity hypothesis is dropped, we can prove that  $f'(x) = g(x)$  for almost every  $x \in [a, b]$ . See [As] or [Bu].

Let  $f_1(x) = \int_{-x}^{\infty} \mu(t, \infty) dt$  and  $f_2(x) = \int_{-\infty}^{-x} \mu(-\infty, t) dt$ . Then  $f_1(x) - f_1(a) = \int_{-x}^{-a} \mu(t, \infty) dt = \int_a^x \mu(-t, \infty) dt \quad \forall a \leq x$ . Thus by the lemma  $f_1$  is differentiable and  $f_1'(t) = \mu(-t, \infty)$  for almost all  $t \in \mathbb{R}$ . Similarly  $f_2'(t) = \mu(-\infty, t)$  for almost all  $t \in \mathbb{R}$ . Thus  $f$  is

differentiable almost everywhere and  $f'(t) = f_1'(t) + f_2'(t)$ .  
Hence  $\mu(-t, \infty) - \mu(-\infty, -t) = \nu(-t, \infty) - \nu(-\infty, -t)$  for a.a.  $t \in \mathbb{R}$ ,  
therefore  $2\mu(-t, \infty) - 1 = 2\nu(-t, \infty) - 1$  and hence  $\mu = \nu$  almost  
everywhere [Lebesgue], that is  $\mu = \nu$  as required.

### 4.3 The ball properties on $X$ and $\mathcal{T}(X)$ .

#### The property (B).

In general (B) is false, even on compact metric spaces, [Da1]. It is even possible to construct two singular measures agreeing on all balls on a compact metric space [Dar]. However Anderson [An] has proved that (B) holds for all finite dimensional spaces, and the theorem below shows that it is the case when  $X$  is an infinite dimensional Hilbert space or  $\mathcal{L}_p$  for  $1 < p < \infty$ , see also [Ch2]. In [Ch1] Christensen proves (B) when there exists a uniform measure  $\mu$  on  $X$ .

In [Din], Dinger considers the question of uniqueness of measures for differing classes of balls. The largest group of positive results are found in [HJ] where Hoffmann-Jorgensen proves that (B) holds for a large class of Banach spaces including  $L_p$ ,  $1 < p \leq \infty$ ,  $L_1$  over a non-atomic measure space,  $C(K)$  and  $c_0$ . His results are contained in a series of corollaries which we reproduce here.

#### Corollary 4.3.1

If  $\mu$  and  $\nu$  are Radon probabilities which coincide on all closed balls, and if  $\|\cdot\|$  is Gateaux differentiable on the unit sphere, then  $\mu = \nu$ . This includes all  $L_p$ -spaces ( $1 < p < \infty$ ).

### Corollary 4.3.2

Let  $E$  be a real Banach algebra satisfying  $\|x\|^2 \leq \|x^2 + y^2\|$   $\forall x, y \in E$ . If  $\mu$  and  $\nu$  are Radon probabilities which coincide on all closed balls of  $E$ , then  $\mu = \nu$ . This includes all the real function algebras (with the sup norm), and in particular  $C(T)$  for  $T$  a topological space,  $c_0(\Gamma)$  for arbitrary index set  $\Gamma$ , and  $L_\infty$  over a general measure space.

### Corollary 4.3.3

Let  $(\Omega, \Sigma, m)$  be a  $\sigma$ -finite non-atomic measure space. If  $\mu$  and  $\nu$  are Radon probabilities which coincide on all closed balls of  $E = L_1(\Omega, \Sigma, m)$ , then  $\mu = \nu$ .

Hoffmann-Jorgensen also remarks that all the results are valid if one supposes that :  $\forall x_0$  in the unit sphere of  $X$  and  $\forall y_0 \in X \exists (a_n) \subseteq \mathbb{R}_+$  so that  $a_n \uparrow \infty$  and for all  $t \in \mathbb{R}$   $\lim_n \mu(B(a_n x_0 + y_0, a_n + t)) = \lim_n \nu(B(a_n x_0 + y_0, a_n + t))$ . So it suffices to assume that  $\mu$  and  $\nu$  coincide on balls of sufficiently large radius and with centre sufficiently far away from 0. He conjectures that the above implies that  $\mu = \nu$  in arbitrary Banach spaces, or that even the following suffices:

(1) <sub>$\epsilon$</sub>   $\mu(B(x, a)) = \nu(B(x, a)) \quad \forall x \in E, \forall 0 \leq a < \epsilon(x)$  where  $\epsilon: E \rightarrow \mathbb{R}$ , with  $\epsilon(x) > 0$  for all  $x \in E$ .

It is easily seen that if  $\mu$  and  $\nu$  are finite atomic measures which coincide on all balls of any Banach space  $X$ , then  $\mu = \nu$ . For suppose that  $\mu = \sum \alpha_i \delta(e_i)$  and  $\nu = \sum \beta_i \delta(f_i)$ . Then  $\mu B(e_1, r) = \nu B(e_1, r)$  for all  $r > 0$ , as  $r \rightarrow 0 \quad \mu B(e_1, r) \rightarrow \alpha_1$  thus there exists  $i$  with  $e_1 = f_i$  and  $\alpha_1 = \beta_i$ . Repeating with  $e_2$  etc we find that  $\mu = \nu$ .

We can approach the problem of whether two probability measures agreeing on all balls can be distinct in the situation of  $X$  being a metric space via the notion of sigma-Q spaces developed by Davies [Da2].

A metric space  $M$  has the property  $Q(h)$ , where  $0 < h < 1$ , if there exists an integer  $k=k(h)$  such that no set of diameter  $d$  in  $M$  contains more than  $k$  points all distant at least  $hd$  apart. This property is then inherited by subspaces. All the properties  $Q(h)$ ,  $0 < h < 1$ , are equivalent, so we shall speak simply of a  $Q$ -space. The union of finitely many  $Q$ -subspaces of a metric space is also  $Q$ , and that the closure of a  $Q$ -subspace of a metric space is also  $Q$ .

It is easy to see that:

Lemma 4.3.4

Let  $d_1$  and  $d_2$  be two Lipschitz equivalent metrics on  $M$ . Then  $M$  is a  $Q$ -space wrt  $d_1$  iff  $M$  is a  $Q$ -space wrt  $d_2$ .

Lemma 4.3.5

Let  $(X, d_1)$  and  $(Y, d_2)$  be  $Q$ -spaces. If  $X$  and  $Y$  are finite dimensional then  $(X \times Y, d)$  is a  $Q$ -space for any metric  $d$  on  $X \times Y$ .

$\mathbb{R}$  is obviously a  $Q$ -space. So by the above  $\mathbb{R}^n$  is also a  $Q$ -space with any metric placed upon it. Thus  $\mathcal{Q}_p^n$  is also a  $Q$ -space. A metric space is sigma- $Q$  if it can be expressed as the union of a sequence of  $Q$ -subspaces. If two probability

measures agree on all balls of a sigma-Q space then they are identical, [Da2]. Hence property (B) is true for all  $\mathbb{R}^n$  and  $\mathcal{Q}_p^n$  and  $\mathcal{Q}_p^n \times [0, \infty)$ .

The property (TB).

We can show that (TB) holds for  $\mathbb{R}$ ,  $\mathcal{Q}_p^n$ ,  $\mathcal{Q}_\infty^n$ ,  $1 \leq n < \infty$ . We shall see later that even if (TB) fails for  $\mathcal{Q}_p$ ,  $1 < p < \infty$ , then we still have uniqueness of marginals.

Proposition 4.3.6

$\mathbb{R}$ ,  $\mathcal{Q}_p^n$ ,  $\mathcal{Q}_\infty^n$  all have (TB) for  $1 < p < \infty$ ,  $n \in \mathbb{N}$ .

Proof

Since for  $X = \mathbb{R}$ ,  $\mathcal{Q}_p^n$ , or  $\mathcal{Q}_\infty^n$ ,  $\mathcal{T}(X) = X$ , we find  $X$  has (TB) iff  $X$  has (B). By the above section  $\mathbb{R}$ ,  $\mathcal{Q}_p^n$ , and  $\mathcal{Q}_\infty^n$  all have (B).

For  $X = \mathcal{Q}_p$ ,  $\mathcal{T}(X) = \mathcal{Q}_p \times [0, \infty)$ . We shall see below that if  $\xi$  and  $\eta$  agree on all sets of the form  $\{t \in \mathcal{T}(\mathcal{Q}_p) : t(x) \leq r\} \forall x, r$ , then their marginals are equal. That is if we disintegrate  $\xi$  into a family of probabilities  $\{\xi_y\}$  on  $[0, \infty)$ ,  $y \in \mathcal{Q}_p$ , and a probability  $\xi_2$  on  $\mathcal{Q}_p$ , then  $\xi_2 = \eta_2$ .

#### 4.4 The uniqueness property for measures on $X$ , $(w_p)$ .

For the case  $X=\mathbb{R}$  or  $\mathbb{C}$   $(w_p)$  holds iff  $p \notin 2\mathbb{N}$ , see [Pl] or [Ru]. Stephenson [St] extended this result to finite dimensional Hilbert spaces, and later the condition of finite dimensionality was removed, see [Ko1] or [Li2]. The situation of  $X=\mathcal{L}_r^{(n)}$ ,  $\mathcal{L}_\infty^{(n)}$  and  $\mathcal{L}_r$  is discussed in [Go&Ko2]. For  $X=\mathcal{L}_r^{(n)}$   $(w_p)$  fails iff  $p/r \in \mathbb{N}$  and one of the following holds:  $p/r < n$ ,  $r$  is even, or both  $r$  and  $p/r$  are odd. For  $X=\mathcal{L}_\infty^{(n)}$   $(w_p)$  holds iff  $p$  is not even for  $\mathbb{C}$ , or  $p+n$  is not odd for  $\mathbb{R}$ . When  $X=\mathcal{L}_r$  we need simply that  $p/r \notin \mathbb{N}$ . This last result is extended in [Li1] to  $L_r$  for  $1 \leq r < \infty$ , the counterexample for  $L_r$  extending to  $\mathcal{L}_r$  and  $\mathcal{L}_r^{(n)}$ . Also in [Li1]  $(w_p)$  is shown to be true  $\forall p > 1$  when  $X=C_0(\Omega)$  for  $\Omega$  a non-compact, locally compact space. For a topological space  $K$  property  $(w_p)$  holds for  $X=C(K)$  whenever  $p \notin \mathbb{N}$ .

#### The property $(w_1)$ .

We shall show that  $(w_1)$  is true for infinite dimensional Hilbert spaces and  $\mathcal{L}_p \times [0, \infty)$  via a series of lemmas below.

Since polynomials are dense in  $\mathcal{C}[0,1]$ , functions of the form  $p(1/s)$  are dense in  $\mathcal{C}[1, \infty]$ . Let  $\mu, \nu$  be measures on  $[1, \infty]$

such that  $\forall k \in \{0, 1, \dots\} \int_1^\infty s^{-k} d\mu(s) = \int_1^\infty s^{-k} d\nu(s)$ . Then

$\int_1^\infty p(1/s) d\mu(s) = \int_1^\infty p(1/s) d\nu(s)$  for all polynomials, hence

$\int_1^\infty f(s) d\mu(s) = \int_1^\infty f(s) d\nu(s)$  for all  $f \in \mathcal{C}[1, \infty]$ , so that  $\mu = \nu$ .

Lemma 4.4.1

Let  $\mu, \nu$  be probability measures on  $[0, \infty)$  such that

$$\int_0^{\infty} t^{1/2} d\mu(t), \int_0^{\infty} t^{1/2} d\nu(t) < \infty \text{ and for all } k \in \{0, 1, \dots\},$$

$$\int_0^{\infty} (t+1)^{1/2-k} d\mu(t) = \int_0^{\infty} (t+1)^{1/2-k} d\nu(t). \text{ Then } \mu = \nu.$$

Proof

Let  $\mu'(A) = \int_A (t+1)^{1/2} d\mu(t)$  for  $A \subseteq [0, \infty)$  and define  $\nu'$

similarly. Then  $\int_0^{\infty} (t+1)^{1/2} (t+1)^{-k} d\mu(t) = \int_0^{\infty} (t+1)^{-k} d\mu'(t)$   
 $= \int_1^{\infty} s^{-k} d\mu_0(s)$  where  $\mu_0 = \mu' \circ T^{-1}$  with  $Tt = t+1$ . Thus  $\mu_0 = \nu_0$ , so that  
 $\mu' = \nu'$ , since  $\int_A (t+1)^{1/2} d(\mu - \nu)(t) = 0 \quad \forall A \subseteq \mathcal{B}[0, \infty)$  and by

considering sets on which  $\mu - \nu$  is positive we have that  $\mu \geq \nu$   
and similarly  $\mu \leq \nu$ , hence  $\mu = \nu$ .

Lemma 4.4.2

Let  $\mu, \nu$  be probability measures on  $[0, \infty)$  such that

$$\int_0^{\infty} t^{1/2} d\mu(t), \int_0^{\infty} t^{1/2} d\nu(t) < \infty \text{ and}$$

$$\int_0^{\infty} (t+\alpha)^{1/2} d\mu(t) = \int_0^{\infty} (t+\alpha)^{1/2} d\nu(t) \text{ for all } \alpha \geq 0. \text{ Then } \mu = \nu.$$

Proof

Then  $\int_0^{\infty} (t+\alpha+1)^{1/2} d\mu(t) = \int_0^{\infty} (t+\alpha+1)^{1/2} d\nu(t)$  for all  $\alpha \geq 0$ . For  
 $0 \leq \alpha < 1$ ,  $\int_0^{\infty} (t+\alpha+1)^{1/2} d\mu(t) = \sum_{k=0}^{\infty} \binom{1/2}{k} \alpha^k \int_0^{\infty} (t+1)^{1/2-k} d\mu(t)$ , and  
therefore  $\int_0^{\infty} (t+1)^{1/2-k} d\mu(t) = \int_0^{\infty} (t+1)^{1/2-k} d\nu(t)$  for all  $k$ .

Theorem 4.4.3

Let  $\xi, \eta$  be probability measures on an infinite dimensional Hilbert space  $\mathcal{H}$ . If  $\int \|y\| d\xi(y), \int \|y\| d\eta(y) < \infty$  and  
 $\int \|x+y\| d\xi(y) = \int \|x+y\| d\eta(y)$  for all  $x$ , then  $\xi = \eta$ .

Proof

Let  $(e_n)$  be an orthonormal sequence in  $\mathfrak{H}$ . Then for all  $n$ ,  $\lambda, x$

$$\int \|\lambda e_n + x + y\| d\xi(y) = \int \|\lambda e_n + x + y\| d\eta(y). \text{ So}$$

$$\|\lambda e_n + x + y\|^2 = \|\lambda e_n\|^2 + 2\langle \lambda e_n | x + y \rangle + \|x + y\|^2$$

$$= \lambda^2 + \|x + y\|^2 + 2\lambda(x_n + y_n) \longrightarrow \lambda^2 + \|x + y\|^2 \text{ as } n \longrightarrow \infty. \text{ Thus for all}$$

$$\lambda, \int [\lambda^2 + \|x + y\|^2]^{1/2} d\xi(y) = \int [\lambda^2 + \|x + y\|^2]^{1/2} d\eta(y).$$

Now let  $x \in \mathfrak{H}$  and define  $\mu_x(A) = \xi\{y \in \mathfrak{H} : \|x + y\|^2 \in A\}$  and  $\nu_x(A) = \eta\{y \in \mathfrak{H} : \|x + y\|^2 \in A\}$ . By considering the measurable map

$$T: \mathfrak{H} \longrightarrow [0, \infty) \\ y \longmapsto \|x + y\|^2$$

we have  $\int_{\mathfrak{H}} (\alpha + \|x + y\|^2)^{1/2} d\xi(y) = \int_{\mathbb{R}_+} (\alpha + t)^{1/2} d\mu_x(t)$

$\forall \alpha \geq 0$ . Hence  $\int_0^\infty (t + \alpha)^{1/2} d\mu_x(t) = \int_0^\infty (t + \alpha)^{1/2} d\nu_x(t)$ , so  $\mu_x = \nu_x$  for all  $x \in \mathfrak{H}$ . Then  $\xi$  and  $\eta$  agree on every ball, since

$$\xi\{y \in \mathfrak{H} : \|x + y\| \leq r\} = \mu_x[0, r] = \nu_x[0, r] = \eta\{y \in \mathfrak{H} : \|x + y\| \leq r\}.$$

If  $\xi$  and  $\eta$  agree on every ball they agree on every set of the form  $\{y \in \mathfrak{H} : (y | z) < 1\}$ , as this is the union of an increasing sequence of balls. Hence the one-dimensional distributions of  $\xi$  and  $\eta$  are the same, and  $\xi$  must coincide with  $\eta$  by a result in [Sch].

