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Non existence of critical scales in the homogenization of the problem with p-Laplace diffusion and nonlinear reaction in the boundary of periodically distributed particles in n-dimensional domains when $p > n$

J.I. Díaz · D. Gómez-Castro ·
A.V. Podolskii · T.A. Shaposhnikova

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Abstract In previous works, the homogenization of the problem with p-Laplace diffusion and nonlinear reaction in the boundary of periodically distributed particles in n-dimensional domains has been studied in the cases where $p \leq n$. The main trait of the cases $p \leq n$ is the existence of a critical size of the particles, for which the nonlinearity arising of the limit problem does not coincide with the non linear term of the microscopic reaction. The main result of this paper proves that in the case $p > n$ there exists no critical size.

Keywords homogenization · p-Laplace diffusion · non-linear boundary reaction · non-critical sizes

1 Introduction

The main goal of this paper is to study the behaviour arising in the homogenization process applied to chemical reactions taking place on fixed-bed nanoreactors, at the microscopic level, on the boundary of the particles

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma} \sigma(u_\varepsilon) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

for a very general type of chemical kinetics (here given by the maximal monotone graph σ of $\mathbb{R}^2$). Here the diffusion is modeled by the quasilinear operator $\Delta_p u_\varepsilon \equiv \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$ with $p > 1$. Notice that $p = 2$ corresponds to the linear diffusion operator, and that $p \neq 2$ appears in turbulent regime flows or non-Newtonian

J.I. Díaz, D. Gómez-Castro
Dpto. Matemática Aplicada & I.M.I.
Universidad Complutense de Madrid
E-mail: jidiaz@ucm.es, E-mail: dgcastro@ucm.es

A.V. Podol'skii, T.A. Shaposhnikova
Moscow State University,
Faculty of Mechanics and Mathematics,
E-mail: originalea@ya.ru, E-mail: shaposh.tan@mail.ru

flows (see [2]). The "normal derivative" must be then understood as $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nu$, where ν is outward unit normal vector on the boundary of the particles $S_\varepsilon \subset \partial\Omega_\varepsilon$. In fact we shall consider the structural assumption

$$n < p < +\infty \text{ and } n \geq 3. \tag{2}$$

In previous works, the cases where $p \leq n$ have been studied (see [3–6, 12, 10, 9] for the details). The main trait of this cases is the existence of a critical size of the particles, for which the non linear term arising of the limit problem does not coincide with the non linear term of the microscopic reaction. If the size of the particles is larger than this critical size then the limit problem is of the form

$$\begin{cases} -\Delta_p u + \mathcal{A}\sigma(u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{3}$$

where $\mathcal{A} > 0$. If the size of the particles is critical, the limit problem becomes

$$\begin{cases} -\Delta_p u + \mathcal{B}|H(u)|^{p-2}H(u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{4}$$

where $\mathcal{B} > 0$ and H is the solution of functional equation depending only on σ, n and the shape of the particle.

The main result of this paper proves that for $p > n$ there exists no critical size. That is, the solution u_ε converges to the homogenized solution u of problem (4) where $\mathcal{A}$ is a constant that will be specified later.

The plan of the rest of the paper is the following: Section 2 will be devoted to the statement of the main results, whilst Sections 3 and 4 are devoted to the proofs.

2 Statement of results

Definition 1 (Perforated domain Ω_ε) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = (-1/2, 1/2)^n$. Denote by G_0 a smooth open set such that $\overline{G}_0 \subset Y$. For $\delta > 0$ and B an open set we define $\delta B = \{x \,|\, \delta^{-1}x \in B\}$. For $\varepsilon > 0$ we define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$. Let $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$ and C_0 is positive number. Define

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j, \tag{5}$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$, $\mathbb{Z}^n$ is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$. It is clear that $\overline{G_j^\varepsilon} \subset Y_j^\varepsilon$. Define

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\; S_\varepsilon = \partial G_\varepsilon,\; \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

It can be checked that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, for some constant $d > 0$, in the sense that $|\Upsilon_\varepsilon|/\varepsilon^{-n} \to d$ as $\varepsilon \to 0$.

