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BLUFFING AS A MIXED STRATEGY

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Abstract

In von Neumann and Morgenstern's simple model of poker, equilibrium has the first player bet with high and low hands, and check with intermediate hands. The second player then calls if his hand is sufficiently high. Betting by the low hands is interpreted as bluffing, and is a pure strategy. Here we show that this equilibrium is nongeneric, in the sense that it ceases to exist if the first player is allowed to choose among many possible bets, rather than just one. Moreover, Newman's solution for this case—which also has pure-strategy bluffing—is shown not to be a sequential equilibrium. However, a modified solution—where low hands bluff using *mixed* strategies—is a sequential equilibrium.

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I.

Several authors (Newman 1959, Cutler 1975, Binmore 1992, Dreef, Borm, and van der Genugten 2003) have used and extended von Neumann and Morgenstern’s (1944, pp. 211–219) simple model of poker where, having randomly drawn hands from the interval $[0, 1]$, a Better decides whether to bet, then a Caller decides whether to call. In equilibrium, the Better’s behavior is characterized by cutoffs a and b such that he bets if his hand x is in the intervals $[0, a)$ or $(b, 1]$ (checking otherwise). The Caller then calls a bet if his hand y is in the interval $(b, 1]$, folds if it is in $[0, a)$, and calls with probability $p_y \in (0, 1)$ otherwise.¹ The betting by the low Better hands is interpreted as bluffing. But the idea of bluffing as a pure strategy—so that a given low hand will *always* bet the same amount—does not correspond with our intuition that bluffing has a random element to it; *sometimes* a poker player will make a “big” bluff with the worst possible hand, but not *all* of the time.

In this note, we show that if the Better is allowed, not just to bet or fold, but to choose among many possible bets (indeed, two will suffice), then von Neumann and Morgenstern’s equilibrium ceases to exist. Newman (1959) analyzed the unlimited-bet case, and found a solution where pure-strategy bluffing still prevails, but with different low hands betting different amounts. However, we show that this solution is not a sequential equilibrium, and offer a modified solution characterized by low hands bluffing with *mixed* strategies.

II.

Each of two risk-neutral individuals, a *Better* and a *Caller*, pays an *ante* of 1 chip from his *bankroll* $m \in \mathbb{N}$ into the *pot* in return for a private *hand*— x for the Better, y for the Caller—dealt uniformly at random from the interval $[0, 1]$. The Better then chooses a *bet* $\beta \in \{l, l + 1, \dots, m\}$ to put into the pot, or he may *check* (bet zero). The Caller finally decides either to *call* (matching the bet) or *fold* (paying no more). If the Caller calls, both players’ hands are revealed and the higher hand receives the pot of $2(\beta + 1)$; if they have the same hand, the pot is split evenly between them. If the Caller folds, the hands remain private and the Better wins the pot of $\beta + 2$.

¹A special case has a cutoff $c \in (a, b)$ such that the Caller calls if and only if his hand exceeds c .

The model of von Neumann and Morgenstern can be thought of as the special case where $l = m$. We consider whether the von Neumann–Morgenstern equilibrium extends to the many-bet case, $l < m$. Let the putative perfect Bayesian equilibrium be characterized by cutoffs $a, b \in [0, 1]$ such that: the Better bets an amount, $\gamma \in \{l, \dots, m\}$ say, if his hand is in the intervals $[0, a)$ or $(b, 1]$, otherwise checking (betting zero); and the Caller with hand y calls a bet of β with probability $p_{\beta, y}$, with $p_{\gamma, y} = 1$ for $y \in (b, 1]$, $p_{\gamma, y} = 0$ for $y \in [0, a)$, and $p_{0, y} = 1$ for $y \in [0, 1]$. Define $p_\gamma := \int_a^b p_{\gamma, y} / (b - a) dy$, $p_\beta := \int_0^1 p_{\beta, y} dy$, $p_\beta^-(x) := \int_0^x p_{\beta, y} dy$ and $p_\beta^+(x) := \int_x^1 p_{\beta, y} dy$.

