Anomalous symmetries of classifiable C*-algebras

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Abstract

This thesis studies the existence and uniqueness of G-kernels on those C*-algebras classified by the Elliott programme. We develop two obstructions to the possible H^3 invariants of a G-kernel. These obstructions arise from studying the unitary algebraic K_1 group and the topological K_0 group of a C*-algebra. As a consequence of these obstructions, we show that any G-kernel on the Jiang-Su algebra has trivial H^3 invariant. Similarly, for finite groups G, any G-kernel on the Cuntz algebra $\mathcal{O}_\infty$ must have trivial H^3 invariant.

We construct multiple examples of G-kernels with non-trivial H^3 invariant and, under a UHF-absorption condition, we classify those G-kernels that have the Rokhlin property on both Kirchberg algebras satisfying the UCT and unital, separable, simple, nuclear, tracially AF C*-algebras that satisfy the UCT. As a follow up to this classification, we study the structure of G-kernels with the Rokhlin property.

Contents

Chapter 1

Introduction

Operator algebras arose as an axiomatisation of the algebra of observables of a quantum system in the work of Murray and von Neumann ([100]). An operator algebra is a subalgebra of the bounded operators on a complex Hilbert space that is closed under taking adjoints. There are two main families: von Neumann algebras, those that are closed under pointwise limits; and C^*-algebras, those that are closed under uniform limits.

Ever since the origins of operator algebras, questions about their structure and classification have been a centrepiece of the theory. The first major result in this direction is Murray and von Neumann's proof that there is a unique approximately finite dimensional type II_1 factor $\mathcal{R}$ ([101]). This result was vastly extended in Connes' seminal work establishing that $\mathcal{R}$ is unique in the a priori larger class of injective II_1 factors. Furthermore, Connes classified all injective factors ([27]) with the exception of one type that was later completed by Haagerup ([57]).

On the other hand, the classification of approximately finite dimensional C^*-algebras (AF-algebras) requires extra information, as was noticed by Glimm in his classification of uniformly hyperfinite C^*-algebras (UHF-algebras) that arise as infinite tensor products of matrix algebras ([52]). Glimm's result was vastly generalised in Elliott's complete classification of AF-algebras by their ordered K-theory groups ([40]). The inquiry for analogous classification results to the Connes–Haagerup classification of injective factors in the setting of C*-algebras, led to the formation of the Elliott classification programme. The ultimate goal of this programme was the classification of simple, separable, amenable C*-algebras through tractable invariants. Recently, this project has achieved a successful resolution; unital, simple, nuclear C*-algebras, with additional necessary regularity properties, are classified by K-theoretic and tracial data (see Theorem 2.7.1). Hereinafter, we will call the C*-algebras satisfying the hypothesis of the classification theorem (Theorem 2.7.1) *classifiable*.

While studying the structure of operator algebras is a compelling endeavor, equally fascinating is the exploration of the structure of their automorphisms. This question is closely interwoven with the classification of algebras themselves, as can be seen in Connes' proof for the uniqueness of the injective II_1 factor ([27]) and recent novel approaches to the classification of C*-algebras ([21]). Studying the automorphisms of an operator algebra is not only natural from a mathematical standpoint but also holds physical significance: an automorphism of an operator algebra can be interpreted as a symmetry of the quantum system it describes.

The automorphisms of an operator algebra A form a group denoted $\mathrm{Aut}(A)$. In [26] Connes classifies automorphisms of $\mathcal{R}$. Building on Connes' work, V. Jones classified finite group actions $G \to \mathrm{Aut}(\mathcal{R})$ ([78]). Subsequently, Ocneanu classified actions of amenable groups on $\mathcal{R}$ ([105]). The classification of amenable group actions on injective factors was completed by Katayama, Sutherland and Takesaki ([82]). In the spirit of these works, there has been extensive research towards classifying group actions on C*-algebras (see [68]), with recent breakthroughs using the powerful machinery of equivariant KK-theory ([48]).

G-kernels and anomalous symmetries

For a unital operator algebra A the group $\mathrm{Aut}(A)$ has a normal subgroup $\mathrm{Inn}(A)$ consisting of *inner automorphisms*. These are of the form $\mathrm{Ad}(u)$ for a unitary u in A, where $\mathrm{Ad}(u)(x) = uxu^*$ for any $x \in A$. The quotient group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ is denoted $\mathrm{Out}(A)$. In physical interpretations of C*-algebra theory, inner automorphisms correspond to *gauge symmetries* of the system described by A, these are regarded as redundant symmetries. Therefore, studying the outer automorphism group is a prominent theme in applications of C*-algebras to theoretical physics (see [19, 20]).

In [26, 28] Connes begins detailed analysis of $\mathrm{Out}(\mathcal{R})$ by classifiying outer automorphisms of $\mathcal{R}$ up to conjugacy in $\mathrm{Out}(\mathcal{R})$.[1] Connes' invariant is composed firstly of the order of the automorphism in $\mathrm{Out}(\mathcal{R})$ and secondly, in the case that the automorphism is of finite order n, an associated n-th root of unity which arises as follows: Let $u \in \mathcal{R}$ be a unitary such that $\theta^n = \mathrm{Ad}(u)$. As $\mathrm{Ad}(u)\theta = \theta^{n+1} = \theta\mathrm{Ad}(u) = \mathrm{Ad}(\theta(u))\theta$ and the centre of $\mathcal{R}$ is trivial, there exists $\lambda \in \mathbb{C}$ such that $\theta(u) = \lambda u$. By applying θ n-times to u, it follows that $\lambda^n = 1$. That the n-th root of unity does not depend on u is shown in [28]. We can now state Connes' classification theorem.

Theorem A. *(Connes cf. [26, 28]) Let θ, $\alpha \in \mathrm{Aut}(\mathcal{R})$. Then θ is conjugate to α in $\mathrm{Out}(\mathcal{R})$ if and only if they have the same order in $\mathrm{Out}(\mathcal{R})$ and associated root of unity. Moreover, for any $n \in \mathbb{N}$ and n-th root of unity γ there exists an automorphism which has order n in $\mathrm{Out}(\mathcal{R})$ and associated n-th root of unity γ. Similarly, there exists an automorphism of infinite order in $\mathrm{Out}(\mathcal{R})$.*

One of the central questions of this thesis is to what extent an analogue of Theorem A can be achieved in the setting of simple, separable, amenable C*-algebras. More generally, this thesis is devoted to the study of group homomorphisms $G \to \mathrm{Out}(A)$ (called *G-kernels*) and the closely related notion of *anomalous actions*. Examples of

[1] An automorphism is called *outer* if it is not an inner automorphism.

G-kernels come from group actions $G \to \mathrm{Aut}(A)$ by further composing with the quotient map $\mathrm{Aut}(A) \to \mathrm{Out}(A)$. However, not all G-kernels come from group actions. For example, Connes' (n, γ) automorphisms of $\mathcal{R}$ for non-trivial n-th roots of unity γ, define $\mathbb{Z}_n$-kernels that can not come from an action of $\mathbb{Z}_n$. If $\overline{\theta} : G \to \mathrm{Out}(A)$ is a G-kernel, one may pick a *lifting* (θ, u) where $\theta : G \to \mathrm{Aut}(A)$ is a lift of the map $\overline{\theta}$ and $u : G \times G \to U(A)$ are unitaries witnessing the failure of the multiplicativity of θ, namely

$$\theta_g \theta_h = \mathrm{Ad}(u_{g,h})\theta_{gh}, \quad \forall g, h \in G. \tag{$\star$}$$

Using the associativity of $\mathrm{Aut}(A)$ and ($\star$) one may associate a 3-cocycle of G with values in the centre of the unitary group of A (see [39, Section 7]). The resulting cohomology class is an invariant of the G-kernel $\overline{\theta}$ known as its H^3 invariant. Non-triviality of the H^3 invariant is an obstruction to lifting the G-kernel to a group action on A. The root of unity computation for Connes' invariants for automorphisms, is an instance of this construction as $H^3(\mathbb{Z}_n, \mathbb{T})$ is isomorphic to the group of n-th roots of unity.

V. Jones and Ocneanu achieve a classification for pointwise outer G-kernels on $\mathcal{R}$ generalising Theorem A.

Theorem B. *([28] for cyclic groups, [78] for finite groups, [105] for amenable groups) Let G be a countable, discrete, amenable group. Then any two pointwise outer G-kernels on $\mathcal{R}$ are classified up to outer conjugacy by their H^3 invariant*

The classification of pointwise outer G-kernels on injective factors was completed by Katayama and Takesaki ([83]). V. Jones further shows that one can achieve any H^3 invariant for G-kernels on $\mathcal{R}$.

Theorem C. *(V. Jones cf. [77, Theorem 2.5]) For any countable, discrete group G and $[\omega] \in H^3(G, \mathbb{T})$ there exists a G-kernel on $\mathcal{R}$ with H^3 invariant $[\omega]$.*

While G-kernels on factors have received considerable attention in the literature, the same cannot be said for simple, separable C*-algebras. This thesis aims to fill this gap by providing a comprehensive and extended analysis of G-kernels on C*-algebras. The research in this thesis is motivated by the fundamental questions regarding the existence and uniqueness of G-kernels on classifiable C*-algebras, building upon the groundbreaking results of Connes, V. Jones, and Ocneanu for G-kernels on $\mathcal{R}$.

In [74] C. Jones introduces the notion of ω-*anomalous actions*. These are closely related to G-kernels. In the case that A has trivial centre (for example if A is simple) an ω-anomalous action of a group G on A coincides with a lift of a G-kernel on A with H^3 invariant $[\omega]$. If A has non-trivial centre, an anomalous action on A is a lift of a G-kernel on A with associated 3-cocycle valued in $\mathbb{T}$ rather than potentially in $Z(U(A))$, making them distinct from lifts of G-kernels (see Section 3.3 for a detailed explanation). This extra condition for anomalous actions is justified by physical interpretations in [74].

Throughout this thesis, we formulate our results in the formalism of anomalous actions. Since we are mainly concerned with simple C*-algebras, this is equivalent to studying G-kernels. However, the formalism of anomalous actions will allow us to define inductive limits and to construct non-trivial G-kernels on simple C*-algebras through anomalous actions on non-simple C*-algebras. Moreover, another benefit of this formalism, is that anomalous actions are easily seen to fit into the framework of actions of C*-tensor categories. This framework provides a unifying framework for studying group actions, anomalous actions and the more general notions of quantum symmetry arising from subfactor theory ([79],[110]).

Main results

My results can be subdivided into two main parts. The first is investigating the existence (or non-existence) of ω-anomalous actions with prescribed 3-cocycle ω. The second is the classification of ω-anomalous actions subject to their existence. For the first question, we will be interested in specific examples of C*-algebras that are of special importance in the classification programme. Two examples that are especially relevant in my work are the Jiang–Su algebra $\mathcal{Z}$ and the UHF-algebras. Due to its role in the classification of C*-algebras, the Jiang–Su algebra $\mathcal{Z}$ is often regarded as the C*-analogue of $\mathcal{R}$ and therefore presents itself as a natural example to consider these questions. Moreover, every classifiable C*-algebra tensorially absorbs $\mathcal{Z}$; showing existence of ω-anomalous actions on $\mathcal{Z}$ with a given ω would imply their existence on any other classifiable C*-algebra.

Although the results of this thesis are expressed in the language of anomalous actions in the main body, all of the results discussed in this introduction will entail anomalous actions on simple C*-algebras and therefore may equivalently be rephrased as results about G-kernels (see Remark 3.3.3). Therefore, for the purpose of this introduction, we rephrase them in terms of G-kernels to emphasise their resemblance to Theorems A, B and C.

Existence

The first main result of this thesis is an obstruction to the possible H^3 invariants of G-kernels on C*-algebras. This obstruction arises as the unitary group of a C*-algebras may have non-trivial abelianisations (see Chapter 4). In the case of $\mathcal{Z}$ and UHF-algebras the abelianisations of their unitary group may be computed through the de la Harpe–Skandalis determinant. The results of the following theorem are contained in my joint work with Evington (see [46]).

Theorem I. *(cf. Theorem 4.2.7, Theorem 4.2.10 and Corollary 4.2.13)*

Let G be a discrete group and $\overline{\theta}$ a G-kernel on $\mathcal{Z}$, then the H^3 invariant of $\overline{\theta}$ must be trivial.

Let G be a discrete group with $H^3(G, \mathbb{Z})$ finitely generated. If $\overline{\theta}$ is a G-kernel on

$\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{n_i}(\mathbb{C})$ then the H^3 invariant of $\overline{\theta}$ has finite order r and $r^\infty | \prod_{i\in\mathbb{N}} n_i$.

For finite groups G, we may apply [74] to construct G-kernels with arbitrary H^3 invariant on the UHF algebra $\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{|G|}$. These existence and non-existence results for G-kernels of UHF algebras allow us to achieve the analogue of the existence part of Theorem A in the setting of UHF algebras.

Theorem II. *(cf. Corollary 6.2.7) Let n_i be a sequence of natural numbers, $n \in \mathbb{N}$ and γ an n-th root of unity. There exists an automorphism of $A = \bigotimes_{i\in\mathbb{N}} \mathbb{M}_{n_i}(\mathbb{C})$ of order $n \in \mathrm{Out}(A)$ with associated n-th root of unity γ if and only if the order of γ appears with infinite multiplicity in $\prod_{i\in\mathbb{N}} n_i$.*

Although the obstruction set out in Chapter 4 restrict the H^3 invariants of G-kernels on fairly general classes of C*-algebras, it has no implications for the existence of G-kernels on simple, infinite C*-algebras. Izumi has shared with me a fascinating idea utilizing topological K-theory that yields a second obstruction to the possible H^3 invariants of G-kernels on C*-algebras, this is fleshed out in Chapter 5. Izumi's obstruction implies that if G is a finite group and θ is a G-kernel on the Cuntz algebra $\mathcal{O}_\infty$, then the H^3 invariant of θ must be trivial. Through thorough analysis of Izumi's obstruction and applying homological techniques, we can achieve divisibility in K-theory as a consequence of the existence of G-kernels with non-trivial H^3 invariants.

Theorem III. *(cf. Corollary 5.4.3) Let G be a finite group and A a simple, separable, unital C*-algebra with $K_1(A) = 0$ and no $|G|$-torsion in $K_0(A)$. Then if $\overline{\theta} : G \to \mathrm{Out}(A)$ is G-kernel on A with $K_0(\overline{\theta}_g) = \mathrm{id}_{K_0(A)}$ for all $g \in G$, then $K_0(A)$ is uniquely divisible by the order of the H^3 invariant of $\overline{\theta}$.*

Using these obstruction results for G-kernels and the explicit constructions of G-kernels performed in Chapter 6, we achieve better understanding of the H^3 invariants that arise for G-kernels on the Cuntz algebras $\mathcal{O}_n$ of [29] (see Section 5.3 and Proposition 6.4.10), unital AF-algebras (see Corollary 5.4.4) and other relevant C*-algebras in the scope of the classification programme.

Classification

After considering the existence of G-kernels we turn to their classification. To achieve this, we take inspiration from the methods used in Theorems A and B. In these theorems, an important role is played by adaptations of Connes' non-commutative Rokhlin Lemma, which yields that outer group actions on $\mathcal{R}$ satisfy a condition called the Rokhlin property that is analogous to properties of ergodic measure preserving actions of amenable groups on probability spaces ([115],[106]). In the C*-setting the analogous property is often not automatic. However, there has been substantial progress in classifying group actions on C*-algebras that satisfy the Rokhlin property (see e.g [59, 60, 66, 67]).

To classify G-kernels, we restrict our scope to finite groups and G-kernels satisfying the Rokhlin property. In [66, 67], Izumi uses the Rokhlin property to boost the classification of Kirchberg algebras ([109, 84]) and unital, simple, nuclear, separable tracially approximately finite dimensional (TAF) algebras ([90]) by their K-theory, to a classification of finite group actions with the Rokhlin property on these classes of C*-algebras by the induced module structure on K-theory ([67, Theorem 4.2, Theorem 4.3]). Assuming that our C*-algebras additionally satisfy a UHF absorbing condition previously considered in [3], we are able to bootstrap Izumi's result to achieve a classification of G-kernels. We state a specific case of our classification result below that makes no reference to TAF algebras by combining succeeding structural results of ([95]) and ([138]). This Theorem will appear in my paper in preparation [51].

Theorem IV. *(cf. Theorem 7.2.2 and Theorem 7.2.3) Let G be a finite group if*

- *A is a Kirchberg algebra satisfying the UCT such that $A \cong A \otimes M_{|G|^\infty}$ or,*

- *A is a unital, separable, simple, nuclear, unique trace C^*-algebra, satisfying the UCT and $A \cong A \otimes M_{|G|^\infty}$*

and $\overline{\theta}, \overline{\alpha}$ are G-kernels on A with the Rokhlin property. Then $\overline{\theta}$ and $\overline{\alpha}$ are conjugate in $\mathrm{Out}(A)$ through an automorphism that preserves K-theory if and only if $K_i(\overline{\theta}_g) = K_i(\overline{\alpha}_g)$ for all $g \in G$ and the H^3 invariants of $\overline{\theta}$ and $\overline{\alpha}$ coincide.

The strategy of Theorem IV is to bootstrap existing classification results for group actions with the Rokhlin property, which are currently only available for Kirchberg algebras or unital, simple, separable TAF-algebras in the work of Izumi. Recent but yet unpublished work of Szabó classifies Rokhlin group actions on all classifiable C*-algebras by using more recent classification results for C*-algebras. With this result in hand, the methods of Theorem IV classify G-kernels with the Rokhlin on any classifiable $M_{|G|^\infty}$-stable C*-algebras by the H^3 invariant and the module structure of the K-theory groups.

Following our classification of G-kernels with the Rokhlin property we proceed to study their structure in more detail. Chapter 8 is devoted to understanding in how far the structure of G-kernels with the Rokhlin property resembles that of group actions with the Rokhlin property. For example we are able to show that a certain lifting criterion for G-kernels of cyclic groups holds generalising [66, Lemma 3.12] (see Corollary 8.1.6). We are also able to compute fixed point algebras of some lifts of G-kernels and show cohomology vanishing type results for their K-theory modules in spirit of [67].

Anomalous actions and tensor categories

In the final chapter of this thesis we discuss C*-tensor categories. These categories are intimately related to Jones' subfactor theory ([79]). As a consequence of decades of work C*-tensor categories are understood as the mathematical object encoding the symmetry of a subfactor ([110, 111, 97, 76]). Adaptations of subfactor theory to C*-algebras ([143, 65, 24]) also allow us to see C*-tensor categories as the mathematical structure encoding the symmetry contained in a finite index inclusion of C*-algebras. In particular, questions about existence and classification of actions of C*-tensor categories on C*-algebras can be understood as looking for analogues of Popa's classification and reconstruction theorems for subfactors ([110, 111]).

In Chapter 9 we show that an ω-anomalous actions of a group G can be understood as an action of the tensor category $\mathbf{Hilb}(G, \omega)$ and vice-versa. Therefore, we are able to rephrase our results of Chapters 4 and 5 as obstructions to the existence of actions of the category $\mathbf{Hilb}(G, \omega)$ on C*-algebras.

Thesis structure

After recalling some useful preliminaries in Chapter 2, we introduce the essentials on G-kernels and anomalous symmetries in Chapter 3. In Chapter 3 we also discuss relevant classification results for group actions on C*-algebras.

In Chapter 4 and 5 we introduce two obstructions to the possible values of three cocycles ω of anomalous actions on a given C*-algebra. Chapter 4 develops an algebraic obstruction that arises from considering the abelianisations of the unitary groups of C*-algebras. This chapter is a fleshed out version of my work with Evington ([46]). In Chapter 5 we develop the details of a topological obstruction whose existence was brought to my attention by Masaki Izumi. We also apply the obstruction to examples of purely infinite C*-algebras and extract general implications of this obstruction.

Chapter 6 is dedicated to constructions of anomalous actions. We recall the general framework introduced by C. Jones in [74] for building anomalous actions on C*-crossed products and apply it in various settings. These methods allow us to build ω-anomalous actions with interesting three cocycles ω on UHF algebras, matrix amplifications of Cuntz algebras and various other relevant C*-algebras.

In Chapter 7 we prove our classification theorem for Rokhlin anomalous actions. We also discuss applications of this result. We then perform a structural analysis of Rokhlin anomalous actions in Chapter 8, where we discuss obstructions to the existence of Rokhlin anomalous actions and the structure of their fixed point algebras. In Chapter 9 we rephrase our results in the context of actions of C*-tensor categories.

Chapter 2

Preliminaries

In this chapter, we will set up our standard notation and recall some background knowledge that will be useful in later chapters. The treatment of the material in Section 2.5 is based on that of [46, Section 1.2] in my joint work with Samuel Evington.

2.1 Assumed knowledge

Throughout this thesis, we assume that the reader knows the basics of topology and category theory, as can be found in [98] and [92]. We may also assume understanding of basic concepts of homological algebra. Our standard reference will be [145].

We will assume familiarity with the basics of operator algebras, as can be found in [99, 15, 119]. In particular we will assume that the reader knows the basics on nuclearity and semidiscreteness as in [15, Chapter 2 and Chapter 3].

2.2 Notational conventions

Throughout this thesis we will usually use the symbols A, B to denote C*-algebras and G, H, K, Γ or Q to denote groups.

Let A be a C*-algebra. We denote by A^+ the positive elements in A, by A_1 the elements in A of norm less than or equal to 1 and by A_1^+ the intersection $A^+ \cap A_1$. We write A^{sa} for the self adjoint elements in A. We denote by $\tilde{A}$ the "forced unitization" of A (see for example [119, Section 1.1.6]). The algebra $\tilde{A}$ coincides with the minimal unitization of A when A is non-unital. We write $M(A)$ for the maximal unitisation of A. If A and B are C*-algebras we will denote by $A \otimes B$ their minimal tensor product. The state space of A will be denoted by $S(A)$; that is the convex topological space, under the weak*-topology, of positive linear functionals on A of norm 1. A positive linear functional $\tau \in S(A)$ is called a *tracial state* (we often refer to it as a *trace*) if $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We denote the space of all tracial states by $T(A)$; we often refer to it as the *tracial state space*. The space $T(A)$ is compact and convex. In fact it is a Choquet simplex (see [122, Theorem 3.1.18]).

For any G-module M we denote the action of $g \in G$ by gm for all $m \in M$. We denote the invariants by $M^G = \{m \in M : gm = m \ \forall g \in G\}$ and the coinvariants by $M_G = M/\langle gm - m : g \in G, m \in M\rangle$.

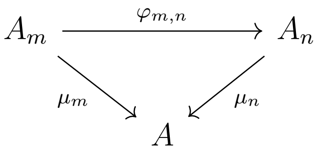
For a category $\mathcal{C}$ we denote by $\mathrm{Obj}(\mathcal{C})$ the class of objects in $\mathcal{C}$ and for any $A, B \in \mathrm{Obj}(\mathcal{C})$ we denote by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ the set of morphisms between them. We will often drop the category $\mathcal{C}$ from the morphism notation and denote $\mathrm{Hom}_{\mathcal{C}}(A, B)$ by $\mathrm{Hom}(A, B)$ instead. We denote the category whose objects are C*-algebras and morphisms are *-preserving homomorphisms by **C*alg**. The full subcategory of separable C*-algebras is $\mathbf{C^*alg}_{\mathrm{sep}}$. We denote the subcategory of **C*alg** whose objects are unital C*-algebras and morphisms are unit preserving *-homomorphisms by $\mathbf{C^*alg}_1$. We write **Ab** for the category of abelian groups.

2.3 Inductive limits

An *inductive sequence* in a category $\mathcal{C}$ is a sequence of pairs (A_n, φ_n) for $n \in \mathbb{N}$ with $A_n \in \mathrm{Obj}(\mathcal{C})$ and $\varphi_n \in \mathrm{Hom}(A_n, A_{n+1})$. For $n > m$ we denote the composition

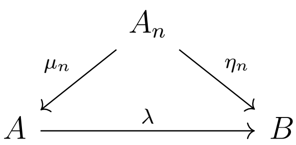
$\varphi_{m,n} = \varphi_{n-1} \ldots \varphi_m \in \mathrm{Hom}(A_m, A_n)$.

An *inductive limit* of (A_n, φ_n) in $\mathcal{C}$ is an object $A \in \mathrm{Obj}(\mathcal{C})$ along with a family of maps $\mu_n \in \mathrm{Hom}(A_n, A)$ such that



(2.3.1)

commutes for all $n \in \mathbb{N}$. We also require a universality condition, that for any other object $B \in \mathrm{Obj}(\mathcal{C})$ and morphisms $\eta_n \in \mathrm{Hom}(A_n, B)$, with the equivalent diagrams to (2.3.1) commuting, there exists a unique morphism $\lambda \in \mathrm{Hom}(A, B)$ such that



(2.3.2)

commutes for all $n \in \mathbb{N}$.

It follows from the definition that an inductive limit, if it exists, is unique in the sense that if (A, μ_n) and (B, η_n) are inductive limits of (A_n, φ_n), then there exists a unique isomorphism $\lambda \in \mathrm{Hom}(A, B)$ making diagrams (2.3.1) and (2.3.2) commute. Due to the uniqueness of inductive limits, we denote the underlying object by $\varinjlim A_n$ and the connection morphisms by $\varphi_{n,\infty} \in \mathrm{Hom}(A_n, \varinjlim A_n)$.

We will mainly be interested in inductive sequences in categories of C*-algebras. In **C*alg** inductive limits always exist. Indeed, for an inductive system (A_n, φ_n) the inductive limit is given by setting $\varinjlim A_n$ to be the completion of

$$\mathcal{A} = \left\{ (a_n) \in \frac{\bigoplus_{l^\infty} A_j}{\bigoplus_{c_0} A_j} : \exists N \in \mathbb{N} \text{ such that } a_{j+1} = \varphi_j(a_j) \ \forall\ j \geq N \right\} \tag{2.3.3}$$

with respect to the norm $\|(a_n)\| = \lim_{n\to\infty} \|a_n\|_{A_n}$. We set the maps $\varphi_{n,\infty}$ to be

$$\varphi_{n,\infty}(a) = \begin{cases} 0 \text{ for } k < n \\ \varphi_{n,k}(a_n) \text{ for } k \geq n \end{cases}$$

for $a = (a_n) \in \mathcal{A}$. A more detailed discussion on inductive limits for C*-algebras can be found for example in [127, Chapter 8].

A specific example of inductive limits which we will use throughout this thesis is that of infinite tensor products. Recall that we use $\otimes$ to denote the minimal tensor product of C*-algebras. Let A_n be a sequence of unital C*-algebras. Define the C*-algebra $\bigotimes_{n\in\mathbb{N}} A_n$ to be the C*-inductive limit arising from the inductive sequence $(\bigotimes_{i=1}^{n} A_i, \varphi_n)$ with

$$\varphi_n(a) = a \otimes 1_{A_{n+1}}, \ a \in \bigotimes_{i=1}^{n} A_i.$$

For example, one could take each A_n to be a full matrix algebra.

Example 2.3.1. A *uniformly hyperfinite algebra* (or in short *UHF-algebra*) is a C*-algebra A such that there exists a sequence of natural numbers n_i for $i \in \mathbb{N}$ with $A \cong \bigotimes_{i\in\mathbb{N}} \mathbb{M}_{n_i}$. Glimm classified UHF algebras by the formal product $\prod_{i\in\mathbb{N}} n_i$ in [52]. What Glimm's classification result yields, is that for any two sequences of natural numbers n_i and m_i, the C*-algebras $\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{n_i}$ and $\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{m_i}$ are isomorphic if and only if the set of powers of primes that divide the product $\prod_{i\in\mathbb{N}} n_i$ and the set of powers of primes that divide the product $\prod_{i\in\mathbb{N}} m_i$ coincide. The formal product $\prod_{i\in\mathbb{N}} n_i$ associated to a UHF algebra A is called its *supernatural number*. When the sequence n_i is constant and equal to $n \in \mathbb{N}$ for an $i \in \mathbb{N}$ we denote the associated UHF algebra by M_{n^∞}.

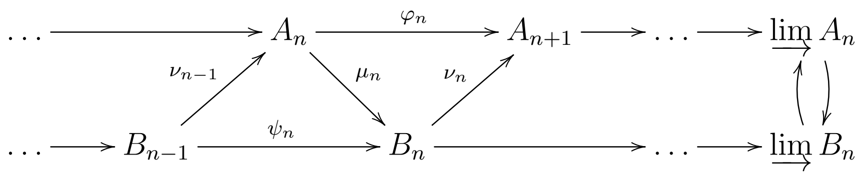
We will be interested in inductive limit C*-algebras whose building blocks are of a specific form.

Example 2.3.2. A C*-algebra A that is the inductive limit of an inductive sequence (A_n, φ_n) of finite dimensional C*-algebras is called an *approximately finite dimensional C*-algebra* or *AF-algebra*. The UHF-algebras of Example 2.3.1 are AF-algebras. However, there are examples of AF-algebras that are not UHF. For example, the compact operators arises as the inductive limit $(\mathbb{M}_n, \varphi_n)$ with $\varphi_n : \mathbb{M}_n \to \mathbb{M}_{n+1}$ defined by $\varphi_n(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.

Example 2.3.3. A *subhomogeneous C*-algebra* A is a C*-algebra such that the ranks of its irreducible representations have an upper bound. By [6, IV.1.4.3] a C*-algebra A is subhomogeneous if and only if it is a C*-subalgebra of $C(X, \mathbb{M}_n)$ for some $n \in \mathbb{N}$ and compact Hausdorff space X. A C*-algebra is called an *approximately subhomogeneous algebra* or *ASH-algebra* if it is the inductive limit of a sequence of subhomogeneous C*-algebras. This class covers AF-algebras but also many other examples of C*-algebras. We will postpone discussion of more general examples of ASH-algebras until later sections.

To show that two inductive limits are isomorphic the following Lemma will be extremely useful.

Lemma 2.3.4 (cf. [119, Exercise 6.8])**.** *Let $\mathcal{C}$ be a category admitting inductive limits and (A_i, φ_i), (B_i, ψ_i) be inductive sequences in $\mathcal{C}$. Suppose there are morphisms $\mu_n : A_n \to B_n$ and $\nu_n : B_n \to A_{n+1}$ such that the following diagram commutes*



(2.3.4)

then there are mutually inverse isomorphisms denoted by the curved arrows that make the whole diagram commutative.

We will call a commuting diagram as in Lemma 2.3.4 an *intertwining*.

2.4 Central sequence algebras

To a C*-algebra A we start by associating another C*-algebra A_∞ which is constructed from sequences in A. The advantage of this is that it allows us to write approximate statements about elements in A as exact statements holding in A_∞. This will not only benefit the presentation of our results but it will also allow us to streamline certain arguments which would otherwise look unnecessarily complicated.

Let A be a C*-algebra, let A_∞ be the quotient C*-algebra

$$A_\infty = \frac{l^\infty(A)}{c_0(A).} \tag{2.4.1}$$

Every element in A_∞ is represented by a bounded sequence of elements $a_n \in A$. For any bounded sequence (a_n) in A we say $(a_n) \in A_\infty$ to refer to the image of the sequence (a_n) under the canonical quotient map. For a C*-subalgebra $B \subset A_\infty$ we may consider the commutant C*-algebra $A_\infty \cap B'$. For example, take $B = A$ where A is embedded in A_∞ by sending $a \in A$ to the constant sequence $a_n = a$ for $n \in \mathbb{N}$. We call $A_\infty \cap A'$ the *central sequence algebra*.

Any *-homomorphism of C*-algebras $\theta : A \to B$ induces a *-homomorphism $\theta : A_\infty \to B_\infty$ through $(a_n) \mapsto (\theta(a_n))$.[1] Therefore, the mapping $A \mapsto A_\infty$ defines a functor from the category of C*-algebras into itself. In particular, any automorphism $\theta \in \mathrm{Aut}(A)$ defines an automorphism of A_∞. If $B \subset A_\infty$ is invariant by both θ and θ^{-1}, the automorphism θ restricts to an automorphism of $A_\infty \cap B'$.

[1] Note the abuse of notation.

2.5 Group homology and cohomology

In this section, we recall some background from group homology and cohomology that will be used throughout this thesis. Details can be found in [11]. As group cohomology will be the most relevant out of the two theories for this thesis, we start by discussing its construction in detail.

2.5.1 Group cohomology

Fix a group G and a $\mathbb{Z}G$-module M. An *n-cochain* is a function $f : G^n \to M$. The set of all n-cochains $C^n(G, M)$ inherits a $\mathbb{Z}G$-module structure from M. The coboundary maps $d^{n+1} : C^n(G, M) \to C^{n+1}(G, M)$ are defined by

$$\begin{aligned} d^{n+1}(f)(g_1, g_2, \dots, g_{n+1}) = {} & g_1 f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \dots, g_n). \end{aligned} \tag{2.5.1}$$

An *n-cocycle* is an n-cochain f satisfying $d^{n+1} f = 0$; the set of all n-cocyles form an abelian group $Z^n(G, M)$ under addition. An *n-coboundary* is an n-cochain f satisfying $f = d^n \eta$ for some $(n-1)$-cochain η; the set of all n-coboundaries forms an abelian group $B^n(G, M)$ under addition.

Since $d^{n+1} \circ d^n = 0$, it follows that $B^n(G, M) \subseteq Z^n(G, M)$. The quotient group $H^n(G, M) = Z^n(G, M)/B^n(G, M)$ is the n-th *cohomology group* of G with *coefficients* in M. An n-cocycle f is called *normalised* if $f(g_1, \dots, g_n) = 1$ whenever any $g_i = 1$ for some $1 \leq i \leq n$. Any n-cocycle is cohomologous to a normalised one (see for example [43, Remark 2.6.3]).

Formula (2.5.1) assumes that M is written additively. In multiplicative notation, the right hand side would be

$$\alpha_{g_1}(f(g_2,\ldots,g_{n+1}))\cdot\prod_{i=1}^{n} f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1})^{\epsilon_i}\cdot f(g_1,\ldots,g_n)^{\epsilon_{n+1}},$$

where $\epsilon_i = (-1)^i$ and where α_g denotes the action of g on M.

For two $\mathbb{Z}G$-modules M' and M and a $\mathbb{Z}G$-module map $\varphi : M' \to M$, one can define a map of abelian groups $\varphi_* : H^n(G, M') \to H^n(G, M)$ by

$$\varphi_*(f)(g_1,\ldots,g_n) = \varphi(f(g_1,\ldots,g_n)).$$

This endows $H^n(G, \cdot)$ with the structure of a covariant functor from the category of $\mathbb{Z}G$-modules to the category of abelian groups. This functor sends short exact sequences of $\mathbb{Z}G$-modules

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0,$$

to long exact sequences. (See [11, Section III.6]).

$$\ldots H^k(G, M') \xrightarrow{\iota_*} H^k(G, M) \xrightarrow{\pi_*} H^k(G, M'') \xrightarrow{\delta} H^{k+1}(G, M')\ldots$$

Similarly, for any group homomorphism $\rho : G \to Q$ and Q-module M one can induce a G module structure on M through $gn = \rho(g)m$ for any $g \in G$ and $m \in M$. Under this induced G-module structure, one can define a map of abelian groups $\rho^* : H^n(Q, M) \to H^n(G, M)$ by

$$\rho^*(f)(g_1,\ldots,g_n) = f(\rho(g_1),\ldots,\rho(g_n)).$$

2.5.2 Group homology

Group cohomology is the dual theory to group homology. For G a group, M a $\mathbb{Z}G$-module and $n \geq 0$ the *n-th homology group* of G with *coefficients* in M is an abelian group denoted by $H_n(G, M)$. One could construct the homology groups in a similar fashion to the construction of Section 2.5.1. However, as we do not require these details we refer the interested reader to [11]. The homology groups satisfy analogous properties to the cohomology groups. For instance, $H_n(G, \cdot)$ is a covariant functor from the category of $\mathbb{Z}G$-modules to the category of abelian groups. The functor $H_n(G, \cdot)$ sends short exact sequences of $\mathbb{Z}G$-modules

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$$

to long exact sequences

$$\ldots H_k(G, M') \xrightarrow{\iota_*} H_k(G, M) \xrightarrow{\pi_*} H_k(G, M'') \xrightarrow{\delta} H_{k-1}(G, M') \ldots$$

In this thesis, M will often be an abelian group endowed with the trivial $\mathbb{Z}G$-module structure, where $gm = m$ (additive notation) or $\alpha_g(m) = m$ (multiplicative notation) for all $g \in G$ and $m \in M$. In this case, the universal coefficient theorem of group cohomology [139, Proposition 11.9.2] holds.

Theorem 2.5.1 (Universal coefficient theorem)**.** *Let M be an abelian group endowed with the trivial $\mathbb{Z}G$-module structure. Then for every $n \in \mathbb{N}$ there is a short exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), M) \to H^n(G, M) \to \operatorname{Hom}_{\mathbb{Z}}(H_n(G, \mathbb{Z}), M) \to 0.$$

Here $\mathbb{Z}$ is endowed with the trivial $\mathbb{Z}G$-module structure. The short exact sequence is natural in all entries.

We refer to [145, Chapter 3] for the definition of $\mathrm{Ext}^1_{\mathbb{Z}}$. It is a useful fact that if M is a divisible abelian group, then $\mathrm{Ext}^1_{\mathbb{Z}}(M', M)$ vanishes for any abelian group M'.

2.5.3 Tate cohomology

In the setting of finite groups one may combine group homology and cohomology into a combined theory called *Tate cohomology*. Fix a finite group G and a $\mathbb{Z}G$-module M, we denote by N the *norm element* of $\mathbb{Z}G$ given by

$$N = \sum_{g \in G} g.$$

The element N yields a map $N : M \to M^G$ by taking any $m \in M$ to Nm. Moreover, as any element of the form $m' = gm - m$ for $g \in G$ and $m \in M$ is in the kernel of N, we get an induced map $\overline{N} : M_G \to M^G$. We may now define the Tate cohomology groups (see [11, Section VI.4]).

Definition 2.5.2. The n-th *Tate cohomology group* of G with coefficients in M is defined by

$$\hat{H}^*(G, M) := \begin{cases} H^n(G, M), & n > 0, \\ \mathrm{Coker}\overline{N}, & n = 0, \\ \ker \overline{N}, & n = -1, \\ H_{-n-1}(G, M), & n < -1. \end{cases}$$

Tate cohomology glues together the long exact sequences for homology and cohomology. A short exact sequence of $\mathbb{Z}G$-modules

$$0 \to M' \xrightarrow{\iota} M \xrightarrow{\pi} M'' \to 0$$

induces a long exact sequence of Tate cohomology

$$\dots \hat{H}^k(G, M') \xrightarrow{\iota^*} \hat{H}^k(G, M) \xrightarrow{\pi^*} \hat{H}^k(G, M'') \xrightarrow{\delta} \hat{H}^{k+1}(G, M') \dots$$

for $k \in \mathbb{Z}$. In Section 8 we will be interested in the following structural property of $\mathbb{Z}G$-modules.

Definition 2.5.3. A $\mathbb{Z}G$-module M is said to be *cohomologically trivial* if $\hat{H}^n(H, M)$ vanishes for all $n \in \mathbb{Z}$ and all subgroups $H \subset G$.

In the following Theorem we combine a few useful results from [11, Section VI.8] which will help us decide when a module is cohomologically trivial.

Theorem 2.5.4. *(cf. [11, VI Theorem 8.7, VI Proposition 8.8]) Let G be a finite group and M be a $\mathbb{Z}G$-module. For every prime $p||G|$ choose a p-Sylow subgroup $G(p) < G$. Then M is cohomologically trivial if and only if it is cohomologically trivial viewed as a $\mathbb{Z}G(p)$-module for every prime p. Moreover, if G itself is a p-group, then M is cohomologically trivial if and only if $\hat{H}^n(G, M)$ vanishes for two consecutive integers.*

2.6 K-theory

The K-theory of a C*-algebra A consists of a pair of abelian groups $K_0(A)$ and $K_1(A)$. This association is functorial, meaning that both K_0 and K_1 are functors from **C*alg** to **Ab**. In this section we discuss some of the basics of K-theory. For more details on K-theory for C*-algebras see [119] and [144].

2.6.1 The K-theory groups

We recall the construction of the functors K_0 and K_1 in the case that A is a unital C*-algebra.

We start by setting up some notation. Let A be a unital C*-algebra and $n \in \mathbb{N}$. Denote by $P_n(A)$ the set of projections in $\mathbb{M}_n(A)$, i.e. $p \in \mathbb{M}_n(A)$ such that $p^2 = p$ and $p = p^*$. We omit the subscript 1 in $P_1(A)$ and write $P(A)$ for the projections in A. We say two projections $p, q \in P_n(A)$ are *Murray–von Neumann equivalent* if there exists a partial isometry $v \in \mathbb{M}_n(A)$ such that $vv^* = p$ and $v^*v = q$. We denote this by $p \sim q$. Denote by $P_\infty(A) = \bigcup_{n=1}^\infty P_n(A)$, where $P_n(A)$ is embedded into $P_{n+1}(A)$ in the top left corner and with a zero in the bottom right entry, i.e. $p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$.

We start by discussing K_0. If $p, q \in P_\infty(A)$, then define their direct sum to be $p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$.

Definition/Proposition 2.6.1 (see [119]). *Let A be a unital C*-algebra, then $P_\infty(A)/\sim$ is an abelian semigroup under the operation $\oplus$. Its Groethendieck completion is an abelian group denoted by $K_0(A)$.*

The Groethendieck completion is a way of producing a group from an abelian semigroup, see [119, Section 3.1.1] for details.

For any $n \in \mathbb{N}$, a projection $p \in P_n(A)$ has an image in $K_0(A)$, denote this image by $[p]_0$. For any element $x \in K_0(A)$, there exists some $n \in \mathbb{N}$ and $p, q \in \mathbb{M}_n(A)$ such that $x = [p]_0 - [q]_0$. (This is a feature of the Groethendieck completion, see [119, Proposition 3.1.7] for details.)

It is worth noting that, as we have passed to the Groethendieck completion of $P_\infty(A)/\sim$ to construct $K_0(A)$, it is no longer necessarily true that if $p, q \in P_\infty(A)$ and $[p]_0 = [q]_0$ then $p \sim q$. However, it is always true that if $[p]_0 = [q]_0$, then there exists some $n \in \mathbb{N}$ such that $p \oplus 1_A^{\oplus n} \sim q \oplus 1_A^{\oplus n}$ (see for example [119, Proposition 3.1.7]). A C*-algebra is said to satisfy *cancellation*, if for any $p, q \in P_\infty(A)$, then $[p]_0 = [q]_0$ implies that $p \sim q$. We will discuss the cancellation property further in Section 2.7.3.

We now discuss K_1. Denote by $U_n(A)$ the group of unitaries in $\mathbb{M}_n(A)$, i.e. $u \in \mathbb{M}_n(A)$ such that $uu^* = u^*u = 1_{\mathbb{M}_n(A)}$. We omit the subscript 1 in $U_1(A)$ and instead write $U(A)$. Set $U_\infty(A) = \bigcup_{n=1}^{\infty} U_n(A)$; this union is taken under the inclusion $U_n(A) \subset U_{n+1}(A)$ via the embedding $u \mapsto u \oplus 1_A$.[2]

Endow $U_\infty(A)$ with the direct limit topology, i.e. $V \subseteq U_\infty(A)$ is open if and only if $V \cap U_n(A) \subseteq U_n(A)$ is open for all $n \in \mathbb{N}$. We can now define K_1.

Definition/Proposition 2.6.2 ([119]). *Let A be a unital C^*-algebra, $K_1(A)$ is the space of path components of $U_\infty(A)$, i.e. $K_1(A) = \pi_0(U_\infty(A))$. It is an abelian group under multiplication.*

For any $n \in \mathbb{N}$, a unitary $u \in U_n(A)$ defines a class in $K_1(A)$; we denote this class by $[u]_1$. In fact, by definition, every class of $K_1(A)$ is represented by a unitary in some matrix amplification of A. Every compact subset $K \subseteq U_\infty(A)$ lies in $U_N(A)$ for some $N \in \mathbb{N}$ by [53, Lemma 1.7]. So continuous paths $[0,1] \to U_\infty(A)$ will factor through $U_N(A)$ for some $N \in \mathbb{N}$. It follows that for any two unitaries $u, v \in U_\infty(A)$, $[u]_1 = [v]_1$ if and only if there exists some $N \in \mathbb{N}$ such that u and v are homotopic in $\mathbb{M}_n(A)$.[3]

Similarly, we could've defined $K_1(A)$ as $\pi_0(Gl_\infty(A))$. This would yield the exact same group as every invertible element in a C*-algebra is homotopic to a unitary through its polar decomposition (see [119, Proposition 2.1.8]).

Let A, B be (unital) C*-algebras and $\varphi : A \to B$ a (unital) *-homomorphism. For any $n \in \mathbb{N}$, the (unital) *-homomorphism φ induces a (unital) *-homomorphism from $\mathbb{M}_n(A)$ to $\mathbb{M}_n(B)$. This homomorphism takes a matrix $(a_{i,j}) \in \mathbb{M}_n(A)$ to the matrix $(\varphi(a_{i,j})) \in \mathbb{M}_n(B)$. We will abuse notation and denote these *-homomorphisms also by φ.

[2]Although we had only defined the operation $\oplus$ for $P_\infty(A)$, its definition makes sense as a map from $\mathbb{M}_n(A) \times \mathbb{M}_m(A) \to \mathbb{M}_{n+m}(A)$.

[3]Two unitaries $u, v \in A$ are said to be homotopic in A ($u \sim_h v$), if there exists a continuous path of unitaries $u(t)$ in A such that $u(0) = u$ and $u(1) = v$.

We are now ready to discuss the functoriality of K-theory. If φ is a unital *-homomorphism, $K_0(\varphi)$ is defined by $K_0(\varphi)([p]_0 - [q]_0) = [\varphi(p)]_0 - [\varphi(q)]_0$ for $p, q \in P_\infty(A)$. Similarly, $K_1(\varphi)$ is defined by $K_1(\varphi)([u]_1) = [\varphi(u)]_1$ for $u \in U_\infty(A)$. This makes $K_0(\cdot)$ and $K_1(\cdot)$ functors from $\mathbf{C^*alg_1}$ to $\mathbf{Ab}$

Remark 2.6.3. In the not necessarily unital case, one needs to take care to extend the functors K_0 and K_1 to functors from the category $\mathbf{C^*alg}$ while still preserving half exactness. We sketch the not necessarily unital case in this remark.

Let A be a not necessarily unital C*-algebra. For a *-homomorphism $\varphi : A \to B$ we denote its unitization by $\tilde{\varphi} : \tilde{A} \to \tilde{B}$; this is defined by $\tilde{\varphi}(a + \lambda 1_{\tilde{A}}) = \varphi(a) + \lambda 1_{\tilde{B}}$. It is routine to check that $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ now defines a unital *-homomorphism. Denote by π_A the canonical homomorphism from $\tilde{A} \to \mathbb{C}$ defined by $\pi_A(a + \lambda 1_{\tilde{A}}) = \lambda$. The groups $K_i(A)$ are defined to be the kernel of the map $K_i(\pi_A) : K_i(A) \to K_i(\mathbb{C})$ for $i = 0, 1$. For a *-homomorphism $\varphi : A \to B$, one can induce a map of abelian groups $K_i(\varphi) : K_i(A) \to K_i(B)$ by setting $K_i(\varphi)(x) := K_i(\tilde{\varphi})(x)$. This definition coincides with our previous definitions when restricted to the category $\mathbf{C^*alg_1}$ ([119, 4.1.2, Proposition 8.1.6]).

We now state a few results about K-theory that we will need in later sections. We start with a useful corollary of Brown's stable isomorphism theorem ([12]).

Proposition 2.6.4. *(Brown cf. [146, Corollary 2.7.18]) Let p be a full projection in a C*-algebra A and $\iota : pAp \to A$ the inclusion.*[4] *Then the maps $K_0(\iota)$ and $K_1(\iota)$ are isomorphisms.*

In Chapter 8 we will require perturbation results for projections and unitaries. These results vaguely say that anything that almost acts like a projection/unitary algebraically, is close to a projection/unitary in norm. The results are folklore (see e.g. [119, Exercise 2.7, Exercise 2.8]) they follow as an application of functional calculus.

[4]An element a in a C*-algebra A is called full if the ideal generated by a is all of A.

Lemma 2.6.5. *Let A be a C^*-algebra. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $a \in A$ satisfies $\|a - a^*\| \leq \delta$ and $\|a^2 - a\| \leq \delta$, then there exists a projection $p \in A$ with $\|a - p\| \leq \varepsilon$.*

Lemma 2.6.6. *Let A be a C^*-algebra. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $a \in A$ satisfies $\|aa^* - 1\| \leq \delta$ and $\|a^*a - 1\| \leq \delta$, then there exists a unitary $u \in A$ with $\|a - u\| \leq \varepsilon$.*

Similarly we will also need to know that close projections and close unitaries are homotopic to one another.

Lemma 2.6.7. *(cf. [119, Proposition 2.2.4]) Let p, q be projections in a C^*-algebra A with $\|p - q\| < 1$, then $p \sim_h q$.*[5]

Lemma 2.6.8. *(cf. [119, Lemma 2.1.4]) Let u, v be unitary elements in a C^*-algebra A with $\|u - v\| < 2$, then $u \sim_h v$.*

Finally, almost Murray von Neumann equivalence of projections implies genuine Murray von Neumann equivalence.

Lemma 2.6.9. *Let p, q be projections in a C^*-algebra A. If there exists $a \in A$ such that $\|aa^* - p\| \leq 1/2$ and $\|a^*a - q\| \leq 1/2$ then $p \sim q$.*

2.6.2 Bott periodicity

In this section we discuss Bott periodicity ([2]). This important theorem of Atiyah and Bott gives an alternative description of the K_0 functor.

Note that $U_\infty(A)$ endowed with the inductive limit topology may not be a topological group, as multiplication may not be jointly continuous. In fact $U_\infty(A)$ will only be a topological group in the inductive limit topology if A is a finite dimensional

[5]In particular $p \sim q$. This follows as homotopic projections are Murray von Neumann equivalent.

C*-algebra (see the discussion in [21, Footnote 58, Section 2.2]). However, any homotopy $[0,1]^2 \to U_\infty(A)$ will factor through $U_N(A)$ for some $N \in \mathbb{N}$. In particular, following for example [9, III.2 Problem 1 and 2], yields that $\pi_1(U_\infty(A))$ is an abelian group.

Theorem 2.6.10 (Bott periodicity)**.** *Let A be a unital C*-algebra. The fundamental group of $U_\infty(A)$, denoted $\pi_1(U_\infty(A))$, is an abelian group under pointwise multiplication.*[6] *Moreover, the map $\varphi : K_0(A) \to \pi_1(U_\infty(A))$ given by $\varphi([p]_0)(t) = pe^{2\pi it} + (1-p)$ for $t \in [0,1]$, extends to a well defined isomorphism of abelian groups.*

Let A be a unital C*-algebra. It is worth noting that, due to the specific form of the isomorphism in Theorem 2.6.10, the Bott isomorphism induces a pointed isomorphism $(K_0(A), [1]) \cong (\pi_1(U_\infty(A)), [\epsilon_1])$ where by ϵ_1 we denote the path $\epsilon_1(t) = e^{2\pi it}$. For a proof of Theorem 2.6.10 see [119]; this proof is based on the proof of Bott and Atiyah in [2].

There are *higher K-functors* K_n for any $n \in \mathbb{N}$ (see [119, Chapter 10]). What Bott periodicity owes its name to, is that it implies that for any $n \in \mathbb{N}$ and C*-algebra A the groups $K_n(A)$ and $K_{n+2}(A)$ are isomorphic.

2.6.3 The Künneth formula and the UCT

In this subsection we discuss the universal coefficient theorem and the Künneth theorems of Rosenberg and Schochet ([121],[124]). The universal coefficient theorem (UCT) provides a way of computing the Kasparov bivariant KK theory groups (introdued in [81]) through the more tractable K-theory groups.

Kasparov's KK-theory is a bifunctor from the category of separable C*-algebras to the category of abelian groups. What this means is that for any two separable C*-algebras A and B, there is an abelian group $KK(A,B)$ and for any two

[6] in fact also concatenation, these two operations are the same by the Eckmann–Hilton argument (see [38, Theorem 1.12])

*-homomorphisms $\varphi : A \to C$ and $\psi : B \to D$, for C, D C*-algebras, one has induced morphisms of abelian groups $KK(\varphi, B) : KK(C, B) \to KK(A, B)$ and $KK(A, \psi)$: $KK(A, B) \to KK(A, D)$ such that $KK(A, \psi) \circ KK(\varphi, B) = KK(\varphi, D) \circ KK(C, \psi)$. This construction is such that $KK(\cdot, B)$ and $KK(A, \cdot)$ are contravariant and covariant functors respectively from $\mathbf{C^*alg}_{\text{sep}}$ to $\mathbf{Ab}$.

The abelian group $KK(A, B)$ is often viewed as a group of generalised morphisms between A and B. For example, any *-homomorphism from A to B defines a class in $KK(A, B)$. In fact, there is an associative product; $KK(A, B) \times KK(B, C) \to KK(A, C)$ that restricts to composition on *-homomorphisms ([81]). This product is called the *Kasparov product*. The Kasparov product allows us to construct the *KK-category*, whose objects are separable C*-algebras and whose morphisms are elements in KK; the composition operation of this category is given by the Kasparov product. The universal coefficient theorem ([121, Theorem 1.17]) states that for any C*-algebra that is isomorphic in the KK-category to an abelian C*-algebra, there is a natural short exact sequence that relates the KK bifunctor, with respect to any other C*-algebra, to the K_0 and K_1 functors. See [5, 23.1.1] for the precise statement of the UCT.

The class of C*-algebras that are isomorphic in the KK-category to an abelian C*-algebra is called the *UCT class* and denoted by $\mathcal{N}$. Many nuclear C*-algebras are known to be contained in $\mathcal{N}$. For example the Cuntz algebras (see Example 2.7.8), UHF-algebras and the Jiang-Su algebra $\mathcal{Z}$ (see Section 2.7). It is a major open problem whether every nuclear C*-algebra is contained in $\mathcal{N}$. Equivalently, whether every nuclear C*-algebra satisfies the conclusion of the UCT or in short *satisfies the UCT* (see [125, Proposition 5.3]). See [14] for a recent survey on the UCT problem. We refer to [5] for more details on $\mathcal{N}$ and the UCT.

We now state the Künneth formula for tensor products which will allow us to compute the K-theory of the tensor product of two C*-algebras.

Theorem 2.6.11 (Künneth formula for tensor products [124]). *Let A, B be C*-algebras with A in $\mathcal{N}$. There are short exact sequences*

$$0 \to \begin{matrix} (K_0(A) \otimes K_0(B)) \\ \oplus \\ (K_1(A) \otimes K_1(B)) \end{matrix} \to K_0(A \otimes B) \to \begin{matrix} \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K_1(B)) \\ \oplus \\ \mathrm{Tor}_1^{\mathbb{Z}}(K_1(A), K_0(B)) \end{matrix} \to 0$$

and

$$0 \to \begin{matrix} (K_0(A) \otimes K_1(B)) \\ \oplus \\ (K_1(A) \otimes K_0(B)) \end{matrix} \to K_1(A \otimes B) \to \begin{matrix} \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K_0(B)) \\ \oplus \\ \mathrm{Tor}_1^{\mathbb{Z}}(K_1(A), K_1(B)) \end{matrix} \to 0$$

that are natural in each entry.[7]

2.7 The classification programme

In analogy with the classification of injective factors, C*-algebraists have worked for decades on a classification of simple, nuclear C*-algebras. The first major classification result for C*-algebras is Elliott's classification of inductive limits of finite dimensional C*-algebras by their ordered K_0 group ([40]).[8] Further classification results for inductive limits of C*-algebras with nice building blocks in [41] and [42] lead for Elliott to conjecture that simple, separable, unital, nuclear C*-algebras are classified by their K-theoretic and tracial data.

A satisfactory classification result for a large class of simple, separable, nuclear C*-algebras was achieved in 2015. This is the culmination of decades of work and the involvement of many, we cite some of the major breakthroughs towards the achievement of this theorem ([40], [41], [42], [84], [109], [90], [148], [55], [43], [138], [23])

[7]See [145] for details on $\mathrm{Tor}_1^{\mathbb{Z}}$.

[8]We have not discussed the order structure of the K_0 group of a C*-algebra yet, see Section 2.7.3.

