Drugs, Guns, and Targeted Competition[⋆]

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Abstract

We consider a dynamic competition game involving three players, in which each player can vary the extent of his competition on a per-rival basis. We call such competition targeted. We show that if the players are myopic, then the weaker players eventually lose the game to their strongest rival. If instead the players are sufficiently far-sighted, then all three players converge in their power and stay in the game. We develop our model in application to drug wars, but the approach of targeted competition can be applied to competition between firms or political parties, or to warfare.

Keywords: targeted competition, dynamic oligopoly, differential games, drug wars
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1. Introduction

Competition lies at the heart of economics and has been extensively studied. However, there are certain competition mechanisms which common in practice but which, to the best of our knowledge, have not been specifically addressed in the literature. These are the mechanisms that provide a competitor with an ability to target his rivals on an individual basis. We call such mechanisms targeted competition. In differentiated product markets firms can target particular rivals through their choice of product attributes. Multiproduct companies can target each other by varying their investments in markets for particular products. Companies sometimes employ comparative advertisement—the practice of running ads that directly compare one's products to those of one's rivals.[2] Unethical practices, e.g. launching fabricated lawsuits, provide additional ways to harm particular competitors. Nor is targeted competition limited to the field of industrial organisation. For instance, political parties and politicians compete through their support for specific programs, and at times governments protect national industries through trade barriers. Finally, warfare, whether locally among crime groups or globally among nations, remains the ultimate example of targeted competition.[3]

Targeted competition poses a strategic consideration that does not arise in non-targeted competition. An individual player can influence the balance of power among his rivals by choosing whom to compete against, which in turn determines how much this player profits or loses in competing with his rivals in the periods to come. In particular, one may intuitively expect the weaker players to direct more resources towards fighting the strongest player than to fighting each other. Otherwise, the strongest player stands a good chance of forcing the weaker players out of the game as time goes by. However, such intuition demands

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[1] The views expressed here are those of the author and do not necessarily represent the views of RBB Economics.

[2] See, e.g., Barigozzi and Peitz (2007).

[3] See Blattman and Miguel (2010) for a review of the literature on civil wars.

further exploration: if one weaker player fights more (allocates more power) against his strongest rival, the remaining weak player has free-riding incentives that can undermine the fight against the strongest player.

In a simplified targeted competition game with three players we show that, when the players are myopic, they prefer to fight more with their weakest opponents. Consequently, the strongest player increases his power and eventually out-competes his weaker rivals. Inversely, if the players are non-myopic and do not discount future profits too much, then the weaker players concentrate more on fighting their strongest opponent. Consequently, the strongest player becomes weaker over time, and all players reach a common power level and their competition persists. This latter result crucially depends on the starting allocation of power: no player shall be too strong initially. If one player is stronger than all his rivals combined (which is the general case in a duopoly), then the stronger player will always win over his rivals.[4]

It is tempting to view the efforts of the weaker players to fight against their strongest rival as a form of tacit collusion. The mechanism at play, however, is conceptually different. Collusive behaviour in repeated games is sustained by the credible threat that other players will punish the one who deviates from the equilibrium. In our game, the equilibrium is a Markov perfect equilibrium. Hence, the strategies do not depend upon past actions and there are no strategies involving retrospective punishment. In this scenario, it is the dynamic structure of the game that pushes the weaker players to fight together against their "common" enemy. If they prefer to fight each other for the sake of immediate gains, then the strongest player will grow in power and his market share will increase at the expense of the rivals. To avoid this loss of future profits, the weaker players fight more against their strongest rival, and so their behaviour resembles tacit collusion.

Many examples of targeted competition have a similar structure to our model. For example, when HP allocates more resources to developing and promoting printing hardware, it targets Canon among others, while when it invests more in its laptop product line, HP targets, say, Sony Inc. Sony, in a similar fashion, can invest more in laptop promotion and target HP, or in photo camera division and target Canon. The same applies to Canon, which can also reallocate its resources so as to fight selected rivals. The most successful company has more resources to invest in product development or more aggressive advertisement campaigns. However, too tough a competition in one of the markets can be exhausting for the corresponding pair of rivals and can lead to the loss to the third competitor in the remaining markets. Similar situations arise in politics. Before and during the World War II, for example, the Western Allies, the Soviet Union and the Axis tried to affect the balance of power among themselves by carefully choosing the timing of participation in major military operations. When one of the rivals conquered important resources (oil, coal or machinery), it grew stronger while weakening its opponent. Many computer and table games (say, RISK) utilise this idea of the balance of power as the main strategic consideration.

There are many real life examples of targeted competition, and each of them requires a bespoke model with a particular specification of instantaneous profits and dynamics of power. In order to develop a formal discussion of targeted competition, we investigate a particular scenario: competition between drug cartels.[5] In our view, this scenario best highlights the nature of targeted competition. On one hand, drug cartels ultimately aim at profits received from drug sales. On the other hand, these organisations do not shy away from re-recruiting or simply eliminating their opponents' members, which is the case, e.g., in conflicts among drug gangs in the U.S. (Levitt and Venkatesh, 2000), or in the ongoing Mexican drug war (Beittel, 2009). Most importantly, this example allows us to come up with a model specification that produces a clean analytic solution. We deem our results applicable to any model with the same strategic considerations and qualitatively similar specifications.

