
Mathematical Modelling of Subglacial Drainage and Erosion

by

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Abstract

The classical theory of channelized subglacial drainage, due originally to Röthlisberger (1972) and Nye (1976), considers water flow in an ice channel overlying a rigid, impermeable bed. At steady flow, creep closure of the channel walls is counteracted by melt-back due to heat dissipation, and this leads to an equilibrium relation between channel water pressure and discharge. More generally, such a balance exhibits an instability that can be used to describe the mechanics of catastrophic flood events known as *jökulhlaups*. In this thesis, we substantiate these developments by exploring a detailed model where the channel is underlain by subglacial till and the flow supports a sediment load. Attention is given to the physics of bed processes and its effect on channel morphology. In particular, we propose a theory in which the channel need not be semi-circular, but has independently evolving depth and width determined by a *local* balance between melting and closure, and in which sediment erosion and deposition is taken into account. The corresponding equilibrium relation indicates a reverse dependence to that in the classical model, justifying the possibility of the subglacial *canals* envisaged by Walder and Fowler (1994). Theoretical predictions for sediment discharge are also derived. Regarding time-dependent flood drainage, we demonstrate how rapid channel widening caused by bank erosion can explain the abrupt recession observed in the flood hydrographs. This allows us to produce an improved simulation of the 1972 jökulhlaup from Grímsvötn, Iceland, and self-consistently, a plausible estimate for the total sediment yield. We also propose a mechanism for the observed flood initiation lake-level at Grímsvötn. These investigations expose the intimate interactions between drainage and sediment transport, which have profound implications on the hydrology, sedimentology and dynamics of ice masses, but which have received little attention.

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which tends to get littered in notes and documents.

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To Maa Maa

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Plate 1 goes here.

Chapter 1

Introduction

1.1 Jökulhlaups

Although glaciers are often perceived to be rather inactive components of the physical landscape, they are capable of exhibiting a wide variety of large-scale fluctuations. The classic example is their *surging* behaviour, where ice flow velocities many times in excess of its normal rate are found, causing rapid advance of the terminus. Lesser known however, are catastrophic floods released by the ice, so called *glacier outburst floods*. These are commonly due to the sudden draining of *ice-dammed lakes* and are observed in many glacier regions of the world. They rank amongst the most voluminous flood events on Earth.

Ice-dammed lakes are large volumes of water trapped by the ice. They may form as *marginal* lakes, when active glaciers block small depressions or neighbouring valleys, in which water accumulates through a combination of surface precipitation and melting at the ice edge. They may also be *subglacial* — situated under the ice. This latter type occurs in regions of significant geothermal heat, such as in Iceland, where one of the best studied examples is that of Grímsvötn beneath the Vatnajökull ice cap (Plate 1). This subglacial lake lies within a volcanic caldera and produces a flood every five to ten years. The high geothermal heat melts ice at the base, thereby increasing the lake volume. At a certain point, the lake-water ‘bursts through’ the seal exerted by the ice dome at the caldera rim and escapes beneath the ice. The resulting flood or *jökulhlaup*¹, then emerges at the ice margin, typically with a duration of several weeks and a peak water discharge of up to $10^4 \text{ m}^3 \text{ s}^{-1}$.

As is the case at Grímsvötn, jökulhlaups from many ice-dammed lake systems tend to be repetitive and recur over a timescale of years, as after the lake drawdown during each flood, there is a slow replenishment of lake-water until the next one occurs. On the other hand, some jökulhlaups are ‘one-off’ events. An important class, characterized by extremely short duration and high peak discharge, are the ones triggered by volcanic eruptions. Here, intense basal melting leads to the rapid

¹Icelandic; literally translated as ‘ice-leap’. The term has been used to describe all kinds of glacier outburst floods, but more correctly refers to those which involve subglacial water drainage.

formation (or refilling) of an ice-dammed lake which precipitates in a flood. The most recent event on Vatnajökull belongs to this type. In October–November 1996, eruption of the subglacial volcano Bárðarbunga generated meltwater that flowed into Grímsvötn; and subsequently, an enormous jökulhlaup lasting one day was unleashed, with a peak flow of $4.5 \times 10^4 \text{ m}^3 \text{ s}^{-1}$. The flood inundated several hundred square kilometres of coastal plain, and incurred damage to roads and utility lines estimated at \$15 million U.S. (see reports by Brandsdóttir (1997), Einarsson *et al.* (1997b) and photo-article by Winter (1997)).

The phenomenal discharge of a jökulhlaup is usually accompanied by high sediment loads. This leads to extensive erosion and deposition, thus evidence of past jökulhlaups may be discerned in the sedimentary record. For instance, Maizels (1989a, 1989b) have identified distinct sequences in the sandur deposits of Sólheimajökull in Southern Iceland, associated with hyper-concentrated sediment flows during eruption-generated floods from Katla. Also, field observation of dramatic erosional features — the Channeled Scablands — in Eastern Washington, U.S.A., led Bretz (1923) to hypothesize that they were due to catastrophic glacier outburst floods of the late Pleistocene. Although his theory was highly controversial, it is now widely recognized that the floods had taken place, and that they were episodic jökulhlaups from a marginal lake located at the present site of Lake Missoula (Bretz, 1969; Baker, 1978). Baker and Milton (1974) suggested that the outflow channels on the surface of Mars, indicative of large-scale drainage activity in the past, might have a similar origin.

Record of the occurrence of jökulhlaups dates back to as early as the 14th century (see Thórarinnsson, 1974), but systematic monitoring of ice-dammed lake systems has not taken place until early this century. Available data on flood frequency, duration and impact are therefore limited. Current fieldwork focuses on various sites in Iceland, Norway, Greenland, and North America. Reviews of Icelandic and Alaskan jökulhlaups are given respectively by Björnsson (1988, 1992) and Post and Mayo (1971). In addition, Walder and Costa (1996) have compiled an up-to-date jökulhlaup database, together with a comprehensive list of references.

1.2 Flood models

Even though jökulhlaups are relatively infrequent and usually occur in uninhabited areas, they have a profound effect on the environment and can sometimes present an acute hydrologic hazard. (This is particularly the case in Iceland). An understanding of these phenomena is needed, especially the way in which their timing and magnitudes may be predicted.

A preliminary investigation by Clague and Mathews (1973) demonstrated the power law relation

$$Q_m = BV_m^k \quad (1.1)$$

between the peak discharge Q_m and the total flood volume V_m for jökulhlaups from ten marginal ice-dammed lakes of the world. B and k are parameters calibrated by

linear regression. This equation may be used to forecast an upper bound for the peak flow. Later studies, based on more extensive and specific data sets (e.g. Björnsson, 1992; Walder and Costa, 1996), indicated that the exponent k depends also on the type of lake systems. It is therefore reasonable to suppose that the value of k reflects the precise flood mechanics, but this has yet to be rigorously justified (Clarke, 1982).

An initial mathematical model of jökulhlaups that considers the underlying physics is due to Nye (1976), and builds on the drainage theory enunciated by Röthlisberger (1972). In Nye’s description, a roughly circular subglacial channel² drains lake-water towards the snout, and is opened rapidly by frictional heat generated by the water flow. There is a positive feedback, as the water flux itself increases due to channel enlargement, and this is taken to be responsible for the ‘swell phase’ of a flood. The increase is subsequently overcome however, by the effect that as the lake is emptied, the channel water pressure decreases, leading to viscous closure of the channel. As a result, the discharge follows a growth–recession profile resembling that of the observed flood hydrographs.

Nye applied this model to the 1972 jökulhlaup from Grímsvötn, and showed that it could simulate well the rising limb of the hydrograph. But later, numerical work by Spring and Hutter (1981b) indicated that the falling limb was less well explained by this theory. Clarke (1982) has also used Nye’s model to simulate outburst floods in Canada. In Part I of this thesis, we are concerned with extending this theory of jökulhlaups. A detailed mathematical treatment of the Nye model will be given in Chapter 2.

1.3 Classical drainage theory

Equilibrium channels

While Nye’s model is based on the time-dependent evolution of the subglacial channel, the earlier theory by Röthlisberger (1972) addressed the *steady flow* situation, in which the rates of channel melting and closure balance each other. This consideration enabled him to derive an equation of the form

$$\left[\Phi + \frac{dN_c}{ds} \right]^{11/8} = cQ^{-1/4}N_c^n, \quad (1.2)$$

where

$$N_c = p_i - p_c, \quad (1.3)$$

which provides a recipe for calculating the water pressure in the channel p_c , for a given water flux distribution Q and drainage topography (cf. his Equation (20); 1972). Here Φ is the total hydraulic potential gradient driving the flow, s is downstream distance, n is a positive constant describing ice rheology, c is a material constant, p_i is the overburden ice pressure, and N_c is the *effective channel pressure*.

²A subglacial channel is one which is situated *below* the ice, i.e., at the interface between basal ice and the underlying substrate.

In general, drainage conditions under glaciers are subjected to a multitude of rapid temporal fluctuations. Although Equation (1.2) is strictly an equilibrium result, it is relevant in predicting the channel water pressure *regime* over a seasonal timescale (or longer), which plays a significant rôle in a number of glaciological phenomena. As we shall see, the derivative dN_c/ds is often negligible, hence (1.2) may be reduced to the relation

$$Q \propto \Phi^{-11/2} N_c^{4n}, \quad (1.4)$$

which implies that the water flux increases with decreasing water pressure or hydraulic gradient. This property is characteristic of channelized drainage in the classical theory.

Models of subglacial drainage

Underneath glaciers and ice sheets, water exists in its liquid phase where the bed is at or above the pressure-melting point of ice, i.e., in *temperate* regions. The general drainage direction is down-gradient of the total hydraulic potential, and is normal to the potential contours at the bed (Shreve, 1972). Channels, however, are not the only form in which water transport is thought to take place. Classical theories have investigated the mathematical models of (i) *sheet flow* (Weertman, 1972); (ii) *channelized flow*, such as that in Röthlisberger or ‘*R*’ channels, which incise upward into ice (described above); and (iii) water flow in small conduits that connect cavities in the bed, known as a *linked-cavity system* (Walder, 1986; Kamb, 1987). These models are described in the following.

Sheet flow is envisaged as a thin film of water at the ice-bed interface, constituting a *distributed system* with high water pressures (close to the overburden ice pressure) and low discharge. Walder (1982) performed a linear stability analysis to show that it is inherently unstable to perturbations in film thickness, due to a heat dissipation feedback. As such, the irregular bed topography of real glaciers may still permit localized sheets of up to several millimetres thick, but these can remain quasi-stable only at low water flux. In regions of moderate to high basal melt-rates, or where there is a substantial amount of water draining to the bed from the ice surface, sheet flow will be unstable to the formation of incipient channels.

Röthlisberger (1972) and Shreve (1972) showed independently that channelized flow tends to become *localized*, forming an arborescent network. Both argued that if there is hydraulic connection between neighbouring channels, then larger channels would grow at the expense of smaller ones as a result of Equation (1.4). On the other hand, Weertman (1972) showed that discrete channels might not be efficient melt-water collectors when the shear stress in basal ice is relatively high; the stress distribution at the interface would be such as to drive water away from the channels. Although this theoretical inconsistency has not been resolved, dye tracing experiments conducted in the field have generally indicated the existence of extensive channel networks under many temperate glaciers, especially during their melt seasons (e.g. Sharp *et al.*, 1993).

As demonstrated by Equation (1.4), the form of the drainage system can exert a certain control over the subglacial water pressure. However, the drainage description in general is greatly complicated by an inter-coupling between water transport and ice dynamics. Notably, high water pressures associated with distributed drainage types can facilitate the lubricating motion between basal ice and bedrock, known as *basal sliding*. In turn, sliding modifies the upper boundary of the drainage system.

The linked-cavity model put forward by Walder (1986) and Kamb (1987) addresses this aspect. As ice slides over a rough bed, it tends to separate from the bed on the downstream sides of bumps, forming water-filled *cavities* linked together by smaller *R*-type channels, or *orifices*. The result is a complex network, in which water flow is controlled predominantly by the orifices and has a strong cross-glacier component. By incorporating the sliding motion into the melting-closure consideration, the authors showed that a stable drainage system can exist, with high sliding velocities and high basal water pressures, in contrast to a channelized system of similar discharge. In addition, the water flux turns out to be an *increasing* function of the water pressure. This is the opposite of that for *R* channels, and favours a system of many interconnected conduits distributed across the bed.

Kamb (1987) proceeded to investigate the stability of the linked-cavity system, and proposed a surge mechanism for valley glaciers based on drainage system *transitions*. (His particular application was the 1982–83 surge of Variegated Glacier, Alaska). Essentially, in the dormant surge phase, channelized flow prevails at the glacier bed with low water pressures and low sliding velocities. Prolonged build-up of ice accumulation then leads to an increased basal shear stress, and a transition to a linked-cavity system up-glacier. This has the effect of lubricating the bed there, and causes the transition front to propagate to the snout (see Hutter and Olunloyo, 1981; McMeeking and Johnson, 1986). The result is the high basal water pressures and fast sliding observed during the active surge phase. Finally, surging terminates because of reduced driving stress and a return transition to channelized flow.

Implications on classical sliding theory

So far, we have introduced basal sliding as a component of the subglacial drainage theory. Conversely, one may regard drainage as a mechanism that underlies the motion of glaciers as well as their hydrology.

From the mathematical viewpoint, solution of the mechanical ice flow problem (e.g. Hutter, 1982; Paterson, 1994) requires the prescription of sufficient boundary conditions at the glacier bed. When basal ice is frozen to bedrock, it is stationary and one imposes a ‘no slip’ boundary condition. But when the ice is temperate, basal sliding is possible, and then the general boundary condition takes the form of the *sliding law*

$$\tau_b = \tau_b(u_b, N), \quad (1.5)$$

in which τ_b and u_b respectively are the shear stress and sliding velocity of basal ice, and N , the effective pressure, is defined as

$$N = p_i - p_b, \quad (1.6)$$

where p_b is the basal (or subglacial) water pressure (cf. Equation (1.3)).

The early sliding theories (Nye, 1969, 1970; Kamb, 1970), based on the regelation/enhanced ice flow mechanism proposed by Weertman (1957), had in fact omitted the effect of N , because they neglected the possibility of a distributed linked-cavity drainage system. Subsequent investigations by Lliboutry (1968, 1979), Iken (1981), and Fowler (1986, 1987) have identified *cavitation* as being important. The basal water pressure is therefore a crucial factor in the sliding law, and in (1.5) N has to be derived from additional consideration of a drainage theory. In general, one seeks a relation of the form

$$N = N(Q, u_b, \Phi), \quad (1.7)$$

where Φ (the total hydraulic gradient) is prescribed, and Q (the water flux) is determined by source terms such as surface and basal melting. Note that N is not necessarily identical to the effective pressure *within* the corresponding flow elements, such as in the channels or cavities, but the two variables are interrelated. Equations (1.5) and (1.7) illustrate the complex nature of the coupling between drainage and ice motion.

1.4 Subglacial sediments

Deformable beds

Glaciers and ice sheets erode their beds. Stress fracturing plucks boulders from the bed, and these are ground up to cobbles, gravel, and finally to a rock flour which gives (as suspended sediment) the outlet rivers their milky colour. The mass of eroded sediment is carried along at the base of the ice as a kind of moving conveyor belt, and is known as *till*. It can form a layer several metres thick, and its deformation can be responsible for the bulk of a glacier's motion, especially if it is water-saturated; this is most likely the case if the ice is temperate.

A deforming basal layer also occurs where an ice sheet overrides sediment, such as in the lowlands of Europe or in the plains of North America during the last glaciation. A modern example is found in the Siple coast of Antarctica, where five *ice streams* (*A* to *E*) exist. They are (except for *C*) fast flowing zones of the ice sheet, and are underlain by several metres of wet deformable sediment (e.g. Alley *et al.*, 1986, 1987; Blankenship *et al.*, 1986). Almost all of their velocity is thought to be due to a sliding motion induced by till deformation.

Part of the interest in the basal till is that the sliding mechanism cannot be identical to that supposed by the classical theory, which assumes a rigid impermeable 'hard' bed. It is important to understand the corresponding sliding theory, for this may be instrumental in explaining why ice streams exist, whether or not ice sheets can surge, and whether that is responsible for large-scale oscillations, such as those observed in the Pleistocene paleoclimatic record, associated with Heinrich events (MacAyeal, 1993). So far, this sliding theory has remained obscure. One of the main

reasons is that there is not yet a satisfactory theory of drainage over deformable sediment beds. We review the relevant developments in the following.

Till rheology and sliding

Till as a wet, granular material is often modelled by a simple realistic rheology law of the type

$$\dot{\epsilon} = A_T \tau^a N^{-b} \quad (1.8)$$

(Boulton and Hindmarsh, 1987), where τ is the applied stress, $\dot{\epsilon}$ is the resulting strain rate, $N = P - p_w$ is the effective pressure of the till, P is the overburden pressure, and p_w is the pore-water pressure. (This definition of N differs from Equation (1.6) since we are dealing with the pore-water *within* the till.) There are further complications to the rheology (e.g. see Clarke, 1987), but Equation (1.8) is probably the most useful form for practical use. Boulton and Hindmarsh's (1987) values, determined from in-situ experiments in till underneath Breidamerkurjökull, Southern Iceland, were $a = 1.33$, $b = 1.8$, and $A_T = 3 \times 10^{-5} \text{ Pa}^{b-a} \text{ s}^{-1}$.

If the till is thin, then one can neglect the vertical variation of the effective pressure due to gravity (this is a crude assumption), and a sliding velocity u_b over a till layer of thickness h_T will give a strain rate

$$u_b/h_T = A_T \tau^a N^{-b}, \quad (1.9)$$

according to (1.8). It follows that the sliding law in this case is

$$\tau_b = \tau = C u_b^r N^s \quad (1.10)$$

(cf. Equation (1.5)), with the constants $C = (h_T A_T)^{-1/a}$, $r = 1/a$, $s = b/a$. This simple calculation indicates that u_b can depend strongly on N , as in the hard bed theory. If we include the effects due to gravity and other till properties, then the derivation is modified, but the sliding relation will still contain N , and hence depend on the subglacial water pressure.

Recently, there has been field evidence suggesting that when the water pressure is sufficiently close to the overburden ice pressure, the shear strength of the ice-till interface itself may be reduced to below that of the underlying sediment. Then sliding may result from a *decoupling* mechanism rather than from the pervasive bed deformation assumed above (Iverson *et al.*, 1995). Obviously, one can also expect N to appear as an integral part of the corresponding sliding law.

Drainage theory for till

By analogy to the hard bed theory, a drainage relation is required for determining N for a sediment bed. There is little to constrain how water drains over subglacial till. One way is by Darcy flow downwards through the till to an underlying aquifer, and outwards beyond the margin. Instead, the till may be underlain by impermeable bedrock, and then meltwater evacuation would be confined to percolation within the till. These porous flow systems have been considered by a number of authors (e.g.

Boulton *et al.*, 1974; Clarke *et al.*, 1984; Lingle and Brown, 1987), but unless the subglacial till/aquifer is thick and is highly permeable, water transport will be inefficient because its capacity will be greatly exceeded by the water supply. (This is especially the case for large, wet-based ice sheets.) In this case, a simple calculation shows that the required flux will lead to negative effective pressures at the bed (Shoemaker, 1986; Alley, 1989), and we infer from this the existence of some kind of flow *along* the ice-till interface.

Two model configurations have been suggested for this. Alley (1989) advocated a patchy Weertman film, in which the water collects in puddles and has a Darcy-type law governing its behaviour. As in the hard bed theory, this flow may be subjected to instability. If that is the case, then channelization could occur. This was investigated by Walder and Fowler (1994). They developed a simple model of channelized drainage over saturated deformable till, and found that for steady flow conditions, two distinct end members are possible: R channels as before, incising upward into basal ice, and also *canals*, which cut into the till. The mechanism governing their behaviour is similar to that in the classical Röthlisberger theory, except that for canals, sediment creep replaces ice closure and sediment erosion replaces ice melting.

A significant result of Walder and Fowler's (1994) analysis is that the end-member states are distinguished by a critical value $\tilde{p}$ of the effective channel pressure. Essentially, for $N_c > \tilde{p}$ (where N_c is the effective channel pressure defined in (1.3)), R channels would exist if the ice surface slope is large enough (~ 0.1), while for $N_c < \tilde{p}$, canals would exist for any slope. Thus for ice sheets or ice streams, which have low surface slopes (~ 0.001), canals would provide the relevant drainage mechanism. Importantly, these are predicted to have high water pressures, consistent with the exceptionally low values of basal effective pressure inferred for active ice streams: e.g. $N \approx 0.5$ bar for Ice Stream B (Alley *et al.*, 1986, 1987; Blankenship *et al.*, 1986). They also have the property that N_c decreases with increasing water flux Q , which provides for a stable distributed channel network. In this case, it is justifiable to interpret N_c (in the canals) as the representative basal effective pressure N .

Glaciological and geomorphological significance

Although the theory of drainage over subglacial till is not well established, it has the following implications. First, the possibility for the basal water pressure to exhibit the property $\partial N_c / \partial Q < 0$ in a distributed flow fashion is a possible source of dynamic instability, much as in the hard bed surge mechanism of Kamb (1987). Preliminary application of a *canal-based* sliding law to simplified ice sheet models has demonstrated that viable mechanisms exist for ice sheet surging and the spontaneous generation of ice streams (Fowler and Johnson, 1995, 1996). Second, one of the direct consequences of drainage and ice motion is glacial *bedform*, of which notable examples include drumlins, eskers, flutings, and tunnel valleys (e.g. Boulton and Hindmarsh, 1987). These sediment features provide valuable constraints when reconstructing the extent and characteristics of past ice masses, but current hypotheses for the rôle of drainage in their formation are controversial. Finally, drainage is also responsible for removing sediment from underneath glaciers and ice sheets, and therefore is one

of the primary processes that determine how the landscape responds to glaciation. Sediment transport, however, has remained a largely unexplored topic in theoretical drainage models.

1.5 Extensions

This thesis is aimed at substantiating the theory of subglacial water transport, with particular emphasis on channelized drainage and its erosional aspect. We shall regard the theory from two different perspectives, so the material has been organized into two parts. Essentially, Part I is concerned with an extension of the time-dependent theory from a (classical) hard bed situation to a sediment bed situation, whereas in Part II, we perform an equivalent extension for the steady flow theory.

Part I: Jökulhlaups

In Chapter 2, we use Nye’s (1976) model as the starting point and set out to examine the mechanics of jökulhlaups. The mathematics of high magnitude, time-dependent flow in the subglacial channel is investigated. (Röthlisberger’s theory is treated as a special case.) Following Nye’s example, we concentrate also on the 1972 Grímsvötn jökulhlaup, as it seems to be typical of the geothermal-induced floods from this lake, and there is sufficient field data.

Our contribution to the theory of jökulhlaups is in identifying the rôle of sediment transport in flood mechanics. As we described earlier, the floodwater is heavily laden with sediment. We provide an argument whereby it is possible to infer the *geometry* of the flood channel from the total sediment yield. In particular, we deduce a cross sectional channel shape that is wide and low, as opposed to roughly circular, as has been assumed in the classical model. This result indicates that the presence of subglacial till is an important factor in determining flood evolution. We discuss also the process of flood initiation. These ideas lead us to generalize the classical theory to the consideration of non-circular channels. A revised model of jökulhlaups is then formulated, by which we are able to produce an improved simulation of the 1972 hydrograph.

Part II: Drainage theory of wide subglacial channels

The central theme in Chapters 3 to 7 is steady flow. Following the flood model proposed in Chapter 2, we consider the detailed description of sediment-floored wide channels (or simply ‘wide channels’). Our main question is how these channels can exist in equilibrium.

Hooke *et al.* (1990) have suggested broad and low ice channels as the cause of why measured basal water pressures are frequently underestimated by the classical theory (e.g. Equation (1.2)). The canals of Walder and Fowler (1994) are also thought to be wide and shallow. Thus, the concept of subglacial channels having high aspect ratios is not entirely new, but the effect of a deformable (and erodible) sediment bed in these channels is not well understood.

In Chapters 3 to 6, we formulate mathematical models for each of the four wide channel processes — ice closure, ice melting, sediment deformation, and sediment erosion — in turn, given ‘typical’ drainage conditions. This refers to channels which measure metres across, and which conduct water flux of the order of cubic metres per second. The exact values are not important as long as the flow is fully turbulent. (Incipient channels are outside the scope of this thesis.) As we shall see, the high aspect ratio assumption allows us to make a number of simplifying approximations. The effect of basal sliding is neglected, except for a specific problem in Chapter 3 concerning the channel width.

The drainage theory is then assembled using the ingredients derived in Chapters 3 to 6. Existing theories have a shortcoming, because they neglect the mechanism that is responsible for determining the width of the channel. In Röthlisberger’s (1972) and Nye’s (1976) model, the hard bed channel is assumed capable of remaining near-circular as it evolves. And for soft beds, Walder and Fowler’s (1994) results are fundamentally due to the assumption that canals, like sub-aerial rivers, choose their depth (of bed incision) by sediment processes. But they circumvent the width specification problem by applying a typical aspect ratio of river channels to their canals, thus *forcing* them to be wide. As it turns out, the stable channel width is found to be crucial in determining the equilibrium properties of the wide channel. This issue is discussed in Chapter 7.

In the same chapter, we develop the steady flow theory for soft bed wide channels, as an extension to Walder and Fowler’s (1994) results. The theory is extended for unsteady flow, with the aim of providing a verification of the jökulhlaup model in Chapter 2. We also explore its implications on sediment erosion and transport, in order to establish the link between the hydrology and sedimentology of glaciers. Conclusions are given in Chapter 8.

Part I

Jökulhlaups

Plate 2 goes here.

Chapter 2

The mechanics of jökulhlaups

2.1 Introduction

Jökulhlaups have not been known to induce glacier surges, despite vast quantities of subglacial water present during the floods. This observation is of fundamental importance, because it suggests that the drainage of flood water is predominantly channelized and not distributed (Nye, 1976). As the flood discharge is exceptionally high, the drainage system is also less susceptible to extraneous variations caused by diurnal fluctuations of melt-water input or interaction with other drainage types. As such, jökulhlaups provide a unique opportunity to study subglacial channels ‘in isolation’.

According to Nye’s (1976) theory, a jökulhlaup is due to an instability of channelized drainage. His mathematical model considers a single subglacial channel that transports water from the ice-dammed lake towards the snout. Two competing processes govern the time evolution of this channel, by respectively enlarging and reducing its cross sectional area. When they are out of balance, model simulation is found to lead to a flood behaviour that resembles a jökulhlaup.

In this chapter, the mechanics of jökulhlaups is investigated with reference to this classical model. Following Nye (1976) and Spring and Hutter (1981b), we focus also on modelling the 1972 jökulhlaup from Grímsvötn. To begin, we give a brief review of Grímsvötn jökulhlaups and present the detailed mathematical formulation of Nye-type models, respectively in Sections 2.2 and 2.3.

Although Nye’s theory is found to be qualitatively plausible, it has a number of drawbacks. In Sections 2.4 and 2.5, we tackle three particular problems concerning flood simulation. The first problem concerns flood initiation. Nye’s theory cannot satisfactorily explain how the pressure seal over the caldera rim at Grímsvötn can be broken (consistently) when the lake-level water pressure is still some six bars below the maximum overburden ice pressure. An understanding of this observation is required if one intends to construct a flood forecasting algorithm.

The second problem is that in order to fit the ‘rising limb’ of the 1972 hydrograph, Nye (1976) was forced to use a value of the Manning roughness which is rather high.

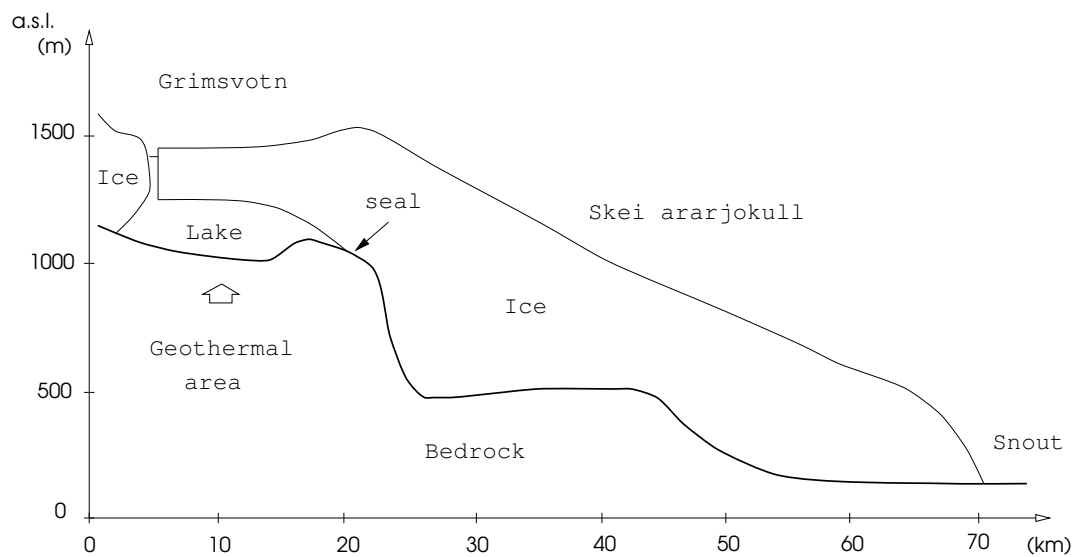


Figure 2.1: (a) A Landsat image of western Vatnajökull (31 January 1973); reproduced from Björnsson (1988). (b) Cross sectional view of Vatnajökull and Grímsvötn along the jökulhlaup drainage route; Björnsson (1974).

(This empirical parameter, used in river mechanics to quantify the flow resistance of a channel, is determined by calibration in his model.) This may be a minor concern, since one is extrapolating stream flow conditions far beyond their normal limits. On the other hand, it may be suggestive of a limitation of the model. A related and more serious problem is the third one. The Nye model is unable to simulate the relatively abrupt shut-off that is observed in the typical Grímsvötn hydrograph, thus the peak discharge is overestimated.

To address these issues, we develop a theory that takes into account the physics of sediment erosion and its effect on the morphology of the flood channel. This includes a schematic initiation mechanism for jökulhlaups, based on the theory of sediment-floored subglacial channels put forward by Walder and Fowler (1994). We also introduce the concept that flood channels need not be (roughly) circular, as has been assumed in the classical model, but have shapes determined by a *local* balance between closure and melting, and by a *lateral* balance between sediment erosion and deformation. The governing equations of this revised model are then formulated, and are used to produce an improved simulation of the 1972 Grímsvötn hydrograph. Our theoretical results are found to be consistent with the observed total sediment yield, and predict a flood channel that has a wide and low cross sectional geometry, reminiscent of the ‘canals’ investigated by Walder and Fowler (1994). We discuss the implications of these findings in Section 2.6.¹

2.2 Jökulhlaups from Grímsvötn

Grímsvötn is a volcanic caldera beneath the Vatnajökull ice cap in south-east Iceland (see Fig. 2.1 and Plate 1). This part of the ice cap is strongly influenced by an underlying geothermal heat source, which melts ice at the base, giving rise to a subglacial lake. The lake has an area of roughly 10 to 30 km², depending on how much water has accumulated, and is almost completely covered by an ice shelf about 200 m thick. Approximately three-quarters of the lake-water is derived from geothermal melting; the rest is due to inflow of surface meltwater and precipitation. As a result of the density change in ice melting, the lake cover sags to form a depression that is visible on the surface. Comprehensive reviews of this geothermal system, including aspects such as ice surface and bedrock topography, long term mass balance, ice dynamics, hydrology, and the characteristics of jökulhlaups, are given by Björnsson (1974, 1988, 1992) and Gudmundsson *et al.* (1995).

Geothermal-related jökulhlaups

Jökulhlaups from Grímsvötn have been reported for the last six hundred years (Thórarinnsson, 1974), and are known to occur at roughly one to three times per decade. Times series data for the lake level since the 1950s indicate that typically, the floods have initiated when the lake surface is about 1430 m above sea level (e.g. Björnsson, 1988, 1992); see Fig. 2.2. This observation has been used to forecast

¹The material in this chapter has been published in Fowler and Ng (1996).

Figure 2.2: The lake level in Grímsvötn over the period 1934–93; reproduced from Gudmundsson et al. (1995).

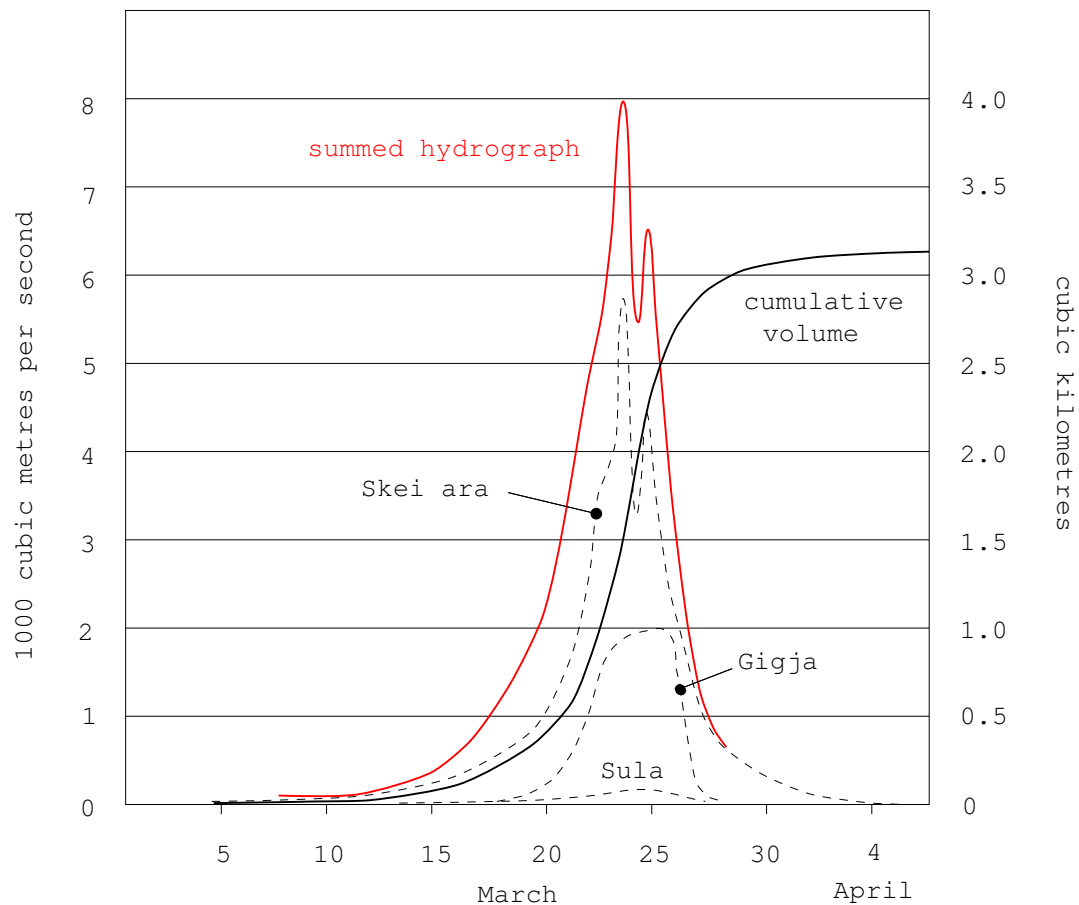


Figure 2.3: Hydrograph of the 1972 jökulhlaup. The discharge of the three main outlet rivers Skeiðará, Gígja and Sula are shown separately by the broken curves; reproduced from Rist (1973).

Figure 2.4: map

the timing of the outbursts to within a time interval of one year. Its underlying mechanism is the subject of Section 2.4.

During a jökulhlaup, drainage of lake-water from Grímsvötn takes place via route-ways under the outlet glacier Skeiðarárjökull to the south (Fig. 2.1 and Plate 2). Flood water issued at the ice margin feeds the major rivers Skeiðará, Súla and Gígjukvísl, which cut through Skeiðarársandur (Fig. 2.4). In the most violent events, the entirety of this coastal outwash plain has been inundated. Hydrograph flood data for this century have been compiled by Thórarinnsson (1974), Rist (1955, 1973, 1976, 1984), and Kristinnsson *et al.* (1986).

We show the flood hydrograph of the 1972 Grímsvötn jökulhlaup in Fig. 2.3, reproduced from Rist (1973). The discharge increased approximately exponentially over the first two to three weeks, but having reached the peak at $8000 \text{ m}^3 \text{ s}^{-1}$, it fell very rapidly. Although this hydrograph displays a subsidiary peak, its characteristic several orders of magnitude growth, followed by a sudden recession, is typical of other Grímsvötn jökulhlaups that are geothermal-related (see Fig. 2.5; excluding the 1938 and 1983 events). In 1972, a total volume of 3 km^3 of lake-water had drained by flood termination, but the lake was not completely empty. The lake surface, having been lowered by about 100 m, was still a substantial distance (about 200 m) above the topographic ridge at the ‘seal’ (labelled in Fig. 2.1b). This is also the case for other Grímsvötn events of this type (Gudmundsson *et al.*, 1995), and indicates that the mechanics of flood evolution is not simply governed by ‘overflowing’ of the lake. As we shall see, this can be explained by the theory proposed by Nye (1976), described in Section 2.3.

Volcanically-triggered jökulhlaups

As we have mentioned before, some outburst floods are triggered by the rapid generation of melt-water during subglacial volcanic eruptions. At Grímsvötn, their occurrence is rare and interrupts the normal flood/refill cycles (see Fig. 2.2). Notable examples took place in 1938, and more recently, in November 1996². These floods tend to contain a high sediment concentration, and have a very short duration, high peak discharge ($> 10^4 \text{ m}^3 \text{ s}^{-1}$), and an extremely rapid rising phase. Their hydrographs can also display a reverse asymmetry to the normal Grímsvötn hydrographs, such as in 1938 (see Fig. 2.5). Generally, these characteristics are observed for volcanic jökulhlaups which originate from elsewhere (e.g. Katla, Mýdalsjökull, South Iceland)³. However, the mechanics of these floods is not at all well understood, and is excluded from our modelling work here.

²The 1983 event was also accompanied by eruptive activity, but is not discussed here; see Björnsson (1988).

³Extensive field data for the 1996 Grímsvötn event is not yet available, but preliminary results have been reported by Einarsson *et al.* (1997b).

Figure 2.5: *Hydrographs of the jökulhlaups on Skeiðarársandur in the period 1922–86; reproduced from Björnsson (1988).*

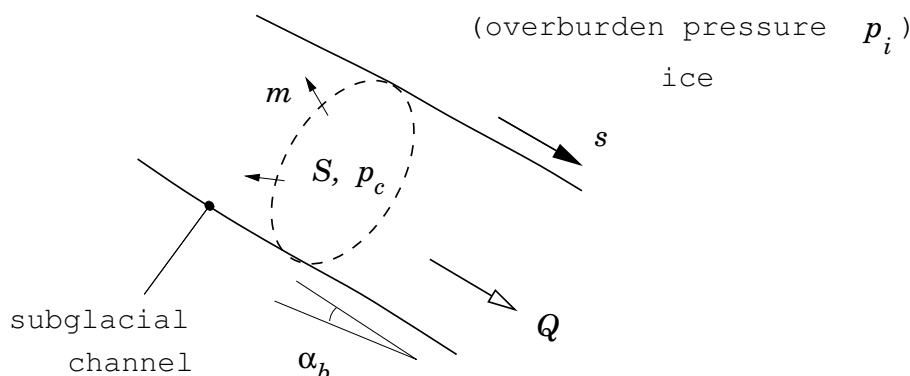


Figure 2.6: *The subglacial channel in the Nye model and our mathematical definitions.*

2.3 The Nye model

2.3.1 Governing equations

In the steady flow theory of Röthlisberger (1972), the subglacial channel is maintained in equilibrium by an exact balance between (i) its rate of enlargement, caused by melt-back of the ice walls, as a result of the frictional heat generated by the turbulent water flow, and (ii) its creep closure velocity, due to the pressure difference between the overlying ice and the water flow. Nye's (1976) model reinstates time dependence into this consideration. His governing equations are now described.

We consider a subglacial channel⁴ of cross sectional area $S(s, t)$, admitting a (closed-conduit) water flux $Q(s, t)$. This is illustrated in Fig. 2.6. The channel is situated at the ice-bed interface, but the presence of this interface is assumed to have no effect on the channel shape, which is assumed to be *cylindrical*. The independent variables are downstream distance s and time t , where $0 < s < l_c$, and l_c is the total channel length, approximately the distance between the ice margin and the lake. We denote the densities of ice and water respectively by ρ_i and ρ_w .

If m is the rate of ice melting (in terms of mass rate per unit channel length), then the mass conservation equation for the water flow is

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_w} - \frac{\partial Q}{\partial s}. \quad (2.1)$$

A second equation for the time derivative of S is given by the kinematic condition

$$\frac{\partial S}{\partial t} = \frac{m}{\rho_i} - K_0 S (p_i - p_c)^n; \quad p_c < p_i, \quad (2.2)$$

which describes the competing effect of ice melting and creep closure on the channel area. p_i and p_c are respectively the overburden ice pressure and the channel water

⁴Alternative terminology such as ice 'conduit' and 'tunnel' is also common in the literature; e.g. see Spring and Hutter (1981a).

pressure, and K_0 and n (≈ 3) are constants derived from Glen's flow law (Nye, 1953). We introduce the effective channel pressure N_c , where

$$N_c = p_i - p_c, \quad (2.3)$$

as in Equation (1.3). This definition is used throughout the text.

Assuming turbulent flow in the channel, conservation of momentum may be described by the empirical Manning formula

$$\rho_w g \sin \alpha_b - \frac{\partial p_c}{\partial s} = \frac{F_1 Q^2}{S^{8/3}} \quad (2.4)$$

(e.g. from Massey, 1989). This equation relates the hydraulic gradient (left-hand side) to flow variables (on the right) via a roughness parameterization. The angle α_b is the local inclination of the channel, and g is gravitational acceleration. F_1 is a property of the channel cross section, given by

$$F_1 = \left(\frac{S}{R_H^2} \right)^{2/3} \rho_w g n'^2, \quad (2.5)$$

where R_H is the channel hydraulic radius (= cross sectional area S / wetted perimeter W), and n' is the roughness coefficient of the ice wall. Since Nye considered circular channels, $F_1 = (4\pi)^{2/3} \rho_w g n'^2$.

Heat energy is dissipated by the water flow. Some of it is stored as internal energy whilst the rest is transferred to the ice wall, where new melt is raised to the channel temperature θ_w after a change of phase. The appropriate energy equation is

$$Q \left(\rho_w g \sin \alpha_b - \frac{\partial p_c}{\partial s} \right) = \rho_w c_w S \frac{d\theta_w}{dt} + mL + mc_w(\theta_w - \theta_i), \quad (2.6)$$

in which θ_i is the melting point, L is the latent heat of fusion and c_w is the specific heat capacity of water.

Finally, we need an equation for the melt-rate m . This is given by considering the boundary layer heat transfer at the wall. Nye used the empirical relationship

$$Nu = 0.023 Re^{0.8} Pr^{0.4} \quad (2.7)$$

for flows in circular tube sections (Dittus and Boelter, 1930)⁵, which is accurate for $10^4 < Re < 1.2 \times 10^5$ and $0.7 < Pr < 100$, with $l_c/D > 60$, l_c and D being the length and diameter of the tube (the channel, in this case). Given the viscosity μ_w and thermal conductivity k_w for water, and the sectional mean velocity $\bar{u} = Q/S$, the Nusselt, Reynolds and Prandtl numbers are defined by

$$Nu = \frac{h_T D}{k_w}, \quad Re = \frac{\bar{u} D \rho_w}{\mu_w}, \quad Pr = \frac{\mu_w c_w}{k_w}. \quad (2.8)$$

⁵Our situation is one where heat transfer takes place between a fluid and its solid phase. The wall temperature is necessarily held *constant* at the melting point. Dittus and Boelter's (1930) relation applies to tube flows in which the temperature *field* is fully developed, therefore it is only a crude approximation.

It follows that the required heat transfer coefficient h_T can be expressed in terms of the flow variables and model constants. Since the rate of heat flow to the wall per unit length of channel is given by

$$h_T \pi D (\theta_w - \theta_i), \quad (2.9)$$

by equating this to the melt-rate, one obtains

$$F_0 \left(\frac{Q}{\sqrt{S}} \right)^{0.8} (\theta_w - \theta_i) = mL + mc_w(\theta_w - \theta_i), \quad (2.10)$$

which is our heat transfer equation. The constant here is

$$F_0 = 0.205 k_w \left(\frac{2\rho_w}{\mu_w \sqrt{\pi}} \right)^{0.8}. \quad (2.11)$$

We have now derived five differential equations (2.1), (2.2), (2.4), (2.6) and (2.10) for the variables Q , S , p_c , m and θ_w , all of which are functions of s and t . (θ_i is a known function of p_c .)

2.3.2 Approximate analytic solutions

In order to derive analytic solutions to model the *rising limb* of the 1972 jökulhlaup, Nye applied the following simplifying assumptions to his equations: (i) The last term in (2.2) is neglected, since channel enlargement dominates closure at this stage. (ii) Field measurements at Grímsvötn and at the ice margin indicate that the water temperature is close to the melting point, so $\theta_w \approx \theta_i \approx 0$ °C (Björnsson, 1988). (iii) The l.h.s. of (2.4) is replaced by its spatial average value, denoted by $\langle -\partial\phi/\partial s \rangle$ (it is equal to $196 \text{ m}^{-2} \text{ s}^{-2}$ for the Grímsvötn topography).

By using these approximations, his governing equations reduce to

$$\begin{aligned} \frac{dS}{dt} &= K_1 S^{4/3}, \\ \frac{dQ}{dt} &= K_2 Q^{5/4}, \end{aligned} \quad (2.12)$$

where

$$K_1 = \frac{1}{\rho_i L \sqrt{F_1}} \left\langle -\frac{\partial\phi}{\partial s} \right\rangle^{3/2}, \quad K_2 = \frac{4}{3\rho_i L F_1^{3/8}} \left\langle -\frac{\partial\phi}{\partial s} \right\rangle^{11/8}. \quad (2.13)$$

The solutions are

$$S = \left(-\frac{3}{K_1 t} \right)^3, \quad Q = \left(-\frac{4}{K_2 t} \right)^4, \quad (2.14)$$

where we have set the integration constants to zero, so that the asymptotes are (arbitrarily) positioned at $t = 0$. (Note that downstream dependence has effectively been removed.) Equation (2.14)₂ agrees remarkably well with hydrographic data for a Manning roughness value $n' = 0.12 \text{ m}^{-1/3} \text{ s}$. This result is illustrated in Fig. 2.7, in which the t -axis has been shifted in order to match field data from Rist (1973).

Figure 2.7: *The total measured discharge in Fig. 2.3 (open circles and broken curve) plotted logarithmically and compared with Nye’s analytic solution (full curve); reproduced from Nye (1976).*

According to Röthlisberger (1972), n' could be expected to range from about $0.01 \text{ m}^{-1/3} \text{ s}$ for a straight smooth pipe to $0.1 \text{ m}^{-1/3} \text{ s}$ for a meandering boulder-strewn torrent at the bed. Although Nye’s estimate lies at the very rough end of this range, his data fit shows that a model of jökulhlaup based on channelized drainage is indeed plausible.

2.3.3 Coupled models

Nye (1976) also discussed how the falling limb may be explained if the last term of Equation (2.2) was retained. Essentially, the lake-level drop during a jökulhlaup causes the effective pressure $p_i - p_c$ to increase through a reduction in p_c . This mechanism may allow viscous closure of the channel to dominate melting eventually, leading to flood recession. If this is the case, then the flood can also terminate before the lake is emptied.

Formally, this effect can be included by coupling his model to an equation that describes the ‘dynamics’ of the lake, namely,

$$\frac{dV}{dt} = -Q(0, t), \quad (2.15)$$

where V is the lake volume. A subsidiary equation relating V to $p_c(0, t)$ is required, and is obtained by hydrostatic consideration. The lake geometry is taken to be known. We discuss the rôle of (2.15) as a boundary condition of the problem in Section 2.3.5.

The fully coupled model, consisting of Equations (2.1), (2.2), (2.4), (2.6), (2.10) and (2.15), is complicated and requires numerical solution. There have been two developments of this type. Spring and Hutter (1981a,b) performed a rigorous mathematical justification of Nye's formulation, then constructed a numerical model to simulate the 1972 event and investigated the sensitivity of model parameters and boundary conditions. Although their momentum and energy equations have included impulsive and kinetic terms, and are based on a different friction parameterization (the Darcy-Weisbach law), their final model does not differ significantly from Nye's. Their best simulated hydrograph is similar to ours, described in Section 2.3.5, and is discussed therein. Clarke (1982) has also investigated Nye's equations, and modified them to take into account the effect of lake geometry and temperature in flood simulations. His dimensionless model, applied to the August 1978 outburst flood from 'Hazard Lake', Yukon Territory, Canada, produced $n' = 0.105 \text{ m}^{-1/3} \text{ s}$, consistent with Nye's estimate for Grímsvötn.

2.3.4 Non-dimensionalization

The solution of the Nye model is facilitated by an appropriate non-dimensionalization, which is now described. To repeat, our equations are

$$\begin{aligned}
 \frac{\partial S}{\partial t} &= \frac{m}{\rho_i} - K_0 S(p_i - p_c)^n, \\
 \frac{\partial S}{\partial t} &= \frac{m}{\rho_w} - \frac{\partial Q}{\partial s} \quad \left(+ \frac{\Omega}{\rho_w} \right), \\
 \rho_w g \sin \alpha_b - \frac{\partial p_c}{\partial s} &= \frac{F_1 Q^2}{S^{8/3}}, \\
 Q \left(\rho_w g \sin \alpha_b - \frac{\partial p_c}{\partial s} \right) &= \rho_w c_w S \frac{d\theta_w}{dt} + mL + mc_w(\theta_w - \theta_i), \\
 F_0 \left(\frac{Q}{\sqrt{S}} \right)^{0.8} (\theta_w - \theta_i) &= mL + mc_w(\theta_w - \theta_i).
 \end{aligned} \tag{2.16}$$

They describe respectively a kinematic condition, conservation of mass, momentum, energy, and heat transfer. Model constants are given in Table 2.1. The value $n' = 0.1 \text{ m}^{-1/3} \text{ s}$ has been assumed, based on the results of Nye (1976) and Clarke (1982). For completeness, we also include an input term in (2.16)₂ to represent water influx from the surface; Ω is the rate of mass gain per unit channel distance.

We first make the substitution

$$p_c = p_i - N_c \tag{2.17}$$

following Equation (2.3). It follows that

$$\rho_w g \sin \alpha_b - \frac{\partial p_c}{\partial s} = \Phi + \frac{\partial N_c}{\partial s}, \tag{2.18}$$

Ice	ρ_i	900	kg m^{-3}
	L	333.5	kJ kg^{-1}
	n	3.0	
	K_0	10^{-24}	$\text{Pa}^{-3} \text{s}^{-1}$
Water	ρ_w	1000	kg m^{-3}
	μ_w	1.787×10^{-3}	$\text{kg m}^{-1} \text{s}^{-1}$
	k_w	0.558	$\text{W m}^{-1} \text{K}^{-1}$
	c_w	4.22	$\text{kJ kg}^{-1} \text{K}^{-1}$
Other constants	g	9.8	m s^{-2}
	F_0	5000	$\text{kg m}^{-3/5} \text{s}^{-11/5} \text{K}^{-1}$
	F_1	530	$\text{kg m}^{-8/3}$

Table 2.1: *Constants used in the Nye model.*

where

$$\Phi = \rho_i g \sin \alpha_s + (\rho_w - \rho_i) g \sin \alpha_b. \quad (2.19)$$

In these equations, Φ is the *total water potential gradient*, and α_s , α_b are the down-stream inclinations of the ice surface and the glacier bed. (Both α_s and α_b are assumed to be small.) Since $\rho_w \approx \rho_i$, we can write

$$\Phi \approx \rho_i g \sin \alpha_s. \quad (2.20)$$

This approximate result is used in the later chapters.

If variables s , t , S , m , Q , N_c , Φ and Ω are non-dimensionalized respectively by s_0 , t_0 , S_0 , m_0 , Q_0 , N_0 , Φ_0 and Ω_0 , and in addition we re-scale the temperature variable $(\theta_w - \theta_i)$ using θ_0 , then the dimensionless model is

$$\begin{aligned}
\frac{\partial S}{\partial t} &= m - S N_c^3, \\
\epsilon \frac{\partial S}{\partial t} &= \epsilon r m - \frac{\partial Q}{\partial s} \quad (+ \Omega), \\
\Phi + \delta \frac{\partial N_c}{\partial s} &= \frac{Q^2}{S^{8/3}}, \\
Q \left(\Phi + \delta \frac{\partial N_c}{\partial s} \right) &= \gamma S \frac{d\theta}{dt} + m(1 + r\gamma\theta), \\
\theta \left(\frac{Q}{\sqrt{S}} \right)^{0.8} &= m(1 + r\gamma\theta),
\end{aligned} \quad (2.21)$$

in which the parameters are given by

$$\begin{aligned}
\epsilon &= \frac{s_0 \Phi_0}{\rho_i L}, \quad \delta = \frac{Q_0^{1/12} F_1^{-1/8}}{(K_0 \rho_i L)^{1/3} s_0 \Phi_0^{13/24}}, \\
\gamma &= \frac{\rho_w c_w}{\rho_i L F_0} Q_0^{1/2} F_1^{3/20} \Phi_0^{17/20}, \quad r = \frac{\rho_i}{\rho_w}.
\end{aligned} \quad (2.22)$$

s_0	50	km
h_0	1	km
t_0	2.22×10^6 (25.7 days)	s
S_0	1.45	m^2
m_0	5.88×10^{-4}	$\text{kg m}^{-1} \text{s}^{-1}$
Q_0	1.00	$\text{m}^3 \text{s}^{-1}$
N_0	7.66×10^5	Pa
Φ_0	196	$\text{kg m}^{-2} \text{s}^{-2}$
Ω_0	0.0200	$\text{kg m}^{-1} \text{s}^{-1}$
θ_0	0.0455	K

Table 2.2: Variable scales used in non-dimensionalizing the Nye model.

The scale relations which we have imposed are

$$\begin{aligned}
S_0 &= \left(\frac{F_1 Q_0^2}{\Phi_0} \right)^{3/8}, \quad m_0 = \frac{Q_0 \Phi_0}{L}, \quad N_0 = \frac{Q_0^{1/12} \Phi_0^{11/24} F_1^{-1/8}}{(K_0 \rho_i L)^{1/3}}, \\
\theta_0 &= F_0^{-1} F_1^{3/20} \Phi_0^{17/20} Q_0^{1/2}, \quad \Omega_0 = \frac{\rho_w Q_0}{s_0}, \\
t_0 &= \rho_i L F_1^{3/8} Q_0^{-1/4} \Phi_0^{-11/8}, \quad \Phi_0 = \frac{\rho_w g h_0}{s_0}.
\end{aligned} \tag{2.23}$$

For Grímsvötn, the horizontal and vertical length scales can simply be taken respectively as the distance and the elevation drop between the lake and the ice margin. Therefore, we put s_0 ($\approx l_c$) = 50 km and h_0 = 1 km. In order to determine the other scales in (2.23), one of them has to be prescribed. We use here a nominal flux scale value $Q_0 = 1.00 \text{ m}^3 \text{s}^{-1}$. With the constants in Table 2.1, we obtain

$$\epsilon = 0.0327, \quad \delta = 0.0782, \quad \gamma = 6.40 \times 10^{-4}, \quad r = 0.9. \tag{2.24}$$

Table 2.2 summarizes the corresponding characteristic scales. Although F_1 has been based on the relatively high n' value derived by Nye (1976) and Clarke (1982), we see from the equations in (2.22) that our parameters are not sensitively dependent on it. Notably, δ remains small if n' is reduced to a more reasonable value, i.e., if F_1 is an order of magnitude smaller.

2.3.5 Analysis and simulation

In this section, the model in (2.21) is used to simulate the 1972 Grímsvötn flood hydrograph. We begin by examining a number of approximations.

If in non-dimensionalizing our model we had prescribed $Q_0 = 1000 \text{ m}^3 \text{s}^{-1}$ (a typical flux scale for a jökulhlaup) instead of $Q_0 = 1 \text{ m}^3 \text{s}^{-1}$, then δ and γ would still

be small, because $\delta \propto Q_0^{1/12}$ and $\gamma \propto Q_0^{1/2}$ according to (2.22). In particular, we now neglect γ , which is at least an order of magnitude smaller than the other parameters. Our equations become

$$\begin{aligned}\frac{\partial S}{\partial t} &= m - SN_c^3, \\ \epsilon \frac{\partial S}{\partial t} &= \epsilon rm - \frac{\partial Q}{\partial s} \quad (+ \Omega), \\ \Phi + \delta \frac{\partial N_c}{\partial s} &= \frac{Q^2}{S^{8/3}}, \\ Q \left(\Phi + \delta \frac{\partial N_c}{\partial s} \right) &= m, \\ \theta \left(\frac{Q}{\sqrt{S}} \right)^{0.8} &= m.\end{aligned}\tag{2.25}$$

The last of these is an explicit equation for θ and is decoupled from the others. Since θ will be of $O(1)$ and θ_0 is only a fraction of a degree (see Table 2.2), the predicted flow temperature will be close to the melting point. This is in agreement with Björnsson's (1988) measurements.

Boundary Conditions

Elimination of m from (2.25)₁ to (2.25)₄ leads to the following first order problem for S , Q and N_c :

$$\begin{aligned}\frac{\partial S}{\partial t} &= \frac{Q^3}{S^{8/3}} - SN_c^3, \\ \frac{\partial Q}{\partial s} &= \epsilon(r-1) \frac{Q^3}{S^{8/3}} + \epsilon SN_c^3 \quad (+ \Omega), \\ \delta \frac{\partial N_c}{\partial s} &= \frac{Q^2}{S^{8/3}} - \Phi.\end{aligned}\tag{2.26}$$

These equations require an initial condition for S , and also two boundary conditions — one for N_c at the snout, and the other for Q (or N_c) at the lake.

For the first boundary condition, we specify $N_c(1, t) = 0$ for $t \geq 0$, since $p_i = p_c = 0$ at the ice margin. Equation (2.15) now provides the other boundary condition. Since p_c at $s = 0$ is the water pressure at the lake end, given the lake geometry, the appropriate dimensionless condition is

$$\left. \frac{\partial N_c}{\partial t} \right|_{s=0} = \zeta Q(0, t),\tag{2.27}$$

where

$$\zeta = \frac{\rho_w g Q_0 t_0}{N_0 A_L},\tag{2.28}$$

and A_L is the (instantaneous) lake surface area. An average value for Grímsvötn is $A_L \approx 30 \text{ km}^2$, so $\zeta \approx 10^{-3}$. (More precisely, one can define the lake geometry by

some function $V(z_w)$, where z_w is the lake surface level measured with respect to a fixed horizontal datum, and then $A_L = dV/dz_w$; e.g. see Clarke, 1982.)

The external specification for our problem is not quite completed. Let us denote $\partial N_c / \partial t|_{s=0}$ by $\chi(t)$. Suppose for the moment χ is given. With the use of (2.27) (and the other initial and boundary conditions which we have discussed), the equations in (2.26) can in principle be solved for Q , N_c and S . Thus, $N_c(0, t)$ depends on $\chi(t)$, i.e., $N_c(0, t) = g[\chi(t)]$, where g is some ‘known’ function imposed by (2.26). We re-define $N_c(0, t)$ as $N_{c0}(t)$ for convenience, so that

$$N_{c0}(t) = g[\chi(t)]. \quad (2.29)$$

The inverse relation of this is

$$N_{c0}'(t) = g^{-1}[N_{c0}(t)], \quad (2.30)$$

which is a first order differential equation. Hence, N_{c0} (and χ) can only be found provided that $N_{c0}(0)$ is given. In other words, we require also the initial lake surface level (or equivalently, $N_c(0, 0)$) to supplement boundary condition (2.27).

Steady Flow

Seasonal fluctuations of channel drainage occur over a timescale in excess of t_0 (see Table 2.2). One can therefore neglect $\partial/\partial t$ in (2.25) and derive the pseudo-steady approximations

$$\begin{aligned} \frac{dQ}{ds} &= \Omega + \epsilon m, \\ Q^{1/4} \left(\Phi + \delta \frac{dN_c}{ds} \right)^{11/8} &= N_c^3, \end{aligned} \quad (2.31)$$

which are essentially the steady flow results of Röthlisberger (1972) (cf. Equation (1.2)). Under quiescent drainage conditions $\Omega \sim 1$ ($\gg \epsilon m$), so Q is determined primarily by the surface melt input Ω . With Q prescribed as such, one can obtain the effective pressure distribution $N_c(s)$ by numerically integrating Equation (2.31)₂ with the boundary condition $N_c(1) = 0$.

If $\delta = 0$, then the solution is straightforwardly

$$N_c = Q^{1/12} \Phi^{11/24}, \quad (2.32)$$

but this no longer satisfies the snout condition. Nevertheless, this solution is valid away from $s = 1$, where downstream variation of N_c is negligible (see Equation (1.4) in Chapter 1). δ is clearly important in explaining *boundary layer* effects. An example of this is given later in Fig. 2.9c. (Indeed, $\delta = 0$ is a singular perturbation approximation that may be investigated by an appropriate re-scaling of s about $s = 1$. One seeks an asymptotic expansion of N_c in the small parameter δ , with the leading order term given by the *outer solution* (2.32). This is omitted here because of space.)

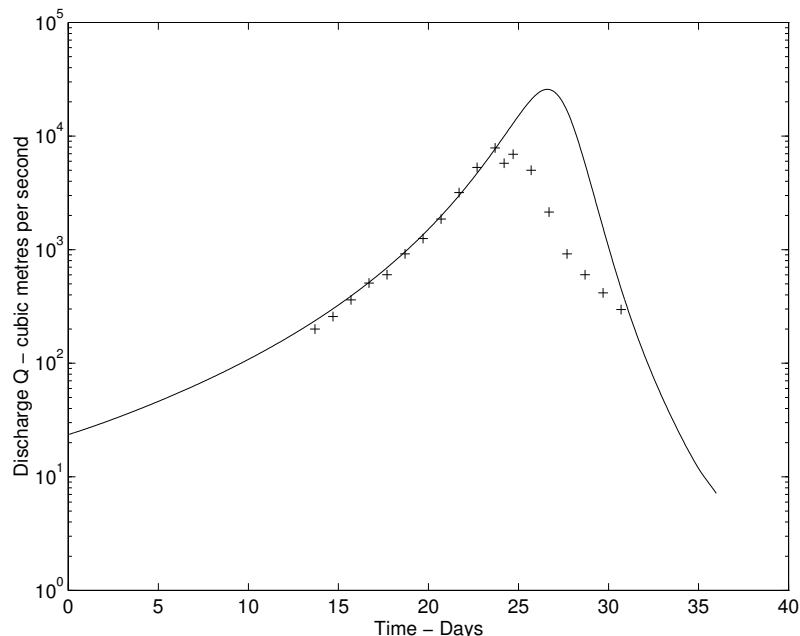


Figure 2.8: *Hydrographic matching between data (crosses) and simulation results of the Nye model (solid line) for the 1972 Grímsvötn jökulhlaup.*

Hydrographic simulation

The equations in (2.26) may be used to simulate the 1972 Grímsvötn hydrograph. We prescribe $\Omega = 0$, and assume the parameter values given by Equations (2.24) and (2.28), based on the Manning roughness value $n' = 0.1 \text{ m}^{-1/3} \text{ s}$. Topography and lake data for Grímsvötn, which provide $\Phi(s)$ and the initial condition $N_c(0, 0)$, are obtained from Björnsson (1974, 1988, 1992).

A suitable numerical scheme for (2.26) consists of a Newton (iteration) method which solves the boundary value problem between Q and N_c , and a simple Euler method which advances the S solution in time. This algorithm is found to converge rapidly. Sufficiently high accuracy is achieved with a (dimensionless) mesh size of $\Delta s = 0.005$ and a (dimensionless) time-step of $\Delta t = 0.002$.

Fig. 2.8 shows our best-fit hydrograph against the flood data compiled by Rist (1973). This result is quite satisfactory and is essentially similar to that obtained by Spring and Hutter (see 1981b, Fig. 6). (Discussions on the model sensitivity tests can be found in their paper.) In Fig. 2.9, we also present selected time samples of the corresponding spatial solutions for Q , S and N_c . The boundary layer of N_c at the snout is clearly visible in Fig. 2.9c.

The general conclusions from our data matching exercise are as follows. In obtaining a close fit to the rising limb of the hydrograph, the model is found to be not capable of reproducing the sudden switch between growth and recession. As a result, the peak discharge is overestimated (by a factor of three or so). The falling limb is

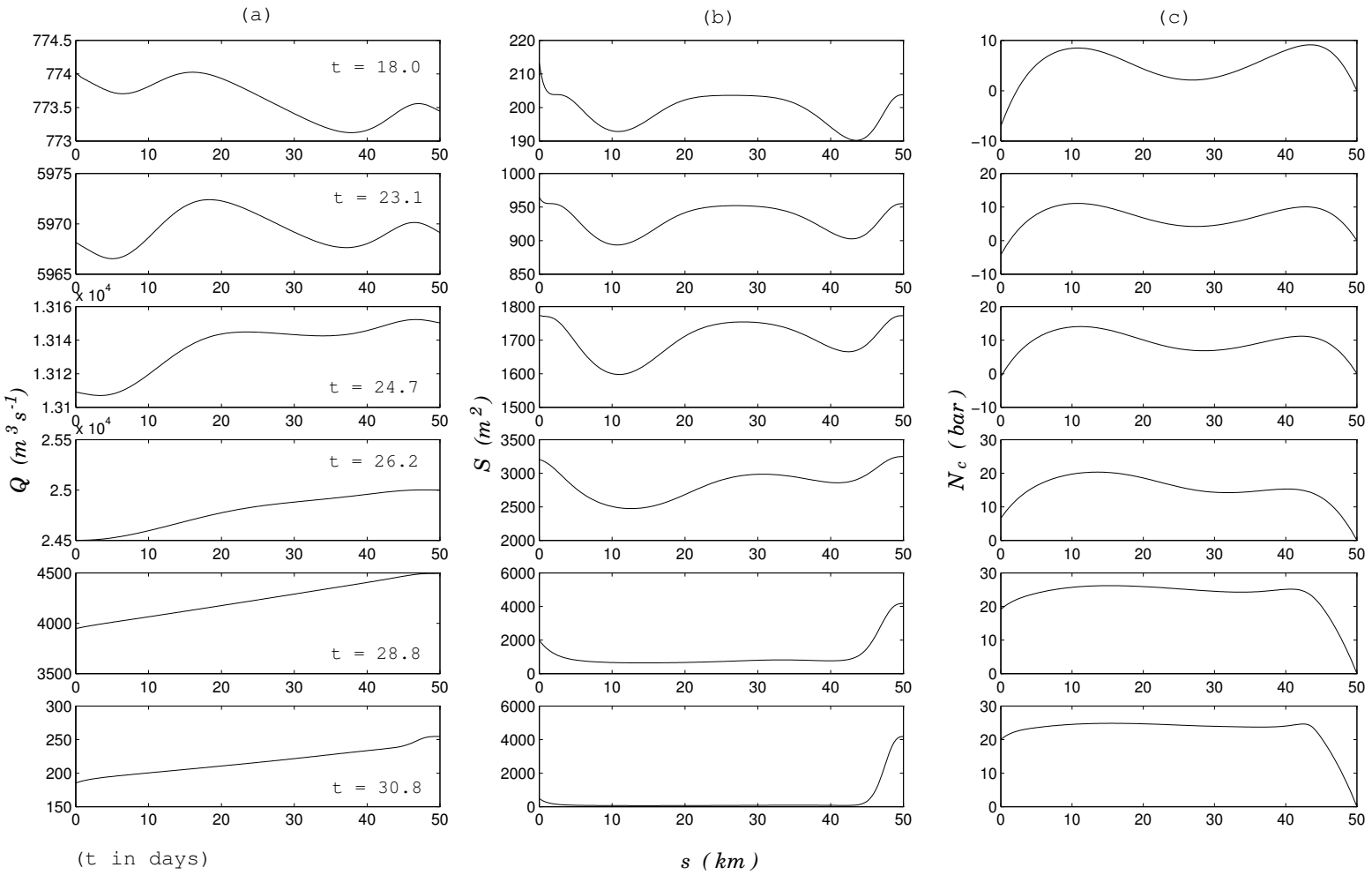


Figure 2.9: Time-sampled solutions of the simulated 1972 Grímsvötn jökulhlaup: (a) water flux Q , (b) channel cross sectional area S , (c) effective pressure N_c .

also poorly simulated. We have experimented with other roughness values n' but this did not resolve the problems (see also Spring and Hutter; 1981b). Thus, closure of the flood channel is apparently more abrupt (or occurs earlier) than can possibly be accounted for by the Nye model, even though the flow law exponent n in Equation (2.2) is already quite large. (High n values promote the sudden takeover of melting by viscous closure.)