Theorem 2.7.1 (Classification Theorem)**.** *The class of infinite dimensional, simple, unital, separable, nuclear $\mathcal{Z}$-stable C*-algebras which satisfy the universal coefficient theorem are classified by K-theory and traces.*[9]

We refer to C*-algebras satisfying the hypothesis of Theorem 2.7.1 as *classifiable C*-algebras*. Notice that, unlike in the Connes–Haagerup classification, notions of amenability, simplicity and separability are not enough for classification. We have discussed the universal coefficient theorem in Section 2.6.3. As stated in Section 2.6.3, it is open whether every nuclear C*-algebra satisfies the UCT. If this were the case, one could remove the UCT as a hypothesis from Theorem 2.7.1. We will discuss $\mathcal{Z}$-stability in Section 2.7.1.

Remark 2.7.2. Apart from the UCT and $\mathcal{Z}$-stability, the classification result of Theorem 2.7.1 is in complete analogy with the classification of amenable factors with separable predual by Connes [27] and Haagerup [57].

The analogy is as follows. Every von Neumann factor has no non-trivial strong operator topology closed ideals and hence is simple as a W*-algebra ([135, Corollary 4.7]). Any von Neumann algebra M with separable predual is represented on a seperable Hilbert space; its unit ball in the strong operator topology is metrisable ([6, I.3.1.4]). Hence, the unit ball of M, and so M itself, is separable in the strong operator topology. For a von Neumann algebra, amenability is equivalent to semidiscreteness ([27, Theorem 6]); the norm version of semidiscreteness coincides with nuclearity on C*-algebras (see [15, Chapter 3.8]).

2.7.1 Jiang–Su stability

The Jiang-Su algebra $\mathcal{Z}$ was introduced in [73] by Jiang and Su as an infinite dimensional, simple, separable, unital, nuclear C*-algebra that has the same K-theory and

[9] See Section 2.7.2 for a precise statement.

traces as the complex numbers. That is $K_0(\mathcal{Z}) = \mathbb{Z}$, $K_1(\mathcal{Z}) = 0$ and $\mathcal{Z}$ has a unique trace. The original construction of the Jiang-Su algebra was as an ASH-algebra with subhomogeneous building blocks of the form

$$\mathcal{Z}_{p,q} = \{f \in C([0,1], \mathbb{M}_{pq}) : f(0) \in \mathbb{M}_p \otimes 1_{\mathbb{M}_q} \text{ and } f(1) \in 1_{\mathbb{M}_p} \otimes \mathbb{M}_q\} \tag{2.7.1}$$

for p, q relatively prime. Jiang and Su chose the connecting maps in such a way that it ensured that its inductive limit $\mathcal{Z}$ had the properties discussed in the previous paragraph. After Jiang and Su's construction, many alternative constructions followed ([120],[94],[123]). The existence of a C*-algebra with the properties of $\mathcal{Z}$ was a counterexample to early versions of the Elliott conjecture, as it had the same K-theoretic and tracial information as $\mathbb{C}$, yet it is infinite dimensional. Its existence implies structural properties on the class of C*-algebras that can be classified by K-theoretic and tracial information. Indeed, by the Künneth formula for tensor products, for any unital C*-algebra A, $K_0(A \otimes \mathcal{Z}) \cong K_0(A)$, $K_1(A \otimes \mathcal{Z}) \cong K_1(A)$ and also $T(A \otimes \mathcal{Z}) \cong T(A)$. The tracial isomorphism is given by sending $\tau \in T(A)$ to $\tau \otimes \tau_{\mathcal{Z}}$, where $\tau_{\mathcal{Z}}$ is the unique trace on $\mathcal{Z}$. This suggests that any invariant solely composed of tracial and K-theoretic information will not distinguish A and $A \otimes \mathcal{Z}$.

Definition 2.7.3. A C*-algebra A is said to be $\mathcal{Z}$ stable if $A \otimes \mathcal{Z} \cong A$.

In fact, further results in [140], clarify that any classification of a class of simple, separable, unital, nuclear C*-algebras outside of the $\mathcal{Z}$-stable case would require much wilder invariants.

2.7.2 The invariant

In this section, we discuss Theorem 2.7.1 in more detail. In particular, we elaborate on what is meant for a C*-algebra to be "classified by K-theory and traces". For this we begin by discussing the trace space $T(A)$ of a C*-algebra A.

Let A, B be unital C*-algebras and $\varphi : A \to B$ a *-homomorphism. One can induce a map $T(\varphi) : T(B) \to T(A)$ by $T(\varphi)(\tau) = \tau \circ \varphi$ for any trace $\tau \in T(B)$. This equips $T(\cdot)$ with the structure of a contravariant functor from $\mathbf{C^*alg}_1$ to the category of convex topological spaces.

We will mainly work with the covariant version of the functor $T(\cdot)$. Let $\mathrm{Aff}(T(A))$ be the space of affine, continuous, real valued functions on the trace space of A. For any unital C*-algebra A the real vector space $\mathrm{Aff}(T(A))$ is an Archimedean order unit space under the order induced by $\mathbb{R}$ pointwise. For a unital *-homomorphism $\varphi : A \to B$ define the unital, positive, continuous map $\mathrm{Aff}(T(\varphi))(f)(\tau) = f(\tau \circ \varphi)$ from $\mathrm{Aff}(T(A))$ to $\mathrm{Aff}(T(B))$. This makes $\mathrm{Aff}(T(\cdot))$ a covariant functor from $\mathbf{C^*alg}_1$ into the category of Archimedean order unit spaces. The functors $\mathrm{Aff}(T(\cdot))$ and $T(\cdot)$ capture the same information (this is explained in [21, Section 2.1]).

We now turn to the classification invariant, consisting of K-theoretic and tracial information, which underlies the statement of Theorem 2.7.1.

Definition 2.7.4. Let A be a unital C*-algebra. The *total invariant* of A is given by

$$KT_u(A) = \{(K_0(A), [1_A]_0), K_1(A), \mathrm{Aff}(T(A)), \rho_A : K_0(A) \to \mathrm{Aff}(T(A))\}.$$

We have seen the first three components of the invariant in previous sections. The *pairing map* ρ_A takes an element $m \in K_0(A)$ and produces an affine function on the trace space of A by $\rho_A(m)(\tau) = \tau(m)$ for each $\tau \in T(A)$. In most accounts of classification, the classification invariant of choice is the Elliott invariant (see for example [117],[55]). Under the assumptions of Theorem 2.7.1, the Elliott invariant captures the same information as the total invariant. In fact, the Elliott invariant carries unnecessary extra information. This is because the order structure on K_0 can be recovered from the pairing map under the assumptions of Theorem 2.7.1. Indeed, if A is a simple, unital, tracial, $\mathcal{Z}$-stable C*-algebra then $K_0(A)$ is a weakly unperforated ordered group by [54]. If moreover A is nuclear (exact is enough), then the positive cone of $K_0(A)$ can be recovered as those elements that are strictly positive under the pairing map (this follows from [127, Theorem 13.3.8] as every state on $K_0(A)$ when A is unital and exact is given by a trace of A [58]).

We now turn to describe the category that $KT_u(\cdot)$ maps into. For the purpose of this exposition we will denote it by **Ell**. The category **Ell** consists of quadruples $\{(G_0, e), G_1, S, \rho : G_0 \to S\}$ with (G_0, e) a pointed abelian group, G_1 an abelian group, S an Archimedean order unit space which is concretely realised as a function space and ρ an additive map sending e to an order unit of S. An morphism Φ from $\{(G_0, e), G_1, S, \rho : G_0 \to S\}$ to $\{(H_0, h), H_1, R, \eta : H_0 \to R\}$ in **Ell** consists of a triple $\Phi = (\phi_0, \phi_1, \psi)$ where $\phi_0 : (G_0, e) \to (H_0, h)$ is a pointed homomorphism of abelian groups, $\phi_1 : G_1 \to H_1$ is a homomorphism of abelian groups and $\psi : S \to R$ is a positive, continuous, unital map such that the diagram

$$\begin{array}{ccc} H_0 & \xrightarrow{\eta} & R \\ {\scriptstyle \varphi_0}\uparrow & & \uparrow{\scriptstyle \psi} \\ G_0 & \xrightarrow{\rho} & S \end{array} \tag{2.7.2}$$

commutes. It is clear that if $\varphi : A \to B$ is a unital *-homomorphism then the map $KT_u(\varphi) : KT_u(A) \to KT_u(B)$ consisting of the triple $(K_0(\varphi), K_1(\varphi), \mathrm{Aff}(T(\varphi)))$ induces a morphism between $KT_u(A)$ and $KT_u(B)$ in **Ell** making $KT_u(\cdot)$ a functor from $\mathbf{C^*alg}_1$ into **Ell**. We can now precisely state the classification theorem.

Theorem 2.7.5 (Precise formulation of Theorem 2.7.1)**.** *Let A, B be simple, unital, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras which satisfy the universal coefficient theorem. Then $A \cong B$ if and only if $KT_u(A) \cong KT_u(B)$. Moreover, for any X in Obj(**Ell**) there exists a simple, unital, separable, nuclear, $\mathcal{Z}$-stable C^*-algebra satisfying the universal coefficient theorem with $KT_u(A) \cong X$.*

2.7.3 A dichotomy

Due to Kirchberg's dichotomy ([117, Theorem 4.1.10]) any classifiable C*-algebra is either purely infinite or stably finite.

Definition 2.7.6. A C*-algebra A is said to be *finite* if no projection in A is Murray von Neumann equivalent to a proper subprojection. A is said to be *stably finite* if $\mathbb{M}_n(A)$ is finite for all $n \in \mathbb{N}$.

Examples of stably finite C*-algebras are AF-algebras (Example 2.3.2), irrational rotation algebras [33, Section VI] and the Jiang-Su algebra $\mathcal{Z}$. Under the assumption of nuclearity (in fact exactness is enough) every unital, simple, stably finite C*-algebra has at least one tracial state [58].

For a unital, stably finite C*-algebra A, its K-theory $K_0(A)$ becomes an *ordered abelian group* by letting the positive cone be $K_0(A)^+ = \{[p]_0 : p \in P(A)\}$ (see [119, Proposition 5.1.5]). Often, extra information is carried in the order structure of the K-theory groups. For example, in Elliott's classification of unital AF-algebras ([40]), the classification invariant is the pointed, ordered K_0-group. Elliott's theorem states that any two unital AF-algebras A and B are isomorphic if and only if $(K_0(A), K_0(A)^+, [1_A]_0) \cong (K_0(B), K_0(B)^+, [1_B]_0)$ i.e. there exists an isomorphism $\varphi : K_0(A) \to K_0(B)$ sending $K_0(A)^+$ to $K_0(B)^+$ and $[1_A]_0$ to $[1_B]_0$.

At the other side of the dichotomy are the purely infinite C*-algebras.

Definition 2.7.7. A simple, unital C*-algebra $A \neq \mathbb{C}$ is said to be *purely infinite* if for every non-zero $a \in A^+$ there exists $x \in A$ such that $1_A = xax^*$.

The motivating examples of simple, purely infinite C*-algebras are the Cuntz algebras introduced in [29].

Example 2.7.8. The *Cuntz algebra* $\mathcal{O}_n$ for $n \in \{2, 3, \dots\}$ are the universal C*-

algebras generated by n isometries s_k for $1 \leq k \leq n$ such that

$$\sum_{k=1}^{n} s_k s_k^* = 1.$$

It is easy to see that the Cuntz algebras are infinite. Indeed the generators s_k are proper isometries. In [29, 30] Cuntz shows that they are simple, purely infinite and $(K_0(\mathcal{O}_n), [1_{\mathcal{O}_n}], K_1(\mathcal{O}_n)) \cong (\mathbb{Z}_{n-1}, 1 \bmod (n-1), 0)$.

Similarly, Cuntz also introduces $\mathcal{O}_\infty$. The C*-algebra $\mathcal{O}_\infty$ is the C*-algebra generated by infinitely many isometries s_k for $k \in \mathbb{N}$ with orthogonal range projections. The C*-algebra $\mathcal{O}_\infty$ is also simple, purely infinite and $(K_0(\mathcal{O}_\infty), [1_{\mathcal{O}_\infty}], K_1(\mathcal{O}_\infty)) \cong (\mathbb{Z}, 1, 0)$.

A simple, purely infinite C*-algebra A has no traces. In fact, the definition implies that any two non-zero projections in A are Murray–von Neumann subequivalent to each other.[10] There are also notions of pure infiniteness for non-simple non-unital C*-algebras ([86]).

Those C*-algebras that are simple, purely infinite, separable and nuclear are called *Kirchberg algebras*. The Kirchberg–Phillips classification theorem ([109]) classifies Kirchberg algebras satisfying the UCT by their K-theory. In fact, the Kirchberg–Phillips classification does more, it says that any two unital Kirchberg algebras which are KK-equivalent, such that the KK-equivalence preserves the class of the unit, are isomorphic. The UCT is used to obtain KK-equivalence of Kirchberg algebras through an isomorphism of their K-groups. For unital Kirchberg algebras Theorem 2.7.1 simplifies to the Kirchberg–Phillips classification theorem.

Theorem 2.7.9. *(Kirchberg–Phillips cf. [109]) Let A, B be unital Kirchberg algebras satisfying the UCT then $A \cong B$ if and only if $(K_0(A), [1_A]_0) \cong (K_0(B), [1_B]_0)$ and $K_1(A) \cong K_1(B)$.*

[10] Let $p, q \in P(A)$, p is said to be Murray–von Neumann subequivalent to q ($p \preceq q$) if there exists a partialy isometry $v \in A$ with $vv^* = p$ and $v^*vq = v^*v$.

For the remaining part of this section we discuss the K-theory groups of simple, purely infinite C*-algebras.

Proposition 2.7.10 (cf. [30])**.** *Let A be a simple, purely infinite C*-algebra. For any two non-zero projections $p, q \in P(A)$, $[p]_0 = [q]_0$ implies that $p \sim q$. Moreover, the map*

$$\{[p]_D \mid p \neq 0 \in P(A)\} \to K_0(A)$$

is a bijection of sets.[11]

Proposition 2.7.10 says that for a simple, purely infinite C*-algebra A, any class in $K_0(A)$ can be represented by a projection in A; one does not need to consider differences of projections in matrix amplifications of A. It also implies that any two non-zero projections that represent the same class in $K_0(A)$ are genuinely Murray von Neumann equivalent in A. We call this second property *cancellation of non-zero projections*. However, cancellation does not in general hold for Kirchberg algebras. This can be seen for example in $\mathcal{O}_2$, as there are non-trivial projections that are equivalent to 0. On the other hand, any simple, nuclear, $\mathcal{Z}$-stable, finite C*-algebra satisfies cancellation (see [118, Theorem 6.7] and [5, Proposition 6.5.1]).

The K_1 group of a purely infinite C*-algebra also behaves nicely.

Proposition 2.7.11 (cf. [30, Theorem 1.9])**.** *Let A be a unital, simple, purely infinite C*-algebra. Then the canonical map $\pi_0(U(A)) \to K_1(A)$ is an isomorphism.*

Remark 2.7.12. If A is unital, separable and simple, it follows from [66, Lemma 2.3](cf. [7, Theorem 1.4 (a)]) that $B = A \otimes \mathcal{O}_\infty$ is unital, simple, purely infinite. By the Künneth formula, $K_0(A) \cong K_0(B)$ and $K_1(A) \cong K_1(B)$. In particular, any unital, simple, separable C*-algebra has a unital, simple, separable, purely infinite C*-algebra with the same K-theory.

[11]For $p \in P(A)$, $[p]_D$ denotes the Murray von Neumann equivalence class of p in A.

We now turn to a particular class of Kirchberg algebras. A unital Kirchberg algebra A is said to be in *Cuntz standard form* if the class $[1_A]_0$ is zero in $K_0(A)$. Any Kirchberg algebra A contains a Kirchberg algebra A^{st} as a corner such that $K_0(A^{st}) = K_0(A)$ and $K_1(A^{st}) = K_1(A)$ but A^{st} is in Cuntz standard form. One can build A^{st} by using pure infiniteness of A to realise a projection $p \in A$ with $p \sim 1_A$ but $p < 1_A$ strictly. The orthogonal projection $1-p$ is a non-zero projection in A with $[1-p]_0 = 0$. By Brown's stable isomorphism theorem [12] the full corner $(1-p)A(1-p)$ is stably isomorphic to A and therefore $K_i((1-p)A(1-p)) \cong K_i(A)$ for $i = 0, 1$ and so $(1-p)A(1-p)$ is an instance of A^{st}. C*-algebras in Cuntz standard form will make an appearance in Chapters 5 and 6.

2.7.4 Tracially AF-algebras

After successfully classifying Kirchberg algebras, the focus of the classification programme shifted to the stably finite setting. This proved to be significantly more complex due to the presence of traces. Early progress was made for C*-algebras arising from inductive limits of well-defined building blocks. A significant breakthrough occurred in [90], Lin achieved a conceptual leap by classifying C*-algebras without relying on an inductive limit decomposition. Lin's classification encompassed C*-algebras that can be locally approximated by finite-dimensional C*-algebras in trace.

Definition 2.7.13. (cf. [90, Section 2]) A simple, unital C*-algebra A is *tracially AF* (*TAF*) if for all $\varepsilon > 0$, $\mathcal{F} \subset A$ finite and $a \in A^+ \setminus \{0\}$, there exists a non-zero projection p and finite dimensional C*-subalgebra $B \subset A$ with $1_B = p$ such that

(i) $\|px - xp\| \leq \varepsilon$ for $x \in \mathcal{F}$,

(ii) $\mathrm{dist}(pxp, B) < \varepsilon$ for $x \in \mathcal{F}$,

(iii) $1-p$ is equivalent to a projection in $\overline{aAa}$.

Under the presence of strict comparison of projections by traces (in the sense of [141, Section 2]), condition (iii) can be replaced with the condition that $\tau(1-p) < \varepsilon$ for every $\tau \in T(A)$. [12]

In [95, Theorem 6.1] it is shown that any unital, separable, simple, nuclear C*-algebra that absorbs a UHF algebra and is quasidiagonal, is a tracially AF C*-algebra. Quasidiagonality is an external approximation property that can be more easily verified than the local approximations of TAF algebras. Moreover, combining Matui–Sato's result with the quasidiagonality theorem of Tikuisis, White and Winter we get the following.

Theorem 2.7.14. *(cf. [95, Theorem 6.1] and [138, Corollary B]) Let A be a unital, separable, simple, nuclear C*-algebra satisfying the UCT such that A has a unique trace and $A \otimes B \cong A$ for some UHF-algebra B. Then A is tracially AF.*

In [90] Lin classifies unital, simple, separable, nuclear TAF algebras satisfying the UCT by their pointed, ordered K-groups.[13]

Theorem 2.7.15. *(Lin cf. [90, Theorem 5.2]) Let A and B be two unital, simple, separable, nuclear TAF algebras satisfying the UCT. Then $A \cong B$ if and only if $((K_0(A), K_0(A)^+, [1_A]_0), K_1(A)) \cong ((K_0(B), K_0(B)^+, [1_B]_0), K_1(B))$*

[12]This property is immediate for many of the C*-algebras we consider. For example by [118, Corollary 4.6] any unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebra satisfies strict comparison of positive elements by traces.

[13]Every unital, simple TAF algebra quasidiagonal and hence stably finite ([89, Theorem 3.4]), so their K_0 group comes with a prescribed order.

Chapter 3

An introduction to anomalous symmetries

In this chapter we set out the technical preliminaries to anomalous actions. The material in Sections 3.2 and 3.3 is based on that of [46, Section 2].

3.1 Symmetries of operator algebras

We start by setting up notation. Let A be a C*-algebra. We denote by $\mathrm{Aut}(A)$ the group of *-automorphisms of A. Any unitary $u \in U(M(A))$ induces an automorphism of A by conjugation which we denote by $\mathrm{Ad}(u)$, such an automorphism is called *inner*. Our convention is that $\mathrm{Ad}(u)(a) = uau^*$ for any $a \in A$. The collection of all inner automorphisms of A form a normal subgroup of $\mathrm{Aut}(A)$; we denote it by $\mathrm{Inn}(A)$. Indeed, observe that if $\alpha \in \mathrm{Aut}(A), u \in U(M(A))$ and $a \in A$

$$\alpha \mathrm{Ad}(u)\alpha^{-1}(a) = \alpha(u\alpha^{-1}(a)u^*) = \mathrm{Ad}(\overline{\alpha}(u))a \tag{3.1.1}$$

for the unique extension $\overline{\alpha}$ of α to $\mathrm{Aut}(M(A))$ (see [16, Proposition 3.8]). From now on we will denote the extension of an automorphism $\alpha \in \mathrm{Aut}(A)$ to an automorphism of $M(A)$ also by α. An automorphism on A is called *outer* if it is not contained in $\mathrm{Inn}(A)$. We denote by $\mathrm{Out}(A)$ the quotient group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$.

Usually, a symmetry of an operator algebra is encoded in a *group action*. That is, there exists a group G and a group homomorphism $G \to \mathrm{Aut}(A)$. In this chapter we discuss anomalous actions, they are a more general notion of symmetry than that of a group action. For a physical motivation of anomalous actions see [74, Section 2.3].

Anomalous actions fit into the framework of quantum symmetries. Quantum symmetries encode actions of higher dimensional objects; examples are those arising from subfactor theory ([97],[110],[45],[76]). There are multiple ways to encode quantum symmetries. The viewpoint that we take is to start with the notion of a group action, replace the acting group by a category equipped with a multiplication operation that need no longer be invertible and $\mathrm{Aut}(A)$ with an appropriate category of symmetries of A that contains non-invertible elements. A precise definition of these generalised symmetries, and how anomalous actions fit into this framework, will be discussed in Chapter 9.

3.2 G-kernels

Before discussing anomalous actions we start by discussing the closely related notion of G-kernels.

Definition 3.2.1. Let G be a group. A *G-kernel* on A is a group homomorphism $\overline{\theta} : G \to \mathrm{Out}(A)$.

Motivated by an analogous concept of the same name in group theory ([39]), the study of G-kernels in the setting of von Neumann algebras was initiated by Nakamura and Takeda ([102, 134]) and developed by Sutherland ([128]) and Jones ([78]).

Fix a G-kernel $\overline{\theta} : G \to \mathrm{Out}(A)$. We now discuss the H^3 invariant associated to a G-kernel. The H^3 invariant of a G-kernel was first defined in [39] for G-kernels on groups (defined as in Definition 3.2.1 but exchanging the C*-algebra A with another group H). The calculations in [39] also work in the setting of Definition 3.2.1, we recall the construction of the H^3 invariant.

For each $g \in G$, choose a lift $\theta_g \in \mathrm{Aut}(A)$ for $\overline{\theta}(g)$. Since $\overline{\theta}$ is a group homomorphism, there exist unitaries $u_{g,h} \in U(M(A))$ such that $\theta_g\theta_h = \mathrm{Ad}(u_{g,h})\theta_{gh}$ for each $g, h \in G$. Given $g, h, k \in G$, we may use associativity in $\mathrm{Aut}(A)$ to compute $\theta_g\theta_h\theta_k$ in two different ways. Firstly, we have

$$\begin{aligned}(\theta_g\theta_h)\theta_k &= \mathrm{Ad}(u_{g,h})\theta_{gh}\theta_k \\ &= \mathrm{Ad}(u_{g,h})\mathrm{Ad}(u_{gh,k})\theta_{ghk} \\ &= \mathrm{Ad}(u_{g,h}u_{gh,k})\theta_{ghk}.\end{aligned} \tag{3.2.1}$$

Secondly, we have

$$\begin{aligned}\theta_g(\theta_h\theta_k) &= \theta_g\mathrm{Ad}(u_{h,k})\theta_{hk} \\ &= \mathrm{Ad}(\theta_g(u_{h,k}))\theta_g\theta_{hk} \\ &= \mathrm{Ad}(\theta_g(u_{h,k}))\mathrm{Ad}(u_{g,hk})\theta_g\theta_h\theta_k \\ &= \mathrm{Ad}(\theta_g(u_{h,k})u_{g,hk})\theta_{ghk}.\end{aligned} \tag{3.2.2}$$

Hence, $\mathrm{Ad}(\theta_g(u_{h,k})u_{g,hk}) = \mathrm{Ad}(u_{g,h}u_{gh,k})$ for all $g, h, k \in G$.

The kernel of the group homomorphism $\mathrm{Ad} : U(M(A)) \to \mathrm{Aut}(A)$ is the centre of the unitary group $Z(U(M(A)))$. Therefore, we may define a function $\omega : G^{\times 3} \to Z(U(M(A)))$ by

$$\omega(g, h, k) = \theta_g(u_{h,k})u_{g,hk}u^*_{gh,k}u^*_{g,h} \tag{3.2.3}$$

for all $g, h, k \in G$.

The group $Z(U(M(A)))$ is abelian, and can be endowed with a $\mathbb{Z}G$-module structure where g acts via $\theta_g|_{Z(U(M(A)))}$. So ω is a 3-cochain with coefficients in $Z(U(A))$.

Moreover, a simple but tedious computation shows that $d\omega = 0$; see [39, Lemma 7.1]. Hence, $\omega \in Z^3(G, Z(U(M(A))))$. The cohomology class $[\omega] \in H^3(G, Z(U(M(A))))$ is the H^3 *invariant* of the G-kernel $\bar{\theta}$, we denote it by $\mathrm{ob}(\overline{\theta})$. It does not depend on the choice of lifts θ_g or the choice of unitaries $u_{g,h}$; see [39, Theorem 7.1].

3.3 Anomalous actions

The main objects of study in this thesis are anomalous actions.

Definition 3.3.1 ([74, Definition 1.1])**.** An *anomalous action* of a countable discrete group G on a C*-algebra A consists of a pair (α, u) where

$$\alpha : G \to \mathrm{Aut}(A)$$

$$u : G \times G \to U(M(A))$$

are a pair of maps such that

$$\alpha_g \alpha_h = \mathrm{Ad}(u_{g,h})\alpha_{gh}, \text{ for all } g, h \in G, \tag{3.3.1}$$

$$\alpha_g(u_{h,k})u_{g,hk}u^*_{gh,k}u^*_{g,h} \in \mathbb{T} \cdot 1_{M(A)}, \text{for all } g, h, k \in G. \tag{3.3.2}$$

We use lowercase notation to improve readability, i.e. $\alpha(g) = \alpha_g$ and $u(g, h) = u_{g,h}$. As explained in Section 3.2 the formula in (3.3.2) defines a 3-cocycle $\omega \in Z^3(G, \mathbb{T})$, this is called the *anomaly* of the action. We often write (α, u) *is a* (G, ω) *action on* A instead of (α, u) is an anomalous action of G on A. We may also say (α, u) is an ω-anomalous action of G on A. We will use $o(\alpha, u)$ to denote the anomaly of (α, u) and $\mathrm{ob}(\alpha, u) = [o(\alpha, u)]$ to denote the class that ω represents in $H^3(G, \mathbb{T})$.

There is a slight change of conventions in Definition 3.3.1 to [74, Definition 1.1] and [46, Definition 2.1]. We make this change to be in tune with the conventions on cocycle actions in the C*-algebra literature (see e.g. [66]). Where a *cocycle action* is an anomalous action with unitary product in (3.3.2) being identically one. Given our conventions in Definition 3.3.1, a (G, ω) action induces an $\overline{\omega}$-anomalous action as in [74, Definition 1.1], this is seen by taking $m_{g,h} = u^*_{g,h}$.

Remark 3.3.2. Suppose ω is a normalized 3-cocycle and (α, u) is a (G, ω) action. Then one can construct a new anomalous action (α', u') such that $\alpha'_g = \alpha_g$, $u'_{g,h} = u_{g,h}$ for all $g, h \in G \setminus \{1_G\}$ and $\alpha'_{1_G} = \mathrm{id}_A$, $u_{g,1_G} = u_{1_G,g} = 1$ for all $g \in G$. The pair (α', u') still satisfies (3.3.1) and condition (3.3.2).

Remark 3.3.3. Suppose $Z(M(A)) = \mathbb{C}$. Then every G-kernel lifts to an ω-anomalous action of G on A, where the cohomology class of ω coincides with the H^3 invariant. This follows from the derivation of the H^3 invariant in Section 3.2 as $Z(U(M(A))) = U(Z(M(A))) = \mathbb{T}$. Conversely, any (G, ω) action (α, u) on A induces a G-kernel $\overline{\alpha}$ by setting $\overline{\alpha}_g = \alpha_g + \mathrm{Inn}(A) \in \mathrm{Out}(A)$. This induced G-kernel has H^3 invariant $[\omega]$.

Moreover, suppose (α, u) is a (G, ω)-action on A with ω' is cohomologous to ω. Then $\omega' = d\lambda \cdot \omega$ for some 2-cochain $\lambda \in C^2(G, \mathbb{T})$. Setting $u'_{g,h} = \lambda(g, h)u_{g,h}$, we have that (α, u') is an ω'-anomalous action of G on A.

Due to Remark 3.3.3, we will often use the terminology of anomalous actions and G-kernels interchangably in the case that $Z(M(A)) = \mathbb{C}$. In this thesis we will mainly be interested in simple C*-algebras, in which case $Z(M(A)) = \mathbb{C}$ is automatic.

Proposition 3.3.4. *Let A be a simple C*-algebra then $Z(M(A)) = \mathbb{C}$.*

Proof. By definition $Z(M(A))$ is an abelian C*-algebra. Therefore, if $Z(M(A))$ is non-trivial there exists two non-trivial orthogonal elements $x, y \in Z(M(A))$. The ideals $\overline{xA}$ and $\overline{yA}$ have trivial intersection as $\overline{xA} \cap \overline{yA} = \overline{xA} \cdot \overline{yA} = 0$. However, as A is simple, $\overline{xA} = A = \overline{yA}$ which is a contradiction. □

Remark 3.3.5. When G is a group and α is a group action on A, the family of automorphisms α_g also define a group action on A_∞. Similarly, if $B \subset A_\infty$ is an

α invariant subalgebra the automorphisms α_g define a group action on $A_\infty \cap B'$. If (α, u) is a anomalous action on a unital C*-algebra A it need not induce an anomalous action on $A_\infty \cap A'$ as $u_{g,h}$ will not be a unitary in $A_\infty \cap A'$. However, the assignment $\alpha_g : (a_n) \mapsto (\alpha_g(a_n))$ induces a group action on $A_\infty \cap A'$. Indeed if $a = (a_n) \in A_\infty \cap A'$ then $\alpha_g \alpha_h(a) = (\alpha_g \alpha_h(a_n)) = (\mathrm{Ad}(u_{g,h})\alpha_{gh}(a_n)) = \alpha_{gh}(a_n)$ as any inner automorphism of A is trivial on central sequences of A.

We now discuss a few constructions which will be useful to keep in mind for later sections. Firstly, if $\varphi : A \to B$ is an isomorphism of C*-algebras and (α, u) is a (G, ω) action on B then the pair $(\varphi^{-1}\alpha\varphi, \varphi^{-1}(u))$ with $(\varphi^{-1}\alpha\varphi)_g = \varphi^{-1}\alpha_g\varphi$ and $\varphi^{-1}(u)_{g,h} = \varphi^{-1}(u_{g,h})$ is a (G, ω) action on A.

If $\rho : G \to Q$ is a homomorphism of groups, one can induce a $(G, \rho^*(\omega))$ action on A from a (Q, ω) action on A. Indeed, suppose (α, u) is a (Q, ω) action on A, define the pair $(\rho^*(\alpha), \rho^*(u))$ by

$$\rho^*(\alpha)_g(a) = \alpha_{\rho(g)}(a),$$
$$\rho^*(u)_{g,h} = u_{\rho(g),\rho(h)}$$

for all $g, h \in G$. It is a straightforward computation that $(\rho^*(\alpha), \rho^*(u))$ is a $(G, \rho^*(\omega))$ action on A.

Another construction that will be useful for us is the tensor product of anomalous actions. If A, B are C*-algebras and (α, u) is a (G, ω) action on A and (β, v) is a (G, ω') action on B then $(\alpha, u) \otimes (\beta, v)$ defined by the pair $(\alpha \otimes \beta, u \otimes v)$ is an anomalous action on $A \otimes B$. It is a routine computation to show that the anomaly of the tensor product $(\alpha, u) \otimes (\beta, v)$ corresponds to the product of 3-cocycles $\omega\omega'$.

3.3.1 Anomalous actions of cyclic groups

We now turn to discuss anomalous actions of cyclic groups. Let $G = \mathbb{Z}_n$ and $\mathbb{T}$ be equipped with the trivial $\mathbb{Z}G$-module structure, using the periodic resolution, it is standard to show that $H^3(\mathbb{Z}_n, \mathbb{T}) \cong \mathbb{Z}_n$. In fact, specific cocycle representatives for the cohomology classes have been computed in [63, Proposition 2.3]. We write the formulas here for convenience. Let $\xi = e^{2\pi i/n}$ and g be the class of $1 + n\mathbb{Z}$ in $\mathbb{Z}_n$; a complete set of normalised cocycle representatives for $H^3(\mathbb{Z}_n, \mathbb{T})$ is given by

$$\lambda_k(g^l, g^r, g^s) = \xi^{\frac{kl(r+s-[r+s])}{n}} \tag{3.3.3}$$

for $0 \leq k, l, r, s \leq n-1$.[1] By Remark 3.3.3, the existence of (G, ω) actions is independent of the choice of representative cocycle in $[\omega]$. In particular, one can use the specific form of the representative cocycles λ_k in $H^3(\mathbb{Z}_n, \mathbb{T})$ as in (3.3.3), to package the information encoded in an anomalous action of a cyclic group only in terms of one unitary and one automorphism.

Lemma 3.3.6. *Let A be a C^*-algebra and $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$ with $k \in \{0, 1, 2, \dots, n-1\}$ such that $[\omega] = [\lambda_k]$. There exists a $(\mathbb{Z}_n, \omega)$ action on A if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(A)$ and a unitary $u \in U(M(A))$ with $\alpha^n = \mathrm{Ad}(u)$ and $\alpha(u) = \xi^k u$.*

Proof. Suppose we have an automorphism α, a unitary u and $0 \leq k \leq n-1$ such that $\alpha^n = \mathrm{Ad}(u)$ and $\alpha(u) = \xi^k u$. We produce a $(\mathbb{Z}_n, \omega)$ action on A by letting $\alpha_{g^m} = \alpha^m$ for $0 \leq m \leq n-1$, $u_{g^i, g^j} = 1$ whenever $i + j < n$ and $u_{g^i, g^j} = u$ whenever $i + j \geq n$ for $0 \leq i, j \leq n-1$. It is a straightforward calculation that (3.3.2) is satisfied for λ_k.

For the reverse direction, suppose (α, u) is a $(\mathbb{Z}_n, \omega)$-action on A. Pick $0 \leq k \leq n-1$ such that $[\lambda_k] = [\omega]$. By Remark 3.3.3, one may assume that (α, u) is a $(\mathbb{Z}_n, \lambda_k)$-action on A. By Remark 3.3.2, as the cocycle is normalized, one may also assume

[1] $r + s$ is the sum in $\mathbb{Z}$ and $[r + s]$ the sum modulo n.

that $u_{g^m,1} = u_{1,g^m} = 1$ and $\alpha_1 = \mathrm{id}_A$ for all $m \in \mathbb{Z}$. In particular, (α, u) consists of $n-1$ automorphisms α_{g^i} for $1 \le i < n$ and $(n-1)^2$ unitaries u_{g^l,g^s} for $1 \le l, s < n$. It follows by induction that $(\alpha_g)^m = \mathrm{Ad}(u_{g,g}u_{g^2,g} \dots u_{g^{m-1},g})\alpha_{g^m}$ for $m \in \mathbb{Z}$. Indeed, the $m = 2$ case is immediate and if it holds for $m-1$ then by the induction hypothesis

$$(\alpha_g)^m = (\alpha_g)^{m-1}\alpha_g = \mathrm{Ad}(u_{g,g}u_{g^2,g} \dots u_{g^{m-2},g})\alpha_{g^{m-1}}\alpha_g = \mathrm{Ad}(u_{g,g}u_{g^2,g} \dots u_{g^{m-1},g})\alpha_{g^m}.$$

In particular, it follows that $\alpha_g^n = \mathrm{Ad}(u_{g,g}u_{g^2,g} \dots u_{g^{n-1},g})$. It remains to show that $\alpha_g(u_{g,g} \dots u_{g^{n-1},g}) = \xi^k u_{g,g} \dots u_{g^{n-1},g}$ in which case the automorphism α_g and the unitary $u_{g,g}u_{g^2,g} \dots u_{g^{n-1},g}$ is the desired pair. Indeed, using the specific form of the cocycle λ_k (see (3.3.3))

$$\xi^k \overset{(3.3.2)}{=} \lambda_k(g, g^{n-1}, g) = \alpha_g(u_{g^{n-1},g})u^*_{g,g^{n-1}}, \tag{3.3.4}$$

$$1 \overset{(3.3.2)}{=} \lambda_k(g, g^j, g) = \alpha_g(u_{g^j,g})u_{g,g^{j+1}}u^*_{g^{j+1},g}u^*_{g,g^j}, \tag{3.3.5}$$

with (3.3.5) holding for all $1 \le j < n-1$. It follows that

$$\begin{aligned}
\alpha_g(u_{g,g}u_{g^2,g} \dots u_{g^{n-1},g}) &= \alpha_g(u_{g,g}u_{g^2,g} \dots u_{g^{n-2},g})\alpha_g(u_{g^{n-1},g}) \\
&\overset{(3.3.4)}{=} \xi^k \alpha_g(u_{g,g}u_{g^2,g} \dots u_{g^{n-2},g})u_{g,g^{n-1}} \\
&= \xi^k \alpha_g(u_{g,g}u_{g^2,g} \dots u_{g^{n-3},g})\alpha_g(u_{g^{n-2},g})u_{g,g^{n-1}} \\
&\overset{(3.3.5)}{=} \xi^k \alpha_g(u_{g,g}u_{g^2,g} \dots u_{g^{n-3},g})u_{g,g^{n-2}}u_{g^{n-1},g}.
\end{aligned}$$

Now repeatedly applying (3.3.5)

$$\alpha_g(u_{g,g} \dots u_{g^{n-1},g}) = \xi^k u_{g,g} \dots u_{g^{n-1},g}.$$

□

As part of the proof of Lemma 3.3.6 we have shown that if A is a C*-algebra with a pair $\alpha \in \operatorname{Aut}(A)$ and $u \in U(M(A))$ such that $\alpha^n = \operatorname{Ad}(u)$ and $\alpha(u) = \gamma u$ for some n-th root of unity γ, then one can induce a $\mathbb{Z}_n$-anomalous action (α, u). When a $\mathbb{Z}_n$-anomalous action is of this form for some automorphism α, we will say it *arises from an automorphism*. In Section 3.5 we show that every anomalous action of a cyclic group arises from an automorphism.

3.3.2 Anomalous actions on $\mathcal{R}$

In [28], Connes considers automorphisms $\alpha \in \operatorname{Aut}(\mathcal{R})$ such that $\alpha^n = \operatorname{Ad}(u)$ for some $u \in U(\mathcal{R})$. He shows that $\alpha(u) = \gamma u$ for some n-th root of unity $\gamma \in \mathbb{C}$. These pairs (α, u), as above, induce anomalous actions on $\mathcal{R}$ (see Lemma 3.3.6).

For any $n \in \mathbb{N}$ and n^{th} root of unity γ, Connes constructs an automorphism s_n^γ with order n in $\operatorname{Out}(\mathcal{R})$ such that $(s_n^\gamma)^n = \operatorname{Ad}(u)$ and $s_n^\gamma(u) = \gamma u$ for some $u \in U(\mathcal{R})$. Connes result is recorded below.

Theorem 3.3.7 (Connes cf. [28, Proposition 1.6])**.** *Fix $n \in \mathbb{N}$, $\mathcal{R} = \overline{\bigotimes}_{i\in\mathbb{N}}(\mathbb{M}_n, \operatorname{tr}_n)$. Let $\pi_i : \mathbb{M}_n \to \mathcal{R}$ be the embedding into the i-th tensor factor, and let $\theta : \mathcal{R} \to \mathcal{R}$ be the endomorphism such that $\theta\pi_i = \pi_{i+1}$ for all $i \in \mathbb{N}$.*

Let γ be an n-th root of unity. Set

$$u = \sum_{j=1}^{n} \gamma^j \pi_1(e_{j,j}) \tag{3.3.6}$$

$$v = \pi_1(e_{n,1})\theta(u) + \sum_{j=1}^{n-1} \pi_1(e_{j,j+1}). \tag{3.3.7}$$

Then the sequence $(\operatorname{Ad}(v\theta(v)\theta^2(v)\cdots\theta^k(v)))_{k=1}^\infty$ converges pointwise in the $\|\cdot\|_2$-norm topology to an automorphism s_n^γ such that $(s_n^\gamma)^n = \operatorname{Ad}(u)$ and $s_n^\gamma(u) = \gamma u$.

Remark 3.3.8. (cf. [28, Proposition 1.6]) For any element of the algebraic tensor product $\bigodot_{i\in\mathbb{N}} \mathbb{M}_n$, Connes shows that the sequence $(\operatorname{Ad}(v\theta(v)\theta^2(v)\cdots\theta^k(v))(x))_{k=1}^\infty$ is eventually constant. It follows that s_n^γ restricts to an automorphism of the UHF algebra $\bigotimes_{i\in\mathbb{N}} \mathbb{M}_n$ and $(\mathrm{Ad}(v\theta(v)\theta^2(v)\cdots\theta^k(v)))_{k=1}^\infty$ converges in the point norm topology on this subalgebra.

As γ ranges over all n-th roots of unity, there is an ω-anomalous $\mathbb{Z}_n$-action on $\mathcal{R}$ for any $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$. Connes result was later generalised by Jones to cover all countable discrete groups.

Theorem 3.3.9 (Jones cf. [77, Theorem 2.5])**.** *Let G be a countable discrete group and $\omega \in Z^3(G, \mathbb{T})$. Then there exists a (G, ω) action on $\mathcal{R}$.*

In Jones' construction, $\mathcal{R}$ is realised as a (twisted) crossed product. We will revisit this construction in Section 6.

3.4 Classification of actions

Following the classification of injective factors there was a great body of work towards classifying symmetries of injective factors. Before we state these results, we discuss the notion(s) of equivalence of group actions on C*-algebras, that usually get considered for classification in the literature.

Definition 3.4.1. Let A, B be C*-algebras and $\alpha : G \to \mathrm{Aut}(A)$, $\beta : G \to \mathrm{Aut}(A)$ be group actions. Then we say that

- α is *conjugate* to β, if there exists an isomorphism $\theta : A \to B$ such that $\alpha_g = \theta\beta_g\theta^{-1}$ for all $g \in G$. We denote it by $\alpha \sim \beta$.
- α is *cocycle conjugate* to β, if there exist unitaries $u_g \in U(M(A))$ for $g \in G$ such that $u_g\alpha(u_h) = u_{gh}$ and the group action $\alpha'_g = \mathrm{Ad}(u_g)\alpha_g$ is conjugate to β. We denote this by $\alpha \simeq \beta$.
- α is *outer conjugate* to β, if the induced G-kernels $\overline{\alpha}$ and $\overline{\beta}$ are conjugate. Or equivalently, if there exists an isomorphism $\theta : A \to B$ such that $\overline{\alpha} = \overline{\theta\beta\theta^{-1}}$.

A family of unitaries u_g for $g \in G$ such that $u_g \alpha_g(u_h) = u_{gh}$ (as used in the definition for cocycle conjugacy) is called an *α-cocycle*. Conjugacy implies cocycle conjugacy which in turn implies outer conjugacy. Usually, conjugacy is too fine of an equivalence relation for classification (see e.g. [68, Section 2.2]). Hence, most classification results for actions are done up to cocycle conjugacy. It also makes sense to talk about outer conjugacy of G-kernels, in fact the setting of G-kernels is a more natural setting for this equivalence relation.

The classification of group actions on injective factors was a major success. In [26] and [28] Connes classified actions of cyclic groups on $\mathcal{R}$, Jones then classified actions of finite groups on $\mathcal{R}$ ([78]) and Ocneanu further generalised these results to the case of discrete amenable groups acting on $\mathcal{R}$ ([105]). We package a special case of these classification results below. An action α of a group G on a C*-algebra A is called *pointwise outer* or *free* if α_g is an outer automorphism for all $g \in G$.

Theorem 3.4.2 ([28],[26],[78],[105])**.** *Let G be a discrete, amenable group. Then any two pointwise outer actions $\alpha, \beta : G \curvearrowright \mathcal{R}$ are cocycle conjugate.*

Connes, Jones and Ocneanu also achieve a classification result for G-kernels.

Theorem 3.4.3 ([28],[26],[78],[105])**.** *Let G be a discrete, amenable group, then any two pointwise outer G-kernels $\overline{\theta}$ and $\overline{\beta}$ on $\mathcal{R}$ are outer conjugate if and only if* $\mathrm{ob}(\overline{\theta}) = \mathrm{ob}(\overline{\beta})$.

The classification of actions of discrete amenable groups on type III factors was completed in [82]; that of G-kernels in [83] (see also [93] for a unified approach).

In analogy to the classification of group actions on injective factors, there has also been substantial work towards classifying group actions on classifiable C*-algebras. We will mainly discuss the case of finite groups.

Uniqueness results, as in Theorem 3.4.2, no longer hold in the case of separable C*-algebras, even for those classifiable C*-algebras that have analogous properties to

$\mathcal{R}$. An example of this lack of uniqueness can be seen in [47] for actions of $\mathbb{Z}_2$ on UHF algebras. Hermann and Jones in [59] and [60] circumvented this problem by only considering actions that have a "Rokhlin type property". The analogous property in the setting of von Neumann algebras is automatic for pointwise outer actions on $\mathcal{R}$ by the results of Connes, Jones and Ocneanu. The Rokhlin property is motivated by the structure of ergodic probability measure preserving (p.m.p) actions of amenable groups on measure spaces. In the case of integer actions the motivating property was shown by Rokhlin in [115]. Rokhlin showed that if $T : X \to X$ is an ergodic p.m.p transformation on a standard probability space X then for any $n \in \mathbb{N}$ there exists a measurable subset E such that $E, TE, T^2, \dots T^{n-1}E$ are pairwise disjoint and their union is almost 1 in measure. Such results were further generalised in [106] for actions of amenable groups on measure spaces.

Many results regarding the classification of group actions on C*-algebras either assume or show a Rokhlin type property (see [68] and references therein). For finite group actions, the Rokhlin property was introduced by Izumi in [66].

Definition 3.4.4. Let G be a finite group and $\alpha : G \curvearrowright A$ an action on a unital C*-algebra A. Then α is said to have the *Rokhlin property* if there exist projections $p_g \in A_\infty \cap A'$ for $g \in G$ such that

$$\sum_{g \in G} p_g = 1_A, \tag{3.4.1}$$

$$\alpha_g(p_h) = p_{gh} \text{ for } g, h \in G. \tag{3.4.2}$$

An example of a group action with the Rokhlin property is the infinite tensor product of the left regular representation

$$\mu_G = \bigotimes_{i=0}^{\infty} \mathrm{Ad}(\lambda_G) \tag{3.4.3}$$

on the UHF algebra $\bigotimes_{i\in\mathbb{N}}\mathcal{B}(l^2(G))\cong\mathbb{M}_{|G|^\infty}$ (see [66, Example 3.2]). Although we have explicitly noted the isomorphism in this case, we will often identify the C*-algebras $\mathcal{B}(l^2(G))$ and $\mathbb{M}_{|G|}$. This isomorphism is immediate from writing a bounded operator in matrix form by choosing a basis for $l^2(G)$.

Remark 3.4.5. It is immediate that any Rokhlin G action α on A is automatically pointwise outer. Otherwise, if α_g is inner for some $g\neq 1_G$, then $\alpha_g=\mathrm{Ad}(u)$ for $u\in U(A)$. For any projection $p\in A_\infty\cap A'$ one has that $\alpha_g(p)=p$. As $\alpha_g(p)$ is not orthogonal to p this contradicts (3.4.1) of Definition 3.4.4.

The Rokhlin property for an action $\alpha : G \curvearrowright A$ can be equivalently characterised as the existence of one projection $p\in A_\infty\cap A'$ such that $\sum_{g\in G}\alpha_g(p)=1_A$. In this case p corresponds to the projection p_{1_G} and the projections p_g of Definition 3.4.4 are recovered as $\alpha_g(p_{1_G})$.

Exploiting the Rokhlin property, in [67, Theorem 4.2 and 4.3] Izumi classifies finite group actions on both Kirchberg algebras that satisfy the UCT and unital, simple, separable, nuclear TAF-C*-algebras that satisfy the UCT (see Sections 2.7.3 and 2.7.4). These actions are classified by the induced module structure at the level of K-theory.

Theorem 3.4.6. *(Izumi cf. [67, Theorem 4.2 and 4.3]) Let G be a finite group and A be a unital Kirchberg algebra satisfying the UCT, or a unital, simple, nuclear, separable TAF-algebra satisfying the UCT. Let $\alpha,\beta : G\to\mathrm{Aut}(A)$ be group actions. If $K_i(\alpha_g)=K_i(\beta_g)$ for all $g\in G$ and $i=0,1$ then there exists $\theta\in\mathrm{Aut}(A)$ such that $\theta\alpha_g\theta^{-1}=\beta_g$ and $K_i(\theta)=\mathrm{id}_{K_i(A)}$ for $i=0,1$.*[2]

Note that Theorem 3.4.6 classifies actions on a C*-algebra A up to conjugacy with the extra condition that the automorphism θ witnessing the conjugacy induces the trivial automorphism in K-theory. We will often call this equivalence relation *K-trivial conjugacy* and denoted by $\simeq_K$.

[2]Note that TAF algebras are called C*-algebras of tracial topological rank zero in [67].

Izumi's classification result relies on the classification of Kirchberg–Phillips (Theorem 2.7.9) and Lin (Theorem 2.7.15). More precisely, it relies on *classification of morphisms*; that the *-homomorphisms between Kirchberg algebras are classified up to approximate unitary equivalence by a K-theoretic invariant (see [67, Theorem 2.7]). Similarly, the *-homomorphisms between unital, simple, separable, nuclear TAF-algebras are classified (see [32, Theorem 1.1]). Recently, Szabó announced a classification of finite group actions with the Rokhlin property on all classifiable C*-algebras by using the more novel classification of morphisms results in [56]. Szabó's result is still unpublished.

Theorem 3.4.7. *(Szabó) Let G be a finite group and A, B classifiable C*-algebras. Suppose α, β are G actions with the Rokhlin property on A, B respectively, then $KT_u(\alpha) \sim KT_u(\beta)$ if and only if $\alpha \sim \beta$.*[3]

3.5 Equivalence relations for anomalous actions

In this section, we discuss equivalence relations for anomalous actions that generalise those discussed in Definition 3.4.1. Before we do so, we start by introducing some notation that will allow us to streamline future definitions.

Definition 3.5.1. Let (α, u) be an anomalous action of a group G on a C*-algebra A. If $v_g \in U(M(A))$ for $g \in G$ the pair (α^v, u^v) with

$$\alpha^v_g = \mathrm{Ad}(v_g)\alpha_g, \quad g \in G,$$

$$u^v_{g,h} = v_g\alpha_g(v_h)u_{g,h}v^*_{gh}, \quad g, h \in G$$

is an anomalous action. We say that (α^v, u^v) is a *unitary perturbation* of (α, u).

[3]As explained in Section 2.7.2, $KT_u(\cdot)$ is a functor. In particular, for an action α on A there is an induced action $KT_u(\alpha)$ on $KT_u(A)$. Therefore, we denote $KT_u(\alpha) \sim KT_u(\beta)$ if there exists an isomorphism $\theta \in \mathrm{Hom}(KT_u(A), KT_u(B))$ such that $\theta KT_u(\alpha_g)\theta^{-1} = KT_u(\beta_g)$ for all $g \in G$.

it is straightforward that $o(\alpha, u) = o(\alpha^v, u^v)$ for any map $v : G \to U(M(A))$.

Definition 3.5.2. Let A, B be C*-algebras, (α, u) be an anomalous action on A and (β, v) be an anomalous action on B. Then we say that

- (α, u) is *conjugate* to (β, v) if there exists an isomorphism $\theta : A \to B$ such that $\alpha_g = \theta\beta_g\theta^{-1}$ and $v_{g,h} = \theta(u_{g,h})$ for all $g, h \in G$. We denote this by $(\alpha, u) \sim (\beta, v)$
- (α, u) is *cocycle conjugate* to (β, v) if there exist unitaries $s_g \in U(M(A))$ for $g \in G$ such that (α^s, u^s) is conjugate to (β, v). We denote this by $(\alpha, u) \simeq (\beta, v)$.
- (α, u) is *outer conjugate* to (β, v) if the G-kernels $\overline{\alpha}$ and $\overline{\beta}$ are conjugate.

We use the same terminology as in Definition 3.4.1. There is no ambiguity in doing so as the definitions laid out in Definition 3.5.2 coincide with those in Definition 3.4.1 when restricted to group actions.

Conjugacy preserves the anomaly of an action. Namely, if $(\alpha, u) \sim (\beta, v)$ then $o(\alpha, u) = o(\beta, v)$. Similarly, the anomaly is preserved under unitary perturbations so if $(\alpha, u) \simeq (\beta, v)$ then $o(\alpha, u) = o(\beta, v)$. This is not the case for outer conjugacy. For example, if (α, u) is an anomalous action of a group G on a C*-algebra A, one may perturb (α, u) by a cochain $\lambda : G \times G \to \mathbb{T}$, to yield an anomalous action $(\alpha, \lambda u)$ which is outer conjugate to (α, u) (as (α, u) and $(\alpha, \lambda u)$ define the same G-kernel). However, $o(\alpha, \lambda u) = d\lambda o(\alpha, u)$ which is not equal to $o(\alpha, u)$ unless λ is a cocycle.

The difference between outer conjugacy and cocycle conjugacy for (G, ω) actions can be measured by 2-cohomology, to see this we introduce an invariant associated to a pair of anomalous actions that yield the same G-kernel.

Let G be a group, $\omega \in Z^3(G, \mathbb{T})$ and $(\alpha, u), (\beta, v)$ be (G, ω) actions on A such that $\alpha_g = \beta_g \mod \mathrm{Inn}(A)$ for any $g \in G$.[4] Then there exists $s_g \in U(M(A))$ for $g \in G$

[4]Or equivalently that the G-kernels $\overline{\alpha}$ and $\overline{\beta}$ are equal.

such that $\beta_g = \mathrm{Ad}(s_g)\alpha_g$. Moreover, for all $g, h \in G$

$$\begin{aligned}\beta_g\beta_h &= \mathrm{Ad}(s_g)\alpha_g\mathrm{Ad}(s_h)\alpha_h \\ &= \mathrm{Ad}(s_g\alpha_g(s_h))\alpha_g\alpha_h \\ &= \mathrm{Ad}(s_g\alpha_g(s_h)u_{g,h})\alpha_{gh} \\ &= \mathrm{Ad}(s_g\alpha_g(s_h)u_{g,h}s^*_{gh})\beta_{gh} \\ &= \mathrm{Ad}(u^s_{g,h})\beta_{gh}.\end{aligned}$$

So $\mathrm{Ad}(u^s_{g,h}) = \mathrm{Ad}(v_{g,h})$ for all $g, h \in G$. Therefore, there exists $\lambda_{g,h} \in Z(U(M(A)))$ such that

$$\lambda_{g,h} = u^s_{g,h}v^*_{g,h} \tag{3.5.1}$$

Proposition 3.5.3. *Let G be a group, $\omega \in Z^3(G, \mathbb{T})$ and $(\alpha, u), (\beta, v)$ be (G, ω) actions on a C^*-algebra A with $\alpha_g = \beta_g \mod \mathrm{Inn}(A)$ for all $g \in G$. For any $s_g \in U(M(A))$ such that $\mathrm{Ad}(s_g)\alpha_g = \beta_g$ for $g \in G$ the 2-cochain $\lambda : G \times G \to Z(U(M(A)))$ defined by*

$$\lambda_{g,h} = u^s_{g,h}v^*_{g,h}$$

is a 2-cocycle. The cohomology class $[\lambda] \in H^2(G, Z(U(M(A))))$ is independent of the choice of unitaries s_g. We denote it by $\mathrm{ob}((\alpha, u), (\beta, v))$.