We can now extend this result to the space  $\mathcal{L}_p \times [0, \infty)$ :

Lemma 4.4.4

Let  $\mu, \nu$  be probability measures on  $[0, \infty)$  such that

$$\int_0^\infty t^{1/p} d\mu(t), \int_0^\infty t^{1/p} d\nu(t) < \infty \text{ and}$$

$$\int_0^\infty (t + \alpha)^{1/p} d\mu(t) = \int_0^\infty (t + \alpha)^{1/p} d\nu(t) \text{ for all } \alpha \geq 0. \text{ Then } \mu = \nu.$$

### Proof

Then  $\int_0^\infty (t+\alpha+1)^{1/p} d\mu(t) = \int_0^\infty (t+\alpha+1)^{1/p} dv(t)$  for all  $\alpha \geq 0$ . For

$$0 \leq \alpha < 1, \quad \int_0^\infty (t+\alpha+1)^{1/p} d\mu(t) = \sum_{k=0}^\infty \binom{1/p}{k} \alpha^k \int_0^\infty (t+1)^{1/p-k} d\mu(t),$$

therefore  $\int_0^\infty (t+1)^{1/p-k} d\mu(t) = \int_0^\infty (t+1)^{1/p-k} dv(t)$  for all  $k$ .

Then as before with  $1/p$  in place of  $1/2$ , we can show that  $\mu = \nu$ .

### Theorem 4.4.5

Let  $\xi, \eta$  be probability measures on an  $\mathcal{L}_p \times [0, \infty) = B$ . If  $\int_B \|y\| d\xi(y), \int_B \|y\| d\eta(y) < \infty$  and  $\int_B \|x+y\| d\xi(y) = \int_B \|x+y\| d\eta(y)$  for all  $x \in B$ , then  $\xi = \eta$ .

### Proof

For  $\lambda \geq 0$ , choose a sequence  $x^{(n)} \in \mathcal{L}_p$  as before, so that  $\|x^{(n)} + z\|_p^p \longrightarrow \|x+z\|_p^p + \lambda^p$ . So

$$\|(x^{(n)}, \alpha) + (y, \beta)\|^p = \|x^{(n)} + y\|^p + (\alpha + \beta)^p \longrightarrow \lambda^p + \|x+y\|^p + (\alpha + \beta)^p \quad \text{as } n \longrightarrow \infty.$$

Thus for all  $\lambda$ , and all  $x \in B$

$$\int [\lambda^p + \|x+y\|^p]^{1/p} d\xi(y) = \int [\lambda^p + \|x+y\|^p]^{1/p} d\eta(y).$$

Now let  $x \in B$  and define  $\mu_x(A) = \xi\{y \in B : \|x+y\|^p \in A\}$  and

$\nu_x = \eta\{y \in B : \|x+y\|^p \in A\}$ . Then

$$\int_B (\alpha + \|x+y\|^p)^{1/p} d\xi(y) = \int_{\mathbb{R}_+} (\alpha + t)^{1/p} d\mu_x(t) \quad \forall \alpha \geq 0. \quad \text{Hence}$$

$$\int_0^\infty (t+\alpha)^{1/p} d\mu_x(t) = \int_0^\infty (t+\alpha)^{1/p} d\nu_x(t), \quad \text{so } \mu_x = \nu_x \text{ for all } x \in B. \quad \text{So } \xi$$

and  $\eta$  agree on every ball, since

$$\xi\{y \in B : \|x+y\| \leq r\} = \mu_x[0, r] = \nu_x[0, r] = \eta\{y \in B : \|x+y\| \leq r\}. \quad \text{Then since the}$$

ball property (B) holds for  $\mathcal{L}_p$  and  $\mathbb{R}$ , we have  $\xi = \eta$ .

Although we can represent  $\mathcal{T}(\mathcal{Q}_p)$  as  $\mathcal{Q}_p \times [0, \infty)$ , Theorem 4.4.5 is not sufficient to show that the uniqueness property for random measures on types,  $(U_1)$ , holds for  $\mathcal{Q}_p$ , as we shall see in the next section.

#### 4.5 The uniqueness property for measures on types, $(U_1)$ .

Example: Does  $X=\mathcal{Q}_p$ ,  $1 < p < \infty$ , have  $(U_1)$ ?

We now look at the example of  $X=\mathcal{Q}_p$ , and ask whether  $\mathcal{Q}_p$  possesses  $(U_1)$ . We show that this reduces to deciding whether measures on  $\mathcal{Q}_p \times [0, \infty)$  are uniquely determined by knowledge of all  $\mu\{(x, \beta) : \|x-y\|^p + \beta^p \leq r\}$ . We prove positive partial results, firstly that in the event of  $(U_1)$  failing for  $\mathcal{Q}_p$ , we still have equality for marginals; and secondly we have equality for measures in the case of finite dimensional  $\mathcal{Q}_p$ .

Consider now a uniformly integrable type on  $L_1(\Omega; \mathcal{Q}_p)$  for  $1 < p < \infty$ . We have seen that this can be represented by the

$$\text{formula } \tau(g) = \int_{\Omega} \int_{\mathcal{Q}_p \times [0, \infty)} [\|g(\omega) + y\|^p + \beta^p]^{1/p} \cdot d\xi_{\omega}(y, \beta) \cdot dP(\omega)$$

where  $g \in L_1(\Omega; \mathcal{Q}_p)$  and  $\xi \in \pi_1(\mathcal{Q}_p \times [0, \infty))$ .

Since there are Fréchet differentiable points in  $\mathcal{Q}_p \times [0, \infty)$ , by Theorem 4.2.1, we can consider the following

$$\text{situation. Let } \xi, \eta \in \mathcal{P}(\mathcal{Q}_p \times [0, \infty)) \text{ be such that for all } x \in \mathcal{Q}_p \\ \int_{\mathcal{Q}_p \times [0, \infty)} [\|x+y\|^p + \beta^p]^{1/p} \cdot d\xi(y, \beta) = \int_{\mathcal{Q}_p \times [0, \infty)} [\|x+y\|^p + \beta^p]^{1/p} \cdot d\eta(y, \beta)$$

and that these integrals are finite.

Lemma 4.5.1

For all  $x \in \mathcal{Q}_p$ ,  $\alpha \in [0, \infty)$  we have

$$\int_{\mathcal{Q}_p \times [0, \infty)} [\|x+y\|^p + \beta^p + \alpha]^{1/p} \cdot d\xi(y, \beta) = \int_{\mathcal{Q}_p \times [0, \infty)} [\|x+y\|^p + \beta^p + \alpha]^{1/p} \cdot d\eta(y, \beta)$$

Proof

For all  $x \in \mathcal{Q}_p$ ,  $\alpha \in [0, \infty)$  we can find a sequence  $\{x^{(n)}\}$  of elements of  $\mathcal{Q}_p$  with  $x = (\text{pointwise}) \lim_n x^{(n)}$

$$\|x^{(n)} - x\|_p \longrightarrow \alpha \quad \text{as } n \longrightarrow \infty.$$

Thus  $\|x^{(n)} + y\|^p \longrightarrow \|x + y\|^p + \alpha^p \quad \forall y \in \mathcal{Q}_p$ . Then

$$\int_{\mathcal{Q}_p \times [0, \infty)} [\|x^{(n)} + y\|^p + \beta^p]^{1/p} \cdot d\xi(y, \beta) \longrightarrow \int_{\mathcal{Q}_p \times [0, \infty)} [\|x + y\|^p + \beta^p + \alpha^p]^{1/p} \cdot d\xi(y, \beta) \quad \text{as } n \longrightarrow \infty.$$

Lemma 4.5.2

$\xi$  and  $\eta$  agree on sets of the form  $\{(y, \beta) : (\|x+y\|^p + \beta^p)^{1/p} \leq r\}$   $\forall x \in \mathcal{Q}_p$  and  $\forall r \geq 0$ .

Proof

Define  $\mu_x(F) = \xi\{(y, \beta) : (\|x+y\|^p + \beta^p)^{1/p} \in F\}$ , and  $\nu_x$  similarly.

We then have  $\int_0^\infty [\alpha + t^p]^{1/p} d\mu_x(t) = \int_0^\infty [\alpha + t^p]^{1/p} d\nu_x(t) \quad \forall x \in \mathcal{Q}_p, \forall \alpha$ .

Then  $\mu_x = \nu_x \quad \forall x$  by a result of W.Linde [Li1]. Thus in particular  $\mu_x[0, r] = \nu_x[0, r]$ , so that  $\xi$  and  $\eta$  agree on sets of the form  $\{(y, \beta) : (\|x+y\|^p + \beta^p)^{1/p} \leq r\}$ . So in terms of types  $\xi$  and  $\eta$  agree on sets of the form  $\{t : t(x) \leq r\}$ , cf the ball property for  $\mathcal{T}(X)$ , (TB) defined at the start of the chapter.

We can disintegrate  $\xi$  into  $\xi_y$  and  $\xi_2$ , where for each  $y \in \mathcal{Q}_p$ ,  $\xi_y$  is a probability measure on  $[0, \infty)$ , and the map  $y \mapsto \xi_y$  is measurable, and  $\xi_2$  is a probability measure on  $\mathcal{Q}_p$ , see section 2.4. With this disintegration we have

$$\int_{\mathcal{Q}_p \times [0, \infty)} f \cdot d\xi = \int_{\mathcal{Q}_p} \int_0^\infty f(y, \beta) \cdot d\xi_y(\beta) \cdot d\xi_2(y) \quad \text{for all } \xi\text{-measurable,}$$

integrable  $f$ . Similarly split  $\eta$  into  $\eta_y$  and  $\eta_2$ .

### Lemma 4.5.3

$\{(y, \beta) : \langle y, z \rangle > s\} = \bigcup_{R=1}^\infty \{(y, \beta) : (\beta^p + \|y - (s+R)\tilde{z}\|^p)^{1/p} < R\}$  where  $z \in \mathcal{Q}_q$ ,  $1/q + 1/p = 1$ ,  $\tilde{z} = z^{q/p}$  and  $s \geq 0$ .

### Proof

A hyperplane  $M$  in  $\mathcal{Q}_p$  is defined by  $M = \{y \in \mathcal{Q}_p : f(y) = s\}$  for  $s \in \mathbb{R}$  and  $f \in (\mathcal{Q}_p)^*$ . That is  $M = \{y \in \mathcal{Q}_p : \langle y, z \rangle = s\}$  for some  $z \in \mathcal{Q}_q$  and  $s \geq 0$ . As we have seen the measure of all sets  $\{y \in \mathcal{Q}_p : \langle y, z \rangle > s\}$  will determine the measure on  $\mathcal{Q}_p$  via Fourier transforms.

$\langle s\tilde{z}, z \rangle = s \sum \tilde{z}_n z_n = s \|z\|^q = s$  as  $\|z\| = 1$ . Thus  $s\tilde{z}$  is the closest element of  $\{y \in \mathcal{Q}_p : \langle y, z \rangle \geq s\}$  to 0.

$$(i) \quad \{y \in \mathcal{Q}_p : \langle y, z \rangle > s\} = \bigcup_{R=1}^\infty \{y : \|y - (s+R)\tilde{z}\| < R\}.$$

a) Let  $x, u \in U = \{u : u = p\tilde{z} \text{ } p \geq 0\}$  be such that  $(x-y) \wedge x = 0$  and  $(x-u) \wedge (u-y) = 0$ . Let  $f = \|y-u\|$ ,  $\delta = \|x-u\|$  and  $d = \|y-x\|$ . If  $d < \lambda = \|x - s\tilde{z}\|$  then  $\|x-y\| = \|y - (s+\lambda)\tilde{z}\| = d < \lambda$ . So assume  $d \geq \lambda$ . Then  $f^p = d^p + \delta^p$ . So choose  $u$  such that  $\delta = d - \lambda + 1 > d - \lambda \geq 0$ , thus  $\lambda + \delta > f$  so that  $y \in \text{ball}[(s+\lambda+\delta)\tilde{z}; \lambda+\delta]$ .

b) Suppose  $y \notin \text{LHS}$ . Then  $\langle y, z \rangle \leq s$ , so as in a) we can construct a ball, centre  $-r\tilde{z}$ , which contains  $y$ ;  $\exists r$  with  $\|y+r\tilde{z}\| < r+s$ .

Then by construction this ball and the ball  $\{y : \|y - (s+R)\tilde{z}\| < R\}$

are distinct, and are separated by the line  $\langle y, z \rangle = s$ . So that  $y \notin \text{RHS}$ .

$$(ii) \bigcup_{R=1}^{\infty} \{(y, \beta) : \|y - (s+R)\tilde{z}\| < R\} = \bigcup_{R=1}^{\infty} \{(y, \beta) : (\beta^p + \|y - (s+R)\tilde{z}\|^p)^{1/p} < R\}.$$

Assume  $(y, \beta) \in \text{LHS}$ . Then  $\exists R$  with  $\|y - (s+R)\tilde{z}\| < R$  thus

$$(\beta^p + \|y - (s+R)\tilde{z}\|^p)^{1/p} < (R^p + \beta^p)^{1/p} = S \text{ so that}$$

$$(\beta^p + \|y - (s+S)\tilde{z}\|^p)^{1/p} < S \text{ i.e. } (y, \beta) \in \text{RHS}.$$

If  $(y, \beta) \in \text{RHS}$  then  $\exists R$  with  $\|y - (s+R)\tilde{z}\| < (R^p - \beta^p)^{1/p}$  so that  $(y, \beta) \in \text{LHS}$ .

#### Corollary 4.5.4

$$\xi_2 = \eta_2 = \lambda \text{ (say).}$$

#### Proof

$$\begin{aligned} \xi_2 \{y \in \mathcal{Q}_p : \langle y, z \rangle > s\} &= \xi \{(y, \beta) : \langle y, z \rangle > s\} \\ &= \bigcup_{R=1}^{\infty} \xi \{(y, \beta) : (\beta^p + \|y - (s+R)\tilde{z}\|^p)^{1/p} < R\} \\ &= \bigcup_{R=1}^{\infty} \eta \{(y, \beta) : (\beta^p + \|y - (s+R)\tilde{z}\|^p)^{1/p} < R\} \\ &= \eta_2 \{y \in \mathcal{Q}_p : \langle y, z \rangle > s\}. \end{aligned}$$

Hence  $\xi_2 = \eta_2$ .

#### Example: Does $c_0$ have $(U_1)$ ?

We now aim to apply the methods that we used for  $X = \mathcal{Q}_p$  to the case when  $X = c_0$ . In this situation however we cannot prove equality of marginals. Rather we establish that the problem reduces to the question of whether measures on  $\mathcal{Q}_\infty \times [0, \infty)$  are

uniquely determined by knowledge of all

$\mu\{(y, \beta) \in \mathcal{Q}_\infty \times [0, \infty) : \max\{\|x+y\|, \beta\} \leq r\}$  for  $x \in c_0$ ,  $r \geq 0$ .

Let  $\tau$  be a uniformly integrable type on  $L_1(\Omega; c_0)$ . Then for  $g \in L_1(\Omega; c_0)$  we have  $\tau(g) = \int_{\Omega} \int_{\mathcal{T}(c_0)} t(g(\omega)) d\xi_\omega(t) dP(\omega)$ . Suppose

that  $t$  is a type on  $c_0$ , with  $t(x) = \lim_n \|x+y^n\|$  where  $x, y^n \in c_0$ .

Then for some  $(y, \beta) \in \mathcal{Q}_\infty \times [0, \infty)$ ,  $t(x) = \max\{\|x+y\|, \beta\} \forall x \in c_0$ . If

$y^n \rightarrow y \in \mathcal{Q}_\infty$  pointwise and  $\|y^n - y\|_\infty \rightarrow \beta$  then

$\beta \geq \inf\{\|x+y\| : x \in c_0\} = \|y\|_{\mathcal{Q}_\infty/c_0} = \limsup_n |y_n|$ . So that we can

represent  $\mathcal{T}(c_0)$  as  $L = \{(y, \beta) : y \in \mathcal{Q}_\infty, \beta \geq \|y\|\}$ . Let

$T: \mathcal{T}(c_0) \rightarrow \mathcal{Q}_\infty \times \mathbb{R}^+$  and  $F = T^{-1}$  be the obvious maps between the  
 $t \mapsto (y, \beta)$

spaces.

Are these maps measurable? Now  $\tau_{(y, \beta)}(x) > \lambda$  iff

$y \in \bigcup_n \{z \in \mathcal{Q}_\infty : |z_n + x_n| > \lambda\}$  or  $\exists q \in \mathbb{Q}^+$  with  $\tau > q$  and  $\limsup |y_n| \geq \lambda - q$ ,

where  $\tau_{(y, \beta)}(x) = \|x+y\| \vee (\tau + d(y, c_0))$ . Let  $G = \{\tau : \tau(x) > \lambda\}$ , then

$T(G) = \bigcup_n \{(y, \tau) : |y_n + x_n| > \lambda\} \cup \bigcup_{q \in \mathbb{Q}_+} [\bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \{y \in \mathcal{Q}_\infty : |y_n| \geq \lambda - q\} \times (q, \infty)]$

$= H$ , say. We need a topology on  $\mathcal{Q}_\infty \times \mathbb{R}^+$  so that this map is a

measurable map, that is can we find a topology in which both

sides are Borel sets. If so then  $T^{-1}H = G$ , thus  $T$  would be Borel

measurable wrt the Borel  $\sigma$ -field on  $\mathcal{T}(c_0)$  and the Borel

$\sigma$ -field on  $\mathcal{Q}_\infty \times \mathbb{R}^+$ . The pointwise topology inherited from

$\mathbb{R}^{\mathbb{N}} \times [0, \infty)$  will suffice. Then  $H$  is a Borel set in the pointwise

topology, and in fact a  $F_{\sigma\delta\sigma}$ -set.