Insofar as Nye's formulation is concerned, the most restrictive assumption seems to be that of a circular cross sectional geometry. Its necessity is implied by the empirical heat transfer law (2.7) and the form of the closure term in (2.2). (The momentum equation (2.4) can cope with a non-circular geometry by way of a variable hydraulic radius.) On the other hand, it is intuitive that a channel with an oblique (e.g. elliptical) cross section would attain a much higher closure rate in the direction normal to its major axis. For a given *wall* roughness, this channel would also present a greater hydraulic resistance than one that is circular but has the same cross sectional area. This may explain the high value of n' required of Nye's (1976) solution. On this basis, we suppose that relaxation of the 'circular' assumption may lead to an improved flood simulation. This concept is explored in Section 2.5.

2.4 Flood initiation

The most obvious reason for an outburst is if the effective pressure at the seal (see Fig. 2.1b) decreases to zero. This is known as the *flotation condition*. The ice barrier is then 'lifted', and the flood can initiate much as a fracture propagates in an elastic medium. However, calculations by Björnsson (1988) indicate that the Grímsvötn seal is habitually broken when the effective pressure is about six to eight bars. The observed initiation lake level, at about 1430 m a.s.l. (Fig. 2.2), is therefore some sixty metres below that required for flotation. More accurately, one might suppose that the critical condition is that the effective *normal* stress is zero, thus deviatoric stresses at the seal could imply a non-zero critical effective pressure. Nye (1976) suggested how these stresses can arise from the cantilever effect of the floating ice shelf, but they are unlikely to be large enough to account for the observed discrepancy (Björnsson, 1988).

Alternatively, the critical initiation pressure may be an inherent property of lake-channel coupling. If this is the case, then one might be able to identify it from a mathematical model which describes the long term dynamics of the system. In this section, we derive such a model by reducing Nye's equations. We are concerned with a time-span associated with the recurring flood cycles themselves.

The 'reduced' model is subsequently found to be inadequate, indicating that other factors are responsible for producing the observed pressure anomaly. One possibility is the presence of erodible, deformable till in the flood path. In relation to this, we propose a schematic seal breaking mechanism for Grímsvötn, based on the theoretical results of Walder and Fowler (1994). While our mechanism is speculative, it provides the first step in tackling the problem of flood initiation.

2.4.1 Flood cycles

The reduced model

To model a succession of jökulhlaups, one can incorporate into Nye's equations the slow regeneration of lake volume due to geothermal melting and inflow from the surface. We take the dimensionless model in (2.25). Since our interest is in simulating the time history of the floods, let us remove s -dependence by using the fact that $\delta \ll 1$ and $\epsilon \ll 1$. As we have already demonstrated, neglect of δ is a singular approximation which disqualifies us from satisfying $N_c = 0$ at $s = 1$; and with $\epsilon \rightarrow 0$, we obtain $Q \approx Q(t)$. By putting $\Phi = 1$, the equations reduce to

$$\frac{dQ}{d\tau} = Q^\beta - QN_c^3, \quad (2.33)$$

in which $\beta = 5/4$. The time axis has been re-scaled by the substitution $\tau = 4t/3$.

Equation (2.33) is the most basic model that describes the subglacial channel. As before, it is coupled to a 'lake continuity' equation. An appropriate dimensional version of this is

$$\frac{dV}{dt} = -Q + m_L, \quad (2.34)$$

where we have included an influx term m_L (in $\text{m}^3 \text{s}^{-1}$); cf. Equation (2.15). To non-dimensionalize this equation, we use our previous scales for t , Q and N_c defined in (2.23), and then our dimensionless coupled model is

$$\begin{aligned} \frac{dQ}{d\tau} &= Q^\beta - QN_c^3, \\ \frac{dN_c}{d\tau} &= \zeta(Q - \nu), \end{aligned} \quad (2.35)$$

where

$$\nu = \frac{m_L}{Q_0}, \quad \zeta = \frac{\rho_w g Q_0 \tau_0}{N_0 A_L} \quad (2.36)$$

(cf. Equation (2.28)). These parameters describe respectively the lake inflow and the lake geometry. The new characteristic time τ_0 is equal to $3t_0/4$. As in Sections 2.3.4 and 2.3.5, Q_0 is to be prescribed, and we use the approximate value $A_L = 30 \text{ km}^2$ for Grímsvötn.

Phase plane analysis

The solutions of (2.35) may be examined on the Q - N_c phase plane in the region $Q \geq 0$. We set $d/d\tau = 0$ in both equations to study their stationary point(s). If the lake-level is constant, then $dN_c/d\tau = 0$; $Q = \nu$ implies an exact balance between lake inflow and outflow. If $dQ/d\tau = 0$, we find $Q = N_c^{3/(\beta-1)} = N_c^{12}$ (since $\beta = 5/4$), which is the steady flow result of Röthlisberger (1972) (cf. Equation (2.32)). The two curves $Q = \nu$ and $Q = N_c^{12}$ intersect at the critical point $(\nu^{1/12}, \nu)$ and separate the phase plane into four regions. The phase trajectories have respectively infinite and zero gradients on these curves. In addition, they do not cross the N_c -axis because $Q_\tau \rightarrow 0$ as $Q \rightarrow 0$. There are no other asymptotic boundaries on this phase plane.

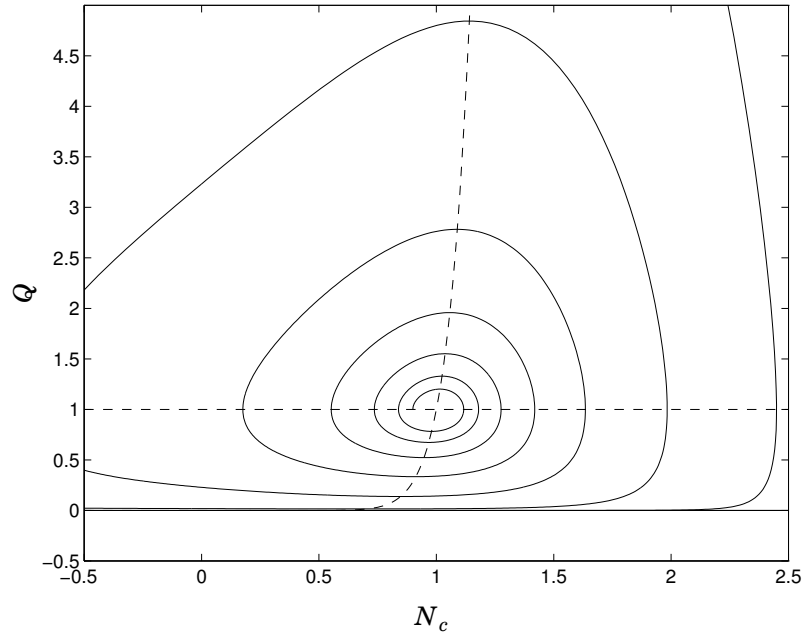


Figure 2.10: Phase plane solution of the equations in (2.35) with $\zeta = \nu = 1$. The dashed lines represent $Q = \nu$ (horizontal straight line) and $Q = N_c^{12}$ (curve).

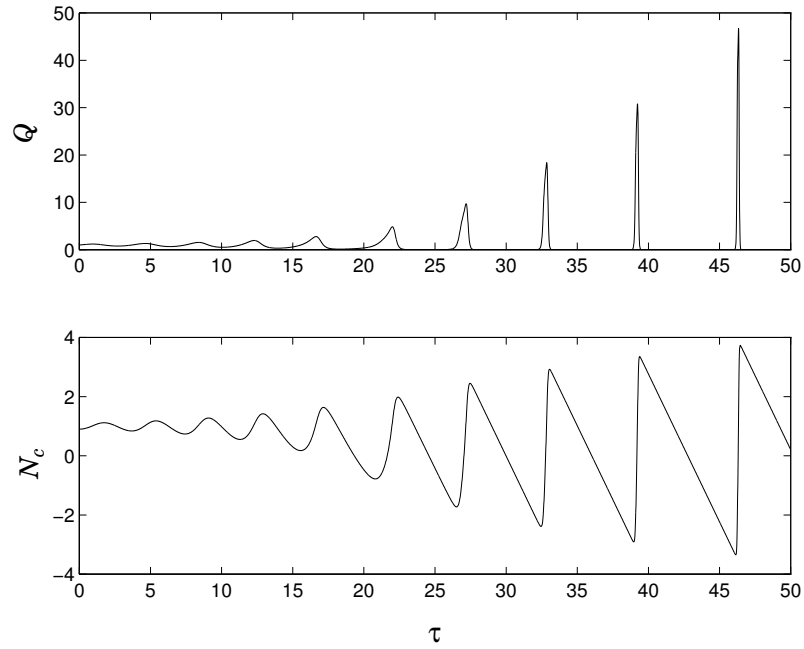


Figure 2.11: The time solutions corresponding to the phase trajectory in Fig. 2.10.

Close to $(\nu^{1/12}, \nu)$, the local linearized system have eigenvalues λ which are solutions of the characteristic equation

$$\lambda^2 - (\beta - 1)\nu^{\beta-1}\lambda + 3\zeta\nu^{\frac{2\beta+1}{3}} = 0. \quad (2.37)$$

The critical point is therefore always *unstable*, due to $\beta (= 5/4) > 1$, and it is a spiral node for

$$\zeta > \frac{\nu^{-2/3}}{192}. \quad (2.38)$$

When (2.38) is satisfied, the phase portrait is a diverging clockwise spiral and may be used to explain a succession of jökulhlaups. To demonstrate this, we have numerically integrated (2.35) by using the simultaneous Runge-Kutta method, with nominal parameters $\zeta = \nu = 1$ and initial values close to $(1, 1)$. The calculated phase trajectory and the corresponding time traces are shown in Figs. 2.10 and 2.11.

The model can also produce limit cycles or converging trajectories, if respectively $\beta = 1$ or $0 < \beta < 1$. As $\beta > 1$, we have $Q_\tau > 0$ and $Q_\tau < 0$ respectively above and below the Röthlisberger curve $Q = N_c^{3/(\beta-1)}$ (e.g. see Fig. 2.10), so steady flow is *unstable* against perturbations in Q if N_c (the lake-level) is held constant. This occurs for instance when $\zeta = 0$, which corresponds to an infinite lake area. We shall invoke this result in Section 2.4.2.

Inequality (2.38) implies that for a given ν , there is a critical lake area below which the solutions are *oscillatory*. In term of jökulhlaups, this defines the condition required for the repetition of flood events. Essentially, the water flow enlarges the channel by melting, but the corresponding drop in lake-level reduces the water pressure, which counteracts channel expansion. If A_L is small enough, this feedback is significant and leads to a seesaw effect in both Q and N_c . This property has been discussed by Nye (1976) in the context of intergranular vein-flow in glacier ice. He investigated a coupled system where $\beta = 3/2$.

Application to jökulhlaups

To simulate flood cycles, we choose the flux scale $Q_0 = 1000 \text{ m}^3 \text{ s}^{-1}$, and then $N_0 = 13.6 \text{ bar}$, $\tau_0 = 3.43 \text{ days}$, and $\zeta = 7.11 \times 10^{-2}$. Björnsson's (1988) estimate for the refilling rate at Grímsvötn is $6.6 \times 10^{11} \text{ kg a}^{-1}$, hence $m_L = 20.9 \text{ m}^3 \text{ s}^{-1}$, and $\nu = 2.09 \times 10^{-2}$. These values satisfy the condition in (2.38). We apply the initial conditions $N_c(0) = 6 \text{ bar}$ (from Björnsson's observation) and $Q(0) = 20.9 \text{ m}^3 \text{ s}^{-1}$ (a nominal value that is equal to m_L).

The simulated solution is shown in Fig. 2.12. The spiral is now severely distorted due to the size of ζ and ν . We have terminated the simulation before the end of the first flood cycle, because extended computing time is needed as Q becomes infinitesimal. The final part of the trajectory may be investigated by introducing the large-time approximations $N_c = O(1)$ and $Q \ll \nu \ll 1$. The model equations are then

$$\begin{aligned} dQ/d\tau &\approx -QN_c^3, \\ dN_c/d\tau &\approx -\zeta\nu, \end{aligned} \quad (2.39)$$

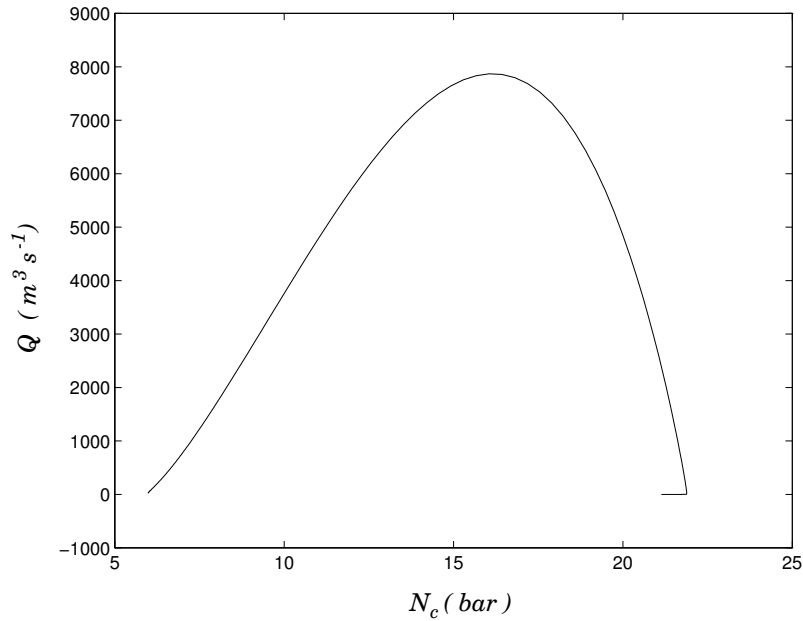


Figure 2.12: *Dimensional phase portrait for the simulated solutions of a Grímsvötn jökulhlaup.*

which have the analytic solutions

$$\begin{aligned} N_c &\approx -\zeta\nu\tau + \text{constant}, \\ \ln Q &\approx \frac{N_c^4}{4\zeta\nu} + \text{constant}. \end{aligned} \quad (2.40)$$

According to (2.40), both N_c and Q will continue to decrease at flood recession. However, Q will recover eventually when the trajectory crosses the curve $Q = N_c^{12}$, as in the trial model (Fig. 2.10). This crossing over signifies the beginning of the next flood cycle, but since $Q \ll 1$, this happens only when N_c has nearly vanished. Our reduced model therefore predicts a zero initiation effective pressure for the next flood, which is clearly incompatible with observations at Grímsvötn. (In fact, N_c will subsequently become *negative*). We describe an alternative approach to this problem in the next section.

2.4.2 Initiation mechanism

As one walks across the Skeiðarársandur, it is immediately apparent that a significant feature of the jökulhlaups is the enormous amount of sediments which they wash down. This sediment may originate from Grímsvötn entirely, but a more likely possibility is that it is (partly) due to the erosion of a subglacial till layer. If this is true, then a till layer may also be present at the seal region, and in this case, one would expect it to influence the way in which the drainage path is opened up at flood initiation.

The picture of the subglacial channel that we are building up involves sediment erosion as a fundamental constituent, and one which is as important as ice melting. As such, this is suggestive of the drainage theory proffered by Walder and Fowler (1994). As we described before, they suggested that two types of steady flow drainage path could occur: channels cut into basal ice, or canals cut into till sediments. At high ice surface slopes R channels provide the stable drainage type, while at lower slopes, canals would be the only possibility. Moreover, there was a critical effective channel pressure $\tilde{p}$ (estimated at 8 bar, but a value that would depend on sediment properties) such that $N_c < \tilde{p}$ for canals, and $N_c > \tilde{p}$ for R channels. In this section, we use this framework to suggest a mechanism whereby the seal can be broken at an effective pressure $N \approx 6$ to 8 bar, thus addressing the first of our conundrums.

Quiescent effective pressure distribution

At the end of a jökulhlaup, the effective pressure at the seal is quite high because the lake-level is low. When the channel is shut down, we suppose that drainage from the lake still operates, but at a vastly reduced level. At the least, we expect a slow percolation through the sediments at the base. When the water flux is of a reasonable size, such a flow would be unstable to channelization (cf. Walder, 1982), but for very low flow rates, the leakage may be of Darcy type. This is specifically the case in the seal region (Fig. 2.1b), where the surface is above the firn line, and there is little surface meltwater reaching the bed.

At lower altitudes however, we can expect surface melt penetration to lead to a larger base flow, and thus for channels to form. With this picture in mind, we can estimate the down-glacier distribution of the effective pressure $N(s)$, by assuming a region of Darcy flow at the seal (near the top end) and an R channel elsewhere (on Skeiðarárjökull).

Let us assume that the transition between these two regions occurs at 8 km from the lake. The lower part of the N -distribution is obtained by solving the dimensional form of Röthlisberger's Equation (2.31)₂, with the boundary condition $N_c = 0$ at the snout. To mimic the melt-water input, we prescribe a channel water flux Q that increases linearly with downstream distance s , from zero at the transition point to the typical observed value $100 \text{ m}^3 \text{ s}^{-1}$ at the ice margin (Björnsson, 1988).

For the seal region itself, we apply Darcy's law

$$\frac{Q_d}{A_d} = -\frac{k_T}{\mu_w} \frac{d}{ds}(p_w + \rho_w g z_b), \quad (2.41)$$

in which k_T is the till permeability, μ_w is the viscosity of water, and z_b is the bedrock topography of the Grímsvötn system. The upper part of the N -distribution is given by $N = p_i - p_w$, where p_w denotes the pore-water pressure in the till. As a first approximation, the percolation cross section A_d and water flux Q_d in the till may be taken to be stationary, whence (2.41) reduces to

$$\frac{d}{ds}(p_w + \rho_w g z_b) = \text{constant}. \quad (2.42)$$

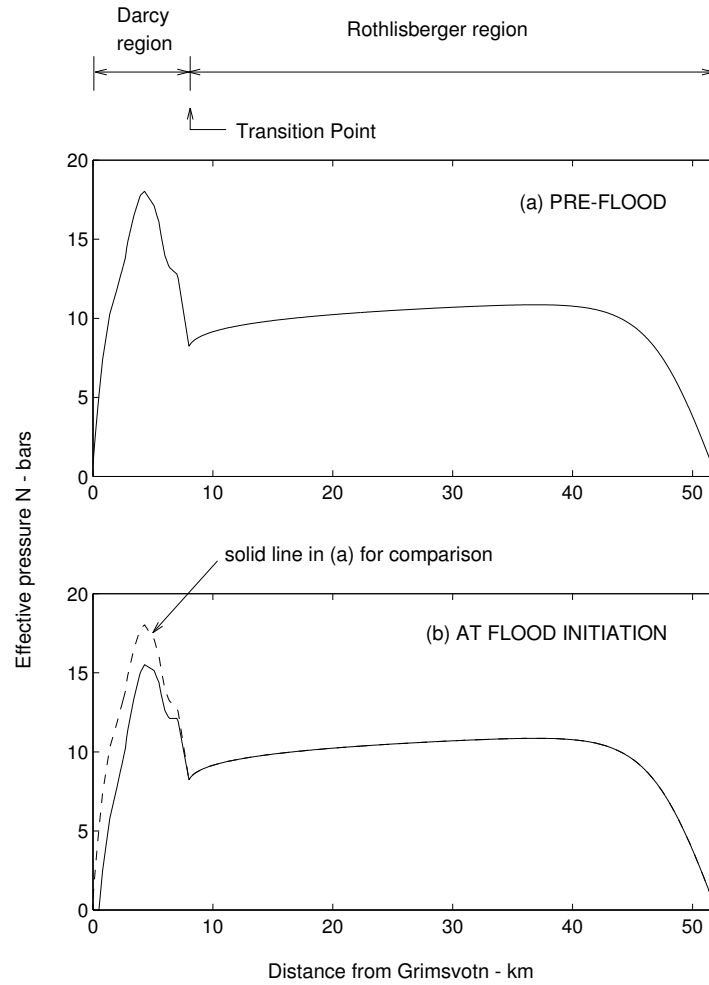


Figure 2.13: *Computed distributions of the effective pressure N via Darcy-Röthlisberger flow at (a) the pre-flood, low lake-level 1374.5 m, and (b) the typical flood initiation lake-level 1430 m.*

This equation requires *two* boundary values for p_w , namely, the hydrostatic lake-level water pressure, and the transition point channel water pressure (obtained via $p_c = p_i - N_c$). Note that the Darcy- and channel- parts of the calculation decouple because $Q_d \ll Q$.

Fig. 2.13 shows the calculated effective pressure distributions at pre-flood and flood initiation lake-levels. We see the typical Röthlisberger profile downstream, with N ($= N_c$) approximately constant and a boundary layer at the snout to adjust N down to zero (Section 2.3.5). In the seal region N has a maximum, and as the lake refills to the observed initiation level, the maximum is reduced to a value of approximately 15 bar. This critical value differs from Björnsson's (1988) estimate (6 to 8 bar) because he neglects Darcy leakage at the seal. (His calculation is based on hydrostatic equilibrium alone, for which the r.h.s. of (2.42) is zero.)

Breaking the seal

The distinction between the Darcy and Röthlisberger regions is not only one of water flux. The ice surface slope over the lower reaches is about 0.02, while at the seal it is much smaller (and is zero at the ice divide). According to Walder and Fowler (1994), the style of drainage depends strongly on the ice surface slope $\sin \alpha_s$. Their steady flow results are summarized in Fig. 2.14a,b.

Specifically, if $\sin \alpha_s$ is very low (e.g. at the seal), then Darcy flow is unstable for high enough water flux, and canals may exist with $N < \tilde{p}$; R channels are not feasible (see Fig. 2.14a,c). At higher slope (e.g. on the flank of Skeiðarárjökull), both R channels and canals are viable, and then R channels are preferred (Fig. 2.14b,c).

Our mechanism for breaking the seal is then, in a crude way, the following: in between floods, Darcy flow at the seal has $\max N > \tilde{p}$ (Fig. 2.14a, point A), and no transition to a canal is possible, while lower down, the surface slope is higher, so R channel drainage can take place (Fig. 2.14b, point B). (This is consistent with our assumption used in the effective pressure calculation; see Fig. 2.13.) However, as the lake refills, the N -maximum in the seal region drops until when $N = \tilde{p}$, a canal is viable, and we envisage a transition to a canal system at the seal (point A $\rightarrow$ A'). This transition effectively acts like opening a tap, and the flood proceeds. Fig. 2.13 indicates that $\tilde{p} \approx 15$ bar.

Our model is rather discursive at this stage. It relies on the heavily theoretical Walder-Fowler description, which itself only describes the steady state, and it cannot reliably describe the spatial variation of the evolving flow. We suppose that the transition from percolation to canal flow would involve erosion of sediments at the seal, in much the same way as ‘piping’ occurs in a porous soil (see Jones, 1981). Conjecturally, the Darcy-Röthlisberger transition point would become unstable and ‘eat’ its way upstream towards the lake. This would happen at a critical value of N in the region of $\tilde{p}$. Once a canal is initiated, the water flux Q is initially too small to cause any change in the lake-level, hence N is held fixed. According to Section 2.4.1, this condition may then cause Q to increase in an unstable fashion, initiating the flood (Fig. 2.14a, from point A' onwards).

The critical effective pressure

Walder and Fowler (1994) derived theoretically the equation

$$\tilde{p} = \left[\frac{\rho_s K_s l_s^2 A_T n^n}{\rho_i K_i l_i^2 A_I a^{a-b}} \right]^{1/(n+b-a)}, \quad (2.43)$$

in which ρ_i and ρ_s are the densities of ice and sediment, K_s and K_i are $O(1)$ shape factors, $l_s \approx l_i$ are characteristic lengths, and A_I , A_T , a , b and n ($= 3$) are rheology constants for ice and for subglacial till (see Equation 4.4, Walder and Fowler, 1994). They used $a = 1.33$, $b = 1.8$ and $A_T = 3 \times 10^{-5} \text{ Pa}^{b-a} \text{ s}^{-1}$, obtained for till beneath Breidamerkurjökull, an outlet glacier of Vatnajökull (Boulton and Hindmarsh, 1987). This leads to $\tilde{p} \approx 8$ bar.

Equation (2.43) shows that $\tilde{p}$ is not sensitively dependent on the deformability of the till A_T , but A_T does in fact vary over a range of several orders of magnitude

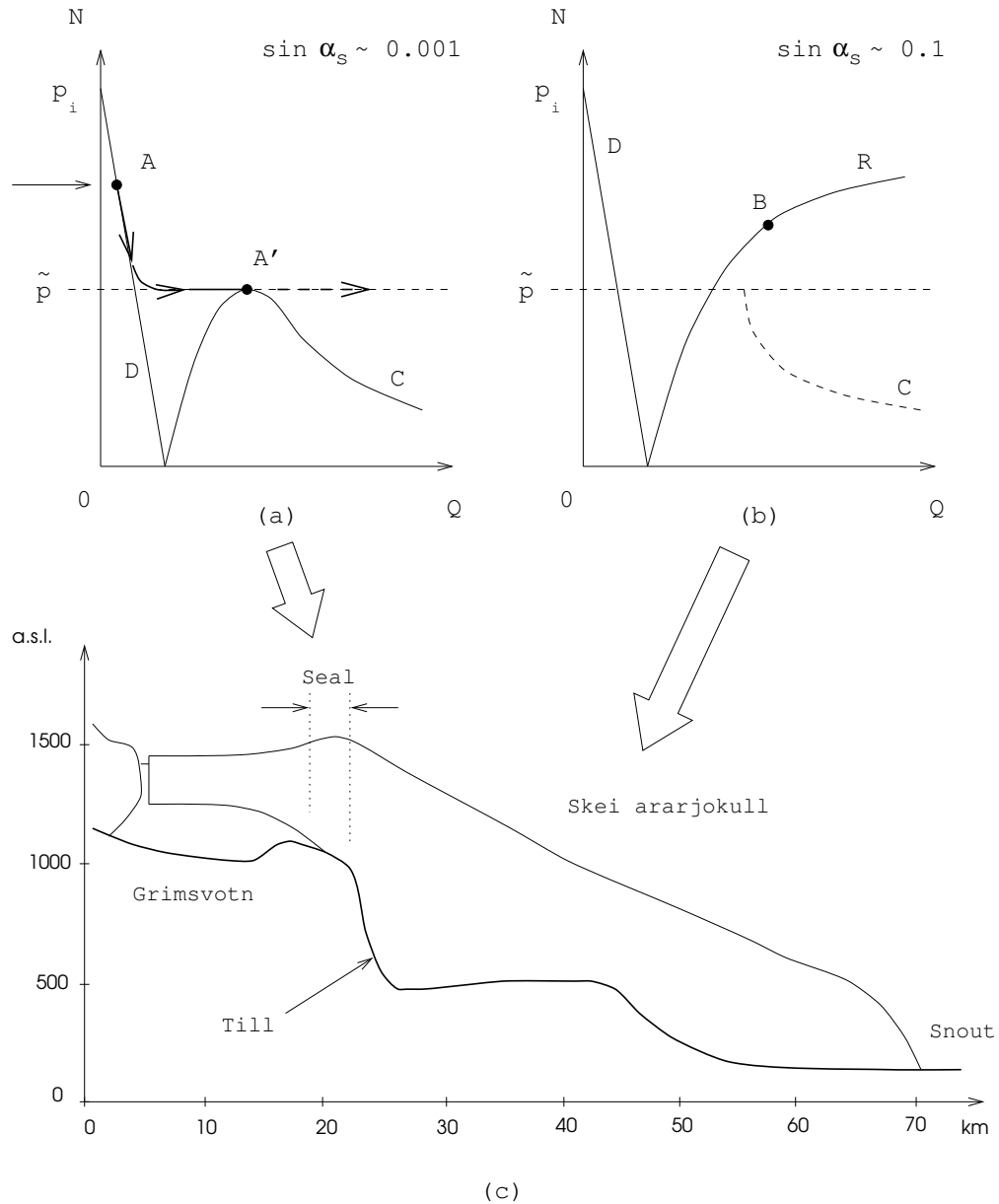


Figure 2.14: Our seal breaking mechanism as shown on the N - Q diagrams at (a) low surface slopes, and (b) high surface slopes. These describe the characteristics of subglacial channels respectively at the seal and under Skeiðarjökull (c). Labels D , R , and C refer to steady flow solution branches for Darcy, Röthlisberger and canal drainage styles. (The R -branch is equivalent to the Röthlisberger curve in Fig. 2.10.) Adapted from Fig. 4; Walder and Fowler (1994).

(Chapter 5). In particular, Boulton and Hindmarsh's (1987) value is generally thought to be rather low. By increasing A_T by a factor of 10, we obtain a more typical value, consistent with that obtained for Ice Stream B (Alley *et al.*, 1986); then $\tilde{p} \approx 16$ bar. Thus, our inferred value from Fig. 2.13 (≈ 15 bar) is not unreasonable.

Finally, if the till is very much non-deformable ($A_T \rightarrow 0$), then we have essentially a hard bed situation, which does not permit canal drainage. In this case, Equation (2.43) predicts flood initiation at the flotation lake-level, with $\tilde{p} = 0$. This has been observed for a number of jökulhlaups (e.g. Clarke, 1982; Sturm and Benson, 1985), and points toward further investigation of the effect of subglacial till in ice-dammed lake systems.

2.5 A wide channel model

Hitherto, we have the notion that the sediment till is an important component in the mechanics of Grímsvötn jökulhlaups, and also we have given a tentative explanation of the problems of Nye-type models, in terms of the subglacial channel geometry. It seems clear that the flood model should address both of these issues in a fundamental way.

The total estimated suspended sediment yield from the 1972 jökulhlaup is 2.4×10^{10} kg (Tómasson *et al.*, 1980). Taking a typical sediment density 2650 kg m^{-3} , this corresponds to an average area of 180 m^2 along the 50 km long subglacial drainage pathway. If this quantity is eroded during the flood, the observed figure strongly suggests that the flood channel is very wide, rather than circular, as envisioned in the Nye theory. For instance, an excavated till depth of two metres would imply a channel width of approximately 100 m. This channel cannot possibly be semi-circular, because its cross sectional area would greatly exceed that predicted by the Nye model (about 10^3 m^2 at peak discharge).

Having raised the concept of wide channels, we may point out that there is in fact no *a priori* reason to assume circular or semi-circular channels, and in principle one ought to determine depth and width separately. In this section we provide one such model. If the channel is wide, then we can expect its effective roughness to be large, which could explain the anomalous value of n' determined by Nye (1976).

Suppose now we imagine the passage of a flood down a wide subglacial channel, excavating a till layer one or two metres deep. As the water-saturated till sediments are eroded from the sides, lateral expansion of the channel will enhance the creep closure of the ice 'roof', and we suggest (and will show) that this accelerated closure is capable of causing a sudden switch from erosion and melting on the rising limb of the hydrograph to shutdown on the falling limb. In this sudden switch, the width of the channel is crucial.

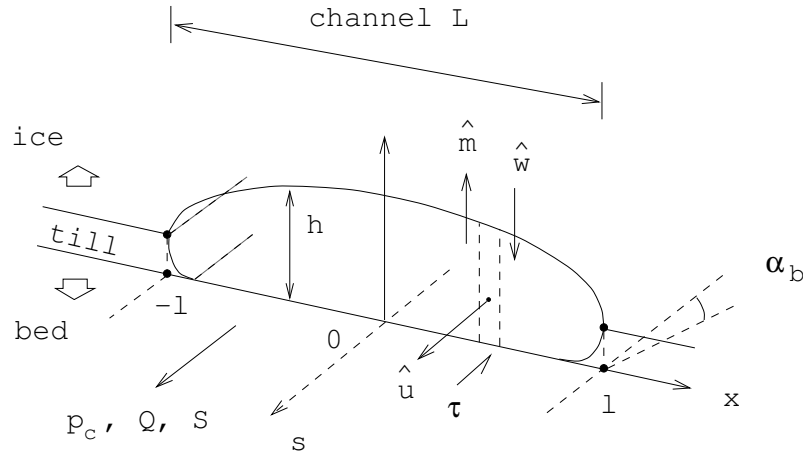


Figure 2.15: *Schematic diagram of the wide channel showing the mathematical symbols used in our formulation. The four thermo-mechanical channel processes of interest here are vertical ice melting and closure, and lateral erosion and creep inflow of till sediments.*

2.5.1 Governing equations

Our wide channel formulation is illustrated by the schematic diagram shown in Fig. 2.15. From our preceding analysis of the Nye model (Section 2.3.5), we suppose that the downstream variation of Q will be negligible, and hence that drainage during jökulhlaups can be ascertained by analyzing the opening and closure of the channel independently of downstream distance s . Also, the boundary layer behaviour at the snout may be neglected. (This is analogous to discarding δ and ϵ in Section 2.4.1.)

Let x be the cross-stream coordinate, and let $h(x, t)$ be the channel depth. The channel is defined to be in $-l \leq x \leq l$, and in general $l = l(t)$. In the Nye model, we effectively (and arbitrarily) assume $l = h$. The relaxation of this assumption requires an extra relation to describe the width $l(t)$.

Our model is based on the limit $h \ll l$. If $h \sim l$, then we regain a Nye-type model. If $h \ll l$, the ice roof and the sediment bed are approximately parallel to each other, then a local balance (in x) leads to the dimensional equations

$$\begin{aligned} \frac{\partial h}{\partial t} &= \frac{\hat{m}}{\rho_i} - \hat{w}, \\ h\Phi &= \hat{\tau}, \\ \hat{\tau} &= \frac{1}{8}f^*\rho_w\hat{u}^2, \\ h\hat{u}\Phi &= \hat{m}L, \end{aligned} \tag{2.44}$$

where $\hat{m}$ is the (mass) melt-rate of ice per unit area, $\hat{w}$ is the roof closure velocity, $\hat{u}$ is the (depth-averaged) water flow velocity, $\hat{\tau}$ is the shear stress at the walls (given

via the friction factor f^*), and Φ is the total driving gradient as before, given by

$$\Phi = \rho_i g \sin \alpha_s + (\rho_w - \rho_i) g \sin \alpha_b; \quad (2.45)$$

cf. Equation (2.18). Superscripts $\hat{\cdot}$ denote *local* variables which are functions of both x and t .

The equations in (2.44) respectively describe depth evolution for h , momentum conservation, a friction parameterization, and energy conservation. The combination (2.44)₂ and (2.44)₃ provides an (local) alternative to the Manning equation, based on the Darcy-Weisbach law used by Walder and Fowler (1994). More specifically, $\hat{\tau}$ represents the total shear stress at the roof and at the bed; hence, one can think of f^* as an extension of the friction factor f used in river mechanics, and thus $f^* \approx 2f$.

For completeness, we can write down also a mass conservation equation, which takes the global integrated form

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = M + \frac{1}{\rho_w} \int_L \hat{m} \, dx, \quad (2.46)$$

where L is the interval $(-l, l)$, and M is a prescribed surface-derived melt-water supply rate (equivalent to Ω/ρ_w in the Nye model; Equation (2.16)). However, this equation is mainly relevant for describing downstream variations, and is omitted in the current model. The channel cross section S and water flux Q are given by

$$S = \int_L h \, dx, \quad Q = \int_L h \hat{u} \, dx. \quad (2.47)$$

Note that a point form of (2.46) is possible, but that it requires a prescription of any lateral transport which occurs within the channel.

The depth evolution equation

In (2.44) and (2.45), there are five equations for the six unknowns h , $\hat{m}$, $\hat{w}$, $\hat{\tau}$, $\hat{u}$ and Φ . The missing equation is provided by the *closure relation* between $\hat{w}$ and N_c ($= p_i - p_c$). With a non-linear ice rheology, explicit derivation of the closure rate is not easily possible, but we can adopt the methods of linear elasticity as used for crack problems by England (1971), for example, to show that for constant viscosity ice, $\hat{w}$ is determined by inverting

$$N_c = -\frac{2\eta_i}{\pi} \oint_L \frac{\partial \hat{w}}{\partial \zeta} \frac{d\zeta}{\zeta - x}, \quad (2.48)$$

where η_i is the ice viscosity, and $\oint$ denotes a principal value integral. If N_c is independent of x , then

$$\hat{w} = \frac{N_c}{2\eta_i} \sqrt{l^2 - x^2}. \quad (2.49)$$

The derivation of this result and justification of a constant cross-channel effective pressure are given in Chapters 3 and 4.

By analogy to the closure of circular ice channels, we now extend (2.49) and allow a non-linear flow law of Glen type by putting $\eta_i^{-1} = N_c^2 \eta_0^{-1}$, where η_0 is a viscosity parameter. (We assume $1/(2\eta_0) \approx K_0$, where K_0 is Nye's closure constant.) From (2.44), we then obtain

$$\frac{\partial h}{\partial t} = \frac{(h\Phi)^{3/2}}{\rho_i L} \sqrt{\frac{8}{f^* \rho_w}} - \frac{N_c^3}{2\eta_0} \sqrt{l^2 - x^2}, \quad (2.50)$$

and as before, this has to be coupled to a lake inlet boundary condition relating N_c and Q . However, we see that $l(t)$ is still undetermined, so we have to prescribe a separate width evolution equation. This is the main difference between our model and the Nye model.

A parameterized model

Equation (2.50) is not completely satisfactory, as it precludes lateral sediment transport, which may be essential for determining the channel depth consistently (e.g. see Parker, 1978). This problem is discussed in detail in Chapter 6 and Section 7.4, but for the moment, we circumvent it by adopting a 'lumped' parameter version of (2.50), which however maintains the independence of $l(t)$. We simply write

$$\frac{dh}{dt} = \frac{\Phi^{3/2}}{\rho_i L} \sqrt{\frac{8}{f^* \rho_w}} h^{3/2} - \frac{N_c^3}{2\eta_0} l, \quad (2.51)$$

where we omit the x -dependence of the melting and closure terms, such that all our variables are now functions of time only. Supplementing this is the lake continuity equation that was described in Sections 2.3.5 and 2.4.1:

$$\frac{dN_c}{dt} = \frac{\rho_w g}{A_L} (Q - m_L), \quad (2.52)$$

where m_L is the lake refilling rate, and A_L its effective area.

It remains to prescribe the half-width $l(t)$. We suppose that bank closure occurs via inward creep of the till, while erosion outwards occurs via bank failure and sediment uptake as suspended load in the flood water. We seek the rate of bank erosion, for which there are no ready empirical relations. Here, a plausible assumption is a direct dependence of the erosion rate on the stream power (i.e. a shear stress $\times$ flow velocity product), thus we propose

$$\frac{dl}{dt} \approx \frac{K_l}{\rho_s} \bar{u}^3, \quad (2.53)$$

where K_l is an erosion parameter, ρ_s is the sediment density, and $\bar{u} = Q/S$. The form of this equation resembles the Meyer-Peter and Müller (1948) relation for bedload transport, and is justified in Chapter 7 (see Section 7.4.2). From Equations (2.44) and (2.47), we find

$$\begin{aligned} S &= 2lh, \\ Q &= 2lh^{3/2} \sqrt{\frac{8\Phi}{f^* \rho_w}}, \end{aligned} \quad (2.54)$$

so the erosional part of dl/dt is given by

$$\frac{dl}{dt} \approx \frac{K_l}{\rho_s} \left(\frac{8\Phi}{f^* \rho_w} \right)^{3/2} h^{3/2}. \quad (2.55)$$

The inward creep of till may be inferred from related work by Fowler and Walder (1993). Most simply, the creep velocity is $dl/dt \approx -lN_c/\eta_T$, where η_T is the till viscosity. But if we adopt instead Boulton and Hindmarsh's (1987) rheology law (1.8), then following Fowler and Walder (1993), we propose that the non-linear response is

$$\frac{dl}{dt} \approx -A_T \frac{N_c^a}{N_\infty^b} l \quad (2.56)$$

(see also Section 7.2). The constants A_T , a and b have been given in Section 1.4. In Equation (2.56), N_∞ is the effective pore-water pressure of the till at the far field, and in a jökulhlaup, we would expect $N_\infty \approx \text{constant}$, and equal to its pre-flood value. Our model for the evolution of l is then

$$\frac{dl}{dt} = \frac{K_l}{\rho_s} \left(\frac{8\Phi}{f^* \rho_w} \right)^{3/2} h^{3/2} - \frac{A_T}{N_\infty^b} l N_c^a. \quad (2.57)$$

Finally, the parameterization of the depth h facilitates the calculation of a bulk channel roughness for comparison with the Nye model. The appropriate conversion formula is

$$n' = R_H^{1/6} \left(\frac{f}{8g} \right)^{1/2}, \quad (2.58)$$

where R_H (the hydraulic radius) $\approx h/2$ for the present case. Since $f \approx f^*/2$, the effective Manning roughness of the wide channel is

$$n' \approx \sqrt{\frac{f^* h^{1/3}}{2^{13/3} g}}. \quad (2.59)$$

Alternatively, we can derive this equation directly from (2.4), (2.5) and (2.44).

Non-dimensionalization

We non-dimensionalize Equations (2.51), (2.52), (2.54)₂ and (2.57) by writing the variables h , N_c , l , t , Q , and Φ in terms of the scales h_0 , N_0 , l_0 , t_0 , Q_0 , Φ_0 , where

$$\begin{aligned} \frac{l_0}{t_0} &= \frac{K_l}{\rho_s} \left(\frac{8\Phi_0}{f^* \rho_w} \right)^{3/2} h_0^{3/2}, & \frac{N_0}{t_0} &= \frac{\rho_w g Q_0}{A_L}, \\ \frac{h_0}{t_0} &= \frac{\Phi_0^{3/2}}{\rho_i L} \left(\frac{8}{f^* \rho_w} \right)^{1/2} h_0^{3/2} = \frac{N_0^3 l_0}{2\eta_0}, & Q_0 &= l_0 h_0^{3/2} \left(\frac{8\Phi_0}{f^* \rho_w} \right)^{1/2}, \end{aligned} \quad (2.60)$$

taking $\Phi_0 = 196 \text{ kg m}^{-2} \text{ s}^{-2}$ for Grímsvötn. The dimensionless model is then

$$\begin{aligned} \frac{dh}{dt} &= \Phi^{3/2} h^{3/2} - l N_c^3, \\ \frac{dN_c}{dt} &= Q - \lambda = 2l \Phi^{1/2} h^{3/2} - \lambda, \\ \frac{dl}{dt} &= \Phi^{3/2} h^{3/2} - \mu l N_c^a, \end{aligned} \quad (2.61)$$

where $\Phi = 1$, and

$$\lambda = \frac{m_L}{Q_0}, \quad \mu = \frac{A_T}{N_\infty^b} t_0 N_0^a. \quad (2.62)$$

In addition to the constants in Table 2.1, we use $\rho_s = 2650 \text{ kg m}^{-3}$, $\eta_0 = 5 \times 10^{23} \text{ Pa}^3 \text{ s}$, and $A_T = 3 \times 10^{-5} \text{ Pa}^{b-a} \text{ s}^{-1}$, $a = 1.33$, $b = 1.8$ (Boulton and Hindmarsh, 1987). Estimates for Grímsvötn are A_L (assumed constant) = 30 km^2 , $N_\infty = 6 \text{ bar}$ (see footnote⁶), and $m_L = 20.9 \text{ m}^3 \text{ s}^{-1}$. We also prescribe a typical friction factor $f = 0.05$ from river flows (Richards, 1982), therefore $f^* = 0.1$.

Taking a nominal erosion constant $K_l = 10^{-3} \text{ kg m}^{-5} \text{ s}^2$, the equations in (2.60) give

$$\begin{aligned} h_0 &= 15.1 \text{ m}, & N_0 &= 10.4 \text{ bar}, & l_0 &= 137 \text{ m}, \\ t_0 &= 1.00 \times 10^5 \text{ s (1.2 days)}, & Q_0 &= 3.19 \times 10^4 \text{ m}^3 \text{ s}^{-1}, \end{aligned} \quad (2.63)$$

and then the parameters are

$$\lambda \approx 10^{-3}, \quad \mu \approx 0.01. \quad (2.64)$$

The approximate value of K_l used here has been selected by the constraint imposed by the observed sediment budget (see later). Its order-of-magnitude size is also confirmed in Section 7.4.2.

Given the initial conditions $h(0)$, $N_c(0)$ and $l(0)$, the equations in (2.61) may be numerically integrated by the simultaneous Runge-Kutta method. Since $\mu \ll 1$, we can expect that widening of the channel by lateral erosion, described by $(2.61)_3$, will accelerate the closure term in $(2.61)_1$ near the maximum flood stage. This behaviour can be examined at leading order accuracy. On the rising limb of a hydrograph, melting dominates closure in the channel depth description, so Equation $(2.61)_1$ gives $h \sim (-t)^{-2}$. By using the other two equations, we now obtain $dl/dt \sim (-t)^{-3}$, $l \sim (-t)^{-2}$, and $N_c \sim (-t)^{-4}$. It follows that $lN_c^3 \sim (-t)^{-14}$, so closure will overtake melting very suddenly, causing a sudden switch to flood recession.

2.5.2 Flood simulation

In Figs. 2.16 and 2.17, we show the results of computing a flood hydrograph using the model in (2.61), where we have chosen the parameter K_l and a combination of $l(0)$ and $N_c(0)$ to provide good fits to the rising limb *and* the peak discharge of the 1972 Grímsvötn jökulhlaup. An initial channel depth $h(0) = 1 \text{ m}$ is specified. We obtain the values $N_c(0) = 7 \text{ bar}$, $l(0) = 6.42 \text{ m}$, and $K_l = 1.6 \times 10^{-4} \text{ kg m}^{-5} \text{ s}^2$. The last of these corresponds to a channel widening rate of approximately 10 metres per day, which is plausible (see Fig. 2.17a). The simulated solutions are found to be extremely

⁶In Section 2.4.2 we find that the seal value of N may be quite high ($\approx 15 \text{ bar}$). Here we use the lower, static estimate of Björnsson (6 bar, 1988) to maximize the effect of creep. Model simulation in Section 2.5.2 shows that the resulting creep term is still negligible.

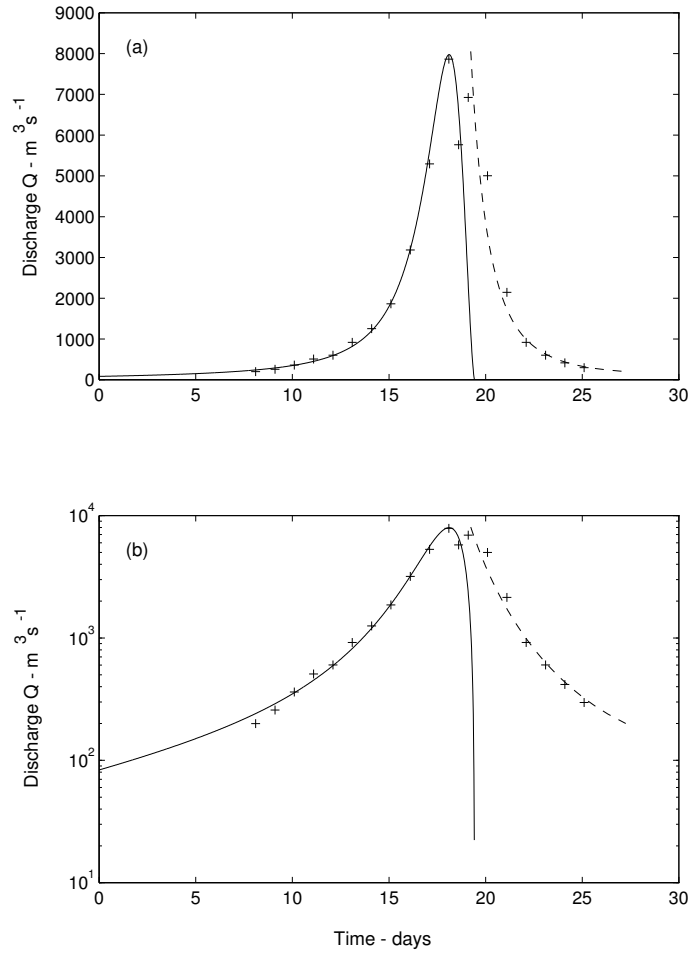


Figure 2.16: A simulated hydrograph (solid line) together with hydrographic data points (crosses; from Rist, 1973) of the 1972 Grímsvötn jökulhlaup: (a) linear discharge scale; (b) logarithmic discharge scale. The dashed lines are the simulated recession solution of Equation (2.67).

sensitive to the initial conditions. This is caused by rapid divergence of the phase trajectories (for h , N_c , l), as has been observed in the Nye model (Section 2.4.1).

We see from Fig. 2.16 that the model is able to simulate the hydrograph quite well, and in particular the sharp cut-off is mimicked. This is due to accelerated closure of the rapidly widening channel. The effective Manning roughness calculated from Equation (2.59) takes an average value of about $0.03 \text{ m}^{-1/3} \text{ s}$ (Fig. 2.18), more normal than the value inferred from Nye's (1976) work. (n' decays to zero at the end because the channel depth vanishes. This is purely an artefact of the conversion in (2.59)).

During the flood, the channel aspect ratio remains high ($2l/h \approx 10$, from Fig. 2.17a,b), therefore the wide channel assumptions used in formulating our model are valid. The predicted lake-level drop is approximately 70 m, given by the change in

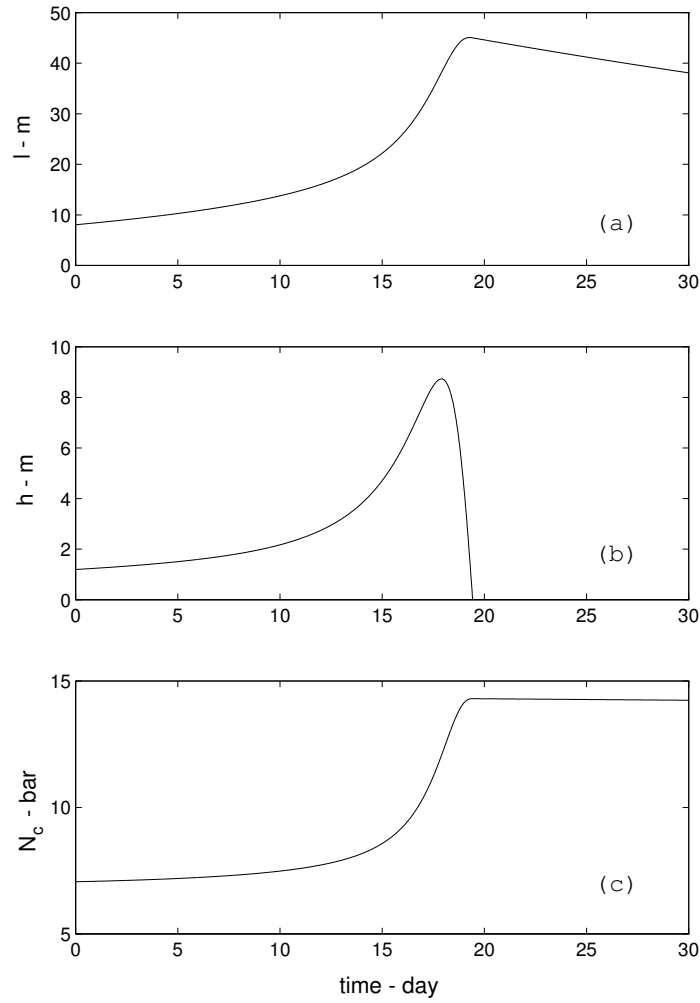


Figure 2.17: Results of (a) channel half-width l , (b) channel depth h and (c) effective pressure N_c from the flood simulation shown in Fig. 2.16.

$N_c/\rho_w g$ shown in Fig. 2.17c. This is lower than the observed value (≈ 100 m) but is probably due to the over-steep simulated recession limb.

Finally, the simulation indicates a maximum channel width of about 90 m. If a till layer 2 metres thick is eroded in the flood, then we arrive at the estimated figure of 180 m^3 of sediments eroded per unit length of channel. We give further discussion of these results in Section 2.6.

The recession limb

It will be seen that we have improved the maximum discharge estimate at the expense of producing a very sharp fall off in water flux. A first attempt to rectify this is to reincorporate the spatial dependence of the falling discharge. By analogy with Equation (2.21)₂, the mass conservation law corresponding to (2.46) can be written

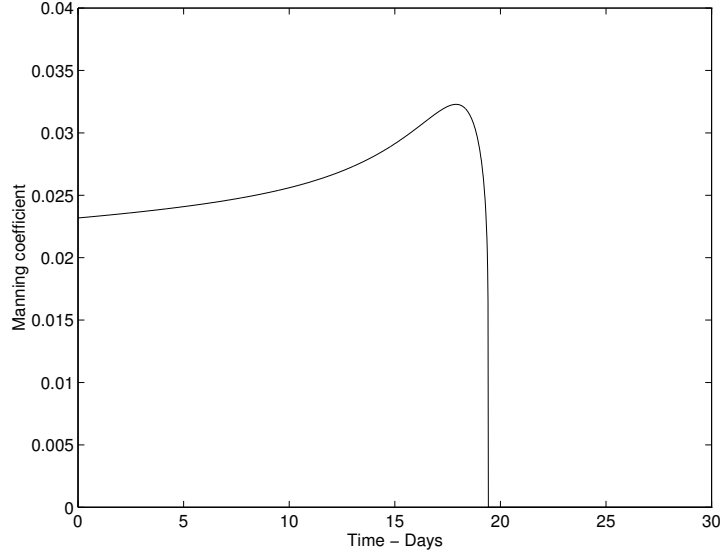


Figure 2.18: The Manning roughness coefficient n' (in $m^{-1/3} s$) of the flood channel calculated from Fig. 2.17.

dimensionlessly as

$$\epsilon \frac{\partial(lh)}{\partial t} + \frac{\partial(lh^{3/2})}{\partial s} = \epsilon r l h^{3/2} + M, \quad (0 \leq s \leq 1), \quad (2.65)$$

in which the parameter ϵ is defined by Equation (2.22)₁. The surface melt run-off is estimated at $M \approx 2 \times 10^{-3}$ (the scale here is $2Q_0/s_0$), corresponding to a $100 \text{ m}^3 \text{ s}^{-1}$ base flow at the terminus. The point here is that, even if the inlet flow shuts off instantaneously, an expansion or rarefaction wave may propagate down the channel, so that h (and hence the water flux) will shut off more gradually. Indeed, we can derive an explicit solution based on the method of characteristics for this decay (Carrier and Pearson, 1988).

At flood termination, l changes little, so the shutdown in h is described by (ignoring the channel melting term)

$$\epsilon \frac{\partial h}{\partial t} + \frac{\partial(h^{3/2})}{\partial s} = \overline{M}, \quad (2.66)$$

where $\overline{M} = M/l \lesssim 10^{-2}$ (since $l = O(1)$). Suppose that $h = 1$ for $s > 0$ at $t = 0$, and then $h = 0$ for $s = 0$ at $t > 0$ — this corresponds essentially to the result in Fig. 2.17b after peak discharge. The solution at the outlet $s = 1$ is then $h = 1 + (\overline{M}t/\epsilon)$ for $0 < t \lesssim 2\epsilon/3$, when the expansion wave reaches the outlet, after which

$$\overline{M} = h^{3/2} - \left(h - \frac{\overline{M}t}{\epsilon}\right)^{3/2}, \quad (2.67)$$

until $t = \epsilon/\overline{M}^{1/3}$, when the base flow with $h = \overline{M}^{2/3}$ resumes.

In Fig. 2.16, we have simulated the descending limb of the hydrograph using Equation (2.67), with $\overline{M} = 2.5 \times 10^{-3}$ and $\epsilon = 2$. Although this data fit is satisfactory, the value of ϵ required is abnormally high (cf. Equation (2.24)), and can only be achieved if the flood channel has an effective tortuosity factor of 60.

Clearly, the observed recession cannot be explained by an expansion wave alone. A possible alternative is that the flood channel does not close up simultaneously along its whole width, as predicted by our ‘lumped’ model, but instead it breaks up (progressively) into smaller drainage filaments which collapse at a lower rate. This may happen because the closure velocity $\hat{w}$ is itself a non-uniform function of x (see Equation (2.49)). In addition, we can expect a substantial amount of sediment deposition as the water flux decreases. This process is likely to be accompanied by an effect known as *braiding*, commonly observed in river channels, leading to a distributed stream flow pattern (e.g. Richards, 1982). However, the precise description of these instabilities will require a more complicated model that can cope with lateral variations in the channel.

2.6 Discussion

In this chapter, we have given a detailed mathematical investigation of the mechanics of jökulhlaups, based on the classical theory proposed by Nye (1976). Essentially, his model can be simplified to allow its simulation as a set of coupled differential equations, but it cannot explain the sharp turn-around observed in the 1972 Grímsvötn flood hydrograph. Also, his theory lacks a satisfactory mechanism of flood initiation.

In order to address these problems, we consider how the presence of subglacial sediments might have a crucial rôle in determining the observed flood behaviour. The triggering mechanism at Grímsvötn is known to operate when the lake pressure is some 6 to 8 bars below the maximum ice overburden pressure. To explain this, we invoke the fledgling theory of basal drainage through a till layer, given by Walder and Fowler (1994), and suggest that Darcy flow in the seal region becomes unstable when the effective pressure is low enough, leading to canals and subsequent flood conditions. While this is plausible, the transition process itself (which concerns how incipient channels are formed) is not well understood, and awaits further investigation.

Next, we provide a revised simulation model based on the interaction of a till layer with the evolving flood channel. If the restriction of a circular cross section is removed, then separate equations are required for the channel depth h and half-width l . We propose an ice closure equation for the depth, and suppose that width is governed by erosion and inward creep of sediments. As a result, we are able to produce an improved simulated hydrograph, with a reasonable value of Manning roughness. In particular, our model can describe both the magnitude and timing of the peak discharge.

The argument for a high aspect ratio flood channel relies on the hypothesis of an extensive till layer, with a thickness of the order of one metre. Boulton and

Hindmarsh's (1987) till at Breidamerkurjökull is roughly one metre deep, and similar values are observed elsewhere along the margin of Vatnajökull, but there has been no direct evidence for the existence of such deposit in the jökulhlaup flood path. Seismic survey similar to that conducted on Ice Stream B (Blankenship *et al.*, 1986) or borehole investigations would help to resolve this issue.

Given this limitation, our simulation results indicate that the total suspended sediment yield can be modelled. This is encouraging not so much regarding prediction (since the actual till thickness is still unknown), but in the sense that given a realistic range of till thicknesses (e.g. 1 to 10 m), the observed yield falls within the simulated yield range. Here, the implicit assumption is that most of the sediment originates from till erosion, and not from Grímsvötn itself. Again, further field investigations are necessary for justifying this.

If the till is eroded during jökulhlaups, there is the question of whether a requisite amount of sediments needs to be replenished in the period between two successive events. We suppose that this requirement is not critical, because the floods are initiated at the weakest point of the seal, where erodible sediment is available. Therefore, it is also unlikely that the flood channel can be confined entirely to a sediment-free region, and avoid expanding into neighbouring soft bed areas. On the other hand, the seal region does have a finite width (of roughly 2 km; Björnsson, 1988), so there is the possibility of sediment exhaustion if flood events are unusually frequent. In this case, our theory would predict an initiation lake-level that is closer to the flotation figure.

In Section 2.5, the governing equations that describe the wide subglacial channel has been introduced, but mainly from a physical perspective. In Part II, we examine a more rigorous mathematical description of the ice and sediment processes of this channel, with the aim of extending the soft bed drainage theory initiated by Walder and Fowler (1994). Although the emphasis there is on steady flow, we shall provide also the validation of our flood model.

Part II

Drainage Theory of Wide Subglacial Channels

Plate 3 goes here.

Chapter 3

Ice closure

3.1 Introduction

We consider the situation shown in Fig. 3.1, where a wide channel exists at the ice-till interface, in general with parts cut into both the ice and the sediment. The channel conducts a water flow, at a pressure that generally differs from the ice overburden pressure (and thus also the far field till pressure). In particular we expect $N_c (= p_i - p_c) > 0$, so the channel tends to close by ice deformation, but this is opposed by melt-back due to turbulent flow dissipation. Similarly, it is recognized that wet till also creeps, and in this case the channel floor may be maintained by erosion and fluvial transport of sediment (Walder and Fowler, 1994).

Our model is predicated on the effectiveness of these processes in governing channel evolution. Following Section 2.5, we adopt an essentially two-dimensional description, with the channel cross section occupying the x - y plane. Separate equations can be constructed for the total depth $h(x, t)$ and half-width $l(t)$, since $h \ll l$. Here t denotes time, $h(\pm l) = 0$, and for convenience we assume a symmetry in $-l \leq x \leq l$ by asserting $h(x, t) = h(-x, t)$. Downstream variations (in s) will be reincorporated later.

For the special case of a hard bed (Fig. 3.2a), the depth evolution equation is

$$\frac{\partial h}{\partial t} = \frac{\hat{m}}{\rho_i} - \hat{w}, \quad (3.1)$$

where $\hat{m}(x, t)$ and $\hat{w}(x, t)$ are respectively the melt-rate and the closure velocity of the ice as before (see Equation (2.44)₁). An associate evolution equation for l can be derived, based on the condition that $h(\pm l) = 0$. The determination of $\hat{w}$ and $\hat{m}$ is the subject of the current chapter and Chapter 4.

Note that Equation (3.1) is also valid if the channel is sediment-floored, but only in describing the ice incision. In this case, an additional depth equation for the bed is required, and we suppose that a change in l would be due to the combined effect of the ice and the till (such as in Section 2.5). This extension is discussed in Chapters 5 to 7, but for the moment the basic description in (3.1) is sufficient.

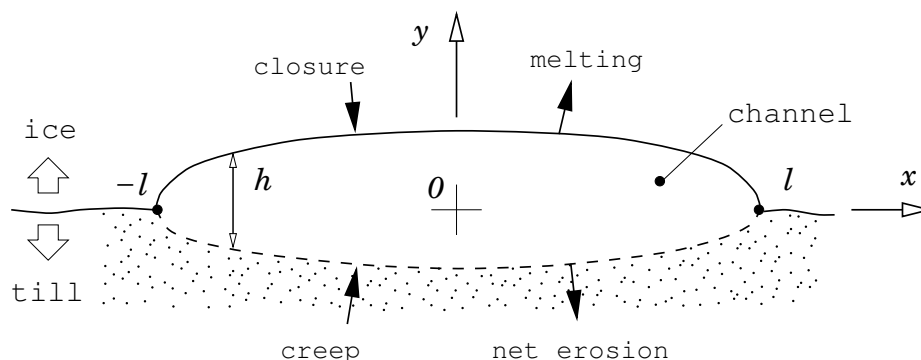


Figure 3.1: *An idealized subglacial wide channel at the ice-till interface, and the processes which govern its evolution.*

The ice closure problem

In this chapter, we seek the closure relation between $\hat{w}$ and N_c (the effective channel pressure), where N_c is a given function of x (and t). With $h \ll l$, an approximate solution is obtained by calculating the ice flow in the upper half-plane (see Fig. 3.2b). The flow is assumed to be pseudo-steady, thus $\partial/\partial t = 0$.

For ice obeying Glen's law (with $n = 3$), Nye's (1953) closure solution for a cylindrical channel of radius R is

$$\frac{dR}{dt} = -\frac{1}{2}K_0RN_c^n \quad (3.2)$$

(see Equation (2.2)), but given our more complicated geometry, the equivalent analytic solution is not easily feasible. To circumvent this, our approach here is to first solve the problem for ice of constant viscosity η_i (corresponding to $n = 1$), then extend the result heuristically for a non-linear rheology. (According to the form of (3.2), a suitable extension is made by the substitution $\eta_i^{-1} = N_c^2\eta_0^{-1}$, where η_0 is a constant related to K_0 .)

In Sections 3.2 and 3.3, the linear ice flow problem is tackled by using complex variable methods developed for 2-D elastic crack problems (England, 1971). (These techniques are also used in Chapter 5, where we consider the analogous problem of till deformation.) In addition to the closure velocity $\hat{w}$ (Section 3.4), we derive the stress distribution at the ice-bed interface (Section 3.5), and we investigate whether the channel width can evolve if basal sliding is allowed (Section 3.6). These results will become useful when we include the effect of till sediments.

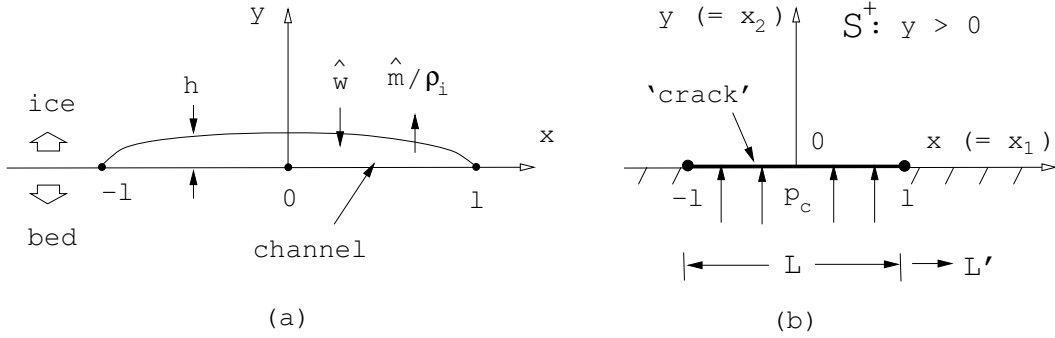


Figure 3.2: (a) Definition diagram of the basic subglacial wide channel, excluding sediment processes. (b) The deforming domain of the ice closure problem.

3.2 Viscous flow model

We treat ice as incompressible with constant density ρ_i . Under the general orthogonal coordinate system (x_1, x_2, x_3) , an appropriate ‘slow flow’ model is given by the mass and momentum conservation equations

$$\begin{aligned} \frac{\partial u_k}{\partial x_k} &= 0, \\ \frac{\partial \sigma_{ij}}{\partial x_j} + \rho_i F_i &= 0, \end{aligned} \quad (3.3)$$

where u_i is the ice velocity, σ_{ij} is the stress tensor, and F_i is the body force per unit mass. (We use the index notation; subscripts i, j and k take the direction values 1, 2 or 3, and repeated indices denote summation.)

To model linear isotropic ice, we use the constitutive relation

$$\sigma_{ij} = -p\delta_{ij} + \eta_i \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (3.4)$$

where η_i is the ice viscosity, p is the mean compressive stress ($= -\sigma_{kk}/3$), and δ is the Kronecker delta function: $\delta_{ij} = 0$ for $i \neq j$; $\delta_{ij} = 1$ for $i = j$. Substitution of σ_{ij} into (3.3)₂ and subsequent reduction with (3.3)₁ lead to

$$-\frac{\partial p}{\partial x_i} + \eta_i \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} \right) + \rho_i F_i = 0. \quad (3.5)$$

If we define vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$, Equations (3.3)₁ and (3.5) can be written more concisely as

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ -\nabla p + \eta_i \nabla^2 \mathbf{u} + \rho_i \mathbf{F} &= 0. \end{aligned} \quad (3.6)$$

This is a common form of the Stokes Equations.