Proof. First we check that λ is a 2-cocycle. Indeed for $g, h, k \in G$ using the fact that

λ is valued in the centre we have that

$$\begin{aligned}
\beta_g(\lambda_{h,k})\lambda_{g,hk}\overline{\lambda_{gh,k}\lambda_{h,k}} &= \overline{\omega_{g,h,k}}\beta_g(v_{h,k})v_{g,hk}v_{gh,k}^*v_{g,h}^*\beta_g(\lambda_{h,k})\lambda_{g,hk}\overline{\lambda_{gh,k}\lambda_{h,k}} \\
&= \overline{\omega_{g,h,k}}(\beta_g(\lambda_{h,k}v_{h,k}))(\lambda_{g,hk}v_{g,hk})(\lambda_{gh,k}v_{gh,k})^*(\lambda_{g,h}v_{g,h})^*) \\
&= \overline{\omega_{g,h,k}}(\beta_g(u_{h,k}^s)u_{g,hk}^s(u_{gh,k}^s)^*(u_{g,h}^s)^*) \\
&= \overline{\omega_{g,h,k}}(\mathrm{Ad}(s_g)\alpha_g(u_{h,k}^s)u_{g,hk}^s(u_{gh,k}^s)^*(u_{g,h}^s)^*) \\
&= 1.
\end{aligned}$$

The last equality holds as the anomaly associated to (α^s, u^s) is also ω. It remains to check that the cohomology class of λ is independent of the choice of unitaries s_g such that $\mathrm{Ad}(s_g)\alpha_g = \beta_g$. If s'_g for $g \in G$ is another family of unitaries such that $\mathrm{Ad}(s'_g)\alpha_g = \beta_g$ there exists a family of unitaries $r_g \in Z(U(M(A)))$ such that $s'_g = r_g s_g$. Denote by λ' the 2-cocycle defined by s' namely

$$\begin{aligned}
\lambda'_{g,h} &= u_{g,h}^{s'}v_{g,h}^* \\
&= s'_g\alpha_g(s'_h)u_{g,h}(s'_{gh})^*v_{g,h}^*.
\end{aligned}$$

Substituting $s'_g = r_g s_g$ and using the fact that $r_g \in Z(U(M(A)))$ then

$$\lambda'_{g,h} = \lambda_{g,h}r_g\alpha_g(r_h)r_{gh}^*$$

for all $g, h \in G$. Therefore, $\lambda' = \lambda dr$ for the 1-cochain $r : G \to Z(U(M(A)))$. Therefore the cohomology classes of λ and λ' are equal and the 2-cohomology invariant is well-defined. □

Fix a group G, $\omega \in Z^3(G, \mathbb{T})$ and C*-algebras A and B. Suppose (α, u) and (β, v) are outer conjugate (G, ω) actions on A and B respectively. By definition, there exists $s_g \in U(M(A))$ and $\varphi : A \to B$ an isomorphism such that $\alpha_g^s = \varphi^{-1}\beta_g\varphi$ for all $g \in G$.

The failure of the pair φ and s constituting a cocycle conjugacy between (α, u) and (β, v) is measured by the multiplicative difference between $u^s_{g,h}$ and $\varphi^{-1}(v_{g,h})$. This is precisely the 2-cocycle associated to the pair (α^s, u^s) and $(\varphi\beta\varphi^{-1}, \varphi^{-1}(v))$ as in (3.5.1). If $\text{ob}((\alpha, u), (\varphi\beta\varphi^{-1}, \varphi^{-1}(v)))$ vanishes, we can boost the outer conjugacy between (α, u) and (β, v) to a cocycle conjugacy. We see an instance of this in the following corollary.

Corollary 3.5.4. *Let G be a group such that $H^2(G, Z(U(M(A))))$ is trivial, $\omega \in Z^3(G, \mathbb{T})$ and A, B be C^*-algebras. Let (α, u) and (β, v) be (G, ω) actions on A and B respectively then if (α, u) and (β, v) are outer conjugate they are cocycle conjugate. Moreover, if (α, u) and (β, v) are anomalous actions on the same C^*-algebra A, then if they induce the same G-kernel on A, (β, v) is a unitary perturbation (α, u).*

Proof. As (α, u) and (β, v) are outer conjugate, there exists an isomorphism $\varphi : A \to B$ and unitaries $s_g \in U(M(A))$ for $g \in G$ such that

$$\alpha^s_g = \varphi^{-1}\beta_g\varphi \tag{3.5.2}$$

for all $g \in G$. If the unitaries $\varphi^{-1}(v_{g,h})$ and $u^s_{g,h}$ are equal for all $g, h \in G$, then (α, u) and (β, v) are cocycle conjugate. Although this might not be the case, we will use the invariant of Proposition 3.5.3 to show that we may tweak s_g by centre valued unitaries, to ensure that the equality described in the previous sentence holds. The anomalous actions (α, u) and $(\varphi^{-1}\beta\varphi, \varphi^{-1}(v))$ have the same anomaly and induce the same G-kernels so the 2-cochain

$$\lambda_{g,h} = u^s_{g,h}\varphi^{-1}(v^*_{g,h}). \tag{3.5.3}$$

for $g, h \in G$ is a 2-cocycle by Proposition 3.5.3. As $H^2(G, Z(U(M(A))))$ is trivial, the cohomology class $[\lambda] \in H^2(G, Z(U(M(A))))$ is trivial. Therefore, there is a 1-cochain

$r : G \to Z(U(M(A)))$ such that $\lambda = dr$. Now $(\alpha^{r^*s}, u^{r^*s})$ is conjugate through φ to (β, v). Indeed, as $r_g \in Z(U(M(A))) = \ker(\mathrm{Ad})$ for all $g \in G$ and $\mathrm{Ad}(s_g)\alpha_g = \varphi^{-1}\beta_g\varphi$, then $\mathrm{Ad}(r_g^* s_g)\alpha_g = \varphi^{-1}\beta_g\varphi$ for $g \in G$. Moreover, by (3.5.3) we get that for $g, h \in G$,

$$\begin{aligned} u_{g,h}^{r^*s} &= r_g^* s_g \alpha_g(r_h^* s_h) u_{g,h} r_{gh} s_{gh}^* \\ &= \overline{\lambda_{g,h}} u_{g,h}^s \\ &= \varphi^{-1}(v_{g,h}). \end{aligned}$$

The final statement is clear by setting $\varphi = \mathrm{id}_A$ in the argument. □

We may now rephrase Lemma 3.3.6.

Corollary 3.5.5. *Let $n \in \mathbb{N}$ and $\lambda_k \in Z^3(\mathbb{Z}_n, \mathbb{T})$ as in (3.3.3) for $0 \le k \le n-1$. Denote by g the element $1 + n\mathbb{Z}$ in $\mathbb{Z}_n$. Let (α, u) be a $(\mathbb{Z}_n, \lambda_k)$ action on a C*-algebra A. Then (α, u) is a unitary perturbation of the $(\mathbb{Z}_n, \lambda_k)$ action on A given by the pair (α', u') defined by*

$$\alpha'_{g^k} = (\alpha_g)^k \tag{3.5.4}$$

$$u'_{g^k, g^l} = \begin{cases} 1 \text{ for } l + k < n \\ u_{g,g} u_{g^2,g} \dots u_{g^{n-1},g} \text{ for } l + k \ge n \end{cases} \tag{3.5.5}$$

with $0 \le l, k \le n-1$. Therefore, every $(\mathbb{Z}_n, \lambda_k)$ action is a unitary perturbation of an anomalous action that arises from an automorphism in the sense of Section 3.3.1.

Proof. That (α', u') yields an anomalous action is shown in Lemma 3.3.6. By construction α and α' induce the same $\mathbb{Z}_n$-kernels. Moreover, the cohomology group $H^2(\mathbb{Z}_n, \mathbb{T})$ is trivial for any $n \in \mathbb{N}$ ([15, pg 58]). Therefore, by Corollary 3.5.4, (α', u') is a unitary perturbation of (α, u). □

Remark 3.5.6. Let G be a group and $\overline{\theta}$ be a G-kernel on a C*-algebra A. For any lift (θ, u) of $\overline{\theta}$ one has an associated 3-cocycle in $Z^3(G, Z(U(M(A))))$ through the computations of Section 3.2. If (θ, u) and (α, v) are arbitrary lifts of $\overline{\theta}$ with the same associated 3-cocycle one may also define a 2-cohomology invariant $\mathrm{ob}((\theta, u), (\alpha, v))$ as in Proposition 3.5.3. This follows from observing that the computations we have performed did not require the products $\theta_g(u_{h,k})u_{g,hk}u^*_{gh,k}u^*_{g,h}$ and $\alpha_g(v_{h,k})v_{g,hk}v^*_{gh,k}v^*_{g,h}$ to be contained in $\mathbb{T}$. This last condition is the only extra requirement for a lift of a G-kernel to be an anomalous action.

Chapter 4

An algebraic K-theory obstruction

In this chapter, we discuss an obstruction to the possible values of anomalies for anomalous symmetries on C*-algebras. This obstruction arises from considering the unitary algebraic K_1 groups for C*-algebras. The content of this section is a fleshed out version of the work undertaken by Evington and myself in [46].

We start by recalling the definition of the unitary algebraic K_1 group of a unital C*-algebra.

Definition 4.0.1. Let A be a unital C*-algebra. The *unitary algebraic K_1-group* of A is defined as the abelianisation of $U_\infty(A)$,

$$K_1^{\mathrm{alg}}(A) = \frac{U_\infty(A)}{[U_\infty(A), U_\infty(A)]}. \tag{4.0.1}$$

A couple of variants of the algebraic K_1-group are possible. Firstly, one can replace unitary groups with general linear groups throughout (see for example [61]). Secondly, one can define *Hausdorffised* unitary algebraic K_1 by replacing the commutator subgroup $[U_\infty(A), U_\infty(A)]$ by its closure in the direct limit topology on $U_\infty(A)$ (see for example [137]). In this chapter, we will mainly be interested in K_1^{alg} although in some cases other variants may be referred to.

Example 4.0.2. $K_1^{\mathrm{alg}}(\mathbb{C}) \cong \mathbb{T}$. We compute this by using the determinant. Indeed, $\det : U_\infty(\mathbb{C}) \to \mathbb{T}$ is a group homomorphism and $[U_\infty(\mathbb{C}), U_\infty(\mathbb{C})]$ is contained in the kernel of det. We show that $\ker(\det) = [U_\infty(\mathbb{C}), U_\infty(\mathbb{C})]$. Let $n \in \mathbb{N}$ and $u \in U_n(\mathbb{C})$ such that $\det(u) = 1$. Without loss of generality, we may assume that u is diagonal. Otherwise there exists a unitary $v \in U_n(\mathbb{C})$ and a diagonal matrix d such that $u = vdv^*$. Hence, $ud^{-1} \equiv 1_{\mathbb{M}_n(\mathbb{C})} \mod [U_\infty, U_\infty]$ so $u \equiv d \mod [U_\infty, U_\infty]$. Let $u = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{T}$ for $1 \le i \le n$. As $\det(u) = 1$ we also have that $\prod_{i=1}^n \lambda_i = 1$. For $\lambda, \mu \in \mathbb{T}$,

$$\begin{pmatrix} \lambda\mu & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\mu} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \mod [U_\infty, U_\infty].$$

Hence, by applying this equality repeatedly to any 2×2 block down the diagonal, it follows that

$$\begin{aligned} 1_{\mathbb{M}_n(\mathbb{C})} &= \mathrm{diag}(\prod_{i=1}^n \lambda_i, 1, \ldots, 1) \\ &\equiv \mathrm{diag}(\lambda_1, \prod_{i=2}^n \lambda_i, 1, \ldots, 1) \mod [U_\infty, U_\infty] \\ &\equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \mod [U_\infty, U_\infty] \\ &\equiv u \mod [U_\infty, U_\infty]. \end{aligned}$$

So the kernel of the determinant is $[U_\infty(A), U_\infty(A)]$. It follows from the isomorphism theorem that $K_1^{\mathrm{alg}}(\mathbb{C}) \cong \mathbb{T}$.

4.1 The de la Harpe–Skandalis determinant

Let A be a unital C*-algebra. There is a canonical surjection $\kappa_A : K_1^{\mathrm{alg}}(A) \twoheadrightarrow K_1(A)$, which is typically not injective. For example, $K_1(\mathbb{C}) = 0$ as the unitary groups $U_n(\mathbb{C})$

are path connected, whereas by Example 4.0.2, $K_1^{\text{alg}}(\mathbb{C}) \cong \mathbb{T}$. To gain information about the kernel of κ_A for arbitrary A, one needs variants of the determinant for arbitrary C*-algebras.

Denote by $U_\infty^{(0)}(A)$ the path component of 1_A in $U_\infty(A)$. Given a tracial state $\tau \in T(A)$, de la Harpe and Skandalis define a group homomorphism

$$\Delta_\tau : U_\infty^{(0)}(A) \to \frac{\mathbb{R}}{\tau_*(K_0(A))},$$

where τ_* is the induced state on $K_0(A)$. We outline the construction below, full details can be found in [35] or [34].

Suppose $u \in U_n^{(0)}(A)$. Let $\xi : [0,1] \to U_n(A)$ be a smooth path with $\xi(0) = 1_A$ and $\xi(1) = u$.[1] Then

$$\Delta_\tau(u) = \frac{1}{2\pi i}\int_0^1 (\tau \otimes \mathrm{Tr}_n)(\xi'(t)\xi(t)^{-1})\,\mathrm{d}t + \tau_*(K_0(A)). \tag{4.1.1}$$

It is not immediately clear that Δ_τ is independent of the choice of path ξ. However, de la Harpe and Skandalis prove this in [35, Lemme 1]. This result follows from Bott periodicity as the difference of two paths with the same end point yield a loop around the identity in $U_\infty(A)$. Bott periodicity implies precisely that loops around the identity, up to homotopy, come from elements in $K_0(A)$. A direct computation using the product rule yields that

$$(\xi_1\xi_2)'(t) = \xi_1'(t)\xi_2(t) + \xi_1(t)\xi_2'(t). \tag{4.1.2}$$

Combining this with the trace identity for τ, we get that Δ_τ is a group homomorphism.

In the case that $u = \exp(2\pi i h)$ for some self-adjoint $h \in \mathbb{M}_n(A)$. Taking ξ to be

[1] By [119, Proposition 2.1.6] u can be written as a product of exponentials $u = \exp(ih_1)\cdots\exp(ih_r)$ for $h_i \in A^{sa}$. Therefore, there exists a path of the form $\xi(t) = \exp(ith_1)\cdots\exp(ith_r)$. This path is C^∞-smooth.

the path $\xi(t) = \exp(2\pi i t h)$

$$\begin{aligned}\Delta_\tau(u) &= \frac{1}{2\pi i}\int_0^1 \tau(2\pi i h \exp(2\pi i t h)\exp(2\pi i t h)^{-1})dt + \tau(K_0(A)) \\ &= \tau(h) + \tau(K_0(A))\end{aligned}$$

In particular, for $x \in \mathbb{R}$, we have

$$\Delta_\tau(e^{2\pi i x}1_A) = x + \tau_*(K_0(A)). \tag{4.1.3}$$

Combining the de la Harpe–Skandalis determinants Δ_τ for all $\tau \in T(A)$, we obtain a group homomorphism

$$\Delta_A : U_\infty^{(0)}(A) \to \frac{\mathrm{Aff}(T(A))}{\rho_A(K_0(A))}. \tag{4.1.4}$$

See Section 2.7.2 for an explanation on the notation used above. We call this map the *universal* de la Harpe–Skandalis determinant.

Remark 4.1.1. In [35], de la Harpe and Skandalis carry out their construction with any continuous linear map $\tau : A \to E$ into a Banach space E satisfying the trace identity $\tau(ab) = \tau(ba)$. The universal de la Harpe–Skandalis determinant can also be obtained by this method starting with the universal trace $\mathrm{Tr} : A \to \mathrm{Aff}_{\mathbb{C}}(T(A))$. By [31, Proposition 2.7], we have $\mathrm{Ker}(\mathrm{Tr}) = \overline{\mathrm{span}}\{ab - ba : a, b \in A\}$, and the universal trace can alternatively be viewed as taking the quotient of A by this closed subspace.

Using the universal de la Harpe Skandalis deteminant, it follows from a result of Ng and Robert ([103]), that the kernel of κ_A is completely determined when A is a classifiable C*-algebra.

Theorem 4.1.2 (cf. [103, Theorem 1.1]). *Let A be a unital, simple, separable, exact and $\mathcal{Z}$-stable* C*-*algebra with $T(A) \neq \emptyset$. There is a short exact sequence*

$$0 \to \frac{\mathrm{Aff}(T(A))}{\rho_A(K_0(A))} \xrightarrow{(\bar{\Delta}_A)^{-1}} K_1^{\mathrm{alg}}(A) \xrightarrow{\kappa_A} K_1(A) \to 0. \tag{4.1.5}$$

Proof. The K_1-class of a unitary $u \in U_\infty(A)$ is its path component in $U_\infty(A)$. So

$$\mathrm{Ker}(\kappa_A) = \frac{U_\infty^{(0)}(A)}{[U_\infty(A), U_\infty(A)]^{(0)}}$$

Moreover, for any $u, v \in U_N(A)$, we have

$$[u, v] \oplus 1_A \oplus 1_A = [u \oplus u^* \oplus 1_A, v \oplus 1_A \oplus v^*]. \tag{4.1.6}$$

Therefore,

$$\mathrm{Ker}(\kappa_A) = \frac{U_\infty^{(0)}(A)}{[U_\infty^{(0)}(A), U_\infty^{(0)}(A)]}. \tag{4.1.7}$$

It remains to show that the universal de la Harpe–Skandalis determinant

$$\Delta_A : U_\infty^{(0)}(A) \to \frac{\mathrm{Aff}(T(A))}{\rho_A(K_0(A))} \tag{4.1.8}$$

is surjective and that its kernel is precisely $[U_\infty^{(0)}(A), U_\infty^{(0)}(A)]$.

For every $f \in \mathrm{Aff}(T(A))$, there is a self-adjoint $h \in A$ with $\tau(h) = f(\tau)$ for all $\tau \in T(A)$ by [91, Theorem 9.3], which builds on results of Cuntz and Pedersen ([31]). Then $\Delta_A(\exp(2\pi i h)) = f + \rho(K_0(A))$. Therefore, Δ_A is surjective.

By construction $\mathrm{Ker}(\Delta_A) \supseteq [U_\infty^{(0)}(A), U_\infty^{(0)}(A)]$. The reverse inclusion follows from [103, Theorem 1.1] since A is a simple, separable, pure C*-algebra with stable rank one and where all quasitraces are traces. (Pureness is a consequences of $\mathcal{Z}$-stability; see [147, Proposition 2.7]. Stable rank one follows from $\mathcal{Z}$-stability when A is stably finite; see [118, Theorem 6.7]. All quasitraces are traces as A is exact; see [58].) □

Remark 4.1.3. By [36, Theorem 3], every unitary in a unital, simple, *infinite* C*-algebra in the path component of the identity is a product of commutators. Hence, the canonical surjection $\kappa_A : K_1^{\text{alg}}(A) \twoheadrightarrow K_1(A)$ is an isomorphism. With Theorem 4.1.2 this facilitates the computation of $K_1^{\text{alg}}(A)$ for any classifiable C*-algebra.

Given a unital *-homomorphism $f : A \to B$. There is a well defined group homomorphism $K_1^{\text{alg}}(f) : K_1^{\text{alg}}(A) \to K_1^{\text{alg}}(B)$ given by $[u]_{K_1^{\text{alg}}(A)} \mapsto [f(u)]_{K_1^{\text{alg}}(B)}$. In particular, $K_1^{\text{alg}}(\cdot)$ is a covariant functor from $\mathbf{C^*alg}_1$ to $\mathbf{Ab}$. The same is true for $K_1(\cdot)$ and $\text{Aff}(T(\cdot))$ (see Section 2.6 and Section 2.7.2).

The short exact sequence (4.1.5) is natural, in the sense that a morphism of unital C*-algebras will induce a morphism between the corresponding short exact sequences. For every A, the short exact sequence (4.1.5) will split since $\text{Aff}(T(A))$ is a divisible group. However, the splitting is not natural.

4.2 The obstruction

We now showcase the obstruction to the existence of anomalous actions arising from K_1^{alg}. The obstruction arises as the unitary group of a C*-algebra can have non-trivial abelian quotients. This is not the case for $\mathcal{R}$, as every unitary in $\mathcal{R}$ can be written as a product of commutators (see for example [10]).

We start by discussing the general obstruction, we then apply it to $\mathcal{Z}$ and to UHF-algebras. We finish the section by proving a general theorem that applies under more general assumptions. The following is the key observation. We apply this proposition in the cases that Γ is the unitary group or its connected component of the identity.

Proposition 4.2.1. *Let G be a group, $\omega \in Z^3(G, \mathbb{T})$ and (α, u) be a (G, ω) action on a C*-algebra A. Suppose $\Gamma < U(M(A))$ is an α invariant subgroup containing $\mathbb{T}$ and $u_{g,h}$ for all $g, h \in G$ and $q : \Gamma \to M$ is a group homomorphism with M a $\mathbb{Z}G$-module satisfying $q(\alpha_g(v)) = g \cdot q(v)$ for $v \in \Gamma$ and $g \in G$. Then $[q \circ \omega] = 0$ in $H^3(G, M)$.*

Proof. Let $g, h, k \in G$. Since addition in M is commutative, applying q to (3.2.3) yields

$$\begin{aligned} q(\omega(g,h,k)1_A) &= q(\alpha_g(u_{h,k})) + q(u_{g,hk}) - q(u_{gh,k}) - q(u_{g,h}) \\ &= g \cdot q(u_{h,k}) - q(u_{gh,k}) + q(u_{g,hk}) - q(u_{g,h}) \\ &= d\eta(g,h,k), \end{aligned} \tag{4.2.1}$$

where η is the 2-cochain defined by $\eta(g,h) = q(u_{g,h})$. □

In order to make use of Proposition 4.2.1, we need a candidate for the homomorphism q. This is where the unitary algebraic K_1 group enters the picture. By construction, $K_1^{\mathrm{alg}}(A)$ is an abelian group and every automorphism of A induces an automorphism of $K_1^{\mathrm{alg}}(A)$ with inner automorphisms acting trivially. Hence, an anomalous action on A gives rise to a $\mathbb{Z}G$-module structure on $K_1^{\mathrm{alg}}(A)$.

The reason for working with $K_1^{\mathrm{alg}}(A)$ instead of $K_1(A)$ is that scalars always have trivial K_1 class but can have non-trivial K_1^{alg} class. This is necessary in order to get a non-trivial conclusion.

Theorem 4.2.2. *Let G be a group and $\omega \in Z^3(G, \mathbb{T})$. Let (α, u) be an (G, ω) action on a unital* C**-algebra A. View $K_1^{\mathrm{alg}}(A)$ as a $\mathbb{Z}G$-module where g acts via $K_1^{\mathrm{alg}}(\alpha_g)$. Then the 3-cocycle given by $(g,h,k) \mapsto [\omega(g,h,k)1_A]_{K_1^{\mathrm{alg}}}$ is trivial in $H^3(G, K_1^{\mathrm{alg}}(A))$.*

Proof. The result follows from Proposition 4.2.1, taking $M = K_1^{\mathrm{alg}}(A)$ with the induced $\mathbb{Z}G$-module structure and $q : U(A) \to K_1^{\mathrm{alg}}(A)$ to be the map $u \mapsto [u]_{K_1^{\mathrm{alg}}}$. □

Remark 4.2.3. The proof of Proposition 4.2.1 still works in the case that ω may be valued in $Z(U(M(A)))$. Therefore, this also leads to obstructions to the possible values of H^3 invariants for G-kernels on A.

Theorem 4.1.2 can compute K_1^{alg} for classifiable C*-algebras and extract information about the $\mathbb{Z}G$-action. It will be easier to extract obstructions to anomalous actions when the unitaries of the anomalous action are connected to the identity. In many of the examples we will consider, the C*-algebra A will have a connected unitary group, so all the anomalous actions will be of this form. However, to allow us to work in greater generality we introduce the following definition.

Definition 4.2.4. Let G be a group and A a unital C*-algebra. We say an anomalous action (α, u) of G on A is *connected* if $u_{g,h} \in U^{(0)}(A)$ for all $g, h \in G$.

Remark 4.2.5. If A is a unital C*-algebras with trivial $K_1(A)$ and A has stable rank one, it follows from [114, Theorem 10.10] and [114, Corollary 4.10] that $U^{(0)}(A) = U(A)$ and hence every anomalous action on A is connected. By Kirchberg's dichotomy, any classifiable C*-algebra is either stably finite or purely infinite. By virtue of [118, Theorem 6.7] any stably finite, classifiable C*-algebra has stable rank one. Moreover, any simple, unital, purely infinite C*-algebra with trivial K_1 has a path connected unitary group by Proposition 2.7.11. Therefore, if A is a classifiable C*-algebra with trivial K_1, any anomalous action on A is connected.

The following special case of Proposition 4.2.1 will be important to achieve obstructions to the existence of anomalous actions.

Corollary 4.2.6. *Let G be a group and $\omega \in Z^3(G, \mathbb{T})$. Let A be a unital C*-algebra and (α, u) be a connected (G, ω) action on A. Let $\tau \in T(A)$ be invariant under α_g for all $g \in G$.*[2] *Then $[\Delta_\tau \circ \omega] = 0$ in $H^3(G, \mathbb{R}/\tau_*(K_0(A)))$, where*

$$\Delta_\tau : U^{(0)}_\infty(A) \to \frac{\mathbb{R}}{\tau_*(K_0(A))} \tag{4.2.2}$$

is the de la Harpe–Skandalis determinant with respect to the trace τ, and the abelian group $\mathbb{R}/\tau_*(K_0(A))$ has the trivial $\mathbb{Z}G$-module structure.

Proof. As $u_{g,h} \in U^{(0)}(A)$ for $g, h \in G$ we may apply Proposition 4.2.1 with $q = \Delta_\tau$. The fact that $\Delta_\tau(\alpha_g(v)) = \Delta_\tau(v)$ follows from (4.1.1) since τ is invariant under α_g. □

[2] A trace τ is said to be invariant under an automorphism α if $\tau \circ \alpha = \tau$.

We can now apply the obstruction. We first apply it to the Jiang-Su algebra $\mathcal{Z}$. It makes sense to start by asking whether there is a (G, ω) action on $\mathcal{Z}$ for a given $\omega \in Z^3(G, \mathbb{T})$. Indeed, in the case that (α, u) is a (G, ω) action on $\mathcal{Z}$, then one could induce a (G, ω) action on any classifiable C*-algebra A as $(\mathrm{id}_A \otimes \alpha, 1_A \otimes u)$ is a (G, ω) action on $A \otimes \mathcal{Z} \cong A$.

Theorem 4.2.7. *Let G be a discrete group and $\omega \in Z^3(G, \mathbb{T})$. There exists a (G, ω) action on $\mathcal{Z}$ if and only if $[\omega] = 0$ in $H^3(G, \mathbb{T})$.*

Proof. The Jiang–Su algebra $\mathcal{Z}$ has a unique tracial state τ (and therefore τ is invariant under all automorphisms). Moreover, $K_0(\mathcal{Z}) \cong \mathbb{Z}$ with the isomorphism given by τ_*. Also $K_1(\mathcal{Z}) = 0$ (see [73, Theorem 1]). Therefore, as observed in Remark 4.2.5, any anomalous action on $\mathcal{Z}$ is connected.

Suppose there exists an ω-anomalous action of G on $\mathcal{Z}$. Then, by Corollary 4.2.6, we have $[\Delta_\tau \circ \omega] = 0$ in $H^3(G, \mathbb{R}/\mathbb{Z})$. However, Δ_τ restricted to $Z(U(\mathcal{Z})) = \mathbb{T}$ is an isomorphism by (4.1.3). Hence, $[\omega] = 0$ in $H^3(G, \mathbb{T})$.

The converse follows by Remark 3.3.3 as every group acts on $\mathcal{Z}$. □

When G is countable, one can also ensure that there exists a free action of G on $\mathcal{Z}$. That is, that there is an action $\alpha : G \to \mathrm{Aut}(\mathcal{Z})$ such that $\alpha_g \notin \mathrm{Inn}(A)$ for any $g \in G$. Indeed, the action of G on $\mathcal{Z} \cong \bigotimes_{g \in G} \mathcal{Z}$ defined by permuting the tensor factors is free. We will use this action of G on $\mathcal{Z}$ in later sections, we will informally call a group action that arises from permuting in a tensor product decomposition a *Bernoulli shift.*

We now turn to discuss the existence of (G, ω) actions on UHF algebras. We begin with a preliminary result that is of independent interest. Given an anomalous action on A, the following lemma will allow us, under certain conditions, to induce anomalous actions on corners of A. Before we state this preliminary lemma we recall the definition of approximate unitary equivalence.

Definition 4.2.8. Let A, B be separable C*-algebras and $\alpha, \beta : A \to B$ be *-homomorphisms. We say α and β are *approximately (multiplier) unitary equivalent* if there exists a sequence of unitaries $u_n \in U(M(B))$ such that

$$\lim_{n\to\infty} \|u_n\alpha(a)u_n^* - \beta(a)\| = 0$$

for all $a \in A$. We denote this by $\alpha \approx_{a.u} \beta$. If $\varphi : A \to A$ is a homomorphism such that $\varphi \approx_{a.u} \mathrm{id}_A$ we say φ is *approximately inner.*

Lemma 4.2.9. *Let G be a group and $\omega \in Z^3(G, \mathbb{T})$. Let A be a* C*-*algebra with cancellation of non-zero projections. Then any (G, ω) action (α, u) on A which preserves the K_0-class of a non-zero projection $p \in P(A)$ induces a (G, ω) action (α', u') on pAp. Moreover;*

(i) if α preserves every K_0 class so does α'.

(ii) If A is separable and α_g is approximately inner for every $g \in G$, then so is α'_g.[3]

Proof. Suppose (α, u) is an ω-anomalous action of G on A. Since A has the cancellation property, there exist partial isometries $v_g \in A$ such that $v_g v_g^* = p$ and $v_g^* v_g = \alpha_g(p)$ for each $g \in G$. Define

$$\alpha'_g = \mathrm{Ad}(v_g) \circ \alpha_g|_{pAp}, \tag{4.2.3}$$

$$u'_{g,h} = v_g \alpha_g(v_h) u_{g,h} v_{gh}^*. \tag{4.2.4}$$

[3]The second part of this Lemma also works in the non-separable setting by allowing for a notion of approximate unitary equivalence which involves nets rather than sequences.

Then $u'_{g,h} \in U(pAp)$ and we have

$$\begin{aligned}\mathrm{Ad}(u'_{g,h})\alpha'_{gh} &= \mathrm{Ad}(v_g\alpha_g(v_h)u_{g,h})\alpha_{gh} && (4.2.5)\\ &= \mathrm{Ad}(v_g\alpha_g(v_h))\alpha_g\alpha_h \\ &= \mathrm{Ad}(v_g)\alpha_g\mathrm{Ad}(v_h)\alpha_h \\ &= \alpha'_g\alpha'_h && (4.2.6)\end{aligned}$$

Computing the 3-cocycle $\omega'(g,h,k)$ associated to (α', u') using (3.2.3), we find that

$$\begin{aligned}\omega'(g,h,k) &= \alpha'_g(u'_{h,k})u'_{g,hk}u'^*_{gh,k}u'^*_{g,h} && (4.2.7)\\ &= v_g\alpha_g(v_h\alpha_h(v_k))\alpha_g(u_{h,k})\alpha_g(v^*_{hk})v^*_g \cdot v_g\alpha_g(v_{hk})u_{g,hk}v^*_{ghk} \\ &\quad\cdot v_{ghk}u^*_{gh,k}\alpha_{gh}(v^*_k)v^*_{gh} \cdot v_{gh}u^*_{g,h}\alpha_g(v^*_h)v^*_g \\ &= v_g\alpha_g(v_h)\alpha_g\alpha_h(v_k)\alpha_g(u_{h,k})u_{g,hk}u^*_{gh,k}\alpha_{gh}(v^*_k)u^*_{g,h}\alpha_g(v^*_h)v^*_g \\ &= v_g\alpha_g(v_h)\alpha_g\alpha_h(v_k)\omega(g,h,k)\alpha_g\alpha_h(v^*_k)\alpha_g(v^*_h)v^*_g \\ &= \omega(g,h,k)v_g\alpha_g(v_h\alpha_h(p)v^*_h)v^*_g \\ &= \omega(g,h,k)p\end{aligned}$$

Therefore, (α', u') is an (G, ω) action on pAp.

We now show (i). Suppose α preserves every K_0-class. As pAp is unital, it suffices to show that for any $n \in \mathbb{N}$ and (non-zero) projection $q \in \mathbb{M}_n(pAp)$ the classes $K_0(\alpha'_g)[q]_0 = [q]_0$. Fix $q \in \mathbb{M}_n(pAp)$. Firstly, as α preserves every K_0-class $K_0(\alpha_g)[q]_0 = [q]_0$. So, by cancellation of non-zero projections, there exists $s_g \in \mathbb{M}_n(A)$ such that $s_gs_g^* = \alpha_g(q)$ and $s_g^*s_g = q$ for $g \in G$. Denote the matrix in $\mathbb{M}_n(pAp)$ with entries v_g on the diagonal and 0's elsewhere by $\mathrm{diag}(v_g, v_g, \ldots, v_g)$. The partial isometries $r_g = \mathrm{diag}(v_g, v_g, \ldots, v_g)s_g$ for $g \in G$ are contained in $\mathbb{M}_n(pAp) =$

$\operatorname{diag}(p, p, \ldots, p)\mathbb{M}_n(A) \operatorname{diag}(p, p, \ldots, p)$ and satisfy $r_g^* r_g = q$ and $r_g r_g^* = \alpha'_g(q)$. Indeed,

$$\begin{aligned} r_g r_g^* &= \operatorname{diag}(v_g, v_g, \ldots, v_g) s_g s_g^* \operatorname{diag}(v_g^*, v_g^*, \ldots, v_g^*) \\ &= \operatorname{diag}(v_g, v_g, \ldots, v_g) \alpha_g(q) \operatorname{diag}(v_g^*, v_g^*, \ldots, v_g^*) \\ &= \alpha'_g(q) \end{aligned}$$

and

$$\begin{aligned} r_g^* r_g &= s_g^* \operatorname{diag}(v_g^* v_g^*, \ldots, v_g^*) \operatorname{diag}(v_g, v_g, \ldots, v_g) s_g \\ &= s_g^* \operatorname{diag}(\alpha_g(p), \alpha_g(p), \ldots, \alpha_g(p)) s_g \\ &= s_g^* (\alpha_g(\operatorname{diag}(p, p, ..., p) s_g s_g^*) s_g \\ &= s_g^* \alpha_g(q) s_g \\ &= s_g^* s_g. \end{aligned}$$

We now turn to (ii). If α_g is approximately inner, then for each $g \in G$ then there exists a sequence of unitaries $w_n^g \in U(M(A))$ such that $\alpha_g(x) = \lim_{n\to\infty} \operatorname{Ad}(w_n^g)(x)$ for all $x \in A$. The product $r_n(g) = v_g w_n^g p$ is contained in pAp. Pick finite increasing sets $F_n \subset A$ containing p such that $\bigcup_{n\in\mathbb{N}} F_n$ is dense in A and $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$ as $n \to \infty$. We may pass to a subsequence to ensure that $\|w_n^g x w_n^{g*} - \alpha_g(x)\| \leq \varepsilon_n$ for any $x \in pF_n p$. Note that

$$\|r_n(g) r_n(g)^* - p\| \leq \|v_g w_n^g p w_n^{g*} v_g^* - v_g \alpha_g(p) v_g^*\| \leq \|w_n^g p w_n^{g*} - \alpha_g(p)\| \leq \varepsilon_n.$$

Similarly, $\|r_n(g)^* r_n(g) - p\| \leq \varepsilon_n$. Let $\delta_n > 0$ such that $\delta_n \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$ one can choose ε_n small enough to ensure that there exist unitaries $u_n(g) \in U(pAp)$ such that $\|r_n(g) - u_n(g)\| \leq \delta_n/2$; this follows from Lemma 2.6.6. Let V be a finite set in pAp and $\epsilon > 0$. Let $n \in \mathbb{N}$ large enough such that $V \subset pF_n p$ and

$\max_{f\in V}(\delta_n, \|f\|\delta_n, \varepsilon_n) < \epsilon$. For $f \in V$

$$\begin{aligned}\|\mathrm{Ad}(u_n(g))f - \mathrm{Ad}(v_g)\alpha_g(f)\| &\le \|\mathrm{Ad}(u_n(g))f - \mathrm{Ad}(r_n(g))f\| \\ &\quad + \|\mathrm{Ad}(r_n(g))f - \mathrm{Ad}(v_g)\alpha_g(f)\| \\ &\le 3\varepsilon\end{aligned}$$

as required. □

Theorem 4.2.10. *Let G be a finite group and $\{n_k\}_{k\in\mathbb{N}}$ a sequence of natural numbers. Suppose there is a (G,ω) action on $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$. Let r be the order of $[\omega]$ in $H^3(G,\mathbb{T})$. Then r^∞ divides the supernatural number $\prod_{k\in\mathbb{N}} n_k$.*

Proof. Let G be a finite group and $\omega \in Z^3(G,\mathbb{T})$ with r its order in $H^3(G,\mathbb{T})$ (see Remark 4.2.11). Let A be the UHF algebra $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ with supernatural number $\mathfrak{n} = \prod_{k\in\mathbb{N}} n_k$. Then A has a unique tracial state τ, which is therefore invariant under all automorphisms. As A is AF, $U^{(0)}(A) = U(A)$, so any anomalous action on A is connected. The K_0 group of A is isomorphic via τ_* to the subgroup $Q(\mathfrak{n}) \subseteq \mathbb{R}$ generated by $\{\frac{1}{n} : n \in \mathbb{N}, n \,|\, \mathfrak{n}\}$.

Suppose there exists an ω-anomalous action of G on A. By Corollary 4.2.6, we have $[\Delta_\tau \circ \omega] = 0$ in $H^3(G, \mathbb{R}/Q(\mathfrak{n}))$. The short exact sequence of modules

$$0 \to \frac{Q(\mathfrak{n})}{\mathbb{Z}} \xrightarrow{\ \iota\ } \mathbb{T} \xrightarrow{\ \Delta_\tau\ } \frac{\mathbb{R}}{Q(\mathfrak{n})} \to 0, \tag{4.2.8}$$

where $\iota(x) = e^{2\pi i x}$, induces a long exact sequence of cohomology groups. Therefore, since $\Delta_{\tau *}[\omega] = 0$ in $H^3(G, \mathbb{R}/Q(\mathfrak{n}))$, we have that $[\omega] = \iota_*(\eta)$ for some $\eta \in H^3(G, Q(\mathfrak{n})/\mathbb{Z})$.

Every element in $Q(\mathfrak{n})/\mathbb{Z}$ has order dividing the supernatural number $\mathfrak{n}$. Since G is finite, the same is true for elements of $C^3(G, Q(\mathfrak{n})/\mathbb{Z})$ and so for elements of $H^3(G, Q(\mathfrak{n})/\mathbb{Z})$. Therefore, r divides the supernatural number $\mathfrak{n}$.

An inductive argument, based on Lemma 4.2.9, now shows that in fact r^∞ divides $\mathfrak{n}$. Suppose r^k divides $\mathfrak{n}$. Then there exists a projection $p \in A$ of trace r^{-k}. As A is a UHF algebra, A has the cancellation property and all automorphisms of A act trivially on $K_0(A)$. Therefore, we may apply Lemma 4.2.9 to obtain a (G,ω) action on pAp. Since pAp is a UHF algebra with supernatural number $r^{-k}\mathfrak{n}$, we can apply the argument above to pAp to get that r divides $r^{-k}\mathfrak{n}$. Hence, r^{k+1} divides $\mathfrak{n}$. □

Note that in the proof of Theorem 4.2.7 it would be sufficient to use the Hausdorffised version of unitary algebraic K_1. However, for the proof of Theorem 4.2.10 the non-Hausdorffised version of unitary algebraic K_1 is required.

Remark 4.2.11. It is a standard result in group cohomology that, for a finite group G, every element in $H^3(G,\mathbb{T})$ has order dividing $|G|$; see for example [11, III. Corollary 10.2]. This further restricts the possible values of r in Theorem 4.2.10. In particular, it follows from 4.2.10 that if $|G|$ is coprime to the supernatural number $\prod_{k\in\mathbb{N}} n_k$, then for any (G,ω) action on $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ we have that $[\omega] = 0$ in $H^3(G,\mathbb{T})$.

We end this section with a general result for unital C*-algebras which encompasses both Theorem 4.2.7 and Theorem 4.2.10.

Theorem 4.2.12. *Let G be a group, $\omega \in Z^3(G,\mathbb{T})$ and A be a unital C*-algebra. Let (α,u) be a connected (G,ω) action on A and $\tau \in T(A)$ be invariant under α_g for all $g \in G$. Suppose $[\omega]$ has finite order r in $H^3(G,\mathbb{T})$. Then $\frac{1}{r} \in \tau_*(K_0(A))$. If $[\omega]$ has infinite order, then $\tau_*(K_0(A))$ is dense in $\mathbb{R}$.*

Proof. Consider the short exact sequence of abelian groups

$$0 \to \frac{\tau_*(K_0(A))}{\mathbb{Z}} \xrightarrow{\quad\iota\quad} \mathbb{T} \xrightarrow{\ \Delta_\tau\ } \frac{\mathbb{R}}{\tau_*(K_0(A))} \to 0. \tag{4.2.9}$$

Applying the universal coefficient theorem for group cohomology (Theorem 2.5.1) we obtain the following commuting diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{\mathbb{Z}}(H_2(G,\mathbb{Z}),\frac{\mathbb{R}}{\tau_*(K_0(A))}) & \hookrightarrow & H^3(G,\frac{\mathbb{R}}{\tau_*(K_0(A))}) & \xrightarrow{\rho_1} & \mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}),\frac{\mathbb{R}}{\tau_*(K_0(A))}) \\
\uparrow & & \uparrow{\scriptstyle \Delta_{\tau*}} & & \uparrow{\scriptstyle \Delta_{\tau*}} \\
\mathrm{Ext}^1_{\mathbb{Z}}(H_2(G,\mathbb{Z}),\mathbb{T}) & \hookrightarrow & H^3(G,\mathbb{T}) & \xrightarrow{\rho_2} & \mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}),\mathbb{T})
\end{array}$$

where the rows are short exact sequences.

Notice that the Ext terms vanish as $\mathbb{T}$ and $\mathbb{R}/\tau_*(K_0(A))$ are divisible groups. In particular, both ρ_1 and ρ_2 are isomorphisms and so, the order of $\rho_2([\omega])$ and that of $[\omega]$ is the same. Moreover, $\Delta_{\tau*}([\omega]) = 0$ by Corollary 4.2.6. As the diagram commutes, we deduce that $\rho_2([\omega])$ is a group homomorphism $f : H_3(G,\mathbb{Z}) \to \mathbb{T}$ that takes values in $\tau_*(K_0(A))/\mathbb{Z}$. Since the group operation of $\mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}),\mathbb{T})$ is pointwise multiplication, the order of f is the same as the exponent of the group $\mathrm{Im}(f) \subseteq \mathbb{T}$.

Suppose $[\omega]$ has finite order r. The only subgroup of $\mathbb{T}$ with exponent r is the group of r-th roots of unity, so $\mathrm{Im}(f)$ is this subgroup. Since f takes values in $\tau_*(K_0(A))/\mathbb{Z}$, this means that $\frac{1}{r} \in \tau_*(K_0(A))$. Suppose $[\omega]$ has infinite order. Then $\mathrm{Im}(f)$ is an infinite subgroup of $\mathbb{T}$. All such subgroups are dense. It follows that $\tau_*(K_0(A))/\mathbb{Z}$ is dense in $\mathbb{T}$. Therefore, $\tau_*(K_0(A))$ is dense in $\mathbb{R}$. □

In Chapter 6, we will see that the conclusion of Theorem 4.2.12 can fail for anomalous actions that are not connected.

In the process of proving Theorem 4.2.12, we showed that after identifying $H^3(G,\mathbb{T})$ with $\mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}),\mathbb{T})$, if there exists a (G,ω) action on A admitting an invariant trace τ, then $[\omega]$ comes from $\mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}),\tau(K_0(A))/\mathbb{Z})$. If G is infinite this gives more information than the conclusion of Theorem 4.2.12. We illustrate this in the following corollary.

Corollary 4.2.13. *Let G be a discrete group with $H_3(G,\mathbb{Z})$ finitely generated, $\omega \in Z^3(G,\mathbb{T})$ and $n_k \in \mathbb{N}$ for $k \in \mathbb{N}$. If there exists a (G,ω) action on $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}(\mathbb{C})$, then the cocycle ω is of finite order r and r^∞ divides the supernatural number $\prod_{k\in\mathbb{N}} n_k$.*

Proof. Suppose there exists a (G, ω) action on $A = \bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}(\mathbb{C})$. As A has a unique trace, it is invariant under the action. It follows from the proof of Theorem 4.2.12 that $[\omega]$ has the same order as an element $\rho_2([\omega]) \in \mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}), \mathcal{Q}(\mathfrak{n})/\mathbb{Z})$, where $\mathfrak{n} = \prod_{k\in\mathbb{N}} n_k$ and $\mathcal{Q}(\mathfrak{n})$ is as in the proof of Theorem 4.2.10. By hypothesis $H_3(G,\mathbb{Z})$ is finitely generated and so any element in $\mathrm{Hom}_{\mathbb{Z}}(H_3(G,\mathbb{Z}), \mathcal{Q}(\mathfrak{n})/\mathbb{Z})$ is of finite order. In particular, $[\omega]$ has finite order. The remaining claim follows from Theorem 4.2.12. □

For many finitely generated groups G the homology group $H_3(G,\mathbb{Z})$ is finitely generated. This is the case for the lattices $\mathbb{Z}^n$, Poly-$\mathbb{Z}$ groups (these are groups which have a subnormal series with each step being an extension by $\mathbb{Z}$) or more generally any group whose classifying space is compact. However, there are finitely presented groups G with $H^3(G,\mathbb{Z})$ not finitely generated ([126]).

Remark 4.2.14. Izumi has informed me that if A is a UHF algebra of type n^∞ then there exists a $(\mathbb{Z}^3, \omega)$ action on A whenever $[\omega] \in \mathbb{Z}[\frac{1}{n}] \subset \mathbb{T} = H^3(\mathbb{Z}^3, \mathbb{T})$.[4] This result is currently unpublished. Combining Izumi's result with Corollary 4.2.13, there exists a $(\mathbb{Z}^3, \omega)$ action on the UHF algebra $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}(\mathbb{C})$ if and only if the order of $[\omega]$ is a finite number r and $r^\infty | \mathfrak{n}$ where $\mathfrak{n} = \prod_{k\in\mathbb{N}} n_k$.

Proposition 4.2.1 may also be used to observe that if A is a non-unital C*-algebra and (α, u) is an anomalous action on A with $u \in U(\tilde{A})$, then $\mathrm{ob}(\alpha, u)$ is trivial. Indeed, applying Proposition 4.2.1 with $q : U(\tilde{A}) \to \mathbb{T}$ the restriction of the quotient map $\tilde{A} \twoheadrightarrow \tilde{A}/A \cong \mathbb{C}$ we obtain that $[\omega] = 0$. This precisely is the reason we allow unitaries in the multiplier algebra rather than unitaries in the minimal unitisation in definitions 3.2.1 and 3.3.1.

[4]The group $\mathbb{Z}[\frac{1}{n}]$ is the additive subgroup of $\mathbb{R}$ generated by fractions of the form $(\frac{1}{n})^k$ for all $k \in \mathbb{N}$.

Chapter 5

Izumi's K_0 obstruction

In Chapter 4, we introduced an obstruction to the existence of anomalous actions that arose from considering the algebraic K_1 group of C*-algebras. This obstruction only allowed us to extract implications about the existence of anomalous actions in the case that the C*-algebra acted upon had traces (see for instance Theorem 4.2.12). The reason for this, is that the starting point of our algebraic K_1 group obstructions was the implication that if (α, u) was an ω-anomalous action on A then the three cocycle $K_1^{\text{alg}}(\omega_{g,h,k})$ was trivial (Proposition 4.2.1). However, the algebraic K_1 group of a unital, infinite C*-algebra A corresponds to the topological K_1 group of A by [36]. Therefore $K_1^{\text{alg}}(\omega_{g,h,k}) = K_1(\omega_{g,h,k})$ is always trivial.[1]

In this chapter, we discuss an obstruction to the existence of anomalous actions of finite groups that has a topological flavour rather than the algebraic flavour of the obstruction in Chapter 4. This obstruction will work without the need for a trace and so restricts the possible anomalies that arise on infinite C*-algebras. An instance of this obstruction can already be seen when one applies it to the C*-algebras $\mathcal{O}_n$ (see Theorem 5.3.1). For example, this implies that any anomalous action of a finite group on $\mathcal{O}_\infty$ has trivial anomaly (Corollary 5.3.2). In general, we may use

[1]The three cocycle ω is scalar valued and the K_1 class of any scalar is the same as the K_1 class of the unit.

this obstruction to deduce K-theoretic divisibility for some C*-algebras that admit non-trivial anomalous actions (see Theorem 5.4.2). Even though the results of this chapter will recover the results for anomalous actions on $\mathcal{Z}$ and UHF algebras of Chapter 4 when the acting group is finite, they will not yield any obstruction if the acting group is infinite. This implies that the obstruction set out in this chapter and that of Chapter 4 are seperate and should be used together.

This obstruction was pointed out to me by Masaki Izumi and builds on his previous work with Matui in which they introduce invariants for cocycle actions ([69, Section 7]). Section 5.1 and Theorem 5.3.1 are based on a two page note that Izumi shared with me by email as a follow up to one of my talks. I would like to thank Izumi for sharing this with me.

5.1 Izumi's invariant

Throughout this section, A will be a unital C*-algebra and G a group. We start by setting up some notation and conventions. A *path* in a C*-algebra will be a continuous map $f : [0,1] \to A$. If f and g are paths in A we will say they are *homotopic* if f and g have the same endpoints and there is an endpoint preserving homotopy between f and g. In this chapter, we will be concerned with paths of unitaries. Two paths $f, g : [0,1] \to U(A)$ will be called homotopic if they are homotopic within the unitary group $U(A)$. If $\tilde{u}$ is a path in $U(A)$ we let $\tilde{u}^*$ the path given by $\tilde{u}^*(t) = \tilde{u}(t)^*$ for $t \in [0,1]$. We denote by $\tilde{u}^{\mathrm{rev}}$ the path which starts at $\tilde{u}(1)$ and ends at $\tilde{u}(0)$ by reversing the path $\tilde{u}$. When defined, we denote the concatenation of two paths $\tilde{u}$ and $\tilde{v}$ by $\tilde{u} \star \tilde{v}$ and the pointwise product by $\tilde{u}\tilde{v}$. Also, let e_u be the path which is constant at the element $u \in U(A)$.

We start with a lemma that will allow us to commute homotopy classes of paths of unitaries. A slightly less general result is used in the proof of [69, Lemma 7.5].

Lemma 5.1.1 (cf. [69, Lemma 7.5])**.** *Let A be a unital C^*-algebra and $v, w : [0,1] \to U^{(0)}(A)$ paths into the path component of the identity in $U(A)$. Then*

(i) if $v(0) \in Z(U(A))$ and $w(1) \in Z(U(A))$ then vw and wv are homotopic as paths from $v(0)w(0)$ to $v(1)w(1)$ in $U(A)$.

(ii) if v, w are paths such that $v(0) = 1$ and $v(1) = w(1)^*z$ for $z \in Z(U(A))$ then vw and wv are homotopic as paths from $w(0)$ to z in $U(A)$.

Proof. The proof follows from standard techniques see [96, Problem (3) Chapter 7] for a similar exercise. We start with showing (i). First, by performing the paths v and w in half the time, we may assume that up to passing to homotopic paths $v(t) = v(0)$ for all $0 \leq t \leq 1/2$ and $v(t) = w(1)$ for all $1/2 \leq t \leq 1$. Now $vw = wv$ pointwise.

Now let v, w, z as in the hypothesis of (ii). Let $\rho : 1 \to w(1)$ be a continuous path. Then $w \sim_h (w \star \rho^{\text{rev}}) \star \rho$ and $v \sim_h (v \star (\rho^* z)^{\text{rev}}) \star \rho^* z$. The paths $w \star \rho^{\text{rev}}$ and $v \star (\rho^* z)^{rev}$ satisfy the conditions of (i). Applying (i) we get that

$$wv \sim_h ((w \star \rho^{\text{rev}}) \star \rho)((v \star (\rho^* z)^{\text{rev}}) \star \rho^* z) \sim_h (v \star (\rho^* z)^{\text{rev}})(w \star \rho^{\text{rev}}) \star \rho^* z \rho \sim_h wv. \qquad \square$$

Let A be a unital C*-algebra. Recall that we call an anomalous action (α, u) on A connected if $u_{g,h} \in U^{(0)}(A)$ for all $g, h \in G$. To any connected anomalous action, we will start by associating a cohomological invariant.

Lemma 5.1.2. *Let G be a group, A be a unital C^*-algebra and (α, u) a connected anomalous action of G on A. The set*

$$K_0^{\#}(A) := \{f : [0,1] \to U(A) : f(0) = 1, f(1) \in \mathbb{T}\} /_{\sim_h} \tag{5.1.1}$$

is an abelian group under pointwise multiplication of paths and a G-module under the action $g \cdot f = \alpha_g(f)$.

Proof. That $K_0^{\#}(A)$ is abelian follows from (i) Lemma 5.1.1. To show that $K_0^{\#}(A)$ is a G-module, note that $\alpha_g\alpha_h(f) = \mathrm{Ad}(u_{g,h})\alpha_{gh}(f)$ for any unitary valued path f and $g, h \in G$. By hypothesis $u_{g,h}$ are in $U^{(0)}(A)$, pick paths $\tilde{u}_{g,h}$ from 1 to $u_{g,h}$, these paths give homotopies $\alpha_g\alpha_h(f) \sim_h \alpha_{gh}(f)$. □

Hereinafter, we will denote by $[\epsilon_\lambda]$ for $\lambda \in \mathbb{R}$ the path homotopy class of the scalar valued unitary path $\epsilon_\lambda : [0,1] \to U^{(0)}(A)$ defined through $\epsilon_\lambda(t) = e^{2\pi i\lambda t}$. Note that any connected anomalous action (α, u) on a unital C*-algebra A, induces a module structure on $\pi_1(U(A))$ by sending a class of a loop of unitaries f to the class of the loop $\alpha_g(f)$.

Proposition 5.1.3. *Let G be a discrete group, A a unital C*-algebra and (α, u) be a connected anomalous action of G on A. There is an isomorphism of G-modules*

$$\frac{\mathbb{R} \oplus \pi_1(U(A))}{\mathbb{Z}(-1, [\epsilon_1])} \cong K_0^{\#}(A)$$

with $\mathbb{R}$ carrying the trivial action.

Proof. Consider the mapping

$$\mathbb{R} \oplus \pi_1(U(A)) \xrightarrow{\varphi} K_0^{\#}(A)$$
$$(\lambda, f) \to [\epsilon_\lambda f].$$

It is clear that φ is a well-defined G-module morphism. We begin by showing surjectivity. Any representative path f of an element of $K_0^{\#}(A)$ has $f(1) = e^{2\pi i\lambda}$ for some $\lambda \in \mathbb{R}$ so $\epsilon_{-\lambda}f$ is a loop at 1_A in $U(A)$ and satisfies $\varphi(\lambda, [\epsilon_{-\lambda}f]) = [f]$. We now compute the kernel of φ. Suppose $\varphi(\lambda, f) = 0$, then $\epsilon_\lambda f \sim_h e_{1_A}$, and so $f(1) = e^{-2\pi i\lambda}$. As f is a loop at 1_A, $\lambda = n$ for some $n \in \mathbb{Z}$ and hence $f \sim_h \epsilon_{-n}$. Therefore, the kernel of φ is given by $\mathbb{Z}(-1, [\epsilon_1])$. The result follows by the isomorphism theorem. □

Let (α, u) be a connected anomalous action of a group G on a unital C*-algebra A. Pick paths $\tilde{u}_{g,h}$ such that $\tilde{u}_{g,h}(0) = 1_A$ and $\tilde{u}_{g,h}(1) = u_{g,h}$ for all $g, h \in G$. For $g, h, k \in G$ let

$$\tilde{\omega}_{g,h,k} := \alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{h,k}. \tag{5.1.2}$$

As (α, u) is an anomalous action $\tilde{\omega}_{g,h,k}(1) \in \mathbb{T}$ and so $\tilde{\omega}_{g,h,k}$ defines an element in $K_0^{\#}(A)$ for any $g, h, k \in G$.