Thus  $T(\mathcal{T}(c_0)) = L$ , and  $T^{-1}(L) = \mathcal{T}(c_0)$ . So that

$$\begin{aligned} \tau(g) &= \int_{\Omega} \int_{\mathcal{T}(c_0)} t(g(\omega)) d\xi_\omega(t) dP(\omega) \\ &= \int_{\Omega} \int_L \max\{\|g(\omega) + y\|, \beta\} d\xi_\omega(y, \beta) dP(\omega). \end{aligned}$$

Proposition 4.5.5

Suppose  $\tau$  has two associated random measures

$$\xi, \eta \in \pi_1(\mathcal{L}_\omega \times [0, \infty)). \text{ Then } \forall g \int_{\mathcal{L}_\omega \times \mathbb{R}_+} \max\{\|g(\omega) + y\|, \beta\} d\xi_\omega(y, \beta) = \int_{\mathcal{L}_\omega \times \mathbb{R}_+} \max\{\|g(\omega) + y\|, \beta\} d\eta_\omega(y, \beta) \quad \text{a.e. [P].}$$

Proof

Since there are Fréchet differentiable points in  $c_0$ , by Theorem 4.2.1 we have  $\int_{\mathcal{T}(c_0)} \tau(g(\omega)) d\xi_\omega(\tau) = \int_{\mathcal{T}(c_0)} \tau(g(\omega)) d\eta_\omega(\tau)$  a.e. [P]. Since the maps  $T$  and  $F$  are measurable,  $\int_{\mathcal{T}(c_0)} \tau(g(\omega)) d\xi_\omega(\tau) = \int_L \max\{\|g(\omega) + y\|, \beta\} d\xi_\omega \circ T^{-1}(y, \beta)$ , let us denote  $\xi_\omega \circ T^{-1}$  by  $\xi_\omega$  as well. Thus if  $G \subseteq L^c$  then  $\xi[G] = \xi\{\tau : T\tau \in G\} = \xi(\emptyset) = 0$ , so that  $\xi[L] = 1$ . Thus  $\int_{\mathcal{L}_\omega \times \mathbb{R}_+} \max\{\|g(\omega) + y\|, \beta\} d\xi_\omega(y, \beta) = \int_L \max\{\|g(\omega) + y\|, \beta\} d\xi_\omega(y, \beta)$ . Hence  $\forall x \in c_0$   $\int_{\mathcal{L}_\omega \times \mathbb{R}_+} \max\{\|g(\omega) + y\|, \beta\} d\xi_\omega(y, \beta) = \int_{\mathcal{L}_\omega \times \mathbb{R}_+} \max\{\|g(\omega) + y\|, \beta\} d\eta_\omega(y, \beta)$  a.e. [P].

Now if probabilities  $\xi$  and  $\eta$  on  $\mathcal{T}(c_0)$  are distinct, will the probabilities  $\xi \circ T^{-1}$  and  $\eta \circ T^{-1}$  be distinct in  $\mathcal{L}_\omega \times [0, \infty)$ ? The answer is yes. Since if we suppose that  $\xi \circ T^{-1} = \eta \circ T^{-1}$ , then for all  $A \subseteq \mathcal{L}_\omega \times [0, \infty)$   $\xi\{\tau : \tau \in T^{-1}A\} = \eta\{\tau : \tau \in T^{-1}A\}$ . Then  $\xi = \eta$  if and only if for every  $B \subseteq \mathcal{T}(c_0)$  there exists  $F$  with  $T^{-1}F = B$ . Let  $F = TB$ , then  $B \subseteq T^{-1}F$ . Suppose  $\exists \tau \notin B$  such that  $T\tau \in TB$ , say  $T\tau = T\sigma$ ; then  $\tau(x) = \max\{\|x + y\|, \beta\} = \sigma(x)$ . Thus  $\tau = \sigma \in B$ . Hence  $\xi = \eta$  whenever  $\xi \circ T^{-1} = \eta \circ T^{-1}$  as required

Now suppose that we are given two distinct probabilities  $\xi$  and  $\eta$  on  $\mathcal{Q}_\infty \times [0, \infty)$ , with  $\forall x \in c_0 \int_{\mathcal{Q}_\infty \times [0, \infty)} \max\{\|x+y\|, \beta\} d\xi(y, \beta) =$

$\int_{\mathcal{Q}_\infty \times [0, \infty)} \max\{\|x+y\|, \beta\} d\eta(y, \beta)$ . Then by an approximation argument for

any  $\alpha \geq 0$  there exists a sequence  $x_n$  in  $c_0$  such that

$\|x_n + y\| \longrightarrow \max\{\|x+y\|, \alpha\}$ , therefore for all  $x$  in  $c_0$  and  $\alpha \geq 0$

$$\int_{\mathcal{Q}_\infty \times [0, \infty)} \max\{\|x+y\|, \beta, \alpha\} d\xi(y, \beta) = \int_{\mathcal{Q}_\infty \times [0, \infty)} \max\{\|x+y\|, \beta, \alpha\} d\eta(y, \beta).$$

#### Lemma 4.5.6

$\xi$  and  $\eta$  agree on sets of the form

$$\{(y, \beta) \in \mathcal{Q}_\infty \times [0, \infty) : \max\{\|x+y\|, \beta\} \leq r\} \quad \forall x \in c_0 \quad \forall r \geq 0.$$

#### Proof

Let  $\mu_x(F) = \xi\{(y, \beta) : \max\{\|x+y\|, \beta\} \in F\}$ , and define  $\nu_x$  similarly. Then we find that  $\forall \alpha \geq 0$  and all  $x$

$$\int_0^\infty \max(t, \alpha) d\mu_x(t) = \int_0^\infty \max(t, \alpha) d\nu_x(t). \quad \text{Then by W.Linde [Li1]}$$

$\mu_x = \nu_x$  for all  $x \in c_0$ . In particular  $\mu_x[0, r] = \nu_x[0, r]$  so that for all  $r \geq 0$  and all  $x \in c_0$ ,

$$\xi\{(y, \beta) : \max\{\|x+y\|, \beta\} \leq r\} = \eta\{(y, \beta) : \max\{\|x+y\|, \beta\} \leq r\}$$

We can now disintegrate  $\xi$  into  $\xi_y$  and  $\xi_2$ , where for each  $y \in \mathcal{Q}_\infty$ ,  $\xi_y$  is a probability measure on  $[0, \infty)$  and  $\xi_2$  is a probability measure on  $\mathcal{Q}_\infty$ , see section 2.4. Then

$$\int_{\mathcal{Q}_\infty \times [0, \infty)} f \cdot d\xi = \int_{\mathcal{Q}_\infty} \int_0^\infty f(y, \beta) d\xi_y(\beta) d\xi_2(y) \quad \text{for all measurable,}$$

integrable  $f$ . Similarly split  $\eta$  into  $\eta_y$  and  $\eta_2$ . Knowledge of

all image measures  $\xi_2 \circ \phi^1$  determines  $\xi_2$  for linear functions

$\phi$ . These are given by  $\xi_2\{x \in X : \langle x, \phi \rangle \geq s\}$  for  $s \in \mathbb{R}^+$  and  $\phi \in (\mathcal{Q}_\infty)^* \cong ba$

or  $(\mathcal{Q}_1 \times (c_0)^0)_1$ . Let  $sz$  be the closest vector in

$\{y \in \mathcal{Q}_\infty : \langle y, \phi \rangle \geq s\}$  to zero, so that  $\|sz\| = s$ .

Lemma 4.5.7

$\{(y, \beta) : \langle y, \phi \rangle > s\} = \bigcup_{R=1}^{\infty} \{(y, \beta) : \max\{\beta, \|y - (s+R)z\|\} < R\}$  where  $s \geq 0$ .

Proof

(i)  $\{y \in \mathcal{Q}_{\infty} : \langle y, \phi \rangle > s\} = \bigcup_{R=1}^{\infty} \{y : \|y - (s+R)z\| < R\}$ .

(a) Let  $y \in \{y : \langle y, \phi \rangle > s\}$ .

Let  $x, u \in U = \{u : u = pz \ p \geq 0\}$  be such that  $(x-y) \wedge x = 0$  and  $(x-u) \wedge (u-y) = 0$ . That is  $[x, y]$  is 'perpendicular' to  $[0, z]$ . Let  $f = \|y - u\|$ ,  $\delta = \|x - u\|$  and  $d = \|y - x\|$ . If  $d < \lambda = \|x - sz\|$  then  $\|x - y\| = \|y - (s+\lambda)z\| = d < \lambda$ . So assume  $d \geq \lambda$ . Then  $f = \|y - u\| = \max\{\|y - x\|, \|x - u\|\} = \max\{d, \delta\}$ . Let  $\delta = d - \lambda + 1 > d - \lambda \geq 0$ . Thus  $\max\{d, \delta\} < \lambda + \delta$  so that  $\|y - u\| = f < \lambda + \delta$ , i.e.  $y \in \text{ball}[u; \lambda + \delta]$ , i.e.  $y \in \text{ball}[(s + \lambda + \delta)z; \lambda + \delta]$ . Thus  $y \in \text{RHS}$

(b) Suppose  $y \notin \text{LHS}$  and  $\|y\| < s$ . Then  $\|u - y\| \geq |\|u\| - \|y\|| > |s + R - s| = R$ . Thus  $y \notin \text{RHS}$ .

Suppose  $y \notin \text{LHS}$  then  $\langle y, \phi \rangle \leq s$ , so as in (a) we can construct a ball, centre  $-rz$ , which contains  $y$ ;  $\exists r$  with  $y \in \{y : \|y + rz\| < r + s\}$ . Then by construction this ball and the ball  $\{y : \|y - (s+R)z\| < R\}$  are distinct, and are separated by the hyperplane  $\langle y, \phi \rangle = s$ . So that  $y \notin \text{RHS}$ .

(ii)  $\bigcup_{R=1}^{\infty} \{(y, \beta) : \|y - (s+R)z\| < R\} = \bigcup_{R=1}^{\infty} \{(y, \beta) : \max\{\beta, \|y - (s+R)z\|\} < R\}$ .

Assume  $(y, \beta) \in \text{LHS}$ . Then there exists an  $R$  with  $\|y - (s+R)z\| < R$ , thus  $\max\{\beta, \|y - (s+R)z\|\} < \max\{R, \beta\}$ , set  $R' = \max\{R, \beta\} + 1$  then  $(y, \beta) \in \text{RHS}$ .

If  $(y, \beta) \in \text{RHS}$  then there exists an  $R$  with  $\max\{\|y - (s+R)z\|, \beta\} < R$  so that  $\|y - (s+R)z\| < R$  and hence  $(y, \beta) \in \text{LHS}$ .

We would now like to prove that  $\xi_2$  and  $\eta_2$  are equal. In particular we wish to show that

$\xi_2\{y \in \mathcal{Q}_\infty : \langle y, \phi \rangle > s\} = \eta_2\{y \in \mathcal{Q}_\infty : \langle y, \phi \rangle > s\}$ . We know that

$\xi\{(y, \beta) : \max\{\|x+y\|, \beta\} \leq r\} = \eta\{(y, \beta) : \max\{\|x+y\|, \beta\} \leq r\}$  for all  $x \in c_0$ ,

and not all  $x \in \mathcal{Q}_\infty$ . If our  $z$  was in  $c_0$ , then we would be done

and  $\xi_2$  would be equal to  $\eta_2$ . Though in general this is not

so, and there does not seem easy way to get from

$\xi\{(y, \beta) : \max\{\|x+y\|, \beta\} \leq r\} = \eta\{(y, \beta) : \max\{\|x+y\|, \beta\} \leq r\}$  for all  $x \in c_0$ ,

to  $\xi\{(y, \beta) : \max\{\|z+y\|, \beta\} \leq r\} = \eta\{(y, \beta) : \max\{\|z+y\|, \beta\} \leq r\}$  for all

$z \in \mathcal{Q}_\infty$ .

Example: Does  $L_2$  possess  $(U_1)$ ?

We show by arguing as before that even if  $(U_1)$  fails for  $L_2$  we still have equality of marginals. Let  $\tau$  be a uniformly integrable type on  $L_1(L_2)$ . Suppose that there exists distinct random measures  $\xi$  and  $\eta$  with for all  $h \in L_1(L_2)$ ,

$$\tau(h) = \int_{\Omega} \int_{\mathcal{T}(L_2)} t[h(\omega)] d\xi_{\omega}(t) dP(\omega) = \int_{\Omega} \int_{\mathcal{T}(L_2)} t[h(\omega)] d\eta_{\omega}(t) dP(\omega).$$

$L_2$  is Fréchet differentiable everywhere, by Theorem 4.2.1,

$$\int_{\mathcal{T}(L_2)} t[h(\omega)] d\xi_{\omega}(t) = \int_{\mathcal{T}(L_2)} t[h(\omega)] d\eta_{\omega}(t) \quad \text{a.e.}[P].$$

So suppose that we have  $\int_{\mathcal{T}(L_2)} t(f) d\xi(t) = \int_{\mathcal{T}(L_2)} t(f) d\eta(t)$  for

all  $f \in L_2$ , with  $\xi, \eta \in \mathcal{P}(\mathcal{T}(L_2))$ . Then in  $L_2$  we can find a

sequence  $x_n$  such that  $\forall y \in L_2, \|x_n + y\|^2 \rightarrow \|x + y\|^2 + \beta^2$  for a given

$(x, \beta) \in L_2 \times \mathbb{R}_+$ . Thus  $\forall f \in L_2$  and  $\forall \beta \in \mathbb{R}_+$

$$\int_{L_2 \times \mathbb{R}_+} [\|f+g\|^2 + \alpha^2 + \beta^2]^{1/2} d\xi(g, \alpha) = \int_{L_2 \times \mathbb{R}_+} [\|f+g\|^2 + \alpha^2 + \beta^2]^{1/2} d\eta(g, \alpha).$$

$f \in L_2$  define  $\mu_f \in \mathcal{P}([0, \infty))$  by  $\mu_f[F] = \xi\{(g, \alpha) : [\|f+g\|^2 + \alpha^2]^{1/2} \in F\}$

where  $F \subseteq \mathbb{R}_+$ . Defining  $\nu_f$  similarly we have

$$\int_0^\infty (t^2 + \beta^2)^{1/2} d\mu_f(t) = \int_0^\infty (t^2 + \beta^2)^{1/2} d\nu_f(t).$$

By a result of Linde [Li1],  $\mu_f = \nu_f$  for all  $f$ . Thus in particular  $\mu_f[0, r] = \nu_f[0, r]$

for all  $r \geq 0$ , so that  $\xi$  and  $\eta$  agree on all sets of the form

$$\{(g, \alpha) : [\|f+g\|^2 + \alpha^2]^{1/2} \leq r\}.$$

Disintegrate  $\xi$  into  $\xi_g$  and  $\xi_2$ . Then if  $\xi_2$  and  $\eta_2$  agree on all sets of the form  $\{f \in L_2 : \langle f, g \rangle > s\}$  they are equal. We can write  $\{g \in L_2 : \langle f, g \rangle > s\} \times [0, \infty)$  as

$$\bigcup_{R=1}^\infty \{(g, \alpha) : [\alpha^2 + \|g - (s+R)f\|^2]^{1/2} < R\},$$

so that by proceeding as in the case of  $\mathcal{Q}_p$ ,  $\xi_2 = \eta_2$ .

Thus 
$$\int_{L_2} \int_{\mathbb{R}_+} [\|f+g\|^2 + \alpha^2 + \beta^2]^{1/2} d\xi_g(\alpha) d\xi_2(g) = \int_{L_2} \int_{\mathbb{R}_+} [\|f+g\|^2 + \alpha^2 + \beta^2]^{1/2} d\eta_g(\alpha) d\xi_2(g)$$
 for all  $\beta \in \mathbb{R}$  and  $g \in L_2$ . If 
$$\int_{\mathbb{R}_+} [\|f+g\|^2 + \alpha^2 + \beta^2]^{1/2} d\xi_g(\alpha) = \int_{\mathbb{R}_+} [\|f+g\|^2 + \alpha^2 + \beta^2]^{1/2} d\eta_g(\alpha)$$
 a.e.  $[\xi_2]$ , then 
$$\int_{\mathbb{R}_+} [\alpha^2 + \gamma^2]^{1/2} d\xi_g(\alpha) = \int_{\mathbb{R}_+} [\alpha^2 + \gamma^2]^{1/2} d\eta_g(\alpha)$$
 a.e.  $[\xi_2]$  and thus by Linde  $\xi_g = \eta_g$  a.e.  $[\xi_2]$ .