Complex variable formulation

We put $x = x_1$, $y = x_2$, and neglect variations in the x_3 -direction by applying a plane flow condition: $u_3 = F_3 = 0$, $\partial/\partial x_3 = 0$. Our problem is one of solving the equations in (3.6) in the domain S^+ : $y > 0$, shown in Fig. 3.2b. In this problem, the channel appears as a ‘crack’ on $y = 0$, $x \in L$, where L denotes the interval $(-l, l)$; we also denote $|x| > l$ by L' .

By making the change of variables (x, y) to $(z, \bar{z})$, where

$$z = x + iy, \quad \bar{z} = x - iy, \quad (3.7)$$

our equations can be translated into a complex variable form if we associate the x -components with the real part, and the y -components with the imaginary part (England, 1971). We facilitate this by defining the complex flow velocity D and body force F , where

$$D = u_1 + iu_2, \quad F = F_1 + iF_2, \quad (3.8)$$

and by using the substitutions

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \quad (3.9)$$

The suitable replacement rules are then

$$\nabla p \rightarrow 2\frac{\partial p}{\partial \bar{z}}, \quad \nabla^2 \mathbf{u} \rightarrow 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial D}{\partial z}\right), \quad (3.10)$$

and (3.6) becomes

$$\begin{aligned} \frac{\partial D}{\partial z} + \overline{\frac{\partial D}{\partial z}} &= 0 \\ -2\frac{\partial p}{\partial \bar{z}} + 4\eta_i \frac{\partial}{\partial \bar{z}}\left(\frac{\partial D}{\partial z}\right) + \rho_i F &= 0. \end{aligned} \quad (3.11)$$

We proceed to solve these equations by following England’s method. For convenience, let us replace the body force term by taking

$$\rho_i F = \frac{\partial V}{\partial \bar{z}}, \quad (3.12)$$

where $V(z, \bar{z})$ is chosen to be real, i.e., $V = \bar{V}$. Equation (3.11)₂ now takes the form

$$\frac{\partial H}{\partial \bar{z}} = 0, \quad (3.13)$$

where $H = -2p + 4\eta_i \partial D / \partial z + V(z, \bar{z})$. Since H and its first partial derivatives are continuous and single-valued in S^+ , (3.13) implies that H is holomorphic in S^+ . It follows that we can write

$$-2p + 4\eta_i \frac{\partial D}{\partial z} + V(z, \bar{z}) = \theta'(z), \quad (3.14)$$

where θ' is a holomorphic function of the complex variable z in the region S^+ . (Here, $'$ denotes differentiation w.r.t. z .)

By using complex conjugation, it is now straightforward to derive from (3.11)₁ and (3.14) (remember p and V are real), that

$$p = -\frac{1}{4}[\theta'(z) + \overline{\theta'(z)}] + \frac{V(z, \bar{z})}{2}, \quad (3.15)$$

and

$$4\eta_i \frac{\partial D}{\partial z} = \frac{1}{2}[\theta'(z) - \overline{\theta'(z)}]. \quad (3.16)$$

Equation (3.16) has the general solution

$$8\eta_i D = \theta(z) - z\overline{\theta'(z)} + \overline{\phi(z)}, \quad (3.17)$$

where $\theta(z)$ and $\phi(z)$ are functions of the form

$$f(z) + \sum_{k=1}^m \frac{\alpha_k}{2\pi i} \ln(z - z_k), \quad (3.18)$$

and $f(z)$ is holomorphic in S^+ . The constants α_k , z_k are associated with the m internal resultant forces in the problem domain, hence θ and ϕ are generally multi-valued functions.

In the current problem, there are clearly no internal resultant forces within S^+ , so $\alpha_k = 0$, and both θ and ϕ are holomorphic functions in S^+ . For compatibility with England's expressions, let us re-define these functions by putting

$$\begin{aligned} \theta(z) &= 4\Omega(z), \\ \phi(z) &= -4\omega(z); \end{aligned} \quad (3.19)$$

such that (3.17) becomes

$$2\eta_i D = 2\eta_i(u_1 + iu_2) = \Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)}. \quad (3.20)$$

$\Omega(z)$ and $\omega(z)$ are known as the *complex potentials*.

Finally, we derive the corresponding stress expressions. Equation (3.4) leads to

$$\begin{aligned} \sigma_{11} + \sigma_{22} &= -2p, \\ \sigma_{11} - \sigma_{22} + 2i\sigma_{12} &= 2\eta_i \left[\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + i \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \right] \\ &= 4\eta_i \frac{\partial D}{\partial \bar{z}}. \end{aligned} \quad (3.21)$$

By substituting p , D , θ and ϕ from (3.15), (3.17) and (3.19), and by using the differentiation rules

$$\frac{\partial}{\partial \bar{z}}[\overline{\Omega'(z)}] = \overline{\Omega''(z)}, \quad \frac{\partial}{\partial \bar{z}}[\overline{\omega(z)}] = \overline{\omega'(z)}, \quad (3.22)$$

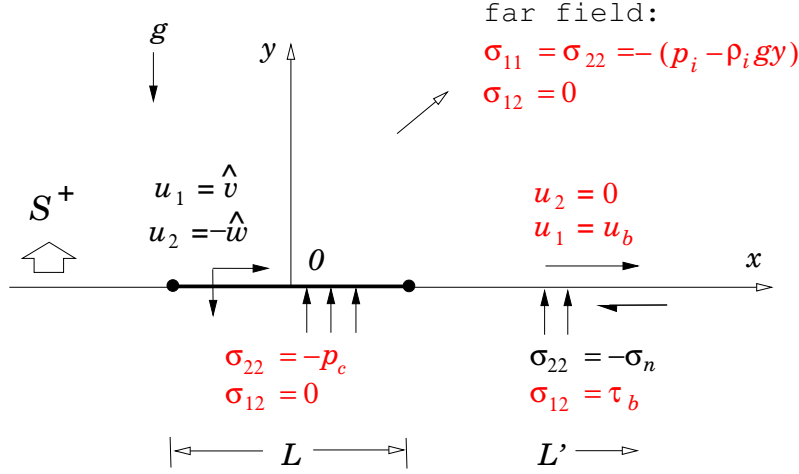


Figure 3.3: *The crack closure problem. (Boundary and far field conditions are shown in red.)*

we obtain

$$\begin{aligned}\sigma_{11} + \sigma_{22} &= 2[\Omega'(z) + \overline{\Omega'(z)}] - V(z, \bar{z}), \\ \sigma_{11} - \sigma_{22} + 2i\sigma_{12} &= -2[z\overline{\Omega''(z)} + \overline{\omega'(z)}].\end{aligned}\quad (3.23)$$

Further subtraction gives

$$\sigma_{22} - i\sigma_{12} = \Omega'(z) + \overline{\Omega'(z)} + z\overline{\Omega''(z)} + \overline{\omega'(z)} - \frac{V(z, \bar{z})}{2}.\quad (3.24)$$

Equations (3.20), (3.23) and (3.24) complete our complex variable model. Solution of the ice flow problem now amounts to finding the complex potentials Ω and ω in S^+ , under boundary constraints imposed at the far field and on $y = 0$. As we shall see, the half-plane geometry allows our required solutions to be expressed in terms of integral relations on L and L' . This is particular convenient since we are less interested in the flow field within S^+ itself.

3.3 The crack problem

Fig. 3.3 illustrates our problem setup. With y pointing upwards, the body force is given by $F = -ig$, hence direct integration of Equation (3.12) produces

$$V = -\rho_i g i(\bar{z} - z).\quad (3.25)$$

Note that V has been chosen to be real as required, and

$$\lim_{z \rightarrow 0^+} [V] = \lim_{z \rightarrow 0^-} [V] = 0,\quad (3.26)$$

where $z \rightarrow 0^+$ and $z \rightarrow 0^-$ denote approaching the x -axis respectively from above and from below.

Boundary conditions

We prescribe a vanishing shear stress away from the channel, and the glaciostatic pressure $p_i - \rho_i g y$ as $|x| \rightarrow \infty$, where p_i is the ice overburden pressure; so we require

$$\begin{aligned}\sigma_{11} + \sigma_{22} &= -2(p_i - \rho_i g y) \\ &= -(p_i + \rho_i g z i) - \overline{p_i + \rho_i g z i}, \\ \sigma_{11} - \sigma_{22} + 2i\sigma_{12} &= 0,\end{aligned}\tag{3.27}$$

as $|z| \rightarrow \infty$. Inside the channel, the water flow has a pressure p_c (a function of x), but it cannot exert a significant shear stress on the ice; therefore, stress continuity at the crack is ensured by choosing

$$\sigma_{22} - i\sigma_{12} = -p_c, \quad \text{on } y = 0, \ x \in L.\tag{3.28}$$

We also define

$$u_1 + iu_2 = \hat{v} - i\hat{w}, \quad \text{on } y = 0, \ x \in L,\tag{3.29}$$

where $\hat{v}$ and $\hat{w}$ are respectively the (unknown) velocities of ‘slip’ and closure of the ice roof.

If the basal ice is frozen to the bed, then the condition outside the channel (on $y = 0$) would be one of ‘no slip’; but in general, this would depend on the nature of the coupling between the ice and the bed. For completeness, let us allow the possibility of *lateral sliding* to occur at the interface, with velocity u_b and basal traction τ_b . The corresponding boundary conditions are then

$$\left. \begin{aligned}\sigma_{22} - i\sigma_{12} &= -\sigma_n - i\tau_b \\ u_1 + iu_2 &= u_b\end{aligned} \right\} \quad \text{on } y = 0, \ x \in L'.\tag{3.30}$$

σ_n is the normal contact stress (compressive) at the bed.

A reflection-based reformulation

We apply boundary condition (3.28) to the stress equation (3.24), by writing

$$\Omega'^+(x) + \overline{\Omega'^+(x)} + x\overline{\Omega''^+(x)} + \overline{\omega'^+(x)} = -p_c, \quad \text{for } x \in L.\tag{3.31}$$

Let us consider also the complex conjugate

$$\overline{\Omega'^+(x)} + \Omega'^+(x) + x\Omega''^+(x) + \omega'^+(x) = -p_c, \quad \text{for } x \in L.\tag{3.32}$$

(The notation used here is $\Omega^+(x) = \lim_{y \rightarrow 0^+} \Omega(z)$.) The first step of exploiting the half-plane geometry involves addition and subtraction of these equations, which give, respectively,

$$\begin{aligned}[2\Omega'(z) + z\Omega''(z) + \omega'(z)]^+ + [\overline{2\Omega'(z) + z\Omega''(z) + \omega'(z)}]^+ &= -2p_c, \\ [z\Omega''(z) + \omega'(z)]^+ - [\overline{z\Omega''(z) + \omega'(z)}]^+ &= 0, \quad \text{for } x \in L.\end{aligned}\tag{3.33}$$

Next, we construct two functions $F(z)$ and $G(z)$, where

$$F(z) = \begin{cases} \Omega(z), & \text{for } y > 0, \\ \overline{\Omega(\bar{z})}, & \text{for } y < 0, \end{cases} \quad (3.34)$$

and

$$G(z) = \begin{cases} \omega(z), & \text{for } y > 0, \\ \overline{\omega(\bar{z})}, & \text{for } y < 0. \end{cases} \quad (3.35)$$

According to the Schwarz reflection principle (Carrier, Krook and Pearson, 1983), F and G are analytic everywhere in the complex plane except possibly on $y = 0$. Also, (3.34) and (3.35) imply that

$$\begin{aligned} F^+(x) &= \lim_{z \rightarrow 0^+} [\Omega(z)] = \Omega^+(x), \\ F^-(x) &= \lim_{z \rightarrow 0^-} [\overline{\Omega(\bar{z})}] = \overline{\Omega^+(x)}, \end{aligned} \quad (3.36)$$

and similarly for G . The equations in (3.33) can therefore be written in the form

$$\begin{aligned} [2F'(z) + zF''(z) + G'(z)]^+ + [2F'(z) + zF''(z) + G'(z)]^- &= -2p_c, \\ [zF''(z) + G'(z)]_\pm^+ &= 0, \quad \text{for } x \in L, \end{aligned} \quad (3.37)$$

where $[\]_\pm^+$ denotes the jump across $y = 0$.

So far, we have used only one of the four (complex) boundary conditions on $y = 0$. Application of the other Equations (3.29), (3.30)₁ and (3.30)₂ leads to paired relations of the same type, as follows:

$$\begin{aligned} [2F'(z) + zF''(z) + G'(z)]^+ + [2F'(z) + zF''(z) + G'(z)]^- &= -2\sigma_n, \\ [zF''(z) + G'(z)]_\pm^+ &= 2i\tau_b, \quad \text{for } x \in L', \end{aligned} \quad (3.38)$$

$$[F(z) - zF'(z) - G(z)]^+ + [F(z) - zF'(z) - G(z)]^- = \begin{cases} 4\eta_i \hat{v} & \text{for } x \in L, \\ 4\eta_i u_b & \text{for } x \in L', \end{cases} \quad (3.39)$$

$$[F(z) + zF'(z) + G(z)]_\pm^+ = \begin{cases} -4\eta_i i\hat{w} & \text{for } x \in L, \\ 0 & \text{for } x \in L'. \end{cases} \quad (3.40)$$

Riemann-Hilbert problems

Note that on the left-hand sides of Equations (3.37) to (3.40), the bracketed terms (in $[\]$) fall into a distinct pattern, and in particular we have

$$\begin{aligned} \frac{d}{dx} [F(z) - zF'(z) - G(z)]^{+/-} &= -[zF''(z) + G'(z)]^{+/-}, \\ \frac{d}{dx} [F(z) + zF'(z) + G(z)]^{+/-} &= [2F'(z) + zF''(z) + G'(z)]^{+/-}. \end{aligned} \quad (3.41)$$

This motivates the definition of the sectionally holomorphic functions

$$\begin{aligned} \Phi(z) &= 2F'(z) + zF''(z) + G'(z), \\ \Theta(z) &= zF''(z) + G'(z), \end{aligned} \quad (3.42)$$

so that (3.37) to (3.40) may now be interpreted as the jump conditions for Φ and Θ across $y = 0$.

The behaviour of Φ and Θ at the far field can be ascertained by applying the conditions in (3.27) to Equation (3.23). We obtain

$$\begin{aligned} 2[\Omega'(z) + \overline{\Omega'(z)}] + \rho_i g i(\bar{z} - z) &\sim -p_i - \rho_i g z i - p_i + \rho_i g \bar{z} i, \\ -2[z\overline{\Omega''(z)} + \overline{\omega'(z)}] &\sim 0, \quad \text{as } |z| \rightarrow \infty, \end{aligned} \quad (3.43)$$

which reduce to

$$\left. \begin{aligned} \Omega'(z) &\sim -p_i/2 \\ \omega'(z) &\sim 0 \end{aligned} \right\} \quad \text{as } |z| \rightarrow \infty. \quad (3.44)$$

It follows that

$$\left. \begin{aligned} \Phi(z) &\sim -p_i \\ \Theta(z) &\sim 0 \end{aligned} \right\} \quad \text{as } |z| \rightarrow \infty. \quad (3.45)$$

Finally, our results from (3.37) to (3.40), and (3.45), are summarized as follows:

$$\begin{aligned} x \in L : \quad &\begin{cases} \Phi^+ - \Phi^- = -4\eta_i d\hat{w}/dx, \\ \Phi^+ + \Phi^- = -2p_c, \end{cases} \\ x \in L' : \quad &\begin{cases} \Phi^+ - \Phi^- = 0, \\ \Phi^+ + \Phi^- = -2\sigma_n, \end{cases} \\ \Phi &\sim -p_i, \quad \text{as } |z| \rightarrow \infty, \end{aligned} \quad (3.46)$$

and

$$\begin{aligned} x \in L : \quad &\begin{cases} \Theta^+ - \Theta^- = 0, \\ \Theta^+ + \Theta^- = -4\eta_i d\hat{v}/dx, \end{cases} \\ x \in L' : \quad &\begin{cases} \Theta^+ - \Theta^- = 2i\tau_b, \\ \Theta^+ + \Theta^- = -4\eta_i du_b/dx, \end{cases} \\ \Theta &\sim 0, \quad \text{as } |z| \rightarrow \infty. \end{aligned} \quad (3.47)$$

These equation groups define so-called Riemann-Hilbert problems for the functions $\Phi(z)$ and $\Theta(z)$, and can be solved by using standard methods described by Carrier, Krook and Pearson (1983), and England (1971). We describe this next.

Integral relations

Due to (3.46)₃, $\Phi(z)$ is analytically continued across $y = 0$, $x \in L'$, so $\Phi(z)$ is holomorphic in the entire complex plane cut along L . The forward Hilbert problem defined by (3.46)_{1,5} leads to

$$\Phi(z) = -\frac{2\eta_i}{\pi} \int_L \frac{d\hat{w}}{d\zeta} \frac{d\zeta}{\zeta - z} - p_i. \quad (3.48)$$

If we substitute this solution back into (3.46)₂ and (3.46)₄, then we obtain

$$\begin{aligned} N_c &= -\frac{2\eta_i}{\pi} \int_L \frac{d\hat{w}}{d\zeta} \frac{d\zeta}{\zeta - x}, & \text{for } x \in L, \\ -\sigma_n &= -\frac{2\eta_i}{\pi} \int_L \frac{d\hat{w}}{d\zeta} \frac{d\zeta}{\zeta - x} - p_i, & \text{for } x \in L', \end{aligned} \quad (3.49)$$

where $\int$ denotes a principal value integral. These equations relate the closure velocity $\hat{w}$ to the externally applied normal stress on $y = 0$ (inside and outside the channel, respectively). For instance when $N_c(x) = 0$, Equation (3.49)₁ leads to $d\hat{w}/d\zeta = 0$ and $\hat{w} = 0$ (due to $\hat{w}(\pm l) = 0$; see footnote¹), and (3.49)₂ gives $\sigma_n = p_i$. As one would expect, this corresponds to the situation where there is effectively no channel. More generally, a positive $N_c(x)$ will lead to a non-zero closure rate, and then (3.49)₂ describes how σ_n departs from the glaciostatic pressure p_i . The determination of $\hat{w}(x)$ and $\sigma_n(x)$ is described in Sections 3.4 and 3.5.

We can examine (3.47) by a similar procedure. In this case, $\Theta(z)$ is holomorphic in the plane cut along L' , given by

$$\Theta(z) = \frac{1}{\pi} \int_{L'} \tau_b \frac{d\zeta}{\zeta - z}, \quad (3.50)$$

and the corresponding integral equations are

$$\begin{aligned} -2\eta_i \frac{d\hat{w}}{dx} &= \frac{1}{\pi} \int_{L'} \tau_b \frac{d\zeta}{\zeta - x}, & \text{for } x \in L, \\ -2\eta_i \frac{du_b}{dx} &= \frac{1}{\pi} \int_{L'} \tau_b \frac{d\zeta}{\zeta - x}, & \text{for } x \in L'. \end{aligned} \quad (3.51)$$

If there is no slip at the bed ($u_b = du_b/dx = 0$), τ_b is identically zero and $\hat{v}$ is a constant ($= 0$, since $\hat{v}(\pm l) = u_b(\pm l) = 0$; again, see footnote¹). In Section 3.6, we investigate the solution of (3.51)₂ when sliding is allowed.

Note that there is no need for us to evaluate $\Phi(z)$ and $\Theta(z)$ themselves, and in addition (3.49) and (3.51) are completely decoupled from each other. This allows the closure and sliding problems to be studied independently.

3.4 Closure velocity

Inversion

We seek the solution of the singular integral equation (3.49)₁, written here in the form

$$-\frac{1}{\pi} \int_L \frac{d\hat{w}}{d\zeta} \frac{d\zeta}{\zeta - x} = f_1(x), \quad \text{for } x \in L, \quad (3.52)$$

¹The precise argument used to show this is detailed in Appendix A.

where

$$f_1(x) = \frac{N_c(x)}{2\eta_i}, \quad (3.53)$$

and $N_c(x)$ is given. If we let

$$g_1(\zeta) = i \frac{d\hat{w}}{d\zeta} \quad (3.54)$$

and define the Cauchy integral

$$G_1(z) = \frac{1}{2\pi i} \int_L \frac{g_1(\zeta)}{\zeta - z} d\zeta, \quad (3.55)$$

then the corresponding Plemelj formulae are

$$\begin{aligned} G_1^+(x) - G_1^-(x) &= g_1(x), \\ G_1^+(x) + G_1^-(x) &= -f_1(x), \quad \text{for } x \in L. \end{aligned} \quad (3.56)$$

This defines an alternative form of the closure problem.

To determine g_1 (and hence $\hat{w}$), it is first necessary to solve the backward Hilbert problem (3.56)₂. We use a Plemelj function $\chi(z)$ which satisfies

$$\chi^+(x) + \chi^-(x) = 0, \quad \text{for } x \in L, \quad (3.57)$$

and put $K(z) = G_1(z)/\chi(z)$, so that K , G_1 and χ are all sectionally holomorphic functions in the plane cut along L . The Plemelj equations then become

$$\begin{aligned} K^+(x) + K^-(x) &= \frac{g_1(x)}{\chi^+(x)}, \\ K^+(x) - K^-(x) &= \frac{-f_1(x)}{\chi^+(x)}, \quad \text{for } x \in L. \end{aligned} \quad (3.58)$$

of which the second equation yields

$$K(z) = -\frac{1}{2\pi i} \int_L \frac{f_1(\zeta)}{\chi^+(\zeta)} \frac{d\zeta}{\zeta - z} + P(z). \quad (3.59)$$

Here $P(z)$ is a polynomial function chosen to fix the behaviour of $K(z)$ as $|z| \rightarrow \infty$.

Back-substitution of $K(z)$ into (3.58)₁ leads to the equation

$$i \frac{d\hat{w}}{dx} = -\frac{\chi^+(x)}{\pi i} \int_L \frac{f_1(\zeta)}{\chi^+(\zeta)} \frac{d\zeta}{\zeta - x} + 2P\chi^+(x), \quad \text{for } x \in L. \quad (3.60)$$

We allow g_1 ($= i\hat{w}'$) to have integrable singularities at the ends $\pm l$, in order that $G_1(z)$ exists (by the definition in Equation (3.55)). We also choose

$$\chi(z) = \frac{1}{\sqrt{z^2 - l^2}}, \quad (3.61)$$

with $\chi \sim 1/z$ at ∞ , so that the branch we are taking has

$$\chi^+(x) = \frac{-i}{\sqrt{l^2 - x^2}}. \quad (3.62)$$

Since $G_1(z) \sim 1/z$ at ∞ , we deduce that $K (= G_1/\chi)$ must be bounded at large $|z|$, hence P is a constant in Equation (3.59). Moreover, there is a symmetry condition for the channel shape, which ensures that both $f_1(x)$ and $\hat{w}(x)$ are even functions of x (and the first two terms in (3.60) odd); therefore, we find $P = 0$. The problem now reduces to the equation

$$\frac{d\hat{w}}{dx} = \frac{\chi^+(x)}{2\eta_i\pi} \int_L \frac{N_c(\zeta)}{\chi^+(\zeta)} \frac{d\zeta}{\zeta - x} \quad (3.63)$$

for $\hat{w}(x)$, with the end conditions $\hat{w}(\pm l) = 0$.

Solution for constant effective channel pressure

As we shall see in Chapter 4 (Section 4.3), N_c may be assumed constant across the channel. In this case, the principal value integral in (3.63) can be evaluated directly in terms of the finite Hilbert transform H , defined by

$$H[\phi(\zeta)](x) = \frac{1}{\pi} \oint_{-1}^1 \frac{\phi(\zeta)}{\zeta - x} d\zeta. \quad (3.64)$$

Here we let

$$I = \int_L \frac{1}{\chi^+(\zeta)} \frac{d\zeta}{\zeta - x} = \pi i l H[\sqrt{1 - \zeta^2}](x/l), \quad (3.65)$$

for which standard results give $H[\sqrt{1 - \zeta^2}](x) = -x$ for $|x| < 1$ (Tricomi, 1957). Thus $I = -\pi i x$, and Equation (3.63) becomes

$$\frac{d\hat{w}}{dx} = -\frac{N_c}{2\eta_i} \frac{x}{\sqrt{l^2 - x^2}}, \quad \text{for } x \in L, \quad (3.66)$$

consistent with our earlier assumption of g_1 being integrable at the ends. (This result can also be derived by applying the residue theorem to a Cauchy integral based on (3.63) (England, 1971).) Direct integration leads to the ellipsoidal closure velocity distribution

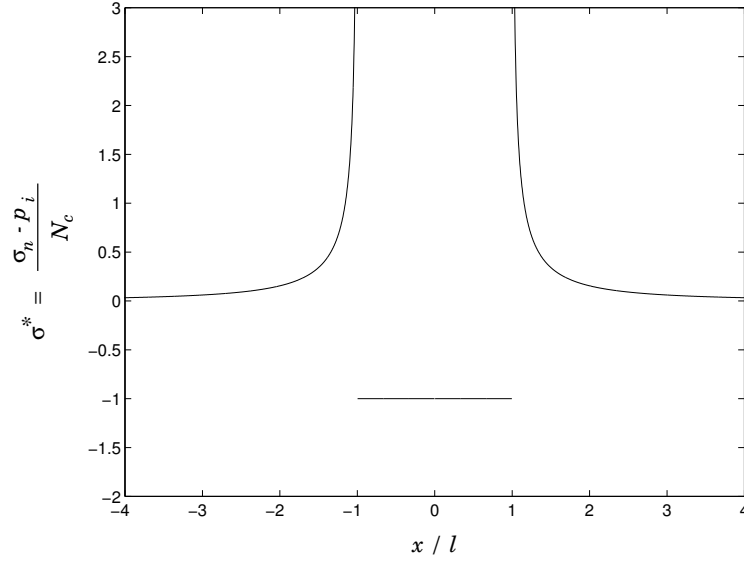
$$\hat{w}(x) = \frac{N_c}{2\eta_i} \sqrt{l^2 - x^2} \quad (3.67)$$

that was used in Section 2.5.1 (see Equation (2.49)).

3.5 Normal stress outside channel

By substituting Equation (3.63) into Equation (3.49)₂, we obtain

$$\sigma_n - p_i = \frac{1}{\pi^2} \int_L \frac{\chi^+(\zeta)}{\zeta - x} \left[\int_L \frac{N_c(t)}{\chi^+(t)} \frac{dt}{t - \zeta} \right] d\zeta, \quad \text{for } x \in L'. \quad (3.68)$$

Figure 3.4: *The dimensionless normal stress function σ^* .*

With N_c constant, this equation can be simplified and written in the form

$$\sigma^*(x) = \frac{\sigma_n - p_i}{N_c} = -\frac{1}{\pi} \int_L \frac{\zeta}{(\zeta - x)\sqrt{l^2 - \zeta^2}} d\zeta, \quad \text{for } x \in L', \quad (3.69)$$

where $\sigma^*(x)$ is a dimensionless measure of the (ice-bed) normal contact stress. Note that this definition for σ^* can be extended into $x \in L$, by interpreting $\int$ as a principal value integral when $|x| < l$ (this follows directly from Equation (3.49)₁). We can anticipate $\sigma^*(x) = -1$ for $x \in L$.

We utilize the Hilbert transform again, by rewriting (3.69) as

$$\begin{aligned} \sigma^*(x) &= -\frac{1}{\pi} \int_L \frac{(\zeta - x) + x}{(\zeta - x)\sqrt{l^2 - \zeta^2}} d\zeta \\ &= -\frac{1}{\pi} \left[\int_L \frac{d\zeta}{\sqrt{l^2 - \zeta^2}} + \frac{x}{l} \int_{-1}^1 \frac{d\zeta}{\sqrt{1 - \zeta^2}(\zeta - x/l)} \right] \\ &= -1 - \frac{x}{l} H[(1 - \zeta^2)^{-1/2}](x/l). \end{aligned} \quad (3.70)$$

A standard result from Tricomi (p. 175, 1957) is that

$$H[(1 - \zeta^2)^{-1/2}](x) = \begin{cases} 0 \\ -\text{sgn}(x) \times (x^2 - 1)^{-1/2}, \end{cases} \quad \text{for } \begin{cases} |x| < 1 \\ |x| > 1, \end{cases} \quad (3.71)$$

therefore we find $\sigma^* = -1$ for $x \in L$ (as expected), and

$$\sigma^*(x) = -1 + \frac{|x|}{\sqrt{x^2 - l^2}}, \quad \text{for } x \in L'. \quad (3.72)$$

According to these results (plotted in Fig. 3.4), σ_n approaches p_i far away from the channel, but it increases rapidly towards the margins and is undefined at $x = \pm l$. This contact stress distribution is analogous to that for semi-circular subglacial channels, derived by Weertman (1972).

3.6 Basal sliding

We have not discussed how the channel width l would evolve. One possibility is that l evolves as a result of sliding at the ice-bed interface. In this case, the basal velocity is given by solving Equation (3.51)₂ in conjunction with a prescribed *sliding law* that describes the contact coupling.

The sliding law generally takes the form of a mixed boundary condition on L' , e.g. $\tau_b = \tau_b(u_b, N)$, where N is an effective pressure defined for water occurring at the interface (see Equations (1.5) and (1.6)). This makes the solution of (3.51)₂ very difficult. However, we argue in the following that the only (realistic) possibility is the trivial solution $\tau_b \equiv u_b \equiv 0$.

We use the fact that the equations of sliding and closure are decoupled. In particular, u_b is independent of the channel water pressure. We also expect $u_b = 0$ when $N_c = 0$, because the channel is then redundant (see Section 3.3). These suggest that $u_b (= \tau_b) = 0$ for all values of N_c , on the condition that the sliding problem has no other solutions. If this is true, the effect of (lateral) sliding may be neglected from our subsequent channel width description.

In this section, we provide two methods of proving this uniqueness of the trivial solution, given the simplest sliding law

$$\tau_b = C u_b \tag{3.73}$$

(independent of N), in which $C (> 0)$ is a constant friction coefficient. Equation (3.51)₂ reduces to the singular integro-differential equation

$$\frac{du_b}{dx} = -\lambda \int_{L'} \frac{u_b(\zeta)}{\zeta - x} d\zeta, \quad \text{for } x \in L', \tag{3.74}$$

where λ (also a constant) $= C/2\pi\eta_i$. To ensure physical consistency, we assume that $u_b(x)$ is differentiable and bounded on L' , with $u_b \rightarrow 0$ as $|x| \rightarrow \infty$. The first proof relies on physical consideration, whereas the second one is based on the mathematical properties of (3.74). (See also footnote².)

²While there are no standard analytical methods for solving equations of the type (3.74), we show in Appendix B that it can be transformed to

$$f'(x) = \lambda \int_L \frac{f(\zeta)}{\zeta - x} d\zeta, \quad \text{for } |x| < L, \tag{3.75}$$

which has the same form but has a continuous domain and path of integration. Delves and Walsh (p. 265, 1974) has derived this equation from their ‘dock problem’; they expect a solution other than $f(x) \equiv 0$, and have discussed the corresponding numerical solution technique. Our transformation,

3.6.1 Proof of uniqueness (i)

Energy integrals

Let there be fluid flow in some region V_0 bounded by a closed surface S_0 . The Stokes equations in (3.5) may be rewritten in the form

$$-\frac{\partial(\Pi u_i)}{\partial x_i} + \eta_i u_i \nabla^2 u_i = 0, \quad (3.76)$$

where Π is a deviatoric pressure defined by

$$\Pi = p - \rho_i \int F_i dx_i. \quad (3.77)$$

With the use of the identity

$$u_i \nabla^2 u_i = \frac{\partial}{\partial x_j} \left(u_i \frac{\partial u_i}{\partial x_j} \right) - \left(\frac{\partial u_i}{\partial x_j} \right)^2, \quad (3.78)$$

integration of (3.76) over V_0 and application of the divergence theorem yields

$$-\oint_{S_0} \Pi u_i n_i dS + \eta_i \oint_{S_0} u_i \frac{\partial u_i}{\partial x_j} n_j dS - \eta_i \int_{V_0} \left(\frac{\partial u_i}{\partial x_j} \right)^2 dV = 0, \quad (3.79)$$

where n_i denotes the outward unit normal vector on S_0 . This equation describes the (global) conservation of energy, and relates the work done on S_0 (the first two terms) to the viscous dissipation within V_0 (the last term).

The difference problem

For any given $N_c(x)$ distribution, let us suppose that our closure problem (together with the possibility of basal sliding) has two flow solutions $\mathbf{u}_A$ and $\mathbf{u}_B$, where by hypothesis $\mathbf{u}_A \neq \mathbf{u}_B$ in S^+ . Since our equations are linear, a ‘difference’ problem motivated for $\mathbf{u} = \mathbf{u}_A - \mathbf{u}_B$ would have the boundary conditions

$$\begin{aligned} \sigma_{12} &= \sigma_{22} = 0, & \text{for } x \in L, \\ \sigma_{12} &= C u_1, \quad u_2 = 0, & \text{for } x \in L', \end{aligned} \quad (3.80)$$

on $y = 0$ (see Equations (3.28) to (3.30)). In addition, Equation (3.49)₁ implies that $\hat{w}_A(x) \equiv \hat{w}_B(x)$ ($= \hat{w}$), so

$$u_2 = 0, \quad \text{on } y = 0, \quad x \in L. \quad (3.81)$$

We can also examine the stresses and velocities at the far field. Both original problems for $\mathbf{u}_A$ and $\mathbf{u}_B$ have finite (and identical) ice flux Q across L , given by $Q = \int_L \hat{w}(x) dx$. Hence, by conserving mass in S^+ , we obtain

$$u_1, u_2 \propto \frac{Q}{z} \quad \text{as } |z| \rightarrow \infty. \quad (3.82)$$

together with the proofs given in the following sections, implies that if such eigen-solution for f exists, it cannot be bounded. See Appendix B.

It follows that the difference problem has

$$u_1, u_2 \sim \frac{1}{z^2} \quad \text{as } |z| \rightarrow \infty. \quad (3.83)$$

For the stresses, Equations (3.23), (3.24) and (3.44) show that in the original problems, the direct components $\rightarrow p_i$ and the shear components $\sim 1/z$ at the far field, hence

$$\sigma_{ij}, \frac{\partial u_i}{\partial x_j} \sim \frac{1}{z} \quad \text{as } |z| \rightarrow \infty, \quad \forall i, j = 1, 2 \quad (3.84)$$

in the difference problem. Since this implies $p (= -\sigma_{kk}/3) \rightarrow 0$, we also have $\Pi \sim 0$ as $|z| \rightarrow \infty$.

Application

We subject Equation (3.79) to the conditions in (3.80), (3.81), (3.83) and (3.84), by defining V_0 to be the upper-half plane S^+ . The boundary S_0 then consists of the x -axis ($L + L'$) and a semi-circular circuit at infinity C_∞ . Since Πu_i and $u_i \partial u_i / \partial x_j \ll 1/z$ as $|z| \rightarrow \infty$, the line integrals over C_∞ vanish; and with $u_2 = 0$ on $y = 0$, we are left with

$$\int_{L+L'} u_1 \frac{\partial u_1}{\partial x_2} dx + \int_{S^+} \left(\frac{\partial u_i}{\partial x_j} \right)^2 dV = 0. \quad (3.85)$$

On $y = 0$, we also have $\partial u_2 / \partial x_1 = 0$, hence (3.4) and (3.80) lead to

$$\eta_i \frac{\partial u_1}{\partial x_2} = \sigma_{12} = \begin{cases} 0 & \text{for } x \in L, \\ C u_1, & \text{for } x \in L'. \end{cases} \quad (3.86)$$

Equation (3.85) now reduces to

$$C \int_{L'} u_1^2 dx + \eta_i \int_{S^+} \left(\frac{\partial u_i}{\partial x_j} \right)^2 dV = 0. \quad (3.87)$$

Here the two integrands are necessarily positive, which implies that $u_1 = 0$ on L' and $\partial u_i / \partial x_j = 0$ everywhere in S^+ . Consequently, the difference solution must be identically zero, and the original problem must have a unique solution (i.e. the trivial solution $u_b = \tau_b = 0$).

3.6.2 Proof of uniqueness (ii)

Real functions formulation

Suppose

$$U(z) = \frac{1}{2\pi i} \int_{L'} \frac{u_b(\zeta)}{\zeta - z} d\zeta, \quad (3.88)$$

so that U is analytic in V_L (the plane cut along L'), and

$$U(z) \sim 1/z \quad \text{as } |z| \rightarrow \infty. \quad (3.89)$$

The associated Plemelj formulae allows Equation (3.74) to be written as

$$U'^+(x) - U'^-(x) = -\lambda\pi i [U^+(x) + U^-(x)], \quad \text{for } x \in L' \quad (3.90)$$

(see also Appendix B). If we define

$$U(z) = p(x, y) + iq(x, y), \quad (3.91)$$

where p and q are real functions of positions, then p and q satisfy the Cauchy-Riemann relations; we have

$$U'(z) = p_x + iq_x = q_y - ip_y \quad (3.92)$$

and

$$\nabla^2 p = \nabla^2 q = 0 \quad \text{for } (x, y) \in V_L. \quad (3.93)$$

Now, by substituting U and U' into (3.89) and (3.90) and equating real and imaginary parts, we obtain

$$\begin{aligned} p(x, y), q(x, y) &\rightarrow 0 \quad \text{as } x^2 + y^2 \rightarrow \infty, \\ x \in L' : \quad \begin{cases} -(q_x^+ - q_x^-) = \lambda\pi(p^+ + p^-) = p_y^+ - p_y^-, \\ p_x^+ - p_x^- = \lambda\pi(q^+ + q^-) = q_y^+ - q_y^-. \end{cases} \end{aligned} \quad (3.94)$$

These are essentially the boundary conditions for the Laplace problems in (3.93).

Integral method

The method here is related to the one in Section 3.6.1. We define a region V_1 with the boundary S_1 , where S_1 consists of L'_+ and L'_- , the circular circuit at infinity C_∞ , and the circular arcs C_{-l} and C_{+l} with radii ϵ . This is illustrated in Fig. 3.5. In the limit as $\epsilon \rightarrow 0$, $V_1 \rightarrow V_L$.

Since $\nabla^2 p = 0$ in V_1 , we may write

$$\int_{V_1} p \nabla^2 p \, dV = 0. \quad (3.95)$$

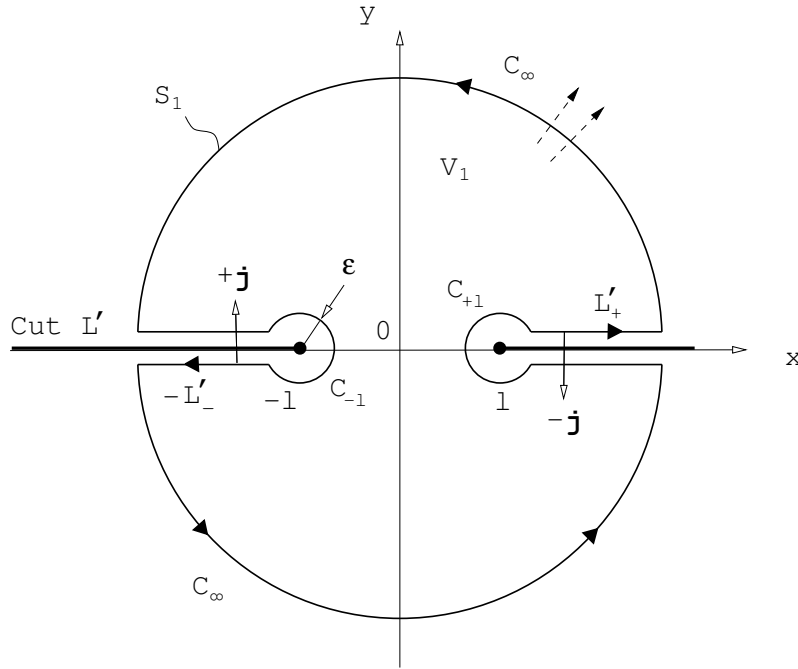
By applying the vector identity $p \nabla^2 p = \nabla \cdot (p \nabla p) - |\nabla p|^2$, followed by the divergence theorem, this equation becomes

$$\oint_{S_1} p \nabla p \cdot \hat{\mathbf{n}} \, dS = \int_{V_1} |\nabla p|^2 \, dV, \quad (3.96)$$

in which $\hat{\mathbf{n}}$ is the normal vector pointing out from S_1 . On L'_+ and L'_- , $\hat{\mathbf{n}}$ is respectively $-\mathbf{j}$ and $+\mathbf{j}$; and on $C_{\pm l}$, $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a local radial vector at each end of the cut L' .

We now take the limit as $\epsilon \rightarrow 0$. Due to (3.94)₁, C_∞ does not contribute to the line integral, therefore

$$\lim_{\epsilon \rightarrow 0} \int_{C_{\pm l}} p \nabla p \cdot \hat{\mathbf{n}} \, dS - \int_{L'_+ L'_-} p \frac{\partial p}{\partial y} dx = \int_{V_L} |\nabla p|^2 \, dV. \quad (3.97)$$


 Figure 3.5: The region V_1 bounded by the contour S_1 .

Let us label the integrals in this equation respectively by I_1 , I_2 and I_3 and examine them in turn.

(i): If we define the local cylindrical coordinate system (r, θ) about each end of L' , then

$$I_1 = \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{2\pi} \left[p \frac{\partial p}{\partial r} \right]_{r=\epsilon}^{(\text{end } -l)} + \left[p \frac{\partial p}{\partial r} \right]_{r=\epsilon}^{(\text{end } +l)} d\theta. \quad (3.98)$$

According to Muskhelishvili (p. 74-75, 1953), since $u_b(\pm l)$ is bounded, we have in the neighbourhood of the ends

$$U(z) = \mp \frac{u_b(c)}{2\pi i} \ln(z - c) + \Phi_0(z), \quad (3.99)$$

where the upper sign is taken for $c = -l$, the lower for $c = +l$, and $\Phi_0(z)$ is a bounded holomorphic function tending to definite limits when $z \rightarrow \pm l$ along any path. Thus,

$$p = \text{Re}(U) = \mp \frac{u_b(\mp l)}{2\pi} \arg(z \pm l) + \text{Re}(\Phi_0) \quad (3.100)$$

near the ends, and we obtain

$$\begin{aligned} I_1 = \lim_{\epsilon \rightarrow 0} \epsilon \int_0^{2\pi} & \left[\left(\text{Re}(\Phi_0) - \frac{u_b(-l)}{2\pi} \theta \right) \frac{\partial [\text{Re}(\Phi_0)]}{\partial r} \right]_{r=\epsilon}^{(\text{end } -l)} \\ & + \left[\left(\text{Re}(\Phi_0) + \frac{u_b(+l)}{2\pi} \theta \right) \frac{\partial [\text{Re}(\Phi_0)]}{\partial r} \right]_{r=\epsilon}^{(\text{end } +l)} d\theta. \end{aligned} \quad (3.101)$$

Since Φ_0 is analytic, the integrand here is bounded as $\epsilon \rightarrow 0$, so $I_1 = 0$.

(ii): From (3.94)₂, it is straightforward to derive

$$\lambda^2 \pi^2 (p_+^2 - p_-^2) + (p_{y+}^2 - p_{y-}^2) = 2\lambda\pi(p^+ p_y^+ + p^- p_y^-), \quad \text{for } x \in L' \quad (3.102)$$

(we put $+/-$ signs as subscripts to avoid confusion with the powers when appropriate), therefore

$$\begin{aligned} I_2 &= -\left[\int_{-\infty}^{-l} + \int_l^{\infty} p^+ p_y^+ + p^- p_y^- dx \right] \\ &= -\frac{1}{2} \left[\int_{-\infty}^{-l} + \int_l^{\infty} \lambda\pi(p_+^2 - p_-^2) + \frac{p_{y+}^2 - p_{y-}^2}{\lambda\pi} dx \right]. \end{aligned} \quad (3.103)$$

To determine the integrand, we can examine the boundary value problem for p . From (3.93) and (3.94), it is

$$\nabla^2 p = 0 \quad \text{in } V_L : \quad \begin{cases} p \rightarrow 0 & \text{as } x^2 + y^2 \rightarrow \infty, \\ p_y^+ - \lambda\pi p^+ = p_y^- + \lambda\pi p^- & \text{for } x \in L'. \end{cases} \quad (3.104)$$

Given a solution to this problem $p_1(x, y)$, the function

$$p_2(x, y) = p_1(x, -y) \quad (3.105)$$

has

$$\frac{\partial p_2}{\partial y} = -p_{1y}(x, -y), \quad (3.106)$$

whereby, the second boundary condition in (3.104) (for p_1) may be written as

$$-p_{2y}^- - \lambda\pi p_2^- = -p_{2y}^+ + \lambda\pi p_2^+, \quad \text{for } x \in L'. \quad (3.107)$$

It follows that p_2 is also a solution. As (3.104) is linear, this implies that the even and odd components (in y) of p_1 are also solutions. In particular, since u_b is real, it is easy to show that Equations (3.88) and (3.91) lead to an odd solution (in y) for p , i.e., of the form

$$\frac{1}{2}[p_1(x, y) - p_1(x, -y)]. \quad (3.108)$$

Hence, we have $p_+^2 = p_-^2$, $p_{y+}^2 = p_{y-}^2$, and $I_2 = 0$.

(iii): With $I_1 = I_2 = 0$, Equation (3.97) reduces to

$$I_3 = \int_{V_L} |\nabla p|^2 dV = 0. \quad (3.109)$$

Since p vanishes at the far field, all solutions of p must be identically zero in V_L , i.e., $p \equiv 0$. By using the Cauchy-Riemann relations $p_x = q_y$, $q_x = -p_y$, we also deduce that $q \equiv 0$, $U(z) = p + iq = 0$, and then the Plemelj equation

$$u_b(x) = U^+(x) - U^-(x), \quad \text{for } x \in L' \quad (3.110)$$

gives $u_b \equiv 0$ on L' . This completes the proof.

3.7 Discussion

In this chapter, an ice flow problem has been solved for the wide channel under the assumption of constant viscosity ice. We obtain the closure velocity across the channel, which can be generalized (heuristically) to the result

$$\hat{w}(x) = \frac{N_c^3}{2\eta_0} \sqrt{l^2 - x^2}, \quad (3.111)$$

where η_0 is a rheological constant. This extension (based on the exact closure solutions for cylindrical ice channels) is crude, but allows us to circumvent the full non-linear problem, the solution of which is feasible only with numerical computation. Note that the ellipsoidal velocity solution in (3.111) is likely to become inaccurate close to the channel margins, because our model has been based on a linear approximation of the channel as a ‘crack’.

The corresponding ice-bed interface stress distribution displays a singular behaviour near the margins (exceeding p_i), compensating the low pressure effect of the channel, where $p_c < p_i$. This ensures that an overall vertical force balance is maintained (Weertman, 1972).

In addition, we argue that (lateral) basal sliding can be neglected from our subsequent channel width consideration. However, the validity of this result does depend on the assumed linearity of our model. If the sliding law is non-linear, then in general multiple ice flow solutions are possible, and this could lead to a non-zero sliding velocity at the ice-bed interface.

The relevance of these results with regard to channel evolution is deferred until the end of Chapter 4.

Chapter 4

Ice melting

4.1 Introduction

The channel water flow that is under current consideration is *turbulent*. This term refers to a motion in which an irregular fluctuation (or mixing motion) is superimposed upon the main stream. There exists a continuous transport of energy from the main flow into progressively smaller ‘eddies’, but these are so extremely complex in detail that a complete theoretical treatment of them has so far proved impossible. In the classical R  thlisberger-Nye theory (Chapter 2), the resulting heat transfer causing the ice wall to melt is modelled by an empirical law determined for cylindrical conduits. Here we seek a corresponding melt-rate expression for wide subglacial channels. Our starting point is a mathematical model of turbulent flow.

The thermo-mechanical equations for flow velocity and temperature within the channel are given in Sections 4.2 and 4.3. We are restricted to a time-averaged description, in which an *eddy viscosity parameterization* is employed to account for the turbulent nature of the flow. As we shall see, the high channel aspect ratio $2l/h$ facilitates an approximation, whereby the experimental results for parallel (flat-plate) flows may be utilized. In particular, it allows us to prescribe the eddy viscosity from the semi-empirical theory due to Prandtl (Schlichting, 1960).

In Section 4.4, we look for model solutions through a singular approximation of our equations based on $\delta \ll 1$, where

$$\delta = \frac{h}{l} \tag{4.1}$$

is an aspect ratio reciprocal. (A typical value under consideration is $\delta = 0.1$.) This small parameter is used extensively in Part II. Subsequently, the local melt-rate $\hat{m}$ is given by

$$\hat{m} = \frac{\hat{q}_i}{L} \sqrt{1 + h_x^2}, \tag{4.2}$$

where L is the latent heat of fusion, and $\hat{q}_i$ is the *normal* heat flux at the roof boundary, calculated from the temperature field. This is an exact relation for the hard bed channel that is being investigated, because h there represents the ice incision. We

note however, that (4.2) is also approximately correct for a sediment-floored wide channel (Chapter 7).

The complete channel description must include separate equations for h and l . Since by definition $h(\pm l) = 0$, it follows that

$$\frac{dl}{dt} = \lim_{x \rightarrow l} \left[-\frac{\partial h}{\partial t} / \frac{\partial h}{\partial x} \right]. \quad (4.3)$$

This enables the determination of the channel widening rate (or its ‘tip velocity’), given $\partial h / \partial t$. For the hard bed case, we can substitute $\partial h / \partial t (= \hat{m} / \rho_i - \hat{w})$ from Equation (3.1). We shall find, however, that this leads to problems: although (4.3) is a formal expression, the parallel flow approximations used to derive $\hat{m}$ are not necessarily valid at the channel margins. In order to derive a tip velocity estimate, we provide an alternative method based on modifying the eddy viscosity function (Section 4.5). The implications of our results are discussed in Section 4.6.

4.2 Turbulent flow model

A classical continuum model to describe steady incompressible turbulent water flow consists of the mass, momentum and energy conservation equations (written in the index notation)

$$\begin{aligned} \frac{\partial u_i}{\partial x_i} &= 0, \\ \rho_w u_j \frac{\partial u_i}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho_w g_i, \\ \rho_w c_w u_j \frac{\partial T}{\partial x_j} &= -\frac{\partial q_j}{\partial x_j} + \Psi, \end{aligned} \quad (4.4)$$

where u_i , p , T are mean values of the flow velocity (vector), pressure and temperature, and g_i is the body force per unit mass (e.g. Schlichting, 1960). τ_{ij} is the deviatoric stress tensor, given by the sum of the laminar and turbulent parts

$$\tau_{ij}^l = \mu_w \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{and} \quad \tau_{ij}^t = -\rho_w \overline{u_i' u_j'}. \quad (4.5)$$

The latter is commonly known as the Reynolds stress, and is due to the fluctuating velocity components u_i' . (Thus, $u_i + u_i'$ represents the total velocity.) The bar $\overline{}$ denotes time averaging, hence $\overline{u_i' u_j'}$ is a second order correlation. Similarly, the heat flux vector has two components: we have

$$q_j = q_j^l + q_j^t = -k_w \frac{\partial T}{\partial x_j} + \rho_w c_w \overline{u_j' T'}, \quad (4.6)$$

where k_w is the thermal conductivity of water and T' is the temperature fluctuation. In (4.4)₃, Ψ is the internal heat dissipation, given by

$$\Psi = \frac{\mu_w}{2} \left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 + \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)^2} \right]. \quad (4.7)$$

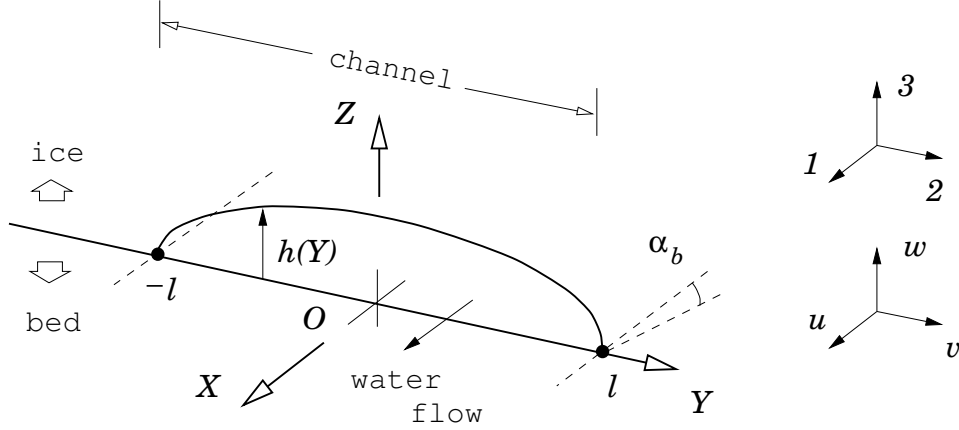


Figure 4.1: The coordinate system and notations used in the channel model of Sections 4.2 and 4.3.

The turbulent terms have to be parameterized so as to close the problem in (4.4). A standard method is to define the *eddy viscosities* for momentum and heat ϵ_m and ϵ_h , through the following relations (Schlichting, 1960):

$$\begin{aligned} \overline{u_i' u_j'} &= -\epsilon_m \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), & \overline{u_j' T'} &= -\epsilon_h \frac{\partial T}{\partial x_j}, \\ \mu_w \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right)^2} &= \rho_w \epsilon_m \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2. \end{aligned} \quad (4.8)$$

It is found for most turbulent flows that $\rho_w \epsilon_m \gg \mu_w$ and $\rho_w c_w \epsilon_h \gg k_w$. As a result, the laminar terms may be neglected, and then Equations (4.4)₂ and (4.4)₃ become

$$\begin{aligned} u_j \frac{\partial u_i}{\partial x_j} &= -\frac{1}{\rho_w} \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\epsilon_m \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + g_i, \\ u_j \frac{\partial T}{\partial x_j} &= \frac{\partial}{\partial x_j} \left(\epsilon_h \frac{\partial T}{\partial x_j} \right) + \frac{\epsilon_m}{2c_w} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2. \end{aligned} \quad (4.9)$$

Channel equations

We consider a fully-developed flow in a straight subglacial channel that is inclined at the (bed) angle α_b . For the moment, let us adopt the coordinate system shown in Fig. 4.1, where $X = x_1$, $Y = x_2$, $Z = x_3$ (and the associated velocity components u , v , w) lie respectively in the downstream, cross-stream and normal directions. It follows that $\partial u_i / \partial X = 0$, $(g_1, g_2, g_3) = (g \sin \alpha_b, 0, -g \cos \alpha_b)$, and our model equations reduce to

$$\begin{aligned} \frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z} &= 0, \\ v \frac{\partial u}{\partial Y} + w \frac{\partial u}{\partial Z} &= -\frac{1}{\rho_w} \frac{\partial p}{\partial X} + g \sin \alpha_b + \frac{\partial}{\partial Y} \left(\epsilon_m \frac{\partial u}{\partial Y} \right) + \frac{\partial}{\partial Z} \left(\epsilon_m \frac{\partial u}{\partial Z} \right), \end{aligned} \quad (4.10)$$

$$\begin{aligned} v \frac{\partial v}{\partial Y} + w \frac{\partial v}{\partial Z} &= -\frac{1}{\rho_w} \frac{\partial p}{\partial Y} + \frac{\partial}{\partial Y} \left(2\epsilon_m \frac{\partial v}{\partial Y} \right) + \frac{\partial}{\partial Z} \left[\epsilon_m \left(\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y} \right) \right], \\ v \frac{\partial w}{\partial Y} + w \frac{\partial w}{\partial Z} &= -\frac{1}{\rho_w} \frac{\partial p}{\partial Z} - g \cos \alpha_b + \frac{\partial}{\partial Y} \left[\epsilon_m \left(\frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z} \right) \right] + \frac{\partial}{\partial Z} \left(2\epsilon_m \frac{\partial w}{\partial Z} \right), \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} u \frac{\partial T}{\partial X} + v \frac{\partial T}{\partial Y} + w \frac{\partial T}{\partial Z} &= \frac{\partial}{\partial X} \left(\epsilon_h \frac{\partial T}{\partial X} \right) + \frac{\partial}{\partial Y} \left(\epsilon_h \frac{\partial T}{\partial Y} \right) + \frac{\partial}{\partial Z} \left(\epsilon_h \frac{\partial T}{\partial Z} \right) \\ &\quad + \frac{\epsilon_m}{c_w} \left\{ 2 \left[\left(\frac{\partial v}{\partial Y} \right)^2 + \left(\frac{\partial w}{\partial Z} \right)^2 \right] \right. \\ &\quad \left. + \left(\frac{\partial u}{\partial Y} \right)^2 + \left(\frac{\partial u}{\partial Z} \right)^2 + \left[\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y} \right]^2 \right\}. \end{aligned} \quad (4.12)$$

We specify no slip at the channel boundaries; for convenience, we also assume that both the ice (roof) and the bed are at the melting point. Therefore, the boundary conditions are

$$u(X, Y, 0) = u(X, Y, h) = T(X, Y, 0) = T(X, Y, h) = 0, \quad \text{for } |Y| \leq l. \quad (4.13)$$

Note that fixing the bed temperature may not be as realistic as fixing the heat flux there, but for our purpose this approximation will suffice. The downstream hydraulic gradient in (4.11)₁ will be prescribed. As we shall see, this eliminates the need to impose boundary conditions for p (see later), so these are omitted here.

4.3 Non-dimensionalization

4.3.1 Mass and momentum equations

We assign the scales $D_0, l_0, h_0, [p], U, V, W, \epsilon_0$ for the coordinates X, Y, Z , pressure p , velocities u, v, w , and eddy viscosity ϵ_m . (We put $l_0 = l$.) If we take

$$W = \delta V \quad (4.14)$$

where

$$\delta = \frac{h_0}{l_0} (\ll 1) \quad (4.15)$$

(cf. Equation (4.1)), then re-scaling of the mass equation leads to (4.10) as before. The momentum equations become

$$\begin{aligned} \frac{UV}{l_0} \left(v \frac{\partial u}{\partial Y} + w \frac{\partial u}{\partial Z} \right) &= -\frac{[p]}{\rho_w D_0} \frac{\partial p}{\partial X} + g \sin \alpha_b + \frac{\epsilon_0 U}{h_0^2} \left[\delta^2 \frac{\partial}{\partial Y} \left(\epsilon_m \frac{\partial u}{\partial Y} \right) + \frac{\partial}{\partial Z} \left(\epsilon_m \frac{\partial u}{\partial Z} \right) \right], \\ \frac{V^2}{l_0} \left(v \frac{\partial v}{\partial Y} + w \frac{\partial v}{\partial Z} \right) &= -\frac{[p]}{\rho_w l_0} \frac{\partial p}{\partial Y} + \frac{\epsilon_0 V}{h_0^2} \left\{ \delta^2 \frac{\partial}{\partial Y} \left(2\epsilon_m \frac{\partial v}{\partial Y} \right) + \frac{\partial}{\partial Z} \left[\epsilon_m \left(\frac{\partial v}{\partial Z} + \delta^2 \frac{\partial w}{\partial Y} \right) \right] \right\}, \\ \frac{VW}{l_0} \left(v \frac{\partial w}{\partial Y} + w \frac{\partial w}{\partial Z} \right) &= -\frac{[p]}{\rho_w h_0} \frac{\partial p}{\partial Z} - g \cos \alpha_b + \frac{\epsilon_0 V}{h_0 l_0} \left\{ \frac{\partial}{\partial Y} \left[\epsilon_m \left(\delta^2 \frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z} \right) \right] \right. \end{aligned}$$

$$+\frac{\partial}{\partial Z}\left(2\epsilon_m\frac{\partial w}{\partial Z}\right)\}, \quad (4.16)$$

in which the variables are now dimensionless. In the following, we discuss the appropriate choice for the scales in these equations.

The pressure scale

Since the channel is wide and low, we presume that the pressure distribution p is essentially ‘hydrostatic’ (uniform with Z). Thus, we anticipate that dimensionally

$$p \approx p(Z = h) + \rho_w g \cos \alpha_b (h - Z), \quad (4.17)$$

where, by definition

$$p(Z = h) \equiv p_c(X, Y) = p_i(X) - N_c(X, Y), \quad (4.18)$$

and p_c and N_c are the pressure variables used in Chapters 2 and 3. The overburden ice pressure p_i is effectively a function of X only. Regarding the $\partial p/\partial Y$ and $\partial p/\partial Z$ terms in (4.16), the scale $[p]$ should therefore be associated with the effective pressure N_c , and not p itself (which is relatively close to p_i). We choose

$$[p] = [N_c] = N_0, \quad (4.19)$$

where $N_0 \sim 1$ to 10 bar is typical for channelized subglacial drainage (see Chapter 2).

Downstream momentum balance

An approximate P.D.E. for u may be derived from Equation (4.16)₁, based on the balance between the driving hydraulic gradient and the resulting turbulent flow stress. By using (4.17) and (4.18), we first write (dimensionally)

$$-\frac{1}{\rho_w} \frac{\partial p}{\partial X} + g \sin \alpha_b \approx \frac{1}{\rho_w} \left(\Phi + \frac{\partial N_c}{\partial X} \right), \quad (4.20)$$

where Φ is the prescribed gradient $\rho_w g \sin \alpha_b - \partial p_i/\partial X$, already defined in Equation (2.19). If we impose

$$\Phi_0 = \frac{\rho_w \epsilon_0 U}{h_0^2} \quad (= [\Phi] = \Phi), \quad (4.21)$$

and in addition assume that

$$D_0 \gg \frac{N_0}{\Phi_0}, \quad (4.22)$$

then (4.16)₁ reduces to the dimensionless equation

$$\frac{\partial}{\partial Z} \left(\epsilon_m \frac{\partial u}{\partial Z} \right) + \delta^2 \frac{\partial}{\partial Y} \left(\epsilon_m \frac{\partial u}{\partial Y} \right) + 1 \approx R_t \left(v \frac{\partial u}{\partial Y} + w \frac{\partial u}{\partial Z} \right). \quad (4.23)$$

Here R_t is an inertia : turbulent stress ratio, given by

$$R_t = \frac{\delta V}{U} R_\epsilon, \quad (4.24)$$

and

$$R_\epsilon = \frac{Uh_0}{\epsilon_0} \quad (4.25)$$

is a Reynolds number based on the eddy viscosity scale.

The eddy viscosity function

In the central part of the wide channel, both the ice and bed boundaries have low curvature and are nearly parallel to each other. Consequently, the vertical velocity distribution (at fixed Y) approximates that in a turbulent flow between a pair of parallel flat plates, and this enables us to prescribe Prandtl's equation

$$\epsilon_m = L_h^2 \left| \frac{\partial u}{\partial Z} \right| \quad (4.26)$$

to parameterize the eddy viscosity (Schlichting, 1960). L_h is a *mixing length* function that has a universal function form, with the length scale

$$[L_h] = \beta h_0, \quad (4.27)$$

where β (~ 0.1 ; e.g. Schlichting, 1960; Fig. 20.5) is a dimensionless function of the flow Reynolds number and the wall roughness. These equations suggest that a suitable scale relation is

$$\epsilon_0 = \beta^2 h_0 U, \quad (4.28)$$

which provides a second equation for determining ϵ_0 and U . (We also obtain $R_\epsilon = 1/\beta^2$.) It is important to note that approximation (4.26) will break down where the velocity contours cease to be parallel (e.g. at the channel margins). This is discussed in Section 4.5.

Alternatively, a scale relation (for U) may be derived from the friction parameterization used in Section 2.5: Equations (2.44)_{2,3} lead to

$$\Phi_0 h_0 = \frac{1}{8} f^* \rho_w U^2. \quad (4.29)$$

The solutions of (4.21) with (4.28) (and also with (4.29)) are

$$U = \frac{1}{\beta} \sqrt{\frac{\Phi_0 h_0}{\rho_w}} = \sqrt{\frac{8\Phi_0 h_0}{f^* \rho_w}}, \quad \epsilon_0 = \beta \sqrt{\frac{\Phi_0}{\rho_w}} h_0^{3/2} = \sqrt{\frac{\Phi_0 f^*}{8\rho_w}} h_0^{3/2}. \quad (4.30)$$

We find the equivalence relation $f^* = 8\beta^2$, consistent with typical values of β (~ 0.1) and f^* (≈ 0.01 to 0.1 ; Chapter 2).

Secondary motion

It is desirable to decouple (4.23) from the other two momentum equations. This reduces the complexity of our problem, but a necessary condition is that

$$R_t = \frac{\delta V}{\beta^2 U} \ll 1. \quad (4.31)$$

If $\delta \approx 0.1$, then we require $V/U \ll 0.1$. (Remember that $W = \delta V$ from (4.14).) Thus, the question is whether or not the secondary flow within the channel cross-section is weak and negligible.

We assume that this is indeed the case, on the basis that in long straight channels, it is commonly observed that the velocities v and w are non-zero, but they are at most several percent of the streamwise velocity u (e.g. Demuren and Rodi, 1984)¹. In Section 4.4, we shall therefore set R_t to zero and discard the inertial terms (r.h.s.), and then Equation (4.23) becomes a closed problem for u .

The argument given above is not rigorous, as we have not established the ratio V/U from any fundamental derivation. It is worth noting however, that there is to date no satisfactory explanation for the observed secondary flow strength. This is a well recognized problem in fluid mechanics. Brundrett and Baines (1964) have reduced Equations (4.11)₂ and (4.11)₃ to a vorticity conservation equation (for the Y – Z plane), and thus identified the Reynolds stress gradients as the cause of secondary motion. However, the parameterization of these turbulence terms is problematic. Notably, simple isotropic formulations (such as our eddy viscosity model) have been shown to be incapable of reproducing secondary motion (Nakayama and Chow, 1983). Some experimental effort has been directed to ascertain more sophisticated parameterizations for turbulence (e.g. the ‘ κ – ϵ ’ model; Nakayama and Chow, 1983; Demuren and Rodi, 1984), but the models show inconsistencies, and require too excessive an amount of calibration in order to be useful here.

Lubrication approximation

Armed with $\delta \ll 1$, $V/U \ll 1$ (and later, $R_t \ll 1$), we write Equations (4.16)₂ and (4.16)₃ in the form

$$\begin{aligned} \frac{\partial p}{\partial Y} &= \frac{\Phi_0 l_0}{N_0} \frac{V}{U} \left[(\text{stress terms}) + R_t \times (\text{inertial terms}) \right], \\ \frac{\partial p}{\partial Z} &= -\frac{\rho_w g h_0}{N_0} \cos \alpha_b + \delta^2 \frac{\Phi_0 l_0}{N_0} \frac{V}{U} \left[(\text{stress terms}) + R_t \times (\text{inertial terms}) \right]. \end{aligned} \quad (4.32)$$

With the nominal values $N_0 = 10^5$ Pa, $\Phi_0 = 10^2$ kg m^{–2} s^{–2}, $h_0 = 1$ m, $l_0 = 10$ m, $\beta = 0.1$, typical scales obtained from (4.30) are

$$U \approx 3 \text{ m s}^{-1}, \quad \epsilon_m \approx 0.03 \text{ m}^2 \text{ s}^{-1}, \quad (4.33)$$

and also we find $\Phi_0 l_0 / N_0 = 0.01$. It follows that

$$p_Y \approx 0, \quad p_Z \approx -\frac{\rho_w g h_0}{N_0} \cos \alpha_b \approx -0.1 \cos \alpha_b, \quad (4.34)$$

which confirms our earlier assumption of the pressure distribution. Moreover, Equations (4.34)₁ and (4.18) imply an effective pressure N_c that is essentially *constant* across the channel. This result has been used in Section 3.4 (see also Section 5.2.2).

¹The secondary motion here is purely turbulence-driven. We neglect *curved* channels, in which secondary motion is also possible, but is due to centrifugal forces and may have higher values.

