

Stability Analysis for a Class of Partial Differential Equations via Semidefinite Programming

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Abstract—This paper studies scalar integral inequalities in one-dimensional bounded domains with polynomial integrands. We propose conditions to verify the integral inequalities in terms of differential matrix inequalities. These conditions allow for the verification of the inequalities in subspaces defined by boundary values of the dependent variables. The results are applied to solve integral inequalities arising from the Lyapunov stability analysis of partial differential equations. Examples illustrate the results.

Index Terms—Distributed Parameter Systems, PDEs, Stability Analysis, Sum of Squares.

I. INTRODUCTION

The need for accurate models to study dynamical systems [1], [2], [4], [30] have driven research efforts towards partial differential equation (PDE) systems - equations involving derivatives with respect to more than one independent variable. Such infinite dimensional system models are challenging to study both analytically and numerically. Conventional numerical approaches to study PDEs rely on spectral or spatial discretization and use tools developed for Ordinary Differential Equations (ODEs) [10], [7]. Numerical methods which do not require finite-dimensional approximations are needed may prevent conservatism in the system analysis.

Stability analysis of PDE systems can be performed with the Lyapunov's second method, which was extended to infinite dimensional systems in [6], [17]. One advantage of this method is that it does not require calculation of semi-groups [16], but instead requires the construction of a Lyapunov Functional (LF). *A priori* choices for LF structures for a given system are difficult to make. In particular cases, for instance when the PDE contains energy-preserving non-linearities such as convection terms, taking the energy of the state as the LF simplifies the stability analysis [25]. The Lyapunov conditions for PDE systems take the form of integral inequalities, therefore methods for computing the LFs require the solution of integral inequalities.

This paper presents a method to verify integral inequalities with integrands given by polynomials in the dependent variables. The polynomial structure allows for a quadratic-like representation of the integrand and we formulate conditions for the positivity of integral expressions in

terms of differential matrix inequalities. The proposed method is then applied to study the stability of systems of inhomogeneous PDEs, using weighted \mathcal{H}^q norms (on Hilbert spaces) as LF candidates.

For the case of integrands that are polynomial in the *independent* variables as well, the differential matrix inequality formulated by the proposed method becomes a polynomial matrix inequality. We then exploit the semidefinite programming (SDP) formulation [21] of optimization problems with linear objective functions and polynomial (sum-of-squares (SOS)) constraints [3] to obtain a numerical solution to the integral inequalities. Several analysis and feedback design problems have been studied using polynomial optimization: the stability of time-delay systems [23], synthesis control laws [24], [29], applied to optimal control law design [15] and system analysis [11]. In the context of PDEs, preliminary results have been presented in [19], where a stability test was formulated as a SOS programme (SOSP) and in [27] where quadratic integrands were studied.

The paper is organised as follows. Section II defines the problem, illustrates how we can write polynomial integrands in quadratic-like forms and proposes a method to verify integral inequalities by studying matrix inequalities associated with quadratic-like representation of the integrands. In Section III, some results for the stability analysis of PDE systems are summarised. Two examples illustrate the application of the proposed method to the stability analysis of PDEs in Section IV. Finally, Section V concludes the paper and gives directions for future research.

Notation. Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{R}_{> 0}$ and \mathbb{R}^n denote the field of reals, non-negative reals, positive reals and the n -dimensional Euclidean space, respectively. The sets of natural numbers and positive natural numbers are denoted \mathbb{N}_0, \mathbb{N} . The closure of set Ω is denoted $\bar{\Omega}$. The boundary $\partial\Omega$ of set Ω is defined as $\bar{\Omega} \setminus \Omega$ with \setminus denoting set subtraction. The ring of polynomials, the ring of positive polynomials, and the ring of sum-of-squares polynomials on a real variable x are denoted $\mathcal{R}[x]$, $\mathcal{P}[x]$ and $\Sigma[x]$ respectively. The ring of SOS square matrices of dimension n , i.e., matrices $M(x) \in \mathcal{R}^{n \times n}[x]$ satisfying $M(x) = \sum_{i=1}^{n_M} N_i^T(x) N_i(x)$ with $N_i(x) \in \mathbb{R}^{d_i \times n}$, is denoted $\Sigma^{n \times n}[x]$. For $n, k \in \mathbb{N}$, define the matrix $K \in \mathbb{N}^{n \times \sigma(n, k)}$, $\sigma(n, k) := \binom{n+k-1}{n-1} = \frac{(n+k-1)!}{(n-1)!k!}$, of which the columns satisfy $\sum_{i=1}^n K_{ij} = k, \forall j$, without repetition. The multi-index notation is used to define the vector of all monomials of degree $k \in \mathbb{N}$ on vector $w = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$, as $w^{\{k\}} := \begin{bmatrix} (w^{K_{(\cdot, 1)}})^T & \dots & (w^{K_{(\cdot, \sigma(n, k))}})^T \end{bmatrix}$ where $w^{K_{(\cdot, j)}} = \prod_{i=1}^n w_i^{K_{ij}}$. The number of terms in $w^{\{k\}}$ is hence given by $\sigma(n, k)$. For instance, with $n = 2$ and $k = 2$ one has $K = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, and $w^{\{2\}} = (w_1^2, w_1 w_2, w_2^2)$. Define the vector containing all monomials in w up to degree k as $\eta^k(w) := \begin{bmatrix} 1 & (w^{\{1\}})^T & \dots & (w^{\{k\}})^T \end{bmatrix}^T$. The set of continuous vector functions, mapping $\Omega \subset \mathbb{R}$ into \mathbb{R}^n , which are k -times differentiable and have continuous derivatives is denoted $\mathcal{C}^k(\Omega)$. For $p(x) \in \mathcal{C}^1(\Omega)$, the d -