We note however that the "a.e.  $[\xi_2]$ " is necessary here. Since if we define  $\xi_g = \begin{cases} \mu & \text{if } g=G \\ 0 & \text{o.w.} \end{cases}$  and  $\eta_g = \begin{cases} \mu & \text{if } h=H \\ 0 & \text{o.w.} \end{cases}$ , where  $G \neq H$  are elements of  $L_2$  and  $\mu$  is a non-trivial probability on  $\mathbb{R}_+$ , and  $\xi_2 = \delta_F$  where  $F$  is distinct from  $G$  and  $H$ . Then

$$\int_{L_2} \int_{\mathbb{R}_+} [\|f+g\|^2 + \alpha^2]^{1/2} d\xi_g(\alpha) d\xi_2(g) = \int_{L_2} \int_{\mathbb{R}_+} [\|f+g\|^2 + \alpha^2]^{1/2} d\eta_g(\alpha) d\eta_2(g) = 0,$$

but  $\xi_2\{g : \xi_g = \eta_g\} \neq 0$ , so that  $\xi_g \neq \eta_g$  but  $\xi_g = \eta_g$  a.e.  $[\xi_2]$ . Of course in this situation  $\xi$  doesn't define a type.

Part 2 : Failure of the uniqueness properties.

We turn our attention to an investigation of situations in which the various uniqueness properties fail. In particular we show that  $(U_q)$  fails for  $[\mathcal{Q}_r \oplus Z]_p$  whenever  $q/p, p/r \in \mathbb{N}$ , and that  $\mathbb{R}$  has  $(U_p)$  iff  $p \notin 2\mathbb{N}$ . It is then shown that if  $X$  has  $(nw_1)$ , then  $\mathcal{Q}_1^n$  embeds in  $X$ . We consider also the implications for a functional norm  $\varphi$  if  $(Y \oplus Z)_\varphi$  fails to have  $(2w_1)$ .

4.6 The failure of the uniqueness property for types,  $(U_p)$ .

Proposition 4.6.1

The properties  $(w_1)$ ,  $(u_1)$  and  $(U_1)$  all fail for  $(\mathcal{Q}_1^2 \oplus Z)_1$ .

Proof

We show that  $(2w_1)$  fails, and the result follows trivially. Let  $X = (\mathcal{Q}_1^2 \oplus Z)_1$ , so every  $x \in X$  has a unique decomposition into  $x = y + z$  with  $y \in \mathcal{Q}_1^2$ ,  $z \in Z$  and  $\|x\| = \|y\| + \|z\|$ .

Let  $m$  be any probability measure on  $Z$ . Let  $\mu = 1/2[\delta(0,0) + \delta(1,1)]$  and  $\nu = 1/2[\delta(0,1) + \delta(1,0)]$ . Then for all  $(u, v) \in X$ ,  $\int_X \|(u, v) + (x, y)\| d(\mu \times m)(x, y) = 1/2\{|u_1| + |u_2| + |u_1 + 1| + |u_2 + 1|\} = \int_X \|(u, v) + (x, y)\| d(\nu \times m)(x, y)$ .

Collecting together some facts about Banach spaces containing  $\mathcal{Q}_1$  helps to classify spaces where failure of uniqueness is likely.

1.  $c_0$  embeds in  $X^*$  iff  $X = \mathcal{Q}_1 \oplus Z$ .

2. Every bounded sequence has a weakly Cauchy subsequence iff  $\mathcal{Q}_1 \hookrightarrow X$  iff  $B_{X^{**}}$  is weak\*-sequentially compact.
3. If  $X^*$  has an equivalent smooth norm then  $\mathcal{Q}_1 \hookrightarrow X$ .
4. If  $X$  is very smooth or  $X^{**}$  is smooth then  $X^*$  has RNP, and then  $\mathcal{Q}_1 \hookrightarrow X$ .
5.  $X^*$  has weak RNP iff  $\mathcal{Q}_1 \hookrightarrow X$ .

Proposition 4.6.2

Whenever  $q/p \in \mathbb{N}$ ,  $(U_q)$  fails for  $\mathcal{Q}_p$ .

Proof

Consider a finite signed measure  $\nu \neq 0$  on  $[0, \infty)$  with

$\int_0^\infty u^k d|\nu|(u) < \infty$  and  $\int_0^\infty u^k d\nu(u) = 0$  for  $k=0, 1, \dots$ . For example, put

$d\nu(u) = \exp(-u^{1/4}) \sin(u^{1/4}) du$  see [Lil]. Then  $\nu \neq 0$ , but

$$\int_0^\infty (s+u)^n d\nu(u) = \sum_{j=0}^n \binom{n}{j} s^j \int_0^\infty u^{n-j} d\nu(u) = 0 \quad \forall n \in \mathbb{N} \quad \forall s > 0.$$

By the Jordan-Hahn decomposition theorem, we can find

measures  $\nu^+$  and  $\nu^-$  on  $[0, \infty)$  with  $\nu = \nu^+ - \nu^-$ , given by

$\nu^+(A) = \sup\{\nu(B) : B \subseteq A, B \in \sigma\text{-field}\}$  and

$\nu^-(A) = \inf\{\nu(B) : B \subseteq A, B \in \sigma\text{-field}\}$ . Then since

$\nu[0, \infty) = \int_0^\infty u^0 d\nu(u) = 0$ ,  $\nu^+[0, \infty) = \nu^-[0, \infty)$  we can take  $\nu^+$  and  $\nu^-$  to

be probability measures. Since  $q/p \in \mathbb{N}$ ,  $\int_0^\infty (s+\alpha)^{q/p} d\nu(\alpha) = 0$ , that

$$\text{is } \int_0^\infty (s+\alpha)^{q/p} d\nu^+(\alpha) = \int_0^\infty (s+\alpha)^{q/p} d\nu^-(\alpha) \quad \forall s.$$

Let  $\nu_1^+$  be the measure on  $[0, \infty)$  given by

$\nu_1^+[0, r] = \nu^+[0, r^p]$ . Similarly define  $\nu_1^-$  by  $\nu_1^-[0, r] = \nu^-[0, r^p]$

Then  $\int_0^\infty (s+\alpha^p)^{q/p} d\nu_1^+(\alpha) = \int_0^\infty (s+\alpha)^{q/p} d\nu^+(\alpha) = \int_0^\infty (s+\alpha^p)^{q/p} d\nu_1^-(\alpha)$

$$\forall s \geq 0, \text{ thus } \int_0^\infty (\|x+e_1\|^p + \alpha^p)^{q/p} dv_1^+(\alpha) = \int_0^\infty (\|x+e_1\|^p + \alpha^p)^{q/p} dv_1^-(\alpha)$$

for  $e_1 = (1, 0, 0, \dots)$  and all  $x$  in  $\mathcal{Q}_p$ .

Define  $\xi_2 = \eta_2 = \delta(e_1)$ , and  $\xi_y = \begin{cases} v_1^+ & \text{if } y = e_1 \\ 0 & \text{o.w.} \end{cases}$

$\eta_y = \begin{cases} v_1^- & \text{if } y = e_1 \\ 0 & \text{o.w.} \end{cases}$ . Then  $\forall x \int_{\mathcal{Q}_p} \int_0^\infty [\|x+y\|^p + \alpha^p]^{q/p} d\xi_y(\alpha) d\xi_2(y) =$

$$\int_0^\infty [\|x+e_1\|^p + \alpha^p]^{q/p} dv_1^+(\alpha) = \int_0^\infty [\|x+e_1\|^p + \alpha^p]^{q/p} dv_1^-(\alpha) =$$

$$\int_{\mathcal{Q}_p} \int_0^\infty [\|x+y\|^p + \alpha^p]^{q/p} d\eta_y(\alpha) d\eta_2(y).$$

Let  $\xi$  be the measure on  $\mathcal{Q}_p \times [0, \infty)$  giving rise to the disintegration  $\xi_y$  and  $\xi_2$ . So that

$$\xi(A) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} \chi_A(y, \alpha) d\xi(y, \alpha) = \int_{\mathcal{Q}_p} \int_0^\infty \chi_A(y, \alpha) d\xi_y(\alpha) d\xi_2(y) =$$

$$\int_0^\infty \chi_A(e_1, \alpha) dv_1^+(\alpha) = v_1^+\{\alpha : (e_1, \alpha) \in A\}.$$

Similarly let

$$\eta(A) = v_1^-\{\alpha : (e_1, \alpha) \in A\}.$$

Finally let  $\xi, \eta \in \pi(\mathcal{Q}_p \times [0, \infty))$  be the constant random measures  $\xi$  and  $\eta$  respectively, i.e.  $\xi_\omega = \xi$  and  $\eta_\omega = \eta$ . So that

$$\forall f \in L_q(\mathcal{Q}_p) \int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} [\|f(\omega) + y\|^p + \alpha^p]^{q/p} d\xi_\omega(y, \alpha) dP(\omega) =$$

$$\int_{\Omega} \int_{\mathcal{Q}_p \times \mathbb{R}_+} [\|f(\omega) + y\|^p + \alpha^p]^{q/p} d\eta_\omega(y, \alpha) dP(\omega).$$

Since  $v \neq 0$ ,  $v_1^+ \neq v_1^-$  and thus  $\xi \neq \eta$  a.e.  $[P]$ .

### Proposition 4.6.3

If  $q/p \notin \mathbb{N}$ , then even if  $(U_q)$  fails for  $\mathcal{Q}_p$  we still have equality of marginals.

### Proof

Suppose that there exists distinct measures  $\xi$  and  $\eta$  on  $\mathcal{Q}_p \times \mathbb{R}_+$  such that for all  $x \in \mathcal{Q}_p$

$$\int_{\mathcal{Q}_p \times \mathbb{R}_+} [\|x+y\|^p + \alpha^p]^{q/p} d\xi(y, \alpha) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} [\|x+y\|^p + \alpha^p]^{q/p} d\eta(y, \alpha).$$

Then proceeding as before, we find

that  $\int_{\mathcal{Q}_p \times \mathbb{R}_+} [\|x+y\|^p + \alpha^p + \beta]^{q/p} d\xi(y, \alpha) = \int_{\mathcal{Q}_p \times \mathbb{R}_+} [\|x+y\|^p + \alpha^p + \beta]^{q/p} d\eta(y, \alpha)$  (\*)

for all  $\beta \geq 0$ . We then similarly define

$\mu_x[F] = \xi\{(y, \alpha) : \|x+y\|^p + \alpha^p \in F\}$  and  $\nu_x[F] = \eta\{(y, \alpha) : \|x+y\|^p + \alpha^p \in F\}$ .

Then (\*) becomes  $\int_0^\infty (\beta+t)^{q/p} d\mu_x(t) = \int_0^\infty (\beta+t)^{q/p} d\nu_x(t) \quad \forall \beta \geq 0$  and

each  $x$ . Suppose  $q/p \notin \mathbb{N}$ , then by a result of W Linde [Li1],

$\mu_x = \nu_x$ . In particular  $\xi\{(y, \alpha) : (\|x+y\|^p + \alpha^p)^{1/p} \leq r\} = \mu_x[0, r^p] =$

$\nu_x[0, r^p] = \eta\{(y, \alpha) : (\|x+y\|^p + \alpha^p)^{1/p} \leq r\}$ . We are now in the same

situation as we were with  $L_1(\mathcal{Q}_p)$ . So, as in section 4.5, we

see that the marginal distributions  $\xi_2$  and  $\eta_2$  agree a.e. [P].

#### Theorem 4.6.4

The property  $(U_q)$  fails for  $[\mathcal{Q}_r \oplus Z]_p$  whenever  $q/p, p/r \in \mathbb{N}$ .

#### Proof

Let  $X = (\mathcal{Q}_r \oplus Z)_p$ . Let  $x = y + z$ , then  $\|x\| = (\|y\|^p + \|z\|^p)^{1/p}$ . Then every  $t \in \mathcal{T}(X)$  has a decomposition  $t(x) = [t_1(y)^p + t_2(z)^p]^{1/p}$ , where  $t_1 \in \mathcal{T}(Y)$ ,  $t_2 \in \mathcal{T}(Z)$ . Letting  $U$  be the map from  $\mathcal{T}(Y) \times \mathcal{T}(Z)$  to  $\mathcal{T}(X)$ , we have  $\int_{\mathcal{T}(X)} t(x)^q d\mu \circ U^{-1}(t) = \int_{\mathcal{T}(Y) \times \mathcal{T}(Z)} [t_1(y)^p + t_2(z)^p]^{q/p} d\mu(t_1, t_2)$ .

From the preceding results there exist distinct measures  $\xi$  and  $\eta$  on  $\mathcal{Q}_r \times [0, \infty)$  with  $\int_{\mathcal{Q}_r \times \mathbb{R}_+} [\|u+y\|^r + \alpha^r]^m d\xi(u, \alpha) = \int_{\mathcal{Q}_r \times \mathbb{R}_+} [\|u+y\|^r + \alpha^r]^m d\eta(u, \alpha)$  for all  $m \in \mathbb{N}$ , and  $\int_{\mathcal{Q}_r \times \mathbb{R}_+} d\xi(u, \alpha) = \int_{\mathcal{Q}_r} \int_0^\infty d\xi_u(\alpha) d\xi_2(u) = \int_0^\infty d\xi_{e_1}(\alpha) = \nu_1^+[0, \infty) = \nu_1^-[0, \infty) = \int_{\mathcal{Q}_r \times \mathbb{R}_+} d\eta(u, \alpha)$ . If  $q/p$  and  $p/r$  are integers, then so is  $q/r - jp/r$  for all  $0 \leq j \leq q/p$ . Thus  $\int_{\mathcal{Q}_r \times \mathbb{R}_+} [(\|u+y\|^r + \alpha^r)^{p/r} + s]^{q/p} d\xi(u, \alpha) =$

$$\begin{aligned} & \sum_{j=0}^{q/p} \binom{q/p}{j} s^j \int_{\mathcal{Q}_r \times \mathbb{R}_+} [\|u+y\|^r + \alpha^r]^{q/r - j p/r} d\xi(u, \alpha) \\ &= \int_{\mathcal{Q}_r \times \mathbb{R}_+} [(\|u+y\|^p + \alpha^p)^{p/r} + s]^{q/p} d\eta(u, \alpha). \end{aligned}$$

Let  $m = \delta_0$ , or any point mass on  $\mathcal{T}(Z)$ . Let  $\mu = \xi \times m$  and  $\nu = \eta \times m$ , which are then probabilities on  $\mathcal{Q}_r \times [0, \infty) \times \mathcal{T}(Z)$ . Then

$$\begin{aligned} & \forall y \in \mathcal{Q}_r, z \in Z \int_{\mathcal{Q}_r \times \mathbb{R}_+ \times \mathcal{T}(Z)} [(\|u+y\|^r + \alpha^r)^{p/r} + t_2(z)^p]^{q/p} d\mu(u, \alpha, t_2) \\ &= \int_{\mathcal{T}(Z)} \int_{\mathcal{Q}_r \times \mathbb{R}_+} [(\|u+y\|^r + \alpha^r)^{p/r} + t_2(z)^p]^{q/p} d\xi(u, \alpha) dm(t_2) \\ &= \int_{\mathcal{Q}_r \times \mathbb{R}_+} [(\|u+y\|^r + \alpha^r)^{p/r} + \|z\|^p]^{q/p} d\xi(u, \alpha) \\ &= \int_{\mathcal{Q}_r \times \mathbb{R}_+} [(\|u+y\|^r + \alpha^r)^{p/r} + \|z\|^p]^{q/p} d\eta(u, \alpha) \\ &= \int_{\mathcal{Q}_r \times \mathbb{R}_+ \times \mathcal{T}(Z)} [(\|u+y\|^r + \alpha^r)^{p/r} + t_2(z)^p]^{q/p} d\nu(u, \alpha, t_2). \end{aligned}$$

Letting  $\mu, \nu: \Omega \longrightarrow \mathcal{P}\{\mathcal{Q}_r \times [0, \infty) \times \mathcal{T}(Z)\}$  be  $\mu_\omega = \mu$  and  $\nu_\omega = \nu$ . Then

$$\begin{aligned} & \int_{\Omega} \int_{\mathcal{T}(X)} t(x)^q d\mu \circ U^1(t) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(Y) \times \mathcal{T}(Z)} [t_1(y)^p + t_2(z)^p]^{q/p} d\mu(t_1, t_2) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{Q}_r \times \mathbb{R}_+ \times \mathcal{T}(Z)} [(\|u+y\|^r + \alpha^r)^{p/r} + t_2(z)^p]^{q/p} d\mu(u, \alpha, t_2) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{Q}_r \times \mathbb{R}_+ \times \mathcal{T}(Z)} [(\|u+y\|^r + \alpha^r)^{p/r} + t_2(z)^p]^{q/p} d\nu(u, \alpha, t_2) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{T}(X)} t(x)^q d\nu_\omega \circ U^1(t) dP(\omega). \end{aligned}$$

All that remains to check is that  $\mu \circ U^1 \neq \nu \circ U^1$  a.e.  $[P]$ .