There are three cartels and three markets for drugs in our model. Each market is supplied by a different pair of cartels. Further, cartels fight each other in the markets they share. Each drug cartel is characterised by its power: the number of men (or women) that the cartel commands. We assume that the amount of drugs that a cartel can sell in a regional market is proportional to the manpower deployed in that market. Moreover, the amount of damage that a cartel inflicts on its regional rival is also proportional to the cartel's

[4]This case, where one player is stronger than his rivals combined, is trivial given our model setup. We therefore omit a formal analysis.

[5]We use the expression "a drug cartel" because it is an ubiquitous expression for drug syndicates. In this context no collusive pricing is implied.

local manpower. Lending anecdotal support to this set up, Levitt and Venkatesh (2000) have studied a gang in which footsoldiers were used for both drug distribution and fighting rival gangs. Finally, any cartel can freely reallocate its manpower and, in doing so, can either optimise its immediate profits or target particular rivals.

Any model of targeted competition should have the following two properties: 1) there should be three or more players—otherwise, the competition cannot be targeted; and 2) the analysis should be dynamic, as the aforementioned strategic considerations about the balance of power can only be studied in a dynamic setting. The closest matching strand of the literature then is that of dynamic oligopoly models. Though many scenarios of dynamic competition have been studied—inventories (Kirman and Sobel, 1974), sticky prices (Fershtman and Kamien, 1987), evolution of sales (Dockner and Jørgensen, 1988), capacity adjustment costs (Reynolds, 1991), varying profit opportunities (Ericson and Pakes, 1995), capital accumulation (Cellini and Lambertini, 1998), collusive behaviour (Fershtman and Pakes, 2000), etc.—targeted competition has not been a part of these analyses.

However, there are two related games that have been studied in the literature: Colonel Blotto games (see, e.g., Roberson, 2006; Hart, 2008; Arad and Rubinstein, 2012) and truel games (Kilgour, 1971).

A Colonel Blotto game is a game between two players who share several battlefields. Each player divides his army between battlefields, each battlefield is won by the larger force, and the player who wins more battlefields wins the game. There are two main differences from our game: 1) there are three players in our game; 2) our game is dynamic: the winner is not determined at once, rather the winner of each round becomes stronger and the game continues. The game of targeted competition that we study can be viewed as a game with three players and three battlefields in which each pair of players shares a battlefield and in which there is no battlefield that is shared by all three players. From this perspective, our game is in part similar to Colonel Blotto games because the players are able to choose how to divide their power against their opponents.

A truel game is an extension of a duel game. There are three players, each with a gun. In each round, each player chooses whom to shoot, and his chance of killing his opponent depends upon his skill; if two or more players are still alive, the game continues. As in our game, each player chooses his opponent, and killing a certain player influences the killer's chance of survival in the rounds to come. The main differences are: 1) in our game, the payoff from the game is the discounted sum of the payoffs from each round, so that each round is valuable, whereas in a truel game the payoff is 1 if the player survives and 0 otherwise; 2) in our game, if the player is "shot," he does not die at once but rather becomes relatively weaker; and 3) in a truel game a player chooses to fight either one opponent or the other, whereas in our game a player chooses *how much* to fight one opponent and *how much* to fight the other (a continuous choice).

So, our game has structural similarities to those of Colonel Blotto and truel games, but we find that the differences described above make our modelling approach more suitable for the study of economic interactions characterised by a flow of interim payoffs.

The rest of the paper is organised as follows. The next section presents a model of targeted competition, which is based on the scenario of drug wars between drug cartels. Section three discusses the implications of the model for the case of myopic cartels and for the case of forward-looking cartels. The last section concludes. All formal proofs are located in the appendix.

2. Setup

There are three drug cartels labelled 1, 2, and 3, which are involved in a lasting armed conflict over their regional markets. Each cartel i at time $t \in [0, \infty)$ is characterised by its manpower $x_i(t)$, which the cartel can employ for competition against its rivals. For brevity, we refer to x_i as the *power* of cartel i.

Denote $x = (x_1, x_2, x_3)$. At any time t, the powers of the cartels, $x(t)$, are common knowledge. The initial state $x(0)$ is normalised so that $\sum_i x_i(0) = 1$ (later on we will see that $\sum_i x_i(t) = 1$ for any t). We also assume that no cartel is too strong to start with. Formally, $x(0) \in X$, where

$$X = \left\{ x \in \mathbb{R}^3 \,\middle|\, \sum_i x_i = 1,\ x_i < \frac{7}{17}\ \forall i \right\}.$$

The reason for the restriction $x_i(0) < \frac{7}{17}$ is a technical one. Under this restriction, the equilibrium actions (which we are to analyse later on) are interior solutions, and the whole problem is analytically tractable. If one considers a more natural restriction that $x_i(0) < \frac{1}{2}$, then one needs to derive a numerical solution to a system of differential equations. We avoid the difficulty of solving the problem numerically by considering a smaller region for x.

