PROFINITE RIGIDITY AND FREE GROUPS

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Abstract. We discuss the extent to which groups such as free groups are determined by their finite images.

1. Introduction

Groups are the mathematical objects that encode symmetry in all contexts: no matter what category of objects X one may be studying, and no matter what sort of maps one may be allowing, the invertible maps from X to itself (i.e. the automorphisms of X) form a group. Thus, in all manner of contexts, one finds reasons to study groups of automorphisms $\mathrm{Aut}(X)$ in order to elucidate the nature of the underlying object X. According to one's nature, one might also be drawn to the study of groups themselves. When this is case, it is natural to reverse the passage from X to $\mathrm{Aut}(X)$: given a group Γ, one seeks objects X such that Γ acts as a group of automorphisms of X; one hopes to illuminate the nature of Γ by observing it in action. Actions on different kinds of objects provide different insights into the nature of Γ, and one quickly learns that the quality of the insights that one gains depends heavily on the nature of both the group and the object on which it is acting.

In all contexts, the groups that have the most unconstrained range of actions are, as the name suggests, *free groups*: associated to any set S one has the free group $F(S)$ whose elements are finite products of the elements of S (and formal inverses) subject to no constraints other than the axioms of a group. Free groups will play a central role in our discussion.

When exploring the symmetries of an object X that interests you, it is natural to grasp at a description of $\mathrm{Aut}(X)$ by seeking (i) a set S of elementary operations that, when performed in suitable combinations, account for all of the symmetries of X, and (ii) a set of rules R describing how different combinations of these elementary operations are related: this leads to the notion of a presentation of a group $\Gamma = \langle S \mid R\rangle$. One can regard a presentation as a concise description of how Γ can be realised as a quotient of the free group $F(S)$. At the beginning of the twentieth century, mathematicians, foremost among them Max Dehn, realised that it is extremely hard to unravel the nature of a group by examining a presentation of it in isolation, even if both of the sets S and R are finite. This insight brings us back to the search for actions: rather than struggling to understand a group Γ as a quotient of a free group described by a finite presentation, one should try to unravel the nature of Γ by exploring how it can act on different kinds of objects.

The most primitive objects to consider are finite sets. Actions on finite sets capture only the finite images of groups, so the power of such actions to explain the nature of Γ is limited by the answer to the fundamental question: to what extent is Γ determined by its set of finite quotients? This compelling question has re-emerged with different emphases throughout the history of group theory, and in recent years it has been animated by a rich interplay with geometry and low-dimensional topology.

The finite images of Γ are encoded in its profinite completion $\widehat{\Gamma}$, a compact topological group that is the inverse limit of the directed system of finite quotients of Γ: if $N < M$ then $\Gamma/N \to \Gamma/M$. For finitely generated groups Γ and Λ, the set of finite images of Γ will be the same as the set of

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finite images of Λ if and only if $\widehat{\Lambda}$ and $\widehat{\Gamma}$ are isomorphic as topological groups; the reader unfamiliar with profinite groups will therefore lose little by reading the statement $\widehat{\Lambda} \cong \widehat{\Gamma}$ as an equality of sets of finite quotients.

If Γ has elements that do not survive in any finite quotient (see Section 4 for examples), then one cannot hope to recover Γ by studying $\widehat{\Gamma}$. Thus it is natural to restrict attention to *residually finite groups*, i.e. groups where every finite subset injects into some finite quotient. The most basic recognition question then becomes: which finitely generated, residually finite groups Γ are *profinitely rigid* in the sense that if Λ is finitely generated and residually finite, then $\widehat{\Lambda} \cong \widehat{\Gamma}$ implies $\Lambda \cong \Gamma$.