Now since  $\xi \neq \eta$ ,  $\exists A \subseteq \mathcal{Q}_r \times [0, \infty)$  such that  $\xi(A) \neq \eta(A)$ . Let  $B = \mathcal{T}(Z)$ , then  $\mu(A \times B) = \xi(A)m(B) = \xi(A) \neq \eta(A) = \nu(A \times B)$ . Since the decomposition  $x = (y, z)$  is unique,  $U(t_1, t_2) = U(s_1, s_2)$  iff  $t_i = s_i$ . Thus  $\mu \circ U^1 \neq \nu \circ U^1$  as required.

Suppose  $p=r$ . Then for each  $f$  we can find a sequence  $f_n$  with  $\|f_n(\omega)+y\|^{p+\alpha^p} \longrightarrow \|f(\omega)+y\|^{p+\alpha^p+s^p}$  for all  $s \geq 0$ . Then

$$\begin{aligned} & \int_{\Omega} \int_{\mathcal{L}_p \times \mathbb{R}_+} [\|f(\omega)+y\|^{p+\alpha^p+s^p}]^{q/p} d\xi_{\omega}(y, \alpha) dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{L}_p \times \mathbb{R}_+} [\|f(\omega)+y\|^{p+\alpha^p+s^p}]^{q/p} d\eta_{\omega}(y, \alpha) dP(\omega). \text{ When } m=\delta_0 \text{ we have} \\ & \int_{\Omega} \int_{\mathcal{L}_p \times \mathbb{R}_+ \times \mathcal{Z}} [\|f(\omega)+y\|^{p+\alpha^p+t_2(z)^p}]^{q/p} dm_{\omega} \times \xi_{\omega} dP(\omega) \\ &= \int_{\Omega} \int_{\mathcal{L}_p \times \mathbb{R}_+ \times \mathcal{Z}} [\|f(\omega)+y\|^{p+\alpha^p+t_2(z)^p}]^{q/p} dm_{\omega} \times \eta_{\omega} dP(\omega). \end{aligned}$$

#### Theorem 4.6.5

$\mathbb{R}$  has  $(U_p)$  iff  $p \notin 2\mathbb{N}$ .

#### Proof

Let  $\tau \in \mathcal{T}(L_p(\mathbb{R}))$  be a uniformly integrable type represented by  $\xi$  and  $\eta$ . Then

$$\begin{aligned} \tau(f) &= \int_{\Omega} \int_{\mathbb{R}} |t+f(\omega)|^p d\xi_{\omega}(t) dP(\omega) = \int_{\Omega} \int_{\mathbb{R}} |t+f(\omega)|^p d\eta_{\omega}(t) dP(\omega), \text{ thus} \\ \int_{\mathbb{R}} |t+f(\omega)|^p d\xi_{\omega}(t) &= \int_{\mathbb{R}} |t+f(\omega)|^p d\eta_{\omega}(t) \text{ a.e. } [P]. \text{ If } p \in 2\mathbb{N} \text{ then} \end{aligned}$$

choose distinct measures  $\mu_1$  and  $\nu_1$  on  $[0, \infty)$  with

$$\int_0^{\infty} |t+s|^p d\mu_1(t) = \int_0^{\infty} |t+s|^p d\nu_1(t) \text{ for all } s \geq 0. \text{ Then define}$$

$$\mu = \begin{cases} \mu_1 & \text{on } [0, \infty) \\ 0 & \text{on } (-\infty, 0] \end{cases} \text{ and define } \nu \text{ similarly. Let } \xi_{\omega} = \mu \text{ and } \eta_{\omega} = \nu,$$

then  $\xi \neq \eta$  so we have non-uniqueness whenever  $p \in 2\mathbb{N}$ .

If  $p \notin 2\mathbb{N}$  then by W.Linde [Li1] we know that

$$\int_{\mathbb{R}} |t+s|^p d\xi_{\omega}(t) = \int_{\mathbb{R}} |t+s|^p d\eta_{\omega}(t) \quad \forall s \in \mathbb{R} \text{ a.e. } [P] \text{ will imply that}$$

$\xi_{\omega} = \eta_{\omega}$  a.e.  $[P]$  as required.

Theorem 4.6.6

The property  $(U_q)$  fails for  $\mathcal{Q}_p^n$  iff  $q/p \in \mathbb{N}$  and one of the following holds:  $q/p < n$ ,  $p$  is even, or both  $p$  and  $q/p$  are odd.

Proof

Consider a uniformly integrable type on  $L_q(\mathcal{Q}_p^n)$  given by  $\tau(f) = \int_{\Omega} \int_{\mathcal{T}(\mathcal{Q}_p^n)} t[f(\omega)]^q d\xi_{\omega}(t) dP(\omega) = \int_{\Omega} \int_{\mathcal{T}(\mathcal{Q}_p^n)} t[f(\omega)]^q d\eta_{\omega}(t) dP(\omega)$ . Then

as  $\mathcal{Q}_p^n$  contains a Fréchet differentiable point, by Theorem

$$4.2.1, \quad \int_{\mathcal{T}(\mathcal{Q}_p^n)} t[f(\omega)]^q d\xi_{\omega}(t) = \int_{\mathcal{T}(\mathcal{Q}_p^n)} t[f(\omega)]^q d\eta_{\omega}(t) \quad \text{a.e.}[P].$$

Let us suppose that there are distinct measures  $\xi$  and  $\eta$  such that for all  $x \in \mathcal{Q}_p^n$ ,  $\int_{\mathcal{T}(\mathcal{Q}_p^n)} t[x]^q d\xi(t) = \int_{\mathcal{T}(\mathcal{Q}_p^n)} t[x]^q d\eta(t)$ . We represent  $\mathcal{T}(\mathcal{Q}_p^n)$  as  $\mathcal{Q}_p^n$ , since if  $t \in \mathcal{T}(\mathcal{Q}_p^n)$  then

$t(x) = \lim_m \|x + u^m\| = \lim_m (\sum_1^n |x_i + u_i^m|^p)^{1/p} = \|x + u\|$  where  $u_i = \lim_m u_i^m$ , so that  $u \in \mathcal{Q}_p^n$ . Hence  $\int_{\mathcal{Q}_p^n} \|x + u\|^q d\xi(u) = \int_{\mathcal{Q}_p^n} \|x + u\|^q d\eta(u) \quad \forall x \in \mathcal{Q}_p^n$ . Then by

[Go&Ko2] uniqueness fails iff  $q/p \in \mathbb{N}$  and at least one of the following holds: (i)  $q/p < n$  (ii)  $p$  is even (iii)  $p$  and  $q/p$  are both odd.

So  $\mathcal{Q}_p^n$  has  $(U_1)$  if  $p > 1$ ,  $\mathcal{Q}_p^n$  has  $(U_q)$  if  $q/p \notin \mathbb{N}$ . Also  $(U_1)$  fails for  $\mathcal{Q}_1^n \quad \forall n$ ,  $(U_{kp})$  fails for  $\mathcal{Q}_p^n$  if  $p$  is even or  $k < n$ ,  $(U_{kp})$  fails for  $\mathcal{Q}_p^n$  if  $p$  and  $k$  are odd.

4.7 Failure of the uniqueness property for measures on X.

Let  $X$  be a Banach space. Let us consider the property  $(w_1)$ . Let  $\mu, \nu$  be probabilities on  $X$ , with  $\forall x \in X$

$$\int_X \|x + y\| d\mu(y) = \int_X \|x + y\| d\nu(y) \quad (*). \quad \text{We can ask the following}$$

question, can we find other examples of Banach spaces and

distinct measures  $\mu$  and  $\nu$  such that  $(*)$  holds? Linde [Li1] has shown that if so  $X$  can't be  $L_p$ ,  $p > 1$ ; or  $C_0(\Omega)$  where  $\Omega$  is a locally compact but non-compact space. It has also been shown that  $(*)$  cannot hold on a Hilbert space, see [Ko1] and [Li2]. Does  $(*)$  imply anything about the structure of the Banach space  $X$ ? We suspect the answer is yes, and will show that quite strong statements about  $X$  can be made if  $\mu$  and  $\nu$  are both atomic measures.

In section 4.8 we assume that  $(2w_1)$  fails for  $X$ , and we show that we can construct a copy of  $\mathcal{Q}_1^2$  inside  $X$ . We will then extend this result to prove that the failure of  $(nw_1)$  for  $X$  implies the existence of  $\mathcal{Q}_1^n$  inside  $X$  in section 4.9.

We first assume  $(2w_1)$  fails with distinct probabilities  $\mu$  and  $\nu$  and calculate the minimum dimension of the closed linear span of their supports.

Let  $\mu = a\delta_e + b\delta_f$  and  $\nu = c\delta_g + d\delta_h$  be distinct probabilities on  $X$ . Assume  $(*)$  holds. Thus  $a+b=c+d=1$ . Then  $\forall x$

$a\|x+e\| + b\|x+f\| = c\|x+g\| + d\|x+h\|$   $(**)$ . Then all four elements  $e, f, g$  and  $h$  are distinct. We can ask what is the dimension of  $Y = \text{closed linear span of } \{e, f, g, h\}$ . Suppose  $\dim Y = 1$ , then  $f = \lambda e$ ,  $g = \alpha e$ ,  $h = \beta e$  for some  $\lambda, \alpha, \beta \neq 1$ . Then putting  $x=0, -e, -\lambda e, -\alpha e, -\beta e$

successively in  $(**)$  we find that  $a + |\lambda|b = c|\alpha| + d|\beta|$  and

$$1. \quad b\|(\lambda-1)e\| = c\|(\alpha-1)e\| + d\|(\beta-1)e\| \quad \Rightarrow \quad b|1-\lambda| = c|\alpha-1| + d|\beta-1|$$

$$2. \quad a\|(\lambda-1)e\| = c\|(\alpha-\lambda)e\| + d\|(\beta-\lambda)e\| \quad \Rightarrow \quad a|1-\lambda| = c|\alpha-\lambda| + d|\beta-\lambda|$$

$$3. \quad a\|(\alpha-1)e\| + b\|(\alpha-\lambda)e\| = d\|(\beta-\alpha)e\| \quad \Rightarrow \quad a|1-\alpha| + b|\alpha-\lambda| = d|\beta-\alpha|$$

$$4. \quad a\|(\beta-1)e\| + b\|(\lambda-\beta)e\| = c\|(\beta-\alpha)e\| \quad \Rightarrow \quad a|1-\beta| + b|\lambda-\beta| = c|\beta-\alpha|$$

$$(1&2) \Rightarrow |1-\lambda| = c|\alpha-1| + c|\alpha-\lambda| + d|\beta-\lambda| + d|\beta-1|$$

$$(3&4) \Rightarrow |\alpha-\beta| = a|\alpha-1| + a|1-\beta| + b|\alpha-\lambda| + b|\beta-\lambda|$$

$$(2&4) \Rightarrow b|1-\lambda| + a|1-\beta| + b|\lambda-\beta| = c|\alpha-1| + c|\alpha-\beta| + d|\beta-1|$$

$$(1&3) \Rightarrow a|1-\alpha| + b|1-\lambda| + b|\lambda-\alpha| = c|\alpha-1| + d|\alpha-\beta| + d|\beta-1|$$

Assume for the moment that  $\lambda, \alpha, \beta \geq 0$ .

(i) Suppose  $\lambda, 1 \geq \alpha, \beta$ . Then  $|\alpha-\beta| = 2a + 2\lambda b - \alpha - \beta = 2c\alpha + 2d\beta - \alpha - \beta$ .

$$\alpha \geq \beta \Rightarrow \alpha = c\alpha + d\beta = c\alpha + d\alpha \Rightarrow 0 = d(\beta - \alpha) \text{ thus } \alpha = \beta.$$

$$\beta \geq \alpha \Rightarrow \beta = c\alpha + d\beta = c\beta + d\beta \Rightarrow 0 = c(\beta - \alpha) \text{ thus } \alpha = \beta.$$

(ii) Suppose  $\lambda, 1 \leq \alpha, \beta$ . Then  $|1-\lambda| = 2\alpha c + 2\beta d - 1 - \lambda = 2a + 2d\lambda - 1 - \lambda$ .

$$1 \geq \lambda \Rightarrow 1 = a + \lambda b = a + b \Rightarrow 1 = \lambda. \quad \lambda \geq 1 \Rightarrow \lambda = a + \lambda b = \lambda a + \lambda b \Rightarrow \lambda = 1.$$

(iii) Suppose  $\lambda, \alpha \leq 1, \beta$ .

If  $\beta \leq 1$  then  $b(1-\lambda) + a(1-\beta) + b(\beta-\lambda) = c(1-\alpha) + c(\beta-\alpha) + d(1-\beta)$  thus

$$1 - 2\lambda b + \beta(b-a) = 1 - 2\alpha c + \beta(c-d) \text{ so that } b\beta - \lambda b = c\beta - \alpha c \text{ whence}$$

$$b(1-\lambda) + a(1-\beta) = c(1-\alpha) + d(1-\beta) \Rightarrow a\beta + b\lambda = \alpha c + \beta d = a + \lambda b \Rightarrow \beta = 1, \text{ a}$$

contradiction.

If  $\beta \geq 1$  then  $b(1-\lambda) + a(\beta-1) + b(\beta-\lambda) = c(1-\alpha) + c(\beta-\alpha) + d(\beta-1)$  thus

$$-2\lambda b + \beta + b - a = -2\alpha c + \beta + c - d \text{ so that } 2b - 2\lambda b - 1 = 2c - 2\alpha c - 1 \text{ whence}$$

$$-\beta + b\lambda + a = \alpha c - \beta + d = \alpha c + \beta d \Rightarrow \beta = 1, \text{ a contradiction.}$$

(iv) If  $1, \beta \leq \lambda, \alpha$  use part (iii).

(v) Suppose  $\lambda, \beta \leq 1, \alpha$ .

If  $\alpha \leq 1$  then  $b(1-\lambda) + a(1-\alpha) + b(\alpha-\lambda) = c(1-\alpha) + d(\alpha-\beta) + d(1-\beta)$  thus

$$-2\lambda b + \alpha(b-a) = -2\beta d + \alpha(d-c) \text{ so that } b(\alpha-\lambda) = d(\alpha-\beta) \text{ whence}$$

$$b(1-\lambda) + a(1-\alpha) = c(1-\alpha) + d(1-\beta) \Rightarrow a\alpha + b\lambda = \alpha c + \beta d = a + \lambda b \Rightarrow \alpha = 1, \text{ a}$$

contradiction.

If  $\alpha \geq 1$  then  $b(1-\lambda) + a(\alpha-1) + b(\alpha-\lambda) = c(\alpha-1) + d(\alpha-\beta) + d(1-\beta)$  thus

$$\alpha - 2\lambda b + \beta + b - a = \alpha - 2\beta d - c + d \text{ so that } 2b - 2\lambda b - 1 = 2d - 2\beta d - 1 \text{ whence}$$

$$b\lambda + a = c + \beta d = \alpha c + \beta d \Rightarrow \alpha = 1, \text{ a contradiction.}$$

(vi) If  $1, \alpha \leq \lambda, \beta$  use part (v).

If any of  $\lambda, \alpha, \beta$  are negative, for example  $\lambda$ , consider the equivalent equality with  $\|x+\lambda e\|$  replaced by  $\|-x-\lambda e\|$ , out of which will arise the same set of equalities as before. Thus  $\dim Y \neq 1$ , so that  $\dim Y \geq 2$ . We can illustrate this with some simple examples.

### Examples

1.  $X = \mathbb{R} \oplus \mathbb{R}$ . Let  $\mu = 1/2[\delta(0,0) + \delta(1,1)]$  and  $\nu = 1/2[\delta(0,1) + \delta(1,0)]$ . Then  $\int_X \|(x,y) + (u,v)\| d\mu(u,v) = \int_X \|(x,y) + (u,v)\| d\nu(u,v)$  for all  $x, y \in \mathbb{R}$ . Then  $Y = \mathbb{R}^2$ , so that  $\dim Y = 2$ .
2.  $X = \mathcal{Q}_1 \times [0, \infty)$ . Let  $\mu = 1/2[\delta(e_1, 0) + \delta(e_2, 1)]$  and  $\nu = 1/2[\delta(e_1, 1) + \delta(e_2, 0)]$ . Then  $X$  is infinite dimensional and  $Y = \text{span}\{(e_1, 0), (e_2, 0), (0, 1)\} = \mathcal{Q}_1^2 \times [0, \infty)$ , so that  $\dim Y = 3$ .
3. We can see that with  $X = \mathbb{R}$ ,  $(**)$  can never hold even with arbitrary finite 2-atomic measures.

### 4.8 Embedding $\mathcal{Q}_1^2$ in $X$ when $(2w_1)$ fails.

We now construct a copy of  $\mathcal{Q}_1^2$  inside  $X$  assuming  $(2w_1)$  fails. We do this by finding elements  $u$  and  $v$  such that  $[u, v]$  is isometric to  $\mathcal{Q}_1^2$ .