There are three regional markets for drugs, and each market is served by a different pair of competing cartels. Each cartel can freely allocate its power among its markets, thus targeting particular rivals. Let y_{ij} denote the amount of power that cartel i allocates to the market shared with cartel j. Further, let $y_1 = (y_{12}, y_{13})$, $y_2 = (y_{21}, y_{23})$, $y_3 = (y_{31}, y_{32})$ and $y = (y_1, y_2, y_3)$.

Here and throughout the paper we use a common notation: index i always denotes a cartel, index $j \neq i$ denotes a rival of cartel i. We use the pair ij to subscript variables relevant to both cartels: e.g., how much cartel i fights cartel j or how many drugs are sold in total in the market that cartels i and j share (the indices are not commutative in the first case and, by definition, they are commutative in the second case).

In the following analysis we focus on Markov strategies, under which the choices of y_i by each cartel i depend only on the current state x and not on the past actions of the cartels. We choose Markov strategies because of the objective of the paper—to study whether forward-looking behaviour can produce collusive outcomes, but without the usual means of sustaining collusion (such as trigger strategies). Moreover, it is appealing to consider Markov strategies for several other reasons. First, an equilibrium in Markov strategies is also an equilibrium in a game with non-Markov strategies. Second, suppose that a game involving general strategies has multiple equilibria, one of which is a Markov equilibrium, then one way to select an equilibrium is to explore whether there is a focal point (Schelling, 1960). If simplicity makes a focal point, then the Markov equilibrium is selected. There are also other reasons, both theoretical and practical, for opting for Markov strategies; see the introduction to Maskin and Tirole (2001).

Each cartel uses all of its power against its opponents and the amount of power used cannot be negative. Given Markov strategies $y(x)$, we have

$$\begin{aligned} & y_i(x) \in Y_i(x), \\ & Y_i(x) = \left\{ y_i \,\middle|\, y_{ij} \geq 0, \sum\nolimits_j y_{ij} = x_i \right\}. \end{aligned} \tag{1}$$

The markets are identical and each market shared by cartels i and j has the following inverse demand for drugs:

$$p_{ij} = a - b q_{ij},$$

where $a > 0$, $b > 0$. In each market the cartels engage in Cournot competition. The amount of drugs that cartel i can supply to the market shared with cartel j is strictly proportional to the presence of cartel i on that market, i.e. to y_{ij}. Hence, $q_{ij} = y_{ij} + y_{ji}$ and the instantaneous profit that cartel i receives from the market shared with cartel j is given by

$$\varphi(y_{ij}, y_{ji}) = (a - b(y_{ij} + y_{ji}))y_{ij}.$$

We further assume that $a > 2b$, which guarantees that marginal profits are always positive and that, consequently, more power is always beneficial.

Let $\pi_i(y)$ denote total instantaneous profit of cartel i. We have

$$\pi_i(y) = \sum_{j \neq i} \varphi(y_{ij}, y_{ji}).$$

Power is not a factor in the instantaneous profit function per se. However, becoming more powerful will yield higher future profits because one can use more power to compete against one's rivals, thus improving the outcomes of the future rounds of competition.

In each market each cartel recruits new members. The number of new recruits by cartel i in the market it shares with cartel j is proportional to the current manpower of cartel i operating in that market, y_{ij}. On the other hand, all three cartels are engaged in an armed conflict against each other in the markets they share. The corresponding losses of cartel i in the market shared with cartel j are proportional to the amount of manpower allocated by cartel j against it, y_{ji}. For simplicity, we assume the recruiting and losing speeds to be equal. (Consequently, the total number of cartel members is constant over time.) Formally,

$$\dot{x}_i(t) = k \sum_{j \neq i} (y_{ij} - y_{ji}), \tag{2}$$

where $k > 0$ is the proportionality coefficient. From $\sum_i x_i(0) = 1$ and from (2) it follows that $\sum_i x_i(t) = 1$ for all t. Eq. (2) can further be interpreted as a re-recruiting of members by the stronger cartel from the weaker cartel on a specific market of operations.

If $x(t)$ reaches the boundary of X, the game ends. T denotes the ending time. Formally,

$$T = \inf\{t \geq 0 \,|\, x(t) \notin X\}.$$

If the game never ends, we write $T = \infty$.

If the game ends, each cartel i receives a terminal profit S_i, the strongest cartel wins, and the weaker cartels lose:

$$S_i(x) = \begin{cases} M(x) \text{ if } x_i > x_j \ \forall j \neq i, \\ 0 \text{ otherwise,} \end{cases}$$

where $M(x) \geq 0$ but is otherwise arbitrary.[6]

The rationale for ending the game if the boundary of X is approached is as follows. If one of the cartels becomes sufficiently strong, it is reasonable to expect that this cartel eventually out-competes its rivals. To simplify the game, we stop it at this moment and assign a non-negative (but otherwise arbitrary) profit M to the strongest cartel and a zero profit to the weaker cartels.

The profit from the whole game is the discounted stream of the instantaneous profits plus the discounted terminal profit, so the profit for cartel i is

$$U_i = \int_0^T e^{-rt} \pi_i(y(x(t)))dt + e^{-rT} S_i(x(T)), \tag{3}$$

where r is an instantaneous discount rate. Alternatively, r can be viewed as a hazard rate for the cartels' leaders. As long as a new leader is elected whenever the previous leader is assassinated (or otherwise quits the game), our specification continues to hold.