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th derivative of p with respect to variable x is denoted $\partial_x^d p$. For $u \in \mathcal{C}^\alpha(\Omega)$, $\alpha \in \mathbb{N}_0$, define $D^\alpha u = v_\alpha(u) := (u, \partial_x u, \dots, \partial_x^\alpha u)$, α is the order of $D^\alpha u$. The variable u is called the *dependent variable* and $x \in \Omega$, the *independent variable*. For notational convenience, depending on the context we may suppress the dependence of $u(t, x)$ on t (by writing $u(x)$) or x (by writing $u(t)$) or both (by writing u). The set of functions satisfying $u \in \mathcal{C}^\alpha(\Omega)$ for which $\|u\|_\alpha^2 = \int_\Omega (D^\alpha u(x))^T (D^\alpha u(x)) dx < \infty$ is denoted $\mathcal{H}^\alpha(\Omega)$. For a matrix $P(x) > 0 \forall x \in \Omega$, define the norm $\|u\|_{(\alpha, P(x))} := (\int_\Omega (D^\alpha u(x))^T P(x) (D^\alpha u(x)) dx)^{\frac{1}{2}}$. The set of real symmetric matrices is denoted $\mathbb{S}^n = \{A \in \mathbb{R}^{n \times n} | A = A^T\}$. For $A \in \mathbb{S}^n$, denote $A \geq 0$ ($A > 0$) if A is positive semidefinite (definite). The ceil function is denoted $\lceil \cdot \rceil$. The Jacobian of a vector field is denoted by ∇ , the Laplacian is denoted ∇^2 and the cross-product is denoted by the symbol \times .

II. INTEGRAL INEQUALITIES WITH POLYNOMIAL INTEGRANDS

In this section, we study inequalities given by polynomials on the dependent variables evaluated at the boundaries of the domain of integration and integral terms with polynomial integrands on the dependent variables.

Consider the integral inequality

$$f_b(v_{\theta-1}(u(1)), v_{\theta-1}(u(0))) + \int_\Omega f_i(x, v_\theta(u(x))) dx \geq 0,$$

with $\Omega = [0, 1]$, $f_b \in \mathcal{R}[v_{\theta-1}(u(1)), v_{\theta-1}(u(0))]$, $f_i(\cdot, v_\theta) \in \mathcal{R}[v_\theta]$ (f_i is a polynomial on its second argument). In order to simplify the exposition, let us define

$$v_{\theta-1}^b(u) := \begin{bmatrix} (v_{\theta-1}(u(1)))^T & (v_{\theta-1}(u(0)))^T \end{bmatrix}^T.$$

For $\max(\deg(f_b), \deg(f_i)) = k$, we can express the polynomials f_b and f_i as in the quadratic-like forms

$$f_b(v_{\theta-1}^b(u)) = \left(\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b(u)) \right)^T F_b \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b(u)) \quad (1)$$

$$f_i(x, v_\theta(u)) = \left(\eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)) \right)^T F_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)) \quad (2)$$

with symmetric matrix $F_b \in \mathbb{S}^{\sigma(2n\theta, \lceil \frac{k}{2} \rceil) \times \sigma(2n\theta, \lceil \frac{k}{2} \rceil)}$ and symmetric matrix function $F_i : \Omega \rightarrow \mathbb{S}^{\sigma(n\theta, \lceil \frac{k}{2} \rceil) \times \sigma(n\theta, \lceil \frac{k}{2} \rceil)}$. The dependent variable $u(x)$ is assumed to belong to sets of the form

$$\mathcal{B}(B) := \{u \in \mathcal{C}^{\theta-1}(\Omega) \mid B v_{\theta-1}^b(u) = 0\}, \quad (3)$$

with $B \in \mathbb{R}^{n_b \times 2n\theta}$, where n_b is the number of constraints on the boundary.