Suppose that for all  $x \in X$   $a\|x+e\| + b\|x+f\| = c\|x+g\| + d\|x+h\|$  where  $0 < a, b, c, d < 1$  with  $a+b=c+d=1$ .

Let  $\|\cdot\|$  be Gateaux differentiable at a point  $x \in X$ . Then for  $R \in \mathbb{R}$ ,  $a\|Rx+e\|+b\|Rx+f\|-(a+b)\|Rx\|=c\|Rx+g\|+d\|Rx+h\|-(c+d)\|Rx\|$   
 $=\frac{a\|x+e/R\|-\|x\|}{1/R} + \frac{b\|x+f/R\|-\|x\|}{1/R} \longrightarrow a\rho'(x;e)+b\rho'(x;f)$   
 $=\rho'(x;ae+bf)$  as  $|R| \longrightarrow \infty$ . Thus  $\rho'(x;ae+bf)=\rho'(x;cg+dh)$  at all points of Gateaux differentiability of  $X$ .

Lemma 4.8.1

Suppose  $Y$  is a finite dimensional Banach space. Suppose that for all points of Gateaux differentiability  $x \in Y$   $\rho'(x;u)=\rho'(x;v)$ . Then  $u=v$ .

Proof

If  $\rho'(x;y)=0$  then  $x \perp y$  that is  $\|x\| \leq \|x+\lambda y\|$  for all scalars  $\lambda$ , see [Di]. If  $\rho'(x;w)=0$  for all  $x \in G(Y)$ , then  $x \perp w \forall x \in G(Y)$ . If  $G(Y)$  is dense in  $Y$ , then  $\exists w_i \in G(Y)$  with  $w_i \longrightarrow w$ . Then  $\|w_i\| \longrightarrow \|w\|$  and  $\|w_i+\lambda w\| \longrightarrow \|w+\lambda w\|$ , and hence  $\|w\| \leq \|w+\lambda w\| = |1+\lambda| \|w\| \forall \lambda$  and in particular for  $\lambda=-1$  which is absurd unless  $w=0$ .

If  $\rho'(x;u)=\rho'(x;v)$  then let  $w=u-v$ .  $Y$  is separable since it is finite dimensional and hence there exists a dense  $G_\delta$ -set of Fréchet differentiable points in  $Y$ . Thus  $w=0$  as required.

We can assume wlog that  $h=0$ , (consider  $e'=e-h$  etc.). Then  $\forall x$   $a\|x+e\|+b\|x+f\|=c\|x+g\|+d\|x\|$ , let  $u=ae$  and  $v=bf$  then since  $ae+bf=cg$ ,  $u+v=cg$ . Hence  $\forall x$   $\|ax+u\|+\|bx+v\|=\|cx+u+v\|+d\|x\|$   
To show that we have a copy of two dimensional  $\mathcal{Q}_1$  inside  $Y$  it will suffice to show that for some  $u,v$  we have  $\|u+v\|=\|u\|+\|v\|=\|u-v\|$ .

Let  $x=0$  in the above. Then  $\|u+v\|=\|u\|+\|v\|$ . Putting  $x=-v/a$  we find that  $\|u-v\|=\|u+(1-c/a)v\|+(d/a-|1-b/a|)\|v\|$ . If  $a=b=c=d$  then this would be  $\|u-v\|=\|u\|+\|v\|$  as we required.

Lemma 4.8.2

Suppose that there exists  $w$  with  $w=\lambda u+(1-\lambda)v$  and  $\|w\|=\lambda\|u\|+(1-\lambda)\|v\|$ . Then  $\|\alpha u+(1-\alpha)v\|=\alpha\|u\|+(1-\alpha)\|v\|$  for all  $\alpha\in(0,1)$ .

Proof

Assume for the moment that  $\|u\|=\|v\|=1$ . Suppose that there exists  $\lambda$  with  $\|\lambda u+(1-\lambda)v\|=\lambda\|u\|+(1-\lambda)\|v\|$ . Suppose that wlog there exists  $\alpha<\lambda$  with  $w=\alpha u+(1-\alpha)v$  and  $\|w\|<\alpha\|u\|+(1-\alpha)\|v\|=1$ . Let  $0<\epsilon<\min\{1-\|w\|,(\lambda-\alpha)/(1-\lambda)\}<1$ ,  $k=\frac{1}{1-\epsilon(1-\lambda)/(1-\alpha)}$  and  $\beta=k(1-\lambda)/(1-\alpha)$ . Then  $k>1$ , and  $\beta\in(0,1)$ .

Then  $\|w+\epsilon u\|<1$ , and since the unit sphere is convex,  $\forall\beta\in(0,1)\|\beta(w+\epsilon u)+(1-\beta)u\|\leq 1$ . Now  $\beta(w+\epsilon u)+(1-\beta)u=(\alpha\beta+\epsilon\beta-\beta+1)u+\beta(1-\alpha)v=[k\lambda+(1+\epsilon\beta-k)]u+k(1-\lambda)v$ . Since  $k>1$  by construction,  $1+\epsilon\beta-k=0$ , thus  $\|\beta(w+\epsilon u)+(1-\beta)u\|=k\|\lambda u+(1-\lambda)v\|>1$  a contradiction. Therefore  $\|\alpha u+(1-\alpha)v\|=\alpha\|u\|+(1-\alpha)\|v\|$  for all  $\alpha\in(0,1)$ .

If  $\|u\|\neq\|v\|$  then let  $\mathcal{Q}=\{z:\|z\|\leq\chi\|u\|+(1-\chi)\|v\|\text{ for some }0<\chi<1\}$ . This set is convex. Let

$0<\epsilon<\min\{\frac{\alpha\|u\|+(1-\alpha)\|v\|-\|w\|}{\|u\|},(\lambda-\alpha)/(1-\lambda),1\}<1$ ,  $k=\frac{1}{1-\epsilon(1-\lambda)/(1-\alpha)}$

and  $\beta=k(1-\lambda)/(1-\alpha)$ . So  $k>1$  and  $\beta\in(0,1)$ . Then  $w+\epsilon u, u\in\mathcal{Q}$ , so  $\beta(w+\epsilon u)+(1-\beta)u\in\mathcal{Q}$  but

$\|\beta(w+\epsilon u)+(1-\beta)u\|=k\|\lambda u+(1-\lambda)v\|>\lambda\|u\|+(1-\lambda)\|v\|$  a contradiction

Since we have  $\|u+v\|=\|u\|+\|v\|$  we can apply the lemma with  $w=1/2(u+v)$  to get  $\|\lambda u+(1-\lambda)v\|=\lambda\|u\|+(1-\lambda)\|v\|$  for all  $\lambda\in(0,1)$ . For  $r>0$  put  $\lambda=1/(1+r)<1$ , so that  $r=(1-\lambda)/\lambda$ . Thus  $\|u+rv\|=\|u\|+r\|v\|$  for all  $r>0$ , and hence  $\|\alpha u+\beta v\|=\alpha\|u\|+\beta\|v\|$  for all  $\alpha$  and  $\beta$  of the same sign.

If  $\|u-v\|=\|u\|+\|v\|$  put  $w=-u$  then  $\|u-rv\|=\|w+rv\|$ . Thus  $\|w+v\|=\|w\|+\|v\|$  and we apply the above lemma to get  $\|w+rv\|=\|w\|+r\|v\|$ , and therefore  $\forall\alpha,\beta\in\mathbb{R}\ \|\alpha u+\beta v\|=|\alpha|\|u\|+|\beta|\|v\|$ .

Let  $x=-v$ . Then  $\|u-av\|+\|(1-b)v\|=\|u+(1-c)v\|+d\|v\|$ . We can assume that  $c=1$ (by normalization), then  $\|u-av\|=\|u\|+(1-b+d)\|v\|=\|u\|+a\|v\|$ . Thus  $\forall\alpha,\beta\in\mathbb{R}$   
 $\|\alpha u+\beta v\|=|\alpha|\|u\|+|\beta|\|v\|$ .

Hence  $[u,v]$  is isometric to  $\mathcal{Q}_1^2$ .

The next result shows that we have tight restrictions upon the choice of discrete measures  $\mu$  and  $\nu$ . Namely that  $\mu=1/2[\delta_e+\delta_f]$  and  $\nu=1/2[\delta_g+\delta_h]$ .

### Lemma 4.8.3

If  $\forall x\ \|ax+u\|+\|bx+v\|=\|cx+u+v\|+d\|x\|$  then  $a=b=c=d$ .

### Proof

Wlog  $c=1\geq d$  and  $a\leq b\leq c=1$ . Thus  $\forall x\ \|ax+u\|+\|bx+v\|=\|x+u+v\|+d\|x\|$  whence  $\|u\|+\|v\|=\|u+v\|$ , and so as above  $\alpha\|u\|+\beta\|v\|=\|\alpha u+\beta v\|$  for all  $\alpha,\beta\geq 0$ . Let  $x=-v$ , then  $\forall x\ \|u-av\|+(b-1)\|v\|=\|u\|+d\|v\|$  so that  $\|u-av\|=\|u\|+(1+d-b)\|v\|=\|u\|+a\|v\|$ . Then if  $\lambda=(a+1)^{-1}$   
 $\|\lambda u-(1-\lambda)v\|=\|(a+1)^{-1}u-a(a+1)^{-1}v\|=(a+1)^{-1}\|u\|+a(a+1)^{-1}\|v\|$   
 $=\lambda\|u\|+(1-\lambda)\|v\|$ . Thus  $\|\alpha u-\beta v\|=\alpha\|u\|+\beta\|v\|$  for all  $\alpha,\beta\geq 0$ .

Let  $x = \lambda u + \mu v$ . Then  $\|ax + u\| + \|bx + v\| = |a\lambda + 1| \|u\| + |a\mu| \|v\| + |b\lambda| \|u\| + |b\mu + 1| \|v\| = f(\lambda, \mu)$  and  $\|x + u + v\| + c\|x\| = |c\lambda| \|u\| + |c\mu| \|v\| + |\lambda + 1| \|u\| + |\mu + 1| \|v\| = g(\lambda, \mu)$ .  $f$  and  $g$  are identical functions from  $\mathbb{R}^2$  to  $\mathbb{R}$ , and so will be differentiable at the same points of  $\mathbb{R}^2$ .  $f$  is differentiable at all points except when  $\lambda = -1/a, 0$  or  $\mu = -1/b, 0$ ; and  $g$  is differentiable at all points except  $\lambda = -1, 0$  or  $\mu = -1, 0$ . Hence  $a = b = 1$ , and so  $c = d = 1$ .

We have already observed that if  $X = (\mathcal{Q}_1^2 \oplus Z)_1$ , then there exist 2-atomic measures  $\mu, \nu$  such that  $(w_1)$  fails. We have now seen that if such measures exist then  $X$  contains an isometric copy of  $\mathcal{Q}_1^2$ . It is natural to ask whether  $X$  is the  $\mathcal{Q}_1$ -direct sum of this subspace and some complement  $Z$ .

Proposition 4.8.4

Suppose that  $\forall x \ \|x + u\| + \|x + v\| = \|x + u + v\| + \|x\|$ . Then  $\forall x \in X \ \forall \alpha, \beta \in \mathbb{R}$   
 $\|x + \alpha u\| + \|x + \beta v\| = \|x + \alpha u + \beta v\| + \|x\| \quad (*)$ .

Proof

Certainly  $(*)$  holds for  $\alpha, \beta = 0, 1, -1$ . Suppose  $(*)$  is true for fixed  $\alpha$  and  $\beta = 1$ . Thus  $\|x + \alpha u\| + \|x + v\| = \|x + \alpha u + v\| + \|x\|$  and  $\|x + \alpha u\| + \|x + (\alpha + 1)u + v\| = \|x + \alpha u + v\| + \|x + (\alpha + 1)u\|$ , so that  $\|x + (\alpha + 1)u\| + \|x + v\| = \|x + (\alpha + 1)u + v\| + \|x\|$ . Similarly starting with  $\alpha = 1$  and  $\beta$  fixed yields  $(*)$  for  $\alpha = 1$  and  $\beta + 1$ .

Suppose for fixed  $\alpha$  and  $\beta$   $(*)$  holds. Then since  $\|x + (\alpha + 1)u\| + \|x + \alpha u + \beta v\| = \|x + (\alpha + 1)u + \beta v\| + \|x + \alpha u\|$  we find that  $\|x + (\alpha + 1)u\| + \|x + \beta v\| = \|x + (\alpha + 1)u + \beta v\| + \|x\|$ . Hence  $(*)$  is true for  $\alpha, \beta \in \mathbb{N}$ . Let  $p, q, r, s \in \mathbb{N}$ .

$$\frac{1}{qs} [\|qsx+psu\|+\|qsx+rqv\|] = \|x+p/qu\|+\|x+r/sv\| = \|x+p/qu+r/sv\|+\|x\|$$

$$= \frac{1}{qs} [\|qsx+psu+rqv\|+\|qsx\|]. \text{ So that } (*) \text{ holds for all } \alpha, \beta \in \mathbb{Q}_+.$$

For  $\alpha, \beta \in \mathbb{R}_+$ , find  $\alpha_n, \beta_n \in \mathbb{Q}_+$  with  $\alpha_n \rightarrow \alpha$  and  $\beta_n \rightarrow \beta$ . Therefore  $\|x+\alpha_n u\|+\|x+\beta_n v\| \rightarrow \|x+\alpha u\|+\|x+\beta v\|$  and  $\|x+\alpha_n u+\beta_n v\|+\|x\| \rightarrow \|x+\alpha u+\beta v\|+\|x\|$ . We can then easily extend the argument to all  $\alpha, \beta \in \mathbb{R}$ .

#### 4.9 Embedding $\mathcal{Q}_1^n$ in $X$ when $(nw_1)$ fails.

We now show that we can embed  $\mathcal{Q}_1^n$  in  $X$  by assuming that the uniqueness property  $(nw_1)$  fails in  $X$ .

So assume that for all  $x \in X$   $b_1 \|x+f_1\|+b_2 \|x+f_2\| = \sum_1^n a_i \|x+g_i\|$  where  $0 < a_i, b_i < 1$  with  $b_1+b_2 = \sum a_i = 1$ .

Let  $\|\cdot\|$  be Gateaux differentiable at a point  $x \in X$ . Then for  $R \in \mathbb{R}$ ,  $b_1 \|Rx+f_1\|+b_2 \|Rx+f_2\| - (b_1+b_2) \|Rx\| = \sum a_i \|Rx+g_i\| - \sum a_i \|Rx\| \rightarrow \rho'(x; b_1 f_1 + b_2 f_2)$  as  $|R| \rightarrow \infty$ . Thus by the lemma above

$$b_1 f_1 + b_2 f_2 = \sum a_i g_i. \text{ Thus let us consider}$$

$$\|b_1 x + \sum e_i\| + b_2 \|x\| = \sum_1^n \|a_i x + e_i\|.$$

To show that we have a copy of  $n$ -dimensional  $\mathcal{Q}_1$  inside  $Y$  it suffices to show that for some set  $\{u_i\}$  of  $n$  elements we have  $\|\sum \alpha_i u_i\| = \sum \alpha_i \|u_i\|$ , for all scalars  $\alpha_i \in \mathbb{R}$ . Let  $x=0$  in the above. Then  $\|\sum e_i\| = \sum_1^n \|e_i\|$ .

Suppose that  $\exists i, j$  and a  $\lambda \in (0, 1)$  with  
 $\|\lambda e_i + (1-\lambda)e_j\| < \lambda\|e_i\| + (1-\lambda)\|e_j\|$ . Then as before  $\forall \alpha \in (0, 1)$   
 $\|\alpha e_i + (1-\alpha)e_j\| < \alpha\|e_i\| + (1-\alpha)\|e_j\|$ . But  $\sum\|e_i\| = \|\sum e_i\| \leq$   
 $\|e_1\| + \dots + \|e_1 + e_j\| + \dots + \|e_n\| < \sum\|e_i\|$ , thus for all  $i, j$  and all  $\lambda$   
 $\|\lambda e_i + (1-\lambda)e_j\| = \lambda\|e_i\| + (1-\lambda)\|e_j\|$ .

Lemma 4.9.1

For all scalars  $s_i$  we have  $\sum |s_i| \|e_i\| = \|\sum s_i e_i\|$ .

Proof

Let  $0 < \alpha_i < 1$  be scalars with sum one. Then  $\sum_1^n \alpha_i \|e_i\| =$   
 $\sum^{n-2} \alpha_i \|e_i\| + (1 - \sum^{n-2} \alpha_i) \left\{ \frac{\|\alpha_{n-1} e_{n-1}\|}{1 - \sum^{n-2} \alpha_i} + \frac{\|\alpha_n e_n\|}{1 - \sum^{n-2} \alpha_i} \right\} =$   
 $\sum^{n-2} \alpha_i \|e_i\| + \|\alpha_{n-1} e_{n-1} + \alpha_n e_n\| = \dots = \|\sum \alpha_i e_i\|$ .