It is obvious that finite groups are profinitely rigid and is easy to see that finitely generated abelian groups are as well, but one quickly struggles to find further examples. The study of groups that are not profinitely rigid owes much to a paper of Serre [25] from 1964. He constructed pairs of smooth complex projective varieties that are Galois conjugate but are not homeomorphic. The fundamental groups in each pair are not isomorphic, but the profinite completions of these groups (being the étale fundamental group of their common scheme) are the same. Other illuminating examples come from the work of Stebe [26]: he described pairs of integer matrices $\phi_1, \phi_2 \in \mathrm{SL}(2, \mathbb{Z})$ that are not conjugate but do have conjugate images in $\mathrm{SL}(2, \mathbb{Z}/m)$ for every positive integer m; from this it follows easily that the mapping tori $\mathbb{Z}^2 \rtimes_{\phi_1} \mathbb{Z}$ and $\mathbb{Z}^2 \rtimes_{\phi_2} \mathbb{Z}$ are not isomorphic but their profinite completions are. The essence of these examples was stripped down to its bare essentials by Baumslag [5] who showed that profinite rigidity can fail even for finite extensions of $\mathbb{Z}$ (see Section 4).

These examples are sobering and cause one to reflect on the proof that $\mathbb{Z}^r$ is profinitely rigid. The key to this argument is the observation that if Γ satisfies a group law – in this case the law $\forall x, y\, (xy = yx)$ – then $\widehat{\Lambda} \cong \widehat{\Gamma}$ will imply that Λ satisfies the same law, provided it is residually finite. In such cases, the question of absolute profinite rigidity reduces to a question of *relative* profinite rigidity, where one asks if $\widehat{\Gamma}$ distinguishes Γ from all other groups in a restricted class, for example the class of groups that satisfy a given law or a more geometric condition. Once one has manouevred into such a relative context, one might use a classification of groups in that class to identify examples of profinitely rigid groups: for example, the free nilpotent group of fixed class on a fixed number of generators is profinitely rigid (although many other nilpotent groups are not).

The pursuit of relative profinite rigidity theorems has provided a focal point for a rich body of research in recent years, particularly in geometric contexts. This includes many settings in which the groups are *full-sized* in the sense that they contain non-abelian free subgroups and hence do not satisfy a law. Thus, for example, a finitely generated free group can be distinguished from any other lattice in a connected Lie group [8], or from any other residually-free group [8, 29], by its finite quotients. But such relative theorems do not lead to absolute profinite rigidity, because in the absence of a group law it is extremely difficult to rule out the possible existence of an utterly exotic Λ, finitely generated and residually finite, with $\widehat{\Lambda} \cong \widehat{\Gamma}$. One has to contend with the possibility that Λ shares few of the familiar characteristics of Γ: even if Γ is familiar to you as a group of matrices, say, why should Λ have such a representation? This is a much wilder context than that considered by Grothendieck [16], who considered pairs of residually finite groups $\iota : H \hookrightarrow G$ such that H is not isomorphic to G but ι nevertheless induces an isomorphism $\widehat{H} \to \widehat{G}$ – see Section 4.

The paucity of our knowledge about (absolute) profinite rigidity is illustrated most starkly by the fact that the following fundamental challenge remains open.

Conjecture 1.1. *If a finitely generated, residually finite group Γ has the same finite quotients as a free group of rank r, then Γ is a free group of rank r.*

As far as I am aware, the first person to ask explicitly whether free groups are profinitely rigid was Remeslennikov [20, Question 5.48]. This remains the central challenge in the field, but in recent years there has been significant progress on related matters. In the following sections I shall describe a sample of this progress, staying close to Conjecture 1.1 and highlighting some related open problems.

Dedication: Many of the results that I am about to describe rely on the modern understanding of groups that act by isometries on spaces of negative and non-positive curvature, a central theme of my book with André Haefliger [10]. The success of that book project owed a great deal to the patient and thoughtful stewardship of Dr Catriona Byrne. It is with great pleasure and gratitude, therefore, that I dedicate this essay to her on the occasion of her retirement.