We study the following problem

Problem 1. Check whether the integral inequality

$$f_b(v_{\theta-1}^b(u)) + \int_\Omega f_i(x, v_\theta(u)) dx \geq 0, \quad (4)$$

holds for all $u \in \mathcal{H}^\theta$ satisfying $u \in \mathcal{B}(B)$.

For a given polynomial f_b , the representation (1) may be non-unique and is taken as an element of the set

$$\begin{aligned} \mathcal{F}_b(k, \theta) = & \left\{ F_b + G_b : \mathbb{S}^{\sigma(n\theta, \lceil \frac{k}{2} \rceil) \times \sigma(n\theta, \lceil \frac{k}{2} \rceil)} \right. \\ & \mid f_i = (\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b(u)))^T F_b \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b(u)), \\ & \left. 0 = (\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b(u)))^T G_b \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b(u)) \right\}. \end{aligned} \quad (5)$$

Similarly, for a given function f_i , the set of quadratic-like representation (2) is taken as an element of the set

$$\begin{aligned} \mathcal{F}_i(k, \theta) = & \left\{ F_i(x) + G_i(x) : \Omega \rightarrow \mathbb{S}^{\sigma(n\theta, \lceil \frac{k}{2} \rceil) \times \sigma(n\theta, \lceil \frac{k}{2} \rceil)} \right. \\ & \mid f_i = (\eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)))^T F_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)), \\ & \left. 0 = (\eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)))^T G_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)) \right\}. \end{aligned} \quad (6)$$

The following example illustrates the above definitions.

Example 1. Consider $f_i(x, (u, \partial_x u)) = x^2 \partial_x u + 2u(\partial_x u)^2$, $u \in \{u \mid u(0) = u(1)\}$, yielding $k = 3$, $\theta = 1$, and $v_\theta(u) = (u, \partial_x u)$, $\eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u)) = (1, u, \partial_x u, u^2, u \partial_x u, (\partial_x u)^2)$. The quadratic representation (2) and the set (3) are obtained with $v_{\theta-1}^b = [u(1) \ u(0)]$,

$$F_i(x) = \begin{bmatrix} 0 & 0 & \frac{x^2}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{x^2}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

The set $\mathcal{F}_i(k, \theta)$ is defined by matrices $G_i(x)$ as

$$G_i(x) = \begin{bmatrix} 0 & 0 & 0 & g_1(x) & g_2(x) & g_3(x) \\ 0 & -2g_1(x) & -g_2(x) & 0 & 0 & 0 \\ 0 & -g_2(x) & -2g_3(x) & 0 & 0 & 0 \\ g_1(x) & 0 & 0 & 0 & 0 & g_4(x) \\ g_2(x) & 0 & 0 & 0 & -2g_4(x) & 0 \\ g_3(x) & 0 & 0 & g_4(x) & 0 & 0 \end{bmatrix}.$$

However, a complete quadratic representation of the integral expression (4), must also account for the differential relation among the entries of $v_\theta(u)$ and is characterized by the set \mathcal{I} as in (7). Notice that the set is obtained by a straightforward application of the fundamental theorem of calculus, which allows us to introduce matrices H_b related to the values at the boundary and matrix functions $H_i(x) \in \mathcal{C}^1(\Omega)$.

The example below illustrates matrices H_b and $H_i(x)$ for an element of set (7).

Example 2. Consider (4) with $f_b = 2u(1)\partial_x u(0)$, $f_i(x, (u, \partial_x u)) = \sin^2(x)u\partial_x u + 2u\partial_x^2 u$, yielding $k = 2$, $\theta = 2$, and $v_\theta(u) = (u, \partial_x u, \partial_x^2 u)$. Since the expression is homogeneous of degree $k = 2$, we replace the inhomogeneous vector $\eta^{\lceil \frac{k}{2} \rceil}(v_\theta(u))$ by a homogeneous vector $(v_\theta(u))^{\{\frac{k}{2}\}}$ to obtain the quadratic expressions with $(v_{\theta-1}^b)^{\{1\}} = (u(1), \partial_x u(1), u(0), \partial_x u(0))$, $(v_\theta(u))^{\{1\}} = (u, \partial_x u, \partial_x^2 u)$. The representation (2) is defined by