Proposition 1 *In any such equilibrium, $l = m$.*

Proof. For the equilibrium to exist, hand $x = a$ of the Better must be indifferent between betting γ and checking:

$$\begin{aligned} 2a - \gamma(1 - b) - \gamma p_\gamma(b - a) + 2(1 - p_\gamma)(b - a) &= 2a \\ \Rightarrow a &= \frac{(\gamma + 2)(p_\gamma a + (1 - p_\gamma)b) - \gamma}{2}, \end{aligned} \quad (1)$$

as must hand $x = b$:

$$\begin{aligned} 2a - \gamma(1 - b) + 2(1 - p_\gamma)(b - a) + (\gamma + 2)p_\gamma(b - a) &= 2b \\ \Rightarrow b &= \frac{1 + p_\gamma a + (1 - p_\gamma)b}{2}. \end{aligned} \quad (2)$$

Together (1) and (2) imply that

$$\begin{aligned} 2b - 1 &= \frac{2a + \gamma}{\gamma + 2} \\ b &= \frac{a + \gamma + 1}{\gamma + 2}. \end{aligned} \quad (3)$$

For the Caller, hands $y \in [a, b]$ must be indifferent between calling γ and folding:

$$\begin{aligned} (2\gamma + 2) \Pr(x < y \mid \gamma) + 0 \Pr(x > y \mid \gamma) - \gamma &= 0 \\ \frac{\Pr(x < y) \Pr(\gamma \mid x < y)}{\Pr(\gamma)} &= \frac{\gamma}{2\gamma + 2}, \end{aligned}$$

by Bayes' Law. Hence,

$$\begin{aligned}\frac{a}{a+1-b} &= \frac{\gamma}{2\gamma+2} \\ a &= \frac{(1-b)\gamma}{\gamma+2}.\end{aligned}$$

Together with (3), this implies that

$$a = \frac{\gamma}{(\gamma+4)(\gamma+1)} < \frac{1}{2},$$

and hence,

$$b = \frac{\gamma^2 + 4\gamma + 2}{(\gamma+4)(\gamma+1)} > \frac{1}{2}.$$

(1) then implies that

$$p_\gamma = \frac{1}{\gamma+1}.$$

Returning to the Better, hands $x \in [0, a)$ must prefer betting γ to any other β :

$$\begin{aligned}2a - \gamma(1-b) - \gamma p_\gamma(b-a) + 2(1-p_\gamma)(b-a) &\geq 2(1-p_\beta) + (\beta+2)p_\beta^-(x) - \beta p_\beta^+(x) \\ \Rightarrow (\gamma+2)(p_\gamma a + (1-p_\gamma)b) - \gamma &\geq 2(1-p_\beta) + (\beta+2)p_\beta^-(0) - \beta p_\beta^+(0) \\ p_\beta &\geq \frac{(\gamma+2)(1-p_\gamma a - (1-p_\gamma)b)}{\beta+2}.\end{aligned}$$

And hands $x \in [b, 1]$ must prefer betting γ to any other β :

$$\begin{aligned}2a - \gamma(1-x) + (\gamma+2)(x-b) + (\gamma+2)p_\gamma(b-a) + 2(1-p_\gamma)(b-a) &\geq \\ 2(1-p_\beta) + (\beta+2)p_\beta^-(x) - \beta p_\beta^+(x) & \\ \Rightarrow (\gamma+2) - \gamma(1-p_\gamma)b - \gamma p_\gamma a &\geq 2(1-p_\beta) + (\beta+2)p_\beta^-(1) - \beta p_\beta^+(1) \\ p_\beta &\leq \frac{\gamma(1-p_\gamma a - (1-p_\gamma)b)}{\beta}.\end{aligned}$$