4.3.2 Energy equation

Finally, we introduce the additional scales $[T] = \theta_0$ and $[\epsilon_h] = \epsilon_0$ (since $\epsilon_h \approx \epsilon_m$; see later). By choosing

$$U^2 = c_w \theta_0, \quad (4.35)$$

non-dimensionalization of Equation (4.12) leads to

$$\begin{aligned} \frac{\delta}{\beta^2} \left[\frac{l_0}{D_0} \left(u \frac{\partial T}{\partial X} \right) + \frac{V}{U} \left(v \frac{\partial T}{\partial Y} + w \frac{\partial T}{\partial Z} \right) \right] \\ = \left(\delta \frac{l_0}{D_0} \right)^2 \frac{\partial}{\partial X} \left(\epsilon_h \frac{\partial T}{\partial X} \right) + \delta^2 \frac{\partial}{\partial Y} \left(\epsilon_h \frac{\partial T}{\partial Y} \right) + \frac{\partial}{\partial Z} \left(\epsilon_h \frac{\partial T}{\partial Z} \right) \\ + \epsilon_m \left\{ 2 \left(\frac{\delta V}{U} \right)^2 \left[\left(\frac{\partial v}{\partial Y} \right)^2 + \left(\frac{\partial w}{\partial Z} \right)^2 \right] \right. \\ \left. + \delta^2 \left(\frac{\partial u}{\partial Y} \right)^2 + \left(\frac{\partial u}{\partial Z} \right)^2 + \left(\frac{V}{U} \right)^2 \left[\frac{\partial v}{\partial Z} + \delta^2 \frac{\partial w}{\partial Y} \right]^2 \right\}. \end{aligned} \quad (4.36)$$

We assume a distance scale $D_0 \sim 10^4$ m, such that $l_0/D_0 \ll V/U$ ($\ll 1$); this allows us to discard the X -derivatives, and is consistent with the condition $D_0 \gg 10^3$ m imposed by Equation (4.22). We also suppose that $V/U < \delta$ (~ 0.1) — if $\delta \approx V/U$ then we have effectively a sheet flow (e.g. Walder, 1982) — whereby (4.36) may be approximated as

$$\frac{\partial}{\partial Z} \left(\epsilon_h \frac{\partial T}{\partial Z} \right) + \delta^2 \frac{\partial}{\partial Y} \left(\epsilon_h \frac{\partial T}{\partial Y} \right) + \epsilon_m \left[\left(\frac{\partial u}{\partial Z} \right)^2 + \delta^2 \left(\frac{\partial u}{\partial Y} \right)^2 \right] \approx R_t \left(v \frac{\partial T}{\partial Y} + w \frac{\partial T}{\partial Z} \right). \quad (4.37)$$

Normal heat flux

Given the (dimensional) temperature field $T(Y, Z)$, the heat flux vector is $\mathbf{q} = -\rho_w c_w \epsilon_h \nabla T$, so the heat flow to the ice is $\hat{q}_i = \mathbf{q} \cdot \hat{\mathbf{n}}$ (evaluated at $Z = h$), where $\hat{\mathbf{n}}$ is the unit normal vector $(-h_Y \hat{\mathbf{Y}} + \hat{\mathbf{Z}})/\sqrt{1 + h_Y^2}$. We obtain (in dimensional terms)

$$\hat{q}_i(Y) = -\frac{\rho_w c_w \epsilon_h}{\sqrt{1 + h_Y^2}} \left(-h_Y \frac{\partial T}{\partial Y} + \frac{\partial T}{\partial Z} \right) \Big|_{Z=h}. \quad (4.38)$$

If we define the flux scale

$$[\mathbf{q}] = [\hat{q}_i] = \rho_w c_w \beta^2 U \theta_0, \quad (4.39)$$

then the relevant dimensionless model is

$$\begin{aligned} \mathbf{q} &= -\epsilon_h \left(\delta \frac{\partial T}{\partial Y} \hat{\mathbf{Y}} + \frac{\partial T}{\partial Z} \hat{\mathbf{Z}} \right), \\ \hat{q}_i(Y) &= -\frac{\epsilon_h}{\sqrt{1 + (\delta h_Y)^2}} \left(-\delta^2 h_Y \frac{\partial T}{\partial Y} + \frac{\partial T}{\partial Z} \right) \Big|_{Z=h}. \end{aligned} \quad (4.40)$$

Our formulation is now complete.

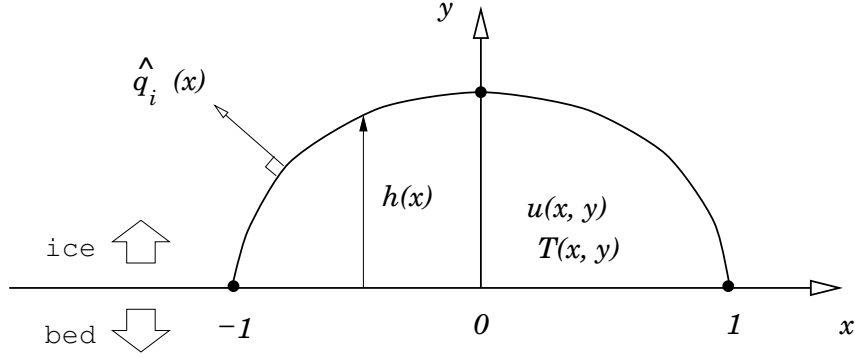


Figure 4.2: *Mathematical symbols and coordinate system of the dimensionless model in Section 4.4.*

4.4 Singular perturbation problem

We revert to our usual 2-D coordinate system, taking x ($= Y$) and y ($= Z$) to be respectively in the cross-stream and normal directions (Fig. 4.2; see also Chapter 3). To summarize, our model is one where the cross sectional pressure distribution $p(x, y)$ is nearly constant (and hydrostatic), and downstream dependence has been removed. The key dimensionless equations are

$$\begin{aligned} \frac{\partial}{\partial y} \left(\epsilon_m \frac{\partial u}{\partial y} \right) + \delta^2 \frac{\partial}{\partial x} \left(\epsilon_m \frac{\partial u}{\partial x} \right) + 1 &\approx R_t \left(v \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial y} \right), \\ \frac{\partial}{\partial y} \left(\epsilon_h \frac{\partial T}{\partial y} \right) + \delta^2 \frac{\partial}{\partial x} \left(\epsilon_h \frac{\partial T}{\partial x} \right) + \epsilon_m \left[\left(\frac{\partial u}{\partial y} \right)^2 + \delta^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] &\approx R_t \left(v \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial y} \right), \\ u(x, 0) = u(x, h) = T(x, 0) = T(x, h) &= 0, \quad \text{for } |x| \leq 1, \end{aligned} \quad (4.41)$$

in which δ and R_t are small parameters, $\epsilon_{m,h}$ are functions of position (x, y) , and $h(x)$ is given. The normal heat flux to the ice is given by

$$\hat{q}_i(x) = - \frac{\epsilon_h}{\sqrt{1 + (\delta h_x)^2}} \left(-\delta^2 h_x \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} \right) \Big|_{y=h}. \quad (4.42)$$

Currently, we have $\beta \sim 0.1$, $V/U \sim 0.01$, such that $R_t \sim \delta$ ($\ll 1$). In (4.41), neglect of the inertial terms (on the right, by taking $R_t = 0$) would correspond to negligible secondary flow, whereas neglect of the x -derivatives (on the left, by taking $\delta = 0$) would correspond to negligible lateral diffusion. Here we apply only the former approximation (despite $R_t \gg \delta^2$), the reason being that the latter is *singular* — the x -derivatives may ‘blow up’ at the margins $x = \pm 1$, thus become responsible for boundary layers² where u and T are adjusted to zero. If this is the case, these derivatives will also be important in determining the velocity and temperature

²The boundary layer thickness is dependent on the end behaviour of $h(x)$, and is inferred later in Section 4.4.2.

contours at the tip, which control the melt-rate there. On the other hand, secondary circulation causes distortion of the contours (Schlichting, 1960), but we suppose that the subsequent effect on heat transfer is minor.

Our model now reduces to the pair of elliptic equations

$$\frac{\partial}{\partial y} \left(\epsilon_m \frac{\partial u}{\partial y} \right) + \delta^2 \frac{\partial}{\partial x} \left(\epsilon_m \frac{\partial u}{\partial x} \right) + 1 = 0, \quad (4.43)$$

$$\frac{\partial}{\partial y} \left(\epsilon_h \frac{\partial T}{\partial y} \right) + \delta^2 \frac{\partial}{\partial x} \left(\epsilon_h \frac{\partial T}{\partial x} \right) + \epsilon_m \left[\left(\frac{\partial u}{\partial y} \right)^2 + \delta^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] = 0 \quad (4.44)$$

for $u(x, y)$ (downstream flow velocity) and $T(x, y)$ (flow temperature), with the boundary conditions shown in (4.41). We investigate their solutions in the following sections.

4.4.1 The outer solutions

Leading order terms

Since $\delta \ll 1$, let us consider

$$\begin{aligned} u &= u_0 + \delta^2 u_1 + \delta^4 u_2 + \dots \\ T &= T_0 + \delta^2 T_1 + \delta^4 T_2 + \dots \end{aligned} \quad (4.45)$$

as the perturbation expansion solutions of (4.43) and (4.44). The leading order equations are

$$\begin{aligned} \frac{\partial}{\partial y} \left(\epsilon_m \frac{\partial u_0}{\partial y} \right) + 1 &= 0, \quad \frac{\partial}{\partial y} \left(\epsilon_h \frac{\partial T_0}{\partial y} \right) + \epsilon_m \left(\frac{\partial u_0}{\partial y} \right)^2 = 0, \\ \text{with } u_0 = T_0 &= 0 \quad \text{at } y = 0, \quad y = h(x). \end{aligned} \quad (4.46)$$

As described before, $h'(x) \ll 1$ in the central part of the channel allows us to prescribe Prandtl's equation (4.26), written here as

$$\epsilon_m = L_h^2 \left| \frac{\partial u_0}{\partial y} \right| \quad (4.47)$$

(dimensional). The *mixing length* L_h (a function of y) is a measure of the typical distance over which the fluid particles are transported by turbulent mixing (Schlichting, 1960). It is scaled to the depth h , and its form can only be determined from experiments: close to the boundaries $L_h \approx \kappa \Delta$, where Δ is the normal distance from the wall, $\kappa = 0.4$ is the von Kármán constant; and L_h reaches its maximum at $y = h/2$. See Fig. 4.3a.

To incorporate L_h more conveniently, we introduce the normalized vertical coordinate

$$\zeta = \frac{y}{h(x)}, \quad 0 \leq \zeta(x, y) \leq 1, \quad (4.48)$$

and also the dimensionless function Λ , defined by

$$\frac{L_h}{h} = \beta \Lambda(\zeta). \quad (4.49)$$

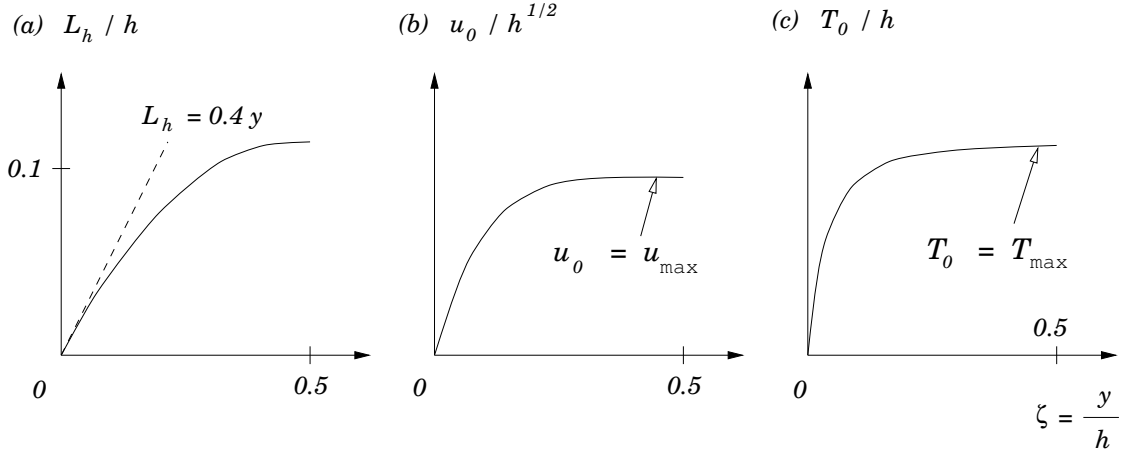


Figure 4.3: Typical distributions of (a) the mixing length, (b) the flow velocity, and (c) the flow temperature as functions of depth in an idealized parallel channel.

Then, re-scaling of this equation and (4.47), with $[y] = [h] = h_0$, $[u_0] = U$, $[L_h] = \beta h_0$ and $\epsilon_0 = \beta^2 h_0 U$, leads to

$$L_h(\zeta) = h\Lambda(\zeta), \quad \epsilon_m = \frac{L_h^2}{h} \left| \frac{\partial u_0}{\partial \zeta} \right| = h\Lambda^2 \left| \frac{\partial u_0}{\partial \zeta} \right|, \quad (4.50)$$

and Equation (4.46)₁ (already dimensionless) becomes

$$\frac{1}{h} \frac{\partial}{\partial \zeta} \left(\Lambda^2 \frac{\partial u_0}{\partial \zeta} \left| \frac{\partial u_0}{\partial \zeta} \right| \right) + 1 = 0, \quad u_0 = 0 \text{ at } \zeta = 0, 1. \quad (4.51)$$

On noting the symmetry of $\Lambda(\zeta)$ and $u_0(\zeta)$ about $\zeta = 1/2$, straightforward integration produces

$$u_0(x, \zeta) = \left(\frac{h}{2} \right)^{1/2} \int_{\zeta_r}^{\zeta} \frac{\sqrt{1-2\eta}}{\Lambda(\eta)} d\eta \quad (4.52)$$

and

$$\epsilon_m(x, \zeta) = h^{3/2} \Lambda(\zeta) \sqrt{\frac{1-2\zeta}{2}}, \quad (4.53)$$

for $\zeta_r \leq \zeta \leq \frac{1}{2}$, where $\zeta_r (\ll 1) > 0$.

In (4.52), $\zeta_r \neq 0$ is necessary in order that the integral converge (because $\Lambda(\zeta)$ is linear as $\zeta \rightarrow 0$). This lower limit of integration at ζ_r , where the velocity is equal to zero, is generally a function of the total flow depth and depends on the nature of the boundary. For a ‘hydraulically smooth’ wall, ζ_r corresponds to the thickness of the ‘laminar sub-layer’, dimensionally, $\approx 5\mu_w/(\rho_w \tilde{u})$, where $\tilde{u} = \sqrt{\Phi h/2\rho_w}$ is the local shear velocity of the flow (Schlichting, 1960; also see Section 6.2). It follows that $\zeta_r \propto h^{-3/2}$, and then the dependence of u_0 on h is very complicated. However, our flow situation is most likely to be ‘hydraulically rough’, with local protrusions at

the ice boundary reaching outside the (very thin) laminar sub-layer, contributing to a form drag. In this case, we can plausibly suppose that ζ_r ($\approx$ constant) $\ll 1$, which leads to $u_0(x, \zeta) \propto [h(x)]^{1/2}$ in (4.52). This is consistent with the Darcy-Weisbach friction formula (2.44)₃.

The leading order temperature solution may also be derived if ϵ_h is known. We invoke the *Reynolds analogy*, which asserts an equivalence between the turbulent transfer of heat and momentum (Schlichting, 1960), i.e.,

$$\epsilon_h = \epsilon_m = \epsilon. \quad (4.54)$$

(From experiments $\epsilon_h/\epsilon_m = 1$ to 1.6; Martinelli, 1947.) Written in composite form, Equation (4.46)₂ is then

$$\frac{\partial q_0}{\partial \zeta} = \frac{\epsilon}{h} \left(\frac{\partial u_0}{\partial \zeta} \right)^2 = \frac{1}{\Lambda} \left[\frac{h(1-2\zeta)}{2} \right]^{3/2}, \quad q_0\left(\frac{1}{2}\right) = 0; \quad (4.55)$$

$$q_0 = -\frac{\epsilon}{h} \frac{\partial T_0}{\partial \zeta} = -h^{1/2} \Lambda \sqrt{\frac{1-2\zeta}{2}} \frac{\partial T_0}{\partial \zeta}, \quad T_0(\zeta_r) = 0; \quad (4.56)$$

in which q_0 is the (vertical) dimensionless heat flux. The solutions are

$$q_0(x, \zeta) = -\left(\frac{h}{2}\right)^{3/2} \int_{\zeta}^{\frac{1}{2}} \frac{(1-2\eta)^{3/2}}{\Lambda(\eta)} d\eta, \quad (4.57)$$

$$T_0(x, \zeta) = \frac{h}{2} \int_{\zeta_r}^{\zeta} \frac{d\zeta}{\Lambda(\zeta)\sqrt{1-2\zeta}} \int_{\zeta}^{\frac{1}{2}} \frac{(1-2\eta)^{3/2}}{\Lambda(\eta)} d\eta, \quad (4.58)$$

for $\zeta_r \leq \zeta \leq \frac{1}{2}$, where T_0 (q_0) is symmetrical (anti-symmetrical) about $\zeta = 1/2$. Typical distributions of u_0 and T_0 are shown in Fig. 4.3b,c.

Higher order terms

One can proceed to calculate the higher order terms of the expansions in (4.45). For example, in terms of the independent variables x and ζ , the equation for u_1 is

$$\frac{1}{h^2} \frac{\partial}{\partial \zeta} \left(\epsilon \frac{\partial u_1}{\partial \zeta} \right) = - \left(\frac{\partial}{\partial x} - \frac{\zeta h_x}{h} \frac{\partial}{\partial \zeta} \right) \left[\epsilon \left(\frac{\partial u_0}{\partial x} - \frac{\zeta h_x}{h} \frac{\partial u_0}{\partial \zeta} \right) \right],$$

$$u_1(x, 0) = u_1(x, 1) = 0. \quad (4.59)$$

As we have $u_0 \propto h^{1/2}$ and $\epsilon \propto h^{3/2}$, here the source term (r.h.s.) goes as $(\epsilon u_{0x})_x \propto (h^2)_{xx}$. If the approximate local representation of the channel tip profile is

$$h(x) \propto (1 \mp x)^\nu, \quad \text{as } x \rightarrow \pm 1, \quad (4.60)$$

where ν (> 0) is a shape exponent, then the source term will be singular (at $x = \pm 1$) if $\nu < 1$. In this case, u_1 ($\propto h^{1/2}(h^2)_{xx}$) is more singular than u_0 . In fact, the higher order terms also become progressively more singular, rendering the expansion for $u(x, \zeta)$ invalid at the margins (at the ‘tips’). Thus clearly, (4.45)₁ applies only away from $x = \pm 1$ and is an *outer* solution. (The same is found to be true for the

T -expansion.) On the other hand, both expansions are uniformly valid in the whole of $|x| \leq 1$ when $\nu \geq 1$, then there are no end boundary layers. This illustrates the singular nature of the approximation $\delta \rightarrow 0$ for $\nu < 1$.

The dimensionless melt-rate

By substituting the temperature solution into (4.2) and (4.42), and by defining the scale

$$[\hat{m}] = \frac{[\hat{q}_i]}{L} = \frac{1}{L} \sqrt{\frac{8}{f^* \rho_w}} (\Phi_0 h_0)^{3/2}, \quad (4.61)$$

we find, to leading order,

$$\hat{m}(x) \sim \left(\frac{h}{2}\right)^{3/2} \int_{\zeta_r}^{\frac{1}{2}} \frac{(1-2\eta)^{3/2}}{\Lambda(\eta)} d\eta. \quad (4.62)$$

This expression, which has an error of $O(\delta^2)$, is valid everywhere for $\nu \geq 1$, and in the outer region (the central part of the channel) for $0 < \nu < 1$. The definite integral $\int_{\zeta_r}^{\frac{1}{2}} (1-2\eta)^{3/2} / \Lambda(\eta) d\eta$ is an $O(1)$ constant. Next, we investigate the corresponding problem near the margins given $\nu < 1$.

4.4.2 The inner problem

To derive the boundary layer equations, the model has to be re-scaled in the vicinity of $x = \pm 1$. Without loss of generality, we prescribe the channel shape

$$h(x) = (1 - x^2)^\nu, \quad \nu > 0 \quad (4.63)$$

(a hyper-ellipse) to mimic the end behaviour supposed by (4.60): $0 < \nu < 1$ refers to a ‘round’ tip profile, $\nu > 1$ refers to a ‘pointy’ tip profile, and the tip is ‘wedge-shaped’ for $\nu = 1$ (see Fig. 4.4).

We consider the range $0 < \nu < 1$, and as before we let $\epsilon_m = \epsilon_h = \epsilon$, but here we assume

$$\epsilon = h^{3/2} \Lambda(\zeta) \sqrt{\frac{|1-2\zeta|}{2}}, \quad \text{for } 0 < \zeta < 1, \quad (4.64)$$

following the parallel approximation result in (4.53). It is important to remember that this result may become invalid at the ends.

The attribute of (4.64) that is most relevant to the re-scaling process (and the boundary layer structure) is its x -dependence through h . In relation to this, let us invoke a ‘toy’ model by replacing ϵ by $\bar{\epsilon}$, where

$$\bar{\epsilon} = h^{3/2}, \quad (4.65)$$

thereby removing the algebraic complications associated with ζ . ($\Lambda \sqrt{|1-2\zeta|}/2$ is an $O(1)$ function.) Here $\bar{\epsilon}$ is effectively a depth-averaged value for ϵ , representative of its cross-stream variation.

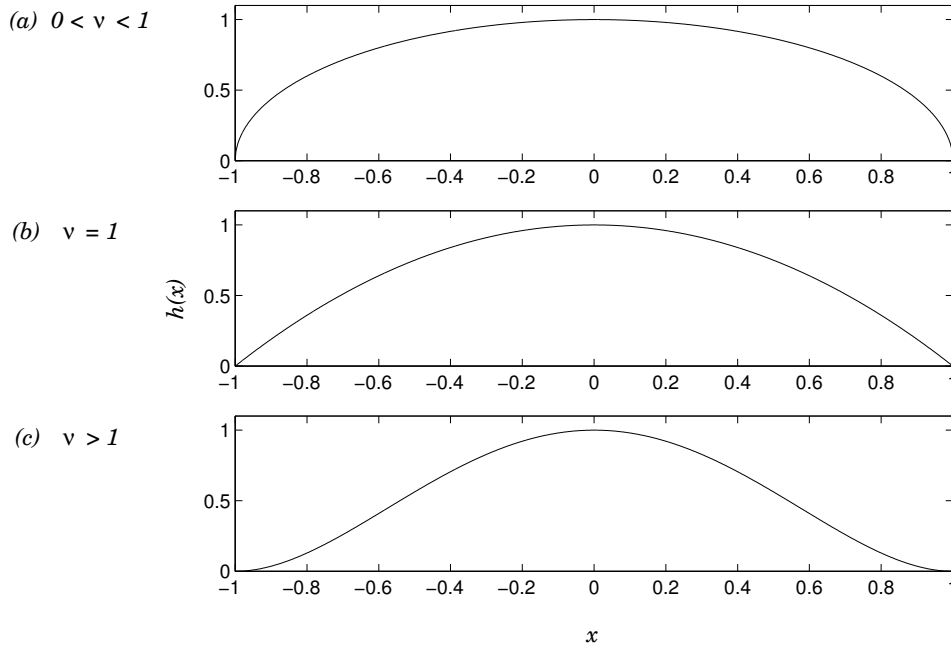


Figure 4.4: The channel depth function $h(x)$ for three different ranges of ν values, with the corresponding (a) round, (b) wedged-shaped, and (c) pointy tip profiles.

Our model now reduces to

$$\begin{aligned} \frac{\partial}{\partial y} \left(h^{3/2} \frac{\partial u}{\partial y} \right) + \delta^2 \frac{\partial}{\partial x} \left(h^{3/2} \frac{\partial u}{\partial x} \right) + 1 &= 0, \\ \frac{\partial}{\partial y} \left(h^{3/2} \frac{\partial T}{\partial y} \right) + \delta^2 \frac{\partial}{\partial x} \left(h^{3/2} \frac{\partial T}{\partial x} \right) + h^{3/2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \delta^2 \left(\frac{\partial u}{\partial x} \right)^2 \right] &= 0, \\ u = T = 0, \quad \text{on } y = 0, \quad y = h(x), \quad \text{for } |x| \leq 1. \end{aligned} \quad (4.66)$$

Owing to symmetry, the solutions will be even functions of x , thus either end may be considered. Accordingly, we define the inner variables

$$x^* = \frac{1 \pm x}{[x]}, \quad y^* = \frac{y}{[y]}, \quad u^* = \frac{u}{[u]}, \quad T^* = \frac{T}{[T]} \quad (4.67)$$

for the ends $x = \mp 1$, where $[x]$, $[y]$, $[u]$ and $[T]$ are determined from scaling at the distinguished limit. In $(4.66)_1$, the leading order outer solution is $u \propto h^{1/2}$, so the singular x -derivative is $\propto \delta^2 (h^2)_{xx}$, which becomes significant when $(1 \pm x) \sim \delta^{\frac{1}{1-\nu}}$. It follows that the appropriate length scale is

$$[x] = \delta^{\frac{1}{1-\nu}} \quad (4.68)$$

(this is the boundary layer thickness); and then, we have

$$[y] = \delta^{\frac{\nu}{1-\nu}}, \quad [u] = \delta^{\frac{\nu}{2(1-\nu)}}, \quad [T] = \delta^{\frac{\nu}{1-\nu}}. \quad (4.69)$$

Re-scaling of (4.66) now leads to

$$\begin{aligned}
x^{\star \frac{3\nu}{2}} (2 - \delta^{\frac{1}{1-\nu}} x^{\star})^{\frac{3\nu}{2}} \frac{\partial^2 u^{\star}}{\partial y^{\star 2}} + \frac{\partial}{\partial x^{\star}} \left[x^{\star \frac{3\nu}{2}} (2 - \delta^{\frac{1}{1-\nu}} x^{\star})^{\frac{3\nu}{2}} \frac{\partial u^{\star}}{\partial x^{\star}} \right] + 1 &= 0, \\
x^{\star \frac{3\nu}{2}} (2 - \delta^{\frac{1}{1-\nu}} x^{\star})^{\frac{3\nu}{2}} \frac{\partial^2 T^{\star}}{\partial y^{\star 2}} + \frac{\partial}{\partial x^{\star}} \left[x^{\star \frac{3\nu}{2}} (2 - \delta^{\frac{1}{1-\nu}} x^{\star})^{\frac{3\nu}{2}} \frac{\partial T^{\star}}{\partial x^{\star}} \right] \\
+ x^{\star \frac{3\nu}{2}} (2 - \delta^{\frac{1}{1-\nu}} x^{\star})^{\frac{3\nu}{2}} \left[\left(\frac{\partial u^{\star}}{\partial y^{\star}} \right)^2 + \left(\frac{\partial u^{\star}}{\partial x^{\star}} \right)^2 \right] &= 0, \\
u^{\star} = T^{\star} = 0 \quad \text{on} \quad y^{\star} = 0, \quad y^{\star} = (2x^{\star})^{\nu} + O(\delta^{\frac{1}{1-\nu}}), &\quad (4.70)
\end{aligned}$$

which is a regular perturbation problem for $u^{\star}$ and $T^{\star}$. If we pose the inner expansions

$$\begin{aligned}
u^{\star} &= u_0^{\star} + \delta^{\frac{1}{1-\nu}} u_1^{\star} + \delta^{\frac{2}{1-\nu}} u_2^{\star} + \dots, \\
T^{\star} &= T_0^{\star} + \delta^{\frac{1}{1-\nu}} T_1^{\star} + \delta^{\frac{2}{1-\nu}} T_2^{\star} + \dots,
\end{aligned} \quad (4.71)$$

then the leading order problem is

$$\begin{aligned}
(2x^{\star})^{\frac{3\nu}{2}} \frac{\partial^2 u_0^{\star}}{\partial y^{\star 2}} + \frac{\partial}{\partial x^{\star}} \left[(2x^{\star})^{\frac{3\nu}{2}} \frac{\partial u_0^{\star}}{\partial x^{\star}} \right] + 1 &= 0, \\
(2x^{\star})^{\frac{3\nu}{2}} \frac{\partial^2 T_0^{\star}}{\partial y^{\star 2}} + \frac{\partial}{\partial x^{\star}} \left[(2x^{\star})^{\frac{3\nu}{2}} \frac{\partial T_0^{\star}}{\partial x^{\star}} \right] + (2x^{\star})^{\frac{3\nu}{2}} \left[\left(\frac{\partial u_0^{\star}}{\partial y^{\star}} \right)^2 + \left(\frac{\partial u_0^{\star}}{\partial x^{\star}} \right)^2 \right] &= 0, \\
u_0^{\star} = T_0^{\star} = 0 \quad \text{on} \quad y^{\star} = 0, \quad y^{\star} = (2x^{\star})^{\nu}. &\quad (4.72)
\end{aligned}$$

These equations are elliptic, and contain degenerate x -derivatives due to ϵ ($\propto h^{3/2}$) vanishing at the ends. The value of ν (< 1) is also non-integer. As such, the solution of (4.72) is notoriously difficult.

Given this difficulty, and the fact that the validity of (4.64) may itself break down at the ends (particularly for this $\nu < 1$ case, corresponding to a round tip), the exact inner solution here is probably not worth pursuing. In the next section, we shall return to estimate the tip value of the normal heat flux (and melt-rate) via a different but much cruder method.

4.5 Width evolution: hard bed

4.5.1 Derivation

Equations (3.1), (4.2) and (4.3) imply that for a hard bed subglacial channel, the margin velocity is given (in dimensional terms) by

$$\frac{dl}{dt} = \lim_{x \rightarrow l} \left[- \frac{\hat{q}_i \sqrt{1 + h_x^2}}{\rho_i L h_x} + \frac{\hat{w}}{h_x} \right], \quad (4.73)$$

in which $\hat{q}_i$ is the normal heat flux (discussed in Section 4.4), and $\hat{w}$ is the generalized non-linear closure rate

$$\hat{w}(x) = \frac{N_c^3}{2\eta_0} (l^2 - x^2)^{1/2} \quad (4.74)$$

derived in Chapter 3. As before, we suppose a channel depth profile of the form $h(x) = A_1(l^2 - x^2)^\nu$, where $A_1 (= h_0/l_0^{2\nu})$ is constant (cf. Equation (4.63)). We seek the behaviour of dl/dt as a function of ν .

Pointy/wedge-shaped tip: $\nu \geq 1$

For $\nu \geq 1$, we have from Section 4.4.1 $\hat{q}_i(x) \propto h^{3/2} \propto (l^2 - x^2)^{3\nu/2}$. Substituting this into (4.73) gives

$$\frac{dl}{dt} = \lim_{x \rightarrow l} [-A_2(l-x)^{\frac{3}{2}-\nu}] = \begin{cases} 0 & 1 \leq \nu < 3/2 \\ -A_3 & \nu = 3/2 \\ -\infty & \nu > 3/2, \end{cases} \quad (4.75)$$

where

$$A_2 = \frac{N_c^3(2l)^{\frac{1}{2}-\nu}}{2A_1\nu\eta_0}, \quad A_3 = \frac{N_c^3l}{6\delta\eta_0}. \quad (4.76)$$

Therefore, the margin can retreat if it is ‘pointy’ enough ($\nu \geq 3/2$), or remain stationary if it is not ($1 \leq \nu < 3/2$). In particular, dl/dt is (negative) infinite when $\nu > 3/2$, but this has the effect of ‘blunting’ the tip (reducing ν). It follows that $\nu = 3/2$ represents a critical profile, at which the tip will retreat in a *stable* manner, and to which a tip with $\nu < 3/2$ must attain before it can retreat. Note that the limiting velocity A_3 is a factor of $(3\delta)^{-1}$ greater than the vertical closure rate at the channel centre.

Round tip: $0 < \nu < 1$

We fail to obtain an exact expression for $\hat{q}_i$ in the tip region. We may infer however, that the vanishing eddy viscosity ($\propto h^{3/2}$) assumed in Section 4.4.2 will lead to vanishing heat flux there, through the rôle of ϵ as a heat diffusivity. The question is whether this is true. Suppose it is, then we can write

$$\lim_{x \rightarrow l} \hat{q}_i(x) \propto (l-x)^\alpha, \quad \text{for } 0 < \nu < 1, \quad (4.77)$$

where α is some positive constant; but then evaluation of (4.73) leads to

$$\frac{dl}{dt} = 0, \quad \text{for } 0 < \nu < 1, \quad (4.78)$$

which is implausible, because it predicts a channel that is unable to widen out at all. We conclude that $\hat{q}_i(x \rightarrow \pm l)$ ought to be non-zero for at least some values of ν (< 1), thus allowing tip-advance. In the following, we propose a model for this by examining more closely the physics of turbulent mixing near the margins.

A geometry-based model for ϵ

Let us consider the eddy viscosity function. In Section 4.4.2 it was assumed that $\epsilon \propto h^{3/2}$. As we have noted before, the validity of this approximation is questionable at the margins, where the velocity contours may depart very much from those in an idealized parallel flow. This is especially the case if $0 < \nu < 1$.

We can attempt a crude extension of the ‘depth-averaged’ eddy viscosity function $\bar{\epsilon}$ at the tip, by using a generalized form of Prandtl’s equation (4.47):

$$\epsilon = L^2 \left| \frac{\partial u}{\partial n} \right|. \quad (4.79)$$

Here L is the mixing length as before, and $\partial u / \partial n$ is the velocity gradient normal to the velocity contours. To construct $\bar{\epsilon}$ from (4.79), we need to consider (i) the distance over which ϵ is to be averaged, and (ii) the overall magnitude of $|\partial u / \partial n|$.

Clearly, $\bar{\epsilon}$ should be allowed to vanish only if $|\partial u / \partial n| = 0$ throughout the averaging distance concerned, or if the averaging distance itself vanishes. This is because L is non-zero except at the walls. If $\nu > 1$, then $\bar{\epsilon} \propto h^{3/2}$ seems to be a reasonable approximation, because the tip boundaries converge in a parallel fashion, with u (and hence $|\partial u / \partial n|$) $\rightarrow 0$ at the ends — this is consistent with a vanishing value of $\bar{\epsilon}$. On the other hand if $0 < \nu < 1$, it is more difficult to establish whether $|\partial u / \partial n|$ (and/or the averaging distance) $\rightarrow 0$ at the tip, and $\bar{\epsilon}$ may well be non-zero for certain ν -values³. (Remember that in both cases, $\bar{\epsilon} \propto h^{3/2}$ is a good approximation for the central (outer) part of the channel.)

Given the preceding discussion, we propose the (dimensional) extended form

$$\bar{\epsilon} = \epsilon_0 (H/h_0)^{\frac{3}{2}} \quad (4.80)$$

for the eddy viscosity distribution, where h_0 , ϵ_0 are scales defined previously, and $H(x)$ is the normal distance from the top boundary (x, h) to the bed $y = 0$, shown in Fig. 4.5a. We regard Equation (4.80) as more appropriate than (the dimensional form of) Equation (4.65), because it takes into account the effect of the local slope $h'(x)$ on the flow contours and on the averaging distance. It also ensures a smooth transition of $\bar{\epsilon}$ from its tip value to $\propto h^{3/2}$ in the central region. In the dimensionless form, this model for $\bar{\epsilon}$ is

$$\begin{aligned} \bar{\epsilon} &= H^{3/2}, \\ H &= h \sqrt{1 + (\delta h')^2}, \end{aligned} \quad (4.81)$$

where $h = (1 - x^2)^\nu$ as before. Straightforward expansion shows that

$$H \sim h + O(\delta^2), \quad (\delta \ll 1), \quad (4.82)$$

which is valid everywhere for $\nu > 1$, but invalid in the tip region $(1 \pm x) \sim \delta^{\frac{1}{1-\nu}}$ for $0 < \nu < 1$. In the latter case, (4.82) is therefore the outer expansion. By re-scaling x and H with

$$[x] = \delta^{\frac{1}{1-\nu}}, \quad [H] = [y] = \delta^{\frac{\nu}{1-\nu}} \quad (4.83)$$

(as in Section 4.4.2), the corresponding inner expansion is

$$H^*(x^*) \sim (2x^*)^\nu \sqrt{1 + 4\nu^2 (2x^*)^{2(\nu-1)}} + O(\delta^{\frac{1}{1-\nu}}). \quad (4.84)$$

³We are concerned with the turbulent bulk flow, so the effect of laminar viscosity μ_w in the wall sub-layer and at acute corners has been ignored.

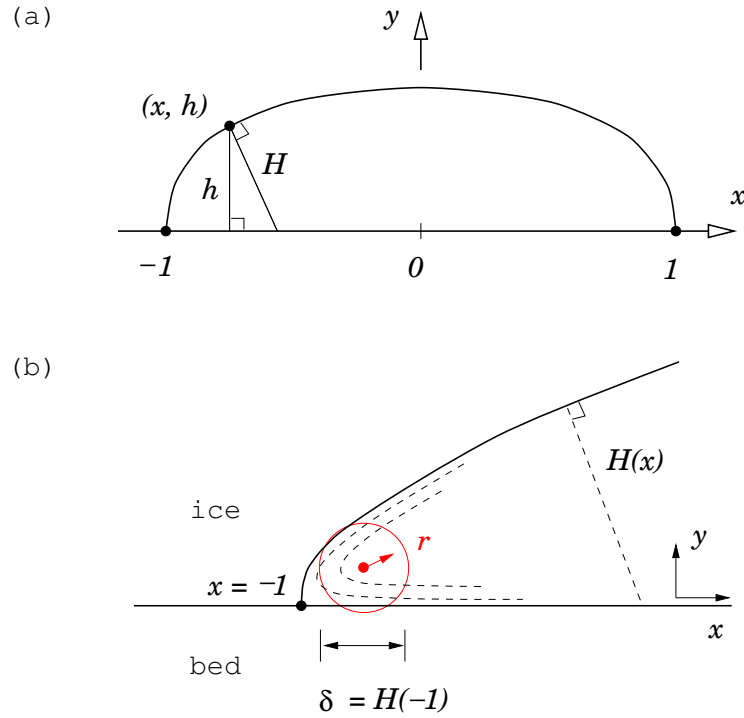


Figure 4.5: (a) The definition of $H(x)$. (b) Flow approximation using the turbulent flow in a ‘tip circle’ (red) of radius δ , for the limiting case $\nu = 1/2$.

At leading order, (4.84) has the local behaviour

$$\begin{aligned} H^*(x^* \rightarrow 0) &\approx 2^{2\nu} \nu x^{*2\nu-1} \\ &= \begin{cases} 0 & 1/2 < \nu \leq 1 \\ 1 & \nu = 1/2 \\ +\infty & 0 < \nu < 1/2, \end{cases} \end{aligned} \quad (4.85)$$

or, when written with the scales of the outer region,

$$H(x \rightarrow \pm 1) \approx \begin{cases} 0 & 1/2 < \nu \leq 1 \\ \delta & \nu = 1/2 \\ +\infty & 0 < \nu < 1/2. \end{cases} \quad (4.86)$$

As such, our revised model is essentially the same as before when $\nu > 1$, but it is capable of predicting a non-zero tip value of $\bar{\epsilon}$ when $0 < \nu < 1$. In particular, $\bar{\epsilon}$ vanishes for $\nu > 1/2$, is finite for $\nu = 1/2$ and blows up for $\nu < 1/2$.

Tip velocity estimate: $0 < \nu < 1$

We suppose that $\hat{q}_i(x \rightarrow 0)$ behaves in a similar way as $\bar{\epsilon}$. If $\nu < 1/2$, then the *normal* melt-rate at the tip is very high, but this has the effect of sharpening the tip (increasing ν). Thus theoretically, $\nu = 1/2$ represents a critical and stable tip

advance profile, to which a tip with $\nu > 1/2$ must build up before it can advance. (This mirrors the stable retreat scenario when $\nu = 3/2$.)

For this limiting case, the tip velocity may be estimated by using a flow model based on (4.66). If the original eddy viscosity coefficients ($h^{3/2}$) are replaced by $H^{3/2}$, where $H(x)$ is substituted from (4.81) (with $\nu = 1/2$), the resulting problem is still too difficult to solve. Instead, we can adopt a representative eddy viscosity by putting

$$H = \delta, \quad (4.87)$$

(based on the local approximation (4.86)₂), and then approximate the tip flow by a cylindrical turbulent flow of radius $H/2$ (see Fig. 4.5b). This is very crude but allows an order-of-magnitude melt-rate to be derived.

In terms of the radial coordinate r , an approximate model is therefore

$$\begin{aligned} \nabla_r^2 u_c + 1 &= 0, \\ \nabla_r^2 T_c + |\nabla_r u_c|^2 &= 0, \quad u_c = T_c = 0 \quad \text{on } r = \delta/2, \end{aligned} \quad (4.88)$$

where subscript c denotes the ‘tip circle’. Here the scales are $[u_c] = \delta^{-3/2}U$, $[T_c] = \delta^{-3}\theta_0$ and $[r] = h_0$, and the solutions are

$$u_c = \frac{1}{4} \left(\frac{\delta^2}{4} - r^2 \right), \quad T_c = \frac{1}{64} \left(\frac{\delta^4}{16} - r^4 \right). \quad (4.89)$$

Consequently, the normal melt-rate (scaled to $[\hat{m}]$ as before) is given by

$$m_c = -\frac{1}{\delta^{3/2}} \frac{\partial T_c}{\partial r_c} \Big|_{r=\delta/2} = \frac{\delta^{3/2}}{128}, \quad (4.90)$$

Note that this value is smaller than that which occurs at the channel centre, by a factor of roughly $10^{-2}\delta^{3/2}$ (cf. relation (4.62), where $\hat{m}$ is $O(1)$). The corresponding tip advance velocity estimate is

$$\frac{dl}{dt} = \frac{m_c[\hat{m}]}{\rho_i} = \frac{\delta^3}{64\rho_i L} \sqrt{\frac{2}{f^*\rho_w}} (\Phi_0 l)^{3/2}, \quad \text{for } \nu = \frac{1}{2}. \quad (4.91)$$

4.5.2 Results

We have shown that ice creep closure or heat dissipation melting can lead to retreat or advance of the margins, respectively, and these processes operate independently. In particular, *stable* tip advance (A) occurs at $\nu = 1/2$, *stable* tip retreat (R) occurs at $\nu = 3/2$, with limiting velocities

$$i_R = -\frac{N_c^3 l}{6\delta\eta_0}, \quad i_A \approx \frac{\delta^3}{64\rho_i L} \sqrt{\frac{2}{f^*\rho_w}} (\Phi l)^{3/2}. \quad (4.92)$$

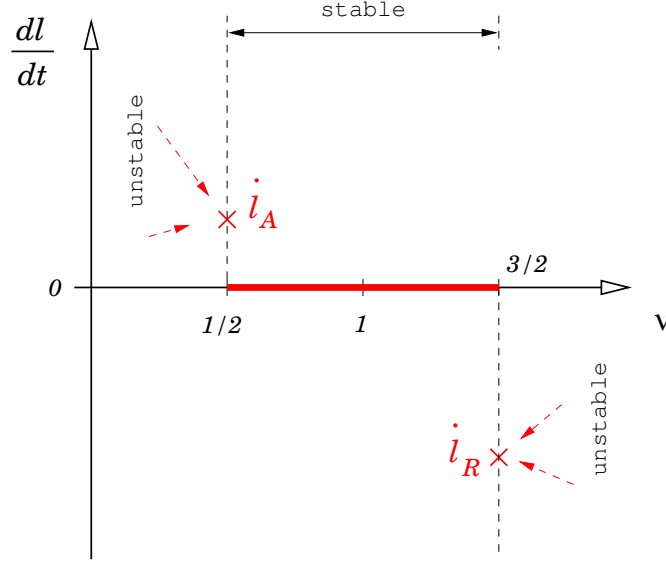


Figure 4.6: The tip velocity dl/dt as a function of tip shape exponent ν .

(Note that $\dot{l}_A \ll \hat{m}/\rho_i$.) The tip is found to be stationary in the intermediate range, and profiles with $\nu > 3/2$ or $\nu < 1/2$ are unattainable. Thus, we have

$$\frac{dl}{dt} = \begin{cases} +\infty & 0 < \nu < 1/2 \\ \dot{l}_A & \nu = 1/2 \\ 0 & \text{for } 1/2 < \nu < 3/2 \\ \dot{l}_R & \nu = 3/2 \\ -\infty & \nu > 3/2. \end{cases} \quad (4.93)$$

These results are summarized in Fig. 4.6, where we show dl/dt as a function of the tip shape parameter ν . If we use the values $N_c = 10^5$ Pa, $l = 10$ m, $\delta = 0.1$, $\eta_0 = 5 \times 10^{23}$ Pa s, $\Phi_0 = 10^2$ kg m $^{-2}$, $f^* = 0.1$, then typical velocities are

$$\dot{l}_R \approx -3 \times 10^{-8} \text{ m s}^{-1}, \quad \dot{l}_A \approx 2 \times 10^{-10} \text{ m s}^{-1}. \quad (4.94)$$

Although $dl/dt = 0$ for $1/2 < \nu < 3/2$, the tip profile can build up to either limit via depth evolution, as described by Equation (3.1). In other words, the tip profile has to get ‘shallow’/‘steep’ enough before it may retreat/advance (see Fig. 4.7). This kind of *waiting time* behaviour is physically plausible, and is encountered also in the mathematics that describes the spreading of (thin) viscous droplets (Lacey *et al.*, 1982). Another situation where there is a tip propagation/tip shape interdependence is found in the analysis of micro-cracks (or valves) in stressed elastic media (e.g. Study Group participants, 1995).

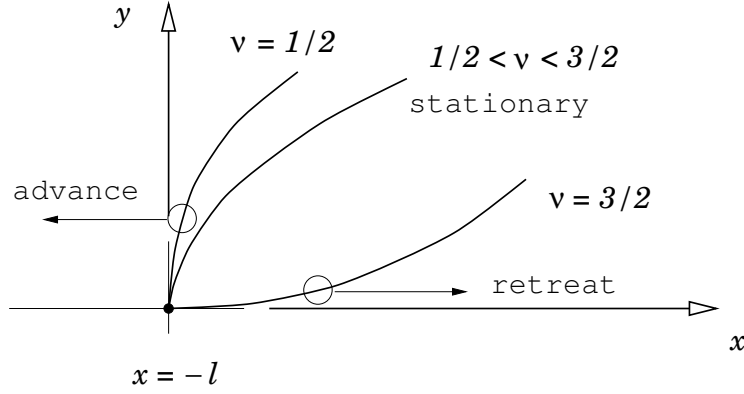


Figure 4.7: *Tip propagation behaviour: (i) $\nu \geq 3/2$, retreat; (ii) $1/2 < \nu < 3/2$, stationary; (iii) $\nu \leq 1/2$, advance.*

4.6 Discussion

In this chapter, the problem of calculating the melt-rate along the ice roof boundary has been tackled with a 2-D channelized water flow model, together with a Prandtl-type parameterization to describe turbulence. As before, we concentrate on a hard bed subglacial wide channel.

The high channel aspect ratio ($\delta \ll 1$) facilitates a perturbation approximation, by which we are able to show that the effective pressure $N_c (= p_i - p_c)$ is essentially constant within the channel cross section. Under the same formulation, determination of flow velocity and temperature involves solving a pair of degenerate elliptic P.D.E.s, with appropriate no slip and melting temperature boundary conditions.

1. Depth evolution

The (dimensional) upward melting velocity, calculated from the temperature solution, is given by

$$\hat{m}(x) \approx \frac{C[\Phi h(x)]^{3/2}}{L\sqrt{\rho_w f^*}}, \quad (4.95)$$

where C is an $O(1)$ constant. This expression has an $O(\delta^2)$ accuracy and applies where $h'(x)$ is small ($\ll 1$). If we consider simple channel profiles in particular, with the tip behaviour

$$\lim_{x \rightarrow \pm l} [h(x)] \propto (l \mp x)^\nu \quad (4.96)$$

where ν (a positive shape parameter) < 1 , then (4.95) is valid in the central part of the channel (as an outer solution), but not within boundary layers at the channel margins. The typical boundary layer thickness is $[h]\delta^{\frac{1}{1-\nu}}$; the corresponding inner problem is difficult to solve. These properties are due to (i) the approximation $\delta \rightarrow 0$ being singular, and (ii) the fact that Prandtl's parameterization breaks down where

$h'(x) \gtrsim 1$. When $\nu \geq 1$, there are no boundary layers and (4.95) is valid everywhere in $|x| < l$.

Having considered the derivation of both $\hat{m}$ (here) and $\hat{w}$ (Equation (3.111), in Chapter 3), we obtain the depth evolution equation

$$\frac{\partial h}{\partial t} = \frac{C(\Phi h)^{3/2}}{\rho_i L \sqrt{f^* \rho_w}} - \frac{N_c^3}{2\eta_0} \sqrt{l^2 - x^2}, \quad (4.97)$$

which is consistent with Equation (2.50) used in the flood model of Chapter 2. (There, we effectively assume that $C = 2\sqrt{2}$.) This equation is applicable where $\hat{m}$ is valid, as described above.

2. Tip migration

We predict that lateral channel closure or expansion can occur as a result of ice deformation or ice melting, respectively. A fundamentally important result is that these processes are mutually exclusive. Equations (4.92) and (4.93) provide a basic model of channel width evolution. We note that these results have been based on a ‘crack’ linearization (Chapter 3) and an approximation of the tip flow heat transfer (Section 4.5). The precise physics at the margins awaits further investigation.

3. Equilibrium channel

According to Equation (4.97), steady flow ($\partial h / \partial t = 0$) can be maintained when the depth profile is

$$h(x) = h(0)[1 - (x/l)^2]^\nu, \quad \nu = 1/3, \quad (4.98)$$

where

$$h(0) = \left(\frac{\rho_i L \sqrt{f^* \rho_w}}{2\eta_0 C} \right)^{2/3} \frac{N_c^2 l^{2/3}}{\Phi}. \quad (4.99)$$

Thus, the equilibrium channel shape has the form supposed in Sections 4.4.2 and 4.5. (Let us take the nominal value $C = 2\sqrt{2}$, following Equation (2.50).) Note that the requirement of a stable width is not violated (in Section 4.5, $\nu = 1/3$ would imply channel expansion), because (4.98) is only an outer approximation. In addition, these two results are independent of whether the substrate is hard or soft, since h here is the total channel depth.

By assuming a wide flow, we can integrate the flow velocity across the channel — using Equations (2.44)_{2,3} and (2.47) — to find the (approximate) subsidiary equation

$$Q = \sqrt{\frac{8\Phi}{\rho_w f^*}} \int_{-l}^l h^{3/2} dx \quad (4.100)$$

where Q is the downstream water flux as before. Then, performing the integration leads to the equilibrium relation

$$Q = \frac{\pi \rho_i L N_c^3 l^2}{4\eta_0 \Phi}. \quad (4.101)$$

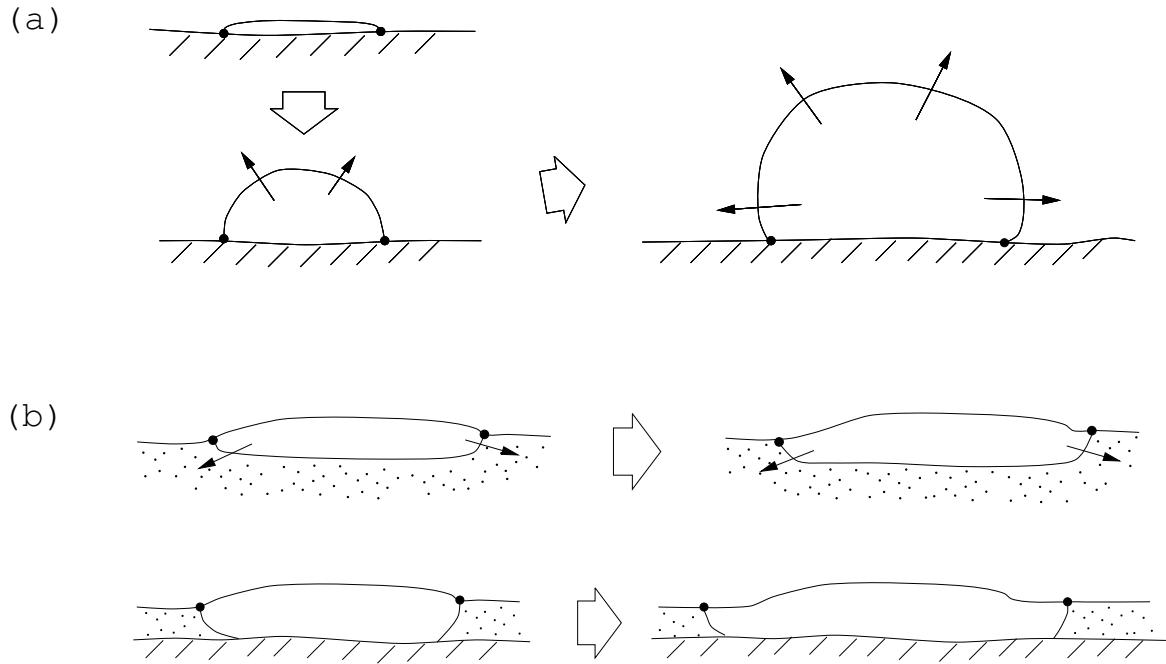


Figure 4.8: Channel widening processes by: (a) conversion to a classical *R*-type channel; (b) undercut or lateral sediment erosion.

For a hard bed, Equation (4.101) is analogous to the classical Röthlisberger (1972) result, but l is unconstrained. Given that lateral expansion/retreat processes cannot occur simultaneously (discussed above), and that the lateral basal sliding velocity is zero (Chapter 3), there appears to be no specific (balance) mechanism that would determine an ‘equilibrium’ channel width as such. We suppose that l will depend solely on evolution history. With l constant, Equation (4.101) gives $Q \propto \Phi^{-1} N_c^3$ — the discharge increases with decreasing water pressure and/or hydraulic gradient — thus a hard bed wide channel is of Röthlisberger type (cf. Equation (1.4)).

4. Extension

Regarding time-dependent width evolution, there is the problem of a negligible expansion rate (since $\dot{l}_A \ll \hat{m}/\rho_i$). We suggest two scenarios in which channel widening is likely occur by other (more efficient) means. One possibility would be by converting first to a semi-circular *R*-type channel (in a flood, for example; see Fig. 4.8a). For deformable beds, an undercutting effect caused by till erosion by the water flow can also increase l (Fig. 4.8b). Indeed, an equilibrium channel width may exist in this case, but to ascertain this, we would have to consider the sediment processes and their interaction, and then our model in (4.97) will be modified. This is the subject of Chapters 5, 6 and 7.

Chapter 5

Sediment creep

5.1 Introduction

Hitherto, our description has been limited to the case of a rigid, impermeable bed. If the subglacial channel is underlain by a layer of ‘till’, then sediment can interact with the water flow through uptake as bedload and suspended load. The channel may enlarge as a result, but under typical conditions the till is water-saturated, with a pressure that is greater than p_c , so there is an opposite effect due to sediment creep closure. Here we are concerned with determining the nature of this motion.

Alley (1992) addressed the question of whether a low-pressure channel could co-exist with an adjacent (thin) till layer that is rapidly deforming towards it. By using a 1-D force balance model, he showed that sediment creep flow under a Coulomb-Bingham rheology is limited to a narrow zone by the channel margins, and this leads to ‘pinch-out’ of the till over bedrock. Thus, the channel can exist with low water pressures (in the classical sense), since it is isolated from till farther away.

If the till is much thicker in comparison to the size of the channel, then pinch-out may not occur, but the closure velocity of the till incision would still be important in the drainage description. To obtain this velocity, Walder and Fowler (1994) have reformulated Nye’s (1953) cylindrical ice closure problem (mentioned in Section 3.1) for sediment that has the rheology proposed by Boulton and Hindmarsh (1987). Their mathematical derivation is detailed in Fowler and Walder (1993).

In this chapter, a steady state sediment creep model based on Fowler and Walder’s (1993) formulation is investigated. We relax the restriction imposed by the cylindrical geometry, by considering a situation where the ice-till interface is taken into account, and where the thickness of the deforming layer, d , is externally prescribed. Two end possibilities will be examined: $d \ll l$, and $d \gg l$, which correspond respectively to the cases of shallow and deep deforming layers. As in Chapter 3, the assumption of a wide channel allows the use of a crack-type approximation.

Our mathematical model is described in the next section. In Section 5.3 we use it to investigate the general properties of a shallow deforming layer; and in Sections 5.4 to 5.6, we present analytic solutions for linear viscosity, low permeability tills.

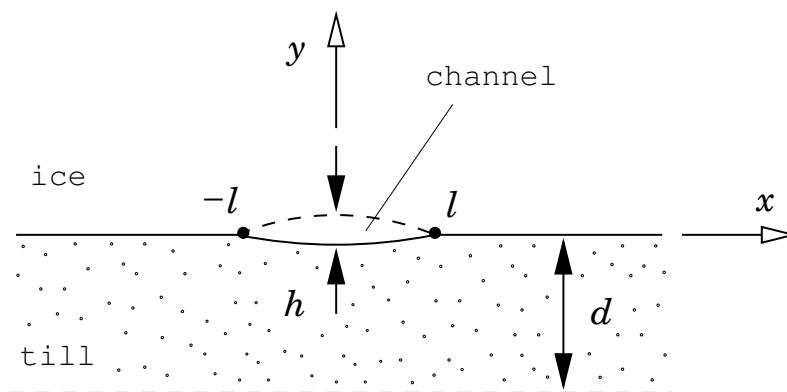


Figure 5.1: *Geometrical setup of the sediment creep closure problem.*

We then discuss our results in Section 5.7, in relation to bed evolution, subglacial morphology and sediment transport. Attention is paid to the problem considered by Alley (1992), and to the formation process of ‘tunnel valleys’ envisaged by Boulton and Hindmarsh (1987).

5.2 Mathematical model

5.2.1 Rheology and geometry

We consider the 2-D situation in Fig. 5.1, which shows a subglacial channel of half-width l and depth h overlying a till. We use the space coordinates defined previously. The channel admits a water flow at pressure p_c , and is taken to be long and ‘wide’ (with $\delta = h/l \ll 1$) so that its bed effectively lies on $y = 0$. Due to the far field overburden ice pressure p_i , where $p_i > p_c$, the till deforms.

The flow law

The motion of porous granular material is often highly complicated. More specifically, the till here is assumed to be saturated, and supports a pore-water (Darcy) flow at a pressure p_w ; in turn, this pressure governs its deformability by modifying the (average) contact stress transmitted within the sediment matrix. p_w is generally a function of position (x, y) , so determination of creep velocity requires also the pore pressure distribution to be calculated, i.e., we have a two-phase coupled problem.

Mathematically, this coupling is conveniently described through the use of the *effective pore pressure* N , defined by

$$N = P - p_w, \quad (5.1)$$

where P is the overburden pressure in the till. If we take the Boulton-Hindmarsh (1987) rheology law

$$\dot{\epsilon} = A_T \tau^a N^{-b} \quad (a, b > 0) \quad (5.2)$$

(from Section 1.4), then (5.1) implies that the apparent till viscosity, given by

$$\eta_T = A_T^{-1} \tau^{1-a} N^b, \quad (5.3)$$

decreases with increasing water pressure, and vice versa. This behaviour is consistent with physical expectation. The parameters determined by Boulton and Hindmarsh (1987) are

$$A_T = 3 \times 10^{-5} \text{ Pa}^{b-a} \text{ s}^{-1}, \quad a = 1.33, \quad b = 1.8. \quad (5.4)$$

In this chapter, we adopt the values of a and b shown here, but A_T will be chosen according to a nominal viscosity figure $\eta_T = 10^{10} \text{ Pa s}$ (see Section 5.2.3). A typical range of η_T inferred from field observations is 10^9 to 10^{11} Pa s (e.g. Paterson, 1994).

While (5.2) is probably the most useful form for practical use, one physically inappropriate inference from it is that the strain rate becomes infinite (or the viscosity tends to zero) as $N \rightarrow 0$. This would be appropriate at the sediment-water interface, where the mixture is freely erodible; but elsewhere, it seems unlikely that the resistance would be negligible even at flotation ($N = 0$), since till deformation still requires cobbles and clasts to move past each other. Therefore, the value $N = 0$ is strictly admissible only at unconfined boundaries.

The deforming domain

We should point out that there are further complications to the rheology (refer to Clarke, 1987; Fowler and Walder, 1993). For instance, a more sophisticated model might include a *yield stress* τ_y , given by the Coulomb relation

$$\tau_y = c_0 + N \tan \phi_0, \quad (5.5)$$

where c_0 is a cohesion due essentially to the clay fraction of the till, and ϕ_0 is an internal friction angle (Section 6.4.1). The Coulomb yield criterion asserts that below failure, when $\tau < \tau_y$, the till behaves as an elastic solid, and above, when $\tau \geq \tau_y$, as a viscous fluid obeying a stress-strain rate relationship similar to (5.2).

c_0 is typically small ($\sim 0.1 \text{ bar}$), but inferred values of τ_y from field data are often significant and may not be negligible ($\sim 1 \text{ bar}$; Paterson, 1994). In this case the extent of the *deforming domain* has to be determined as part of the solution, i.e., as a free boundary. The resulting problem is extremely difficult. On the other hand, the normal situation is one where the till conducts basal meltwater towards the bed, so N (and hence τ_y) increases with depth in the till, and this leads to a roughly horizontal interface between an upper deforming (A-) layer and a lower non-deforming (B-) layer. Although this picture may be less accurate near a channel, it motivates the definition of a fixed boundary problem, where we neglect τ_y and instead *specify* the thickness of the deforming layer d (see Fig. 5.1). In this specification, the lower boundary of our domain may either be the bedrock interface or the A/B transition.

We can now introduce a parameter

$$\gamma = \frac{d}{l} \quad (5.6)$$

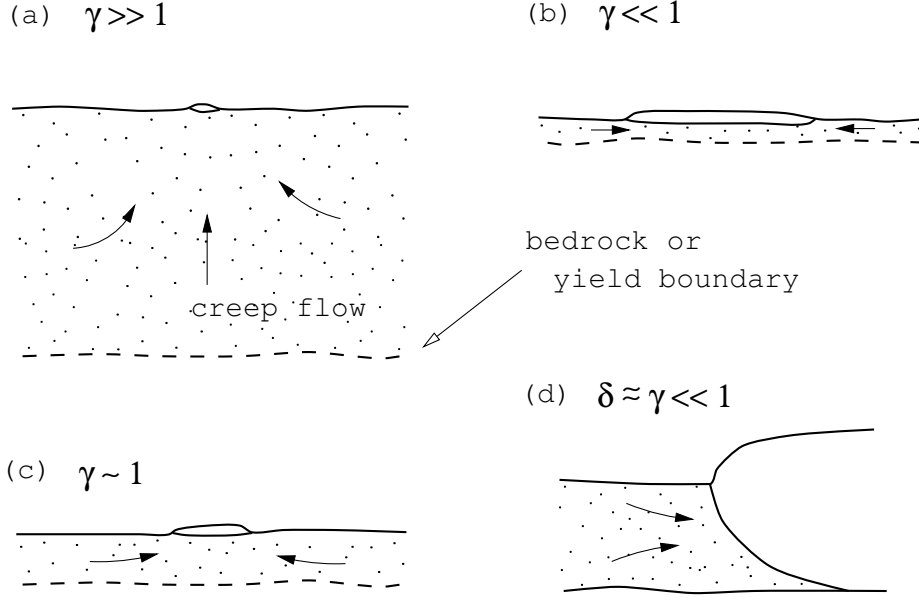


Figure 5.2: Four scenarios of subglacial till deformation into an overlying wide channel: (a) deep till, (b) shallow till, (c) intermediate case, (d) ‘disjoint’ case.

to define the aspect ratio of our creep problem. The limits $\gamma \gg 1$ and $\gamma \ll 1$ correspond respectively to the ‘deep till’ and ‘shallow till’ scenarios shown in Fig. 5.2a,b, investigated in Sections 5.3 to 5.6. Other possibilities include the intermediate case $\gamma \sim 1$ (Fig. 5.2c), and the ‘disjoint’ situation discussed by Alley (1992), where the till has been eroded down to bedrock (Fig. 5.2d).

5.2.2 Formulation

Rheology in tensor form

Our model follows closely the one proposed by Fowler and Walder (1993). We first extend flow law (5.2) for more general deformation of a compressible two-phase medium. (We use the usual tensor notation.) Given that the creep motion is caused by that part of the stress tensor (σ_{ij}) transmitted in the sediment phase only, i.e., the *effective stress* σ'_{ij} , defined by

$$\sigma'_{ij} = \sigma_{ij} + p_w \delta_{ij} \quad (5.7)$$

(Clarke, 1987), an isotropic formulation leads to the relations

$$\begin{aligned} \frac{\tau_{ij}}{2\eta_T} &= \dot{\epsilon}_{ij} - \frac{1}{3}(\nabla \cdot \mathbf{u})\delta_{ij}, \\ \dot{\epsilon}_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\ \tau_{ij} &= \sigma'_{ij} - \frac{1}{3}\sigma'_{kk}\delta_{ij}, \end{aligned} \quad (5.8)$$

where $\mathbf{u} = (u_1, u_2, u_3) = (u, v, w)$ is the till velocity vector, $\dot{\epsilon}_{ij}$ is the corresponding strain rate, and τ_{ij} is *deviatoric* shear stress. We take η_T from Equation (5.3), wherein τ is now interpreted as the second stress invariant; thus,

$$2\tau^2 = \tau_{ij}\tau_{ij}. \quad (5.9)$$

Since $P = -\sigma_{kk}/3$, the definitions in (5.7) and (5.8)₃ reduce to

$$\tau_{ij} = \sigma_{ij} + P\delta_{ij}. \quad (5.10)$$

Mass and momentum conservation

Let $\mathbf{U} = (U, V, W)$ be the water velocity *relative* to the till matrix, and let $\mathbf{g}$ be vector gravity. Darcy's law may be written as

$$n_s \mathbf{U} = -\frac{k_T}{\mu_w} (\nabla p_w - \rho_w \mathbf{g}), \quad (5.11)$$

in which k_T , the permeability, will depend mainly on till composition, with a wide range of values from 10^{-19} m^2 (clay rich) to 10^{-13} m^2 (coarse gravel). We take k_T as prescribed. The till porosity n_s is a decreasing function of N (e.g. see Clarke, 1987), but its precise functional form is not important here. We introduce a *compressibility* index β_V , with the definition

$$\frac{dn_s}{dN} = n_s'(N) = -\beta_V. \quad (5.12)$$

Typically $\beta_V N \sim 0.1$, according to Fowler and Walder (1993).

Now, conservation of mass for each phase requires that

$$\begin{aligned} \text{Solid :} \quad & -\frac{\partial n_s}{\partial t} + \nabla \cdot [(1 - n_s)\mathbf{u}] = 0, \\ \text{Liquid :} \quad & \frac{\partial n_s}{\partial t} + \nabla \cdot [n_s(\mathbf{u} + \mathbf{U})] = 0. \end{aligned} \quad (5.13)$$

We have ignored internal communiton and losses due to washing out of fines. The momentum equation for the till is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = 0, \quad (5.14)$$

in which ρ is the bulk density, given by

$$\rho = n_s \rho_w + (1 - n_s) \rho_s. \quad (5.15)$$

(ρ_s is the sediment density.) Equations (5.1), (5.3), (5.8)_{1,2}, (5.9) to (5.15) complete our model.