$$F_b = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, F_i(x) = \begin{bmatrix} 0 & -\frac{\sin^2(x)}{2} & 1 \\ -\frac{\sin^2(x)}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathcal{I} := \left\{ (\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b))^T (F_b + H_b) \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b) + \int_{\Omega} \left[(\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta}(u)))^T F_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta}(u)) + \frac{d}{dx} \left((\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}(u)))^T H_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}(u)) \right) \right] dx \right. \\ \left. | F_b \in \mathcal{F}_b(k, \theta), F_i \in \mathcal{F}_i(k, \theta), (\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b))^T H_b \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}^b) + \left[(\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}(u(x))))^T (H_i(x)) \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}(u(x))) \right]_{x=0}^1 = 0 \right\} \quad (7)$$

$$\frac{d}{dx} \left((v_{\theta-1}(u))^T H_i(x) (v_{\theta-1}(u)) \right) = (v_{\theta})^T \begin{bmatrix} \frac{d}{dx} h_{11}(x) & \frac{d}{dx} h_{12}(x) + h_{11}(x) & \frac{1}{2} h_{12}(x) \\ \frac{d}{dx} h_{12}(x) + h_{11}(x) & \frac{d}{dx} h_{22}(x) + h_{12}(x) & h_{22}(x) \\ \frac{1}{2} h_{12}(x) & h_{22}(x) & 0 \end{bmatrix} (v_{\theta}) \quad (8)$$

and the terms characterising the multiplicity of the integral as described by (7) are obtained with (8) and with

$$(v_{\theta-1}^b)^T H_b (v_{\theta-1}^b) = (v_{\theta-1}^b)^T \begin{bmatrix} -H_i(1) & 0 \\ 0 & H_i(0) \end{bmatrix} (v_{\theta-1}^b).$$

Remark 1. Further to the non-uniqueness associated to the algebraic relations in the vector describing the quadratic representation, which is characterised by the sets \mathcal{F}_b , \mathcal{F}_i , the Fundamental Theorem of Calculus exposes the non-uniqueness of the integral expression associated to the differential relations of the elements in $v_{\theta}(u)$, characterising the set (7).

In order to simplify the presentation of the next result, let us introduce the function $\bar{H}_i(x)$, which satisfies

$$\frac{d}{dx} \left((\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}(u)))^T H_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta-1}(u)) \right) = (\eta^{\lceil \frac{k}{2} \rceil}(v_{\theta}(u)))^T \bar{H}_i(x) \eta^{\lceil \frac{k}{2} \rceil}(v_{\theta}(u)) \quad (9)$$

and allows us to denote the quadratic form in the integrand of (7) in terms of matrix $F_i(x) + \bar{H}_i(x)$. The quadratic-like characterization of the integrand in terms of the algebraic and the differential relations leads to conditions for integral inequalities in terms of matrix inequalities as follows:

Theorem 1. If there exist $F_b \in \mathcal{F}_b$, $F_i(x) \in \mathcal{F}_i$, satisfying (1)-(2), and $H_i(x) \in \mathcal{C}^1(\Omega)$, yielding H_b as in (7) and $\bar{H}_i(x)$ as in (9) such that

$$(\eta^{\bar{k}}(v_{\theta-1}^b))^T (F_b + H_b) \eta^{\bar{k}}(v_{\theta-1}^b) \geq 0 \quad \forall u \in \mathcal{B}(B), \quad (10)$$

$$F_i(x) + \bar{H}_i(x) \geq 0 \quad \forall x \in \Omega, \quad (11)$$

where $\bar{k} = \lceil \frac{k}{2} \rceil$, then inequality (4) holds in the subspace defined by $\mathcal{B}(B)$.

Proof: For given polynomials f_b and f_i satisfying $k = \max(\deg(f_b), \deg(f_i))$ we can express an integral expresion as in (4), say $\phi(u)$, using the quadratic forms as defined in (1)-(2) with $F_b \in \mathcal{F}_b(k, \theta)$, $F_i(x) \in \mathcal{F}_i(k, \theta)$, which gives

$$\begin{aligned} \phi(u) &= f_b(v_{\theta-1}^b) + \int_{\Omega} f_i(x, v_{\theta}(u)) dx, \\ &= (\eta^{\bar{k}}(v_{\theta-1}^b))^T F_b \eta^{\bar{k}}(v_{\theta-1}^b) \\ &\quad + \int_{\Omega} (\eta^{\bar{k}}(v_{\theta}(u)))^T F_i(x) \eta^{\bar{k}}(v_{\theta}(u)) dx. \end{aligned}$$