2. Full-sized groups that are profinitely rigid

The group of orientation-preserving isometries of real hyperbolic space is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, in dimension 2, and $\mathrm{PSL}(2,\mathbb{C})$ in dimension 3. Thus the lattices in these Lie groups are the fundamental groups of finite-volume, orientable hyperbolic orbifolds in these dimensions; the orbifold is compact if the lattice is cocompact, and the orbifold is a manifold if the lattice has no non-trivial elements of finite order. The proof of the following theorem relies on many aspects of the modern understanding of such orbifolds, including the arithmetic naturally associated to them and various consequences of the work of Agol and Wise showing that lattices in $\mathrm{PSL}(2,\mathbb{C})$ act nicely on CAT(0) cube complexes (see [1]): more precisely, these groups are virtually special in the sense of [18]. Among other things, this last theorem implies that finitely generated, discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are good in the sense of Serre, an important property in many results concerning profinite rigidity: Γ is *good* if for any finite $\mathbb{Z}\Gamma$-module M, the map $H^*(\widehat{\Gamma},M)\to H^*(\Gamma,M)$ induced by $\Gamma\hookrightarrow\widehat{\Gamma}$ is an isomorphism.

Theorem 2.1 ([11], [12])**.** *There exist arithmetic lattices in* $\mathrm{PSL}(2,\mathbb{C})$ *and* $\mathrm{PSL}(2,\mathbb{R})$ *that are profinitely rigid in the absolute sense.*

For the moment, only finitely many lattices in $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{R})$ are known to be profinitely rigid. Each of the examples in $\mathrm{PSL}(2,\mathbb{R})$ is a triangle group, i.e. the group of symmetries $\Delta(p,q,r)$ of a tiling of the hyperbolic plane by geodesic triangles with vertex angles π/p, π/q, π/r. The least-area example to which the current techniques apply is $\Delta(2,3,8)$. The examples in $\mathrm{PSL}(2,\mathbb{C})$ include both cocompact and non-cocompact lattices; some of the cocompact examples have torsion and some do not. The cocompact examples include the fundamental group of the Weeks manifold, the unique compact hyperbolic 3-manifold of smallest volume. The non-cocompact examples include the Bianchi group $\mathrm{PSL}(2,\mathbb{Z}[\omega])$, where ω is a primitive cube root of unity, and the non-cocompact lattice of minimal covolume.

Conjecture 2.2. *All lattices in* $\mathrm{PSL}(2,\mathbb{C})$ *and* $\mathrm{PSL}(2,\mathbb{R})$ *are profinitely rigid in the absolute sense.*

For lattices $\Gamma_1,\Gamma_2<\mathrm{PSL}(2,\mathbb{R})$ we know that $\widehat{\Gamma}_1\cong\widehat{\Gamma}_2$ implies $\Gamma_1\cong\Gamma_2$, since relative profinite rigidity has been established in this context [8]. For lattices in $\mathrm{PSL}(2,\mathbb{C})$ this is unknown but Liu [21] proved that for each lattice Γ, only finitely many other lattices can have the same profinite completion as Γ, up to isomorphism: in the terminology of [17], the *profinite genus* of Γ among lattices in $\mathrm{PSL}(2,\mathbb{C})$ is finite. It is unknown if the genus of Γ among all finitely generated (or finitely presented) residually finite groups is finite (a weaker form of Conjecture 2.2). The profinite completion distinguishes the fundamental groups of 3-manifolds that are hyperbolic from those which are not [30]. Many non-hyperbolic 3-manifolds can be distinguished from all others by the profinite completions of their fundamental groups (e.g. [28]), but the examples of Stebe described above $\mathbb{Z}^2\rtimes_\phi\mathbb{Z}$ show that not all 3-manifold groups enjoy this property.

Important elements of the proof of Theorem 2.1 extend to lattices in other Lie groups (cf. [22]), but other aspects, particularly the control on finitely generated subgroups, do not. Correspondingly, it is unknown whether groups such as $\mathrm{SL}(3,\mathbb{Z})$ are profinitely rigid, and profinite rigidity is known to fail for certain lattices in other semisimple Lie groups [2], even in rank one [27].