In this geometry we consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = 0 & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega, \end{cases} \tag{6}$$

where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε and σ is a nondecreasing function such that $\sigma(0) = 0$ and $f \in L^{p'}(\Omega)$. In this paper we will be interested in the case $p > n$ and $\alpha > 1$.

We define $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ as the closure in $W^{1,p}$ of $\{f \in \mathcal{C}^\infty(\overline{\Omega}_\varepsilon) : f|_{\partial\Omega} = 0\}$.

Definition 2 Let (Ω_ε) be a sequence of domains $\Omega_\varepsilon \subset \Omega \subset \mathbb{R}^n$ and $\partial\Omega \subset \partial\Omega_\varepsilon$ where Ω is bounded. We say that the sequence has a uniformly bounded sequence of extension operators in $W^{1,p}$ if there exists a sequence (P_ε) where:

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega) \tag{7}$$

where $P_\varepsilon u|_{\Omega_\varepsilon} = u_\varepsilon$ for every $u \in W^{1,p}(\Omega_\varepsilon)$ and there exists $K_p > 0$ independent of ε such that

$$\|\nabla P_\varepsilon u\|_{L^p(\Omega)} \le K_p \|\nabla u\|_{L^p(\Omega_\varepsilon)}, \qquad \text{for every } \varepsilon > 0. \tag{8}$$

Applying the techniques in [8] we can prove that

Lemma 1 *The sequence (Ω_ε) has an uniformly bounded sequence of extension operators.*

We will use the existence of a Poincaré constant for $W_0^{1,p}(\Omega)$, $C_{p,\Omega}$, such that

$$\|v\|_{L^p(\Omega)} \le C_{p,\Omega} \|\nabla v\|_{L^p(\Omega)}, \qquad v \in W_0^{1,p}(\Omega). \tag{9}$$

In fact we can also show the following, which is seldom stated

Theorem 1 *Let $p > 1$. If there exists a sequence of uniformly bounded extension operators in $W_0^{1,p}$ then there exists a uniform Poincaré constant for $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. In particular, if (8) holds and $C_{p,\Omega}$ is a Poincaré constant for $W_0^{1,p}(\Omega)$, then, $K_p C_{p,\Omega}$ is a Poincaré constant for $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.*

Proof We simply indicate that

$$\|v\|_{L^p(\Omega_\varepsilon)} \le \|P_\varepsilon v\|_{L^p(\Omega)} \le C_\Omega \|\nabla P_\varepsilon v\|_{L^p(\Omega)} \le C_\Omega K_p \|\nabla v\|_{L^p(\Omega_\varepsilon)} \tag{10}$$

which concludes the proof.

Our aim is to prove the following results

Theorem 2 *Let $n < p < +\infty, \alpha > 1$, σ be a continuous nondecreasing function such that $\sigma(0) = 0$, u_ε be the solution of (6) and let*

$$\gamma^* = \alpha(n-1) - n. \tag{11}$$

Then, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ where $u \in W_0^{1,p}(\Omega)$ is the unique weak solution of one of the following problem

1. *If $\gamma = \gamma^*$ then*
$$\begin{cases} -\Delta_p u + \mathcal{A}\sigma(u) = f, & \Omega, \\ u = 0 & \partial\Omega \end{cases} \tag{12}$$
where $\mathcal{A} = C_0^{n-1}|\partial G_0|$.
2. *If $\gamma < \gamma^*$ then*
$$\begin{cases} -\Delta_p u = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{13}$$

Lemma 2 *Let $n < p < +\infty, \alpha > 1$ and $\sigma \equiv 0$. Then $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ where u is the unique solution of* (13) *(equivalently* (12) *for $\sigma \equiv 0$).*

Theorem 3 *Let $n < p < +\infty, \alpha > 1$, $\gamma > \gamma^*$ and $\sigma \in \mathcal{C}^1(\mathbb{R})$ nondecreasing function such that $\sigma(0) = 0$. Then, there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and*
$$\sigma(u(x)) = 0, \ a.e.\ x \in \Omega. \tag{14}$$
In other words, $u(x) \in \sigma^{-1}(0)$ for a.e. $x \in \Omega$.