Two-dimensional model

We write $x_1 = x$, $x_2 = y$, and apply the plane flow condition $w = W = 0$, $\partial/\partial x_3 = 0$. A pseudo-steady approximation of our model ($\partial/\partial t \approx 0$) then leads to the following equations.

$$\begin{aligned}
 \text{Sediment framework :} \quad & N = P - p_w, \\
 & \rho = n_s \rho_w + (1 - n_s) \rho_s, \\
 & n_s'(N) = -\beta_V, \\
 & \eta_T = A_T^{-1} \tau^{1-a} N^b, \\
 & \tau = \eta_T \sqrt{\frac{4}{3}[(u_x)^2 - u_x v_y + (v_y)^2] + (u_y + v_x)^2},
 \end{aligned} \tag{5.16}$$

$$\begin{aligned}
 \text{Water flow :} \quad & n_s U = -\frac{k_T}{\mu_w} \frac{\partial p_w}{\partial x}, \\
 & n_s V = -\frac{k_T}{\mu_w} \left(\frac{\partial p_w}{\partial y} + \rho_w g \right),
 \end{aligned} \tag{5.17}$$

$$\begin{aligned}
 \text{Mass conservation :} \quad & u_x + v_y + \frac{\partial}{\partial x}(n_s U) + \frac{\partial}{\partial y}(n_s V) = 0, \\
 & u N_x + v N_y = \frac{1 - n_s}{\beta_V} \left[\frac{\partial}{\partial x}(n_s U) + \frac{\partial}{\partial y}(n_s V) \right],
 \end{aligned} \tag{5.18}$$

$$\begin{aligned}
 \text{Till momentum :} \quad & \frac{2}{3} \frac{\partial}{\partial x} [\eta_T (2u_x - v_y)] + \frac{\partial}{\partial y} [\eta_T (u_y + v_x)] = P_x, \\
 & \frac{\partial}{\partial x} [\eta_T (u_y + v_x)] + \frac{2}{3} \frac{\partial}{\partial y} [\eta_T (2v_y - u_x)] = P_y + \rho g.
 \end{aligned} \tag{5.19}$$

These equations describe the coupled problem of till deformation and water flow, and have independent variables x and y . In (5.18), the first equation is the sum of the two equations in (5.13), and the second equation has been derived using (5.13)₂, (5.16)₃ and (5.18)₁.

To solve the problem set out in Fig. 5.1, boundary conditions for the stresses, velocities and pore-water pressure are required. We impose (i) vanishing deformation at the far field, (ii) no slip and no through-flow conditions at the ice and bedrock interfaces (the latter excludes water flux input due to basal ice melting or groundwater flow), (iii) a far field pore-water pressure p_∞ , and (iv) a channel pressure p_c that is constant (justified in Section 4.3.1). The shear stress exerted by the channel flow on the till is negligible.

For the moment, we also assume that the ice-till interface is subjected to a given normal contact stress $\sigma_n(x)$ (where $\sigma_n \rightarrow p_i$ as $|x| \rightarrow \infty$), and we allow it to deform vertically. There is a choice of either $u = 0$ or $\sigma_{xy} = 0$ as the remaining condition for this boundary, decided later in Section 5.6. Summarizing these, we have:

$$L \ (y = 0, \ |x| \leq l) : \quad \sigma_{xy} = 0,$$

$$- \sigma_{yy} = p_w = p_c; \quad (5.20)$$

$$\begin{aligned} L' \ (y = 0, |x| > l) : \quad & \sigma_{yy} = -\sigma_n(x), \\ & U = V = 0, \\ & u = 0 \text{ or } \sigma_{xy} = 0; \end{aligned} \quad (5.21)$$

$$B \ (y = -d) : \quad u = v = U = V = 0; \quad (5.22)$$

$$\begin{aligned} \text{As } |x| \rightarrow \infty : \quad & p_w \rightarrow p_\infty, \\ & u, v \rightarrow 0, \\ & \sigma_{xx} = \sigma_{yy} = -p_i. \end{aligned} \quad (5.23)$$

Note that the problem boundaries have been labelled L , L' and B for ease of identification.

5.2.3 Non-dimensionalization

The equations in (5.16) to (5.19) may be non-dimensionalized with the length scales $[x] = l$, $[y] = \tilde{\gamma}l$, together with other assignments shown in Table 5.1, where $\tilde{\gamma}$ is a parameter chosen to effect the desired depth scale. Accordingly, an appropriate dimensionless model is

$$\begin{aligned} N &= P - p_w, \\ \rho &= 1 - n_s(1 - \beta), \\ n_s'(N) &= -\kappa, \\ \eta_T &= \tau^{1-a} N^b, \\ \tau &= \eta_T \sqrt{(u_y + \tilde{\gamma}^2 v_x)^2 + \frac{4\tilde{\gamma}^2}{3} [(u_x)^2 - u_x v_y + (v_y)^2]}, \end{aligned} \quad (5.24)$$

$$\begin{aligned} n_s U &= -\frac{\partial p_w}{\partial x}, \\ \tilde{\gamma}^2 n_s V &= -\frac{\partial p_w}{\partial y} - \tilde{\gamma} \alpha \beta, \end{aligned} \quad (5.25)$$

$$\begin{aligned} u_x + v_y + \frac{\kappa \Pi}{\tilde{\gamma}^2} \left[\frac{\partial}{\partial x} (n_s U) + \frac{\partial}{\partial y} (n_s V) \right] &= 0, \\ u N_x + v N_y &= \frac{\Pi(1 - n_s)}{\tilde{\gamma}^2} \left[\frac{\partial}{\partial x} (n_s U) + \frac{\partial}{\partial y} (n_s V) \right], \end{aligned} \quad (5.26)$$

$$\begin{aligned} P_x &= \frac{\partial}{\partial y} [\eta_T (u_y + \tilde{\gamma}^2 v_x)] + \frac{2\tilde{\gamma}^2}{3} \frac{\partial}{\partial x} [\eta_T (2u_x - v_y)], \\ P_y &= -\tilde{\gamma} \alpha \rho + \tilde{\gamma}^2 \left\{ \frac{\partial}{\partial x} [\eta_T (u_y + \tilde{\gamma}^2 v_x)] + \frac{2}{3} \frac{\partial}{\partial y} [\eta_T (2v_y - u_x)] \right\}. \end{aligned} \quad (5.27)$$

variable	scale
x	l
y	$\tilde{\gamma}l$
n_s	1
ρ, ρ_w, ρ_s	ρ_s
η_T	$[\eta]$
u	$[u]$
v	$\tilde{\gamma}[u]$
U	$[U]$
V	$\tilde{\gamma}[U]$
N, P, p_w	$[N]$
τ, σ_{ij}	$\tilde{\gamma}[N]$

Table 5.1: *The scale definitions used in our non-dimensionalization procedure.*

In these equations, the parameters are

$$\alpha = \frac{\rho_s g l}{[N]}, \quad \beta = \frac{\rho_w}{\rho_s}, \quad \kappa = \beta_V [N], \quad \Pi = \frac{[\eta] k_T}{\kappa \mu_w l^2}, \quad (5.28)$$

where κ represents till compressibility and Π ($\propto [\eta] k_T$) is a permeability/deformability ratio. (The effect of different values of Π is described in Section 5.3.) The boundary conditions have the same form as those in (5.20) to (5.23), and we have chosen characteristic scales which satisfy the relations

$$[\eta] = A_T^{-1} \tilde{\gamma}^{1-a} [N]^{1-a+b}, \quad [u] = \frac{\tilde{\gamma}^2 l [N]}{[\eta]}, \quad [U] = \frac{k_T [N]}{l \mu_w}. \quad (5.29)$$

If we use the constants $\rho_s = 2650 \text{ kg m}^{-3}$, $k_T = 10^{-19}$ (clay) to 10^{-13} m^2 (gravel), $a = 1.33$ and $b = 1.8$ (from Equation (5.4)), and prescribe the nominal values $[\eta] = 10^{10} \text{ Pa s}$, $l = 10 \text{ m}$, $[N] = 1 \text{ bar}$, then the dimensionless parameters are

$$\begin{aligned} \alpha &\approx 3, & \beta &\approx 0.4, \\ \kappa &\approx 0.1 \quad (\text{following Fowler and Walder, 1993}), \\ \Pi &\approx 10^{-7} \text{ (clay) to } 10^{-1} \text{ (gravel)}, \end{aligned} \quad (5.30)$$

and the last two equations in (5.29) give

$$[u] = 10^{-4} \tilde{\gamma}^2 \text{ m s}^{-1}, \quad [U] \approx 10^{-12} \text{ to } 10^{-6} \text{ m s}^{-1}. \quad (5.31)$$

Shortly later, we shall see that a suitable range is $\tilde{\gamma} \sim 10^{-1}$ to 10^0 . Hence, our rescaling procedure leads to plausible velocity scales, and also $A_T \sim 10^{-3} \text{ Pa}^{b-a} \text{ s}^{-1}$ via Equation (5.29)₁. In comparison with (5.4), this value indicates that the subglacial till investigated by Boulton and Hindmarsh (1987) is at the stiff end of the spectrum, as has already been pointed out in Section 2.4.2. (Note that our N scale is consistent with their measured range of effective pressures, which is in the region of 0.5 bar.)

5.3 Shallow till approximation

If the till is deep ($\gamma \gg 1$), then the natural depth scale to use is $[y] = l$, therefore $\tilde{\gamma} = 1$. This problem is essentially a hybrid of that investigated by Fowler and Walder (1993; they have $\delta = h/l \sim 1$ also), and is not discussed here. On the other hand, the obvious choice for a shallow till ($\gamma \ll 1$) would be to take $[y] = d$, thus

$$\tilde{\gamma} = \gamma \ll 1. \quad (5.32)$$

(A typical value is $\gamma = 0.1$.) In this case, the smallness of γ ($\equiv \tilde{\gamma}$) facilitates a low order approximation of the model, stated as follows:

$$\tau = \eta_T |u_y| + O(\gamma), \quad \eta_T = \tau^{1-a} N^b, \quad n_s'(N) = -\kappa, \quad (5.33)$$

$$n_s U = -\frac{\partial p_w}{\partial x}, \quad -\gamma \alpha \beta = \frac{\partial p_w}{\partial y} + O(\gamma^2), \quad (5.34)$$

$$P_x = \frac{\partial}{\partial y}(\eta_T u_y) + O(\gamma^2), \quad P_y = -\gamma \alpha \rho + O(\gamma^2), \quad (5.35)$$

and

$$\begin{aligned} u_x + v_y + \frac{\kappa \Pi}{\gamma^4} [N_{yy} + \gamma^2 (N_{xx} - P_{xx})] &= 0, \\ u N_x + v N_y &= \frac{\Pi(1 - n_s)}{\gamma^4} [N_{yy} + \gamma^2 (N_{xx} - P_{xx})], \end{aligned} \quad (5.36)$$

where $N = P - p_w$. These equations have been derived by perturbation expansion of (5.24) to (5.27), in the limit as $\gamma \rightarrow 0$, followed by subsequent inter-substitutions. Without solving them, it is possible to infer the general characteristics of the solutions. Equations (5.33)₃ and (5.34)₁ for the unknowns n_s and U decouple readily from the rest.

Velocity field

The equations in (5.33) to (5.35) describe how the till is being ‘squeezed’ towards the channel, and define the problem for $u(x, y)$ given N . To $O(\gamma)$ accuracy, (5.34)₂ and (5.35)₂ imply that both the pore pressure and the overburden stress are effectively static, with negligible variations in the y -direction. In particular as $\gamma \rightarrow 0$, p_w , P and N are functions of x only, so the problem reduces to

$$N^{b/a} \frac{\partial}{\partial y} (|u_y|^{1/a-1} u_y) = \frac{dP}{dx}, \quad (5.37)$$

with $u = 0$ on B and another boundary condition on $y = 0$. On L , this condition is $u_y = 0$ (shear-free), and since $P_x = 0$, the outer solution in $|x| < 1$ is just $u = 0$.

In $|x| > 1$, we have instead $u = 0$ or $u_y = 0$ on L' , and $P = \sigma_n$. Since $1/a \approx 1$, Equation (5.37) leads to a near parabolic vertical velocity profile. This solution is driven by the applied stress gradient σ_n' , with the pressure field N modulating the local viscosity.

A typical σ_n distribution has vanishing gradient at large $|x|$ (where $\sigma_n \rightarrow p_i$), and exhibits a positive singular behaviour at the channel margins (e.g. Weertman, 1972; see Section 3.5). In addition, N has to change from its far field value $N_\infty (= p_i - p_\infty)$ to zero on L . (This is discussed next.) We therefore expect the creep flow to be concentrated in the neighbourhood of $(\pm 1, 0)$, and decay with distance from these points. These results are summarized in Fig. 5.3a.

Pressure field and permeability limits

By taking a nominal (small) aspect ratio $\gamma = 0.1$, we obtain

$$10^{-3} \text{ (clay)} < \frac{\Pi}{\gamma^4} < 10^3 \text{ (gravel)}. \quad (5.38)$$

Thus, the parameter Π/γ^4 can be small or large depending on the composition of the till. As we now demonstrate, the associated extremal limits have very different implications on the solutions of (5.36).

Equations (5.36)₁ and (5.36)₂ describe respectively mass conservation and convective-diffusion of N , and have the boundary conditions (i) $v = 0$ on B , (ii) $\partial N/\partial y = 0$ (no through-flow) on L' and B , (iii) $N \rightarrow N_\infty$ (+ve) as $|x| \rightarrow \infty$, and (iv) $N = 0$ on L . If $\Pi/\gamma^4 \ll 1$, then they simplify to

$$u_x + v_y \approx 0, \quad uN_x + vN_y \approx 0, \quad (5.39)$$

which describe essentially an incompressible till, with N advected along streamlines. This result is consistent with the low efficiency of Darcy drainage in a low permeability till.

More precisely, (5.39) is a singular approximation which breaks down near L' and B , and also where N changes rapidly. However, boundary layers for N will be ‘weak’, for it is N_y (and not N) that is prescribed; see b.c. (ii) above. It is therefore possible to infer the outer N -distribution from its boundary values and the flow structure (u, v) . (v is determined by solving (5.39)₁, given u .) Essentially, the leading order solutions in $|x| < 1$ are $u = v = 0$, with $N = 0$. This region is separated from $|x| > 1$ by a *transition* layer of thickness of $O(\sqrt{\Pi}/\gamma)$, in which the neglected (diffusive) x -derivatives become important, N and P change rapidly, and the sediment flow field is complicated. The till velocity must decay exponentially as we move into $|x| < 1$. A plausible outer solution in $|x| > 1$ is $N = N_\infty$. These results are illustrated in Fig. 5.3b. We describe further development in Section 5.6.4.

The high permeability case ($\Pi/\gamma^4 \gg 1$) may similarly be examined, but the corresponding approximation indicates that the till is diffusion-dominant. (5.36) reduces to the Poisson’s Equation

$$N_{yy} + \gamma^2 N_{xx} = \gamma^2 P_{xx}, \quad (5.40)$$

which is now independent of till velocities. Solution of this equation provides a smooth transition of length of $O(\gamma)$ between the outer solutions at the channel bed ($N = 0$) and away from the channel ($N = N_\infty$).

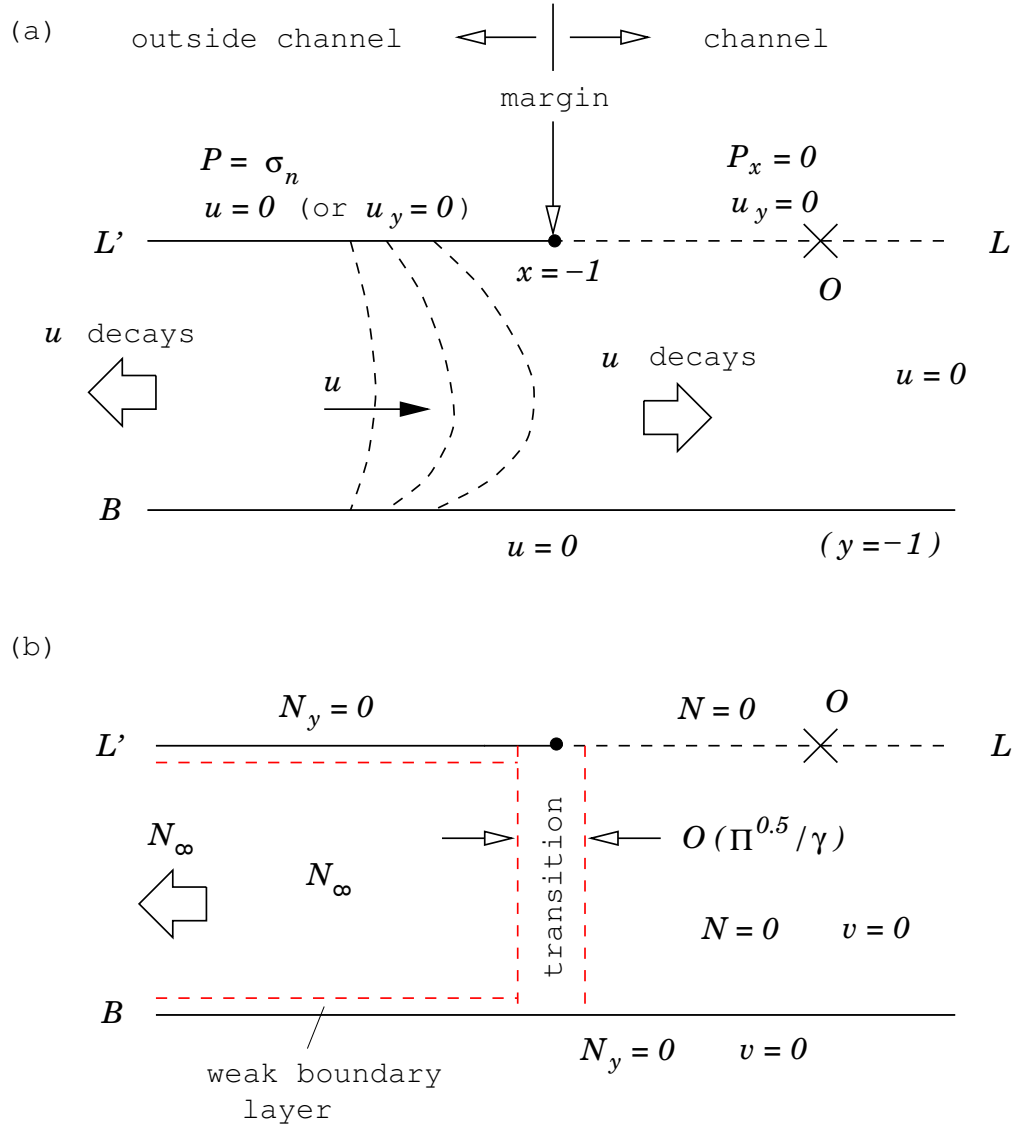


Figure 5.3: Structure of the shallow till solutions as inferred from the lubrication approximation $\gamma \rightarrow 0$: (a) velocity distribution, (b) effective pressure distribution. Red dashed lines indicate the likely positions of the boundary/transition layers. The results are symmetrical about $x = 0$.

Remembering that our width scale is l , the cases $\Pi/\gamma^4 \ll 1$ and $\Pi/\gamma^4 \gg 1$ respectively give (dimensional) transition lengths of $O(\sqrt{\Pi}d/\gamma^2)$ and $O(d)$. The former is much less than d , whereas the latter is independent of till properties. At the distinguished limit $\Pi/\gamma^4 \sim 1$, little information could be deduced concerning the solution structure, because of the two-way coupling between till motion and pore-water percolation.

5.4 Analytic solution: preliminaries

Up to now, the main difficulty in our model has been due to the Boulton-Hindmarsh (1987) law (5.2). In the following sections, we present explicit creep flow solutions for the case where this rheology has been replaced by a constant till viscosity η_T . This is a *linear* approximation, and in order to decouple fully the problem for u and v , we further assume a *low permeability* till, i.e., $\Pi/\gamma^4 \ll 1$, which requires $k_T \lesssim 10^{-16} \text{ m}^2$. This incompressibility condition (discussed in Section 5.3) leads to the dimensional model

$$\begin{aligned} u_x + v_y &= 0, \\ \eta_T \nabla^2 u &= P_x, \\ \eta_T \nabla^2 v &= P_y + \rho g, \end{aligned} \tag{5.41}$$

with boundary conditions given by (5.20) to (5.23) as before. These are just the Stokes Equations of Section 3.2 (cf. Equations (3.6)).

There are numerous ways of solving these equations, but given our domain geometry, and the fact that only solutions on $y = 0$ (the closure velocities) are sought, by far the simplest method involves the use of England's (1971) complex variable formulation, introduced in Chapter 3. Essentially, this allows (5.41) to be rewritten as

$$\begin{aligned} 2\eta_T(u + iv) &= \Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)}, \\ \sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= -2[z\overline{\Omega''(z)} + \overline{\omega'(z)}], \\ \sigma_{yy} - i\sigma_{xy} &= \Omega'(z) + \overline{\Omega'(z)} + z\overline{\Omega''(z)} + \overline{\omega'(z)} - \frac{V(z, \bar{z})}{2}, \end{aligned} \tag{5.42}$$

where $i^2 = -1$, $z = x + iy$, $V = \rho g i(z - \bar{z})$, and the over-bar $\bar{}$ denotes complex conjugation. The problem now is to find the two holomorphic functions $\Omega(z)$ and $\omega(z)$ (or their associate functions) in $-d \leq y \leq 0$, under constraints imposed by the boundary and far field conditions. In Sections 5.5 and 5.6, we investigate the deep till ($\gamma \gg 1$) and shallow till ($\gamma \ll 1$) solutions based on this formulation.

5.5 Stokes flow: deep till

When $d \gg l$, the lower boundary B may be neglected altogether, and the domain becomes the half-plane $y < 0$. We can couple this problem to an analogous (linear, incompressible) problem for the ice that occupies the upper half-plane and has corresponding viscosity η_i . The boundary conditions on L' (in (5.21)) become redundant, but there is now a requirement of stress and velocity *continuity* across the ice-till interface. This is illustrated in Fig. 5.4, where subscripts i, T, b represent the ice, the till, the basal interface, and $\hat{w}_i$ and $\hat{w}_T$ denote closure velocities towards the channel.

Riemann-Hilbert problems

The complex variable derivation has already been detailed in Sections 3.2 and 3.3, and is not repeated here. In the current problem, its application leads directly to the four Riemann-Hilbert problems

$$\begin{aligned} x \in L : \quad & \begin{cases} \Phi^+ - \Phi^- = -4i\eta_i \hat{w}'_i \\ \Phi^+ + \Phi^- = -2p_c \end{cases} & \begin{cases} \Theta^+ - \Theta^- = 0 \\ \Theta^+ + \Theta^- = -4\eta_i u'_i \end{cases} \\ x \in L' : \quad & \begin{cases} \Phi^+ - \Phi^- = 4i\eta_i v'_b \\ \Phi^+ + \Phi^- = -2\sigma_n, \end{cases} & \begin{cases} \Theta^+ - \Theta^- = 2i\tau_b \\ \Theta^+ + \Theta^- = -4\eta_i u'_b, \end{cases} \\ |z| \rightarrow \infty : \quad & \Phi \sim -p_i, \quad \Theta \sim 0, \end{aligned} \quad (5.43)$$

and

$$\begin{aligned} x \in L : \quad & \begin{cases} \phi^+ - \phi^- = -4i\eta_T \hat{w}'_T \\ \phi^+ + \phi^- = -2p_c \end{cases} & \begin{cases} \theta^+ - \theta^- = 0 \\ \theta^+ + \theta^- = -4\eta_T u'_T \end{cases} \\ x \in L' : \quad & \begin{cases} \phi^+ - \phi^- = -4i\eta_T v'_b \\ \phi^+ + \phi^- = -2\sigma_n, \end{cases} & \begin{cases} \theta^+ - \theta^- = -2i\tau_b \\ \theta^+ + \theta^- = -4\eta_T u'_b, \end{cases} \\ |z| \rightarrow \infty : \quad & \phi \sim -p_i, \quad \theta \sim 0, \end{aligned} \quad (5.44)$$

where $'$ denotes the x -derivative, and the pairs $\Phi(z), \Theta(z)$ and $\phi(z), \theta(z)$ are sectionally holomorphic functions in $S^+ \cup S^-$ (cf. the equations in (3.46) and (3.47)). As before, the required solutions are obtained via integral equations valid on $y = 0$.

Results from Θ and θ

If we define $M = \Theta + \theta$, then $M^+ \equiv M^-$ on $y = 0$, and $M \sim 0$ as $|z| \rightarrow \infty$. These imply that $M(z) \equiv 0$, and therefore

$$\begin{aligned} \eta_i u'_i + \eta_T u'_T &= 0, & \text{for } x \in L, \\ u'_b(x) &= u_b(x) = 0, & \text{for } x \in L', \end{aligned} \quad (5.45)$$

where we have imposed a zero translation condition at the far field.

Going back to (5.44)₂, with $u'_b = 0$, let us now define the function

$$F(z) = \begin{cases} \theta(z) & \text{for } y > 0 \\ -\theta(z), & \text{for } y < 0, \end{cases} \quad (5.46)$$

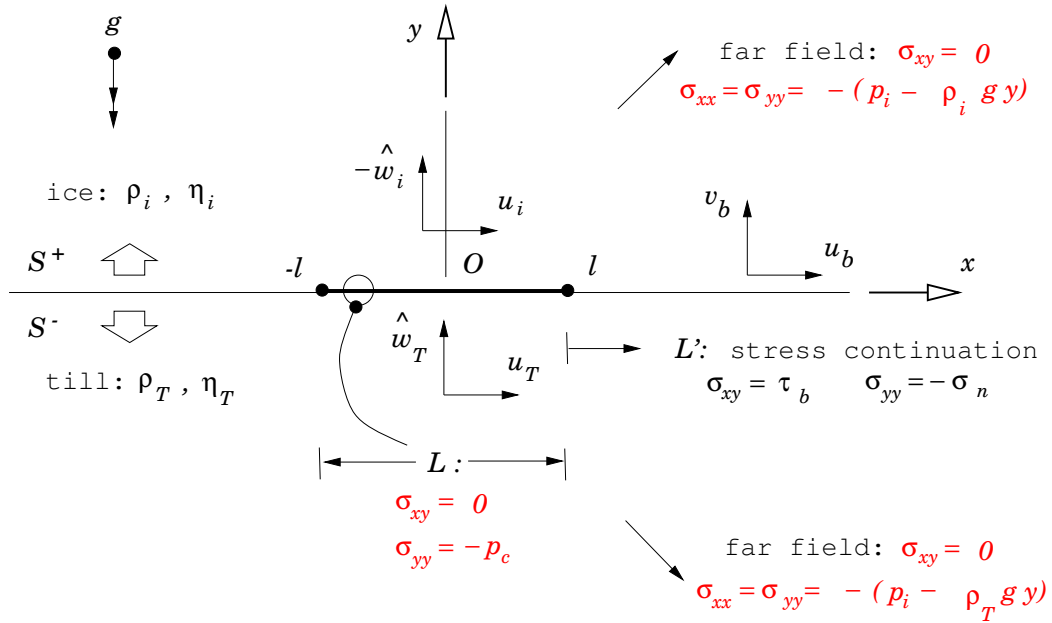


Figure 5.4: The coupled ice-till problem in Section 5.5. (Boundary and far field conditions are shown in red.)

such that $F \sim 0$ as $|z| \rightarrow \infty$, and

$$x \in L : \begin{cases} F^+ - F^- &= -4\eta_T u_T' \\ F^+ + F^- &= 0, \end{cases} \quad x \in L' : \begin{cases} F^+ - F^- &= 0 \\ F^+ + F^- &= -2i\tau_b. \end{cases} \quad (5.47)$$

The corresponding integral equation pair is

$$\begin{aligned} 0 &= \int_L \frac{du_T}{d\zeta} \frac{d\zeta}{\zeta - x}, & \text{for } x \in L, \\ \tau_b &= -\frac{2\eta_T}{\pi} \int_L \frac{du_T}{d\zeta} \frac{d\zeta}{\zeta - x}, & \text{for } x \in L'. \end{aligned} \quad (5.48)$$

Equation (5.48)₁ has the general solution $u_T' \propto (l^2 - x^2)^{-1/2}$ (Tricomi, 1957), and we require $u_T(\pm l) = u_b(\pm l) = 0$ by continuity. It is easy to show that the unknown multiplicative constant is zero, and hence that $u_T(x) = 0$ for $x \in L$. (This argument is similar to the one given in Appendix A.) As this implies $F(z) \equiv 0$, we also find $\tau_b(x) = 0$ on L' , and from (5.45)₁ $u_i(x) = 0$ on L .

Results from Φ and ϕ

We use a similar method, by evaluating

$$N(z) = \begin{cases} \Phi(z) - \phi(z) & \text{for } y > 0 \\ \phi(z) - \Phi(z), & \text{for } y < 0. \end{cases} \quad (5.49)$$

It follows that $N(z) \equiv 0$ (or equivalently $\Phi = \phi$), and in addition

$$\begin{aligned} \eta_i \hat{w}'_i &= \eta_T \hat{w}'_T, & \text{for } x \in L, \\ v'_b &= 0, & \text{for } x \in L'. \end{aligned} \quad (5.50)$$

Given that $v_b(\pm\infty) = 0$ (no far field translation), we have $v_b(x) = 0$ on L' . Both $(5.43)_1$ and $(5.44)_1$ are now identical in form to (3.46), so they lead to the integral equations given in (3.49) (Section 3.3), supplemented by $(5.50)_1$. Since $N_c (= p_i - p_c)$ is constant, we obtain (as in Sections 3.4 and 3.5)

$$\hat{w}_i(x) = \frac{N_c}{2\eta_i} \sqrt{l^2 - x^2}, \quad \hat{w}_T(x) = \frac{N_c}{2\eta_T} \sqrt{l^2 - x^2}, \quad \text{for } x \in L, \quad (5.51)$$

and

$$\frac{\sigma_n(x) - p_i}{N_c} = -1 + \frac{|x|/l}{\sqrt{(x/l)^2 - 1}}, \quad \text{for } x \in L' \quad (5.52)$$

(see Equation (3.72)). These solutions show that the creep closure velocities are ellipsoidal, and that the normal contact stress follows a Weertman-type (1972) distribution (Fig. 3.4). Importantly, the other solutions on $y = 0$, $x \in L'$ are $u_b = v_b = \tau_b = 0$. Therefore, the ice-till interface remains *stationary* in this fully coupled problem, despite different values of ice and sediment viscosities.

5.6 Stokes flow: shallow till

If $d \ll l$, then according to Section 5.3 the till velocities should vanish in the bulk of $|x| < l$. It is therefore sufficient to consider one end of the channel only. We shift the origin to the (left-hand) channel margin by replacing x with $x + l$, and let us re-scale the problem using $[x] = [y] = d$, $[u] = [v] = N_c d / \eta_T$, and $[P] = [\sigma] = N_c$. The Stokes equations become

$$\begin{aligned} u_x + v_y &= 0, \\ \nabla^2 u &= P_x, \\ \nabla^2 v &= P_y + g^*, \end{aligned} \quad (5.53)$$

where $g^* = \rho g d / N_c$. We use the complex variable formulation in (5.42) as before, putting $\eta_T = 1$ and replacing ρg of the body force term (V) by g^* .

The revised geometry is shown in Fig. 5.5. The till now occupies the region S : $-1 \leq y \leq 0$, which has finite depth. As a result, the half-plane techniques used in Section 5.5 are no longer applicable, and here we are restricted to considering sediment deformation alone.

With the ice domain neglected, suitable boundary conditions must be imposed to approximate the coupling at the ice-till interface L' . These include the normal stress σ_n and one of $u = 0$ or $\sigma_{xy} = 0$, as described in (5.21). A reasonable assumption is to take σ_n from Equation (5.52), which is an exact result for deep till, also typical

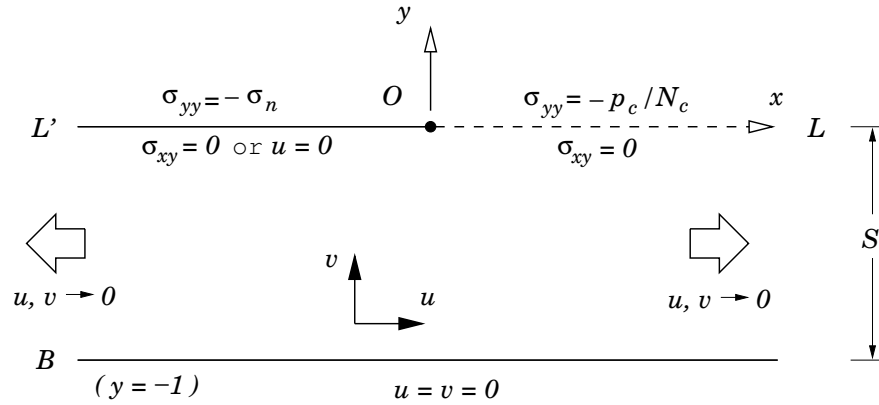


Figure 5.5: The boundary conditions of the shallow till problem in Section 5.6.

in classical hard bed theories (see Section 3.5). Since this is a first approximation, the choice for the remaining condition is less critical. We apply $\sigma_{xy} = 0$ because it is more convenient (see later). Our boundary conditions are summarized in Fig. 5.5.

To solve this ‘infinite strip’ problem, we use a method developed by Tiffen and Semple (1965), based on taking the *Fourier transform* of the equations in (5.42). This leads to closed-form integral solutions which are difficult to invert, but we can overcome this problem by making numerical approximations. Note that (5.53) can also be written as the biharmonic equation

$$\nabla^4 \psi = 0, \quad (5.54)$$

where ψ is a stream function, defined by $u = \psi_y$, $v = -\psi_x$. As such, the discontinuous conditions on $y = 0$ are suggestive of the Wiener-Hopf technique, where one takes the Fourier transform of (5.54) instead of its complex variable representation (Carrier, Krook and Pearson, 1983). A classic example of this is found in the problem of an ice-strip partly sliding over and partly adhering to its bed (Hutter and Olunloyo, 1980). For our purpose however, Tiffen and Semple’s method turns out to be more concise and is adopted herein.

5.6.1 The transformed problem

Fundamental equations

Taking the complex variable equations (5.42)_{1,3}, with $\eta_T = 1$ and with the body force term neglected¹:

$$\begin{aligned} \sigma_{yy} - i\sigma_{xy} &= \Omega'(z) + \overline{\Omega'(z)} + z\overline{\Omega''(z)} + \overline{\omega'(z)}, \\ 2(u + iv) &= \Omega(z) - z\overline{\Omega'(z)} - \overline{\omega(z)}, \end{aligned} \quad (5.55)$$

¹The gravity term, given by $V(z, \bar{z})$, produces a linear contribution to the direct stresses but has no effect on the velocity results sought here.

we introduce the functions

$$\lambda(z) = \Omega'(z), \quad \psi(z) = \Omega'(z) + z\Omega''(z) + \omega'(z), \quad (5.56)$$

which are more convenient for dealing with the strip problem than Ω and ω . It follows that we can write

$$\begin{aligned} \sigma_{yy} + i\sigma_{xy} &= \overline{\lambda(z)} - 2iy\lambda'(z) + \psi(z), \\ -2\frac{\partial}{\partial x}(u - iv) &= -\overline{\lambda(z)} - 2iy\lambda'(z) + \psi(z). \end{aligned} \quad (5.57)$$

Our problem is to determine the functions λ and ψ , which are holomorphic in S , subject to the boundary conditions in Fig. 5.5. Subsequently, the creep closure velocity is obtained by evaluating v on L .

As a shorthand to the equations in (5.57), let us define

$$p_r(z, \bar{z}) = \begin{cases} \sigma_{yy} + i\sigma_{xy} \\ -2\frac{\partial}{\partial x}(u - iv), \end{cases} \quad \chi_r = \begin{cases} 1 \\ -1, \end{cases} \quad \text{for } r = \begin{cases} 1 \\ 2. \end{cases} \quad (5.58)$$

We proceed to take the Fourier transform of the non-analytic functions

$$p_r(z, \bar{z}) = \chi_r \overline{\lambda(z)} - 2iy\lambda'(z) + \psi(z), \quad r = 1, 2. \quad (5.59)$$

Multiplication of (5.59) by e^{izs} and integration w.r.t. z along any line in the strip parallel to the real axis gives the equation

$$\int_{-\infty}^{\infty} p_r(z, \bar{z}) e^{izs} dz = \chi_r e^{-2ys} \int_{-\infty}^{\infty} \overline{\lambda(\bar{z})} e^{i\bar{z}s} d\bar{z} - 2iy \int_{-\infty}^{\infty} \lambda'(z) e^{izs} dz + \int_{-\infty}^{\infty} \psi(z) e^{izs} dz, \quad (5.60)$$

where s — the transform space variable — is taken to be real, $\overline{\lambda(\bar{z})}$ is the holomorphic function of $\bar{z}$ equal to $\overline{\lambda(z)}$, and $d\bar{z} = dz$ along the line of integration. As λ and ψ are holomorphic along the line of integration, we make the assumption that

$$\lim_{x \rightarrow \pm\infty} e^{izs} \lambda(z) = 0, \quad \lim_{x \rightarrow \pm\infty} e^{izs} \psi(z) = 0, \quad (5.61)$$

so that the complex Fourier inversion formula may be applied in the strip. The conditions in (5.61) are justified by the solutions given below, and are consistent with the fact that the till velocities vanish at the far field.

The crux of this method hinges on the fact that the transform of a holomorphic function of z is a function of s only, which enables us to write

$$\begin{aligned} P_r(s, y) &= \int_{-\infty}^{\infty} p_r(z, \bar{z}) e^{izs} dz, & \Psi(s) &= \int_{-\infty}^{\infty} \psi(z) e^{izs} dz, \\ \Lambda(s) &= \int_{-\infty}^{\infty} \lambda(z) e^{izs} dz, & \Lambda^*(s) &= \int_{-\infty}^{\infty} \overline{\lambda(\bar{z})} e^{i\bar{z}s} d\bar{z} \end{aligned} \quad (5.62)$$

for the integrals in (5.60), the notation indicating that Λ , Λ^* and Ψ are independent of y . By definition

$$\Lambda^*(s) = \overline{\Lambda(-s)} \quad (5.63)$$

and

$$\int_{-\infty}^{\infty} \lambda'(z) e^{izs} dz = -is\Lambda(s), \quad (5.64)$$

therefore Equation (5.60) becomes

$$P_r(s, y) = \chi_r e^{-2ys} \overline{\Lambda(-s)} - 2ys\Lambda(s) + \Psi(s), \quad r = 1, 2. \quad (5.65)$$

Alternatively, we could have isolated the real and imaginary parts of (5.57) before taking its Fourier transform. In this case, we use the additional relations

$$\int_{-\infty}^{\infty} \overline{\psi(z)} e^{izs} dz = e^{-2ys} \overline{\Psi(-s)}, \quad \int_{-\infty}^{\infty} \overline{\lambda'(z)} e^{izs} dz = -ise^{-2ys} \overline{\Lambda(-s)}, \quad (5.66)$$

and then the resulting transforms are

$$\begin{aligned} 2ie^{-ys} \int_{-\infty}^{\infty} \sigma_{xy} e^{ixs} dx &= -(1 + 2ys)\Lambda(s) + e^{-2ys}(1 - 2ys)\overline{\Lambda(-s)} + \Psi(s) - e^{-2ys}\overline{\Psi(-s)}, \\ 2e^{-ys} \int_{-\infty}^{\infty} \sigma_{yy} e^{ixs} dx &= (1 - 2ys)\Lambda(s) + e^{-2ys}(1 + 2ys)\overline{\Lambda(-s)} + \Psi(s) + e^{-2ys}\overline{\Psi(-s)}, \\ -4e^{-ys} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{ixs} dx &= -(1 + 2ys)\Lambda(s) - e^{-2ys}(1 - 2ys)\overline{\Lambda(-s)} + \Psi(s) + e^{-2ys}\overline{\Psi(-s)}, \\ 4ie^{-ys} \int_{-\infty}^{\infty} \frac{\partial v}{\partial x} e^{ixs} dx &= (1 - 2ys)\Lambda(s) - e^{-2ys}(1 + 2ys)\overline{\Lambda(-s)} + \Psi(s) - e^{-2ys}\overline{\Psi(-s)}. \end{aligned} \quad (5.67)$$

Boundary conditions

The next step is to determine $\Lambda(s)$ and $\Psi(s)$ from (5.65) and (5.67) by applying the boundary conditions on $y = 0$ and $y = -1$. Let us define the Fourier transforms

$$\begin{aligned} F_1(s) &= 2 \int_{-\infty}^{\infty} \sigma_{xy}|_{y=0} e^{ixs} dx, & F_2(s) &= 2 \int_{-\infty}^{\infty} \sigma_{yy}|_{y=0} e^{ixs} dx, \\ G_1(s) &= -4 \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}|_{y=0} e^{ixs} dx, & G_2(s) &= 4 \int_{-\infty}^{\infty} \frac{\partial v}{\partial x}|_{y=0} e^{ixs} dx, \\ G_b(s) &= -2 \int_{-\infty}^{\infty} \frac{\partial}{\partial x}(u - iv)|_{y=-1} e^{ixs} dx. \end{aligned} \quad (5.68)$$

Substituting these into (5.65) and (5.67), with the correct assignments of y and χ_r , leads to

$$e^s G_b(s) = -e^{2s} \overline{\Lambda(-s)} + 2s\Lambda(s) + \Psi(s) \quad (5.69)$$

and

$$\begin{aligned}
iF_1(s) &= -\Lambda(s) + \overline{\Lambda(-s)} + \Psi(s) - \overline{\Psi(-s)}, \\
F_2(s) &= \Lambda(s) + \overline{\Lambda(-s)} + \Psi(s) + \overline{\Psi(-s)}, \\
G_1(s) &= -\Lambda(s) - \overline{\Lambda(-s)} + \Psi(s) + \overline{\Psi(-s)}, \\
iG_2(s) &= \Lambda(s) - \overline{\Lambda(-s)} + \Psi(s) - \overline{\Psi(-s)}.
\end{aligned} \tag{5.70}$$

Note that $F_1(s) = \overline{F_1(-s)}$ (and also for F_2 , G_1 and G_2), because the stresses and velocities here are real quantities.

Given the function G_b and any two of F_1 , F_2 , G_1 , G_2 , Equations (5.69) and (5.70) serve to determine Λ and Ψ . If the entire stress/velocity field is required, λ and ψ may be obtained by evaluating the Fourier inversions

$$\lambda(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Lambda(s) e^{-isz} ds, \quad \psi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(s) e^{-isz} ds. \tag{5.71}$$

However, this is not necessary because we seek only v on $y = 0$, and that is given directly via the inverse transform of $G_2(s)$.

We apply the boundary conditions shown in Fig. 5.5. Straightforwardly, no slip on B implies that $G_b(s) = 0$. The normal stress $\sigma_{yy}(x, 0)$ is given, so $F_2(s)$ is a known function. Thus,

$$\begin{aligned}
-e^{2s} \overline{\Lambda(-s)} + 2s\Lambda(s) + \Psi(s) &= 0, \\
\Lambda(s) + \overline{\Lambda(-s)} + \Psi(s) + \overline{\Psi(-s)} &= F_2(s).
\end{aligned} \tag{5.72}$$

As described before, the remaining boundary condition on $y = 0$ consists of $\sigma_{xy} = 0$ for $x > 0$, and two choices for $x < 0$: $\sigma_{xy} = 0$, or $u = 0$. With regard to our current formulation the former possibility would suffice, but the latter gives rise to a mixed² condition that is partially encapsulated in *each* of F_1 and G_1 — the resulting problem is much more difficult. Here we choose the former, putting $\sigma_{xy} = 0$ on $L \cup L'$, so $F_1(s) = 0$, and our last relation is

$$-\Lambda(s) + \overline{\Lambda(-s)} + \Psi(s) - \overline{\Psi(-s)} = 0. \tag{5.73}$$

Addition of (5.72)₂ and (5.73) followed by elimination of $\Psi(s)$ with (5.72)₁ now leads to the equation

$$2(1 + e^{2s}) \overline{\Lambda(-s)} - 4s\Lambda(s) = F_2(s). \tag{5.74}$$

Taking the conjugate of (5.74) and replacing s by $-s$ yields another equation

$$2(1 + e^{-2s}) \Lambda(s) + 4s \overline{\Lambda(-s)} = \overline{F_2(s)}, \tag{5.75}$$

²*Mixed* in the sense that the type of boundary condition in $x > 0$ differs from that in $x < 0$.

so that, when $\overline{\Lambda(-s)}$ is eliminated between the two, we obtain

$$\Lambda(s) = \frac{1 + e^{2s} - 2s}{2[(1 + e^{-2s})(1 + e^{2s}) + 4s^2]} F_2(s) = \frac{e^s \cosh s - s}{4(\cosh^2 s + s^2)} F_2(s). \quad (5.76)$$

By re-substituting this into (5.72)₁, it follows that

$$\Psi(s) = \frac{e^s \cosh s + s(2s - 1)}{4(\cosh^2 s + s^2)} F_2(s). \quad (5.77)$$

The function $G_2(s)$ is given directly by substituting Λ and Ψ into Equation (5.70)₄.

Normal stress transform $F_2(s)$

The ice-till contact stress is taken from Equation (5.52). Using our current length scale d and stress scale N_c ($= p_i - p_c$), we find the local behaviour $\sigma_n(x) \rightarrow (-2\gamma x)^{-1/2} + p_c/N_c + O(\sqrt{\gamma})$ as $x \rightarrow 0_-$, and also $\sigma_n(x) \rightarrow p_i/N_c$ as $x \rightarrow -\infty$. Therefore, an appropriate input stress function for $x < 0$ is

$$\sigma_{yy}(x, 0) = -\sigma_n(x) = -\frac{p_i}{N_c} - (-2\gamma x)^{-1/2}. \quad (5.78)$$

For $x > 0$, we have in the channel $\sigma_{yy}(x, 0) = -p_c/N_c$.

The constant part $-p_c/N_c$ acting on both sides L and L' has no effect on the flow and can be taken out. Since our problem is linear and has homogeneous boundary conditions (elsewhere), it is natural to split the applied normal stress into two parts:

$$(a) : \sigma_{yy}(x, 0) = \begin{cases} -1 \\ 0, \end{cases} \quad (b) : \sigma_{yy}(x, 0) = \begin{cases} -(-2\gamma x)^{-1/2} \\ 0, \end{cases} \quad \text{for } \begin{matrix} x < 0 \\ x > 0, \end{matrix} \quad (5.79)$$

and superpose the results. These components describe respectively a ‘step’ input and a ‘singular’ input. By using standard tables, we obtain the corresponding Fourier transforms

$$\begin{aligned} (a) : \quad F_2(s) &= -2\pi \left[\delta(s) + \frac{1}{\pi i s} \right], \\ (b) : \quad F_2(s) &= \begin{cases} -(1+i)\sqrt{-\pi/\gamma s} \\ -(1-i)\sqrt{\pi/\gamma s}, \end{cases} \quad \text{for } \begin{matrix} s < 0 \\ s > 0, \end{matrix} \end{aligned} \quad (5.80)$$

where $\delta(s)$ is the delta function. We have derived all the necessary ingredients for the inversion of $G_2(s)$.

5.6.2 The inversion integral

Substitution of Λ and Ψ into (5.70)₄ and the use of (5.68)₄ produce the transform pair

$$G_2(s) = iK(s)F_2(s) = 4 \int_{-\infty}^{\infty} \frac{\partial v}{\partial x} \Big|_{y=0} e^{ixs} dx, \quad (5.81)$$

$$\frac{\partial v}{\partial x} \Big|_{y=0} = \frac{i}{8\pi} \int_{-\infty}^{\infty} F_2(s) K(s) e^{-isx} ds, \quad (5.82)$$

in which the *transfer function* is

$$K(s) = \frac{s - \sinh s \cosh s}{\cosh^2 s + s^2}. \quad (5.83)$$

(And $v|_{y=0} \rightarrow 0$ as $|x| \rightarrow \infty$.) In general, the entire velocity field $v(x, y)$ may also be obtained via (5.67)₄, using the expression

$$\begin{aligned} \frac{\partial v}{\partial x} = & -\frac{i}{8\pi} \int_{-\infty}^{\infty} \frac{F_2(s)}{\cosh^2 s + s^2} \left\{ \cosh s [\sinh(y+1)s - ys \cosh(y+1)s] \right. \\ & \left. + s[s(y+1) \sinh(ys) - \cosh(ys)] \right\} e^{-isx} ds, \end{aligned} \quad (5.84)$$

but evaluation of this is extremely tedious. As our concern here is the till surface velocity, we consider the particular case $y = 0$ where (5.84) reduces more simply to (5.82).

The transfer function $K(s)$

The Fourier inversion in (5.82) is still problematic. $K(s)$ has an infinite number of poles on the complex s -plane, located at the roots of the transcendental equations

$$\cosh s \pm is = 0. \quad (5.85)$$

This renders analytical methods such as contour integration improbable, thus we are forced to represent $K(s)$ by an approximate function, which we denote by $K^*(s)$.

As $K(s)$ is odd, it is convenient to do this for $s > 0$ first and then extend the result for $s < 0$ by reflection; however, this may introduce significant error associated with the curve-fit at the origin. In particular, $K^*(s)$ may not be adequately ‘smooth’ at $s = 0$; as we shall see, this can lead to incorrect behaviour of $v(x, 0)$ at the far field. In order to select a suitable $K^*(s)$, let us examine the properties of the transform pair.

We use the simple unit step (5.79)₁ to illustrate. $F_2(s)$ as given by (5.80)₁ consists of a non-flow component $\delta(s)$ due to the mean stress, and a flow component $1/s$ arising from differential stress. Since the latter is anti-symmetric (in x), with $v(\pm\infty, 0) = 0$, it follows that

$$v(0, 0) = 0, \quad \int_{-\infty}^{\infty} v(x, 0) dx = 0. \quad (5.86)$$

(The second of these is a statement of mass conservation.) Now if we apply the transform property $\mathcal{F}[f'(x)] = -is\mathcal{F}[f(x)]$, then Equation (5.81) leads to the pair

$$\begin{aligned} v(x, 0) &= -\frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{F_2(s)K(s)}{s} e^{-isx} ds, \\ \frac{F_2(s)K(s)}{s} &= -4 \int_{-\infty}^{\infty} v(x, 0) e^{ixs} dx. \end{aligned} \quad (5.87)$$

By setting $x = 0$ and $s = 0$ in these equations as appropriate, the conditions in (5.86) become

$$K(0) = 0, \quad \lim_{s \rightarrow 0} \left[\frac{K(s)}{s} \right] = 0, \quad \lim_{s \rightarrow 0} \left[\frac{K(s)}{s^2} \right] = 0, \quad (5.88)$$

which are automatically satisfied by (5.83). The approximating function $K^*(s)$ must also possess these properties.

Keeping the step input as before, a more general constraint on $K^*(s)$ is revealed by examining the large- x behaviour of the Fourier integral in (5.82). Let us define

$$\begin{aligned} I(x) &= \frac{-i}{2} \int_{-\infty}^{\infty} F_2(s) K(s) e^{-isx} ds \\ &= \int_{-\infty}^{\infty} \frac{K(s)}{s} e^{-isx} ds \quad \left(\propto \frac{\partial v}{\partial x} \Big|_{y=0} \right) \end{aligned} \quad (5.89)$$

and find its asymptotic expansion as $x \rightarrow \infty$ in powers of $1/x$. After one integration by parts, we obtain

$$I(x) = \left[\frac{e^{-isx} K(s)}{-ixs} \right]_{-\infty}^{\infty} + \frac{1}{ix} \int_{-\infty}^{\infty} \left(\frac{K(s)}{s} \right)' e^{-isx} ds, \quad (5.90)$$

on the condition that $K(s)/s$ is continuous. Since $\lim_{s \rightarrow \pm\infty} [K(s)/s] = 0$, and that $\int_{-\infty}^{\infty} |(K(s)/s)'| ds$ exists, the boundary terms vanish and application of the Riemann-Lebesgue lemma (Bender and Orszag, 1978) implies that the integral here $\rightarrow 0$ as $x \rightarrow \infty$. It follows that $I(x) \ll 1/x$ as $x \rightarrow \infty$. Repeating integration by parts for (5.90) gives

$$I(x) = \frac{1}{ix} \left[\frac{e^{-isx}}{-ix} \left(\frac{K(s)}{s} \right)' \right]_{-\infty}^{\infty} - \frac{1}{x^2} \int_{-\infty}^{\infty} \left(\frac{K(s)}{s} \right)'' e^{-isx} ds, \quad (5.91)$$

where similarly we can argue that $I(x) \ll 1/x^2$. As $K(s)/s$ is infinitely differentiable, this process may be continued indefinitely, implying that $I(x)$ (and $v(x, 0)$) is exponentially small as $x \rightarrow \infty$.

Clearly, if $K(s)$ is approximated by a function $K^*(s)$ that has a discontinuous derivative at $s = 0$ at some finite order, then our results for I (and v) will only *algebraically* approximate the true solution at large x . Specifically, if $d^n/ds^n [K^*(s)/s]$ is continuous up to and including $n = m$, then $I(x) \sim 1/x^{m+2}$ and $v(x, 0) \sim 1/x^{m+1}$. It follows that $K^*(s)$ should be made as smooth as possible at the origin (by maximizing m).

One can postulate similar results for the singular stress input (5.80)₂, but the corresponding asymptotic expressions are difficult to derive. In the following, we simply use the step input as a tuning device for $K^*(s)$. At the least, it is necessary to ensure that the total (approximate) creep flux $\int_0^{\infty} v(x, 0) dx$ remains bounded, hence the minimum requirement is $m = 1$ — this is equivalent to the conditions in (5.88). To achieve an acceptable level of accuracy, we shall actually impose $m = 2$, i.e., we require $K^*(s)$ to have the Taylor expansion (about $s = 0$)

$$K^*(s) = -\frac{2s^3}{3} + \text{higher order terms.} \quad (5.92)$$

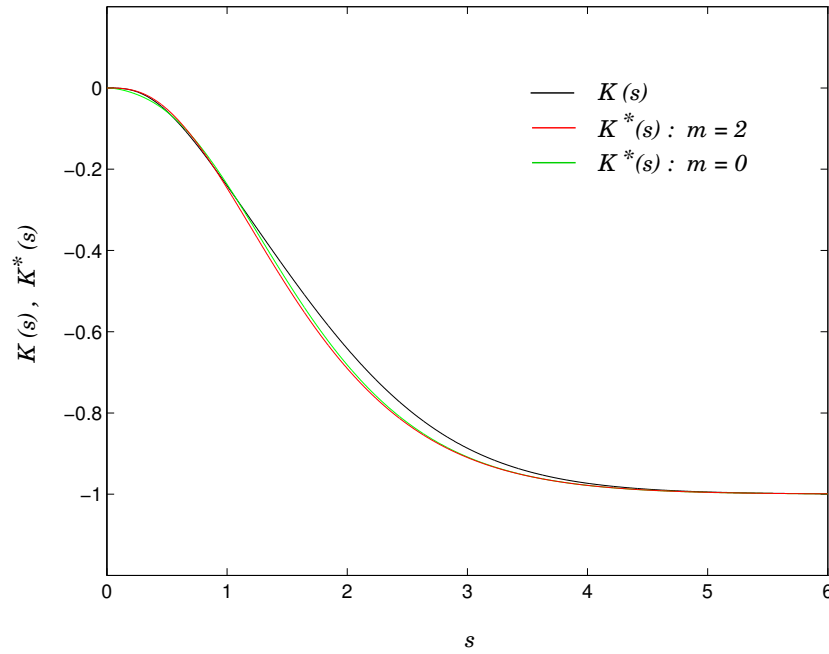


Figure 5.6: The transfer function $K(s)$ and the best-fit approximating function $K^*(s)$ at the approximation orders $m = 2$ and $m = 0$.

The approximating function $K^*(s)$

We follow the method described by Sneddon (p. 476, 1951), of representing $K^*(s)$ in the form $\sum p_n s^n e^{-q_n s}$, where p_n and q_n are constants. From (5.83), we find $K(0) = 0$, and $K(s) \sim -1 + 4s(s+1)e^{-2s}$ as $s \rightarrow \infty$. An appropriate expression that has this behaviour is

$$K^*(s) = \begin{cases} (e^{-2s} - 1) + c_1(e^{-c_2 s} - e^{-2s}) + (4s^2 + c_3 s)e^{-2s} + c_4 s(e^{-c_5 s} - e^{-c_6 s}), & s > 0, \\ (1 - e^{2s}) + c_1(e^{2s} - e^{c_2 s}) + (c_3 s - 4s^2)e^{2s} + c_4 s(e^{c_5 s} - e^{c_6 s}), & s < 0, \end{cases} \quad (5.93)$$

where $c_1, \dots, c_6$ are constants, $c_2, c_5, c_6 > 2$, $c_4 > 0$, and

$$c_3 = 4. \quad (5.94)$$

Substituting (5.93) into (5.92) gives us the three additional relations

$$\begin{aligned} \lim_{s \rightarrow 0} \left[\frac{K^*(s)}{s} \right] = 0 & \Rightarrow c_1 = \frac{2 - c_3}{2 - c_2}, \\ \lim_{s \rightarrow 0} \left[\frac{K^*(s)}{s^2} \right] = 0 & \Rightarrow c_4(c_5 - c_6) = \frac{c_1}{2}(c_2 - 2)^2 + 2, \\ \lim_{s \rightarrow 0} \left[\frac{K^*(s)}{s^3} \right] = -2/3 & \Rightarrow c_1(8 - c_2^3) + 12c_3 - 56 + 3c_4(c_5^2 - c_6^2) = -4, \end{aligned} \quad (5.95)$$

so there are two degrees of freedom in choosing c_1 to c_6 . In Fig. 5.6, we show our best-fit result by plotting K and K^* in $s \geq 0$. Suitable values are

$$[c_1, c_2, c_3, c_4, c_5, c_6] = [20, 2.1, 4, 10, 2.424, 2.214]. \quad (5.96)$$

For comparison, we also include the case where conditions (5.94) and (5.95)_{2,3} are relaxed (by setting $c_4 = 0$), corresponding to an approximation order $m = 0$. Then we would anticipate a non-integrable velocity response for a step stress input; in this case, the suitable c -values are

$$[c_1, c_2, c_3] = [-1.2, 4, -0.4]. \quad (5.97)$$

5.6.3 Inversion results

(a) Unit step input

We replace $K(s)$ by $K^*(s)$. Equation (5.82), with F_2 taken from (5.80)₁, becomes

$$\frac{\partial v_a}{\partial x} = -\frac{1}{2\pi} \int_0^\infty \frac{K^*(s)}{s} \cos xs \, ds. \quad (5.98)$$

By using the standard integrals (see Gradshteyn and Ryzhik, 1980)

$$\begin{aligned} \int_0^\infty e^{-cs} \cos xs \, ds &= \frac{c}{x^2 + c^2}, & \int_0^\infty s e^{-cs} \cos xs \, ds &= \frac{c^2 - x^2}{(x^2 + c^2)^2}, & (c > 0), \\ \int_0^\infty \frac{e^{-bs} - e^{-cs}}{s} \cos xs \, ds &= \frac{1}{2} \ln \frac{x^2 + c^2}{x^2 + b^2}, & (b \geq 0, c > 0), \\ \int_0^\infty \frac{e^{-cs} - 1}{s} \cos xs \, ds &= \ln \left| \frac{x}{\sqrt{x^2 + c^2}} \right|, & (c > 0), \end{aligned} \quad (5.99)$$

we obtain

$$\begin{aligned} \frac{\partial v_a}{\partial x} &= -\frac{1}{2\pi} \left[\ln \left| \frac{x}{\sqrt{x^2 + 4}} \right| + \frac{c_1}{2} \ln \frac{x^2 + 4}{x^2 + c_2^2} \right. \\ &\quad \left. + \frac{4(4 - x^2)}{(4 + x^2)^2} + \frac{2c_3}{x^2 + 4} + c_4 \left(\frac{c_5}{x^2 + c_5^2} + \frac{c_6}{x^2 + c_6^2} \right) \right], \end{aligned} \quad (5.100)$$

so that further integration, with the use of $v_a(\pm\infty) \rightarrow 0$ and (5.95)₁, leads to

$$\begin{aligned} v_a(x) &= -\frac{1}{2\pi} \left\{ x \left[\ln \left| \frac{x}{\sqrt{x^2 + 4}} \right| + \frac{c_1}{2} \ln \frac{x^2 + 4}{x^2 + c_2^2} \right] + c_1 c_2 \left[\tan^{-1} \left(\frac{x}{2} \right) - \right. \right. \\ &\quad \left. \left. \tan^{-1} \left(\frac{x}{c_2} \right) \right] + \frac{4x}{x^2 + 4} + c_4 \left[\tan^{-1} \left(\frac{x}{c_5} \right) - \tan^{-1} \left(\frac{x}{c_6} \right) \right] \right\}. \end{aligned} \quad (5.101)$$

In Fig. 5.7, this approximate solution is plotted by using the c -values in (5.96) and (5.97). It displays the expected far field behaviour: $v_a \sim 1/x$ (non-integrable) when $m = 0$, and $v_a \sim 1/x^3$ (integrable) when $m = 2$. Integrating once more produces the volume flux function

$$\begin{aligned} q_a(x_0) &= 2\pi \int_0^{x_0} v_a(x) dx = \\ &= -\frac{c_1}{4} [(x_0^2 + 4) \ln(x_0^2 + 4) - (x_0^2 + c_2^2) \ln(x_0^2 + c_2^2) + 2c_2^2 \ln c_2] \end{aligned}$$

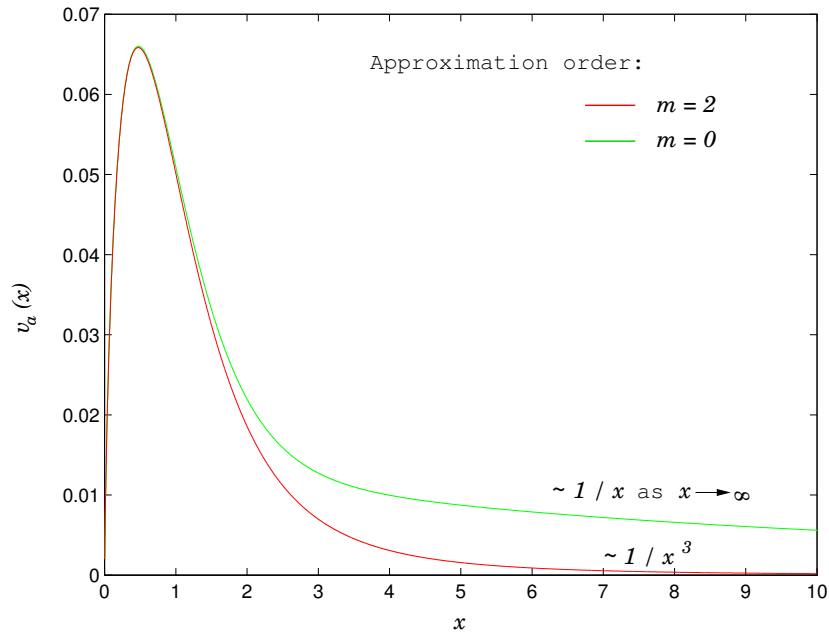


Figure 5.7: The creep velocity solution $v_a(x)$ at approximation orders $m = 2$ and $m = 0$ for the unit step stress input $(5.80)_1$.

$$\begin{aligned}
& -\frac{x_0^2}{2} \ln x_0 + \frac{x_0^2 - 4}{4} \ln(x_0^2 + 4) + 2(c_1 + 1) \ln 2 \\
& -c_1 c_2 \left\{ x_0 \left[\tan^{-1}\left(\frac{x_0}{2}\right) - \tan^{-1}\left(\frac{x_0}{c_2}\right) \right] + 2 \ln \frac{2}{\sqrt{x_0^2 + 4}} - c_2 \ln \frac{c_2}{\sqrt{x_0^2 + c_2^2}} \right\} \\
& -c_4 \left\{ x_0 \left[\tan^{-1}\left(\frac{x_0}{c_5}\right) - \tan^{-1}\left(\frac{x_0}{c_6}\right) \right] + c_5 \ln \frac{c_5}{\sqrt{x_0^2 + c_5^2}} - c_6 \ln \frac{c_6}{\sqrt{x_0^2 + c_6^2}} \right\}, \quad (5.102)
\end{aligned}$$

shown in Fig. 5.8. The ‘ $m = 2$ ’ solution has a total volume flux given by $q_a(\infty) \approx 0.689$.

(b) Singular stress input

With $F_2(s)$ taken from $(5.80)_2$, it is easier to calculate the velocity by using $(5.87)_1$ directly. The corresponding inversion formula is

$$v_b(x) = \frac{1}{8\sqrt{\pi\gamma}} \left[\int_{-\infty}^0 \frac{(1+i)K^*(s)}{s\sqrt{-s}} e^{-isx} ds + \int_0^{\infty} \frac{(1-i)K^*(s)}{s\sqrt{s}} e^{-isx} ds \right]. \quad (5.103)$$

If we apply the change of variable $t = \sqrt{-s}$ for $s < 0$ and $t = \sqrt{s}$ for $s > 0$, then substitution of $K^*(s)$ from (5.93) leads to

$$\begin{aligned}
v_b(x) = \frac{1}{2\sqrt{\pi\gamma}} \int_0^{\infty} (\cos xt^2 - \sin xt^2) & \left[\frac{e^{-2t^2} - 1}{t^2} + \frac{c_1(e^{-c_2 t^2} - e^{-2t^2})}{t^2} \right. \\
& \left. + (4t^2 + c_3)e^{-2t^2} + c_4(e^{-c_5 t^2} - e^{-c_6 t^2}) \right] dt. \quad (5.104)
\end{aligned}$$

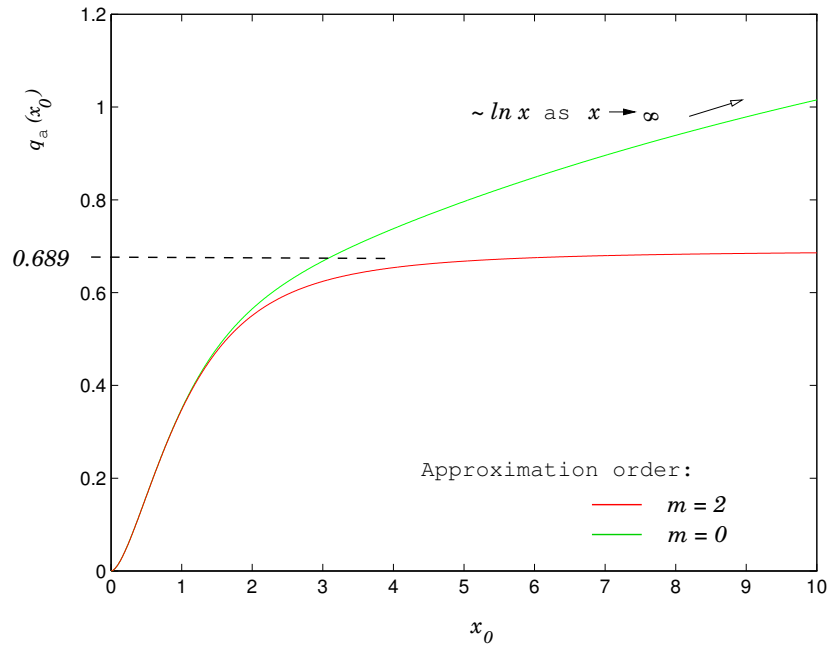


Figure 5.8: The volume flux function $q_a(x_0)$ at approximation orders $m = 2$ and $m = 0$ for the unit step stress input $(5.80)_1$.