Lemma 5.1.4. *Let (α, u) be a connected anomalous action on a unital C*-algebra A and $\tilde{\omega}$ be as in (5.1.2). Then $\tilde{\omega} \in Z^3(G, K_0^{\#}(A))$. Moreover, the class $[\tilde{\omega}] \in H^3(G, K_0^{\#}(A))$ is a well defined invariant of (α, u), we denote it by $\widetilde{\mathrm{ob}}(\alpha, u)$.*

Proof. The proof follows [69, Lemma 8.1]. We write the argument for completeness. Throughout this proof we will repeatedly apply homotopies. The main homotopies we will use are those laid out in Lemma 5.1.1. We will also use that if u is a unitary in $U^{(0)}(A)$ and $[f] \in K_0^{\#}(A)$ then $[\mathrm{Ad}(u)f] = [f]$. Following the proof may be a bit tedious so we will colour code the paths on which we preform a homotopy operation the line before we do so. Firstly we check that $\tilde{\omega}$ is a 3-cocycle. Let $g, h, k \in G$, then

$$\begin{aligned}
&g \cdot \tilde{\omega}_{h,k,l}\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [\alpha_g\alpha_h(\tilde{u}_{k,l})\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l}\tilde{u}^*_{h,k})]\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [u_{g,h}\alpha_{gh}(\tilde{u}_{k,l})u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l}\tilde{u}^*_{h,k})]\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [u^*_{g,h}u_{g,h}\alpha_{gh}(\tilde{u}_{k,l})u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l}\tilde{u}^*_{h,k})u_{g,h}]\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [\alpha_{gh}(\tilde{u}_{k,l})u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l}\tilde{u}^*_{h,k})u_{g,h}][\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}\alpha_{gh}(\tilde{u}^*_{k,l})]\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l}\tilde{u}^*_{h,k})u_{g,h}]\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l}\tilde{u}^*_{h,k})u_{g,h}u^*_{g,h}]\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l})\alpha_g(\tilde{u}^*_{h,k})]\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k}
\end{aligned}$$

We evaluate the last item colour coded in red at 1. We will show that this is equal to the adjoint of the evaluation at 1 of the blue colour coded path modulo scalars. Therefore these paths satisfy the hypothesis of Lemma 5.1.1 (ii). Indeed

$$\begin{aligned} u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l})(1) &= u_{g,h}u_{gh,k}u_{ghk,l}u^*_{gh,kl}u^*_{g,h}\alpha_g(u_{h,kl}u^*_{hk,l}) \\ &= u_{g,h}u_{gh,k}(\omega^{-1}_{gh,k,l}u^*_{gh,k}\alpha_{gh}(u_{k,l}))u^*_{g,h}\alpha_g(u_{h,kl}u^*_{hk,l}) \\ &= \omega^{-1}_{gh,k,l}\alpha_g(\alpha_h(u_{k,l})u_{h,kl}u^*_{hk,l}) \\ &= \omega^{-1}_{gh,k,l}\omega_{h,k,l}\alpha_g(u_{h,k}) \end{aligned}$$

as required. We may now apply Lemma 5.1.1 (ii) to swap the red and the blue colour coded paths. We continue with our computation of the 3-cocycle formula for $\tilde{\omega}$,

$$\begin{aligned} &g\cdot\tilde{\omega}_{h,k,l}\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\ &= [\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\alpha_g(\tilde{u}_{h,kl}\tilde{u}^*_{hk,l})][\alpha_g(\tilde{u}_{hk,l})\tilde{u}_{g,hkl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}]\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\ &= [\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\alpha_g(\tilde{u}_{h,kl})\tilde{u}_{g,hkl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}]\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k}. \end{aligned}$$

We wish to apply Lemma 5.1.1 (ii) again to swap the red and blue colour coded paths. To do this we need to check that the evaluations at 1 of the red and blue paths are adjoint to each other up to an element in the centre of the unitary group. We start by calculating the evaluation at 1 of the blue path,

$$\begin{aligned} \alpha_g(\tilde{u}_{h,kl})\tilde{u}_{g,hkl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}(1) &= \alpha_g(u_{h,kl})u_{g,hkl}u^*_{ghk,l}u^*_{g,hk} \\ &= (\omega_{g,h,kl}u_{g,h}u_{gh,kl})u^*_{ghk,l}u^*_{g,hk}. \end{aligned}$$

Now evaluating the red path at 1 we have

$$\begin{aligned}\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}(1) &= \alpha_g(u^*_{h,k})u_{g,h}u_{gh,k}u_{ghk,l}u^*_{gh,kl}u^*_{g,h} \\ &= (\omega^{-1}_{g,h,k}u_{g,hk}u^*_{gh,k})u_{gh,k}u_{ghk,l}u_{gh,kl}u^*_{g,h} \\ &= \omega^{-1}_{g,h,k}u_{g,hk}u_{ghk,l}u^*_{gh,kl}u^*_{g,h}.\end{aligned}$$

So the red and blue paths evaluated at 1 are indeed adjoint up to a scalar. We apply Lemma 5.1.1 (ii) and proceed with our computation of the 3-cocycle identity for $\tilde{\omega}$,

$$\begin{aligned}&g \cdot \tilde{\omega}_{h,k,l}\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\ &= [\alpha_g(\tilde{u}_{h,kl})\tilde{u}_{g,hkl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}][\tilde{u}_{g,h}\tilde{u}_{gh,kl}\tilde{u}^*_{g,hkl}\alpha_g(\tilde{u}^*_{h,kl})]\tilde{\omega}_{g,h,k} \\ &= [\tilde{u}_{g,h}\tilde{u}_{gh,kl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}]\tilde{\omega}_{g,h,k}.\end{aligned}$$

Once more, evaluating the blue path at 1 yields

$$\begin{aligned}\tilde{u}_{gh,kl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}(1) &= u_{gh,kl}u^*_{ghk,l}u^*_{g,hk}(\omega^{-1}_{g,h,k}u_{g,hk}u^*_{gh,k}) \\ &\quad u_{gh,k}u_{ghk,l}u^*_{gh,kl}u^*_{g,h} \\ &= \omega^{-1}_{g,h,k}u^*_{g,h}.\end{aligned}$$

Which is, up to a scalar factor, the adjoint of the evaluation at one of the red path. We apply Lemma 5.1.1 (ii) to get that

$$\begin{aligned}&g \cdot \tilde{\omega}_{h,k,l}\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\ &= [\tilde{u}_{gh,kl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})u_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}u^*_{g,h}\tilde{u}_{g,h}]\tilde{\omega}_{g,h,k}.\end{aligned}$$

As the orange path ends at 1_A and $\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}$ starts at 1_A Lemma 5.1.1 (i) allows us to swap these paths. Our three cocycle computation is complete with need of one final application of Lemma 5.1.1 (ii) to the red and blue colour coded paths below,

$$
\begin{aligned}
&g \cdot \tilde{\omega}_{h,k,l}\tilde{\omega}^{-1}_{gh,k,l}\tilde{\omega}_{g,hk,l}\tilde{\omega}^{-1}_{g,h,kl}\tilde{\omega}_{g,h,k} \\
&= [\tilde{u}_{gh,kl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})u_{g,h}(u^*_{g,h}\tilde{u}_{g,h})\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}]\tilde{\omega}_{g,h,k} \\
&= [\tilde{u}_{gh,kl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})\tilde{u}_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}][\alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{g,h}]. \\
&= [\tilde{u}_{g,h}\tilde{u}_{gh,k}\tilde{u}_{ghk,l}\tilde{u}^*_{gh,kl}\tilde{u}_{gh,kl}\tilde{u}^*_{ghk,l}\tilde{u}^*_{g,hk}\alpha_g(\tilde{u}^*_{h,k})][\alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{g,h}] \\
&= 1.
\end{aligned}
$$

It remains to check that the cohomology class of $\tilde{\omega}$ is independent of the path of unitaries chosen. This will establish that the class $\widetilde{\mathrm{ob}}(\alpha, u)$ is well defined. It will follow from repeated use of Lemma 5.1.1 (i). We will use colour coding to denote the paths to which we apply Lemma 5.1.1 (i). For $g, h \in G$, let $\hat{u}_{g,h}$ be another path of unitaries with $\hat{u}_{g,h}(0) = 1$ and $\hat{u}_{g,h}(1) = u_{g,h}$ and $\hat{\omega}_{g,h,k}$ its associated 3-cocycle, then

$$
\begin{aligned}
\tilde{\omega}_{g,h,k}\hat{\omega}^{-1}_{g,h,k} &= [\alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{g,h}\hat{u}_{g,h}\hat{u}_{gh,k}\hat{u}^*_{g,hk}\alpha_g(\hat{u}^*_{h,k})] \\
&= [\alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\hat{u}_{gh,k}\hat{u}^*_{g,hk}\alpha_g(\hat{u}^*_{h,k})][\tilde{u}^*_{g,h}\hat{u}_{g,h}] \\
&= [\alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\hat{u}^*_{g,hk}\alpha_g(\hat{u}^*_{h,k})][\tilde{u}^*_{gh,k}\hat{u}_{gh,k}][\tilde{u}^*_{g,h}\hat{u}_{g,h}] \\
&= [\alpha_g(\tilde{u}_{h,k})\alpha_g(\hat{u}^*_{h,k})][\tilde{u}_{g,hk}\hat{u}^*_{g,hk}][\tilde{u}^*_{gh,k}\hat{u}_{gh,k}][\tilde{u}^*_{g,h}\hat{u}_{g,h}] \\
&= (d\eta)_{g,h,k}
\end{aligned}
$$

with η a 2-cochain valued in $K_0^{\#}(A)$ defined by $\eta(g,h) = [\tilde{u}_{g,h}\hat{u}_{g,h}]$ for $g, h \in G$. □

The invariant defined in Lemma 5.1.4 yields the following obstruction to the existence of anomalous actions.

Theorem 5.1.5. *Let A be a unital C^*-algebra and (α, u) be a connected (G, ω) action on A. Denote by ev_1 the map from $K_0^{\#}(A) \to \mathbb{T}$ given by $ev_1([f]) = f(1)$. Then $[\omega]$*

is in the image of the map $H^3(G, K_0^{\#}(A)) \xrightarrow{ev_{1*}} H^3(G, \mathbb{T})$.

Proof. Let $\widetilde{\mathrm{ob}}(\alpha, u)$ as in Lemma 5.1.4. Then $ev_{1*}(\widetilde{\mathrm{ob}}(\alpha, u)) = [\omega]$. $\square$

5.2 The invariant for G-kernels

The invariant of Lemma 5.1.4 has a counterpart for G-kernels. We discuss this in this section.

Lemma 5.2.1. *Let G be a group, A a unital C*-algebra with connected unitary group and $\overline{\theta}$ a G-kernel on A. Choose a lifting (θ, u) of $\overline{\theta}$. The set*

$$K_0^{\$}(A) := \{f : [0,1] \to U(A) \mid f(0) = 1, f(1) \in Z(U(A))\} \big/ {\sim_h} \tag{5.2.1}$$

is an abelian group under pointwise multiplication of paths and a G-module under the action $g \cdot f = \theta_g(f)$. The G-module structure is independent of the choice of lift of $\overline{\theta}$.

Proof. The fact that it is a G-module follows exactly as in Lemma 5.1.2. Any other lift of $\overline{\theta}$ is given by $\mathrm{Ad}(v_g)\theta_g$ for unitaries $v_g \in U(A)$, $g \in G$. As the unitary group is connected $\mathrm{Ad}(v_g)f \sim_h f$ for any representative path f in $K_0^{\$}(A)$. Therefore the G-module structures defined by θ_g and $\mathrm{Ad}(v_g)\theta_g$ for $g \in G$ are equal. $\square$

Let $\overline{\theta} : G \to \mathrm{Out}(A)$ be a G-kernel of a unital C*-algebra A and (θ, u) a lifting of $\overline{\theta}$ as in Section 3.2. Let $\omega \in Z^3(G, Z(U(A)))$ be the 3-cocycle associated to (θ, u). We proceed exactly as in Section 5.1. Pick paths $\tilde{u}_{g,h}$ such that $\tilde{u}_{g,h}(0) = 1$ and $\tilde{u}_{g,h}(1) = u_{g,h}$ and define

$$\tilde{\omega}(g, h, k) := \alpha_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{h,k} \;\forall g, h, k \in G. \tag{5.2.2}$$

Lemma 5.2.2. *Let G be a group, A a unital C*-algebra with connected unitary group and $\overline{\theta}$ be a G-kernel on A. Let $\tilde{\omega}$ be as in (5.2.2). Then $\tilde{\omega} \in Z^3(G, K_0^{\$}(A))$. Moreover,* *the class* $[\tilde{\omega}] \in H^3(G, K_0^{\$}(A))$ *is a well defined invariant of* $\overline{\theta}$*, we denote it by* $\widetilde{\mathrm{ob}}(\overline{\theta})$*.*

Proof. The calculation of Lemma 5.1.4 works verbatim to show that $\tilde{\omega}$ is a 3-cocycle and that its cohomology class is independent of the unitary paths chosen. It remains to check that the cohomology class $[\tilde{\omega}]$ is independent of the lift (θ, u) of $\overline{\theta}$.

Firstly, we show that if we choose a lifting (θ, u) of $\overline{\theta}$ and tweak $u_{g,h}$ by unitaries $\lambda_{g,h} \in Z(U(A))$, the associated 3-cocycles are cohomologous. Indeed, pick paths $\tilde{u}_{g,h}, \tilde{\lambda}_{g,h} \in U(A)$ for each $g, h \in G$ with $\tilde{u}_{g,h}(0) = \tilde{\lambda}_{g,h}(0) = 1_A$, $\tilde{u}_{g,h}(1) = u_{g,h}$ and $\tilde{\lambda}_{g,h}(1) = \lambda_{g,h}$. Note that the paths $\tilde{\lambda}_{g,h}$ need not be valued in the centre of the unitary group, even though $\lambda_{g,h} \in Z(U(A))$. Let $\tilde{\omega}$ be the associated 3-cocycle to (θ, u) and $\tilde{\omega}'$ the associated 3-cocycle to $(\theta, \lambda u)$. As in the proof Lemma 5.1.4 we will be required to apply (ii) Lemma 5.1.1 repeatedly. We will colour code the paths to which we apply the Lemma before we perform the homotopy;

$$
\begin{aligned}
\tilde{\omega}'_{g,h,k} &= [\theta_g(\tilde{\lambda}_{h,k})\theta_g(\tilde{u}_{h,k})\tilde{\lambda}_{g,hk}\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{\lambda}^*_{gh,k}\tilde{u}^*_{g,h}\tilde{\lambda}^*_{g,h}] \\
&= [\theta_g(\tilde{u}_{h,k})\tilde{\lambda}_{g,hk}\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{\lambda}^*_{gh,k}\tilde{u}^*_{g,h}][\theta_g(\tilde{\lambda}_{h,k})][\tilde{\lambda}^*_{g,h}] \\
&= [\tilde{\lambda}^*_{gh,k}\tilde{u}^*_{g,h}\theta_g(\tilde{u}_{h,k})\tilde{\lambda}_{g,hk}\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}][\theta_g(\tilde{\lambda}_{h,k})][\tilde{\lambda}^*_{g,h}] \\
&= [\tilde{u}^*_{g,h}\theta_g(\tilde{u}_{h,k})\tilde{\lambda}_{g,hk}\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}][\theta_g(\tilde{\lambda}_{h,k})][\tilde{\lambda}^*_{gh,k}][\tilde{\lambda}^*_{g,h}] \\
&= [\tilde{\lambda}_{g,hk}\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{g,h}\theta_g(\tilde{u}_{h,k})][\theta_g(\tilde{\lambda}_{h,k})][\tilde{\lambda}^*_{gh,k}][\tilde{\lambda}^*_{g,h}] \\
&= \tilde{\omega}_{g,h,k}(d\tilde{\lambda})_{g,h,k}.
\end{aligned}
$$

So $\tilde{\omega}$ and $\tilde{\omega}'$ are cohomologous. Now let (θ', u') be another lift of $\overline{\theta}$. Then there exist unitaries $v_g \in U(A)$ and $\lambda_{g,h} \in Z(U(A))$ for $g, h \in G$ such that $\theta'_g = \mathrm{Ad}(v_g)\theta_g$ and $u'_{g,h} = \lambda_{g,h}v_g\theta_g(v_h)u_{g,h}v^*_{gh}$. Due to the preceeding computation, we may assume that $\lambda_{g,h} = 1$ for every $g, h \in G$. Pick continuous paths $\tilde{v}_g \in U(A)$ with $\tilde{v}_g(0) = 1, \tilde{v}_g(1) = v_g$. Let $\tilde{u}'_{g,h} = \tilde{v}_g\theta_g(\tilde{v}_h)\tilde{u}_{g,h}\tilde{v}^*_{gh}$ and $\tilde{\omega}'_{g,h,k} = \theta'_g(\tilde{u}'_{h,k})\tilde{u}'_{g,hk}\tilde{u}'^*_{gh,k}\tilde{u}'^*_{g,h}$, these constitute paths which start at 1 and such that $\tilde{u}'_{g,h}(1) = u'_{g,h}$ and $\tilde{\omega}'_{g,h,k}(1) \in Z(U(A))$. We have that

$$\begin{aligned}\tilde{\omega}'_{g,h,k} &= [\theta'_g(\tilde{u}'_{h,k})\tilde{u}'_{g,hk}\tilde{u}'^*_{gh,k}\tilde{u}'^*_{g,h}] \\ &= [v_g\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k)\tilde{u}_{h,k}\tilde{v}^*_{hk})v^*_g\tilde{v}_g\theta_g(\tilde{v}_{hk})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\theta_{gh}(\tilde{v}^*_k)\tilde{u}^*_{g,h}\theta_g(\tilde{v}^*_h)\tilde{v}^*_g]\end{aligned}$$

As the colour coded path ends at 1_A and $\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k)\tilde{u}_{h,k}\tilde{v}^*_{hk})$ starts at 1_A we may apply (i) Lemma 5.1.1 to swap the two paths. Therefore, applying (i) Lemma 5.1.1 once more to the orange colour coded path below

$$\begin{aligned}\tilde{\omega}'_{g,h,k} &= [v_g(v^*_g\tilde{v}_g)\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k))\theta_g(\tilde{u}_{h,k})\theta_g(\tilde{v}^*_{h,k})\theta_g(\tilde{v}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\theta_{gh}(\tilde{v}^*_k)\tilde{u}^*_{g,h}\theta_g(\tilde{v}^*_h)\tilde{v}^*_g] \\ &= [\tilde{v}_g\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k))\theta_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\theta_{gh}(\tilde{v}^*_k)\tilde{u}^*_{g,h}\theta_g(\tilde{v}^*_h)\tilde{v}^*_g] \\ &= [\tilde{v}_g\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k))\theta_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}u^*_{g,h}\theta_g\theta_h(\tilde{v}^*_k)u_{g,h}\tilde{u}^*_{g,h}\theta_g(\tilde{v}^*_h)\tilde{v}^*_g] \\ &= [\tilde{v}_g\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k))\theta_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}u^*_{g,h}(u_{g,h}\tilde{u}^*_{g,h})\theta_g\theta_h(\tilde{v}^*_k)\theta_g(\tilde{v}^*_h)\tilde{v}^*_g] \\ &= [\tilde{v}_g\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k))\theta_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{g,h}\theta_g\theta_h(\tilde{v}^*_k)\theta_g(\tilde{v}^*_h)\tilde{v}^*_g].\end{aligned}$$

We may swap the red and blue colour coded paths by (ii) Lemma 5.1.1 to get that

$$\begin{aligned}\tilde{\omega}'_{g,h,k} &= [\theta_g(\tilde{u}_{h,k})\tilde{u}_{g,hk}\tilde{u}^*_{gh,k}\tilde{u}^*_{g,h}\theta_g\theta_h(\tilde{v}^*_k)\theta_g(\tilde{v}^*_h)\tilde{v}^*_g\tilde{v}_g\theta_g(\tilde{v}_h\theta_h(\tilde{v}_k))] \\ &= \tilde{\omega}_{g,h,k}\end{aligned}$$

as required. □

One may apply Lemma 5.2.2 to yield the G-kernel counterpart to the obstruction laid out in Theorem 5.1.5.

Corollary 5.2.3. *Let G be a group, A be a unital C^*-algebra such that $U^{(0)}(A) = U(A)$ and $\overline{\theta}$ a G-kernel on A. Denote by ev_1 the map $ev_1 : K_0^{\$}(A) \to Z(U(A))$ defined by $ev_1(f) = f(1)$. Then $\mathrm{ob}(\overline{\theta}) \in \mathrm{im}(ev_{1*})$.*

5.3 Implications for Cuntz algebras

We start by applying the obstruction developed in Section 5.1 to the Cuntz algebras $\mathcal{O}_n$ for $n \in \{2, \dots, \infty\}$. Note that any anomalous action on $\mathcal{O}_n$ is connected as the unitary group of $\mathcal{O}_n$ is connected by [30, Corollary 3.12].

Theorem 5.3.1. *Let G a group and $\omega \in Z^3(G, \mathbb{T})$. If there exists a (G, ω) action on $\mathcal{O}_n$ with $n \in \mathbb{N} \setminus \{1\}$ then $[\omega] \in (n-1)H^3(G, \mathbb{T})$. If instead $n = \infty$ then $[\omega] \in kH^3(G, \mathbb{T})$ for every $k \in \mathbb{N}$.*

Proof. Firstly, suppose that $n \in \mathbb{N} \setminus \{1\}$. By [136, Theorem 4.3] there is a pointed isomorphism $(\pi_1(U(\mathcal{O}_n)), [\epsilon_1]) \cong (\pi_1(U_\infty(\mathcal{O}_n)), [\epsilon_1])$. This composed by the Bott isomorphism (see Theorem 2.6.10) yields a pointed isomorphism $(\pi_1(U(\mathcal{O}_n)), [\epsilon_1]) \cong (K_0(\mathcal{O}_n), [1_{\mathcal{O}_n}]_0)$. The pointed group $(K_0(\mathcal{O}_n), [1_{\mathcal{O}_n}]_0)$ has been computed by Cuntz (see [30, Theorem 3.7]) to be isomorphic to $(\mathbb{Z}_{n-1}, [1])$. We will denote the class 1 in $\mathbb{Z}_{n-1}$ by s. By Lemma 5.1.2, $K_0^{\#}(\mathcal{O}_n) \cong \frac{\mathbb{R} \oplus \mathbb{Z}_{n-1}}{\mathbb{Z}(-1,s)}$ equipped with the trivial G-module structure, the triviality of the G-module structure follows as every automorphism of $\mathcal{O}_n$ is approximately inner ([116, Theorem 3.6]). The map $\varphi : K_0^{\#}(\mathcal{O}_n) \to \mathbb{T}$ given by $\varphi(\lambda, s^k) = e^{\frac{2\pi i(\lambda+k)}{n-1}}$ is an isomorphism and the map ψ making the diagram

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\psi} & \mathbb{T} \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ K_0^{\#}(\mathcal{O}_n) & \xrightarrow{ev_1} & \mathbb{T} \end{array}$$

commute is uniquely defined by $\psi(\mu) = \mu^{n-1}$ for any $\mu \in \mathbb{T}$. By Theorem 5.1.5, $[\omega]$ is in the image of the map $ev_{1*} : H^3(G, K_0^{\#}(\mathcal{O}_{n-1})) \to H^3(G, \mathbb{T})$. Equivalently $[\omega]$ is in the image of ψ_* that, as the group operation of $H^3(G, \mathbb{T})$ is pointwise multiplication, corresponds to the subgroup $(n-1)H^3(G, \mathbb{T})$ (written additively) or $H^3(G, \mathbb{T})^{n-1}$ (written multiplicatively).

We now consider the case $n = \infty$. Firstly, as in the case $n \in \mathbb{N} \setminus \{1\}$, a combination of [149] and Bott periodicity implies that $(\pi_1(U(\mathcal{O}_\infty)), [\epsilon_1]) \cong (K_0(\mathcal{O}_\infty), [1_{\mathcal{O}_\infty}])$. Cuntz also computed in [30, Corollary 3.11] that $(K_0(\mathcal{O}_\infty), [1_{\mathcal{O}_\infty}]_0) \cong (\mathbb{Z}, 1)$. Therefore by Lemma 5.1.2, $K_0^{\#}(\mathcal{O}_\infty) \cong \frac{\mathbb{R}\oplus\mathbb{Z}}{\mathbb{Z}(-1,1)}$ (as in the $n \in \mathbb{N} \setminus \{1\}$ case this also has the trivial G-module structure by [116]) . The map $\varphi : K_0^{\#}(\mathcal{O}_\infty) \to \mathbb{R}$ given by $\varphi(\lambda, k) = \lambda - k$ is an isomorphism and the map ψ making the diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\psi} & \mathbb{T} \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\ K_0^{\#}(\mathcal{O}_\infty) & \xrightarrow{ev_1} & \mathbb{T} \end{array} \tag{5.3.1}$$

is uniquely defined as $\psi(\mu) = e^{2\pi i\mu}$ for $\mu \in \mathbb{R}$. Hence, Theorem 5.1.5 implies that $[\omega]$ is in the image of $ev_{1*} : H^3(G, K_0^{\#}(\mathcal{O}_\infty)) \to H^3(G, \mathbb{T})$ or equivalently (by (5.3.1)) in the image of $\psi_* : H^3(G, \mathbb{R}) \to H^3(G, \mathbb{T})$. Any cocycle in the image of ψ_* is of the form $[\omega] = [e^{2\pi i\eta}]$ for a 3-cocycle $\eta \in Z^3(G, \mathbb{R})$. For $k \in \mathbb{N}$, let $\nu \in Z^3(G, \mathbb{T})$ be given by $\nu_{g,h,l} = e^{\frac{2\pi i\eta_{g,h,l}}{k}}$ for $g, h, l \in G$. Then $[\nu]^k = [\omega]$ (written multiplicatively) or $k[\nu]$ (written additively). □

For any finite group G the cohomology group $H^3(G, \mathbb{T})$ is $|G|$-torsion (see [11, III. Corollary 10.2]). Thus it is an immediate corollary of Theorem 5.3.1 that there are no anomalous actions with non-trivial anomalies of finite groups on $\mathcal{O}_\infty$.

Corollary 5.3.2. *Let G be a finite group. There exists a (G, ω) action on $\mathcal{O}_\infty$ if and only if $[\omega] = 0 \in H^3(G, \mathbb{T})$.*

Remark 5.3.3. In Theorem 5.3.1 we used that $(K_0(A), [1_A]) \cong (\pi_1(U(A)), [\epsilon_1])$ is a pointed isomorphism of abelian groups when $A = \mathcal{O}_n$. This holds more generally for any $\mathcal{Z}$-stable, unital C*-algebra A by [72, Theorem A].

In fact not only does there not exist any finite group anomalous action on $\mathcal{O}_\infty$ but there also does not exist any on $\mathcal{O}_\infty \otimes \mathbb{K}$ or $\mathcal{O}_\infty^{st}$. Recall that $\mathcal{O}_\infty^{st}$ is the cut down of $\mathcal{O}_\infty$ by a non-zero projection with K_0 class zero (see Section 2.7.3).

Proposition 5.3.4. *Let G be a finite group. There exists a (G, ω) action on $\mathcal{O}_\infty^{st}$ or $\mathcal{O}_\infty \otimes \mathbb{K}$ if and only if $[\omega] = 0 \in H^3(G, \mathbb{T})$.*

Proof. The if statement is trivial, we show the only if statement. Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$. We start by showing that if there exists a (G, ω) action on $\mathcal{O}_\infty^{st}$ then $[\omega] = 0$. Suppose (α, u) is an ω-anomalous action of G on $\mathcal{O}_\infty^{st}$ with $\mathrm{ob}(\alpha, u)$ non-trivial. Then $\pi : g \mapsto K_0(\alpha_g) \in \mathrm{Aut}_{f.o}(K_0(\mathcal{O}_\infty^{st}, [1]_0))$ for $g \in G$ defines a group homomorphism.[2] The subgroup $H := \{g \in G : \ K_0(\alpha_g) = \mathrm{id}_{K_0(\mathcal{O}_\infty)}\} < G$ fits into an exact sequence

$$0 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \mathrm{Aut}_{f.o}(K_0(\mathcal{O}_\infty^{st}), [1]_0). \tag{5.3.2}$$

We will start by showing that we can reduce to the case that $H = G$. The group $(K_0(\mathcal{O}_\infty^{st}), [1]_0) \cong (\mathbb{Z}, 0)$ and so the group $\mathrm{Aut}_{f.o}(K_0(\mathcal{O}_\infty^{st}), [1]_0) \cong \mathbb{Z}_2$. To be more precise, the group $\mathrm{Aut}_{f.o}(K_0(\mathcal{O}_\infty^{st}), [1]_0)$ is generated by an element Φ of order two which sends a projection $[p]_0$ representing the class of $1 \in \mathbb{Z} \cong K_0(\mathcal{O}_\infty^{st})$ to the class $-[p]_0$. By [4, Theorem 4.3] there exists $\beta \in \mathrm{Aut}(\mathcal{O}_\infty^{st})$ such that $\beta^2 = \mathrm{id}_{\mathcal{O}_\infty^{st}}$ and $K_0(\beta) = \Phi$. We may now define a group action θ of G on $\mathcal{O}_\infty^{st}$ as follows

$$\theta_g(a) = \begin{cases} a, \ \text{if } \pi(g) = \mathrm{id}_{K_0(\mathcal{O}_\infty^{st})} \\ \beta(a), \ \pi(g) = \Phi. \end{cases}$$

By the Kirchberg-Phillips classification theorem (Theorem 2.7.9) and the Künneth formula for tensor products (Theorem 2.6.11) the isomorphism $\overline{\varphi} : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ given by $\overline{\varphi}(n, m) = nm$ lifts to an isomorphism φ from $\mathcal{O}_\infty^{st} \otimes \mathcal{O}_\infty^{st}$ to $\mathcal{O}_\infty^{st}$ with $K_0(\varphi) = \overline{\varphi}$. Therefore, the tensor product anomalous action $(\varphi(\alpha \otimes \theta)\varphi^{-1}, \varphi(u \otimes 1))$ of $\mathcal{O}_\infty^{st}$ has anomaly ω. We will now preform a computation to deduce that the induced action in K_0 given by $\varphi(\alpha \otimes \theta)\varphi^{-1}$ is trivial. In this computation we will naturally identify

[2] For a unital C*-algebra A, the group $\mathrm{Aut}_{f.o}(K_0(A), [1]_0)$ is the group of finite order automorphisms of the K_0 group of A that preserve the class of the unit.

$K_0(\mathcal{O}_\infty^{st} \otimes \mathcal{O}_\infty^{st})$ and $K_0(\mathcal{O}_\infty^{st}) \otimes K_0(\mathcal{O}_\infty^{st})$. We are free to do so by the Künneth formula for tensor products. Now, for $g \in G$ such that $\pi(g) = \Phi$

$$\begin{aligned} K_0(\varphi(\alpha_g \otimes \theta_g)\varphi^{-1})([p]_0) &= K_0(\varphi)(K_0(\alpha_g) \otimes K_0(\theta_g))([p]_0 \otimes [p]_0) \\ &= K_0(\varphi)(\Phi([p]_0) \otimes \Phi([p]_0)) \\ &= K_0(\varphi)((-[p]_0) \otimes (-[p]_0)) \\ &= K_0(\varphi)([p]_0 \otimes [p]_0) \\ &= [p]_0. \end{aligned}$$

So for $g \in G$ such that $\pi(g) = \Phi$ the automorphism $K_0(\varphi(\alpha_g \otimes \theta_g)\varphi^{-1}) = \mathrm{id}_{K_0(\mathcal{O}_\infty^{st})}$. Similarly, for $g \in G$ such that $\pi(g) = \mathrm{id}_{K_0(\mathcal{O}_\infty^{st})}$ both $K_0(\theta_g)$ and $K_0(\alpha_g)$ are trivial and therefore $K_0(\varphi(\alpha_g \otimes \theta_g)\varphi^{-1}) = \mathrm{id}_{K_0(\mathcal{O}_\infty^{st})}$. Replacing (α, u) by $(\varphi(\alpha \otimes \theta)\varphi^{-1}, \varphi(u \otimes 1))$ we may assume that the group $G = H$ or equivalently that $K_0(\alpha_g) = \mathrm{id}_{K_0(\mathcal{O}_\infty^{st})}$ for all $g \in G$. Therefore, the class $[p]_0$ is fixed by $K_0(\alpha_g)$ for $g \in G$ and by Lemma 4.2.9 the anomalous action (α, u) induces an ω-anomalous action on the corner $p\mathcal{O}_\infty^{st}p$. However, as $p\mathcal{O}_\infty^{st}p$ is a full corner in $\mathcal{O}_\infty^{st}$ the inclusion $p\mathcal{O}_\infty^{st}p \xrightarrow{\iota} \mathcal{O}_\infty^{st}$ induces an isomorphism in K-theory (Proposition 2.6.4). In particular, the pointed K_0 group of $p\mathcal{O}_\infty^{st}p$ is isomorphic to $(\mathbb{Z}, 1)$ and the K_1 group is trivial. So $p\mathcal{O}_\infty^{st}p$ has the same (pointed) K-theory groups as $\mathcal{O}_\infty$ and by Theorem 2.7.9 the C*-algebras $p\mathcal{O}_\infty^{st}p$ and $\mathcal{O}_\infty$ are isomorphic. However, as the first admits an ω-anomalous action so does the second. By Corollary 5.3.2 the cohomology class $[\omega]$ is trivial.

We now turn to the case of (G, ω) actions on $\mathcal{O}_\infty \otimes \mathbb{K}$. If (α, u) is an ω-anomalous action on $\mathcal{O}_\infty \otimes \mathbb{K}$ then $K_0(\alpha_g)[p]_0 = [p]_0$ for any non-zero projection $p \in P(\mathcal{O}_\infty \otimes \mathbb{K})$ whose K_0 class is zero. By Lemma 4.2.9 (α, u) induces a (G, ω) action on $p(\mathcal{O}_\infty \otimes \mathbb{K})p$ which is isomorphic to $\mathcal{O}_\infty^{st}$ by Theorem 2.7.9. Therefore, $[\omega] = 0$. □

5.4 Divisibility in K-theory

In this section, we have a closer look at the obstruction defined in Section 5.1. The goal is to achieve divisibility in K-theory as a consequence of the existence of anomalous actions. This is a step towards characterising for which C*-algebras A there exists a (G, ω) action for a fixed group G and 3-cocycle $\omega \in Z^3(G, \mathbb{T})$.

Lemma 5.4.1. *Let G be a finite group, $\omega \in Z^3(G, \mathbb{T})$ of order r in $H^3(G, \mathbb{T})$ and A be a unital C*-algebra. If (α, u) is a connected (G, ω) action on A such that there is no $|G|$ torsion in $K_0(A)_G$ then $[1_A]_0 \in K_0(A)_G$ is r-divisible.*

Proof. Let $K_0^{\#}(A)$ be as in Section 5.1. Let $\pi : K_0^{\#}(A) \to K_0^{\#}(A)_G$ be the quotient map. The map ev_1 descends to a well defined map on $K_0^{\#}(A)_G$ and so

$$\begin{array}{ccc} H^3(G, K_0^{\#}(A)) & \xrightarrow{ev_{1*}} & H^3(G, \mathbb{T}) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ H^3(G, K_0^{\#}(A)_G) & \xrightarrow{ev_{1*}} & H^3(G, \mathbb{T}) \end{array}$$

commutes. By Theorem 5.1.5 the class $[\omega]$ is in the image of $ev_{1*} : H^3(G, K_0^{\#}(A)) \to H^3(G, \mathbb{T})$ and so, $[\omega]$ is also in the image of $ev_{1*} : H^3(G, K_0^{\#}(A)_G) \to H^3(G, \mathbb{T})$. As $K_0^{\#}(A)_G$ has the trivial G-module structure, the universal coefficient theorem (see Theorem 2.5.1) applies. This yields the following commuting diagram:

$$\begin{array}{ccc} H^3(G, K_0^{\#}(A)_G) & \xrightarrow{ev_{1*}} & H^3(G, \mathbb{T}) \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ \mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), K_0^{\#}(A)_G) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), \mathbb{T}). \end{array} \tag{5.4.1}$$

With $\varphi : \mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), K_0^{\#}(A)_G) \to \mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), \mathbb{T})$ the map induced by the module map ev_1. Note that η is an isomorphism as $\mathbb{T}$ is a divisible group. Therefore, the order of $\eta([\omega])$ is also r. The order of the function $\eta([\omega]) \in \mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), \mathbb{T})$ is the same as the exponent of the group $\mathrm{Im}(\eta([\omega])) \subset \mathbb{T}$. So, as the only subgroup of the circle which is of exponent r is the subgroup of r-th roots of unity, there exists $x \in H_3(G, \mathbb{Z})$ such that $\eta([\omega])(x) = e^{2\pi i/r}$. We will now preform a diagram chase to complete the proof. As $[\omega]$ is in the image of ev_{1*} the commutativity of diagram (5.4.1) establishes the existence of a function $f \in \mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), K_0^{\#}(A)_G)$ such that $\varphi(f) = \eta([\omega])$. The map φ is induced by ev_1 so $\varphi(f)(x) = ev_1(f(x)) = e^{2\pi i/r}$. By Lemma 5.1.2 there is an isomorphism of G-modules

$$K_0^{\#}(A) \cong \frac{\mathbb{R} \oplus \pi_1(U(A))}{\mathbb{Z}(-1, [\epsilon_1])}.$$

Under this isomorphism, ev_1 corresponds to exponentiating the $\mathbb{R}$ component. Using this picture of $K_0^{\#}(A)$, $f(x) = [(1/r, a)]$ for some $a \in \pi_1(U(A))_G$. But f is in $\mathrm{Hom}_{\mathbb{Z}}(H_3(G, \mathbb{Z}), K_0^{\#}(A)_G)$ and the order of x divides the size of the group $|G|$ (see [11, Corollary 10.2]). Hence, denoting by k the order of $x \in H_3(G, \mathbb{Z})$ the equation $0 = f(kx) = kf(x) = k[(1/r, a)]$ holds. In particular, there exists $s \in \mathbb{Z}$ such that $k/r = -s$ and $ka = s[\epsilon_1]$ and so

$$rka = -k[\epsilon_1]. \tag{5.4.2}$$

Now note that the canonical homomorphism $\iota : \pi_1(U(A)) \to \pi_1(U_\infty(A))$ and the Bott isomorphism $\varphi : \pi(U_\infty(A)) \to K_0(A)$ are G-equivariant (the equivariance of the first map is by definition, the equivariance of the second map can be deduced directly from the form of the Bott isomorphism, see Theorem 2.6.10). Therefore $\varphi \circ \iota$ descends to a map from $\pi_1(U(A))_G$ to $K_0(A)_G$ and maps $[\epsilon_1]$ to $[1_A]_0$. Hence applying $\varphi \circ \iota$ to (5.4.2) we have that $k\varphi \circ \iota(ra) = -k[1_A]_0$. As $K_0(A)_G$ has no $|G|$ torsion it has no k torsion and dividing through by k we have that $r(\varphi \circ \iota(-a)) = [1_A]_0$ with $\varphi \circ \iota(-a) \in K_0(A)_G$ as required. □

Theorem 5.4.2. *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$ of order r in $H^3(G, \mathbb{T})$. Let A be a unital C^*-algebra with no $|G|$ torsion in $K_0(A)$ and cancellation of non-zero* *projections. If (α, u) is a connected (G, ω) action on A inducing the trivial G-module structure on $K_0(A)$, then $K_0(A)$ is uniquely r-divisible.*

Proof. As $K_0(A)$ has no $|G|$ torsion it has no r-torsion, so it suffices to show that $K_0(A)$ is r-divisible. Suppose there exists an element $x \in K_0(A)$ which is not r-divisible. As A is unital, x is given by a difference of projections $x = [p]_0 - [q]_0$ for $p, q \in P(\mathbb{M}_n(A))$ for some $n \in \mathbb{N}$. Without loss of generality $[p]_0$ is not r-divisible. The induced action on $K_0(A)$ is trivial, so $[\alpha_g(p)]_0 = [p]_0$ for all $g \in G$. As A has cancellation of non-zero projections, it follows from Lemma 4.2.9 that you have an induced (G, ω) action on the corner $p\mathbb{M}_n(A)p$. This action similarly induces the trivial G-module structure on $K_0(p\mathbb{M}_n(A)p)$. So, the hypothesis of Lemma 5.4.1 is satisfied, and it follows that $[p]_{p\mathbb{M}_n(A)p}$ is r-divisible. Hence there exists $c \in K_0(p\mathbb{M}_n(A)p)$ such that

$$[p]_{p\mathbb{M}_n(A)p} = rc.$$

It follows that $[p]_{\mathbb{M}_n(A)} = r\iota_*(c)$ with $\iota : p\mathbb{M}_n(A)p \to \mathbb{M}_n(A)$ the inclusion. As the inclusion $A \to \mathbb{M}_n(A)$ induces an isomorphism in K_0 the class $[p]_0$ is also r divisible in $K_0(A)$. □

Corollary 5.4.3. *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$ of order r in $H^3(G, \mathbb{T})$. Let A be a simple, separable, unital C^*-algebra with no $|G|$ torsion in $K_0(A)$. If (α, u) is a connected (G, ω) action on A inducing the trivial G-module structure on $K_0(A)$ then $K_0(A)$ is r-divisible.*

Proof. The (G, ω) action $\alpha_g \otimes \mathrm{id}_{\mathcal{O}_\infty}$ on $A \otimes \mathcal{O}_\infty$ for $g \in G$ induces a connected anomalous action on a simple purely infinite C*-algebra (see Remark 2.7.12). The K_0 group of $A \otimes \mathcal{O}_\infty$ is isomorphic to that of A by the Künneth theorem for tensor products (see Theorem 2.6.11) and the induced module structure on $K_0(A \otimes \mathcal{O}_\infty)$ is trivial due to naturality of the Künneth exact sequence. Therefore, without loss of generality, we may assume that A is simple purely infinite. As every simple purely infinite C*-algebra has cancellation of non-zero projections (see Section 2.7.3) the result follows by Theorem 5.4.2. □

We will see in Chapter 6 that the torsion assumption in Proposition 5.4.2 can not be dropped (see for example the discussion succeeding Proposition 6.4.10).

Under the assumptions of Theorem 5.4.2, if one further has that A falls within a class of C*-algebras classified by K-theory and traces, then the existence of a (G, ω) action on A, that acts trivially on K-theory, will imply a UHF-stability condition.

Corollary 5.4.4. *Let G be a finite group. Let A be a classifiable C*-algebra with $K_1(A) = 0$ with no $|G|$-torsion in $K_0(A)$, or alternatively let A be a unital AF-algebra. If there exists a (G, ω) action on A inducing the trivial G module structure on $K_0(A)$, with r the order of $[\omega]$, then $A \otimes \mathbb{M}_{r^\infty} \cong A$.*

Proof. We first show the result in the case that A is a classifiable C*-algebra with $K_1(A) = 0$ and no $|G|$-torsion in $K_0(A)$. Note that $A \otimes \mathbb{M}_{r^\infty}$ is also classifiable. By Theorem 2.7.1 A is isomorphic to $A \otimes \mathbb{M}_{r^\infty}$ if and only if $KT_u(A) \cong KT_u(A \otimes \mathbb{M}_{r^\infty})$. Let (α, u) be a (G, ω) action on A with $[\omega]$ of order r. As $K_1(A) = 0$ it follows from Remark 4.2.5 that (α, u) is connected. Therefore, by Corollary 5.4.3 $K_0(A)$ is uniquely r-divisible. The result will follow if we show that for any unital C*-algebra A with $K_1(A) = 0$ and $K_0(A)$ uniquely r-divisible, then $KT_u(A) \cong KT_u(A \otimes \mathbb{M}_{r^\infty})$. This is well known to experts (see for example [3, Example 4.4]). However, as the author could not find a proof in the literature we provide a proof for completeness.

By the Künneth Theorem for tensor products (Theorem 2.6.11) $K_0(A \otimes \mathbb{M}_{r^\infty}) \cong K_0(A) \otimes \mathbb{Z}[1/r]$ and $K_1(A \otimes \mathbb{M}_{r^\infty}) \cong \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), \mathbb{Z}[1/r])$. The $\mathrm{Tor}_1^{\mathbb{Z}}$ term vanishes as $\mathbb{Z}[1/r]$ has no torsion and hence is a flat module. So $K_1(A \otimes \mathbb{M}_{r^\infty})$ vanishes. We show that the homomorphism of abelian groups

$$\psi : K_0(A) \to K_0(A) \otimes \mathbb{Z}[1/r]$$
$$a \to a \otimes 1$$

is a bijection. As $K_0(A)$ is uniquely r-divisible then for any $a \in K_0(A)$ there is a well defined element $r^{-1}a \in K_0(A)$ and $\varphi(r^{-1}a) = r^{-1}a \otimes 1 = a \otimes r^{-1}$ yielding surjectivity of ψ. To show injectivity of ψ consider the homomorphism of abelian groups $\varphi : K_0(A) \otimes \mathbb{Z}[1/r] \to K_0(A)$ defined on elementary tensors by $\varphi(a \otimes k/r^j) = kr^{-j}a$. It is clear that $\varphi \circ \psi = \mathrm{id}_{K_0(A)}$ and so ψ is injective. Therefore, $K_0(A) \cong K_0(A \otimes \mathbb{M}_{r^\infty})$. There is also an isomorphims of trace spaces $T(A) \cong T(A \otimes \mathbb{M}_{r^\infty})$ as $\mathbb{M}_{r^\infty}$ has a unique trace. The isomorphism is given by sending a trace $\tau \in T(A)$ to $\tau \otimes \tau_{r^\infty}$ with τ_{r^∞} the unique trace on $\mathbb{M}_{r^\infty}$. Under the identification of $K_0(A \otimes \mathbb{M}_{r^\infty})$ with $K_0(A)$ the pairing maps on A and $A \otimes \mathbb{M}_{r^\infty}$ also coincide.

We now turn to the case that A is a unital AF-algebra. As $A \otimes \mathbb{M}_{r^\infty}$ is also a unital AF-algebra, by Elliott's classification theorem ([40]) it suffices to show that $K_0(A)$ and $K_0(A \otimes \mathbb{M}_{r^\infty})$ are isomorphic as ordered groups. This is equivalent to showing that $KT_u(A) \cong KT_u(A \otimes \mathbb{M}_{r^\infty})$. Indeed, every AF-algebra has unperforated K_0 group so the order structure of $K_0(A)$ can be recovered from the traces. (See [119, Proposition 7.2.8] for a proof that the K_0 group of an AF-algebra is unperforated and [119, Section 5.2] for how the order structure is recovered from the traces in this case.) By Theorem 5.4.2 we have that $K_0(A)$ is uniquely r-divisible and hence $KT_u(A) \cong KT_u(A \otimes \mathbb{M}_{r^\infty})$ as shown in the previous part of the proof. □

We will see in Chapter 6 that for any finite group G and 3-cocycle $\omega \in Z^3(G, \mathbb{T})$ there exists a (G, ω) action on $\mathbb{M}_{|G|^\infty}$. This comes close to a characterisation of the existence of (G, ω) actions which act trivially on K_0 for those C*-algebras which satisfy the hypothesis of Corollary 5.4.4.

Chapter 6

Explicit constructions

Building ω-anomalous actions with non-trivial 3-cocycle ω is not an easy task. The mere non-triviality of the cohomology class of ω implies that, unlike in the case of group actions, there exist no trivial ω-anomalous actions on C*-algebras. In fact, it is clear from the obstructions layed out in Chapters 4 and 5 that whether or not a C*-algebra admits an anomalous action is an intricate question.

As a counterpart to the obstruction results, in this chapter we take on the task of building anomalous actions on C*-algebras. As discussed in Section 3.3.2 anomalous actions have been constructed on $\mathcal{R}$. In the setting of C*-algebras, anomalous actions have been constructed on continuous trace C*-algebras in [20] and on abelian C*-algebras in [74]. Actually, the work of [74] gives a systematic way of building anomalous actions on C*-crossed products. General constructions of actions of unitary tensor categories on C*-algebras can also be used to yield anomalous actions on C*-algebras ([104],[111],[64]).

In this chapter, we are primarily interested in producing anomalous actions on classifiable C*-algebras. We start by considering anomalous actions on UHF algebras; we provide two constructions. The first construction is based on the general framework of [74]. The second construction is established through an inductive limit of anomalous actions on finite dimensional C*-algebras. These two constructions will be shown to yield equivalent actions as a corollary of our classification results of Chapter 7. In the remaining part of the chapter, we restrict to discussing anomalous actions of cyclic groups. Section 6.4 introduces a particular method for constructing anomalous actions of cyclic groups on C*-algebras. This technique was introduced by the author and Evington in [46]. Even though it can now be inferred from the framework of [74] it was developed independently. Applying this method we construct anomalous actions of cyclic groups on irrational rotation algebras, Bunce–Deddens algebras, crossed products of strongly self absorbing C*-algebras by Bernoulli shifts and matrix amplifications of Cuntz algebras. The first construction of anomalous actions on UHF algebras (Theorem 6.2.3), the actions on irrational rotation algebras (Proposition 6.4.2) and on Bunce–Deddens algebras (Proposition 6.4.4) were contained in my joint work with Evington (see [46]).

Before we get going with our constructions we begin with a preliminary section in which we discuss some properties of interest that anomalous actions may satisfy.

6.1 Structural properties of anomalous actions

Throughout this chapter, we will not only be interested in examples of anomalous actions but also in structural properties that these examples may satisfy. Firstly, it is natural to consider the following generalisation of the Rokhlin property for finite group actions (Definition 3.4.4).

Definition 6.1.1. Let G be a finite group and (α, u) an anomalous action on a unital C*-algebra A. Then (α, u) is said to have the *Rokhlin property*, if there exist projections $p_g \in A_\infty \cap A'$ for $g \in G$ such that

$$\sum_{g \in G} p_g = 1_A, \tag{6.1.1}$$

$$\alpha_g(p_h) = p_{gh} \text{ for } g, h \in G. \tag{6.1.2}$$

Example 6.1.2. Let $n \in \mathbb{N}$ and γ be an n-th root of unity. The $\mathbb{Z}_n$ anomalous action on the UHF algebras $A = M_{n^\infty}$ induced by the automorphism s_n^γ (see Theorem 3.3.7 and Remark 3.3.8) has the Rokhlin property. To show this it suffices to construct a projection $p \in A_\infty \cap A'$ such that $\sum_{i=0}^{n-1} (s_n^\gamma)^i(p) = 1_A$. Indeed the projections $(s_n^\gamma)^i(p)$ for $0 \leq i \leq n-1$ would satisfy conditions (6.1.1) and (6.1.2) of Definition 6.1.1.

Denote by $r_i : \mathbb{M}_n \to \bigotimes_{i \in \mathbb{N}} \mathbb{M}_n$ for $i \in \mathbb{N}$ the unital embedding of $\mathbb{M}_n$ into the i-th tensor product factor. Our candidate projection is $p = (p_k) = (r_k(e_{1,1}))$. Firstly, p belongs to $A_\infty \cap A'$ as any finite set of $M_{n^\infty} = \bigotimes_{i \in \mathbb{N}} \mathbb{M}_n$ is approximately contained in finitely many tensor factors. Moreover, it is a straightforward computation to show that $s_n^\gamma(r_k(e_{i,i})) = r_k(e_{i+1,i+1})$ for any $1 \leq i \leq n$. (Where the addition in the subscripts is understood mod n.) Therefore, the k-th entry of a representative sequence for $\sum_{i=0}^{n-1} (s_n^\gamma)^i(p)$ is given by $\sum_{i=0}^{n-1} (s_n^\gamma)^i(r_k(e_{1,1})) = \sum_{i=1}^{n} r_k(e_{i,i}) = 1_A$ so $\sum_{i=0}^{n-1} (s_n^\gamma)^i(p) = 1_A$.

It also makes sense to define the Rokhlin property for G-kernels through considering an arbitrary lift. This property is independent of the lift chosen. Indeed, if there exist projections $p_g \in A_\infty \cap A'$ satisfying conditions (6.1.1) and (6.1.2) for a family of automorphisms α_g, the conditions also hold by replacing α_g with $\mathrm{Ad}(v_g)\alpha_g$ for any family of unitaries v_g for $g \in G$. This follows as the projections p_g are contained in $A_\infty \cap A'$ and hence invariant under inner automorphisms of A.

Definition 6.1.3. Let G be a finite group and A a unital C*-algebra. A G-kernel $\overline{\theta}$ of A has the *Rokhlin property* if for some/any lift (θ, u) there exist projections $p_g \in A_\infty \cap A'$ for $g \in G$ such that

$$\sum_{g \in G} p_g = 1_A, \tag{6.1.3}$$

$$\theta_g(p_h) = p_{gh} \text{ for } g, h \in G. \tag{6.1.4}$$

Remark 6.1.4. The Rokhlin property for a finite group action α on unital C*-algebra A can be reformulated as the existence of an equivariant *-homomorphism $\Phi : (C(G), \lambda_g) \to (A_\infty \cap A', \alpha_g)$. If δ_g are the point masses of $C(G)$ their image $\Phi(\delta_g)$ yield Rokhlin projections. Similarly, one can define an equivariant *-homomorphism from a set of Rokhlin projections by sending δ_g to p_g. This equivalence of formulations also trivially holds in the generality of anomalous actions.

To introduce the next property, we start by discussing inductive limits of G-C*-algebras and of unital (G, ω)-C*-algebras. Let G be a group, we denote the category whose objects are pairs (A, α) with A a C*-algebra and $\alpha : G \to \mathrm{Aut}(A)$ a group action on A and whose morphisms are equivariant *-homomorphisms by G-**C*alg**. Suppose $(A_n, \alpha_n, \varphi_n)$ is an inductive sequence in G-**C*alg**, then (A_n, φ_n) is an inductive sequence in **C*alg** with limit $(A, \varphi_{n,\infty})$ (see Section 2.3). The map $\alpha_g((a_n)) = (\alpha_n(g)(a_n))$ is a well defined bounded *-automorphism on the dense *-subalgebra $\mathcal{A} \subset A$ defined in (2.3.3). Therefore, the continuous extension of α_g to all of A defines an action on A. The maps $\varphi_{n,\infty}$ are α_n to α equivariant and it is a routine check that $(A, \alpha, \varphi_{n,\infty})$ defines the inductive limit of $(A_n, \alpha_n, \varphi_n)$ in G-**C*alg**.

Remark 6.1.5. In [133] Szabó considers a more general notion of morphism that he calls an *cocycle morphism*. In his work, Szabó considers the category whose objects are G-C*-algebras and whose morphisms are cocycle morphisms and shows that this category admits inductive limits. This generalises the discussion proceeding this remark.

Similarly, inductive limits exist in the categories of (G,ω)-C*-algebras. Let $\omega \in Z^3(G,\mathbb{T})$, we let (G,ω)-$\mathbf{C^*alg}_1$ be the category whose objects are triples (A,α,u) with A a unital C*-algebra and (α,u) a (G,ω)-action on A (we call this a (G,ω)-C*-algebra) and $\mathrm{Hom}((A,\alpha,u)),(B,\beta,v)))$ is given by α to β equivariant unital *-homomorphisms $\varphi : A \to B$ such that $\varphi(u_{g,h}) = v_{g,h}$ for $g,h \in G$. Let $(A_n,\alpha_n,u_n,\varphi_n)$ be an inductive sequence in (G,ω)-$\mathbf{C^*alg}_1$. Let $(A,\varphi_{n,\infty})$ be the inductive limit of (A_n,φ_n) in $\mathbf{C^*alg}$. Once more $\alpha_g((a_n)) = (\alpha_n(g)(a_n))$ for $(a_n) \in \mathcal{A}$ extend to automorphisms of A for any $g \in G$. Setting $u_{g,h} = \varphi_{1,\infty}(u_1(g,h))$ the pair (α,u) defines a (G,ω) action on A. The data $(A,\alpha,u,\varphi_{n,\infty})$ can be seen to be the inductive limit of $(A_n,\alpha_n,u_n,\varphi_n)$. This brings us to the following definition.

Definition 6.1.6. Let A be a unital AF-C*-algebra and (α,u) be a (G,ω)-action on A. We say α is a *strict AF-action* if there exists an inductive limit (A_n,φ_n) consisting of finite dimensional C*-algebras A_n and (G,ω) actions (α_n,u_n) on A_n such that

1. $\varphi_n\alpha_n = \alpha_{n+1}\varphi_n, \ \forall n \in \mathbb{N}$,
2. $\varphi_n(u_n) = u_{n+1}, \forall n \in \mathbb{N}$,
3. $(A,\alpha,u) = \varinjlim(A_n,\varphi_n,\alpha_n,u_n)$.

Remark 6.1.7. If G is finite, a (G,ω) strict AF-action is an instance of an *AF-action* of the fusion category $\mathrm{Hilb}(G,\omega)$ as introduced in [25, Definition 4.8]. (See Section 9.2 to see how (G,ω) actions fit within the framework of [25].) We now translate the notion of an AF-action of [25] to the setting of anomalous actions by applying the results of Chapter 9. In [25] the authors construct inductive limits in the category whose objects are unital (G,ω)-C*-algebras and whose morphisms are given by pairs $(\varphi,\mathfrak{v}) : (\alpha,u) \to (\beta,v)$ with $\varphi : A \to B$ a unital *-homomorphism and $\mathfrak{v} : G \to U(A)$

unitaries such that $\mathrm{Ad}(\mathfrak{v}_g)\varphi\alpha_g = \beta_g\varphi$ and $\varphi(u_{g,h}) = \mathfrak{v}_{gh}^* v_{g,h}\alpha_g(\mathfrak{v}_h)\mathfrak{v}_g$ for $g, h \in G$. Therefore, an anomalous *AF-action* is one which is cocycle conjugate to an inductive limit, in this generalised category of (G, ω)-actions, of anomalous actions on finite dimensional C*-algebras.