Following the definition of set \mathcal{I} in (7) and the definition of \bar{H}_i in (9) we obtain

$$\begin{aligned} \phi(u) &= (\eta^{\bar{k}}(v_{\theta-1}^b))^T (F_b + H_b) \eta^{\bar{k}}(v_{\theta-1}^b) \\ &\quad + \int_{\Omega} (\eta^{\bar{k}}(v_{\theta}(u)))^T (F_i(x) + \bar{H}_i(x)) \eta^{\bar{k}}(v_{\theta}(u)) dx. \end{aligned} \quad (12)$$

Hence, if the boundary term satisfies (10), and the integral term satisfies (11) then $\phi(u) \geq 0 \quad \forall u \in \mathcal{B}(B)$. ■

Remark 2. Inequality (11) is a differential matrix inequality since the elements $\bar{H}_i(x)$ involve continuously differentiable functions and their derivatives.

A. Semidefinite programming formulation

Whenever matrix $F_i(x)$ is a polynomial on variable x and we impose polynomial dependence of $\bar{H}_i(x)$ on variable x , inequality (11) can be addressed by a straightforward application of Putinar's Positivstellensatz [14, Theorem 2.14]. Note that the set $\Omega = [0, 1]$, can be described as the semi-algebraic set $\{x | \omega(x) := x(1-x) \geq 0\}$.

Corollary 1. For $F_i(x) + \bar{H}_i(x) \in \mathcal{R}^{n_d \times n_d}[x]$, if there exists $N(x) \in \Sigma^{n_M \times n_M}[x]$ such that

$$F_i(x) + \bar{H}_i(x) - N(x)\omega(x) \in \Sigma^{n_M \times n_M}[x] \quad (13)$$

then (11) holds.

Remark 3. If the coefficients of $F_i(x)$ and $\bar{H}_i(x)$ depend affinely in unknown parameters and the degree of $N(x)$ is fixed, checking whether (13) holds can be cast as a feasibility test of a convex set of constraints, an SDP, whose dimension depends on the degree of $F_i(x) + \bar{H}_i(x)$ and $N(x)$ and on the dimension of matrix $F_i(x) + \bar{H}_i(x)$ which depends on the degree k and the order θ as in (1)-(2).

The formulation of inequalities (10) and (11) is possible thanks to the application of the Fundamental Theorem of Calculus to characterize the set of quadratic-like representations of an integral inequality, as described by the set (7). The terms introduced in the integrand by matrix H_i do not affect the value of the integral and allow for a test for positivity based on the positivity of the matrices in the quadratic-like representation. This is reminiscent of the quadratic representation that is used in sum-of-squares when checking positivity of a polynomial. Also, for polynomial expressions, the algebraic relations in the quadratic representation of integrand polynomials are here defined in terms of functions, (see (6)) instead of scalars. With the solution to (10) it is possible to verify inequalities in subspaces as in (3), incorporating boundary values of the dependent variables.

III. STABILITY ANALYSIS FOR PARTIAL DIFFERENTIAL EQUATIONS

This section recalls some results for the stability analysis of PDEs and formulates integral inequalities for the convergence of norms of the solutions of PDEs with polynomial dependence on the dependent variables. Results extending Lyapunov's second method to PDE systems can be found in [17], [20], [31] and [12]. The formulation detailed in the latter is recalled below:

Definition 1. A nonlinear semi-group on a compact normed space C is a family of maps $\{S(t) \mid C \rightarrow C, t \geq t_0\}$ such that

- for each $t \geq t_0$, $S(t)$ is continuous from C to C ,
- for each $u \in C$, the mapping $t \rightarrow S(t)u$ is continuous,
- $S(0)$ is the identity on C ,
- $S(t)(S(\tau)u) = S(t+\tau)u$ for all $u \in C$ and all $t, \tau \geq 0$.

Definition 2. Let $\{S(t), t \geq t_0\}$ be a nonlinear semi-group on C and for any $u \in C$, let $Y(u) = \{S(t)u, t \geq t_0\}$ be the orbit through u . We say u is an equilibrium point if $Y(u) = \{u\}$.

An orbit $Y(u)$ is *stable* if for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $t \geq t_0$, $\|S(t)u - S(t)v\|_C < \epsilon$ whenever $\|u - v\|_C < \delta(\epsilon)$, $v \in C$, where $\|\cdot\|_C$ is the norm defined on C . An orbit is *uniformly asymptotically stable* if it is stable and also there is a neighbourhood $D = \{v \in C \mid \|u - v\|_C < r\}$ such that $\|S(t)u - S(t)v\|_C \rightarrow 0$ as $t \rightarrow \infty$, uniformly for $v \in D$ ¹. Similarly, it is *exponentially stable* if there exist $\nu, \gamma > 0$ such that

$$\|S(t)u - S(t)v\|_C \leq \gamma \|u - v\|_C e^{-\nu(t-t_0)},$$

for all $t \geq t_0$ and all $u, v \in C$.