Therefore, p_β must lie in the range

$$\begin{aligned} \frac{(\gamma + 2)(1 - p_\gamma a - (1 - p_\gamma)b)}{\beta + 2} &\leq p_\beta \leq \frac{\gamma(1 - p_\gamma a - (1 - p_\gamma)b)}{\beta} \\ \Leftrightarrow \frac{2(\gamma + 2)^2}{(\beta + 2)(\gamma + 4)(\gamma + 1)} &\leq p_\beta \leq \frac{2\gamma(\gamma + 2)}{\beta(\gamma + 4)(\gamma + 1)}. \end{aligned} \quad (4)$$

For this range to be nonempty, it must be the case that

$$\begin{aligned} \frac{\gamma + 2}{\beta + 2} &\leq \frac{\gamma}{\beta} \\ \Leftrightarrow \beta &\leq \gamma, \end{aligned}$$

i.e. that γ is the maximum bet m .

Hands $x \in [a, b]$ must then prefer checking to betting any $\beta \in \{l, \dots, m - 1\}$:

$$\begin{aligned} 2x &\geq 2(1 - p_\beta) + (\beta + 2)p_\beta^-(x) - \beta p_\beta^+(x) \\ \Leftrightarrow p_\beta^-(x) &\leq \frac{(\beta + 2)p_\beta - 2(1 - x)}{2(\beta + 1)}. \end{aligned}$$

And since $p_\beta^+(b) \leq 1 - b$, it must be the case that $p_\beta^-(b) \geq p_\beta - 1 + b$. Hence, for $p_\beta^-(b)$ to exist, we require that

$$\begin{aligned} p_\beta - 1 + b &\leq \frac{(\beta + 2)p_\beta - 2(1 - b)}{2(\beta + 1)} \\ \Leftrightarrow p_\beta &\leq \frac{2(\gamma + 2)}{(\gamma + 4)(\gamma + 1)}. \end{aligned}$$

But since $(\gamma + 2)/(\beta + 2) > 1$ for any $\beta \in \{l, \dots, m - 1\}$, this contradicts the lower bound in (4) unless $l = \gamma = m$. ■

III.

This raises the obvious question of what *is* the equilibrium in the many-bets setting. Of course, given the model's continuous hand space, we are not guaranteed the existence of an equilibrium, but since the continuum of hands is merely a convenient approximation of the discrete reality, we should expect there to be one. The closest answer we have to this question in the literature is provided by Newman (1959), who provides a

solution for the modified setting where hands are drawn from the *open* interval $(0, 1)$, and any nonnegative real number may be chosen as a bet.

Newman's solution involves the Better checking if his hand lies in the interval $(1/7, 4/7)$, and betting β with either hand $(1/7)(1 - 3\xi^2 + 2\xi^3)$ or $1 - (3/7)\xi^2$, where $\xi = 2/(\beta + 2)$; the Caller then calls the bet β if and only if his hand exceeds $1 - (6/7)\xi$. An increasing amount $\beta(x)$ is thus bet by hands in the interval $[4/7, 1)$, with an asymptote at 1, and these bets are mimicked by hands in the interval $(0, 1/7)$, so that the Caller knows that the bet β was made by one of two possible hands, but does not know which one; he is then indifferent at a cutoff somewhere in between the two possible hands. A modified form of von Neumann and Morgenstern's pure-strategy bluffing thus carries over to the many-bets setting, by Newman's account.

However, Newman's solution is not a sequential equilibrium. To see this, note that for the Caller, the cutoff hand $y = 1 - (6/7)\xi$ must be indifferent between calling β and folding:

$$(2\beta + 2) \Pr(x < y \mid \beta) + 0 \Pr(x > y \mid \beta) - \beta = 0$$

$$\frac{\Pr(x < y) \Pr(\beta \mid x < y)}{\Pr(\beta)} = \frac{\beta}{2\beta + 2},$$

by Bayes' Law. The left-hand side of this expression is undefined, since $\Pr(\beta \mid x < y) = \Pr(\beta) = 0$; hence the necessity of using Kreps and Wilson's (1982) sequential equilibrium. Some delicacy arises, however, from the infinite sets of possible hands and bets, and so we employ Myerson and Reny's (2011) "basic sequential equilibrium."