Although AF-actions are more natural from the categorical standpoint, we will not require them in this thesis and so we restrict ourselves to strict AF-actions. The difference is that we require morphisms in our category to be on the nose equivariant rather than equivariant up to isomorphism, which in the categorical picture is witnessed by the $\mathfrak{v}$. Moreover, although strict AF-actions are preserved under conjugacy, they may not be preserved under unitary perturbations. Allowing for perturbations up to cocycle conjugacy in the definition of AF-actions is motivated categorically as cocycle conjugacy coincides with natural isomorphism of functors.

6.2 Anomalous actions on UHF algebras: construction 1

We begin this section with a review of Vaughan Jones' construction of anomalous actions on $\mathcal{R}$ using twisted crossed products and its recent adaptation to the C*-setting by Corey Jones in [74]. We will then use this construction to build anomalous actions on UHF algebras.

We assume that the reader is familiar with the construction of the crossed product(s) of a C*-algebra A by a discrete group G with respect to an action $\alpha : G \to \mathrm{Aut}(A)$. (A good treatment of the subject can be found in [15].)

The crossed product construction can be generalised by twisting the multiplication in G by a 2-cocyle $c \in Z^2(G, \mathbb{T})$, i.e. the canonical unitaries $\{v_g : g \in G\}$ in the multiplier algebra of a twisted crossed product satisfy

$$v_g v_h = c(g,h) v_{gh} \quad \text{for } g,h \in G, \tag{6.2.1}$$

$$v_g a v_g^* = \alpha_g(a) \quad \text{for } g \in G,\ a \in A. \tag{6.2.2}$$

We write $A \rtimes_{\alpha,c}^{\mathrm{alg}} G$ for the *algebraic* twisted crossed product, whose elements can be viewed as (finite) formal sums $\sum_{g\in G} a_g v_g$ with $a_g \in A$. As in the non-twisted case, there are two natural choices of completions: the reduced twisted crossed product $A \rtimes_{\alpha,c}^{\mathrm{r}} G$ and the maximal twisted crossed product $A \rtimes_{\alpha,c}^{\max} G$; see [17] or [107] for full details. Note that when c is trivial, we recover the usual crossed products.

We can now state an existence theorem for anomalous actions, due to Corey Jones ([74]), which in turn is based on Vaughan Jones' construction in the von Neumann setting ([77]). Notice that some formulas in this exposition appear different to those contained in [74]. This is a consequence of our choice of conventions for anomalous actions.

Theorem 6.2.1. *Suppose we have the following data:*

- *a group G and $[\omega] \in H^3(G,\mathbb{T})$ with a normalised representative $\omega \in Z^3(G,\mathbb{T})$;*
- *a group Γ and a surjective homomorphism $\rho : \Gamma \twoheadrightarrow G$ with kernel K;*
- *a normalised cochain $c \in C^2(\Gamma,\mathbb{T})$ such that $dc = \rho^*(\omega)$;*
- *a* C*-*algebra B and an action $\pi : \Gamma \to \mathrm{Aut}(B)$.*

Then there exists an ω-anomalous action of G on the reduced twisted crossed product $B \rtimes_{\pi,\bar{c}}^{\mathrm{r}} K$, where $\bar{c}_{g,h} = \overline{c_{g,h}}$ for $g,h \in K$ is the 2-cocycle given by restricting the pointwise conjugate of $c \in C^2(\Gamma,\mathbb{T})$ to K.[1]

[1]We use G to denote the acting group. In [74] the acting group is denoted by Q instead.

A detailed proof of Theorem 6.2.1 can be found in [74, Theorem 3.1]. We provide a brief outline on how the anomalous action is constructed.

The anomalous action is in fact defined on the algebraic twisted crossed product and shown to extend to the reduced twisted crossed product.[2] The automorphisms $\theta_g \in \mathrm{Aut}(B \rtimes^{\mathrm{r}}_{\pi,\bar{c}} K)$ are given by

$$\theta_g\left(\sum_{k\in K} a_k v_k\right) = \sum_{k\in K} c_{\hat{g}k\hat{g}^{-1},\hat{g}}\overline{c_{\hat{g},k}}\pi_{\hat{g}}(a_k)v_{\hat{g}k\hat{g}^{-1}}, \tag{6.2.3}$$

where $g \mapsto \hat{g}$ is a choice of set theoretic section of $\rho : \Gamma \twoheadrightarrow G$. The unitaries $u_{g,h} \in U(M(B \rtimes^{\mathrm{r}}_{\pi,\bar{c}} K))$ are given by

$$u_{g,h} = \overline{c_{\hat{g},\hat{h}}}c_{\mu(g,h),\widehat{gh}}v_{\mu(g,h)}, \tag{6.2.4}$$

where $\mu : G \times G \to K$ is defined by $\hat{g}\hat{r} = \mu(g,r)\widehat{gr}$.

In order to access Theorem 6.2.1, we will also need the following lemma of a cohomological nature. This lemma also goes back to Vaughan Jones ([77]) with the additional observations in the case when G is finite due to Corey Jones ([74]).

Lemma 6.2.2. *Let G be a group and $[\omega] \in H^3(G,\mathbb{T})$ with a normalised representative $\omega \in Z^3(G,\mathbb{T})$. There exist*

- *a group Γ,*
- *a surjective homomorphism $\rho : \Gamma \twoheadrightarrow G$ with abelian kernel K,*
- *a normalised 2-cochain $c \in C^2(\Gamma,\mathbb{T})$ such that $dc = \rho^*(\omega)$.*

Moreover, when G is finite, then K can be chosen to be a finite abelian group whose order is a power of $|G|$, and c can be chosen such that $c|_K = 1$.

[2] The action also extends by universality to the full twisted crossed product.

Proof. See [77, Lemma 2.3] for the general case and [74, Lemma 3.7] for the technical improvements when G is finite. We observe that, in the proof of [74, Lemma 3.7], K is defined to be the quotient of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \mathbb{Z}_{|G|})$ by a copy of $\mathbb{Z}_{|G|}$. Hence, the order of K is $|G|^{|G|-1}$. $\square$

Theorem 6.2.3. *Let G be a finite group and let $\omega \in Z^3(G, \mathbb{T})$ be any 3-cocycle. There exists a (G, ω) action on the UHF algebra $\mathbb{M}_{|G|^\infty}$. We denote this action by s_G^ω.*

Proof. As any 3-cocycle is cohomologous to a normalised 3-cocycle we may assume that ω is normalised (see Remark 3.3.3). Let Γ, K, $\rho : \Gamma \twoheadrightarrow G$, $[\omega]$ and $c \in C^2(\Gamma, \mathbb{T})$ be as in the conclusion of Lemma 6.2.2 where $|K|$ is a power of $|G|$ and $c|_K = 1$. Note that $|\Gamma| = |K||G|$ is finite. Let $\lambda_\Gamma : \Gamma \to U(\mathcal{B}(\ell^2(\Gamma)))$ denote the left regular representation of Γ. Write $\mathrm{Ad}(\lambda_\Gamma)$ for the induced action on $\mathcal{B}(\ell^2(\Gamma))$ given by $\mathrm{Ad}(\lambda_\Gamma)_\gamma(T) = \lambda_\Gamma(\gamma)T\lambda_\Gamma(\gamma)^*$ for all $T \in \mathcal{B}(\ell^2(\Gamma))$ and $\gamma \in \Gamma$. Let $B = \bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(\Gamma))$ and let $\pi = \mathrm{Ad}(\lambda_\Gamma)^{\otimes\infty}$.

Applying Theorem 6.2.1, we obtain an ω-anomalous action of G on $B \rtimes^{\mathrm{r}}_{\pi,\bar{c}} K$. The remainder of this proof consists of demonstrating that this twisted crossed product is in fact isomorphic to the UHF algebra with supernatural number $|G|^\infty$.

First, we observe that, since $c|_K = 1$, the twisted crossed product $B \rtimes^{\mathrm{r}}_{\pi,\bar{c}} K$ is in fact not twisted in this case. Moreover, as K is finite, there is no distinction to be made between the algebraic, the reduced and the full crossed products. Therefore, we shall simplify our notation and write $B \rtimes_\pi K$ instead of $B \rtimes^{\mathrm{r}}_{\pi,c} K$.

Next, we consider the restriction of λ_Γ to K. By decomposing Γ into right K-cosets, we see that $\lambda_\Gamma|_K$ is equivalent to $|\Gamma/K| = |G|$ copies of the left regular representation $\lambda_K : K \to \mathcal{B}(\ell^2(K))$. Hence, $\lambda_\Gamma|_K$ is equivalent to $\lambda_K \otimes 1_{\ell^2(G)}$ where $1_{\ell^2(G)}$ denotes the trivial representation of K on the Hilbert space $\ell^2(G)$.

It follows that we have an equivariant isomorphism of C*-algebras

$$\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(\Gamma)) \cong \bigotimes_{j\in\mathbb{N}} (\mathcal{B}(\ell^2(K)) \otimes \mathcal{B}(\ell^2(G)), \tag{6.2.5}$$

where K acts by π on the left hand side and by $\sigma := \mathrm{Ad}(\lambda_K \otimes 1_{\ell^2(G)})^{\otimes\infty}$ on the right hand side.

Taking crossed products, we have

$$\begin{aligned} B \rtimes_\pi K &\cong \left(\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(K)) \otimes \mathcal{B}(\ell^2(G)\right) \rtimes_\sigma K \\ &\cong \left(\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(K)) \rtimes_{\mathrm{Ad}(\lambda_K)^{\otimes\infty}} K\right) \otimes \mathcal{B}(\ell^2(G))^{\otimes\infty}. \end{aligned} \tag{6.2.6}$$

Since $\mathcal{B}(\ell^2(G))^{\otimes\infty}$ is a UHF algebra with supernatural number $|G|^\infty$, it suffices show that the crossed product $\bigotimes_{k\in\mathbb{N}} \mathcal{B}(\ell^2(K)) \rtimes_{\mathrm{Ad}(\lambda_K)^{\otimes\infty}} K$ is isomorphic to a UHF algebra with supernatural number $|K|^\infty$. This follows from a combination of deep results in the literature as explained in the proof of [46, Theorem C]. However, we will compute this directly for the purpose of this exposition.

Since K is a finite abelian group it is isomorphic to a product of cyclic groups $K = K_1 \times \cdots \times K_r$. As the left regular regular representation λ_K is equivalent to the tensor product $\lambda_{K_1} \otimes \cdots \otimes \lambda_{K_r}$ of the respective left regular representations

$$\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(K)) \rtimes_{\mathrm{Ad}(\lambda_K)^{\otimes\infty}} K \cong \bigotimes_{i=1}^{r} \bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(K_i)) \rtimes_{\mathrm{Ad}(\lambda_{K_i})^{\otimes\infty}} K_i. \tag{6.2.7}$$

Therefore, it suffices to show that each $\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(K_i)) \rtimes_{\mathrm{Ad}(\lambda_{K_i})^{\otimes\infty}} K_i$ is isomorphic to a UHF algebra with supernatural number $|K_i|^\infty$.

The final step required for this proof is to explicitly compute the crossed product $\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(\mathbb{Z}_m)) \rtimes \mathbb{Z}_m$, where the action is $\mathrm{Ad}(\lambda_{\mathbb{Z}_m})^{\otimes\infty}$. As this action leaves the finite-dimensional subalgebras $A_n = \bigotimes_{j=1}^n \mathcal{B}(\ell^2(\mathbb{Z}_m))$ invariant, we have

$$\bigotimes_{j\in\mathbb{N}} \mathcal{B}(\ell^2(\mathbb{Z}_m)) \rtimes \mathbb{Z}_m = \lim_{n\to\infty} A_n \rtimes_{\mathrm{Ad}(\lambda_{\mathbb{Z}_m})^{\otimes n}} \mathbb{Z}_m. \tag{6.2.8}$$

Each crossed product in the inductive limit on the right hand side is a finite-dimensional C*-algebra, we compute the Brattelli diagram of this inductive limit.

We let $u \in \mathcal{B}(\ell^2(\mathbb{Z}_m))$ be the image of the left regular representation at the generator $1+m\mathbb{Z}$ of $\mathbb{Z}_m$, and write v for the canonical unitary in the crossed products $A_n \rtimes \mathbb{Z}_m$ coming from $1+m\mathbb{Z}$. (Note that v is common to all stages of the inductive limit.) We have

$$vav^* = (u^{\otimes n})a(u^{\otimes n})^* \tag{6.2.9}$$

for all $a \in A_n$. In particular, v and $u^{\otimes n}$ commute.

Any element $x \in A_n \rtimes \mathbb{Z}_m$ may be written as a sum $x = \sum_{i=0}^{m-1} a_i v^i$ with $a_i \in A_n$. Let $a \in A_n$. We compute that

$$\begin{aligned} ax - xa &= \sum_{i=0}^{m-1} aa_i v^i - a_i v^i a \\ &= \sum_{i=0}^{m-1} [aa_i - a_i(u^i)^{\otimes n} a (u^{-i})^{\otimes n}]v^i. \end{aligned} \tag{6.2.10}$$

Thus, x commutes with a if and only if $a_i(u^i)^{\otimes n}$ commutes with a for $i = 0, \dots m-1$. Since $Z(A_n) \cong \mathbb{C}$, we see that the only possible central elements of $A_n \rtimes \mathbb{Z}_m$ are $\sum_{i=0}^{m-1} c_i (u^{-i})^{\otimes n} v^i$ where $c_i \in \mathbb{C}$. Since v and $u^{\otimes n}$ commute these elements are indeed central.

Therefore, $Z(A_n \rtimes \mathbb{Z}_m)$ has dimension m and is isomorphic to the group algebra of $\mathbb{Z}_m$ with the element $(u^{-1})^{\otimes n} v$ representing the object $1+m\mathbb{Z}$ in the group algebra. In particular, Fourier theory establishes that the minimal central projections of $A_n \rtimes \mathbb{Z}_m$

are

$$p_\xi^{(n)} = \frac{1}{m}\sum_{i=0}^{m-1} \xi^i (u^{-i})^{\otimes n} v^i, \tag{6.2.11}$$

where ξ ranges over the set of m-th roots of unity.

We now consider the dual action of $\widehat{\mathbb{Z}_m}$ on $A_n \rtimes \mathbb{Z}_m$. By definition, $\alpha \in \widehat{\mathbb{Z}_m} = \mathrm{Hom}(\mathbb{Z}_m, \mathbb{T})$ fixes elements in A_n and maps v to $\alpha(1+m\mathbb{Z})v$. Hence, α maps $p_\xi^{(n)}$ to $p_{\alpha(1+m\mathbb{Z})\xi}^{(n)}$. Therefore, $\widehat{\mathbb{Z}_m}$ acts transitivity on the minimal central projections of $A_n \rtimes \mathbb{Z}_m$. Consequently, the matrix algebras $p_\xi^{(n)}(A_n \rtimes \mathbb{Z}_k)p_\xi^{(n)}$ are mutually isomorphic. Since $\dim(A_n \rtimes \mathbb{Z}_m) = m \dim A_n$, we conclude that $A_n \rtimes \mathbb{Z}_m$ is isomorphic to $\bigoplus_{j=1}^m \mathbb{M}_{m^n}$.

To compute the Bratelli diagram, we need to consider the inclusion of $A_n \rtimes \mathbb{Z}_m$ into $A_{n+1} \rtimes \mathbb{Z}_m$. Let ξ_1 and ξ_2 be any m-th roots of unity. We compute that

$$p_{\xi_1}^{(n)} p_{\xi_2}^{(n+1)} = \frac{1}{m^2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \xi_1^i \xi_2^j (u^{-i})^{\otimes n} (u^{-j})^{\otimes n+1} v^{i+j}. \tag{6.2.12}$$

Applying the canonical conditional expectation $E_{A_{n+1}} : A_{n+1} \rtimes \mathbb{Z}_m \to A_{n+1}$, we get

$$\begin{aligned} E_{A_n+1}(p_{\xi_1}^{(n)} p_{\xi_2}^{(n+1)}) &= \frac{1}{m^2}\sum_{i=0}^{m-1} \xi_1^i \xi_2^{-i} (u^{-i})^{\otimes n} (u^i)^{\otimes n+1} \\ &= \frac{1}{m^2}\sum_{i=0}^{m-1} \xi_1^i \xi_2^{-i} (1^{\otimes n} \otimes u^i) \\ &\neq 0, \end{aligned} \tag{6.2.13}$$

since $1, u, \ldots, u^{m-1}$ are linearly independent in $\mathcal{B}(\ell^2(\mathbb{Z}_m))$. Therefore, $p_{\xi_1}^{(n)} p_{\xi_2}^{(n+1)} \neq 0$.

It follows that the Bratelli diagram for the inclusion $A_n \rtimes \mathbb{Z}_m$ in $A_{n+1} \rtimes \mathbb{Z}_m$ is a complete bipartite graph $K_{m,m}$. The multiplicity of each edge must be one by dimension counting. Therefore, the Bratelli diagram for the inductive limit is of the form

$$\mathbb{M}_m^{\oplus m} \xrightarrow{K_{m,m}} \mathbb{M}_{m^2}^{\oplus m} \xrightarrow{K_{m,m}} \mathbb{M}_{m^3}^{\oplus m} \xrightarrow{K_{m,m}} \mathbb{M}_{m^4}^{\oplus m} \xrightarrow{K_{m,m}} \cdots . \tag{6.2.14}$$

The AF algebra with this Bratelli diagram at each stage is known to be the UHF algebra with supernatural number m^∞. (See for example [33, Example III.2.4], where the $m = 2$ case is analysed in detail.) □

Proposition 6.2.4. *For any finite group G and $\omega \in Z^3(G, \mathbb{T})$ the action s_G^ω has the Rokhlin property.*

Proof. Let $B = \mathbb{M}_{|G|^\infty}$. We adopt the same notation as in the proof of Theorem 6.2.3. Moreover, denote by $r_i : \mathcal{B}(l^2(\Gamma)) \to B$ the unital embedding into the i-th tensor factor. Let e_K in $\mathcal{B}(l^2(\Gamma))$ be the projection onto $l^2(K)$, that is

$$e_K(\sum_{\gamma\in\Gamma} \mu_\gamma \gamma) = \sum_{\gamma\in K} \mu_\gamma \gamma$$

for any complex scalars μ_γ. Let $p_n = r_n(e_K)$ for $n \in \mathbb{N}$. Note that the projection $p = (p_n) \in B_\infty$ commutes with any constant sequence of elements in B. This follows as for any finite set F of B there exists a large enough $m \in \mathbb{N}$ such that F is approximately contained in the subalgebra $\bigotimes_{i=0}^m \mathcal{B}(l^2(\Gamma)) \subset B$. Moreover, considering p as a projection in $(B \rtimes K)_\infty$ it will also commute with the subalgebra $C^*(K) \subset (B \rtimes K)_\infty$. Indeed, e_K is invariant under $\mathrm{Ad}(\lambda_\Gamma)_k$ for any $k \in K$ and therefore for any $n \in \mathbb{N}$ and $k \in K$

$$\begin{aligned} v_k p_n v_k^* &= \mathrm{Ad}(\lambda_\Gamma)_k^{\otimes\infty}(r_n(e_K)) \\ &= r_n(\mathrm{Ad}(\lambda_\Gamma)_k e_K)) \\ &= r_n(e_K) \\ &= p_n \end{aligned}$$

Therefore, $p \in (B \rtimes K)_\infty \cap (B \rtimes K)'$.

We claim that the projections $p_g := s^\omega_G(g)(p) = (s^\omega_G(g)(p_n))_{n\in\mathbb{N}}$ form a set of Rokhlin projections. We start by showing that the sum $\sum_{g\in G} s^\omega_G(g)(p) = 1$. Let $n \in \mathbb{N}$ and $g \in G$, then as the cocycle c is normalised it follows from (6.2.3) that

$$\begin{aligned} s^\omega_G(g)(p_n) &= \pi_{\hat{g}}(p_n) \\ &= \mathrm{Ad}(\lambda_\Gamma)^{\otimes\infty}_{\hat{g}}(p_n) \\ &= \mathrm{Ad}(\lambda_\Gamma)^{\otimes\infty}_{\hat{g}}(r_n(e_K)) \\ &= r_n(\mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)). \end{aligned} \tag{6.2.15}$$

The maps r_n are unital so it suffices to show that $\sum_{g\in G} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K) = 1_{\mathcal{B}(l^2(\Gamma))}$. To see this, let $\gamma \in \Gamma$, $g \in G$ and $\delta_\gamma \in l^2(\Gamma)$ the point mass at γ then

$$\begin{aligned} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)(\delta_\gamma) &= \lambda_\Gamma(\hat{g})e_K\lambda_\Gamma(\hat{g}^{-1})(\delta_\gamma) \\ &= \lambda_\Gamma(\hat{g})e_K(\delta_{\hat{g}^{-1}\gamma}) \\ &= \begin{cases} \delta_\gamma \text{ if } \gamma \in \hat{g}K, \\ 0 \text{ otherwise.} \end{cases} \end{aligned} \tag{6.2.16}$$

The left K cosets are pairwise disjoint and cover the whole group Γ. Therefore, it follows that $\sum_{g\in G} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)(\delta_\gamma) = \delta_\gamma$ for every $\gamma \in \Gamma$. As the operators $\sum_{g\in G} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)$ and $\mathrm{id}_{\mathcal{B}(l^2(\Gamma))}$ coincide on a spanning set of $l^2(\Gamma)$, these operators are equal.

It remains to show that if for any $h, g \in G$ the projections $s^\omega_G(h)p_g = p_{hg}$. This is immediate since $s^\omega_G(h)p_g = s^\omega_G(h)s^\omega_G(g)p = \mathrm{Ad}(u_{h,g})s^\omega_G(hg)p = \mathrm{Ad}(u_{h,g})p_{hg} = p_{hg}$ where the last equality in the chain holds as p_{hg} commutes with $B \rtimes K$. □

As a corollary of Theorem 6.2.3 one can build (G, ω) actions for G a finite group on $\mathcal{O}_2$, $\mathcal{O}_{|G|}$ or any UHF algebra whose associated supernatural number is divisible by $|G|^\infty$ by tensoring on the action of Theorem 6.2.3. (See Section 3.3 for a discussion on the tensor product of anomalous actions.)

Similarly, one can also construct a (G,ω) action on the Jacelon–Razak C*-algebra $\mathcal{W}$ ([70]) as it absorbs $M_{|G|^\infty}$ for any finite group G. If G is an infinite, countable group and $\omega \in Z^3(G,\mathbb{T})$ one can also use Corey Jones' theory to construct (G,ω) actions on both $\mathcal{O}_2$ and $\mathcal{W}$.

Corollary 6.2.5. *Let G be a countable discrete group and $\omega \in Z^3(G,\mathbb{T})$ then there exists a (G,ω) action on $\mathcal{O}_2$.*

Proof. By Kirchberg's $\mathcal{O}_2$ absorption theorem (see [85]), it suffices to construct a (G,ω) action on a unital, simple, separable, nuclear C*-algebra B.

Let A be a simple, separable, unital C*-algebra with an outer action α of G on A. For instance, let A be a tensor product of some infinite dimensional, simple, separable, unital C*-algebra indexed over G and α the Bernoulli shift action. By Lemma 6.2.2 there exists a short exact sequence of groups $K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G$ such that $\pi^*(\omega) = dc$ for some 2-cochain $c \in C^2(G,\mathbb{T})$ and K is abelian. By Theorem 6.2.1 there exists a (G,ω) action on $B = A \rtimes_{\alpha,\overline{c}} K$. The C*-algebra B is unital and separable, it is simple by [87] or [18] and nuclear as K is amenable. $\square$

Before we proceed to consider the case of $\mathcal{W}$, we need to introduce some terminology. Let A be a C*-algebra with a unique trace τ and π_τ its GNS representation, we say an automorphism $\varphi \in \mathrm{Aut}(A)$ is *strongly outer* if the extension of φ to an automorphism of $\pi_\tau(A)''$ is an outer automorphism. Similarly, for a group G, we say an anomalous/group G-action α is strongly outer if α_g is strongly outer for all $g \in G$.

Corollary 6.2.6. *Let G be a countable discrete group and $\omega \in Z^3(G,\mathbb{T})$ then there exists a (G,ω) action on $\mathcal{W}$.*

Proof. By the results of [22] and [55] (see [22, Corollary D]), if A is a simple, separable, nuclear, monotracial C*-algebra then $A \otimes \mathcal{W} \cong \mathcal{W}$. Therefore, it suffices to construct a (G,ω) action on some simple, separable, nuclear, monotracial C*-algebra.

Let $A = \bigotimes_{g\in G} \mathcal{Z}$ and α the Bernoulli shift action of G on A. By Lemma 6.2.2 there exists a short exact sequence of groups $K \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G$ such that $\pi^*(\omega) = dc$ for some 2-cochain $c \in C^2(G, \mathbb{T})$ and K is abelian. By Theorem 6.2.1 there exists a (G, ω) action on $B = A \rtimes_{\alpha,\bar{c}} K$. The C*-algebra B is separable, simple and nuclear. That B is monotracial follows from [137, Theorem 5.4] as α is strongly outer.[3] (Note that [137, Theorem 5.4] is only stated for genuine crossed products, the result will hold for twisted crossed products by circle valued 2-cocycles σ by following the same argument.) □

In some cases, our construction and obstruction results so far, allow us to characterise the existence of anomalous actions on certain classes of C*-algebras. For example, a consequence of Theorem 4.2.10 and Connes' construction ([28]) we can characterise the existence of anomalous actions of finite cyclic groups on UHF algebras.

Corollary 6.2.7. *Let G be a finite cyclic group, $\omega \in Z^3(G, \mathbb{T})$ and r be the order of ω. Then there exists a (G, ω) action on a UHF algebra $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ if and only if $r^\infty | \prod_{k\in\mathbb{N}} n_k$.*

Proof. The only if direction follows from Theorem 4.2.10. We turn to the if direction. Let $G = \mathbb{Z}_m$ and ξ an m-th root of unity of order r. We take the viewpoint of Lemma 3.3.6, it suffices to construct an automorphism α of $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ and a unitary v such that $\alpha^m = \mathrm{Ad}(v)$ and $\alpha(v) = \xi v$. First suppose that m and r have the same prime factors. By Theorem 3.3.7 (see also Remark 3.3.8), there is an automorphism α of $\mathbb{M}_{m^\infty}$ and a unitary v such that $\alpha^m = \mathrm{Ad}(v)$ and $\alpha(v) = \xi v$. As $\mathbb{M}_{m^\infty} \cong \mathbb{M}_{r^\infty}$ we can induce a $(\mathbb{Z}_m, \omega)$-action on $\mathbb{M}_{r^\infty}$ for any $[\omega]$ of order r. As $r^\infty | \prod_{k\in\mathbb{N}} n_k$ one may induce an action on $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k} \cong \bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k} \otimes \mathbb{M}_{r^\infty}$ by acting on the tensor copy of $\mathbb{M}_{r^\infty}$.

[3] That the Bernoulli shift on $\mathcal{Z}$ is strongly outer is folklore, see e.g. [49, Proposition 4.2]. I could not find a proof in the literature. However, it follows as in the proof of Proposition 9.3.6 as the GNS closure of the Bernoulli shift on $\mathcal{Z}$ is the Bernoulli shift on $\mathcal{R}$.

For arbitrary m decompose $m = st$ for $s, t \in \mathbb{N}$ with $t^\infty = r^\infty$ and $\gcd(s, r) = 1$. By Theorem 3.3.7, there is an automorphism $\theta \in \operatorname{Aut}(\mathbb{M}_{r^\infty})$ and u in $U(\mathbb{M}_{r^\infty})$ such that $\theta^t = \operatorname{Ad}(u)$ and $\theta(u) = \xi u$. Then θ will satisfy $\theta^m = (\theta^t)^s = \operatorname{Ad}(u^s)$ and $\theta(u^s) = \xi^s u$. The function $\xi \mapsto \xi^s$ is a bijection on the set of m-th roots of unity of order r (as $\gcd(r, s) = 1$). Therefore, there exist $(\mathbb{Z}_m, \omega)$ actions on $\mathbb{M}_{m^\infty} \cong \mathbb{M}_{r^\infty}$ for any $m \in \mathbb{N}$ and $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$ of order r. The result follows as previously. $\square$

One may prove the if direction of Corollary 6.2.7 more directly. Indeed, using the same notation as in the hypothesis of Corollary 6.2.7, after close analysis of the representative cocycles of cyclic groups in (3.3.3), there exists a surjection $\rho : G \to \mathbb{Z}_r$ such that $\omega = \rho^*(\eta)$ for some $\eta \in Z^3(\mathbb{Z}_r, \mathbb{T})$ of order $r \in H^3(\mathbb{Z}_r, \mathbb{T})$. Also, there exists a $(\mathbb{Z}_r, \eta)$-action on $\mathbb{M}_{r^\infty}$ and hence on $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ as $r^\infty | \prod_{k\in\mathbb{N}} n_k$. This $(\mathbb{Z}_r, \eta)$ action induces a (G, ω) action on $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ through ρ (as explained in Section 3.3).

In the case that the order of $[\omega]$ is the size of the group $|G|$ we may use the obstruction results of Chapter 5 along with the existence of (G, ω) actions on UHF algebras to characterise the existence of (G, ω) actions on Cuntz algebras.

Corollary 6.2.8. *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$ with $[\omega]$ of order $|G|$. There exists a (G, ω) action on $\mathcal{O}_n$ for $n \geq 2$ if and only if* $\gcd(|G|, n - 1) = 1$.

Proof. Suppose there is a (G, ω) action on $\mathcal{O}_n$. By Theorem 5.3.1 it follows that $[\omega] = (n-1)[\eta]$ for some $[\eta] \in H^3(G, \mathbb{T})$. Any element of $H^3(G, \mathbb{T})$ is annihilated by $|G|$ (see [11, III. Corollary 10.2]) so $\big(|G|/\gcd(|G|, n-1)\big)[\omega] = \big(|G|(n-1)/\gcd(|G|, n-1)\big)[\eta] = 1$. Therefore $|G| = |G|/\gcd(|G|, n-1)$ and so $\gcd(|G|, n-1) = 1$.

If $\gcd(|G|, n-1) = 1$, then $\mathcal{O}_n \otimes \mathbb{M}_{|G|^\infty} \cong \mathcal{O}_n$ by combining Theorem 2.7.1 and Theorem 2.6.11 (see the proof of Corollary 5.4.4). So the anomalous action $\mathrm{id}_{\mathcal{O}_n} \otimes s_G^\omega$ induces a (G, ω) action on $\mathcal{O}_n$. $\square$

6.3 Anomalous actions on UHF algebras: construction 2

In this section we give another construction of a (G,ω) action on the UHF algebra $\mathbb{M}_{|G|^\infty}$. Our classification results in Chapter 7 will imply that this action is equivalent to that of Theorem 6.2.3. Unlike the construction performed in 6.2.3 this construction will be visibly compatible with a Bratteli diagram of $\mathbb{M}_{|G|^\infty}$. In particular, it is a strict AF-action in the sense of Definition 6.1.6. The existence of a strict AF ω-anomalous G action on $\mathbb{M}_{|G|^\infty}$ follows from an adaptation of the Ocneanu compactness argument to the C*-setting ([104]). However, in this case, it is easier to construct a strict AF-action explicitely through the inductive limit.

A key idea for this construction is to make use of the canonical (G,ω) action on $C(G)$. The existence of this action is well known to experts, we record this in the proposition below.

Proposition 6.3.1 (cf. [8]). *Let G be a countable discrete group and $\omega \in Z^3(G,\mathbb{T})$. The pair (λ_G, u) with*

$$\lambda_G(g)(f)(h) = f(g^{-1}h) \tag{6.3.1}$$

$$u_{g,h}(k) = \omega_{k^{-1},g,h} \tag{6.3.2}$$

for $g,h,k \in G$ and $f \in C(G)$ induces a (G,ω) action on $C(G)$.

Proof. The left regular representation λ_G is an action of G on an abelian C*-algebra. Therefore for any $g,h \in G$ one has that $\lambda_G(g)\lambda_G(h) = \mathrm{Ad}(u_{g,h})\lambda_G(gh)$. Moreover, for $g,h,k,l \in G$

$$\begin{aligned}\lambda_G(g)(u_{h,k})u_{g,hk}u^*_{gh,k}u^*_{g,h}(l) &= \omega_{(g^{-1}l)^{-1},h,k}\omega_{l^{-1},g,hk}\overline{\omega_{l^{-1},gh,k}\omega_{l^{-1},g,h}} \\ &= \omega_{g,h,k}\end{aligned}$$

by applying the 3-cocycle formula. □

For any finite group G and circle valued 3-cocycle ω, the (G,ω) action from Proposition 6.3.1 induces an action on the first layer of a Bratelli diagram for $\mathbb{M}_{|G|^\infty}$. The task will consist of inductively constructing anomalous actions on further layers of the Bratteli diagram that restrict to the action on the preceeding layer.

Proposition 6.3.2. *Let G be a finite group and $\omega \in Z^3(G,\mathbb{T})$, then there exists a strict AF ω-anomalous G-action with the Rokhlin property on $\mathbb{M}_{|G|^\infty}$. We denote this action by θ_G^ω.*

Proof. In this proof we will use the symbols $g, h, k, x, y, x_i, y_i, s_i$ for $i \in \mathbb{N}$ to denote elements of the group G. Let $A_n = C(G) \otimes \bigotimes_{i=1}^{n-1} \mathcal{B}(l^2(G))$ for $n \in \mathbb{N}$, where by convention $A_1 = C(G)$. For $f \in C(G)$, let $M_f \in \mathcal{B}(l^2(G))$ be the multiplication operator by f. Consider the *-homomorphisms $\varphi_n : A_n \to A_{n+1}$ defined by $\varphi_n(f \otimes T) = 1 \otimes M_f \otimes T$ for $f \in C(G)$ and $T \in \bigotimes_{i=1}^{n-1} \mathcal{B}(l^2(G))$.

The inductive system (A_n, φ_n) has an inductive limit (we write the limit by A) which is known to be isomorphic to $\mathbb{M}_{|G|^\infty}$. Indeed, the Bratelli diagram of this AF-algebra is easily seen to be the complete bipartite graph on $|G|$-vertices, it is common knowlegde that this coincides with the UHF-algebra of type $|G|^\infty$ (see [33, Example III.2.4] for the case $|G| = 2$) We construct a (G,ω) action on each finite dimensional algebra A_n such that the actions commute with the inclusion maps φ_n. This will induce a strict AF ω-anomalous G action on $\mathbb{M}_{|G|^\infty}$ by the universal property of the inductive limit (see Section 6.1).

To be precise, we construct a family of maps $\theta_n : G \to \mathrm{Aut}(A_n)$ and $u_n : G \times G \to U(A_n)$ such that:

1. $\theta_n(g)\theta_n(h) = \mathrm{Ad}(u_n(g,h))\theta_n(gh)$,

2. $\omega_{g,h,k} = \theta_n(g)(u_n(h,k))u_n(g,hk)u_n(gh,k)^*u_n(g,h)^*$,

3. $\varphi_n(u_n(g,h)) = u_{n+1}(g,h)$,

4. $\varphi_n\theta_n(g) = \theta_{n+1}(g)\varphi_n$,

for all $n \in \mathbb{N}$. To build this we will consider the group actions $\theta'_n : G \to \mathrm{Aut}(A_n)$ defined by $\theta'_n(g) = \lambda_G(g) \otimes \bigotimes_{i=1}^{n-1} \mathrm{Ad}(\lambda_G)_g$ where λ_G is the left regular representation of G. Note that $\varphi_n\theta'_n(g) = \theta'_{n+1}(g)\varphi_n$. To take into account the anomaly, we will tweak θ'_n by suitable diagonal operators $d_n \in \mathrm{Aut}(A_n)$ and ensuring that (1) and (2) hold. To define d_n we start by introducing some notation. Let $\delta_k \in C(G)$ be the point mass at k i.e.

$$\delta_k(g) = \begin{cases} 1 \text{ if } g = k, \\ 0 \text{ otherwise.} \end{cases}$$

Let $e_{g,h} \in \mathcal{B}(l^2(G))$ be defined by

$$e_{g,h}(f)(k) = \begin{cases} f(g) \text{ if } k = h, \\ 0 \text{ otherwise,} \end{cases}$$

for $f \in l^2(G)$. We now let

$$\theta_n(g) = d_n(g)\theta'_n(g)$$

with $d_n(g)$ defined inductively

$$\begin{aligned} d_1(g) =& \mathrm{id}_{A_1}, \\ d_2(g)(\delta_k \otimes e_{x_1,y_1}) =& \omega_{x_1^{-1},g,g^{-1}k}\overline{\omega_{y_1^{-1},g,g^{-1}k}}(\delta_k \otimes e_{x_1,y_1}), \end{aligned}$$

and

$$d_n(g)(\delta_k \otimes e_{x_1,y_1} \otimes \cdots \otimes e_{x_{n-1},y_{n-1}})$$
$$= \omega_{x_{n-1}^{-1},g,g^{-1}x_{n-2}} \overline{\omega_{x_{n-3}^{-1},g,g^{-1}x_{n-2}} \omega_{y_{n-1}^{-1},g,g^{-1}y_{n-2}}} \omega_{y_{n-3}^{-1},g,g^{-1}y_{n-2}}$$
$$(d_{n-2}(g)(\delta_k \otimes e_{x_1,y_1} \cdots \otimes e_{x_{n-3},y_{n-3}}) \otimes e_{x_{n-2},y_{n-2}} \otimes e_{x_{n-1},y_{n-1}})$$

for all $n > 2$ with the convention that $x_0 = y_0 = k$. As we have defined $d_n(g)$ on a spanning set of A_n, $d_n(g)$ extend to linear maps from A_n to itself. In fact each $d_n(g)$ is an endomorphism of A_n. First, it is clear that they preserve the *-operation. To show the multiplicativity, it is sufficient to check on the spanning set. We show this by induction. For the case $n = 2$ the only non-trivial statement is that

$$d_2(g)(\delta_k \otimes e_{x_1,y_1}) d_2(g)(\delta_k \otimes e_{y_1,y_2}) = d_2(g)(\delta_k \otimes e_{x_1,y_2}).$$

Expanding the left hand side we have that it is given by

$$d_2(g)(\delta_k \otimes e_{x_1,y_1}) d_2(g)(\delta_k \otimes e_{y_1,y_2})$$
$$= \omega_{x_1^{-1},g,g^{-1}k} \overline{\omega_{y_1^{-1},g,g^{-1}k}} \omega_{y_1^{-1},g,g^{-1}k} \overline{\omega_{y_2^{-1},g,g^{-1}k}} (\delta_k \otimes e_{x_1,y_2})$$
$$= \omega_{x_1^{-1},g,g^{-1}k} \overline{\omega_{y_2^{-1},g,g^{-1}k}} (\delta_k \otimes e_{x_1,y_2})$$

which coincides with the right hand side. To show that $d_n(g)$ is multiplicative for $n > 2$ it suffices to show that

$$d_n(g)(\delta_k \otimes e_{x_1,y_1} \otimes \cdots \otimes e_{x_{n-1},y_{n-1}}) d_n(g)(\delta_k \otimes e_{y_1,s_1} \otimes .... \otimes e_{y_{n-1},s_{n-1}})$$
$$= d_n(g)(\delta_k \otimes e_{x_1,s_1} \otimes .... \otimes e_{x_{n-1},s_{n-1}}).$$

This follows immediately from the induction hypothesis and a direct computation of the left hand side (as in the case for $n = 2$). Notice that each $d_n(g)$ fixes elements of the form $\delta_k \otimes e_{x_1,x_1} \otimes e_{x_2,x_2} \dots e_{x_{n-1},y_{n-1}}$.

Let (θ_1, u_1) be the (G,ω) action on A_1 defined in Proposition 6.3.1. Now, let $u_n(g,h) = \varphi_{1,n}(u_1(g,h))$ and $\theta_n(g) = d_n(g)\theta'_n(g)$. For the remaining part of the proof we check that (θ_n, u_n) satisfy (1)-(4) for all $n \in \mathbb{N}$. We will repeatedly use the 3-cocycle formula during the calculations, instead of commenting on this every time, we will instead colour code the parts of our equations to which we apply the 3-cocycle formula.

We start by showing (1). Firstly,

$$\begin{aligned}\theta_n(g)\theta_n(h) &= d_n(g)\theta'_n(g)d_n(h)\theta'_n(h)\\ &= d_n(g)\theta'_n(g)d_n(h)\theta'_n(g)^{-1}\theta'_n(gh)\\ &= d_n(g)[g \cdot d_n(h)]\theta'_n(gh)\end{aligned}$$

denoting $g \cdot d_n(h) = \theta'_n(g)d_n(h)\theta'_n(g)^{-1}$. It is clear that (1) holds for all $n \in \mathbb{N}$ if and only if $d_n(g)g \cdot d_n(h)d_n(gh)^{-1} = \mathrm{Ad}(u_n(g,h))$ on A_n for all $n \in \mathbb{N}$. This holds trivially for $n = 1$. For $n = 2$ it follows from the 3-cocycle formula that

$$\begin{aligned}&d_2(g)g \cdot d_2(h)d_2(gh)^{-1}(\delta_{ghk} \otimes e_{x_1,y_1})\\ =&d_2(g)g \cdot d_2(h)(\delta_{ghk} \otimes e_{x_1,y_1})\overline{\omega_{x_1^{-1},gh,k}}\omega_{y_1^{-1},gh,k}\\ =&d_n(g)(\delta_{ghk} \otimes e_{x_1,y_1})\overline{\omega_{x_1^{-1},gh,k}}\omega_{y_1^{-1},gh,k}\omega_{x_1^{-1}g,h,k}\overline{\omega_{y_1^{-1}g,h,k}}\\ =&(\delta_{ghk} \otimes e_{x_1,y_1})\color{red}{\overline{\omega_{x_1^{-1},gh,k}}\omega_{x_1^{-1}g,h,k}\omega_{x_1^{-1},g,hk}}\color{blue}{\overline{\omega_{y_1^{-1},g,hk}}\omega_{y_1^{-1},gh,k}\overline{\omega_{y_1^{-1}g,h,k}}}\\ =&(\delta_{ghk} \otimes e_{x_1,y_1})\omega_{g,h,k}\omega_{x_1^{-1},g,h}\overline{\omega_{g,h,k}\omega_{y_1^{-1},g,h}}\\ =&\mathrm{Ad}(\varphi_1(u_1(g,h))(\delta_{ghk} \otimes e_{x_1,y_1}).\end{aligned}$$

We now proceed with an inductive argument for arbitrary n. We assume that (1) holds for $n-2$, preforming a similar computation to the case $n=2$;

$$\begin{aligned}
&d_n(g)g\cdot d_n(h)d_n(gh)^{-1}(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}})\\
=&\big(\mathrm{Ad}(u_{n-2}(g,h))((\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}})\\
&\overline{\omega_{x_{n-1}^{-1},gh,x_{n-2}}}\omega_{x_{n-1}^{-1}g,h,x_{n-2}}\omega_{x_{n-1}^{-1},g,hx_{n-2}}\omega_{x_{n-3}^{-1},gh,x_{n-2}}\overline{\omega_{x_{n-3}^{-1}g,h,x_{n-2}}\omega_{x_{n-3}^{-1},g,hx_{n-2}}}\\
&\overline{\omega_{y_{n-1}^{-1},gh,y_{n-2}}}\omega_{y_{n-1}^{-1}g,h,y_{n-2}}\omega_{y_{n-1}^{-1},g,hy_{n-2}}\omega_{y_{n-3}^{-1},gh,y_{n-2}}\overline{\omega_{y_{n-3}^{-1}g,h,y_{n-2}}\omega_{y_{n-3}^{-1},g,hy_{n-2}}}\\
=&\big(\mathrm{Ad}(u_{n-2}(g,h))((\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}})\\
&\omega_{g,h,x_{n-2}}\omega_{x_{n-1}^{-1},g,h}\overline{\omega_{g,h,x_{n-2}}\omega_{x_{n-3}^{-1},g,h}\omega_{g,h,y_{n-2}}\omega_{y_{n-1}^{-1},g,h}}\omega_{g,h,y_{n-2}}\overline{\omega_{y_{n-3}^{-1},g,h}}\\
=&\big(\mathrm{Ad}(u_{n-2}(g,h))(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}}\big)\\
&\omega_{x_{n-1}^{-1},g,h}\overline{\omega_{x_{n-3}^{-1},g,h}\omega_{y_{n-1}^{-1},g,h}}\omega_{y_{n-3}^{-1},g,h}\\
=&(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-1},y_{n-1}})\omega_{x_{n-1}^{-1},g,h}\overline{\omega_{y_{n-1}^{-1},g,h}}\\
=&\mathrm{Ad}(u_n(g,h))(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}}).
\end{aligned}$$

For (4) it suffices to show that $\varphi_n d_n(g)=d_{n+1}(g)\varphi_n$. For $n=1$

$$\begin{aligned}
d_2(g)\varphi_1(\delta_k)&=\sum_{r\in G}d_2(g)(\delta_r\otimes e_{k,k})\\
&=(1\otimes e_{k,k})\\
&=\varphi_1 d_1(g)(\delta_k)
\end{aligned}$$

as d_1 is the identity map. The case $n=2$ follows too

$$\begin{aligned}
d_3(g)\varphi_2(\delta_k\otimes e_{x,y})&=\sum_{r\in G}d_3(g)(\delta_r\otimes e_{k,k}\otimes e_{x,y})\\
&=(1\otimes e_{k,k}\otimes e_{x,y})\omega_{x^{-1},g,g^{-1}k}\overline{\omega_{y^{-1},g,g^{-1}k}}\\
&=\varphi_2 d_2(g)(\delta_k\otimes e_{x,y}).
\end{aligned}$$

Assuming that the case $n-2$ holds, we now argue by induction,

$$\begin{aligned}
&d_{n+1}(g)\varphi_n(\delta_k \otimes e_{x_1,y_1}\cdots \otimes e_{x_{n-1},y_{n-1}})\\
=&d_{n+1}(g)(\varphi_{n-3}(\delta_k \otimes e_{x_1,y_1}\cdots \otimes e_{x_{n-3},y_{n-3}}) \otimes e_{x_{n-2},y_{n-2}} \otimes e_{x_{n-1},y_{n-1}})\\
=&(d_{n-2}(g)\varphi_{n-3}(\delta_k \otimes e_{x_1,y_1}\cdots \otimes e_{x_{n-3},y_{n-3}}) \otimes e_{x_{n-2},y_{n-2}} \otimes e_{x_{n-1},y_{n-1}})\\
&\omega_{x_{n-1}^{-1},g,g^{-1}x_{n-2}}\overline{\omega_{x_{n-3}^{-1},g,g^{-1}x_{n-2}}\omega_{y_{n-1}^{-1},g,g^{-1}y_{n-2}}}\omega_{y_{n-3}^{-1},g,g^{-1}y_{n-2}}\\
=&(\varphi_{n-2}d_{n-2}(g)(\delta_k \otimes e_{x_1,y_1}\cdots \otimes e_{x_{n-3},y_{n-3}}) \otimes e_{x_{n-2},y_{n-2}} \otimes e_{x_{n-1},y_{n-1}})\\
&\omega_{x_{n-1}^{-1},g,g^{-1}x_{n-2}}\overline{\omega_{x_{n-3}^{-1},g,g^{-1}x_{n-2}}\omega_{y_{n-1}^{-1},g,g^{-1}y_{n-2}}}\omega_{y_{n-3}^{-1},g,g^{-1}y_{n-2}}\\
=&\varphi_n d_n(g)(\delta_k \otimes e_{x_1,y_1}\cdots \otimes e_{x_{n-1},y_{n-1}})
\end{aligned}$$

Condition (3) is immediate. It remains to show that (2) holds for arbitrary n. This follows from (2) for the case $n=1$ and from (4). For $n \in \mathbb{N}$

$$\begin{aligned}
&\theta_n(g)(u_n(h,k))u_n(g,hk)u_n(gh,k)^*u_n(g,h)^*\\
&= \theta_n(g)(\varphi_{1,n}(u_1(h,k)))\varphi_{1,n}(u_1(g,hk))\varphi_{1,n}(u_1(gh,k)^*)\varphi_{1,n}(u_1(g,h)^*)\\
&= \varphi_{1,n}(\theta_1(g)(u_1(h,k))u_1(gh,k)u_1(g,hk)^*u_1(g,h)^*)\\
&= \omega_{g,h,k}\varphi_{1,n}(1_{A_1})\\
&= \omega_{g,h,k}.
\end{aligned}$$

To show that θ_G^ω has the Rokhlin property we construct a family of Rokhlin projections. The projections $\delta_g \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}} \in Z(A_n)$ satisfy $\theta_n(g)(\delta_h \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}}) = \delta_{gh}\otimes\mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}}$ and $\sum_{g\in G}\delta_g\otimes\mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}} = \mathrm{id}_{A_n}$. Note also that for any $\varepsilon > 0$ and finite set $\mathcal{F} \subset \mathbb{M}_{|G|^\infty}$ there exists $n \in \mathbb{N}$ such that $\mathrm{dist}(\mathcal{F}, A_m) < \varepsilon$ for $m \geq n$. Therefore, the projections $p_g \in A_\infty$ with n-th coordinate given by $\varphi_{n,\infty}(\delta_g \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}})$ for $g \in G$ satisfy the conditions of Definition 6.1.1. □

Remark 6.3.3. In the case that $\omega = 1$ the construction in Proposition 6.3.2 greatly simplifies. Indeed, $d_n(g)$ is the identity automorphism and $u_n(g,h)$ is the unit for all $g,h \in G$ and $n \in \mathbb{N}$. Therefore, θ^1_G restricts to the group action $\theta_n = \lambda_G \bigotimes_{i=0}^{n-1} \mathrm{Ad}(\lambda_G)$ on each A_n. We denote θ^1_G simply by θ_G. The action θ_G is conjugate to the infinite tensor product action μ_G of (3.4.3). To show this we will use Lemma 2.3.4. The G-C*-algebra $(\mathbb{M}_{|G|^\infty}, \mu_G)$ is the inductive limit of (B_n, μ_n, ϕ_n) with $B_n = \bigotimes_{i=1}^{n} \mathcal{B}(l^2(G))$, $\mu_n = \mathrm{Ad}(\lambda_G)^{\otimes n}$ and $\phi_n(M) = \mathrm{id}_{\mathcal{B}(l^2(G))} \otimes M$ for any $M \in B_n$. The maps $\eta_n : A_n \to B_n$ and $\nu_n : B_n \to A_{n+1}$ defined by $\eta_n(f \otimes M) = M_f \otimes M$ and $\nu_n(N) = 1 \otimes N$ for $f \in C(G)$, $M \in \bigotimes_{i=1}^{n-1} \mathcal{B}(l^2(G))$ and $N \in \bigotimes_{i=1}^{n} \mathcal{B}(l^2(G))$ are easily seen to produce an intertwining between $(A_n, \theta_n, \varphi_n)$ and (B_n, μ_n, ϕ_n). Hence by Lemma 2.3.4 the actions θ_G and μ_G are conjugate.

Let $\omega \in Z^3(\mathbb{Z}_2, \mathbb{T})$ be the standard representative of the non-trivial element of $H^2(\mathbb{Z}_2, \mathbb{T})$ (see (3.3.3)). After identifying $\mathcal{B}(l^2(\mathbb{Z}_2))$ with $\mathbb{M}_2$ and $C(\mathbb{Z}_2)$ with $\mathbb{C} \oplus \mathbb{C}$ under a choice of basis, one can intertwine the two Bratelli diagrams for $\mathbb{M}_{2^\infty}$ in a similar fashion to Remark 6.3.3 to show that $\theta^\omega_{\mathbb{Z}_2}(1+2\mathbb{Z})$ and Connes' automorphism s^γ_2 of Section 3.3.2 for $\gamma = -1$ are conjugate. However, it is not immediate whether the automorphisms $\theta^{\lambda_1}_{\mathbb{Z}_3}(1+3\mathbb{Z})$ and $\theta^{\lambda_2}_{\mathbb{Z}_3}(1+3\mathbb{Z})$ for the standard representative cocycles $\lambda_1, \lambda_2 \in Z^3(\mathbb{Z}_3, \mathbb{T})$ (see 3.3.3 for the formula of the cocycles λ_1 and λ_2) are conjugate to s^ξ_3 and $s^{\xi^2}_3$ respectively, with ξ the primitive third root of unity. It will follow from a later classification result (Theorem 7.2.3) that the actions induced by s^γ_n and $\theta^{\lambda_k}_{\mathbb{Z}_n}$ for $\gamma = e^{2\pi i k/n}$ are cocycle conjugate.

6.4 Anomalous actions of finite cyclic groups

In this section, we present a method for building $(\mathbb{Z}_n, \omega)$ actions on C*-algebras that arise as crossed products by the integers. This method was independently discovered by Corey Jones (see for example [74, Corollary 3.6]). Let A be a C*-algebra, throughout this section we will denote by v the unitary generating $C^*(\mathbb{Z}) \subset U(M(A \rtimes \mathbb{Z}))$.

We take the viewpoint of Lemma 3.3.6. To construct anomalous actions of $\mathbb{Z}_n$ on a C*-algebra $B = A \rtimes \mathbb{Z}$ for all possible 3-cocycles $\omega \in Z^3(G, \mathbb{T})$, it is sufficient to construct for any n-th root of unity ξ an automorphism $\alpha \in \mathrm{Aut}(B)$ and a unitary $u \in U(M(B))$ such that $\alpha^n = \mathrm{Ad}(u)$ and $\alpha(u) = \xi u$. Our method is abstracted in the following lemma.

Lemma 6.4.1. *Let A be a C*-algebra, $\alpha \in \mathrm{Aut}(A)$ and ξ an n-th root of unity. If there exists an automorphism $\beta \in \mathrm{Aut}(A)$ such that $\beta^n = \alpha$ then the mapping*

$$\phi(\sum_{k=1}^{m} a_k v^k) = \sum_{k=1}^{m} \xi^k \beta(a_k) v^k \tag{6.4.1}$$

defines an automorphism of $A \rtimes_\alpha \mathbb{Z}$ such that $\phi^n = \mathrm{Ad}(v)$ and $\phi(v) = \xi v$.

Proof. That the formula in (6.4.1) defines an automorphism of $A \rtimes \mathbb{Z}$ follows from the universal property of the crossed product. By construction $\phi^n(a) = \beta^n(a) = \alpha(a) = \mathrm{Ad}(v)a$ for $a \in A$ and $\phi^n(v) = v = \mathrm{Ad}(v)(v)$ so $\phi^n = \mathrm{Ad}(v)$ on $A \rtimes_\alpha \mathbb{Z}$. Moreover, ϕ is defined in a way such that $\phi(v) = \xi v$. □

We start by considering anomalous actions on the rotation algebras $\mathcal{A}_\theta$. For $\theta \in \mathbb{R}$ the rotation algebra $\mathcal{A}_\theta$ is the universal C*-algebra generated by two unitaries u and v such that

$$e^{2\pi i\theta} vu = uv$$

Proposition 6.4.2 (cf. [74, Corollary 3.6])**.** *Let $\theta \in \mathbb{R}$ and $\mathcal{A}_\theta$ be the rotation algebra. For any $n \in \mathbb{N}$ and $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$ there exists a $(\mathbb{Z}_n, \omega)$ action on $\mathcal{A}_\theta$.*

Proof. The rotation algebra $\mathcal{A}_\theta$ is isomorphic to $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ (see e.g. [33, Section VI]) with $\alpha(f)(t) = f(e^{-2\pi i\theta} t)$. For any $n \in \mathbb{N}$ the automorphism $\beta(f)(t) = f(e^{-2\pi i\frac{\theta}{n}} t)$ satisfies $\beta^n = \alpha$. The result follows from Lemma 6.4.1. □

Remark 6.4.3. In the case that the angle θ is irrational the rotation algebra $\mathcal{A}_\theta$ is simple. Therefore, Proposition 6.4.2 induces a $\mathbb{Z}_n$-kernel on $\mathcal{A}_\theta$ with lifting obstruction $[\omega]$ for arbitrary $[\omega] \in H^3(\mathbb{Z}_n, \mathbb{T})$ and $n \in \mathbb{N}$.

Moreover, still under the assumption of irrational θ, $\mathcal{A}_\theta$ has a unique trace τ and $K_0(A_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta \subseteq \mathbb{R}$ with the isomorphism induced by the trace τ (see e.g. [33, Example VIII.5.1]). However, Proposition 6.4.2 does not contradict Theorem 4.2.12 since the anomalous action constructed in Proposition 6.4.2 is not connected.