3. Restrictions on the nature of profinitely-free groups

As we discussed earlier, a basic challenge that one faces when trying to settle a challenge such as Conjecture 1.1 is that one starts out knowing essentially nothing about the nature of Γ if $\widehat{\Gamma} \cong \widehat{F}_r$. This challenge is fertile because it forces one to find ways of extracting information about free groups from their finite quotients alone. For example, one might ask if Γ must be finitely presented, or hyperbolic (in the sense of Gromov), or linear, or residually-free? None of these properties is known. On the other hand, one can prove that Γ must have the same nilpotent quotients as F_r, that it must satisfy a version of the Freiheitsatz, and that it cannot have a finitely generated normal subgroup of infinite index other than $\{1\}$; see [13]. A theorem proved recently by Jaikin-Zapirain [19] is particularly intriguing because it is the first to place Γ in a class of groups where there is a reasonable hope of establishing a classification theorem that might allow one to resolve Conjecture 1.1: he proves that a finitely generated, residually finite group Γ with $\widehat{\Gamma} \cong \widehat{F}_r$ must be residually nilpotent, hence *parafree* in the sense of Baumslag [4]. He also proves that any finitely generated, residually finite group with the same profinite completion as a surface group must be residually nilpotent.

4. Failure of profinite rigidity close to free groups

The relatives of free groups that we consider here are virtually free groups, hyperbolic groups, direct products of free groups, and 3-manifold groups.

It is easy to see that if N is finite then $N \times \mathbb{Z}$ is profinitely rigid. Examples of Baumslag [5] show that this rigidity fails if one replaces $N \times \mathbb{Z}$ with a semidirect product. Moreover, as explained in [14], one can modify Baumslag's construction to exhibit pairs of non-isomorphic groups H_1 and H_2 that have the same finite quotients and have the same finite index in a group $N \times \mathbb{Z}$ with N finite. To see this, we consider $G_1 = (\mathbb{Z}/25) \rtimes_\alpha \mathbb{Z}$ and $G_2 = (\mathbb{Z}/25) \rtimes_\beta \mathbb{Z}$, where, in multiplicative notation, $\alpha \in \mathrm{Aut}(\mathbb{Z}/25)$ is $\alpha(x) = x^6$ and $\beta(x) = x^{11}$. Noting that α and β generate the same cyclic subgroup of order 5 in $\mathrm{Aut}(\mathbb{Z}/25)$, one can prove by direct argument that $G_1 \not\cong G_2$ but $\widehat{G}_1 \cong \widehat{G}_2$.

Let $N = (\mathbb{Z}/25) \rtimes_\alpha (\mathbb{Z}/5) = \langle x, y \rangle$, where x generates the first factor and the generator y in the second factor acts as α. Let t be a generator for the second factor of $N \times \mathbb{Z}$. Then $H_1 = \langle x, yt \rangle$ and $H_2 = \langle x, y^2 t \rangle$ both have index 5 in $N \times \mathbb{Z}$, and $H_1 \cong G_1$ while $H_2 \cong G_2$.

By taking free products of copies of these groups, we see that there are virtually free groups of every finite rank that are not profinitely rigid (cf. [17]). In contrast, Conjecture 2.2 posits that free products of finite cyclic groups are profinitely rigid. Also, if Conjecture 1.1 is true, every finitely generated, residually finite group with the same finite images as a virtually free group must itself be virtually free.

Free groups are hyperbolic and 1-dimensional. Thus, when exploring the limits of Conjecture 1.1, one might wonder about hyperbolic groups of dimension 2. To explain why Conjecture 1.1 fails in this setting, we need a supply of finitely presented groups that are infinite but do not map onto any non-trivial finite group. There are various ways to construct such groups. To be explicit, we consider the following family from [9]; one knows that these groups are infinite because they are amalgamated free products of groups that have infinite abelianisation.

$$B_p = \langle a, b, \alpha, \beta \mid ba^{-p}b^{-1}a^{p+1},\ \beta\alpha^{-p}\beta^{-1}\alpha^{p+1},\ [bab^{-1}, a]\beta^{-1},\ [\beta\alpha\beta^{-1}, \alpha]b^{-1} \rangle.$$