Thus, our setup is a differential game with simultaneous play (see Dockner et al., 2000) and we restrict our attention to Markov strategies. The strategies are functions $y(x)$ satisfying (1), the state variables x evolve according to (2), and the objective functions are given by (3).

Alternatively, a repeated game with intervals of length Δt can be set up in a similar fashion. The solution to the repeated game, which can be obtained using the one-stage deviation principle (Fudenberg and Tirole, 1991, Sec. 4.2), yields our solution as $\Delta t \to 0$. However, the analysis of the repeated game is more tedious than the analysis of its differential counterpart. Thus, we have opted for a differential game.

3. Analysis

We consider two cases: a case with myopic cartels and a general case. In both cases, we look for Markov perfect equilibria (MPE) and analyse the resulting equilibrium dynamics.

[6] If the game ends and two cartels are equally strong, they both lose. This assumption is made for the sake of simplicity and does not change the results.

The cartels are myopic if they focus on current profits. So, for a myopic cartel i the profit of the game at time t is

$$U_i(t) = \pi_i(y(x(t))).$$

The dynamics of the myopic case are summarised in the following proposition (we limit our attention to a general initial state in which one of the cartels is strictly stronger than the rest).

Proposition 1. *Suppose, without loss of generality, that $x_1(0) > x_2(0)$, $x_1(0) > x_3(0)$. If $r = \infty$ (myopic cartels), then there exists a unique MPE. This MPE is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10}$$

and its dynamics are given by

$$\dot{x}_i(t) = \frac{9k}{5}\left(x_i(t) - \frac{1}{3}\right).$$

In this equilibrium the game ends and the strongest cartel wins: $T < \infty$, $x_1(T) > x_2(T)$, $x_1(T) > x_3(T)$.

So, myopic cartels fight more against each other than against their strongest rival. This result occurs because investing in a market shared with a weaker opponent yields higher immediate profits. Consequently, the strongest cartel wins.

Proposition 1 serves as a benchmark for our setup. We are interested to know whether long term strategic considerations induce the weaker cartels to balance the power of their strongest rival. If the model was specified in such a way that even myopic cartels balanced their strongest opponent, then our question would be ill-posed. As it stands now, the question is well-posed and we proceed with its discussion.

When the cartels value their future profits highly enough (i.e., when r is sufficiently small), the weaker cartels have incentives to fight their strongest opponent more, so as to balance his future power and thus earn higher profits in the long run. However, this intuition is incomplete. If one of the weaker cartels spends its power to balance the strongest cartel, then the other weak cartel has incentives to free-ride, and to fight its weaker rival rather than its stronger rival. These free-riding incentives might preclude the existence of an equilibrium with converging power levels. The following proposition resolves this ambiguity.

Proposition 2. *If $r < \frac{4k}{3}$, then there exists a unique MPE with linear symmetric strategies such that $x(t) \to \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ as $t \to \infty$. If $r \geq \frac{4k}{3}$, then no such equilibria exist.*[7] *The equilibrium in question is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{c(x_k - x_j)}{2},$$

where

$$c = \frac{1}{18}\left(5\frac{r}{k} - 14 - \sqrt{25\left(\frac{r}{k}\right)^2 - 176\frac{r}{k} + 304}\right),$$

and the equilibrium dynamics are given by

$$\dot{x}_i(t) = \frac{3k(c+1)}{2}\left(x_i(t) - \frac{1}{3}\right).$$

[7] As the proposition says, we limit our attention to equilibrium strategies that are linear and symmetric, and that lead the cartels towards the centre. However, we do not restrict non-equilibrium strategies in any way: i.e., we consider all possible deviations from equilibrium strategies.

For visual purposes, coefficient c can be viewed as a linear function of r/k, because $c \approx 9/16 \cdot r/k - 7/4$, and the error is no more than 0.0036 for $r/k \in (0, 4/3)$.

So, for a sufficiently small r there is an MPE in which the weaker cartels balance the power of the strongest cartel. In this equilibrium each cartel allocates more power against its stronger rival (because $c < -1$). Further, the strongest cartel becomes weaker over time, while the weaker cartels become stronger. Consequently, the power levels of all three cartels converge. Finally, the more the cartels value their future profits, the faster the speed of convergence is (c is more negative for smaller values of r).

For the same model parameters, can there be another equilibrium such that, instead of counter-balancing each other, the stronger cartels fight the weaker cartel and hence the weaker cartel eventually becomes so weak that the game ends? While we do not have a complete answer, Proposition 3 partially settles this question by showing that at least no such equilibria in linear strategies exist. The proposition also hints at why the complete answer is challenging: if such an alternative equilibrium does exist, it must be non-linear.

Proposition 3. *If $r < \infty$, then there are no MPE in linear symmetric strategies such that $T < \infty$ (i.e., such that the game ends in a finite time due to one of the cartels becoming sufficiently strong).*

4. Concluding Remarks

Stackelberg (1952) has argued that a duopoly can never achieve equilibrium in price/quantity-setting strategies. Moreover, the actors in a duopoly will fight for leadership, and consequently, either one of them will become much stronger in economic terms, or they will find it beneficial to collude.