Consider then the finite approximation to Newman's game where the hands are drawn uniformly at random from a grid $\Theta \triangleq \{\theta, 2\theta, \dots, 1 - \theta\}$, $\theta > 0$, and any bet may be chosen from the grid $\Delta \triangleq \{\delta, 2\delta, \dots, m - \delta, m\}$, $\delta > 0$. Now consider the totally mixed strategy profile σ where the Better chooses the bet $\tilde{\beta} \in \Delta$ closest to Newman's $\beta(x)$ with probability $1 - \varepsilon$, $\varepsilon > 0$, but puts probability $\delta\varepsilon/m$ on each of his other possible bets and checking. Then, to satisfy Myerson and Reny's ε -approximate equilibrium, the Caller indifference condition above becomes the ε -indifference condition,

$$\frac{\frac{y}{1-\theta} \Pr(\tilde{\beta} \mid x < y)}{\Pr(\tilde{\beta})} \geq \frac{\tilde{\beta}}{2\tilde{\beta} + 2} - \varepsilon,$$

which, as $\theta, \delta, \varepsilon \rightarrow 0$ and $m \rightarrow \infty$, implies that

$$\begin{aligned} y &= \frac{\beta}{\beta + 1} \\ 1 - \frac{6}{7}\xi &= \frac{\beta}{\beta + 1} \\ \Leftrightarrow \quad \beta &= \frac{2}{5}, \end{aligned}$$

contradicting β 's variability. Hence, Newman's solution is not a basic sequential equilibrium.

However, a modification of Newman's solution *is* a basic sequential equilibrium. To see this, note that the lowest that the Caller's cutoff hand can be in Newman's solution is $1/7$, the lower limit of the interval of hands with which the Better checks. But then, given any bet β , the Better's expected payoff from betting β is the same with any hand below $1/7$; these hands win against the same folded Caller hands and lose against the same calling hands. All of these hands receive expected payoff

$$1 - \frac{6}{7}\xi - \frac{6}{7}\xi(\beta + 1) = -\frac{5}{7}$$

from any bet β . We are thus free to have them mix with the probabilities required for consistency of Caller assessments, i.e.

$$\begin{aligned} \frac{\frac{1/7/\theta]{\theta}}{1-\theta} \Pr\left(\hat{\beta} \mid x < \frac{1}{7}\right)}{\frac{1/7/\theta]{\theta}}{1-\theta} \Pr\left(\hat{\beta} \mid x < \frac{1}{7}\right) + \frac{\theta}{1-\theta}} &\geq \frac{\hat{\beta}}{2\hat{\beta} + 2} + \varepsilon \\ \Leftrightarrow \quad [1/7\theta] \Pr\left(\hat{\beta} \mid x < \frac{1}{7}\right) &\geq \frac{\hat{\beta} + \hat{\varepsilon}}{(\hat{\beta} + 2 - \hat{\varepsilon})}, \quad \hat{\beta} \in \Delta, \end{aligned}$$

where $\hat{\varepsilon} = (2\hat{\beta} + 2)\varepsilon$, and Newman's strategies are otherwise unchanged. Given Δ , there exists a value of θ sufficiently small that we can satisfy these conditions (with the conditional probability measure summing to $1 - \varepsilon$, and the remaining ε measure placed on checking), giving a basic sequential equilibrium. There will be multiple mixed-strategy equilibria satisfying the conditions in this case, but a particularly simple one would involve all hands below $1/7$ using the same mixed strategy $\{p_{\hat{\beta}}\}_{\hat{\beta} \in \Delta}$, placing positive and increasing probability on all bets. This provides a basis for the view of bluffing as a mixed, rather than a pure, strategy.

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