By applying the Rips construction to this presentation ([10, p.224]), we obtain a short exact sequence $1 \to N \to \Gamma \to B_p \to 1$ with N finitely generated and Γ a residually finite, hyperbolic group with a 2-dimensional classifying space. It is easy to imagine that $\widehat{B}_p = 1$ might imply that $\widehat{N} \cong \widehat{\Gamma}$, and using the fact that $H_2(B_p, \mathbb{Z}) = 0$ one can prove that this is indeed the case (see [9]). With [7, Theorem A] in hand, one can modify this argument to prove the following result, in which I emphasise that hyperbolicity is in the sense of Gromov, in contrast to Conjecture 2.2.

Theorem 4.1. *There exist residually finite, (Gromov) hyperbolic groups* Γ *of dimension* 2 *with uncountably non-isomorphic subgroups* $\iota_H : H \hookrightarrow \Gamma$ *such that* $\hat{\iota}_H : \widehat{H} \to \widehat{\Gamma}$ *is an isomorphism. Moreover, one can arrange for infinitely many of these subgroups to be finitely generated.*

Because the second homology group $H_2(B_p, \mathbb{Z})$ is trivial, B_p can also serve as Q in the following criterion, which originates in the work of Platonov and Tavgen [23] and was adapted in [3] and [9].

Proposition 4.2 ([23])**.** *Let* $f : G \to Q$ *be an epimorphism of groups, with* G *finitely generated and* Q *finitely presented. Suppose that* $\widehat{Q} = 1$ *and* $H_2(Q, \mathbb{Z}) = 0$*. Then, the fibre product* $P = \{(g, h) \mid f(g) = f(h)\} < G \times G$ *is finitely generated and* $P \hookrightarrow G \times G$ *induces an isomorphism* $\widehat{P} \overset{\cong}{\to} \widehat{G \times G}$*.*

By applying the above criterion to epimorphisms $F \to Q$ from a free group, Platonov and Tavgen showed that *the direct product of two non-abelian free groups is not profinitely rigid,* the second part of the following theorem. The first part can be proved by applying a similar template of proof to suitable sequences of quotients $F \to Q_1 \to Q_2 \to \dots$. The third part follows from the fact that a finitely presented subgroup of $F \times F$ must be of finite index if it maps onto both factors and intersects each non-trivially [6].

Theorem 4.3. *Let* F *be a finitely generated, non-abelian free group.*

(1) *There exist uncountably many non-isomorphic groups* H *such that* $H \hookrightarrow F \times F$ *induces an isomorphism* $\widehat{H} \cong \widehat{F \times F}$*;*
(2) *infinitely many of these groups* H *are finitely generated.*
(3) *There does not exist a finitely presented subgroup* $H \neq F \times F$ *such that* $H \hookrightarrow F \times F$ *induces an isomorphism* $\widehat{H} \cong \widehat{F \times F}$*.*

I deliberately phrased this result in a way that emphasizes the importance of *finiteness properties* in the context of profinite rigidity. For the moment, it is unclear what role finiteness properties might play in Conjecture 1.1. In particular, it is possible that the conjecture is false for finitely generated groups but true if one assumes that Γ is finitely presented. In this vein, Alan Reid, Ryan Spitler and I recently proved [15] that there exist finitely presented, residually finite groups that are profinitely rigid amongst all *finitely presented*, residually finite groups, but have infinite genus among *finitely generated*, residually finite groups. Our examples are direct products $G \times G$ where G is the fundamental group of a certain type of Seifert fibre space (a 3-manifold foliated by circles); the centre of G is infinite cyclic and $G/Z(G)$ is isomorphic to one of the triangle groups $\Delta < \mathrm{PSL}(2, \mathbb{R})$ covered by Theorem 2.1. With this recent result in mind, the reader should compare Theorem 4.3 with:

Conjecture 4.4. *Let* F *and* F' *be finitely generated free groups. If a finitely presented, residually finite group* Γ *has the same finite quotients as* $F \times F'$*, then* Γ *is isomorphic to* $F \times F'$*.*

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