"Duopoly is an unstable market form not only in the sense that price is apt to be indeterminate, but much more because it is unlikely to remain as a market form for any length of time. The inherent contradictions in the duopolistic situation press for a solution through the adoption of another market form—monopoly."

We have studied targeted competition as applied to drug cartels. If there are only two drug cartels, then the strongest one will indeed become a monopolist. However, we show that if there are three drug cartels, and if these cartels are sufficiently forward looking, then there exists an equilibrium in which every cartel competes more against its stronger rival. Consequently, the cartels converge in power and their oligopolistic rivalry persists—the system does not degenerate into monopoly.

Appendix

Because the nature of the players is irrelevant for formal proofs, in this appendix we refer to the three cartels as players. In the following proofs we denote the best response strategies with $\tilde{y}$ and the equilibrium strategies with $\hat{y}$.

Proposition 1. *Suppose, without loss of generality, that $x_1(0) > x_2(0)$, $x_1(0) > x_3(0)$. If $r = \infty$ (myopic players), then there exists a unique MPE. This MPE is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10}$$

and its dynamics are given by

$$\dot{x}_i(t) = \frac{9k}{5}\left(x_i(t) - \frac{1}{3}\right).$$

In this equilibrium the game ends and the strongest player wins: $T < \infty$, $x_1(T) > x_2(T)$, $x_1(T) > x_3(T)$.

Proof. Maximising $U_i(t)$ in (y_{ij}, y_{ik}) w.r.t. $y_{ij} + y_{ik} = x_i$ gives a unique best response

$$\tilde{y}_{ij}(x) = \begin{cases} 0 & \text{if} \quad \gamma_{ij}(x) < 0, \\ \gamma_{ij}(x) & \text{if} \quad 0 \le \gamma_{ij}(x) \le x_i, \\ x_i & \text{if} \quad \gamma_{ij}(x) > x_i, \end{cases}$$

where

$$\gamma_{ij}(x) = \frac{x_i}{2} + \frac{y_{ki}(x) - y_{ji}(x)}{4}.$$

Without loss of generality suppose that player i is the strongest player at time t, i.e. $x_i(t) \geq x_j(t)$ and $x_i(t) \geq x_k(t)$. Then, having $y_j \in Y_j(x)$ and $y_k \in Y_k(x)$, we obtain

$$0 < \frac{x_i}{2} - \frac{x_j}{4} \leq \gamma_{ij}(x) \leq \frac{x_i}{2} + \frac{x_k}{4} < x_i. \tag{4}$$

Thus, $\tilde{y}_{ij}(x) = \gamma_{ij}(x)$ and $\tilde{y}_{ik}(x) = \gamma_{ik}(x)$.

Consider player j. Given (4) and having $x \in X$, we get

$$\gamma_{ji}(x) \geq \frac{x_j}{2} - \frac{1}{4}\left(\frac{x_i}{2} + \frac{x_k}{4}\right) \geq \frac{1}{2} \cdot \frac{3}{17} - \left(\frac{1}{8} \cdot \frac{7}{17} + \frac{1}{16} \cdot \frac{7}{17}\right) = \frac{3}{272} > 0.$$

By symmetry, $\gamma_{jk}(x) < x_j$. Analogously, $\gamma_{jk}(x) > 0$ and $\gamma_{ji}(x) < x_j$. Thus, $\tilde{y}_{ji}(x) = \gamma_{ji}(x)$ and $\tilde{y}_{jk}(x) = \gamma_{jk}(x)$.

Exactly the same reasoning applies to player k. So, all the best responses are interior in the equilibrium. Intersecting the best responses we obtain the unique equilibrium point

$$\hat{y}_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10}. \tag{5}$$

Because we are considering Markov strategies, (5) constitutes a unique Markov perfect equilibrium.

Consider an arbitrary i. Plugging (5) into (2) and using $x_1 + x_2 + x_3 = 1$ yields

$$\dot{x}_i(t) = \frac{9k}{5}\left(x_i(t) - \frac{1}{3}\right).$$

Therefore,

$$\dot{x}_1(t) - \dot{x}_i(t) = \frac{9k}{5}\left(x_1(t) - x_i(t)\right). \tag{6}$$

X is bounded, and $x_1(0) > x_i(0)$ for $i \in \{2, 3\}$. It then follows from (6) that $x(t)$ reaches the boundary of X at some time T and that $x_1(T) > x_i(T)$ for $i \in \{2, 3\}$. □

Proposition 2. *If $r < \frac{4k}{3}$, then there exists a unique MPE with linear symmetric strategies such that $x(t) \to \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ as $t \to \infty$. If $r \geq \frac{4k}{3}$, then no such equilibria exist. The equilibrium in question is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{c(x_k - x_j)}{2},$$

where

$$c = \frac{1}{18}\left(5\frac{r}{k} - 14 - \sqrt{25\left(\frac{r}{k}\right)^2 - 176\frac{r}{k} + 304}\right),$$

and the equilibrium dynamics are given by

$$\dot{x}_i(t) = \frac{3k(c+1)}{2}\left(x_i(t) - \frac{1}{3}\right).$$

Proof. An equilibrium strategy $\hat{y}_{ij}$ is linear if

$$\hat{y}_{ij}(x) = \alpha_{ij}x_i + \beta_{ij}x_j + \gamma_{ij}x_k + \delta_{ij}, \tag{7}$$

where α_{ij}, β_{ij}, γ_{ij} and δ_{ij} are arbitrary but constant coefficients.