By defining the definite integrals (taking their values from Gradshteyn and Ryzhik, 1980)

$$\begin{aligned}
 C_0(x, k) &= \int_0^\infty e^{-kt^2} \cos xt^2 dt = \frac{\sqrt{\pi}}{2(x^2 + k^2)^{1/4}} \cos \left[\frac{1}{2} \tan^{-1} \left(\frac{x}{k} \right) \right], \\
 S_0(x, k) &= \int_0^\infty e^{-kt^2} \sin xt^2 dt = \frac{\sqrt{\pi}}{2(x^2 + k^2)^{1/4}} \sin \left[\frac{1}{2} \tan^{-1} \left(\frac{x}{k} \right) \right], \\
 C_1(x, k) &= \int_0^\infty t^2 e^{-kt^2} \cos xt^2 dt = \frac{\sqrt{\pi}}{4(x^2 + k^2)^{3/4}} \cos \left[\frac{3}{2} \tan^{-1} \left(\frac{x}{k} \right) \right], \\
 S_1(x, k) &= \int_0^\infty t^2 e^{-kt^2} \sin xt^2 dt = \frac{\sqrt{\pi}}{4(x^2 + k^2)^{3/4}} \sin \left[\frac{3}{2} \tan^{-1} \left(\frac{x}{k} \right) \right], \quad (5.105)
 \end{aligned}$$

and by using the result

$$\int_0^\infty \frac{e^{-\alpha t^2} - e^{-\beta t^2}}{t^2} (\cos xt^2 - \sin xt^2) dt = 2[(x + \beta)C_0(x, \beta) - (x + \alpha)C_0(x, \alpha) + (x - \beta)S_0(x, \beta) - (x - \alpha)S_0(x, \alpha)], \quad (5.106)$$

Equation (5.104) may be written in the form

$$v_b(x) = \frac{1}{2\sqrt{\pi\gamma}} \left\{ -2c_1[(x + c_2)C_0(x, c_2) + (x - c_2)S_0(x, c_2)] \right\}$$

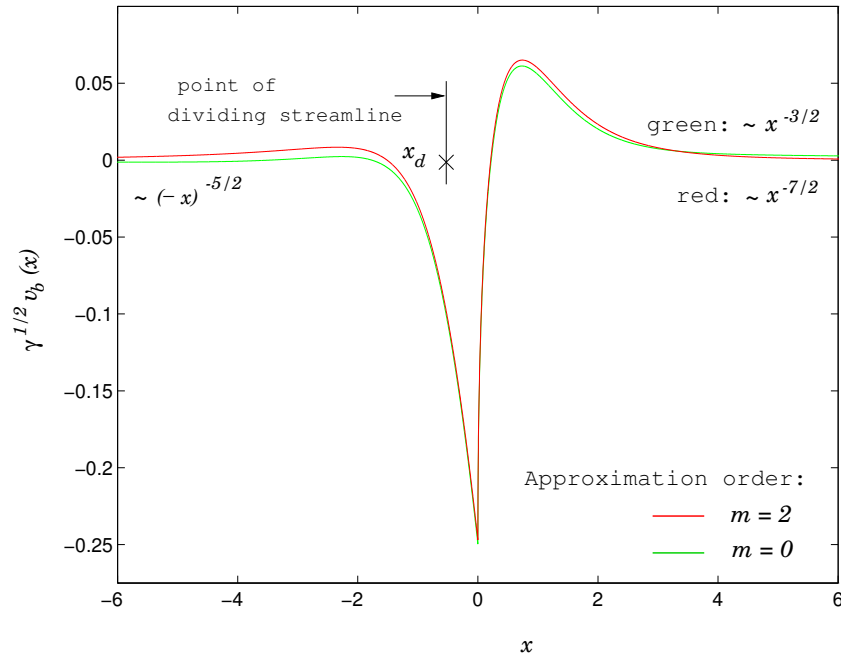


Figure 5.9: The creep velocity solution $v_b(x)$ at approximation orders $m = 2$ and $m = 0$ for the singular stress input $(5.80)_2$. Note that the vertical axis here is $\gamma^{1/2}v_b$.

$$\begin{aligned}
& + [2(x+2)(c_1-1) + c_3]C_0(x, 2) + [2(x-2)(c_1-1) - c_3]S_0(x, 2) \\
& + 2x[S_0(x, 0) + C_0(x, 0)] + 4[C_1(x, 2) - S_1(x, 2)] \\
& + c_4[C_0(x, c_5) - S_0(x, c_5) - C_0(x, c_6) + S_0(x, c_6)] \}. \quad (5.107)
\end{aligned}$$

After substitution and manipulation, we obtain

$$\begin{aligned}
v_b(x) = \frac{1}{2\sqrt{\gamma}} \Big\{ & - \frac{c_1}{(x^2 + c_2^2)^{1/4}} [(x + c_2) \cos \theta_{c_2} + (x - c_2) \sin \theta_{c_2}] \\
& + \frac{2(x+2)(c_1-1) + c_3}{2(x^2 + 4)^{1/4}} \cos \theta_2 + \frac{2(x-2)(c_1-1) - c_3}{2(x^2 + 4)^{1/4}} \sin \theta_2 \\
& + \mu(x)\sqrt{2x} + \frac{(\cos 3\theta_2 - \sin 3\theta_2)}{(x^2 + 4)^{3/4}} \\
& + \frac{c_4}{2} \left[\frac{\cos \theta_{c_5} - \sin \theta_{c_5}}{(x^2 + c_5^2)^{1/4}} - \frac{\cos \theta_{c_6} - \sin \theta_{c_6}}{(x^2 + c_6^2)^{1/4}} \right] \Big\}, \quad (5.108)
\end{aligned}$$

where

$$\theta_{c_i} = \frac{1}{2} \tan^{-1} \left(\frac{x}{c_i} \right), \quad \theta_2 = \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right), \quad (5.109)$$

and $\mu(x)$ is the unit step function $\mu = 1$ for $x > 0$, $\mu = 0$ for $x < 0$.

We use the c -values in (5.96) corresponding to the (more accurate) $m = 2$ approximation. The velocity solution (5.108), plotted in Fig. 5.9, is asymmetric, and has

the interesting feature that $v_b(0) \neq 0$. Local expansion about $x = 0$ gives

$$\sqrt{\gamma}v_b \approx \begin{cases} -0.25 + \sqrt{x/2} & \text{as } x \rightarrow 0_+ \\ -0.25 - 0.35x & \text{as } x \rightarrow 0_-, \end{cases} \quad (5.110)$$

which contains a square-root closure behaviour in $x > 0$, also found in the deep till solution (see end of Section 5.5). At the far field, we have $v_b \sim x^{-7/2}$ and $v_b \sim (-x)^{-5/2}$ respectively as $x \rightarrow +\infty$ and $-\infty$.

As before, the $m = 0$ solution is included for comparison (shown in green, Fig. 5.9). The decay as $x \rightarrow +\infty$ is less severe, with $v_b \sim x^{-3/2}$. (The exponent is increased by 2, as in the step input case.) Thus, our calculated velocity responses seem to support the hypothesis that the exact solution decays exponentially towards the channel centre. This would provide an appropriate matching condition with the outer velocity solution (see Section 5.3).

Superposition

Finally, the total (dimensionless) creep velocity is given by the expression

$$v_c(x) = v_a(x) + v_b(x), \quad (5.111)$$

where v_a and v_b are taken from (5.101) and (5.108) (with $m = 2$). These solutions indicate that till deformation occurs close to the channel margin(s), within a distance of several till thicknesses from it. There is subsidence of the ice-till interface (L') just outside the channel, and a corresponding in-creek of sediment into the channel. Deformation is virtually absent in the central part of the channel bed. (v_c decays to one-tenth of maximum at $|x| \approx 3$). This *localized* creep behaviour is consistent with the prediction of Section 5.3.

Since the functions plotted in Figs. 5.7 and 5.9 have similar magnitudes, $\gamma \ll 1$ implies that creep flow is dominated by the singular stress component; the step stress response v_a is negligible, and thus $v_c \approx v_b$. Accordingly, the (dimensionless) net sediment flux, obtained by numerical integration, is

$$q_c \approx \int_0^\infty v_b(x, 0) dx \approx \frac{0.09}{\sqrt{\gamma}}. \quad (5.112)$$

With the values assumed earlier in Sections 5.2 and 5.3 — $\eta_T = 10^{10}$ Pa s, $l = 10$ m, $\gamma = 0.1$, $N_c = 1$ bar — we obtain

$$\begin{aligned} [v_c] &= \frac{N_c d}{\eta_T} \approx 10^{-5} \text{ m s}^{-1} \quad (\text{or } 0.01 \text{ mm s}^{-1}), \\ [q_c] &= \frac{N_c d^2}{\eta_T} \approx 10^{-5} \text{ m}^2 \text{ s}^{-1}. \end{aligned} \quad (5.113)$$

These velocity and flux scales are quite reasonable.

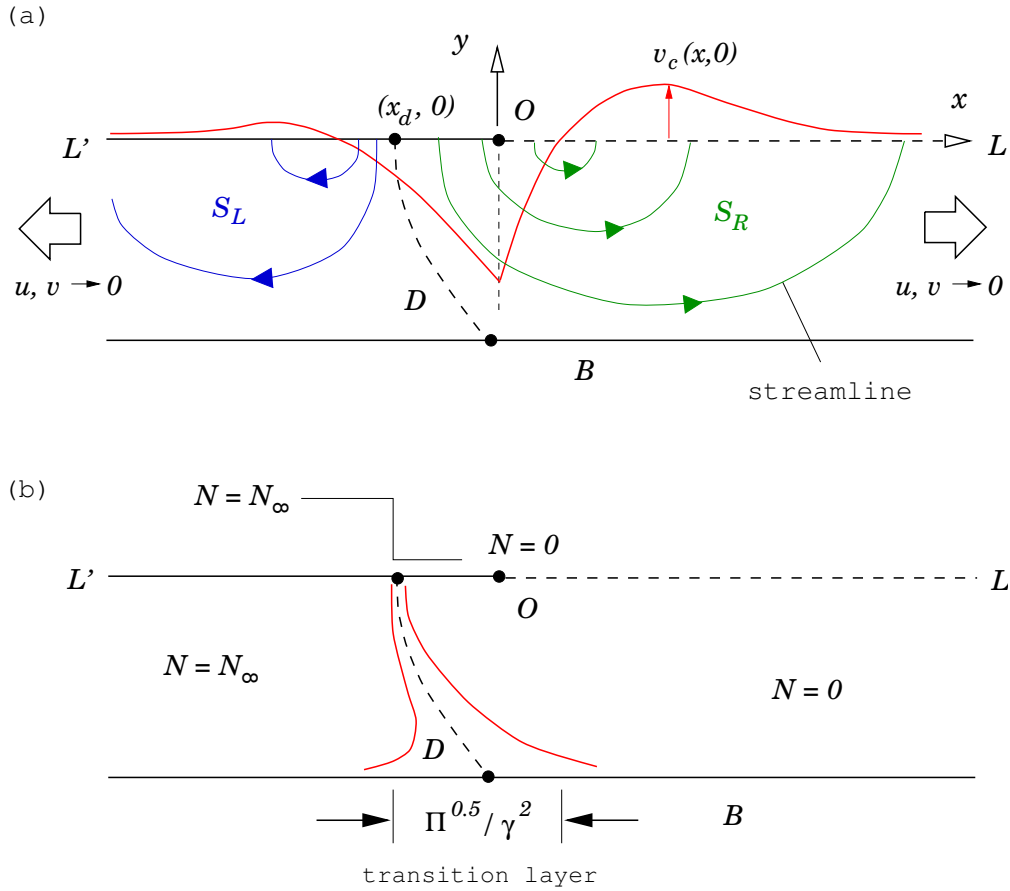


Figure 5.10: (a) Streamlines of the creep flow inferred from the surface velocity $v_c(x,0)$. The inclination of the dividing streamline is hypothetical. (b) Leading order solution of the effective pore pressure distribution. S_L and S_R denote two advection cells, in which N -values are essentially constant. The transition layer smooths out the discontinuity that originates on $y = 0$, and expands by sideways diffusion as one moves along D .

5.6.4 Flow structure and effective pressure

Having derived the till surface velocity, it is now possible to infer the rest of the flow field and the effective pressure distribution in $-1 \leq y \leq 0$. We apply the results of Section 5.3, using the fact that our current problem is a special case of the shallow till model, with Boulton-Hindmarsh exponents $a = 1$, $b = 0$ (constant viscosity), and $\Pi/\gamma^4 \ll 1$ (low permeability).

Given the velocity solution (5.108) (Fig. 5.9), there exists a value $x = x_d$ such that

$$\int_{-\infty}^{x_d} v_b(x) dx = \int_{x_d}^{\infty} v_b(x) dx = 0; \quad (5.114)$$

$x_d \approx -0.57$ from numerical integration. By using mass conservation, it is easy to establish that the flow must consist of two ‘cells’. A reasonable supposition is that the

streamline which emanates from $(x_d, 0)$ (which we label D) meets the lower boundary B . We sketch the likely streamline pattern in Fig. 5.10a, showing the cells S_L and S_R , separated by D , the dividing streamline, the inclination of which is unknown. In this sketch we have assumed that $u(x, 0)$ is small.

In Section 5.3, we found that the (outer) effective pressure solutions should be $N = 0$ below the channel (in $x > 0$) and $N = N_\infty$ away from the channel (in $x < 0$), due to a simple flow advection argument; the transition region is close to the channel margin. It is immediately obvious that the high- N and low- N regions are respectively S_L and S_R . Consequently, the pressure transition must take place in the neighbourhood of D , with a dimensional width scale $\sqrt{\Pi}d/\gamma^2$ deduced before. Since diffusion is important within this transition layer, we would expect the layer to widen out along D . We have sketched the N -distribution in Fig. 5.10b.

5.7 Discussion

In this chapter, the motion of sediment creep closure in the wide channel has been investigated, using a model based on Fowler and Walder's (1993) theory for circular channels. Here we summarize our results and explore some of their implications.

1. Shallow, Boulton-Hindmarsh till

We show that for a shallow deforming sediment layer, where $\gamma = d/l \ll 1$, localized creep flow occurs in the neighbourhood of the channel margins, driven by the normal contact stress gradient at the ice-till interface. This is accompanied by transition of the effective pore-water pressure from its relatively high far field value to the low value in the channel. Therefore, there is generally a complicated coupling between sediment motion and water percolation near the margins.

Water movement in the till depends on the efficiency of percolation, but also on advection, caused by the bulk till motion itself. Accordingly, the N -transition length scale, which we denote by Δ , is found to depend critically on a dimensionless parameter Π/γ^4 — a permeability/deformability ratio. For a high permeability till (where $\Pi/\gamma^4 \gg 1$) we obtain $\Delta \sim d$, independent of till properties, whereas for a low permeability till (where $\Pi/\gamma^4 \ll 1$) Δ is small ($\sim \sqrt{\Pi}d/\gamma^2$). The former situation is one where sediment deformation has little effect on the water pressure. In the latter, Darcy drainage is negligible, so the pore-water is transported in bulk, at essentially the same effective pressure; as a result, we have a sharp transition by the channel margins where N has to be brought down to zero.

In both of these cases, the central part of the channel (bed) is found to be stationary and passive, with vanishing effective pore pressure; there the sediment has very low strength and is prone to erosion by the water flow. Thus, our shallow till results are applicable also to the 'disjoint' situation discussed by Alley (1992) (Fig. 5.2d). (However, Alley neglected water percolation in his model.) At the margins, we suppose that the bed would adopt a certain bank profile, governed by the processes of sediment erosion and deposition. This is investigated in Chapter 6.

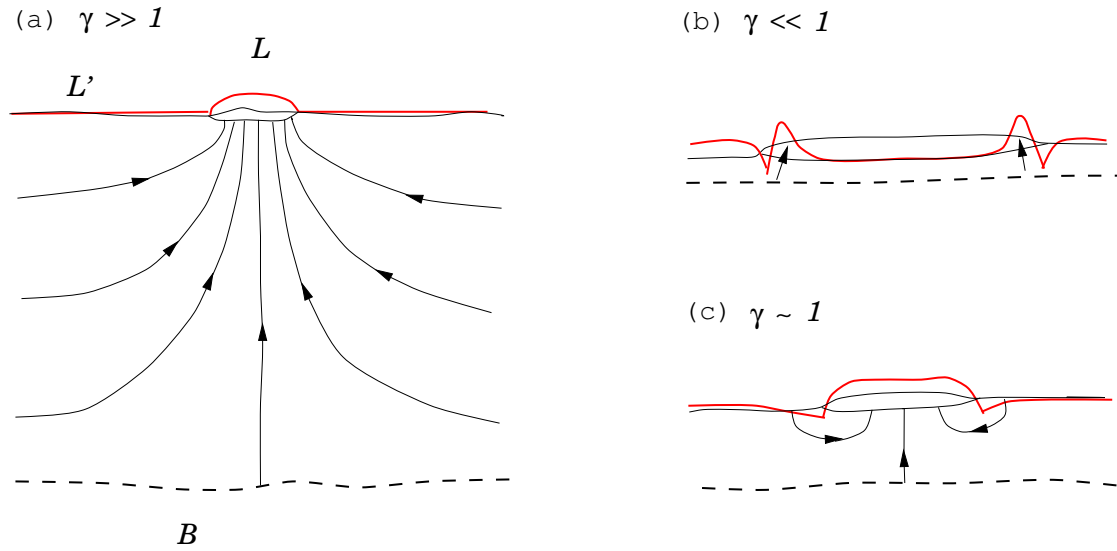


Figure 5.11: Schematic diagram showing (i) the closure velocity profile (red line, on L), (ii) the inferred creep flow structure (arrowed lines), and (iii) the deformation of the ice-till interface (red line, on L'), for (a) a deep till, (b) a shallow till, and (c) the intermediate case.

2. Linear, low permeability till

In Sections 5.4 to 5.6, our calculations have focussed on the special case of a linear (constant viscosity), low permeability ($\Pi/\gamma^4 \ll 1$) till, for which the creep problem admits analytic solutions.

(a) Effect of geometry

For a shallow till ($\gamma \ll 1$), the creep localization behaviour reported above has partly been confirmed by solving a Stokes flow problem. The velocity solution at the till surface indicates that creep flow is accompanied by subsidence of the ice-till interface just outside the channel. This result is similar to that obtained by Alley (1992), who has assumed also a linear rheology (together with a yield stress). However, his model considers only the far field driving stress as the input. This is equivalent to the step component of our input stress distribution. We show that if γ is small (corresponding to a wide channel), then the singular component is much more important, dominating the step component by a factor of $\gamma^{-1/2}$. In this case, Alley's predictions would underestimate the creep and pinch-out velocities.

We can also solve the Stokes Equations for a deep till ($\gamma \gg 1$), for the case where there is full coupling between the ice and the till. In contrast, the solutions indicate an ellipsoidal closure velocity across the channel, as in the ice closure problem (Chapter 3), and a stationary ice-till interface; the sediment must originate from the far field. A Weertman-type contact stress is derived as part of the solution. We suppose that

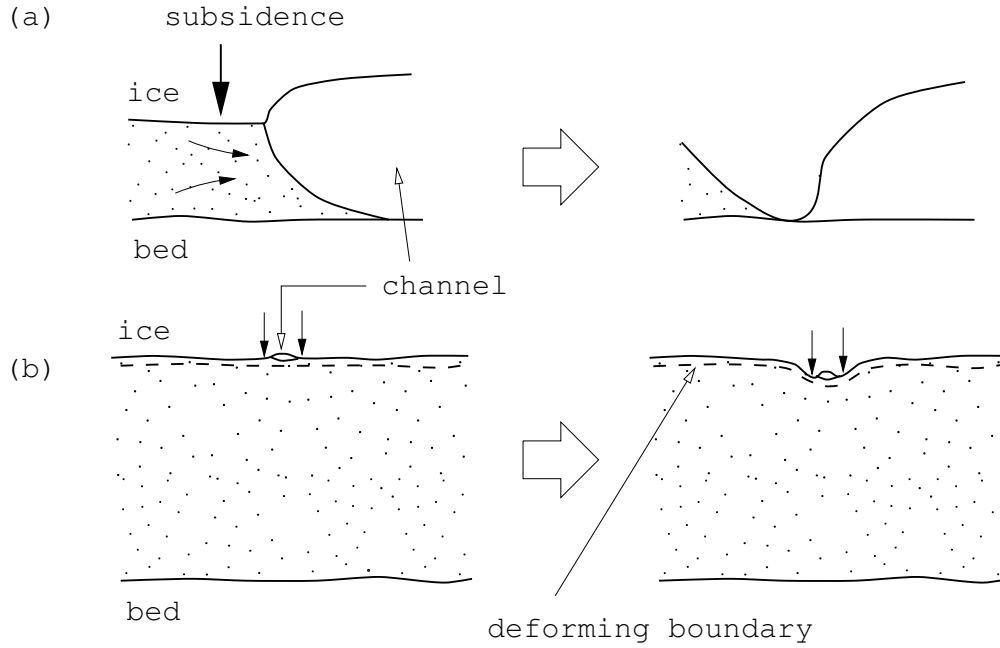


Figure 5.12: (a) The ‘pinch-out’ process envisaged by Alley (1992). (b) Formation of a tunnel valley in a deep sediment deposit, due to subsidence acceleration induced by a shallow deforming layer.

the solution for the $\gamma \sim 1$ case (Fig. 5.2c) is somewhat between the two extremes we have described. Our results are summarized in Fig. 5.11.

(b) Total sediment flux

By integrating the closure velocity across the channel, we obtain

$$\begin{aligned}
 Q_c &\approx \frac{\pi N_c d^2}{4\eta_T \gamma^2} & (\text{Deep: } \gamma \gg 1), \\
 Q_c &\approx \frac{N_c d^2}{5\eta_T \sqrt{\gamma}} & (\text{Shallow: } \gamma \ll 1),
 \end{aligned} \tag{5.115}$$

where Q_c is the total sediment flux due to creep closure (per unit downstream channel distance; i.e., in $\text{m}^2 \text{s}^{-1}$). Hence, given the assumption of *steady flow* and a constant deforming depth d , a large channel (where $\gamma \ll 1$) would be a more efficient sediment collector than a small channel ($\gamma \gg 1$). On the other hand, a large channel can also become isolated due to rapid sediment thinning (see below).

(c) Sediment pinch-out and tunnel valleys

If a shallow deforming till is underlain immediately by bedrock, then subsidence of the ice-till interface will (eventually) lead to pinch-out at the margins. This can lead to channel isolation, as has been suggested by Alley (1992); see Fig. 5.12a. Although

the problem geometry is altered as this process takes place, a nominal pinch-out timescale from Section 5.6 is $4\eta_T\sqrt{\gamma}/N_c$. By taking $\eta_T = 10^{10}$ Pa s, $N_c = 1$ bar as before, this is approximately $5\sqrt{\gamma}$ days. Since $\gamma \ll 1$, pinch-out is predicted to occur rapidly adjacent to wide channels.

Sediment thinning can have a different consequence in thick sediment deposits. Boulton and Hindmarsh (Fig. 14b; 1987) proposed that a subsiding ice-till interface could cause the channel to migrate downwards, leading to the formation of a *tunnel valley*. At first sight, our deep till solutions seem to imply a zero thinning rate. However, this result applies strictly when $d/l \rightarrow \infty$. In reality d may be large but must be finite; therefore, in practice there is a small but non-zero subsidence velocity over a large distance from the channel, as can be deduced from mass conservation. This process would be responsible for tunnel valley formation.

Here we suggest a mechanism whereby this process may be accelerated. Even if the till is deep, the actual *deforming* till thickness may be small in comparison to the channel width, and then the subsidence velocity would be high. Theoretically, we require the interface between the A- and B-horizons to migrate downwards at the same rate, so the viability of this mechanism depends on how the effective pore-water pressure distribution near the channel is coupled to the yield surface position; see Fig. 5.12b. This constitutes a free-boundary problem that is not investigated in the current work.

3. Non-linear Extension

One of our main objectives is to derive the sediment creep closure velocity, which would be important in describing channel depth evolution (Chapter 7). To avoid numerical solution of the full non-linear problem, here we propose an extension of our results for a Boulton-Hindmarsh till rheology. This approach is essentially similar to the one used in Chapter 3.

For a low permeability till, our linear (deep/shallow) solutions indicate that the typical strain rate is given by

$$\dot{\epsilon} = N_c/\eta_T, \quad (5.116)$$

where effectively, N_c may be interpreted as the driving stress of the deformation τ . This identification is reasonable since N_c enters our problem as the scale for σ_n ; see Equation (5.52). In addition, the effective pore-pressure distribution consists of a transition between two (outer) regions, with N_∞ as the pressure scale. Thus clearly, N_∞ would provide the relevant viscosity-controlling parameter if the till is non-linear. Currently, we have

$$\eta_T = (A_T\tau^{a-1}N^{-b})^{-1} \quad (5.117)$$

from Equation (5.3), where $a = 1$, $b = 0$. To reactivate a non-linear rheology, we simply relax these restrictions by using Boulton and Hindmarsh's (1987) exponents, and we let $\tau = N_c$, and $N = N_\infty$. It follows that

$$\frac{N_c}{\eta_T} \longrightarrow \frac{A_T N_c^a}{N_\infty^b}, \quad \text{where } a = 1.33, \ b = 1.8. \quad (5.118)$$

The velocity scale for our shallow till solution would become $A_T N_c^a d / N_\infty^b$. This has the same form as Fowler and Walder's radial closure velocity — $\propto R_0 (p_i - p_c)^a (p_i - p_\infty)^{-b}$, where R_0 is the channel radius — derived as a low permeability asymptotic solution of the non-linear model (see Equation (3.48), 1993).

The extension given in (5.118) will be used directly in the model of Chapter 7, although here we must emphasize the heuristic nature of its derivation. Obviously, the actual closure velocity (for either the deep/shallow till situation) will have a different functional form from our linear solution, but we can assume that they are qualitatively similar. We do not have an expression for high permeability tills.

Finally, we point out that the explicit shallow till solution in Section 5.6 (and the inferences that followed) has relied on a Weertman-type stress distribution. We are effectively assuming that the ice motion can cope with the deforming ice-till interface. The validity of this will depend on the relative stiffness of the ice and the sediment. We propose that if η_i / η_T is small, then our current results are accurate (for a linear till). This is unlikely unless the effective pore-pressure in the till is high. On the other hand if η_i / η_T is large, then the actual contact stress is probably less singular than the one assumed here. Therefore, our $\gamma \ll 1$ solution sets an upper bound to the sediment creep velocity. (This point becomes relevant in Chapter 7.) The linear hybrid problem, where ice occupying the upper-half plane is coupled to sediment in an infinite strip, awaits further investigation.

Chapter 6

Sediment erosion and deposition

6.1 Introduction

Under certain conditions, the action of water flow over a non-cohesive bed material results in the motion of sediment particles. Differential transport of the particles can give rise to a bed topography that changes with time. An everyday example of this is found in ‘sub-aerial’ river channels, for which the shape of the container (the river planform and cross section) evolves continuously in response to variations of the water flux that it is carrying. In this respect, there is little difference between a river channel and the wide channel that is being considered.

Sediment transport can take place in several modes, depending on the magnitude of the shear stress exerted by the flow on the bed, and on the characteristics of bed material. The stress has to exceed a certain value — the critical Shields stress — in order to initiate particle motion. Once this is surpassed, particles can remain in close proximity to the bed, sliding, rolling and saltating over it; this is known as *bedload*. If the imposed stress is high, then particles will be lifted into the flow as *suspended load*. In general, this consists of the lighter grains when a range of particle sizes is present. (In this case, the lightest particles may even lose contact with the bed altogether. This constitutes a *washload* which contributes to only a small proportion of the total quantity of sediment in transport.)

The processes of bedload and suspension are dynamical: different particles are set into motion, are lifted up, and fall back to the bed at the same time. The collective near-bed motion is best described in terms of *erosion* and *deposition*, and the net effect of these is displacement of the bed boundary. One of the most fundamental modelling problems is in determining the *equilibrium* cross sectional shape of river channels which are straight and self-formed. In this case, the problem is in identifying how a zero *net* rate of erosion can be achieved at the bed perimeter.

Einstein (1972) first noted that suspended load is continuously being deposited on the banks of active river channels. If a river is neither widening or narrowing, then its bank material has to be scoured at the same rate. The relevant mechanisms were provided and formulated into a mathematical model by Parker (1978). He

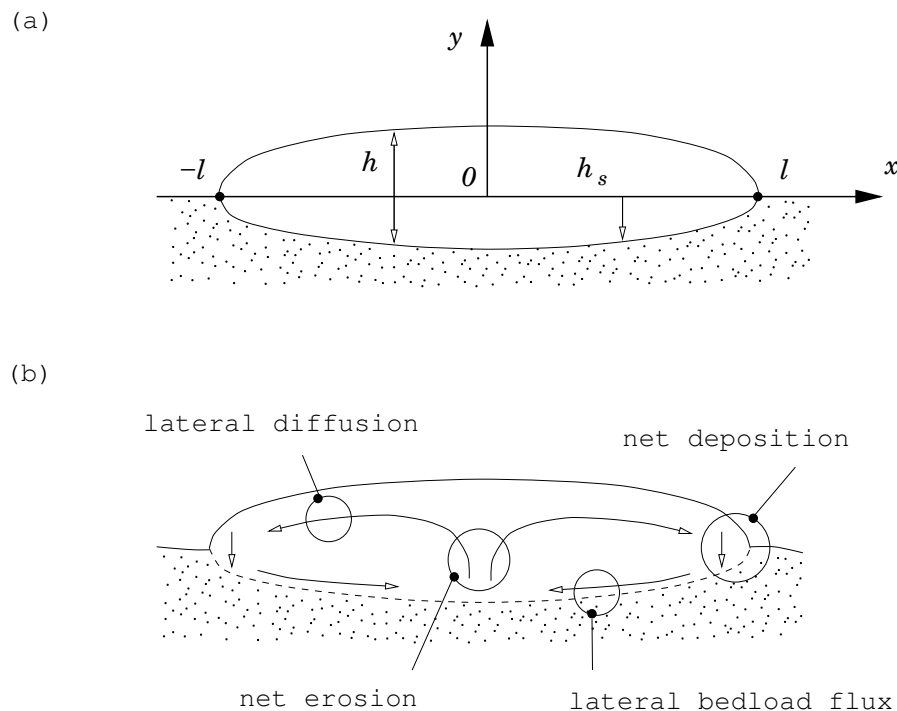


Figure 6.1: Schematic diagram showing the sediment-floored wide subglacial channel: (a) definition of mathematical symbols; (b) the sediment circulation required to maintain the bed in equilibrium.

proposed that bank scour is due to bedload particles being pulled downslope towards the channel centre by gravity. To compensate this effect, there must be an equal and opposite sediment flux towards the banks, and that is explained by lateral (turbulent) diffusion of suspended sediment in the flow.

Clearly, a similar circulation of sediment is operative also in a wide subglacial channel, but the corresponding model would have to take into account the other processes described in Chapters 3 to 5. This problem is tackled in Chapter 7. In the current chapter, we set up the necessary groundwork for that by first studying bed erosion and deposition alone; hence, our focus is essentially on river channels.

Parker's model (1978) assumes *non-cohesive* sediment particles with a uniform grain size. Typical subglacial till has negligible cohesion (Section 5.2.1), but is characterized by bimodal or polymodal grain size distributions (e.g. Drewry, 1986; Menzies, 1995). This complicating factor is removed here by the use of a representative grain diameter in the model. Note that this approximation introduces no conceptual difficulty. In a single component model, distinction between the two transport modes becomes one of proximity to the bed: bedload particles collide frequently with the bed, whereas suspended particles do not. Parker has also neglected secondary currents in the water flow. This is consistent with the assumptions made in Chapter 4.

In Section 6.2, we begin by describing our sediment bed model, following closely the semi-empirical formulation of Parker (1978). The equations are applied to the particular case of an equilibrium river channel in Section 6.3, where also Parker's solutions are examined. Although this investigation is not strictly part of the final drainage theory, it will help us establish the fundamental properties of wide channels in Chapter 7.

Back in Section 4.6, we put forward a notional equilibrium drainage theory for hard beds, but there we lack a recipe for the channel width (and its time evolution). The same problem does not appear in the river model, because the total water flux can be taken as a prescribed quantity. As we shall see however, this constraint cannot be applied to the subglacial channel. In order to overcome this difficulty, we argue that an additional condition, imposed by the physics of sediment motion, ought to be satisfied at the channel margins. We derive this condition in Section 6.4, using a sophisticated model introduced by Kovacs and Parker (1994). A short discussion is given in Section 6.5.

6.2 Mobile bed formulation

Let us consider the wide subglacial channel as before, in the (essentially) 2-D description; but here we define $h(x, t)$ to be the *total* water depth (see Fig. 6.1). t denotes time. We are interested in determining the incision depth of the channel in the sediment bed, $h_s(x, t)$, taken positive downwards. The bed is assumed to consist of non-cohesive sediment particles with (a representative) diameter D_s and density ρ_s .

Given a water flow in the channel, the bed surface is subjected to a shear stress τ_b , given approximately by

$$\tau_b \approx \frac{1}{2} \Phi h. \quad (6.1)$$

(The pre-factor $1/2$ appears in the case of the subglacial channel, but is omitted for a river; see Section 2.5.1). This stress has an associate *shear velocity*

$$\tilde{u} = (\tau_b / \rho_w)^{1/2} \quad (6.2)$$

describing the corresponding level of turbulence. $\tilde{u}$ measures the ability of the flow to support suspended sediment transport.

As assumed in previous chapters, the channel is symmetrical and becomes shallower as one moves from the centre towards the banks, i.e., h decreases from its maximum value at $x = 0$ to zero at $x = \pm l$. Since Equations (6.1) and (6.2) imply that $\tilde{u} \propto h^{1/2}$, one can expect a similar reduction of suspended sediment load in the same directions. The lateral concentration gradients that result will drive a net *diffusion* of suspended sediment to the banks, thus *overloading* them. The excess sediment must be deposited, as there the water flow is unable to maintain it in suspension, and this causes a decrease in the depth h_s . Likewise, export of sediment from the centre leaves

it *underloaded*, causing an increase in h_s . Equilibrium can be achieved if there is a return of sediment to the channel centre, owing to the component of bedload directed downslope under the effect of gravity. This sediment cycle is illustrated schematically in Fig. 6.1b. In the following, we consider the ingredient processes required of this description, and then build up the general mathematical model.

6.2.1 Bedload transport

Downstream flux

If $\tau_b > \tau_c$, where τ_c is the critical Shields stress (this is small; see Section 6.4.1), then particle motion is initiated, and a standard equation to predict bedload transport is

$$\frac{q_b}{(RgD_s)^{1/2}D_s} = 8 \left(\frac{\tau_b - \tau_c}{\rho_w RgD_s} \right)^{3/2} \quad (6.3)$$

(e.g. Richards, 1982), where q_b is the bedload transport rate, expressed as a volume flux per unit width of the bed (with units $\text{m}^2 \text{s}^{-1}$), and R is the density ratio $\rho_s/\rho_w - 1$. This has the Meyer-Peter and Müller (1948) form, but is applicable only to flat horizontal beds. We assume that it is valid also for sediment beds that have small slope and curvature.

Under certain flow conditions, flat beds are known to be unstable to the formation of *dunes*. Parker (1978) has extended (6.3) for the case of a dune covered bed, by using the experimentally derived conversion law

$$(\tau_b - \tau_c) \rightarrow \frac{0.4}{\rho_w g R D_s} \tau_b^2. \quad (6.4)$$

In this case, the resulting bedload equation has a τ_b^3 dependence.

Lateral flux

We require a similar expression for the *lateral* flux q_{bl} . Clearly, q_{bl} must vanish when the lateral bed slope is zero, since we are neglecting secondary flow currents. Parker (1978) asserted that it is proportional to both q_b and the lateral slope, hence he proposed

$$q_{bl} = \frac{q_b}{\mu_c} \frac{\partial h_s}{\partial x}, \quad (6.5)$$

where μ_c is a dynamic friction factor. This is probably the simplest realistic relation that may be used.

6.2.2 Suspended load transport

Diffusion

Let $c(x, y, t)$ denote the volumetric suspended sediment concentration within the cross section, and v_s be the particle *settling velocity* (i.e., the mean terminal velocity of sediment particles in water, falling under gravity alone). Neglecting secondary

flow, the sediment (volume) fluxes in the lateral (x) and normal (y) directions are respectively

$$-\epsilon_x \frac{\partial c}{\partial x}, \quad -\left(\epsilon_y \frac{\partial c}{\partial y} + v_s c\right), \quad (6.6)$$

where ϵ_x and ϵ_y (functions of both x and y , and also t) are the turbulent eddy diffusivities in the corresponding directions.

The general specification of ϵ_x and ϵ_y is a difficult problem, but for a wide sub-glacial channel, it is justifiable to set them both equal to the local depth-averaged value determined in Chapter 4 (see Equations (4.65) and (4.30)₂); therefore, we put

$$\epsilon_x = \epsilon_y = \epsilon = \sqrt{\frac{\Phi f^*}{8\rho_w}} h^{3/2}. \quad (6.7)$$

With f^* replaced by f , this approximation is also applicable to the river case. Parker (1978) employed slightly different but *constant* values for ϵ_x and ϵ_y :

$$\epsilon_x \sim \epsilon_y \approx 0.1 \tilde{u} h, \quad (6.8)$$

where $\tilde{u}$ and h are evaluated at $x = 0$. Since that $f \sim 0.01$ to 0.1 , we have $\sqrt{f/8} \sim 0.1$. By substituting for $\tilde{u}$ from (6.1) and (6.2), it is easy to show that Parker's values are close to the average predicted by (6.7).

Erosion and deposition

Conservation of suspended sediment requires that

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left(\epsilon \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left(\epsilon \frac{\partial c}{\partial y} + v_s c \right). \quad (6.9)$$

The top surface boundary condition here is that of vanishing flux in the y -direction:

$$-\left(\epsilon \frac{\partial c}{\partial y} + v_s c\right) \Big|_{y=h-h_s} = 0; \quad (6.10)$$

and let us assume a bottom surface condition of the form

$$-\epsilon \frac{\partial c}{\partial y} \Big|_{y=-h_s} = v_s E. \quad (6.11)$$

In this last equation, $v_s E$ is effectively the *erosion rate* (into suspension), where E is a dimensionless parameter. Following Parker (1978), let us prescribe the relation

$$E = 0.092 \left(\frac{\tau_b - \tau_c}{\rho_w g R D_s} \right)^{3/2}, \quad (6.12)$$

found to be approximately valid for $D_s \sim 0.1$ mm. This equation seems reasonable as it predicts an erosion rate that is proportional (locally) to the stream power measure $\tau_b \tilde{u}$ (remember that τ_c is small). We assume that it is applicable also to a moderately

sloping bed. If we use the conversion law (6.4) for dune beds, then (6.12) gives $E \propto \tau_b^3$; Parker (1978) used this particular form in his model.

Likewise, one can define a *rate of deposition* (from suspension), equal to $v_s D$, where D is dimensionless. Since the deposition rate is simply given by the product $v_s c|_{y=-h_s}$, we have

$$D = c|_{y=-h_s}. \quad (6.13)$$

The derivation of $c|_{y=-h_s}$ will be given shortly.

Depth integrated form

Our formulation is simplified by integrating (6.9) in the y -direction, with the use of boundary conditions (6.10) and (6.11). This allows us to use a *depth integrated* concentration, defined by

$$\zeta = \int_{-h_s}^{h-h_s} c \, dy, \quad (6.14)$$

as our suspended sediment variable. (More precisely, ζ is the *suspended sediment content*, that is, the total volume of sediment per unit bed area suspended in the vertical water column.) If the bed is moderately curving, the approximate integration result is

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial q_{sl}}{\partial x} + v_s(E - D), \quad (6.15)$$

where q_{sl} , the depth integrated lateral suspended flux, is

$$q_{sl} = - \int_{-h_s}^{h-h_s} \epsilon \frac{\partial c}{\partial x} \, dy \approx -\epsilon \frac{\partial \zeta}{\partial x}. \quad (6.16)$$

In (6.15), $E > D$ describes an underloaded water column, whereas $E < D$ describes an overloaded one.

Finally, we seek an expression for D in terms of ζ , to be used in Equation (6.15). In order to do this, we first solve (approximately, using the depth-averaged value of ϵ_y) the steady state problem in the case of a flat bed, then apply the results to a gently sloping/curving bed. By putting $\partial/\partial t = \partial/\partial x = 0$, Equations (6.9) and (6.10) reduce to

$$\epsilon \frac{dc}{dy} + v_s c = 0. \quad (6.17)$$

If we pose the boundary condition $c|_{y=-h_s} = c_b$, then the solution is

$$c(y) = c_b \exp \left[-\frac{v_s}{\epsilon} (y + h_s) \right], \quad (6.18)$$

and performing the integral in (6.14) leads to

$$\zeta = \frac{c_b \epsilon}{v_s} \left(1 - e^{-\frac{v_s h}{\epsilon}} \right). \quad (6.19)$$

Given that the typical values we shall be using are $v_s \sim 10^{-1} \text{ m s}^{-1}$, $h \sim 1 \text{ m}$, $f^* \lesssim 0.1$, and $\Phi \lesssim 100 \text{ kg m}^{-2} \text{ s}^{-2}$, it is found that $e^{-v_s h/\epsilon} \lesssim 0.1$. Hence, Equation (6.19) reduces approximately to

$$\zeta \approx \frac{c_b \epsilon}{v_s} \quad (6.20)$$

(this is valid also for rivers; see Parker, 1978), and this gives us

$$D = \frac{v_s}{\epsilon} \zeta. \quad (6.21)$$

Equation (6.21) encapsulates the expected physical behaviour that the deposition rate should depend directly on both the sediment fall velocity and the (local) suspended sediment content, but inversely on turbulence intensity (which promotes upward diffusion of sediment).

6.2.3 Kinematic condition

Formally, the bed surface $y = -h_s$ is defined to be at the bottom of the bedload layer. Given the lateral bedload flux q_{bl} , and the erosion and deposition rates $v_s E$ and $v_s D$, the correct equation describing bed surface evolution is

$$\frac{\partial h_s}{\partial t} = \frac{1}{1 - n_s} \left[\frac{\partial q_{bl}}{\partial x} + v_s (E - D) \right], \quad (6.22)$$

in which n_s is the *bed sediment porosity*. As described in Chapter 5, n_s can generally vary *within* the bed. But here we are referring to its value at the bed interface, where the effective pore pressure N is practically zero; therefore, we shall assume that n_s is constant. Typical values are 0.2 to 0.4 (Paterson, 1994).

6.2.4 Non-dimensionalization

Model equations

We have derived all the necessary equations to describe the mobile bed. In (6.22), q_{bl} , E , and D are to be substituted respectively from Equations (6.3) and (6.5), (6.12), and (6.21). The last of these (for D) requires ζ to be determined from Equation (6.15), hence the full problem involves solving a pair of *coupled* equations, for both the bed surface $h_s(x, t)$ and the suspended sediment content $\zeta(x, t)$. (Remember also that l is a function of t .) To summarize, our equations are

$$\begin{aligned} \frac{\partial h_s}{\partial t} &= \frac{1}{1 - n_s} \left[\frac{\partial q_{bl}}{\partial x} + v_s (E - D) \right], \\ \frac{\partial \zeta}{\partial t} &= -\frac{\partial q_{sl}}{\partial x} + v_s (E - D), \quad (\zeta \geq 0); \end{aligned} \quad (6.23)$$

where

$$q_{bl} = \frac{8(\tau_b - \tau_c)_{0+}^{3/2}}{\mu_c R \rho_w^{3/2} g} \frac{\partial h_s}{\partial x}, \quad q_{sl} = -\epsilon \frac{\partial \zeta}{\partial x},$$

$$E = 0.092 \left(\frac{\tau_b - \tau_c}{\rho_w g R D_s} \right)_{0+}^{3/2}, \quad D = \frac{v_s}{\epsilon} \zeta, \quad (6.24)$$

and

$$\tau_b = \frac{\Phi h}{2}, \quad \epsilon = \sqrt{\frac{\Phi f^*}{8 \rho_w}} h^{3/2}, \quad \tau_c \approx 0. \quad (6.25)$$

The total hydraulic gradient Φ is specified. The smallness of τ_c is justified later in Section 6.4.1. The subscript notation which we introduce here, defined by $(x)_{0+} = \max(x, 0)$, ensures that $q_{bl} = E = 0$ when $\tau_b < \tau_c$. In addition, the suspended sediment concentration cannot possibly be negative, therefore $\zeta \geq 0$.

The equations above define the basic model for investigating the bed incision of subglacial channels (Chapter 7); a modified version will also be applied to rivers (Section 6.3). Their validity lies in the assumption that the bed (solution) has relatively low slope and curvature. For subsequent work, let us specify the following sediment properties:

$$D_s = 5 \times 10^{-4} \text{ m}, \quad \rho_s = 2650 \text{ kg m}^{-3}, \quad v_s = 5 \times 10^{-2} \text{ m s}^{-1}, \\ \mu_c = 0.8, \quad R = 1.65, \quad n_s = 0.3. \quad (6.26)$$

In the subglacial channel model, we shall use the exact form of the equations as shown. The water depth $h(x, t)$ is given by the depth evolution equation derived in Chapters 3 and 4. This additional coupling is expected to complicate the problem.

For the river, we replace f^* by f and the pre-factor in $(6.25)_1$ by unity. A simplification is available, due to the condition that

$$h \equiv h_s \quad (\geq 0). \quad (6.27)$$

Also, the expressions for q_{bl} and E are modified for a *dune covered bed*, following Parker (1978). Summing up, the necessary replacements in the river model are

$$\tau_b = \Phi h, \quad q_{bl} = \frac{2.02 \tau_b^3}{\mu_c \rho_w^3 (gR)^{5/2} D_s^{3/2}} \frac{\partial h_s}{\partial x}, \quad E = 0.0233 \left(\frac{\tau_b}{\rho_w g R D_s} \right)^3. \quad (6.28)$$

Boundary conditions

Given the flux definitions in (6.24) and (6.28), the pair in (6.23) are second order non-linear diffusion equations. Their solution requires the prescription of initial conditions for each of h_s , ζ and also l . At steady flow these are discarded, and a *constraint* for the channel width l is imposed instead. We describe this in Section 6.3 and Chapter 7.

Additionally, two (symmetric) pairs of boundary conditions are required. One can simply use the fact that the channel depth, by definition, must vanish at its margins. Thus, the first two conditions are just

$$h_s = 0, \quad \text{at } x = \pm l. \quad (6.29)$$

As for the remaining pair, the integral definition of ζ in (6.14) implies that

$$\zeta = 0, \quad \text{at } x = \pm l, \quad (6.30)$$

due to the vanishing depth.

There is, however, an alternative to using (6.30): that is, the lateral suspended flux must vanish at the ends, giving

$$q_{sl} = -\epsilon \frac{\partial \zeta}{\partial x} = 0, \quad \text{at } x = \pm l. \quad (6.31)$$

This is physically realistic, because while suspended sediment can be deposited downward onto the bed, it cannot diffuse *into* the margins. Parker (1978) has in fact chosen to apply conditions (6.29) and (6.31) and not (6.30) in solving his river equations. In Section 6.3, we shall follow his example, but then return to discuss this issue at the end of that section.

Dimensionless model: subglacial channels

Let us define the following variable scales:

$$\begin{aligned} [h] &= [h_s] = h_0, & [\zeta] &= \zeta_0, & [x] &= [l] = l_0, \\ [t] &= t_s & (\text{a sediment timescale}), \\ [\tau_b] &= [\tau_c] = \tau_0, & [\epsilon] &= \epsilon_0, & [\Phi] &= \Phi_0. \end{aligned} \quad (6.32)$$

If we impose the scale relations

$$\tau_0 = \Phi_0 h_0, \quad \epsilon_0 = \sqrt{\frac{\Phi_0 f^*}{8\rho_w}} h_0^{3/2} \quad (\text{see Equation (4.30)}_2), \quad (6.33)$$

$$\frac{h_0}{t_s} = \frac{8h_0}{\mu_c R g l_0^2} \left(\frac{\tau_0}{\rho_w} \right)^{3/2}, \quad 0.092 \left(\frac{\tau_0}{\rho_w g R D_s} \right)^{3/2} = \frac{v_s \zeta_0}{\epsilon_0}, \quad (6.34)$$

then the corresponding dimensionless form of (6.23) to (6.25) is

$$\begin{aligned} \frac{\partial h_s}{\partial t} &= \frac{1}{1-n_s} \left\{ \frac{\partial}{\partial x} \left[(\tau_b - \tau_c)_{0+}^{3/2} \frac{\partial h_s}{\partial x} \right] + \kappa_s \left[(\tau_b - \tau_c)_{0+}^{3/2} - \frac{\zeta}{\epsilon} \right] \right\}, \\ \beta_s \frac{\partial \zeta}{\partial t} &= \lambda^2 \frac{\partial}{\partial x} \left(\epsilon \frac{\partial \zeta}{\partial x} \right) + (\tau_b - \tau_c)_{0+}^{3/2} - \frac{\zeta}{\epsilon}, \quad (\zeta \geq 0), \end{aligned} \quad (6.35)$$

$$\tau_b = \frac{\Phi h}{2}, \quad \epsilon = \Phi^{1/2} h^{3/2}, \quad \tau_c \approx 0, \quad (6.36)$$

in which the parameters are given by

$$\lambda = \frac{\epsilon_0}{l_0 v_s}, \quad \kappa_s = \frac{v_s^2 \zeta_0 t_s}{\epsilon_0 h_0}, \quad \beta_s = \frac{\epsilon_0}{v_s^2 t_s}. \quad (6.37)$$

The boundary conditions have the same forms as those in (6.29) to (6.31):

$$(i) \ h_s = 0, \quad (ii) \ \zeta = 0, \quad (iii) \ \epsilon \frac{\partial \zeta}{\partial x} = 0, \quad \text{at } x = \pm l. \quad (6.38)$$

Given that each of Φ_0 , h_0 and l_0 is prescribed, the equations in (6.34) produce the characteristic scales

$$t_s = \frac{\mu_c R \rho_w^{3/2} g l_0^2}{8(\Phi_0 h_0)^{3/2}}, \quad \zeta_0 = \frac{0.092 h_0^3 \Phi_0^2}{v_s \rho_w^2 (g R D_s)^{3/2}} \sqrt{\frac{f^*}{8}}; \quad (6.39)$$

and consequently, explicit expressions for the dimensionless parameters are

$$\lambda = \frac{h_0^{3/2}}{l_0 v_s} \sqrt{\frac{\Phi_0 f^*}{8 \rho_w}}, \quad \kappa_s = \frac{0.092 \mu_c v_s l_0^2}{8 h_0 D_s^{3/2} \sqrt{g R}}, \quad \beta_s = \frac{h_0^3 \sqrt{8 f^*}}{\mu_c R g} \left(\frac{\Phi_0}{\rho_w v_s l_0} \right)^2. \quad (6.40)$$

Dimensionless model: dune-bedded rivers

We define the same scales as before, using (6.32). A similar re-scaling procedure, applied to Equations (6.23) to (6.27) (with appropriate replacements shown in (6.28)), leads to the dimensionless model

$$\begin{aligned} \frac{\partial h_s}{\partial t} &= \frac{1}{1 - n_s} \left[\frac{\partial}{\partial x} \left(\tau_b^3 \frac{\partial h_s}{\partial x} \right) + \kappa_s \left(\tau_b^3 - \frac{\zeta}{\epsilon} \right) \right], \quad (h_s \geq 0), \\ \beta_s \frac{\partial \zeta}{\partial t} &= \lambda^2 \frac{\partial}{\partial x} \left(\epsilon \frac{\partial \zeta}{\partial x} \right) + \tau_b^3 - \frac{\zeta}{\epsilon}, \quad (\zeta \geq 0), \\ \tau_b &= \Phi h_s, \quad \epsilon = \Phi^{1/2} h_s^{3/2}, \end{aligned} \quad (6.41)$$

where

$$\begin{aligned} \lambda &= \frac{h_0^{3/2}}{l_0 v_s} \sqrt{\frac{\Phi_0 f}{8 \rho_w}}, \quad \kappa_s = \frac{0.0233 \mu_c v_s l_0^2}{2.02 D_s^{3/2} h_0 \sqrt{g R}}, \\ \beta_s &= \frac{2.02 h_0^{9/2}}{\mu_c (g R)^{5/2} D_s^{3/2} (v_s l_0)^2} \left(\frac{\Phi_0}{\rho_w} \right)^{7/2} \sqrt{\frac{f}{8}}. \end{aligned} \quad (6.42)$$

The boundary conditions are the same as before, and the characteristic scales assumed here include the two in (6.33), together with

$$t_s = \frac{\mu_c \rho_w^3 (g R)^{5/2} D_s^{3/2} l_0^2}{2.02 (\Phi_0 h_0)^3}, \quad \zeta_0 = \frac{0.0233 h_0^{9/2}}{v_s (g R D_s)^3} \left(\frac{\Phi_0}{\rho_w} \right)^{7/2} \sqrt{\frac{f}{8}}. \quad (6.43)$$

Water flux

We shall be interested also in the total water flux Q . From Equation (4.100), an approximate expression is $Q = (8\Phi/\rho_w f^*)^{1/2} \int_{-l}^l h^{3/2} dx$. (For rivers, f^* and h are replaced respectively by f and h_s .) If we define

$$[Q] = Q_0 = \sqrt{\frac{8\Phi_0}{\rho_w f^*}} l_0 h_0^{3/2} \quad (6.44)$$

following (2.60)₄, then the corresponding dimensionless form is

$$Q = \Phi^{1/2} \int_{-l}^l h^{3/2} dx. \quad (6.45)$$

6.3 Equilibrium river channels

We examine Parker's (1978) river model in this section. Taking the dimensionless equations in (6.41), we put $\partial/\partial t = 0$, $\Phi = 1$; we also replace (6.41)₄ by

$$\epsilon = 1, \quad (6.46)$$

which corresponds essentially to Parker's approximation in Equation (6.8). After some rearrangement, the model equations become

$$\begin{aligned} \frac{d}{dx} \left(h_s^3 \frac{dh_s}{dx} - \kappa_s \lambda^2 \frac{d\zeta}{dx} \right) &= 0, \\ \lambda^2 \frac{d^2 \zeta}{dx^2} &= \zeta - h_s^3, \end{aligned} \quad (6.47)$$

in which $h_s \geq 0$, $\zeta \geq 0$. Parker used boundary conditions (i) and (iii) of (6.38), thus

$$h_s = 0, \quad \frac{d\zeta}{dx} = 0, \quad \text{at } x = \pm l \quad (6.48)$$

(cf. Equations (24) to (27); Parker, 1978).

Up to now, the only prescribed scale in the problem is

$$\Phi_0 = \rho_w g \sin \alpha, \quad (6.49)$$

where α is the downstream inclination of the river. To proceed, let us define $l = 1$ (dimensionless), thereby assigning the width scale l_0 to be the actual channel half-width. As such, l_0 will have to be determined as part of the solution to the problem. In order to determine also an appropriate depth scale h_0 , we put

$$\kappa_s \lambda^2 \equiv 1; \quad (6.50)$$

then, the first two equations in (6.42) lead to

$$h_0 = 26.3 (gR)^{1/4} D_s^{3/4} \sqrt{\frac{\rho_w v_s}{\mu_c \Phi_0 f}}. \quad (6.51)$$

If we specify the typical (river) values $f = 0.1$ and $\sin \alpha = 0.01$, then the scales calculated from Equations (6.33), (6.43), (6.44), (6.49) and (6.51) are

$$\begin{aligned} \Phi_0 &= 98 \text{ kg m}^{-2} \text{ s}^{-2}, \quad h_0 = 0.445 \text{ m}, \quad \tau_0 = 43.6 \text{ kg m}^{-1} \text{ s}^{-2}, \\ \epsilon_0 &= 1.04 \times 10^{-2} \text{ m}^2 \text{ s}^{-1}, \quad \zeta_0 = 0.762 \text{ m}, \quad Q_0 = 0.831 l_0 \text{ m}^3 \text{ s}^{-1}, \end{aligned} \quad (6.52)$$

and by using (6.42)₁, we also obtain

$$\lambda = \frac{0.208}{l_0}. \quad (6.53)$$

Subsequently, if l_0 turns out to be much greater than h_0 , then the parameter λ will be small.

It is immediately obvious that Equation (6.47)₁ can be integrated. Since $h_s = d\zeta/dx = 0$ at either ends, the constant of integration is zero, provided that the depth solution has the local behaviour

$$h_s(x) \propto (1 \pm x)^\nu \quad \text{at } x = \mp 1, \text{ where } \nu \geq 1/4. \quad (6.54)$$

Anticipating that this is true (and with $\kappa_s \lambda^2 = 1$), our model reduces to

$$\begin{aligned} h_s^3 \frac{dh_s}{dx} &= \frac{d\zeta}{dx}, & \lambda^2 \frac{d^2\zeta}{dx^2} &= \zeta - h_s^3, & (h_s, \zeta \geq 0); \\ \text{b.c.s: } h_s(\pm 1) &= 0, & \frac{d\zeta}{dx} \Big|_{x=1} &= 0. \end{aligned} \quad (6.55)$$

The first equation here describes bed equilibrium as a balance between lateral bed-load flux (l.h.s.) and suspended diffusion flux (r.h.s.). The second equation describes suspended sediment conservation in the flow: sediment gain by the water column due to lateral diffusion (l.h.s.) equals loss by net deposition onto the bed (r.h.s.). Note that one of the four original boundary conditions has been used.

We can perform one more integration if we make a change of variable, by letting

$$G(x) = [h_s(x)]^4, \quad (6.56)$$

where $G(x) \geq 0$ in $|x| \leq 1$. Equation (6.55)₁ then becomes $dG/dx = 4d\zeta/dx$. For convenience, let us define also

$$h_s(0) = H, \quad (6.57)$$

such that

$$G(0) = H^4. \quad (6.58)$$

After some manipulation, we arrive at the equations

$$\begin{aligned} \frac{G}{4} &= \zeta - A, & \lambda^2 \frac{d^2G}{dx^2} &= \zeta - G^{3/4}, \\ \text{b.c.s: } G(\pm 1) &= 0, & G_x(1) &= 0. \end{aligned} \quad (6.59)$$

There are sufficient boundary conditions to determine the constant A and the two integration constants of (6.59)₂. The parameter λ is still an unknown however, so an extra condition is required. This is provided by prescribing the water flux Q through Equations (6.45) and (6.52)₆; i.e., we use the relation

$$\frac{Q \text{ (dimensional)}}{0.831l_0} = \int_{-1}^1 h_s^{3/2} dx \quad (6.60)$$

to derive l_0 once the functional form of h_s has been calculated. The use of this particular constraint is plausible, since the river should only be able to ‘decide’ its (equilibrium) cross section if the discharge is given.

6.3.1 Parker's (1978) solution

If we anticipate that $\lambda^2 \ll 1$, the equations in (6.59) may be solved approximately by using a singular perturbation method. The *outer solutions*, valid in the central region of the channel, are given straightforwardly by the constants

$$G = H^4, \quad \zeta = H^3; \quad (6.61)$$

therefore we find

$$A = H^3 - H^4/4. \quad (6.62)$$

These solutions are correct to all orders in their asymptotic expansions (in λ^2). Essentially, they ensure that the balance between erosion and deposition is satisfied.

The inner variable appropriate for the (right) bank region is

$$X = \frac{1-x}{\lambda}, \quad (X \geq 0). \quad (6.63)$$

Then, (6.59) becomes

$$\begin{aligned} \frac{G_i}{4} = \zeta_i - A, \quad \frac{1}{4} \frac{d^2 G_i}{dX^2} = \zeta_i - G_i^{3/4}, \\ \text{b.c.s: } G_i(0) = 0, \quad \frac{dG_i}{dX}(0) = 0, \end{aligned} \quad (6.64)$$

where subscript i denotes the inner region, and limit matching with the outer solution requires that

$$G_i \rightarrow H^4, \quad \zeta_i \rightarrow H^3, \quad \text{as } X \rightarrow \infty. \quad (6.65)$$

This *inner problem* can be solved directly, determining H as well. We eliminate ζ_i from the pair in (6.64), then multiply both sides of the resulting equation by dG_i/dX . A further integration w.r.t. X produces

$$\frac{1}{8}(G_{i,X})^2 = \frac{1}{8}G_i^2 - \frac{4}{7}G_i^{7/4} + \left(H^3 - \frac{H^4}{4}\right)G_i + B, \quad (6.66)$$

in which B is a constant. Now, the solution of this equation that satisfies the boundary conditions in (6.64) is

$$\int_0^{G_i} \frac{dG_i}{\sqrt{G_i^2 - \frac{32}{7}G_i^{7/4} + 2H^3(4-H)G_i}} = X, \quad (6.67)$$

but we also require $G_i \rightarrow H^4$ as $X \rightarrow \infty$ according to (6.65)₁. The equivalent requirement here is that the integrand blows up (and that the integral converges) as this happens. It follows that

$$H = \frac{24}{7}. \quad (6.68)$$

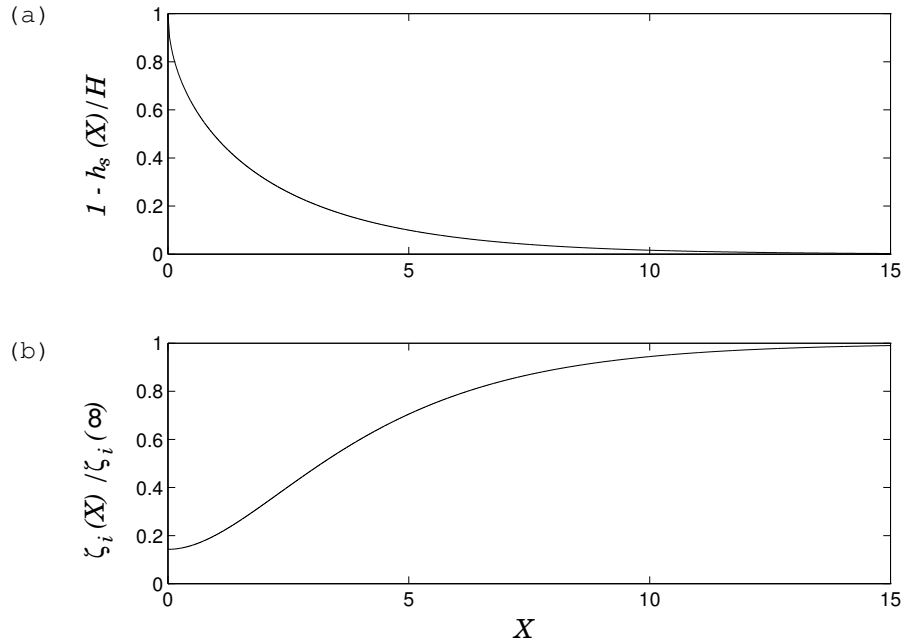


Figure 6.2: *Parker's (1978) boundary layer solutions for a river channel: (a) normalized sediment bed profile, (b) normalized suspended sediment content. X is dimensionless distance measured from the margins.*

It is convenient to express the complete inner solutions, obtained from Equations (6.64)₁, (6.67) and (6.68), in the form

$$\begin{aligned} \zeta_i(X) &= \frac{H^4}{4} \left[\xi(X) + \frac{1}{6} \right], \\ \sqrt{3} \int_0^\xi \frac{d\xi}{\sqrt{\xi + 3\xi^2 - 4\xi^{7/4}}} &= X \quad (\text{in } X \geq 0) \end{aligned} \quad (6.69)$$

(cf. Equations (33) to (35); Parker, 1978), in which

$$\xi(X) = \frac{G_i(X)}{H^4} = \left[\frac{h_s(X)}{H} \right]^4, \quad \text{for } 0 \leq \xi \leq 1. \quad (6.70)$$

Here (6.69)₁ is an equation for the suspended sediment content ζ ($= \zeta_i$). The variable ξ is the fourth power of the normalized channel depth, and has its functional form defined via the integral relation (6.69)₂. In Fig. 6.2, we plot the relevant solutions through the numerical evaluation of this integral. Note that $\xi \propto X^2$ as $X \rightarrow 0$, so $h_s \propto (1 \mp x)^{1/2}$ as $x \rightarrow \pm 1$; this is consistent with the requirement in (6.54). The solutions also predict *non-zero* ζ at the margins (where $\xi = 0$), in discord with the (unused) boundary condition (6.30).

These results — essentially the same as Parker's (1978) — imply that a wide river channel (for which $\lambda^2 \ll 1$) will have a central, flat bed region, where essentially the

lateral fluxes vanish and a net deposition-erosion balance takes place. The maximum depth is $h_s \approx 24h_0/7$ m. With our current model constants, this value is approximately 1.56 m. The bank regions occupy a small fraction ($\approx 5\lambda$) of the total channel width $2l_0$, and have the profile shown in Fig. 6.2a. Equation (6.42)₁ shows that λl_0 is a constant, hence the (dimensional) bank widths are dependent on sediment properties and channel slope, but not on Q .

For this asymptotic solution to be reasonably accurate, one may expect that the banks would have to be far apart so that they do not ‘interact’ with each other (see the next section); i.e. λ is required to be small in (6.53). A maximum value may be 0.1. This corresponds to an l_0 -value of at least 2 m, so the prescribed water flux has to exceed $20 \text{ m}^3 \text{ s}^{-1}$ (estimated from Equation (6.60)). Parker (1978) has discussed the application of this model to field data.

6.3.2 Non-linear oscillator approach

Generally, Q may not be large enough to warrant $\lambda \ll 1$, then the equations in (6.59) require a different treatment. Here we describe one such alternative. Rewritten in a slightly modified form, the model is

$$\begin{aligned} \frac{G}{4} &= \zeta - A, & \frac{\lambda^2}{4} \frac{d^2 G}{dx^2} &= \frac{G}{4} - G^{3/4} + A, & (\zeta, G \geq 0), \\ \text{b.c.s: } G(\pm 1) &= 0, & G_x(1) &= 0, \end{aligned} \quad (6.71)$$

in which λ is defined by (6.53), and A is an unknown constant as before. The constraint due to prescribed Q is given by (6.60).

Since in (6.71)₁ $G(1) = 0$ and $\zeta(1) \geq 0$, we must have $A \geq 0$. Also, remember that $G(x)$ is an even function in $|x| \leq 1$. This symmetry allows us to solve the problem just in $0 \leq x \leq 1$, with the revised boundary conditions

$$G(1) = 0, \quad G_x(0) = 0, \quad G_x(1) = 0. \quad (6.72)$$

Let us concentrate on the second equation (for G). Multiplying by G_x and then integrating w.r.t. x lead to

$$\frac{\lambda^2}{8} (G_x)^2 + V = k, \quad (6.73)$$

where k is a constant, and V is a function of G , given by

$$V(G) = -\frac{G^2}{8} + \frac{4}{7} G^{7/4} - AG \quad (6.74)$$

(cf. Equation (6.66)). The point here is that (6.73) has the standard form of an equation which describes a *non-linear oscillator*¹, and particularly, it is possible to seek the exact solution (in integral form) in this case. This allows us to identify the effect of λ as it decreases and also to verify Parker’s solution.

¹By analogy V is the corresponding potential function of ‘position’ G .

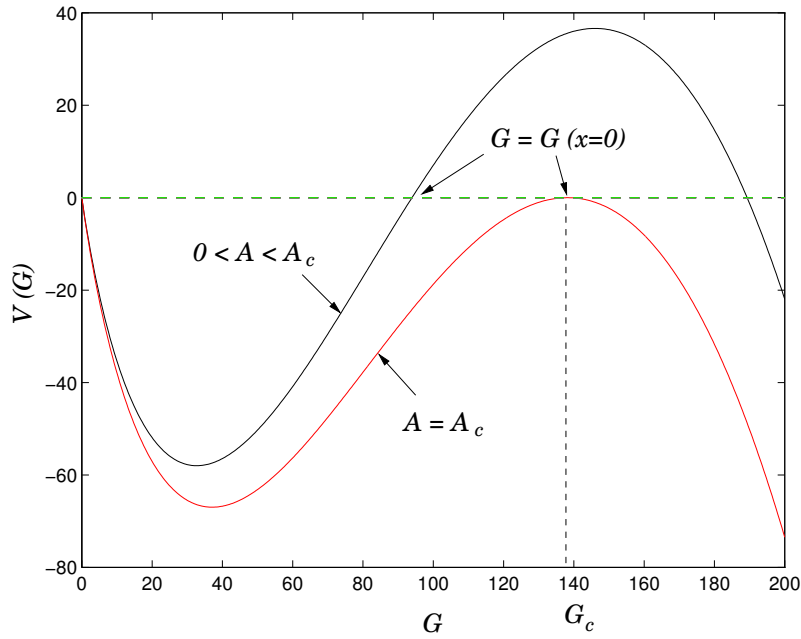


Figure 6.3: The potential function $V(G)$, for $0 < A < A_c$ (black) and $A = A_c$ (red).