Remark 1 In this setting $(p > n)$ there exists no critical exponent α^*. This is quite natural since, for $p < n$ the critical exponent results $\alpha^* = \frac{n}{n-p}$. The case $p = n$ was done in [9].

We will use the following comparison result, which will be proved later

Lemma 3 *Let $p > 2$ and let $u_\varepsilon, \hat{u}_\varepsilon$ be the solutions of* (6) *with σ and $\hat\sigma$ continuous functions. Then,*
$$\|\nabla(u_\varepsilon - \hat{u}_\varepsilon)\|_{L^p(\Omega_\varepsilon)}^{p-1} \le C\varepsilon^{\frac{\gamma^*-\gamma}{p}} \|\sigma - \hat\sigma\|_{\mathcal{C}(\mathbb{R})}. \tag{15}$$

Remark 2 Since any function $v \in W^{1,p}(\Omega), p > n$ is Hölder with the estimate
$$|v(x) - v(y)| \le C|x-y|^{1-\frac{n}{p}} \|\nabla v\|_{L^p(\Omega)}, \qquad \text{if } [x,y] \subset \Omega \tag{16}$$
where $[x,y] = \{\lambda x + (1-\lambda y) : \lambda \in [0,1]\}$, we have that $(P_\varepsilon u_\varepsilon)$ is uniformly Hölder continuous, and therefore (u_ε) is also uniformly Hölder continuous.

We need some information on the traces on S_ε. We can compute the following lemma, analogous to results in [8] which, for the proof, points to [7].

Lemma 4 *Let $p > n$ and for $u \in W^{1,p}(Y_\varepsilon)$ where $Y_\varepsilon = \varepsilon Y \setminus \overline{a_\varepsilon G_0}$. Then,*
$$\int_{a_\varepsilon S_0} |u|^p dS \le K \left(a_\varepsilon^{n-1} \varepsilon^{-n} \int_{Y_\varepsilon} |u|^p dx + a_\varepsilon^{n-1} \varepsilon^{p-n} \int_{Y_\varepsilon} |\nabla u|^p dx \right) \tag{17}$$
where K is independent of ε.

Remark 3 In particular, if $a_\varepsilon = C_0\varepsilon^\alpha$ we have

$$\int_{a_\varepsilon S_0} |u|^p dS \le K\left(\varepsilon^{\gamma^*}\int_{Y_\varepsilon}|u|^p dx + a_\varepsilon^{n-1}\varepsilon^{p-n}\int_{Y_\varepsilon}|\nabla u|^p dx\right) \tag{18}$$

This explains the choice of γ^*. If $p < n$ then $a_\varepsilon^{n-1}\varepsilon^{p-n}$ is replaced by a_ε^{p-1}. In that case

$$\frac{a_\varepsilon^{p-1}}{\varepsilon^{\gamma^*}} = C_0^{p-1}\varepsilon^{\alpha(p-n)+n}. \tag{19}$$

For the cases $p < n$ this exponent is the one that produces the appearance of a critical case, which corresponds to $\alpha = \frac{n}{n-p}$. In the case $p = n$ a similar expression exists, but is more self-involved (see [9]).

The following result will be instrumental in the proof. Nonetheless it has a great intrisic mathematical value.