These strategies are symmetric if 1) $\hat{y}_{ij} = \hat{y}_{ik}$ whenever $x_j = x_k$, and 2) $\hat{y}_{ij} = \hat{y}_{ji}$ whenever $x_i = x_j$. Imposing these symmetry conditions on (7) and using $\sum_i x_i = 1$ and $y_{ij} + y_{ik} = x_i$, yields

$$\alpha_{ij} + \delta_{ij} = \frac{1}{2}, \quad \beta_{ij} + \delta_{ij} = \beta_{ji} + \delta_{ji} = -(\gamma_{ij} + \delta_{ij}).$$

Let $c = 2(\gamma_{ij} + \delta_{ij})$. Then

$$\hat{y}_{ij}(x) = \frac{x_i + c(x_k - x_j)}{2}. \tag{8}$$

From $\sum_i x_i(t) = 1$, from (2) and from (8), it follows that

$$\dot{x}_i(t) = \frac{3k(c+1)}{2}\left(x_i(t) - \frac{1}{3}\right). \tag{9}$$

So, $x_i(t) \to \frac{1}{3}$ for all i as $t \to \infty$ if and only if $c < -1$.

To prove the proposition, we need to show that for $r < \frac{4k}{3}$ there exists a unique $c < -1$ such that (8) constitutes an MPE, and that for $r \geq \frac{4k}{3}$ no such c exists.

To investigate when (8) constitutes an MPE, we need to investigate when $\hat{y}_i$ is the best response to $\hat{y}_j$ and $\hat{y}_k$. So, we fix the strategies of players j and k at $\hat{y}_j$ and $\hat{y}_k$ and consider different strategies of player i. Given the strategies of players j and k, all possible strategies of player i can be divided into two classes: those strategies that never end the game ($T = \infty$)—let this be class $\mathcal{A}$, and those that eventually do ($T < \infty$)—class $\mathcal{B}$. We proceed as follows. First, we restrict the strategies of player i to class $\mathcal{A}$ and investigate when the strategy $\hat{y}_i$, as given by (8), is indeed the best response strategy. Then, we extend our results to $\mathcal{A} \cup \mathcal{B}$.

So, let the strategies of player i be restricted to class $\mathcal{A}$. Let us compute the value function V of player i if every player follows strategy $\hat{y}$ and if the game starts at $x(0) = x$. Solving (9) gives us

$$x_i(t) = \left(x_i - \frac{1}{3}\right)e^{3k(c+1)/2\cdot t} + \frac{1}{3}. \tag{10}$$

Let $z_i = x_i - \frac{1}{3}$. As $x_1 + x_2 + x_3 = 1$, we obtain $z_1 + z_2 + z_3 = 0$ and

$$\hat{y}_{ij}(x) = \frac{x_i + c(x_k - x_j)}{2} = \frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}.$$

Then (using $\sum_i z_i = 0$ where appropriate)

$$\begin{aligned}
&\pi_i(\hat{y}(z)) = (a - b(\hat{y}_{ij} + \hat{y}_{ji}))\hat{y}_{ij} + (a - b(\hat{y}_{ik} + \hat{y}_{ki}))\hat{y}_{ik} = \\
&\left(a - b\left(\frac{z_i + c(z_k - z_j)}{2} + \frac{z_j + c(z_k - z_i)}{2} + \frac{1}{3}\right)\right) \cdot \left(\frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}\right) + \\
&\left(a - b\left(\frac{z_i + c(z_j - z_k)}{2} + \frac{z_k + c(z_j - z_i)}{2} + \frac{1}{3}\right)\right) \cdot \left(\frac{z_i + c(z_j - z_k)}{2} + \frac{1}{6}\right) = \\
&\left(a - \frac{b}{3}\right)\left(z_i + \frac{1}{3}\right) - \frac{b(3c-1)}{2}\left(z_k\left(\frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}\right) + z_j\left(\frac{z_i + c(z_j - z_k)}{2} + \frac{1}{6}\right)\right) = \\
&\qquad \frac{b(3c-1)}{4}z_i^2 + \frac{12a + b(3c-5)}{12}z_i + \frac{3a-b}{9} - \frac{bc(3c-1)}{4}(z_k - z_j)^2.
\end{aligned} \tag{11}$$