Equation (6.74) shows that $V(0) = 0$. By applying the boundary conditions to (6.73) and (6.74), we obtain $k = 0$ and

$$V = 0 \quad \text{when} \quad G = G(x = 0), \quad (6.75)$$

where $V(G) \leq 0$ in the interval $0 \leq G \leq G(x = 0)$. Hence, $G(x = 0)$ is the first positive zero of V ; this is consistent with the fact that $A \geq 0$. We sketch the general form of $V(G)$ in Fig. 6.3.

The solution $G(x)$ for $0 \leq x \leq 1$ may now be expressed in the integral form

$$\frac{2\sqrt{2}}{\lambda}(1-x) = \int_0^G \frac{dG}{\sqrt{-V(G)}} = \int_0^G \frac{dG}{\sqrt{G^2/8 - \frac{4}{7}G^{7/4} + AG}}, \quad (6.76)$$

where $0 \leq G \leq G(x = 0)$. Of course, the final requirement of this solution is that $G = G(x = 0)$ when $x = 0$. If we define the definite integral

$$f(A) = \int_0^{G(0)} \frac{d\eta}{\sqrt{-V(\eta)}} = \int_0^{G(0)} \frac{d\eta}{\sqrt{\eta^2/8 - \frac{4}{7}\eta^{7/4} + A\eta}}, \quad (6.77)$$

then this requirement is

$$f(A) = \frac{2\sqrt{2}}{\lambda}, \quad (6.78)$$

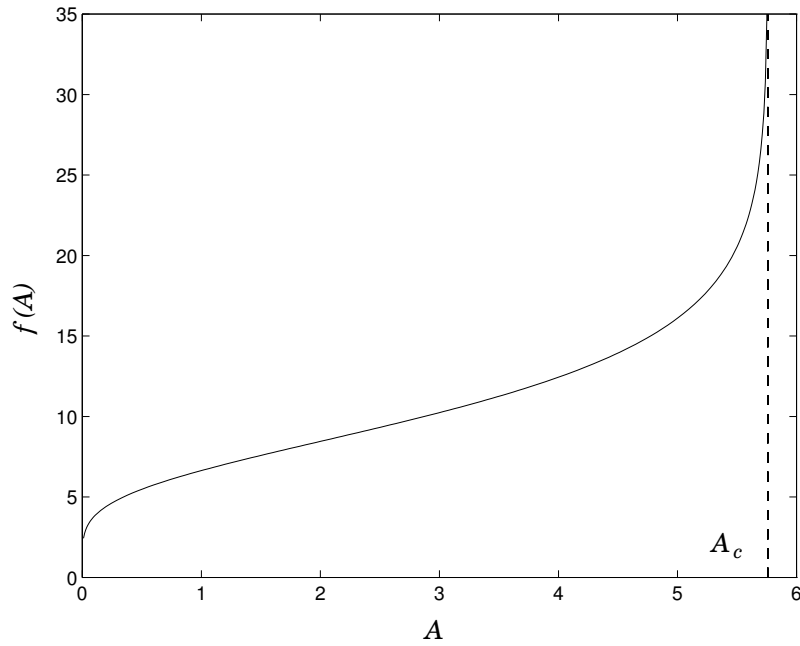


Figure 6.4: The function $f(A)$, derived from numerical integration.

which provides a relation between A and λ . Since $G(0)$ is the first zero of the denominator in the integrand (due to (6.75)), it is not difficult to evaluate $f(A)$ by numerical integration. We plot the computed function in Fig. 6.4. Here A_c is the critical value of A at which the $V(G)$ curve touches the positive G -axis (see Fig. 6.3; red line), and we denote the corresponding G -value by G_c . These critical values, calculated from (6.74), are

$$A_c = \frac{1}{7} \left(\frac{24}{7} \right)^3, \quad G_c = \left(\frac{24}{7} \right)^4. \quad (6.79)$$

If $A > A_c$, then V has no positive zeros and the integrals in (6.76) and (6.77) diverge.

The river problem now consists of Equations (6.76) to (6.78), and the constraint

$$Q \text{ (dimensional, in m}^3 \text{ s}^{-1}\text{)} = \frac{0.173}{\lambda} \int_{-1}^1 [G(x)]^{3/8} dx, \quad (6.80)$$

derived from (6.53), (6.56) and (6.60). Given Q , these equations define the exact inverse problem for λ , A and $G(x)$, which can be solved by an iteration method.

Let us seek a small- λ approximation of this inverse problem. When $\lambda \ll 1$, $f(A)$ is very large, and then Figs. 6.3 and 6.4 show that $A \approx A_c$, $G(0) \approx G_c$. Since $G(0) = H^4$ from Equation (6.58), we obtain $H \approx 24/7$, in agreement with Parker's estimate. We have also computed numerically the normalized depth solutions $1 - h_s/H$ by using Equation (6.76), for a range of λ values; see Fig. 6.5a. To facilitate comparison with Parker's result, the same solutions are plotted in Fig. 6.5b, but with $(1 - x)/\lambda$ ($= X$ in Section 6.3.1) as the horizontal axis. This demonstrates how Parker's boundary layer solution becomes a progressively better approximation as λ decreases.

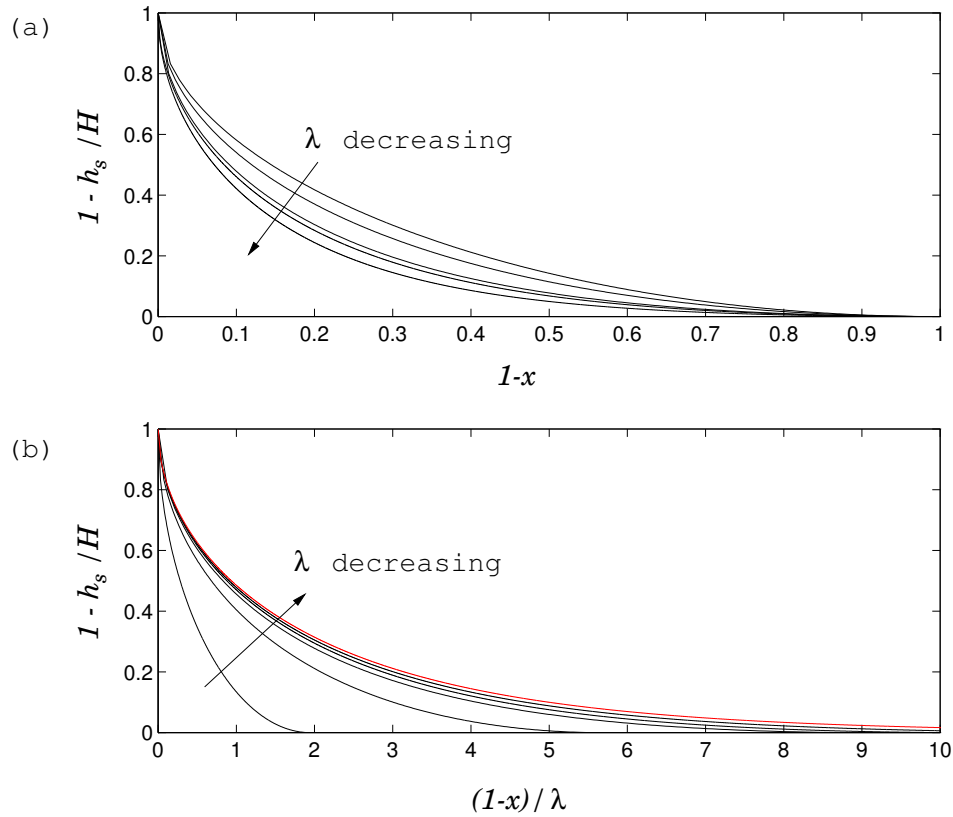


Figure 6.5: (a) Computed normalized river channel cross sections for λ -values 0.51, 0.18, 0.11, 0.10, and 0.08, plotted against $1-x$. (b) The same solutions plotted against $(1-x)/\lambda$. Parker's (1978) boundary layer solution from Fig. 6.2a is shown in red for comparison.

6.3.3 Remarks

Having analysed the equilibrium river model in detail, we observe that the solutions are still not entirely realistic, in that they predict at the margins (i) a non-zero sediment content ζ and (ii) an infinite bed slope dh_s/dx .

In (6.30), the boundary condition $\zeta = 0$ is based on the assumption that at the ends $x = \pm 1$, the suspended sediment concentration c remains finite whilst h vanishes. Although this is reasonable, the *depth-integrated* definition of ζ itself must become grossly inaccurate at the margins — in terms of describing the deposition and lateral diffusion of sediment. Consequently, it is hardly surprising that not all three boundary conditions in (6.29) to (6.31) can be satisfied. This problem is unavoidable unless one tackles the full 2-D formulation. From now on, we therefore neglect anomalous $\zeta(\pm 1)$ values as long as they are quite small. For instance, our preceding solutions indicate that $\zeta(\pm 1) \approx \zeta(0)/7$ for a wide river; this is reasonable.

The second problem is probably more serious, because it strongly violates the low slope assumptions used in formulating the model, and also because physically,

non-cohesive sediment would collapse if it is allowed to rest at an angle greater than the angle of repose². q_{bl} must vanish at $x = \pm 1$ in an equilibrium situation. Given the current form of the lateral bedload flux:

$$q_{bl} \propto \tau_b^3 \frac{dh_s}{dx} \propto h_s^3 \frac{dh_s}{dx} \quad (\text{for a dune-bed}), \quad (6.81)$$

this occurs if h_s satisfies (6.54). However, that does *not* preclude $dh_s/dx \rightarrow \pm\infty$ at the ends, which happens when $1/4 \leq \nu < 1$. Since dominant balance (e.g. in Equation (6.71)₁) implies that $\nu = 1/2$, Parker's model will always predict an infinite bank slope.

By reinstating a bedload description which includes the *critical Shields stress* τ_c (e.g. Equation (6.24)₁), our problem may indeed be overcome. One reason is that τ_c depends on the bed slope:

$$\tau_c = \tau_c(dh_s/dx). \quad (6.82)$$

This would introduce an additional degree of freedom, overlooked in Parker's model. More specifically, although τ_c is small, it is comparable to the flow stress τ_b at the margins, so it will be important in determining the lateral bed flux there. Note that the use of (6.82) necessitates

$$\tau_c \geq 0 \quad (6.83)$$

to be prescribed also. This condition ensures that there is no incipient failure anywhere on the bed surface, and is an essential equilibrium requirement.

6.4 The bank condition

Elucidation of the rôle of critical Shields stress τ_c is motivated by a more immediate reason. In the river model, an improved bedload formulation would lead to a revised bank profile prediction, but it can be expected that the overall channel dimensions would remain much the same as before. This is due to the way the water flux serves as an integral constraint. On the other hand, this constraint seems to be inadequate for the subglacial channel case. In Section 4.6, we have already shown that the steady drainage equation takes the dimensional form

$$Q = Q(N_c, \Phi, l) \quad (6.84)$$

(see Equation (4.101)), but the channel width l is unknown. In this case, bed processes at the margins could be of critical importance, provided that the channel is underlain by till. We propose that the application of a revised lateral bed flux term (such as Equation (6.24)₁), together with (6.82) and (6.83), i.e.,

$$q_{bl} = \frac{8(\tau_b - \tau_c)_{0+}^{3/2}}{\mu_c R \rho_w^{3/2} g} \frac{\partial h_s}{\partial x}, \quad \tau_c = \tau_c(dh_s/dx) \geq 0, \quad (6.85)$$

²The steep banks often observed in real river channels are due to the presence of cohesive (clay-like) sediment or the effect of vegetation.

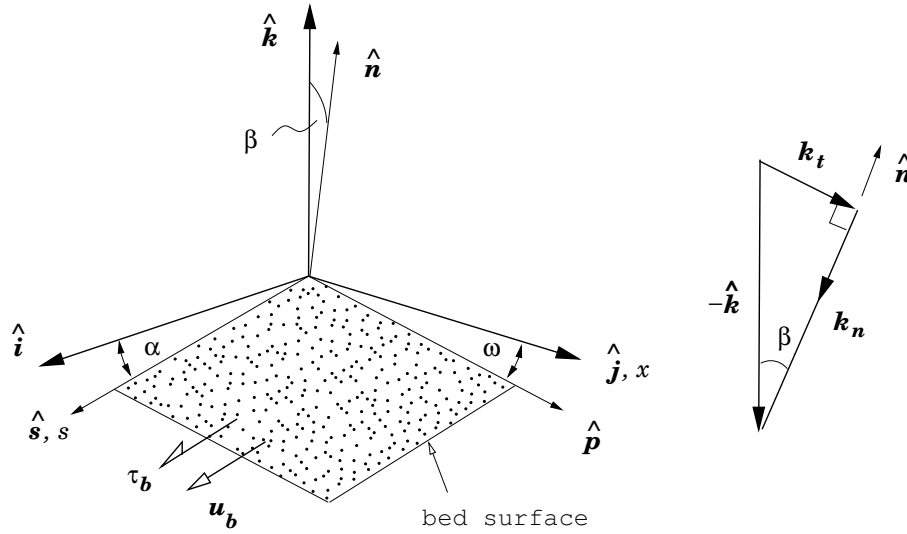


Figure 6.6: The vector coordinate system used in the Shields stress/bedload formulation.

ought to determine l . This investigation is undertaken in Chapter 7.

Here we are concerned with the derivation and the physical basis of (6.85)₂. It is necessary to consider the 3-D situation of a given water flow *past* an inclined sediment bed. In the following, we adopt the *vectorial* description of τ_c and q_{bl} given by Kovacs and Parker (1994). This model is fully mechanistic and applies smoothly up to the angle of repose. We begin by describing the first-principle derivation of their equations.

6.4.1 Threshold of motion

Generalized vector coordinate system

The grain threshold τ_c of a bed of coarse uniform grains, experimentally determined by Shields (1936), is given by the equation

$$\frac{\tau_c}{\rho_w g R D_s} \approx 0.045, \quad (6.86)$$

but this semi-empirical result is valid only for a plane horizontal bed. To cope with an arbitrarily sloping bed, we use the vector system shown in Fig. 6.6, defined *locally* for any element on the bed surface. Let $\hat{\mathbf{n}}$ denote the unit normal vector to the surface and let $\hat{\mathbf{k}}$ represent the unit upward vertical vector. The vector $-\hat{\mathbf{k}}$ may be decomposed into a component normal and a component tangential to the bed plane; these are given by

$$\mathbf{k}_n = -(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}, \quad \mathbf{k}_t = -\hat{\mathbf{k}} + (\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}. \quad (6.87)$$

The bed slope is then

$$\tan \beta = |\mathbf{k}_t|/|\mathbf{k}_n|, \quad (6.88)$$

where β is the bed slope angle.

Suppose now that water is flowing over the bed and exerts a shear stress $\boldsymbol{\tau}_b$ on the surface. By definition, this stress is tangential to the bed, i.e., $\boldsymbol{\tau}_b \cdot \hat{\mathbf{n}} = 0$, therefore, its unit direction vector

$$\hat{\mathbf{s}} = \frac{\boldsymbol{\tau}_b}{|\boldsymbol{\tau}_b|} \quad (6.89)$$

lies in the bed plane. We can also define a unit vector $\hat{\mathbf{p}}$ in the *lateral* direction (also tangential to the bed), such that the projections of $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ back onto the horizontal plane are respectively $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ form the right-orthogonal set. Then, we can write

$$\cos \alpha = \hat{\mathbf{s}} \cdot \hat{\mathbf{i}}, \quad \cos \omega = \hat{\mathbf{p}} \cdot \hat{\mathbf{j}}, \quad (6.90)$$

where α and ω are respectively the downstream and lateral bed slope angles. Note that the directions of $\hat{\mathbf{j}}$ and $\hat{\mathbf{s}}$ here coincide respectively with our x - (cross-stream) and s - (downstream) axes (Section 2.5). In general, $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ are not necessarily orthogonal to each other.

Force balance and grain threshold

For turbulent flows consisting of a fully rough turbulent inner layer near the bed surface, bedload motion is contained within this inner layer and extends only a few grain diameters above the bed. The representative bed shear stress $\boldsymbol{\tau}_b$, given at the top of the thin bedload layer, has associated with it an *effective* near-bed flow velocity $\mathbf{u}_b$. This velocity, assumed parallel to $\boldsymbol{\tau}_b$, is evaluated by means of the rough logarithmic law of the wall. Following Kovacs and Parker (1994), we can write

$$\mathbf{u}_b = u_b \hat{\mathbf{s}}, \quad \boldsymbol{\tau}_b = \tau_b \hat{\mathbf{s}}, \quad (6.91)$$

and

$$\frac{u_b}{\tilde{u}} = a_1^{1/2}, \quad (6.92)$$

where τ_b and $\tilde{u}$ have already been defined in (6.1) and (6.2), and the dimensionless constant a_1 (where $a_1^{1/2} \approx 10$) embodies the relevant properties of the inner layer, such as bed roughness and the characteristic depth of the bedload.

We apply these definitions to a force balance consideration for an average sediment particle saltating within the bedload layer. If the particle moves with a steady (mean) velocity $\mathbf{v}_p$, then $\mathbf{v}_p$ must be tangential to the bed, but it is not necessarily parallel to $\mathbf{u}_b$. Let us define a relative velocity

$$\mathbf{u}_r = \mathbf{u}_b - \mathbf{v}_p. \quad (6.93)$$

In vector notations then, all the different forces acting on the particle *in the bed plane* are

$$\begin{aligned} \text{Fluid drag :} \quad & \mathbf{F}_d = \frac{\pi}{2} \rho_w c_d (D_s/2)^2 |\mathbf{u}_r| \mathbf{u}_r, \\ \text{Immersed weight:} \quad & \mathbf{F}_g = \frac{4\pi}{3} \rho_w g R (D_s/2)^3 \mathbf{k}_t, \\ \text{Coulomb friction:} \quad & \mathbf{F}_c = -\frac{4\pi}{3} \mu_c \rho_w g R (D_s/2)^3 |\mathbf{k}_n| \hat{\mathbf{t}}_p, \end{aligned} \quad (6.94)$$

where c_d stands for drag coefficient, the unit vector

$$\hat{\mathbf{t}}_p = \frac{\mathbf{v}_p}{|\mathbf{v}_p|} \quad (6.95)$$

defines the direction of particle motion, and μ_c is the dynamic friction factor (see Equation 6.5). From experiments, μ_c is approximately constant and equal to the *static* Coulomb friction factor μ_s associated with incipient slope failure. It is assumed here for simplicity that $\mu_c = \mu_s$. Hence, following the definition in Section 5.2.1, the angle of repose is given by

$$\phi_0 = \tan^{-1}(\mu_s) = \tan^{-1}(\mu_c). \quad (6.96)$$

For typical values of ϕ_0 ranging from 30° to 40° , we obtain $0.58 < \mu_c < 0.84$.

Since $\mathbf{v}_p$ denotes a steady velocity, there is no net force on the particle (and on the bulk bedload). Equilibrium requires that

$$\mathbf{F}_d + \mathbf{F}_g + \mathbf{F}_c = 0. \quad (6.97)$$

This equation reduces with the aid of (6.94) and (6.95) to the following dimensionless (*) form:

$$\frac{1}{a_1} |\mathbf{u}_r^*| \mathbf{u}_r^* = \sigma \tau_{co} \left(|\mathbf{k}_n| \hat{\mathbf{t}}_p - \frac{\mathbf{k}_t}{\mu_c} \right), \quad (6.98)$$

for which we have introduced the scales u_0 and τ_0 respectively for velocities and shear stress, with

$$\tau_0 = \rho_w u_0^2. \quad (6.99)$$

(τ_0 is defined in Section 6.2.4.) The parameters here are given by the relations

$$\sigma = \frac{\rho_w g R D_s}{\tau_0}, \quad \tau_{co} = \frac{4\mu_c}{3a_1 c_d}. \quad (6.100)$$

(With the scales in (6.52), $\sigma \approx 0.2$.) Under the same scales, the supplementary equations become

$$|\mathbf{u}_b^*| \mathbf{u}_b^* = a_1 \boldsymbol{\tau}_b^*, \quad \mathbf{u}_r^* = \mathbf{u}_b^* - \mathbf{v}_p^*, \quad \hat{\mathbf{t}}_p = \frac{\mathbf{v}_p^*}{|\mathbf{v}_p^*|}. \quad (6.101)$$

Next, we show that the parameter τ_{co} is in fact the dimensionless measure of the critical Shields stress for horizontal beds.

Generalized critical Shields stress

At the threshold of motion, force equilibrium on the particle is satisfied with $\mathbf{v}_p^* = 0$. Thus $\mathbf{u}_r^* = \mathbf{u}_b^*$, and Equation (6.98) becomes

$$\frac{1}{a_1} |\mathbf{u}_{bo}^*| \mathbf{u}_{bo}^* = \sigma \tau_{co} \left(|\mathbf{k}_n| \hat{\mathbf{t}}_{po} - \frac{\mathbf{k}_t}{\mu_c} \right), \quad (6.102)$$

in which subscript $_o$ refers to zero particle motion. Also, the shear stress τ_b^* attains its critical value τ_c^* ($= \tau_c^* \hat{\mathbf{s}}$), so (6.102) and (6.101)₁ yield, upon reduction,

$$\left| \frac{\tau_c^*}{\sigma \tau_{co}} \hat{\mathbf{s}} + \frac{\mathbf{k}_t}{\mu_c} \right| = |\mathbf{k}_n|. \quad (6.103)$$

This equation brings out rather clearly that for any point on an inclined sediment bed, the (dimensionless) critical Shields stress τ_c^* is direction dependent. For instance, τ_c^* is greater when moving grains upslope ($\mathbf{k}_t/|\mathbf{k}_t| = -\hat{\mathbf{s}}$) than when moving grains downslope ($\mathbf{k}_t/|\mathbf{k}_t| = +\hat{\mathbf{s}}$).

Several special cases are embedded in (6.103). The simplest is that of a horizontal bed, for which $\mathbf{k}_t = 0$, and $\mathbf{k}_n = 1$. It follows that

$$\tau_c^* \equiv \sigma \tau_{co} \quad \left(= \frac{\rho_w g R D_s}{\tau_0} \tau_{co} \right). \quad (6.104)$$

This equation confirms our claim on τ_{co} , since it has the same form as Equation (6.86), with $\tau_{co} \approx 0.045$. If we use the sediment values in (6.26), then (dimensionally) $\tau_c \approx 0.4$ Pa, therefore τ_c is small.

More generally, one can resolve (6.103) into its Cartesian components, using our earlier definitions of downstream and lateral bed slopes, then solve for τ_c^* . The result is

$$\tau_c^* = \frac{\sigma \tau_{co}}{\mu_c} \left[\sqrt{\frac{\mu_c^2 - \cos^2 \alpha \tan^2 \omega}{\sec^2 \alpha + \tan^2 \omega}} - \sin \alpha \right]. \quad (6.105)$$

In the case of vanishing lateral slope ($\omega = 0$), this reduces to

$$\tau_c^* = \sigma \tau_{co} \cos \beta \left(1 - \frac{\tan \beta}{\mu_c} \right), \quad \text{where } \beta = \alpha; \quad (6.106)$$

and in the case of vanishing downstream slope ($\alpha = 0$), a similar reduction leads to

$$\tau_c^* = \sigma \tau_{co} \cos \beta \sqrt{1 - \left(\frac{\tan \beta}{\mu_c} \right)^2}, \quad \text{where } \beta = \omega. \quad (6.107)$$

These relations demonstrate the dependence of τ_c^* (or τ_c) on the bed slope. If the bed slopes down or across the flow direction, the grain threshold is maximum for a horizontal bed, but it decreases (non-linearly) to zero as the slope approaches the angle of repose.

6.4.2 Bedload transport flux

The vectorial bedload transport flux $\mathbf{q}_b$ is conveniently expressed in terms of the product equation

$$\mathbf{q}_b = \xi \mathbf{v}_p, \quad (6.108)$$

where $\mathbf{v}_p$ is the (mean) particle velocity, and ξ is the volume of particles participating as bedload per unit bed area, i.e., the *bedload content*. In order to calculate ξ (and

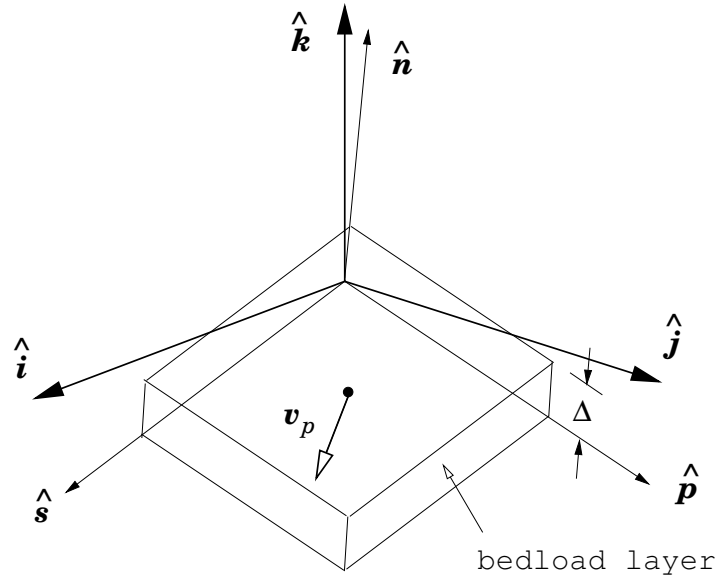


Figure 6.7: *The elementary parallelepiped of bedload sediment used in the force balance consideration.*

hence $\mathbf{q}_b$), let us consider a definable effective bedload layer above the bed, shown in Fig. 6.7. As this thin layer contains moving sediment particles, its bulk density is greater than water alone, and can be taken to be equal to $\eta\rho_w$, where $\eta > 1$.

Force balance

Immediately above the bed surface element, the bedload layer forms an elementary parallelepiped with height Δ normal to the bed (this is the actual bedload layer thickness). Assuming steady motion, force balance *on the sediment phase* of this (thin) control volume can be expressed by the equation

$$\boldsymbol{\tau}_b + \mathbf{w}_g + \mathbf{f}_c = \boldsymbol{\tau}_B, \quad (6.109)$$

where

$$\mathbf{w}_g = \rho_w g \Delta (\eta - 1) \mathbf{k}_t, \quad \mathbf{f}_c = -\mu_c \rho_w g \Delta (\eta - 1) |\mathbf{k}_n| \hat{\mathbf{t}}_p. \quad (6.110)$$

The interpretation of the terms here is as follows. $\boldsymbol{\tau}_b$ and $\boldsymbol{\tau}_B$ are the fluid shear stresses acting respectively on the top of the bedload layer and on the bed surface. We assume that the net momentum input to the fluid phase from gravity is negligible, because it is partially balanced by pressure differences, and because Δ is small. As a result, the deficit $\boldsymbol{\tau}_b - \boldsymbol{\tau}_B$ has to equalize the sum of (i) the Coulomb friction felt by the grains, exerted by the bed surface (which acts in the $-\hat{\mathbf{t}}_p$ direction), and (ii) the sediment weight component in the bed plane (which acts in the $\mathbf{k}_t$ direction).

Now, mass conservation within the control volume requires that

$$\eta\rho_w = \eta_s\rho_s + \eta_w\rho_w, \quad (6.111)$$

where η_s and η_w are the volume fractions of sediment and water, respectively, in the bedload. By definition $\eta_s + \eta_w = 1$, therefore $\eta - 1 = R\eta_s$. Also, the sediment content may now be written as

$$\xi = \eta_s \Delta. \quad (6.112)$$

With these definitions, Equations (6.109) and (6.110) can be re-scaled and simplified, giving

$$\boldsymbol{\tau}_b^* + \sigma \xi^* (\mathbf{k}_t - \mu_c |\mathbf{k}_n| \hat{\mathbf{t}}_p) = \boldsymbol{\tau}_B^*. \quad (6.113)$$

The stress scale is τ_0 as before, and we have introduced D_s as the scale of the sediment content, thus

$$\xi^* = \frac{\xi}{D_s}. \quad (6.114)$$

The Bagnold condition

According to Bagnold (1956), when bedload transport takes place on a horizontal bed, the basal stress τ_B^* should become equal to the Shields critical value τ_{co} . (An equivalent statement is that the bed-surface/bedload interface is found at the depth where the grains are at the threshold of motion.) Although this condition is scalar in nature, it can be generalized to the vectorial case. The simplest hypothesis is that the downstream component of $\boldsymbol{\tau}_B^*$ equals the critical value (Kovacs and Parker, 1994), i.e.,

$$\boldsymbol{\tau}_B^* \cdot \hat{\mathbf{s}} = \tau_c^*. \quad (6.115)$$

Applying this condition to Equation (6.113) now allows ξ^* to be obtained, as

$$\xi^* = \frac{\tau_b^* - \tau_c^*}{\sigma(\mu_c |\mathbf{k}_n| \hat{\mathbf{t}}_p - \mathbf{k}_t) \cdot \hat{\mathbf{s}}}, \quad (6.116)$$

and this completes the derivation for ξ . To summarize, our general bedload transport predictor consists of (6.116), with (6.108) in its dimensionless form

$$\mathbf{q}_b^* = \xi^* \mathbf{v}_p^*. \quad (6.117)$$

The flux scale is

$$[q] = D_s [u] = D_s (\tau_0 / \rho_w)^{1/2}. \quad (6.118)$$

6.4.3 Application

To apply the bedload predictor, the critical stress τ_c^* and the particle velocity $\mathbf{v}_p^*$ have to be computed first. Given the slope orientation and sediment properties, τ_c^* is given straightforwardly by Equation (6.105). However, the determination of $\mathbf{v}_p^*$ and its associate direction vector $\hat{\mathbf{t}}_p$ is difficult. The relevant equations (taken from (6.98) and (6.101)) are

$$\begin{aligned} \mathbf{u}_r^* &= \mathbf{u}_b^* - \mathbf{v}_p^*, & \hat{\mathbf{t}}_p &= \mathbf{v}_p^* / |\mathbf{v}_p^*|, \\ |\mathbf{u}_r^*| \mathbf{u}_r^* &= a_1 \sigma \tau_{co} \left(|\mathbf{k}_n| \hat{\mathbf{t}}_p - \frac{\mathbf{k}_t}{\mu_c} \right), \end{aligned} \quad (6.119)$$

where

$$\mathbf{u}_b^* = u_b^* \hat{\mathbf{s}} = (a_1 \tau_b^*)^{1/2} \hat{\mathbf{s}}. \quad (6.120)$$

Although the applied stress τ_b^* is prescribed, these equations are not amenable to general analytical solution. The method based on vector component resolution is found to lead to two coupled sixth order algebraic equations, which can be tackled only by numerical solution (Kovacs and Parker, 1994).

Here, we shall not attempt a complete reformulation of our bedload model, but instead, we derive a low slope approximation (where $\tan \omega \ll \mu_c$) and also investigate what happens near the angle of repose. The corresponding results are sufficient for use in Chapter 7. The downstream slope is assumed to be negligible ($\alpha = \alpha_b$ for our subglacial channel). By using Fig. 6.6, we find that

$$\frac{dh_s}{dx} = \tan \omega \cos \alpha \quad (6.121)$$

(here h_s and x are dimensional); with $\alpha \approx 0$, we also have $\hat{\mathbf{s}} \approx \hat{\mathbf{i}} (\perp \hat{\mathbf{p}})$, $\beta \approx \omega$, $|\mathbf{k}_n| \approx \cos \omega$, and $\mathbf{k}_t \approx \hat{\mathbf{p}} \sin \omega$.

Low slope approximation

When $\mu_c \gg \tan \omega \approx 0$, one can anticipate particle motion to be predominantly in the downstream direction. Consider

$$\mathbf{v}_p^* \approx U_s \hat{\mathbf{s}} + U_p \hat{\mathbf{p}}, \quad (6.122)$$

where U_s , U_p are respectively the downstream and lateral velocity components, and let us assume $U_p \ll U_s$ ($\sim u_b^*$). Substituting this into (6.119) leads to the following approximation:

$$(u_b^* - U_s)^2 \hat{\mathbf{s}} - U_p (u_b^* - U_s) \hat{\mathbf{p}} \approx a_1 \sigma \tau_{co} \left[\cos \omega \hat{\mathbf{t}}_p - \frac{\sin \omega}{\mu_c} \hat{\mathbf{p}} \right], \quad (6.123)$$

where

$$\hat{\mathbf{t}}_p \approx \hat{\mathbf{s}} + \frac{U_p}{U_s} \hat{\mathbf{p}} \quad (\approx \hat{\mathbf{s}}). \quad (6.124)$$

If we proceed to equate terms in $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ in Equation (6.123), then to leading order (and taking $\cos \omega \approx 1$), we obtain

$$\begin{aligned} U_s &\approx u_b^* - \sqrt{a_1 \sigma \tau_{co}} \quad (\approx \sqrt{a_1 \tau_b^*}), \\ U_p &\approx \sqrt{a_1 \sigma \tau_{co}} \frac{\tan \omega}{\mu_c}. \end{aligned} \quad (6.125)$$

These approximations are valid as long as $\tau_b^* \gg \sigma \tau_{co}$, which is typically the case under stream flow conditions. They describe that the drag force on the bedload is balanced by frictional force downstream, and by its weight component in the lateral direction. Since $\tan \omega \ll \mu_c$, these results are consistent with our original assumption $U_p \ll U_s$.

Having estimated $\mathbf{v}_p^*$, application of (6.116) and (6.117) leads to the predicted bedload flux

$$\begin{aligned} \text{downstream:} \quad \mathbf{q}_b^* \cdot \hat{\mathbf{s}} &\approx \frac{\tau_b^* - \tau_c^*}{\sigma \mu_c} \sqrt{a_1 \tau_b^*}, \\ \text{cross-stream:} \quad \mathbf{q}_b^* \cdot \hat{\mathbf{p}} &\approx \sqrt{a_1 \sigma \tau_{co}} \frac{\tau_b^* - \tau_c^*}{\sigma \mu_c^2} \tan \omega. \end{aligned} \quad (6.126)$$

Note that if (6.126)₁ is modified slightly by replacing τ_b^* with $(\tau_b^* - \tau_c^*)$, it is essentially equivalent to Equation (6.3), with $\sqrt{a_1}/\mu_c \approx 8$. On the other hand, since $\tan \omega \approx dh_s/dx$, the second result here is less similar to the lateral flux equation (6.5) proposed by Parker (1978). Nevertheless, (6.126) is a good approximation only when $\tan \omega \ll \mu_c$. Therefore, the use of Parker-type flux terms is probably not too unreasonable at low/intermediate bed slopes.

Condition at the margins

The situation is different at high slopes. We show here why an infinite bank slope solution is inappropriate. Let us consider a channel at *steady flow*. The banks must be kept at equilibrium. This means that the margins (located at $x = \pm l$) are neither advancing or receding, and have vanishing lateral bedload flux ($q_{bl} = 0$).

Since the flow contains suspended sediment, there is continuous particle deposition onto the banks, even within (an arbitrarily) close neighbourhood of the margins. As a result, q_{bl} can be zero at $x = \pm l$, but is non-zero elsewhere. It follows that the threshold of motion is attained everywhere along the bed perimeter:

$$\tau_b^* \geq \tau_c^*, \quad \text{in } |x| \leq l. \quad (6.127)$$

τ_c^* is non-negative (due to (6.83)), and $\tau_b^* \propto h$ according to (6.25)₁.

Now the flow depth h vanishes at the margins, hence we have $0 = \tau_b^* \geq \tau_c^* \geq 0$, which implies that

$$\tau_c^* \equiv 0 \quad \text{at } x = \pm l; \quad (6.128)$$

that is, the sediment particles at the margins are at their threshold of motion. This result determines a *slope condition* via Equation (6.105), giving

$$\mu_c^2 - \cos^2 \alpha \tan^2 \omega = \tan^2 \alpha + \sin^2 \alpha \tan^2 \omega. \quad (6.129)$$

When $\alpha \approx 0$, this reduces approximately to

$$\tan \omega \approx \mp \mu_c \left(\approx \frac{dh_s}{dx} \right), \quad \text{at } x = \pm l. \quad (6.130)$$

In other words, the channel margins must rest (approximately) at the angle of repose. This ‘bank condition’ is central to the argument in Chapter 7. Its dimensionless form has $h_0/l_0 \times dh_s/dx$ replacing dh_s/dx .

Finally, it is necessary to check that q_{bl} (or $\mathbf{q}_b^* \cdot \hat{\mathbf{p}}$) also vanishes at the end. Both $\mathbf{v}_p^*$ and $\mathbf{u}_b^*$ vanish at $x = \pm l$, so we require $|\mathbf{k}_n| \hat{\mathbf{t}}_p = \mathbf{k}_t/\mu_c$ on the r.h.s. of (6.119). When $\alpha = 0$, we have $|\mathbf{k}_n| = \cos \omega$, $|\mathbf{k}_t| = \sin \omega$, where $\hat{\mathbf{t}}_p \parallel \mathbf{k}_t$ (since the direction of ‘incipient’ sediment motion is down the banks). This is consistent with Equation (6.130).

6.5 Discussion

In this chapter, the basic equations describing mobile bed evolution — (6.23) to (6.25) — have been established by using a model proposed by Parker (1978). By reducing them to study the special case of a river channel, we identify potential problems associated with the application of boundary conditions, and with the effect of Shields critical stress τ_c in controlling lateral bedload transport/bed equilibrium at the margins.

The next step is to integrate these equations with the ice evolution description of Chapters 3 and 4, and the creep velocity results of Chapter 5, in a general drainage model for wide channels. (The non-dimensionalization procedure has already been initiated in Section 6.2.4.) This final construction will be based on insights gained previously into the nature of the four ingredient processes. Most importantly, we shall propose that the neglect of τ_c , though acceptable in river models, must in the sub-glacial case be backed up by a suitable adjustment to the boundary conditions. This adjustment involves the use of the ‘bank condition’ derived in Section 6.4, and is absolutely essential in order to ensure a consistent model for determining an equilibrium channel width. The full argument is given in Chapter 7.

Chapter 7

The drainage model

7.1 Introduction

At the beginning of Part II (Section 3.1), we stated the hypothesis proposed by Walder and Fowler (1994), concerning how water transport in our wide subglacial channel could be maintained through the interaction of various processes — namely, through an ice melting-closure balance and a sediment erosion-creep balance (see Fig. 3.1). Having considered each of the ingredient processes in detail, we now develop a mathematical model to describe this, and also to describe (more generally) how the channel would evolve in time. The model equations are given in Section 7.2.

Here our initial emphasis is on the conditions underlying *steady flow*. The central requirement is that of *channel equilibrium*: both the roof and the bed interfaces are to remain stationary under the coupled action of the ice-sediment processes. Although this is conceptually simple, its consequences are not immediately obvious, due to an asymmetry between the upper and the lower halves of the channel. The full investigation is given in Section 7.3. We derive the steady flow solutions of the model and establish the corresponding drainage characteristics. Most importantly, this includes an extension of the classical Röthlisberger’s (1972) relation for deformable subglacial beds. We also investigate the equilibrium shape of the channel and explore the issue of sediment transport. Our results are found to complement those of Walder and Fowler (1994) in some respects, but there are important differences concerning the distinction between R channels and canals. In particular, we propose that an erosion-creep balance for the channel bed is not strictly correct.

In Section 7.4, we briefly consider the application of our model to unsteady flows, with the jökulhlaup model proposed in Chapter 2 in mind. Further treatment of time-dependent drainage however, is outside the scope of this thesis and will have to be tackled elsewhere. Finally, our findings in Part II are summarized by way of an extended discussion, in Section 7.5. We provide general conclusions in Chapter 8.

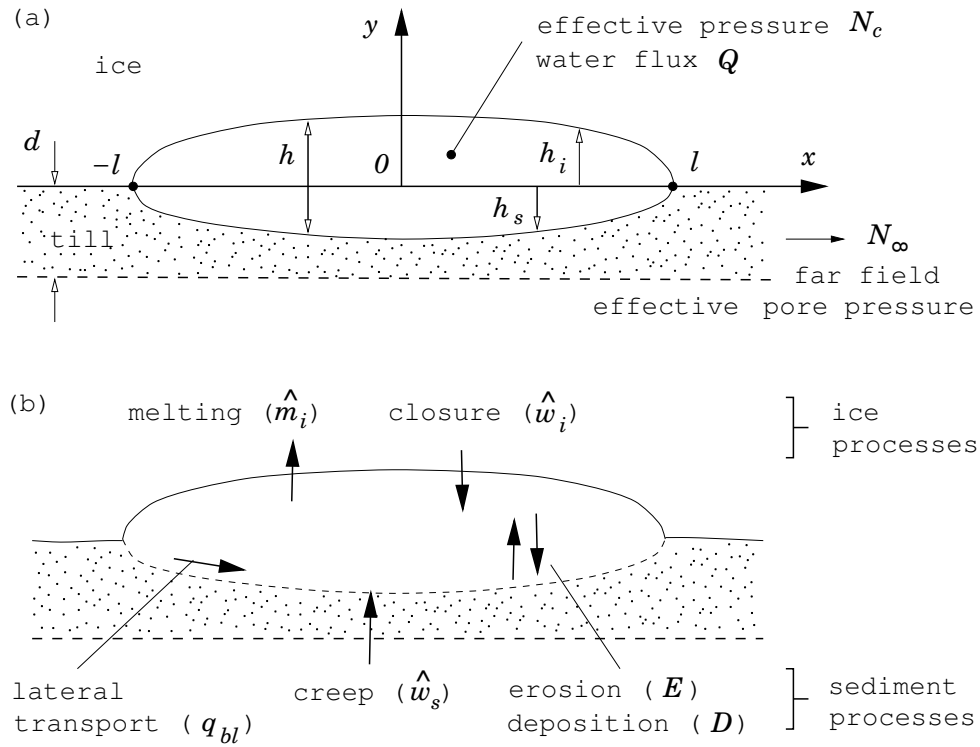


Figure 7.1: (a) Mathematical symbols and (b) the ice and sediment processes in the wide subglacial channel.

7.2 Model formulation

7.2.1 Governing equations

Fig. 7.1 shows a schematic diagram of our wide channel. Armed with the results derived in Chapters 3 to 6, we can now assemble the time-dependent equations to describe its evolution. In addition to our previous mathematical symbols, we define h_i to be the ice incision part of the channel depth, such that

$$h = h_i + h_s, \quad (h \geq 0), \quad (7.1)$$

the subscripts i and s denoting ice and sediment respectively. By definition

$$h(\pm l) = 0. \quad (7.2)$$

For completeness, we also reincorporate downstream (s -) dependence¹. The appropriate evolution equations are then

¹We shall write the s -derivatives explicitly as $\partial/\partial s$ to avoid confusion with subscript s .

$$\frac{\partial h_i}{\partial t} = \frac{\hat{m}_i}{\rho_i} - \hat{w}_i, \quad (7.3)$$

$$\frac{\partial h_s}{\partial t} = \frac{1}{1 - n_s} \left[\frac{\partial q_{bl}}{\partial x} + \frac{\partial q_b}{\partial s} + v_s(E - D) \right] - [\hat{w}_s]_D, \quad (7.4)$$

in which

$$\hat{m}_i = \frac{C(\Psi h)^{3/2}}{L\sqrt{f^*\rho_w}}, \quad \hat{w}_i = \frac{N_c^3}{2\eta_0} \sqrt{l^2 - x^2}, \quad \hat{w}_s = \frac{A_T N_c^a}{2N_\infty^b} \sqrt{l^2 - x^2}, \quad (7.5)$$

and

$$q_b = \frac{8(\tau_b - \tau_c)_{0+}^{3/2}}{\rho_w^{3/2} g R}, \quad q_{bl} = \frac{q_b}{\mu_c} \frac{\partial h_s}{\partial x},$$

$$E = 0.092 \left(\frac{\tau_b - \tau_c}{\rho_w g R D_s} \right)_{0+}^{3/2}, \quad D = \frac{v_s}{\epsilon} \zeta. \quad (7.6)$$

Other subsidiary relations that we require are

$$\tau_b = \frac{\Psi h}{2}, \quad \epsilon = \sqrt{\frac{\Psi f^*}{8\rho_w}} h^{3/2}, \quad (7.7)$$

$$\frac{\tau_c}{\rho_w g R D_s} = \frac{\tau_{co}}{\mu_c} \left[\cos \alpha_b \sqrt{\frac{\mu_c^2 - (\partial h_s / \partial x)^2}{1 + (\partial h_s / \partial x)^2}} - \sin \alpha_b \right] \quad (\geq 0). \quad (7.8)$$

To describe mass conservation of the water and sediment phases, we have

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{1}{\rho_w} \int_{-l}^l \hat{m}_i \, dx + M, \quad (7.9)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial(\hat{u}\zeta)}{\partial s} = \frac{\partial}{\partial x} \left(\epsilon \frac{\partial \zeta}{\partial x} \right) + \frac{\partial}{\partial s} \left(\epsilon \frac{\partial \zeta}{\partial s} \right) + v_s(E - D), \quad (\zeta \geq 0), \quad (7.10)$$

where

$$S = \int_{-l}^l h \, dx, \quad Q = \int_{-l}^l h \hat{u} \, dx, \quad \hat{u} = \sqrt{\frac{8\Psi}{\rho_w f^*}} h^{1/2}. \quad (7.11)$$

Note that as before, it is the total flow depth that determines the melt-rate $\hat{m}_i$, shear stress τ_b and eddy viscosity ϵ . The closure terms $\hat{w}_{i,s}$ assume non-linear ice/till rheology; $[\]_D$ denotes ‘deep till only’ (see later).

Our governing equations have been adapted from Equations (2.46), (4.97), (6.23) to (6.25) (and their supplements), and have the (x -) boundary conditions

$$h_s = 0, \quad \epsilon \frac{\partial \zeta}{\partial x} = 0, \quad \text{at } x = \pm l. \quad (7.12)$$

We use the material constants shown in Table 2.1, Boulton and Hindmarsh’s (1987) creep properties in Equation (5.4), the sediment properties defined in Equation (6.26), and a critical Shields constant $\tau_{co} = 0.045$ (Section 6.4.1). We also assume α_b to be

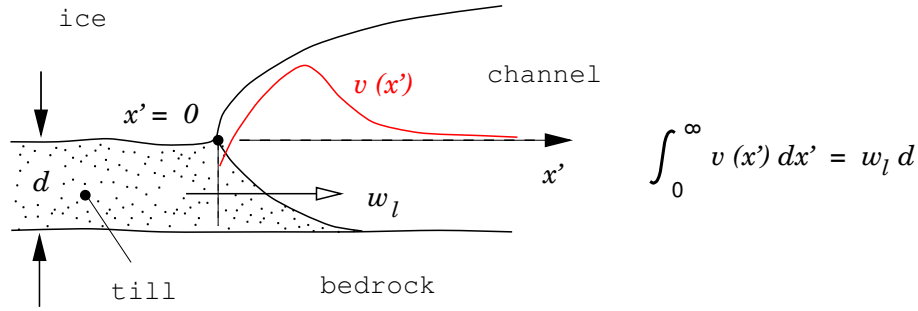


Figure 7.2: Definition diagram for the lateral closure velocity w_l .

small ($\sin \alpha_b \lesssim 0.1$), and $\eta_0 = 5 \times 10^{23} \text{ Pa}^3 \text{ s}$, $C = 2\sqrt{2}$, $f^* = 0.1$, in order to be consistent with Section 2.5.

Generally, the far field effective pore pressure N_∞ , downstream channel slope α_b and melt-water input rate M are prescribed functions of s . We put $M = 0$ for convenience², and an equation for Ψ is given shortly. Therefore, there are enough equations (twenty) for the unknowns h_i , h_s , h , $\hat{m}_i$, $\hat{w}_i$, $\hat{w}_s$, q_{bl} , q_b , E , D , τ_b , τ_c , ϵ , $\hat{u}$, ζ (which are functions of x , s and t), and Ψ , N_c , l , Q , S (which are functions of s and t). Equation (7.2) is an implicit definition of the channel width, in the sense that an equation for l or $\partial l / \partial t$ may be derived from it if necessary. This is discussed in Sections 7.3 and 7.4.

Downstream variation

The s -derivatives in Equations (7.4) and (7.10) describe downstream advection and diffusion of sediment. We have also taken the general definition of the (total) hydraulic gradient

$$\Psi = \Phi + \frac{\partial N_c}{\partial s} \quad (7.13)$$

from (2.18); as before, $\Phi(s)$ is given by Equation (2.19) (or approximately, by (2.20)). Subsequently, we show that the downstream variations may be neglected. This allows us to adopt the 2-D approach as in the earlier chapters.

Sediment creep closure

In (7.4), the closure term $[\hat{w}_s]_D$ is ‘switched on’ only in the case of a deep (deforming) till layer, for which $d \gg l$. We assume this because creep flow is limited to the channel margins if the till is shallow ($d \ll l$) (see Chapter 5). In this case, the corresponding velocity function is given by (5.111), and we can represent the overall creep motion by a *lateral closure velocity* w_l for the channel width, averaged over till

²The effect of subsidiary/englacial channels draining melt-water into (our) main channel is not considered here. In addition, we assume that the till conducts a negligible water flux into the channel, because of its low permeability.

thickness d . This is illustrated in Fig. 7.2. Taking Equations (5.112) and (5.113)₂, a simple mass conservation consideration leads to

$$w_l \approx \frac{A_T N_c^a \sqrt{dl}}{10 N_\infty^b}. \quad (7.14)$$

This result is incorporated into a width evolution equation in Section 7.4.

Limitations

Being aware of the assumptions and approximations on which our model is based will help us interpret its solutions correctly. We repeat the most relevant ones here. The channel is taken to be wide, i.e., $\delta = h/l \ll 1$, which permits a local balance consideration at each cross-stream position x . Consequently, we have shown or argued that:

- Secondary currents in the water flow can be neglected (Section 4.3.1).
- The channel effective pressure is essentially constant within the cross section, so that $N_c = N_c(s, t)$ (Section 4.3.1).
- The formulae for $\hat{m}_i$, τ_b , ϵ , and $\hat{u}$ are strictly leading order approximations valid in the central part of the channel. Particularly, the first three may not in reality vanish at the margins, but the resulting effect (especially in melting the tip) will be negligible. Equation (7.11)₂ for Q is also a leading order approximation. In all of these terms the errors are of $O(\delta^2)$ (Chapter 4).

In addition, we note that:

- $\hat{w}_i$, $\hat{w}_s$ and w_l are based on heuristic extension of the solutions to the corresponding linearized creep problems, and they are potentially inaccurate at the margins due to a crack approximation; the basal till is assumed to have a low permeability (Chapters 3 and 5). w_l is an upper bound velocity estimate (Section 5.7).
- The model is liable to produce a non-zero suspended sediment content at $x = \pm l$, where the depth-integrated definition of ζ breaks down (Section 6.3.3).
- The flux equations in (7.6) are assumed to be applicable to a sediment bed that is moderately sloping/curving (Section 6.2).

7.2.2 Non-dimensionalization

The rescaling procedure outlined in Section 6.2.4 is now extended (see Equations (6.32) to (6.40)). We define the scales

$$\begin{aligned} [h_i] = [h_s] = [h] &= h_0, & [x] = [l] &= l_0, \\ [s] &= s_0, & [t] &= t_s \quad (\text{a sediment timescale}), \end{aligned}$$

$$\begin{aligned}
[\Psi] &= [\Phi] = \Phi_0, & [N_c] &= [N_\infty] = [N], \\
[\tau_b] &= [\tau_c] = \tau_0, & [\epsilon] &= \epsilon_0, & [\hat{u}] &= u_0, \\
[\zeta] &= \zeta_0, & [S] &= S_0, & [Q] &= Q_0,
\end{aligned} \tag{7.15}$$

and impose the scale identities in (6.33), (6.39) and (6.44), together with

$$\frac{(2\Phi_0 h_0)^{3/2}}{\rho_i L \sqrt{f^* \rho_w}} = \frac{[N]^3 l_0}{2\eta_0}, \quad S_0 = h_0 l_0, \quad u_0 = \sqrt{\frac{8\Phi_0 h_0}{\rho_w f^*}}, \quad \frac{S_0}{t_s} = \frac{Q_0}{s_0}. \tag{7.16}$$

Then, non-dimensionalization of Equations (7.1) to (7.13) leads to (after substitution)

$$\beta_i \frac{\partial h_i}{\partial t} = (\Psi h)^{3/2} - N_c^3 \sqrt{l^2 - x^2}, \tag{7.17}$$

$$\begin{aligned}
\frac{\partial h_s}{\partial t} &= \frac{1}{1 - n_s} \left\{ \frac{\partial}{\partial x} [(\tau_b - \tau_c)_{0+}^{3/2} \frac{\partial h_s}{\partial x}] + \frac{\xi \mu_c}{\delta} \frac{\partial}{\partial s} [(\tau_b - \tau_c)_{0+}^{3/2}] + \right. \\
&\quad \left. \kappa_s [(\tau_b - \tau_c)_{0+}^{3/2} - \frac{\zeta}{\epsilon}] \right\} - \frac{\kappa_c N_c^a}{N_\infty^b} \sqrt{l^2 - x^2},
\end{aligned} \tag{7.18}$$

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = \frac{r}{\beta_i} \int_{-l}^l (\Psi h)^{3/2} dx, \tag{7.19}$$

$$\beta_s \left[\frac{\partial \zeta}{\partial t} + \frac{\partial (\hat{u} \zeta)}{\partial s} \right] = \lambda^2 \left[\frac{\partial}{\partial x} \left(\epsilon \frac{\partial \zeta}{\partial x} \right) + \xi^2 \frac{\partial}{\partial s} \left(\epsilon \frac{\partial \zeta}{\partial s} \right) \right] + (\tau_b - \tau_c)_{0+}^{3/2} - \frac{\zeta}{\epsilon}, \tag{7.20}$$

with the subsidiary relations

$$h = h_i + h_s \quad (\geq 0), \quad \Psi = \Phi + \nu \frac{\partial N_c}{\partial s}, \tag{7.21}$$

$$\tau_b = \frac{\Psi h}{2}, \quad \hat{u} = (\Psi h)^{1/2}, \quad \epsilon = \Psi^{1/2} h^{3/2}, \tag{7.22}$$

$$\tau_c = \frac{\sigma \tau_{co}}{\mu_c} \left[\cos \alpha_b \sqrt{\frac{\mu_c^2 - \delta^2 (\partial h_s / \partial x)^2}{1 + \delta^2 (\partial h_s / \partial x)^2}} - \sin \alpha_b \right] \quad (\geq 0), \tag{7.23}$$

$$S = \int_{-l}^l h dx, \quad Q = \Psi^{1/2} \int_{-l}^l h^{3/2} dx, \tag{7.24}$$

where

$$h(\pm l) = 0. \tag{7.25}$$

The re-scaled boundary conditions remain identical in form to (7.12). Equations (7.17) to (7.20) respectively describe roof evolution, bed evolution, mass continuity of the water flow, and conservation of suspended sediment.

In the above, the dimensionless parameters are given by

$$\begin{aligned}
\beta_i &= \frac{\rho_i L \sqrt{f^* \rho_w}}{C \Phi_0^{3/2} h_0^{1/2} t_s}, & \xi &= \frac{l_0}{s_0}, & \kappa_s &= \frac{v_s^2 \zeta_0 t_s}{\epsilon_0 h_0}, \\
\kappa_c &= \frac{A_T l_0 t_s [N]^{a-b}}{2 h_0}, & r &= \frac{\rho_i}{\rho_w}, & \beta_s &= \frac{\epsilon_0}{v_s^2 t_s}, \\
\lambda &= \frac{\epsilon_0}{l_0 v_s}, & \nu &= \frac{[N]}{s_0 \Phi_0}, & \sigma &= \frac{\rho_w g R D_s}{\Phi_0 h_0},
\end{aligned} \tag{7.26}$$

and explicit equations for the variable scales are

$$\begin{aligned}\tau_0 &= \Phi_0 h_0, & u_0 &= \sqrt{\frac{8\Phi_0}{\rho_w f^\star}} h_0^{1/2}, & \epsilon_0 &= \sqrt{\frac{\Phi_0 f^\star}{8\rho_w}} h_0^{3/2}, \\ t_s &= \frac{\mu_c R \rho_w^{3/2} g l_0^2}{8(\Phi_0 h_0)^{3/2}}, & s_0 &= \frac{\rho_w g R \mu_c l_0^2}{\Phi_0 h_0 \sqrt{8f^\star}}, & \zeta_0 &= \frac{0.092 h_0^3 \Phi_0^2}{v_s \rho_w^2 (g R D_s)^{3/2}} \sqrt{\frac{f^\star}{8}}, \\ S_0 &= h_0 l_0, & Q_0 &= \sqrt{\frac{8\Phi_0}{\rho_w f^\star}} l_0 h_0^{3/2}, & [N] &= \left[\frac{2\eta_0 (2\Phi_0 h_0)^{3/2}}{\rho_i L l_0 \sqrt{f^\star} \rho_w} \right]^{1/3}.\end{aligned}\quad (7.27)$$

We also use the formal definition

$$\delta = \frac{h_0}{l_0}. \quad (7.28)$$

(Subsequently, it is necessary to check that the actual channel aspect ratio is high; see Section 7.3.2.) If Φ_0 , h_0 and l_0 are prescribed, then Equation (7.27) is sufficient for determining all the scales. With the nominal values $\Phi_0 = 10^2 \text{ kg m}^{-2} \text{ s}^{-2}$, $h_0 = 1 \text{ m}$, $l_0 = 10 \text{ m}$, we obtain

$$\begin{aligned}\tau_0 &= 100 \text{ Pa}, & u_0 &= 2.83 \text{ m s}^{-1}, & \epsilon_0 &= 3.54 \times 10^{-2} \text{ m}^2 \text{ s}^{-1}, \\ t_s &= 5.11 \times 10^3 \text{ s } (\approx 1.4 \text{ hrs.}), & s_0 &= 14.5 \text{ km}, & \zeta_0 &= 2.83 \text{ m}, \\ S_0 &= 10 \text{ m}^2, & Q_0 &= 28.3 \text{ m}^3 \text{ s}^{-1}, & [N] &= 4.55 \text{ bar},\end{aligned}\quad (7.29)$$

and

$$\begin{aligned}\beta_i &= 208, & \xi &= 6.92 \times 10^{-4}, & \kappa_s &= 1.02 \times 10^3, \\ \kappa_c &= 1.68 \times 10^{-3}, & r &= 0.9, & \beta_s &= 2.77 \times 10^{-3}, \\ \lambda &= 7.07 \times 10^{-2}, & \nu &= 0.315, & \sigma &= 8.09 \times 10^{-2}, & \delta &= 0.1.\end{aligned}\quad (7.30)$$

In the following section, the relevance of some of these parameters is identified. We also derive a simplified model based on removing small terms.

7.2.3 Approximations

Timescales

t_s is a timescale associated with bed evolution. In Equation (7.20), $\beta_s \ll 1$ implies that the suspended sediment distribution relaxes to equilibrium very quickly. Consequently we neglect the l.h.s. of this equation. λ^2 is also small, but the associate x -derivative is retained because it may account for marginal boundary layers.

In (7.17), $\beta_i \gg 1$ indicates that the ice roof evolves over a long timescale and is effectively stationary in a time t_s . The derivative $\partial h_i / \partial t$ ($\sim 1/\beta_i$) becomes important only when the r.h.s. is very much out of balance, then the channel is far from being in equilibrium. (This happens during a jökulhlaup; see Section 7.4). Under normal conditions $h \sim 1$, the melt-water flux term in (7.19) is also negligible.

Parameters ξ and κ_c

$\xi \ll \delta \ll 1$ allows us to discard the s -derivatives in (7.18) and (7.20). This corresponds to neglecting downstream advection and diffusion of sediments. For the moment we retain the creep term in (7.18) despite $\kappa_c \ll 1$, since it may become important at low values of far field effective pore-water pressure (when $N_\infty \lesssim 0.2$ bar; see Section 7.3.2). This term is also responsible for the global sediment budget of the channel (Section 7.3.3).

Net erosion

In Equation (7.18), the terms representing suspended sediment erosion and deposition, i.e.,

$$\left[(\tau_b - \tau_c)_{0+}^{3/2} - \frac{\zeta}{\epsilon} \right], \quad (7.31)$$

is preceded by a large coefficient $\kappa_s \gg 1$. However, the combination is still of $O(1)$, because (7.31) itself is only of $O(\lambda^2)$ according to Equation (7.20). In fact $\kappa_s \lambda^2 \approx 5$. It follows that in (7.18), this net erosion term is as important as lateral bedload transport in determining how the bed evolves. Evidently, the erosion and deposition rates themselves are much greater than their difference. This is consistent with our picture that sediment suspended in the water column is essentially in *dynamic* equilibrium.

Downstream boundary layer

The parameter ν has the same rôle as in the Nye (1976) model in explaining pressure boundary layers. Since its effect is not critical to our subsequent discussion, we put $\nu \rightarrow 0$, thereby reducing (7.21)₂ to $\Psi = \Phi$. This singular approximation disqualifies us from satisfying the snout value of N_c . (Refer to Section 2.3.5).

In order to restore the 2-D formulation of the earlier chapters, we proceed to remove s -dependence altogether by neglecting Equation (7.19), which describes how the water flux varies downstream. An equation for Q (more precisely, $Q(t)$) is now (7.24)₂. Equation (7.24)₁ (for $S(t)$) also decouples from the model.

The bank condition

Our final approximation concerns the critical Shields stress definition in Equation (7.23). Inclusion of τ_c in the model leads to great difficulties because of the non-linearity it introduces in (7.18) and (7.20). With $\sigma\tau_{co} \approx 4 \times 10^{-3}$, τ_c can reasonably be neglected, but it clearly becomes important near the margins where $\tau_b (\propto h)$ decays to $O(\sigma\tau_{co})$. In Chapter 6, we deduced that at *steady flow*, the effect of this is to pin down the bank edge at (approximately) the angle of repose, such that

$$\delta \frac{dh_s}{dx} \approx \mp \mu_c, \quad \text{at } x = \pm l. \quad (7.32)$$

This is an implicit requirement which arises from the functional form of (7.23), and the fact that τ_c must vanish at $x = \pm l$ (Section 6.4.3). Therefore, if we insist on setting $\tau_c = 0$, which is a valid approximation elsewhere in the channel, then we ought to ensure that h_s behaves in the correct manner as the margins are approached. In

Section 7.3, we show that this can be achieved by applying (7.32) as an extra pair of boundary conditions.

The approximate model

After making the appropriate simplifications, our dimensionless model reduces to

$$\beta_i \frac{\partial h_i}{\partial t} = (\Phi h)^{3/2} - N_c^3 \sqrt{l^2 - x^2}, \quad (7.33)$$

$$\frac{\partial h_s}{\partial t} = \frac{1}{1 - n_s} \frac{\partial}{\partial x} \left(\tau_b^{3/2} \frac{\partial h_s}{\partial x} - \kappa_s \lambda^2 \epsilon \frac{\partial \zeta}{\partial x} \right) - \frac{\kappa_c N_c^a}{N_\infty^b} \sqrt{l^2 - x^2}, \quad (7.34)$$

$$\lambda^2 \frac{\partial}{\partial x} \left(\epsilon \frac{\partial \zeta}{\partial x} \right) = \frac{\zeta}{\epsilon} - \tau_b^{3/2}, \quad (7.35)$$

$$\text{b.c.s : } h_s = 0, \quad \epsilon \frac{\partial \zeta}{\partial x} = 0, \quad \text{at } x = \pm l; \quad (7.36)$$

in which $N_c(t)$, Φ , N_∞ are given, and

$$\begin{aligned} h &= h_i + h_s \quad (\geq 0), & h(\pm l) &= 0, \\ \tau_b &= \frac{\Phi h}{2}, & \epsilon &= \Phi^{1/2} h^{3/2}, & Q &= \Phi^{1/2} \int_{-l}^l h^{3/2} dx. \end{aligned} \quad (7.37)$$

At steady flow the time derivatives are neglected, but we impose the bank conditions

$$\frac{dh_s}{dx} = \mp \frac{\mu_c}{\delta} \quad \text{at } x = \pm l. \quad (7.38)$$

7.3 Equilibrium subglacial channels

7.3.1 Preliminaries

Before performing the mathematical derivation, we can gain a qualitative understanding of how an equilibrium state may be achieved in the channel. The argument presented here is based on the work of Chapters 4 and 6. We use Fig. 7.1 and the equations in (7.33) to (7.38) for reference.

Ice processes

We begin by summarizing the results of Section 4.6. Roof evolution is due to the competing effects of melting (which depends on h) and ice deformation (which depends on N_c and l). At steady flow these processes balance each other, according to Röthlisberger's theory (Chapter 2). This condition leads to a uniquely defined function for the total depth via (7.33):

$$h(x) = \frac{N_c^2}{\Phi} (l^2 - x^2)^{1/3}, \quad (7.39)$$

but it places no restriction on $h_s(x)$, so the bed position is left unspecified as far as roof equilibrium is concerned. We also lack a recipe for the channel width. As a

result, the corresponding drainage equation that relates water flux Q and effective channel pressure N_c is incomplete. Substitution of (7.39) into (7.37)₅ gives us

$$Q = \frac{\pi N_c^3 l^2}{2\Phi} \quad (7.40)$$

(cf. Equation (6.84)), but l is unknown³. We propose that both $h_s(x)$ and l also exhibit unique equilibrium values, but these are governed by sediment bed processes instead. This is discussed below.

Suspended sediment

Here the equilibrium requirement concerns the sediment distribution $\zeta(x)$ in the channel, described by (7.35). This equation is already in a form that is independent of time, due to an approximation made earlier concerning its short relaxation timescale. Importantly, the rates of sediment deposition, erosion and lateral diffusion depend only on ζ and h , but not on h_s , therefore a knowledge of l and $h(x)$ is sufficient for determining $\zeta(x)$. This problem consists of Equations (7.35), (7.37)_{3,4}, and boundary condition (7.36)₂.

Bed processes

The remaining primary unknown is $h_s(x)$, which is given by solving Equation (7.34) with $\partial/\partial t$ set to zero. In an equilibrium river, the bed profile h_s (equal to h) ensures that the net flux of sediment falling out of suspension is carried away laterally, as increments in bedload everywhere along the interface (Chapter 6). At first sight, the corresponding problem for our subglacial channel appears to be more complicated because the till also deforms, as shown by the last term of (7.34). But in fact, the bed stress τ_b and diffusivity ϵ here does *not* depend on h_s , as $h \neq h_s$ in this case. Consequently h_s enters this problem only in providing the slope to drive lateral bedload transport, and its determination is greatly simplified. (This is in marked contrast to the river case; Section 6.3.2.) Given l and $h(x)$, we can find τ_b , ϵ and ζ , so (7.34) is a straightforward linear second-order O.D.E. for $h_s(x)$. The boundary conditions are given by (7.36)₁. (In addition, (7.37)₁ is an explicit equation for $h_i(x)$.)

The width problem

Let us denote the *functional form* of $h(x)$ by f , that is,

$$h(x) = h(0) f\left(\frac{x}{l}\right), \quad \text{with } |x| \leq l, \quad h(0) > 0; \quad (7.41)$$

where f is a symmetric function satisfying $f(0) = 1$, $f(\pm 1) = 0$. For the moment, let us assume f is given. So far, our statement is that by imposing the requirement of stationarity to (7.34) and (7.35), $\zeta(x)$, $h_s(x)$ and also the drainage equation can be determined *if* both l and $h(0)$ are prescribed. We have identified one relation between l and $h(0)$, given by (7.39). The question is whether the sediment processes can provide a further constraint.

³Equations (7.39) and (7.40) are the dimensionless form of Equations (4.98), (4.99) and (4.101).