For scalars  $r_i \geq 0$ ,  $\sum r_i \|e_i\| = \sum r_i \left\{ \frac{r_1 \|e_1\|}{\sum r_i} + \dots + \frac{r_n \|e_n\|}{\sum r_i} \right\} = \|\sum r_i e_i\|$ .

Suppose that  $\exists i, j$  and a  $\lambda \in (0, 1)$  with  
 $\|\lambda e_i - (1-\lambda)e_j\| < \lambda\|e_i\| + (1-\lambda)\|e_j\|$ . Then as before  $\forall \alpha \in (0, 1)$   
 $\|\alpha e_i - (1-\alpha)e_j\| < \alpha\|e_i\| + (1-\alpha)\|e_j\|$ . In particular  
 $\|e_i - e_j\| < \|e_i\| + \|e_j\|$ . Assume wlog that  $i=1$  and  $j=2$ . Let  $x = -e_2$ .  
Then  $\|e_1 + (1-b_1)e_2 + e_3 + \dots + e_n\| + b_2\|e_2\| = \|e_1 - a_1 e_2\| + \|(1-a_2)e_2\| + \dots$   
 $\|e_n - a_n e_2\|$ . So  $(1-b_1+b_2)\|e_2\| < (a_1+a_3+\dots+a_n+1-a_2)\|e_2\|$ , i.e.  
 $a_2 < b_1$ . By considering  $x = -e_1$ , we find that  $a_1 < b_1$ . Thus if  
 $a_i \geq b_1$  then  $\forall \alpha \in (0, 1) \|\alpha e_i - (1-\alpha)e_j\| = \alpha\|e_i\| + (1-\alpha)\|e_j\|$ . In  
particular this works if  $a_i = a_j = b_1$ .

If  $a_1 < b_1$  then let  $0 > \lambda \geq -1/b_1 > -1/a_1$ . Then  $\lambda b_1 + 1, \lambda a_1 + 1 \geq 0$ .  
Let  $x = \lambda e_1$ . Then  $\|b_1 \lambda e_1 + e_1 + \dots + e_n\| + b_2 \|\lambda e_1\| =$

$$(b_1\lambda + 1 + b_2|\lambda|)\|e_1\| + \sum_2 \|e_i\| = \sum \|e_i + a_i\lambda e_1\|$$

$$< (a_1\lambda + 1)\|e_1\| + \sum_2 \|e_i\| + (a_2 + \dots + a_n)|\lambda|\|e_1\|. \text{ Thus}$$

$$1 + b_1\lambda - b_2\lambda < 1 + a_1\lambda - (a_2 + \dots + a_n)\lambda, \text{ that is } b_1 - b_2 < a_1 - (a_2 + \dots + a_n) \text{ so}$$

$b_1 < a_1$ , a contradiction. So  $\forall \alpha \in (0, 1)$

$$\|\alpha e_i - (1-\alpha)e_j\| = \alpha\|e_i\| + (1-\alpha)\|e_j\|.$$

Then we find that for all scalars  $s_i \in \mathbb{R}$ ,

$$\sum |s_i| \|e_i\| = \|\sum s_i e_i\| \text{ as required.}$$

#### Lemma 4.9.2

Suppose that for all scalars  $s_i \in \mathbb{R}$ ,  $\sum |s_i| \|e_i\| = \|\sum s_i e_i\|$  then  $a_i = b_1 = 1/n$  for all  $i$ , and  $b_2 = (n-1)/n$ .

#### Proof

Let  $x = \sum \lambda_i e_i$ . Then  $\|b_1 x + e_1\| + \|b_2 x\| = \sum |b_1 \lambda_i + 1| \|e_i\| + \sum |b_2 \lambda_i| \|e_i\| = f(\lambda_1, \dots, \lambda_n)$  and  $\sum \|a_i x + e_i\| = \sum \{ |a_i \lambda_i| \|e_i\| + \dots + |a_i \lambda_n| \|e_n\| + |a_i \lambda_i + 1| \|e_i\| \} = g(\lambda_1, \dots, \lambda_n)$ .  $f$  and  $g$  are identical functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ , and so will be differentiable at the same points.  $f$  is differentiable at all points except when  $\lambda_i = 0$  or  $-1/b_1$ ; and  $g$  is differentiable at all points except  $\lambda_i = -1/a_i$  or  $0$ . Hence  $a_i = b_1 = 1/n$ , and  $b_2 = (n-1)/n$ .

#### Proposition 4.9.3

Suppose that  $\forall x \sum \|x + e_i\| = \|x + \sum e_i\| + (n-1)\|x\|$ . Then  $\forall x \in X \forall \alpha_i \in \mathbb{R}$   
 $\sum \|x + \alpha_i e_i\| = \|x + \sum \alpha_i e_i\| + (n-1)\|x\| \quad (*)$ .

Proof

Certainly  $(*)$  holds for  $\alpha_i = 0, 1, -1$ . Suppose  $(*)$  is true for fixed  $\alpha_i$ . Thus  $\sum \|x + e_i + \alpha_i e_i\| = \|x + e_i + \sum \alpha_i e_i\| + (n-1)\|x + e_i\|$  and  $\sum \|x + e_i + \alpha_i e_i\| = \|x + (\alpha_i + 1)e_i\| + \sum_2^n \{\|x + e_i\| + \|x + \alpha_i e_i\| - \|x\|\}$ . So that  $\|x + (\alpha_i + 1)e_i\| + \sum_2^n \|x + \alpha_i e_i\| = \|x + (\alpha_i + 1)e_i + \sum_2^n \alpha_i e_i\| + (n-1)\|x\|$ . Hence  $(*)$  is true for  $\alpha_i \in \mathbb{N}$ , by induction.

Let  $p_i, q_i \in \mathbb{N}$ .

$$\sum \|x + p_i e_i / q_i\| = \frac{1}{\prod q_i} \left[ \sum_i \|x \prod_j q_j + p_i (\prod_{j \neq i} q_j) e_i\| \right] =$$

$$\frac{1}{\prod q_i} \left[ (n-1) \|x \prod_j q_j\| + \|x + \sum_i p_i (\prod_{j \neq i} q_j) e_i\| \right] = (n-1) \|x\| + \|x + \sum p_i e_i / q_i\|.$$

So that  $(*)$  holds for all  $\alpha_i \in \mathbb{Q}_+$ . We can then easily extend the argument to all  $\alpha, \beta \in \mathbb{R}$ .

We have seen that if  $(2w_1)$  fails, then there is a copy of  $\mathcal{Q}_1^2$  inside  $X$ . One can ask whether a similar equality would force a copy of  $c_0$  in  $X$ . We can prove the following.

Proposition 4.9.4

Suppose there exists  $e, f, g, h$  (all distinct) with  $\max\{\|x + e\|, \|x + f\|\} = \max\{\|x + g\|, \|x + h\|\}$  for all  $x \in X$ . Then  $\|e + f\| = \|e - f\| = \max\{\|e\|, \|f\|\}$ .

Proof

By translating by  $-h$  (consider  $x-h$ ) we can consider the situation in which for all  $x \in X$ ,  $\max\{\|x + e\|, \|x + f\|\} = \max\{\|x + g\|, \|x\|\}$ .

Then  $\max\{\|Rx+e\|, \|Rx+f\|\} - \|Rx\| =$   
 $\max\left[\frac{\|x+e/R\| - \|x\|}{1/R}, \frac{\|x+f/R\| - \|x\|}{1/R}\right] \longrightarrow \max[\rho'(x;e), \rho'(x;f)]$  as  
 $R \longrightarrow \infty$ .

Thus  $\max[\rho'(x;e), \rho'(x;f)] = \max[\rho'(x;g), 0]$  for all  $x \in X$ .  
 So  $\|g\| = \max\{\|e\|, \|f\|\}$ . If  $g=e+f$ , then  $\|e+f\| = \max\{\|e\|, \|f\|\}$  and so  
 by considering  $x=-f$  we find that  $\|e-f\| = \max\{\|e\|, \|f\|\}$ .

If for a fixed  $x$ ,  $\rho'(x;g) < 0$ , then  
 $0 = \max[\rho'(x;e), \rho'(x;f)] = \rho'(x;e)$  wlog. Thus  $\rho'(-x;g) > 0$  so that  
 $\rho'(-x;g) = \max[\rho'(-x;e), \rho'(-x;f)] = \rho'(-x;f)$ . Thus  
 $\rho'(x;f) = \rho'(x;g)$  if  $\rho'(x;e) = 0$ . If  $0 < \rho'(x;g) = \rho'(x;e)$  say, then  
 $0 = \max[\rho'(-x;e), \rho'(-x;f)] = \rho'(-x;f)$  thus  $0 = \rho'(x;f)$ . So  $\rho'(x;g)$   
 equals one of  $\rho'(x;e)$  and  $\rho'(x;f)$  and the other is zero. Thus  
 $\rho'(x;g) = \rho'(x;e) + \rho'(x;f) = \rho'(x;e+f)$ , then by a previous result  
 $g=e+f$ . So  $\max\{\|x+e+f\|, \|x\|\} = \max\{\|x+e\|, \|x+f\|\}$ . Then  
 $\|e-f\| = \max\{\|f\|, \|e\|\} = \|e+f\|$ .

We thus have a partition of the Gateaux-differentiable  
 points of  $X$  into a disjoint union of  $\mathfrak{R} = \{x: \rho'(x;f) = 0\}$  and  
 $\mathfrak{S} = \{x: \rho'(x;e) = 0\}$ .

Let  $x, y \in X$ . We say  $x$  is orthogonal to  $y$ , written  $x \perp y$  see  
 [Di], whenever  $\|x\| \leq \|x + \lambda y\|$  for all scalars  $\lambda$ . If  $\phi \in X^*$  then  $x \perp y$   
 for each  $y \in \ker \phi$  iff  $|\phi(x)| = \|\phi\| \cdot \|x\|$ . With  $\phi(y) = \rho'(x; y)$  we see  
 that  $\rho'(x; y) = 0$  implies  $x \perp y$ . Hence  $\mathfrak{R} = \{x: x \perp f\}$  and  $\mathfrak{S} = \{x: x \perp e\}$ .

4.10 The structure of  $(Y\oplus Z)_\phi$  when  $(2w_1)$  fails.

We now consider the conditions that the failure of  $(2w_1)$  in  $(Y\oplus Z)_\phi$  impose on the norm of  $X$ , and in particular we show that if  $(2w_1)$  fails, then  $\phi$  is essentially an  $\mathcal{Q}_1$ -norm, that is  $\phi(\alpha, \beta) = \alpha k_1 + \beta k_2$  for constants  $k_1, k_2 \geq 0$ .

Let  $X = (Y\oplus Z)_\phi$  be a functional direct sum, that is  $\|(y, z)\| = \phi(\|y\|, \|z\|)$ . Then from the axioms of the norm we have  $\forall y, r \in Y, \forall z, s \in Z, \phi(\|y+r\|, \|z+s\|) \leq \phi(\|y\|, \|z\|) + \phi(\|r\|, \|s\|)$ ,  $|\alpha| \phi(\|y\|, \|z\|) = \phi(|\alpha| \|y\|, |\alpha| \|z\|)$ , and  $\phi(\|y\|, \|z\|) = 0$  iff  $y = z = 0$ . Thus  $\phi: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , and  $c\phi(\alpha, \zeta) = \phi(c\alpha, c\zeta) \forall c \geq 0$ . Hence  $\phi(\alpha, 0) = \alpha\phi(1, 0) = \alpha k_1$ , say. Similarly  $\phi(0, \beta) = \beta k_2$ .

Suppose  $\int_X \|(y, z) + x\| d\mu(x) = \int_X \|(y, z) + x\| d\nu(x) \forall (y, z) \in X$ , where  $\mu = [a\delta(e, 0) + b\delta(f, 0)]$  and  $\nu = [c\delta(g, 0) + d\delta(h, 0)]$ , with  $0 < a, b, c, d < 1$  and  $a + b = c + d = 1$ . Then  $a\phi(\|y+e\|, \|z\|) + b\phi(\|y+f\|, \|z\|) = c\phi(\|y+g\|, \|z\|) + d\phi(\|y+h\|, \|z\|)$  for all  $y \in Y$  and  $z \in Z$ .

Let  $z = 0$ . Then  $a\|y+e\| + b\|y+f\| = c\|y+g\| + d\|y+h\|$  for all  $y \in Y$ . By the above we know that  $a = b = c = d = 1$  and  $\|y+u\| + \|y+v\| = \|y+u+v\| + \|y\|$  for  $u = e$  and  $v = f$ . From this we find that for  $\alpha, \beta \in \mathbb{R}$   $\phi(\|y+\alpha u\|, \|z\|) + \phi(\|y+\beta v\|, \|z\|) = \phi(\|y+\alpha u + \beta v\|, \|z\|) + \phi(\|y\|, \|z\|)$ . Let  $y = 0$  so that  $\phi(\alpha, \zeta) + \phi(\beta, \zeta) = \phi(\alpha + \beta, \zeta) + \phi(0, \zeta)$  for all  $\alpha, \beta, \zeta \geq 0$ . In fact we can assume that  $\phi(\alpha, \zeta) + \phi(\beta, \zeta) = \phi(\gamma, \zeta) + \phi(\delta, \zeta)$  for all  $\alpha, \beta, \gamma, \delta, \zeta \geq 0$  whenever  $\alpha + \beta = \gamma + \delta$ .

Then  $\phi(\alpha, \zeta) + \phi(\beta, \zeta) = \phi(\alpha + \beta, \zeta) + \zeta k_2$ , so that  
 $\phi(2\alpha, \zeta) + \zeta k_2 = 2\phi(\alpha, \zeta)$ . Hence  $\phi(\alpha, \zeta) = 1/2\phi(2\alpha, \zeta) + 1/2\zeta k_2 =$   
 $1/4\{\phi(2^2\alpha, \zeta) + \zeta k_2\} + \zeta k_2 = 1/2^3\phi(2^3\alpha, \zeta) + 7/2^3\zeta k_2$   
 $= \dots = 2^{-n}\phi(2^n\alpha, \zeta) + (n-1)/n\zeta k_2$ . So  
 $\phi(\alpha, \zeta) = \lim_n 2^{-n}\phi(2^n\alpha, \zeta) + \zeta k_2 = \lim_n \phi(\alpha, 2^{-n}\zeta) + \zeta k_2$ .

If  $z_i \longrightarrow z$  in  $Z$  then  $\forall y, |\phi(\|y\|, \|z\|) - \phi(\|y\|, \|z_i\|)| =$   
 $|\|(y, z_i)\| - \|(y, z)\|| \leq \|(0, z - z_i)\| = \phi(0, \|z - z_i\|) = \|z - z_i\| k_2 \longrightarrow 0$  as  
 $i \longrightarrow \infty$ . So if  $\zeta_i \longrightarrow \zeta$  in  $\mathbb{R}$ , then  $\forall \alpha \phi(\alpha, \zeta_i) \longrightarrow \phi(\alpha, \zeta)$ . Hence  
 $\phi(\alpha, \zeta) = \lim_n \phi(\alpha, 2^{-n}) + \zeta k_2 = \phi(\alpha, 0) + \zeta k_2 = \alpha k_1 + \zeta k_2$ . So  $\phi$  is forced to  
behave like an  $\mathcal{Q}_1$ -norm as expected.

We can show that  $a=b=c=d$  by the following. Let us  
suppose that  $\forall \alpha, \beta, \gamma, \delta, \zeta \geq 0$   $a\phi(\alpha, \zeta) + b\phi(\beta, \zeta) = c\phi(\delta, \zeta) + d\phi(\gamma, \zeta)$   
whenever  $\alpha + \beta = \delta + \gamma$ . Then  $\phi(\alpha, \zeta) = \frac{c}{a+b}\phi(2\alpha, \zeta) + \frac{d}{a+b}\zeta k_2 =$   
 $\frac{c}{a+b} \left[ \frac{c}{a+b}\phi(2^2\alpha, \zeta) + \frac{d}{a+b}\zeta k_2 \right] + \frac{d}{a+b}\zeta k_2 =$   
 $c^n\phi(2^n\alpha, \zeta) + d\zeta k_2 \{1 + c + \dots + c^{n-1}\}$ . So  
 $\phi(\alpha, \zeta) = \lim_n c^n\phi(2^n\alpha, \zeta) + \frac{d}{1-c}\zeta k_2 = \lim_n \phi(2^n c^n \alpha, c^n \zeta) + \zeta k_2$ .

If  $y_i \longrightarrow y$  and  $z_i \longrightarrow z$  then,  $|\phi(\|y\|, \|z\|) - \phi(\|y_i\|, \|z_i\|)| =$   
 $|\|(y_i, z_i)\| - \|(y, z)\|| \leq \|(y - y_i, z - z_i)\| \leq \|(y - y_i, 0)\| + \|(0, z - z_i)\|$   
 $= \|y - y_i\| k_1 + \|z - z_i\| k_2 \longrightarrow 0$  as  $i \longrightarrow \infty$ . So that if  $c > 1/2$  then  
 $2^n c^n \longrightarrow \infty$  which cannot happen. If  $c < 1/2$  then  $2^n c^n \longrightarrow 0$ , so that  
 $\phi(\alpha, 0) = \alpha k_1 = 0 \forall \alpha$ . Thus  $c = 1/2$ , and  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ . Similarly we  
can show that  $a=b=c=d$ .