Proposition 1 *Let $p > n$, $\alpha > 1$, $\gamma^* = \alpha(n-1) - n$ and $v_\varepsilon \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. Then*

$$\varepsilon^{-\gamma^*}\int_{S_\varepsilon} v_\varepsilon dS \to \mathcal{A}\int_{\Omega} v dS \tag{20}$$

where

$$\mathcal{A} = C_0^{n-1}|\partial G_0|. \tag{21}$$

This result does not hold if $p < n$, and this causes the appearance of a term known as *strange term*, first noticed by Cioranescu and Murat for the linear problem [1], and which has been well documented also in the nonlinear case (see, e.g., [6, 12]).

The technique for the proof of this result uses the following auxiliary result. Define function $M_\varepsilon(x)$ as Y_ε — periodic solution of the boundary value problem

$$\begin{cases} \Delta_p m_\varepsilon = \mu_\varepsilon, \; x \in Y_\varepsilon = \varepsilon Y \setminus \overline{a_\varepsilon G_0}; \\ \partial_{\nu_p} m_\varepsilon = 1, \; x \in \partial(a_\varepsilon G_0) = S_\varepsilon^0; \\ \partial_{\nu_p} m_\varepsilon = 0, \; x \in \partial Y_\varepsilon \setminus S_\varepsilon^0; \end{cases} , \quad \mu_\varepsilon = \frac{C_0^{n-1}\varepsilon^{\alpha(n-1)-n}|\partial G_0|}{1-(a_\varepsilon\varepsilon^{-1})^n|G_0|},$$
$$\text{and} \int_{Y_\varepsilon} m_\varepsilon(x)dx = 0. \tag{22}$$

This has the nice property of allowing us to write, for any test function $\varphi \in W^{1,p}(Y_\varepsilon)$

$$-\int_{Y_\varepsilon}|\nabla m_\varepsilon|^{p-2}\nabla m_\varepsilon\nabla\varphi dx + \int_{S_\varepsilon^0}\varphi dS = \mu_\varepsilon\int_{Y_\varepsilon}\varphi dx. \tag{23}$$

Denote by P_ε^j the center of the ball $G_\varepsilon^j = P_\varepsilon^j + \varepsilon^\alpha G_0$. Let T_ε^j denote the ball of radius $\varepsilon/4$ centered at the point P_ε^j. Let $M_\varepsilon^j = m_\varepsilon(x - P_j^\varepsilon)$ be the solution of the boundary value problem. We will use the following fact, which we will prove later

Lemma 5 *The following estimate holds*

$$\|\nabla M_\varepsilon\|_{L^p(\cup_j Y_\varepsilon^j)} \le C(a_\varepsilon\varepsilon^{-1})^{\frac{n-1}{p-1}} \tag{24}$$

3 Proof of Proposition 1

Proof (of Lemma 5) Setting in (23) $\varphi = m_\varepsilon$ and applying Theorem 1, Lemma 4 and the definition of $m_\varepsilon(x)$, we obtain

$$\begin{aligned}
\|\nabla m_\varepsilon\|^{p^2}_{L^p(Y_\varepsilon)} &\le \left(\left| \int_{S^0_\varepsilon} m_\varepsilon dS \right| + \mu_\varepsilon \left| \int_{Y_\varepsilon} m_\varepsilon dx \right| \right)^p \\
&\le \left(\int_{S^0_\varepsilon} 1 dS \right)^{p-1} \|m_\varepsilon\|^p_{L^p(S^0_\varepsilon)} \le C_1 a_\varepsilon^{(n-1)(p-1)} \|m_\varepsilon\|^p_{L^p(S^0_\varepsilon)} \\
&\le C_2 a_\varepsilon^{(n-1)(p-1)} \left(a_\varepsilon^{n-1} \varepsilon^{-n} \|m_\varepsilon\|^p_{L^p(Y_\varepsilon)} + a_\varepsilon^{n-1} \varepsilon^{p-n} \|\nabla m_\varepsilon\|^p_{L^p(Y_\varepsilon)} \right) \\
&\le C_3 \left(a_\varepsilon^{p(n-1)} \varepsilon^{p-n} + a_\varepsilon^{p(n-1)} \varepsilon^{p-n} \right) \|\nabla m_\varepsilon\|^p_{L^p(Y_\varepsilon)} \qquad (25) \\
&\le C_4 a_\varepsilon^{p(n-1)} \varepsilon^{p-n} \|\nabla m_\varepsilon\|^p_{L^p(Y_\varepsilon)}, \qquad (26)
\end{aligned}$$