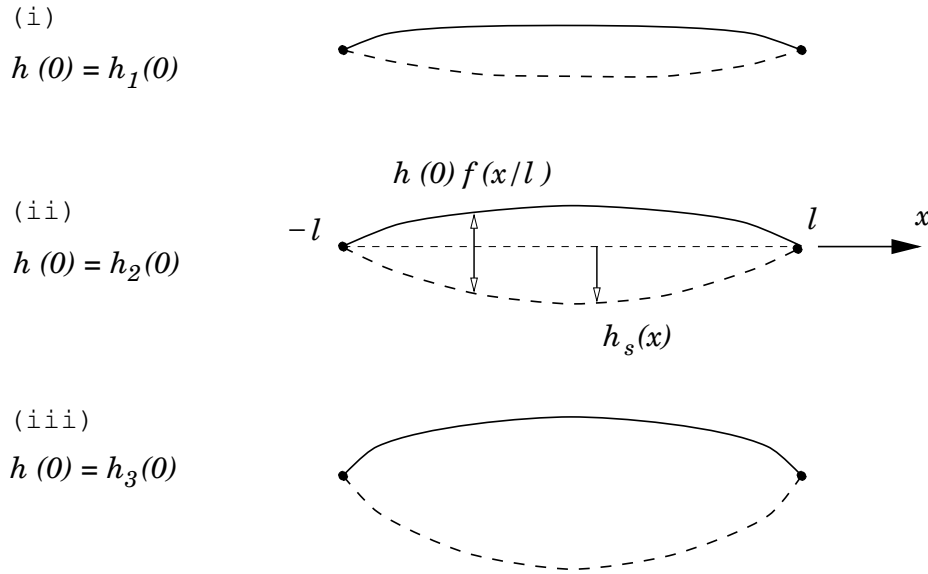


Figure 7.3: If ζ and h_s are stationary, then for given l and function shape f , the bed function $h_s(x)$ depends on the value of $h(0)$. We illustrate this dependence with the sequence of cases shown in (i), (ii) and (iii), for which $h_1(0) < h_2(0) < h_3(0)$.

Here we establish that for a given l , there is a unique value of $h(0)$ that is physically consistent with the condition of *bed equilibrium*. Essentially, if we vary $h(0)$ at fixed l , then in order to keep $\partial\zeta/\partial t = \partial h_s/\partial t = 0$, the bed function $h_s(x)$ has to vary accordingly. This is illustrated in Fig. 7.3. Let us denote the resulting *family* of functions h_s by $g(x, h(0))$. Bed equilibrium requires that (i) there is no incipient slope failure anywhere in $|x| \leq l$, i.e., $\tau_c \geq 0$, and (ii) bedload motion is initiated everywhere in $|x| < l$, such that $\tau_b \geq \tau_c$, where $\tau_b \propto h$. These imply $\tau_c(\pm l) = 0$, therefore $h_s(x)$ must satisfy the bank conditions in (7.38) (Section 6.4.3). Since g is a ‘continuum’ family of functions, we can now argue that out of the many possibilities from g , there is only one function $h_s(x)$ (and hence, one corresponding value of $h(0)$) which displays the required marginal bed slope (see Fig. 7.3). In our original model, formal application of (i) and (ii) would lead to this selection of h_s automatically. Clearly, if τ_c is neglected as in the current model, then one must apply (7.38) as a substitute in order to determine $h_s(x)$.

Finally, we bring back roof equilibrium into our consideration. Comparison of Equations (7.39) and (7.41) leads to

$$f(X) = (1 - X^2)^{1/3}, \quad h(0) = \frac{N_c^2 l^{2/3}}{\Phi}. \quad (7.42)$$

As $h(0)$ is determined by l (by bed equilibrium), but also independently $h(0)$ is related to l via (7.42)₂, the channel width l must depend uniquely on N_c , Φ and other model constants. Thus, Equation (7.40) takes the form $Q = Q(N_c, \Phi, l)$ where $l = l(N_c, \Phi)$.

It follows that a R  thlisberger-type relation of the form $Q = Q(N_c, \Phi)$ is indeed viable for a soft bed wide channel. In addition, the channel geometry — defined by l , $h(x)$, $h_s(x)$ — is uniquely related to the value of N_c (or Q).

In Section 7.3.2, we perform the detailed derivation using the approximate model, with (7.38) replacing (7.37)₂. ($h(\pm l) = 0$ has already been used where we infer $\tau_c(\pm l) = 0$.) Note that there are eight equations for the eight unknowns h , h_s , h_i , l , τ_b , ϵ , ζ , Q , so the problem has not been over-specified.

7.3.2 Steady flow results

Method of solution

With $\partial/\partial t = 0$, the equations in (7.33) to (7.37) reduce upon substitution to

$$(\Phi h)^{3/2} = N_c^3 \sqrt{l^2 - x^2}, \quad (7.43)$$

$$\lambda^2 \Phi^{1/2} \frac{d}{dx} \left(h^{3/2} \frac{d\zeta}{dx} \right) = \frac{\zeta}{\Phi^{1/2} h^{3/2}} - \left(\frac{\Phi h}{2} \right)^{3/2},$$

$$\text{b.c. } h^{3/2} \frac{d\zeta}{dx} = 0, \quad \text{at } x = \pm l, \quad (7.44)$$

$$\Phi^{1/2} \frac{d}{dx} \left[h^{3/2} \frac{d}{dx} \left(\frac{\Phi h_s}{2\sqrt{2}} - \kappa_s \lambda^2 \zeta \right) \right] = \kappa_c (1 - n_s) \frac{N_c^a}{N_\infty^b} \sqrt{l^2 - x^2},$$

$$\text{b.c. } h_s = 0, \quad \text{at } x = \pm l, \quad (7.45)$$

and

$$Q = \Phi^{1/2} \int_{-l}^l h^{3/2} dx, \quad h_i = h - h_s. \quad (7.46)$$

Equations (7.43) and (7.46)₁ lead directly to the results in (7.39) and (7.40), and (7.46)₂ is an explicit expression for h_i . We omit the bank conditions for the moment.

The problems for ζ and h_s can easily be solved by using the change of variable x to θ , where $x = l \sin \theta$ and $-\pi/2 \leq \theta \leq \pi/2$. We have

$$\sqrt{l^2 - x^2} = l \cos \theta, \quad h(\theta) = \frac{N_c^2 (l \cos \theta)^{2/3}}{\Phi}, \quad [h(x)]^{3/2} \frac{d}{dx} \longrightarrow \frac{N_c^3}{\Phi^{3/2}} \frac{d}{d\theta}, \quad (7.47)$$

so (7.44) and (7.45) become

$$\lambda^2 \frac{d^2 \zeta}{d\theta^2} - \frac{\Phi^2}{N_c^6} \zeta = -\frac{\Phi l^2}{2\sqrt{2}} \cos^2 \theta,$$

$$\text{b.c. } d\zeta/d\theta = 0 \quad \text{at } \theta = \pm\pi/2, \quad (7.48)$$

$$\frac{d^2}{d\theta^2} \left(\frac{\Phi h_s}{2\sqrt{2}} - \kappa_s \lambda^2 \zeta \right) = \frac{\kappa_c \Phi (1 - n_s) l^2}{N_c^{3-a} N_\infty^b} \cos^2 \theta,$$

$$\text{b.c. } h_s(\pm\pi/2) = 0. \quad (7.49)$$

The solutions, expressed in terms of the original variable x , are found to be

$$\zeta(x) = \frac{N_c^6 l^2}{4\sqrt{2}\Phi} \left[1 + \frac{1 - 2(x/l)^2}{1 + 4\lambda^2 \Phi^{-2} N_c^6} \right], \quad (7.50)$$

$$h_s(x) = \frac{\kappa_s \lambda^2 N_c^6 (l^2 - x^2)}{\Phi^2 + 4\lambda^2 N_c^6} + \frac{\kappa_c (1 - n_s) l^2}{\sqrt{2} N_c^{3-a} N_\infty^b} \left\{ [\sin^{-1}(x/l)]^2 + \left(\frac{x}{l}\right)^2 - 1 - \frac{\pi^2}{4} \right\}. \quad (7.51)$$

According to (7.50), the predicted sediment distribution ζ is highest at the channel centre and decreases towards the margins. In particular, $\zeta(x)$ exhibits no boundary layer behaviour near $x = \pm l$, even though $\lambda^2 \ll 1$. The ratio $\zeta(\pm l)/\zeta(0)$ ($\approx 2\lambda^2 \Phi^{-2} N_c^6$) is also small; thus although $\zeta(\pm l)$ is non-zero, its value is acceptable. A related problem is that the deposition rate D in (7.44) ($\propto \zeta/h^{3/2}$) is infinite at the margins. Nevertheless, D itself is integrable w.r.t. x , so at the least, the total deposition flux across the channel is bounded. Parker (1978) circumvented this problem by assuming a constant value of ϵ (Section 6.3.1).

The channel width and associated results

We proceed to apply the bank conditions in (7.38). A problem exists because $h_s(x)$, as given by (7.51), has infinite end slopes caused by sediment creep closure — the κ_c -term. However, it has already been pointed out that this term is a poor representation near $x = \pm l$, due to a crack approximation. We therefore suppose that its slope contribution ought to be finite. On this basis, $\kappa_c \ll 1$ allows us to neglect this term, provided that the far field effective pore pressure is not too small, i.e., $\kappa_c N_\infty^{-b} \ll \kappa_s \lambda^2$, or dimensionally, $N_\infty \gtrsim 0.2$ bar. This condition is not too restrictive under typical subglacial drainage regimes.

Putting κ_c to zero, application of (7.38) leads to the width relation

$$l = \frac{\mu_c (\Phi^2 + 4\lambda^2 N_c^6)}{2\delta \kappa_s \lambda^2 N_c^6} \approx \frac{\mu_c \Phi^2}{2\delta \kappa_s \lambda^2 N_c^6}. \quad (7.52)$$

By substituting this into (7.39), (7.51) and (7.40) (again, neglecting λ^2 and κ_c as appropriate), we obtain

$$h(x) = \left(\frac{\mu_c}{2\delta \kappa_s \lambda^2} \right)^{2/3} \left[1 - \left(\frac{x}{l} \right)^2 \right]^{1/3} \Phi^{1/3} N_c^{-2}, \quad (7.53)$$

$$h_s(x) = \frac{\mu_c^2}{4\delta^2 \kappa_s \lambda^2} \left[1 - \left(\frac{x}{l} \right)^2 \right] \Phi^2 N_c^{-6}, \quad (7.54)$$

$$Q = \frac{\pi}{2} \left(\frac{\mu_c}{2\delta \kappa_s \lambda^2} \right)^2 \Phi^3 N_c^{-9} \quad (7.55)$$

respectively. Here, the first two equations define the channel cross section, and the last one constitutes the steady flow relation which we seek. Further elimination of N_c from (7.52) and (7.55) provides an alternative width relation

$$l = \left(\frac{8\delta \kappa_s \lambda^2}{\pi^2 \mu_c} \right)^{1/3} Q^{2/3}, \quad (7.56)$$

which (curiously) is independent of Φ . The fact that l increases with Q is physically reasonable. With the typical parameter values $\mu_c = 0.8$, $\kappa_s \lambda^2 \approx 5.1$, and $\delta = 0.1$, Equations (7.52) to (7.56) lead to the approximate results

$$\begin{aligned} l &\approx \Phi^2 N_c^{-6} \approx Q^{2/3}, & h(0) &\approx \Phi^{1/3} N_c^{-2}, \\ h_s(0) &\approx 3\Phi^2 N_c^{-6}, & Q &\approx \Phi^3 N_c^{-9}. \end{aligned} \quad (7.57)$$

In our original formulation, we had assumed a ‘deep till’ situation, but since then the effect of sediment deformation has been found to be entirely negligible, whether till is deep or not (because $\kappa_c \ll 1$). Hence, the results above are appropriate as long as the channel does not incise down to the bedrock (such as in Fig. 7.1). The necessary condition is that $d/h_0 > h_s(0)$. For instance, d has to be at least several metres when $\Phi \approx N_c \approx 1$, according to (7.57)₃.

Consistency with high aspect ratio assumption

When $\Phi \approx N_c \approx 1$, both l and $h(0)$ are close to unity. The predicted aspect ratio of the channel is then large ($\approx 2/\delta = 20$), and our wide channel approximations are justified. More generally, suppose that we tolerate a worst-case (minimum) aspect ratio k_1 , for which a reasonable value⁴ is $k_1 = 4$. The corresponding dimensionless requirement is then

$$\frac{h(0)}{l} \lesssim \frac{2}{k_1 \delta}, \quad (7.58)$$

and by using (7.52) to (7.56), we obtain

$$Q^{5/9} N_c \gtrsim \frac{k_1 \delta}{2} \left(\frac{\pi}{2}\right)^{5/9} \left(\frac{\mu_c}{2\delta \kappa_s \lambda^2}\right)^{7/9} \quad (\approx 0.21 \text{ when } k_1 = 4). \quad (7.59)$$

This relation defines the domain of validity of our solutions, and is included in Fig. 7.4 a,b (lower red curves, labelled ‘AR’). The scales for Q and N_c are found in (7.29).

Consistency with suspended sediment content

When dimensionally $\zeta(x) \approx h(x)$, the channel is choked up with suspended sediment, and our solutions will no longer be applicable. Therefore, a necessary condition is that $\zeta \lesssim k_2 h$, where k_2 is a small constant between 0 and 1. This leads to a second validity relation. A reasonable worst-case value of k_2 is 0.3.⁵

Due to the functional forms of $\zeta(x)$ and $h(x)$ given in (7.50) and (7.53), it is actually sufficient to impose

$$\frac{\zeta(0)}{h(0)} \lesssim k_2. \quad (7.60)$$

In dimensionless terms this is

$$\frac{\zeta(0)}{h(0)} \lesssim k_3, \quad \text{where } k_3 = \frac{k_2 h_0}{\zeta_0} \quad (\approx 0.1). \quad (7.61)$$

⁴Remember that the error terms are of $O(k_1^{-2})$.

⁵More accurately, turbid flows with $\zeta > k_2 h$ are still possible, but they have significantly different rheology and are described as *hyper-concentrated*. The k_2 value used here is based on a sediment volume fraction of 30% (Baker *et al.*, 1988).

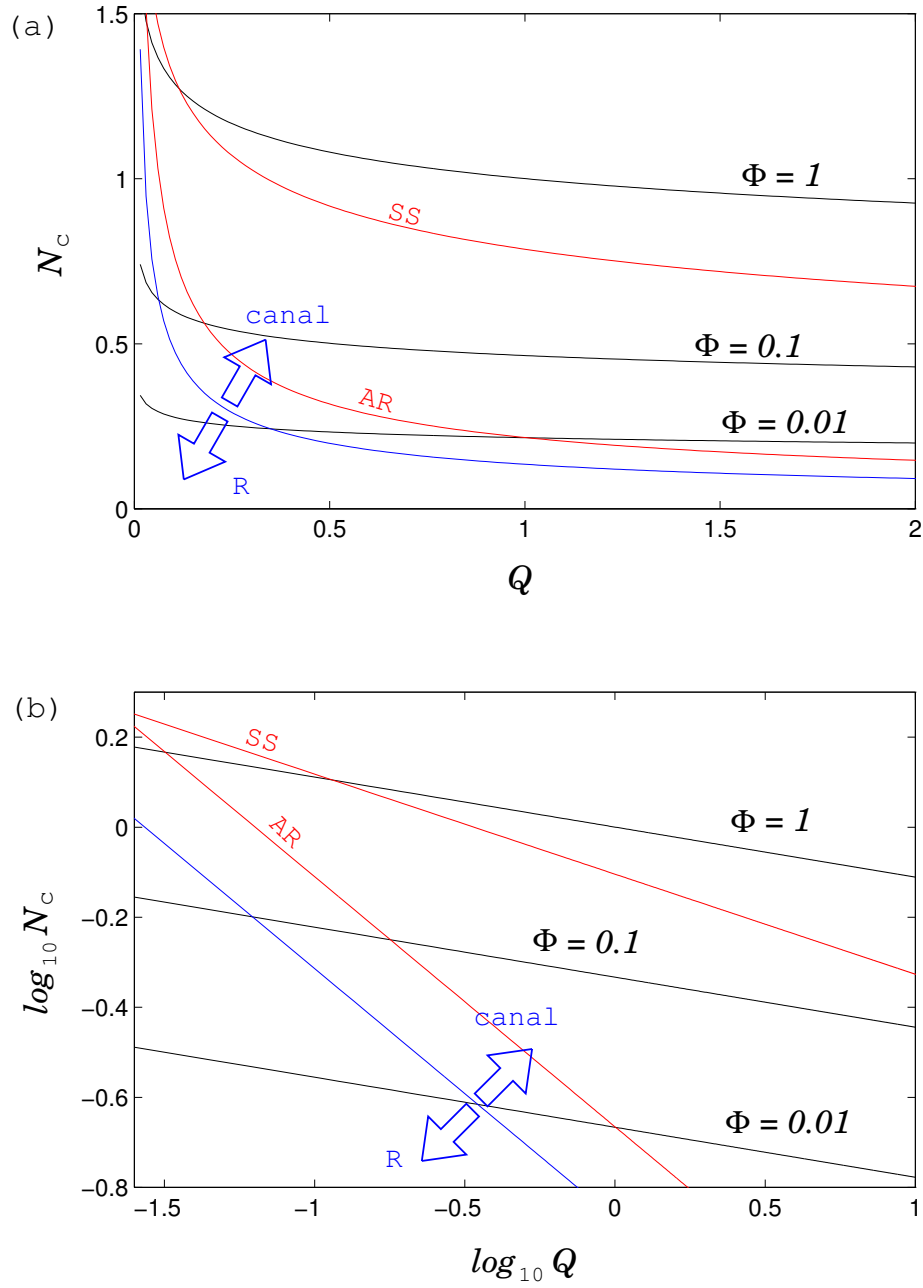


Figure 7.4: Dimensionless steady flow results for sediment-floored wide channels, plotted on the N_c - Q plane in (a) linear scales and (b) logarithmic scales: (i) drainage relations for Φ values 1, 0.1 and 0.01 (black), (ii) the region of validity of our results (between red curves), and (iii) the bounding curve that distinguishes between R -type and canal-type geometries (blue). The scales here are $[N_c] = 5 \text{ bar}$, $[Q] = 30 \text{ m}^3 \text{ s}^{-1}$, and $[\Phi] = 100 \text{ kg m}^{-2} \text{ s}^{-2}$ (from Equation (7.29)). Thus, typically Φ ($\approx \rho_i g \sin \alpha_s / [\Phi]$) is > 0.1 for valley glaciers, and $\lesssim 0.1$ for ice sheets. ($\sin \alpha_s$ is the ice surface slope.)

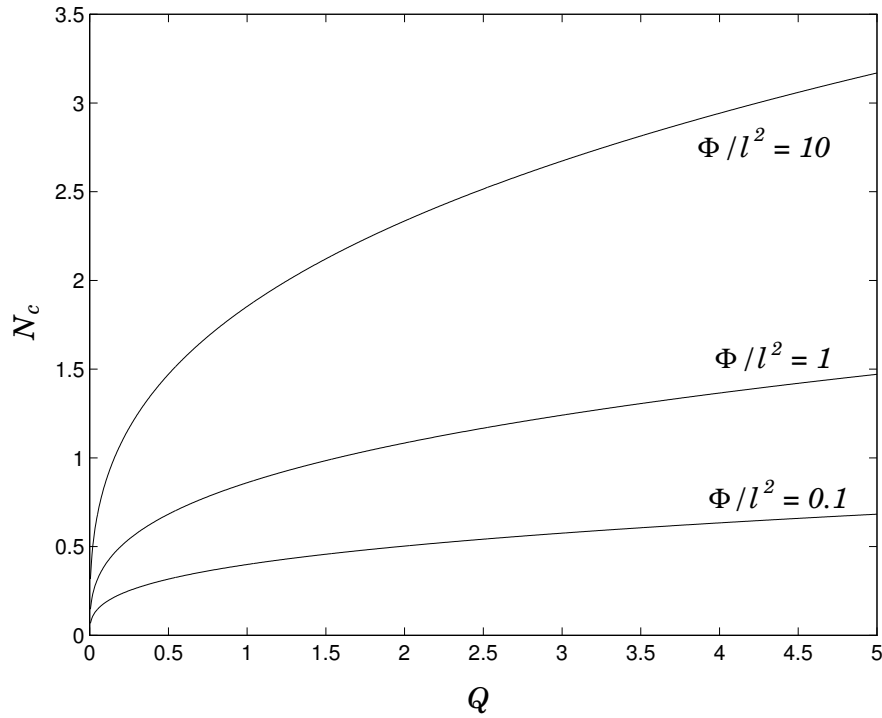


Figure 7.5: *Dimensionless steady flow results for hard bed wide channels, plotted on the N_c - Q plane for values of $\Phi/l^2 = 10$, 1, and 0.1.*

With the aid of Equations (7.55) and (7.56), and $\lambda^2 \ll 1$, we derive

$$Q^{2/9} N_c \lesssim 2^{1/24} k_3^{1/4} \left(\frac{\pi^2 \mu_c}{\delta \kappa_s \lambda^2} \right)^{1/9}. \quad (7.62)$$

This inequality has also been included in Fig. 7.4 a,b (upper red curves, labelled ‘SS’).

Drainage relation: soft bed vs. hard bed

Equation (7.55) encapsulates the steady flow characteristics of our sediment-floored wide channel. Similarly, Equation (7.40) provides the hard bed equivalent — here the width l is unknown, but is essentially held constant as long as the channel is wide, due to $dl/dt \approx 0$; in Section 4.6, we have described a possible mechanism by which l would be able to change in this case.

To repeat, our dimensionless drainage relations are as follows:

$$Q = \frac{\pi}{2} \left(\frac{\mu_c}{2\delta \kappa_s \lambda^2} \right)^2 \Phi^3 N_c^{-9} \quad (\text{soft bed}), \quad (7.63)$$

$$Q = \frac{\pi N_c^3 l^2}{2\Phi} \quad (\text{hard bed, } l \text{ constant}). \quad (7.64)$$

We plot these results respectively in Figs. 7.4 and 7.5, where values of Φ and Φ/l^2 are varied to generate the corresponding drainage surfaces (shown in black). We use

the scale and parameter definitions of (7.29) and (7.30). (A validity relation is also applicable for the hard bed case. It is identical to (7.58) with $h(0)$ taken from (7.42)₂.) The general dimensional form of (7.63) and (7.64) is

$$Q = \frac{2^{18}\pi g R v_s^2}{0.092^2} \left(\frac{\eta_0 D_s \Phi}{\rho_i L N_c^3} \right)^3 \quad (\text{soft bed}), \quad (7.65)$$

$$Q = \frac{\pi \rho_i L N_c^3 l^2}{4 \eta_0 \Phi} \quad (\text{hard bed, } l \text{ constant}). \quad (7.66)$$

In these equations, the imposed hydraulic gradient is given approximately by $\Phi \approx \rho_i g \sin \alpha_s$, taken from (2.20), where $\sin \alpha_s$ is the ice surface slope.

Since we have

$$Q \propto \Phi^{c_1} N_c^{c_2}, \quad \text{where} \quad \begin{cases} c_1 > 0, c_2 < 0 & \text{for soft bed,} \\ c_1 < 0, c_2 > 0 & \text{for hard bed,} \end{cases} \quad (7.67)$$

we can also write

$$\begin{aligned} \frac{\partial Q}{\partial N_c} < 0, \quad \frac{\partial Q}{\partial \Phi} > 0, & \quad (\text{soft bed}), \\ \frac{\partial Q}{\partial N_c} > 0, \quad \frac{\partial Q}{\partial \Phi} < 0, & \quad (\text{hard bed}). \end{aligned} \quad (7.68)$$

These results indicate that the hard bed channel is essentially of R  thlisberger-type (Section 4.6). On the other hand, the soft bed channel has the characteristics of the canals hypothesized by Walder and Fowler (1994), with discharge increasing with water pressure and driving gradient. (Their Equation (4.22) gives $c_1 = 2$, $c_2 = -3$.) According to this model, the distinction between the two *types* of drainage characteristics depends not on the pressure regime (as is proposed by Walder and Fowler, 1994), but on the nature of the underlying substrate. If the channel is underlain by a significant quantity of erodible sediment, then it behaves like a canal. This result is of fundamental importance and is discussed in Section 7.5.

Cross section: R channels vs. canals

We consider exclusively a *soft bed* wide channel. Whether it incises predominantly in the ice or in the sediment may be ascertained by evaluating the ratio $h_s(0)/h(0)$. Equations (7.53) and (7.54) give

$$\frac{h_s(0)}{h(0)} = \left(\frac{\mu_c}{2\delta} \right)^{4/3} \frac{\Phi^{5/3} N_c^{-4}}{(\kappa_s \lambda^2)^{1/3}}. \quad (7.69)$$

If $h_s(0)/h(0) \approx 0$, then the channel cross section is that of an *R* channel incising upwards; but if $h_s(0)/h(0) \approx 1$, the channel resembles a canal incising down. These scenarios are illustrated in Fig. 7.6 a,b. The ratio can also exceed unity, and then the roof extends below the level of the ice-till interface, as shown in Fig. 7.6c. Note that this property is purely *geometrical* and is independent of the drainage characteristics of the channel (always canal-type, in this case).

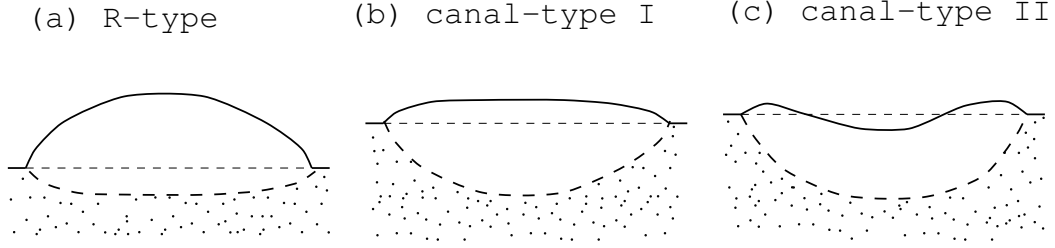


Figure 7.6: *Generic cross sectional shapes, due to (a) $0 \approx h_s(0)/h(0) < 1/2$, (b) $1/2 < h_s(0)/h(0) \approx 1$, and (c) $h_s(0)/h(0) > 1$. (The vertical length scale has been exaggerated for convenience.)*

For a given Φ , we can define a dimensionless critical effective pressure $\tilde{p}_W$, such that an R channel geometry is predicted when $N_c \gg \tilde{p}_W$, and a canal geometry is predicted when $N_c \ll \tilde{p}_W$. According to (7.69), $N_c = \tilde{p}_W$ corresponds $h_s(0)/h(0) = 1/2$. (The subscript W denotes a wide channel.) This is similar to Walder and Fowler's (1994) definition described in Section 2.4.2; their critical value $\tilde{p}$ given by Equation (2.43) is approximately 8 bar.

Equation (7.69) now leads to

$$\tilde{p}_W = \left(\frac{\mu_c}{\delta}\right)^{1/3} \left(\frac{\Phi^5}{2\kappa_s\lambda^2}\right)^{1/12} \approx 1.65\Phi^{5/12}, \quad (7.70)$$

which, unlike Walder and Fowler's $\tilde{p}$ value, is not a model constant. It is also useful to express $\tilde{p}_W$ as a function of Q . We do this by first eliminating Φ between (7.69) and (7.63), and then derive

$$\tilde{p}_W = \left[\frac{1}{2^{12}(\kappa_s\lambda^2)^7} \left(\frac{\delta}{\mu_c}\right)^2 \left(\frac{\pi}{Q}\right)^5 \right]^{1/9} \approx 0.135Q^{-5/9}, \quad (7.71)$$

so that

$$\begin{aligned} Q^{5/9}N_c &\ll 0.135, & \text{for an } R \text{ channel geometry,} \\ Q^{5/9}N_c &\gg 0.135, & \text{for a canal geometry.} \end{aligned} \quad (7.72)$$

This result has been incorporated into Fig. 7.4. The bounding curve (shown in blue) has the same form as Equation (7.59) but lies to its left. It follows that a canal *geometry* is the only possibility within the region of validity of our model.

Recap

The theory that we propose here addresses the steady flow properties of wide channels for both hard bed and sediment bed scenarios. The drainage regime (flow variables and geometry) is found to be uniquely definable given any two of Q , Φ and N_c ; for a hard bed, the channel width l has to be prescribed also. As in Walder

and Fowler's (1994) theory, we predict the possibility of both *R*-type and canal-type *drainage characteristics* (Figs. 7.4 and 7.5) and *cross sections* (Fig. 7.6). In particular, our model indicates that soft bed channels have necessarily the geometry *and* characteristics of canals, regardless of the ice surface slope; this is true within certain validity limits. Hard bed channels belong to the *R* category. These results are not entirely compatible with those of Walder and Fowler (1994) however. They assert that at high values of Φ , soft bed channels can also exhibit *R*-type characteristics when N_c exceeds the critical value $\tilde{p}$; see Fig. 2.14b. Although their model is intended for (near-) circular channels also, it is based on much simpler assumptions and has poorly defined validity limits. Essentially, our current investigation serves to elucidate Walder and Fowler's predictions at the high aspect ratio limit. We provide further discussion in Section 7.5.

7.3.3 Sediment transport

Theoretical rating curves

Having established the equilibrium drainage properties, we can also construct an estimate for the sediment discharge at steady flow. As far as we know, the following derivation produces the first subglacial analogue of a sediment transport rating relation.

We consider a soft bed wide subglacial channel. The total sediment flux Q_T consists of suspended load and bedload. Let us denote these respectively by Q_s and Q_b , then

$$Q_T = Q_s + Q_b, \quad (7.73)$$

and it follows from Section 7.2.1 that (dimensionally)

$$Q_s = \int_{-l}^l \hat{u} \zeta \, dx, \quad Q_b = \int_{-l}^l q_b \, dx. \quad (7.74)$$

By choosing a flux scale $[Q_s]$ where $[Q_s] = u_0 \zeta_0 l_0$ ($\approx 80 \text{ m}^3 \text{ s}^{-1}$), non-dimensionalization of (7.74) yields

$$Q_s \approx \int_{-l}^l (\Phi h)^{1/2} \zeta \, dx, \quad Q_b \approx \kappa_b \int_{-l}^l (\Phi h)^{3/2} \, dx, \quad (7.75)$$

where

$$\kappa_b = \frac{8\rho_w \sqrt{gR} D_s^{3/2} v_s}{0.092 \Phi_0 h_0^2}. \quad (7.76)$$

Under our current scales in (7.29), $\kappa_b \approx 1.95 \times 10^{-3}$, hence the suspended load contribution dominates⁶, and $Q_T \approx Q_s$. (The form of (7.76) indicates that Q_b can become important, but only at very low flow gradients.) Substitution for h , ζ and

⁶This is consistent with the fact that bedload transport is generally neglected in channelized drainage considerations.

l from Equations (7.50), (7.52) and (7.53) (with $\lambda^2 \ll 1$, as before), with the use of (7.55), now leads to

$$Q_T \approx 2^{-1/2} \Phi^{4/3} \left(\frac{Q}{\pi} \right)^{13/9} \left(\frac{\mu_c}{\delta \kappa_s \lambda^2} \right)^{4/9} \int_{-1}^1 (1 - X^2)^{7/6} dX, \quad (7.77)$$

$$Q_T \approx 2^{5/6} N_c^4 \left(\frac{Q}{\pi} \right)^{17/9} \left(\frac{\mu_c}{\delta \kappa_s \lambda^2} \right)^{-4/9} \int_{-1}^1 (1 - X^2)^{7/6} dX. \quad (7.78)$$

(Numerical integration gives $\int_{-1}^1 (1 - X^2)^{7/6} dX \approx 1.275$.) The validity of these relations awaits confirmation by field observation.

Sediment source: the downstream problem

Given a steady flow scenario, one might ask: ‘Where does the sediment come from in the first place?’ We can investigate this question by considering the effect of downstream variation in the model. Taking our original equations from Section 7.2.2, with $\partial/\partial t = 0$, $\tau_c = 0$, substitution of (7.17) into (7.19) and (7.20) into (7.18) leads to

$$\begin{aligned} \frac{dQ}{ds} &= \frac{r N_c^3}{\beta_i} \int_{-l}^l \sqrt{l^2 - x^2} dx \\ &= \frac{\pi r N_c^3 l^2}{2\beta_i}, \end{aligned} \quad (7.79)$$

and

$$\begin{aligned} \frac{\partial}{\partial x} \left[\tau_b^{3/2} \frac{\partial h_s}{\partial x} - \kappa_s \lambda^2 \epsilon \frac{\partial \zeta}{\partial x} \right] + \frac{\partial}{\partial s} \left\{ \kappa_s \left[\beta_s \hat{u} \zeta - \xi^2 \lambda^2 \epsilon \frac{\partial \zeta}{\partial s} \right] + \frac{\xi \mu_c}{\delta} \tau_b^{3/2} \right\} \\ = \frac{\kappa_c (1 - n_s) N_c^a}{N_\infty^b} \sqrt{l^2 - x^2} \end{aligned} \quad (7.80)$$

respectively. By integrating the latter equation across the channel, with $\xi/\delta \ll \kappa_s \beta_s$ (and applying the appropriate zero-flux conditions at the ends), we obtain

$$\frac{d}{ds} \int_{-l}^l \hat{u} \zeta dx = \frac{\pi \kappa_c (1 - n_s) l^2 N_c^a}{2 \kappa_s \beta_s N_\infty^b} \approx \frac{dQ_T}{ds}. \quad (7.81)$$

Thus clearly, the (slow) increment of sediment load downstream is due to sediment creep closure. (On the other hand, we showed in Section 7.3.2 that this process has little effect in determining the equilibrium channel cross section.) In these equations, l is given by (7.52), with Ψ replacing Φ , i.e.,

$$l = \frac{\mu_c}{2 \delta \kappa_s \lambda^2 N_c^6} \left(\Phi + \nu \frac{dN_c}{ds} \right)^2. \quad (7.82)$$

One can in principle determine all the subglacial channel variables (in the s -direction), by solving a coupled O.D.E. problem defined by Equations (7.79), (7.81), (7.82) and (7.77) (again with Φ replaced), with boundary values for Q , Q_T and N_c (e.g. $N_c(\text{snout}) = 0$). However, this is deferred to further work because of space.

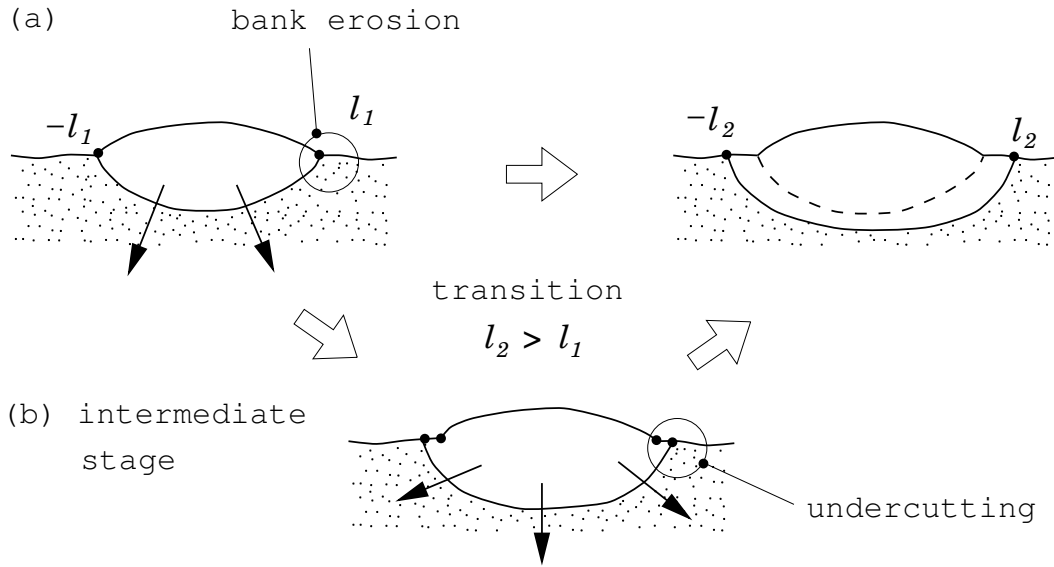


Figure 7.7: Widening process in a soft bed wide subglacial channel. (The vertical length scale has been exaggerated.)

7.4 Width evolution: soft bed

We have not addressed the stability of the equilibrium state itself. This would involve constructing a phase plane (more precisely, a phase ‘space’ with axes Q , N_c , l ... etc.) similar to the one in Section 2.4.1 for the Nye model. Essentially, a set of steady flow conditions can be identified as a critical point on the phase space, and one is interested in the properties of the phase trajectories around it. The first obstacle in such analysis lies in the derivation of an equation to describe *width evolution*. In the most general terms this problem is quite difficult, thus here we shall limit ourselves to a basic, introductory investigation. The following sections examine the mathematical description of an expanding channel width ($\partial l / \partial t > 0$) in the deep till and shallow (disjoint) till situations respectively. In the latter case, we shall concentrate on deriving an approximate width equation for jökulhlaups. The aim is to connect up the flood model of Part I with our current drainage model.

7.4.1 Mathematical description

Back in Section 4.6, we concluded that the ice processes alone cannot result in efficient widening of the channel. If the bed consists of till, then a transition between two equilibrium states — induced for example by changes in Q , N_c , or both — would be accompanied by a change in l , as implied by Equations (7.52) and (7.56). Suppose l increases, then the transition must cause a net removal of sediment from the margins. This can occur via a *bank erosion* mechanism, and is illustrated in Fig. 7.7a.

We are interested in the time-dependent description of this process. As a minimal requirement, the corresponding equations ought to indicate the possibility of propagating free boundaries, located at $x = \pm l$. To ascertain this, we investigate a reduction of the model in (7.33) to (7.38). Notably, we neglect till deformation, and $\beta_i \gg 1$ implies that the roof remains passive, so h_i can be taken as a prescribed function of x . Therefore, the basic equations are

$$\begin{aligned} (1 - n_s) \frac{\partial h_s}{\partial t} &= \Phi^{1/2} \frac{\partial}{\partial x} \left[h^{3/2} \frac{\partial}{\partial x} \left(\frac{\Phi h_s}{2\sqrt{2}} - \kappa_s \lambda^2 \zeta \right) \right], \\ \lambda^2 \Phi^{1/2} \frac{\partial}{\partial x} \left(h^{3/2} \frac{\partial \zeta}{\partial x} \right) &= \frac{\zeta}{\Phi^{1/2} h^{3/2}} - \left(\frac{\Phi h}{2} \right)^{3/2}, \\ \text{b.c.s : } h_s &= 0, \quad h^{3/2} \frac{\partial \zeta}{\partial x} = 0, \quad \text{at } x = \pm l, \end{aligned} \quad (7.83)$$

where

$$h = h_i + h_s. \quad (7.84)$$

As shown in Fig. 7.7b, bank erosion clearly involves *undercutting* at the ice-till interface. Without loss of generality, the problem may therefore be further simplified by assuming $h_i(x) = 0$, and $h(x) \equiv h_s(x)$. We also discard the l.h.s. of (7.83)₂, thus neglecting possible boundary layers of ζ . The solution of that equation is then

$$\zeta \approx \frac{\Phi^2 h^3}{2\sqrt{2}}. \quad (7.85)$$

We use this outer approximation because it will not affect the particular model property that is being investigated.

Substitution of ζ into the rest of (7.83) now produces

$$\begin{aligned} 2\sqrt{2}(1 - n_s) \frac{\partial h_s}{\partial t} &= \Phi^{3/2} \frac{\partial}{\partial x} \left[h_s^{3/2} (1 - 3\kappa_s \lambda^2 \Phi h_s^2) \frac{\partial h_s}{\partial x} \right], \\ \text{b.c.s : } h_s &= 0, \quad h_s^{7/2} \frac{\partial h_s}{\partial x} = 0, \quad \text{at } x = \pm l. \end{aligned} \quad (7.86)$$

($\kappa_s \lambda^2 (> 0)$ is an $O(1)$ constant.) Given an appropriate initial condition, this constitutes a non-linear diffusion problem for $h_s(x, t)$. Here the diffusion coefficient is positive where $h_s (> 0)$ decays to less than $(3\kappa_s \lambda^2 \Phi)^{-1/2}$. It is also degenerate at the margins since h_s vanishes there. (The second boundary condition should therefore be replaced by $h_s^{3/2} \partial h_s / \partial x = 0$.) This is characteristic of the class of parabolic equations that describes the spreading of a viscous droplet under gravity (e.g. Fowler, 1997). Thus indeed, the solution of (7.86) has the potential of exhibiting an $O(1)$ tip advance velocity. (Viewed upside down, our channel is analogous to the droplet cross section.) The short evolution timescale ($t_s \approx 1.4$ hrs) indicates that the widening mechanism could be efficient. We note that this type of equation is also found in the problem of (2-D) ice sheet evolution (e.g. p. 85; Fowler, 1997).

Although it is possible to solve Equation (7.86), in order to derive a accurate estimate for dl/dt it is necessary to use the full model in Section 7.2.2, with as few

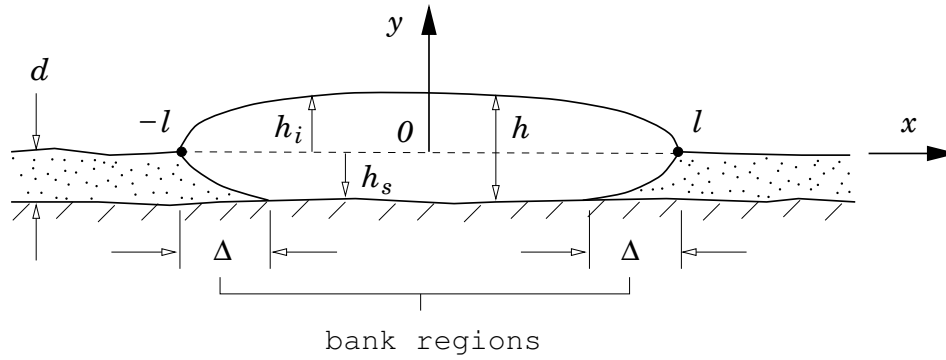


Figure 7.8: *Schematic diagram of the flood channel described in Section 7.4.2. The variables here are in dimensional form.*

approximations as possible (especially at the margins). Extensive numerical computation is then unavoidable. In this connection, Kovacs and Parker (1994) have used their vectorial bedload formulation (described in Section 6.4) to simulate the widening process in a gravel river channel. Their work provides a valuable starting point for further investigation.

7.4.2 Jökulhlaups

In this section we attempt to justify the use of the width equation (2.57) in our model of jökulhlaups. We take the full dimensionless model equations (7.17) to (7.25). During a flood, the channel is envisaged to be wide, and incise (for most of its width) down to the base of the till (Section 2.5). It is also far from equilibrium, with melting dominating closure, except at the very end of the flood when the flow rate recedes. Given these conditions, we can write

$$h \approx h_i + d_0 \quad (\gg 1), \quad (7.87)$$

where $d_0 (= d/h_0)$ is the dimensionless till thickness; see Fig. 7.8. It follows that $\partial h_i / \partial t \approx \partial h / \partial t$, and that both τ_c and the sediment closure (κ_c) term may be neglected. If we also ignore downstream variations (following Section 2.5), then the model becomes

$$\frac{\partial h_i}{\partial t^*} = (\Phi h)^{3/2} - N_c^3 \sqrt{l^2 - x^2}, \quad (7.88)$$

$$(1 - n_s) \frac{\partial h_s}{\partial t^*} = \beta_i \left\{ \frac{\partial}{\partial x} \left[\frac{(\Phi h)^{3/2}}{2\sqrt{2}} \frac{\partial h_s}{\partial x} \right] + \kappa_s \left[\frac{(\Phi h)^{3/2}}{2\sqrt{2}} - \frac{\zeta}{\Phi^{1/2} h^{3/2}} \right] \right\}, \quad (7.89)$$

$$\frac{\partial \zeta}{\partial t^*} = \frac{\beta_i}{\beta_s} \left[\lambda^2 \Phi^{1/2} \frac{\partial}{\partial x} \left(h^{3/2} \frac{\partial \zeta}{\partial x} \right) + \frac{(\Phi h)^{3/2}}{2\sqrt{2}} - \frac{\zeta}{\Phi^{1/2} h^{3/2}} \right], \quad (7.90)$$

$$Q = \Phi^{1/2} \int_{-l}^l h^{3/2} dx, \quad (7.91)$$

in which $N_c(t^*)$ is obtained via a lake continuity consideration (Section 2.3.5). t has been re-scaled with $1/\beta_i$, i.e., $t^* = t/\beta_i$, so that the timescale is now associated with roof evolution, given by $t_s\beta_i$ (≈ 12 days). The boundary conditions are

$$h_s = 0, \quad h^{3/2} \frac{\partial \zeta}{\partial x} = 0, \quad \text{at } x = \pm l \quad (7.92)$$

as before. Upon further simplification, these equations ought to reduce to the ‘lumped’ model in (2.61)_{1,3}. It is immediately obvious that on removing x -dependence, Equation (7.88) leads directly to (2.61)₁. It remains for us to derive the evolution equation for l .

Width equation

Since we have a ‘disjoint’ shallow till, decrease of l results from sediment creep closure in the *lateral* direction (Chapter 5). In Section 7.2.1 (Equation (7.14)), we have already introduced a closure velocity

$$w_l \approx \frac{A_T N_c^a \sqrt{dl}}{10 N_\infty^b}. \quad (7.93)$$

Although this has a slightly different form from (2.56), non-dimensionalization leads to

$$\frac{dl}{dt^*}_{(-)} \approx -\kappa_l N_c^a N_\infty^{-b} \sqrt{dl} \quad (7.94)$$

(subscript $_{(-)}$ denotes a reduction in l), where $\kappa_l (= \beta_i \kappa_c \delta^{3/2}/5 \approx 2 \times 10^{-3})$ is small. Therefore, this closure term is negligible as in Section 2.5.

Next we consider bank erosion, which causes an increase in l . In the central part of the channel $h_s \equiv d$ (constant); there is no lateral bedload transport because the bed here is non-erodible. In fact with $h \gg 1$, the water column can support an arbitrary amount of suspended sediment. Under this condition the transport capacity of bedload is insignificant.

In contrast, the till has not been eroded to its base close to the margins. We can seek a bank velocity estimate by using (7.89) and (7.90), which are applicable in these regions. These equations respectively describe bed evolution and the suspended sediment distribution. In particular, the former is responsible for determining the width (and the profile) of the ‘bank regions’⁷. Without going into the detailed solution, let us suppose (very crudely) that the resulting bank profiles have width Δ , where Δ is some constant multiple of d . This is illustrated in Fig. 7.8.

Integration of (7.90) w.r.t. x , together with the use of (7.92)₂, now leads to

$$\frac{dZ}{dt^*} \approx \int_{-l}^{-(l-\Delta_0)} + \int_{l-\Delta_0}^l \frac{(\Phi h)^{3/2}}{2\sqrt{2}} - \frac{\zeta}{\Phi^{1/2} h^{3/2}} dx, \quad (7.95)$$

⁷This moving-boundary problem is difficult, and is closely related to the description in Section 7.4.1.

where

$$Z = \int_{-l}^l \zeta \, dx, \quad \Delta_0 = \frac{\Delta}{l_0} \quad (\ll l). \quad (7.96)$$

Since Z represents essentially the total sediment volume in transport (per unit length of channel), Equation (7.95) describes the rate at which till sediments are incorporated into the flow. It follows that by using mass conservation, we can write

$$\frac{dZ}{dt^*} = \frac{2d(1-n_s)}{\zeta_0} \frac{dl}{dt^*_{(+)}}. \quad (7.97)$$

(Subscript $_{(+)}$ denotes widening.) With $h \gg 1$, $\Phi \sim 1$, and $\zeta \sim h$ (to preclude a hyper-concentrated flow), we obtain

$$\frac{dl}{dt^*_{(+)}} \approx \frac{\zeta_0}{d(1-n_s)} \int_{l-\Delta_0}^l \frac{(\Phi h)^{3/2}}{2\sqrt{2}} \, dx \approx \frac{(\Phi \bar{h})^{3/2} \zeta_0}{2\sqrt{2}(1-n_s)l_0} \left(\frac{\Delta}{d} \right), \quad (7.98)$$

where $\Delta/d = O(1)$, and $\bar{h}(t)$ can be taken as the mean (or lumped) value of the channel depth. Equation (7.98) has the same functional form as Equation (2.55), and this completes our derivation. Note that if we rewrite (7.98) in the dimensional form used in Chapter 2, then the corresponding erosion coefficient K_l is given by

$$K_l = \frac{\rho_s \zeta_0}{2\sqrt{2}(1-n_s)t_s \beta_i} \left(\frac{\rho_w f^*}{8\Phi_0 h_0} \right)^{3/2} \left(\frac{\Delta}{d} \right). \quad (7.99)$$

Under the current scales, we find $K_l \approx 5 \times 10^{-4} (\Delta/d) \, \text{kg m}^{-5} \text{ s}^2$. This is in reasonable order-of-magnitude agreement with the calibrated value determined in Section 2.5 ($1.6 \times 10^{-4} \, \text{kg m}^{-5} \text{ s}^2$).

7.5 Discussion

In this chapter, a generalized wide channel drainage model has been proposed, based on the ingredients detailed in Chapters 3 to 6. A novel feature of this model is that it takes into account the cross-stream (x -) dependence of the individual ice/sediment processes within the channel, thus allowing its cross sectional shape to evolve freely with time. Importantly, this eliminates the need to presuppose a circular or semi-circular geometry (as in the classical Röthlisberger-Nye theory; Chapter 2), or the bed incision geometry (as in the Walder-Fowler theory; 1994).

A direct consequence is that the channel shape is self-determined at steady flow. In Section 7.3, we have elucidated the precise process interactions required to maintain this condition, for both cases of a hard bed and a soft (erodible, deformable) sediment bed. The use of appropriate equilibrium constraints leads to a unique model solution and enables the derivation of a drainage relation between water flux Q and effective pressure N_c .

1. Equilibrium soft bed channels

In a soft bed channel, a melting-closure balance at the roof determines the total depth $h(x)$ (Chapters 3 and 4), and similarly an erosion-deposition-lateral bedload balance gives us the bed position $h_s(x)$ (Chapter 6). A bank condition is also necessary in order to determine the equilibrium width l and complete the cross section specification. A technical detail concerning this condition is that it is applicable only if the model supports a sufficient number of degrees of freedom, as in the present case (Section 7.3.1). (A counter example is given in Section 6.3.3, where a similar bedload formulation as the one used here leads to an infinite bank slope in Parker's (1978) river model, disregarding the physics of incipient particle motion.)

Having derived the steady flow solutions, it is found that the corresponding channel has the geometry and drainage characteristics of *canals*, summarized in Figs. 7.4, 7.6b,c. The drainage relation has the form

$$Q \propto \Phi^3 N_c^{-9} \quad (7.100)$$

(from Equation (7.65)). Deformation due to sediment creep plays a negligible rôle in determining these cross section based properties, because the erosion/deposition timescale is short (giving $\kappa_c \ll 1$). This is where our theory differs fundamentally from that of Walder and Fowler (1994), who assert an erosion-creep balance for the bed instead. However, we find that the small (but non-zero) creep flux into the channel does contribute to the overall sediment budget, as it has to be carried away downstream, adding to the total transport load. The channel cross section has to adjust accordingly (Section 7.3.3).

We also derive a theoretical sediment rating curve, of the form $Q_T \propto \Phi^{4/3} Q^{13/9}$ (Equation (7.77)). Remarkably, the power law dependence predicted here is in excellent agreement with the field-estimated temporal discharge and suspended sediment load of the 1972 and 1976 Skeiðará jökulhlaups. The data-calibrated Q -exponent is approximately 1.4 (see Figs. 3 to 5, Tómasson, 1980). This is an indication that the suspension dynamics in our model may at least be qualitatively realistic, even though flood drainage is far from equilibrium and the actual transport load would incorporate a large range of particle sizes. Section 7.3.3 provides a framework for extended modelling.

It is important to point out that our model equations have been based on some questionable assumptions, particularly concerning the ice and sediment rheology. A main criticism would be that the non-linear closure terms are constructed from the results of linearized problems (Chapters 3 and 5). Numerical solution is required to justify such heuristic extensions. In addition, we assume a low permeability till ($k_T \lesssim 10^{-16} \text{ m}^2$), which demands the presence of a high fraction of silt or finer particles, and there are other limitations related to whether the till is 'deep' or 'shallow'. Nevertheless, these do not detract from the general statement given above, regarding how the balance of processes is achieved within the sediment-floored channel, and the subsequent drainage characteristics. For instance, the ellipsoidal shape of the ice closure velocity is not critically important. Rather, it is the functional dependence

$\hat{w}_i \propto N_c^3 l$ that is responsible for the form of Equation (7.39). This dependence is plausible and consistent with Nye's closure formulation (Equation (2.2)). We have also shown that sediment creep is negligible when the effective till pressure is greater than a fraction of a bar, which is normally the case.

2. Equilibrium hard bed channels

The fact that the channel is underlain by sediment is crucial in the preceding description. If we replace the soft bed with a hard bed, then there is no longer an equilibrium constraint on the channel width, and the drainage relation given by (7.66) has the form

$$Q \propto \Phi^{-1} N_c^3 l^2, \quad (l \text{ constant}), \quad (7.101)$$

predicting Röthlisberger-type characteristics (Section 4.6, and Fig. 7.5). We propose that l will depend on evolution history. This implies that two hard bed channels of equal discharge can differ substantially in water pressure if their widths are different. This has been pointed out by Hooke *et al.* (1990).

3. R channels vs. canals

An important consequence of the current theory is that the drainage type is solely dependent on the nature of the underlying substrate. We predict a one-to-one correspondence between R -type (canal-type) characteristics and a hard bed (soft bed) substrate. (The channel geometry turns out to be irrelevant in this connection.) On the one hand, this is very much in favour of Walder and Fowler (1994), who first proposed the possibility of soft bed channelized drainage exhibiting $\partial Q / \partial N_c < 0$. On the other hand, this raises the question of whether deformable beds can really support *both* R channel and canal drainage. This 'dual' prediction by Walder and Fowler hinges on the net-erosion-creep balance which they assume for the bed incision, but we have shown that this balance is inappropriate. At this stage however, our model is heavily restricted by a high aspect ratio assumption, so we cannot rule out the possibility of soft bed R -type characteristics.

So far, we have considered exclusively an idealized hard bed, or a deep till bed. The latter is realistic for certain glaciers — ice streams in particular (e.g. Ice Stream B, Antarctica; Alley *et al.*, 1986, 1987) — but the intermediate 'disjoint' case is also conceivable and is probably much more common, especially underneath alpine glaciers where sediment is available in thinner layers. Neither the theory presented here nor Walder and Fowler's theory can ascertain the property of the corresponding drainage channels. If these are wide, they are likely to become isolated over a timescale of days or longer, due to sediment pinch-out at the margins (Section 5.7; and Alley, 1992). Then, we suppose that our hard bed results are applicable.

4. Lateral evolution: outlook

The current theory has indicated how the width of our subglacial wide channel can reduce over time: (i) by ice closure over a hard bed (see Equation (4.92)₁), and (ii) by in-creep of a shallow till layer (Equation (7.14)). However, the mechanism of width expansion is perhaps not as well understood. Our initial conclusion is that

ice melting is relatively inefficient (Chapter 4), and that bank erosion would be the dominant process for a soft bed. An exploratory investigation of this process is provided in Section 7.4. Although we are able to verify (very crudely) the jökulhlaup model of Chapter 2, it is clear that there are still major uncertainties concerning the mathematical description of sediment–flow interaction. This difficulty has to be overcome if one were to derive a realistic bank velocity estimate.

Plate 4 goes here.

Chapter 8

Conclusions

This thesis has examined theoretically the process of channelized water drainage underneath glaciers and ice sheets. We substantiate the classical models developed by Röthlisberger (1972) and Nye (1976), which consider water flow in a channel overlying a rigid, impermeable bed, to the case where the channel is underlain by subglacial till and the flow supports a sediment load. The channel is assumed to be wide and low. As such, this work is very much in the spirit of a recent theory proposed by Walder and Fowler (1994), but here we provide a much more detailed description of the ice and sediment processes within the channel. Time-dependent drainage has also been considered in the context of jökulhlaup mechanics. Through these, we are able to elucidate some of the intimate interactions that exist between channelized drainage and sediment transport, which have received relatively little attention. On a larger scale, these interactions have far-reaching implications on the hydrology, sedimentology and dynamics of ice masses.

In Part I (Chapter 2), we introduce a primitive drainage model to address various problems concerning the simulation of Grímsvötn jökulhlaups, using the 1972 event as a typical example. These catastrophic flood events can be explained in terms of an instability of channelized flow, which arises from a lake-coupling feedback, and the competition between ice melting and ice closure (Nye, 1976). Our mathematical description extends this fundamental concept by allowing an evolving channel depth, and also an independently evolving channel width caused by lateral erosion and creep of a thin till layer. The cross section aspect ratio can therefore vary with time.

The presence of till sediments is crucial in this model. In particular, we demonstrate how rapid widening of the flood channel can lead to a sudden roof collapse, coincident with the abrupt recession observed after peak discharge. This mechanism enables us to produce an improved simulated flood hydrograph, and self-consistently, a plausible order-of-magnitude estimate for the total sediment yield. We then propose a tentative explanation for the observed flood initiation lake-level, based on the steady flow theory of Walder and Fowler (1994). Essentially, we associate ‘seal breaking’ with a transition from Darcy leakage to an incipient canal that completes the flood path leading from Grímsvötn. Again, this process involves some kind of erosion of the basal till.

The nature of the subglacial substrate has an equally profound effect in governing the properties of the channel at steady flow. In Part II, we seek an equilibrium theory for wide channels by refining our earlier drainage model. We investigate detailed mathematical models for each of the four ice-sediment processes — ice melting, ice closure, sediment creep, sediment erosion and deposition — in turn, taking into account their cross-stream variation within the channel (Chapters 3 to 6). Importantly, we identify the precise balance that is required to maintain the channel in equilibrium, and show that there is indeed a drainage relation analogous to that derived by Röthlisberger (1972), for both cases of a soft bed and a hard bed (Chapter 7). Hard bed channels are predicted to behave like R channels, with water pressure p_c decreasing with increasing water flux Q , favouring an arborescent drainage network, whereas soft bed channels have the reverse characteristics, typical of the distributed canal drainage proposed by Walder and Fowler (1994). Although this theory is somewhat limited by a high aspect ratio assumption, it provides a rigorous justification for the possibility of a canal-type pressure–flux relation (where $\partial Q/\partial p_c > 0$). This is currently thought to be a crucial ingredient in various large-scale instabilities associated with deformable beds, such as ice sheet surges and ice stream formation (Fowler and Johnson, 1995, 1996). Having said this, we point out that there has so far been no direct evidence of canals, so their existence remains controversial. Engelhardt and Kamb (1997) have recently shown that a canal drainage system is compatible with borehole field data obtained on Ice Stream B, West Antarctica.

A number of possibilities for further investigation arise from this work. A significant difference between Walder and Fowler’s (1994) theory and ours is that they indicate the possibility of both R channels and canals for a soft bed, with a certain (constant) effective pressure demarcating the two regimes. This feature is invoked in the seal breaking argument of Part I, but is not predicted in Part II. Moreover, neither theory is strictly applicable at the very low discharge conditions at flood initiation. Clearly, the process of incipient channelization at the ice-till interface ought to be investigated. This might involve a combination of a flow instability similar to that considered by Walder (1982) and a piping mechanism (Jones, 1981).

There are other questions concerning Grímsvötn jökulhlaups, especially in relation to flood initiation. In Chapter 2 we note that the simulated hydrographs are very sensitive to the initial conditions which we prescribe; this is also true in Nye’s model. If our models do indeed provide a reasonable description, then this suggests that the actual flood events are also sensitive to the conditions at initiation (the incipient channel geometry for example). Yet, the observed flood behaviour is well delineated by the Clague-Mathews relation (Equation (1.1); see Fig. 16, Björnsson, 1992). The origin of this global description is unclear. In addition, the recent jökulhlaup in November 1996 was triggered at an anomalously high lake-level (≈ 1510 m a.s.l.; Einarsson *et al.*, 1997a). At present, there are no models for the mechanics of this type of volcanically-triggered flood.

The process of sediment entrainment by channelized drainage has been considered by various workers. Notably, Collins (1996) has proposed a conceptual grid-based

model to address the way channels can capture sediment by lateral migration; and N. Arnold (personal communication, 1998) has attempted to simulate numerically the (seasonal) sediment yield of the Haut Glacier d'Arolla, Switzerland, by adding an erosion term to Spring and Hutter's (1981b) equations. However, these models lack a local description of how the sediment is eroded, deposited and interacts with channel evolution. Our drainage model in Chapter 7 provides a first description of this type. In relation to this, we have established an understanding of the basic sediment transport characteristics of soft bed channels. This aspect is open to further examination. For instance, we have mentioned briefly the possibility of a meandering or a braided network (Chapter 2). We can also consider the effect of 'armouring', the process whereby the channel bed is prevented from erosion due to the deposition of coarse clasts. These are common phenomena encountered in drainage over mobile beds (e.g. rivers). Finally, we have not investigated whether our theory can be related to the formation of eskers, which are sinuous ridges of sediment deposited by channelized drainage. Obviously, these extensions will require further refinement to the model, in order that it can describe both time-dependent and downstream effects accurately.

Appendix A

Equation (3.49)₁, with $N_c = 0$

This problem amounts to solving the equation

$$\oint_L \frac{f_1(\zeta)}{\zeta - x} d\zeta = 0, \quad \text{for } x \in L, \quad (\text{A.1})$$

where $f_1 = \hat{w}'$. According to Tricomi (1957), this eigen-problem has solutions of the form

$$f_1(x) = c_1(l^2 - x^2)^{-1/2}, \quad (\text{A.2})$$

where c_1 is a constant. Thus, unless $c_1 \equiv 0$, we have either $f_1 > 0$ or $f_1 < 0$ in $|x| \leq l$, such that

$$\hat{w}(+l) - \hat{w}(-l) \quad \left(= \int_L f_1(x) dx \right) \quad \neq 0 \quad (\text{A.3})$$

— which contradicts the given conditions $\hat{w}(\pm l) = 0$. It follows that the appropriate solution has $c_1 \equiv 0$, therefore $\hat{w}(x) = 0$ on L .

(Note: the result (A.2) can be arrived at by considering the Plemelj properties of the function

$$F_1(z) = \frac{1}{2\pi i} \int_L \frac{f_1(\zeta)}{\zeta - z} d\zeta, \quad (\text{A.4})$$

sectionally holomorphic in the plane cut along L . Equation (A.1) implies that F_1 is a basic Plemelj function (of the form $1/\sqrt{z^2 - l^2}$) for this domain. See also the next problem.)

Equation (3.51)_{1,2}, with $u_b = 0$

The situation here is similar to the one above. We have

$$\begin{aligned} 0 &= \oint_{L'} \frac{f_2(\zeta)}{\zeta - x} d\zeta, \quad \text{for } x \in L'; \text{ (cf. Equation (A.1))}, \\ \frac{d\hat{v}}{dx} &\propto \int_{L'} \frac{f_2(\zeta)}{\zeta - x} d\zeta, \quad \text{for } x \in L, \end{aligned} \quad (\text{A.5})$$

in which $f_2 = \tau_b$, and the end conditions are $\hat{v}(\pm l) = 0$. If we define the function

$$F_2(z) = \frac{1}{2\pi i} \int_{L'} \frac{f_2(\zeta)}{\zeta - z} d\zeta, \quad (\text{A.6})$$

sectionally holomorphic in the plane cut along L' (with $F_2 \sim 1/z$ as $|z| \rightarrow \infty$), then (A.5) written in Plemelj form is

$$\begin{aligned} 0 &= F_2^+ + F_2^-, & \text{for } x \in L', \\ \frac{d\hat{v}}{dx} &\propto F_2(x), & \text{for } x \in L. \end{aligned} \quad (\text{A.7})$$

The first equation here implies that $F_2(z)$ is a basic Plemelj function, i.e., $1/\sqrt{z^2 - l^2}$. Choosing the branch where $F_2 \rightarrow 1/z$ at ∞ , we find

$$F_2(x) = \frac{-i}{\sqrt{l^2 - x^2}}, \quad \text{for } x \in L. \quad (\text{A.8})$$

More generally, we can let $F_2(x) = c_2/\sqrt{l^2 - x^2}$ where c_2 is a real constant. Equation (A.7)₂ now takes the form $d\hat{v}/dx \propto c_2/\sqrt{l^2 - x^2}$. As in the previous problem, the multiplicative constant has to be zero in order to satisfy end conditions (of v). It follows that $F(z)$ and hence τ_b ($= f_2 = F_2^+ - F_2^-$ on L') are identically zero.

Appendix B

Domain transformation of Equation (3.74)

Let us define

$$U(z) = \frac{1}{2\pi i} \int_{L'} \frac{u_b(\zeta)}{\zeta - z} d\zeta, \quad (\text{B.1})$$

so that U is analytic in the complex plane cut along L' , and $U \sim O(1/z)$ at infinity. The Plemelj formulae are

$$\begin{aligned} U^+(x) + U^-(x) &= \frac{1}{\pi i} \oint_{L'} \frac{u_b(\zeta)}{\zeta - x} d\zeta, \\ U^+(x) - U^-(x) &= u_b(x), \quad \text{for } x \in L', \end{aligned} \quad (\text{B.2})$$

hence we can write (3.74) as

$$[U'(z) + \lambda\pi i U(z)]^+ - [U'(z) - \lambda\pi i U(z)]^- = 0, \quad \text{for } x \in L'. \quad (\text{B.3})$$

If we define the function

$$W(z) = \begin{cases} U'(z) + \lambda\pi i U(z), & \text{for } y > 0, \\ U'(z) - \lambda\pi i U(z), & \text{for } y < 0, \end{cases} \quad (\text{B.4})$$

then Equation (B.3) simplifies to

$$W^+(x) - W^-(x) = 0, \quad \text{for } x \in L'. \quad (\text{B.5})$$

Since $U^+ = U^-$ and $U'^+ = U'^-$ on L , and also that U and U' vanish at infinity, we obtain

$$\begin{aligned} x \in L : \quad \begin{cases} W^+(x) + W^-(x) &= 2U'(x), \\ W^+(x) - W^-(x) &= 2\lambda\pi i U(x); \end{cases} \\ W(z) &\sim 0, \quad \text{as } |z| \rightarrow \infty. \end{aligned} \quad (\text{B.6})$$

According to (B.5), $W(z)$ is a sectionally holomorphic function in the plane cut along L , we therefore suppose

$$W(z) = \frac{1}{2\pi i} \int_L \frac{f(\zeta)}{\zeta - z} d\zeta, \quad (\text{B.7})$$

where $f(x)$ is a function defined on L . Comparison of the corresponding Plemelj formulae with (B.6) now leads to

$$\begin{aligned} \frac{1}{\pi i} \oint_L \frac{f(\zeta)}{\zeta - x} d\zeta &= 2U'(x), \\ f(x) &= 2\lambda\pi i U(x), \quad \text{for } x \in L, \end{aligned} \quad (\text{B.8})$$

where it follows that the integro-differential equation for $f(x)$ is

$$\frac{df}{dx} = \lambda \oint_L \frac{f(\zeta)}{\zeta - x} d\zeta, \quad \text{for } x \in L, \quad (\text{B.9})$$

identical to (3.75).

Bounded solutions to Equation (B.9)?

Suppose there exists a bounded function $f(x)$ satisfying (B.9), then Equation (B.8)₂ implies that $U(z)$ is bounded at the ends $x = \pm l$. This is due to a result derived by Muskhelishvili (p. 74-75, 1953), stated in Equation (3.99) in the main text. If U is bounded, then according to (B.2)₂ u_b is also bounded at the ends. However, we prove in Sections 3.6.1 and 3.6.2 that in this case, the only possibility is that u_b (and also $U(z)$) $\equiv 0$, hence that $f(x) \equiv 0$ for all $x \in L$.

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