We now turn to the Bunce–Deddens algebras (see [33, Section V.3]). These algebras arise as the crossed product of a Cantor space by an odometer action. Like the UHF algebras, Bunce–Deddens algebras are classified up to isomorphism by a supernatural number $\mathfrak{n} = \prod_{k\in\mathbb{N}} n_k$. In the odometer construction, n_k is the number of values on the k-th dial of the odometer (see [33, Theorem VIII.4.1]). The Bunce–Deddens algebra $B_\mathfrak{n}$ has a unique trace τ, $K_0(B_\mathfrak{n}) = Q(\mathfrak{n})$ and $K_1(B_\mathfrak{n}) = \mathbb{Z}$.

Proposition 6.4.4. *Let $\mathfrak{n} = \prod_{k\in\mathbb{N}} n_k$ be a supernatural number. Let $B_\mathfrak{n}$ be the corresponding Bunce–Deddens algebra. For any $m \in \mathbb{N}$, which is coprime to $\mathfrak{n}$ and any $\omega \in Z^3(\mathbb{Z}_m, \mathbb{T})$ there exists a $(\mathbb{Z}_m, \omega)$ action on $B_\mathfrak{n}$.*

Proof. Let $X = \prod_{k\in\mathbb{N}} X_k$ where X_k is a discrete topological space with n_k points. Let $\alpha \in \mathrm{Homeo}(X)$ be the odometer map and $\alpha^* \in \mathrm{Aut}(C(X))$ be the induced automorphism. Then $B_\mathfrak{n} \cong C(X) \rtimes_{\alpha^*} \mathbb{Z}$.

The automorphism α has an m-th root whenever m is coprime to $\mathfrak{n}$. In the case where $\mathfrak{n} = p^\infty$, this is just the observation that m is invertible in the ring of p-adic integers. In general, we work in the topological ring R that arises as the inverse limit of the system

$$\cdots \to \frac{\mathbb{Z}}{n_4 n_3 n_2 n_1 \mathbb{Z}} \to \frac{\mathbb{Z}}{n_3 n_2 n_1 \mathbb{Z}} \to \frac{\mathbb{Z}}{n_2 n_1 \mathbb{Z}} \to \frac{\mathbb{Z}}{n_1 \mathbb{Z}}. \tag{6.4.2}$$

We identify X with the underlying topological space of R and α with addition by 1_R. Since the image of $m \cdot 1_R$ is invertible at each stage of the system, it is invertible in

R. The result follows by Lemma 6.4.1. □

We now turn our attention to C*-algebras that arise as crossed products by Bernoulli shifts on infinite tensor products.

Proposition 6.4.5. *Let A be a unital C*-algebra isomorphic to $\bigotimes_{i\in\mathbb{Z}} A$ and let σ be the right shift on the tensor product. That is, after identifying A with $\bigotimes_{i\in\mathbb{Z}} A$, the automorphism σ is given by $\sigma(\bigotimes_{i\in\mathbb{Z}} a_i) = \bigotimes_{i\in\mathbb{Z}} a_{i-1}$ on elementary tensors of finite support. For any $n \in \mathbb{N}$ and any $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$ there exist a $(\mathbb{Z}_n, \omega)$ action on $A \rtimes \mathbb{Z}$.*

Proof. As $A \cong \bigotimes_{i\in\mathbb{Z}} A$ it follows that $A \cong \bigotimes_{i\in\mathbb{Z}}(\overset{n}{\underset{s=1}{\otimes}} A)$. Let $a_{i,s} \in A$ for $i \in \mathbb{Z}$ and $1 \leq s \leq n$ with $a_{i,0} := a_{i-1,n}$. Consider the automorphism $\beta \in \mathrm{Aut}(A)$ defined by $\beta(\bigotimes_{i\in\mathbb{Z}}(\otimes_{s=1}^{n} a_{i,s})) = \bigotimes_{i\in\mathbb{Z}}(\otimes_{s=1}^{n} a_{i,s-1})$. Then $\beta^n = \sigma$, the result follows from Lemma 6.4.1. □

Relevant examples of C*-algebras isomorphic to their infinite tensor product are given by the strongly self absorbing C*-algebras of [142]. Examples of these are UHF algebras of infinite type, $\mathcal{Z}$ and $\mathcal{O}_\infty$.[4] Proposition 6.4.5 yields $(\mathbb{Z}_n, \omega)$ actions on $\mathcal{Z} \rtimes \mathbb{Z}$, $\mathcal{O}_\infty \rtimes \mathbb{Z}$ and $\mathbb{M}_{k^\infty} \rtimes \mathbb{Z}$ for any $k, n \in \mathbb{N}$ and $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$.

For any finite group G, in [67, Lemma 5.2] Izumi builds a Kirchberg algebra D in the UCT class with $K_0(D) \cong \mathbb{Z}^{|G|}$, $K_1(D) = 0$ and a G-action α on D with the Rokhlin property, such that $K_0(D)$ and the group ring $\mathbb{Z}[G]$ are isomorphic as G-modules. Izumi uses these Rokhlin actions to characterise the pair of modules that arise as the K-theory groups of a Kirchberg algebra in the UCT class, with the module structure arising from a Rokhlin action on the underlying Kirchberg algebras (see Theorem 8.2.3).

If G is a cyclic group and $\omega \in Z^3(G, \mathbb{T})$, we can also construct a Kirchberg algebra in the UCT class D and a (G, ω) action on D with the Rokhlin property

[4] Recall that a UHF algebra is said to be of infinite type if it is an infinite tensor product of the same matrix algebra i.e. the supernatural number associated to it is n^∞ for some $n \in \mathbb{N}$.

such that $K_0(D) \cong \mathbb{Z}[G]$ and $K_1(D) = 0$ as G-modules. By the Kirchberg–Phillips classification theorem, D will coincide with Izumi's construction. We first recall the notion of approximate representable actions which is dual to the Rokhlin property by [66, Lemma 3.8].

Definition 6.4.6. Let G be a finite abelian group and A a unital C*-algebra. A G action α on A is called *approximately representable* if there exists unitaries $u_g \in A_\infty$ fixed by α such that

$$\alpha_g(a) = u_g a u_g^*, \quad a \in A, \quad g \in G \tag{6.4.3}$$

and $u_g u_h = u_{gh}$.

In the following proof we denote by σ_n the automorphism of $\mathcal{O}_\infty \cong \bigotimes_{k=1}^{n} \mathcal{O}_\infty$ given by $\sigma_n(a_1 \otimes a_2 \otimes \dots a_n) = a_n \otimes a_1 \otimes \dots \otimes a_{n-1}$ for $a_i \in \mathcal{O}_\infty$.

Proposition 6.4.7. *Let $n \in \mathbb{N}$. For any $1 \le m \le n$ the assignments*

$$\alpha_m(\sum_{k=0}^{n-1} a_k v^k) = \sum_{k=0}^{n-1} e^{2\pi i k/n} \sigma_{n^2}^m(a_k) v^k$$

for $a_k \in \mathcal{O}_\infty$ and v the unitary coming from $1+n\mathbb{Z}$ in $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$ define automorphisms of $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$. The automorphisms α_m satisfy $\alpha_m^n = \mathrm{Ad}(v^m)$ and $\alpha_m(v^m) = e^{2\pi i m/n} v^m$. Hence, the automorphisms α_m induce $\mathbb{Z}_n$ anomalous actions on $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$. These anomalous actions have the Rokhlin property and as $\mathbb{Z}_n$-modules $K_0(\mathcal{O}_\infty \rtimes_\sigma \mathbb{Z}_n) \cong \mathbb{Z}[\mathbb{Z}_n]$ and $K_1(\mathcal{O}_\infty \rtimes_\sigma \mathbb{Z}_n) = 0$.

Proof. Notice that α_m are automorphisms by the universal property of the crossed product. Moreover $\alpha_m^n(a) = \sigma_{n^2}^{mn}(a) = \sigma_n^m(a) = \mathrm{Ad}(v^m)(a)$ for $a \in \mathcal{O}_\infty$ and $\alpha_m^n(v) = (e^{2\pi i k/n})^n v = v = \mathrm{Ad}(v^m)(v)$ so $\alpha_m^n = \mathrm{Ad}(v^m)$ on $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$. Moreover, $\alpha_m(v^m) =$

$e^{2\pi i m/n}v^m$. Denote by g the class of $1+n\mathbb{Z}$ in $\mathbb{Z}_n$. As in Lemma 3.3.6 the mappings

$$\alpha_m(g^k) = \alpha_m^k, \quad 0 \le k \le n-1$$
$$u_m(g^i, g^j) = \begin{cases} 1, & i+j=n \\ v^m, & i+j>n \end{cases}$$

define anomalous actions of $\mathbb{Z}_n$ on $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$.

We now turn to the module structure of the K-theory groups of $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$ under α_m. It follows from [130, Corollary 6.9] that the inclusion $\iota : C^*(\mathbb{Z}_n) \hookrightarrow \mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$ is a KK-equivalence. Therefore, $K_0(\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n) = \mathbb{Z}^n$, $K_1(\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n) = 0$. Moreover, standard Fourier theory establishes a family of generators for the K_0 classes of $C^*(\mathbb{Z}_n)$, through the inclusion ι the generators of $K_0(\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n)$ are given by the projections

$$p_\xi = \frac{1}{n}\sum_{i=0}^{n-1} \xi^i v^i \tag{6.4.4}$$

where ξ ranges over the set of n-th roots of unity. To understand the module structure of $K_0(\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n)$, we compute α_m evaluated at the generating projections p_ξ. The automorphism α_m restricts to the dual action of $\hat{\mathbb{Z}}_n$ on $C^*(\mathbb{Z}_n)$. So $\alpha_m(p_\xi) = p_{e^{2\pi i/n}\xi}$ for any n-th root of unity ξ. Therefore, as a $\mathbb{Z}_n$ module $K_0(\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n) \cong \mathbb{Z}[\mathbb{Z}_n]$ (note that the module structure is independent of m).

It remains to show that the anomalous actions defined by α_m have the Rokhlin property. By [130, Corollary 6.10] the $\mathbb{Z}_{n^2}$ action on $\mathcal{O}_\infty$ given by σ_{n^2} is approximately representable. The reason α_m has the Rokhlin property is due to the duality between approximate representability and the Rokhlin property ([66, Lemma 3.8]). We proceed to show this by following the argument in [66, Lemma 3.8]. By approximate representability of the $\mathbb{Z}_{n^2}$ action given by σ_{n^2} (see Definition 6.4.6) there exists a unitary $u \in (\mathcal{O}_\infty)_\infty$ fixed by σ_{n^2} such that for all $a \in \mathcal{O}_\infty$

$$\sigma_{n^2}(a) = uau^* \tag{6.4.5}$$

and $u^{n^2} = 1$. As u is fixed by σ_{n^2} it is fixed by $\sigma_n = \sigma_{n^2}^n$ and hence commutes with v. Define a unital *-homomorphism φ by

$$\begin{aligned} \varphi : C^*(\mathbb{Z}_n) &\to (\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n)_\infty \\ g^k &\mapsto (u^{-n}v)^k. \end{aligned}$$

for $0 \leq k \leq n-1$. The Fourier transform induces an isomorphism $F : C^*(\mathbb{Z}_n) \to C(\hat{\mathbb{Z}}_n)$ with $\hat{\mathbb{Z}}_n \cong \mathbb{Z}_n$ the dual group. Moreover, F is equivariant from the dual action on $\mathrm{C}^*(\mathbb{Z}_n)$ to the (left) regular representation of $\hat{\mathbb{Z}}_n$ on $C(\hat{\mathbb{Z}}_n)$. Therefore, by Remark 6.1.4, it suffices to show that the image of φ commutes with $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$ and that φ is equivariant from $(C^*(\mathbb{Z}_n), \chi)$ to $(\mathcal{O}_\infty \rtimes \mathbb{Z}_n, \alpha)$ with χ the dual action. We start by showing that the image of φ commutes with $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$. To show this, it suffices to show that $\varphi(g)$ commutes with any element in the algebraic crossed product $\mathcal{O}_\infty \rtimes_{\sigma_n}^{\mathrm{alg}} \mathbb{Z}_n$. Let $a_k \in \mathcal{O}_\infty$ for $1 \leq k \leq n$ then

$$\begin{aligned} \varphi(g) \sum_{k=0}^{n-1} a_k v^k = u^{-n} v \sum_{k=0}^{n-1} a_k v^{-1} v v^k &= \sum_{k=0}^{n-1} u^{-n} \sigma_n(a_k) v^k v \\ &= \sum_{k=0}^{n-1} u^{-n} \sigma_n(a_k) u^n u^{-n} v^k v \\ &= \sum_{k=0}^{n-1} a_k u^{-n} v^k v \\ &= \sum_{k=0}^{n-1} a_k v^k u^{-n} v \\ &= \sum_{k=0}^{n-1} a_k v^k \varphi(g). \end{aligned}$$

It remains to show the equivariance of φ. Let $0 \leq k, l \leq n-1$ then

$$\begin{aligned}\varphi(\chi(g^l)(g^k)) &= e^{2\pi ikl/n} u^{-nk} v^k \\ &= e^{2\pi ikl/n} \sigma_{n^2}^{ml}(u^{-nk}) v^k \\ &= \alpha_m^l(u^{-nk} v^k) \\ &= \alpha_m(g^l)(\varphi(g^k)).\end{aligned}$$ □

Corollary 6.4.8. *For any $n \in \mathbb{N}$ and $\omega \in Z^3(G, \mathbb{T})$ there exists a $(\mathbb{Z}_n, \omega)$ action with the Rokhlin property on a unital Kirchberg algebra D in the UCT class such that $K_0(D) \cong \mathbb{Z}[\mathbb{Z}_n]$ and $K_1(D) = 0$ as $\mathbb{Z}_n$-modules.*

Proof. This follows immediately from Proposition 6.4.7 as $\mathcal{O}_\infty \rtimes_\sigma \mathbb{Z}_n$ is nuclear, it is simple by [87] and purely infinite as a consequence of [71, Theorem 3]. Moreover, $\mathcal{O}_\infty \rtimes_\sigma \mathbb{Z}_n$ satisfies the UCT as the inclusion $\iota : C^*(\mathbb{Z}_n) \hookrightarrow \mathcal{O}_\infty \rtimes \mathbb{Z}_n$ is a KK-equivalence by [130, Corollary 6.9]. □

We finish this section by discussing the existence of anomalous actions of cyclic groups on matrix amplifications of Cuntz algebras. In Corollary 6.2.8 we discussed the existence of anomalous actions on $\mathcal{O}_n$ through tensorial absorption of UHF algebras. We now construct actions on matrix amplifications of Cuntz algebras that can not arise in this form. We begin by recalling a construction of Cuntz ([29]).

Let $n \in \mathbb{N}$, $B_n = \mathbb{M}_{n^\infty} \otimes \mathbb{K}$ and τ the unique semifinite trace on B_n normalised such that $\tau(1 \otimes e_{11}) = 1$. Let θ be the automorphism of B_n with the property that $\tau(\theta(x)) = \frac{1}{n}\tau(x)$ for any $x \in B_n^+$. Precisely, θ is defined by the composition

$$\mathbb{M}_{n^\infty} \otimes \mathbb{K} \cong \mathbb{M}_{n^\infty} \otimes (\mathbb{M}_n \otimes \mathbb{K}) = \big(\mathbb{M}_{n^\infty} \otimes \mathbb{M}_n\big) \otimes \mathbb{K} \cong \mathbb{M}_{n^\infty} \otimes \mathbb{K}.$$

In [29, Section 2] it is shown that $B_n \rtimes_\theta \mathbb{Z} \cong \mathcal{O}_n \otimes \mathbb{K}$. We will need to understand the K-theory of $B_n \rtimes_\theta \mathbb{Z}$ in detail. Therefore we summarise the Pimsner–Voiculescu exact sequence associated to $B_n \rtimes_\theta \mathbb{Z}$ following [33, VIII.5.3]. (See for example Theorem [33, Theorem VIII.5.1] for a statement of the Pimsner–Voiculescu exact sequence.)

Denote by $\iota : B_n \to B_n \rtimes_\theta \mathbb{Z}$ the inclusion. As $K_1(\mathbb{M}_{n^\infty}) = 0$, the Pimsner–Voiculescu exact sequence reduces to the following exact sequence

$$0 \to K_1(B_n \rtimes_\theta \mathbb{Z}) \to \mathbb{Z}\left[\frac{1}{n}\right] \xrightarrow{\mathrm{id}_* - \theta_*} \mathbb{Z}\left[\frac{1}{n}\right] \xrightarrow{\iota_*} K_0(B_n \rtimes_\theta \mathbb{Z}) \to 0. \tag{6.4.6}$$

The automorphism θ_* is uniquely determined by where it sends the class $[1 \otimes e_{11}]_0$. As $\tau_* : K_0(B_n) \to \mathbb{Z}\left[\frac{1}{n}\right]$ is an isomorphism and θ scales τ by $\frac{1}{n}$, the map θ_* is given by multiplication by $\frac{1}{n}$. Therefore, $\mathrm{id}_* - \theta_*$ is the map of multiplication by $\frac{n-1}{n}$. So $\mathrm{id}_* - \theta_*$ is injective and (6.4.6) reduces to

$$0 \to \mathbb{Z}\left[\frac{1}{n}\right] \xrightarrow{\times \frac{n-1}{n}} \mathbb{Z}\left[\frac{1}{n}\right] \xrightarrow{\iota_*} K_0(B_n \rtimes_\theta \mathbb{Z}) \to 0. \tag{6.4.7}$$

Appealing to the isomorphism theorem, $K_0(B_n \rtimes_\theta \mathbb{Z}) \cong \mathbb{Z}_{n-1}$. Moreover, $[\iota(1 \otimes e_{11})]_0$ coincides with the class of $1 + (n-1)\mathbb{Z}$ in $K_0(B_n \rtimes_\theta \mathbb{Z})$ under this isomorphism.

Lemma 6.4.9. *Let $k \in \mathbb{N}$ and ξ a k-th root of unity. For every $m \in \mathbb{N}$ there exists an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_{m^k} \otimes \mathbb{K})$ and a unitary $u \in U(M(\mathcal{O}_{m^k} \otimes \mathbb{K}))$ such that*

$$\alpha^k = \mathrm{Ad}(u)$$
$$\alpha(u) = \xi u.$$

Proof. We adopt the notation introduced in the previous few paragraphs. As $\mathcal{O}_{m^k} \otimes \mathbb{K}$ is isomorphic to $B_{m^k} \rtimes_\theta \mathbb{Z}$, it is sufficient to show that the second algebra admits such a pair (α, u). The automorphism β of B_{m^k} given by

$$\Big(\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{m^k}\Big) \otimes \mathbb{K} \cong \Big(\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{m^k}\Big) \otimes (\mathbb{M}_m \otimes \mathbb{K}) = \Big(\bigotimes_{i\in\mathbb{N}} \mathbb{M}_{m^k} \otimes \mathbb{M}_m\Big) \otimes \mathbb{K} \cong \bigotimes_{i\in\mathbb{N}} \mathbb{M}_{m^k} \otimes \mathbb{K}$$

satisfies $\tau(\beta(x)) = \frac{1}{m}\tau(x)$ for any $x \in B^+_{m^k}$. By construction $\beta^k = \theta$. Define $\alpha \in \mathrm{Aut}(B_{m^k} \rtimes_\theta \mathbb{Z})$ by

$$\alpha(\sum_{i=1}^{m} a_i v^i) = \sum_{i=1}^{m} \beta(a_i) v^i$$

for any $m \in \mathbb{N}$ and $\sum_{i=1}^m a_i v^i \subset B_{m^k} \rtimes_\theta \mathbb{Z}$. The pair (α, v) satisfy the requirements of the proposition (see Lemma 6.4.1 for a similar computation). □

Recall a unital, simple, purely infinite C*-algebra is in Cuntz standard form if its class of the unit $[1]_0$ is zero. We may combine Lemma 6.4.9 and Lemma 4.2.9 to induce actions on both matrix amplifications of Cuntz algebras and Cuntz algebras in Cuntz standard form.

Proposition 6.4.10. *Let $k \in \mathbb{N}$ and $\omega \in Z^3(\mathbb{Z}_k, \mathbb{T})$. For any $m \in \mathbb{N}$ there exists a $(\mathbb{Z}_k, \omega)$ action on $\mathcal{O}^{st}_{m^k}$ and on $\mathcal{O}_{m^k} \otimes \mathbb{M}_{\frac{m^k-1}{\gcd(m^{k-1}-1, m^k-1)}}$. In particular, there is a $(\mathbb{Z}_2, \omega)$ action on $\mathcal{O}_{m^2} \otimes \mathbb{M}_{m+1}$.*

Proof. Firstly, by Lemma 6.4.9 there exists a $(\mathbb{Z}_k, \omega)$ action (α, u) on $\mathcal{O}_{m^k} \otimes \mathbb{K}$. To prove this proposition, we will be required to understand the automorphism $\alpha_* \in \mathrm{Aut}(K_0(\mathcal{O}_{m^k} \otimes \mathbb{K}))$. It will be convenient for us to switch to the crossed product presentation and work instead with $B_{m^k} \rtimes_\theta \mathbb{Z}$. By abuse of notation we denote by (α, u) the $\mathbb{Z}_n$ anomalous action on $B_{m^k} \rtimes_\theta \mathbb{Z}$ constructed in the proof of Lemma 6.4.9.

Before we proceed, recall from the proof of Lemma 6.4.9 that β is the automorphism of B_{m^k} with the property that $\tau(\beta(x)) = \frac{1}{m}x$ for every $x \in B^+_{m^k}$ and that $\alpha(a) = \beta(a)$ for any $a \in B_{m^k}$. To make sense of α_* it suffices to know the image under α_* of a generator of $K_0(B_{m^k} \rtimes_\theta \mathbb{Z})$. We have computed in the paragraphs preceeding Lemma 6.4.9 that such a generator is given by $[\iota(1 \otimes e_{11})]_0$. The automorphism β scales the trace by $\frac{1}{m}$, it follows that $[\beta(1 \otimes e_{11})]_0 = \frac{1}{m} \in \mathbb{Z}\left[\frac{1}{m^k}\right]$ and $\alpha_*[\iota(1 \otimes e_{11})]_0 = \iota_*[\beta(1 \otimes e_{11})]_0 = \iota_* \frac{1}{m}$. Moreover, by (6.4.7), the map of multiplication by $\frac{m^k-1}{m^k}$ is contained in the kernel of ι_*. Therefore, $\iota_*(\frac{1}{m}) = \iota_*(\frac{1}{m} + \frac{m^k-1}{m^k}(m^{k-1})) =$

$\iota_*(m^{k-1}) = m^{k-1}\iota_*[1 \otimes e_{11}]_0$. So $\alpha_* \in \mathrm{Aut}(B_{m^k} \rtimes_\theta \mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}_{m^k-1})$ is multiplication by m^{k-1}. And thus, α_* is multiplication by m^{k-1} considered as an automorphism of $K_0(\mathcal{O}_{m^k} \otimes \mathbb{K})$ too.

Denote by $m_0 = \frac{m^k-1}{\gcd(m^{k-1}-1, m^k-1)}$. The elements $[0]$ and $[m_0]$ in $\mathbb{Z}_{m^k-1}$ are fixed under the map of multiplication by m^{k-1}. Indeed

$$[m_0 m^{k-1}] = [m_0(m^{k-1}-1)] + [m_0] = (m^k-1)[\frac{m^{k-1}-1}{\gcd(m^{k-1}-1, m^k-1)}] + [m_0] = [m_0].$$

Pick a non-zero projection p in $\mathcal{O}_{m^k}$ such that $[p \otimes e_{11}]_0 = [0]$. We may now apply Lemma 4.2.9 to induce $(\mathbb{Z}_k, \omega)$ actions on $(p \otimes e_{11})(\mathcal{O}_{m^k} \otimes \mathbb{K})(p \otimes e_{11}) \cong \mathcal{O}^{st}_{m^k}$. Similarly, $[1 \otimes (e_{11} + e_{22} + \ldots + e_{m_0 m_0})]_0 = [m_0]$ and applying Lemma 4.2.9 it follows that there exist $(\mathbb{Z}_k, \omega)$ actions on $(1 \otimes (e_{11} + \ldots + e_{m_0 m_0}))(\mathcal{O}_{m^k} \otimes \mathbb{K})(1 \otimes (e_{11} + \ldots + e_{m_0 m_0})) \cong \mathcal{O}_{m^k} \otimes \mathbb{M}_{m_0}$.

For the observation in the case that $k = 2$, note that $m^2 - 1 = m(m-1) + (m-1)$. Therefore $\gcd(m^2-1, m-1) = m-1$ and so $m_0 = m+1$. □

Proposition 6.4.10 allows us to construct interesting anomalous actions. For instance, there exists a $(\mathbb{Z}_2, \omega)$ action on $\mathcal{O}_9 \otimes \mathbb{M}_4$ for the non-trivial 3-cocycle ω. However, there does not exist a $(\mathbb{Z}_2, \omega)$ action for non-trivial ω on $\mathcal{O}_9$ due to Theorem 5.3.1. This example also establishes that the torsion assumption in Theorem 5.4.2 is necessary. Indeed, $\mathcal{O}_9 \otimes \mathbb{M}_4$ is unital, has cancellation of non-zero projections and all its automorphisms are approximately inner (the last of these follows from [116]). However, its K_0 group is $\mathbb{Z}_8$ which is not 2-divisible.

Chapter 7

Classification

In this chapter, we consider the classification of anomalous actions on C*-algebras. Building on work of Connes in [28], we develop an abstract method for reducing the problem of classifying anomalous actions to that of classifying cocycle actions (see Lemma 7.2.1). Recall that a cocycle action is a specific type of anomalous action, a pair (α, u) as in Definition 3.3.1 with the further restriction that the unitary product in (3.3.2) is the trivial 3-cocycle. As discussed in Section 3.4 there are multiple results for the classification of group actions on C*-algebras, these classification results may also entail classification for cocycle actions of groups on C*-algebras through cohomology vanishing type results (see e.g. [66, Lemma 3.12]). The main idea for our classification is that if a C*-algebra A tensorially absorbs a strongly self-absorbing C*-algebra $\mathcal{D}$, that admits anomalous actions with arbitrary anomaly, then one may cancel out the anomaly of any (G, ω) action on A by tensoring with a $(G, \overline{\omega})$ action on $\mathcal{D}$ to reduce to a cocycle action. This allows to boost classification results for cocycle actions to classification of anomalous group actions by further adding the anomaly to the invariant.

In Chapter 6, we have shown that if G is a finite group and $\omega \in Z^3(G, \mathbb{T})$, then there exists a (G, ω) action on $\mathbb{M}_{|G|^\infty}$. This existence result, allows us to use our strategy to boost Izumi's classification of Rokhlin actions of finite groups on unital Kirchberg algebras satisfying the UCT and unital, separable, simple, nuclear TAF-algebras that satisfy the UCT ([66],[67]) to a classification of anomalous actions on these C*-algebras, if one further restricts to those that tensorially absorb a particular UHF-algebra. A crucial step to execute this strategy is a model action absorption result which we show in Section 7.1. We finish the chapter by discussing an application of the classification results of [25]. We show that in some cases, every AF-anomalous action has the Rokhlin property. The content of this chapter will appear in my article in preparation [51].

7.1 Absorption of model action

In this subsection we show that any Rokhlin anomalous action of a finite group G, on an $\mathbb{M}_{|G|^\infty}$-stable C*-algebra, absorbs the action μ_G of (3.4.3) up to cocycle conjugacy. The methods utilised in this chapter are an adaptation of Vaughan Jones' work ([78]) to the C*-setting. We start by recalling strongly self-absorbing C*-algebras.

Definition 7.1.1. (Toms–Winter cf. [142, Definition 1.3]) A unital, separable C*-algebra $\mathcal{D}$ is called *strongly self-absorbing* if there exists an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ and unitaries $u_n \in U(\mathcal{D} \otimes \mathcal{D})$ with

$$\lim_{n\to\infty} \|\varphi(a) - u_n(a \otimes 1_{\mathcal{D}})u_n^*\| = 0$$

for all $a \in \mathcal{D}$

It is easy to see that UHF-algebras of infinite type are strongly self-absorbing. However, it is much harder to see that $\mathcal{Z}$ ([73]), $\mathcal{O}_2$ ([117, Theorem 5.2.1]) and $\mathcal{O}_\infty$ ([117, Theorem 7.2.6]) are strongly self-absorbing.

In his work [131, 132, 129], Szabó establishes the theory of strongly self-absorbing

C*-dynamical systems as an equivariant version of strongly self-absorbing C*-algebras. We recall the main definition.

Definition 7.1.2. Let G be a locally compact group. A G action γ on a separable, unital C*-algebra $\mathcal{D}$ is called *strongly self-absorbing* if there exists an equivariant isomorphism $\varphi : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$ and unitaries $u_n \in U(\mathcal{D} \otimes \mathcal{D})$ fixed by $\gamma \otimes \gamma$ with

$$\lim_{n\to\infty} \|\varphi(a) - u_n(a \otimes 1_{\mathcal{D}})u_n^*\| = 0$$

for all $a \in \mathcal{D}$.

The relevant example of a strongly self-absorbing action for this section is μ_G. That μ_G is strongly self-absorbing follows as a consequence of [131, Example 5.1].

In [131, Theorem 3.7] Szabó shows equivalent conditions for a cocycle action to tensorially absorb a strongly self-absorbing action. Although Szabó's theory only treats the case of cocycle actions absorbing a given strongly self-absorbing group action, many of the arguments follow by the same methods when replacing cocycle actions by anomalous actions that may have non-trivial anomaly. The proofs of [131, Lemma 2.1, Theorem 2.6] and [131, Theorem 3.7, Corollary 3.8] for example, make no use of the anomaly associated to (α, u) and (β, w) being trivial. Under this observation, we can state a specific case of [131, Corollary 3.8].

Theorem 7.1.3 (cf. [131, Theorem 2.8])**.** *Let A and $\mathcal{D}$ be separable, unital C*-algebras and G a finite group. Assume $(\alpha, u) : G \curvearrowright A$ is an anomalous action. Let $\gamma : G \curvearrowright \mathcal{D}$ be a strongly self-absorbing group action. If there exists an equivariant and unital *-homomorphism*

$$(\mathcal{D}, \gamma) \to (A_\infty \cap A', \alpha_\infty),$$

then (A, α, u) is cocycle conjugate to $(A \otimes \mathcal{D}, \alpha \otimes \gamma, u \otimes 1_{\mathcal{D}})$ through a map $\varphi : A \to A \otimes \mathcal{D}$ that is approximately unitarily equivalent to the first factor embedding $\mathrm{id}_A \otimes 1_{\mathcal{D}}$.

We still require a few more results before we can achieve the model action absorption. These are based on known results in the setting of group actions. As the proofs are sufficiently short, we include them to clarify that the arguments also work for anomalous actions.

Lemma 7.1.4 (cf. [62, Theorem 3.3]). *Let A be a unital C*-algebra, G a finite group and (α, u) an anomalous action of G on A with the Rokhlin property. If B is a separable α-invariant C*-subalgebra of A_∞ containing A and there exists a unital homomorphism $M \to A_\infty \cap B'$ for some separable C*-algebra M, then there exists a unital homomorphism $M \to (A_\infty \cap B')^\alpha$.*

Proof. Fix a unital homomorphism $\psi : M \to A_\infty \cap B'$. By the Rokhlin property there exist projections $p_g \in A_\infty \cap A'$ such that $\sum_{g \in G} p_g = 1$ and $\alpha_g(p_h) = p_{gh}$. By a standard reindexing argument, one may additionally assume that each p_g commmute with both $\cup_{g \in G} \alpha_g(\psi(M))$ and B. Now consider the unital *-homomorphism $\varphi : M \to A_\infty \cap B'$ given by

$$\varphi(m) = \sum_{g \in G} \alpha_g(\psi(m)) p_g.$$

For $m \in M$ and $k \in G$

$$\begin{aligned} \alpha_k(\varphi(m)) &= \sum_{g \in G} \alpha_k\left(\alpha_g(\psi(m))\right) p_{kg} \\ &= \sum_{g \in G} \mathrm{Ad}(u_{k,g})(\alpha_{kg}(\psi(m))) p_{kg} \\ &= \varphi(m). \end{aligned}$$

In the last line we have used that each α_g is an automorphism of A_∞ and B is α invariant so α maps $A_\infty \cap B'$ into itself. Hence as $A \subset B$, we have that $A_\infty \cap B' \subset A_\infty \cap A'$ and any inner automorphism of A acts trivially on the image of $\alpha_g|_{A_\infty \cap B'}$ for any $g \in G$. Therefore, φ defines a unital *-homomorphism into $(A_\infty \cap B')^\alpha$. □

In the next Lemma we use $[a,b]$ to denote the additive commutator $ab-ba$. Also, recall that if α is a action of a group G on a unital C*-algebra A, an α-cocycle is a family of unitaries $v_g \in U(A)$ for $g \in G$ such that $v_g\alpha_g(v_h) = v_{gh}$.

Lemma 7.1.5 (cf. [60, Lemma III.1]). *Let A be a unital C*-algebra and G a finite group. Let (α, u) be an anomalous action of G on A with the Rokhlin property. Let B be a separable α-invariant C*-subalgebra of A_∞ containing A. For any α-cocycle v_g for the action induced by α on $A_\infty \cap B'$ there exists $U \in A_\infty \cap B'$ with $U^*\alpha_g(U) = v_g$.*

Proof. By the Rokhlin property there exists a family of projections $p_g \in A_\infty \cap A'$ such that $\sum_{g\in G} p_g = 1$ and $\alpha_g(p_h) = p_{gh}$ holds for all $g,h \in G$. As in the previous lemma, one may additionally ensure that $\|[p_g, v_h]\| = 0$ and $\|[p_g, b]\| = 0$ for all $g,h \in G$ and $b \in B$. Consider $U = \sum_{g\in G} v_g p_g \in A_\infty \cap B'$. Then $UU^* = 1 = U^*U$ and also

$$\begin{aligned}
U\alpha_g(U^*) &= \sum_{h,k} v_h p_h p_{gk} \alpha_g(v_k^*) \\
&= \sum_{k} p_{gk} v_{gk} \alpha_g(v_k^*) \\
&= \sum_{k} p_{gk} v_{gk} v_{gk}^* v_g \\
&= \sum_{k} v_g p_{gk} \\
&= v_g,
\end{aligned}$$

as required. □

The proof of the next lemma is based on the proof of [78, Proposition 3.4.1].

Lemma 7.1.6. *Let G be a finite group and (α, u) be an anomalous action with the Rokhlin property on a unital, separable C*-algebra A such that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$. Then there exists a G-equivariant unital embedding*

$$(\mathbb{M}_{|G|^\infty}, \mu_G) \to (A_\infty \cap A', \alpha).$$

Proof. To prove this we inductively construct equivariant *-homomorphisms ϕ_n : $(\mathcal{B}(l^2(G)), \mathrm{Ad}(\lambda_G)) \to (A_\infty \cap A', \alpha)$ for $n \in \mathbb{N}$ with commuting images. Then the map defined by $a_1 \otimes \cdots \otimes a_n \otimes \cdots \longmapsto \prod_{i\in\mathbb{N}} \phi_i(a_i)$ will induce a μ_G to α equivariant map into $A_\infty \cap A'$.

As $A \cong A \otimes \mathbb{M}_{|G|^\infty}$ there exists a unital embedding $\mathbb{M}_{|G|} \to A_\infty \cap A'$. Hence by Lemma 7.1.4 this induces a unital embedding $\mathbb{M}_{|G|} \to (A_\infty \cap A')^\alpha$, or equivalently a collection of matrix units $(e'_{g,h})_{g,h\in G}$ in $(A_\infty \cap A')^\alpha$. Consider the permutation unitary $v_g = \sum_h e'_{gh,h}$. This gives a unitary representation of G on $(A_\infty \cap A')^\alpha$. Indeed,

$$\begin{aligned} v_g v_k &= \sum_{h',h\in G} e'_{gh,h} e'_{kh',h'} \\ &= \sum_{h'\in G} e'_{gkh',h'} \\ &= v_{gk}. \end{aligned}$$

In particular v_g is an α-cocycle on $A_\infty \cap A'$. Therefore, by Lemma 7.1.5 there exists a unitary $u \in A_\infty \cap A'$ such that $u\alpha_g(u^*) = v_g$. Now $f_{g,h} = u^* e'_{g,h} u$ for $g, h \in G$ is a set of matrix units such that

$$\begin{aligned} \alpha_k(f_{g,h}) &= \alpha_k(u^*) e'_{g,h} \alpha_k(u) \\ &= u^* v_k e'_{g,h} v_k^* u \\ &= u^* \Big(\sum_{h',h''\in G} e'_{kh',h'} e'_{g,h} e'_{h'',kh''} \Big) u \\ &= u^*(e'_{kg,kh}) u \\ &= f_{kg,kh}. \end{aligned}$$

For $g, h \in G$, let $e_{g,h} \in \mathcal{B}(l^2(G))$ be defined by

$$e_{g,h}(f)(k) = \begin{cases} f(g) \text{ if } k = h, \\ 0 \text{ otherwise}, \end{cases}$$

for $f \in l^2(G)$ and $k \in G$. The elements $e_{g,h}$ form a set of matrix units for $\mathcal{B}(l^2(G))$. Define a *-homomorphism

$$\begin{aligned} \phi_1 : \mathcal{B}(l^2(G)) &\to A_\infty \cap A' \\ e_{g,h} &\mapsto f_{g,h}. \end{aligned}$$

It is straightforward that this defines an $\mathrm{Ad}(\lambda_G)$ to α equivariant *-homomorphisms.

Suppose $\phi_1, \phi_2, \ldots, \phi_n : (\mathcal{B}(l^2(G)), \mathrm{Ad}(\lambda_G)) \to (A_\infty \cap A', \alpha)$ are equivariant maps with commuting images. Pick a unital embedding of $\mathbb{M}_{|G|}$ into the C*-algebra $A_\infty \cap C^*(A, \mathrm{im}(\phi_1), \ldots, \mathrm{im}(\phi_n))'$ (As $A \otimes \mathbb{M}_{|G|^\infty} \cong A$ there exists a unital embedding of $\mathbb{M}_{|G|}$ into $A_\infty \cap A'$, moreover by reindexing one can assure that the image of this embedding also commutes with the images of ϕ_i for $1 \leq i \leq n$). As $\mathrm{im}(\phi_i)$ is G-invariant for all $1 \leq i \leq n$, it follows that α induces an action of G on $A_\infty \cap C^*(A, \mathrm{im}(\phi_1), \ldots \mathrm{im}(\phi_n))'$. By Lemma 7.1.4 there exists a unital embedding $\mathbb{M}_{|G|} \to (A_\infty \cap C^*(A, \mathrm{im}(\phi_1), \ldots \mathrm{im}(\phi_n))')^\alpha$. Repeating the argument for the case when $n = 1$ but replacing $A_\infty \cap A'$ by $A_\infty \cap C^*(A, \mathrm{im}(\phi_1), \ldots \mathrm{im}(\phi_n))'$ now yields a unital equivariant homomorphism

$$\phi_{n+1} : (B(l^2(G)), \mathrm{Ad}(\lambda_G)) \to (A_\infty \cap C^*(A, \mathrm{im}(\phi_1, \ldots \phi_n))', \alpha).$$

Considering ϕ_{n+1} as a unital equivariant homomorphism into $A_\infty \cap A'$ the induction argument is complete. □

We have collected all the necessary ingredients to prove the model action absorption.

Proposition 7.1.7. *Let G be a finite group and A a unital, separable C*-algebra such that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$. Let α be a (G, ω) action on A with the Rokhlin property. Then the anomalous actions (α, u) and $(\alpha \otimes \mu_G, u \otimes 1_{\mathbb{M}_{|G|^\infty}})$ are cocycle conjugate through an isomorphism that is approximately unitarily equivalent to $\mathrm{id}_A \otimes 1_{\mathbb{M}_{|G|^\infty}}$.*

Proof. By Lemma 7.1.6 there exists a G-equivariant unital embedding $(\mathbb{M}_{|G|^\infty}, \mu_G) \to (A_\infty \cap A', \alpha)$. Thus, by Theorem 7.1.3 the result follows. □

7.2 Classification

We now discuss the abstract approach to bootstrapping the classification of group actions on a given class of C*-algebras, to a classification of anomalous actions. This method is a generalisation of that used by Connes in [28, Section 6].

Before proceeding with the result, we set up notation. For a group G, we say "(α, u) is an anomalous G-action on A" and "(A, α, u) is an anomalous G-C*-algebra" interchangeably. Let F be a functor whose domain category is **C*alg**. We say F is *invariant under approximate unitary equivalence* if $F(\alpha) = F(\theta)$ whenever $\alpha \approx_{a.u} \theta$. We also say that F satisfies *existence of isomorphisms* restricted to a subcategory $\mathcal{C} \subset$ **C*alg**, if whenever $\Phi \in \mathrm{Hom}(F(A), F(B))$ is an isomorphism for $A, B \in \mathcal{C}$, then there exists an isomorphism $\varphi : A \to B$ in $\mathcal{C}$ with $F(\varphi) = \Phi$. The sort of functors with these properties are those used in the classification of C*-algebras. For example, the functor consisting of pointed K_0 and K_1 is invariant under approximate unitary equivalence, it also satisfies existence of isomorphisms when restricted to the category of unital Kirchberg algebras satisfying the UCT (see [109]). Similarly, the functor KT_u of Section 2.7 is invariant under approximate unitary equivalence and satisfies existence of isomorphisms when restricted to classifiable C*-algebras by Theorem 2.7.1.

If F is invariant under unitary equivalence, an anomalous action (A,α,u) induces a G-action on $F(A)$ through the automorphisms $F(\alpha_g)$. If (A,α,u) and (B,β,v) are anomalous actions, we say the induced actions $F(\alpha_g)$ and $F(\beta_g)$ are *conjugate* if there exists an isomorphism $\Phi : F(A) \to F(B)$ with $\Phi F(\alpha_g)\Phi^{-1} = F(\beta_g)$ for all $g \in G$. We denote this by $F(\alpha) \sim F(\beta)$.

Let (A,α,u) and (A,β,v) be two anomalous G-C*-algebras, we denote $(\alpha,u) \simeq_F (\beta,v)$ if $(\alpha,u) \simeq (\beta,v)$ through an automorphism θ with $F(\theta) = \mathrm{id}_{F(A)}$. This notion recovers K-trivial cocycle conjugacy of Section 3.4 when F is taken to be the functor consisting of $K_0 \oplus K_1$. Finally, if $\mathfrak{R}$ is a class of anomalous G-C*-algebras, we will say $\mathfrak{R}$ is *closed under conjugacy*, if whenever $(A,\alpha,u) \in \mathfrak{R}$ and $\varphi : A \to B$ is an isomorphism in **C*alg** then $(B,\varphi\alpha\varphi^{-1},\varphi(u)) \in \mathfrak{R}$.

Lemma 7.2.1. *Let G be a group, $\mathcal{D}$ a strongly self absorbing C*-algebra and $\mathfrak{R}$ a class of anomalous G-C*-algebras that is closed under conjugation. Let F be a functor with domain category the category of C*-algebras which is invariant under approximate unitary equivalence and satisfies existence of isomorphisms for C*-algebras in $\mathfrak{R}$. Suppose further that,*

(A1) there exists a G-action $(\mathcal{D},\mu_G,1)$ such that if $(A,\alpha,u) \in \mathfrak{R}$, then $(A,\alpha,u) \simeq (A\otimes\mathcal{D},\alpha\otimes\mu_G,u\otimes 1)$ through an automorphism that is approximately unitarily equivalent to $\mathrm{id}_A \otimes 1_{\mathcal{D}}$*;*

(A2) if there exists a (G,ω) action in $\mathfrak{R}$ for some $\omega \in Z^3(G,\mathbb{T})$, then there exist a (G,ω) and $(G,\overline{\omega})$ action $(\mathcal{D},s^{\omega}_G,u^{\omega})$ and $(\mathcal{D},s^{\overline{\omega}}_G,u^{\overline{\omega}})$ respectively such that $(\mathcal{D},s^{\overline{\omega}}_G,u^{\overline{\omega}}) \otimes (\mathcal{D},s^{\omega}_G,u^{\omega}) \simeq (\mathcal{D},\mu_G,1)$ and for any (G,ω)-action $(A,\alpha,u) \in \mathfrak{R}$, $(A,\alpha,u) \otimes (\mathcal{D},s^{\overline{\omega}}_G,u^{\overline{\omega}}) \in \mathfrak{R}$;

(A3) for cocycle actions $(A,\alpha,u),(B,\beta,v) \in \mathfrak{R}$ (i.e. $o(\alpha,u) = o(\beta,v) = 1$), $F(\alpha) \sim F(\beta)$ if and only if $\alpha \simeq \beta$.

Then, if (A,α,u) and (B,β,v) in $\mathfrak{R}$, $(A,\alpha,u) \simeq (B,\beta,v)$ if and only if $F(\alpha) \sim F(\beta)$ and $o(\alpha,u) = o(\beta,v)$.

With the same hypothesis but replacing (A3) with the condition that

(A3') for cocycle actions (A,α,u) and (A,β,v) in $\mathfrak{R}$, $(A,\alpha,u) \simeq_F (A,\beta,v)$ if and only if $F(\alpha_g) = F(\beta_g)$ for all $g \in G$,

then if (A,α,u) and (A,β,v) in $\mathfrak{R}$, $(A,\alpha,u) \simeq_F (A,\beta,v)$ if and only if $o(\alpha,u) = o(\beta,v)$ and $F(\alpha_g) = F(\beta_g)$ for every $g \in G$.

Proof. First we show that if (A1)-(A3) hold and (A,α,u), (B,β,v) are anomalous actions in $\mathfrak{R}$, then $(A,\alpha,u) \simeq (B,\beta,v)$ if and only if $F(\alpha) \sim F(\beta)$ and $o(\alpha,u) = o(\beta,v)$. If $(A,\alpha,u) \simeq (B,\beta,v)$, it is clear that $o(\alpha,u) = o(\beta,v)$ and also that $F(\alpha) \sim F(\beta)$ as F is trivial when evaluated at inner automorphisms. We now turn to the converse. Suppose $F(\alpha) \sim F(\beta)$ and $o(\alpha,u) = o(\beta,v)$. First note that this implies that also $F(\alpha \otimes \mathrm{id}_{\mathcal{D}}) \sim F(\beta \otimes \mathrm{id}_{\mathcal{D}})$. Indeed, by (A1) let $\phi_A : A \to A \otimes \mathcal{D}$ and $\phi_B : B \to B \otimes \mathcal{D}$ be isomorphisms which are approximately unitarily equivalent to the first factor embeddings and $\Phi : F(A) \to F(B)$ be an isomorphism such that $\Phi F(\alpha_g)\Phi^{-1} = F(\beta_g)$ for $g \in G$. Note $F(\alpha_g \otimes \mathrm{id}_{\mathcal{D}})F(\phi_A) = F(\alpha_g \otimes \mathrm{id}_{\mathcal{D}})F(\mathrm{id}_A \otimes 1_{\mathcal{D}}) = F(\alpha_g \otimes 1_{\mathcal{D}}) = F(\phi_A)F(\alpha_g)$ (and similarly replacing A by B). Hence we compute that

$$\begin{aligned}
F(\alpha_g \otimes \mathrm{id}_{\mathcal{D}})F(\phi_A)\Phi F(\phi_B)^{-1} &= F(\phi_A)F(\alpha_g)\Phi F(\phi_B)^{-1} \\
&= F(\phi_A)\Phi F(\beta_g)F(\phi_B)^{-1} \\
&= F(\phi_A)\Phi F(\phi_B)^{-1}F(\beta_g \otimes \mathrm{id}_{\mathcal{D}})
\end{aligned}$$

it follows that $F(\phi_B)\Phi F(\phi_A)^{-1}$ conjugates $F(\alpha_g \otimes \mathrm{id}_{\mathcal{D}})$ to $F(\beta_g \otimes \mathrm{id}_{\mathcal{D}})$ for all $g \in G$.

Now, by hypothesis we have that

$$\begin{aligned}(A,\alpha,u) &\overset{(A1)}{\simeq} (A\otimes\mathcal{D},\alpha\otimes\mu_G,u\otimes 1_{\mathcal{D}})\\ &\overset{(A2)}{\simeq} (A\otimes(\mathcal{D}\otimes\mathcal{D}),\alpha\otimes(\mu_G^{\overline{\omega}}\otimes\mu_G^{\omega}),u\otimes(u^{\overline{\omega}}\otimes u^{\omega}))\\ &= ((A\otimes\mathcal{D})\otimes\mathcal{D},(\alpha\otimes\mu_G^{\overline{\omega}})\otimes\mu_G^{\omega},(u\otimes u^{\overline{\omega}})\otimes u^{\omega})\\ &\overset{(A3),(A2)}{\simeq} ((B\otimes\mathcal{D})\otimes\mathcal{D},(\beta\otimes\mu_G^{\overline{\omega}})\otimes\mu_G^{\omega},(v\otimes u^{\overline{\omega}})\otimes u^{\omega})\\ &= (B\otimes(\mathcal{D}\otimes\mathcal{D}),\beta\otimes(\mu_G^{\overline{\omega}}\otimes\mu_G^{\omega}),v\otimes(u^{\overline{\omega}}\otimes u^{\omega}))\\ &\overset{(A2)}{\simeq} (B\otimes\mathcal{D},\beta\otimes\mu_G,v\otimes 1_{\mathcal{D}})\\ &\overset{(A1)}{\simeq} (B,\beta,v).\end{aligned} \tag{7.2.1}$$

Where in the third isomorphism we have used (A3) for the cocycle actions $(A\otimes\mathcal{D},\alpha_g\otimes\mu_G^{\overline{\omega}},u\otimes u^{\overline{\omega}})$ and $(B\otimes\mathcal{D},\beta_g\otimes\mu_G^{\overline{\omega}},v\otimes u^{\overline{\omega}})$. The reason we may apply (A3) in this setting is that $\mu_G^{\overline{\omega}}$ is approximately inner (every automorphism of an strongly self-absorbing C*-algebra is approximately inner) and hence our previous computation shows that $F(\alpha_g\otimes\mu_G^{\overline{\omega}})=F(\alpha_g\otimes\mathrm{id}_{\mathcal{D}})\sim F(\beta_g\otimes\mathrm{id}_{\mathcal{D}})=F(\beta_g\otimes\mu_G^{\overline{\omega}})$ as required for the application of (A3).

Now suppose that we replace condition (A3) with (A3'). We will show that under the hypothesis of the lemma, (A3') implies (A3). Therefore, the cocycle conjugacies in (7.2.1) still hold. Then we compute the isomorphisms that induce the cocycle conjugacies in (7.2.1) and show that their composition is the identity after applying F. Let (A,α,u) and (B,β,v) be cocycle actions in $\mathfrak{R}$. Suppose $F(\alpha)\sim F(\beta)$. There exists an isomorphism $\Phi\in\mathrm{Hom}(F(A),F(B))$ such that $\Phi F(\beta_g)\Phi^{-1}=F(\alpha_g)$ for all $g\in G$. By existence of isomorphisms for F, there exists a *-isomorphism $\varphi:B\to A$ with $F(\varphi)=\Phi$. Therefore $F(\varphi\beta_g\varphi^{-1})=F(\alpha_g)$ for all $g\in G$. By (A3') one has that $(A,\alpha,u)\simeq_F(A,\varphi\beta\varphi^{-1},\varphi(v))\simeq(B,\beta,v)$.

Set $A=B$ in (7.2.1). Reading from top to bottom in (7.2.1), denote by φ_1, φ_2, φ_3, φ_4 and φ_5 the isomorphisms inducing each of the conjugacies. Note that $\varphi_5=\varphi_1^{-1}$

and $\varphi_4 = \varphi_2^{-1}$. By (A1), $\varphi_1 \approx_{a.u} \mathrm{id}_A \otimes 1_{\mathcal{D}}$. Moreover, $\varphi_2 \approx_{a.u} \mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$ by ([142, Corollary 1.12]). Denote by φ the isomorphism inducing the cocycle conjugacy from $(A\otimes\mathcal{D}, \alpha\otimes\mu_G^{\overline{\omega}}, u\otimes u^{\overline{\omega}})$ to $(A\otimes\mathcal{D}, \beta\otimes\mu_G^{\overline{\omega}}, u\otimes u^{\overline{\omega}})$ which satisfies $F(\varphi) = F(\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}})$. We may use the functoriality of F and its invariance under approximate unitary equivalence to see that

$$\begin{aligned} F(\varphi_5\varphi_4\varphi_3\varphi_2\varphi_1) =& F(\mathrm{id}_A \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}})^{-1} F(\varphi \otimes \mathrm{id}_{\mathcal{D}}) F(\mathrm{id}_A \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}}) \\ &= F(\mathrm{id}_A \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}})^{-1} F(\mathrm{id}_A \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathrm{id}_{\mathcal{D}}) F(\mathrm{id}_A \otimes 1_{\mathcal{D}} \otimes 1_{\mathcal{D}}) \\ &= \mathrm{id}_{F(A)} . \end{aligned}$$

□

We now prove our classification theorems. Recall that two anomalous actions (α, u) and (β, v) on a C*-algebra A are K-trivially cocycle conjugate if they are cocycle conjugate and the automorphism φ performing the conjugacy satisfies $K_i(\varphi) = \mathrm{id}_{K_i(A)}$ for $i = 0, 1$. We denote this by $(\alpha, u) \simeq_K (\beta, v)$

Theorem 7.2.2. *Let G be a finite group and A be a unital Kirchberg algebra satisfying the UCT with $A \cong A \otimes \mathbb{M}_{|G|^\infty}$. If (α, u), (β, v) are anomalous actions of G on A with the Rokhlin property then $(\alpha, u) \simeq_K (\beta, v)$ if and only if $o(\alpha, u) = o(\beta, v)$ and $K_i(\alpha_g) = K_i(\beta_g)$ for all $g \in G$ and $i = 0, 1$.*

Proof. We check that the hypothesis of Lemma 7.2.1 is satisfied. Let $\mathcal{D} = \mathbb{M}_{|G|^\infty}$, F be the functor consisting of the pointed K_0 group direct sum the K_1 group and $\mathfrak{R}$ the class of Rokhlin anomalous G-actions on unital Kirchberg algebras satisfying the UCT. That F satisfies existence of isomorphisms follows from [109]. Condition (A1) follows from Proposition 7.1.7. For any $\omega \in Z^3(G, \mathbb{T})$, we have actions $(\mathcal{D}, s_G^\omega, u^\omega)$ by Theorem 6.2.3. That $(\mathcal{D}, s_G^{\overline{\omega}}, u^{\overline{\omega}}) \otimes (\mathcal{D}, s_G^\omega, u^\omega) \simeq (\mathcal{D}, \mu_G, 1)$ follows from [60, Theorem III.6] combined with [65, Lemma 3.12] as the actions $(\mathcal{D}, s_G^\omega, u^\omega)$ have the Rokhlin property (and hence property $\mathcal{R}_\infty$) by Proposition 6.2.4. Therefore, (A2) is also satisfied. Finally (A3') is satisfied by Izumi's classification result [67, Theorem 4.2] and that every cocycle action with the Rokhlin property is a unitary perturbation of a group action [67, Lemma 3.12]. □

Theorem 7.2.3. *Let G be a finite group and A be a unital, simple, nuclear TAF-algebra in the UCT class such that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$ and $(\alpha, u), (\beta, v)$ are anomalous actions on A with the Rokhlin property, then $(\alpha, u) \simeq_K (\beta, v)$ if and only if $o(\alpha, u) = o(\beta, v)$ and $K_i(\alpha_g) = K_i(\beta_g)$ for all $g \in G$.*

Proof. We apply Lemma 7.2.1 with $\mathcal{D} = \mathbb{M}_{|G|^\infty}$, $\mathfrak{R}$ the class of Rokhlin anomalous actions on $\mathbb{M}_{|G|^\infty}$-stable unital, simple, separable, nuclear TAF-algebras satisfying the UCT and F the functor consisting of ordered K_0 and K_1. Firstly, F satisfies existence of isomorphisms by [90]. (A1) holds by Proposition 7.1.7. (A2) holds for the same reason as in the proof of Theorem 7.2.2. Condition (A3') follows from a combination of [67, Theorem 4.3] and [66, Lemma 3.12]. □

More generally, we may use Szabó's unpublished classification result (Theorem 3.4.7) to classify Rokhlin anomalous actions of G on classifiable $\mathbb{M}_{|G|^\infty}$-stable C*-algebras.