Let $m = 3k(c+1)/2$, then (10) gives $z_i(t) = z_i e^{mt}$. So,

$$V_i(z) = \int_0^\infty e^{-rt}\pi_i(\hat{y}(z(t)))dt =$$
$$\int_0^\infty e^{-rt}\left(\frac{b(3c-1)}{4}\left(z_i e^{mt}\right)^2 + \frac{12a+b(3c-5)}{12}z_i e^{mt} + \frac{3a-b}{9} - \frac{bc(3c-1)}{4}\left(z_k e^{mt} - z_j e^{mt}\right)^2\right)dt =$$
$$\frac{b(3c-1)}{4}\frac{1}{r-2m}z_i^2 + \frac{12a+b(3c-5)}{12}\frac{1}{r-m}z_i + \frac{3a-b}{9}\frac{1}{r} - \frac{bc(3c-1)}{4}\frac{1}{r-2m}(z_k - z_j)^2. \quad (12)$$

Plugging in $z_i = x_i - \frac{1}{3}$ and $m = 3k(c+1)/2$ gives

$$V_i(x) = \int_0^\infty e^{-rt}\pi_i(\hat{y}(x(t)))dt = c_1\left(x_i - \frac{1}{3}\right)^2 + c_2\left(x_i - \frac{1}{3}\right) + c_3 + c_4(x_k - x_j)^2, \quad (13)$$

where

$$\begin{cases} c_1 = \dfrac{b(3c-1)}{4(r-3k(c+1))}, \\ c_2 = \dfrac{12a+b(3c-5)}{6(2r-3k(c+1))}, \\ c_3 = \dfrac{3a-b}{9r}, \\ c_4 = -\dfrac{bc(3c-1)}{4(r-3k(c+1))}. \end{cases}$$

Now consider the Hamilton-Jacobi-Bellman equations:

$$\hat{y}_i(x) \in \operatorname{Arg}\max_{y_i \in Y_i(x)}\left(\pi_i(y_i, \hat{y}_{-i}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(y_i, \hat{y}_{-i}(x))\right), \quad (14)$$

$$rV_i(x) = \pi_i(\hat{y}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(\hat{y}(x)). \quad (15)$$

If these equations are satisfied for all $x \in X$, then $\hat{y}_i$ is the best response to $\hat{y}_{-i}$ (when the strategies of player i are limited to class $\mathcal{A}$ so that $x(t)$ never leaves X)—see Dockner et al. (2000, chapters 3 and 4).

Eq. (15) is automatically satisfied given the way V is constructed. We now check Eq. (14). Let

$$g(y_i, x) = \pi_i(y_i, \hat{y}_{-i}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(y_i, \hat{y}_{-i}(x)).$$

Using (8), (13) and the definitions for π_i and f_i to expand $g(y_i, x)$ and maximising the result w.r.t. $y_{ij} + y_{ik} = x_i$ gives us

$$\tilde{y}_{ij}(x) = \frac{x_i + d(x_k - x_j)}{2}, \quad (16)$$

$$d = \frac{1-c}{4} - \frac{ck(3c-1)}{2(r-3k(c+1))}. \quad (17)$$

Strategy $\hat{y}_i$ is the best response strategy if (8) coincides with (16), i.e., if $c = d$. Using (17) to expand the equation $c = d$ and simplifying it yields

$$9c^2 + \left(14 - 5\frac{r}{k}\right)c + \left(\frac{r}{k} - 3\right) = 0. \quad (18)$$

Eq. (18) has real roots only when $\frac{r}{k} \in (-\infty, 76/25] \cup [4, \infty)$. It is straightforward to show the following. If $\frac{r}{k} \geq 4$, then both roots are strictly positive. If $\frac{r}{k} \leq 76/25$, then one of the roots is always strictly positive, while the other root,

$$c^* = \frac{1}{18}\left(5\frac{r}{k} - 14 - \sqrt{25\left(\frac{r}{k}\right)^2 - 176\frac{r}{k} + 304}\right),$$

satisfies $c^* < -1$ if and only if $\frac{r}{k} < \frac{4}{3}$.[8]

So, if $\frac{r}{k} < \frac{4}{3}$, then there is a unique $c < -1$, defined by $c = c^*$, such that (8) constitutes an MPE. If $\frac{r}{k} \geq \frac{4}{3}$, then no such c exists. For $0 \leq \frac{r}{k} < \frac{4}{3}$ we further obtain that $-\frac{7}{4} < c^* < -1$. It is straightforward to verify that $\hat{y}_i(x) \in Y_i(x)$ for such c^* and for $x \in X$.

To finish the proof, it only remains to show that (8), with $c = c^*$, constitutes an equilibrium also in class $\mathcal{B}$.

Fix c at c^* and consider an arbitrary strategy $\grave{y}_i(x) \in \mathcal{B}$. Under a class $\mathcal{B}$ strategy, the game ends at some T (which is determined by $\grave{y}_i(x)$). Let

$$y_i^n(x,t) = \begin{cases} \grave{y}_i(x) & \text{if} \quad t \leq T - \epsilon_n, \\ \hat{y}_i(x) & \text{if} \quad t > T - \epsilon_n, \end{cases}$$

where ϵ_n is a sequence, $\epsilon_n > 0$ and $\lim_{n\to\infty} \epsilon_n = 0$. This strategy $y_i^n(x,t)$ belongs to $\mathcal{A}$; therefore, it gives the same or a lower profit than the best response strategy $\hat{y}_i(x)$, i.e.