Finally, we have the following inequality

$$\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)} \le K a_\varepsilon^{\frac{n-1}{p-1}} \varepsilon^{\frac{p-n}{p(p-1)}}. \qquad (27)$$

Hence, since $\# \Upsilon_\varepsilon \le C\varepsilon^{-n}$ we get the estimate

$$\|\nabla M_\varepsilon\|_{L^p(\cup_j Y^j_\varepsilon)} \le C(a_\varepsilon \varepsilon^{-1})^{\frac{n-1}{p-1}}, \qquad (28)$$

which concludes the proof.

Remark 4 Notice that from (25) to (26) we apply that $p > n$. In the case $p < n$ the other term is dominant, and hence the comparison is $\|\nabla M_\varepsilon\|_{L^p} \le C(a_\varepsilon \varepsilon^{-1})^{\frac{n}{p}}$ (see [8]).

Let $M^j_\varepsilon(x)$ be a restriction of function $M_\varepsilon(x)$ on Y^j_ε. Using the definition of $M^j_\varepsilon(x)$, we can make the following transformations

$$\begin{aligned}
\varepsilon^{-\gamma} \int_{S_\varepsilon} v_\varepsilon dS &= \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int_{Y^j_\varepsilon} div(|\nabla M^j_\varepsilon|^{p-2} \nabla M^j_\varepsilon v_\varepsilon) dx = \\
&= \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int_{Y^j_\varepsilon} |\nabla M^j_\varepsilon|^{p-2} \nabla M^j_\varepsilon \nabla v_\varepsilon dx + \\
&\quad + \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int_{Y^j_\varepsilon} (\Delta_p M^j_\varepsilon) v_\varepsilon dx = \\
&= \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \int_{Y^j_\varepsilon} |\nabla M^j_\varepsilon|^{p-2} \nabla M^j_\varepsilon \nabla v_\varepsilon dx + \\
&\quad + \varepsilon^{-\gamma} \sum_{j \in \Upsilon_\varepsilon} \mu_\varepsilon \int_{Y^j_\varepsilon} v_\varepsilon dx \qquad (29)
\end{aligned}$$

Using (28), we get

$$\varepsilon^{-\gamma}\int_{\Omega_\varepsilon}|\nabla M_\varepsilon|^{p-1}|\nabla v_\varepsilon|dx \le C\varepsilon^{-\gamma}\left(\int_{\Omega_\varepsilon}|\nabla M_\varepsilon|^p dx\right)^{\frac{p-1}{p}}$$
$$\le C\varepsilon^{-\alpha(n-1)+n}a_\varepsilon^{n-1}\varepsilon^{1-n} = C\varepsilon. \tag{30}$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j}|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j\cdot\nabla v_\varepsilon dx = 0 \tag{31}$$

and, finally, we use the fact (see [13]) that, since $v_\varepsilon \rightharpoonup v$ in $W^{1,2}(\Omega)$ we have

$$\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon\int_{Y_\varepsilon^j}v_\varepsilon dx \to C_0^{n-1}|\partial G_0|\int_\Omega v dx. \tag{32}$$

Remark 5 Notice that, for $p < n$ estimate (30) transform into $C\varepsilon^{\frac{1}{p}(n-\alpha(n-p))}$ producing the appearance of a critical α (see [8]).