Theorem 7.2.4. *Let G be a finite group. Let A and B be unital, simple, separable, nuclear, $\mathbb{M}_{|G|^\infty}$-stable C*-algebras satisfying the UCT and (α, u), (β, v) be anomalous G-actions with the Rokhlin property on A and B respectively. Then $(\alpha, u) \simeq (\beta, v)$ if and only if $KT_u(\alpha) \sim KT_u(\beta)$ and $o(\alpha, u) = o(\beta, v)$.*

Proof. We apply Lemma 7.2.1 with $\mathcal{D} = \mathbb{M}_{|G|^\infty}$, $\mathfrak{R}$ the class of Rokhlin anomalous actions on $\mathbb{M}_{|G|^\infty}$-stable unital, simple, separable, nuclear C*-algebras satisfying the UCT and $F = KT_u$. Firstly, F satisfies existence of isomorphisms by Theorem 2.7.1. (A1) holds by Proposition 7.1.7. (A2) holds as in the proof of Theorem 7.2.2. (A3) follows from a combination of Szabó's classification (Theorem 7.1.3) and [66, Lemma 3.12]. □

We have shown a classification of anomalous actions on some classes of simple C*-algebras by the anomaly and the induced action on K-theory. Such a classification also implies a classification of G-kernels, we illustrate it by using Theorem 7.2.4, the same argument may also be used to rewrite the results of Theorems 7.2.2 and 7.2.3.

Corollary 7.2.5. *Let A and B be unital, simple, separable, nuclear, $\mathbb{M}_{|G|^\infty}$-stable C*-algebras satisfying the UCT and $\overline{\alpha}$, $\overline{\beta}$ be G-kernels with the Rokhlin property on A and B respectively. Then $\overline{\alpha}$ and $\overline{\beta}$ are outer conjugate if and only if $KT_u(\overline{\alpha}) \sim KT_u(\overline{\beta})$ and* $\mathrm{ob}(\overline{\alpha}) = \mathrm{ob}(\overline{\beta})$.

Proof. The forward direction is clear. To show the reverse direction, pick lifts (α, u) of $\overline{\alpha}$ and (β, v) of $\overline{\beta}$ such that $o(\alpha, u) = o(\beta, v)$. This can be done as $\mathrm{ob}(\alpha, u) = \mathrm{ob}(\beta, v)$. As (α, u) and (β, v) satisfy the hypothesis of Theorem 7.2.4, it follows that $(\alpha, u) \simeq (\beta, v)$ and so $\overline{\alpha}$ and $\overline{\beta}$ are outer conjugate. □

For a finite group G, we may combine the result of Theorem 7.2.3 with Theorem 2.7.14 to achieve the classification of anomalous actions of G on unital, separable, simple, nuclear, unique trace $\mathbb{M}_{|G|^\infty}$-stable C*-algebras satisfying the UCT thus obtaining Theorem IV (see also the proof of Corollary 7.2.5 and Theorem 7.2.2).

As an application of Theorem 7.2.3, we get that for any finite group G and 3-cocycle $\omega \in Z^3(G, \mathbb{T})$ the actions s_G^ω of Theorem 6.2.3 and θ_G^ω of Proposition 6.3.2 are cocycle conjugate. Indeed, θ_G^ω has the Rokhlin property and similarly s_G^ω has the Rokhlin property by Proposition 6.2.4. Therefore, as θ_G^ω is a strict AF-action (see Definition 6.1.6) s_G^ω is an AF-action in the sense of [25] (see Remark 6.1.7).

We finish this chapter by studying to what extent Rokhlin anomalous actions on AF-algebras are AF-actions and vice versa. To do this, we will require results of [25]. In [25], the authors associate an invariant to any AF-action F, of a fusion category $\mathcal{C}$, on a AF-algebra A. Vaguely, this invariant consists of the K_0-groups of all Q-system extensions of A by F and all natural maps between these extensions. The authors also show that any two AF-actions on AF-algebras A and B are equivalent if and only if their invariants are isomorphic. As observed in [25, Section 5.1], if the acting category $\mathcal{C}$ is *torsion-free* (see [1, Definition 3.7]), the invariant of [25] simplifies to just the module structure of $K_0(A)$ under the action of the fusion ring of $\mathcal{C}$. We apply this when the acting category is $\mathbf{Hilb}(G, \omega)$ and the action is induced by an anomalous action (α, u) as explained in [46, Proposition 5.6]. The fusion ring of $\mathbf{Hilb}(G, \omega)$ is just $\mathbb{Z}[G]$ and the module structure of $K_0(A)$ is just given by $K_0(\alpha_g)$.

Corollary 7.2.6. *Let G be a finite group and A a simple, unital AF-algebra such that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$. Let (α, u) be a (G, ω)-action on A with $K_0(\alpha_g) = \mathrm{id}_A$ for all $g \in G$. If (α, u) has the Rokhlin property, then it is an AF-action. Moreover, if $[\omega|_H] \neq 0$ for any non-trivial subgroup $H < G$ then the converse holds.*

Proof. If (α, u) has the Rokhlin property, then by Theorem 7.2.3 it is cocycle conjugate to the AF ω-anomalous G-action $\mathrm{id}_A \otimes \theta_G^\omega$ on A. Therefore (α, u) is AF as (by definition) being AF is preserved under cocycle conjugacy (see Remark 6.1.7).

We now consider the converse statement. An AF ω-anomalous G action (α, u) induces an AF-action of the fusion category $\mathbf{Hilb}(G, \omega)$ in the sense of [25] (to see how a (G, ω)-action induces a $\mathbf{Hilb}(G, \omega)$ action see [46, Proposition 5.6], that this will be AF is discussed in Remark 6.1.7). By the hypothesis on ω, the fusion category $\mathbf{Hilb}(G, \omega)$ is torsion free, so as $K_0(\alpha_g) = \mathrm{id}_A$ and $K_0(\mathrm{id}_A \otimes \theta_G^\omega) = \mathrm{id}_A$, then [25, Theorem A] yields that the AF ω-anomalous G actions induced by (α, u) and $\mathrm{id}_A \otimes \theta_G^\omega$ are cocycle conjugate. As the Rokhlin property is preserved under cocycle conjugacy, (α, u) has the Rokhlin property. □

Remark 7.2.7. One may drop the hypothesis that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$ in Corollary 7.2.6 if one instead assumes that the H^3 class of the anomaly ω of (α, u) has order $|G|$.[1] Indeed, then A will automatically absorb $\mathbb{M}_{|G|^\infty}$ as a consequence of Corollary 5.4.4.

[1]If one assumes this, it is also automatic that $[\omega|_H] \neq 0$ for any subgroup $H < G$.

The behavior observed in the converse of Corollary 7.2.6 is quite different from the behaviour of group actions. It was already observed in [47] that there exist AF-actions of $\mathbb{Z}_2$ on $\mathbb{M}_{2^\infty}$ which do not have the Rokhlin property.

Chapter 8

Anomalous actions with the Rokhlin property

In this chapter, we aim to investigate the extent to which the structural properties and K-theoretical restrictions demonstrated in [66, 67] in the setting of cocycle actions of finite groups with the Rokhlin property also apply in the setting of finite group anomalous actions with the Rokhlin property.

In Section 8.1 we restrict our attention to anomalous actions of cyclic groups. Motivated by a notion studied by Connes for automorphisms of II_1 factors ([28]), we introduce automorphisms of minimal period for C^*-algebras with trivial centre. These automorphisms are particularly well behaved. For example, we are able to compute their fixed point algebras in certain settings. We show that every anomalous action of a cyclic group with the Rokhlin property is a unitary perturbation of an automorphism of minimal period. In the case that G is a cyclic group, this result generalises Izumi's proof that every cocycle action of G with the Rokhlin property is a unitary perturbation of a G-action ([66, Lemma 3.12]). Our computation of the fixed point algebras for automorphisms of minimal period allows us to show that the counterpart of a unitary cohomology vanishing result of Izumi does not hold in the setting of anomalous actions (see Proposition 8.2.9).

After discussing automorphisms of minimal period, we discuss the K-theoretical restrictions to the existence of Rokhlin anomalous actions. This is motivated by the work of Izumi in [66, 67] which we briefly recall in Section 8.2.1.

8.1 Automorphisms of minimal period

Fix a unital C*-algebra A with trivial centre. In this section we study automorphisms of minimal period as introduced by Connes in [28, Theorem 2.5]. In [28] Connes classifies finite order automorphisms of $\mathcal{R}$. Connes associates to any automorphism $\alpha \in \mathrm{Aut}(\mathcal{R})$ that is of finite order in the outer automorphism group, a pair of invariants $(p_0(\alpha), \gamma(\alpha))$. The value $p_0(\alpha)$ is defined to be the order of $\overline{\alpha}$ in $\mathrm{Out}(\mathcal{R})$. The invariant $\gamma(\alpha)$ is a $p_0(\alpha)$-th root of unity. The invariant $\gamma(\alpha)$ arises from taking a unitary u such that $\alpha^{p_0(\alpha)} = \mathrm{Ad}(u)$ and computing the multiplicative difference $\alpha(u)u^* = \gamma(\alpha)$. These invariants also make sense for automorphisms of arbitrary C*-algebra with trivial centre. In fact, Connes' root of unity computation is an instance of the computation performed in Section 3.2 by setting $G = \mathbb{Z}_{p_0(\alpha)}$ and considering the G-kernel

$$\begin{aligned} \mathbb{Z}_{p_0(\alpha)} &\to \mathrm{Out}(A) \\ k + p_0(\alpha)\mathbb{Z} &\mapsto \alpha^k + \mathrm{Inn}(A) \end{aligned}$$

for any $0 \leq k \leq p_0(\alpha) - 1$. The root of unity is associated to the G-kernel as an example of a H^3 invariant computation for G-kernels (as discussed in Section 3.3, there is a set of representative 3-cocycles for $H^3(\mathbb{Z}_{p_0(\alpha)}, \mathbb{T})$ that are uniquely determined by the $p_0(\alpha)$-th roots of unity).

For any root of unity γ, we will also denote by $o(\gamma)$ the order of γ in the circle. We may now recall the definition of an automorphism of minimal period.

Definition 8.1.1. (cf. [28, Theorem 2.5]) Let A be a C*-algebra with trivial centre and α be an automorphism of A. Then α is said to be of *minimal period* if α is an automorphism of finite period and its period is equal to $p_0(\alpha)o(\gamma(\alpha))$.

The reason for this terminology is that if $\alpha \in \mathrm{Aut}(A)$ is an automorphism of finite order, then it also has finite order as an element of $\mathrm{Out}(A)$. Therefore, there exists some unitary $u \in U(A)$ such that $\alpha^{p_0(\alpha)} = \mathrm{Ad}(u)$ and $\alpha(u) = \gamma(\alpha)u$. We want to consider the smallest number p such that α^p could be potentially equal id_A. Firstly, as $\overline{\alpha^p} = \overline{\mathrm{id}_A} \in \mathrm{Out}(A)$ we know that $p_0(\alpha)|p$. Therefore, there exists some $j \in \mathbb{N}$ such that $p = p_0(\alpha)j$. Evaluating $\alpha^{p_0(\alpha)j}$ at u we have $\alpha^{p_0(\alpha)j}(u) = \alpha^j\mathrm{Ad}(u)(u) = \alpha^j(u) = \gamma(\alpha)^j u$. Thus, if $\alpha^p = \mathrm{id}_A$, then p needs to be a multiple of $p_0(\alpha)o(\gamma(\alpha))$. The automorphism α is of minimal period precisely when the order of α is $p_0(\alpha)o(\gamma(\alpha))$.

We have seen a few examples of automorphisms of minimal period in previous sections.

Example 8.1.2. The automorphisms $s_n^\gamma \in \mathrm{Aut}(\mathbb{M}_{n^\infty})$ of Theorem 3.3.7 (see also Remark 3.3.8) are of minimal period for any $n \in \mathbb{N}$ and n-th root of unity γ. Indeed, fix $n \in \mathbb{N}$ and γ an n-th root of unity. Using the notation of Theorem 3.3.7 the automorphism s_n^γ is built such that there exists a unitary $u \in U(\mathbb{M}_{n^\infty})$ with $(s_n^\gamma)^n = \mathrm{Ad}(u)$, $s_n^\gamma(u) = \gamma u$ and $u^{o(\gamma)} = 1$. Also, by construction $p_0(s_n^\gamma) = n$ and $\gamma(s_n^\gamma) = \gamma$. Therefore, $(s_n^\gamma)^{o(\gamma(s_n^\gamma))p_0(s_n^\gamma)} = (s_n^\gamma)^{o(\gamma)n} = \mathrm{Ad}(u^{o(\gamma)}) = \mathrm{id}_{\mathbb{M}_{n^\infty}}$.

Example 8.1.3. Fix $n, m \in \mathbb{N}$ and $\gamma = e^{2\pi i m/n}$. In Proposition 6.4.7 we construct automorphisms α_m on the C*-algebra $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$ with σ_n the automorphism given by shifting the indexing set in the tensor product decomposition $\bigotimes_{i=1}^{n}\mathcal{O}_\infty \cong \mathcal{O}_\infty$. These automorphisms satisfy $\alpha_m^n = \mathrm{Ad}(v^m)$ and $\alpha_m(v^m) = \gamma v^m$ for v the canonical unitary coming from $1 + n\mathbb{Z}$ in $\mathcal{O}_\infty \rtimes_{\sigma_n} \mathbb{Z}_n$.

To compute Connes' invariants for α_m, notice that by Proposition 6.4.7 the anomalous action of $\mathbb{Z}_n$ induced by the pair α_m and v^m has the Rokhlin property. Therefore

α_m^j is outer for any $j < n$. Moreover, as $\alpha_m^n = \mathrm{Ad}(v^m)$, the invariant $p_0(\alpha_m) = n$. Also $\alpha_m(v^m) = \gamma v^m$ by construction, so $\gamma(\alpha_m) = \gamma$. We now check that $\alpha_m^{o(\gamma)n}$ is the identity. Note that $\alpha_m^n(v) = v$, so to check that $\alpha_m^{o(\gamma)n} = \mathrm{id}_{\mathcal{O}_\infty \rtimes \mathbb{Z}_n}$ it suffices to check whether the equality $\alpha_m^{o(\gamma)n}|_{\mathcal{O}_\infty} = \mathrm{id}_{\mathcal{O}_\infty}$ holds. As $\gamma = e^{2\pi i m/n}$ we have that $o(\gamma) = n/\gcd(m,n)$. Therefore, for $a \in \mathcal{O}_\infty$

$$\alpha_m^{o(\gamma)n}(a) = \sigma_{n^2}^{o(\gamma)nm}(a) = \sigma_{n^2}^{n^2(m/\gcd(m,n))}(a) = a. \qquad \square$$

The automorphisms of minimal period that we consider in Examples 8.1.2 and 8.1.3 come from anomalous actions of cyclic groups. If this is the case we will say that the anomalous action (α, u) of $\mathbb{Z}_n$ on A *arises* from an automorphism of minimal period (see Section 3.3.1 where we use the same terminology from cyclic anomalous actions coming from automorphisms of finite order in $\mathrm{Out}(A)$).

In [66, Lemma 3.12] Izumi shows that if (α, u) is a cocycle action with the Rokhlin property on A, then there is a unitary perturbation (α^s, u^s) such that α^s is a group action of G and $u^s_{g,h} = 1$ for all $g, h \in G$. In the remaining part of this subsection, we are interested in understanding to what extent the analogous statement is true for anomalous actions with the Rokhlin property.

First, we recall that up to replacing (α, u) by $(\alpha, \lambda u)$ for some circle valued 2-cochain λ and performing a unitary perturbation, every anomalous action is induced by an automorphism α such that $\alpha^n = \mathrm{Ad}(u)$ and $\alpha(u) = \gamma u$ for some unitary $u \in U(A)$ and n-th root of unity γ (see Proposition 3.3.6 and Corollary 3.5.5). If $\gamma = 1$, the pair (α, u) defines a cocycle action on A. In this case [66, Lemma 3.12] implies that there exists a unitary $V \in U(A)$ such that $\mathrm{Ad}(V)\alpha$ is an automorphism of order n. However, when γ is a non-trivial root of unity the automorphism $\mathrm{Ad}(V)\alpha$ can not be of order n. We show that, assuming the Rokhlin property, one may choose V such that $\mathrm{Ad}(V)\alpha$ is of minimal period. Before we do this, we need a few lemmas.

The first is a technical lemma that we will require in the proof of Lemma 8.1.5. For $l > 0$ and $\lambda \in \mathbb{C}$ we denote by $B_l(\lambda)$ the open ball of radius l centred around λ.

Lemma 8.1.4. *Let A be a unital C^*-algebra, $n \in \mathbb{N}$, $\alpha \in \mathrm{Aut}(A)$, $u \in U(A)$ and γ an n-th root of unity such that*

$$\alpha^n = \mathrm{Ad}(u),$$
$$\alpha(u) = \gamma u$$

and the induced $\mathbb{Z}_n$ anomalous action (α, u) has the Rokhlin property. If there exists some $0 < l < \pi/o(\gamma)$ with $\sigma(u) \subset \bigcup_{1 \leq j \leq o(\gamma)} B_l(\gamma^j)$, then for any $\varepsilon > 0$ there exists $v \in U(A)$ satisfying

$$\|(v\alpha(v)\dots\alpha^{n-1}(v)u)^{o(\gamma)} - 1\| \leq \varepsilon, \tag{8.1.1}$$

$$\|v - 1\| \leq \varepsilon + o(\gamma)l, \tag{8.1.2}$$

$$\sigma(v\alpha(v)\dots\alpha^{n-1}(v)u) \subset \bigcup_{1 \leq j \leq o(\gamma)} B_\varepsilon(\gamma^j). \tag{8.1.3}$$

Proof. By assumption, the unitary u has spectral gaps at the midpoints of $\mathbb{T}$ between neighbouring $o(\gamma)$-th roots of unity. Therefore, the characteristic functions with respect to the arcs $B_{\pi/o(\gamma)}(\gamma^j) \cap \mathbb{T}$ for every $1 \leq j \leq o(\gamma)$ are continuous when restricted to $\sigma(u)$. Let $p_j = \chi_{B_{\pi/o(\gamma)}(\gamma^j)}$ for $1 \leq j \leq o(\gamma)$ through functional calculus. The projections p_j are pairwise orthogonal and

$$\sum_{j=1}^{o(\gamma)} p_j = 1. \tag{8.1.4}$$

Let $e_0, e_1 \dots e_{n-1} \in A_\infty \cap A'$ be a family of Rokhlin projections for (α, u) and $r = \sum_{k=1}^{o(\gamma)} \gamma^k p_k \in U(A)$. Set

$$s = ru^* e_0 + (1 - e_0) \in U(A_\infty). \tag{8.1.5}$$

Then using that the e_i are pairwise orthogonal and commute with A we have that

$$s\alpha(s)\dots\alpha^{n-1}(s) = (ru^*e_0 + \sum_{\substack{j=0\\ j\neq 0}}^{n-1} e_j)(\gamma^{-1}\alpha(r)u^*e_1 + \sum_{\substack{j=0\\ j\neq 1}}^{n-1} e_j)\dots(\gamma^{1-n}\alpha^{n-1}(r)u^*e_{n-1} + \sum_{j=0}^{n-2} e_j)$$
$$= \sum_{j=0}^{n-1} \gamma^{-j}\alpha^j(r)u^*e_j.$$

As $r^{o(\gamma)} = 1$, it follows that

$$(s\alpha(s)\dots\alpha^{n-1}(s)u)^{o(\gamma)} = 1. \tag{8.1.6}$$

By the spectral mapping theorem, it follows that the spectrum of $s\alpha(s)\dots\alpha^{n-1}(s)u$ is contained in the $o(\gamma)$-th roots of unity. Moreover,

$$\begin{aligned}
\|s-1\| &= \|(\sum_{k=1}^{o(\gamma)} \gamma^k p_k)u^*e_0 - e_0\| \\
&\leq \|(\sum_{k=1}^{o(\gamma)} \gamma^k p_k)u^* - 1\| \\
&\leq \sum_{k=1}^{o(\gamma)} \|\gamma^k p_k - up_k\| \qquad (8.1.7)\\
&= \sum_{k=1}^{o(\gamma)} \sup_{z\in B_l(\gamma^k)\cap\mathbb{T}} \|\gamma^k - z\| \\
&\leq o(\gamma)l.
\end{aligned}$$

The second last equality holds as p_k are the spectral projections of u onto the subset of the spectrum contained in $B_l(\gamma^k)$. As s is a unitary in A_∞, one may use Lemma 2.6.6 to write $s = (s_n)$ for a sequence of unitaries $s_n \in U(A)$. As conditions (8.1.6) and (8.1.7) hold for s and $\sigma(s)$ is contained in the set of $o(\gamma)$-th roots of unity, then for any $\varepsilon > 0$ one may pick n large enough such that

$$\|(s_n\alpha(s_n)\dots\alpha^{n-1}(s_n)u)^{o(\gamma)} - 1\| \leq \varepsilon,$$

$$\|s_n - 1\| \leq \varepsilon + o(\gamma)l,$$

$$\sigma(s_n\alpha(s_n)\dots\alpha^{n-1}(s_n)u) \subset \bigcup_{1\leq j\leq o(\gamma)} B_\varepsilon(\gamma^j)$$

as required. □

Lemma 8.1.5. *Let A be a unital C^*-algebra, $n \in \mathbb{N}$, $\alpha \in \mathrm{Aut}(A)$, $u \in U(A)$ and γ an n-th root of unity such that*

$$\alpha^n = \mathrm{Ad}(u),$$

$$\alpha(u) = \gamma u.$$

If the induced $\mathbb{Z}_n$ anomalous action (α, u) has the Rokhlin property, then there is a unitary $W \in U(A)$ such that $(\mathrm{Ad}(W)\alpha)^{o(\gamma)n} = \mathrm{id}_A$.

Proof. Let (α, u) be as in the hypothesis and $e_0, \dots, e_{n-1} \in A_\infty \cap A'$ be Rokhlin projections for α. First, notice that if we replace α by $\mathrm{Ad}(r)\alpha$, the n-th power $(\mathrm{Ad}(r)\alpha)^n = \mathrm{Ad}(r\alpha(r)\dots\alpha^{n-1}(r)u)$ and $\mathrm{Ad}(r)\alpha(r\alpha(r)\dots\alpha^{n-1}(r)u) = \gamma r\alpha(r)\dots\alpha^{n-1}(r)u$. Moreover, the projections e_i for $0 \leq i \leq n-1$ are also Rokhlin projections for the anomalous action defined by the pair $(\mathrm{Ad}(r)\alpha, r\alpha(r)\dots\alpha^{n-1}(r)u)$. Therefore, for any $r \in U(A)$, the pair $(\mathrm{Ad}(r)\alpha, r\alpha(r)\dots\alpha^{n-1}(r)u)$ still satisfy the hypothesis of the Lemma. We will use this observation regularly throughout this proof.

We now proceed with the proof of the lemma. The strategy is similar to the proof of [66, Lemma 3.12]. We build a sequence of unitaries $v_m \in U(A)$ such that $\lim\limits_{m\to\infty} v_m v_{m-1}\dots v_1$ converges to a unitary $W \in U(A)$ and $(\mathrm{Ad}(W)\alpha)^{o(\gamma)n} = \mathrm{id}_A$. We first set

$$s' = u^* e_0 + (1 - e_0) \in U(A_\infty).$$

Exactly as in the computation for the unitary defined in (8.1.5), it is clear that

$$(s'\alpha(s') \dots \alpha^{n-1}(s')u)^{o(\gamma)} = 1.$$

As $s' \in U(A_\infty)$ by Lemma 2.6.6 one may pick a sequence of unitaries $s'_n \in U(A)$ such that $s = (s'_n)$. Choosing $n \in \mathbb{N}$ sufficiently large, one can find some unitary $r = s'_n \in U(A)$ with the difference between 1 and $(r\alpha(r) \dots \alpha^{n-1}(r)u)^{o(\gamma)}$ being arbitrarily small. So for any $1 > \delta > 0$ there exists $r_\delta \in U(A)$ such that

$$\|(r_\delta \alpha(r_\delta) \dots \alpha^{n-1}(r_\delta)u)^{o(\gamma)} - 1\| < \delta. \tag{8.1.8}$$

As the spectral radius of a normal element is bounded by its norm, (8.1.8) implies that $\sigma((r_\delta \alpha(r_\delta) \dots \alpha^{n-1}(r_\delta)u)^{o(\gamma)}) \subset B_\delta(1) \cap \mathbb{T}$ and by the spectral mapping theorem

$$\begin{aligned}\sigma(r_\delta \alpha(r_\delta) \dots \alpha^{n-1}(r_\delta)u)^{o(\gamma)} &= \sigma((r_\delta \alpha(r_\delta) \dots \alpha^{n-1}(r_\delta)u)^{o(\gamma)}) \\ &\subset B_\delta(1) \cap \mathbb{T}.\end{aligned}$$

We may hence pick δ sufficiently small to ensure that

$$\sigma(r_\delta \alpha(r_\delta) \dots \alpha^{n-1}(r_\delta)u) \subset \cup_{j=1}^{o(\gamma)} B_{\pi/2o(\gamma)}(\gamma^j) \cap \mathbb{T}. \tag{8.1.9}$$

Set $\alpha_1 = \alpha$, $u_1 = u$ and $v_1 = r_\delta$. Subsequently, let $\alpha_2 = \mathrm{Ad}(v_1)\alpha_1$ and $u_2 = v_1 \alpha(v_1) \dots \alpha^{n-1}(v_1)u_1$. By (8.1.9) and the discussion at the beginning of this proof, the pair (α_2, u_2) satisfies the hypothesis of Lemma 8.1.4. Therefore, there exists

$v_2 \in U(A)$ such that

$$\|(v_2\alpha_2(v_2)\dots\alpha_2^{n-1}(v_2)u_2)^{o(\gamma)} - 1\| \leq \frac{\pi}{o(\gamma)4},$$

$$\|v_2 - 1\| \leq \frac{\pi}{4o(\gamma)} + \frac{\pi}{2},$$

$$\sigma(v_2\alpha_2(v_2)\dots\alpha_2^{n-1}(v_2)u_2) \subset \bigcup_{1\leq j\leq o(\gamma)} B_{\pi/4o(\gamma)}(\gamma^j).$$

Suppose for $m \geq 2$ there is $\alpha_m \in \mathrm{Aut}(A)$, $u_m \in U(A)$ such that $\alpha_m^n = \mathrm{Ad}(u_m)$, $\alpha_m(u_m) = \gamma u_m$ and the anomalous actions (α_m, u_m) has the Rokhlin property and v_m such that

$$\|(v_m\alpha_m(v_m)\dots\alpha_m^{n-1}(v_m)u_m)^{o(\gamma)} - 1\| \leq \frac{\pi}{o(\gamma)2^m}, \tag{8.1.10}$$

$$\|v_m - 1\| \leq \frac{\pi}{2^m o(\gamma)} + \frac{\pi}{2^{m-1}}, \tag{8.1.11}$$

$$\sigma(v_m\alpha_m(v_m)\dots\alpha_m^{n-1}(v_m)u_m) \subset \bigcup_{1\leq j\leq o(\gamma)} B_{\pi/2^m o(\gamma)}(\gamma^j). \tag{8.1.12}$$

Set $\alpha_{m+1} = \mathrm{Ad}(v_m)\alpha_m$ and $u_m = v_m\alpha_m(v_m)\dots\alpha_m^{n-1}(v_m)u_m$. Applying Lemma 8.1.4 we there exists $v_{m+1} \in U(A)$ such that

$$\|(v_{m+1}\alpha_{m+1}(v_{m+1})\dots\alpha^{n-1}(v_{m+1})u_{m+1})^{o(\gamma)} - 1\| \leq \frac{\pi}{o(\gamma)2^{m+1}},$$

$$\|v_{m+1} - 1\| \leq \frac{\pi}{2^{m+1}o(\gamma)} + \frac{\pi}{2^m},$$

$$\sigma(v_{m+1}\alpha_{m+1}(v_{m+1})\dots\alpha_{m+1}^{n-1}(v_{m+1})u_{m+1}) \subset \bigcup_{1\leq j\leq o(\gamma)} B_{\pi/2^{m+1}o(\gamma)}(\gamma^j).$$

Continue this construction inductively. Let $w_m = v_m v_{m-1}\dots v_1$ then by (8.1.11), for any $m \geq 2$,

$$\|w_{m+1} - w_m\| \leq \|v_m - 1\| \leq \frac{\pi}{2^{m+1}o(\gamma)} + \frac{\pi}{2^m}.$$

The bound achieved for $\|w_{m+1} - w_m\|$ constitutes a summable sequence and hence

(w_m) is Cauchy. Therefore w_m converges to a unitary $\lim_{m\to\infty} w_m = W$. Moreover, for any $a \in A$, $(\mathrm{Ad}(W)\alpha)^{o(\gamma)n}(a) = \mathrm{Ad}((W\alpha(W)\dots\alpha^{n-1}(W)u)^{o(\gamma)})a = a$ by (8.1.10). ☐

First, we consider an immediate corollary of Lemma 8.1.5 for G-kernels of cyclic groups.

Corollary 8.1.6. *Let $n \in \mathbb{N}$, A be a unital C^*-algebra with trivial centre and $\overline{\theta} : \mathbb{Z}_n \to \mathrm{Out}(A)$ be a $\mathbb{Z}_n$-kernel with the Rokhlin property. Let $k|n$ be the order of the H^3 invariant of $\overline{\theta}$ and $\pi : \mathbb{Z}_{nk} \to \mathbb{Z}_n$ the canonical surjection. The $\mathbb{Z}_{nk}$ kernel $\pi^*\overline{\theta}$ lifts to an action of $\mathbb{Z}_{nk}$ on A.*

Proof. Let θ be an automorphism lifting $\overline{\theta}_{1+n\mathbb{Z}}$ then θ^j lifts $\overline{\theta}_{j+n\mathbb{Z}}$ for $0 \leq j \leq n-1$. As $\overline{\theta}$ is a $\mathbb{Z}_n$-kernel with H^3 invariant of order k, there exists a unitary u such that $\theta^n = \mathrm{Ad}(u)$ and a k-th root of unity γ with $\theta(u) = \gamma u$. By Lemma 8.1.5 there exists a unitary $W \in U(A)$ such that $(\mathrm{Ad}(W)\theta)^{kn} = \mathrm{id}_A$. The automorphism $\mathrm{Ad}(W)\theta$ also lifts $\overline{\theta}_{1+n\mathbb{Z}} = \pi^*\overline{\theta}_{1+nk\mathbb{Z}}$ and so $(\mathrm{Ad}(W)\theta)^j$ lifts $\pi^*\overline{\theta}_{j+nk\mathbb{Z}}$ for all $1 \leq j \leq nk-1$. So $j + kn\mathbb{Z} \mapsto (\mathrm{Ad}(W)\theta)^j$ is a $\mathbb{Z}_{kn}$ action lifting $\pi^*\overline{\theta}$. ☐

We now rephrase Corollary 8.1.6 to the language of anomalous actions. For anomalous actions we also need to keep track of the unitaries. This makes the argument more tedious. Recall that if (α, u) is an anomalous action of a group G on A and $v_g \in U(A)$ for $g \in G$, the unitary perturbation (α^v, u^v) is defined by $\alpha^v_g = \mathrm{Ad}(v_g)\alpha_g$ and $u^v_{g,h} = v_g\alpha_g(v_h)u_{g,h}v^*_{gh}$.

Corollary 8.1.7. *Let $n \in \mathbb{N}$ and $\omega \in Z^3(\mathbb{Z}_n, \mathbb{T})$. Let (α, u) be a $(\mathbb{Z}_n, \omega)$ action with the Rokhlin property on a unital C^*-algebra A with trivial centre. Up to potentially replacing $u_{g,h}$ with $\lambda_{g,h}u_{g,h}$ for $g, h \in \mathbb{Z}_n$ with some 2-cochain $\lambda : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T}$, there exists a family of unitaries $v_g \in U(A)$ for $g \in \mathbb{Z}_n$ such that (α^v, u^v) arises from an automorphism of minimal period.*

Proof. Let (α, u) be a $(\mathbb{Z}_n, \omega)$ action on A. By replacing u by λu for some 2-cochain $\lambda : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{T}$ we may assume that (α, u) is a $(\mathbb{Z}_n, \lambda_k)$ anomalous action for some $0 \leq k \leq n-1$, where λ_k is the 3-cocycle from (3.3.3) (see Remark 3.3.3). Therefore, by Corollary 3.5.5 some unitary perturbation of (α, u) arises from an automorphism. Namely, there exists a pair $\beta \in \mathrm{Aut}(A)$, $v \in U(A)$ and unitaries s_g for $g \in \mathbb{Z}_n$ such that $\beta^n = \mathrm{Ad}(v)$, $\beta(v) = e^{2\pi i k/n} v$ and $(\alpha^s, u^s) = (\beta, v)$ (here we have denoted by (β, v) the anomalous action induced by the automorphism β and unitary v, see Section 3.3.1). As (α, u) has the Rokhlin property, and the Rokhlin property is preserved by unitary perturbations, (β, v) has the Rokhlin property. By Lemma 8.1.5 there exists $W \in U(A)$ such that $(\mathrm{Ad}(W)\beta)^{\gcd(k,n)n} = \mathrm{id}_A$. Let

$$r_{k+n\mathbb{Z}} = W\beta(W)\ldots\beta^{k-1}(W)$$

for any $0 \leq k \leq n-1$. It follows from a straightforward computation that (β^r, u^r) arises from the automorphism $\mathrm{Ad}(W)\beta$. Therefore, (α^{rs}, u^{rs}) arises from the automorphism $\mathrm{Ad}(W)\beta$. To complete the proof it suffices to check that $\mathrm{Ad}(W)\beta$ is of minimal period. We know that $(\mathrm{Ad}(W)\beta)^{\gcd(k,n)n} = \mathrm{id}_A$. Therefore, $\mathrm{Ad}(W)\beta$ is of minimal period if and only if $\gcd(k,n)n = o(\gamma(\mathrm{Ad}(W)\beta))p_0(\mathrm{Ad}(W)\beta)$. Since β has the $\mathbb{Z}_n$ Rokhlin property, β^j is not an inner automorphism for any $1 \leq j < n$. Moreover, $\beta^n \in \mathrm{Inn}(A)$. So $p_0(\beta) = p_0(\mathrm{Ad}(W)\beta) = n$. Also, as the n-th root of unity invariant of Connes is preserved under unitary conjugation $o(\gamma(\mathrm{Ad}(W)\beta)) = o(e^{2\pi i k/n}) = \gcd(k,n)$. □

I suspect that Corollary 8.1.6 holds in greater generality. This brings me to pose the following problem.

Problem 8.1.8. *Let G be a finite group, A be a unital C^*-algebra with trivial centre and $\overline{\theta} : G \to \mathrm{Out}(A)$ be a G-kernel on A with the Rokhlin property and H^3 invariant $[\omega]$. If $\pi : \Gamma \twoheadrightarrow G$ is a surjection with $\pi^*[\omega] = 1$ does $\pi^*\overline{\theta}$ lift to a Γ action on A?*

8.1.1 The fixed point algebra of an automorphism of minimal period.

We now discuss the fixed point algebras for automorphisms of minimal period. These computations will be relevant in later sections (see Proposition 8.2.9). Through studying fixed point algebras of minimal period we will be able compute the fixed point algebras of the automorphisms $s_n^\gamma \in \mathrm{Aut}(\mathbb{M}_{n^\infty})$ introduced in Section 3.3.

Lemma 8.1.9. *Let A be a unital C^*-algebra with trivial centre and $\alpha \in \mathrm{Aut}(A)$ be an automorphism of minimal period. Then there exists a partition of unity consisting of projections $r_0, r_1, \ldots, r_{o(\gamma(\alpha))-1} \in P(A)$ such that*

(i) $\alpha^{p_0(\alpha)} = \mathrm{Ad}\left(\sum_{j=0}^{o(\gamma(\alpha))-1} \gamma(\alpha)^j r_j\right)$,

(ii) $\alpha(r_i) = r_{i-1}$, *with the addition understood modulo* $o(\gamma(\alpha))$,

(iii) The map

$$\mathrm{id}_A \oplus \alpha \oplus \alpha^2 \cdots \oplus \alpha^{o(\gamma)-1} : (r_0 A r_0)^{\alpha^{o(\gamma(\alpha))}} \longrightarrow A^\alpha$$

is an isomorphism.

Proof. To simplify the notation of the proof we let $n = p_0(\alpha)$, $k = o(\gamma(\alpha))$ and $\gamma = \gamma(\alpha)$. We start by showing (i). By assumption there exists a unitary $u \in U(A)$ such that $\alpha^n = \mathrm{Ad}(u)$ and $\alpha(u) = \gamma u$. As α is of minimal period, we have that $\mathrm{id}_A = \alpha^{nk} = \mathrm{Ad}(u^k)$ and so u^k is contained in the centre of A. As A has trivial centre, it follows that $u^k = \lambda$ for some $\lambda \in \mathbb{T}$. Therefore, by replacing u by $\lambda^{-k}u$, we may assume that $u^k = 1$. We claim that the spectrum $\sigma(u)$ coincides with the k-th roots of unity. This claim combined with the spectral theorem will imply (i).

If $\lambda \in \sigma(u)$ as $u^k = 1$ then $\lambda^k = 1$. Therefore, λ is an k-th root of unity. We now show that the spectrum contains all k-th roots of unity. The spectrum is a non-empty subset of the complex plane, so there exists some k-th root of unity ω

such that $u - \omega$ is non-invertible. Therefore, $\alpha^j(u-\omega) = \gamma^j u - \omega$ is non-invertible and so $u - \gamma^{-j}\omega$ is non-invertible for any $j \in \mathbb{N}$. As the action of the group of k-th roots of unity on itself is transitive, all the k-th roots of unity are contained in $\sigma(u)$. Letting $r_0, r_1, \ldots, r_{k-1} \in P(A)$ be the spectral projections with respect to $1, \gamma, \ldots, \gamma^{k-1}$ respectively, we have $u = \sum_{j=1}^{k-1} \gamma^j r_j$.

Equation (ii) follows as

$$\sum_{j=0}^{k-1} \gamma^j \alpha(r_j) = \alpha(u) = \gamma u = \sum_{j=0}^{k-1} \gamma^{j+1} r_j.$$

Therefore the projections $\alpha(r_j) = r_{j-1}$ for $0 \leq j \leq k$.

We now turn to (iii). Firstly, note that if $a \in A^\alpha$ then $\alpha^n(a) = \mathrm{Ad}(u)(a) = a$ so we have the containment $A^\alpha \subset A^{\mathrm{Ad}(u)}$. Moreover, if $a \in A^{\mathrm{Ad}(u)}$ then $\mathrm{Ad}(u)\alpha(a) = \alpha(\mathrm{Ad}(u)(a)) = \alpha(a)$ as $\alpha(u) = \gamma u$. Therefore, α restricts to an automorphism of $A^{\mathrm{Ad}(u)}$ and we have an equality $A^\alpha = (A^{\mathrm{Ad}(u)})^\alpha$.

We will start by computing the fixed point algebra $A^{\mathrm{Ad}(u)}$. The fixed point algebra $A^{\mathrm{Ad}(u)}$ is given by elements in A commuting with u. Namely, it coincides with the algebra $C^*(u)' \cap A$. By the spectral theorem $C^*(u) = C^*(r_0, r_1, \ldots, r_{k-1})$ and so $A^{\mathrm{Ad}(u)} = C^*(r_0, r_1, \ldots, r_{k-1})' = r_0Ar_0 \oplus r_1Ar_1 \oplus \cdots \oplus r_{k-1}Ar_{k-1}$. We may now use (ii) to compute that $A^\alpha = (A^{\mathrm{Ad}(u)})^\alpha = \{a \oplus \alpha(a) \oplus \cdots \oplus \alpha^{k-1}(a) | a \in (r_0Ar_0)^{\alpha^k}\}$. Thus the map

$$\mathrm{id}_A \oplus \alpha \oplus \alpha^2 \oplus \cdots \oplus \alpha^{k-1} : (r_0Ar_0)^{\alpha^k} \longrightarrow A^\alpha$$

is an isomorphism of C*-algebras. □

If α is an automorphism of minimal period such that its associated root of unity $\gamma(\alpha)$ is a primitive $p_0(\alpha)$-th root of unity, the fixed point algebra of α is a corner in A.

Corollary 8.1.10. *Let A be a unital C^*-algebra with trivial centre and $\alpha \in \mathrm{Aut}(A)$ be an automorphism of minimal period such that $p_0(\alpha) = o(\gamma(\alpha))$. Then there exists a projection $r \in P(A)$ such that $rAr \cong A^\alpha$. In particular, if A is simple, then $K_0(A) \cong K_0(A^\alpha)$.*

Proof. The first part follows immediately from Lemma 8.1.9 (iii) as

$$\alpha^{o(\gamma(\alpha))} = \alpha^{p_0(\alpha)} = \mathrm{Ad}\Bigg(\sum_{j=0}^{o(\gamma(\alpha))-1} \gamma(\alpha)^j r_j\Bigg)$$

fixes any element in r_0Ar_0. Denote by $\varphi : r_0Ar_0 \to A^\alpha$ an isomorphism. When A is simple, r_0 is a full projection. By Proposition 2.6.4 the map $K_0(\iota) : K_0(r_0Ar_0) \to K_0(A)$ induced by the inclusion $\iota : r_0Ar_0 \to A$ is an isomorphism. Therefore, $K_0(\varphi)K_0(\iota)^{-1}$ is an isomorphism from $K_0(A)$ to $K_0(A^\alpha)$. □

Even in the case that $p_0(\alpha) \neq o(\gamma(\alpha))$, Lemma 8.1.9 can be useful to compute the fixed point algebra of automorphisms of minimal period. We show an instance of this for UHF algebras.

Theorem 8.1.11. *Let A be a UHF algebra and $\alpha \in \mathrm{Aut}(A)$ an automorphism of minimal period. If the induced $\mathbb{Z}_{p_0(\alpha)}$-kernel $\overline{\alpha}$ has the Rokhlin property then $A^\alpha \cong A$.*

Proof. We let $n = p_0(\alpha)$, $k = o(\gamma(\alpha))$ and $\gamma = \gamma(\alpha)$. Firstly, by Lemma 8.1.9 there is a partition of unity r_j for $0 \leq j \leq k-1$ such that $\alpha^n = \mathrm{Ad}(\sum_{j=0}^{k-1} \gamma^j r_j)$ and $A^\alpha \cong (r_0Ar_0)^{\alpha^k}$. It suffices to show that $A \cong (r_0Ar_0)^{\alpha^k}$.

First we show that $A \cong r_0Ar_0$. Every automorphism of a UHF algebra is approximately inner, so by Lemma 8.1.9 (ii) the projections r_j for $0 \leq j \leq k-1$ are Murray von Neumann equivalent. Therefore, $\tau(r_j) = 1/k$ for every $0 \leq j \leq k-1$. The $\mathbb{Z}_n$-kernel induced by α has anomaly of order k, so by Theorem 4.2.10 the supernatural number $\mathfrak{n}$ of A is formally divided by k^∞. In particular, r_0Ar_0 is a UHF algebra with supernatural number $k^{-1}\mathfrak{n} = \mathfrak{n}$. So $A \cong r_0Ar_0$ by Glimm's classification ([52]).

We now show that $r_0Ar_0 \cong (r_0Ar_0)^{\alpha^k}$. For this we make use of the Rokhlin property. As $\overline{\alpha}$ is a $\mathbb{Z}_n$-kernel with the Rokhlin property, there is a projection $q \in A_\infty \cap A'$ such that

$$\sum_{j=0}^{n-1} \alpha^j(q) = 1.$$

The automorphism $\alpha^k|_{r_0Ar_0}$ satisfies $(\alpha^k|_{r_0Ar_0})^{n/k} = \mathrm{Ad}(\sum_{j=0}^{k-1} \gamma^j r_j)|_{r_0Ar_0} = \mathrm{id}_A$. Therefore, the automorphism $\alpha^k|_{r_0Ar_0}$ is at most of order n/k. Consider the $\mathbb{Z}_{n/k}$ action on r_0Ar_0 given by $j + \frac{n}{k}\mathbb{Z} \mapsto (\alpha^k|_{r_0Ar_0})^j$ for $0 \leq j < n/k$. This group action inherits the Rokhlin property from $\overline{\alpha}$. To see this consider the projection

$$q' = \sum_{j=0}^{k-1} \alpha^j(q) r_0$$

contained in $(r_0Ar_0)_\infty \cap (r_0Ar_0)'$. This projection satisfies the Rokhlin conditions for the action induced by $\alpha^k|_{r_0Ar_0}$ as

$$\begin{aligned} \sum_{j=0}^{\frac{n}{k}-1} \alpha^{kj}(q') &= r_0 \sum_{j=0}^{\frac{n}{k}-1} \left(\sum_{i=0}^{k-1} \alpha^{kj+i}(q) \right) \\ &= r_0. \end{aligned}$$

So $j + \frac{n}{k}\mathbb{Z} \mapsto (\alpha^k|_{r_0Ar_0})^j$ is a Rokhlin action of $\mathbb{Z}_{n/k}$ on the UHF algebra r_0Ar_0. By [66, Theorem 3.13] the inclusion $\iota : (r_0Ar_0)^{\alpha^k} \to (r_0Ar_0)$ induces an ordered isomorphism $K_0(\iota) : K_0((r_0Ar_0)^{\alpha^k}) \to K_0(r_0Ar_0)^{K_0(\alpha^k)}$. Similarly, by [60] the fixed point algebra $(r_0Ar_0)^{\alpha^k}$ is an AF-algebra. By Elliott's classification theorem for AF algebras ([40]) the AF algebras r_0Ar_0 and $(r_0Ar_0)^{\alpha^k}$ are isomorphic as required. □

One can apply Theorem 8.1.11 to compute the fixed point algebras of the automorphisms s_n^γ introduced in Section 3.3.2 (see also Example 6.1.2). It is shown that s_n^γ are of minimal period in Example 8.1.2 and that they have the Rokhlin property in Example 6.1.2. Thus they satisfy the hypothesis of Theorem 8.1.11. It follows that the fixed point algebras $(\mathbb{M}_{n^\infty})^{s_n^\gamma} \cong \mathbb{M}_{n^\infty}$ for every $n \in \mathbb{N}$ and n-th root of unity γ

8.2 Cohomology vanishing

In [66, 67] Izumi shows K-theoretical constraints to the existence of Rokhlin actions of finite groups on C*-algebras. These come in the form of Tate cohomology vanishing for the acting group with coefficients in the K-theory modules (see Section 2.5.2 for Tate cohomology). In this section we study to what extent these constraints hold in the setting of anomalous actions with the Rokhlin property.

Throughout this section, whenever we consider a G-C*-algebra (A, α) or a (G, ω)-C*-algebra (A, α, u), we will view the K-theory groups of A as G-modules through α. We start by recalling Izumi's results.

8.2.1 Cohomology vanishing for group actions

First we recall some notation. Let n be a natural number and Γ be an abelian group, we denote by $n\Gamma = \{ng : g \in \Gamma\}$, ${}_n\Gamma = \{g \in \Gamma : ng = 0\}$ and $\Gamma_n = \Gamma/n\Gamma$.

In [67, Theorem 3.3] Izumi exhibits the following restriction for modules that arise as K-theory groups of simple, unital C*-algebras with the module structure induced by a Rokhlin group action.

Theorem 8.2.1. *(Izumi cf. [67, Theorem 3.3]) Let G be a finite group and α a Rokhlin action of G on a unital, simple C*-algebra A. The modules $K_i(A)$, ${}_nK_i(A)$ and $K_i(A)_n$ are cohomologically trivial for all $i = 0, 1$ and $n \in \mathbb{N}$.*[1]

The conditions of Theorem 8.2.1 are packaged in the following definition.

Definition 8.2.2. (Izumi cf. [66, Definition 3.8]) Let G be a finite group. A G-module M is called *completely cohomologically trivial* if M, ${}_nM$ and M_n are cohomologically trivial for all $n \in \mathbb{N}$.

[1]See Definition 2.5.3.

Theorem 8.2.1 is an obstruction to the existence of Rokhlin actions of groups on C*-algebras. A simple example of this obstruction is in the case that A is the UHF-algebra $\mathbb{M}_{3^\infty}$ and G is the cyclic group of order two. The C*-algebra $\mathbb{M}_{3^\infty}$ does not admit a Rokhlin action of $\mathbb{Z}_2$. This follows from Theorem 8.2.1 as any automorphism on $\mathbb{M}_{3^\infty}$ is approximately inner, so acts trivially on K-theory. Moreover, the cohomology group $\hat{H}^2(\mathbb{Z}_2, K_0(\mathbb{M}_{3^\infty})) \cong \hat{H}^2(\mathbb{Z}_2, \mathbb{Z}[1/3]) \neq 0$ ([11, pg 58]).

In fact, the obstruction of Theorem 8.2.1 was shown to be the unique obstruction to lifting a group action on the K-theory groups of a unital Kirchberg algebra A that satisfies the UCT to a Rokhlin action on A.

Theorem 8.2.3. *(Izumi cf. [67, Corollary 5.4]) Let A be a unital Kirchberg algebra satisfying the UCT and G a finite group. Let $\overline{\alpha}^0$ be a G-action on $K_0(A)$ fixing the class $[1_A]_0$ and $\overline{\alpha}^1$ be a G-action on $K_1(A)$ such that $K_i(A)$ are completely cohomologically trivial for $i = 0, 1$. Then there exists a G-action on A with the Rokhlin property such that $K_i(\alpha_g) = \overline{\alpha}^i_g$ for all $g \in G$ and $i = 0, 1$.*

Proof. This is just a reformulation of [67, Corollary 5.4], we include an argument to explain how to acquire this reformulation.

Let A, $\overline{\alpha}^0$ and $\overline{\alpha}^1$ be as in the hypothesis. By [67, Corollary 5.4] there exists a unital Kirchberg algebra B satisfying the UCT and β a Rokhlin action of G on B such that there are isomorphisms

$$\Phi_0 : (K_0(B), [1_B]_0) \to (K_0(A), [1_A]),$$

$$\Phi_1 : K_1(B) \to K_1(A)$$

with $\Phi_i K_i(\beta_g) = \overline{\alpha}^i_g \Phi_i$ for all $g \in G$ and $i = 0, 1$. By Kirchberg–Phillips classification ([109]), there exists an isomorphism $\varphi : B \to A$ with $K_i(\varphi) = \Phi_i$ for $i = 0, 1$. The Rokhlin action α on A given by $\alpha_g = \varphi\beta_g\varphi^{-1}$ for $g \in G$ is such that $K_i(\alpha_g) = \overline{\alpha}^i_g$ for all $g \in G$ and $i = 0, 1$. □

Theorem 8.2.1 is not only relevant as a tool to characterise the existence of Rokhlin actions on unital Kirchberg algebras that satisfy the UCT. It is also crucially used in Izumi's classification theorem (Theorem 8.2.3).

A crucial ingredient for the proof of Theorem 8.2.1 is the following K-theoretical consequence of the Rokhlin property for group actions.

Theorem 8.2.4. *(Izumi cf. [66, Theorem 3.13]) Let A be a simple, unital C*-algebra and α be a Rokhlin action of a finite group G on A. Let A^α be the fixed point algebra and $\iota : A^\alpha \to A$ the inclusion. Then*

$$K_i(\iota) : K_i(A^\alpha) \to K_i(A)^G$$

is an isomorphism for $i = 0, 1$. Moreover, the map

$$K_i(A) \xrightarrow{\sum_{g\in G} K_i(\alpha_g)} K_i(A)^G$$

is surjective for $i = 0, 1$.

8.2.2 K-theoretic restrictions to Rokhlin anomalous actions

We now turn to discuss K-theoretical obstructions to the existence of Rokhlin anomalous actions. Recall that if M is a G-module, the norm map $N : M \to M^G$ is defined by $m \mapsto \sum_{g\in G} gm$ for $m \in M$. We start with a lemma that shows the surjectivity of the norm map for anomalous actions with the Rokhlin property. The proof follows standard techniques (see e.g. the proof of [66, Theorem 3.13]). For any $\varepsilon > 0$, in the following proof we will denote by $O(\varepsilon)$ any expression which is polynomial in ε with constant term 0.

Lemma 8.2.5. *Let G be a finite group, A be a simple, unital C^*-algebra and (α, u) be an anomalous action of G on A that has the Rokhlin property. Then the mapping $\sum_{g\in G} K_i(\alpha_g) : K_i(A) \to K_i(A)^G$ is a surjection for $i = 0, 1$.*

Proof. It is sufficient to show this for A simple, purely infinite. Indeed, we claim that the map $\sum_{g\in G} K_i(\alpha_g) : K_i(A) \to K_i(A)^G$ is surjective if and only if the map

$$K_i(A \otimes \mathcal{O}_\infty) \xrightarrow{\sum_{g\in G} K_i(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_\infty})} K_i(A \otimes \mathcal{O}_\infty)^G$$

is surjective. To see this, notice that by the Künneth theorem for tensor products (Theorem 2.6.11), there are isomorphisms $\varphi_i : K_i(A) \to K_i(A \otimes \mathcal{O}_\infty)$ such that

$$\begin{array}{ccc} K_0(A) & \xrightarrow{K_i(\alpha_g)} & K_0(A) \\ \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_i} \\ K_0(A \otimes \mathcal{O}_\infty) & \xrightarrow{K_i(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_\infty})} & K_0(A \otimes \mathcal{O}_\infty) \end{array}$$

commutes for all $g \in G$ and $i = 0, 1$. So $\varphi_i|_{K_0(A)^G} : K_0(A)^G \to K_0(A \otimes \mathcal{O}_\infty)^G$ is an isomorphism for $i = 0, 1$. Thus, using commutivity of the diagram, $\sum_{g\in G} K_i(\alpha_g)$ corestricted to $K_i(A)^G$ is surjective if and only if $\sum_{g\in G} K_i(\alpha_g \otimes \mathcal{O}_\infty)$ corestricted to $K_i(A \otimes \mathcal{O}_\infty)^G$ is surjective. Moreover, $A \otimes \mathcal{O}_\infty$ is simple, purely infinite (see Remark 2.7.12)

Similarly, it is sufficient to show the statement for $i = 0$. Indeed, by the Künneth formula, the surjectivity of the norm map for $i = 1$ for the action (α, u) on A is equivalent to the surjectivity of the norm map for the case $i = 0$ for the Rokhlin action $(\alpha \otimes \mathrm{id}_{\mathcal{P}_\infty}, u \otimes 1)$ with $\mathcal{P}_\infty$ the simple, purely infinite C*-algebra with $K_0(\mathcal{P}_\infty) = \{0\}$ and $K_1(\mathcal{P}_\infty) = \mathbb{Z}$.

Let $[p] \in K_0(A)^G$ be non-zero then $[p] = [\alpha_g(p)]$. By cancellation of non-zero projections there exist partial isometries $v_g \in A$ for $g \in G$ such that $p = v_g v_g^*$ and $\alpha_g(p) = v_g^* v_g$. For any $\varepsilon > 0$ and finite set $\mathcal{F} \subset A$ containing v_g,$v_g v_h^*$ and $\alpha_g(p)$ for all

$g, h \in G$ pick projections $e_g \in A$ for $g \in G$ such that

$$\sum_{g \in G} e_g = 1,$$

$$\|\alpha_g(e_h) - e_{gh}\| \leq \varepsilon, \quad \forall g, h \in G,$$

$$\|[x, e_g]\| \leq \varepsilon, \quad \forall g \in G, x \in \mathcal{F}.$$

Let $f = e_1 p$. Note that,

$$\|f^2 - f\| = \|e_1 p e_1 p - e_1 p\| \leq \|p e_1 - e_1 p\| \leq \varepsilon.$$

In particular, for any $\delta > 0$, provided ε is sufficiently small, there exists a projection q in A with $\|f - q\| \leq \delta$ (see Lemma 2.6.5). Now let $w = \sum_{g \in G} e_g v_g$. Then

$$\begin{aligned}\|ww^* - p\| &= \|\sum_{g,h \in G} e_g v_g v_h^* e_h - p\| \\ &= O(\varepsilon) + \|\sum_{g \in G} e_g p - p\| \\ &= O(\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\|w^* w - \sum_{g \in G} \alpha_g(q)\| &= \|\sum_{g \in G} (v_g^* e_g v_g - \alpha_g(q))\| \\ &= O(\varepsilon) + \|\sum_{g \in G} (e_g \alpha_g(p) - \alpha_g(q))\| \\ &= O(\varepsilon) + O(\delta) + \|\sum_{g \in G} (e_g \alpha_g(p) - \alpha_g(e_1)\alpha_g(p))\| \\ &= O(\varepsilon) + O(\delta).\end{aligned}$$

Therefore, choosing ε and δ sufficiently small, it follows from Lemma 2.6.9 that

$\sum_{g\in G}\alpha_g(q)\sim p$ and hence $[p]_0\in\operatorname{im}(\sum_{g\in G}K_0(\alpha_g))$. $\square$

We can now show cohomological triviality of the K-theory modules for Rokhlin anomalous action. This will follow exactly as in [66, Theorem 3.3]. Let N be the norm map of a module M, recall that we denote by $\overline{N}$ the induced map by N from M_G to M^G (see Section 2.5.3).