Definition 3. Let $\{S(t), t \geq t_0\}$ be a nonlinear semi-group on C . A Lyapunov function is a continuous real-valued function V on C such that

$$\frac{d}{dt}V(u) = \lim_{t \rightarrow 0^+} \frac{V(S(t)u) - V(u)}{t} \leq 0, \quad (14)$$

for all $u \in C$.

Theorem 2. (Lyapunov Theorem for Nonlinear Semi-groups, [12, Theorem 4.1.4]), Let $\{S(t), t \geq t_0\}$ be a nonlinear semi-group, and let 0 be an equilibrium point in C . Suppose V is a Lyapunov function on C which satisfies $V(0) = 0$, and $V(u) \geq \alpha_1 \|u\|_C$ for $\alpha_1 > 0$ and $u \in C$. Then, 0 is stable. In addition, if $\partial_t V(u) \leq -\alpha_2 \|u\|_C$ for $\alpha_2 > 0$, then 0 is uniformly asymptotically stable.

For the proof of the above theorem, refer to [12, p.84]. In [5, Theorem 5.1.3], necessary and sufficient conditions for a linear operator defining a PDE to be the infinitesimal generator of exponentially stable strongly continuous semigroups on a Hilbert space is given in terms of the existence of a positive operator satisfying a Lyapunov equation. Such a theorem is based on the results in [6]. The extension of this result for more general spaces has

¹i.e. $\forall \epsilon > 0, \exists T > 0 : t > T$ s.t. $\|S(t)u - S(t)v\|_C < \epsilon, \forall v \in D$.

been provided in [22]. The above Theorem establishes conditions for the stability and uniform asymptotic stability of *nonlinear semigroups*. Please refer to [16] for conditions on nonlinear operators to generate semigroups of nonlinear contractions. The remaining of this section proposes a sufficient condition for the existence of Lyapunov function for PDEs defined by nonlinear polynomial operators. In what follows, we present the class of PDE systems and Lyapunov functionals studied in this paper. Consider the following PDE system

$$\partial_t u = F(x, D^\alpha u), \quad u(t_0, x) = u_0(x) \in \mathcal{M} \subset \mathcal{H}^q(\Omega) \quad (15)$$

where $q \in \mathbb{N}_0$. Let $F(x, D^\alpha u) = \mathcal{N}u$, where \mathcal{N} is a nonlinear (polynomial) operator defined on \mathcal{M} , a closed subset of $\mathcal{H}^q(\Omega)$. Continuous solutions to the PDE (15) exist in \mathcal{M} and are unique, provided \mathcal{N} generates a nonlinear semigroup of contractions.

The following theorem is a Lyapunov result for the exponential convergence of the \mathcal{H}^q norm of the solutions to (15).

Theorem 3. Suppose there exist a functional $V \in \mathcal{C}^1$, with $V(0) = 0$, and scalars $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ such that

$$c_1 \|u\|_q^2 \leq V(u) \leq c_2 \|u\|_q^2 \quad (16)$$

$$\frac{dV(u)}{dt} \leq -c_3 \|u\|_q^2 \quad (17)$$

then, the \mathcal{H}^q norm of the trajectories of (15) satisfy

$$\|u(t, x)\|_q^2 \leq \frac{c_2}{c_1} \|u_0(x)\|_q^2 e^{-\frac{c_3}{c_1}(t-t_0)} \quad (18)$$

where $u_0 = u(t_0, x)$.

Proof: From (16), (17), one has $\frac{dV(u)}{V(u)} \leq -\frac{c_3}{c_1}$.

Since $\frac{dV(u)}{V(u)} = \frac{d(\ln(V(u)))}{dt}$, the integral in time of the above expression over $[t_0, t]$ yields $\ln(V(u(t))) - \ln(V(u(t_0))) \leq -\frac{c_3}{c_1}(t - t_0)$ which gives $V(u(t)) \leq V(u(t_0))e^{-\frac{c_3}{c_1}(t-t_0)}$. Finally (18) is obtained by applying the bounds of (16) on the above inequality. ■

Consider candidate Lyapunov functionals of the form

$$V(u) = \frac{1}{2} \int_{\Omega} (D^q u)^T P(x) (D^q u) \, dx, \quad P(x) > 0 \, \forall x \in \Omega. \quad (19)$$

That is, $V(u) = \frac{1}{2} \|u\|_{(q, P(x))}^2$, the squared $P(x)$ -weighted \mathcal{H}^q -norm. The following lemma states the equivalence of the weighted norm and the \mathcal{H}^q -norm.