$$\int_0^\infty e^{-rt}\pi_i(\hat{y}(x(t)))dt \geq \int_0^\infty e^{-rt}\pi_i(y^n(x(t)))dt = \int_0^{T-\epsilon_n} e^{-rt}\pi_i(\grave{y}(x(t)))dt + \int_{T-\epsilon_n}^\infty e^{-rt}\pi_i(\hat{y}(x(t)))dt.$$

Taking the limit as $n \to \infty$ yields

$$\int_0^\infty e^{-rt}\pi_i(\hat{y}(x(t)))dt \geq \int_0^T e^{-rt}\pi_i(\grave{y}(x(t)))dt + V_i(x(T)).$$

On the other hand, the profit from employing strategy $\grave{y}_i(x)$ is

$$\int_0^T e^{-rt}\pi_i(\grave{y}(x(t)))dt + S_i(x(T)).$$

Therefore, if $S_i(x(T)) \leq V_i(x(T))$, then $\hat{y}_i$ is the optimal strategy in class $\mathcal{B}$ as well.

Eq. (9), as derived from from (2), (8) and $\sum_i x_i(t) = 1$, still holds even if player i has an arbitrary strategy $\grave{y}_i(x)$ (but given that other players have equilibrium strategies). Consequently, (10) holds as well.

As $x(0) \in X$, it follows from the definition of X that $x_i(0) < \frac{7}{17}$. Given (10) and using $c < -1$, we then also have that $x(T) < \frac{7}{17}$. At the same time, $x(T)$ belongs to the boundary of X. So, if it was true that $x_i(T) > x_j(T)$ for all $j \neq i$, then it would also be true that $x_i(T) = \frac{7}{17}$. Because it is not, we have established that $x_i(T) \leq x_j(T)$ for at least some $j \neq i$. Therefore, $S_i(x(T)) = 0$. At the same time, based on $\varphi(\hat{y}_{ij}(x), \hat{y}_{ji}(x)) > 0$, we have that $V_i(x(T)) > 0$.

Thus, $S_i(x(T)) \leq V_i(x(T))$ and $\hat{y}_i(x)$, with $c = c^*$, is the best response in class $\mathcal{B}$ as well.

Essentially, a weaker player can choose a strategy that will lead him to reach the boundary of X, but it is not optimal for him to do so. The strongest player may prefer to reach the boundary if he is still the strongest player when he does so, but he cannot achieve this result if his rivals are using equilibrium strategies. □

Proposition 3. *If $r < \infty$, then there are no MPE in linear symmetric strategies such that $T < \infty$ (i.e., such that the game ends in a finite time due to one of the players becoming sufficiently strong).*

[8]The fact that we get rational numbers here (76/5, 4, 4/3) is a peculiar coincidence, and it came as a surprise to the authors.

Proof. Consider the proof of Proposition 2. Equations (7) through (11) remain unchanged, because their derivation does not depend on whether $T = \infty$ or $T < \infty$ but only on the condition that the equilibrium strategies are linear and symmetric.

Without loss of generality, suppose that $x_1 \geq x_2 > x_3$. Then, given (10), player 1 is the first player to attain the boundary $x_i = 7/17$, at which point the game stops. Having $z_i = x_i - \frac{1}{3}$ and $m = 3k(c+1)/2$, we obtain:

$$z_1 e^{-mT} = \frac{7}{17}$$

or

$$T(z) = \frac{1}{m}\big(\ln(7/17) - \ln z_1\big). \tag{19}$$

The stopping time depends on the current power of player 1.

Analogously with Eq. (12), but accounting for $T(z) < \infty$, we have

$$V_i(z) = \int_0^{T(z)} e^{-rt}\pi_i(\hat{y}(z(t)))dt =$$

$$\int_0^{T(z)} e^{-rt}\left(\frac{b(3c-1)}{4}\Big(z_i e^{mt}\Big)^2 + \frac{12a + b(3c-5)}{12} z_i e^{mt} + \frac{3a-b}{9} - \frac{bc(3c-1)}{4}\Big(z_k e^{mt} - z_j e^{mt}\Big)^2\right) dt =$$

$$\frac{b(3c-1)}{4} \cdot \frac{1 - e^{(2m-r)T(z)}}{r - 2m} z_i^2 + \frac{12a + b(3c-5)}{12} \cdot \frac{1 - e^{(m-r)T(z)}}{r - m} z_i +$$

$$\frac{3a-b}{9} \cdot \frac{1 - e^{-rT(z)}}{r} - \frac{bc(3c-1)}{4} \cdot \frac{1 - e^{(2m-r)T(z)}}{r - 2m} (z_k - z_j)^2.$$

So, given (19), the value function $V_i(z)$ is no longer linear-quadratic in z but involves rational powers of z, e.g.

$$e^{-rT(z)} = (7/17)^{-\frac{r}{m}} \cdot z_1^{\frac{r}{m}}.$$

Consequently, Eq. 14 can never deliver a fixed point that is linear in x, which is a requirement imposed on the equilibrium in question. Hence, no such equilibrium exists. In summary, if the players go to the boundary and the game ends, then the ending time is not fixed but depends on the current powers of the players, and that prevents the existence of a linear equilibrium. □

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