We now prove various results, each of which gives a sufficient condition on  $\varphi$  to force it to be an  $\mathcal{Q}_1$ -norm.

Lemma 4.10.1

Let  $0 < u_i, r_i < 1$  and  $\sum_1^n u_i = \sum_1^n r_i = 1$ . Suppose that whenever  $\alpha_i, \beta_i \geq 0$  and  $\sum \alpha_i = \sum \beta_i$  we have  $\sum_1^n u_i \phi(\alpha_i, \zeta) = \sum_1^n r_i \phi(\beta_i, \zeta)$  for all  $\zeta \geq 0$ . Then  $u_i = r_i = 1/n$  and  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ .

Proof

Put  $\alpha_i = \alpha$  and  $\beta_1 = n\alpha, \beta_j = 0$ . Then  $\phi(\alpha, \zeta) = r_1 \phi(n\alpha, \zeta) + \sum_2^n r_i \zeta k_2 = r_1 [r_1 \phi(n^2\alpha, \zeta) + \sum_2^n r_i \zeta k_2] + \sum_2^n r_i \zeta k_2 = r_1^j \phi(n^j \alpha, \zeta) + \sum_2^n r_i \zeta k_2 \{1 + r_1 + \dots + r_1^{j-1}\}$ . So  $\phi(\alpha, \zeta) = \lim_j r_1^j \phi(n^j \alpha, \zeta) + \zeta k_2 = \phi(\lim_j r_1^j n^j \alpha, 0) + \zeta k_2 = \lim_j r_1^j n^j \alpha k_1 + \zeta k_2$ . Thus  $\alpha = \lim_j r_1^j n^j \alpha$  so  $r_1 = 1/n$ , and  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ . Similarly we can find that  $u_i = r_i = 1/n$ .

Lemma 4.10.2

Let  $0 < u_i, r_i < 1$  and  $\sum_1^n u_i = \sum_1^n r_i = 1$ . Suppose that whenever  $\alpha_i, \beta_i, \zeta_i, \xi_i \geq 0$  and  $\sum \alpha_i = \sum \beta_i, \sum \zeta_i = \sum \xi_i$  we have  $\sum_1^n u_i \phi(\alpha_i, \zeta_i) = \sum_1^n r_i \phi(\beta_i, \xi_i)$ . Then  $u_i = r_i = 1/n$  and  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ .

Proof

Put  $\zeta_i = \xi_i = \zeta$ . Then apply the above lemma to get the result.

Lemma 4.10.3

Let  $0 < u_i, r_i < 1$  and  $\sum_1^n u_i = \sum_1^n r_i = 1$ . Suppose that whenever  $\alpha_i, \beta_i \geq 0$  and  $\sum \alpha_i u_i = \sum \beta_i r_i$  we have  $\sum_1^n u_i \phi(\alpha_i, \zeta) = \sum_1^n r_i \phi(\beta_i, \zeta)$  for all  $\zeta \geq 0$ . Then  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ .

### Proof

Put  $\alpha_i = \alpha$  and  $\beta_1 = \alpha/r_1$ ,  $\beta_j = 0$ ,  $j \neq 1$ . Then  $\sum \alpha_i u_i = \sum \beta_i r_i$ . Thus  
$$\phi(\alpha, \zeta) = r_1 \phi(\alpha/r_1, \zeta) + \sum_2^n r_i \zeta k_2 = \phi(\alpha, r_1^j \zeta) + \sum_2^n r_i \zeta k_2 \{1 + r_1 + \dots + r_1^{j-1}\}$$
$$\longrightarrow \phi(\alpha, 0) + \phi(0, \zeta) = \alpha k_1 + \zeta k_2, \quad \text{as } j \longrightarrow \infty.$$

### Lemma 4.10.4

Let  $0 < u_i, r_i < 1$  and  $\sum_1^n u_i = \sum_1^n r_i = 1$ . Suppose that whenever  $\alpha_i, \beta_i, \zeta_i, \xi_i \geq 0$  and  $\sum \alpha_i u_i = \sum \beta_i r_i$ ,  $\sum \alpha_i \zeta_i = \sum \beta_i \xi_i$  we have  
$$\sum_1^n u_i \phi(\alpha_i, \zeta_i) = \sum_1^n r_i \phi(\beta_i, \xi_i).$$
 Then  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ .

### Proof

Put  $\zeta_i = \xi_i = \zeta$  for any  $\zeta \geq 0$ . Then apply the previous lemma to prove that  $\phi(\alpha, \zeta) = \alpha k_1 + \zeta k_2$ .

### Lemma 4.10.5

Suppose that  $\|(y, z)\| = \phi(y, z)$  is a norm on  $X = Y \oplus Z$ . Suppose that  $\forall z \in Z$   $\phi(x, z) + \phi(y, z) = \phi(u, z) + \phi(v, z)$  whenever  $x + y = u + v$  for  $x, y, u, v \in Y$ . Then  $\phi(y, z) = \phi(y, 0) + \phi(0, z) = \|(y, 0)\| + \|(0, z)\|$ .

### Proof

As before  $\phi(x, z) = 2^{-n} \phi(2^{-n} x, z) + (n-1) \phi(0, z) / n$ . So  
$$\phi(x, z) = \lim_n \phi(x, 2^{-n} z) + \phi(0, z) = \phi(x, 0) + \phi(0, z) = \|(y, 0)\| + \|(0, z)\|.$$

### Example

Suppose that  $\|(y, z)\| + \|(y + e_1 + e_2, z)\| = \|(y + e_1, z)\| + \|(y + e_2, z)\|$  for all  $(y, z) \in Y \oplus Z$ . Let  $X = Y \oplus Z$  and  $x = (y, z)$ ,  $e = (e_1, 0)$ ,  $f = (e_2, 0)$ . Then  $\|x\| + \|x + e + f\| = \|x + e\| + \|x + f\|$  for all  $x \in X$ , thus  
$$\|x\| + \|x + \alpha e + \beta f\| = \|x + \alpha e\| + \|x + \beta f\|$$
 for all  $\alpha, \beta \geq 0$ . That is  
$$\|(y, z)\| + \|(y + \alpha e_1 + \beta e_2, z)\| = \|(y + \alpha e_1, z)\| + \|(y + \beta e_2, z)\|$$
 for all

$(y, z) \in Y \oplus Z$ . If  $Y \equiv [e_1, e_2]$  then  $\|(r, z)\| + \|(s, z)\| = \|(u, z)\| + \|(v, z)\|$  whenever  $r+s=u+v$ , so apply the lemma above to get  $\|(y, z)\| = \|(y, 0)\| + \|(0, z)\|$ .

### Example

Consider  $X \oplus Y \oplus Z = [e_1, e_2] \oplus [e_3, e_4] \oplus Z$  with the functional norm  $\|(x, y, z)\| = \phi(\|x\|, \|y\|, \|z\|)$ . Let

$\mu = 1/2[\delta(e_1, e_3, 0) + \delta(e_1, e_4, 0) + \delta(e_2, e_3, 0) + \delta(e_2, e_4, 0)]$  and

$\nu = 1/2[\delta(e_1+e_2, e_3+e_4, 0) + \delta(0, e_3+e_4, 0) + \delta(e_1+e_2, 0, 0) + \delta(0, 0, 0)]$ ,

giving  $\phi(\|x+e_1\|, \|y+e_3\|, \|z\|) + \phi(\|x+e_1\|, \|y+e_4\|, \|z\|) +$

$\phi(\|x+e_2\|, \|y+e_3\|, \|z\|) + \phi(\|x+e_2\|, \|y+e_4\|, \|z\|) =$

$\phi(\|x+e_1+e_2\|, \|y+e_3+e_4\|, \|z\|) + \phi(\|x\|, \|y+e_3+e_4\|, \|z\|) +$

$\phi(\|x+e_1+e_2\|, \|y\|, \|z\|) + \phi(\|x\|, \|y\|, \|z\|),$

so that whenever  $\sum \alpha_i = \sum \gamma_i$  and  $\sum \beta_i = \sum \delta_i$

$\phi(\alpha_1, \beta_1, \zeta) + \phi(\alpha_1, \beta_2, \zeta) + \phi(\alpha_2, \beta_1, \zeta) + \phi(\alpha_2, \beta_2, \zeta) =$

$\phi(\gamma_1, \delta_1, \zeta) + \phi(\gamma_2, \delta_1, \zeta) + \phi(\gamma_1, \delta_2, \zeta) + \phi(\gamma_2, \delta_2, \zeta) \quad \forall \zeta \geq 0.$

Thus  $\phi(\alpha, \beta, \gamma) = 4^{-1}\phi(4\alpha, \beta, \zeta) + 3/4\phi(0, \beta, \zeta) = \dots$

$= 4^{-n}\phi(4^n\alpha, \beta, \zeta) + (4^n-1)\phi(0, \beta, \zeta)/4^n \longrightarrow \lim_n \phi(\alpha, \beta 4^{-n}, \zeta 4^{-n})$

$+ \phi(0, \beta, \zeta) = \phi(\alpha, 0, 0) + \phi(0, \beta, \zeta)$  and

$\phi(0, \beta, \zeta) = 4^{-n}\phi(0, 4^n\beta, \zeta) + (4^n-1)\phi(0, 0, \zeta)/4^n \longrightarrow$

$\lim_n \phi(0, \beta, 4^{-n}\zeta) + \phi(0, 0, \zeta) = \phi(0, \beta, 0) + \phi(0, 0, \zeta)$ . Thus

$\phi(\alpha, \beta, \gamma) = \phi(\alpha, 0, 0) + \phi(0, \beta, 0) + \phi(0, 0, \zeta) = k_1\alpha + k_2\beta + k_3\zeta.$

This is easily extended to the situation when

$X = (Y_1 \oplus \dots \oplus Y_n \oplus Z)_\phi$  with  $\phi$  a functional norm and  $Y_i \equiv [e_{2i-1}, e_{2i}]$ .

Then under obvious assumptions  $\phi(\alpha_1, \dots, \alpha_n, \zeta) = \sum_1^n \alpha_i k_i + \zeta k_{n+1}$ .

Although a functional norm had to essentially be an  $\mathcal{Q}_1$ -norm, this is not so in general, as we can see by the following examples.

Examples.

1. Let  $Z$  be any real vector space, and let  $\|\cdot\|_1, \|\cdot\|_2$  be norms on  $Z \oplus \mathbb{R}$ . Define a norm on  $Z \oplus \mathbb{R} \oplus \mathbb{R} = Z \oplus [e_1, e_2]$  by

$$\|z + \lambda_1 e_1 + \lambda_2 e_2\| = \|(z, \lambda_1)\|_1 + \|(z, \lambda_2)\|_2. \text{ Then if } x = z + \lambda_1 e_1 + \lambda_2 e_2,$$

$$\|x\| + \|x + e_1 + e_2\| = \|(z, \lambda_1)\|_1 + \|(z, \lambda_2)\|_2 + \|(z, \lambda_1 + 1)\|_1 + \|(z, \lambda_2 + 1)\|_2 =$$

$$(\|(z, \lambda_2)\|_2 + \|(z, \lambda_1 + 1)\|_1) + (\|(z, \lambda_1)\|_1 + \|(z, \lambda_2 + 1)\|_2) =$$

$\|x + e_1\| + \|x + e_2\|$ . So by setting  $\mu = 1/2[\delta(0) + \delta(e_1 + e_2)]$  and  $\nu = 1/2[\delta(e_1) + \delta(e_2)]$  we have non-uniqueness. Thus

$X = (Z \oplus [e_1, e_2])^\wedge$ , but  $\|\cdot\|$  is not necessarily the  $\mathcal{Q}_1$ -norm.

For example if  $X = \mathbb{R}^3$  with  $\|\cdot\|_1$  and  $\|\cdot\|_2$  being 2-norms, then  $\|(z, \lambda_1, \lambda_2)\|_X = \sqrt{(z^2 + \lambda_1^2)} + \sqrt{(z^2 + \lambda_2^2)}$ .

2. Let  $\|\cdot\|_1$  be a norm on  $Z \oplus \mathbb{R}$ , and  $\|\cdot\|_2$  be a norm on  $\mathbb{R}$ . Let  $\|z + \lambda_1 e_1 + \lambda_2 e_2\| = \|z + \lambda_1 e_1\|_1 + \|\lambda_2 e_2\|_2$ , then  $\mu$  and  $\nu$  as above will provide non-uniqueness but with  $\|\cdot\|$  not being an  $\mathcal{Q}_1$ -norm.

Let  $X = Z \oplus [e_1, e_2]$ . Suppose that by

$\|z + \lambda_1 e_1 + \lambda_2 e_2\|_X = \phi(\|z + \lambda_1 e_1\|_1, \|z + \lambda_2 e_2\|_2)$ . Then we can show that

$\phi$  is an  $\mathcal{Q}_1$ -norm. We have  $\forall z \in Z; \forall \alpha, \beta \in \mathbb{R}$

$$\phi(\|z\|, \|z\|) + \phi(\|(z, \alpha)\|, \|(z, \beta)\|) =$$

$$\phi(\|(z, \alpha)\|, \|(z, 0)\|) + \phi(\|(z, 0)\|, \|(z, \beta)\|). \text{ Put } z=0,$$

$$\phi(\|(0, \alpha)\|, \|(0, \beta)\|) = \phi(\|(0, \alpha)\|, 0) + \phi(0, \|(0, \beta)\|), \text{ so that } \forall A, B \geq 0,$$

$\phi(A, B) = \phi(A, 0) + \phi(0, B)$ . Since  $\phi$  defines a norm,  $\phi(A, 0) = Ak_1$  and  $\phi(0, B) = Bk_2$ , where  $k_1, k_2 \geq 0$ . So that

$$\|z + \lambda_1 e_1 + \lambda_2 e_2\|_X = k_1 \|z + \lambda_1 e_1\|_1 + k_2 \|z + \lambda_2 e_2\|_2.$$

Similarly if  $\|z + \lambda_1 e_1 + \lambda_2 e_2\|_X = \phi(\|z + \lambda_1 e_1\|_1, \|\lambda_2 e_2\|_2)$  or  $\phi(\|\lambda_1 e_1\|_1, \|z + \lambda_2 e_2\|_2)$  we can show that  $\phi$  is an  $\mathcal{Q}_1$ -norm.

#### Lemma 4.10.6

Suppose that  $\forall x \in X, \forall \alpha, \beta \in \mathbb{R}$ ,

$$\|x\| + \|x + \alpha e_1 + \beta e_2\| = \|x + \alpha e_1\| + \|x + \beta e_2\|. \text{ Then}$$

$$\|e_1\| + \|e_2\| = \rho'(e_1; e_2) + \rho'(e_2; e_1).$$

#### Proof

Let  $x=0, \alpha=\beta=1$ , then  $\|e_1 + e_2\| = \|e_1\| + \|e_2\|$  so that

$$\|x\| + \|x + \alpha e_1 + \beta e_2\| - \|\alpha e_1 + \beta e_2\| = \|x + \alpha e_1\| - \alpha \|e_1\| + \|x + \beta e_2\| - \beta \|e_2\| =$$

$$\frac{\|x/\alpha + e_1\| - \|e_1\|}{1/\alpha} + \frac{\|x/\beta + e_2\| - \|e_2\|}{1/\beta} \longrightarrow \rho'(e_1; x) + \rho'(e_2; x) \text{ as}$$

$\alpha, \beta \longrightarrow \infty$ . If  $\alpha = \beta$ , then LHS  $\longrightarrow \|x\| + \rho'(e_1 + e_2; x)$ . Hence  $\forall x \in X$

$$\|x\| + \rho'(e_1 + e_2; x) = \rho'(e_1; x) + \rho'(e_2; x). \text{ In particular if } x = ae_1 + be_2$$

$$\|ae_1 + be_2\| + \rho'(e_1 + e_2; ae_1 + be_2) = a\rho'(e_1; e_1) + a\rho'(e_2; e_1) + b\rho'(e_1; e_2) +$$

$$b\rho'(e_2; e_2). \text{ Let } a=b \text{ so that}$$

$$\|e_1\| + \|e_2\| + \|e_1 + e_2\| = \|e_1\| + \|e_2\| + \rho'(e_1; e_2) + \rho'(e_2; e_1) \text{ that is}$$

$$\|e_1\| + \|e_2\| = \rho'(e_1; e_2) + \rho'(e_2; e_1).$$

Thus 2-atomic non-uniqueness is false for  $c_0$ .

For example if  $X = c_0, \|e_1\| = \|e_2\| = 1$ . Let  $i$  be the unique index (if it exists) such that  $\|x\| = |x_i|$ , then

$\rho'(x; y) = (\text{sign } x_i) \cdot y_i$ , so  $\rho'(e_i, e_j) = 0$  for  $i \neq j$ , see chapter 3.

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