4 Proof of Theorem 2

First, let us prove the auxiliary lemma

Proof (of Lemma 3) By considering the difference of weak formulations we can write, for the test function $u_1 - u_2$

$$\int_\Omega(|\nabla u_2|^{p-2}\nabla u_2 - |\nabla u_1|^{p-2}\nabla u_1)\cdot\nabla(u_2-u_1)dx$$
$$+\,\varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma_2(u_2)-\sigma_2(u_1))(u_2-u_1)dS$$
$$=\varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma_1(u_1)-\sigma_2(u_1))(u_2-u_1)dS. \tag{33}$$

For $p \ge 2$ it is true that (see [11] or [2, Lemma 4.10])

$$\|\nabla(u_1-u_2)\|^p_{L^p(\Omega_\varepsilon)} \le \left|\varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma_2(u_1)-\sigma_1(u_1))(u_2-u_1)dS\right| \tag{34}$$
$$\le \varepsilon^{-\gamma}|S_\varepsilon|^{\frac{1}{p'}}\|\sigma_2-\sigma_1\|_\infty\|u_1-u_2\|_{L^p(S_\varepsilon)} \tag{35}$$
$$\le C\varepsilon^{-\frac{\gamma}{p}}\|\sigma_2-\sigma_1\|_\infty\|u_1-u_2\|_{L^p(S_\varepsilon)}, \tag{36}$$

since $|S_\varepsilon| \le C\varepsilon^{-\gamma}$. By applying Lemma 4 we deduce that

$$\|\nabla(u_1-u_2)\|^p_{L^p(\Omega_\varepsilon)} \le K\varepsilon^{-\frac{\gamma}{p}}\|\sigma_1-\sigma_2\|_\infty\varepsilon^{\frac{\gamma^*}{p}}$$
$$\times\left(\|u_1-u_2\|_{L^p(\Omega_\varepsilon)}+\|\nabla(u_1-u_2)\|_{L^p(\Omega_\varepsilon)}\right). \tag{37}$$

Applying the uniform Poincaré inequality we deduce

$$\|\nabla(u_1-u_2)\|^p_{L^p(\Omega_\varepsilon)} \le K\varepsilon^{\frac{\gamma^*-\gamma}{p}}\|\sigma_1-\sigma_2\|_\infty\|\nabla(u_1-u_2)\|_{L^p(\Omega_\varepsilon)}. \qquad (38)$$

which concludes the proof.

We consider the weak formulation

$$\int\limits_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla v dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(u_\varepsilon)vdS = \int\limits_{\Omega_\varepsilon} fvdx, \qquad \forall v\in W^{1,p}_0(\Omega). \quad (39)$$

Since u_ε is a weak solution and $p>n$ we have that

$$\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \le \|f\|_{L^p(\Omega_\varepsilon)} \qquad (40)$$

Therefore (u_ε) is a bounded sequence in $W^{1,p}(\Omega_\varepsilon)$. Hence $(P_\varepsilon u_\varepsilon)$ is a uniformly Hölder sequence in Ω, and therefore uniformly bounded

$$\|u_\varepsilon\|_{\mathcal{C}(\Omega)} \le \|P_\varepsilon u_\varepsilon\|_{\mathcal{C}(\Omega)} \le C, \qquad \text{for some } C>0. \qquad (41)$$

Hence we have that

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla v dx \to \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx, \qquad (42)$$

$$\int_{\Omega_\varepsilon} fvdx \to \int_\Omega fvdx. \qquad (43)$$

Proof (of Theorem 2) First let us assume that $\gamma<\gamma^*$. Let $u_{\varepsilon,0}$ be the solution corresponding to $\sigma=0$. Then

$$\|u_\varepsilon-u_{\varepsilon,0}\|_{W^{1,p}(\Omega_\varepsilon)} \le \varepsilon^{\frac{\gamma-\gamma^*}{p(p-1)}}\|\sigma\|^{\frac{1}{p-1}}_{\mathcal{C}(K)} \qquad (44)$$

where K is a compact such that $\|P_\varepsilon u_\varepsilon\|_{L^\infty}, \|P_\varepsilon u_{\varepsilon,0}\|_{L^\infty}\in K\subset\mathbb{R}$. Then $P_\varepsilon u_\varepsilon \rightharpoonup u_0$ the solution of (13) by applying Lemma 2.