Theorem 8.2.6. *Let G be a finite group and (α,u) a Rokhlin anomalous action of G on a unital, simple C*-algebra A. The modules $K_i(A)$ are cohomologically trivial for all $i=0,1$.*

Proof. By the Künneth formula (Theorem 2.6.11) there are $K_i(\alpha_g)$ to $K_i(\alpha_g\otimes\mathrm{id}_{\mathcal{O}_\infty})$ equivariant isomorphisms $\varphi_i: K_i(A)\to K_i(A\otimes\mathcal{O}_\infty)$ for $i=0,1$. Therefore, it suffices to prove the statement for A simple, purely infinite through replacing (A,α,u) by $(A\otimes\mathcal{O}_\infty,\alpha\otimes\mathrm{id},u\otimes 1)$ if necessary (see Remark 2.7.12).

Due to Theorem 2.5.4, it will suffice to show that $\hat{H}^0(G,K_i(A))$ and $\hat{H}^1(G,K_i(A))$ vanish for $i=0,1$. In fact, it will suffice to do this for the case $i=1$. Indeed, let $\mathcal{P}_\infty$ be the unital, simple, purely infinite C*-algebra with $K_0(\mathcal{P}_\infty)\cong 0$ and $K_1(\mathcal{P}_\infty)\cong\mathbb{Z}$. By the Künneth formula, there are $K_{i+1}(\alpha_g)$ to $K_i(\alpha_g\otimes\mathrm{id})$ equivariant isomorphisms $\psi_i: K_{i+1}(A)\to K_i(A\otimes\mathcal{P}_\infty)$ for $i=0,1$. Therefore, if we show that $\hat{H}^0(G,K_1(A))$ and $\hat{H}^1(G,K_1(A))$ vanish for all anomalous actions with the Rokhlin property on simple, purely infinite A, then $\hat{H}^0(G,K_1(A\otimes\mathcal{P}_\infty))$ and $\hat{H}^1(G,K_1(A\otimes\mathcal{P}_\infty))$ vanish. Equivalently $\hat{H}^0(G,K_0(A))$ and $\hat{H}^1(G,K_0(A))$ vanish. Summarising, it suffices to show that $\hat{H}^0(G,K_1(A))$ and $\hat{H}^1(G,K_1(A))$ vanish for any anomalous Rokhlin action (α,u) on a unital, simple, purely infinite C*-algebra A.

Firstly, $\hat{H}^0(G,K_1(A))=\operatorname{Coker}(\overline{N})$ with N the norm map for $K_1(A)$. By Lemma 8.2.5 the map N is surjective. Therefore $\overline{N}$ is surjective and $\hat{H}^0(G,K_1(A))$ vanishes. We now show that $\hat{H}^1(G,K_1(A))=H^1(G,K_1(A))$ vanishes. This follows exactly as in the proof of Theorem [67, Theorem 3.3], we include the argument for completeness.

Let $c \in Z^1(G, K_1(A))$, namely a map $c : G \to K_1(A)$ such that

$$gc(h) - c(gh) + c(g) = 0, \quad g, h \in G. \tag{8.2.1}$$

As A is simple, purely infinite we may pick $u_g \in U(A)$ for $g \in G$ such that $[u_g] = c(g)$. Then (8.2.1) may be rephrased as $[u_g] = [u_{gh}\alpha_g(u_h^*)]$. Therefore, there exists a homotopy of unitaries $v_{g,h}^{(t)}$ for $g, h \in G$ with

$$v_{g,h}^{(0)} = u_{gh}\alpha_g(u_h^*)$$

and

$$v_{g,h}^{(1)} = u_g.$$

Also choose $0 = t_0 < t_1 < ... < t_m = 1$ such that

$$\|v_{g,h}^{(t_i)} - v_{g,h}^{(t_{i+1})}\| < \frac{1}{2|G|}, \quad \forall g, h \in G, 1 \leq i \leq m.$$

Let $\epsilon > 0$ and $\mathcal{F}$ a finite set in A containing u_h for all $h \in G$. Then as a consequence of the Rokhlin property, one can find a partition of unity of projections $\{e_g\}_{g\in G}$ such that

$$\|e_{gh} - \alpha_g(e_h)\| \leq \varepsilon, \quad \forall g, h \in G,$$

$$\|[e_g, x]\| \leq \varepsilon, \quad \forall g \in G, \quad x \in \mathcal{F}.$$

Let

$$u = \sum_{h\in G} e_h u_h^*$$

and

$$w_g^{(t_i)} = \sum_{h\in G} e_{gh} v_{g,h}^{(t_i)}, \quad g \in G, \ 1 \leq i \leq m.$$

Then it is easy to see that

$$\begin{aligned}
&\|uu^* - 1\| \leq O(\epsilon),\\
&\|u^*u - 1\| \leq O(\epsilon),\\
&\|w_g^{(t_i)} w_g^{(t_i)*} - 1\| \leq O(\varepsilon), \quad \forall g \in G,\ 1 \leq i \leq m,\\
&\|w_g^{(t_i)} w_g^{(t_i)*} - 1\| \leq O(\varepsilon), \quad \forall g \in G,\ 1 \leq i \leq m,\\
&\|u^*\alpha_g(u) - w_g^{(0)}\| \leq O(\varepsilon), \quad \forall g \in G,\\
&\|w_g^{(t_i)} - w_g^{(t_i)}\| < 1/2, \quad \forall g \in G, \quad 1 \leq i \leq m.
\end{aligned}$$

The first four equations ensure that for ε small enough u and $w_g^{(t_i)}$ are close to unitaries in A (see Lemma 2.6.6) and in particular are invertible for all $g \in G$ and $1 \leq i \leq m$. Moreover, the last two equations ensure that by passing to close unitaries, one may further choose ε small enough so that for every $g \in G$ one has homotopies $u^{-1}\alpha_g(u) \sim_h w_g^{(0)} \sim_h w_g^{(t_1)} \sim_h \cdots \sim_h w_g^{(1)} = u_g$ (see Lemma 2.6.8). Therefore, one has the following chain of equivalences for K_1 classes $g[u] - [u] = [w_g^{(0)}] = [w_g^{(1)}] = \cdots = [u_g] = c(g)$. So the cochain $[u]$ is such that $d[u]_g = c(g)$ for all $g \in G$, and c is a coboundary. □

In Theorem 8.2.6 we show cohomological triviality of the K-theory modules for any Rokhlin anomalous action on a simple, unital C*-algebra. In the group action case, Izumi can establish complete cohomological triviality (see Theorem 8.2.1). Izumi's proof is reliant on the maps $K_i(\iota) : K_i(A^\alpha) \to K_i(A)^G$ being an isomorphism for $i = 0, 1$ (see Theorem 8.2.4). For the remainder of this section we show that this isomorphism does not hold, even if we restrict ourselves to Rokhlin automorphisms of minimal period. However, different methods may still show complete cohomological triviality of $K_i(A)$, it would be useful to understand whether this is still the case.

Problem 8.2.7. *Let G be a finite group and (α, u) a Rokhlin anomalous action of G on a unital, simple C*-algebra A. Are the modules $K_i(A)$ completely cohomologically trivial?*

8.2.3 The case of automorphisms of minimal period

In this subsection we explore to what extent the analogous result to Theorem 8.2.4 holds for automorphisms of minimal period. Firstly, we note that the inclusion from the fixed point algebra into A, still induces a surjection from the K-theory of the fixed point algebra to the fixed points in K-theory.

Proposition 8.2.8. *Let A be a simple, unital C*-algebra, α be an automorphism of minimal period such that its induced anomalous action has the Rokhlin property and $\iota : A^\alpha \to A$ the inclusion. Then $K_i(\iota) : K_i(A^\alpha) \to K_i(A)^{K_i(\alpha)}$ is a surjection for $i = 0, 1$.*

Proof. First, we may assume that A is simple, purely infinite. Indeed, if the result holds for all simple, purely infinite C*-algebras then for any simple, unital A one has that the inclusion $\iota' : (A \otimes \mathcal{O}_\infty)^{\alpha\otimes\mathrm{id}} \to A \otimes \mathcal{O}_\infty$ is such that $K_i(\iota')$ is surjective for $i = 0, 1$. As $(A \otimes \mathcal{O}_\infty)^{\alpha\otimes\mathrm{id}} = A^\alpha \otimes \mathcal{O}_\infty$, the map $\iota' = \iota \otimes \mathrm{id}_{\mathcal{O}_\infty}$. A diagram chase using the Künneth formula (as in the proof of Lemma 8.2.5) yields that $K_i(\iota)$ is surjective for $i = 0, 1$. Similarly, it is sufficient to show the surjectivity of $K_0(\iota)$. The surjectivity of $K_1(\iota)$ will follow by replacing (A, α) with $(A \otimes \mathcal{P}_\infty, \alpha \otimes \mathrm{id})$. Note that both $A \otimes \mathcal{O}_\infty$ and $A \otimes \mathcal{P}_\infty$ are simple, purely infinite by Remark 2.7.12.

Let $k = o(\gamma(\alpha))$ and $n = p_0(\alpha)$. By Lemma 8.1.9 there is a unitary $u \in U(A)$ such that $\alpha^n = \mathrm{Ad}(u)$ with $u = \sum_{i=0}^{k-1} \gamma(\alpha)^i r_i$ for some partition of unity of projections r_i such that $\alpha(r_i) = r_{i-1}$ modulo k. We start by showing that for any projection p in A, such that $[p]_0 \in K_0(A)^{K_0(\alpha)}$, there is a projection f in $A \cap C^*(u)'$ with $[f]_0 = [p]_0$. Let $[p]_0 \in K_0(A)^{K_0(\alpha)}$, by Lemma 8.2.5 the map $\sum_{j=0}^{n-1} K_0(\alpha^j) : K_0(A) \to K_0(A)^{K_0(\alpha)}$

is a surjection. So there exists a projection $q \in A$ with $[p]_0 = \sum_{r=0}^{n-1}[\alpha^j(q)]_0$. As A is simple, r_0 is a full projection. By Proposition 2.6.4 the inclusion $r_0Ar_0 \to A$ induces an isomorphism in K-theory. Let q' in r_0Ar_0 be a projection with $[q']_0 = [q]_0 + [\alpha^k(q)]_0 + \ldots [\alpha^{n-k}(q)]_0$ (note we don't need to go to matrix amplifications of r_0Ar_0 as corners of simple, purely infinite C*-algebras are simple, purely infinite). The projection $f = \sum_{j=0}^{k-1} \alpha^j(q')$ commutes with u and $[f]_0 = [p]_0$ as required.

Therefore, it suffices to show that for any class $[p]_0 \in K_0(A)^{K_0(\alpha)}$ such that $p \in A \cap C^*(u)'$, there exists another projection $q \in A^\alpha$ such that q is equivalent to p. As $K_0(p) = K_0(\alpha(p))$ and A has cancellation of non-zero projections, there exist partial isometries $v_j \in A$ for $1 \leq j < n$ such that $p = v_jv_j^*$ and $\alpha^j(p) = v_j^*v_j$. For any $\delta > 0$ and finite set $\mathcal{F} \subset A$ containing $\{\alpha^s(v_l), \alpha^l(p), p, u\}$ for all $0 \leq l, s < n$, the Rokhlin property yields projections e_j for $1 \leq j \leq n$ such that

$$\|\alpha^i(e_j) - e_{j+i \bmod n}\| \leq \delta,$$

$$\|[e_j, x]\| \leq \delta, \quad \forall x \in \mathcal{F}$$

and
$$\sum_{j=0}^{n-1} e_j = 1.$$

A simple computation shows that for some constant C depending only on n

$$\|\alpha^j(e_1)\alpha^i(e_1)\| \leq C\delta; \tag{8.2.2}$$

$$\|1 - \sum_{i=0}^{n-1} \alpha^i(e_1)\| \leq C\delta; \tag{8.2.3}$$

$$\|[\alpha^j(e_1), \alpha^i(p)]\| \leq C\delta; \tag{8.2.4}$$

$$\|[\alpha^j(e_1), u^l]\| \leq C\delta, \tag{8.2.5}$$

for all $0 \leq i \neq j < n-1$ and $1 \leq l < k$. Let $e = \sum_{j=0}^{n-1} \alpha^j(e_1p)$ and $v = e_1p +$

$\sum_{j=1}^{n-1} \alpha^j(e_1)v_j$. By the estimates, there exists a constant D dependent on n such that

$$\|vv^* - p\| \leq D\delta, \tag{8.2.6}$$

$$\|v^*v - e\| \leq D\delta. \tag{8.2.7}$$

Similarly, applying (8.2.2)-(8.2.5) and that p commutes with u there is a constant C_1 only depending on n such that

$$\|e - \alpha(e)\| \leq \delta, \tag{8.2.8}$$

$$\|e^2 - e\| \leq C_1\delta, \tag{8.2.9}$$

$$\|e - e^*\| \leq C_1\delta. \tag{8.2.10}$$

Let $m = nk$. As α is of period m, $E_\alpha = \frac{1}{m}\sum_{j=0}^{m-1} \alpha^j : A \to A^\alpha$ is a conditional expectation. By (8.2.8) and (8.2.9) there is a constant K dependent on m such that:

$$\|E_\alpha(e) - e\| \leq \|E_\alpha(e) - \frac{\sum_{i=0}^{n-1} \alpha^i(e)}{n}\| + \|\frac{\sum_{i=0}^{n-1} \alpha^i(e)}{n} - e\| \leq K\delta,$$
$$\|E_\alpha(e)^2 - E_\alpha(e)\| \leq \|E_\alpha(e)^2 - E_\alpha(e)e\| + \|E_\alpha(e)e - e^2\| + \|e^2 - e\| \leq K\delta.$$

Therefore $E_\alpha(e)$ is an element in A^α which is $K\delta$ close to satisfying the defining properties of a projections. Similarly, by (8.2.10) and (8.2.9) e is $c\delta$ close to satisfying the defining properties of a projection for some constant c. One may choose δ sufficiently small to apply Lemma 2.6.5. Therefore, with a sufficient small choice of $\delta > 0$ there exist projections $q \in A^\alpha$ and $q_0 \in A$ with

$$\max\{\|E_\alpha(e) - q\|, \|E_\alpha(e) - e\|, \|vv^* - p\|, \|v^*v - e\|\} < \frac{1}{4},$$
$$\|q_0 - e\| < \frac{1}{8}.$$

Notice $\|q - q_0\| = \|q - E_\alpha(e)\| + \|E_\alpha(e) - e\| + \|e - q_0\| < 1$ and by Lemma 2.6.7 $q \sim q_0$. Similarly, $\|v^*v - q_0\| \leq \|v^*v - e\| + \|e - q_0\| < 1/2$ and by Lemma 2.6.9 $q_0 \sim p$. Combining these equivalences $q \sim p$ with $q \in A^\alpha$ as required. □

However, even in the case that α is an automorphism of minimal period, the Rokhlin property does not imply the injectivity of $K_0(\iota) : K_0(A^\alpha) \to K_0(A)^{K_0(\alpha)}$.

Proposition 8.2.9. *Let α_1 be the automorphism of minimal period of $\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2$ described in Proposition 6.4.7 (see also Example 8.1.3). Let ι be the inclusion map $\iota : (\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2)^{\alpha_1} \to \mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2$. Then $K_0(\iota)$ is not injective.*

Proof. Denote by v the canonical unitary corresponding to $1 + 2\mathbb{Z}$ in $\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2$. Let $p = \frac{1+v}{2} \in P(\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2)$. Consider the following composition of maps

$$K_0(\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2) \to K_0(p(\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2)p) \xrightarrow{1+K_0(\alpha_1)} K_0((\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2)^{\alpha_1}) \xrightarrow{K_0(\iota)} K_0(\mathcal{O}_\infty \rtimes_{\sigma_2} \mathbb{Z}_2)^{K_0(\alpha_1)}.$$

Reading from left to right, the first map is an isomorphism as p is a full projection (Proposition 2.6.4). By Proposition 6.4.7 the anomalous action induced by α_1 has the Rokhlin property. So the second map is an isomorphism by Corollary 8.1.10. Hence, $K_0(\iota)$ is injective if and only if the composition from $K_0(\mathcal{O}_\infty \rtimes \mathbb{Z}_2) \to K_0(\mathcal{O}_\infty \rtimes \mathbb{Z}_2)^{K_0(\alpha)}$ is injective. However, $K_0(\mathcal{O}_\infty \rtimes \mathbb{Z}_2) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_0(\mathcal{O}_\infty \rtimes \mathbb{Z}_2)^{K_0(\alpha_1)} \cong \mathbb{Z}$ (see Proposition 6.4.7). There is no injection of groups from $\mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$. □

Chapter 9

Implications in the setting of tensor categories

In this chapter, we rephrase anomalous actions as actions of unitary tensor categories (UTC's). These categories are equipped with a multiplication operation that weakly resembles that of groups. Unitary tensor categories represent one of the several axiomatisations of the mathematical structure underlying the symmetry present in a subfactor ([97, 76, 110]).

For a group G and $\omega \in Z^3(G, \mathbb{T})$ we relate ω-anomalous actions of G to actions of the unitary tensor category $\mathbf{Hilb}(G, \omega)$. This reformulation will allow us to reword the existence and obstruction results for anomalous actions of Chapters 4, 5 and 6, as existence and obstruction results for actions of the category $\mathbf{Hilb}(G, \omega)$ on C*-algebras. For example, we will show that $\mathbf{Hilb}(G, \omega)$ only acts on $\mathcal{Z}$ when $[\omega] = 0 \in H^3(G, \mathbb{T})$. The results of this section yield obstructions to the existence of actions of unitary tensor categories on C*-algebras, that go beyond the previously known restrictions of K-theoretic nature on the dimensions of the simple objects of the acting category ([80],[65, Section 4],[24, Proposition 5.2]).

The correspondence between ω-anomalous actions and actions of $\mathbf{Hilb}(G, \omega)$ was first pointed out in my work with Evington in the unital case (see [46, Section 5]). As we require this correspondence for possibly non-unital C*-algebras, we extend this result to the generality of σ-unital C*-algebras in Proposition 9.3.1. As a corollary of Proposition 9.3.1, we may apply one of our findings of Section 5.1 to show that for any finite group G and $\omega \in Z^3(G, \mathbb{T})$ the category $\mathbf{Hilb}(G, \omega)$ acts on $\mathcal{O}_\infty$ only if $[\omega] = 0$. The remainder of Section 9.2 and Section 9.3 are a fleshed out version of [46, Section 5].

9.1 Hilbert bimodules

Before we discuss tensor categories, we swiftly recall some basics on Hilbert bimodules. This will be a fast paced introduction to the necessary concepts; more detailed accounts of the theory of Hilbert modules and bimodules can be found for instance in [88] or [112]. First we recall the notion of a (right)-Hilbert module introduced by Paschke ([108]).

Definition 9.1.1. Let X be a vector space over $\mathbb{C}$ and B a C^*-algebra, we call X a *(right)-Hilbert B-module* if X is a right B-module equipped with a function $\langle \cdot, \cdot \rangle_B : B \times B \to B$ such that

(i) $\langle \cdot, \cdot \rangle$ is left conjugate linear and right linear;

(ii) for any $x, y \in X$ and $b \in B$ one has that $\langle x, yb \rangle = \langle x, y \rangle b$.

(iii) for any $x \in X$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$;

(iv) $\langle x, y \rangle = \langle y, x \rangle^*$ for each $x, y \in X$;

(v) X is complete with respect to the norm given by $\|\langle x, x \rangle\|^{1/2}$.

We call $\langle \cdot, \cdot \rangle$ with properties (i)-(iv) a *right B-inner product* on X. A left-Hilbert B-module is defined similarly but instead equipping a vector space X over $\mathbb{C}$ with a left B-action and a left B-inner product. In this thesis the default will be right-Hilbert modules.

For Hilbert B-modules X and Y we denote the vector space of algebraically adjointable operators from X to Y by $\mathcal{L}(X,Y)$. These operators will automatically be bounded module maps and the algebra $\mathcal{L}(X,X) = \mathcal{L}(X)$ is a C*-algebra ([112, Section 2.2]). If $x, y \in X$ we can construct a *rank one operator* $|x\rangle\langle y| \in \mathcal{L}(X)$ defined by $|x\rangle\langle y|(z) = x\langle y, z\rangle$ for all $z \in X$. The closure of the span of rank one operators in $\mathcal{L}(X)$ is denoted by $\mathcal{K}(X)$. We now turn to Hilbert bimodules.

Definition 9.1.2. Let A, B be C^*-algebras, a *(right)-Hilbert A-B-bimodule* is a (right)-Hilbert B-module X equipped with a non-degenerate $*$-homomorphism $\lambda : A \to \mathcal{L}(X)$. The homomorphism λ is called the *left action* of A on X and is often denoted by ax or $a \cdot x$ instead of $\lambda(a)x$.[1]

If X and Y are right-Hilbert A-B-bimodules we denote by $\mathrm{Hom}(X,Y)$ the vector space of right adjointable operators that commute with the left A-action. So, $\mathrm{Hom}(X,X) := \mathcal{L}(X) \cap A'$ is a C*-algebra. We call $T \in \mathrm{Hom}(X,Y)$ such that $TT^* = \mathrm{id}_Y$ and $T^*T = \mathrm{id}_X$ a *unitary* and say X, Y are *unitarily equivalent* if there exists a unitary $T \in \mathrm{Hom}(X,Y)$.

Similarly, there is the notion of a left-Hilbert A-B-bimodule. A left-Hilbert A-B bimodule is a left-Hilbert A-module with an adjointable right B-action. Our default Hilbert bimodules will be right-Hilbert bimodules.

If X is a Hilbert A-B-bimodule and Y is a Hilbert B-C-bimodule we may form their *internal tensor product* $X \otimes Y$ that is a Hilbert A-C-bimodule. We sketch this construction and refer to [88, Section 4] for details. To perform the internal tensor product one starts by considering the algebraic tensor product of vector spaces $X \odot Y$.

[1] The reason we include "(right)" in the naming of these bimodules is to emphasise that the bimodules only carry a right inner product. Later on we will also be intersted in bimodules which carry left inner products.

We identify the elements of the form $xb \odot y$ with $x \odot by$ to form the quotient

$$V = X \odot Y/\mathrm{Span}\{xb \odot y - x \odot by : \ x \in X, \ y \in Y, \ b \in B\}.$$

We denote the image of the elementary tensor $x \odot y$ of $X \odot Y$ in V under the canonical quotient map by $x \otimes y$. One may define a right C-action and a right C-inner product on V by

$$(x \otimes y)b = x \otimes yb,$$

$$\langle x \otimes y, z \otimes w\rangle = \langle x, \langle y, z\rangle w\rangle,$$

for any $x, z \in X$ and $z, w \in Y$. It follows that V equipped with this C-action and C-valued inner product satisfies (i)-(iv) of Definition 9.1.1. We produce a Hilbert C-module $X \otimes Y$ by completing V under the norm defined by the inner product. Moreover, one can induce a left A action on $X \otimes Y$ through

$$a(x \otimes y) = ax \otimes y$$

for all $a \in A$, $x \in X$ and $y \in Y$. This equips $X \otimes Y$ with the structure of a Hilbert A-C-bimodule. The following family of bimodules will be relevant to us in later sections.

Example 9.1.3. (cf. [13, Section 3]) Let A be a C*-algebra and $\theta \in \mathrm{Aut}(A)$. Let A_θ be the Hilbert A-A-bimodule with underlying vector space A, where the A-actions and A-inner product are given by

$$a \cdot x \cdot b = ax\theta(b), \tag{9.1.1}$$

$$\langle x, y\rangle = \theta^{-1}(x^*y). \tag{9.1.2}$$

Suppose $\theta, \phi \in \mathrm{Aut}(A)$. If A is unital, any morphism $f \in \mathrm{Hom}(A_\theta, A_\phi)$ must be given by right multiplication with $f(1)$, and $f(1)$ must intertwine the right A-actions. It follows that A_θ and A_ϕ are unitary isomorphic if and only if there is a unitary $u \in A$ with $\theta = \mathrm{Ad}(u)\phi$ and an isomorphism from A_θ to A_ϕ is given by $a \mapsto au$ for $a \in A$. Similarly, in the case that A is non-unital, it is shown in [13, Corollary 3.2] that if $A_\theta \cong A_\phi$ then there is a unitary u in $M(A)$ with $\theta = \mathrm{Ad}(u)\phi$ and an isomorphism from A_θ to A_ϕ is given by $a \mapsto au$ for $a \in A$. Moreover, $\mathrm{Hom}(A_\theta, A_\theta) = Z(M(A))$. We also have $A_\theta \otimes A_\phi \cong A_{\theta\circ\phi}$ via the unitary isomorphism $J(x \otimes y) = x\theta(y)$. We denote the bimodule A_{id_A} simply by A.

A Hilbert A-B-bimodule X is called *invertible* if there exists a Hilbert B-A-bimodule Y such that $X \otimes Y \cong B$ and $Y \otimes X \cong A$. By [37, Lemma 2.4], invertible bimodules coincide with *imprimitivity bimodules*. Imprimitivity bimodules were introduced by Rieffel to study Morita equivalence for C*-algebras ([113]). Recall that a (right)-Hilbert A-module is called *full* if $\mathrm{Span}\{\langle x, y\rangle : x, y \in X\}$ is dense in A.

Definition 9.1.4. Let A, B be C^*-algebras an *imprimitivity A-B-bimodule* is a complex vector space X which is a full right-Hilbert A-B-bimodule and a full left-Hilbert A-B bimodule. Moreover

(i) Denoting by ${}^L\langle\cdot,\cdot\rangle$ the left inner product and by $\langle\cdot,\cdot\rangle^R$ the right inner product one has ${}^L\langle x, y\rangle z = x\langle y, z\rangle^R$ for $x, y, z \in X$;

(ii) For $x \in X$, $a \in A$ and $b \in B$ one has $(a \cdot x) \cdot b = a \cdot (x \cdot b)$.

For an imprimitivity A-B bimodule X one can define its *contragrediant* bimodule $\overline{X} = \{\overline{x} : x \in X\}$ with $\mathbb{C}$-vector space structure given by $\overline{x} + \overline{y} = \overline{x+y}$ and $\lambda\overline{x} = \overline{\lambda x}$ for any $x, y \in X$ and $\lambda \in \mathbb{C}$. The left and right actions of $\overline{X}$ are defined by $b\overline{x}a = \overline{a^* x b^*}$ and its left and right inner products are defined by ${}^L\langle\overline{x}, \overline{y}\rangle = \langle x, y\rangle^R$ and $\langle\overline{x}, \overline{y}\rangle^R = {}^L\langle x, y\rangle$. The contragrediant bimodule $\overline{X}$ is clearly an imprimitivity

B-A-bimodule and the maps

$$\begin{aligned}\Phi :& X \otimes \overline{X} \to A && (9.1.3)\\ & x \otimes \overline{y} \mapsto {}^{L}\langle x, y\rangle && (9.1.4)\end{aligned}$$

and

$$\begin{aligned}\Psi :& \overline{X} \otimes X \to B && (9.1.5)\\ & \overline{x} \otimes y \mapsto \langle x, y\rangle^{R} && (9.1.6)\end{aligned}$$

are bimodule isomorphisms. Conversely, every invertible bimodule X is an imprimitivity bimodule and its inverse is isomorphic to $\overline{X}$ by [37, Lemma 2.4].

For $\theta \in \mathrm{Aut}(A)$, the bimodules A_θ of Example 9.1.3 are imprimitivity. This follows as A_θ is invertible with inverse $A_{\theta^{-1}}$. One could also check that it satisfies the conditions of Definition 9.1.4 taking the left inner product ${}^{L}\langle a, b\rangle = ab^*$ for $a, b \in A$.

Two C*-algebras A and B are called *Morita equivalent* if there exists an imprimitivity A-B-bimodule. A relevant example of an imprimitivity bimodules is the Morita equivalence between a σ-unital C*-algebra A and its stabilisation.

Example 9.1.5. (cf. [144, Example 15.1.7]) Let A be a C*-algebra. We let H_A be the Hilbert A-module with underlying vector space

$$H_A := \{(x_n) : x_n \in A \text{ for } n \in \mathbb{N} \text{ and } \sum_{n\in\mathbb{N}} x_n^* x_n \text{ converges}\}$$

and right A-action and right A-inner products given by

$$(x_n)a = (x_n a),$$

$$\langle (x_n), (y_n)\rangle_A = \sum_{n\in\mathbb{N}} x_n^* y_n,$$

for $(x_n), (y_n) \in H_A$ and $a \in A$. The right Hilbert A-module H_A is full. By Proposition [112, Proposition 3.8] taking ${}_{K(H_A)}\langle (x_n), (y_n)\rangle = |(x_n)\rangle\langle (y_n)|$ for $(x_n), (y_n) \in H_A$ equips H_A with the structure of a imprimitivity $K(H_A)$-A-bimodule. A computation shows that $K(H_A) \cong A \otimes \mathbb{K}$ (see e.g. [144, Example 15.2.11]). Therefore H_A is an imprimitivity $A \otimes \mathbb{K}$-A-bimodule.

9.2 C* tensor categories and their actions

In this section we recall the notion of a C*-tensor category and what it means for one to act on a C*-algebra. All categories that we shall consider will be $\mathbb{C}$-*linear categories*, meaning that the space of morphisms $\mathrm{Hom}(X, Y)$ between any two objects of the category is endowed with a $\mathbb{C}$-vector space structure and the composition of morphisms is a bilinear mapping.

Definition 9.2.1. ([50]; see also [75, Section 2]) A C*-*category* is a $\mathbb{C}$-linear category $\mathcal{C}$ equipped with a conjugate linear map $^* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(Y, X)$ for every $X, Y \in \mathrm{Obj}(\mathcal{C})$ such that

1. $\phi^{**} = \phi$ for all $\phi \in \mathrm{Hom}(X, Y)$;

2. $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ for all $\psi \in \mathrm{Hom}(X, Y), \phi \in \mathrm{Hom}(Y, Z)$;

3. The function $\|\cdot\| : \mathrm{Hom}(X, Y) \to [0, \infty]$ given by

$$\|\phi\|^2 = \sup\{\lambda > 0 : \phi^* \circ \phi - \lambda\, \mathrm{id}_X \text{ is not invertible}\}$$

   is a complete norm on $\mathrm{Hom}(X, Y)$;

4. $\|\phi \circ \psi\| \leq \|\phi\|\|\psi\|$ for all $\psi \in \mathrm{Hom}(X, Y), \phi \in \mathrm{Hom}(Y, Z)$;

5. $\|\phi^* \circ \phi\| = \|\phi\|^2$ for all $\phi \in \mathrm{Hom}(X, Y)$;

6. For all $\phi \in \mathrm{Hom}(X,Y)$, $\phi^* \circ \phi$ is a positive element of the C*-algebra $\mathrm{Hom}(X,X)$.

A C*-*functor* $F : \mathcal{C} \to \mathcal{D}$ between C*-categories is required to be $\mathbb{C}$-linear on morphisms and *-preserving. A natural isomorphism $\nu : F \to G$ is said to be *unitary* if $\nu_x \in \mathrm{Hom}(F(X), G(X))$ satisfies $\nu_X^* \circ \nu_X = 1_{F(X)}$ and $\nu_X \circ \nu_X^* = 1_{G(X)}$ for all $X \in \mathcal{C}$.

A simple example of a C*-category is **Hilb**, whose objects are finite-dimensional Hilbert spaces over $\mathbb{C}$ and morphism are linear maps. The map $\phi \mapsto \phi^*$ is the Hilbert space adjoint. More generally, for any C*-algebra A, the (right)-Hilbert A-modules and adjointable maps form a C*-category (see for example [88]).

We now consider tensor product structures on a C*-category.

Definition 9.2.2. (see for example [44, 75]) A C*-*tensor category* is a C*-category $\mathcal{C}$ together with a $\mathbb{C}$-linear bifunctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object $1_{\mathcal{C}} \in \mathrm{Obj}(\mathcal{C})$ and unitary natural isomorphisms

$$\begin{aligned} \alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \qquad (9.2.1) \\ \lambda_X &: (1_{\mathcal{C}} \otimes X) \to X, \\ \rho_X &: (X \otimes 1_{\mathcal{C}}) \to X, \end{aligned}$$

such that $(\phi \otimes \psi)^* = (\phi^* \otimes \psi^*)$ and the following diagrams commute for any $X, Y, Z, W \in \mathrm{Obj}(\mathcal{C})$

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{\alpha_{W,X,Y} \otimes \mathrm{id_Z}} & & \searrow^{\alpha_{W \otimes X,Y,Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle \alpha_{W,X \otimes Y,Z}} & & & & \downarrow{\scriptstyle \alpha_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\mathrm{id_W} \otimes \alpha_{X,Y,Z}} & & W \otimes (X \otimes (Y \otimes Z)), \end{array} \qquad (9.2.2)$$

$$\begin{array}{ccc} (X \otimes 1_{\mathcal{C}}) \otimes Y & \xrightarrow{\alpha_{X,1,Y}} & X \otimes (1_{\mathcal{C}} \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} & \\ & X \otimes Y. & \end{array} \tag{9.2.3}$$

The C*-category **Hilb** can be endowed with the additional structure of a C*-tensor category by taking $- \otimes -$ to be the Hilbert space tensor product, 1_{Hilb} to be the 1-dimensional Hilbert space $\mathbb{C}$, and taking

$$\begin{aligned} \alpha_{X,Y,Z} &: (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z), \\ \lambda_X &: 1_{\mathbb{C}} \otimes x \mapsto x, \\ \rho_X &: x \otimes 1_{\mathbb{C}}, \mapsto x \end{aligned} \tag{9.2.4}$$

for X, Y, Z Hilbert spaces and x, y, z in X, Y, Z respectively. We now state the additional examples of C*-tensor categories that will be relevant for the remainder of the thesis.

Example 9.2.3. In the following examples, G is a discrete group, $\omega \in Z^3(G, \mathbb{T})$ is a 3-cocyle, and A is a C*-algebra.

1. **Hilb**(G): The objects are finite-dimensional, G-graded Hilbert spaces, i.e. finite-dimensional Hilbert spaces X with a decomposition $X = \bigoplus_{g \in G} X_g$. The morphisms are linear maps that preserve the G-grading. The tensor product is the usual Hilbert space tensor product with the G-grading defined by $(X \otimes Y)_g = \bigoplus_{h \in G} X_h \otimes Y_{h^{-1}g}$. The remaining structure is the same as for **Hilb**.

2. **Hilb**(G, ω): Defined exactly the same as **Hilb**(G) except that the associators are now given by

$$\alpha_{X,Y,Z} : (x \otimes y) \otimes z \mapsto \omega(g, h, k)\, x \otimes (y \otimes z)$$

for $x \in X_g$, $y \in Y_h$, $z \in Z_k$ and g, h, k in G.

3. $\mathbf{Bim}(A)$: The objects are (right)-Hilbert A-A-bimodules. The morphisms are the adjointable right A module maps that commute with the left A-action. The tensor product is the internal tensor product.

Finally, we recall the notion of a C*-tensor functor.

Definition 9.2.4. A C*-*tensor functor* $(F, J) : \mathcal{C} \to \mathcal{D}$ is a C*-functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(1_{\mathcal{C}}) \cong 1_{\mathcal{D}}$ together with unitary natural isomorphisms $J_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ such that the following diagram commutes

$$\begin{array}{ccc}
(F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\alpha_{F(X),F(Y),F(Z)}} & F(X) \otimes (F(Y) \otimes F(Z)) \\
\downarrow{\scriptstyle J_{X,Y} \otimes \mathrm{id}_{F(Z)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(X)} \otimes J_{Y,Z}} \\
F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\
\downarrow{\scriptstyle J_{X \otimes Y,Z}} & & \downarrow{\scriptstyle J_{X,Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(\alpha_{X,Y,Z})} & F(X \otimes (Y \otimes Z)).
\end{array} \tag{9.2.5}$$

We are particularly interested in *fully faithful* C*-tensor functors $F : \mathcal{C} \to \mathcal{D}$, i.e. functors for which the the induced map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is an isomorphism for all $X, Y \in \mathrm{Obj}(\mathcal{C})$. We say the C*-tensor category $\mathcal{C}$ *acts* on a unital C*-algebra A if there exists a C*-tensor functor $\mathcal{C} \to \mathbf{Bim}(A)$, we denote this by $\mathcal{C} \curvearrowright A$.

There is also the notion of *unitary tensor category* which more closely describes the symmetries that arise in subfactor theory (see e.g. [24, Section 2] for a definition). Loosely speaking, a unitary tensor category $\mathcal{C}$ is a C*-tensor category which additionally satisfies a notion of duality that weakly resembles both invertibility for groups and duality for representations. We do not define unitary tensor categories as we will not need the technicalities of its definition, but we do remark that the categories $\mathbf{Hilb}(G, \omega)$ are unitary tensor categories for any group G and 3-cocycle ω.

9.3 Anomalous actions as actions of C*-tensor categories

We now consider the relationship between anomalous actions of a group G on a C*-algebra A and an action of $\mathbf{Hilb}(G,\omega)$ on A. Before we do so we set up some notation. We write $\mathbb{C}_g$ for the Hilbert space $\mathbb{C}$ viewed as as G-graded Hilbert space that is homogeneous of degree g.

Recall that H_A is the canonical imprimitivity $A\otimes\mathbb{K}$-A-bimodule of Example 9.1.5 and $\overline{H_A}$ its contragrediant bimodule. Denote by $\Phi : \overline{H_A} \otimes H_A \to A$ the canonical isomorphism (see (9.1.3)). We write $(\mathrm{Ad}(H_A), K) : \mathbf{Bim}(A) \to \mathbf{Bim}(A \otimes \mathbb{K})$ for the equivalence of C*-tensor categories that sends a Hilbert A-A bimodule $X \mapsto H_A \otimes X \otimes \overline{H_A}$, a morphism of Hilbert A-A bimodules $T \mapsto \mathrm{id}_{H_A} \otimes T \otimes \mathrm{id}_{\overline{H_A}}$ and $K_{X,Y} = \mathrm{id}_{H_A\otimes X} \otimes \Phi \otimes \mathrm{id}_{Y\otimes\overline{H_A}}$. We denote by $(\mathrm{Ad}(H_A), K)^{-1}$ a choice of inverse functor.

Proposition 9.3.1. *Let G be a discrete group, $\omega \in Z^3(G,\mathbb{T})$, and A a σ-unital C*-algebra. An ω-anomalous action (θ, u) of G on $A\otimes\mathbb{K}$, induces an action of $\mathbf{Hilb}(G,\omega)$ on A given by composing the functor $(F, J) : \mathbf{Hilb}(G,\omega) \to \mathbf{Bim}(A\otimes\mathbb{K})$ such that*

$$F(\mathbb{C}_g) = (A \otimes \mathbb{K})_{\theta_g}, \tag{9.3.1}$$

$$J_{\mathbb{C}_g,\mathbb{C}_h}(x \otimes y) = x\theta_g(y)u_{g,h}, \tag{9.3.2}$$

with the equivalence of C-tensor categories $(\mathrm{Ad}(H_A), K)^{-1} : \mathbf{Bim}(A\otimes\mathbb{K}) \to \mathbf{Bim}(A)$. The action of $\mathbf{Hilb}(G,\omega)$ is fully faithful whenever A is simple and (θ, u) is free. Conversely, any action of $\mathbf{Hilb}(G,\omega)$ on A induces an ω-anomalous action of G on $A \otimes \mathbb{K}$.*

Proof. As $(\mathrm{Ad}(H_A), K)^{-1}$ is an equivalence, $(\mathrm{Ad}(H_A), K)^{-1}\circ(F, J)$ will be an action of $\mathbf{Hilb}(G,\omega)$ on A if and only if (F, J) is an action of $\mathbf{Hilb}(G,\omega)$ on $A\otimes\mathbb{K}$. Similarly,

$(\mathrm{Ad}(H_A), K)^{-1} \circ (F, J)$ is fully faithful if and only if (F, J) is fully faithful. So it suffices to construct an action $(F, J) : \mathbf{Hilb}(G, \omega) \to \mathbf{Bim}(A \otimes \mathbb{K})$ and check the fully faithfulness conditions for (F, J).

We define F on a general object $X = \bigoplus_{g\in G} X_g$ of $\mathbf{Hilb}(G, \omega)$ by $F(X) = \bigoplus_{g\in G} X_g \otimes_{\mathbb{C}} (A \otimes \mathbb{K})_{\theta_g}$. On a general morphisms $f = \bigoplus_{g\in G} f_g \in \mathrm{Hom}(X, Y)$, we define $F(f) = \bigoplus_{g\in G} f_g \otimes_{\mathbb{C}} \mathrm{id}_{(A\otimes\mathbb{K})_{\theta_g}}$.

Since (θ, u) is an ω-anomalous action we have

$$u_{g,h}\theta_{gh}(a) = \theta_g(\theta_h(a))u_{g,h}, \tag{9.3.3}$$

$$\omega(g, h, k)u_{g,h}u_{gh,k} = \theta_g(u_{h,k})u_{g,hk}, \tag{9.3.4}$$

for $a \in A \otimes \mathbb{K}$ and $g, h, k \in G$. It follows that $J_{\mathbb{C}_g,\mathbb{C}_h}(x \otimes y) = x\theta_g(y)u_{g,h}$ defines a unitary isomorphism of bimodules $(A\otimes\mathbb{K})_{\theta_g} \otimes (A\otimes\mathbb{K})_{\theta_h} \cong (A\otimes\mathbb{K})_{\theta_{gh}}$, and the monoidal structure axiom (9.2.5) holds when $X = \mathbb{C}_g$, $Y = \mathbb{C}_h$, $Z = \mathbb{C}_k$. The definition of the tensorator $J_{X,Y}$ for general objects $X = \bigoplus_{g\in G} X_g$ and $Y = \bigoplus_{h\in G} Y_h$ is uniquely determined by naturality. It has the form $J_{X,Y} = \bigoplus_{g,h\in G} \mathrm{id}_{X_g} \otimes_{\mathbb{C}} \mathrm{id}_{Y_h} \otimes_{\mathbb{C}} J_{\mathbb{C}_g,\mathbb{C}_h}$. Hence the monoidal structure axiom remains valid by naturality.

Suppose A is simple and (θ, u) is a free anomalous action. Then by freeness we have that $\mathrm{Hom}((A \otimes \mathbb{K})_{\theta_g}, (A \otimes \mathbb{K})_{\theta_h}) = 0$ whenever $g \neq h$, and that $\mathrm{Hom}((A \otimes \mathbb{K})_{\theta_g}, (A \otimes \mathbb{K})_{\theta_g}) \cong Z(M(A \otimes \mathbb{K}))$; see Example 9.1.3. Moreover, as A is simple so is $A \otimes \mathbb{K}$ and $Z(M(A \otimes \mathbb{K})) = \mathbb{C}$. It follows that F is fully faithful when restricted to the objects $\mathbb{C}_g$ for $g \in G$. As any object in $\mathbf{Hilb}(G, \omega)$ is a finite direct sum of $\mathbb{C}_g$ the functor F is fully faithful.

For the converse, suppose $\mathbf{Hilb}(G, \omega)$ acts on A. Then by composing with $(\mathrm{Ad}(H_A), K)$ there exists a C*-tensor functor $(F, J) : \mathbf{Hilb}(G, \omega) \to \mathbf{Bim}(A \otimes \mathbb{K})$. As $F(\mathbb{C}_g) \otimes F(\mathbb{C}_{g^{-1}}) \cong F(\mathbb{C}_{g^{-1}}) \otimes F(\mathbb{C}_g) \cong F(1_{\mathbb{C}}) \cong 1_{\mathrm{Bim}(A\otimes\mathbb{K})}$ the bimodules $F(\mathbb{C}_g)$ are invertible for all $g \in G$ and so are a self-Morita equivalence by [37, Lemma 2.4]. As $A \otimes \mathbb{K}$ is stable and σ-unital [13, Corollary 3.5] implies the existence of $\theta_g \in \mathrm{Aut}(A \otimes \mathbb{K})$ and unitary isomorphisms $F(\mathbb{C}_g) \cong (A \otimes \mathbb{K})_{\theta_g}$ for $g \in G$. Let $g, h \in G$. Then

$$(A \otimes \mathbb{K})_{\theta_{gh}} \cong F(\mathbb{C}_g \otimes \mathbb{C}_h) \cong F(\mathbb{C}_g) \otimes F(\mathbb{C}_h) \cong (A \otimes \mathbb{K})_{\theta_g \theta_h}. \tag{9.3.5}$$

This unitary isomorphism of bimodules must be implemented by right multiplication by a unitary $u_{g,h} \in U(M(A \otimes \mathbb{K}))$ satisfying (9.3.3). Moreover, (9.3.4) holds since F and J satisfy the monoidal structure axiom (9.2.5). Hence, (θ, u) is an ω-anomalous action of G on $A \otimes \mathbb{K}$. □

Proposition 9.3.1 allows us to translate the constructions of Chapter 6 into the framework of actions of $\mathbf{Hilb}(G, \omega)$. Indeed, any anomalous action (θ, u) on a C*-algebra A or a matrix amplification of A (see e.g. Proposition 6.4.10) induces an action on $A \otimes \mathbb{K}$ by considering $(\theta \otimes \mathrm{id}_{\mathbb{K}}, u \otimes 1_{B(\mathcal{H})})$. Moreover, under the light of Proposition 9.3.1 one can immediately rephrase Proposition 5.3.4.

Corollary 9.3.2. *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$. Then* $\mathbf{Hilb}(G, \omega)$ *acts on $\mathcal{O}_\infty$ if and only if $[\omega] = 0 \in H^3(G, \mathbb{T})$.*

To access our obstruction results for anomalous actions on $\mathcal{Z}$ and UHF-algebras we will need to take a closer look at the invertible bimodules over these C*-algebras. Recall that an object X in a C*-tensor category $\mathcal{C}$ is *invertible* if there exists another object $Y \in \mathcal{C}$ with $X \otimes Y \cong 1_{\mathcal{C}} \cong Y \otimes X$. We shall say that X has *finite order* if $X^{\otimes N} \cong 1_{\mathcal{C}}$ for some $N \in \mathbb{N}$.

Lemma 9.3.3. *Let A be a unital C^*-algebra with the cancellation property. If $\mathrm{Aut}(K_0(A), K_0(A)_+)$ is trivial, then every invertible element $X \in$* $\mathbf{Bim}(A)$ *is unitary isomorphic to a bimodule of the form A_θ for some $\theta \in \mathrm{Aut}(A)$.*

Similarly, if $\mathrm{Aut}(K_0(A), K_0(A)_+)$ has no non-trivial elements of finite order, then every invertible element $X \in$ $\mathbf{Bim}(A)$ *of finite order is unitary isomorphic to a bimodule of the form A_θ for some $\theta \in \mathrm{Aut}(A)$.*

Proof. Let $X \in \mathbf{Bim}(A)$ be an invertible bimodule. Then X is a self-Morita equivalence of A by [37, Lemma 2.4]. Then $Y = \mathrm{Ad}(H_A)(X)$ is a self-Morita equivalence of $A \otimes \mathbb{K}$. As $A \otimes \mathbb{K}$ is stable and σ-unital, $Y \cong (A \otimes \mathbb{K})_\theta$ for some $\theta \in \mathrm{Aut}(A \otimes \mathbb{K})$ by [13, Corollary 3.5]. Let $\theta_* \in \mathrm{Aut}(K_0(A), K_0(A)_+)$ be the induced automorphism.

Suppose $\mathrm{Aut}(K_0(A), K_0(A)_+)$ is trivial. Then $\theta_* = \mathrm{id}_{K_0(A)}$. Therefore, $[\theta(1_A \otimes e_{11})]_{K_0(A)} = [1_A \otimes e_{11}]_{K_0(A)}$. Since A has the cancellation property, there exists a partial isometry $v \in A \otimes \mathbb{K}$ with $v^*v = 1 \otimes e_{11}$ and $vv^* = \theta(1 \otimes e_{11})$. The series

$$u = \sum_{i=1}^{\infty} \theta(1 \otimes e_{i1}) v (1 \otimes e_{1i}) \tag{9.3.6}$$

converges in the strict topology on $M(A \otimes \mathbb{K})$ and defines a unitary $u \in U(M(A \otimes \mathbb{K}))$ such that $\theta(1 \otimes e_{ij}) = \mathrm{Ad}(u)(1 \otimes e_{ij})$ for all $i, j \in \mathbb{N}$. It follows that $\mathrm{Ad}(u^*)\theta$ fixes $1 \otimes \mathbb{K}$, and so is of the form $\theta' \otimes \mathrm{id}_\mathbb{K}$ for some $\theta' \in \mathrm{Aut}(A)$. Hence, $Y \cong (A \otimes \mathbb{K})_{\theta' \otimes \mathrm{id}_\mathbb{K}}$, and we have $X \cong \mathrm{Ad}(\overline{H_A})(Y) \cong A_{\theta'}$, as required.

If $X^{\otimes N} \cong 1_{\mathrm{Bim}(A)}$, then in the above argument θ^N is an inner automorphism, so $\theta_*^N = \mathrm{id}_{K_0(A)}$. Hence, in this case, it suffice to know that $\mathrm{Aut}(K_0(A), K_0(A)_+)$ has no non-trivial elements of finite order. □

We are now ready to rephrase some of our results from Chapter 4.

Corollary 9.3.4. *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$. Then* $\mathbf{Hilb}(G, \omega)$ *acts on $\mathcal{Z}$ if and only if $[\omega] = 0 \in H^3(G, \mathbb{T})$.*

Proof. Suppose that there exists an action $(F, J) : \mathbf{Hilb}(G, \omega) \to \mathbf{Bim}(\mathcal{Z})$. As $(K_0(\mathcal{Z}), K_0(\mathcal{Z})^+) = (\mathbb{Z}, \mathbb{Z}^+)$, there are no non-trivial elements of $\mathrm{Aut}(K_0(\mathcal{Z}), K_0(\mathcal{Z})_+)$. Moreover, $\mathcal{Z}$ is simple, unital and has the cancellation property by virtue of having stable rank one (see [5, Proposition 6.5.1]). Hence, by Lemma 9.3.3, up to unitary isomorphism all invertible bimodules in $\mathbf{Bim}(\mathcal{Z})$ are of the form $\mathcal{Z}_\theta$ for some $\theta \in \mathrm{Aut}(\mathcal{Z})$.

Since (J, F) is a C*-tensor functor, we have that $F(\mathbb{C}_g)$ is invertible with inverse $F(\mathbb{C}_{g^{-1}})$ for each $g \in G$. Therefore, there exists $\theta_g \in \mathrm{Aut}(\mathcal{Z})$ such that $F(\mathbb{C}_g) \cong \mathcal{Z}_{\theta_g}$ for all $g \in G$. One may repeat the argument of the converse of Proposition 9.3.1 to deduce that there exists an ω-anomalous action of G on $\mathcal{Z}$. By Theorem 4.2.7, $[\omega] = 0$ in $H^3(G, \mathbb{T})$. □

Corollary 9.3.5. *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$. There exists an action of* $\mathbf{Hilb}(G, \omega)$ *on* $\bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ *only if, letting r be the order of $[\omega] \in H^3(G, \mathbb{T})$, r^∞ divides the supernatural number $\prod_{k\in\mathbb{N}} n_k$.*

Proof. Let $A = \bigotimes_{k\in\mathbb{N}} \mathbb{M}_{n_k}$ be a UHF algebra with supernatural number $\mathfrak{n} = \prod_{k\in\mathbb{N}} n_k$. Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$. Let $(F, J) : \mathbf{Hilb}(G, \omega) \to \mathbf{Bim}(A)$ be C*-tensor functor.

The ordered group $K_0(A)$ is isomorphic to the subgroup $Q(\mathfrak{n}) \subseteq \mathbb{Q}$ generated by $\{\frac{1}{n} : n \in \mathbb{N}, n \,|\, \mathfrak{n}\}$ with the order inherited from $\mathbb{Q}$. Since $\mathbb{Q}$ is uniquely divisible, any automorphism of $Q(\mathfrak{n})$ is determined by the image of 1. It follows that the only automorphism of $Q(\mathfrak{n})$ with finite order is multiplication by -1, which doesn't preserve the order structure. Hence, $\mathrm{Aut}(K_0(A), K_0(A)_+)$ has no non-trivial elements of finite order. Since UHF algebras are simple, unital and have the cancellation property, we may use Lemma 9.3.3 to deduce that, up to unitary isomorphism, all invertible bimodules of finite order in $\mathbf{Bim}(A)$ are of the form A_θ for some $\theta \in \mathrm{Aut}(A)$.

Since (J, F) is a C*-tensor functor, we have that $F(\mathbb{C}_g)$ is invertible with inverse $F(\mathbb{C}_{g^{-1}})$ for each $g \in G$. Moreover, as G is finite, $F(\mathbb{C}_g)$ has finite order for each $g \in G$. Therefore, there exist automorphisms $\theta_g \in \mathrm{Aut}(A)$ such that $F(\mathbb{C}_g) \cong A_{\theta_g}$. The same argument as in the converse of Proposition 9.3.1 yields that there is an ω-anomalous action of G on A. By Theorem 4.2.10, if r is the order of $[\omega]$ then r^∞ divides the supernatural number $\mathfrak{n}$. □

Proposition 9.3.1 allows us to convert the anomalous actions of Chapter 6 into actions of the category $\mathbf{Hilb}(G,\omega)$. However, we are more interested in the existence of fully faithful actions of $\mathbf{Hilb}(G,\omega)$. Proposition 9.3.1 also establishes that full faithfulness of the $\mathbf{Hilb}(G,\omega)$ action induced by an anomalous action (θ,u) is equivalent to freeness of (θ,u). We finish this chapter by discussing that, under the presence of $\mathcal{Z}$-stability, one can always produce a free anomalous action from one which may not necessarily be free.

Let (α,u) be any (G,ω)-action on a simple C*-algebra A. Set $K=\alpha^{-1}(\mathrm{Inn}(A))$, one has a short exact sequence

$$0\to K\xrightarrow{\iota}G\xrightarrow{\pi}G/K\to 0.$$

Choosing a set theoretic lift $q\mapsto\hat{q}$ to the quotient map π we may define a G/K-kernel on A by

$$\theta: G/K\to \mathrm{Out}(A)$$
$$q\mapsto[\alpha_{\hat{q}}].$$

The construction of θ is such that the pullback $\pi^*(\theta)=\alpha$. Denoting by $[\omega']$ the H^3 invariant of θ, it follows that $\pi^*[\omega']=[\omega]$. If the group G is finite and (α,u) is not free $|G/K|<|G|$. Hence as $\pi^*[\omega']=[\omega]$ and the order of $[\omega']$ divides $|G/K|$ by [11, III. Corollary 10.2], the order of $[\omega]$ also divides $|G/K|$. This observation shows that the freeness of (α,u) is automatic whenever $[\omega]$ has order $|G|$. More generally, if $[\omega]$ does not vanish when restricted to any subgroup of G, any ω-anomalous action of the group G will be free.

The following tensor product trick can be used to obtain free anomalous actions on $\mathcal{Z}$-stable C*-algebras.

Proposition 9.3.6. *Let (θ,u) be a (G,ω) action of a countable discrete group G on a C*-algebra A. Let $\alpha: G\curvearrowright\bigotimes_{g\in G}\mathcal{Z}$ be the Bernoulli shift action. Set $\theta'_g=\theta_g\otimes\alpha_g$*

and $u'_{g,h} = u_{g,h} \otimes 1_{\mathcal{Z}}$*. Then* (θ', u') *is a free* (G, ω) *action of* G *on* $A \otimes (\bigotimes_{g \in G} \mathcal{Z})$*.*

Proof. Since α is a G action, it follows immediately that (θ', u') is a (G, ω) action. It remains to show that (θ', u') is free.

Let $(z_n)_{n=1}^{\infty}$ be a central sequence in $\mathcal{Z}$ where each z_n is a self-adjoint element with spectrum $[-1, 1]$. Set $x_n = 1_A \otimes z_n \otimes (\bigotimes_{g \neq 1_G} 1_{\mathcal{Z}})$. Then

$$\begin{aligned} \|\theta'_g(x_n) - x_n\| &= \|z_n \otimes 1_{\mathcal{Z}} - 1_{\mathcal{Z}} \otimes z_n\|_{\mathcal{Z} \otimes \mathcal{Z}} \\ &= \sup_{s,t \in [-1,1]} |s - t| \\ &= 2 \end{aligned} \tag{9.3.7}$$

for all $n \in \mathbb{N}$ and $g \neq 1_G$. However, $\lim_{n \to \infty} \|\phi(x_n) - x_n\| = 0$ for all inner automorphisms ϕ, since z_n is a central sequence in $\mathcal{Z}$. □

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