Lemma 1. If $P(x) > 0 \, \forall x \in \Omega$, then the norms $\|u\|_{(q, P)}$ and $\|u\|_q$ are equivalent.

The proof consists of showing that the inequalities $\sqrt{\lambda_m} \|u\|_q \leq \|u\|_{(q, P)} \leq \sqrt{\lambda_M} \|u\|_q$ hold, with $\lambda_m > 0$, $\lambda_M > 0$ satisfying $\lambda_m I \leq P(x) \leq \lambda_M I, \forall x \in \Omega$.

Remark 4. For $q_1 < q_2$, the space \mathcal{H}^{q_1} is embedded in \mathcal{H}^{q_2} [8, Sec 5.6]. Therefore, stability in \mathcal{H}^{q_2} -norm implies stability in \mathcal{H}^{q_1} -norm, but the converse does not necessarily hold.

Proposition 1. *If there exists a function $P(x)$ and positive scalars ϵ_1, ϵ_2 such that*

$$\int_{\Omega} [(D^q u)^T P(x)(D^q u) - \epsilon_1 (D^q u)^T (D^q u)] dx \geq 0 \quad (20)$$

$$- \int_{\Omega} [(D^q u)^T P(x) F(x, D^q u) + F^T(x, D^q u) P(x)(D^q u) + \epsilon_2 (D^q u)^T (D^q u)] dx \geq 0 \quad (21)$$

then the \mathcal{H}^q norm of solutions to (15) satisfy (18) with $c_1 = \min_{\Omega}(\lambda_{\min}(P(x)))$, $c_2 = \max_{\Omega}(\lambda_{\max}(P(x)))$, and $c_3 = \epsilon_2$.

The verification of inequalities (20), (21) is a sufficient condition to verify the conditions of Theorem 3, since it restricts the class of functions in (16), (17) to be as in (19).

Inequalities (20), (21) are integral inequalities such as the ones studied in Section II. The sets $\mathcal{B}(B)$ as in (3) associated to the inequalities are defined by the domain of the PDE operators. The results of Section II-A can therefore be applied to (20)-(21) whenever the integrands are polynomial functions. In this context, $P(x)$ is a variable parameterized in terms of polynomial coefficients. For such a parameterization a tradeoff must be accounted for: as the class of parameterized functions is increased by adding elements elements of the basis (increasing the degree of a polynomial $P(x)$) the computational burden required to solve (13) increases.

IV. EXAMPLES

In this section we present solutions to inequalities (20), (21) by solving the SOS constraints obtained with the formulation of Corollary 1, (the constraints are cast as SDP with SOSTOOLS [18] and solved with SeDuMi [26]). The exponential stability of the \mathcal{L}_2 norm of the solution of a simple hyperbolic equation is studied below.

Example A) Consider the equation

$$\partial_t u = -\partial_x u \quad x \in [0, 1], t > 0 \quad u(t, 0) = 0, \quad (22)$$

which gives $\mathcal{B} = \left\{ u \in \mathcal{C}^2(\Omega) \mid \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = 0 \right\}$. Given $\lambda > 0$, let $E_p = \frac{1}{2} \int_{\Omega} p(x) u^2(x) dx$, be the candidate function to certify $-\lambda E_p - \dot{E}_p = \int_{\Omega} -\frac{1}{2} \lambda p(x) u^2(x) + p(x) u \partial_x u(x) dx \geq 0$ (proving the exponential stability with convergence rate λ). We apply Theorem 1 to the Lyapunov inequalities in Proposition 1 to obtain

$$p(x) > 0 \quad \forall x \in \Omega \quad (23a)$$

$$\begin{bmatrix} u(1) \\ u(0) \end{bmatrix}^T \begin{bmatrix} -h(1) & 0 \\ 0 & h(0) \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} > 0 \quad \forall \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \in \mathcal{B} \quad (23b)$$

$$M(x) = \frac{1}{2} \begin{bmatrix} -\lambda p(x) - \frac{d}{dx} h(x) & (h(x) - p(x)) \\ (h(x) - p(x)) & 0 \end{bmatrix} \geq 0 \quad \forall x \in \Omega. \quad (23c)$$

Solve (23c) by imposing $h(x) = -p(x)$ and by solving the differential equation $\partial_x h(x) + \lambda h(x) = 0$, to obtain

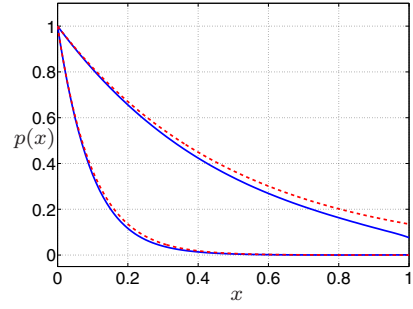


Fig. 1: Weighting functions proving exponential stability for convergence rates $\lambda \in \{2, 10\}$. The red dotted curves depict the solution $p(x) = e^{-\lambda x}$. The solid blue lines correspond to the polynomials obtained by solving (24).