Assume that $\gamma=\gamma^*$. We start by considering $\sigma\in\mathcal{C}^1(\mathbb{R})$. Since the solutions are uniformly bounded and continuous, we have that

$$\|\sigma'(u_\varepsilon)\|_{\mathcal{C}(S_\varepsilon)} \le \|\sigma'(u_\varepsilon)\|_{\mathcal{C}(\Omega)} \le C \qquad (45)$$

since σ' is continuous. Notice that $\sigma(P_\varepsilon u_\varepsilon)=P_\varepsilon(\sigma(u_\varepsilon))$ on $\overline{\Omega}_\varepsilon$. Hence

$$\|\nabla(\sigma(u_\varepsilon))\|_{L^p(\Omega_\varepsilon)} \le \|\sigma'(u_\varepsilon)\|_{\mathcal{C}(\Omega)}\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C. \qquad (46)$$

Therefore there exists $\hat\sigma\in W^{1,p}(\Omega)$ such that $P_\varepsilon\sigma(u_\varepsilon)\rightharpoonup\hat\sigma$. Since $p>n$ the convergence is also in the sense of $\mathcal{C}(\Omega)$, and therefore $\hat\sigma=\sigma(u)$. We conclude, hence, that, for $v\in W^{1,p}$ we have

$$\varepsilon^{-\gamma^*}\int_{S_\varepsilon}\sigma(u_\varepsilon)vdS \to \mathcal{A}\int_\Omega \sigma(u)vdx. \qquad (47)$$

Then, limit becomes

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx + \mathcal{A}\int_\Omega \sigma(u) v dx = \int_\Omega f v dx, \qquad \forall v \in W_0^{1,p}(\Omega). \tag{48}$$

Let $\sigma \in \mathcal{C}(\Omega)$. Let us consider an approximating sequence $\sigma \in \mathcal{C}^1$, $\sigma^{-1}(0) = 0$ and $\sigma_\delta \to \sigma$ in $\mathcal{C}([-M, M])$ as $\delta \to 0$ where $\|P_\varepsilon u_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} < M$ for all $\varepsilon > 0$. We have that

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_{W^{1,p}}^{p-1} \le C\|\sigma_\delta - \sigma\|_{\mathcal{C}([-M,M])}. \tag{49}$$

Passing to the limit we have that

$$\|u - u_\delta\|_{W^{1,p}}^{p-1} \le C\|\sigma_\delta - \sigma\|_{\mathcal{C}([-M,M])}, \tag{50}$$

where u_δ satisfies (12), with σ_δ instead of σ. As $\delta \to 0$ the sequence $u_\delta \to w$ where w is the solution of (12). Therefore, due to (50) we have that $u = w$, which concludes the proof.

Proof (of Theorem 3) If $\gamma > \gamma^*$ we write

$$\varepsilon^{\gamma-\gamma^*}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla v dx + \varepsilon^{-\gamma^*}\int_{S_\varepsilon} \sigma(u_\varepsilon) v dS = \varepsilon^{\gamma-\gamma^*}\int_{\Omega_\varepsilon} f v dx, \tag{51}$$

for all $v \in W_0^{1,p}(\Omega)$. Hence, in the limit

$$\mathcal{A}\int_\Omega \sigma(u) v \, dx = 0, \qquad \forall v \in W_0^{1,p}(\Omega). \tag{52}$$

That is $\sigma(u(x)) = 0$ for a.e. $x \in \Omega$.

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