$h(x) = -e^{-\lambda x}$, which satisfies (23a) (i.e. $-h(x) = p(x) > 0$). Notice that with $p(x) = -h(x) = e^{-\lambda x}$ the inequality of (23c) holds for all $x \in \mathbb{R}$. Inequality (23b) is expressed as $-h(1)u^2(1) > 0$, which clearly holds since $-h(1) = e^{-\lambda} > 0$. The inequalities then hold for any $\lambda > 0$ which proves the exponential stability of the \mathcal{L}_2 norm of the solution for *any* convergence rate. This is expected as, for bounded domains, the equation presents *finite-time* stability.

The inequalities in (20)-(21) were also formulated with polynomial weighting function $p(x)$, with $q = 0$ (giving $D^q u = u$) and $\epsilon_2 = \lambda$. Theorem 1 is applied to the resulting inequalities and we use the Positivstellensatz to formulate the SOS as in Corollary 1

$$\text{Find } p, h, N \quad \text{subject to} \quad \begin{aligned} M(x) + N(x)x(x-1) &\in \Sigma^{2 \times 2}[x], \\ N(x) &\in \Sigma^{2 \times 2}[x]. \end{aligned} \quad (24)$$

Solutions to the above inequalities were obtained for $\lambda \in (0, 10]$ (the value $\lambda^* = 10$ was solved with $\deg(p(x)) = \deg(h(x)) = 30$). The numerical results provide polynomial Lyapunov certificates for the \mathcal{L}_2 stability of the solutions of (22). A comparison of the solution $p(x) = e^{-\lambda x}$ and the numerical solutions are depicted in Figure 1. \square

In the next example we study the stability of a nonlinear, inhomogeneous PDE.

Example B) (System of nonlinear inhomogeneous PDEs) Consider the following PDE

$$\partial_t u = R^{-1} \nabla^2 u + C(x)u + u \times (D(x) \nabla u) \quad (25)$$

where $u(\cdot, x) : \mathbb{R} \rightarrow \mathbb{R}^3$, $R > 0$ and

$$C(x) = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ x & -x^2 & 0 \end{bmatrix}; D(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-\frac{1}{2} + x) \end{bmatrix} \quad (26)$$

subject to the boundary conditions $u(0) = u(1) = 0$. In order to obtain the asymptotic stability bound in terms of the parameter R a straightforward approach is to take energy as the Lyapunov function. In this case, the nonlinear terms, given by $u \times (D(x) \nabla u) = (-\frac{1}{2} + x) [u_2 \partial_x u_3 - u_1 \partial_x u_3]^T$, are removed from the expression of the time-derivative of the energy and the

remaining expression is influenced only by the linear terms $R^{-1}\nabla^2 u + C(x)u$. This choice results in conservatism since the stability may not be certified by the energy of the state taken as the Lyapunov function. Instead consider a weighted \mathcal{L}_2 energy as a LF candidate and solve (20), (21) to obtain $P(x)$, the weighting function. In order to illustrate this solution, we compute the largest value of parameter R for which stability could be certified both with energy and the weighted \mathcal{L}_2 norm.

The results are depicted in Table I.

TABLE I: Stability intervals for parameter $R \in (0, R^*]$ for different degrees of $P(x)$.

$\deg(P(x))$	0, $P(x) = I$	1	2	3
R^*	6.3	10.5	17.5	21

V. CONCLUSION

This paper proposed tests for positivity of integral expressions with integrands that are polynomial on the dependent variables. A set of quadratic-like representations for the polynomial integrand is obtained with the Fundamental Theorem of Calculus. This way, the value of the dependent variables at the boundary of the integration domain can be accounted for. The positivity of the integral is then studied by analyzing the matrices associated to the quadratic-like representation. Under the assumption that the integrand is also polynomial on the independent variable, matrix positivity tests are cast as SDPs. We then studied integral inequalities from Lyapunov stability conditions for PDEs. The proposed Lyapunov functionals are given by weighted \mathcal{H}^q -norms. Two examples were presented to illustrate the effectiveness of the proposed approach. Future work will cover the problems of synthesis of in-domain and boundary controllers, [13], [9] and the extension to integrals over domains of two spatial dimensions, of which [28] reports some preliminary results.

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